Onset of oscillations in nonsoliton pulses in nonlinear dispersive fibers *

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Abstract

We study the modulation equations for the amplitude and phase of smoothed rectangular pulse initial data for the defocusing nonlinear Schrodinger (NLS) equation in the semiclassical limit, and show that these equations exhibit shock formation. In this way we identify and explain one source for the onset of pulse oscillations in nonlinear fibers whose transmission is modeled by the semiclassical NLS equation. The onset of pulse ripples predicted here develops on the leading and trailing slopes of a smooth pulse, as a consequence of shock formation in the modulation equations. This mechanism for the onset of pulse ripples is distinct, both in the location and timescale, from the scenario pursued by Kodama and Wabnitz [11]: a piecewise linear pulse evolves for distances $O(1)$ down the fiber, beyond which oscillations develop associated with the vanishing of the upper step of the pulse [10]. Here we show that the scenario in [11] is correct, but *specific to pure rectangular pulses*; any smoothing of this data fails to obey their scenario, but rather is described by the results presented here. That is, the semiclassical limit of the NLS equation is highly unstable with respect to smooth regularizations of rectangular data. In our analysis, the onset of oscillations is associated with the location of the maximum gradient of the pulse slopes, and onset occurs on the pulse slopes, at short distances down the fiber proportional to the inverse of this maximum gradient. Explicit upper and lower bounds on the initial shock location are derived. We thereby deduce the onset for this source of pulse degradation scales linearly with the pulse width, and scales with the reciprocal square root of the fiber nonlinear coefficient, the pulse power, and the fiber dispersion coefficient.

1 Introduction

Recently there has developed an interest in the propagation of non-soliton pulses in nonlinear dispersive fibers [16], [15]. As is well known [19], [7], the nonlinear wave equation which describes the evolution of the wave envelope as a function of time t and length z along the fiber is

$$iQ_z - \frac{k''}{2}Q_{tt} + \gamma|Q|^2Q = 0, \quad (1)$$

where t is the "retarded time", k'' is the fiber dispersion, and γ is the fiber nonlinearity coefficient.

In [15] Marcuse developed a simple model based on rectangular pulse initial data in order to approximate the evolution of the root mean square (RMS) width of smooth pulses. In the same paper, the model predictions for rectangular pulses were compared with numerical simulations of smooth pulses. The evidence presented indicated that although the rectangular initial data is not preserved, the evolution of the RMS width predicted under this assumption is remarkably accurate. Numerical evidence also showed that the general shape of the amplitude function (a "bump") is preserved over long distances (10,000km) albeit with pulse ripples, or oscillations, on the leading and trailing slopes.

The phenomenon of the development of oscillations in pulse-type data is familiar both in the optics community (see [20], and [1] and references therein) and in integrable theory, in particular in the context of small dispersion limits of integrable odes and pdes [13],[23]-[25], [2], [17], [3], [9]. The canonical example for this phenomenon is a WKB approximation; see for example [13] for the KdV equation, [3] for the Toda lattice, and [2] for the defocusing NLS equation. Kodama and Wabnitz [11] made the important observation that this standard WKB approximation may be relevant for the

evolution of pulse data for the NLS equation.

In order to connect to the rigorous results for the small dispersion limit of the NLS equation, one must scale the equations consistent with the experiment and deduce the semi-classical NLS equation. Kodama and Wabnitz introduced one such scaling in [11] and we now introduce an alternative scaling motivated by the experiments of Marcuse.

The first step in the scaling process involves selection of the 3 physical scales necessary to nondimensionalize the NLS pde (1): a lengthscale z_0 , a timescale t_0 , and a power P_0 . We emphasize that there is arbitrariness in how one selects these scales, but the predictions we ultimately make for real experiments are independent of this a priori choice. These physical scales simply instantiate a unit of distance, time and power. The experimental data, consisting of the fiber material characteristics and the specific pulse characteristics, will be consistently built into these characteristic scales. Finally then, our predictions for the location of onset of pulse ripples are translated to physical units in terms of the fiber properties (the dispersion and nonlinearity coefficients) and the input pulse characteristics (the pulse power and width).

We select P_0 to be the power of the initial pulse, which defines a dimensionless field envelope q ,

$$q = \frac{Q}{\sqrt{P_0}}. \quad (2)$$

In terms of dimensionless coordinates $Z = \frac{z}{z_0}$, $T = \frac{t}{t_0}$, the equation for q is

$$\frac{i}{z_0 \gamma P_0} q_Z - \frac{1}{2} \frac{k''}{\gamma P_0 t_0^2} q_{TT} + |q|^2 q = 0. \quad (3)$$

Now *we pick z_0 to be the total transmission length*, so that the dimensionless fiber domain is $0 \leq Z \leq 1$. The dimensionless quantity $(z_0 \gamma P_0)^{-1}$ is

a familiar aspect ratio in nonlinear optics which we shall label as a distinguished parameter, ϵ :

$$\epsilon = \frac{1}{\gamma P_0 z_0} = \frac{L_{NL}}{z_0}; \quad (4)$$

the lengthscale $L_{NL} = \frac{1}{\gamma P_0}$ is referred to as the nonlinear lengthscale in fiber optics.

Finally, we choose the timescale t_0 to be the initial pulse width. Defining

$$\delta = \frac{k''}{\gamma P_0 t_0^2} \quad (5)$$

we obtain

$$i\epsilon q_Z - \frac{\delta}{2} q_{TT} + |q|^2 q = 0. \quad (6)$$

For applicability of the semi-classical integrable theory, two independent conditions must be satisfied.

Condition 1. The parameter ϵ should be small.

Here our scaling choices become critical. L_{NL} is fixed by the material coefficient γ and power P_0 of the initial pulse; thus, the nonlinear lengthscale is experimentally determined. We further choose z_0 as the total transmission length, so that ϵ is defined consistently: for ϵ to be small, the transmission length must be order(s) of magnitude greater than the fiber nonlinear lengthscale.

Condition 2. The parameter δ should be $O(\epsilon^2)$.

