Entropy of Transformations that Preserve an Infinite Measure

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Abstract

RACHEL LOUISE BAYLESS: Entropy of Transformations that Preserve an Infinite Measure
(Under the direction of Jane Hawkins)

In this dissertation we study transformations that preserve an infinite measure, with a focus on functions which preserve Lebesgue measure on the real line. More specifically, we investigate measure-theoretic properties of rational R-functions of negative type. We prove all rational R-functions of negative type are conservative, exact, ergodic, rationally ergodic, pointwise dual ergodic, and quasi-finite. We also explicitly construct the wandering rates and return sequences for all rational R-functions of negative type. The primary topic of study, however, is entropy of transformations preserving an infinite measure. We provide a method of computing the Krengel entropy for all rational R-functions of negative type. We also provide complete isomorphism invariants for c-isomorphisms between degree two rational R-functions of negative type.

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Table of Contents

List of Figures

CHAPTER 1

INTRODUCTION

In this dissertation we study the dynamics of transformations that preserve an infinite measure. In particular, we investigate the measure-theoretic properties of rational functions which preserve Lebesgue measure on the real line. While the literature on measure-theoretic properties of finite-measure-preserving transformations is well elaborated, there is not always a clear analogue of these properties for infinite-measure-preserving systems. In particular, entropy does not have a clear extension to transformations that preserve an infinite measure. The entropy of a system measures the amount of information gained with each application of an experiment or transformation. Higher entropy corresponds to more disorder and less predictable systems. The classical Kolmogorov-Sinai definition of entropy relies heavily on the ability to associate probabilities to possible events or outcomes. Thus, there is no universal analogue of entropy for infinite-measure-preserving systems. Different possibilities have been given independently by Krengel [**Kre**], Parry [**Par**], and Roy [**Roy**]. Two of these definitions have been around since the late 1960's, but there exist very few examples where any of these entropies have been computed explicitly. In this dissertation we provide a method of computing the Krengel entropy for an entire class of rational maps which preserve Lebesgue measure on the real line. Furthermore, we prove that these transformations are quasi-finite, so the three definitions of Krengel, Parry, and Roy coincide.

The transformations of interest in this thesis are a negative variant of R-functions. An R-function is an analytic map on the upper-half plane, $\mathbb{R}^{2+} = \{x + iy : y > 0\}$, that leaves $\mathbb{R}^{2+}$ invariant. R-functions have been studied in harmonic analysis under various names including Herglotz functions or Nevanlinna functions. The name R-function, however, dates back to Kac and Krein [**KK**]. No single source provides a complete history of R-functions, so in Chapter 3 we give a detailed description of the context in which such functions arise. This work is primarily concerned with dynamics of one-dimensional maps, so we impose an extra condition on R-functions. Given an R-function, f, we require that for Lebesgue almost every $x \in \mathbb{R}$, we have $\lim_{y \to 0} f(x + iy) = F(x)$, where $F : \mathbb{R} \to \mathbb{R}$ is a measurable map. Such maps F which are called the boundary functions associated to R-functions.

This dissertation provides an in depth study of the boundary functions associated to rational R-functions of negative type. These are rational maps, $T : \mathbb{R} \to \mathbb{R}$ such that $T = -F$, where F is the boundary function associated to an R-function to $\mathbb{R}$. In [**Let**] Letac proved that all rational R-functions (of both positive and negative type) preserve Lebesgue measure. Furthermore, the measure-theoretic properties of R-functions of positive type have been studied by Aaronson in [**Aar2**].

In Chapter 2 we give a brief introduction to classical measure-theoretic dynamical properties such as conservativity, ergodicity, and exactness. We also detail some nonstandard properties which arise only when considering infinite-measure-preserving systems. These properties are rational ergodicity and pointwise dual ergodicity. In Chapter 3 we give an overview of Aaronson's results on R-functions of positive type to provide context for our work on the negative case.

In Chapter 4 we prove that all rational R-functions of negative type are conservative, exact, and ergodic with respect to Lebesgue measure. We further prove that all rational R-functions of negative type are rationally ergodic and pointwise dual ergodic. These results are less restrictive than the aforementioned results of Aaronson for R-functions of positive type.

In Chapter 5 we provide a method of computing the Krengel entropy for all rational R-functions of negative type. The method is modeled after Rohlin's formula for entropy of expanding interval maps preserving an absolutely continuous probability measure. Furthermore, in Chapter 4 we show that all rational R-functions of negative type are quasi-finite, which implies that the three definitions of entropy coincide.

The usefulness of entropy in ergodic theory arises from the fact that it is an isomorphism invariant. That is, if two probability-preserving transformations are isomorphic, then they have the same entropy. For infinite-measure-preserving systems, however, there exist less restrictive isomorphisms called c-isomorphisms. In fact, if two infinite-measure-preserving transformations are c-isomorphic where $c \neq 1$, then they do not necessarily have the same Krengel entropy. In Chapter 6 we provide complete invariants (involving entropy) for c-isomorphisms between quadratic rational R-functions of negative type. We also give preliminary results on 1-isomorphism invariants for cubic rational R-functions of negative type.

Finally, in Chapter 7 we discuss open problems and possible future directions. In particular, we lay the framework for proving that R-functions of negative type (which are not necessarily rational) are exact.

CHAPTER 2

BACKGROUND

2.1. Preliminary Definitions

We begin with a few basic definitions of classical properties in ergodic theory. We use $(X, \mathcal{B}, m)$ to denote a measure space X together with a σ-algebra of measurable sets, $\mathcal{B}$, for a measure, m. We assume throughout that the measure m is σ-finite. We use $(X, \mathcal{B}, m, T)$ to denote a σ-finite measure space $(X, \mathcal{B}, m)$ together with a transformation $T : X \to X$ such that $T^{-1}\mathcal{B} \subseteq \mathcal{B}$. In this dissertation we consider only *nonsingular* systems $(X, \mathcal{B}, m, T)$. That is, given $A \in \mathcal{B}$, we have $m(T^{-1}A) = 0$ if and only if $m(A) = 0$. In fact, we are primarily interested in measure-preserving transformations (defined below) which is stronger than nonsingular.

Definition 2.1.1. A measurable function $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ is called *measure-preserving* if $m(T^{-1}A) = m(A)$ for all $A \in \mathcal{B}$.

We study the dynamical properties of measure-preserving transformations. In particular, we focus on transformations which preserve an infinite measure. That is, we consider measure-preserving systems $(X, \mathcal{B}, m, T)$ such that $m(X) = \infty$. We call such systems infinite-measure-preserving. A few of the most commonly studied measure-theoretic properties are defined below.

Definition 2.1.2. A nonsingular system $(X, \mathcal{B}, m, T)$ is *ergodic* if for every $A \in \mathcal{B}$ such that $T^{-1}A = A$ we have $m(A) = 0$ or $m(A^c) = 0$.

In other words, T is ergodic if the only invariant sets are trivial or the entire space. The next property also involves the preimages of sets and is closely related to ergodicity.

Definition 2.1.3. A nonsingular system $(X, \mathcal{B}, m, T)$ is *exact* if

$$\cap_{n>0} T^{-n}\mathcal{B} = \{\emptyset, X\} \bmod m. \tag{2.1.1}$$

Equivalently, a nonsingular system $(X, \mathcal{B}, m, T)$ is exact if $A \in \mathcal{B}$ such that

$$T^{-n}(T^n(A)) = A \quad \text{for all } n > 0, \tag{2.1.2}$$

implies $m(A) = 0$ or $m(A^c) = 0$.

The following classical result can be found in [**Roh2**], and is given here for completeness.

Lemma 2.1.4. *Let T be a nonsingular transformation of $(X, \mathcal{B}, m)$. If T is exact, then T is ergodic.*

In general the converse of Lemma 2.1.4 does not hold. For example, if T is invertible, then T is not exact. Furthermore, Eigen and Hawkins have constructed noninvertible shift maps which are ergodic but not exact (see [**EH**]).

Another classical property of interest is related to the recurrence of points to positive measure sets.

Definition 2.1.5. A set $A \in \mathcal{B}$ is called *wandering* for the nonsingular system $(X, \mathcal{B}, m, T)$ if the sets $\{T^{-i}A\}_{i=0}^{\infty}$ are pairwise disjoint.

Definition 2.1.6. A nonsingular system $(X, \mathcal{B}, m, T)$ is called *conservative* if there does not exist a wandering set of positive measure.

We note that a measure-preserving system $(X, \mathcal{B}, m, T)$ such that $m(X) < \infty$ is automatically conservative. We give an intuitive argument for the proof. If $m(A) > 0$

and $m(T^{-1}A) = m(A)$, then A only has so much room to "wander" throughout a finite space X. On the other hand, if $m(X) = \infty$, then conservativity of T is not automatic, because the inverse images of A won't necessarily fill the entire infinite space. Conservativity is, however, tied to the existence of measurable sets which do sweep out the entire space X.

DEFINITION 2.1.7. Let $(X, \mathcal{B}, m, T)$ be a nonsingular system. A set $A \in \mathcal{B}$ is called a *sweep-out set* for T if

$$\bigcup_{n=0}^{\infty} T^{-n}A = X \mod m. \tag{2.1.3}$$

Equivalently, A is a sweep-out set if for almost every $x \in X$ there exists an n_x such that $T^{n_x}(x) \in A$.

The following theorem relates the existence of sweep-out sets to conservativity of measure-preserving transformations.

THEOREM 2.1.8 (Maharam's Recurrence Theorem, [**Mah**]). *Suppose $(X, \mathcal{B}, m, T)$ is a measure-preserving system. If there exists a sweep-out set $A \in \mathcal{B}$ with $m(A) < \infty$, then T is conservative.*

Finally, we introduce the notion of an isomorphism between measure-preserving transformations.

DEFINITION 2.1.9 (Isomorphic). Let $(X_1, \mathcal{B}_1, m_1, T_1)$ and $(X_2, \mathcal{B}_2, m_2, T_2)$ be two measure-preserving systems. Suppose we have two sets $M_1 \in \mathcal{B}_1$ and $M_2 \in \mathcal{B}_2$ with $m_1(X_1 \setminus M_1) = 0$ and $m_2(X_2 \setminus M_2) = 0$ such that $T_1(M_1) \subseteq M_1$ and $T_2(M_2) \subseteq M_2$. We say $(X_1, \mathcal{B}_1, m_1, T_1)$ is *isomorphic* to $(X_2, \mathcal{B}_2, m_2, T_2)$ (or T_1 is isomorphic to T_2) if there exists an invertible map $\phi : M_1 \to M_2$ such that for all $A \in \mathcal{B}_2|_{M_2}$,

(1) $\phi^{-1}(A) \in \mathcal{B}_1|_{M_1}$,

(2) $m_1(\phi^{-1}(A)) = m_2(A)$, and

(3) $(\phi \circ T_1)(x) = (T_2 \circ \phi)(x)$ for all $x \in M_1$.

We will denote this situation by $\phi : T_1 \to T_2$, and ϕ is called an *isomorphism.*

It is clear that measure-theoretic properties such as conservativity, exactness, and ergodicity are invariant under isomorphism.

2.2. The Induced Transformation and Return-Time Sets

As stated above, we are primarily interested in studying measure-theoretic properties of transformations preserving an infinite measure. One technique that will be used throughout the following sections is inducing on a set of finite measure. The induced transformation provides a way to study the dynamics of transformations that preserve an infinite measure by looking only at a finite piece of the space.

Let $(X, \mathcal{B}, m, T)$ be a nonsingular system. Given $A \in \mathcal{B}$ let $\tilde{A} \subseteq A$ be the set defined by $\tilde{A} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^{-i}A$. That is, $\tilde{A}$ is the set of points in X which "hit" A infinitely often under iteration of T. For $x \in \tilde{A}$ define $\phi_A(x) = \min\{n : T^n(x) \in A\}$. That is, $\phi_A(x)$ is the *first-hitting-time* of x to A. If $x \in A$, then $\phi_A(x)$ is often referred to as the *first-return-time* of x to A. The *induced transformation*, $T_A : A \to A$, is defined by

$$\begin{aligned} T_A(x) &= T^{\phi_A(x)}(x) \text{ for } x \in \tilde{A} \\ T_A(x) &= x \text{ for } x \notin \tilde{A}. \end{aligned}$$

We note that if T is a conservative transformation and A is a sweep-out set for T, then by Definition 2.1.7 we have $A = \tilde{A}$. Letting $\mathcal{B}|_A = \{B \cap A : B \in \mathcal{B}\}$ and $m|_A(B) = m(A \cap B)$, we have the following classical result.

THEOREM 2.2.1. *Suppose $(X, \mathcal{B}, m, T)$ is a measure-preserving system. If A is a sweep-out set for T, then T_A is a measure-preserving transformation of $(A, \mathcal{B}|_A, m|_A)$.*

If $(X, \mathcal{B}, m, T)$ is an infinite-measure-preserving system and A is a sweep-out set with $m(A) < \infty$, then inducing on A yields a finite-measure-preserving system $(A, \mathcal{B}|_A, m|_A, T_A)$. We can often deduce information about the original infinite-measure-preserving system, $(X, \mathcal{B}, m, T)$, from the dynamics of the finite-measure-preserving system $(A, \mathcal{B}|_A, m|_A, T_A)$. For example, we have the following classical theorem which can be found in [**AW**].

THEOREM 2.2.2. *If $(A, \mathcal{B}|_A, m|_A, T_A)$ is ergodic, then $(X, \mathcal{B}, m, T)$ is also ergodic.*

In subsequent sections we will discuss similar results for studying the behavior of $(X, \mathcal{B}, m, T)$ via $(A, \mathcal{B}|_A, m|_A, T_A)$. Thus, it is important to know when sweep-out sets exist for infinite-measure-preserving systems. The following theorem says that if $(X, \mathcal{B}, m, T)$ is conservative and ergodic, then every positive-measure set is a sweep-out set.

THEOREM 2.2.3. *Let $(X, \mathcal{B}, m, T)$ be an infinite-measure-preserving system. If T is conservative and ergodic, then every $A \in \mathcal{B}$ such that $m(A) > 0$ is a sweep-out set for T.*

PROOF. Let $A \in \mathcal{B}$ such that $m(A) > 0$. Set

$$C_A = \{x \in X : \sum_{n=1}^{\infty} (\mathbb{1}_A \circ T^n)(x) = \infty\}. \tag{2.2.1}$$

We have that C_A is invariant for T. Therefore, by ergodicity, $m(C_A) = 0$ or $m(C_A^c) = 0$. However, by conservativity of T, we have that $A \subseteq C_A$, so $C_A = X$ mod m. That is, almost every $x \in X$ hits A infinitely often under iteration of T, so A is a sweep-out set. □

For the rest of this section, we assume $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, measure-preserving system, and $A \in \mathcal{B}$ with $0 < m(A) < \infty$ is a sweep-out set for T. We develop some notation to describe precise hitting-times to A.

Let $\mathfrak{A}$ denote the *first-return partition* of A. That is, $\mathfrak{A} = \{A_k\}$, where

$$A_k = \{x \in A : \phi_A(x) = k\} = A \cap T^{-k}A \setminus \bigcup_{j=1}^{k-1} T^{-j}A. \tag{2.2.2}$$

Let $\mathfrak{B} = \{B_k\}$ be a similar partition of A^c. That is,

$$B_k = \{x \in A^c : \phi_A(x) = k\} = A^c \cap T^{-k}A \setminus \bigcup_{j=1}^{k-1} T^{-j}A = T^{-k}A \setminus \bigcup_{j=0}^{k-1} T^{-j}A. \tag{2.2.3}$$

It is useful to view the action of T on the sets A_k and B_k as a two-story tower (see Figure 2.1).

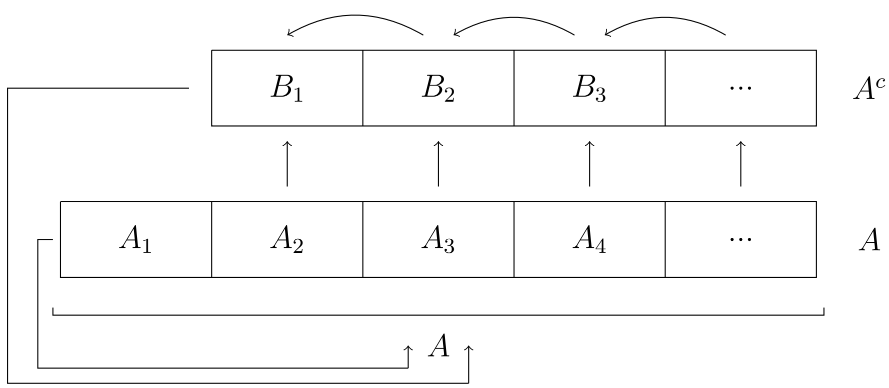

FIGURE 2.1. How the atoms of $\mathfrak{A}$ and $\mathfrak{B}$ move under T.

We now further develop the notation and make a few observations that will be helpful in subsequent sections. We define another collection of sets $\{D_n\}_{n\geq 0}$ by setting

$$D_0 = A \quad \text{and} \quad D_n = \{x \in A : \phi_A(x) > n\} \text{ for } n \geq 1. \tag{2.2.4}$$

We note that $D_n = \cup_{k=n+1}^{\infty} A_k$ and the $\{D_n\}_{n\geq 0}$ are nested such that $D_{n+1} \subset D_n$. Furthermore, we can relate D_n and B_n via the following result.

LEMMA 2.2.4. *Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure-preserving system. Given the sets $\{D_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 1}$ defined as above we have*

$$m(D_n) = m(B_n) \quad \textit{for all } n \geq 1. \tag{2.2.5}$$

Proof. By the definition of $\{A_n\}$ and $\{D_n\}$ we have

$$m(A) = m\left(\bigcup_{k=1}^{n} A_k\right) + m\left(\bigcup_{k=n+1}^{\infty} A_k\right) = \sum_{k=1}^{n} m(A_k) + m(D_n). \tag{2.2.6}$$

We also have

$$T^{-1}A = (T^{-1}A \cap A) \cup (T^{-1}A \setminus A) = A_1 \cup B_1. \tag{2.2.7}$$

Therefore,

(2.2.8)

$$T^{-2}A = T^{-1}A_1 \cup T^{-1}B_1 = T^{-1}A_1 \cup (T^{-1}B_1 \cap A) \cup (T^{-1}B_1 \setminus A) = T^{-1}A_1 \cup A_2 \cup B_2.$$

Thus, repeated application of (2.2.7) yields

$$T^{-n}A = T^{-(n-1)}A_1 \cup T^{-(n-2)}A_2 \cup ... \cup T^{-1}A_{n-1} \cup A_n \cup B_n, \tag{2.2.9}$$

which is a disjoint union. By assumption m is an invariant measure for T, so

$$\begin{aligned} m(T^{-n}A) &= m\left(\left(\bigcup_{k=1}^{n} T^{-(n-k)}A_k\right) \cup B_n\right) \\ &= \sum_{k=1}^{n} m(T^{-(n-k)}A_k) + m(B_n) \\ &= \sum_{k=1}^{n} m(A_k) + m(B_n) \\ &= \sum_{k=1}^{n} m(A_k) + m(D_n) \\ &= m(A), \end{aligned}$$

where the last two lines come from the definition of D_n and the observation in (2.2.6).

□

The following lemma gives a way to construct an invariant measure for T from an invariant measure for the induced transformation. It will be important in our study of entropy in Chapter 5. The statement can be found in [**Yur**], however no proof is given there.

