Generalized Twistor Spaces for Hyperkähler and Quaternionic Kähler Manifolds

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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2013

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Abstract

REBECCA GLOVER: Generalized Twistor Spaces for Hyperkähler and Quaternionic Kähler Manifolds
(Under the direction of Justin Sawon)

Generalized complex geometry is a newly emerging field that unites two areas of geometry, symplectic and complex, revealing surprising new aspects in both. Largely motivated by physics, it provides a mathematical context for studying certain string theoretic topics. Since it is a relatively new field, mathematicians are still learning how known geometric objects fit into the realm of generalized complex geometry. One such object is Penrose's twistor space. In this dissertation, we study the generalization of twistor theory for K3 surfaces, hyperkähler, and quaternionic Kähler manifolds.

We use generalized complex geometry to construct a manifold fibered over $\mathbb{C}P^1 \times \mathbb{C}P^1$ that arises from a family of complex and symplectic structures on a K3 surface. We call this manifold a *generalized twistor space*. After proving that it has an integrable generalized complex structure, we describe properties on this space analogous to those in classical twistor theory. We then extend this construction to all hyperkähler manifolds of higher dimension. Finally, we consider the quaternionic Kähler analogue of this generalized twistor space. We produce a candidate for a generalized almost complex structure on the space and conjecture that the structure is integrable.

To my Dad, who helped me discover my love for mathematics.

Acknowledgements

I would first like to thank my advisor, Justin Sawon. His patience with me was unyielding, and I am grateful for his comments, thorough explanations, and for the opportunities he gave me during my time at UNC. I would also like to thank the professors at UNC who taught and guided me, especially Pat Eberlein, Shrawan Kumar, Richard Rimanyi, and Jonathan Wahl. My committee members all deserve thanks for their willingness and constructive comments. I would also like to thank Marco Gualtieri, Gil Cavalcanti, and Nigel Hitchin for clarifications and suggestions. Part of this research was completed while I visited the Hausdorff Institute of Mathematics in Bonn, Germany. I am indebted to them for their incredible hospitality. I am also grateful to the professors at Santa Clara University, especially Rick Scott for his inspiring help and guidance.

I could not have completed these past five years without the immense support of my family and friends. In particular, I would like to thank my fellow graduate students, including Emily and Lauren for yoga sessions and walks, and Swarnava and Dan for encouragement, math help, and coffee breaks. I am also grateful to Jenny for lunch dates and Caroline for commiserating with me on the hardships of graduate school. My family was my cheering section, never doubting my abilities and always encouraging me, even from far away. I would like to thank Katie, Rick and Liz for for their love and support, and my parents for their financial and emotional help in getting me to this point. Last, but certainly not least, I would like to thank Tim for his faith, love, and support. I am eternally grateful to him for consistently believing that I was better than I thought, listening to my daily rants, and saving me from being lost in a world of twistors, horned serpents, and holom subbundles.

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Introduction

This dissertation details the generalization of twistor theory for various quaternionic manifolds in the context of generalized geometry. Generalized complex geometry was first introduced by Hitchin [**28**] and further studied by Gualtieri [**23, 25**] and Cavalcanti [**13**] as a geometric structure that would encompass several types of geometry in one. In particular, it simultaneously generalizes complex and symplectic geometry, two branches of mathematics that had very little interplay for over a century. It is the study of a structure defined on the direct sum of the tangent bundle and the cotangent bundle of a manifold. As Gualtieri explains in his thesis, generalized complex geometry provides a geometric interpretation of several older concepts studied by physicists such as branes, B-fields, and bi-Hermitian data coming from supersymmetry. Since it is a relatively new field of mathematics, many objects already explored for various known geometric structures have not yet been examined in the language of generalized complex geometry. One such area of particular interest is the twistor construction.

Classical twistor theory

Twistor theory was developed about forty years ago by physicist Roger Penrose [**35**] as a map from objects in Minkowski space-time to a 3-dimensional complex projective space called the twistor space. The twistor space construction was successful in transforming differential equations that defined non-linear gravitons to more easily solvable linear differential equations in complex space. Since then, mathematicians have expanded on this construction, developing the theory for all Riemannian 4-manifolds and later in higher dimensions [**1, 41**]. The twistor space construction does not extend 'nicely' to all Riemannian manifolds of dimension $n > 4$. However, for quaternionic manifolds, in particular hyperkähler and quaternionic Kähler manifolds, the twistor theory is quite interesting.

Let M be a hyperkähler manifold. The twistor construction as described in [**27**] begins with a $\mathbb{C}P^1$-family of complex structures on M. The bundle of these structures results in a complex manifold Z called the *twistor space*, where $Z \cong M \times \mathbb{C}P^1$ as smooth manifolds. The fibration $Z \to \mathbb{C}P^1$ is holomorphic. As we will describe in Section 2.1, certain holomorphic data on Z can be constructed from the underlying structure on M. Further, given a manifold Z that fibers holomorphically over $\mathbb{C}P^1$ with these properties, we can reconstruct a hyperkähler manifold M such that Z is the twistor space of M.

The twistor space becomes even more interesting when we examine how vector bundles on M relate to vector bundles on Z. Consider the manifold M as a compact Riemannian manifold with metric g and let V be a vector bundle on M equipped with a connection ∇. Then ∇ is said to be a *Yang-Mills connection* if it is an absolute minimum of the Yang-Mills functional

$$\frac{1}{8\pi^2}\int_X |R|^2\omega$$

where R is the curvature of the connection and ω is the volume form on M [**1**]. On a hyperkähler manifold, every self-dual connection (a connection such that the curvature R is $SU(2)$-invariant) on a complex bundle satisfies this condition and is therefore Yang-Mills [**32**]. These self-dual bundles with connection (V, ∇) on M, under some stability conditions, can be pulled back to holomorphic vector bundles with connection on Z. Further, holomorphic vector bundles on Z that are trivial in the fiber ($\mathbb{C}P^1$) direction can be realized as the pullback of self-dual bundles on M. This bijection between bundles is called the *twistor correspondence*. Since holomorphic vector bundles on a complex manifold are well-understood, this provided a tool for understanding the solutions to the Yang-Mills equations in this setting. In the case where $\dim_{\mathbb{R}} M = 4$, the twistor correspondence was particularly enlightening, as self-dual bundles correspond to instantons, which have physical applications.

When M is quaternionic Kähler, similar results hold. In particular, the twistor space Z is a complex manifold, even though in general M does not have a global complex structure. Additionally, the twistor space is Kähler for many examples of quaternionic Kähler manifolds, such as $\mathbb{H}P^n$ [**7**]. The twistor correspondence described above holds for these manifolds as well [**11**]. As twistor spaces in these geometric settings have proven to be so interesting in the past, it is a natural next step to see how they fit into more modern geometries, such as generalized complex geometry.

Generalized complex geometry

Let M be a $2n$-dimensional manifold. The bundle $TM \oplus T^*M$ has a natural non-degenerate inner product on it, as well as a bracket called the Courant bracket. A generalized complex structure is defined as an endomorphism

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$$

such that $\mathcal{J}^2 = -1$ and $\mathcal{J}$ is orthogonal with respect to the inner product with some integrability conditions; $\mathcal{J}$ is integrable when the $+i$-eigenbundle L is involutive with respect to the Courant bracket. Generalized complex structures can arise from complex or symplectic structures, which we will describe in more detail in Chapter 1.

This relatively new field has shed light on several topics in both physics and geometry. It has provided a mathematical framework for studying string theoretic topics such as supersymmetric sigma models and branes [**31, 33**]. Since mirror symmetry, a rich mathematical subject with physical applications, provides a link between complex and symplectic geometry, it is natural to ask how generalized complex geometry fits into this duality between manifolds. In fact, mathematicians and physicists have been studying mirror symmetry in this context with some enlightening results [**4, 16**]. Further, generalized complex geometry has revealed some interesting new aspects of other areas of geometry such as Poisson varieties and SKT structures. See, for example [**12, 24, 29, 36**].

Main results: the generalized twistor space

In this dissertation, we examine the role that generalized complex geometry plays in twistor theory. Since hyperkähler and quaternionic Kähler manifolds have such rich symplectic and complex data, we focus on their twistor spaces within this generalized geometry setting. Our main results are restricted to the twistor theory for K3 surfaces and their higher-dimensional analogues, hyperkähler manifolds, although we conjecture that a similar result holds for quaternionic Kähler manifolds.

Let M be hyperkähler and let Z be its twistor space. Consider the fiber product

$$\mathcal{X} = Z \times_M Z.$$

We call this manifold the *generalized twistor space* of M. It is a $\mathbb{C}P^1 \times \mathbb{C}P^1$-bundle over M such that every point $p \in \mathcal{X}$ defines a generalized complex structure on M. In Chapter 3, we prove the following theorem, first for K3 surfaces in four dimensions (Theorem 3.2.1) and then for all hyperkähler manifolds of dimension $4n$ (Theorem 3.5.2).

THEOREM 0.0.1. *Let M be a hyperkähler manifold. Then the generalized twistor space has an integrable generalized complex structure.*

We then prove that this generalized twistor space has certain properties analogous to those of the twistor space in the classical setting (Theorem 3.5.3).

THEOREM 0.0.2. *Let $\mathcal{X}$ be the generalized twistor space of a hyperkähler manifold M. Then:*

(1) $\mathcal{X}$ is a smooth fiber bundle

$$\pi : \mathcal{X} \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

and a reduction of generalized complex manifolds.

(2) The bundle admits a family of sections that are generalized complex factor submanifolds, each with generalized normal bundle isomorphic to

$$\mathbb{C}^{2n} \otimes (\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)).$$

(3) There is a pure spinor representing the generalized complex structure on $\mathcal{X}$ given by

$$\Psi = (-i(\alpha - \beta))^n \exp\left(\tfrac{i}{2}\left(\tfrac{i(\alpha+\beta)}{\alpha-\beta}\omega_I + \tfrac{i}{\alpha-\beta}\sigma - \tfrac{i\alpha\beta}{\alpha-\beta}\bar{\sigma}\right)\right) \wedge (d\alpha \wedge d\beta),$$

where $\sigma = \omega_J + i\omega_K$ that defines a structure of complex type along the diagonal, and of type 2 everywhere else.

(4) $\mathcal{X}$ has a real structure τ compatible with the above and inducing an antipodal map on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

The resulting generalized complex manifold $\mathcal{X}$ is an interesting example of a generalized complex structure in the sense that it does not come from a symplectic or a complex structure. Further, this example does not seem to arise from previously known constructions of generalized complex manifolds such as Poisson deformations [**25**] or blow-ups [**15**]. Examples such as these are essential to the development of this new field. The generalized twistor space also provides a first step towards developing a generalization of twistor theory, with the potential to help us understand more about the bundles on this space through a type of 'generalized twistor correspondence'.

At the end of this dissertation, we provide the initial construction for a generalized twistor space of a quaternionic Kähler manifold. For a manifold M that is quaternionic Kähler, we define the space and give a candidate for a generalized almost complex structure on it. We then conjecture that this defines an integrable structure. The construction of this candidate depends on a new proof of the integrability of the complex structure on the classical twistor space.

Overview

The first chapter of the thesis explains the main objects needed for generalized complex geometry, following [**25**]. We introduce the natural inner product on the space $T \oplus T^*$. We then develop the bracket on this space and describe a natural structure called a Courant algebroid that arises from this bracket. Section 1.2 builds on the previous section as we introduce a generalized almost complex structure on $T \oplus T^*$, and give conditions for it to be integrable. We discuss the various ways a generalized complex structure can be defined and describe an integer-valued function over the manifold called type. Finally, we give examples and state the main theorems from generalized complex geometry needed for this dissertation.

In Chapter 2, we define and explain the properties of of hyperkähler manifolds and K3 surfaces. We then detail their twistor theory. In Section 2.2, we provide the background to study generalized complex geometry on K3 surfaces and hyperkähler manifolds. We present another way to define a generalized complex structure, by an element of the spin representation on the Clifford algebra $\mathrm{CL}(T_p \oplus T_p^*)$. We describe the almost complex structure on the twistor space using this notion of spinors and give a proof that this structure is integrable. In the last two sections, we define generalized Kähler and generalized K3 structures on a manifold.

In Chapter 3, we present our main results. We build the generalized twistor space of a K3 surface M by constructing a family of generalized complex structures on M in three ways, and prove that these constructions are equivalent. We then prove that the generalized twistor space for a K3 surface is an integrable generalized complex manifold of dimension 8 using the pure spinor defining it. In Sections 3.3 and 3.4, we prove that there are analogous properties on the generalized twistor space to the classical case. Finally, we extend our results to all hyperkähler manifolds of dimension $4n$.

In Chapter 4, we define quaternionic Kähler manifolds and explain their twistor theory. We describe the contact geometry of the twistor space. We then present a new proof of the integrability of the complex structure on the twistor space for a 4-dimensional quaternionic Kähler manifold by using the contact form and generalized complex geometry. At the end of the chapter, we extend these results to all quaternionic Kähler manifolds of dimension $4n$.

In the final chapter of this dissertation, we define the generalized twistor space for a quaternionic Kähler manifold. We provide a generalized almost complex structure on this space, and conjecture that it is integrable. Lastly, we present a candidate for the pure spinor representing this generalized almost complex structure for a four-dimensional quaternionic Kähler manifold, in the hopes that it will aid us in eventually proving integrability of this structure.

CHAPTER 1

Generalized complex geometry

Generalized complex geometry is a generalization of complex and symplectic geometry. In this chapter, we give an overview of generalized complex geometry as described by Gualtieri [**25**]. We detail the basic definitions and theorems, providing a context for studying generalized twistor theory.

1.1. Preliminaries

Let M be a real n dimensional manifold. There is a natural inner product on $T \oplus T^*$ (which we denote by $\mathbb{T}$), the direct sum of the tangent bundle and its dual, which is symmetric and non-degenerate, given by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$$

for $X, Y \in C^\infty(T)$, $\xi, \eta \in C^\infty(T^*)$. This inner product has signature (n,n) and $\mathbb{T}$ has structure group $O(n,n)$, a non-compact orthogonal group. Further, at each point $p \in M$, $\mathbb{T}_pM$ has a canonical orientation, coming from the orientation on $\mathbb{R}$ as we can show that $\wedge^{2n}(\mathbb{T}_pM) \cong \mathbb{R}$. Thus, the structure group preserving the inner product and this orientation at a given point $p \in M$ is $SO(n,n)$. We can explain the symmetries of this space by looking at the Lie algebra:

$$\mathfrak{so}(\mathbb{T}_pM) = \{R \;\mid\; \langle Rx, y\rangle + \langle x, Ry\rangle = 0 \;\;\forall\;\; x, y \in \mathbb{T}_pM\}.$$

We decompose this R as

$$R = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix},$$

where $A \in End(T_pM)$, $B : T_pM \to T_p^*M$, and $\beta : T_p^*M \to T_pM$, $B^* = -B$ and $\beta^* = -\beta$. Thus we can think of $B \in C^\infty(\wedge^2 T^*)$ as a 2-form and $\beta \in C^\infty(\wedge^2 T)$ as a bivector, and

$$\mathfrak{so}(\mathbb{T}_pM) = End(T_pM) \oplus \wedge^2 T_p^*M \oplus \wedge^2 T_pM.$$

The symmetries that we will focus on in this thesis are the 2-forms B and the bivectors β, which give rise to what are called the B-field transform and the β-transform. The B-field transform acts by $\exp(B)$ on $C^\infty(\mathbb{T})$ by $X + \xi \mapsto X + \xi + i_X B$, thus acting as a shear transformation, fixing projections onto T and shearing in the T^* direction. On the other hand, the β-transform acts by $\exp(\beta)$ on $C^\infty(\mathbb{T})$ by $X + \xi \mapsto X + \xi + i_\xi \beta$, fixing projections onto T^* and shearing in the T direction.

1.1.1. The Courant bracket. Using this inner product, we can develop a bracket on this space, which is essentially the extension of the Lie bracket to $\mathbb{T}$. This gives $\mathbb{T}$ the structure of a Courant algebroid, which we define below.

Definition 1.1.1. A *Courant algebroid* E is a real vector bundle over M equipped with a bracket $[\cdot,\cdot]$ defined on $C^\infty(E)$, a nondegenerate inner product $\langle\cdot,\cdot\rangle$ and a bundle map $\pi : E \to T$ such that the following conditions are satisfied for $e_1, e_2 \in C^\infty(E)$:

(1) $[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$.
(2) $\pi([e_1,e_2]) = [\pi(e_1),\pi(e_2)]$.
(3) $[e_1, fe_2] = f[e_1,e_2] + (\pi(e_1)f)e_2, \quad f \in C^\infty(M)$.
(4) $\pi(e_1)\langle e_2,e_3\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$.
(5) $[e_1,e_1] = \frac{1}{2}\pi^* d\langle e_1,e_1\rangle$.

A Courant algebroid is called *exact* if $E \cong \mathbb{T}$. In this case, the Courant bracket is defined for $[e_1,e_2] \in C^\infty(E)$ as

$$[e_1,e_2]\cdot\phi = [[d,e_1\cdot],e_2\cdot]\phi \quad \forall \phi \in \Omega^\bullet(M).$$

Note that for $e_1 = X + \xi, e_2 = Y + \eta \in C^\infty(T \oplus T^*)$, this yields the expression

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi.$$

Here, the action on $\phi \in \Omega^\bullet(X)$ is given by

$$(X + \xi) \cdot \phi = i_X \phi + \xi \wedge \phi.$$

Technically, this is called the *Dorfman bracket* on $\mathbb{T}$. The skew-symmetrized version of this bracket is called the Courant bracket. However, we claim that this distinction will not matter in defining a generalized complex structure. In fact, in Section 1.2 we will consider a subbundle of E such that the Dorfman bracket is skew-symmetric on this subbundle.

For the remainder of this paper, we will consider E to be exact. Therefore, we use the terms E and $\mathbb{T}$ interchangeably.

1.1.2. Symmetries of the Courant bracket. Recall that the Lie bracket on sections of the tangent bundle has symmetry group defined by the diffeomorphisms of the manifold M. When we consider the Courant bracket on sections of $\mathbb{T}$, we actually get a larger symmetry group.

PROPOSITION 1.1.1. *(Gualtieri, [**25**]) Let F be an orthogonal automorphism of $\mathbb{T}$, covering the diffeomorphism $\phi : M \to M$, and preserving the Courant bracket. Then $F = \phi_* e^B$ for a unique d-closed 2-form $B \in \Omega^2(M)$.*

By this proposition, symmetries of the Courant bracket are given by both diffeomorphisms and B-field transforms.

1.2. Generalized complex structures

We now define an endomorphism on this space called a generalized complex structure using this natural inner product and bracket on $\mathbb{T}$.

1.2.1. Definition.

DEFINITION 1.2.1. A *generalized almost complex structure* on M is an endomorphism

$$\mathcal{J} : E \to E$$

of an exact Courant algebroid $E \cong \mathbb{T}$ which is orthogonal with respect to the natural inner product and such that $\mathcal{J}^2 = -1$.

A generalized almost complex structure becomes a *generalized complex structure* (is integrable) when its $+i$-eigenbundle $L \subset E \otimes \mathbb{C}$ is involutive, i.e. closed under the Courant bracket. This $+i$-eigenbundle has a special structure due to the nature of the inner product and the Courant bracket on it called a *Dirac structure*.

DEFINITION 1.2.2. A *complex Dirac structure* $L \subset E \otimes \mathbb{C}$ of an exact Courant algebroid is a maximal isotropic subbundle that is closed under the Courant bracket, or integrable. This Dirac structure arises as the $+i$-eigenbundle of a generalized complex structure $\mathcal{J}$.

In fact, under certain conditions, a Dirac structure (a closed maximal isotropic subbundle of $\mathbb{T}_{\mathbb{C}}$) can determine a generalized complex structure on a manifold.

PROPOSITION 1.2.1. *A generalized complex structure is equivalent to a complex Dirac structure $L \subset E \otimes \mathbb{C}$ such that $L \cap \bar{L} = \{0\}$.*

The above proposition gives us two ways to define a generalized complex structure; by the endomorphism $\mathcal{J}$ and by the Dirac structure L. A third alternative to defining it comes from the notion of spinors, which we discuss in Chapter 2.

REMARK. One of the known obstructions to having a generalized complex structure on a manifold is that the manifold must be even dimensional. To see this, take a point $p \in M$ and consider $x \in E$ through p such that $\langle x, x\rangle = 0$. Then $\langle \mathcal{J}x, \mathcal{J}x\rangle = 0$, so the pair $\{x, \mathcal{J}x\}$ span an isotropic subspace S of E_p. If we continue to take $x' \in S^{\perp}$ and add the pairs $\{x', \mathcal{J}x'\}$ to the set S until $S = S^{\perp}$, we end up with a maximal isotropic subspace S of even dimension. So the dimension of M is even.

1.2.2. Type. There is an integer-valued function on the manifold M that can be used to distinguish various maximal isotropic subbundles $L \subset E \otimes \mathbb{C}$. When L is a Dirac structure for a generalized complex structure, this function can be defined in the following two ways.

DEFINITION 1.2.3.

(1) The *type* of a maximal isotropic L_p is the codimension of its projection onto T_pM. There are two connected components in the space of maximal isotropics of $\mathbb{T}M$, determined by whether the type is even or odd.

(2) The *type* of a generalized complex structure $\mathcal{J}$ is the function

$$type(\mathcal{J}) : M \to \mathbb{Z}$$

given by

$$type(\mathcal{J})(p) = \frac{1}{2} dim_{\mathbb{R}} T_p^*M \cap \mathcal{J}(T_p^*M).$$

Possible values of this function could be anything ranging from 0 to n, where n is half of the real dimension of M.