This is a further restriction on the experiment, namely that the *dispersive lengthscale*,

$$L_{Disp.} = \left(\frac{k''}{t_0^2} \right)^{-1}, \quad (7)$$

must be very long, on the order of ϵ^{-1} times the total transmission length z_0 . To evaluate whether this condition is satisfied, one must choose the timescale t_0 consistent with the experiment; we select t_0 as the initial pulse width. Alternatively, one might pick the pulse slope width, but then our prediction will simply scale with that parameter. The primary issue is whether the pulse width or slope width yields the necessary order of δ .

Now we present some scenarios in which we are approximately in the semiclassical regime. Marcuse [15] considered the following numerical values:

$$\begin{aligned}
P_0 &= 1 \text{ mW}, \\
L_{NL} = \frac{1}{\gamma P_0} &= .31 \text{ Mm}, \\
D &= -.05 \frac{\text{ps}}{\text{nm km}}, \\
\lambda &= 1550 \text{ nm}, \\
A_{eff} &= 40 \mu\text{m}^2, \\
n_2 &= 3.2 \times 10^{-20} \frac{\text{m}^2}{\text{W}}, \\
z_{tot} &= 10,000 \text{ km}, \\
t_0 &= 200 \text{ ps}.
\end{aligned}$$

(Recall that $k'' = \frac{-\lambda^2 D}{2\pi c}$, $\gamma = \frac{n_2 \omega_0}{c A_{eff}}$, λ is the wavelength, D is the group velocity dispersion, c is the speed of light in a vacuum, A_{eff} is the effective area of the fiber core, $\omega_0 = \frac{2\pi c}{\lambda}$ is the carrier frequency, and n_2 is the nonlinear coefficient of the fiber.) For these values, we find that $\epsilon = .0310$, and $\delta = 4.942 \times 10^{-4} = 0.5143\epsilon^2$.

In recently installed transoceanic systems, e.g. TPC5 or TAT 12-13, the relevant parameter values are [4]

$$P_0 = 5 \text{ mW},$$

$$\begin{aligned}
L_{NL} = \frac{1}{\gamma P_0} &= 292 km, \\
D &= -.2 \frac{ps}{nm \ km}, \\
\lambda &= 1558 nm, \\
A_{eff} &= 50 \mu m^2, \\
n_2 &= 2.6 \times 10^{-20} \frac{m^2}{W}, \\
z_{tot} &= 500 \ km, \\
t_0 &= 200 \ ps.
\end{aligned}$$

Note while these lines cross the Atlantic Ocean, the defocusing fiber segments are 500 km with a 5 km focusing segment in between [4]. With these values, we find that $\epsilon = 0.5840$, and $\delta = 0.0018 = (.0733\epsilon)^2$.

Remark: a convenient rescaling of t_0 : The above examples of current fiber optic scenarios justify the use of semiclassical theory of the NLS equation to predict the evolution of pulses. As part of our procedure, we now need to change variables, use semiclassical NLS results, and translate them back to dimensional coordinates. In order to simply invoke the semiclassical theory, it is only necessary to rescale the single parameter t_0 , and arrive at the canonical semiclassical NLS equation. In this way, every fiber/initial data scenario approximated by the semiclassical NLS equation is described by the results to follow. One then converts these predictions to dimensional form, and the physical predictions depend only on the fiber and pulse characteristics, not on the choice of t_0 . (The alternative would be to maintain the physical definition of t_0 as the pulse width, and rescale the WKB theory and associated shock estimates to follow. The choice here is much simpler, and transfers to every application described in this paper, as illustrated explicitly in Section 3.)

Rather than choosing t_0 to be the pulse width, *in what follows we will*

choose t_0 so that $\delta \equiv \epsilon^2$, i.e.

$$t_0 = \frac{\sqrt{L_{NL}k''}}{\epsilon}. \quad (8)$$

In the above examples, the new values of t_0 are as follows. For the Marcuse experiment, $t_0 = 143.4\text{ps}$; with this scaling a pulse of width 200ps has a nondimensional width of 1.394, a 400ps pulse has twice that nondimensional width, etc. For TPC5 or TAT 12-13, $t_0 = 14.78\text{ps}$; with this scaling a pulse of width 200ps has a nondimensional width of 13.53. This is the manual of translation for entering the experimental pulse width, along with the fiber characteristics and pulse power, into the canonical semiclassical NLS equation:

$$i\epsilon q_Z - \frac{\epsilon^2}{2}q_{TT} + |q|^2q = 0. \quad (9)$$

In these variables, the WKB initial data is

$$q_0(T) = A_0(T)e^{\frac{i}{\epsilon}S_0(T)}. \quad (10)$$

The rectangular data studied in [15] and [11] can be obtained by setting the phase $S_0(T)$ identically equal to 0, and taking the amplitude $A_0(T)$ to be piecewise constant, i.e.

$$q_0(T) = \left\{ \begin{array}{ll} \sqrt{\rho_0} & \text{for } |T| < 1, \\ 0 & \text{otherwise} \end{array} \right\}. \quad (11)$$

In [11], Kodama and Wabnitz propose that a WKB approximation may be used to explain the evolution of the pulse in more detail. Indeed, the authors re-derived modulation equations for $A(T, Z)$ and $S(T, Z)$ (see [26]), constructed piecewise-linear solutions of the modulation equations for the

rectangular initial data (11), and compared these solutions to numerical simulations of (9).

These results are a plausible scenario given what is now known in integrable dispersive pdes. A rigorous justification of this scenario is not contained in [11]; rigorous results that apply to this problem are contained in the literature. In particular, equation (9) with WKB initial data of the form (10), where the initial functions $A_0(T)$ and $S_0(T)$ are such that $2A_0 \pm S_0$ are both arbitrary single-hump functions, has been studied rigorously as $\epsilon \rightarrow 0$ [9] (see below for further discussion). The equations for the evolution of A and S turn out to be a hyperbolic system of conservation laws, which have analytically [8] and experimentally [2] been shown to exhibit shock formation. In an appendix we apply these rigorous results to validate the results in [11].