Lemma 2.2.5. *Let $(X, \mathcal{B}, m, T)$ be a nonsingular system. Let $A \in \mathcal{B}$ with $0 < m(A) < \infty$ be a sweep-out set, and suppose $\nu_A << m|_A$ is a T_A-invariant measure. Then the following formula gives a T-invariant measure $\mu_{\nu_A} << m$:*

$$\mu_{\nu_A}(E) = \sum_{k=0}^{\infty} \nu_A(D_k \cap T^{-k}E), \quad \text{for all } E \in \mathcal{B}. \tag{2.2.10}$$

Proof. Given $E \in \mathcal{B}$, we have

$$\mu_{\nu_A}(E) = \sum_{k=0}^{\infty} \nu_A(D_k \cup T^{-k}E) = \sum_{k=0}^{\infty} \int_{D_k} \mathbb{1}_{(T^{-k}E)} d\nu_A.$$

Therefore,

$$\begin{aligned}
\mu_{\nu_A}(T^{-1}E) &= \sum_{k=0}^{\infty} \nu_A(D_k \cap T^{-(k+1)}E) = \sum_{k=0}^{\infty} \int_{D_k} \mathbb{1}_{(T^{-(k+1)}E)} d\nu_A \\
&= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \int_{A_j} \mathbb{1}_{(T^{-k}E)} d\nu_A \\
&= \sum_{k=1}^{\infty} \int_{A_k} \mathbb{1}_{(T^{-k}E)} d\nu_A + \sum_{k=2}^{\infty} \sum_{j=k}^{\infty} \int_{A_j} \mathbb{1}_{(T^{-k}E)} d\nu_A \\
&= \nu_A(T_A^{-1}(A \cap E)) + \sum_{k=1}^{\infty} \int_{D_k} \mathbb{1}_{(T^{-k}E)} d\nu_A. \qquad (2.2.11)
\end{aligned}$$

The measure ν_A is invariant for T_A, so we have $\nu_A(T_A^{-1}(A \cap E)) = \nu_A(A \cap E)$. Thus, (2.2.11) becomes

$$\nu_A(A \cap E) + \sum_{k=1}^{\infty} \int_{D_k} \mathbb{1}_{(T^{-k}E)} d\nu_A = \sum_{k=0}^{\infty} \int_{D_k} \mathbb{1}_{(T^{-k}E)} d\nu_A = \mu_{\nu_A}(E). \tag{2.2.12}$$

□

The following lemma says, if the T_A-invariant measure $\nu_A = \mu|_A$ and A is a sweep-out set, then the T-invariant measure obtained from Lemma 2.2.5 is precisely μ. The statement and proof can be found in both [**Aar4**] and [**Yur**].

Lemma 2.2.6. *If A is a sweep-out set, and $\mu|_A$ is T_A invariant, then $\mu_{\mu|_A} = \mu$.*

Proof. Given $n \geq 0$ and $E \in \mathcal{B}$, we have

$$\mu(E) = \mu|_A(A \cap E) + \sum_{k=1}^{n} \mu|_A(D_k \cap T^{-k}E) + \mu\left(\left(\bigcup_{k=0}^{n} T^{-k}A\right)^c \cap T^{-n}E\right). \tag{2.2.13}$$

Since A is a sweep-out set we know $\lim_{n\to\infty} \mu\left(\left(\bigcup_{k=0}^{n} T^{-k}A\right)^c\right) = 0$, so

$$\mu(E) = \mu|_A(A \cap E) + \sum_{k=1}^{\infty} \mu|_A(D_k \cap T^{-k}E) = \mu_{\mu|_A}(E). \tag{2.2.14}$$

□

2.3. The Perron-Frobenius Operator

In this section we develop a few more tools for studying the behavior of dynamical systems which will be used in subsequent sections. We say a nonsingular system $(X, \mathcal{B}, m, T)$ is n-to-1 if for almost every $x \in X$, the set $T^{-1}(x)$ contains precisely n distinct points. Given a nonsingular, n-to-1 system $(X, \mathcal{B}, m, T)$, we call a partition $\mathcal{P} = \{P_i\}_{i=1}^n$ of X a *Rohlin partition* for T if $T : P_i \to X$ is one-to-one and onto for each $i = 1, ..., n$ (see [**Roh1**]). Furthermore, we denote each branch $T|_{P_i}$ by T_i.

Definition 2.3.1. Let $(X, \mathcal{B}, m, T)$ be a nonsingular n-to-1 system, and let $\mathcal{P} = \{P_i\}_{i=1}^n$ be a Rohlin partition of X. We define the *Jacobian* of T by

$$J_T(x) = \sum_{i=1}^{n} \mathbb{1}_{P_i}(x) \frac{dmT_i}{dm}(x). \tag{2.3.1}$$

We note that if $X = \mathbb{R}$, m is Lebesgue measure, and T is piecewise C^1, then $J_T(x) = |T'(x)|$.

Definition 2.3.2. Given a nonsingular n-to-1 system $(X, \mathcal{B}, m, T)$ and $f \in L^1(m)$, we define the *Perron-Frobenius operator* by

$$\mathcal{L}_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{J_T(y)}. \tag{2.3.2}$$

Given a nonsingular n-to-1 system $(X, \mathcal{B}, m, T)$ and a Rohlin partition $\mathcal{P} = \{P_i\}_{i=1}^n$ we let ψ_i denote the inverse of T restricted to P_i. Therefore, $\psi_i = T_i^{-1} : X \to P_i$ is a one-to-one and onto mapping. We can rewrite (2.3.2) as $\mathcal{L}_T f(x) = \sum_{i=1}^n \frac{f(\psi_i(x))}{J_T(\psi_i(x))}$. We note that $T(T^{-1}(x)) = x$ for all $x \in X$, so $T(\psi_i(x)) = x$ for all $i = 1, .., n$ and $x \in X$. Taking the Jacobian of both sides yields $J_T(\psi_i(x)) \cdot J_{\psi_i}(x) = 1$. Therefore, $J_{\psi_i}(x) = \frac{1}{J_T(\psi_i(x))}$. Therefore, for $f \in L^1(m)$ equation (2.3.2) can be rewritten as

$$\mathcal{L}_T f(x) = \sum_{i=1}^n f(\psi_i(x)) \cdot J_{\psi_i}(x). \tag{2.3.3}$$

The following lemma relates the Perron-Frobenius operator to the existence of an invariant measure for T and can be found in [**Haw**].

LEMMA 2.3.3. *Given a nonsingular n-to-1 system $(X, \mathcal{B}, m, T)$, a function $f \in L^1(m)$ satisfies $\mathcal{L}_T f = f$ if and only if the measure ν defined by $f dm = d\nu$ is invariant for T.*

PROOF. Suppose $f \in L^1(m)$ such that $\mathcal{L}_T(f) = f$ and ν is a measure on X defined by $f dm = d\nu$. For $A \in \mathcal{B}$ we have

$$\nu(A) = \int_X \mathbb{1}_A(x) d\nu(x) = \int_X \mathbb{1}_A(x) \cdot f(x) dm(x) = \int_X \mathbb{1}_A(x) \cdot (\mathcal{L}_T f)(x) dm(x). \tag{2.3.4}$$

By the definition of $\mathcal{L}_T$ in (2.3.3) we have that (2.3.4) is equal to

$$\int_X \mathbb{1}_A(x) \cdot \sum_{i=1}^n f(\psi_i(x)) \cdot J_{\psi_i}(x) dm(x). \tag{2.3.5}$$

We let $\psi_i(x) = y$, so the above line becomes

$$\int_X \sum_{i=1}^n \mathbb{1}_{\psi_i A}(y) \cdot f(y) dm(y) = \int_X \mathbb{1}_{T^{-1}A} f(y) dm(y) = \nu(T^{-1}(A)). \tag{2.3.6}$$

The reverse direction is similar. $\square$

Another well-studied operator that is related to the Perron-Frobenius operator is the Koopman operator.

DEFINITION 2.3.4. Given $f \in L^p(m)$, the *Koopman operator* is defined by

$$U_T(f)(x) = (f \circ T)(x). \tag{2.3.7}$$

It is well known that $U_T : L^p(m) \to L^p(m)$ is a linear isometry, but for our purposes we will restrict to the case where $p = \infty$. From functional analysis we have that the dual of $L^1(m)$ is $L^\infty(m)$. The dual pairing $\langle \cdot, \cdot \rangle : L^1(m) \times L^\infty(m) \to \mathbb{R}$ satisfies

$$\langle f, g \rangle = \int_X f(x) \cdot g(x) dm(x), \quad \text{for } (f, g) \in L^1(m) \times L^\infty(m). \tag{2.3.8}$$

Now letting $(f, g) \in L^1(m) \times L^\infty(m)$ and considering the operators $\mathcal{L}_T$ and U_T, we have

$$\langle \mathcal{L}_T f, g \rangle = \int_X (\mathcal{L}_T f)(x) \cdot g(x) dm(x) = \int_X \sum_{i=1}^{n} f(\psi_i(x)) \cdot J_{\psi_i}(x) \cdot g(x) dm(x). \tag{2.3.9}$$

We change variables setting $y = \psi_i(x)$, and (2.3.9) becomes

$$\sum_{i=1}^{n} \int_{\psi_i(X)} f(y) \cdot (g \circ T)(y) dm(y) = \int_X f(y) \cdot (g \circ T)(y) dm(y) = \langle f, U_T g \rangle. \tag{2.3.10}$$

Therefore, the Perron-Frobenius operator is dual to the Koopman operator, and $\mathcal{L}_T$ is often referred to as the *dual operator* (as in [**Aar4**]). We adopt this language in the subsequent sections.

2.4. Rational Ergodicity and Pointwise Dual Ergodicity

We begin by stating the well-known Birkhoff Ergodic Theorem. A proof of this result can be found in any introductory ergodic theory text (see for example [**Wal**] or [**Pet**]).

Theorem 2.4.1. *Suppose $(X, \mathcal{B}, m, T)$ is a system preserving a σ-finite measure, and let $f \in L^1(m)$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = f^* \text{ almost everywhere,} \tag{2.4.1}$$

where $f^* \in L^1(m)$. Furthermore, $f^* \circ T = f^*$ almost everywhere.

We are interested in studying "Birkhoff-like" properties for ergodic transformations preserving an infinite measure. Two such properties are called rational ergodicitiy and pointwise dual ergodicity. Both definitions are due to Aaronson and have been studied in [**Aar1**], [**Aar2**], and [**Aar4**]. These are nonstandard notions, and no single source gives their entire story (including motivation, definition, and relationship to each other). Thus, we give a complete description here.

2.4.1. Rational Ergodicity. If $(X, \mathcal{B}, m, T)$ is an ergodic finite-measure-preserving system, then we have the following consequence of the Birkhoff ergodic theorem. If $A, B \in \mathcal{B}$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} m(B \cap T^{-k}A) = \frac{m(A)m(B)}{m(X)}. \tag{2.4.2}$$

On the other hand, if $m(X) = \infty$, then (2.4.2) is not well defined. We are interested in properties in the flavor of (2.4.2) that are satisfied by transformations preserving an infinite measure.

Let $A \in \mathcal{B}$ with $0 < m(A) < \infty$, and let $a_n(A) = \sum_{k=0}^{n-1} m(A \cap T^{-k}A)$. Let $W(T)$ denote the collection of sets, $A \in \mathcal{B}$, satisfying

$$\lim_{n\to\infty} \frac{1}{a_n(A)}\sum_{k=0}^{n-1} m(B \cap T^{-k}C) = \frac{m(B)m(C)}{m(A)^2}, \tag{2.4.3}$$

for all $B, C \in \mathcal{B} \cap A$. We say that T is *weakly rationally ergodic* if T is ergodic and $W(T) \neq \emptyset$.

We now introduce a stronger condition than weak rational ergodicity. Let $R(T)$ denote the collection of sets for which

$$\sup_{n\geq 1}\int_A\left(\frac{1}{a_n(A)}\sum_{k=1}^{n-1}\mathbb{1}_A\circ T^k\right)^2 dm<\infty. \tag{2.4.4}$$

We say that T is *rationally ergodic* if T is ergodic and $R(T)\neq\emptyset$. We will show that rational ergodicity implies weak rational ergodicity. First, we must catalogue the following well-known property of Hilbert spaces. A proof can be found in most functional analysis text books (see for example [**Wei**]).

THEOREM 2.4.2. *Let $\mathcal{H}$ be a Hilbert space. Every bounded sequence (f_n) in $\mathcal{H}$ contains a weakly convergent subsequence (f_{n_k}).*

Using Theorem 2.4.2 we prove the following lemma, which was originally stated in [**Aar1**]. Recall from Section 2.2 that T_A denotes the induced transformation and $\phi_A(x)$ denotes the first-return-time of x to A.

LEMMA 2.4.3. *Let $A\in R(T)$ and $\omega_n=(1/a_n(A))\sum_{k=0}^{n-1}\mathbb{1}_A\circ T^k$. Then, there exists a subsequence such that $\omega_{n_k}\to\omega$ weakly in $L^2(A)$ and $\omega\circ T_A=\omega$.*

PROOF. Let $A\in R(T)$ and $\omega_n=(1/a_n(A))\sum_{k=0}^{n-1}\mathbb{1}_A\circ T^k$ be as above. Then, by (2.4.4) for all $n\geq 1$ we have $\omega_n\in L^2(A)$ and there exists an M such that $||\omega_n||_2\leq M$. Thus, by Theorem 2.4.2 there exists a subsequence of ω_{n_k} of ω_n such that $\omega_{n_k}\to\omega$ weakly in $L^2(A)$. We abuse notation and write $\omega_n\to\omega$ after passing to the subsequence if necessary. Now, for $x\in A$ we have

$$
\begin{aligned}
(\omega_n \circ T_A)(x) &= \frac{1}{a_n(A)} \sum_{k=0}^{n-1} \mathbb{1}_A(T^k(T_A(x))) \\
&= \frac{1}{a_n(A)} \sum_{k=0}^{n-1} \mathbb{1}_A(T^{k+\phi_A(x)}(x)) \\
&= \frac{1}{a_n(A)} \sum_{k=\phi_A(x)}^{n-1+\phi_A(x)} \mathbb{1}_A(T^k(x)) \\
&= \frac{1}{a_n(A)} \left(\sum_{k=0}^{n-1+\phi_A(x)} \mathbb{1}_A(T^k(x)) - \sum_{k=1}^{\phi_A(x)-1} \mathbb{1}_A(T^k(x)) - \mathbb{1}_A(x) \right)
\end{aligned}
$$

The middle term $\sum_{k=1}^{\phi_A(x)-1} \mathbb{1}_A(T^k(x)) = 0$ since $T^k \notin A$ for all $1 \leq k \leq \phi_A(x) - 1$. Also, $x \in A$, so $\mathbb{1}_A(x) = 1$. Thus, we have,

$$(\omega_n \circ T_A)(x) = \omega_{n+\phi_A(x)}(x) - \frac{1}{a_n(A)}.$$

Taking the limit of both sides we see that $\omega \circ T_A = \omega$ almost everywhere. $\square$

Before proving that rational ergodicity is indeed stronger than weak rational ergodicity we make one final note about each ω_n. We have

$$
\begin{aligned}
\int_A \omega_n dm &= \frac{1}{a_n(A)} \int_A \sum_{k=0}^{n-1} \mathbb{1}_A \circ T^k \\
&= \frac{1}{a_n(A)} \sum_{k=0}^{n-1} m(A \cap T^{-k}A) = 1.
\end{aligned}
$$

Thus, we have that $\int_A \omega_n dm = 1 \; \forall n$.

THEOREM 2.4.4. *If an ergodic measure-preserving system,* $(X, \mathcal{B}, m, T)$, *is rationally ergodic, then it is weakly rationally ergodic.*

PROOF. We will show that $R(T) \subseteq W(T)$. Let $A \in R(T)$, and let $\omega_n = (1/a_n(A)) \sum_{k=0}^{n-1} \mathbb{1}_A \circ T^k$. By Lemma 2.4.3 we have $\omega \circ T_A = \omega$, and the ergodicity of T_A implies that ω is constant a.e. We also know that $\int_A \omega_n dm = 1 \; \forall n$, so $\int_A \omega dm = 1$.

Thus, we have that $\omega = 1/m(A)$ almost everywhere. Further, if $B \in \mathcal{B} \cap A$, then $\int_B \omega_n \to \int_B \omega$. Integrating we see that

$$\text{(2.4.5)} \qquad \frac{1}{a_n(A)} \sum_{k=0}^{n-1} m(B \cap T^{-k}A) \to \frac{m(B)}{m(A)} \quad \text{for all } B \in \mathcal{B} \cap A.$$

The same argument applies to T^{-1} since T is measure-preserving, so

$$\text{(2.4.6)} \qquad \frac{1}{a_n(A)} \sum_{k=0}^{n-1} m(A \cap T^{-k}B) \to \frac{m(B)}{m(A)} \quad \text{for all } B \in \mathcal{B} \cap A.$$

Now choose any $C \in \mathcal{B} \cap A$ and let $\sigma_n = (1/a_n(A)) \sum_{k=0}^{n-1} \mathbb{1}_C \circ T^k$, then $||\sigma_n||_2 \leq M$, for $n \geq 1$. An argument similar to that in the proof of Lemma 2.4.3 shows that $\int_A \sigma_n \to \int_A \sigma$, and σ is constant almost everywhere. Integrating σ_n over A we obtain

$$\begin{aligned} \int_A \sigma_n dm &= \frac{1}{a_n(A)} \int_A \sum_{k=0}^{n-1} \mathbb{1}_C \circ T^k dm \\ &= \frac{1}{a_n(A)} \sum_{k=0}^{n-1} m(T^{-k}C \cap A). \end{aligned}$$

Thus, by (2.4.6) we have $\int_A \sigma_n dm \to \frac{m(C)}{m(A)}$, so

$$\text{(2.4.7)} \qquad \sigma_n \to \frac{m(C)}{m(A)^2}.$$

In particular, integrating both sides of (2.4.7) over $B \in \mathcal{B} \cap A$ yields

$$\text{(2.4.8)} \qquad \frac{1}{a_n(A)} \sum_{k=0}^{n-1} m(B \cap T^{-k}C) \to \frac{m(B)m(C)}{m(A)^2}.$$

□

In [**Aar1**] Aaronson commented that there is no known weakly rationally ergodic transformation which is not rationally ergodic. To our knowledge the question of whether rational ergodicity is strictly stronger than weak rational ergodicity remains an open problem.

2.4.2. Pointwise Dual Ergodicity. Now, we turn our attention to another property of ergodic transformations preserving an infinite measure. We have the following consequence of the Birkhoff ergodic theorem (Theorem 2.4.1). If T is ergodic and $m(X) < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \frac{1}{m(X)} \int f dm, \tag{2.4.9}$$

almost everywhere for all $f \in L^1(m)$. Again, (2.4.9) holds only if $m(X) < \infty$. We are interested in a property in the flavor of (2.4.9) that is well defined when $m(X) = \infty$. We now state the definition of a pointwise dual ergodic transformation, which is a system preserving a σ-finite measure that has a "Birkhoff-like" property.

Definition 2.4.5. A conservative, ergodic, infinite-measure-preserving system $(X, \mathcal{B}, m, T)$ is called *pointwise dual ergodic* if there are constants $a_n(T)$ such that

$$\lim_{n\to\infty} \frac{1}{a_n(T)} \sum_{k=0}^{n-1} \mathcal{L}_T^k f(x) = \int_X f dm,$$

almost everywhere for all $f \in L^1(m)$.

Aaronson proved the following theorem which shows pointwise dual ergodicity is stronger than rational ergodicity.

Theorem 2.4.6 ([**Aar4**]). *Let $(X, \mathcal{B}, m, T)$ be a conservative ergodic measure-preserving system. If T is pointwise dual ergodic, then T is rationally ergodic.*

In order to study infinite-measure-preserving transformations, we often study the transformation on a finite piece of the space. Thus, in infinite ergodic theory we are often interested in the existence of particularly nice sets called Darling-Kac sets.

Definition 2.4.7. Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure-preserving system. A set $A \in \mathcal{B}$ with $0 < m(A) < \infty$ is called a *Darling-Kac* set for T if there exist constants $a_n(A) > 0$ such that

$$\frac{1}{a_n(A)} \sum_{k=0}^{n-1} \mathcal{L}_T^k \mathbb{1}_A \to 1 \quad \text{uniformly for almost every } x \in A. \tag{2.4.10}$$

We will draw a connection between Darling-Kac sets and pointwise dual ergodic transformations (and therefore $a_n(A)$ and $a_n(T)$) via Hurewicz's ergodic theorem. For completeness, we state Hurewicz's ergodic theorem here.