Note that these two definitions of type align. Further, although type must be of fixed parity throughout the manifold, it can jump by an even number from point to point.

REMARK. The B-field transform discussed earlier introduces an interesting fact related to this function: a maximal isotropic that is transformed by a B-field to another does not change type. The B-field acts on a generalized complex structure as

$$\exp(B) \cdot \mathcal{J} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}.$$

Since we know by Section 1.1.2 that B-fields leave the Courant bracket invariant, this useful fact helps determine integrability of a generalized complex structure.

1.2.3. Examples. The following are the two most basic examples of generalized complex structures. As we shall see later, these examples are useful in constructing more complicated structures.

Example 1.2.4. (Complex type) Suppose J is an almost complex structure on M. Consider the endomorphism of $\mathbb{T}$

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

Note that $\mathcal{J}_J^2 = -1$ and the $+i$-eigenbundle is the subset

$$L_J = T^{0,1} \oplus T^{1,0^*}.$$

This subbundle is involutive if J is integrable. Further, the type of $\mathcal{J}_J$ (called complex type) is constant throughout the manifold,

$$\text{type}(\mathcal{J}_J) \equiv n = \frac{1}{2} dim_{\mathbb{R}}(M).$$

In fact, any generalized complex structure of type n is the B-field transform of a complex type structure. We distinguish these two by calling $\mathcal{J}_J$ a generalized complex structure of *complex type* and the B-field transform of it *type* n.

Example 1.2.5. (Symplectic type) Suppose ω is an almost symplectic structure on M and consider the endomorphism of $\mathbb{T}$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Notice again that $\mathcal{J}_\omega^2 = -1$ and we can calculate the $+i$-eigenbundle as

$$L_\omega = \{X - i\omega(X) \ \mid \ X \in T \otimes \mathbb{C}\},$$

which is integrable if and only if $d\omega = 0$, i.e. ω is symplectic. The type of $\mathcal{J}_\omega$ (called symplectic type) is again constant,

$$\operatorname{type}(\mathcal{J}_\omega) \equiv 0.$$

As before, any generalized complex structure of type 0 is the B-field transform of a structure of symplectic type.

In each of these examples, notice that we use that the original structure is complex (respectively symplectic), in order to prove that the generalized complex structure is integrable.

We conclude this section with a theorem by Gualtieri in his thesis that describes the composition of a generalized complex structure $\mathcal{J}$. As we will see in Proposition 2.2.1, at any given point, a generalized complex structure of type k is equivalent to the direct sum of a complex structure of complex dimension k and a symplectic structure of real dimension $2n - 2k$. Gualtieri's Generalized Darboux Theorem proves a stronger result. A regular point is defined as a point $p \in M$ where the type of the generalized complex structure is constant in a neighborhood of p.

THEOREM 1.2.6. *(Generalized Darboux Theorem, Gualtieri* [**25**]*) Any regular point of type k in a generalized complex manifold has a neighborhood which is equivalent to the product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $(\mathbb{R}^{2n-2k}, \omega_0)$.*

In other words, a generalized complex manifold in neighborhood of a regular point can be seen as a holomorphic foliation with symplectic leaves. In [**2**], Bailey proves even more about the local structure of a generalized complex manifold. We note first that as explained by Gualtieri, deformations of generalized complex structures of complex type are determined by complex deformations, B-field and β-transforms. These β-transforms are called Poisson deformations.

Theorem 1.2.7. *(Bailey, [**2**]) Suppose M is a generalized complex manifold that is of complex type along some locus $P \subset M$. Then for any point $p \in P$, there exists a neighborhood of p such that the generalized complex structure $\mathcal{J}$ is equivalent to a Poisson deformation of a complex structure.*

Note that the Poisson tensor β that defines the deformation in the theorem above vanishes along the complex locus.

1.2.4. Interpolating between Symplectic and Complex Types. Let M be a manifold of real dimension $4n$ with complex structure I and holomorphic symplectic structure $\sigma = \omega_J + i\omega_K$, such that σ is a nondegenerate closed $(2,0)$-form (e.g. a hyperkähler manifold or K3 surface, see Chapter 2). Recall that we can build generalized complex structures $\mathcal{J}_I$, a complex type $2n$ structure, or $\mathcal{J}_{\omega_J}$, a symplectic type 0 structure. This example will illustrate a family of generalized complex structures that interpolate between this complex type and symplectic type.

Note that $\omega_J I = I^*\omega_J$, which means

$$\mathcal{J}_{\omega_J}\mathcal{J}_I = -\mathcal{J}_I\mathcal{J}_{\omega_J}.$$

Then the generalized complex structures of complex type and symplectic type anticommute, and we can form the one-parameter family of generalized almost complex structures

$$\mathcal{J}_t = (\cos t)\mathcal{J}_I + (\sin t)\mathcal{J}_{\omega_J}, \quad t \in \left[0, \tfrac{\pi}{2}\right].$$

This is a generalized almost complex structure that is integrable for every $t \in [0, \frac{\pi}{2}]$. We can show this by computing that for $t \in (0, \frac{\pi}{2}]$, $\mathcal{J}_t$ is a B-field transform of the symplectic structure determined by $\omega = (\csc t)\omega_J$, where $B = -(\cot t)\omega_K$. Note that $\mathcal{J}_{\frac{\pi}{2}} = \mathcal{J}_{\omega_J}$ and $\mathcal{J}_0 = \mathcal{J}_I$. Hence, this is a family interpolating between a complex structure and a symplectic structure.

Note. For $k = 1$, as in the case of K3 surfaces, the structures here jump between complex type (type 2) and symplectic type (type 0). A K3 surface may have a generalized complex structure of type 1, but it cannot be written in this form.

This example will be important in Chapter 3 to help us determine an $S^2 \times S^2$-family of generalized complex structures on a K3 surface M. However, first we need to explain the geometry of K3 surfaces and hyperkähler manifolds.

CHAPTER 2

Hyperkähler geometry and K3 surfaces

A Kähler manifold is a complex Riemannian manifold with symplectic data; namely, a non-degenerate d-closed 2-form compatible with the complex structure. Hyperkähler manifolds are a quaternionic extension of this in the sense that they contain a triple of complex structures satisfying certain quaternionic relations and corresponding Kähler forms. Some of the first examples were found by physicists studying supersymmetric sigma models, and later described geometrically using Penrose's twistor space for hyperkähler manifolds in four dimensions, such as K3 surfaces. In [**27**], Hitchin, Karlhede, Lindström, and Roček (HKLR) give a general account of the twistor space construction for hyperkähler manifolds in higher dimensions, and describe how some of these examples can be realized in this setting.

In this chapter, we define hyperkähler manifolds and describe their twistor theory as in [**27**], and introduce the setting for extending this to generalized complex geometry. In the first section, we provide basic definitions and develop the twistor space on a hyperkähler manifold. The second section details the theory needed for defining generalized complex structures on a K3 surface. We describe a third way to describe a generalized complex structure, through the notion of a pure spinor, a mixed form whose annihilator is the Dirac structure L. We then develop the theory of generalized Kähler geometry, and from this, define generalized K3 structures, thus laying the groundwork for generalized twistor theory in Chapter 3.

2.1. Twistor theory for K3 surfaces and hyperkähler manifolds

This section details the fundamentals of classical twistor theory for K3 surfaces, and more generally, hyperkähler manifolds. We begin with some basic definitions and examples of each of these geometric objects.

2.1.1. Definitions.

Definition 2.1.1. A *Kähler manifold* is a complex manifold with complex structure I and compatible Riemannian metric g such that

$$\omega(X,Y) := g(IX,Y)$$

is d-closed, i.e. $d\omega = 0$.

There are many known examples of Kähler manifolds, in particular $\mathbb{C}^n$ with the standard Euclidean metric, $\mathbb{C}P^n$ with the Fubini-study metric, and complex tori T^n with the induced metric from $\mathbb{C}^n$.

Definition 2.1.2. A *K3 surface* M is a compact connected complex surface with

$$H^1(M, \mathcal{O}_M) = 0$$

and trivial canonical bundle, i.e.

$$\Omega^2_M = K_M = \mathcal{O}_M.$$

Well-known examples of K3 surfaces include the double cover of $\mathbb{C}P^2$ branched along a sextic curve and Kummer surfaces [**3**].

Definition 2.1.3. A *hyperkähler manifold* (M,g) is a Riemannian manifold of (real) dimension $4n$ with three complex structures I, J, K and quaternionic relations

$$I^2 = J^2 = K^2 = -Id \ \text{ and } \ IJ = -JI = K.$$

Further, these complex structures are Kähler with respect to the metric g, so that there exist closed 2-forms ω_I, ω_J, and ω_K that are Kähler for I, J, and K, respectively.

NOTE. The definition above implies that a hyperkähler manifold is a complex manifold with a holomorphic symplectic form. In the notation given above, if we choose I as the complex structure for M, this holomorphic symplectic form is given by

$$\sigma = \omega_J + i\omega_K.$$

It is not easy to find examples of hyperkähler manifolds. Some of the most common examples arise as moduli spaces of vector bundles on K3 and abelian surfaces (e.g. the Hilbert scheme of n-points on a K3 surface). Further, due to the Calabi Yau Theorem [**45**] and a result of Bochner [**8**], K3 surfaces have a hyperkähler structure. In fact, K3 surfaces are the only non-flat compact examples of hyperkähler manifolds in dimension 4. They are some of the most interesting examples, and for this reason, we will prove much of the theory of generalized twistor spaces for K3 surfaces before extending it to higher dimensions. However, first we must develop classical twistor theory for hyperkähler manifolds.

2.1.2. Twistor theory. In this section, we follow [**27**] to define the twistor space of a hyperkähler manifold. Let M be a hyperkähler manifold of real dimension $4n$ and metric g. Then, as outlined above, M has complex structures I, J, and K and corresponding Kähler structures ω_I, ω_J, and ω_K. For any $(a, b, c) \in S^2$, $\lambda = aI + bJ + cK$ is a complex structure on M, as

$$(aI + bJ + cK)^2 = -1.$$

Further, for each complex structure λ defined by a point $(a, b, c) \in S^2$, there exists a corresponding holomorphic symplectic form ω_λ. This gives a family of complex structures on M, parametrized by S^2. We consider this S^2 as $\mathbb{C}P^1$ by patching together two copies of the complex plane and call them $U, \tilde{U}$. Let $\zeta, \tilde{\zeta}$ be coordinates for $U, \tilde{U}$, respectively related by $\tilde{\zeta} = \zeta^{-1}$ on $U \cap \tilde{U} \cong \mathbb{C} \setminus \{0\}$. Then the coordinate ζ is defined by the map

$$(a, b, c) = \left(\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{-i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} \right)$$

and similarly for $\tilde{\zeta}$.

DEFINITION 2.1.4. The *twistor space* Z of M is the smooth product $M \times \mathbb{C}P^1$ with almost complex structure defined on the tangent space $T_p Z = T_m M \oplus T_\zeta \mathbb{C}P^1$ for $p = (m, \zeta) \in Z$ as

$$\mathbb{I} = \left(\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}} I + \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}} J - \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} K, I_\zeta \right),$$

where I_ζ is the standard complex structure on $\mathbb{C}P^1$.

In [**27**], HKLR show that this structure $\mathbb{I}$ is an integrable complex structure on Z. The map $\pi : Z \to \mathbb{C}P^1$ gives a holomorphic fiber bundle with holomorphic sections such that the normal bundle is isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$. These sections $\{m\} \times \mathbb{C}P^1$ are called *twistor lines*. Further, on each fiber $\pi^{-1}(\zeta)$, there exists a holomorphic symplectic form ω_ζ given by

$$\omega_\zeta = (\omega_J + i\omega_K) + 2\zeta\omega_I - \zeta^2(\omega_J - i\omega_K). \tag{2.1}$$

Note that this form is holomorphic and quadratic in ζ. Hence, it is a holomorphic section of the vector bundle $\wedge^2 T_F^*(2)$, where $T_F(2)$ is the tangent bundle along the fibers, twisted by $\mathcal{O}(2)$. Let $\tau : Z \to Z$ be the real structure on Z induced by the antipodal map on the unit sphere, a map that sends I_ζ to $-I_\zeta$. All of the holomorphic data defined above are compatible with τ. The following theorem shows that given a complex manifold X with the same holomorphic information, we can reconstruct a hyperkähler manifold M such that X is the twistor space for M.

THEOREM 2.1.5. *(HKLR, [**27**]) Let X be a complex manifold of dimension $2n + 1$ such that:*

(1) X is a holomorphic fiber bundle $\pi : X \to \mathbb{C}P^1$ over the projective line.

(2) the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$.

(3) there exists a holomorphic section ω of $\wedge^2 T_F^*(2)$ defining a symplectic form on each fiber.

(4) X has a real structure τ compatible with the above three conditions and inducing the antipodal map on $\mathbb{C}P^1$.

Then the parameter space of real sections is a real $4n$-dimensional manifold with a natural hyperkähler metric for which X is the twistor space.

We refer the reader to [**27**] for the proof of this theorem.

2.2. Generalized complex geometry on K3 surfaces

Since K3 surfaces have such rich complex and symplectic data, it is a natural next step to examine the generalized geometry of these objects. In order to do this, however, we must present an alternate way of defining generalized complex structures, by pure spinors. A spinor is a mixed form such that the space $\mathbb{T}$ acts on it in a natural way. In fact, we will see that a correct choice of spinor can determine a Dirac structure. In this section, we cite results from Chevalley [**18**] and follow Gualtieri's discussion [**25**] on how they relate to generalized complex geometry.

2.2.1. Spinors of generalized complex structures. The inner product on $\mathbb{T}$ gives rise to a bundle of Clifford algebras $\mathrm{CL}(\mathbb{T})$ defined on the smooth sections of $\mathbb{T}$ by

$$(X+\xi)^2 = \langle X+\xi, X+\xi\rangle \quad \text{for} \quad X+\xi \in C^\infty(\mathbb{T}).$$

Each smooth section of this bundle, $\mathcal{C}$, has a natural spin representation on $C^\infty(\wedge^\bullet T_{\mathbb{C}}^* M)$ such that

$$(X+\xi)\cdot\phi = i_X\phi + \xi\wedge\phi.$$

The elements ϕ for this spin representation $C^\infty(\wedge^\bullet T_{\mathbb{C}}^* M)$ are called *spinors*.

Definition 2.2.1. At each point $\phi \in C^\infty(\wedge^\bullet T^*_{\mathbb{C}} M)$, a nonzero spinor, its *nullspace* at a point $p \in M$ is defined by

$$L_{\phi_p} = \{v \in \mathbb{T}_p M \otimes \mathbb{C} \quad | \quad v \cdot \phi_p = 0\}.$$

A spinor ϕ_p is called *pure* when L_{ϕ_p} is maximally isotropic in $\mathbb{T}_p M \otimes \mathbb{C}$.

By results outlined by Chevalley [**18**], given a vector space V every maximal isotropic subspace $L \subset (V \oplus V^*) \otimes \mathbb{C}$ can be represented by a unique line $K_L \subset \wedge^\bullet V^*_{\mathbb{C}}$ of pure spinors such that L annihilates K_L. A generator for such a spinor line K_L, is a *pure spinor*, ϕ. Thus, given a Dirac structure L_p at any given point $p \in M$, we can pick a representative pure spinor ϕ_p that defines it. In fact, we can define any generalized complex structure on a manifold by a pure spinor ϕ. This mixed form may change throughout the manifold as the generalized complex structure changes type. However, based on Theorem 1.2.6, we note that in a neighborhood of constant type, we can write a pure spinor for a generalized complex structure as the product of a spinor for a complex type structure (along the holomorphic leaf space) and a spinor for a symplectic-type structure (on the leaves of the foliation). The following two propositions help to determine integrability of a generalized almost complex structure defined by a pure spinor.

Proposition 2.2.1. *(Gualtieri, [**25**]) At a given point $p \in M$, with Dirac structure $L \subset E \otimes \mathbb{C}$ for a generalized complex structure, the pure spinor line through p, K_p, can be expressed by a single generator, defined by*

$$\phi_p = \exp(B + i\omega)\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_k$$

where B and ω are the real and imaginary components of a complex 2-form in $\wedge^2 T^* \otimes \mathbb{C}$ and $(\theta_1, \ldots, \theta_k)$ form a basis for $L \cap (T^* \otimes \mathbb{C})$.

In fact, in a neighborhood of $p \in M$, where M is a generalized complex manifold, there always exists a spinor that we can write as above. However, the rank of $L \cap (T^* \otimes \mathbb{C})$ may increase or decrease throughout the neighborhood, as the generalized complex structure changes type.

Proposition 2.2.2. *(Gualtieri, [**25**]) Locally, a generalized almost complex structure given by a pure spinor ϕ is integrable if and only if there exists a local section $X+\xi \in C^\infty(\mathbb{T}_\mathbb{C})$ such that*

$$d\phi = (X+\xi)\cdot\phi.$$

Example 2.2.3. The generalized complex structure coming from a symplectic structure ω as in Example 1.2.5 has spinor $\phi = e^{i\omega}$. On the other hand, for a generalized complex structure induced by complex structure J as in Example 1.2.4, $\phi = \Omega$, where Ω is a holomorphic n-form. In the example given in Section 1.2.4, the spinor for the generalized complex structure $\mathcal{J}_t$ for $\theta \neq 0$ is

$$\phi = \exp(-(\cot t)\omega_K + i(\csc t)\omega_J)$$

where as for $\mathcal{J}_0$, it is $\phi = \omega_J + i\omega_K$, the holomorphic symplectic form for to the complex structure I.

For each of these spinors, $d\phi = 0$, however this is not always the case, as we will see in Chapter 4. A d-closed global spinor ϕ for a generalized complex structure is defined as a *generalized Calabi-Yau structure* [**28**].

Note that the value of the type function at a point p of a generalized complex structure can be found in the spinor representing the structure, as the number k in Proposition 2.2.1. In the example above, a symplectic-type generalized complex structure has type 0 everywhere, as there is no holomorphic component. Conversely, a complex-type generalized complex structure has no symplectic component, and $k = n$, as noted earlier.

2.2.2. The Twistor Space as a generalized complex manifold. In this section, we describe the twistor space Z for a hyperkähler M as a generalized complex manifold. We will use the notion of spinors defined in the previous section.

As in Section 2.1.2, let Z be the smooth product $M \times \mathbb{C}P^1$ with almost complex structure defined on the tangent space $T_pZ = T_mM \oplus T_\zeta\mathbb{C}P^1$ for $p = (m, \zeta) \in Z$ as

$$\mathbb{I} = \left(\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}} I + \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}} J + \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} K, I_\zeta \right),$$

where I_ζ is the standard complex structure on $\mathbb{C}P^1$. From [**27**], we know that for every $\zeta \in \mathbb{C}P^1$, there exists a holomorphic symplectic form

$$\omega_\zeta = (\omega_J + i\omega_K) + 2\zeta\omega_I - \zeta^2(\omega_J - i\omega_K)$$

on M compatible with the above complex structure on T_mM.

Now consider the twistor space Z with generalized almost complex structure $\mathcal{J}_{\mathbb{I}}$. In the following proposition, we define a spinor for this generalized almost complex structure and prove that the structure is integrable. We use the holomorphic symplectic form to define this spinor.

PROPOSITION 2.2.4. *Let M be a hyperkähler manifold of dimension $4n$ with twistor space Z. Then a spinor for the generalized almost complex structure $\mathcal{J}_{\mathbb{I}}$ is*

$$\phi = ((1 - \zeta^2)\omega_J + 2\zeta\omega_I + i(1 + \zeta^2)\omega_K)^n \wedge d\zeta$$

and further, this structure is integrable.

PROOF. It is easy to see that ϕ is a spinor for the generalized almost complex structure $\mathcal{J}_{\mathbb{I}}$, since a spinor for a generalized complex structure of complex type is simply the holomorphic $(2n + 1, 0)$ form for the underlying complex structure. In order to prove integrability, we use Proposition 2.2.2. Note that

$$d\phi = n\omega_\zeta^{n-1} \wedge d\zeta \wedge (-2\zeta\omega_J + 2\omega_I + 2i\zeta\omega_K) \wedge d\zeta = 0,$$

since $d\omega_I = d\omega_J = d\omega_K = 0$. Hence, the spinor ϕ defines an integrable generalized complex structure. $\square$

This construction will be useful in generalizing the twistor space in Chapter 3. First we must explain a little more background material.

2.2.3. Generalized Kähler structures. As Gualtieri explains in [**23**], we can generalize Kähler manifolds within the context of generalized complex geometry.

DEFINITION 2.2.5. A *generalized almost Kähler structure* on M^{2n} is a pair of commuting generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that

$$\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$$

defines a positive definite metric on $\mathbb{T}M$. The pair $\mathcal{J}_1, \mathcal{J}_2$ determines an (integrable) generalized Kähler structure if $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable as generalized complex structures.

EXAMPLE 2.2.6. Given a usual Kähler structure (I, ω), the pair of generalized complex structures $\mathcal{J}_I, \mathcal{J}_\omega$ determine a generalized Kähler structure on M such that the metric is defined by

$$\mathcal{G} = -\mathcal{J}_I\mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix},$$

where g is the compatible Riemannian metric on M.