Kodama and Wabnitz appear to capture quite well the mean behavior of the pulse evolution, in particular the mean profile shape and location. However, similar to Marcuse's work [15], oscillations appear on the slopes of the pulse amplitude $A(T, Z)$ in the numerically computed solution which are not predicted nor resolved by the formulae derived in [11] (see, for example, figure 1 in [11]). Indeed, previously Marcuse [15] noted the onset of oscillations on the leading and trailing slope of the pulse amplitude. Marcuse further emphasized that these slope oscillations do not go away with refined numerical calculations, and therefore "are real".

The issues we address here go beyond the mean profile shape and location. In particular we address the following questions:

For what data do slope oscillations occur?

Is this phenomenon unstable with respect to initial data?

If so, what is the physically relevant initial data?

Here we show *analytically*, in agreement with the numerical experiments of Marcuse, that the onset of slope oscillations are not due to numerical errors, but rather are due to the early (i.e. near $Z = 0$) formation of shocks in the zero-phase modulation equations. As noted earlier, it is now standard behavior that modulation equations both govern the asymptotic behavior of an integrable system as a parameter (the coefficient of dispersion) tends to zero, and are prone to shock formation [13], [23]-[25], [3], [9]. Moreover, beyond shock formation, the ansatz which leads to the modulation equations *is no longer valid*, and fails to describe the asymptotic behavior of the integrable system.

Beyond shock formation, typically additional oscillations develop at the shock front, and spread. These oscillations have wavelength which tends to zero as the coefficient of dispersion tends to zero (typically $O(\epsilon)$, where ϵ is the coefficient of dispersion), while their amplitude is $O(1)$ and independent of the dispersion coefficient. The “post-shock” issues of the generation of pulse ripples, their extent and evolution, are beyond the scope of this paper. Nonetheless, a detailed description of this dispersive shock phenomenon in the defocusing NLS equation has been carried out in [14] and [2]. We mention that in [2], the authors: (a) documented numerically the onset of oscillations in the defocusing NLS equation with WKB-type initial data for small values of ϵ , (b) explained these oscillations in terms of shock formation, and (c) suggested that oscillations observed experimentally by Rothenberg and Grischkowsky [20] are “experimental evidence supporting the semi-classical theory”.

In this paper we rigorously show:

(1) for virtually any “bump” initial data (i.e. for a large class of pulse shapes, including any smoothed rectangular pulse), the underlying zero-phase

modulation equations develop shocks;

(2) the location of onset is associated with the maximum absolute gradient of $\sqrt{A_0(T)}$, while the timescale of onset of oscillations is proportional to the inverse of the maximum absolute gradient of $\sqrt{A_0(T)}$.

These analytical results characterize the onset of slope oscillations as seen in figure 1 of [11].

We repeat that integrable theory further provides a basis for rigorous explanation and precise description of the oscillations themselves, in particular the extent of the region of oscillations for all distances down the fiber beyond initial shocking, the spreading of this region, and its effect on the shape and evolution of the non-oscillatory regions of the pulse [21], [3], [18]. The establishment of this next level of detail for these optical applications remains for the future.

Remark 1. Recently Kodama [10] has suggested that the oscillations appearing in figure 1 of [11] are due to a different phenomenon associated to the zero-phase modulation equations studied in [11], namely the contraction and vanishing of the “step” or upper flat portion of the pulse. In [11] the authors posit a solution of the form (10), taking purely rectangular initial data which is piecewise-linear, i.e. a *weak* solution of (13)-(14). It remains to prove that equation (9) (as a dispersive regularization of (13) - (14)) should select this weak solution as $\epsilon \rightarrow 0$. In the appendix we rigorously prove: for this *special initial data*, the behavior of $q^{(\epsilon)}(T, Z)$ as ϵ tends to zero is indeed described, at least until the step vanishes, by the explicit solution obtained in [11].

This rigorous step is necessary for several reasons. First, the WKB approach and the derivation of modulation equations are approximate. In particular, terms of order ϵ^2 have been neglected, while terms of order ϵ have

been retained. It is a remarkable fact from the Lax-Levermore method that these terms through order ϵ rigorously govern the behavior of the NLS equation (9) as $\epsilon \downarrow 0$. Second, once one admits weak solutions, the uniqueness of solutions of the modulation equations is lost. Therefore a rigorous principle must be established that selects the correct behavior from the infinite family of weak solutions. The Lax-Levermore method provides this selection principle in the form of a maximization problem characterizing the small ϵ limit of the NLS equation (9) [9]. In the appendix we outline these details and establish that the piecewise linear solution solves the Lax-Levermore maximization problem, and therefore that solution of the modulation equations describes the small ϵ limit of (9) with pure rectangular data.

It follows that for pure rectangular initial data, oscillations do not develop before the upper step vanishes. Hence we have the possibility of two distinct sources of oscillations, one from the development of shocks in the modulation equations (established here), and the other from this “step vanishing” phenomenon (pursued by Kodama [10]). It is interesting to note that the evolution of the continuum limit of the Toda lattice with piecewise constant initial data was considered in [17], and the evolution was shown to be piecewise linear, as in [11], until the “upper step” vanished. The evolution was described **for all time**, and it was shown, in particular, that for piecewise constant initial data for the continuum limit of the Toda lattice, *no oscillations develop*. This Toda behavior is in sharp contrast to the semi-classical limit of the defocusing NLS equation suggested by Kodama, and poses an interesting analytical contrast for small dispersion limits.

Remark 2. In practice one cannot produce a pure rectangular pulse, so it is relevant to investigate whether the predictions based on this ideal approximation are robust with respect to smoothing of the pulse. In fact the

predictions are far from robust: *any smooth pulse which approximates a pure rectangular pulse will evolve in such a way as to develop slope oscillations, due to shock formation in the zero-phase modulation equations (13-14)*. It is of engineering relevance to determine how these slope oscillations alter the analytical predictions of mean pulse profile and location in [11], [15], and also to investigate how small ϵ must be for these predictions and those in [11] to apply.

Remark 3. It follows from our work that the distance along the fiber before the development of these oscillations is inversely proportional to the maximum of $|\frac{A'(T,0)}{\sqrt{A(T,0)}}|$. Thus we see that pulses with steep profiles (i.e. good approximations of rectangular pulses) will develop oscillations practically at $Z = 0$, well before the step could possibly vanish. Furthermore, this implies that the zero-phase modulation equations considered in [11], when applied to smoothed rectangular pulses, become invalid for Z beyond the location of shock formation.