THEOREM 2.4.8 (Hurewicz's Ergodic Theorem, [**Hur**]). *Suppose that* $(X, \mathcal{B}, m, T)$ *is a conservative, ergodic, nonsingular system. Then, for all* $f, g \in L^1(m)$ *we have*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathcal{L}_T^k f(x)}{\sum_{k=1}^{n} \mathcal{L}_T^k g(x)} = \frac{\int_X f dm}{\int_X g dm} \quad \textit{for almost every } x \in X. \tag{2.4.11}$$

It is well known that if a conservative, ergodic, measure-preserving transformation T has Darling-Kac sets, then it is pointwise dual ergodic ([**Aar4**]). This result follows from Hurewicz's ergodic theorem letting $g = \mathbb{1}_A$ and $a_n(T) = \frac{a_n(A)}{m(A)}$, where A is a Darling-Kac set. Another consequence of Hurewicz's ergodic theorem is the asymptotic universality of $a_n(T)$.

DEFINITION 2.4.9. Given two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers we write

$$a_n \sim b_n, \quad \text{if } \lim_{n\to\infty} \frac{a_n}{b_n} = 1. \tag{2.4.12}$$

In this case we say $\{a_n\}$ is *asymptotic* to $\{b_n\}$.

From Hurewicz's ergodic theorem, if both A, A' are Darling-Kac sets for T, then

$$a_n(T) = \frac{a_n(A)}{m(A)} \sim \frac{a_n(A')}{m(A')}. \tag{2.4.13}$$

CHAPTER 3

R-FUNCTIONS

3.1. Boole and Generalized Boole Transformations

This dissertation focuses on transformations of the real line. Our primary measure of interest is one-dimensional Lebesgue measure, which we will always denote by λ.

Boole's transformation is defined by $B(x) = x - \frac{1}{x}$. In 1857 Boole showed

$$\int_{\mathbb{R}} g(x)dx = \int_{\mathbb{R}} g\left(B(x)\right)dx, \tag{3.1.1}$$

for all integrable functions g. It is clear that (3.1.1) is equivalent to showing that the transformation B preserves λ. Boole's transformation has become an archetypal example in infinite ergodic theory.

In 1973 Adler and Weiss proved the following theorem.

THEOREM 3.1.1 ([**AW**]). *Boole's transformation is conservative and ergodic with respect to λ.*

Extensions of B called *generalized Boole transformations* have the form

$$G(x) = x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x}, \tag{3.1.2}$$

where $\beta, t_k, p_k \in \mathbb{R}$ and $p_k > 0$ for all $k = 1, ..., N$.

In 1972 the following exercise appeared in a book by Pólya and Szegő ([**PS**]): "Show that $\pm G$ gives a complete characterization of all rational functions preserving λ." Later, in 1977 Letac solved this exercise and proved that a rational function preserves λ if and only if it is $\pm G$ ([**Let**]). This result is discussed in more detail in

Section 3.4. The ergodic properties of generalized Boole transformations were studied by Li and Schweiger in 1978, and they proved the following theorem.

THEOREM 3.1.2 ([**LS**]). *Let G be a generalized Boole transformation. If $\beta = 0$, then G is conservative and ergodic with respect to λ.*

The generalized Boole transformations have been studied under many names, including rational R-functions and rational inner functions. We will refer to transformations in the form of (3.1.2) as *rational R-functions of positive type.* The subsequent sections of this chapter give the history of R-functions from a harmonic analysis point of view. The results presented here provide a larger framework from which generalized Boole transformations arise.

3.2. Functions on the Unit Disc

Let $\mathbb{D}_r = \{z : |z| < r\}$ denote an open disc of radius r centered at 0 in $\mathbb{C}$, and $\mathbb{T}_r = \{z : |z| = r\}$ denote a circle of radius r. When $r = 1$ we may drop the subscript and denote the unit disc by $\mathbb{D} = \{z : |z| < 1\}$ and the unit circle by $\mathbb{T} = \{z : |z| = 1\}$. Also, let $z = x + iy \in \mathbb{C}$ and $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z.

The Poisson integral formula represents a function, $u : \mathbb{D}_r \to \mathbb{R}$, which is harmonic in $\mathbb{D}_r$ and continuous on $\mathbb{T}_r$. If $z \in \mathbb{D}_r$, then we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \Re \left(\frac{re^{i\theta} + z}{re^{i\theta} - z} \right) d\theta. \tag{3.2.1}$$

For more discussion of (3.2.1) see [**Con**] or [**Rem**]. Now, suppose we have an analytic function, $f : \mathbb{D}_r \to \mathbb{C}$, such that $\Re(f(z)) = u(z)$. That is, $f(z) = u(z) + iv(z)$ for some harmonic function $v : \mathbb{D}_r \to \mathbb{R}$. We claim

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta + iv(0). \tag{3.2.2}$$

In order to prove the claim we set $f(z) = u(z) + iv(z)$ and $g(z) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \frac{re^{i\theta}+z}{re^{i\theta}-z} d\theta +$ $iv(0)$, and we show that $f(z) = g(z)$. First, taking the real part of g and comparing it to (3.2.1), we have $\Re(g(z)) = u(z) = \Re(f(z))$. Therefore, by the Cauchy-Riemann equations f and g differ by a purely imaginary constant. Evaluating at $z = 0$, we have $f(0) = u(0) + iv(0) = g(0)$. Thus, $f(0) - g(0) = 0$, and $f(z) = g(z)$ for all $z \in \mathbb{D}_r$. Equation (3.2.2) is known as the Schwarz integral formula (for more discussion see [**Con**] or [**Rem**]).

Our goal is to use the Schwarz integral formula to obtain a general form for analytic functions, $f : \mathbb{D} \to \mathbb{C}$ (not necessarily continuous on $\mathbb{T}$), which have nonnegative real part. First, we must develop a bit more material and notation. Let $\mathcal{M}(\mathbb{T})$ denote the space of all Borel measures on $\mathbb{T}$. For each harmonic function h on $\mathbb{D}$ and $0 < r < 1$ there is a corresponding Borel measure $\tau_r \in \mathcal{M}(\mathbb{T})$ defined by

$$d\tau_r(e^{i\theta}) = \frac{1}{2\pi} h(re^{i\theta}) d\theta. \tag{3.2.3}$$

The following theorem concerning convergence of measures can be found in [**RR**].

THEOREM 3.2.1. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel measures on $\mathbb{T}$. Suppose there exists a constant $M < \infty$ such that $\mu_n(\mathbb{T}) \leq M$, for $n \geq 1$. Then there exists a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ and a Borel measure μ on $\mathbb{T}$ such that $\mu(\mathbb{T}) \leq M$ and*

$$\lim_{k \to \infty} \int_{\mathbb{T}} f d\mu_{n_k} = \int_{\mathbb{T}} f d\mu, \tag{3.2.4}$$

for every continuous complex-valued function f on $\mathbb{T}$.

We have the following theorem concerning functions which are analytic in $\mathbb{D}$ and take values in the right half-plane. The proof has been modified from [**Tsu**] and [**GG**].

THEOREM 3.2.2. *A function, f, is analytic in $\mathbb{D}$ with $\Re(f(z)) > 0$ for all $z \in \mathbb{D}$ if and only if f can be written as*

$$f(z) = i\gamma + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\tau(e^{i\theta}), \tag{3.2.5}$$

where $\gamma \in \mathbb{R}$ and τ is a finite nonnegative measure on $\mathbb{T}$.

Proof. If f admits a representation as in (3.2.5), then for $z \in \mathbb{D}$ where $z = re^{i\psi}$ we have

$$\Re(f(z)) = \Re\left(\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\tau(e^{i\theta})\right) = \int_{\mathbb{T}} \frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2} d\tau(e^{i\theta}) \geq 0. \tag{3.2.6}$$

To prove the other direction, we suppose that f is analytic in $\mathbb{D}$, so $f(z) = u(z)+iv(z)$ where $u, v : \mathbb{D} \to \mathbb{R}$ are harmonic. Thus, for each $r < 1$ f is harmonic on $\mathbb{D}_r$ and continuous on $\mathbb{T}_r$. By the Schwarz integral formula, for each $r < 1$ we have

$$f(z) = iv(0) + \frac{1}{2\pi}\int_{\mathbb{T}} u(re^{i\theta}) \frac{re^{i\theta}+z}{re^{i\theta}-z} d\theta. \tag{3.2.7}$$

Recasting (3.2.7) in the language of measures as in (3.2.3) we have

$$f(z) = iv(0) + \int_{\mathbb{T}} \frac{re^{i\theta}+z}{re^{i\theta}-z} d\tau_r(e^{i\theta}), \tag{3.2.8}$$

where τ_r is a measure on $\mathbb{T}$ such that $d\tau_r(e^{i\theta}) = \frac{1}{2\pi}u(re^{i\theta})d\theta$. Note that $u(re^{it}) > 0$, so the measure τ_r is nonnegative. Now, we show that τ_r is finite on $\mathbb{T}$. That is, $\int_{\mathbb{T}} d\tau_r(e^{i\theta}) < \infty$. By definition of τ_r we have

$$\int_{\mathbb{T}} d\tau_r(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta. \tag{3.2.9}$$

In order to show the right-hand side of (3.2.9) is finite, we apply the Poisson integral formula to u and obtain

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta. \tag{3.2.10}$$

Finally, we recall $u(0) = \Re(f(0)) < \infty$, because f is analytic in $\mathbb{D}$. Therefore, τ_r is a finite measure on $\mathbb{T}$. Considering $f(rz)$ together with (3.2.8) we have

$$f(rz) = iv(0) + \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\tau_r(e^{i\theta}). \tag{3.2.11}$$

Finally, by Theorem 3.2.1 there exists a sequence $\{r_n\} \uparrow 1$ and a finite nonnegative measure τ on $\mathbb{T}$ such that

$$f(z) = iv(0) + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\tau(e^{i\theta}). \tag{3.2.12}$$

$\square$

3.3. Functions on the Upper Half-Plane

Let $\mathbb{R}^{2+}$ denote the upper half-plane in $\mathbb{C}$. That is, $\mathbb{R}^{2+} = \{x + iy : x, y \in \mathbb{R} \text{ and } y > 0\}$.

Definition 3.3.1 (Kac and Krein, [**KK**]). An analytic function $f : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ is called an *R-function.*

Let ϕ be the conformal map taking $\overline{\mathbb{R}^{2+}}$ to $\overline{\mathbb{D}}$ defined by

$$\phi(z) = \frac{z - i}{z + i} \quad \text{and} \quad \phi^{-1}(z) = i\frac{1 + z}{1 - z}. \tag{3.3.1}$$

We will change variables using ϕ in the proof of the following theorem, so for convenience we make a note about the procedure. If τ is a Borel measure on $\mathbb{T} \setminus \{1\}$, define $(\tau \circ \phi)(A) = \tau(\phi(A))$ for every Borel set $A \subseteq \mathbb{R}$, so that $\tau \circ \phi$ is a Borel measure on $\mathbb{R}$. For every $g \in L^1(\tau)$, $g \circ \phi \in L^1(\tau \circ \phi)$ and

$$\int_{\mathbb{T}\setminus\{1\}} g(e^{i\theta}) d\tau(e^{i\theta}) = \int_{\mathbb{R}} (g(\phi(t)) d(\tau \circ \phi)(t). \tag{3.3.2}$$

The following theorem can be found in [**KK**], but no proof is given. We have modified the proof given in [**GG**].

Theorem 3.3.2. *An analytic function, f, is an R-function if and only if it can be represented in the form*

$$f(z) = \beta_f + \alpha_f z + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\mu(t), \tag{3.3.3}$$

where $\beta_f, \alpha_f \in \mathbb{R}$, $\alpha_f \geq 0$, and μ is a finite nonnegative measure on $\mathbb{R}$.

Proof. Let $z = x + iy \in \mathbb{R}^{2+}$. If $f(z) = u(z) + iv(z)$ is of the form (3.3.3), then

(3.3.4)
$$f(z) = \underbrace{\beta_f + \alpha_f x + \int_{\mathbb{R}} \frac{t - x + t^2 x - tx^2 - ty^2}{(t-x)^2 + y^2} d\mu(t)}_{u(z)} + i \underbrace{\left(\alpha_f y + \int_R \frac{y + t^2 y}{(t-x)^2 + y^2} d\mu(t) \right)}_{v(z)}.$$

Thus, we can see that if $y > 0$, then $v(z) > 0$. To prove the other direction we suppose f is an R-function. Let ϕ be as in (3.3.1). We have that $F(z) = -i(f \circ \phi^{-1})(z)$ satisfies the conditions of Theorem 3.2.2, so

$$F(z) = i\gamma + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\tau(e^{i\theta}), \tag{3.3.5}$$

where $\gamma \in \mathbb{R}$ and τ is a finite nonnegative measure on $\mathbb{T}$. Furthermore, $f(z) = i(F \circ \phi)(z)$, so we have

$$\begin{aligned} f(z) &= i \left(i\gamma + \int_{\mathbb{T}} \frac{e^{i\theta} + \frac{z-i}{z+i}}{e^{i\theta} - \frac{z-i}{z+i}} d\tau(e^{i\theta}) \right) \\ &= -\gamma + i \int_{\mathbb{T}\setminus\{1\}} \frac{e^{i\theta} + \frac{z-i}{z+i}}{e^{i\theta} - \frac{z-i}{z+i}} d\tau(e^{i\theta}) + i\tau(\{1\}) \left(\frac{1 + \frac{z-i}{z+i}}{1 - \frac{z-i}{z+i}} \right) \\ &= -\gamma + \alpha z + i \int_{\mathbb{T}\setminus\{1\}} \frac{e^{i\theta} z + ie^{i\theta} + z - i}{e^{i\theta} z + ie^{i\theta} - z + i} d\tau(e^{i\theta}). \end{aligned} \tag{3.3.6}$$

The last line comes from letting $\alpha = \tau(\{1\})$ and noting that $i\frac{1+\frac{z-i}{z+i}}{1-\frac{z-i}{z+i}} = \phi^{-1}(\phi(z)) = z$. Consider the integrand of (3.3.6), and define $g(e^{i\theta}) = \frac{e^{i\theta}z+ie^{i\theta}+z-i}{e^{i\theta}z+ie^{i\theta}-z+i}$. Using the change of variables laid out in (3.3.2) we obtain

$$\begin{aligned} (3.3.6) &= -\gamma + \alpha z + i \int_{\mathbb{R}} \frac{\left(\frac{t-i}{t+i}\right) z + i\left(\frac{t-i}{t+i}\right) + z - i}{\left(\frac{t-i}{t+i}\right) z + i\left(\frac{t-i}{t+i}\right) - z + i} d(\tau \circ \phi)(t) \\ &= -\gamma + \alpha z + \int_{\mathbb{R}} \frac{tz + 1}{t - z} d(\tau \circ \phi)(t). \end{aligned} \tag{3.3.7}$$

Thus, letting $\beta_f = -\gamma$, $\alpha_f = \alpha$, and $\mu = \tau \circ \phi$ yields

$$f(z) = \beta_f + \alpha_f z + \int_{\mathbb{R}} \frac{tz+1}{t-z} d\mu(t). \tag{3.3.8}$$

□

Given an R-function, f, the following lemma gives another way to classify the coefficient α_f.

LEMMA 3.3.3. *If f is an R-function as in (3.3.3), then*

$$\lim_{y\to\infty} \frac{\Im(f(iy))}{y} = \alpha_f. \tag{3.3.9}$$

PROOF. If f is an R-function, then f has the form (3.3.3) and can be written in terms of its real and imaginary parts as in (3.3.4). Therefore, we have

$$\frac{\Im(f(iy))}{y} = \alpha_f + \int_{\mathbb{R}} \frac{1+t^2}{t^2+y^2} d\mu(t). \tag{3.3.10}$$

We will show that the second term on the left-hand side of (3.3.10) approaches 0 as $y \to \infty$. If $y > 1$, then $\frac{1+t^2}{t^2+y^2} < 1$, so for sufficiently large N we have

$$\int_{-\infty}^{-N} \frac{1+t^2}{t^2+y^2} d\mu(t) + \int_{N}^{\infty} \frac{1+t^2}{t^2+y^2} d\mu(t) < \int_{-\infty}^{-N} d\mu(t) + \int_{N}^{\infty} d\mu(t) < \frac{\epsilon}{2}. \tag{3.3.11}$$

On the other hand, if y is sufficiently large, then

$$\int_{-N}^{N} \frac{1+t^2}{t^2+y^2} d\mu(t) < \frac{\epsilon}{2}. \tag{3.3.12}$$

□

Given an R-function, f, the following lemma gives a condition under which we can write f in a simpler form.

LEMMA 3.3.4. *If f is an R-function as in (3.3.3), and $\int_{\mathbb{R}} t^2 d\mu(t) < \infty$, then f can be written in the reduced form*

$$f(z) = \alpha_f z + \beta'_f + \int_{\mathbb{R}} \frac{d\nu(t)}{t-z}, \tag{3.3.13}$$

where $d\nu(t) = (1+t^2)d\mu(t)$ *and* $\beta'_f = \beta_f - \int_{\mathbb{R}} t d\mu(t)$.

PROOF. We have

$$\begin{aligned}
f(z) &= \alpha_f z + \beta_f + \int_{\mathbb{R}} \frac{1+tz}{t-z} d\mu(t) \\
&= \alpha_f z + \beta_f + \int_{\mathbb{R}} \frac{1+t^2-t^2+tz}{t-z} d\mu(t) \\
&= \alpha_f z + \beta_f + \int_{\mathbb{R}} \frac{(1+t^2)-t(t-z)}{t-z} d\mu(t) \\
&= \alpha_f z + \beta_f - \int_{\mathbb{R}} t d\mu(t) + \int_{\mathbb{R}} \frac{1+t^2}{t-z} d\mu(t) \\
&= \alpha_f z + \beta'_f + \int_{\mathbb{R}} \frac{d\nu(t)}{t-z}.
\end{aligned}$$

□

DEFINITION 3.3.5 (Kac and Krein [**KK**]). An R-function, f, is an element of R_0 if f admits the following representation

$$f(z) = \int_{\mathbb{R}} \frac{d\nu(t)}{t-z}, \tag{3.3.14}$$

where ν is a finite nonnegative measure on $\mathbb{R}$.

LEMMA 3.3.6. *If f is an R-function such that* $\lim_{y\to\infty} iyf(iy) = c < \infty$, *then* $f \in R_0$.

PROOF. We have that f is an R-function, so f has form (3.3.3). Writing $f(iy)$ in terms of its real and imaginary parts yields

$$f(iy) = \beta_f + \int_{\mathbb{R}} \frac{t-ty^2}{t^2+y^2} d\mu(t) + i\left(\alpha_f y + \int_R \frac{y+t^2y}{t^2+y^2} d\mu(t)\right). \tag{3.3.15}$$

Therefore,

$$(3.3.16) \qquad iyf(iy) = -\alpha_f y^2 - \int_R \frac{(1+t^2)y^2}{t^2+y^2} d\mu(t) + iy\left(\beta_f + \int_{\mathbb{R}} \frac{t-ty^2}{t^2+y^2} d\mu(t)\right).$$

Convergence of the real part along with Lemma 3.3.3 gives $a_f = 0$. Convergence of the real part also yields $\int_{\mathbb{R}}(1+t^2)d\mu(t) < \infty$. Therefore, we may apply Lemma 3.3.4 to obtain

$$(3.3.17) \qquad f(z) = \beta'_f + \int_{\mathbb{R}} \frac{d\nu(t)}{t-z},$$

where $\beta'_f = \int_{\mathbb{R}} t d\mu(t)$. To see that $\beta'_f = 0$ we consider the imaginary part of (3.3.16). We have

$$(3.3.18) \qquad y\left(\beta_f + \int_{\mathbb{R}} \frac{t-ty^2}{t^2+y^2} d\mu(t)\right) \to 0 \quad \text{as } y \to \infty.$$

Therefore,

$$(3.3.19) \qquad y\left(\beta_f - \int_{\mathbb{R}} \frac{ty^2}{t^2+y^2} d\mu(t)\right) + y\left(\int_{\mathbb{R}} \frac{t}{t^2+y^2} d\mu(t)\right) \to 0 \quad \text{as } y \to \infty.$$

Since the first piece of (3.3.19) converges, we know

$$(3.3.20) \qquad \beta_f - \int_{\mathbb{R}} \frac{ty^2}{t^2+y^2} d\mu(t) \to 0 \quad \text{as } y \to \infty.$$

Hence, $\beta_f = \int_{\mathbb{R}} t d\mu(t)$ and $\beta'_f = 0$. $\square$

We are interested in studying transformations on the real line, so we impose an extra restriction on R-functions.