Recall that the bundle $\mathbb{T}M$ has structure group $O(2n, 2n)$. Further, we can reduce to its maximal compact subgroup $O(2n) \times O(2n)$, which is equivalent to choosing a $2n$-dimensional subbundle C_+ that is positive definite with respect to the inner product. Let C_- be the negative definite orthogonal complement to C_+. Note that $\mathbb{T}M \cong C_+ \oplus C_-$, and this defines a positive definite metric

$$\mathcal{G} = \langle , \rangle|_{C_+} - \langle , \rangle|_{C_-}$$

on $\mathbb{T}M$. If this metric commutes with a generalized complex structure, $\mathcal{G}\mathcal{J} = \mathcal{J}\mathcal{G}$, then $(\mathcal{J}, \mathcal{G}\mathcal{J})$ is a generalized Kähler structure on M.

This determines what is called a bi-Hermitian structure on M. Suppose $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{G})$ is a generalized Kähler structure on M. Then C_+, C_- are stable under $\mathcal{J}_1, \mathcal{J}_2$. Further, considering the map $\pi : \mathbb{T}M \to TM$, there exist isomorphisms

$$\pi : C_\pm \xrightarrow{\sim} TM.$$

Using these isomorphisms, we can project $\mathcal{J}_1$ from $C_\pm$ to TM to induce two almost complex structures J_+, J_- on TM. Further, we get a Riemannian metric g and a two form b on M such that $C_\pm$ is the graph of the map

$$b \pm g : TM \to T^*M.$$

Definition 2.2.7. An *almost bi-Hermitian structure* on a manifold M is a pair of almost complex structures $J_\pm$ with a compatible Riemannian metric g and a 2-form b. An almost bi-Hermitian structure becomes a *bi-Hermitian structure* if $J_\pm$ are integrable.

Thus, any generalized almost Kähler structure automatically determines an almost bi-Hermitian structure on M. These bi-Hermitian structures were originally studied in [**21**] because they describe certain supersymmetries in physics. On the other hand, given an almost bi-Hermitian structure $(J_\pm, g, b)$, as in [**23**], we can reconstruct the generalized almost Kähler structure $\mathcal{J}_1, \mathcal{J}_2$. Let $\omega_\pm = gJ_\pm$. Then we have

$$\mathcal{J}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} -(J_+ + J_-) & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & J_+^* + J_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{2.2}$$

$$\mathcal{J}_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} -(J_+ - J_-) & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & J_+^* - J_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{2.3}$$

and the metric $\mathcal{G}$ is given by

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}.$$

In order to discuss conditions for integrability of a generalized almost Kähler structure using the bi-Hermitian structure, note first that $\mathcal{J}_1$ and $\mathcal{J}_2$ decompose $(\mathbb{T}M) \otimes \mathbb{C}$ into their $\pm i$-eigenbundles

$$(\mathbb{T}M) \otimes \mathbb{C} = L_1 \oplus \overline{L_1} = L_2 \oplus \overline{L_2}.$$

Further, since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, we can decompose L_1 into the $\pm i$-eigenbundles of $\mathcal{J}_2$, so that $L_1 = L_1^+ \oplus L_1^-$. Then we get the decomposition

$$(\mathbb{T}M) \otimes \mathbb{C} = L_1^+ \oplus L_1^- \oplus \overline{L_1^+} \oplus \overline{L_1^-}.$$

Since $\pi : C_\pm \xrightarrow{\sim} TM$, L_1^+ is the $+i$-eigenbundle for J_+ and L_1^- is the $+i$-eigenbundle for J_-. Then integrability of the generalized Kähler structure seems to be linked to integrability of the complex structures $J_\pm$. In fact, given an almost bi-Hermitian structure $(J_\pm, g, b)$ on a manifold M^{2n}, we can construct a generalized almost Kähler structure that is integrable if certain conditions hold.

Proposition 2.2.2. *(Gualtieri,* [**23**]*) Let* $(J_\pm, g, b)$ *be an almost bi-Hermitian structure on a* $2n$*-dimensional manifold* M*. Let* $\mathcal{J}_1, \mathcal{J}_2$ *be a generalized almost Kähler structure defined on* M *as in* (2.2) *and* (2.3)*. Then* $\mathcal{J}_1, \mathcal{J}_2$ *are integrable if and only if* $J_\pm$ *are integrable complex structures and*

$$db(X, Y, Z) = d\omega_+(J_+X, J_+Y, J_+Z) = -d\omega_-(J_-X, J_-Y, J_-Z).$$

Sometimes a generalized almost Kähler structure $\mathcal{J}_1, \mathcal{J}_2$ defined by $(J_\pm, g, b)$ is not integrable under these conditions, but one of the generalized almost complex structures is. The following theorem describes conditions on the almost bi-Hermitian structure so that $\mathcal{J}_1$ is integrable.

Proposition 2.2.3. *(Chen and Nie,* [**17**]*) Given the generalized almost complex structure* $\mathcal{J}_1$ *defined by an almost bi-Hermitian structure* (J_+, J_-, g, b) *as above,* $\mathcal{J}_1$ *is integrable if and only if the following hold:*

$$
\begin{aligned}
-idb(X_+ + X_-, Y_+ + Y_-, Z_+) &= d\omega_+(X_+ + X_-, Y_+ + Y_-, Z_+) \\
&\quad -(\omega_+ + \omega_-)(X_-, [Y_+ + Y_-, Z_+]) \\
&\quad +(\omega_+ + \omega_-)(Y_-, [X_+ + X_-, Z_+]) - Z_+\omega_+(X_-, Y_-) \\
-idb(X_+ + X_-, Y_+ + Y_-, Z_-) &= d\omega_-(X_+ + X_-, Y_+ + Y_-, Z_-) \\
&\quad -(\omega_- + \omega_+)(X_+, [Y_+ + Y_-, Z_-]) \\
&\quad +(\omega_- + \omega_+)(Y_+, [X_+ + X_-, Z_-]) - Z_-\omega_-(X_+, Y_+)
\end{aligned}
$$

where $X_\pm, Y_\pm, Z_\pm \in L_1^\pm$.

2.2.4. Generalized K3 Structures. In what follows, we discuss the generalized complex geometry of K3 surfaces. This will allow us to define the generalized twistor space for a K3 surface in Chapter 3. This topic was introduced in [**31**], as a specialization of Hitchin's discussion on generalized Calabi Yau geometry [**28**].

On a K3 surface M, consider only closed even pure spinors, i.e. even forms $\phi \in C^\infty(\wedge^\bullet T^*_{\mathbb{C}})$ consisting of a zero-form ϕ_0, a 2-form ϕ_2, and a 4-form ϕ_4. Let ϕ, ψ be two such spinors. As Hitchin remarks in [**28**], we can pair ϕ, ψ using the Mukai (or Chevalley) pairing of forms, which we simplify for even spinors as

$$\langle\langle \phi, \psi \rangle\rangle := -\phi_0 \wedge \psi_4 + \phi_2 \wedge \psi_2 - \phi_4 \wedge \psi_0.$$

Let ϕ be an even closed form such that

$$\langle\langle \phi, \phi \rangle\rangle = 0, \quad \langle\langle \phi, \overline{\phi} \rangle\rangle.$$

Note that this is precisely the condition for ϕ to define a *generalized Calabi-Yau structure* on M (see [**28**]). We denote by $P_\phi \subset C^\infty(\wedge^\bullet T^* M)$ the real two-dimensional vector space spanned by the real and imaginary components of ϕ. This plane, along with a natural pointwise orientation, uniquely determines the spinor line K_p for ϕ at each point $p \in M$.

Definition 2.2.8. A *generalized K3 structure* is a pair of pure spinors ϕ, ψ such that

$$\langle\langle \phi, \phi \rangle\rangle = \langle\langle \psi, \psi \rangle\rangle = 0, \tag{2.4}$$

$$\langle\langle \phi, \overline{\phi} \rangle\rangle = \langle\langle \psi, \overline{\psi} \rangle\rangle > 0, \tag{2.5}$$

and such that P_ϕ and $P_{\phi'}$ are pointwise orthogonal.

Note that this last condition implies that

$$\langle\langle \phi, \psi \rangle\rangle = 0.$$

Further, these spinors determine a pair of generalized complex structures $\mathcal{J}, \mathcal{J}'$ that are generalized Kähler on M.

Example 2.2.9. When M is a K3 surface, we can use the hyperkähler structure to write out explicit spinors for a generalized K3 structure on M. Let $\sigma = \omega_J + i\omega_K$ be a holomorphic symplectic 2-form on M and let $\phi = \sigma$. Then by a few simple calculations, we find that $\psi = e^{i\omega_I}$. Note that ϕ is a form of complex type, while ψ is a form of symplectic type. In general, we can find pairs of spinors on generalized K3 surfaces that are both symplectic type, or one is complex and the other is symplectic. In the following chapter, we will expand this example to a family of generalized K3 structures over $S^2 \times S^2$.

CHAPTER 3

Generalized twistor theory for K3 surfaces

In the original twistor theory for K3 surfaces and more generally hyperkähler manifolds, holomorphic data encoded in the twistor space revealed information about the geometric structure of the underlying manifold. In the following chapter, we develop this twistor theory in the context of generalized complex structures in the hope that this will reveal more about this new subject. We construct a *generalized twistor space* of a K3 surface M by looking at a $\mathbb{C}P^1 \times \mathbb{C}P^1$-family of generalized complex structures on M coming from the original structure on the K3, and further generalize it to all hyperkähler manifolds of dimension $4n$. In [**31**], Huybrechts introduces the idea of a family of K3 surfaces parametrized by $\mathbb{C}P^1 \times \mathbb{C}P^1$, but he does not construct the generalized twistor space. The subject was also discussed for hyperkähler manifolds in [**9**], however here we provide a more thorough explanation of the construction. We note that the spinor derived in [**9**] is different from ours; we claim that ours is the correct spinor for the generalized complex structure given. In particular, Bredthauer's spinor is not correct since it does not agree with the spinor for the classical twistor space described in Chapter 2.

The first section of this chapter presents the initial construction. We illustrate three different ways of defining a family of generalized complex structures on a K3 surface M, and then prove that the families are equivalent. In the next section, we use this information to build the generalized twistor space, which we prove is a generalized complex manifold. The generalized twistor space has analogous properties to that of the classical twistor space as given in Theorem 2.1.5, however we must first define the appropriate generalizations to describe these properties. We provide these in Section 3.3. Finally, in Section 3.4, we prove our second main theorem about the generalized twistor space of a K3 surface and its properties. The last section of the chapter extends this construction to all hyperkähler manifolds of dimension $4n$. This is a natural extension to higher dimensions, since our construction depends only on the hyperkähler structure of a K3 surface.

3.1. Preliminaries

In order to construct the generalized twistor space, we first build a family of generalized complex structures on M. We present this construction in three ways. The first is by looking at a family of generalized K3 structures as a quadric in $\mathbb{C}P^3$, which determines the parameter space of this family. The second is by examining a bi-Hermitian structure on M determined by I, J, K and the metric g. This will help us to define our generalized complex structure on the generalized twistor space. The third will be a spinor derivation starting with the 1-parameter family $(\cos\theta)\mathcal{J}_I + (\sin\theta)\mathcal{J}_{\omega_J}$ and transforming it first by $\omega_J \mapsto \cos\phi\omega_J + \sin\phi\omega_K$ and then by the usual twistor family $I \mapsto aI + bJ + cK$. We will use the first approach to obtain the other two, and conclude the section by proving that the latter two constructions are equivalent.

3.1.1. The parameter space of a family of generalized K3 Surfaces. Let (ϕ, ϕ') define a generalized K3 structure on M. Recall that P_ϕ denotes the real vector space spanned by the real and imaginary parts of ϕ, and that ϕ generates a line K_ϕ of spinors defining the underlying generalized complex structure associated to it. Take the pointwise oriented positive four-space $\Pi_{(\phi,\phi')} \subset C^\infty(\wedge^{ev}(T^*M))$ spanned by P_ϕ and $P_{\phi'}$.

Definition 3.1.1. We call the set of all generalized K3 structures (ϕ, ϕ') with fixed Π,

$$T_\Pi = T_{(\phi,\phi')}$$

the *twistor quadric*.

REMARKS.

(1) Note that P_ϕ determines a line of spinors spanned by ϕ. Therefore, we can think of $\Pi_{\mathbb{C}}$ as a vector space $V \otimes \mathbb{C}$ such that the elements of $V \otimes \mathbb{C}$ are pure spinor lines $[\phi] = K$.

(2) The equations defining a generalized K3 structure (2.4), (2.5) show that T_Π is a quadric in $\mathbb{C}P^3 = \mathbb{P}^3(\Pi_{\mathbb{C}})$:

$$T_\Pi = \mathcal{Q}_\Pi = \{[v] \in \Pi \;\mid\; \langle\langle [v], [v] \rangle\rangle = 0, \quad \langle\langle [v], \overline{[v]} \rangle\rangle > 0\}.$$

Further, for any vector space V, there is a natural isomorphism between the Grassmanian of oriented planes $Gr_2^o(V) \cong SO(n)/SO(2) \times SO(n-2)$ and the quadric $\mathcal{Q}_V \subset \mathbb{P}(V \otimes \mathbb{C})$ defined by

$$\mathcal{Q}_V = \{v \;\mid\; v \cdot v = 0, \quad v \cdot \bar{v} > 0\}.$$

Therefore, T_Π is naturally isomorphic to $Gr_2^o(\Pi) \cong S^2 \times S^2 \cong \mathbb{C}P^1 \times \mathbb{C}P^1$.

Suppose we consider the generalized K3 structure on M defined by $(\phi, \phi') = (\sigma, \exp(i\omega_I))$ as in Example 2.2.9, where $\sigma = \omega_J + i\omega_K$. Then, as explained in [**31**], the twistor quadric is determined by

$$\Pi = \langle \omega = \omega_I, \omega_J, \omega_K, 1 - \frac{\omega^2}{2} \rangle.$$

Further, as we will see more explicitly in the next section, the classical twistor family $\{aI + bJ + cK \;\mid\; (a, b, c,) \in S^2\} \cong \mathbb{C}P^1$ sits inside T_Π as a hyperplane section, and is not one of the components of the product $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Although we have determined the parameter space of the twistor family of generalized K3 structures on M, we have not explicitly defined the structures themselves. The following two sections will attempt to demonstrate this.

3.1.2. Bi-Hermitian structure of M. As a K3 surface, M has a bi-Hermitian structure, i.e. a metric g and two Hermitian complex structures $I_\pm$. As described in Section

2.2.3, a generalized almost Kähler structure can be reconstructed from bi-Hermitian data. We use these facts to define two generalized almost complex structures $\mathcal{J}, \mathcal{J}'$ which, up to scaling by a complex factor, will determine a generalized K3 structure.

Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in S^2$ and let $(I_+, I_-, b = 0, g)$ define a bi-Hermitian structure on M, with

$$I_+ = a_1 I + a_2 J + a_3 K, \quad I_- = b_1 I + b_2 J + b_3 K. \tag{3.1}$$

Then as in Section 2.2.3, equations (2.2) and (2.3), we can define

$$\omega_+ := gI_+ = a_1\omega_I + a_2\omega_J + a_3\omega_K, \quad \omega_- := gI_- = b_1\omega_I + b_2\omega_J + b_3\omega_K \tag{3.2}$$

and write

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} -(I_+ + I_-) & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & I_+^* + I_-^* \end{pmatrix}$$

$$\mathcal{J}' = \frac{1}{2}\begin{pmatrix} -(I_+ - I_-) & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & I_+^* - I_-^* \end{pmatrix},$$

an $S^2 \times S^2$-family of generalized almost Kähler structures on M. Using similar techniques as in Section 2.1.2, we consider $S^2 \times S^2 \cong \mathbb{C}P^1 \times \mathbb{C}P^1$ by stereographic projection and rewrite $\mathcal{J}, \mathcal{J}'$ in terms of $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$ with

$$(a_1, a_2, a_3) = \left(\frac{1 - \alpha\bar{\alpha}}{1 + \alpha\bar{\alpha}}, \frac{\alpha + \bar{\alpha}}{1 + \alpha\bar{\alpha}}, -i\frac{\alpha - \bar{\alpha}}{1 + \alpha\bar{\alpha}}\right),$$

$$(b_1, b_2, b_3) = \left(\frac{1 - \beta\bar{\beta}}{1 + \beta\bar{\beta}}, \frac{\beta + \bar{\beta}}{1 + \beta\bar{\beta}}, -i\frac{\beta - \bar{\beta}}{1 + \beta\bar{\beta}}\right).$$

Proposition 3.1.1. *The pair of generalized almost complex structures on M given by $\mathcal{J}, \mathcal{J}'$ are integrable for any point $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$. Further, up to multiplication by a complex factor, $\mathcal{J}, \mathcal{J}'$ define a generalized K3 structure on M for every point $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$.*

Remark. When $\alpha = \beta$ along the diagonal $\Delta = S^2 \cong \mathbb{C}P^1 \subset \mathbb{C}P^1 \times \mathbb{C}P^1$,

$$\text{type}(\mathcal{J})(p) = 2, \quad \text{type}(\mathcal{J}')(p) = 0 \qquad \forall \;\; p \in M.$$

Then $\mathcal{J}$ defines a generalized complex structure of complex type throughout M while $\mathcal{J}'$ defines one of symplectic type. The complex structure underlying $\mathcal{J}$ for every $\alpha = \beta$ gives rise to the usual family of complex structures on M over the diagonal $\Delta = \mathbb{C}P^1$ sitting as the hyperplane section inside $\mathbb{C}P^1 \times \mathbb{C}P^1$. Conversely, along the 'superdiagonal', where $\alpha = -\bar{\beta}^{-1}$, $\mathcal{J}$ is of symplectic type and $\mathcal{J}'$ is of complex type. Outside of these two diagonals, we will see that $\mathcal{J}$ and $\mathcal{J}'$ are B-field transforms of structures of symplectic type, both of type 0.

Before proving Proposition 3.1.1, we must first define the pure spinors for these generalized almost complex structures. This leads us into our third construction of the generalized twistor family.

3.1.3. Construction by Pure Spinors. In order to derive the family of generalized K3 structures using spinors, we recall the generalized complex family from the interpolation example in Section 1.2.4. For $\theta \in [0, \frac{\pi}{2}]$, let $\mathcal{J}_\theta$ denote the 1-parameter family of generalized complex structures

$$\mathcal{J}_\theta = \cos\theta \mathcal{J}_I + \sin\theta \mathcal{J}_{\omega_J}.$$

Introducing another parameter, $\varphi \in [0, 2\pi]$ and extending θ to $[0, \pi]$, there exists a family of generalized almost complex structures written in spherical coordinates over S^2:

$$\mathcal{J}_{\theta,\varphi} = \cos\theta \mathcal{J}_I + \sin\theta\cos\varphi \mathcal{J}_{\omega_J} + \sin\theta\sin\varphi \mathcal{J}_{\omega_K}.$$

Lemma 3.1.2. *For $\theta \neq 0$, $\mathcal{J}_{\theta,\varphi}$ is a B-field transform of a generalized complex structure of symplectic type determined by $\omega = \csc\theta(\cos(\varphi)\omega_J + \sin(\varphi)\omega_K)$, where $B =$*

$-\cot\theta(\cos(\varphi)\omega_K - \sin(\varphi)\omega_J)$. *Thus, $\mathcal{J}_{\theta,\varphi}$ is integrable and corresponds to the pure spinor*

$$\Phi_{\theta,\varphi} = \begin{cases} e^{B+i\omega} & \theta \neq 0 \\ \omega_J + i\omega_K & \theta = 0. \end{cases}$$

Proof. It follows from a basic calculation that for $\theta \neq 0$,

$$e^B \mathcal{J}_{\theta,\varphi} e^{-B} = \begin{bmatrix} 0 & -(\csc\theta(\cos(\varphi)\omega_J + \sin(\varphi)\omega_K))^{-1} \\ \csc\theta(\cos(\varphi)\omega_J + \sin(\varphi)\omega_K) & 0 \end{bmatrix}.$$

Then since the pure spinor for $\mathcal{J}_\omega$ is $\exp\left(i\csc\theta(\cos(\varphi)\omega_J + \sin(\varphi)\omega_K)\right)$, we get the spinor $\Phi_{\theta,\varphi}$ by B-field transform. Further, $\mathcal{J}_{\theta,\varphi}$ is integrable since $dB = 0 = d\omega$.