Remark 4. The question of whether or not “step-vanishing oscillations” actually occur requires further study. For physically relevant data, there is no distinguished upper step, so it is unclear what would be the event corresponding to the piecewise-linear step vanishing. The next remark suggests an alternative source for oscillations on the peak of the pulse amplitude. While a full analytical treatment using integrable methods could resolve this, a direct numerical implementation of the Lax - Levermore minimization problem for the semiclassical limit of the NLS equation (9), analogous to the study [18] of D. McLaughlin and J. Strain is perhaps more efficient.

Remark 5. Following the results of this paper, a very different scenario than that suggested by Kodama (see Remark 1 above) arises to explain the oscillations on the step seen in [11] (assuming ϵ is sufficiently small): Slope

oscillations develop early at the location of steepest gradient of the amplitude; these oscillations spread up and down the slopes and emerge at some $O(1)$ distance down the fiber on the front and back of the top branch of the pulse. Then these pulse oscillations travel towards each other, and eventually collide near the peak of the pulse. This collision, according to integrable theory, corresponds to the generation of higher-phase modulations, with additional temporal complexity. Tian [22] has embarked on a rigorous study of the single phase modulation equations. Kodama [10] has announced similar multiphase modulational behavior as a secondary effect to the step vanishing source of oscillations.

An outline of the remainder of this paper is as follows. We explain briefly how one derives the zero-phase modulation equations for the amplitude $A(T, Z)$ and phase $S(T, Z)$. Then we prove that for any “single-hump” initial data, the solution of these equations develops shocks. A simple argument then shows that *any smoothed rectangular pulse* falls into this class of initial data, and hence *must exhibit shock formation at locations on the leading and trailing slopes of the pulse* associated with the location of the maximum of $|\frac{A'(T,0)}{\sqrt{A(T,0)}}|$, *and on short distances down the fiber* proportional to the inverse of the maximum of $|\frac{A'(T,0)}{\sqrt{A(T,0)}}|$. Furthermore, we predict an explicit interval of transmission, inside of which the semiclassical limit is guaranteed to exhibit onset of shock formation.

Following that, we explain, using intuition from a large body of work in integrable systems, that *these shock phenomena imply the onset of slope oscillations in the amplitude and phase of the solution of the NLS equation (9) for any smoothed rectangular pulse and for small values of ϵ* . The precise nature of the post-shock pulse evolution remains for future study.

This behavior is in sharp contrast to the behavior for the pure rectangular data (11), which is discussed in the appendix. There we prove that for this special initial data, the behavior of $q^{(\epsilon)}(T, Z)$ as ϵ tends to zero is indeed described, at least until the step vanishes, by the explicit solution obtained in [11].

2 Analysis of Zero-Phase Modulation Equations

To obtain the zero-phase modulation equations for $A(T, Z)$ and $S(T, Z)$, we insert the ansatz

$$q(T, Z) = A(T, Z)e^{\frac{i}{\epsilon}S(T, Z)} \quad (12)$$

into equation (9) and expand, matching orders in ϵ . We obtain at order ϵ^0 :

$$-S_Z + \frac{1}{2}S_T^2 + A^2 = 0, \quad (13)$$

and at order ϵ^1 :

$$A_Z - \frac{1}{2}(2A_TS_T + AS_{TT}) = 0. \quad (14)$$

We can place this system in Riemann invariant form by setting

$$\beta = 2A + S_T, \quad (15)$$

$$\alpha = 2A - S_T, \quad (16)$$

for then

$$\begin{aligned} \beta_Z - \left(\frac{3\beta - \alpha}{4}\right)\beta_T &= 0, \\ \alpha_Z - \left(\frac{\beta - 3\alpha}{4}\right)\alpha_T &= 0. \end{aligned} \quad (17)$$

Our assumptions on $A(T, 0)$ and $S(T, 0)$ are that $A, S \in \mathcal{S}(\mathbf{R})$, Schwarz class (i.e. smooth), and that the Riemann invariants α and β should each possess precisely 1 critical point. Note that this is satisfied by the Schwarz class approximation to the initial data (11):

$$A(T, 0) = e^{-CT^2}, \quad (18)$$

$$S(T, 0) = 0. \quad (19)$$

Of course, this will be satisfied by a large class of pulses, and in particular for generic smooth approximations to (11); see for example figure 2 in [11], and the class of pulses considered numerically in [15]. Note in particular that for the smoothed pulse initial data

$$\begin{aligned} A(T, 0) &= A_0(T), \\ S(T, 0) &= 0, \end{aligned} \quad (20)$$

with $A_0 \in \mathcal{S}(\mathbf{R})$ and A_0 possessing precisely one maximum and no other critical points, we have

$$\beta_0(T) = \alpha_0(T) = 2A_0(T). \quad (21)$$

While the above formal calculations begin with the ansatz that as the pulse propagates down the fiber, it can be described by (12), in fact this is fully rigorous. Integrable methods [9] have established that before the time of shock formation in the system (13-14), as ϵ tends to zero, the amplitude $A^{(\epsilon)}(T, Z)$ and phase $S^{(\epsilon)}(T, Z)$ of $q^{(\epsilon)}(T, Z)$ converge to the solution of the hyperbolic system (13-14) with initial data (10). See also the recent work of E. Grenier [6].