Definition 3.3.7. A function, f, is an *inner function* on the upper half-plane, if f is an R-function such that for λ-almost every $x \in \mathbb{R}$ the limit $\lim_{y\to 0} f(x+iy)$ exists and is real. We let $F(x) = \lim_{y\to 0} f(x+iy)$ and call F the *boundary function* associated to f.

We have the following relationship between inner functions and R-functions:

$$\{\text{inner functions of the upper half-plane}\} \subset \{R\text{-functions}\}.$$

The following classical theorem is stated here for completeness, and the proof can be found in [**RR**].

THEOREM 3.3.8 (Fatou's Theorem). *Let*

$$h(z) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-x)^2 + y^2}, \quad y > 0, \tag{3.3.21}$$

where σ is a nonnegative Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{d\sigma(t)}{1+t^2} < \infty$. Suppose σ has the following Lebesgue decomposition

$$d\sigma = H d\lambda + d\sigma_s, \tag{3.3.22}$$

where λ is Lebesgue measure and σ_s is the singular piece. Then,

$$\lim_{z \to x} h(z) = H(x) \quad \text{for almost every} \quad x \in \mathbb{R}. \tag{3.3.23}$$

Using Fatou's Theorem we obtain the following theorem which is a special case of Theorem 3.3.2 and provides a general form for all inner functions of the upper half-plane. The statement can be found in [**Aar4**], but we give a different proof.

THEOREM 3.3.9. *An R-function, f, is an inner function if and only if for every $z \in \mathbb{R}^{2+}$ it can be represented as*

$$f(z) = \beta_f + \alpha_f z + \int_{\mathbb{R}} \frac{1+tz}{t-z} d\mu(t), \tag{3.3.24}$$

where $\beta_f \in \mathbb{R}$, $\alpha_f \geq 0$, and μ is a positive finite measure on $\mathbb{R}$ that is singular with respect to Lebesgue measure.

Proof. If f is an inner function, then f is an R-function. By Theorem 3.3.2 we have that

$$f(z) = \beta_f + \alpha_f z + \int_{\mathbb{R}} \frac{1+tz}{t-z} d\mu(t), \tag{3.3.25}$$

where $\beta_f \in \mathbb{R}$, $\alpha_f \geq 0$, and μ is finite nonnegative measure on $\mathbb{R}$. If we let $d\sigma(t) = (1+t^2)d\mu(t)$, then

$$f(z) = \beta_f + \alpha_f z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t), \tag{3.3.26}$$

where $d\sigma$ defines nonnegative measure on $\mathbb{R}$, such that $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$. Thus, we need only show that σ is singular with respect to Lebesgue measure. We let $d\sigma = H d\lambda + d\sigma_s$ be the Lebesgue decomposition of σ. Letting $z = x+iy$ and considering the imaginary part of f we have

$$\Im(f(x+iy)) = \alpha_f y + \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} d\sigma(t). \tag{3.3.27}$$

We apply Fatou's Theorem to $\Im(f(x+iy))$ and $d\sigma$ to obtain $\lim_{z\to x} \Im(f(z)) = H(x)$ a.e. on $\mathbb{R}$. However, we have that $\lim_{y\to 0} \Im(f(x+iy)) = 0$, because f is inner. Therefore, $H = 0$, and $\sigma = \sigma_s$ is singular.

For the other direction, it is clear that if f has the form in (3.3.24), then f is an inner function. $\square$

3.4. Dynamics of Boundary Functions on the Real Line

We are interested in measure-theoretic dynamical properties of the boundary functions associated to inner functions, so we study maps F such that $F(x) = \lim_{y\to 0} f(x+iy)$ for λ-almost every $x \in \mathbb{R}$, where f is an inner function. We have $F : \mathbb{R} \setminus M \to \mathbb{R}$, where $\lambda(M) = 0$.

The following theorem due to Letac connects inner functions with measurable functions $F : \mathbb{R} \to \mathbb{R}$ preserving the class of Cauchy distributions. If $z = a+ib$, then we denote a *Cauchy distribution* by

$$\sigma_z(t) = \frac{dP_z}{d\lambda}(t) = \frac{b}{\pi\left((t-a)^2+b^2\right)}. \tag{3.4.1}$$

We note that P_z is a measure (sometimes called a *Cauchy measure*) on $\mathbb{R}$ is given by

$$P_z(A) = \int_{\mathbb{R}} \mathbb{1}_A(t)\frac{b}{\pi\left((t-a)^2+b^2\right)}d\lambda(t), \tag{3.4.2}$$

for any measurable set A.

THEOREM 3.4.1 ([**Let**]). *Let $\epsilon = \pm 1$. A measurable function, $F : \mathbb{R} \to \mathbb{R}$, preserves the class of Cauchy distributions if and only if $\epsilon \cdot F$ is a boundary function associated to an inner function. In particular, if f is the inner function corresponding to F and $z \in \mathbb{C}$, then*

$$P_z \circ F^{-1} = P_{f(z)}. \tag{3.4.3}$$

We have the following corollary of Theorem 3.4.1 which was first proved by Letac in [**Let**] and shown again by Aaronson in [**Aar2**]. Our proof follows the outline of Aaronson.

COROLLARY 3.4.2. *If F is the boundary function associated to an inner function, then F preserves λ if and only if $\alpha_F = 1$.*

PROOF. Let f be an inner function as in (3.3.24), and let $F : \mathbb{R} \to \mathbb{R}$ be the boundary function associated to f. We write f in terms of its real and imaginary parts as $f(z) = u(z) + iv(z)$. We consider the action of f on $z = ib$. We note

$$u(ib) = b\int_{\mathbb{R}} \frac{t(1-b^2)}{t^2+b^2}d\mu(t) \quad \text{and} \quad v(ib) = \alpha_F b + \int_{\mathbb{R}} \frac{b(1+t^2)}{t^2+b^2}d\mu(t). \tag{3.4.4}$$

By the dominated convergence theorem we have $\lim_{b\to\infty} u(ib) = \mu(\mathbb{R})$ and $\lim_{b\to\infty} v(ib) - b = 0$.

Thus, we have

$$\frac{u(ib)}{b} \to 0 \quad \text{and} \quad \frac{v(ib)}{b} \to \alpha_T \quad \text{as} \quad b \to \infty. \tag{3.4.5}$$

Suppose $\alpha_F = 1$. For $A \in \mathcal{B}$ we have

$$\pi b P_{ib}(A) = \pi b \int_{\mathbb{R}} \mathbb{1}_A(t) \frac{b}{\pi(t^2 + b^2)} d\lambda(t), \tag{3.4.6}$$

so $\lim_{b\to\infty} \pi b P_{ib}(A) = \lambda(A)$. Also,

$$\pi b P_{f(ib)}(A) = \pi b \int_{\mathbb{R}} \mathbb{1}_A(t) \frac{v(ib)}{\pi\left((t^2 - u(ib)^2) + v(ib)^2\right)}, \tag{3.4.7}$$

so using (3.4.5) we have $\lim_{b\to\infty} \pi b P_{f(ib)}(A) = \lambda(A)$. Finally, by Theorem 3.4.1 we have $P_{ib}(F^{-1}A) = P_{f(ib)}(A)$, so taking the limit as $b \to \infty$ yields $\lambda(F^{-1}A) = \lambda(A)$. For the reverse direction suppose $\alpha_F \neq 1$. We now have that $\lim_{b\to\infty} \pi b P_{f(ib)}(A) = \frac{1}{\alpha_F}\lambda(A)$, so $\lambda(F^{-1}A) = \frac{1}{\alpha_F}\lambda(A)$. □

We are primarily interested in transformations that preserve an infinite measure, so our study of inner functions and their variants will be restricted to the case where $\alpha_F = 1$. Aaronson proved the following theorem for infinite-measure-preserving boundary functions associated to inner functions, which can be written in a reduced form like (3.3.13).

THEOREM 3.4.3 ([**Aar4**]). *Suppose that F is the boundary function associated to an inner function and*

$$F(x) = x + \beta + \int_{\mathbb{R}} \frac{d\nu(t)}{t - x}, \tag{3.4.8}$$

where $\beta \in \mathbb{R}$ and ν is singular and compactly supported on $\mathbb{R}$. If $\beta = 0$, then F is exact and pointwise dual ergodic with respect to λ. The return sequence is given by $a_n(F) \sim \frac{1}{\pi}\sqrt{\frac{2n}{\nu(\mathbb{R})}}$. If $\beta \neq 0$, then F is totally dissipative and non-ergodic with respect to λ.

We note that generalized Boole transformations as in (3.1.2) are boundary functions associated to reduced form inner functions, and therefore they fall into the scope of Theorem 3.4.3. Let δ_x denote a point mass measure with mass 1 at the point x. If G is a generalized Boole transformation, then we can represent G as

$$G(x) = x + \beta + \int_{\mathbb{R}} \frac{d\nu(t)}{t-x}, \quad \text{where } \nu = \sum_{i=1}^{N} p_i \delta_{t_i}. \tag{3.4.9}$$

Thus, Letac's 1977 [**Let**] result (Corollary 3.4.2) completes the exercise of Pólya and Szegő showing that $\pm G$ are the only rational functions preserving λ. Also, Aaronson [**Aar2**] took Li and Schweiger's result (Theorem 3.1.2) further when he proved Theorem 3.4.3, showing that if $\beta = 0$ then generalized Boole transformations are also exact and pointwise dual ergodic.

3.5. R-functions of Negative Type

We study a slight variant of R-functions called R-functions of negative type. As the name suggests we have the following definition.

Definition 3.5.1. An analytic map, h, is an *R-function of negative type* if $h : \mathbb{R}^{2+} \to \mathbb{R}^{2-}$ and $h : \mathbb{R}^{2-} \to \mathbb{R}^{2+}$. That is, h permutes the upper and lower half planes in $\mathbb{C}$.

We are primarily interested in *rational* R-functions of negative type. That is, $h(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials. Holtz and Tyaglov [**HT**] have proved the following theorem which classifies all rational R-functions of negative type.

Theorem 3.5.2. *A function, h, is a rational R-function of negative type if and only if*

$$h(z) = -\alpha z - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - z}, \tag{3.5.1}$$

where $\alpha, \beta, t_k, p_k \in \mathbb{R}$, and $\alpha, p_k > 0$ for $k = 1, ..., N$.

The rest of this dissertation is an in-depth study of the one-dimensional dynamics of the boundary functions associated to rational R-functions of negative type. That is, we study $S : \mathbb{R} \setminus M \to \mathbb{R}$ where $\lambda(M) = 0$ such that $S(x) = \lim_{y \to 0} h(x + iy)$.

We note that if $\alpha = 1$, then rational R-functions of negative type are precisely the negatives of generalized Boole transformations. Both the exercise in Pólya and Szegő [**PS**] and Letac's result [**Let**] (Theorem 3.4.2) imply that all rational R-functions of negative type with $\alpha = 1$ preserve λ. Throughout the rest of this thesis we assume S is the boundary function associated to a rational R-function of negative type, and $\alpha = 1$.

In Future Work (Section 7.1) we lay the framework for using the machinery developed in the first sections of this chapter to extend some of our results to R-functions of negative type which are not necessarily rational.

CHAPTER 4

RATIONAL R-FUNCTIONS OF NEGATIVE TYPE

From now on, when we say S is a rational R-function of negative type we mean that $S : \mathbb{R} \to \mathbb{R}$ is the restriction of a rational map $h : \mathbb{R}^{\pm} \to \mathbb{R}^{\mp}$. We also assume throughout that $\alpha_S = 1$. Therefore, S has the following form

$$S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x}, \tag{4.0.1}$$

where $\beta, t_k, p_k \in \mathbb{R}$, and $p_k > 0$ for $k = 1, ..., N$. We also assume throughout that the poles $\{t_i\}_{i=1}^{N}$ are in ascending order. That is, $t_i < t_{i+1}$ for all $i = 1, ..., N-1$.

4.1. Basic Properties

In this section we catalogue some basic properties of rational R-functions of negative type.

The derivative of any rational R-function of negative type has the following form

$$S'(x) = -1 - \sum_{k=1}^{N} \frac{p_k}{(t_k - x)^2}, \tag{4.1.1}$$

where $t_k, p_k \in \mathbb{R}$ and $p_k > 0$. Thus, $S'(x) < -1$ for all $x \in \mathbb{R}$, and S is everywhere decreasing.

Lemma 4.1.1. *If S is a rational R-function of negative type, then $S : (t_k, t_{k+1}) \to \mathbb{R}$ is one-to-one and onto for $k = 1, ..., N-1$. The same is true outside the smallest and largest poles. That is, both $S : (-\infty, t_1) \to \mathbb{R}$ and $S : (t_N, \infty) \to \mathbb{R}$ are one-to-one and onto mappings.*

Proof. We know S is continuous on $(-\infty, t_1)$, (t_N, ∞), and (t_k, t_{k+1}) for $k = 1, ..., N-1$, and by (4.1.1) S is everywhere decreasing. These observations paired with the following limits yield the result. We have

$$\begin{aligned}
&\lim_{x \to t_k^+} S(x) = \infty \qquad &&\lim_{x \to -\infty} S(x) = \infty \\
&\lim_{x \to t_k^-} S(x) = -\infty \qquad &&\lim_{x \to \infty} S(x) = -\infty.
\end{aligned}$$

□

As a consequence of Lemma 4.1.1 we have that any rational R-function of negative type is an $(N+1)$-to-1 mapping with respect to λ on $\mathbb{R}$. The general shape of any rational R-function of negative type is shown in Figure 4.1.

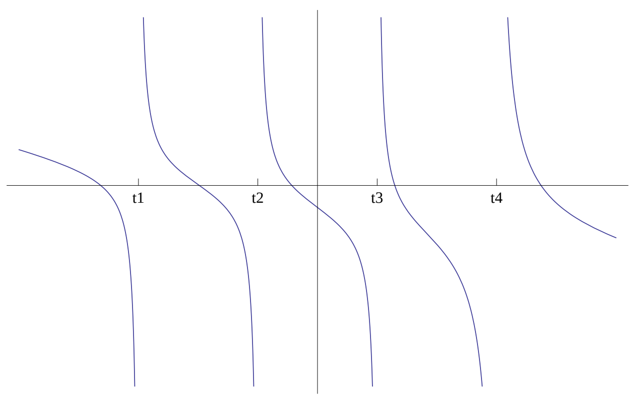

Figure 4.1. An example of S when $N = 4$.

4.2. Exactness and Ergodicity

Recall from Section 2.1 that a non-singular system $(X, \mathcal{B}, m, T)$ is *ergodic* if $A \in \mathcal{B}$ is T-invariant implies $m(A) = 0$ or $m(A^c) = 0$. Furthermore, we say T is exact if $A \in \mathcal{B}$ such that

$$T^{-n}(T^n(A)) = A \quad \text{for all } n > 0, \tag{4.2.1}$$

implies $m(A) = 0$ or $m(A^c) = 0$. By Lemma 2.1.4 if T is exact, then T is ergodic.

Recall that a consequence of Theorem 3.4.3 says that the restriction of a rational R-function of positive type, $F(x) = x + \beta + \sum_{k=1}^{N} \frac{p_k}{t_k - x}$, where $\beta, t_k, p_k \in \mathbb{R}$ and $p_k > 0$, is exact if $\beta = 0$. Otherwise, F is totally dissipative and nonergodic. We will appeal to this result while proving all rational R-functions of negative type are exact, which in contrast to Aaronson's result places no restriction on the constant, β. The following theorem is the main result of this section.

THEOREM 4.2.1. *Let S be a rational R-function of negative type. That is,*

$$S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x},$$

where $t_k, p_k \in \mathbb{R}$ and $p_k > 0$. Then S is exact and ergodic with respect to Lebesgue measure.

Before proving Theorem 4.2.1 we note the following lemma on exactness of a transformation and its iterates

LEMMA 4.2.2. *If $(X, \mathcal{B}, m, T)$ is a nonsingular system such that T^2 is exact with respect to m, then T is exact with respect to m.*

PROOF. If $A \in \mathcal{B}$ is a set for T as in (4.2.1), then $T^{-2n}T^{2n}(A) = A$ for all $n > 0$. Thus, $(T^2)^{-n}(T^2)^n A = A$, so A also has the property in (4.2.1) for T^2. In other words we have

$$\{A : T^{-n}T^n A = A, \text{ for all } n > 0\} \subseteq \{B : (T^2)^{-n}(T^2)^n B = B, \text{ for all } n > 0\}.$$

T^2 is exact, so all B as above have the property that $m(B) = 0$ or $m(B^c) = 0$. Therefore, the set A is such that $m(A) = 0$ or $m(A^c) = 0$, and T is exact. $\square$

The following proposition characterizes the second iterate of a rational R-function of negative type.

PROPOSITION 4.2.3. *Let S be a rational R-function of negative type. That is, $S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x}$, where $\beta, t_k, p_k \in \mathbb{R}$ and $p_k > 0$. Then, S^2 is the restriction of a rational R-function of positive type, and S^2 has form*

$$S^2(x) = x + \sum_{k=1}^{N^2+2N} \frac{\rho_k}{\tau_k - x}, \tag{4.2.2}$$

where $\tau_k, \rho_i \in \mathbb{R}$, $\rho_k > 0$.

The proof of Proposition 4.2.3 will require the following formula for partial fraction decomposition. Given two polynomials $P(x)$ and $Q(x)$ such that $\deg(P) < n$ and $Q(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ where the α_i are distinct, we have

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)} \frac{1}{(x - \alpha_i)}. \tag{4.2.3}$$

PROOF OF PROPOSITION 4.2.3. Suppose S is a rational R-function of negative type as in the proposition. Let S^2 be the second iterate of S. We have,

$$S^2(x) = x + \sum_{k=1}^{N} \frac{p_k}{t_k - x} - \sum_{k=1}^{N} \frac{p_k}{t_k - S(x)}, \tag{4.2.4}$$

where $t_k, p_k \in \mathbb{R}$ and $p_k > 0$. The first two terms of (4.2.4) look like pieces from a rational R-function of positive type and are in the correct form. Considering the third term in (4.2.4) we will show that for a fixed k

$$\frac{-p_k}{t_k - S(x)} = \sum_{j=1}^{N+1} \frac{a_{k,j}}{r_{k,j} - x}, \tag{4.2.5}$$

where $r_{k,j}, a_{k,j} \in \mathbb{R}$ and $a_{k,j} > 0$.