For $\theta = 0$, we have

$$\mathcal{J}_{0,\varphi} = \mathcal{J}_I$$

so the spinor Φ is the holomorphic symplectic 2-form $\omega_J + i\omega_K$. $\square$

To change from $(\theta,\varphi) \in S^2$ to $\xi \in \mathbb{C}P^1$ we use stereographic projection:

$$(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi) = \left(\frac{1-|\xi|^2}{1+|\xi|^2}, \frac{\xi+\bar{\xi}}{1+|\xi|^2}, -i\frac{\xi-\bar{\xi}}{1+|\xi|^2}\right).$$

Then $\cos\theta = \frac{1-|\xi|^2}{1+|\xi|^2}$ and $\sin\theta = \frac{2|\xi|}{1+|\xi|^2}$, which means the generalized complex structure is given by

$$\mathcal{J}_\xi = \frac{1-|\xi|^2}{1+|\xi|^2}\mathcal{J}_I - i\frac{\xi-\bar{\xi}}{1+|\xi|^2}\mathcal{J}_{\omega_J} + \frac{\xi+\bar{\xi}}{1+|\xi|^2}\mathcal{J}_{\omega_K}, \tag{3.3}$$

and the pure spinor is

$$\begin{aligned} \Phi_\xi &= \exp\left(\tfrac{1-|\xi|^2}{2|\xi|^2}\left(-\mathrm{Re}(\xi)\omega_K + \mathrm{Im}(\xi)\omega_J\right) + \tfrac{i(1+|\xi|^2)}{2|\xi|^2}\left(\mathrm{Re}(\xi)\omega_J + \mathrm{Im}(\xi)\omega_K\right)\right) \\ &= \exp\left(\tfrac{1}{2|\xi|^2}(-\mathrm{Re}(\xi) + i\mathrm{Im}(\xi))(\omega_K - i\omega_J) + \tfrac{1}{2}(\mathrm{Re}(\xi) + i\mathrm{Im}(\xi))(\omega_K + i\omega_J)\right) \\ &= \exp\left(\tfrac{1}{2|\xi|^2}(-\bar{\xi})(-i\sigma) + \tfrac{1}{2}\xi i\bar{\sigma}\right) \\ &= \exp\left(\tfrac{i}{2\xi}\sigma + \tfrac{i}{2}\xi\bar{\sigma}\right) = 1 + \left(\tfrac{i}{2\xi}\sigma + \tfrac{i}{2}\xi\bar{\sigma}\right) - \tfrac{1}{4}\sigma\bar{\sigma}, \end{aligned}$$

where $\sigma = \omega_J + i\omega_K$. Multiplying through by $-2i\xi$, we get

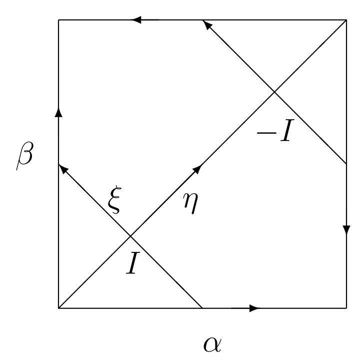
$$\Phi_\xi = -2i\xi + \sigma + \xi^2\bar{\sigma} + i\xi\omega_I^2.$$

Note that when $\xi = 0$, $\Phi_\xi = \sigma$ corresponds to the generalized complex structure $\mathcal{J}_I$, and when $\xi = \infty$, $\Phi_\xi = \bar{\sigma}$ corresponds to generalized complex structure $\mathcal{J}_{-I}$. Further, when $\xi = \exp(i\phi)$, i.e. when $\theta = \frac{\pi}{2}$, we have spinor $\Phi_\xi = \exp\left(i(\cos\varphi\omega_J + \sin\varphi\omega_K)\right)$ and generalized complex structure of symplectic type $\mathcal{J}_{\cos\varphi\omega_J + \sin\varphi\omega_K}$.

Thus, given $I = I_0$ and a choice of holomorphic 2-form σ, we get a generalized complex structure defined by equation (3.3) depending on ξ with spinor

$$\Phi_\xi = (-2i\xi)\exp\left(\frac{i}{2}\left(\frac{\sigma}{\xi} + \xi\bar{\sigma}\right)\right).$$

Note that the parameter ξ gives a direction pointing away from the diagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Let η be a second parameter denoting the direction along the diagonal, as shown in the diagram below. Then Φ_ξ defines a generalized complex structure along the locus $\eta = 0$.



$\mathbb{C}P^1 \times \mathbb{C}P^1$

Moving away from this locus, let

$$I_\eta = \frac{1 - \eta\overline{\eta}}{1 + \eta\overline{\eta}} I + \frac{2\mathrm{Re}(\eta)}{1 + \eta\overline{\eta}} J + \frac{2\mathrm{Im}(\eta)}{1 + \eta\overline{\eta}} K,$$

$\eta \in \mathbb{C}P^1$. We know from [**27**] that there exists a holomorphic symplectic form σ_η corresponding to I_η given by:

$$\sigma_\eta = \frac{\sigma + 2\eta\omega_I - \eta^2\bar{\sigma}}{1 + |\eta|^2} \tag{3.4}$$

Then a pure spinor for the generalized complex structure is

$$\Phi_{\eta,\xi} = -2i\xi \exp\left(\frac{i}{2}\left(\frac{\sigma_\eta}{\xi} + \xi\bar{\sigma_\eta}\right)\right).$$

In order to rewrite this in coordinates $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, we find a linear fractional transformation such that

$$0 \mapsto \eta, \qquad \infty \mapsto -\bar{\eta}^{-1}.$$

Since the isometries of $\mathbb{C}P^1 \cong S^2$ are given by $SU(2)/\pm I_{2\times 2} \cong SO(3)$, we want

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PGL(2, \mathbb{C}) = SU(2, \mathbb{C})/\pm I_{2\times 2}$$

such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \sim \begin{bmatrix} \eta \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \sim \begin{bmatrix} 1 \\ -\bar{\eta} \end{bmatrix}$$

This will determine a map that sends

$$\xi \mapsto \alpha, \qquad -\xi \mapsto \beta,$$

up to rotation by $e^{i\vartheta}$ for some angle ϑ. In order to ensure that $\vartheta = 0$, i.e. that this transformation does not rotate the coordinates, we check that it gives the correct transformation at the level of complex structures. In other words, we check that the matrix $Y \in SO(3)$ such that Y is the image of X under the map $SU(2)/\pm I_{2\times 2} \to SO(3)$ sends $I = I_0 \to I_\eta$ and $\sigma \to \sigma_\eta$.

Recall that the map $SU(2) \xrightarrow{\sim} SO(3)$ can be given by, for $U \in SU(2)$

$$U(x,y) = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Re}(x^2 - y^2) & \mathrm{Im}(x^2 + y^2) & -2\mathrm{Re}(x\bar{y}) \\ -\mathrm{Im}(x^2 - y^2) & \mathrm{Re}(x^2 + y^2) & 2\mathrm{Im}(x\bar{y}) \\ 2\mathrm{Re}(x\bar{y}) & 2\mathrm{Im}(x\bar{y}) & |x|^2 - |y|^2 \end{pmatrix}.$$

Let $x = \frac{1}{\sqrt{1+|\eta|^2}}$ and $y = \frac{\eta}{\sqrt{1+\eta^2}}$. Then after shifting rows and columns, we have the following image $Y \in SO(3)$:

$$Y = \frac{1}{1+|\eta|^2} \begin{pmatrix} 1 - |\eta|^2 & 2\mathrm{Re}(\eta) & -2\mathrm{Im}(\eta) \\ -2\mathrm{Re}(\eta) & \mathrm{Re}(1-\eta^2) & \mathrm{Im}(1+\eta^2) \\ 2\mathrm{Im}(\eta) & -\mathrm{Im}(1-\eta^2) & \mathrm{Re}(1+\eta^2) \end{pmatrix}.$$

We consider this transformation on vectors associated to complex structures, i.e.

$$aI + bJ + cK \rightsquigarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Note that this transformation takes

$$I = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longmapsto I_\eta = \begin{bmatrix} \frac{1-\eta\bar{\eta}}{1+\eta\bar{\eta}} \\ \frac{2\mathrm{Re}(\eta)}{1+\eta\bar{\eta}} \\ \frac{2\mathrm{Im}(\eta)}{1+\eta\bar{\eta}} \end{bmatrix}$$

and further, on the level of 2-forms, we have

$$\begin{bmatrix} \omega_I \\ \omega_J \\ \omega_K \end{bmatrix} = \begin{bmatrix} \omega_I \\ \mathrm{Re}(\sigma) \\ \mathrm{Im}(\sigma) \end{bmatrix} \longmapsto \begin{bmatrix} \omega_\eta \\ \mathrm{Re}(\sigma_\eta) \\ \mathrm{Im}(\sigma_\eta) \end{bmatrix},$$

where σ_η is given by equation (3.4), and ω_η is the Kähler form associated to I_η. Thus, we have the correct map.

Therefore, the matrix transformation is given by

$$X = \frac{1}{\sqrt{1+|\eta|^2}} \begin{bmatrix} 1 & \eta \\ -\bar{\eta} & 1 \end{bmatrix}.$$

Considering the images of $\begin{bmatrix} \xi \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\xi \\ 1 \end{bmatrix}$, we have

$$X \begin{bmatrix} \xi \\ 1 \end{bmatrix} = \frac{1}{\sqrt{1+|\eta|^2}} \begin{bmatrix} \xi+\eta \\ 1-\xi\bar{\eta} \end{bmatrix} \sim \begin{bmatrix} \xi+\eta \\ 1-\xi\bar{\eta} \end{bmatrix}$$

$$X \begin{bmatrix} -\xi \\ 1 \end{bmatrix} = \frac{1}{\sqrt{1+|\eta|^2}} \begin{bmatrix} -\xi+\eta \\ 1+\xi\bar{\eta} \end{bmatrix} \sim \begin{bmatrix} -\xi+\eta \\ 1+\xi\bar{\eta} \end{bmatrix}$$

as elements of $\mathbb{C}P^1$. Hence,

$$\alpha = \frac{\xi+\eta}{1-\xi\bar{\eta}}$$
$$\beta = \frac{-\xi+\eta}{1+\xi\bar{\eta}}.$$

Writing Φ in terms of our new coordinates α and β gives us, up to multiplication by a complex constant,

$$\Phi = \Phi_{\alpha,\beta} = \sigma + (\alpha-\beta)(-i + \frac{i}{2}\omega_I^2) + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma}.$$

There is another spinor Φ', defined in the proposition below, such that the pair Φ, Φ' are pure spinors for the generalized complex structures $\mathcal{J}, \mathcal{J}'$ determined by the bi-Hermitian structure as in Section 3.1.2.

Lemma 3.1.3. *The two families of generalized almost complex structures given by $\mathcal{J}, \mathcal{J}'$ and Φ, Φ' are equivalent. In other words, there are pure spinors for $\mathcal{J}, \mathcal{J}'$ that are, respectively,*

$$\Phi = \sigma + (\alpha-\beta)\left(-i + i\frac{\omega^2}{2}\right) + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma} \tag{3.5}$$

$$\Phi' = \bar{\beta}\left(\sigma + (\alpha + \bar{\beta}^{-1})\left(-i + i\frac{\omega}{2}^2\right) + (\alpha - \bar{\beta}^{-1})\omega_I + \alpha\bar{\beta}^{-1}\bar{\sigma}\right), \tag{3.6}$$

where $\sigma = \omega_J + i\omega_K$ *and* $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$.

Proof. We first note that for $\alpha \neq \beta$, we can write Φ as $\Phi = Ce^{B+i\omega}$, where C is a complex scalar and

$$\omega = \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_I + \frac{\mathrm{Re}(\alpha)(1 + |\beta|^2) - \mathrm{Re}(\beta)(1 + |\alpha|^2)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_J \tag{3.7}$$

$$+\frac{\mathrm{Im}(\alpha)(|\beta|^2 + 1) - \mathrm{Im}(\beta)(|\alpha|^2 + 1)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_K \tag{3.8}$$

$$\begin{aligned} B &= \frac{2\mathrm{Im}(\alpha\bar{\beta})}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_I + \frac{\mathrm{Im}(\alpha)(1 - |\beta|^2) - \mathrm{Im}(\beta)(1 - |\alpha|^2)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_J \\ &\quad + \frac{\mathrm{Re}(\alpha)(|\beta|^2 - 1) - \mathrm{Re}(\beta)(|\alpha|^2 - 1)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_K \end{aligned}$$

The rest of the proof is purely computational; we refer the reader to Appendix A for the details. There, we prove that for $\alpha \neq \beta$, the generalized complex structure for $\mathcal{J}$ is the B-field transform of a generalized complex structure of symplectic type $\mathcal{J}_\omega$, where B and ω are as above.

If $\alpha = \beta$, as noted earlier, $\mathcal{J}$ is of complex type coming from the almost complex structure

$$\lambda = \frac{1 - \alpha\bar{\alpha}}{1 + \alpha\bar{\alpha}}I + \frac{\alpha + \bar{\alpha}}{1 + \alpha\bar{\alpha}}J + \frac{-i(\alpha - \bar{\alpha})}{1 + \alpha\bar{\alpha}}K.$$

The spinor can be given by the holomorphic symplectic form associated to λ

$$\omega_\lambda = \sigma + 2\alpha\omega_I - \alpha^2\bar{\sigma},$$

which is equal to $\Phi|_{\alpha=\beta}$. Hence Φ is a pure spinor defining $\mathcal{J}$.

In order to see that Φ' is a pure spinor for $\mathcal{J}'$, we use the bi-Hermitian structure. Note that in order to get from $\mathcal{J}$ to $\mathcal{J}'$, we simply map the second almost complex structure to its complex conjugate: $I_- \mapsto -I_-$. This induces the antipodal map on the second coordinate of $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, thereby mapping $\Phi \mapsto \Phi'$. The multiplication by $\bar{\beta}$

simply ensures that we have a generalized K3 structure according to the Mukai pairing as we will see in the next proof.

$\square$

Using these spinors, we now prove Proposition 3.1.1.

PROOF. (Proposition 3.1.1) From Lemma 3.1.3, since $d\omega = dB = 0$, $\mathcal{J}$ is a B-field transform of an integrable generalized complex structure. Therefore, outside of the locus where $\alpha = \beta$, $\mathcal{J}$ is integrable. If $\alpha = \beta$, as noted before, $\mathcal{J} = \mathcal{J}_\lambda$, which is integrable since λ is. A similar argument shows that $\mathcal{J}'$ is integrable, however in this case $\mathcal{J}'$ is of complex type along the superdiagonal $\alpha = -\bar{\beta}^{-1}$.

As defined, Φ and Φ' are pure even closed spinors defining $\mathcal{J}, \mathcal{J}'$ respectively, and we can check that

$$\langle\langle \Phi, \Phi \rangle\rangle = \langle\langle \Phi', \Phi' \rangle\rangle = \langle\langle \Phi, \Phi' \rangle\rangle = 0,$$

$$\langle\langle \Phi, \overline{\Phi} \rangle\rangle = \langle\langle \Phi', \overline{\Phi'} \rangle\rangle > 0.$$

Hence, $\mathcal{J}, \mathcal{J}'$ define a generalized K3 structure on M.

$\square$

REMARK. There is a second proof for the integrability of $\mathcal{J}, \mathcal{J}'$ using the bi-Hermitian structure. Note that $\mathcal{J}, \mathcal{J}'$ is a generalized almost Kähler structure satisfying the conditions in Proposition 2.2.2, since $d\omega_+ = d\omega_- = 0$. Hence, $\mathcal{J}$ and $\mathcal{J}'$ determine an integrable generalized Kähler structure on M.

3.2. The generalized twistor space of a K3 surface

Having defined a family of generalized complex structures on a K3 surface M, we can now form the generalized twistor space of M. The construction of this space mirrors that of the classical case for hyperkähler manifolds in Section 2.1.2.

Let $\mathcal{X}$ be the smooth product manifold $M \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with generalized almost complex structure defined at each $p = (m, \alpha, \beta) \in \mathcal{X}$ by

$$\mathcal{K} = (\mathcal{J}, \mathcal{J}_{I_\alpha}, \mathcal{J}_{I_\beta}) : \mathbb{T}_p(M \times \mathbb{C}P^1 \times \mathbb{C}P^1) \to \mathbb{T}_p(M \times \mathbb{C}P^1 \times \mathbb{C}P^1),$$

where $\mathcal{J}_{I_\alpha}$ and $\mathcal{J}_{I_\beta}$ are the generalized complex structures arising from the natural complex structures I_α and I_β on $\mathbb{C}P^1$. Here, α and β simply indicate the distinction between each $\mathbb{C}P^1$. We will show that this structure is integrable, and call the resulting generalized complex manifold $\mathcal{X}$ the *generalized twistor space* of M.

Remark. We can recover the usual twistor space Z sitting as a submanifold inside $\mathcal{X}$ defined by the locus where $\mathcal{K}$ is purely of complex type. As explained in Section 3.1.2, this corresponds to the points where $\alpha = \beta$.

Theorem 3.2.1. *$\mathcal{X}$ is a generalized complex manifold; in other words, the generalized almost complex structure on $\mathbb{T}(\mathcal{X})$ defined by*

$$\mathcal{K} = (\mathcal{J}, \mathcal{J}_{I_\alpha}, \mathcal{J}_{I_\beta})$$

is integrable.

We can write the spinor for this generalized complex structure on $\mathcal{X}$ as follows. Let $\rho = d\alpha \wedge d\beta$ be the spinor for $(\mathcal{J}_{I_\alpha}, \mathcal{J}_{I_\beta})$ on $\mathbb{T}(\mathbb{C}P^1 \times \mathbb{C}P^1)$. Let $\Psi = \Phi \wedge \rho$. It is clear that this spinor corresponds to the generalized complex structure $\mathcal{K}$ above.

Proof. Recall that a generalized almost complex structure defined by a pure spinor ϕ is integrable if and only if $d\phi = (X + \xi) \cdot \phi$ for $X + \xi \in C^\infty(\mathbb{T})$. In fact, we have that $d\Psi = 0$. Note that $d\rho = d(d\alpha \wedge d\beta) = 0$. Further, we have:

$$\begin{aligned} d\Psi &= d\Phi \wedge \rho + (-1)^{|\Phi|}\Phi \wedge d\rho \\ &= (d(\sigma) + d(\alpha - \beta)(i + i\frac{\omega^2}{2}) - (\alpha - \beta)\frac{i}{2}d(\omega^2) \\ &\quad + d(\alpha + \beta)\omega_I - (\alpha + \beta)d\omega_I - d(\alpha\beta)\bar{\sigma} - \alpha\beta d\bar{\sigma}) \wedge \rho \end{aligned}$$

But $d\omega_I = d\omega_J = d\omega_K = 0$, so we have

$$d\Psi = (d(\alpha - \beta)(i + i\frac{\omega^2}{2}) + d(\alpha + \beta)\omega_I - d(\alpha\beta)\bar{\sigma}) \wedge d\alpha \wedge d\beta = 0$$

Hence, $\mathcal{K}$ is integrable.

$\square$

Remarks.

(1) Note that $\mathcal{X}$ is not generalized Kähler. Suppose we let $\mathcal{K}' = (\mathcal{J}', \mathcal{J}_{I_\alpha}, -\mathcal{J}_{I_\beta})$. Then the product $\mathcal{G} = -\mathcal{K}\mathcal{K}'$ is not positive definite on $\mathbb{T}\mathcal{X}$, so $\mathcal{K}$ and $\mathcal{K}'$ do not define a generalized Kähler structure on $\mathcal{X}$. Additionally, there does not seem to be a more suitable generalized complex structure on $\mathbb{C}P^1 \times \mathbb{C}P^1$ that induces a generalized Kähler structure on $\mathcal{X}$.

(2) There is a second proof that $\mathcal{K}$ is integrable using results from [**17**]. In fact, using Proposition 2.2.3, we can show that $\mathcal{K}$ is an integrable generalized complex structure induced by the bi-Hermitian structure

$$((I_\pm, I_\alpha, I_\beta), (\omega_\pm, \omega_\alpha, \omega_\beta), g, b = 0),$$

where $(I_\pm, \omega_\pm)$ is the bi-Hermitian structure on M defined in Section 3.1.2 and $I_\alpha, \omega_\alpha, I_\beta, \omega_\beta$ come from the natural Kähler structure on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

3.3. Generalized complex reduction and submanifolds

In order to more thoroughly describe the properties of the generalized twistor space, we digress into a discussion on reduction and submanifolds of generalized complex objects. Throughout this section, M will denote an even-dimensional manifold.

3.3.1. Reduction of generalized complex structures. The reduction of geometric structures has been studied extensively for many areas of geometry, including symplectic, Kähler, and hyperkähler geometry. In each of these settings, the reduction procedure provides a way to produce new geometric structures from others. Consider, for example, a symplectic manifold (M, ω). Let G be a Lie group acting on M and preserving the non-degenerate 2-form ω. The Lie algebra $\mathfrak{g}$ then acts on sections of TM and there exists an equivariant moment map $\mu : M \to \mathfrak{g}^*$ determined by this action that provides the medium for reduction. The reduced manifold $\mu^{-1}(0)/G$, called the Marsden-Weinstein-Meyer quotient, is in fact a symplectic manifold ([**34**]).

We might ask if we can use this reduction technique on generalized complex manifolds of symplectic type, or more generally, of any type. Extending this algorithm to generalized complex geometry, however, requires some additional set-up. In order to reduce a generalized complex structure, we must first discuss how to extend the Lie group action to sections of the Courant algebroid $E \cong \mathbb{T}M$.

The theory of reduction of generalized complex structures was developed independently by Gualtieri, Cavalcanti, and Bursztyn [**10**] and Stiénon and Xu [**43**]. In this dissertation, we follow the procedure from [**10**].

Definition 3.3.1. Let G be a Lie group acting on a manifold with Lie algebra $\mathfrak{g}$. A *Courant algebra* over $\mathfrak{g}$ is a vector space $\mathfrak{a}$ with a bilinear bracket $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ and a bracket-preserving homomorphism $\pi : \mathfrak{a} \to \mathfrak{g}$ satisfying the Leibniz condition. A Courant algebra is *exact* if π is surjective and $\mathfrak{h} = \ker(\pi)$ is abelian.

Note that for $\mathfrak{g} = C^\infty(TM)$, the Courant algebroid $C^\infty(\mathbb{T}M)$ is an example of a Courant algebra.

Definition 3.3.2. Let G be a connected Lie group acting on a manifold M with action $\psi : \mathfrak{g} \to C^\infty(TM)$. An *extended action* on $E \cong \mathbb{T}M$ is an exact Courant algebra $\mathfrak{a}$ over $\mathfrak{g}$ with a Courant algebra morphism $\rho : \mathfrak{a} \to C^\infty(\mathbb{T}M)$ such that:

(1) $\mathfrak{h}$ acts trivially on $C^\infty(\mathbb{T}M)$, which means it acts by closed 1-forms.
(2) The induced action of $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ on $C^\infty(\mathbb{T}M)$ integrates to a G-action on $\mathbb{T}M$.

Gualtieri, Cavalcanti, and Bursztyn (BCG) also give conditions for when $\mathfrak{g}$ integrates to a G-action on the Courant algebroid; for details, we refer the reader to [**10**].