Following a procedure due to Lax [12], we can establish upper and lower bounds on the distance Z down the fiber of existence of a solution to equations (13-14) with initial data as described above. We remark that this

procedure was also implemented by S. Jin [8] to estimate the shock time for the zero phase modulation equations for NLS. From [12]:

Theorem 1 (*Lax*) *Consider the hyperbolic system in 2 unknowns in Riemann invariant form*

$$r_t + \lambda r_x = 0, \quad (22)$$

$$s_t + \mu s_x = 0, \quad (23)$$

and suppose that the characteristic speed $\lambda(r, s)$ satisfies

$$\lambda_r < 0 \quad (24)$$

for all relevant values of the Riemann invariants r, s . Let $h(r, s)$ solve

$$h_s = \frac{\lambda_s}{\lambda - \mu}, \quad (25)$$

and define

$$\bar{A} = \min_{r, s \in \hat{R}} |\lambda_r e^{-h}|, \quad (26)$$

$$\bar{B} = \max_{r, s \in \hat{R}} |\lambda_r e^{-h}|, \quad (27)$$

$$m = \max_{x \in \hat{X}} |e^{h(r_0(x), s_0(x))} r_{0,x}|, \quad (28)$$

where $r_0(x)$ and $s_0(x)$ are the Riemann invariants at $t = 0$, $\hat{X} \subset \mathbf{R}$, and $\hat{R} \subset \mathbf{R}^2$ is a set of relevant values for the Riemann invariants r and s , i.e.

$$\begin{aligned} \hat{R} = \{ (r, s) : r \in \{r_0(x) : x \in \hat{X}\}, \\ s \in \{s_0(x) : x \in \hat{X}\} \}. \end{aligned} \quad (29)$$

Then a solution to the set of equations (22-23) exists at least until $t_ = (\bar{B}m)^{-1}$, but no solution can exist up to the time $t^* = (\bar{A}m)^{-1}$.*

In the present context, we identify the distance variable Z with t , the variable T is identified with x , $r = \beta$, and $s = \alpha$, and we have

$$\lambda = -\frac{3\beta - \alpha}{4}, \quad (30)$$

$$\mu = -\frac{\beta - 3\alpha}{4}, \quad (31)$$

and hence $\lambda - \mu = -\frac{1}{2}(\alpha + \beta)$. Thus we can take

$$h(\alpha, \beta) = -\frac{1}{2} \log(\alpha + \beta), \quad (32)$$

for the function h in (25), and hence we have

$$\lambda_\beta e^{-h} = -\frac{3}{4} \sqrt{\alpha + \beta}. \quad (33)$$

Now we define

$$m = \max_{T \in \mathbf{R}} \frac{\beta_{0,T}}{\sqrt{\alpha_0 + \beta_0}}, \quad (34)$$

and let the location where this maximum is achieved be denoted T_c .

Lax's theorem employs the characteristic curve for β (i.e. r), and hence we may take β to lie in the so-called range of relevant values

$$\beta_{min} = \beta_0(T_c) \leq \beta \leq \beta_{max}. \quad (35)$$

The Riemann invariant α sits in it's range of relevant values

$$0 \leq \alpha \leq \alpha_{max}. \quad (36)$$

These choices guarantee that the predictions to follow are as sharp as the theorem provides.

Using (33), (35) and (36) together with the fact that $|A_0(T)| \leq 1$, we obtain values for $\bar{A}$ and $\bar{B}$ in (26) and (27):

$$\bar{A} = \frac{3}{4} \sqrt{\beta_{min}}, \quad (37)$$

$$\bar{B} = \frac{3}{2}. \quad (38)$$

Therefore there is a solution to the zero-phase modulation equations (17) at least until the fiber location

$$Z_* = \frac{2}{3m}. \quad (39)$$

However, no solution can exist beyond the fiber location

$$Z^* = \frac{4}{3m\sqrt{\beta_{min}}}. \quad (40)$$

3 Semiclassical Predictions for Fibers

We now pursue predictions for the fiber locations $Z_* = \frac{2}{3m}$, $Z^* = \frac{4}{3m\sqrt{\beta_{min}}}$ for pulse initial data of the form (20) with $S(T) = 0$. This corresponds to initial data for the Riemann invariants,

$$\beta_0(T) = \alpha_0(T) = 2A_0(T). \quad (41)$$

Hence the important quantity m reduces to a simple form:

$$m = \max_{T \in \hat{T}} \left| \frac{A'_0(T)}{\sqrt{A}} \right| = \max_{T \in \hat{T}} \left| 2 \left(\sqrt{A_0(T)} \right)' \right|. \quad (42)$$

We now amplify the distinctions between these predictions for pulse oscillations when we choose rectangular data, smooth nearly rectangular data, and experimental data.

For rectangular data, combining the results of Kodama and Wabnitz and our Appendix, the prediction is that *no pulse oscillations occur up to 10,000 km*.

Both a smooth nearly rectangular pulse and the experimental data used in [15] are captured by the family of “super Gaussian” pulses

$$A_0(T) = e^{-\left(\frac{T}{w}\right)^n}, \quad (43)$$

Figure 1: Plot of e^{-T^n} for $n = 2, 10, 50, \infty$.

where

$$w = \frac{\hat{t}}{2t_0}, \quad (44)$$

t_0 is defined in (8), and $\hat{t}$ is the pulse width. For n very large, (43) provides a good regularization of a rectangular pulse. For moderate values of n , e.g. $n = 10$, members of the family (43) have been used as fits to true pulses in optical fibers [15]. In Figure 1, we plot four members of the family (43), $n = 2, 10, 50, \infty$, for $w = 1$.

For the data (43), we can explicitly compute the quantities $\bar{A}$, $\bar{B}$ and m , and we find:

$$\bar{A} = \frac{3}{2^{3/2}e} e^{1/n} \quad (45)$$

$$\bar{B} = \frac{3}{2} \quad (46)$$

$$m = \frac{2(n-1)}{ew} \left(\frac{e}{2(1 - \frac{1}{n})} \right)^{1/n}. \quad (47)$$

We are now in a position to illustrate two critical conclusions.

First, consider the result for rectangular data compared with a nearly rectangular smooth approximation provided by n large in (43). For $n \gg 1$, $m \sim \frac{2n}{ew}$, $\bar{A} \sim \frac{3}{2^{3/2}e}$, and therefore the interval $[Z_*, Z^*]$ shrinks to a point at the location $Z = 0$. This implies that shocks form arbitrarily close to $Z = 0$ in the rectangular limit ($n \rightarrow \infty$) of the super Gaussian family. Thus we have a stark illustration of *instability with respect to smoothing of rectangular initial data*. The pure rectangular data rigorously does not shock for the full length of the fiber (10,000 km), whereas an arbitrarily close yet smooth approximation to rectangular data shocks instantly.