First, we write out the denominator of the left-hand-side of (4.2.5) and obtain

$$\begin{aligned} t_k - S(x) &= t_k + x + \beta + \sum_{i=1}^{N} \frac{p_i}{t_i - x} \\ &= \frac{(t_k + x + \beta) \prod_{i=1}^{N} (t_i - x) + \sum_{i=1}^{N} p_i \prod_{j \neq i} (t_j - x)}{\prod_{i=1}^{N} (t_i - x)}. \end{aligned} \tag{4.2.6}$$

From Lemma 4.1.1 we have $t_k - S(x) = 0$ has $N+1$ distinct real solutions. That is, $S^{-1}(t_k) = \{r_{(k,1)}, ..., r_{(k,N+1)}\}$. Thus, (4.2.6) becomes

$$\frac{-\prod_{i=1}^{N+1}(r_{k,i} - x)}{\prod_{i=1}^{N}(t_i - x)}. \tag{4.2.7}$$

The negative in the numerator of (4.2.7) comes from the fact that if N is even, then $\text{sign}(x^{N+1}) = +1$, and if N is odd, then $\text{sign}(x^{N+1}) = -1$. Therefore, the entire fraction on the left-hand-side of (4.2.5) can be written

$$\frac{-p_k}{t_k - S(x)} = \frac{p_k \prod_{i=1}^{N}(t_i - x)}{\prod_{i=1}^{N+1}(r_{k,i} - x)} = \frac{-p_k \prod_{i=1}^{N}(x - t_i)}{\prod_{i=1}^{N+1}(x - r_{k,i})}. \tag{4.2.8}$$

Now, to use (4.2.3) we let

$$\frac{P(x)}{Q(x)} = \frac{-p_k \prod_{i=1}^{N}(x - t_i)}{\prod_{i=1}^{N+1}(x - r_{k,i})}, \tag{4.2.9}$$

and note that $Q'(r_{k,j}) = \prod_{i \neq j}^{N+1}(r_{k,j} - r_{k,i})$. Therefore,

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \sum_{j=1}^{N+1} \frac{P(r_{k,j})}{Q'(r_{k,j})} \cdot \frac{1}{(x - r_{k,j})} \\
&= \sum_{j=1}^{N+1} \frac{-p_k \prod_{i=1}^{N}(r_{k,j} - t_i)}{\prod_{i \neq j}(r_{k,j} - r_{k,i})} \cdot \frac{-1}{(r_{k,j} - x)} \\
&= \sum_{j=1}^{N+1} \frac{a_{k,j}}{r_{k,j} - x}, \qquad (4.2.10)
\end{aligned}$$

where $a_{k,j} = \frac{p_k \prod_{i=1}^{N}(r_{k,j} - t_i)}{\prod_{i \neq j}(r_{k,j} - r_{k,i})}$. Finally, we will show that $a_{k,j} > 0$ for each $j = 1, ..., N+1$. We do this by considering the sign of the numerator and denominator separately. First, we consider the denominator of $a_{k,j}$. We assume $\{r_{(k,1)}, ..., r_{(k,N+1)}\}$ are in ascending order, so

$$j > i \implies (r_{k,j} - r_{k,i}) > 0 \quad \text{and} \quad j < i \implies (r_{k,j} - r_{k,i}) < 0. \tag{4.2.11}$$

Therefore, in the denominator of $a_{k,j}$ we have

$$\prod_{i\neq j}(r_{k,j}-r_{k,i}) = \underbrace{\prod_{i=1}^{j-1}(r_{k,j}-r_{k,i})}_{+}\cdot \underbrace{\prod_{i=j+1}^{N+1}(r_{k,j}-r_{k,i})}_{(-1)^{N-j+1}}. \tag{4.2.12}$$

Now, we consider the the numerator of $a_{k,j}$. We know $p_k > 0$. We also know there is exactly one $r_{k,j}$ in each atom of the partition in Lemma 4.1.1. That is, $r_{k,1} \in (-\infty, t_1)$, $r_{k,N+1} \in (t_N, \infty)$, and $r_{k,j} \in (t_{j-1}, t_j)$ for $j = 2, ...N$. Thus, we have the following three cases:

(1) $j > i \implies (r_{k,j} - t_i) > 0$,

(2) $j = i \implies (r_{k,j} - t_i) < 0$, and

(3) $j < i \implies (r_{k,j} - t_i) < 0$.

Therefore, we can count the number of sign changes coming from (1), (2), and (3) in the numerator of $a_{k,j}$ and obtain

$$\prod_{i=1}^{N}(r_{k,j}-t_i) = \underbrace{\prod_{i=1}^{j-1}(r_{k,j}-t_i)}_{+}\cdot \underbrace{(r_{k,j}-t_j)}_{-1}\cdot \underbrace{\prod_{i=j+1}^{N}(r_{k,j}-t_i)}_{(-1)^{N-j}}. \tag{4.2.13}$$

Thus, $\text{sign}(a_{k,j}) = \frac{(-1)^{(N-j+1)}}{(-1)^{(N-j+1)}}$ which is positive. We have shown

$$\begin{aligned} S^2(x) &= x + \sum_{k=1}^{N}\frac{p_k}{t_k - x} + \sum_{k=1}^{N}\sum_{j=1}^{N+1}\frac{a_{k,j}}{r_{k,j}-x} \\ &= x + \sum_{k=1}^{N^2+2N}\frac{\rho_k}{\tau_k - x}, \end{aligned} \tag{4.2.14}$$

where $\tau_k, \rho_k \in \mathbb{R}$ and $\rho_k > 0$. $\square$

Note that Proposition 4.2.3 shows that if S is a rational R-function of negative type, then S^2 is a rational R-function of positive type with constant term 0. We are now ready to prove the main theorem of this section.

Proof of Theorem 4.2.1. Let S be a rational R-function of negative type as in the theorem. That is, $S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x}$ where $\beta, t_k, p_k \in \mathbb{R}$ and $p_k > 0$. Let S^2 the second iterate of S. By Proposition 4.2.3 we have that S^2 is a rational R-function of positive type and can be written

$$S^2(x) = x + \sum_{k=1}^{N^2+2N} \frac{\rho_k}{\tau_k - x}, \tag{4.2.15}$$

where $\tau_k, \rho_k \in \mathbb{R}$ and $\rho_k > 0$. The constant term in S^2 is 0, so we may appeal to Theorem 3.4.3 to obtain S^2 is exact. Therefore, S is exact, by Lemma 4.2.2. Finally, by Lemma 2.1.4 S is also ergodic. □

4.3. Conservativity

Recall from Section 2.1 that a nonsingular system $(X, \mathcal{B}, m, T)$ is *conservative* if there does not exist a wandering set of positive measure. In other words, for all $A \in \mathcal{B}$ with $m(A) > 0$, there exists $k > 0$ such that

$$m(T^{-k}A \cap A) > 0. \tag{4.3.1}$$

Again we consider rational R-functions of negative type, so

$$S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x},$$

where $\beta, t_k, p_k \in \mathbb{R}$ and $p_k > 0$.

Let ω_1, .., ω_{N+1} denote the fixed points of S in ascending order. For the rest of this chapter it is convenient to conjugate S so that $\omega_1 = 0$. That is, $S(0) = 0$ and all other fixed points are positive. Let

$$\phi(x) = x - \omega_1 \quad \text{and} \quad \phi^{-1}(x) = x + \omega_1.$$

Consider the transformation $\tilde{S} = \phi \circ S \circ \phi^{-1}$. We have

$$\tilde{S}(x) = -x - 2\omega_1 - \beta - \sum_{k=1}^{N} \frac{p_k}{(t_k - \omega_1) - x}.$$

We note that $\tilde{S}(0) = (\phi \circ S \circ \phi^{-1})(0) = (\phi \circ S)(\omega_1) = \phi(\omega_1) = 0$. Also, if the poles of S are $\{t_1, ..., t_N\}$ in ascending order, then the poles of $\tilde{S}$ are $t_k - w_1$ for $k = 1, ..., N$ in ascending order. The smallest pole of $\tilde{S}$ is $t_1 - w_1$ which is greater than 0. Furthermore, $\phi : S \to \tilde{S}$ is an isomorphism (as in Definition 2.1.9), so S and $\tilde{S}$ have the same measure theoretic properties. Thus, without loss of generality we may assume that $\omega_1 = 0$ is the smallest fixed point of S, and all other fixed points, $\{\omega_i\}_{i=2}^{N+1}$, as well as all the poles, $\{t_i\}_{i=1}^{N}$, are positive. For the rest of this chapter we assume S has the general shape in Figure 4.2.

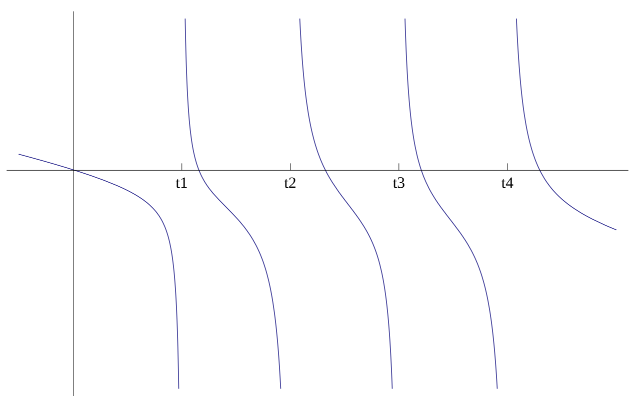

FIGURE 4.2. An example of conjugated S when $N = 4$.

Let q_1, ..., q_{N+1} denote the roots of S. Note that $q_1 = \omega_1 = 0$. We define a partition $\mathcal{Q} = \{Q_1, ..., Q_{N+1}\}$ of $\mathbb{R}$ to be the intervals between the roots. That is, $Q_i = [q_i, q_{i+1})$ for $i = 1, ..., N$ and $Q_{N+1} = (\infty, q_1) \cup [q_{N+1}, \infty)$. The general shape of S along with the partition $\mathcal{Q}$ are depicted in Figure 4.3.

REMARK 4.3.1. Using a process similar to that in Lemma 4.1.1 we know S maps the individual atoms of $\mathcal{Q}$ one-to-one and onto $\mathbb{R}$. That is, $S : Q_i \to \mathbb{R}$ is one-to-one and onto for $i = 1, ..., N + 1$, and $\mathcal{Q}$ is a Rohlin partition for S.

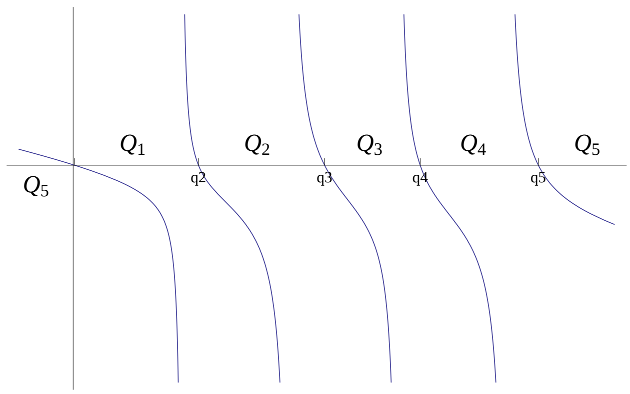

FIGURE 4.3. An example of the $\mathcal{Q}$ partition for S when $N = 4$.

The above remark motivates the following notation. We let ψ_i denote the inverse of S restricted to Q_i. That is

$$\psi_i = S^{-1}|_{Q_i}, \tag{4.3.2}$$

so $\psi_i : \mathbb{R} \to Q_i$ is one-to-one and onto for $i = 1, ..., N+1$. We denote the refinement $Q_{i_1} \cap S^{-1}Q_{i_2} \cap ... \cap S^{-(n-1)}Q_{i_n}$ by $Q_{i_1...i_n}$, and let

$$\psi_{i_1...i_n} = S^{-1}|_{Q_{i_1...i_n}}, \quad \text{so} \quad \psi_{i_1...i_n} = \psi_{i_1...i_{n-1}} \circ \psi_{i_n}. \tag{4.3.3}$$

Note that $\psi_{i_1...i_n} : \mathbb{R} \to Q_{i_1...i_n}$ is one-to-one and onto. We define one more piece of notation and let

$$\psi_{i[k]} = \underbrace{\psi_i \circ \psi_i \circ ... \circ \psi_i}_{k-\text{times}}. \tag{4.3.4}$$

PROPOSITION 4.3.2. *The set* $A = \bigcup_{i=1}^{N} Q_i = [q_1, q_{N+1}]$ *is a sweep-out-set for* S. *That is,* $\bigcup_{n\geq 0} S^{-n}A = \mathbb{R}$ *mod* λ.

Before we prove Proposition 4.3.2 we need the following lemma.

LEMMA 4.3.3. *Let* $\{x_k\}_{k=0}^{\infty}$ *be a sequence of positive numbers. If* c *is a positive constant and* $x_{k+1}^2 \geq x_k^2 + c$, *then* $x_k > \sqrt{\frac{c}{2}} \cdot \sqrt{k}$.

PROOF. If $x_1^2 \geq x_0^2 + c$, then $x_1 \geq (\sqrt{x_0^2+c})\sqrt{1} \geq \sqrt{\frac{c}{2}}\sqrt{1}$. Now we proceed by induction. Assume $x_k > \sqrt{\frac{c}{2}} \cdot \sqrt{k}$. Thus, we have

$$x_{k+1}^2 \geq x_k^2 + c \geq \frac{c}{2} \cdot k + c = \frac{c}{2}(k+2) > \frac{c}{2}(k+1). \tag{4.3.5}$$

Therefore, $x_{k+1} > \sqrt{\frac{c}{2}} \cdot \sqrt{k+1}$. □

We are now ready to prove Proposition 4.3.2.

PROOF OF PROPOSITION 4.3.2. Without loss of generality let $S(x) = -x - \beta - \sum_{i=1}^{N} \frac{p_i}{t_i - x}$, where $\beta, t_i, p_i \in \mathbb{R}$ with $t_i, p_i > 0$ and $S(0) = 0$. Let $\{Q_i\}_{i=1}^{N+1}$ be the partition defined above, let and ψ_i be the inverse of S defined in (4.3.2). We define a sequence $\{z_k\}_{k=0}^{\infty}$ such that

$$z_0 = q_1 = 0 \quad \text{and} \quad z_k = \psi_{(N+1)}(x_{k-1}) = \psi_{(N+1)[k]}(0). \tag{4.3.6}$$

In order to better understand the sequence $\{z_k\}$ it is convenient to define two separate sequences corresponding to the even and odd terms. We will denote the even terms by $\{W_k\}_{k\geq 0}$. That is, $W_0 = 0$ and $W_k = z_{2k}$. The odd terms will be $\{V_k\}_{k\geq 0}$ such that $V_0 = q_{N+1}$ and $V_k = z_{2k+1}$. These two sequences are precisely the endpoints of the union of pullbacks of A, because

$$[W_0, V_0] = A, \tag{4.3.7}$$

$$[W_1, V_0] = A \cup S^{-1},$$

$$[W_1, V_1] = A \cup S^{-1}A \cup S^{-2}A,$$

$$[W_2, V_1] = A \cup S^{-1}A \cup S^{-2}AS^{-3}A, \tag{4.3.8}$$

and so on. In general, we have

$$[W_{\lceil k/2 \rceil}, V_{\lfloor k/2 \rfloor}] = \bigcup_{j=0}^{k} S^{-j}A \tag{4.3.9}$$

where $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the ceiling and floor functions respectively.

In order to show A is a sweep-out-set, we need to show

$$\lim_{k\to\infty} V_k = \infty \quad \text{and} \quad \lim_{k\to\infty} W_k = -\infty. \tag{4.3.10}$$

We will show the first statement in (4.3.10), and the second statement follows by a similar argument. We first note that $V_k = \psi_{(N+1)[2k]}(q_{N+1})$. This implies that $V_k = S^2(V_{k+1})$. By Proposition 4.2.3 we have $S^2(x) = x + \sum_{i=1}^{N^2+2N} \frac{\rho_i}{\tau_i - x}$, where $\tau_i, \rho_i \in \mathbb{R}$ and $\rho_i > 0$. That is, letting $\mathfrak{N} = N^2 + 2N$ we have

$$V_k = V_{k+1} + \sum_{i=1}^{\mathfrak{N}} \frac{\rho_i}{\tau_i - V_{k+1}}. \tag{4.3.11}$$

We will show that

$$V_k \geq c_1 \sqrt{k}, \tag{4.3.12}$$

which implies $V_k \to \infty$ as $k \to \infty$. First, for $k \geq 1$ we have $V_k > 0$ and $V_k \in Q_{N+1}$. Thus, $V_k > \tau_i$ for all $i = 1, ..., \mathfrak{N}$. By (4.3.11) we have that for $k \geq 1$

$$V_k \leq V_{k+1} - \frac{\rho_{\mathfrak{N}}}{V_{k+1}}. \tag{4.3.13}$$

Multiplying both sides by V_{k+1} we have

$$V_k V_{k+1} \leq V_{k+1}^2 - \rho_{\mathfrak{N}}. \tag{4.3.14}$$

Therefore, by the quadratic formula we have

$$4V_{k+1}^2 \geq 2V_k^2 + 2V_k \sqrt{V_k^2 + 4\rho_{\mathfrak{N}}} + 4\rho_{\mathfrak{N}}. \tag{4.3.15}$$

We note that $\sqrt{V_k^2 + \rho_{\mathfrak{N}}} \geq V_k$, so (4.3.15) implies

$$V_{k+1}^2 \geq V_k^2 + \rho_{\mathfrak{N}}. \tag{4.3.16}$$

By Lemma 4.3.3 $V_k \geq \sqrt{\frac{\rho_{\mathfrak{N}}}{2}} \cdot \sqrt{k}$. Thus, $V_k \to \infty$. A similar argument shows $W_k < -c_2\sqrt{n}$, so $W_k \to -\infty$. $\square$

The following theorem is the main result of this section, and says that all rational R-functions of negative type are conservative.

THEOREM 4.3.4. *If S is a rational R-function of negative type, then S is conservative with respect to λ.*

PROOF. Without loss of generality assume $S(x) = -x - \beta - \sum_{k=1}^{N} \frac{p_k}{t_k - x}$ where $\beta, t_k, p_k \in \mathbb{R}$ with $t_k, p_k > 0$, and $S(0) = 0$. By Proposition 4.3.2 the set $A = \bigcup_{i=1}^{N+1} Q_i = [q_1, q_{N+1}]$ is a sweep-out-set for S. Therefore, by Maharam's Recurrence Theorem (Theorem 2.1.8), we have that S is conservative. $\square$

4.4. Wandering Rates

If $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, measure-preserving system, then the wandering rate of a finite-measure set $A \in \mathcal{B}$ measures the amount of X which is "seen" by A after n iterations of T.

DEFINITION 4.4.1. Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure-preserving system. The *wandering rate* of a set $A \in \mathcal{B}$ with $m(A) < \infty$ is the sequence

$$L_A(k) = m\left(\bigcup_{i=0}^{k} T^{-i}A\right). \tag{4.4.1}$$

Let S be a rational R-function of negative type, and let $A = [q_1, q_{N+1}]$ be the same sweep-out-set for S as in Section 4.3. The following Proposition is the main result of this section.

PROPOSITION 4.4.2. *The wandering rate of the set $A = [q_1, q_{N+1}]$ under a rational R-function of negative type, S, is given by*

$$L_A(k) = 2\sqrt{k \sum_{j=1}^{N^2+2N} \rho_j}, \tag{4.4.2}$$

where the ρ_j are the weights on the linear factors coming from the second iterate S^2.

Before proving Proposition 4.4.2 we need a few auxiliary results. Let $\{V_k\}_{k\geq 0}$ be the sequence defined as in Proposition 4.3.2. That is, $V_0 = q_{N+1}$ and $V_k = \psi_{(N+1)[2k]}(q_{N+1})$. In Proposition 4.3.2 we showed $\lim_{k\to\infty} V_k = \infty$. The following lemma provides the precise growth rate of V_k.

LEMMA 4.4.3. *If V_k is the sequence defined as in Proposition 4.3.2, then*

$$V_{k+1} \sim \sqrt{2k \sum_{j=1}^{N^2+2N} \rho_j}, \tag{4.4.3}$$

where the ρ_j are the weights on the linear factors coming from the second iterate S^2.