Remark. In this dissertation, we will primarily use *trivially extended G-actions*, i.e. extensions such that $\mathfrak{a} = \mathfrak{g}$ and $\pi = \mathrm{id}$. In this case we have the commutative diagram given below.

$$\begin{array}{ccc}\mathfrak{g} & \xrightarrow{\text{id}} & \mathfrak{g}\\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\gamma}\\ C^\infty(E) & \longrightarrow & C^\infty(TM)\end{array}$$

An extended action of a connected Lie group G on $E \cong \mathbb{T}M$ gives E the structure of an equivariant G-bundle such that the bracket is preserved by the action. Further, it determines two G-invariant distributions in E: $K = \rho(\mathfrak{a})$ and its orthogonal, $K^\perp$. From these distributions, we define the *large distribution* $\Delta_b := \pi(K + K^\perp) \subset TM$ and the *small distribution* $\Delta_s := \pi(K^\perp) = Ann(\rho(\mathfrak{h})) \subset TM$. The following theorem describes conditions on E to reduce down to an exact Courant algebroid E_{red}.

THEOREM 3.3.3. *(BCG, [**10**]) Let $E \cong \mathbb{T}M$ be an exact Courant algebroid over M and $\rho : \mathfrak{a} \to C^\infty(E)$ be an extended G-action. Let $P \in M$ be a leaf of Δ_b on which G acts freely and properly, and over which $\rho(\mathfrak{h})$ has constant rank. Then the Courant bracket on E descends to*

$$E_{red} = \frac{K^\perp}{K \cap K^\perp|_P}\Big/G$$

and makes it into a Courant algebroid over $M_{red} = P/G$ with surjective anchor. Further, E_{red} is exact if and only if $\pi(K) \cap \pi(K^\perp) = \pi(K \cap K^\perp)$ along P.

EXAMPLE 3.3.4. ([**10**]) Let G act freely and properly on M with infinitesimal action $\gamma : \mathfrak{g} \to C^\infty(TM)$ and consider the trivially extended action such that $\mathfrak{a} = \mathfrak{g}$. Then $K = \gamma(\mathfrak{g})$, $K^\perp = TM \oplus Ann(K)$ and

$$\Delta_s = \Delta_b = TM.$$

So the reduced Courant algebroid is

$$E_{red} = TM/K \oplus Ann(K) = \mathbb{T}(M/G).$$

Proposition 3.3.1. *(BCG, [**10**]) Let $\iota : S \hookrightarrow M$ be a submanifold of a manifold equipped with an exact Courant algebroid E. Then the vector bundle*

$$E_S := \frac{\mathrm{Ann}(TS)^{\perp}}{\mathrm{Ann}(TS)} = \frac{\pi^{-1}(TS)}{\mathrm{Ann}(TS)}$$

inherits the structure of an exact Courant algebroid over S.

Consider a generalized complex structure $\mathcal{J}$ on a manifold M. In order to see how this reduction of Courant algebroids extends to generalized complex structures, we first see how the Dirac structure L for $\mathcal{J}$ reduces. Suppose we are in the context of Theorem 3.3.3 such that E_{red} is exact and suppose further that the action ρ preserves the structure $\mathcal{J}$ on E. Let $\tilde{K} := K \cap (K^{\perp} + T^*M)$, which is isotropic along P. Then we can reduce the Dirac structure

$$L_{red} = \frac{(L \cap \tilde{K}_{\mathbb{C}} + \tilde{K}_{\mathbb{C}})|_P}{\tilde{K}_{\mathbb{C}}|_P} \Big/ G.$$

Proposition 3.3.2. *(BCG, [**10**]) L_{red} determines a generalized complex structure $\mathcal{J}_{red}$ on E_{red} if and only if*

$$L_{red} \cap \overline{L_{red}} = \{0\}$$

which happens if and only if

$$\mathcal{J}\tilde{K} \cap \tilde{K}^{\perp} \subset \tilde{K}$$

over a leaf $P \hookrightarrow M$ of the distribution Δ_b.

If (M, ω) is a symplectic manifold such that $\mathcal{J}_{\omega}$ is a generalized complex structure of symplectic type on M and the extended action comes from a symplectic G-action, this procedure is equivalent to that of symplectic reduction. On the other hand, if (M, I) is a complex manifold and G is a complex Lie group acting holomorphically on M, generalized reduction of $\mathcal{J}_I$ is simply a holomorphic quotient. However, as we will see in Theorem 3.4.1, the reduced generalized complex structure $\mathcal{J}_{red}$ will not necessarily have the same type as $\mathcal{J}$. For example, a generalized complex structure of complex type may reduce down to a generalized complex structure of symplectic type.

3.3.2. Generalized complex submanifolds. As we have already seen in Proposition 3.3.1, there is some interplay between generalized reduction and submanifolds. However, until now we have not formulated a definition for generalized complex submanifolds. In this section, we will attempt to do this. Since generalized complex structures are objects that arise from both symplectic and complex structures, there are many different notions of what a generalized complex submanifold should be (see, for example, [**5**], [**14**], [**25**]). We only discuss a few of these here - those that materialize from properties of the generalized twistor space.

DEFINITION 3.3.5. Let S be a submanifold of a generalized complex manifold $(M, \mathcal{J})$. Let $N^*S = \mathrm{Ann}(TS) \subset T^*M$ be the conormal bundle of S, where the annihilator is determined by the natural inner product on $TM \oplus T^*M$. Then S is called a *generalized complex submanifold* if

$$\mathcal{J}(N^*S) \subset N^*S.$$

REMARKS.

(1) Indeed, using reduction arguments from Section 3.3.1, we can see that S has a generalized complex structure induced by the generalized complex structure on M. Specifically, if E is the Courant algebroid for M, N^*S is a subbundle of $E|_S$. Then $E_{red} := (N^*S)^{\perp}/N^*S$ defines a Courant algebroid on S, and therefore inherits a generalized complex structure, $\mathcal{J}_{red}$. In the case where M is a complex manifold, S is a complex submanifold, and when M is symplectic, S is a symplectic submanifold.

(2) This is a different definition than Gualtieri gives in [**25**]. We call that submanifold a *generalized Lagrangian submanifold*, as it corresponds to a Lagrangian submanifold when $\mathcal{J}$ is of symplectic type.

Note that the twistor space Z sitting inside the generalized twistor space $\mathcal{X}$ is a generalized complex submanifold, since

$$\mathcal{J}(N^*Z) \subset N^*Z.$$

Definition 3.3.6. Suppose $X = M_1 \times M_2$ is the smooth product of manifolds with a generalized complex structure $\mathcal{J}$ that pointwise decomposes into a block matrix $\mathcal{J} = \mathcal{J}_{M_1} \oplus \mathcal{J}_{M_2}$. Then $\mathcal{J}(\mathbb{T}M_i) \subset \mathbb{T}M_i$ and for $m_j \in M_j$, $j \neq i$, the subspace $\{m_j\} \times M_i$ inherits a generalized complex structure. We call the submanifold $M_i \hookrightarrow X$ a *generalized complex factor submanifold*.

Below, we define a generalized tangent and normal bundles for a generalized factor submanifold. Notice that this is a different definition than in [**23**].

Definition 3.3.7. Let $S \hookrightarrow M$ be a generalized complex factor submanifold of a generalized complex manifold $(M, \mathcal{J})$. The *generalized tangent bundle* $\mathcal{T}_S$ is defined as the bundle

$$\mathcal{T}_S := TS \oplus T^*S$$

and the *generalized normal bundle* $\mathcal{N}_S$ is defined as

$$\mathcal{N}_S := NS \oplus N^*S$$

3.4. Properties of the generalized twistor space

After having laid the groundwork for various objects in generalized complex geometry, we can now prove our main theorem on generalized twistor theory for K3 surfaces.

Theorem 3.4.1. *Let $\mathcal{X}$ be the generalized twistor space of a K3 surface M. Then:*

(1) $\mathcal{X}$ is a smooth fiber bundle

$$\pi : \mathcal{X} \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

and a reduction of generalized complex manifolds.

(2) The bundle admits a family of sections that are generalized complex factor submanifolds, each with generalized normal bundle isomorphic to

$$\mathbb{C}^2 \otimes (\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)).$$

(3) There is a pure spinor representing the generalized complex structure on $\mathcal{X}$ given by

$$\Psi = \left(\sigma + (\alpha - \beta)\left(-i + \frac{i}{2}\omega^2\right) + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma}\right) \wedge d\alpha \wedge d\beta$$

that defines a structure of complex type along the diagonal, and of type 2 everywhere else, where $\sigma = \omega_J + i\omega_K$.

(4) $\mathcal{X}$ has a real structure τ compatible with the above and inducing an antipodal map on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Proof. (1) For every point $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, the fiber $\{(\alpha,\beta)\} \times M$ sits inside $\mathcal{X}$ as a generalized complex submanifold. This fiber varies holomorphically in α and β, which we can see easily in the pure spinor determining the generalized complex structure. To see that this is a reduction of generalized complex structures, we use a trivially extended G-action with $\mathfrak{g} = C^\infty(TM)$.

Then we have the commutative diagram:

$$\begin{tikzcd} C^\infty(TM) \arrow[r, "\mathrm{id}"] \arrow[d, hook, "\rho"'] & C^\infty(TM) \arrow[d, hook, "\gamma"'] \\ C^\infty(\mathbb{T}\mathcal{X}) \arrow[r] & C^\infty(T\mathcal{X}) \end{tikzcd}$$

where $\mathbb{T}\mathcal{X} \cong \mathbb{T}M \oplus \mathbb{T}(\mathbb{C}P^1 \times \mathbb{C}P^1)$. Hence $K = \gamma(\mathfrak{g}) = TM$ and

$$K^\perp = TM \oplus T(\mathbb{C}P^1 \times \mathbb{C}P^1) \oplus \mathrm{Ann}(TM) = TM \oplus T(\mathbb{C}P^1 \times \mathbb{C}P^1) \oplus T^*(\mathbb{C}P^1 \times \mathbb{C}P^1).$$

The G-invariant distributions are then given by

$$\Delta_s = \Delta_b = \pi(K^\perp) = \pi(K \oplus K^\perp) = TM \oplus T(\mathbb{C}P^1 \times \mathbb{C}P^1).$$

For a leaf $P \subset M$ of the distribution, we have the reduced Courant algebroid

$$
\begin{aligned}
E_{red} &= \frac{K^{\perp}|_P}{K \cap K^{\perp}|_P}\Big/ G \\
&= \frac{TM + T(\mathbb{C}P^1 \times \mathbb{C}P^1) + T^*(\mathbb{C}P^1 \times \mathbb{C}P^1)}{TM}\Big/ G \\
&= T(\mathbb{C}P^1 \times \mathbb{C}P^1) + \mathrm{Ann}(TM) \\
&= \mathbb{T}(\mathbb{C}P^1 \times \mathbb{C}P^1).
\end{aligned}
$$

Then the Courant algebroid $E \cong \mathbb{T}\mathcal{X}$ reduces to the Courant algebroid on $\mathbb{C}P^1 \times \mathbb{C}P^1$. Further, the Dirac structure L associated to $\mathcal{K}$ on $\mathcal{X}$ reduces to

$$
\begin{aligned}
L_{red} &= \frac{(L \cap (T_{\mathbb{C}}M + T_{\mathbb{C}}(\mathbb{C}P^1 \times \mathbb{C}P^1) + T^*_{\mathbb{C}}(\mathbb{C}P^1 \times \mathbb{C}P^1)) + T_{\mathbb{C}}M)|_P}{T_{\mathbb{C}}M|_P}\Big/ G \\
&= T^{0,1}(\mathbb{C}P^1 \times \mathbb{C}P^1) \oplus T^{1,0^*}(\mathbb{C}P^1 \times \mathbb{C}P^1),
\end{aligned}
$$

which is the Dirac structure associated to the generalized complex structure $\mathcal{J}_{red} = (\mathcal{J}_{I_\alpha}, \mathcal{J}_{I_\beta})$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$. Clearly, $L_{red} \cap \overline{L_{red}} = \{0\}$. Hence we get a generalized complex reduction

$$\mathcal{X} \to \mathbb{C}P^1 \times \mathbb{C}P^1.$$

(2) The sections of the fiber bundle $\mathcal{X} \to \mathbb{C}P^1 \times \mathbb{C}P^1$ are given by $\{m\} \times \mathbb{C}P^1 \times \mathbb{C}P^1$. As before, we call these sections *generalized twistor quadrics*. Since the generalized complex structure $\mathcal{K}$ on $\mathcal{X}$ decomposes as a block diagonal matrix acting separately on M and $\mathbb{C}P^1 \times \mathbb{C}P^1$, we get by definition that the sections of the fiber bundle are generalized complex factor submanifolds of $\mathcal{X}$.

To determine the normal bundle of these sections, note that on a smooth level, the manifold is $M \times S^2 \times S^2$, so the generalized normal bundle of these sections is smoothly isomorphic to $\mathbb{C}^2 \times S^2 \times S^2$. However, because the generalized complex structure $\mathcal{J}$ is dependent on a point $(\alpha, \beta) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, the bundle is twisted over the generalized complex manifold $\mathcal{X}$. We follow [**27**] by looking at the $+i$-eigenbundle of the generalized complex structure on M.

Recall from Section 2.2.3 that given a generalized almost Kähler pair $\mathcal{J}$, $\mathcal{J}'$ as in (2.2) and (2.3), we can decompose

$$(\mathbb{T}M) \otimes \mathbb{C} = L_1^+ \oplus L_1^- \oplus \overline{L_1^+} \oplus \overline{L_1^-}$$

where L_1^+ is the $+i$-eigenbundle of J_+ and L_1^- is the $+i$-eigenbundle for J_-. Then by using the generalized complex structure $\mathcal{J}$ for M in Section 3.1 we get

$$J_+ = a_1 I + a_2 J + a_3 K, \quad J_- = b_1 I + b_2 J + b_3 K$$

which means that $L_1^+ = T_+^{0,1}M$, $L_1^- = T_-^{0,1}M$.

Consider $\mathcal{X}$ as the fiber product $Z \times_M Z$, where Z is the classical twistor space for M. We claim that this is the same space as defined in Section 3.2. Then $\mathcal{J}$ induces an almost complex structure I_+ on one copy of Z and another almost complex structure I_- on the other copy of Z coming from the underlying bi-Hermitian structure on M. Let

$$p_1 : \mathcal{X} = Z \times_M Z \to Z_1, \quad p_2 : \mathcal{X} = Z \times_M Z \to Z_2$$

be projections down to the first and second copies of Z, respectively.

$$\begin{tikzcd}
 & Z \times_M Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
Z_1 \arrow[d, "\pi_1"] \arrow[dr] & & Z_2 \arrow[d, "\pi_2"] \arrow[dl] \\
\mathbb{C}P^1 & M & \mathbb{C}P^1
\end{tikzcd}$$

From [**27**], we know that the normal bundle of each section of $Z_1 \xrightarrow{\pi_1} \mathbb{C}P^1$ is $\mathbb{C}^2 \otimes \mathcal{O}(1)$, and similarly for $Z_2 \xrightarrow{\pi_2} \mathbb{C}P^1$. Pulling back these bundles up to $\mathcal{X}$, we get that the generalized normal bundle of a twistor quadric is isomorphic to

$$\mathbb{C}^2 \otimes (\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)),$$

which we write as $[\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)]^2$.

Note that along the diagonal $\alpha = \beta$, we get that the generalized normal bundle is

$$[\mathcal{O}(1+0) \oplus \mathcal{O}(0+1)]^2 = [\mathcal{O}(1) \oplus \mathcal{O}(1)]^2,$$

which agrees with the classical twistor theory setting, as the generalized normal bundle is the direct sum of the normal bundle and its dual.

(3) Proved above.

(4) Consider the antipodal map on $S^2 \times S^2$:

$$\tau : M \times S^2 \times S^2 \to M \times S^2 \times S^2$$

$$\tau(m, \alpha, \beta) = (m, -\frac{1}{\bar{\alpha}}, -\frac{1}{\bar{\beta}})$$

This map defines a real structure on $\mathcal{X}$, as we can easily see that τ maps the generalized complex structure to its conjugate since

$$\mathcal{K} \longmapsto -\mathcal{K}$$

and up to scalar multiplication,

$$\Phi \longmapsto \bar{\alpha}\bar{\beta}\sigma + (\bar{\alpha} - \bar{\beta})(-i + \frac{i\omega^2}{2}) - (\bar{\beta} + \bar{\alpha})\omega_I - \bar{\sigma} = -\bar{\Phi}.$$

Clearly, all of the generalized holomorphic data defined above is compatible with this real stucture.

□

Along with the properties detailed in the previous theorem, $\mathcal{X}$ is an interesting example of a generalized complex manifold because it is not purely of complex or symplectic type. Further, it is an example of a generalized complex structure that does not seem to be obtained by previously used techniques, including blow-ups [**15**] or Poisson deformations [**25**]. Recall from Theorem 1.2.7 that in a small enough neighborhood of a complex locus, a generalized complex structure is the Poisson deformation of a complex structure. Hence, locally around the complex locus $\pi^{-1}(\Delta) \subset \mathcal{X}$, there is some complex structure

I that can be deformed by a Poisson bivector β to $\mathcal{K}$ (where β vanishes along $\pi^{-1}(\Delta)$). This does not seem to be the case globally.

3.5. Extension to higher dimensions

In this section, we extend our results on K3 surfaces to hyperkähler manifolds. This is a natural next step, considering we used the underlying hyperkähler structure on a K3 surface to determine the family of generalized complex structures. Note that in this section, we do not refer to the generalized hyperkähler structure that M admits. This is simply because a generalized hyperkähler structure is unnecessarily complicated for our situation; by definition, a generalized hyperkähler structure is six generalized complex structures $\mathcal{J}_i$, $i = 1 \dots 6$ and a generalized metric $\mathcal{G}$ with certain bi-quaternionic relations.

3.5.1. Construction and Properties. Given a $4n$-dimensional hyperkähler manifold M, define a family of generalized almost complex structures over $S^2 \times S^2 \cong \mathbb{C}P^1 \times \mathbb{C}P^1$

$$\begin{aligned}
\mathcal{J} &= \frac{1}{2}\begin{pmatrix} -(I_+ + I_-) & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & I_+^* + I_-^* \end{pmatrix} \\
\mathcal{J}' &= \frac{1}{2}\begin{pmatrix} -(I_+ - I_-) & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & I_+^* - I_-^* \end{pmatrix},
\end{aligned}$$

where $I_\pm$, $\omega_\pm$ are defined by (3.1), (3.2) for $(a_1, a_2, a_3), (b_1, b_2, b_3) \in S^2$. However, we are now in dimension $4n$, so the spinor (3.5) must be modified slightly to fit this situation.

Similar to Section 3.1.3, we derive the spinor for $\mathcal{J}$ in terms of coordinates ξ, η as

$$\Phi_{\eta,\xi} = (-2i\xi)^n \exp\left(\frac{i}{2}\left(\frac{\sigma_\eta}{\xi} + \xi\bar{\sigma_\eta}\right)\right).$$

Further, the coordinate transformation from η, ξ to α, β will be the same. Then, up to multiplication by complex constant, we define the spinors as below.

Proposition 3.5.1. *Let M^{4n} be a hyperkähler manifold with generalized almost Kähler structures $\mathcal{J}$, $\mathcal{J}'$. Then the pure spinors for $\mathcal{J}$ and $\mathcal{J}'$, respectively, are*

$$\Phi = (-i(\alpha-\beta))^n \exp\left(\frac{i}{2}\left(\frac{i(\alpha+\beta)}{\alpha-\beta}\omega_I + \frac{i}{\alpha-\beta}\sigma - \frac{i\alpha\beta}{\alpha-\beta}\bar{\sigma}\right)\right) \tag{3.9}$$

and

$$\Phi' = (-i\bar{\beta}(\alpha+\bar{\beta}^{-1}))^n \exp\left(\frac{i}{2}\left(\frac{i(\alpha-\bar{\beta}^{-1})}{\alpha+\bar{\beta}^{-1}}\omega_I + \frac{i}{\alpha+\bar{\beta}^{-1}}\sigma + \frac{i\alpha\bar{\beta}^{-1}}{\alpha+\bar{\beta}^{-1}}\bar{\sigma}\right)\right), \tag{3.10}$$

where $\sigma = \omega_J + i\omega_K$. Further, $\mathcal{J}$, $\mathcal{J}'$ define an integrable generalized Kähler structure on M.

It is not immediately clear that Φ extends to the locus where $\alpha = \beta$ (or Φ' for $\alpha = -\bar{\beta}^{-1}$). If we write $\Phi = \Phi_{\alpha,\beta}$ after expanding the exponential

$$\Phi_{\alpha,\beta} = \sum_{j=0}^{2n} \frac{1}{j!} i^{n-j}(\alpha-\beta)^{n-j}(\sigma + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma})^j,$$

it seems that the terms for $j > n$ contain poles along the diagonal $\alpha = \beta$. However, this is not the case, and before proving Proposition 3.5.1, we will prove the following lemma.