Second, compare the predicted interval within which shocks form for $n = 50$, corresponding to a nearly rectangular smooth pulse; for $n = 10$, used in [15] as an approximation to a realistic optical pulse; and for $n = 2$, a pure Gaussian. From the above formulas for arbitrary n , we find (with w from (44))

$$Z_* = \frac{ew}{3(n-1)} \left(\frac{e}{2(1 - \frac{1}{n})} \right)^{-1/n}, \quad (48)$$

$$Z^* = \frac{e^2 w \sqrt{2}}{3(n-1)} \left(\frac{e^2}{2(1 - \frac{1}{n})} \right)^{-1/n}. \quad (49)$$

We emphasize that this is essentially a two-parameter family of pulse shapes and corresponding predictions for pulse propagation. The parameter n characterizes the pulse shape within this super Gaussian family, whereas the pulse width is parameterized through $w = \frac{i}{2t_0}$. (Recall that the parameter t_0 has been specified to achieve a semiclassical NLS equation.)

Predictions for the experiments of Marcuse [15]

We obtain laboratory predictions for fibers by computing the corresponding dimensional distances, $z_0 Z_*$ and $z_0 Z^*$, where z_0 is the total transmission

length as above:

$$z_* = z_0 Z_* = \frac{e\hat{t}}{6(n-1)} \sqrt{\frac{L_{NL}}{k''}} \left(\frac{e}{2(1-\frac{1}{n})} \right)^{-1/n} \quad (50)$$

$$z^* = z_0 Z^* = \frac{e^2 \hat{t} \sqrt{2}}{6(n-1)} \sqrt{\frac{L_{NL}}{k''}} \left(\frac{e^2}{2(1-\frac{1}{n})} \right)^{-1/n}. \quad (51)$$

One might call the interval (z_*, z^*) a “capture window” inside of which the evolution of the super-Gaussian pulse in the semiclassical limit is guaranteed to develop an infinite gradient (a shock), and we surmise subsequent pulse ripples which require further analysis.

- Note that **this capture window scales with the pulse width $\hat{t}$, nonlinear length L_{NL} , and fiber dispersion k''** , as follows:

$$\hat{t}^1, L_{NL}^{1/2}, (k'')^{-1/2}. \quad (52)$$

- Note further that **this capture window for oscillations is independent of the characteristic timescale t_0 , and the characteristic lengthscale z_0** , as emphasized in the introduction.

From these formulas, we consider Marcuse’s experiments [15], and fix the pulse width $\hat{t} = 400ps$, and compare pulse shapes $n = 50$, $n = 10$, and $n = 2$. We find

$$z_* = 256 \text{ km}, \quad z^* = 966 \text{ km} \quad \text{for } n = 50 \quad (53)$$

$$z_* = 1347 \text{ km}, \quad z^* = 4685 \text{ km} \quad \text{for } n = 10 \quad (54)$$

$$z_* = 7663 \text{ km}, \quad z^* = 17867 \text{ km} \quad \text{for } n = 2. \quad (55)$$

These results suggest that the pulse oscillations in figure 1 of [15] at location $Z = 10,000$ km were first generated somewhere between 1347 and 4685 km.

Note further that the predictions for other pulse widths, say $300ps$ or $200ps$, are immediately deduced from the above estimates through the linear scaling with respect to $\hat{t}$ of the window.

The second conclusion we highlight is the prediction for realistic optical pulses ($n = 10$) viz a viz the prediction for smoothed, nearly rectangular pulses ($n = 50$). The entire capture window for the onset of pulse oscillations scales essentially with n^{-1} , so that larger n (i.e. steeper pulses) corresponds to pulse ripples on shorter transmission lengths. For example, the more realistic pulse value of $n = 10$ is predicted to experience onset of oscillations at transmission lengths around 540% past the nearly rectangular pulse shape of $n = 50$.

Predictions for fibers in use: For the case of the recently installed systems TPC-5 and TAT 12-13, using the pulse shape considered in [15] (here $n = 10$ and the pulse width is $\hat{t} = 200ps$), these calculations predict that a shock should develop somewhere between 326 and 1137 kilometers down the fiber.

We caution again that the connection between shock formation and the onset of oscillations is rigorously correct under the assumption $\epsilon \rightarrow 0$. The connection for fixed (though small) $\epsilon > 0$ is assumed, and will be pursued in future work.

The solution to the zero-phase modulation equations (17) will break down when a shock forms, i.e. when one of β_T or α_T develops a singularity. This will generically occur in only one of the two curves, at the time T_b . Of course, in the case that $S_0 \equiv 0$, shocks will form in both Riemann invariants simultaneously. The location down the fiber of shock formation shall be denoted Z_b .

We have just shown that there exists a distance Z_b down the fiber at

which the zero-phase modulation equations (17) break down due to shock formation. Moreover, we see that for a good approximation to a rectangular pulse, shock formation occurs near $Z = 0$; hence, for any Z beyond this location of shock formation, the zero-phase modulation equations do not possess a well-defined solution.

While the methods used here rely on nothing more than the Riemann invariant form (17), and hence give only bounds on the distance down the fiber for existence of solutions to these equations, further detailed analysis of the zero-phase modulation equations is within the reach of new integrable techniques. Indeed, in [3], the authors considered a completely analogous problem for the continuum limit of the Toda Lattice, and using complete integrability, arrived at the explicit solution of the zero-phase modulation equations for a large class of initial data. The solution was then used to explicitly provide the location and time of shock formation, and the region and spread of the oscillations governed by the single phase modulation equations. Similar results were obtained by Tian [21] and Wright [27] for the case of the KdV equation.

4 Conclusion

The primary mathematical results in this paper are: the modulation equations (13-14) with general smooth pulse initial data must shock; shocks form on the leading and trailing slopes of the pulse at locations associated with the maximum of $|\frac{A'_0(T)}{\sqrt{A_0(T)}}|$, where $A_0(T)$ is the initial pulse amplitude; and, the location down the fiber of shock formation is inversely proportional to the maximum of $|\frac{A'_0(T)}{\sqrt{A_0(T)}}|$.