PROOF. Let $S(x) = -x - \beta - \sum_{j=1}^{N} \frac{p_j}{t_j - x}$, where $\beta, t_j, p_j \in \mathbb{R}$ with $t_j, p_j > 0$, and $S(0) = 0$. By Proposition 4.2.3 we have $S^2(x) = x + \sum_{j=1}^{N^2+2N} \frac{\rho_j}{\tau_j - x}$, where $\tau_j, \rho_j \in \mathbb{R}$ and $\rho_j > 0$. By Proposition 4.3.2 and its proof we have $V_k = V_{k+1} + \sum_{j=1}^{N^2+2N} \frac{\rho_j}{\tau_j - V_{k+1}}$. Therefore,

$$\begin{aligned}
V_{k+1}^2 - V_k^2 &= (V_{k+1} - V_k)(V_{k+1} + V_k) \\
&= \left(-\sum_{j=1}^{N^2+2N} \frac{\rho_j}{\tau_j - V_{k+1}}\right)\left(2V_{k+1} + \sum_{j=1}^{N^2+2N} \frac{\rho_j}{\tau_j - V_{k+1}}\right) \\
&= \underbrace{\left(\sum_{j=1}^{N^2+2N} \frac{-2V_{k+1}\rho_j}{\tau_j - V_{k+1}}\right)}_{(I)} - \underbrace{\left(\sum_{j=1}^{N^2+2N} \frac{\rho_j}{\tau_j - V_{k+1}}\right)^2}_{(II)}. \qquad (4.4.4)
\end{aligned}$$

By Proposition 4.3.2 we have that $\lim_{k\to\infty} V_k = \infty$, so $(I) \to 2\sum_{j=1}^{N^2+2N} \rho_j$ and $(II) \to 0$ as $k \to \infty$. That is, $\lim_{k\to\infty} V_{k+1}^2 - V_k^2 = 2\sum_{j=1}^{N^2+2N} \rho_j$, so

$$V_{k+1}^2 \sim V_k^2 + 2\sum_{j=1}^{N^2+2N} \rho_j. \tag{4.4.5}$$

We also know, $V_k^2 \sim V_{k-1}^2 + 2\sum_{j=1}^{N^2+2N} \rho_j$, so we can rewrite (4.4.5) as

$$V_{k+1}^2 \sim V_{k-1}^2 + 2 \cdot 2 \sum_{j=1}^{N^2+2N} \rho_j. \tag{4.4.6}$$

Continuing in this way yields

$$V_{k+1} \sim \sqrt{2k \sum_{j=1}^{N^2+2N} \rho_j}. \tag{4.4.7}$$

A similar argument also shows that for $\{W_k\}_{k\geq 0}$ defined as in Proposition 4.3.2 we have

$$W_{k+1} \sim -\sqrt{2k \sum_{j=1}^{N^2+2N} \rho_j}. \tag{4.4.8}$$

□

We will use the following lemma on asymptotics in our study of the wandering rates for rational R-functions of negative type.

Lemma 4.4.4. *If $k \in \mathbb{N}$, then*

$$\sqrt{\lceil k/2 \rceil} + \sqrt{\lfloor k/2 \rfloor} \sim \sqrt{2k},$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor functions respectively.

Proof. If k is even, then

$$\sqrt{\lceil k/2 \rceil} + \sqrt{\lfloor k/2 \rfloor} = 2\sqrt{k/2} = \sqrt{2k}.$$

If k is odd, then $k/2 = m.5$ for some $m \in \mathbb{N}$. Therefore,

$$\sqrt{\lceil k/2 \rceil} + \sqrt{\lfloor k/2 \rfloor} = \sqrt{m+1} + \sqrt{m}.$$

Comparing this to $\sqrt{2k} = \sqrt{2(m+1+m)} = \sqrt{4m+2}$, we have

$$\lim_{k\to\infty} \frac{\sqrt{\lceil k/2\rceil} + \sqrt{\lfloor k/2\rfloor}}{\sqrt{2k}} = \lim_{m\to\infty} \sqrt{\frac{m+1}{4m+2}} + \sqrt{\frac{m}{4m+2}} = 1,$$

which completes the proof. □

Proof of Proposition 4.4.2. Let S be a rational R-function of negative type, and S^2 the second iterate of S. Suppose $A = [q_1, q_{N+1}]$ is the sweep-out-set for S as in Proposition 4.3.2. By the definition of wandering rate (Definition 4.4.1) and the proof of Proposition 4.3.2 we have

$$L_A(k) = \bigcup_{j=0}^{k} S^{-j} A = m\left[W_{\lceil k/2\rceil}, V_{\lfloor k/2\rfloor}\right]. \tag{4.4.9}$$

By Lemma 4.4.3 we understand the asymptotics of V_k and W_k, so the right-hand-side of (4.4.9) becomes

$$\begin{aligned}
m\left[W_{\lceil k/2\rceil}, V_{\lfloor k/2\rfloor}\right] &\sim \sqrt{2\lceil k/2\rceil \sum_{j=1}^{N^2+2N} \rho_j} + \sqrt{2\lfloor k/2\rfloor \sum_{j=1}^{N^2+2N} \rho_j} \\
&= \sqrt{2 \sum_{j=1}^{N^2+2N} \rho_j} \cdot \left(\sqrt{\lceil k/2\rceil} + \sqrt{\lfloor k/2\rfloor}\right) \\
&\sim 2\sqrt{k \sum_{j=1}^{N^2+2N} \rho_j}, \qquad (4.4.10)
\end{aligned}$$

where the last step comes from Lemma 4.4.4. □

There are connections between wandering rates and hitting-times of sweep-out sets for conservative transformations. Intuitively, the wandering rate measures the rate at which A sweeps out $\mathbb{R}$ under inverse iteration, while the hitting-time of a point $x \in X$ measures the number of forward iterations required for x to hit A. The rest of this section is devoted to making this connection more precise. Recall from Section 2.2 that given a conservative measure-preserving system, $(X, \mathcal{B}, m, T)$, and a sweep-out-set, $A \in \mathcal{B}$, we let $\phi_A(x)$ denote the first hitting-time of $x \in X$ to A. That is, $\phi_A(x) = \inf\{n : T^n(x) \in A\}$. Furthermore, we can partition A

and A^c into first hitting-time sets, $A_k = \{x \in A : \phi_A(x) = k\}$ (see (2.2.2)) and $B_k = \{x \in A^c : \phi_A(x) = k\}$ (see (2.2.3)). We also define sets D_k (as in (2.2.4)) such that $D_0 = A$ and $D_k = \{x \in A : \phi_A(x) > k\}$.

LEMMA 4.4.5. *Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, measure-preserving system, and $A \in \mathcal{B}$ with $m(A) < \infty$. We have*

$$L_A(n) = \sum_{k=0}^{n} m(D_k). \tag{4.4.11}$$

PROOF. By definition we have $L_A(n) = m\left(\bigcup_{k=0}^{n} T^{-k}A\right)$, and

$$\bigcup_{k=0}^{n} T^{-k}A = A \cup \left(\bigcup_{k=1}^{n} T^{-k}A \setminus \bigcup_{j=0}^{k-1} T^{-j}A\right) = A \cup \bigcup_{k=1}^{n} B_k, \tag{4.4.12}$$

where all unions after the second equals sign are disjoint. Thus, we have

$$L_A(n) = m(A) + \sum_{k=1}^{n} m(B_k). \tag{4.4.13}$$

Recall that in Lemma 2.2.4 we proved $m(B_k) = m(D_k)$ for all $k \geq 1$. Therefore, (4.4.13) equals

$$m(A) + \sum_{k=1}^{n} m(D_k) = \sum_{k=0}^{n} m(D_k). \tag{4.4.14}$$

□

4.5. Pointwise Dual Ergodicity

Recall from Section 2.4.2 that a conservative, ergodic, measure-preserving system $(X, \mathcal{B}, m, T)$ is called pointwise dual ergodic if there are constants $a_n(T)$ such that

$$\lim_{n\to\infty} \frac{1}{a_n(T)} \sum_{k=0}^{n-1} \mathcal{L}_T^k f = \int_X f dm \quad \text{for all } f \in L^1(m).$$

The following proposition says all rational R-functions of negative type are pointwise dual ergodic. The proof uses techniques developed in [**Aar2**], [**Aar3**], and [**Let**].

Proposition 4.5.1. *If S is a rational R-function of negative type, then S is pointwise dual ergodic.*

Proof. Let $S(x) = -x - \beta - \sum_{i=1}^{N} \frac{p_i}{t_i - x}$, where $t_i, p_i \in \mathbb{R}$ with $t_i, p_i > 0$, and $S(0) = 0$. We know $S : \mathbb{R} \to \mathbb{R}$. Recall from Chapter 3 that $S = h|_{\mathbb{R}}$, where $h : \mathbb{C} \to \mathbb{C}$ is a rational map which permutes the upper and lower half planes. Also, recall from Section 3.4 that for $t \in \mathbb{R}$ and $\omega = a + ib \in \mathbb{C}$ the Cauchy distribution σ_ω is defined by

$$\sigma_\omega(t) = \frac{1}{\pi}\Im\left(\frac{1}{t-\omega}\right) \tag{4.5.1}$$

where $\Im(z)$ denotes the imaginary part of z. By Theorem 3.4.1 (originally proved in [**Let**]) we have

$$\mathcal{L}_S(\sigma_\omega) = \sigma_{h(\omega)}. \tag{4.5.2}$$

Therefore,

$$\sum_{k=0}^{n-1} \mathcal{L}_S^k \sigma_\omega(t) = \sum_{k=0}^{n-1} \sigma_{h^k(\omega)}(t) = \frac{1}{\pi}\sum_{k=0}^{n-1} \Im\left(\frac{1}{t - h^k(\omega)}\right). \tag{4.5.3}$$

If $h^k(\omega) = u_k + iv_k$, then (4.5.3) equals

$$\frac{1}{\pi}\sum_{k=0}^{n-1} \Im\left(\frac{1}{t-(u_k+iv_k)}\right) = \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{v_k}{(t-u_k)^2 + v_k^2}. \tag{4.5.4}$$

A calculation (similar to one in [**Aar2**] or [**Aar3**]) shows there exists an $M \in \mathbb{R}$ such that $|u_k| < M$ for all $k \geq 1$, and

$$\lim_{k\to\infty} \frac{v_k}{\sqrt{2k\sum_{i=1}^{N} p_i}} = 1. \tag{4.5.5}$$

Therefore, from (4.5.4) we have

$$\lim_{n\to\infty} \frac{\frac{1}{\pi}\sum_{k=0}^{n-1} \frac{v_k}{(t-u_k)^2+v_k^2}}{\frac{1}{\pi}\sum_{k=0}^{n-1} \frac{v_k}{(u_k)^2+v_k^2}} = \lim_{n\to\infty} \frac{\frac{1}{\pi}\sum_{k=0}^{n-1} \Im\left(\frac{1}{t-h^k(\omega)}\right)}{\frac{1}{\pi}\sum_{k=0}^{n-1} \Im\left(\frac{-1}{h^k(\omega)}\right)} = 1. \tag{4.5.6}$$

That is, we have

$$\sum_{k=0}^{n-1} \mathcal{L}_S^k \sigma_\omega(t) \sim \frac{1}{\pi} \sum_{k=0}^{n-1} \Im\left(\frac{-1}{h^k(\omega)}\right). \tag{4.5.7}$$

From Theorem 4.2.1 we have S is ergodic with repsect to Lebesgue measure. Therefore, if $f \in L^1(\lambda)$, then by Theorem 2.4.8 (Hurewicz's Ergodic Theorem) we have

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \mathcal{L}_S^k f(t)}{\sum_{k=0}^{n-1} \mathcal{L}_S^k \sigma_\omega(t)} = \frac{\int_{\mathbb{R}} f d\lambda}{\int_{\mathbb{R}} \sigma_\omega d\lambda} \quad \text{for almost every } t \in \mathbb{R}. \tag{4.5.8}$$

By definition of the Cauchy distribution, σ_ω we have $\int_{\mathbb{R}} \sigma_\omega d\lambda = 1$. Therefore, if $a_n(S) \sim \frac{1}{\pi}\sum_{k=0}^{n-1} \Im\left(\frac{-1}{h^k(\omega)}\right)$, then

$$\lim_{n\to\infty} \frac{1}{a_n(S)} \sum_{k=0}^{n-1} \mathcal{L}_S^k f(t) = \int_{\mathbb{R}} f d\lambda \tag{4.5.9}$$

almost everywhere, and S is pointwise dual ergodic. $\square$

If T is a pointwise dual ergodic transformation, then the sequence $a_n(T)$ is called a *return sequence* for T. We have that $a_n(T)$ is intimately related to the wandering rate (and therefore return times) of sweep-out-sets. We will exploit this fact and calculate the return sequence for rational R-functions of positive type. First we give two basic definitions.

Definition 4.5.2. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* at ∞ if there exists an $\alpha \in \mathbb{R}$ such that for all $m > 0$ we have

$$\lim_{x\to\infty} \frac{f(xm)}{f(x)} = m^\alpha. \tag{4.5.10}$$

Similarly, a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* at 0 if there exists an $\alpha \in \mathbb{R}$ such that for all $m > 0$ we have

$$\lim_{x\to 0} \frac{f(xm)}{f(x)} = m^{\alpha}. \tag{4.5.11}$$

The constant α is call the index of variation. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *slowly varying* if it is regularly varying with index $\alpha = 0$.

We note that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at ∞ with index α if and only if $f(x) = x^{\alpha} L(x)$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying at ∞. Furthermore, $x \mapsto f(x)$ is regularly varying at 0 if and only if $x \mapsto f(\frac{1}{x})$ is regularly varying at ∞. We also have the notion of regularly varying sequences. A positive sequence $\{y_n\}_{n=1}^{\infty}$ is *regularly varying* at ∞ if there exists an $\alpha \in \mathbb{R}$ such that for all $m > 0$ we have $\lim_{n\to\infty} \frac{y_{nm}}{y_n} = m^{\alpha}$. For more information on regular variation see [**Aar4**] or [**BGT**].

Definition 4.5.3. Given two real sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ such that $y_n, z_n > 0$ for all n, we write

$$y_n \lesssim z_n \quad \text{if} \quad \lim_{n\to\infty} \frac{y_n}{z_n} \leq 1. \tag{4.5.12}$$

The following proposition can be found in [**Aar4**]. We, however, provide a slightly different proof.

Proposition 4.5.4. *Let $(X, \mathcal{B}, m, T)$ be a conservative, pointwise dual ergodic, measure-preserving system, and let A be a Darling-Kac set. Then,*

$$n \lesssim a_n(T) L_A(n) \lesssim 2n. \tag{4.5.13}$$

Furthermore, if $L_A(n)$ is regularly varying at ∞ with index $1 - \alpha$, then

$$a_n(T) \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{n}{L_A(n)}, \tag{4.5.14}$$

where Γ denotes the gamma function. That is, if $\Re(z) > 0$, then $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$.

We will use the following Tauberian theorem in the proof of Proposition 4.5.4, so we state it here for completeness. A continuous version was originally proved in [**Kar**] and can also be found in [**Aar4**]. We use a discrete version which can be found in [**BGT**].

THEOREM 4.5.5 (Karamata's Tauberian Theorem). *Let $\{y_n\}$ be a sequence of positive real numbers. Define*

$$Y(s) = \sum_{n\geq 0} y_n e^{-ns}. \tag{4.5.15}$$

Suppose that for all $s > 0$, we have $Y(s) < \infty$. Let $f(n)$ be slowly varying at ∞, and let $p, \theta \in [0, \infty)$. Further, let $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt$. Then the following statements are equivalent:

(1) $Y(s) \sim \theta \left(\frac{1}{s}\right)^p f\left(\frac{1}{s}\right)$ *as* $s \to 0$

(2) $\sum_{k=0}^{n-1} y_k \sim \frac{\theta}{\Gamma(p+1)} n^p f(n)$ *as* $n \to \infty$.

If $\{y_n\}$ is monotone and $p > 0$, then both (1) and (2) are equivalent to:

(3) $y_n \sim \frac{\theta p}{\Gamma(p+1)} n^{p-1} f(n)$ *as* $n \to \infty$.

We are now in a position to prove Proposition 4.5.4.

PROOF OF PROPOSITION 4.5.4. We first prove (4.5.13). Suppose $(X, \mathcal{B}, m, T)$ is a conservative, pointwise dual ergodic, measure-preserving system. Given $A \in \mathcal{B}$ we define the first return time sets $A_k = \{x \in A : \phi_A(x) = k\}$ (see (2.2.2)). We also define sets D_k (as in (2.2.4)) such that $D_0 = A$ and $D_k = \{x \in A : \phi_A(x) > k\}$. Note that

$$\bigcup_{k=0}^{n} T^{-k}A = \bigcup_{k=0}^{n} T^{-k}D_{n-k}, \tag{4.5.16}$$

because the right-hand-side is the set of all $x \in X$ whose orbit enters A within n applications of T. A point x in the left-hand-side is in the k^{th} set on the right-hand-side if k is the last time before n that the orbit of x passes through A. Passing to characteristic functions and integrating (4.5.16) over A yields

$$\int_A \mathbb{1}_{(\cup_{k=0}^n T^{-k}A)} dm = \int_A \mathbb{1}_{(\cup_{k=0}^n T^{-k}D_{n-k})} dm,$$

so we have

$$\begin{aligned} m(A) &= \sum_{k=0}^{n} \int_X \mathbb{1}_A \cdot \mathbb{1}_{D_{n-k}} \circ T^k dm \\ &= \sum_{k=0}^{n} \int_X \mathcal{L}_T^k \mathbb{1}_A \cdot \mathbb{1}_{D_{n-k}} dm \\ &= \int_A \left(\sum_{k=0}^{n} \mathcal{L}_T^k \mathbb{1}_A \cdot \mathbb{1}_{D_{n-k}} \right) dm. \end{aligned} \tag{4.5.17}$$

If we sum the identities in (4.5.17), then we can obtain the identity in (4.5.13) as follows,

$$\begin{aligned} (N+1)m(A) &= \int_A \left(\sum_{n=0}^{N} \sum_{k=0}^{n} \mathcal{L}_T^k \mathbb{1}_A \cdot \mathbb{1}_{D_{n-k}} \right) dm \\ &\le \int_A \left(\sum_{k=0}^{N} \mathcal{L}_T^k \mathbb{1}_A \right) \left(\sum_{j=0}^{N} \mathbb{1}_{D_j} \right) dm \\ &\le \int_A \left(\sum_{n=0}^{2N} \sum_{k=0}^{n} \mathcal{L}_T^k \mathbb{1}_A \cdot \mathbb{1}_{D_{n-k}} \right) dm \\ &= 2(N+1)m(A). \end{aligned} \tag{4.5.18}$$

The set A is a Darling-Kac set, so we can get a handle on (4.5.18) in the following way

$$\begin{aligned} \int_A \left(\sum_{k=0}^{N} \mathcal{L}_T^k \mathbb{1}_A \right) \left(\sum_{j=0}^{N} \mathbb{1}_{D_j} \right) dm &\sim a_{N+1}(A) \cdot \int_A \left(\sum_{j=0}^{N} \mathbb{1}_{D_j} \right) dm \\ &= a_{N+1}(A) \cdot L_A(N+1). \end{aligned} \tag{4.5.19}$$

Therefore, we have

$$(N+1)m(A) \lesssim a_{N+1}(A) \cdot L_A(N+1) \lesssim 2(N+1)m(A). \tag{4.5.20}$$

Recall that from Hurewicz's ergodic theorem we have $a_n(T) \sim \frac{a_n(A)}{m(A)}$ (see (2.4.13)). Therefore, (4.5.20) is equivalent to (4.5.13).