Lemma 3.5.1. *Φ is smoothly defined for all α, β. In particular, if $j \neq n$ then the jth term in $\Phi = \Phi_{\alpha,\beta}$ is divisible by $(\alpha - \beta)$. Therefore, when $\alpha = \beta$ we have*

$$\Phi_{\alpha,\alpha} = \frac{1}{n!}(\sigma + 2\alpha\omega_I - \alpha^2\bar{\sigma})^n.$$

Proof. The statement is trivial for $j < n$, so assume $j > n$ and write $j = n+k$ with $0 < k \leq n$. Denote $\sigma + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma}$ by $\sigma_{\alpha,\beta}$ and $\sigma + 2\alpha\omega_I - \alpha^2\bar{\sigma}$ by σ_α. Then

$$\sigma_{\alpha,\beta} = \sigma_\alpha + (\alpha-\beta)\tau_\alpha,$$

where

$$\tau_\alpha = -\omega_I + \alpha\bar{\sigma} = -\frac{1}{2}\left(\frac{\partial}{\partial\alpha}(\sigma_\alpha)\right).$$

Now σ_α is a holomorphic two-form with respect to some complex structure, and therefore $\sigma_\alpha^{n+1} = 0$ by degree reasons. Differentiation with respect to α yields $\sigma_\alpha^n \tau_\alpha = 0$. Therefore, up to a constant, the $(n+k)$th term of $\Phi_{\alpha,\beta}$ is

$$\begin{aligned}
(\alpha-\beta)^{-k}\sigma_{\alpha,\beta}^{n+k} &= (\alpha-\beta)^{-k}(\sigma_\alpha + (\alpha-\beta)\tau_\alpha)^{n+k} \\
&= (\alpha-\beta)^{-k}\sum_{l=k+1}^{n+k}\binom{n+k}{l}\sigma_\alpha^{n+k-l}(\alpha-\beta)^l\tau_\alpha^l \\
&= \sum_{l=k+1}^{n+k}\binom{n+k}{l}\sigma_\alpha^{n+k-l}(\alpha-\beta)^{l-k}\tau_\alpha^l,
\end{aligned}$$

which is clearly divisible by $(\alpha-\beta)$. □

Similarly, Φ' extends along the locus $\alpha = -\bar{\beta}^{-1}$.

Proof. (Proposition 3.5.1) The proof that Φ, Φ' are the spinors for $\mathcal{J}, \mathcal{J}'$, respectively, is essentially the same as the proof of Proposition 3.1.1. In particular, for $\alpha \neq \beta$, $\mathcal{J}$ can be written as a B-field transform of a generalized complex structure of symplectic type $\mathcal{J}_\omega$, where

$$\begin{aligned}
\omega &= \frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta})}\omega_I + \frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta})}\omega_J \\
&\quad + \frac{\mathrm{Im}(\alpha)(|\beta|^2+1)-\mathrm{Im}(\beta)(|\alpha|^2+1)}{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta})}\omega_K
\end{aligned}$$

$$\begin{aligned}
B &= \frac{2\mathrm{Im}(\alpha\bar{\beta})}{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta})}\omega_I + \frac{\mathrm{Im}(\alpha)(1-|\beta|^2)-\mathrm{Im}(\beta)(1-|\alpha|^2)}{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta})}\omega_J \\
&\quad + \frac{\mathrm{Re}(\alpha)(|\beta|^2-1)-\mathrm{Re}(\beta)(|\alpha|^2-1)}{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta})}\omega_K.
\end{aligned}$$

The spinor for this generalized complex structure can be expressed by (3.9). If $\alpha = \beta$, the generalized complex structure $\mathcal{J}$ is of complex type with underlying complex structure

$$J_\alpha = \frac{1-\alpha\bar{\alpha}}{1+\alpha\bar{\alpha}}I + \frac{\alpha+\bar{\alpha}}{1+\alpha\bar{\alpha}}J - \frac{i(\alpha-\bar{\alpha})}{1+\alpha\bar{\alpha}}K.$$

On the other hand, by the lemma above for $\alpha = \beta$,

$$\Phi = \Phi_{\alpha,\alpha} = \frac{1}{n!}(\sigma + 2\alpha\omega_I - \alpha^2\bar{\sigma})^n$$

is a holomorphic $(2n, 0)$-form for the complex structure J_α. Thus, $\Phi_{\alpha,\alpha}$ is a pure spinor for the generalized complex structure $\mathcal{J}_{J_\alpha}$.

In order to show integrability, we note that for $\alpha \neq \beta$, $\mathcal{J}$ is clearly integrable as a B-field transform of a symplectic structure (similarly for $\mathcal{J}'$ if $\alpha \neq -\bar{\beta}^{-1}$). Along the locus $\alpha = \beta$, $\mathcal{J}$ is of complex type as in the K3 case, which is integrable since the underlying complex structure is.

All of the above results follow similarly for $\mathcal{J}'$ and Φ'.

$\square$

The *generalized twistor space* $\mathcal{X}$ of M is the smooth product $M \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with generalized almost complex structure defined on $\mathbb{T}_p\mathcal{X} = \mathbb{T}_m M \oplus \mathbb{T}_{(\alpha,\beta)}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ as

$$\mathcal{K} = (\mathcal{J}, \mathcal{J}_{I_\alpha}, \mathcal{J}_{I_\beta}),$$

where I_α, I_β denote the natural complex structures on each copy of $\mathbb{C}P^1$.

THEOREM 3.5.2. *The generalized twistor space for a $4n$-dimensional hyperkähler manifold is a generalized complex manifold; in other words, $\mathcal{K}$ is integrable.*

PROOF. As in the K3 case, let $\Psi = \Phi \wedge d\alpha \wedge d\beta \in \wedge^\bullet T^*\mathcal{X}$ be a pure spinor defining $\mathcal{K}$. Then

$$\begin{aligned} d\Psi &= d\Phi \wedge d\alpha \wedge d\beta \\ &= \left(\sum_{j=0}^{2n} \frac{1}{j!} i^{n-j} (d\alpha - d\beta)^{n-j} (\sigma + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma})^j \right) \wedge d\alpha \wedge d\beta \\ &\quad - \left(\sum_{j=0}^{2n} \frac{1}{j!} i^{n-j} (\alpha - \beta)^{n-j} d(\sigma + (\alpha+\beta)\omega_I - \alpha\beta\bar{\sigma})^j \right) \wedge d\alpha \wedge d\beta \end{aligned}$$

Then using the fact that $d\omega_I = d\omega_J = d\omega_K = 0$, we have

$$d\Psi = 0.$$

Hence Ψ defines an integrable generalized complex structure by Proposition 2.2.2. $\square$

Note that again, $\mathcal{X}$ is not generalized Kähler for the same reasons as in the K3 case. The next theorem states properties of the generalized twistor space of a hyperkähler manifold. We skip this proof, as the proof in dimension 4 extends easily to dimension $4n$.

THEOREM 3.5.3. *Let $\mathcal{X}$ be the generalized twistor space of a hyperkähler manifold M. Then:*

(1) $\mathcal{X}$ is a smooth fiber bundle

$$\pi : \mathcal{X} \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

and a reduction of generalized complex manifolds.

(2) The bundle admits a family of sections that are generalized complex factor submanifolds, each with generalized normal bundle isomorphic to

$$\mathbb{C}^{2n} \otimes (\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)).$$

(3) There is a pure spinor representing the generalized complex structure on $\mathcal{X}$ given by

$$\Psi = (-i(\alpha - \beta))^n \exp\left(\frac{i}{2}\left(\frac{i(\alpha+\beta)}{\alpha-\beta}\omega_I + \frac{i}{\alpha-\beta}\sigma - \frac{i\alpha\beta}{\alpha-\beta}\bar{\sigma}\right)\right) \wedge (d\alpha \wedge d\beta),$$

where σ is the holomorphic symplectic 2-form on M, and Ψ defines a structure of complex type along the diagonal, and of type 2 everywhere else.

(4) $\mathcal{X}$ has a real structure τ compatible with the above and inducing an antipodal map on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

CHAPTER 4

Quaternionic Kähler geometry

Quaternionic Kähler, or QK, manifolds are $4n$-dimensional manifolds with holonomy contained in $Sp(n) \cdot Sp(1)$. They have a natural quaternionic structure, as hyperkähler manifolds do, but in general are not complex manifolds. In fact, QK manifolds are generically not Kähler manifolds at all, despite their name. The twistor space of a QK manifold, however, is a complex manifold. In this chapter, we discuss the basics of quaternionic Kähler geometry. We describe the construction of the twistor space as well as its natural contact geometry. The last section will serve as a transition for examining the generalized twistor space of a QK manifold. We define the generalized complex structure on the twistor space of a QK manifold arising from the natural almost complex structure, and give a new proof of integrability of the almost complex structure using a pure spinor argument.

4.1. Definitions

This section will serve to define QK manifolds and their properties. We will follow [**7**] and [**40**] and refer the reader to these for a more thorough treatment of this subject. Define a group of transformations on $\mathbb{R}^{4n} \cong \mathbb{H}^n$ as

$$Sp(n) \cdot Sp(1) = \{v \longmapsto Avq^*, \quad A \in Sp(n), \quad q \in Sp(1)\}.$$

This is a subgroup of $SO(4n)$ such that

$$Sp(n) \cdot Sp(1) \cong Sp(n) \times Sp(1)/\{\pm I\}.$$

Definition 4.1.1. A quaternionic Kähler manifold (QK manifold) M is an oriented $4n$-dimensional Riemannian manifold such that the holonomy group is contained in $Sp(n) \cdot Sp(1)$.

Remarks.

(1) Definition 4.1.1 only holds for manifolds such that $n \geq 2$. In the case where $n = 1$, this would merely imply that M is an oriented Riemannian manifold, as $Sp(1) \cdot Sp(1) = SO(4)$. Thus, in dimension 4, we add the additional restriction that M must be self-dual and Einstein.

(2) Note that a hyperkähler manifold has holonomy $Sp(n)$. Thus, hyperkähler manifolds are a specific type of QK manifolds; they are those which have zero scalar curvature. Because we have already described twistor theory for hyperkähler manifolds, for the rest of this chapter we will assume that the scalar curvature is nonzero.

Consider a covering of M by open sets U_i. Then for each $p \in U_i$, we have three almost complex structures I, J, K on T_pM with quaternionic relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K, \quad \text{etc.}$$

and corresponding almost Kähler forms

$$\omega_I = g(I\cdot, \cdot), \quad \omega_J = g(J\cdot, \cdot), \quad \omega_K = g(K\cdot, \cdot), \tag{4.1}$$

where g is the metric on M. However, unlike on a hyperkähler manifold, I, J, and K are not necessarily globally defined. Further, they are not integrable and $d\omega_i \neq 0$ for $i = 1, 2, 3$. Instead, we can relate them by using a connection on M.

Let ∇ be a torsion-free (e.g. Levi-Civita) connection on a QK manifold M^{4n}. Then for local almost complex structures I, J, and K,

$$\begin{array}{rlll} \nabla I = & & -\theta^3 J & +\theta^2 K \\ \nabla J = & \theta^3 I & & -\theta^1 K \\ \nabla K = & -\theta^2 I & +\theta^1 J & \end{array}$$

where $\theta^1, \theta^2, \theta^3$ are called the *connection* 1-*forms* on M. Using this formula, we can develop similar relations between $d\omega_i$, ω_i, and θ^i. Further, there is a 4-form on M given by

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

Note that although $\omega_I, \omega_J, \omega_K$ are not d-closed, $d\Omega = 0$. In fact, Ω is globally defined and Ω^n is a non-vanishing volume form.

Example 4.1.2. Consider the manifold

$$\mathbb{H}P^n \cong (\mathbb{H}^{n+1} - \{0\})/\mathbb{H}^*,$$

where $\mathbb{H}^*$ acts on $\mathbb{H}^{n+1}$ on the right. This is a quaternionic Kähler manifold (for explanation, see [**7**]). Note that $\mathbb{H}P^1 \cong S^4$ admits no global almost complex structure and in fact, this is true for all n. Thus, although locally we have almost complex structures I, J, and K, these do not extend to all of $\mathbb{H}P^n$.

In fact, no QK manifold with positive scalar curvature admits a global compatible almost complex structure [**40**]. Symmetric QK manifolds with positive scalar curvature were classified by J.A. Wolf as the so-called Wolf spaces [**44**].

The following proposition is an important fact to note about QK manifolds. Recall that we have already defined a 4-dimensional QK manifold to be Einstein.

Proposition 4.1.3. *([**6**], [**40**]) A quaternionic Kähler manifold (M^{4n}, g) is Einstein for $n \geq 2$.*

4.2. The twistor space of a QK manifold

Let M be a quaternionic Kähler manifold. To construct the twistor space, we follow [**7**].

4.2.1. Construction. Let E be the 3-dimensional vector subbundle of $\mathrm{End}(TM)$ generated by I, J, K on each local chart U_i. Take the S^2-subbundle of E given by

$$Z = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}.$$

Z is called the *twistor space* of M. Let π denote projection down onto M

$$\pi : Z \to M$$

such that a point $z \in Z$ determines an almost complex structure $I_z = aI + bJ + cK$ on $T_{\pi(z)}M$.

The following theorem was proven using representation theory by Salamon in [**40**], however we will follow Bérard-Bergery's proof in [**7**].

THEOREM 4.2.1. *([**7**], [**40**]) The twistor space Z is a complex manifold of real dimension $4n + 2$.*

We define the complex structure on Z as follows. Choose a torsion-free $Sp(n) \cdot Sp(1)$-connection ∇ on M (e.g. the Levi-Civita connection). This connection determines a splitting of the tangent bundle $TZ = \mathcal{H} \oplus \mathcal{V}$ such that $\mathcal{V} = \ker(\pi)$ is the vertical distribution, tangent to the fibers of Z for each $m \in M$, and $\mathcal{H}$ is the supplementary horizontal distribution. For $z \in Z$, the canonical complex structure on S^2 induces a complex structure $\bar{J} : \mathcal{V}_z \to \mathcal{V}_z$. Further, for each $z \in Z$, we have an isomorphism defined by the tangent map

$$\pi_* : \mathcal{H}_z \to T_{\pi(z)}M.$$

We can lift the almost complex structure $I_z : T_{\pi(z)}M \to T_{\pi(z)}M$ to an endomorphism

$$\widehat{J} : \mathcal{H}_z \to \mathcal{H}_z.$$

Then we define the complex structure $\mathbb{J}$ on Z by the block diagonal matrix

$$\mathbb{J} = \begin{bmatrix} \widehat{J} & \\ & \bar{J} \end{bmatrix} : TZ = \mathcal{H} \oplus \mathcal{V} \to TZ. \tag{4.2}$$

The integrability of this almost complex structure depends on the vanishing of the Nijenhuis tensor which is given by

$$N\mathbb{J}(X,Y) = [\mathbb{J}(X), \mathbb{J}(Y)] - \mathbb{J}[X, \mathbb{J}(Y)] - \mathbb{J}[\mathbb{J}(X), Y] - [X,Y].$$

For the proof that $N\mathbb{J}(X,Y) = 0$, we refer the reader to [**7**]. In Section 4.3, we will present an alternate proof of Theorem 4.2.1 using generalized complex geometry.

4.2.1.1. *Example: Quaternionic Projective Space.* The twistor space of $\mathbb{H}P^n$ is $Z = \mathbb{C}P^{2n+1}$ and the S^2-fibration

$$\pi : \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$$

is the quaternionic Hopf fibration. Notice this means that the twistor space is not only a complex manifold, it is Kähler as well. In [**26**], Hitchin proved that $\mathbb{C}P^3$ was one of only two examples of Kählerian twistor spaces over a 4-dimensional manifold.

4.2.2. Contact geometry of the twistor space. The twistor space Z admits an additional structure; it is a complex contact manifold. We first recall some basic facts from contact geometry, directing the reader to [**37**] for a more thorough treatment of this subject.

DEFINITION 4.2.2. A *complex contact manifold* is a complex manifold X of odd complex dimension $2n+1$ together with an open covering $\{U_i\}$ by coordinate neighborhoods such that the following are true:

(1) On each U_i there is a 1-form θ_i with

$$\theta_i \wedge (d\theta_i)^n \neq 0.$$

(2) For every i, j, on $U_i \cap U_j$, there is a non-vanishing transition function f_{ij} : $U_i \cap U_J \to S^1$ such that

$$\theta_j = f_{ij}\theta_i.$$

We note that technically, this is the definition for a *normalized contact structure* on X, however we claim in our situation that this will not make a difference. The *horizontal subbundle* of a complex contact manifold is a holomorphic subbundle

$$\mathcal{H} = \{X \in T_pU_i : \theta_i(X) = 0\}$$

of complex rank $2n$. Further, given the complex contact structure, there exists a unique rank 2 subbundle $\mathcal{V}$ of TX, invariant under the complex structure, such that

$$TX \cong \mathcal{H} \oplus \mathcal{V}.$$

This subbundle is called the *vertical subbundle* of TX and there exists a projection map

$$\pi : TX \to \mathcal{H}$$

such that $\ker(\pi) = \mathcal{V}$. Thus, on each U_i, the contact structure θ_i is a $\mathcal{V}$-valued 1-form vanishing along $\mathcal{H}$. This will be important in determining the contact structure of the twistor space.

NOTE. At every point $p \in X$, the horizontal subspace $\mathcal{H}_p$ is a vector space of complex dimension $2n$ such that $d\theta_i|_{\mathcal{H}} \neq 0$. In fact, the form $d\theta_i|_{\mathcal{H}}$ defines an almost symplectic form on $\mathcal{H}$. Further, there exists an almost complex structure $J : \mathcal{H} \to \mathcal{H}$ compatible with this form. This fact will be illuminated in our discussion of the contact geometry of the twistor space in the next section.

The following proposition illustrates the relationship between QK manifolds and complex contact geometry. The proof can be found in [**40**].

PROPOSITION 4.2.3. *([**40**]) Let M be a quaternionic Kähler manifold with non-zero scalar curvature. The twistor space Z is a complex contact manifold.*

We will present a construction of the complex contact structure on Z in the next section. An alternate construction may be found in [**37**]. Note that the vertical and horizontal sections defined in Section 4.2.1, $\mathcal{V}$ and $\mathcal{H}$, respectively correspond to the vertical and horizontal sections defined above. Thus, the contact structure will be a local $\mathcal{V}$-valued 1-form vanishing on $\mathcal{H}$, where $\mathcal{V} = \ker(\pi)$ for $\pi : Z \to M$ and $TZ = \mathcal{V} \oplus \mathcal{H}$.

4.3. Generalized complex geometry of the twistor space

In this section, we illustrate integrability of the almost complex structure $\mathbb{J}$ on the twistor space of a quaternionic Kähler manifold using generalized complex geometry. This will serve as a first step towards constructing the generalized twistor space. In the first section, we consider the twistor space Z of a 4-dimensional manifold M. We use the complex contact geometry of Z in order to construct a spinor for the generalized complex structure of complex type. This contact geometry was defined in [**37, 40**]. However, since we are in 4-dimensions, we will use Salamon's results on twistor spaces of 4-manifolds from [**42**] in order to describe the contact structure in more detail. In the second section we extend our results to higher dimensions.