Physically, these results establish the following scenario for the evolution

of smooth pulse initial data in nonlinear dispersive fibers governed by the NLS pde (9), with ϵ small. For small but nonzero distances $0 \leq Z \leq Z_b$, the non-oscillatory smooth pulse persists and zero - phase modulation equations (13-14) govern the evolution; at Z_b (explicitly estimated above), the modulation equations break down due to shock formation. (For $Z > Z_b$, we further surmise oscillations develop on the leading and trailing pulse slopes. The characterization of these ripples, their extent, and propagation are open problems.) We further show that for super-Gaussian pulse data, the capture window for the onset of pulse ripples, inside of which Z_b is guaranteed to sit, scales with $\hat{t}^1$, $L_{NL}^{1/2}$, and $(k'')^{-1/2}$, where $\hat{t}$ is the pulse width, L_{NL} is the nonlinear length, and k'' is the fiber dispersion. (Recall that $L_{NL} = (\gamma P_0)^{-1}$, where γ is the fiber nonlinear (Kerr) index and P_0 is the initial pulse power.)

We have further established (in the Appendix) that the results of Kodama and Wabnitz [11] for pure rectangular pulse data are rigorous. That is, a piecewise linear solution of the modulation equations persists and describes the small ϵ limit of the NLS equation (9), void of pulse ripples for $O(1)$ distances down the fiber - at least until the upper step vanishes, and the pulse becomes triangular. However, as explicitly illustrated in Section 3, the evolution of piecewise-linear data is unstable to arbitrary smoothing of rectangular pulses, and therefore fails to detect the breakdown of the WKB ansatz and corresponding zero-phase modulation equations at early distances down the fiber.

As noted, it remains to characterize the evolution of smooth pulses beyond the onset of pulse ripples, or equivalently, beyond the formation of shocks in the modulation equations. We anticipate that the spread and propagation of these oscillations will be governed (at least for some distance beyond Z_b) by single-phase modulation theory, as developed in [5]. It further remains

of engineering relevance to resolve the extent to which the slope oscillations identified here alter the appealing predictions of Kodama and Wabnitz [11] and Marcuse [15] for the evolution of the mean pulse width and center of mass, and also to investigate how small ϵ must be for the predictions of semiclassical theory (here or in [11]) to apply.

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5 Appendix

In this appendix we show that the special piecewise-linear solution to the modulation equations (13)-(14) constructed by Kodama and Wabnitz rigorously governs the behavior as $\epsilon \rightarrow 0$ of the solution to the NLS equation (9) with pure rectangular initial data (11).

We preface these arguments with an explanation of *why* this step is necessary. It must be emphasized that the WKB modulation equations (13)-(14) are formal, based on an ansatz (12), and are approximate for finite small ϵ . In particular, terms of order ϵ^2 are neglected in the WKB approach. It is a fact that as long as the solution to the modulation equations (13) - (14) is smooth (and hence unique), it will govern the behavior of the NLS equation for $\epsilon \downarrow 0$. However, once the solution to the modulation equations is not smooth, one has an infinite family of solutions to choose from, and moreover the quantities neglected in the derivation of the modulation equations are not small. Thus in the presence of shocks or jump discontinuities, one needs

additional information to describe the behavior of the NLS equation. The fundamental importance of the Lax-Levermore method is that the behavior as ϵ tends to 0 is characterized rigorously. The behavior, for example, is quite distinct from that picked out by a zero dissipation limit. For NLS the weak limit as $\epsilon \downarrow 0$ has been rigorously shown to be governed by a maximization problem.

Below we show that the piecewise linear solution of Kodama and Wabnitz solves the maximization problem for rectangular initial data, and therefore governs the behavior as ϵ goes to zero of the NLS equation for this initial data. However, we remark again that we have shown in the body of the text that this solution is unstable to any smoothing of the rectangular data (11), and fails to predict the onset of slope oscillations characterized above.

In [9], the authors considered the NLS equation

$$i\epsilon\partial_t\Psi + \frac{\epsilon^2}{2}\partial_{xx}\Psi + (1 - |\Psi|^2)\Psi = 0, \quad (56)$$

with far-field boundary conditions

$$\Psi(x, t) \approx e^{\pm \frac{i}{\epsilon} S_\infty} \quad \text{as } x \rightarrow \pm\infty, \quad (57)$$

and the initial condition

$$\Psi(x, 0) = \Psi^{(\epsilon)}(x) = A_0(x)e^{\frac{i}{\epsilon} S_0(x)}, \quad (58)$$

for smooth functions $A_0(x)$ and $S_0(x)$ that are independent of ϵ and are such that $\frac{1}{2}S_x \pm A$ each possesses precisely 1 critical point, and $A \rightarrow 1$ as $|x| \rightarrow \infty$. They prove, in particular, that the densities

$$\rho^{(\epsilon)} = |A^{(\epsilon)}(x, t)|^2 - 1, \quad (59)$$

$$\mu^{(\epsilon)} = |A^{(\epsilon)}(x, t)|^2 S_x^{(\epsilon)}(x, t) \quad (60)$$

satisfy

$$L^1 - \lim_{\epsilon \rightarrow 0} \rho^{(\epsilon)} = \rho(x, t), \quad (61)$$

$$L^1 - \lim_{\epsilon \rightarrow 0} \mu^{(\epsilon)} = \mu(x, t), \quad (62)$$

for $0 \leq t \leq t_b$, where t_b is the shock time for the associated zero-phase modulation equations, $\rho(x, t) = |A(x, t)|^2 - 1$, $\mu(x, t) = |A(x, t)|^2 S_x(x, t)$, and (A, S) is a particular solution of the zero-phase modulation equations which are *obtained through a maximization problem in function space*. In particular, when one considers weak solutions of the zero-phase modulation equations (e.g. those considered by Kodama and Wabnitz), it is this maximization problem which **uniquely determines the solution of the modulation equations which governs the NLS dynamics**.

Indeed, Jin, Levermore, and McLaughlin showed that the solution is obtained through the following procedure; we adapt their procedure to the special case of the present simple initial data

$$A_0(x) = \left\{ \begin{array}{ll} 0 & \text{for } x \in [1, \gamma] \\ 1 & \text{otherwise} \end{array} \right\}, \quad (63)$$

$$S_0(x) = 0. \quad (64)$$

Note that this initial data, when modified so that it is symmetric about $x = 0$, is a single pulse centered at $x = 0$ with width 2, and as we will see, the behavior for $x > \gamma$ does not alter the behavior of the pulse. The modifications to the method developed in [9] necessary to include the rectangular data (63)-(64) are minor, and can be found in [17], for the case of the continuum limit of the Toda lattice.