Now, in order to prove the stronger estimate in (4.5.14), we need to more cleverly handle the convolution in (4.5.17). Taking the discrete Laplace transform yields

$$\begin{aligned} m(A)\sum_{n\geq 0} e^{-ns} &= \int_A \sum_{n\geq 0}\left(\sum_{k=0}^{n} \mathcal{L}_T^k \mathbb{1}_A \cdot \mathbb{1}_{D_{n-k}}\right) e^{-ns} dm \\ &= \int_A \left(\sum_{n\geq 0} \mathcal{L}_T^n \mathbb{1}_A \cdot e^{-ns}\right)\left(\sum_{n\geq 0} \mathbb{1}_{D_n} \cdot e^{-ns}\right) dm. \end{aligned} \tag{4.5.21}$$

We want to get a handle on $\sum_{n\geq 0} \mathcal{L}_T^n \mathbb{1}_A \cdot e^{-ns}$, so we note the following identity (similar to Lemma 3.8.4 in [**Aar4**])

$$\sum_{n\geq 0} \mathcal{L}_T^n \mathbb{1}_A \cdot e^{-ns} = (1-e^{-s})\sum_{n\geq 0}\left(\sum_{k=0}^{n} \mathcal{L}_T^k \mathbb{1}_A\right) e^{-ns}. \tag{4.5.22}$$

We know A is a Darling-Kac set, so by definition $\sum_{k=0}^{n} \mathcal{L}_T^k \mathbb{1}_A \sim a_n(A)$. Furthermore, it is clear that $(1-e^{-s}) \sim s$ as $s \to 0$, so (4.5.22) becomes

$$\sum_{n\geq 0} \mathcal{L}_T^n \mathbb{1}_A \cdot e^{-ns} \sim s \sum_{n\geq 0} a_n(A) e^{-ns} \quad \text{as } s \to 0. \tag{4.5.23}$$

Using (4.5.23) combined with the fact that $\sum_{n\geq 0} e^{-ns} \sim \frac{1}{s}$ as $s \to 0$, we can substitute into (4.5.21) to obtain

$$\begin{aligned} \frac{m(A)}{s} &\sim s \cdot \int_A \left(\sum_{n\geq 0} a_n(A) e^{-ns}\right)\left(\sum_{n\geq 0} \mathbb{1}_{D_n} \cdot e^{-ns}\right) dm \\ &= s \cdot \left(\sum_{n\geq 0} a_n(A) e^{-ns}\right)\left(\sum_{n\geq 0} m(D_n) e^{-ns}\right). \end{aligned} \tag{4.5.24}$$

Now, we are in a position to apply Karamata's Tauberian Theorem (Theorem 4.5.5) to the two sums in (4.5.24). By assumption $L_A(n)$ is regularly varying at ∞ with index $1-\alpha$, so we can write $L_A(n) = n^{1-\alpha}\tilde{f}(n)$, where $\tilde{f}(n)$ is slowly varying at ∞. Since $\tilde{f}(n)$ is slowly varying, so is $f(n) = \tilde{f}(n)\cdot\Gamma(2-\alpha)$. From Lemma 4.4.5 we know $L_A(n) = \sum_{k=0}^{n} m(D_k)$. Therefore,

$$L_A(n) = \sum_{k-0}^{n-1} m(D_k) = \frac{(n-1)^{1-\alpha}\cdot f(n)}{\Gamma(2-\alpha)}, \tag{4.5.25}$$

which satisfies part (2) of Theorem 4.5.5 with $\theta = 1$ and $p = 1-\alpha$. Therefore, by part (1) of Theorem 4.5.5 we have

$$\sum_{n\geq 0} m(D_n)e^{-ns} \sim \left(\frac{1}{s}\right)^{1-\alpha} f\left(\frac{1}{s}\right) \quad \text{as } s \to 0. \tag{4.5.26}$$

Using this, we show $\sum_{n\geq 0} a_n(A)e^{-ns}$ also satisfies part (1) of Theorem 4.5.5. From (4.5.24) we have

$$\sum_{n\geq 0} a_n(A)e^{-ns} \sim \frac{m(A)}{s^2}\cdot\frac{1}{\sum_{n\geq 0} m(D_n)e^{-ns}}. \tag{4.5.27}$$

Therefore, substituting from (4.5.26) yields

$$\sum_{n\geq 0} a_n(A)e^{-ns} \sim \frac{m(A)}{s^2}\cdot\frac{1}{\left(\frac{1}{s}\right)^{1-\alpha} f\left(\frac{1}{s}\right)} = m(A)\left(\frac{1}{s}\right)^{1+\alpha}\frac{1}{f(\frac{1}{s})}, \tag{4.5.28}$$

which satisfies part (1) of Theorem 4.5.5 with $\theta_2 = m(A)$, $p_2 = 1+\alpha$, and $f_2 = \frac{1}{f}$. Therefore, since $a_n(A)$ is monotone we have from part (3) of Theorem 4.5.5

$$a_n \sim \frac{m(A)(1+\alpha)}{\Gamma(2+\alpha)} n^\alpha f_2(n). \tag{4.5.29}$$

We know $f = \frac{1}{f_2}$, and by (4.5.25) we have $f(n) = \frac{L_A(n)\Gamma(2-\alpha)}{n^{1-\alpha}}$. Substituting for f_2 in (4.5.29) yields

$$a_n \sim \frac{m(A)(1+\alpha)n^\alpha}{\Gamma(2+\alpha)}\cdot\frac{n^{1-\alpha}}{L_A(n)\Gamma(2-\alpha)}. \tag{4.5.30}$$

Noting that

$$\frac{(1+\alpha)}{\Gamma(2+\alpha)} = \frac{(1+\alpha)}{(1+\alpha)\Gamma(1+\alpha)} = \frac{1}{\Gamma(1+\alpha)}, \tag{4.5.31}$$

the proof is complete. $\square$

Now, let's return to rational R-functions of negative type. Recall that by Proposition 4.4.2 we know the wandering rate, $L_A(n)$ for $A = [q_1, q_{N+1}]$. Thus, we can understand the return times rational R-functions of negative type, S.

LEMMA 4.5.6. *If S is a rational R-function of negative type, then the return seqence, $a_k(T)$ for S is given by*

$$a_k(S) \sim \frac{2}{\pi}\sqrt{\frac{k}{\sum_{j=1}^{N^2+2N} \rho_j}}. \tag{4.5.32}$$

PROOF. Let $S(x) = -x - \beta - \sum_{i=1}^{N} \frac{p_i}{t_i - x}$, where $\beta, t_i, p_i \in \mathbb{R}$ with $t_i, pi > 0$, and $S(0) = 0$. By By Proposition 4.2.3 we have $S^2(x) = x + \sum_{j=1}^{N^2+2N} \frac{\rho_j}{\tau_j - x}$, where $\tau_j, \rho_j \in \mathbb{R}$ with $\rho_j > 0$. Let $A = [q_1, q_{N+1}]$, so by Proposition 4.4.2 we know the wandering rate of A is given by $L_A(k) = 2\sqrt{k \sum_{j=1}^{N^2+2N} \rho_j}$. Note that $L_A(k)$ is regularly varying with index 1/2. Now, Proposition 4.5.4 relates $L_A(k)$ to $a_k(S)$ in the following way $a_k(T) \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \cdot \frac{k}{L_A(k)}$. Therefore,

$$a_k(S) \sim \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \cdot \frac{k}{2\sqrt{k \sum_{j=1}^{N^2+2N} \rho_j}}. \tag{4.5.33}$$

Noting that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ yields the result. $\square$

4.6. Quasi-Finiteness

Before we can state the definition of a quasi-finite transformation, we need some notation. Let $(X, \mathcal{B}, m)$ be a finite-measure space. The entropy of a countable partition, $\alpha = \{a_i\}$, of X is defined to be

$$H(\alpha) = -\sum_{i=0}^{\infty} m(a_i)\log(m(a_i)). \tag{4.6.1}$$

The following definition is due to Krengel and can be found in [**Kre**].

Definition 4.6.1. Let $(X, \mathcal{B}, m, T)$ be a conservative measure-preserving system. The map T is called *quasi-finite* if there exists a sweep-out-set $A \in \mathcal{B}$ with $m(A) < \infty$ such that the first return time partition, $\mathfrak{A} = \{A_k\}_{k\geq 1}$ (as in (2.2.2)), has finite entropy.

A stronger property than quasi-finite is called log lower bounded. The following definition can be found in [**AP**]. Recall from Section 2.2 that for $A \in \mathcal{B}$ we set $\phi_A(x) = \min\{n : T^n(x) \in A\}$.

Definition 4.6.2. Let $(X, \mathcal{B}, m, T)$ be a conservative, ergodic, infinite-measure-preserving system. We set

$$\mathcal{F}_{\log} = \left\{ A \in \mathcal{B} : 0 < m(A) < \infty \text{ and } \int_A \log(\phi_A) dm < \infty \right\}. \tag{4.6.2}$$

The transformation T is called *log-lower bounded* (LLB) if $\mathcal{F}_{\log} \neq \emptyset$.

The following Lemma is stated as a remark in [**AP**] with very little explanation. The details of the proof are, however, outlined in a slightly different context in [**Aar1**]. We combine the information to state and prove the complete result here.

Lemma 4.6.3. *If T is log lower bounded, then T is quasi-finite.*

Proof. Let T be an LLB transformation, and let $A \in \mathcal{F}_{\log}$. By (4.6.2) we have,

$$\begin{aligned} \int_A \log(\phi_A) dm &= \sum_{n=1}^{\infty} \int_{A_n} \log(n) dm \\ &= \sum_{n=1}^{\infty} m(A_n)\log(n) < \infty. \end{aligned} \tag{4.6.3}$$

Thus, we need to show that $\sum_{n=1}^{\infty} m(A_n)\log(n) < \infty$ implies

$$H(\mathfrak{A}) = \sum_{n=1}^{\infty} m(A_n) \log \frac{1}{m(A_n)} < \infty. \tag{4.6.4}$$

Let $\mathcal{C} = \{n \geq 1 : m(A_n) < \frac{1}{(n+1)^2}\}$. Note that the function $x\log(1/x)$ is increasing on $(0, 1/4)$. If $n \in \mathcal{C}$, then

$$m(A_n) \log \frac{1}{m(A_n)} \leq \frac{1}{(n+1)^2} \log \frac{1}{(n+1)^2} \leq \frac{2\log(n+1)}{(n+1)^2}. \tag{4.6.5}$$

If $n \notin \mathcal{C}$ (i.e. $(n+1)^2 \leq \frac{1}{m(A_n)}$), then

$$m(A_n) \log \frac{1}{m(A_n)} \leq m(A_n) 2\log(n+1). \tag{4.6.6}$$

Therefore,

$$\begin{aligned} H(\mathcal{A}) &= \sum_{n\in\mathcal{C}} m(A_n)\log\frac{1}{m(A_n)} + \sum_{n\notin\mathcal{C}} m(A_n)\log\frac{1}{m(A_n)} \\ &\leq \sum_{n=1}^{\infty} \frac{2\log(n+1)}{(n+1)^2} + \sum_{n=1}^{\infty} m(A_n) 2\log(n+1). \end{aligned} \tag{4.6.7}$$

The first sum in (4.6.7) clearly converges, and the second sum converges by our assumption that $A \in \mathcal{F}_{log}$ and (4.6.3). $\square$

Given $A \in \mathcal{B}$ with $m(A) < \infty$, we consider the sets D_n defined in (2.2.4). The following lemma gives a necessary and sufficient condition on $m(D_n)$ under which $A \in \mathcal{F}_{\log}$. The idea for the proof has been adapted from [**Aar1**].

Lemma 4.6.4. *Let T be an LLB transformation, then*

$$A \in \mathcal{F}_{\log} \iff \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n} m(D_k) < \infty. \tag{4.6.8}$$

Proof. We begin by making the observation that

$$\sum_{n=k}^{\infty} \frac{1}{n^2} \sim \frac{1}{k}. \tag{4.6.9}$$

This comes from the fact that we can estimate integral remainders, in the following way

$$\int_{k+1}^{\infty} \frac{1}{x^2} dx \leq \sum_{n=k}^{\infty} \frac{1}{n^2} \leq \int_{k}^{\infty} \frac{1}{x^2} dx$$
$$\implies \frac{1}{k+1} \leq \sum_{n=k}^{\infty} \frac{1}{n^2} \leq \frac{1}{k}.$$

Thus, considering the right-hand-side of (4.6.8) we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n} m(D_k) = \sum_{k=1}^{\infty} m(D_k) \sum_{n=k}^{\infty} \frac{1}{n^2} < \infty \iff \sum_{n=1}^{\infty} \frac{m(D_n)}{n} < \infty. \tag{4.6.10}$$

We now consider

(4.6.11)
$$\sum_{n=1}^{\infty} \frac{m(D_n)}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n+1}^{\infty} m(A_k) = \sum_{k=2}^{\infty} m(A_k) \sum_{n=1}^{k-1} \frac{1}{n} < \infty \iff \sum_{n=1}^{\infty} \log(n) m(A_n) < \infty.$$

The equivalence in (4.6.11) comes from the fact that the sequence $w_k = \log(k) - \sum_{n=1}^{k-1} \frac{1}{n}$ converges. By definition $A \in \mathcal{F}_{\log} \iff \sum_{n=1}^{\infty} \log(n) m(A_n) < \infty$ which completes the proof. $\square$

Proposition 4.6.5. *If S is a rational R-function of negative type, then the set $A = [q_1, q_{N+1}]$ is a quasi-finite set for S.*

Proof. By Lemma 4.6.4 we have $A \in \mathcal{F}_{\log} \iff \sum_{k=1}^{\infty} \frac{L_A(k)}{k^2} < \infty$. By Lemma 4.4.2 $L_A(k) = 2\sqrt{k \sum_{j=1}^{N^2+2N} \rho_j}$, where the ρ_j are the weights on the linear factors coming from the second iterate S^2. Therefore, $A \in \mathcal{F}_{\log}$, so by Lemma 4.6.3 A is quasi-finite set for S. $\square$

We are now ready to state and prove the main theorem of this section.

Theorem 4.6.6. *All rational R-functions of negative type are log-lower bounded and quasi-finite.*

Proof. By Proposition 4.6.5, there exists a a log-lower bounded (and therefore quasi-finite) set for every rational R-function of negative type. $\square$

CHAPTER 5

ENTROPY

5.1. Preliminaries on Entropy

5.1.1. Entropy of Finite-Measure-Preserving Transformations. We begin with the definition of entropy for transformations preserving a finite measure. Let $(X, \mathcal{B}, m)$ be a finite-measure space, and let $\alpha = \{a_i\}$ be a countable partition of X. The entropy of α is defined by

$$H(\alpha) = -\sum_{i=0}^{\infty} m(a_i)\log(m(a_i)). \tag{5.1.1}$$

Let $\alpha_1 = \{a_{(1,i)}\}$ and $\alpha_2 = \{a_{(2,i)}\}$ be two partitions of X. We define their refinement to be

$$\alpha_1 \vee \alpha_2 = \{a_{(1,i)} \cap a_{(2,i)} : a_{(1,i)} \in \alpha_1, a_{(2,i)} \in \alpha_2\}. \tag{5.1.2}$$

If $\alpha_1 = \{a_{(1,i)}\}$, ..., $\alpha_n = \{a_{(n,i)}\}$ are finitely many partitions of X then $\alpha_1 \vee \alpha_2 \vee ... \vee \alpha_n$ denotes their common refinement.

If $\alpha = \{a_i\}$ is a countable partition, and T is a measure-preserving transformation of $(X, \mathcal{B}, m)$, then $T^{-n}\alpha$ denotes the partition $\{T^{-n}a_i\}$. The entropy of T with respect to α is defined by

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H(\alpha \vee T^{-1}\alpha \vee ... \vee T^{-(n-1)}\alpha). \tag{5.1.3}$$

The entropy of the transformation T is defined by

$$h(T) = \sup h(T, \alpha), \tag{5.1.4}$$

where the supremum is taken over all finite partitions α.

5.1.2. Entropy of Infinite-Measure-Preserving Transformations. Krengel was the first to extend the notion of entropy to infinite-measure-preserving transformations (see [**Kre**]). In order to state Krengel's definition, we must first recall the definition of the induced transformation from Section 2.2. That is, if $(X, \mathcal{B}, m, T)$ is a conservative, measure-preserving system and $A \in \mathcal{B}$, then T_A is the induced transformation of T on A (see (2.2.1) for a complete definition). We now state Krengel's definition of entropy for conservative infinite-measure-preserving transformations.

Definition 5.1.1 ([**Kre**]). Let $(X, \mathcal{B}, m, T)$ be a conservative measure-preserving system. Let $A \in \mathcal{B}$ such that $0 < m(A) < \infty$. Define

$$h_{Kr}(T) = \sup_A h(T_A, m|_A). \tag{5.1.5}$$

Taking the supremum over all finite-measure sets, $A \in \mathcal{B}$, to calculate $h_{Kr}(T)$ would be quite cumbersome. Luckily, Krengel also proved the following theorem which provides a simplification of Definition 5.1.1 in the case when A is a sweep-out set.

Theorem 5.1.2 ([**Kre**]). *Let $(X, \mathcal{B}, m, T)$ be a conservative measure-preserving system. If $A \in \mathcal{B}$ such that $0 < m(A) < \infty$, and A is a sweep-out set for T, then*

$$h_{Kr}(T) = h(T_A, m|_A). \tag{5.1.6}$$

We note that Krengel's definition of entropy is equivalent to Abramov's formula for entropy in the finite-measure-preserving case. Also, we have written $m|_A$, to emphasize that we are considering the measure, m, restricted to A (not normalized). Had we instead normalized m on A, then we would need to multiply the right-hand side of (5.1.5) by a factor of $m(A)$.

In 1969 Parry provided a different extension of entropy to transformations preserving an infinite measure (see [**Par**]). Before stating Parry's definition, we need a few definitions concerning conditional entropy.

Let $(X, \mathcal{B}, m, T)$ be a measure-preserving system. Let $\mathcal{C}$ be a sub-σ-algebra of $\mathcal{B}$. If $f \in L^1(m)$, then $d\mu = f dm$ defines a measure such that $\mu(A) = \int_A f dm$. By the Radon-Nikodym Theorem there exists a function $E(f|\mathcal{C})$ such that

$$\int_C E(f|\mathcal{C}) dm = \int_C f dm \quad \text{for all } C \in \mathcal{C}. \tag{5.1.7}$$

We call $E(f|\mathcal{C})$ the conditional expectation of f given $\mathcal{C}$. Now, for $A \in \mathcal{B}$ we define $m(A|\mathcal{C}) = E(\mathbb{1}_A|\mathcal{C})$. If $\alpha = \{a_i\}$ is a countable partition of X, then we define the conditional information of α given $\mathcal{C}$ to be

$$I(\alpha|\mathcal{C}) = -\sum_{a_i \in \alpha} \log(m(a_i|\mathcal{C})) \cdot \mathbb{1}_{a_i}. \tag{5.1.8}$$

Finally, the conditional entropy of α given $\mathcal{C}$ is defined by

$$H(\alpha|\mathcal{C}) = \int_X I(\alpha|\mathcal{C}) dm. \tag{5.1.9}$$

Given a partition α we write $\hat{\alpha}$ to denote the σ-algebra generated by α. That is, elements of $\hat{\alpha}$ are unions of the atoms in α. For more information on the information function and conditional entropy see [**Par**] or [**Pet**]. We now state Parry's definition of entropy for infinite-measure-preserving transformations.

Definition 5.1.3 ([**Par**]). Let $(X, \mathcal{B}, m, T)$ be a system preserving a σ-finite measure. The *Parry entropy* of T is defined by

$$h_{Pa}(T) = \sup\{H(\alpha|\widehat{T^{-1}\alpha})\},$$

where the supremum is taken over all finite partitions α such that $T^{-1}\alpha \leq \alpha$.

The following theorem relates $h_{Kr}(T)$ and $h_{Pa}(T)$, and a proof can be found in [**Par**].