4.3.1. The twistor space as a generalized complex manifold. Let M be a 4-dimensional QK manifold with positive scalar curvature. In this section, we present the twistor space in a slightly different way, although we claim that it defines the same space as in Section 4.2.1. Let the twistor space Z be the bundle of forms

$$Z = \{x\omega_I + y\omega_J + z\omega_K \;\mid\; (x, y, z) \in S^2\}$$

where $\omega_I, \omega_J, \omega_K$ are local 2-forms on M as given in equation (4.1). Recall that these forms satisfy the following relations:

$$
\begin{aligned}
d\omega_I &= && -\theta^3\omega_J && +\theta^2\omega_K \\
d\omega_J &= \theta^3\omega_I && && -\theta^1\omega_K \\
d\omega_K &= -\theta^2\omega_I && +\theta^1\omega_J &&
\end{aligned}
\tag{4.3}
$$

where θ_i are connection 1-forms on M. Let $b^i, i = 1 \ldots 3$ be 1-forms on Z defined by

$$b^1 = dx + y\pi^*\theta^3 - z\pi^*\theta^2$$

$$b^2 = dy - x\pi^*\theta^3 + z\pi^*\theta^1$$

$$b^3 = dz + x\pi^*\theta^2 - y\pi^*\theta^1.$$

We claim that these forms annihilate the horizontal space $\mathcal{H} \subset TZ$ as defined above. Further, define the *curvature 2-forms* by

$$
\begin{aligned}
\Psi_1 &= d\theta^1 + \theta^3 \wedge \theta^2 \\
\Psi_2 &= d\theta^2 + \theta^1 \wedge \theta^3 \\
\Psi_3 &= d\theta^3 + \theta^2 \wedge \theta^1.
\end{aligned}
\tag{4.4}
$$

These curvature 2-forms are related to ω_i in the following way. Let c denote the scalar curvature of M. Then

$$\Psi_1 = \frac{c}{2}\omega_I, \qquad \Psi_2 = \frac{c}{2}\omega_J, \qquad \Psi_3 = \frac{c}{2}\omega_K. \tag{4.5}$$

We define a generalized almost complex structure of complex type on Z by a representative pure spinor. Recall that since $\dim_{\mathbb{R}}(Z) = 6$, this is equivalent to defining a non-vanishing $(3,0)$-form , which we obtain from the contact structure on Z. Let

$$\theta = \frac{-iy}{(1+x)(y+iz)}b^3 + \frac{iz}{(1+x)(y+iz)}b^2 - \frac{1}{(1+x)(y+iz)}b^1. \tag{4.6}$$

We claim that this is a (local) contact 1-form on Z. To see this, consider the vertical space $\mathcal{V}_p$ at a point $p \in Z$. Note that there is an isomorphism $\mathcal{V}_p \xrightarrow{\sim} S^2$. Consider $S^2 \cong \mathbb{C}P^1$ as a complex manifold by stereographic projection

$$(x, y, z) \mapsto \left(\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{2\mathrm{Re}(\zeta)}{1 + \zeta\bar{\zeta}}, \frac{-2\mathrm{Im}(\zeta)}{1 + \zeta\bar{\zeta}} \right).$$

We can write ζ as

$$\zeta = \frac{y - iz}{1 + x}.$$

Then the space of holomorphic 1-forms on S^2 is spanned by $d\zeta$, which we write in coordinates as

$$d\zeta = \frac{-iydz + izdy - dx}{(1 + x)(y + iz)}.$$

Here, we use that

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad xdx + ydy + zdz = 0.$$

Pulling this back to $\mathcal{V}$, the forms dx, dy, dz are twisted by the curvature 1-forms, so we get (4.6), a $\mathcal{V}$-valued 1-form vanishing on $\mathcal{H}$. We can simplify (4.6), so that

$$\theta = d\zeta + i\zeta\theta^1 - \frac{1}{2}i(1 - \zeta^2)\theta^2 - \frac{1}{2}(1 + \zeta^2)\theta^3, \tag{4.7}$$

since

$$\frac{-i(y^2 + z^2)}{(1 + x)(y + iz)} = \frac{-i(y + iz)(y - iz)}{(1 + x)(y + iz)} = -i\zeta$$

$$\begin{aligned} \frac{ixy - z}{(1 + x)(y + iz)} &= \frac{ixy^2 + iz^2 - yz + xyz}{(1 + x)(y + iz)(y - iz)} \\ &= \frac{-yz(1 - x) + i((1 - x)(1 + x) - y^2(1 - x))}{(1 + x)^2(1 - x)} \\ &= \frac{-yz + i + ix - iy^2}{(1 + x)^2} = \frac{1}{2}i(1 - \zeta^2) \end{aligned}$$

$$\begin{aligned}\frac{ixz+y}{(1+x)(y+iz)} &= \frac{-i(-xyz+yz+iy^2+ixz^2)}{(1+x)(y+iz)(y-iz)}\\ &= \frac{-i(yz(1-x)+i((1-x)(1+x)-z^2(1-x)))}{(1+x)^2(1-x)}\\ &= \frac{-i(yz+i((1+x)-z^2))}{(1+x)^2} = \frac{1}{2}(1+\zeta^2).\end{aligned}$$

Consider now $d\theta$:

$$d\theta = d\zeta\wedge(i\theta^1+i\zeta\theta^2-\zeta\theta^3)+i\zeta d\theta^1-\frac{1}{2}i(1-\zeta^2)d\theta^2-\frac{1}{2}(1+\zeta^2)d\theta^3.$$

Using (4.4) and (4.5), this reduces to

$$d\theta = \theta\wedge(i\theta^1+i\zeta\theta^2-\zeta\theta^3)+\tfrac{1}{2}ic(2\zeta\omega_1+(\zeta^2-1)\omega_2+i(\zeta^2+1)\omega_3) = \theta\wedge(i\theta^1+i\zeta\theta^2-\zeta\theta^3)+d\theta|_{\mathcal{H}},$$

where $\mathcal{H}$ is the horizontal subbundle such that $b^1=b^2=b^3=0$. Note the resemblance between $d\theta|_{\mathcal{H}}$ and its hyperkähler analogue in (2.1). In fact, they should be identical up to multiplication by a complex scalar. Recall that on the horizontal subbundle of a complex contact manifold, there exists an almost holomorphic symplectic structure defined by $d\theta|_{\mathcal{H}}$ compatible with an almost complex structure J. This almost complex structure on the horizontal subspace at a point $p\in Z$ can be given by

$$\widehat{J}_p = xI+yJ+zK, \qquad (x,y,z)\in S^2.$$

Further, $d\theta|_{\mathcal{H}} = \frac{1}{2}ic(2\zeta\omega_1+(\zeta^2-1)\omega_2+i(\zeta^2+1)\omega_3)$ is the almost symplectic form compatible with $\widehat{J}_p$. When M is hyperkähler, this is the holomorphic symplectic form on M associated to the complex structure $\widehat{J}_p$. However, since a QK manifold is not a complex manifold, in this case it simply defines a non-vanishing local 2-form compatible with the almost complex structure.

Hence we have that (4.7) is a (local) contact 1-form, such that

$$\theta\wedge d\theta = \theta\wedge d\theta|_{\mathcal{H}} \neq 0$$

is a nonvanishing 3-form. Define a generalized complex structure by the following pure spinor

$$\phi = \theta \wedge d\theta|_H = (d\zeta + i\zeta\theta^1 - \tfrac{1}{2}i(1-\zeta^2)\theta^2 - \tfrac{1}{2}(1+\zeta^2)\theta^3) \wedge \left(-\tfrac{1}{2}ic(\zeta^2\sigma + 2\zeta\omega_I - \bar{\sigma})\right), \tag{4.8}$$

where $\sigma = \omega_2 + i\omega_3$. We claim that ϕ is a 3-form compatible with the almost complex structure $\mathbb{J}$ given by (4.2), so this defines the generalized almost complex structure of complex type

$$\mathcal{J} = \begin{bmatrix} -\mathbb{J} & 0 \\ 0 & \mathbb{J}^* \end{bmatrix}.$$

PROPOSITION 4.3.1. *The generalized almost complex structure on Z given by the above spinor ϕ is integrable.*

REMARK. Although the contact form defined above is only local, we note that the integrability condition given in Proposition 2.2.2 (which we will use to prove the proposition) is also local. Further, since we know that the contact 1-forms patch together nicely with non-vanishing S^1-valued functions, i.e.

$$\theta_i = f_{ij}\theta_j, \qquad f_{ij} : U_i \cap U_j \to S^1,$$

it will be enough to prove that the generalized almost complex structure is locally integrable in order to prove that it is integrable on all of Z.

PROOF. Recall from Proposition 2.2.2 that a generalized complex structure given by a spinor ϕ is integrable if there exists some $X + \xi \in T \oplus T^*$ such that

$$d\phi = (X + \xi) \cdot \phi = i_X\phi + \xi \wedge \phi.$$

Since $\phi \in \Omega^3(Z)$, $d\phi \in \Omega^4(Z)$, so we want to find some $\xi \in T^*Z$ such that

$$d\phi = \xi \wedge \phi.$$

Consider

$$d\phi = d(\theta \wedge d\theta|_{\mathcal{H}}) = d\theta \wedge d\theta|_{\mathcal{H}} - \theta \wedge d(d\theta|_{\mathcal{H}}). \tag{4.9}$$

We have $d\theta|_{\mathcal{H}} \wedge d\theta|_{\mathcal{H}} = 0$, so

$$d\theta \wedge d\theta|_{\mathcal{H}} = \theta \wedge (i\theta^1 + i\zeta\theta^2 - \zeta\theta^3) \wedge d\theta|_{\mathcal{H}} + d\theta|_{\mathcal{H}} \wedge d\theta|_{\mathcal{H}} = \theta \wedge (i\theta^1 + i\zeta\theta^2 - \zeta\theta^3) \wedge d\theta|_{\mathcal{H}}.$$

Now consider the second half of equation (4.9)

$$\begin{aligned}
d(d\theta|_{\mathcal{H}}) &= \frac{1}{2}ic((d\zeta)(2\omega_1 + 2\zeta\omega_2 + 2i\zeta\omega_3) + 2\zeta d\omega_1 + (\zeta^2 - 1)d\omega_2 + i(\zeta^2 + 1)d\omega_3) \\
&= \frac{1}{2}ic(d\zeta(2\omega_1 + 2\zeta\omega_2 + 2i\zeta\omega_3) + (-i(1 + \zeta^2)\theta^2 + (\zeta^2 - 1)\theta^3)\omega_1 \\
&\quad +(i(1 + \zeta^2)\theta^1 - 2\zeta\theta^3)\omega_2 + (-(\zeta^2 - 1)\theta^1 + 2\zeta\theta^2)\omega_3),
\end{aligned}$$

where we use equation (4.3) to obtain the second line. Thus,

$$\begin{aligned}
\theta \wedge d(d\theta|_{\mathcal{H}}) &= d\zeta \wedge (\tfrac{1}{2}ic)((-i(1 + \zeta^2)\theta^2 + (\zeta^2 - 1)\theta^3)\omega_1 + (i(1 + \zeta^2)\theta^1 - 2\zeta\theta^3)\omega_2 \\
&\qquad +(-(\zeta^2 - 1)\theta^1 + 2\zeta\theta^2)\omega_3) \\
&\quad +(i\zeta\theta^1 - \tfrac{1}{2}i(1 - \zeta^2)\theta^2 - \tfrac{1}{2}(1 + \zeta^2)\theta^3) \wedge (\tfrac{1}{2}ic)(d\zeta) \wedge (2\omega_1 \\
&\qquad\quad +2\zeta\omega_2 + 2i\zeta\omega_3) \\
&\quad +(i\zeta\theta^1 - \tfrac{1}{2}i(1 - \zeta^2)\theta^2 - \tfrac{1}{2}(1 + \zeta^2)\theta^3) \wedge ((-i(1 + \zeta^2)\theta^2(\zeta^2 - 1)\theta^3)\omega_1 \\
&\qquad +(i(1 + \zeta^2)\theta^1 - 2\zeta\theta^3)\omega_2 + (-(\zeta^2 - 1)\theta^1 + 2\zeta\theta^2)\omega_3) \\
&= \tfrac{1}{2}ic(d\zeta \wedge ((-2i\zeta\theta^1 - 2i\zeta^2\theta^2 + 2\zeta^2\theta^3)\omega_1 \\
&\qquad +(i(1 - \zeta^2)\theta^1 + i\zeta(1 - \zeta^2)\theta^2 - \zeta(1 - \zeta^2)\theta^3)\omega_2 \\
&\qquad +((\zeta^2 + 1)\theta^1 + \zeta(1 + \zeta^2)\theta^2 + i\zeta(1 + \zeta^2)\theta^3)\omega_3) \\
&\quad +(i\zeta\theta^1 - \tfrac{1}{2}i(1 - \zeta^2)\theta^2 - \tfrac{1}{2}(1 + \zeta^2)\theta^3) \wedge (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3)) \wedge d\theta|_{\mathcal{H}}.
\end{aligned}$$

Simplifying this expression, we get

$$\theta \wedge d(d\theta|_{\mathcal{H}}) = \theta \wedge (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge d\theta|_{\mathcal{H}}.$$

Then equation (4.9) is equal to

$$\begin{aligned} d\phi &= (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge \theta \wedge d\theta|_{\mathcal{H}} - \theta \wedge (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge d\theta|_{\mathcal{H}} \\ &= 2(-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge \theta \wedge d\theta|_{\mathcal{H}} \\ &= \xi \wedge \phi. \end{aligned}$$

Hence, ϕ determines an integrable generalized complex structure on Z. $\square$

4.3.2. Extension to higher dimensions. Although our construction depended on the results on twistor spaces for 4-dimensional manifolds, this proof can be extended to all quaternionic Kähler manifolds of dimension $4n$. We simply need the contact structure of the twistor space, which exists for all QK manifolds. We illustrate the generalization here.

Let M be a QK manifold of real dimension $4n$ with twistor space of dimension $4n+2$

$$Z = \{x\omega_I + y\omega_J + z\omega_K \ \mid \ (x, y, z) \in S^2\}.$$

Then the local contact structure on Z can be given as in the previous section by

$$\theta = d\zeta + i\zeta\theta^1 - \frac{1}{2}i(1-\zeta^2)\theta^2 - \frac{1}{2}(1+\zeta^2)\theta^3,$$

where $\theta^1, \theta^2, \theta^3$ are the connection 1-forms on Z, and ζ gives a complex parameter on $\mathcal{V}_p \cong \mathbb{C}P^1$, $p \in Z$. Then

$$d\theta = \theta \wedge (i\theta^1 + i\zeta\theta^2 - \zeta\theta^3) + \tfrac{1}{2}ic(2\zeta\omega_1 + (\zeta^2-1)\omega_2 + i(\zeta^2+1)\omega_3) = \theta \wedge (i\theta^1 + i\zeta\theta^2 - \zeta\theta^3) + d\theta|_{\mathcal{H}},$$

where again $\mathcal{H}$ is the horizontal subbundle such that $b^1 = b^2 = b^3 = 0$.

Consider Z with a generalized almost complex structure $\mathcal{J}_{\mathbb{J}}$ of complex type determined by $\mathbb{J}$, (4.2). Then we claim that the spinor for $\mathcal{J}_{\mathbb{J}}$ is

$$\phi = \theta \wedge (d\theta)^n = \theta \wedge (d\theta|_{\mathcal{H}})^n.$$

We can compute that this defines an integrable generalized complex structure, since

$$\begin{aligned}
d\phi &= d\theta \wedge (d\theta|_{\mathcal{H}})^n - \theta \wedge d(d\theta|_{\mathcal{H}})^n \\
&= \theta \wedge (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge (d\theta|_{\mathcal{H}})^n - \theta \wedge n(d\theta|_{\mathcal{H}})^{n-1} \wedge d(d\theta|_{\mathcal{H}}) \\
&= \theta \wedge (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge (d\theta|_{\mathcal{H}})^n - n(d\theta|_{\mathcal{H}})^{n-1} \wedge \theta \wedge (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge d\theta|_{\mathcal{H}} \\
&= (-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge \phi - n(i\theta^1 + i\zeta\theta^2 - \zeta\theta^3) \wedge \phi \\
&= (n+1)(-i\theta^1 - i\zeta\theta^2 + \zeta\theta^3) \wedge \phi.
\end{aligned}$$

This proves that the complex structure $\mathbb{J}$ on the twistor space Z is integrable for all QK manifolds M of dimension $4n$.

CHAPTER 5

Generalized twistor space for quaternionic Kähler manifolds

In this chapter, we introduce generalized twistor theory for QK manifolds. Although this is a natural extension, QK manifolds lack the global complex and symplectic structures that hyperkähler manifolds have, so the theory is much more intricate. Instead, we use the complex contact geometry of the classical twistor space.

In the first section, we construct a family of local generalized almost complex structures on a QK manifold of dimension $4n$ using bi-Hermitian data, as in the hyperkähler setting. We then in Section 2 use that family and the classical twistor space Z to describe the generalized twistor space, a bundle of generalized almost complex structures over the QK manifold. We conjecture that the generalized twistor space is a generalized complex manifold, and describe its structure. In the final section of this chapter, we give a candidate for the spinor of the generalized twistor space, using the contact geometry of the classical twistor space.

5.1. The generalized twistor family

Consider a quaternionic Kähler manifold M^{4n} with Riemannian metric g. As before, let U_i be a covering of M and let I,J, and K be local almost complex structures for each U_i, such that g is Hermitian for I, J, and K and $IJ = -JI = K$. Further, let $\omega_I = gI$, $\omega_J = gJ$, and $\omega_K = gK$ be local 2-forms corresponding to the almost complex structures. Then we can construct a local almost bi-Hermitian structure on M given by

$$(g, \quad b = 0, \quad J_+ = x_1 I + y_1 J + z_1 K, \quad J_- = x_2 I + y_2 J + z_2 K)$$

with $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^2$. As in Section 3.1.2, there are two (local) generalized almost complex structures $\mathcal{J}, \mathcal{J}'$ on $\mathbb{T}M$ that can be reconstructed from the almost bi-Hermitian data (g, b, J_+, J_-) such that

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} -(J_+ + J_-) & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & J_+^* + J_-^* \end{pmatrix} \tag{5.1}$$

$$\mathcal{J}' = \frac{1}{2}\begin{pmatrix} -(J_+ - J_-) & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & J_+^* - J_-^* \end{pmatrix} \tag{5.2}$$

where $\omega_\pm = gJ_\pm$. Then $\mathcal{J}, \mathcal{J}'$ is a local generalized almost Kähler structure on $U_i \subset M$ for every pair of points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^2$. Thus this defines an $S^2 \times S^2$-family of local generalized almost Kähler structures over U_i for every i. The bundle of these families will determine our generalized twistor space.

5.2. The generalized twistor space of a QK manifold

In this section, we use the classical QK case to construct the generalized twistor space. Let $\mathcal{X}$ be the space defined by the fiber product

$$\mathcal{X} := Z \times_M Z,$$

where Z is the classical twistor space of M. Note that $\mathcal{X}$ sits as an $S^2 \times S^2$-bundle over M

$$\pi : \mathcal{X} \to M$$

such that each point $z \in \mathcal{X}$ defines a pair of local almost complex structures J_+ and J_- on $T_{\pi(z)}M$ as described in the previous section. These, combined with the Riemannian metric g on M, determine an almost bi-Hermitian structure $(g, b = 0, J_\pm)$ on M for every point $z \in \mathcal{X}$. Hence, $\mathcal{X}$ is an $S^2 \times S^2$-bundle of local almost bi-Hermitian structures on M. Equivalently, we can view $\mathcal{X}$ as a bundle of local generalized almost Kähler structures $\mathcal{J}, \mathcal{J}'$ on M defined as in equations (5.1), (5.2).

Alternatively, we can view this space as a subbundle of the endomorphisms of $\mathbb{T}M$. Let $\mathbb{E}$ be the 6-dimensional vector subbundle of $\mathrm{End}(\mathbb{T}M)$ generated by $\mathcal{J}_I, \mathcal{J}_J, \mathcal{J}_K$ and

$\mathcal{J}_{\omega_1}, \mathcal{J}_{\omega_2}, \mathcal{J}_{\omega_3}$ on each local chart U_i. Then we claim that $\mathcal{X}$ is the $S^2 \times S^2$-subbundle of $\mathbb{E}$ defined by

$$
\begin{aligned}
\mathcal{X} \;\; = \;\; & \{\tfrac{1}{2}((x_1+x_2)\mathcal{J}_I + (y_1+y_2)\mathcal{J}_J + (z_1+z_2)\mathcal{J}_K \\
& \quad +(x_1-x_2)\mathcal{J}_{\omega_I} + (y_1-y_2)\mathcal{J}_{\omega_J} + (z_1-z_2)\mathcal{J}_{\omega_K}) \;\; | \;\; (x_j, y_j, z_j) \in S^2\}.
\end{aligned}
$$

DEFINITION 5.2.1. The $S^2 \times S^2$-subbundle $\mathcal{X}$ is called the *generalized twistor space* of a quaternionic Kähler manifold M.

We define a generalized almost complex structure on $\mathbb{T}\mathcal{X}$ as follows. Choose a torsion-free $Sp(n) \cdot Sp(1)$-connection ∇ on M. The connection induces a splitting of the tangent bundle $T\mathcal{X} = \mathcal{V} \oplus \mathcal{H}$ so that

$$T\mathcal{X} \oplus T^*\mathcal{X} = \mathcal{V} \oplus \mathcal{V}^* \oplus \mathcal{H} \oplus \mathcal{H}^*,$$

where $\mathcal{V}$ is the vertical distribution (tangent to the fibers of $\pi : \mathcal{X} \to M$) and $\mathcal{H}$ is the supplementary horizontal distribution. We define their duals $\mathcal{V}^*$ and $\mathcal{H}^*$ as the forms vanishing on these spaces.

We claim that horizontal transport associated to $\mathcal{H} \oplus \mathcal{H}^*$ preserves the canonical metric of the fibers $S^2 \times S^2$ as well as their orientation, and thus preserves the canonical complex structure on $S^2 \times S^2$. This complex structure induces a generalized complex structure $\overline{\mathcal{J}}$ on $\mathcal{V}_z \oplus \mathcal{V}_z^*$ for each $z \in \mathcal{X}$.

Note that for every $z \in \mathcal{X}$ we have an isomorphism defined by the tangent map

$$\pi_* : \mathcal{H}_z \to T_{\pi(z)}M.$$

Further, each $z \in \mathcal{X}$ defines a local generalized almost complex structure on the space $\mathbb{T}_{\pi(z)}M$ given by equation (5.1). We can lift this generalized structure up to an endomorphism $\widehat{\mathcal{J}}$ on $\mathcal{H}_z \oplus \mathcal{H}_z^*$ by $\pi_* \oplus (\pi_*^{-1})^*$.

Define a natural generalized almost complex structure $\underline{\mathcal{J}}$ on $\mathcal{X}$ by the block diagonal matrix:

$$\underline{\mathcal{J}} = \begin{bmatrix} \widehat{\mathcal{J}} & \\ & \overline{\mathcal{J}} \end{bmatrix} : \mathbb{T}\mathcal{X} = \mathcal{H} \oplus \mathcal{H}^* \oplus \mathcal{V} \oplus \mathcal{V}^* \to \mathbb{T}\mathcal{X} \tag{5.3}$$

Conjecture 5.2.1. *Let M be a quaternionic Kähler manifold of dimension $4n$. Then its generalized twistor space $\mathcal{X}$ is a generalized complex manifold with generalized complex structure $\underline{\mathcal{J}}$ as defined above.*

We would like to prove this conjecture by defining a spinor for the generalized complex structure $\underline{\mathcal{J}}$. In the next section, we derive a candidate for this spinor if $\dim(M) = 4$, and provide some justification on why it is appropriate.

5.3. Candidate for a spinor

Recall in Section 4.3 that we used the contact geometry of the twistor space Z to determine a spinor for the generalized almost complex structure on Z. In this section, we expand on that spinor, using our results from the generalized twistor space of a hyperkähler manifold.