Define

$$\mathbf{A} = \{\text{measurable functions } \eta \text{ on } (0, \pi) : 0 \leq \eta(\theta) \leq (\gamma - 1) \sin \theta\}. \quad (65)$$

Let η^* be the unique solution of the following maximization problem

$$\max_{\eta \in \mathbf{A}} \left[\frac{2}{\pi} \int a(\theta) \eta(\theta) d\theta + \frac{1}{\pi^2} \int \int \log \left| \frac{\sin(\frac{1}{2}(\sigma - \theta))}{\sin(\frac{1}{2}(\sigma + \theta))} \right| \eta(\sigma) \eta(\theta) d\sigma d\theta \right], \quad (66)$$

where

$$a(\theta) = \sin \theta (t \cos \theta - x + \gamma). \quad (67)$$

Then

$$\rho(x, t) = \frac{2}{\pi} \partial_x \int_0^\pi \eta^*(\theta) \sin \theta d\theta, \quad (68)$$

$$\mu(x, t) = -\frac{2}{\pi} \partial_t \int_0^\pi \eta^*(\theta) \sin \theta d\theta. \quad (69)$$

In [9], the authors further showed that a function η^* solves the maximization problem (66) if and only of

$$L\eta^*(\sigma) + a(\sigma) = 0 \quad \text{for } \sigma \in \{\sigma : \eta^* > 0\}, \quad (70)$$

$$L\eta^*(\sigma) + a(\sigma) \leq 0 \quad \text{for } \sigma \in \{\sigma : \eta^* = 0\}, \quad (71)$$

$$L\eta^*(\sigma) + a(\sigma) \geq 0 \quad \text{for } \sigma \in \{\sigma : \eta^* = (\gamma - 1) \sin \sigma\}, \quad (72)$$

where L is the integral operator

$$Lh(\sigma) = \frac{1}{\pi} \int_0^\pi \log \left| \frac{\sin(\frac{1}{2}(\sigma - \theta))}{\sin(\frac{1}{2}(\sigma + \theta))} \right| h(\theta) d\theta. \quad (73)$$

Define

$$f(z; x, t) = i(\gamma - 1) [z - (z^2 - 1)^{1/2}] \quad \text{for } x \leq 1 - t \quad (74)$$

$$\begin{aligned} &= i(\gamma - x) [z - (z^2 - 1)^{1/2}] + \\ &\quad + it [2z(z^2 - 1)^{1/2} - (2z^2 - 1)] \quad \text{for } 1 + 2t < x < \gamma - 2t, \end{aligned} \quad (75)$$

and for $x \in [1 - t, 1 + 2t]$, define

$$f(z; x, t) = i(\gamma - x) [z - R^{1/2}] + \quad (76)$$

$$-it [2z^2 - 1 - R^{1/2} (2z + \beta + 1)] + \quad (77)$$

$$+i(\gamma - 1) [R^{1/2} - (z^2 - 1)^{1/2}],$$

where

$$R = (z - \beta)(z - 1), \quad (78)$$

$$\beta = \frac{1}{3} \left[1 + \frac{2}{t}(1 - x) \right]. \quad (79)$$

The square roots are taken so that $R(z)^{1/2} \approx z$ and $(z^2 - 1)^{1/2} \approx z$ for $z \rightarrow \infty$, and for z real they are evaluated from $\mathbf{C}_+$, i.e. $R(z)^{1/2} = \lim_{\delta \downarrow 0} R(z + i\delta)^{1/2}$.

Now set

$$\eta^*(\sigma; x, t) = \Re f(-\cos(\sigma)) \text{ for } \sigma \in (0, \pi), \quad (80)$$

where $\Re f$ denotes the real part of the function f . One can then show explicitly that $\eta^*(\sigma; x, t)$ solves the maximization problem (66) for all $x \leq \gamma - 2t$, by contour integration and the use of the relationship between the operator L and the classical Hilbert transform operator in the unit disc: $L\eta(\sigma) = \int_0^\sigma H\eta(\sigma') d\sigma'$. A simple calculation using relations (68) and (69) shows that:

for $x < 1 - t$

$$\rho(x, t) = 0, \quad (81)$$

$$\mu(x, t) = 0; \quad (82)$$

for $x \in [1 - t, 1 + 2t]$,

$$\rho(x, t) = \frac{1}{9} \left(\frac{1 - x}{t} + 2 \right)^2 - 1, \quad (83)$$

$$\mu(x, t) = \frac{2}{3} \left(\frac{1 - x}{t} - 1 \right) (\rho(x, t) + 1); \quad (84)$$

and for $x \in (1 + 2t, \gamma - 2t)$,

$$\rho(x, t) = -1, \quad (85)$$

$$\mu(x, t) = 0. \quad (86)$$

Thus

$$A(x, t) = \left\{ \begin{array}{ll} 1 & \text{for } x < 1 - t \\ \frac{1}{3} \left(\frac{1-x}{t} + 2 \right) & \text{for } x \in [1 - t, 1 + 2t] \\ 0 & \text{for } x \in (1 + 2t, \gamma - 2t) \end{array} \right\}, \quad (87)$$

$$S_x = \left\{ \begin{array}{ll} 0 & \text{for } x < 1 - t \\ \frac{2}{3} \left(\frac{1-x}{t} - 1 \right) & \text{for } x \in [1 - t, 1 + 2t] \\ -2 & \text{for } x \in (1 + 2t, \gamma - 2t) \end{array} \right\}, \quad (88)$$

as in [11].

Note that by considering the modification of the initial data so that one obtains a pulse centered at $x = 0$ with width 2, that the behavior for $|x| > \gamma - 2t$ does not alter the behavior of the pulse, and hence the formulae derived here are valid until the upper step has vanished, i.e. until $t = 1$. Kodama [10] has announced further solutions for $t > 1$.

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