Let M be a QK manifold of dimension 4 (e.g. $\mathbb{H}P^1$). Define holomorphic coordinates α and β on $S^2 \times S^2 \cong \mathbb{C}P^1 \times \mathbb{C}P^1$, such that for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^2$,

$$\alpha = \frac{y_1 - iz_1}{1 + x_1}, \qquad \beta = \frac{y_2 - iz_2}{1 + x_2}.$$

Then

$$d\alpha = \frac{-iy_1 dz_1 + iz_1 dy_1 - dx_1}{(1 + x_1)(y_1 + iz_1)}, \qquad d\beta = \frac{-iy_2 dz_2 + iz_2 dy_2 - dx_2}{(1 + x_2)(y_2 + iz_2)}$$

define the space of holomorphic 1-forms on each copy of $S^2 \cong \mathbb{C}P^1$. Pulling these back to $\mathcal{V}$, we get a twisting by the curvature 1-forms θ_i, as in Section 4.3, such that

$$\begin{aligned} p_1^*(d\alpha) &= d\alpha + i\alpha\theta^1 - \frac{1}{2}i(1 - \alpha^2)\theta^2 - \frac{1}{2}(1 + \alpha^2)\theta^3 \\ p_2^*(d\beta) &= d\beta + i\beta\theta^1 - \frac{1}{2}i(1 - \beta^2)\theta^2 - \frac{1}{2}(1 + \beta^2)\theta^3, \end{aligned} \tag{5.4}$$

where $p_1 : \mathcal{V} \to \mathbb{C}P^1$ and $p_2 : \mathcal{V} \to \mathbb{C}P^1$ are the projection maps down to each copy of $\mathbb{C}P^1$. Let ψ_1 and ψ_2 denote $p_1^*(d\alpha)$, $p_2^*(d\beta)$, respectively. Then

$$\begin{aligned}
\psi_1 \wedge \psi_2 &= d\alpha \wedge d\beta + d\alpha \wedge (i\beta\theta^1 - \frac{1}{2}i(1-\beta^2)\theta^2 - \frac{1}{2}(1+\beta^2)\theta^3) \\
&\quad -d\beta \wedge (i\alpha\theta^1 - \frac{1}{2}i(1-\alpha^2)\theta^2 - \frac{1}{2}(1+\alpha^2)\theta^3) \\
&\quad +(i\alpha\theta^1 - \frac{1}{2}i(1-\alpha^2)\theta^2 - \frac{1}{2}(1+\alpha^2)\theta^3) \wedge (i\beta\theta^1 \\
&\qquad -\frac{1}{2}i(1-\beta^2)\theta^2 - \frac{1}{2}(1+\beta^2)\theta^3) \\
&= d\alpha \wedge d\beta + d\alpha \wedge (i\beta\theta^1 - \frac{1}{2}i(1-\beta^2)\theta^2 - \frac{1}{2}(1+\beta^2)\theta^3) \\
&\quad -d\beta \wedge (i\alpha\theta^1 - \frac{1}{2}i(1-\alpha^2)\theta^2 - \frac{1}{2}(1+\alpha^2)\theta^3) \\
&\quad +\frac{1}{2}(\alpha-\beta)((1+\alpha\beta)\theta^1\theta^2 - i(1-\alpha\beta)\theta^1\theta^3 - i(\alpha+\beta)\theta^2\theta^3)
\end{aligned}$$

defines a $\mathcal{V}$-valued 2-form vanishing on $\mathcal{H}$. Note that $\psi_1 \wedge \psi_2$ provides a sort of generalization of the contact 1-form θ in Section 4.3. However, we cannot just simply take the derivative of this form in order to determine the part of the spinor defined along $\mathcal{H}$. Our generalized almost complex structure defined in equation (5.3) is not of constant type; it changes throughout the manifold. Therefore, the spinor should be a mixed form as in the K3 case. In fact, we can use the derivation of the spinor (3.5) to find that an appropriate candidate is

$$\begin{aligned}
\Phi &= (\alpha\beta - 1)\omega_2 + i(\alpha\beta+1)\omega_3 + (\alpha-\beta)(-i + \tfrac{i}{2}\omega^2) + (\alpha+\beta)\omega_1 \\
&= -i(\alpha-\beta) + (\alpha+\beta)\omega_1 + (\alpha\beta-1)\omega_2 + i(\alpha\beta+1)\omega_3 + \tfrac{i}{2}(\alpha-\beta)\omega^2.
\end{aligned} \tag{5.5}$$

PROPOSITION 5.3.1. *There is a pure spinor for the generalized almost complex structure $\underline{\mathcal{J}}$ defined by*

$$\Psi = \psi_1 \wedge \psi_2 \wedge \Phi,$$

where ψ_1, ψ_2, Φ are as in equations (5.4) *and* (5.5).

PROOF. Along the subbundle such that $\alpha = \beta$, note that $\underline{\mathcal{J}}|_{\alpha=\beta}$ is simply the generalized almost complex structure arising from the usual complex structure on the twistor space, and Ψ is just the spinor defined by

$$\Psi = \Psi_{\alpha=\beta} = \psi_1 \wedge \psi_2 \wedge \left(\frac{1}{2} ic(2\zeta\omega_1 + (\zeta^2 - 1)\omega_2 + i(\zeta^2+1)\omega_3) \right).$$

On the other hand, if $\alpha \neq \beta$, we can show as in Appendix A that the generalized almost complex structure $\underline{\mathcal{J}}|_{\mathcal{H}\oplus\mathcal{H}^*} = \widehat{\mathcal{J}}$ is a B-field transform of a generalized almost complex structure determined by a 2-form ω, where

$$\begin{aligned} \omega &= \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_I + \frac{\mathrm{Re}(\alpha)(1+|\beta|^2) - \mathrm{Re}(\beta)(1+|\alpha|^2)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_J \\ &\quad + \frac{\mathrm{Im}(\alpha)(|\beta|^2+1) - \mathrm{Im}(\beta)(|\alpha|^2+1)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_K \end{aligned}$$

$$\begin{aligned} B &= \frac{2\mathrm{Im}(\alpha\bar{\beta})}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_I + \frac{\mathrm{Im}(\alpha)(1-|\beta|^2) - \mathrm{Im}(\beta)(1-|\alpha|^2)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_J \\ &\quad + \frac{\mathrm{Re}(\alpha)(|\beta|^2-1) - \mathrm{Re}(\beta)(|\alpha|^2-1)}{|\alpha|^2 + |\beta|^2 - 2\mathrm{Re}(\alpha\bar{\beta})}\omega_K. \end{aligned}$$

However, B and ω are not closed 2-forms, so this fact does not help prove integrability. Nevertheless, the $\mathcal{H}$-valued spinor Φ should then be the same as in the hyperkähler case, up to multiplication by a complex scalar, which is clearly true. Along the vertical component, ψ_1 and ψ_2 give holomorphic 1-forms defining the generalized complex structure of complex type $\underline{\mathcal{J}}|_{\mathcal{V}\oplus\mathcal{V}^*} = \overline{\mathcal{J}}$. Hence Ψ is a pure spinor defining the generalized almost complex structure $\underline{\mathcal{J}}$ on $\mathcal{X}$. $\square$

In order to prove Conjecture 5.2.1, we would like to show that locally, there exists some element $X + \xi \in \mathbb{T}\mathcal{X}$ such that $d\Psi = (X+\xi)\cdot\Psi$. In order to to this, it is possible that we may have to consider the *twisted Courant bracket* (see [**23**]) on $\mathcal{X}$, or simply define the generalized almost complex structure differently.

If this conjecture is true, it would open up possibilities to extend the generalized twistor space construction to other manifolds. In particular, we could easily extend this to all self-dual 4-manifolds, as we used Salamon's results on the classical twistor space for self-dual 4-manifolds [**42**] to describe the spinor. The next natural question would be to ask whether this construction could extend to higher dimensions. For a quaternionic Kähler manifold, this seems to be true (similar to the hyperkähler case), however, beyond that it is unclear. In the classical setting, the twistor space exists for all quaternionic and hypercomplex manifolds. Essentially, these are quaternionic Kähler and hyperkähler manifolds, respectively, without an underlying Riemannian metric. It is possible that there exists a generalized twistor space for these types of manifolds, however our methods will not work without a metric (since we would not have the generalized almost Kähler structures (5.1), (5.2)). Thus, we would have to use an alternate way of defining a family of generalized almost complex structures on the manifold in this setting.

APPENDIX A

Computation of Proposition 3.1.3

In this appendix, we will provide the missing details of the proof of Lemma 3.1.3. This is equivalent to proving the following proposition.

PROPOSITION A.0.1. *For $\alpha \neq \beta$, the generalized complex structure $\mathcal{J}$ is a B-field transform of the generalized complex structure of symplectic type $\mathcal{J}_\omega$, where*

$$\begin{aligned}\omega &= \frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\omega_I+\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\omega_J \\ &\quad+\frac{\mathrm{Im}(\alpha)(|\beta|^2+1)-\mathrm{Im}(\beta)(|\alpha|^2+1)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\omega_K\end{aligned}$$

$$\begin{aligned}B &= \frac{-i(\alpha\bar{\beta}-\bar{\alpha}\beta)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\omega_I+\frac{\mathrm{Im}(\alpha)(1-|\beta|^2)-\mathrm{Im}(\beta)(1-|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\omega_J \\ &\quad+\frac{\mathrm{Re}(\alpha)(|\beta|^2-1)-\mathrm{Re}(\beta)(|\alpha|^2-1)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\omega_K.\end{aligned}$$

The first step in this process is the following lemma.

LEMMA A.0.2. *For ω as given in equation* (3.7)*,*

$$\begin{aligned}\omega^{-1} &= \frac{|\alpha|^2-|\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}\omega_I^{-1}+\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\omega_J^{-1} \\ &\quad+\frac{\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\omega_K^{-1}.\end{aligned}$$

PROOF. Write

$$\omega = x\omega_I + y\omega_J + z\omega_K, \quad \omega^{-1} = \omega' = d\omega_I^{-1} + e\omega_J^{-1} + f\omega_K^{-1}.$$

Then we want

$$\omega\omega' = (xd+ye+zf)\mathrm{id}^* + xeK^* - xfJ^* - ydK^* + yfI^* + zdJ^* - zeI^* = \mathrm{id}^*,$$

where $\mathrm{id}^* : T^*M \to T^*M$ is the identity map. Here, we use quaternionic relations, e.g. $\omega_J \omega_K^{-1} = -\omega_K \omega_J^{-1} = I^*$. This equation holds if and only if

$$xd + ye + zf = 1 \tag{A.1}$$

and

$$xe - yd = 0, \quad zd - xf = 0 \quad yf - ze = 0. \tag{A.2}$$

For equation (A.1), we have

$$\begin{aligned} xd + ye + zf &= \frac{|\alpha|^4+|\beta|^4+2|\alpha\beta|^2+|\alpha|^2+|\beta|^2|\alpha\beta|^2+|\beta|^2+|\alpha|^2|\alpha\beta|^2-(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)(2\mathrm{Re}(\alpha\bar{\beta}))}{(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)(|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta}))} \\ &= \frac{(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)|\alpha|^2+(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)|\beta|^2-(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)(2\mathrm{Re}(\alpha\bar{\beta}))}{(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)(|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar{\beta}))} \\ &= 1 \end{aligned}$$

and equations (A.2) yield

$$\begin{aligned} xe - yd &= \left(\frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right)\left(\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right) \\ &\quad - \left(\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right)\left(\frac{|\alpha|^2-|\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}\right) = 0 \\ zd - xf &= \left(\frac{\mathrm{Im}(\alpha)(|\beta|^2+1)-\mathrm{Im}(\beta)(|\alpha|^2+1)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right)\left(\frac{|\alpha|^2-|\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}\right) \\ &\quad - \left(\frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right)\left(\frac{\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right) = 0 \\ yf - ze &= \left(\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right)\left(\frac{\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right) \\ &\quad - \left(\frac{\mathrm{Im}(\alpha)(|\beta|^2+1)-\mathrm{Im}(\beta)(|\alpha|^2+1)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right)\left(\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right) = 0. \end{aligned}$$

Then $\omega\omega' = 1$. Similarly, using hyperkähler relations, $\omega'\omega = 1$. $\square$

Using this lemma, we now prove Proposition A.0.1 by proving that $\mathcal{J} = e^B \mathcal{J}_\omega e^{-B}$ for $\alpha \neq \beta$.

PROOF. (Proposition A.0.1) Using stereographic projection, we first write $\mathcal{J}$ in terms of α, β:

$$\begin{aligned}\mathcal{J} \;&=\; \tfrac{|\alpha\beta|^2-1}{(1+|\alpha|^2)(1+|\beta|^2)}\mathcal{J}_I + \tfrac{\mathrm{Re}(\alpha)(1+|\beta|^2)+\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\mathcal{J}_J + \tfrac{\mathrm{Im}(\alpha)(1+|\beta|^2)+\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\mathcal{J}_K \\ &\quad + \tfrac{|\alpha|^2-|\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}\mathcal{J}_{\omega_I} + \tfrac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\mathcal{J}_{\omega_J} + \tfrac{\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\mathcal{J}_{\omega_K}.\end{aligned}$$

Note that if Φ is a spinor for $\mathcal{J}$, then $\mathcal{J} = e^{-B}\mathcal{J}_\omega e^B$, where

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

i.e.

$$\mathcal{J} = \begin{pmatrix} -\omega^{-1}B & -\omega^{-1} \\ B\omega^{-1}B+\omega & B\omega^{-1} \end{pmatrix}.$$

We have by the lemma above that $-\omega^{-1} = -\frac{1}{2}(\omega_+^{-1} - \omega_-^{-1})$. Then we just need to prove

$$\omega^{-1}B = \frac{1}{2}\left(I_+ + I_-\right), \tag{A.3}$$

$$B\omega^{-1}B + \omega = \frac{1}{2}\left(\omega_+ - \omega_-\right), \tag{A.4}$$

and the fact that

$$B\omega^{-1} = \frac{1}{2}\left(I_+^* + I_-^*\right)$$

follows easily from (A.3). Write as above

$$\omega = x\omega_I + y\omega_J + z\omega_K, \quad \omega^{-1} = d\omega_I^{-1} + e\omega_J^{-1} + f\omega_K^{-1},$$

$$B = a\omega_I + b\omega_J + c\omega_K.$$

Then

$$\omega^{-1}B = (da+eb+fc)\mathrm{id} + (fb-ec)I + (dc-fa)J + (ea-db)K$$

so equation (A.3) is equivalent to

$$ad + eb + fc = 0 \tag{A.5}$$

$$fb - ec = \frac{|\alpha\beta|^2-1}{(1+|\alpha|^2)(1+|\beta|^2)} \tag{A.6}$$

$$dc - fa = \frac{\mathrm{Re}(\alpha)(1+|\beta|^2)+\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)} \tag{A.7}$$

$$ea - db = \frac{\mathrm{Im}(\alpha)(1+|\beta|^2)+\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}. \tag{A.8}$$

For equation (A.5), we have:

$$\begin{aligned} ad + be + fc &= \frac{(|\alpha|^2-|\beta|^2)(2\mathrm{Im}(\alpha\bar{\beta}))+(\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2))(\mathrm{Im}(\alpha)(1-|\beta|^2)-\mathrm{Im}(\beta)(1-|\alpha|^2))}{(1+|\alpha|^2)(1+|\beta|^2)} \\ &\quad + \frac{(\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2))(\mathrm{Re}(\alpha)(|\beta|^2-1)-\mathrm{Re}(\beta)(|\alpha|^2-1))}{(1+|\alpha|^2)(1+|\beta|^2)} \\ &= \frac{2(|\alpha|^2-|\beta|^2)\mathrm{Im}(\alpha\bar{\beta})-2\mathrm{Im}(\alpha\bar{\beta})(|\alpha|^2-|\beta|^2)}{(1+|\alpha|^2)(1+|\beta|^2)} = 0. \end{aligned}$$

Considering the coefficients of I, J, and K, we have the following computations. Equation (A.6) yields:

$$\begin{aligned} fb - ec &= \left(\frac{\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right)\left(\frac{\mathrm{Im}(\alpha)(1-|\beta|^2)-\mathrm{Im}(\beta)(1-|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\right) \\ &\quad - \left(\frac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right)\left(\frac{\mathrm{Re}(\alpha)(|\beta|^2-1)+\mathrm{Re}(\beta)(1-|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\right) \\ &= \frac{|\alpha|^2(1-|\beta|^4)+|\beta|^2(1-|\alpha|^4)+\mathrm{Re}(\alpha)\mathrm{Re}(\beta)(2|\alpha\beta|^2-2)-\mathrm{Im}(\alpha)\mathrm{Im}(\beta)(2-2|\alpha\beta|^2)}{(1+|\alpha|^2)(1+|\beta|^2)(|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta))} \\ &= \frac{|\alpha\beta|^2-1}{(1+|\alpha|^2)(1+|\beta|^2)}. \end{aligned}$$

Similarly, for equation (A.7), we have:

$$\begin{aligned} dc - fa &= \left(\frac{|\alpha|^2-|\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}\right)\left(\frac{\mathrm{Re}(\alpha)(|\beta|^2-1)+\mathrm{Re}(\beta)(1-|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}-\bar{\alpha}\beta)}\right) \\ &\quad - \left(\frac{\mathrm{Im}(\alpha)(1+|\beta|^2)-\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right)\left(\frac{\mathrm{Im}(\alpha\bar{\beta})}{|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta)}\right) \\ &= \frac{1}{2}\,\frac{|\alpha|^4(\beta+\bar{\beta})+|\beta|^4(\alpha+\bar{\alpha})+|\alpha|^2(\alpha+\bar{\alpha}-\bar{\alpha}\beta^2-\alpha\bar{\beta}^2)+|\beta|^2(\beta+\bar{\beta}-\alpha^2\bar{\beta}-\bar{\alpha}^2\beta)-(\alpha^2\bar{\beta}+\beta\bar{\alpha}^2+\bar{\alpha}\beta^2+\alpha\bar{\beta}^2)}{(1+|\alpha|^2)(1+|\beta|^2)(|\alpha|^2+|\beta|^2-(\alpha\bar{\beta}+\bar{\alpha}\beta))} \\ &= \frac{\mathrm{Re}(\alpha)(1+|\beta|^2)+\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}. \end{aligned}$$

And equation (A.8) yields:

$$\begin{aligned} ea - db &= \left(\tfrac{\mathrm{Re}(\alpha)(1+|\beta|^2)-\mathrm{Re}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}\right)\left(\tfrac{-i(\alpha\bar\beta-\bar\alpha\beta)}{|\alpha|^2+|\beta|^2-(\alpha\bar\beta+\bar\alpha\beta)}\right) \\ &\quad -\left(\tfrac{|\alpha|^2-|\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}\right)\left(\tfrac{\mathrm{Im}(\alpha)(1-|\beta|^2)-\mathrm{Im}(\beta)(1-|\alpha|^2)}{|\alpha|^2+|\beta|^2-(\alpha\bar\beta+\bar\alpha\beta)}\right) \\ &= -\tfrac{i}{2}\left(\tfrac{(\alpha-\bar\alpha)(1+|\beta|^2)(|\alpha|^2+|\beta|^2-(\alpha\bar\beta+\bar\alpha\beta))+(\beta-\bar\beta)(1+|\alpha|^2)(|\alpha|^2+|\beta|^2-(\alpha\bar\beta+\bar\alpha\beta))}{(|\alpha|^2+|\beta|^2-(\alpha\bar\beta+\bar\alpha\beta))(1+|\alpha|^2)(1+|\beta|^2)}\right) \\ &= \tfrac{\mathrm{Im}(\alpha)(1+|\beta|^2)+\mathrm{Im}(\beta)(1+|\alpha|^2)}{(1+|\alpha|^2)(1+|\beta|^2)}. \end{aligned}$$

Thus the first condition (A.3) is satisfied.

To prove that $B\omega^{-1}B+\omega = \frac{1}{2}(\omega_+ - \omega_-)$, we use the fact that

$$\omega^{-1}B\omega^{-1}B = \left(\frac{1}{2}(I_+ + I_-)\right)^2.$$

Hence this is equivalent to proving

$$\left(\frac{1}{2}(I_+ + I_-)\right)^2 = \omega^{-1}\left(\frac{1}{2}(\omega_+ - \omega_-)\right) - 1. \tag{A.9}$$

For the right-hand side of equation (A.9), note that

$$\left(\tfrac{1}{2}(I_+ + I_-)\right)^2 = -\left(\tfrac{(|\alpha\beta|^2-1)^2+(\mathrm{Re}(\alpha)(1+|\beta|^2)+\mathrm{Re}(\beta)(1+|\alpha|^2))^2+(\mathrm{Im}(\alpha)(1+|\beta|^2)+\mathrm{Im}(\beta)(1+|\alpha|^2))^2}{(1+|\alpha|^2)^2(1+|\beta|^2)^2}\right).$$

This reduces to

$$\begin{aligned} \left(\tfrac{1}{2}(I_+ + I_-)\right)^2 &= -\left(\tfrac{(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)+|\alpha\beta|^2(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2)+(\alpha\bar\beta+\bar\alpha\beta)(1+|\alpha|^2)(1+|\beta|^2)}{(1+|\alpha|^2)^2(1+|\beta|^2)^2}\right) \\ &= -\left(\tfrac{1+|\alpha\beta|^2+\alpha\bar\beta+\bar\alpha\beta}{(1+|\alpha|^2)(1+|\beta|^2)}\right). \end{aligned}$$

On the left-hand side of equation (A.9), we have

$$\begin{aligned} \left(\tfrac{1}{2}(\omega_+ - \omega_-)\right)\omega^{-1} - 1 &= \tfrac{|\alpha|^2+|\beta|^2-2\mathrm{Re}(\alpha\bar\beta)}{(|\alpha|^2+1)(|\beta|^2+1)} - 1 \\ &= \tfrac{-(|\alpha\beta|^2+\alpha\bar\beta+\bar\alpha\beta+1)}{(1+|\alpha|^2)(1+|\beta|^2)} = \left(\tfrac{1}{2}(I_+ + I_-)\right)^2. \end{aligned}$$

This last computation proves the proposition.

$\square$

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