THEOREM 5.1.4. *If T is a conservative measure-preserving transformation of a σ-finite measure space, then $h_{Kr}(T) \geq h_{Pa}(T)$. If T is quasi-finite, then $h_{Kr}(T) = h_{Pa}(T)$.*

Finally, the Poisson suspension, $(X^*, \mathcal{B}^*, m^*, T_*)$, of a system preserving a σ-finite measure, $(X, \mathcal{B}, m, T)$, is a method of associating a probability-preserving transformation to a possibly infinite-measure-preserving system. $(X^*, \mathcal{B}^*, m^*, T_*)$ is a point process in which identical particles propagate according to T, do not interact with one another, and the expected number of particles in each set $E \in \mathcal{B}$ is determined (in a Poisson manner) by $m(E)$. A formal description of the Poisson suspension is given in [**Roy**] and [**JMRdlR**].

DEFINITION 5.1.5 ([**Roy**]). The *Poisson entropy* of an infinite-measure-preserving transformation is defined as the Kolmogorov entropy of the Poisson suspension. That is, $h_{PS}(T) = h(T_*)$.

The following result relates the Poisson entropy to that of Krengel and Parry, and a proof can be found in [**JMRdlR**].

THEOREM 5.1.6. *If T is a conservative measure-preserving transformation of a σ-finite measure space, then $h(T_*) \geq h_{Pa}(T)$, and all three definitions coincide if T is quasi-finite.*

5.2. Krengel Entropy of Rational R-functions of Negative Type

In this section we provide a method of computing the Krengel entropy for all rational R-functions of negative type. The following theorem is the main result of this section.

THEOREM 5.2.1. *If S is a rational R-function of negative type, then*

$$h_{Kr}(S) = \int_{\mathbb{R}} \log |S'(x)| d\lambda(x). \tag{5.2.1}$$

Before proving Theorem 5.2.1 we give a little motivation and history for the integral formula. The following definition can be found in [**Tha**].

DEFINITION 5.2.2. Let $I = [a, b]$ be a closed interval in $\mathbb{R}$. Let $\mathfrak{T}_R(I)$ denote the class of all transformations $T : I \to I$ such that there exists a partition into subintervals $\{I_j : j \in J\}$ satisfying the following properties:

(1) (piecewise differentiable and surjective) $T|_{I_j}$ is C^2 and $\overline{T(I_j)} = I$ for all j. Each I_j contains exactly one fixed point of T.
(2) (expanding) There exists a $\rho > 1$ such that $|T'(x)| \geq \rho$ for all $x \in I_i$.
(3) (Adler's condition) $\left|\frac{T''(x)}{T'(x)^2}\right|$ is bounded on $\bigcup_{j \in J} I_j$.

If $T \in \mathfrak{T}_R(I)$, then T satisfies Renyi's condition, and T preserves an absolutely continuous finite measure, μ ([**Rén**], [**Tha**]).

The following theorem concerning entropy of $T \in \mathfrak{T}_R(I)$ is referred to as Rohlin's entropy formula, and it has been studied in [**Tha**], [**Yur**], and [**PY**].

THEOREM 5.2.3. [**Roh2**] *Let $I = [a, b]$ be a closed interval of $\mathbb{R}$. If $T \in \mathfrak{T}_R(I)$ and μ is invariant for T, then*

$$h(T) = \int_I \log |T'(x)| d\mu(x). \tag{5.2.2}$$

Let S be a rational R-function of negative type. Consider the Rohlin partition $\mathcal{Q} = \{Q_1, ..., Q_{N+1}\}$ for S as in Remark 4.3.1. Recall that $0 = q_1, q_2, ..., q_{N+1}$ are the roots of S, and $Q_i = [q_i, q_{i+1})$ for $i = 1, ..., N$ while $Q_{N+1} = (-\infty, q_1) \cup (q_{N+1}, \infty)$. Also, recall that $S : Q_i \to \mathbb{R}$ is one-to-one and onto. Furthermore, we let ψ_i denote the inverse of S restricted to Q_i, and $\psi_{i_1,...,i_n} = S^{-1}|_{Q_{i_1 \ldots i_n}}$. We define the notation $\psi_{i[k]} = \psi_i^k$ as in (4.3.4).

We know from Proposition 4.3.2 that $A = [q_1, q_{N+1})$ is a sweep-out set for S. Let $\mathfrak{A} = \{A_k\}$ be the first-return partition of A as in (2.2.2). That is, $A_k = \{x \in A : \phi_A(x) = k\}$. Now, we partition each atom A_k into N sets $A_{k,i}$ for $i = 1, .., N$ such that $T^k : A_{k,i} \to A$ is one-to-one and onto. Let $\mathfrak{B} = \{B_k\}$ the hitting-time partition of A^c as in (2.2.3). That is, $B_k = \{x \in A^c : \phi_A(x) = k\}$. Figure 5.1 shows how $A_{k,i}$ and B_k move under ψ maps. Each solid arrow depicts a one-to-one and onto mapping, and the dashed arrows indicate a series of $N - 2$ one-to-one and onto mappings.

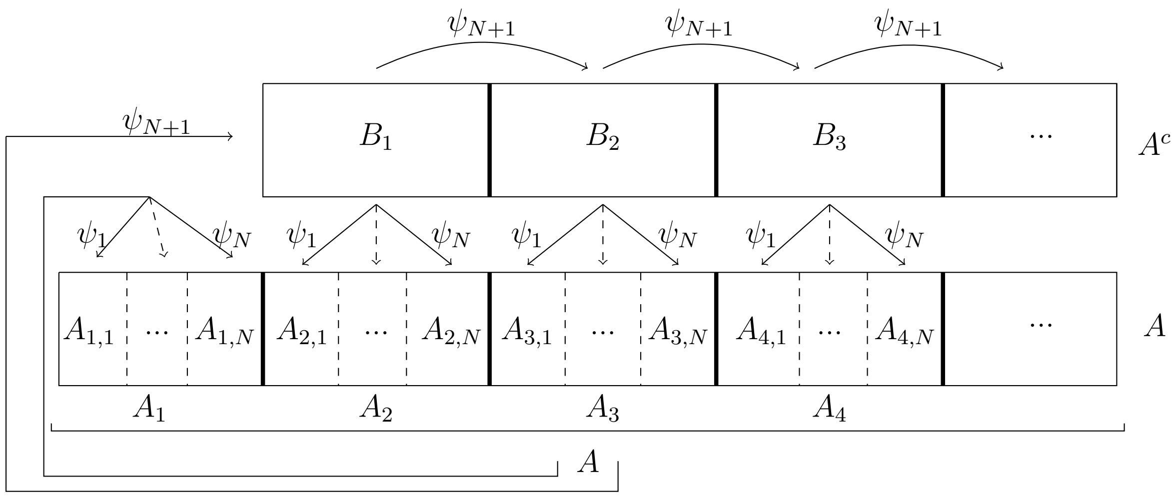

FIGURE 5.1. How hitting-time sets move under ψ_j, $j = 1, ..., N + 1$.

Recall from Section 2.2 the definition of the induced transformation, S_A. We have that S_A is a finite-measure-preserving transformation of $(A, \mathcal{B}|_A, \lambda|_A)$. Our goal is to apply Rohlin's formula to the induced transformation.

LEMMA 5.2.4. *If S is a rational R-function of negative type and $A = [q_1, q_{N+1})$, then the induced transformation, $S_A \in \mathfrak{T}_R(A)$.*

PROOF. We want so show S_A satisfies (1)-(3) of Definition 5.2.2. Consider the partition $A_{k,i}$ defined above. To show (1) we note that $S_A = S^k$ on each $A_{k,i}$, so $S_A : A_{k,i} \to A$ is one-to-one and onto. Furthermore, S is piecewise smooth on $\mathbb{R}$, so

S_A is C^2 on each $A_{k,i}$. To show (2) we note,

$$|S'(x)| = 1 + \sum_{i=1}^{N} \frac{p_i}{(t_i - x)^2}. \tag{5.2.3}$$

We have $|S'(x)| > 1$ for all $x \in \mathbb{R}$, but $|S'(x)| \to 1$ as $x \to \infty$. The set $A = [q_1, q_2]$, however, is bounded away from ∞. Therefore, there exists a constant $\rho > 1$ such that $\inf_{x \in A} |S'(x)| \geq \rho$, so by the chain rule $|S'_A(x)| \geq \rho > 1$ for all $x \in A$. Finally, to show (3) we let $x \in A_{k,i}$ and use the chain rule to obtain

$$\begin{aligned}
\left| \frac{S''_A(x)}{(S'_A(x))^2} \right| &\leq \left| \frac{S''(S^{k-1}(x))}{(S'(S^{k-1}(x)))^2} \right| + \left| \frac{S''(S^{k-2}(x))}{S'(S^{k-1}(x)) \cdot (S'(S^{k-2}(x)))^2} \right| \\
&\quad + \ldots + \left| \frac{S''(x)}{S'(S^{k-1}(x)) \cdot S'(S^{k-2}(x)) \ldots S'(S(x)) \cdot (S'(x))^2} \right| \\
&\leq \sum_{j=1}^{k} \left| \frac{S''(S^{k-j}(x))}{(S'(S^{k-j}(x)))^2} \right|. \qquad (5.2.4)
\end{aligned}$$

A calculation shows $|S''(y)(S'(y))^{-2}|$ is bounded and decreases for large $|y|$ satisfying $|S''(y)(S'(y))^{-2}| \leq M|y|^{-3}$. Since $x \in A_k$, we know $S^{k-j}(x) \in B_j$ (as in (2.2.3)). From our study of A in (4.3.9), we have the following two cases:

(1) If j is even, then $B_j = [V_{(j/2)-1}, V_{j/2}]$
(2) If j is odd, then $B_j = [W_{\lceil j/2 \rceil}, W_{\lfloor j/2 \rfloor}]$.

Therefore, $S^{k-j} \in [V_{(j/2)-1}, V_{j/2}] \cup [W_{\lceil j/2 \rceil}, W_{\lfloor j/2 \rfloor}]$. Considering the right-hand side of (5.2.4) we have

$$\begin{aligned}
\sum_{j=1}^{k} \left| \frac{S''(S^{k-j}(x))}{(S'(S^{k-j}(x)))^2} \right| &\leq M \sum_{j=1}^{k} |W_{\lfloor j/2 \rfloor}|^{-3} + |V_{(j/2)-1}|^{-3} \\
&\leq M \sum_{j=1}^{k} \frac{1}{c_2^3 (\lfloor j/2 \rfloor)^{3/2}} + \frac{1}{c_1^3 (j/2 - 1)^{3/2}}, \qquad (5.2.5)
\end{aligned}$$

where the second line comes from the proof of Proposition 4.3.2. We see that the limit as $k \to \infty$ of (5.2.5) is finite, and independent of k. $\square$

We have the following lemma concerning the Krengel entropy of rational R-functions of negative type.

LEMMA 5.2.5. *If S is a rational R-function of negative type, and $A = [q_1, q_{N+1})$ as above, then*

$$h_{Kr}(S) = \int_A \log |S_A'(x)| d\lambda_A(x). \tag{5.2.6}$$

PROOF. By Lemma 5.2.4 we know $S_A \in \mathfrak{T}_R(A)$, so by Theorem 5.2.3 we have $h(S_A) = \int_A \log |S_A'(x)| d\lambda_A(x)$. Proposition 4.3.2 implies that the set A is a sweep-out set, so by Theorem 5.1.2 we have $h_{Kr}(S) = h(S_A)$. □

Lemma 5.2.5 provides a theoretical way to compute the Krengel entropy of any rational R-function of negative type. The integral, however, is nontrivial. Our ultimate goal is to use Lemma 5.2.5 to prove Theorem 5.2.1, which says the Krengel entropy of any rational R-function of negative type can be computed using Rohlin's formula. The first step is to show $\log |S'|$ is indeed integrable, and we do in the next lemma.

LEMMA 5.2.6. *If S is a rational R-function of negative type, then*

$$\int_{\mathbb{R}} \log |S'(x)| d\lambda(x) < \infty. \tag{5.2.7}$$

PROOF. We have that $S(x) = -x - \beta - \sum_{i=1}^{N} \frac{p_i}{t_i - x}$, so $\{t_1, ..., t_N\}$ are the poles of S. Assume $t_i < t_{i+1}$, so the t_i are in ascending order. We will show the integrability of $\log |S'(x)|$ in three separate pieces.

(1) The integral between the poles is finite. That is,

$$\int_{t_i+\varepsilon}^{t_{(i+1)}-\varepsilon} \log |S'(x)| d\lambda(x) < \infty,$$

for $i = 1, ..., N-1$.

(2) The integral near each pole is finite. That is,

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} \log|S'(x)|d\lambda(x) < \infty,$$

for $i = 1, ..., N$.

(3) The integral outside the smallest and largest poles is finite. That is,

$$\int_{-\infty}^{t_1-\varepsilon} \log|S'(x)|d\lambda(x) + \int_{t_N+\varepsilon}^{\infty} \log|S'(x)|d\lambda(x) < \infty.$$

To show (1) we note that $\log|S'|$ is continuous on the compact set $[t_i+\varepsilon, t_{(i+1)}-\varepsilon]$, so there exists an $M \in \mathbb{R}$ such that $\log|S'(x)| < M$ for all $x \in [t_i + \varepsilon, t_{(i+1)} - \varepsilon]$. Therefore,

$$\int_{t_i+\varepsilon}^{t_{(i+1)}-\varepsilon} \log|S'(x)|d\lambda(x) \leq M \cdot \lambda\left([t_i + \varepsilon, t_{(i+1)} - \varepsilon]\right) < \infty. \tag{5.2.8}$$

Now, to show (2) we let $x \in [t_i - \varepsilon, t_i + \varepsilon]$ and write

$$|S'(x)| = 1 + \sum_{j=1}^{i-1} \frac{p_j}{(t_j - x)^2} + \frac{p_i}{(t_i - x)^2} + \sum_{j=i+1}^{N} \frac{p_j}{(t_j - x)^2}. \tag{5.2.9}$$

If $i \neq j$, then each term $\frac{p_j}{(t_j-x)^2}$ is bounded. Therefore, we need only show

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} \log\left(\frac{p_i}{(t_i - x)^2}\right) d\lambda(x) < \infty, \tag{5.2.10}$$

which is equivalent to showing

$$\int_{t_i-\varepsilon}^{t_i+\varepsilon} \log\left(\frac{1}{(t_i - x)^2}\right) d\lambda(x) < \infty. \tag{5.2.11}$$

In order to show (5.2.11) we change variables using $y = (t_i - x)$, so the integral becomes

$$\int_{\varepsilon}^{-\varepsilon} \log(y^2)d\lambda(y) = 2\lim_{\tau\to 0}\int_{\tau}^{\varepsilon} \log(y^2)d\lambda(y) = 2\lim_{\tau\to 0} -2y + y\log(y^2)\Big|_{\tau}^{\varepsilon}. \tag{5.2.12}$$

A simple computation shows that $2\lim_{\tau\to 0} -2y + y\log(y^2)|_\tau^\varepsilon = 2(-2\varepsilon + \varepsilon\log(\varepsilon^2))$ which is finite for all ε and equals 0 as $\varepsilon \to 0$.

Finally, to prove (3) we give an argument which shows $\int_{t_N+\varepsilon}^{\infty} \log|S'(x)|d\lambda(x) < \infty$, and we comment that the other piece follows in a similar way. First, choose $M > 0$ large enough such that $1+\sum_{j=1}^{N} \frac{p_i}{(t_i-x)^2} \leq \frac{Np_1}{(t_1-x)^2}$ for all $x \in (M,\infty)$. Note that $\log|S'|$ is continuous on the compact set $[t_N+\varepsilon, M]$. Thus, it is bounded and integrable. Now, we need only show

$$\int_M^\infty \log\left(1+\sum_{j=1}^{N}\frac{p_i}{(t_i-x)^2}\right) d\lambda(x) < \infty, \tag{5.2.13}$$

which by our choice of M is equivalent to showing

$$\int_M^\infty \log\left(1+\frac{1}{x^2}\right) d\lambda(x) < \infty. \tag{5.2.14}$$

We note $\log(1+x) < x$ for all $x \in \mathbb{R}$, so $\log\left(1+\frac{1}{x^2}\right) < \frac{1}{x^2}$. Since $\frac{1}{x^2}$ is integrable, we have shown (5.2.14), and thus completed the proof of (3). $\square$

Now that we have $\log|S'|$ is integrable, our goal is to prove the following proposition, which is the main tool that will be used in the proof of Theorem 5.2.1. It says that if S is a rational R-function of negative type, then the expression in Rohlin's formula for the induced transformation, S_A, is equal to that in Rohlin's formula for the original transformation, S.

Proposition 5.2.7. *If S is a rational R-function of negative type and $A = [q_1, q_{N+1})$, then*

$$\int_A \log|S_A'(x)|d\lambda|_A(x) = \int_{\mathbb{R}} \log|S'(x)|d\lambda(x). \tag{5.2.15}$$

The main idea in the proof of Proposition 5.2.7 is to exploit the fact that if $x \in A_k$ as in (2.2.2), then $S_A(x) = S^k(x)$. Intuitively, we will unravel the chain rule on the atoms of the first-return partition via change of variables. Before proving Proposition

5.2.7 we recall some notation from Section 2.2. Let $D_k = \{x \in A : \phi_A(x) > k\}$ as in (2.2.4). The following observation will be key in keeping track of how sets move under S_A.

REMARK 5.2.8. We have $A = \bigcup_{i=1}^{N} \psi_i(\mathbb{R})$ and $A^c = \psi_{N+1}(\mathbb{R})$. We also have that

$$D_k = \bigcup_{i=1}^{N} \psi_{i,(N+1)[k-1]}(A^c) \tag{5.2.16}$$

The following lemma will be used in the proof of Proposition 5.2.7.

LEMMA 5.2.9. *Let $x \in A^c$, then*

$$\sum_{k=1}^{\infty} \sum_{j=1}^{N} J_{\psi_{j,(N+1)[k-1]}}(x) = 1. \tag{5.2.17}$$

PROOF. By Lemmas 2.2.5 and 2.2.6, we have for any $C \in \mathcal{B}$

$$\lambda(C) = \sum_{k=0}^{\infty} \lambda|_A(D_k \cap S^{-k}C). \tag{5.2.18}$$

Writing D_k in terms of ψ functions as in (5.2.16) yields

$$\begin{aligned} \lambda(C) &= \lambda|_A(D_0 \cap C) + \sum_{k=1}^{\infty} \lambda|_A \left(\bigcup_{j=1}^{N} \psi_{j,(N+1)[k-1]}(A^c) \cap S^{-k}C \right) \\ &= \lambda|_A(D_0 \cap C) + \sum_{k=1}^{\infty} \lambda|_A \left(\bigcup_{j=1}^{N} \psi_{j,(N+1)[k-1]}(A^c \cap C) \right) \\ &= \lambda|_A(D_0 \cap C) + \sum_{k=1}^{\infty} \sum_{j=1}^{N} \lambda|_A \left(\psi_{j,(N+1)[k-1]}(A^c \cap C) \right). \end{aligned} \tag{5.2.19}$$

Therefore, for $x \in A^c$

$$1 = \frac{d\lambda}{d\lambda}(x) = \sum_{k=1}^{\infty} \sum_{j=1}^{N} \frac{d\lambda|_A}{d\lambda}(\psi_{j,(N+1)[k-1]}(x)) \cdot J_{\psi_{j,(N+1)[k-1]}}(x). \tag{5.2.20}$$

We know $\psi_{j,(N+1)[k-1]}(x) \in A$, so $\frac{d\lambda|_A}{d\lambda}(\psi_{j,(N+1)[k-1]}(x)) = 1$ for all k. Therefore,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{N} J_{\psi_{j,(N+1)[k-1]}}(x) = 1. \tag{5.2.21}$$

$\square$

We are now ready to prove Proposition 5.2.7.

Proof of Proposition 5.2.7. Let S be a rational R-function of negative type and $A = [q_1, q_{N+1})$. By the definition of S_A and A_k we have that $A = \bigcup_{k=1}^{\infty} A_k$, and for $x \in A_k$, $S_A(x) = S^k(x)$. Therefore,

$$\int_A \log|S_A'(x)|d\lambda|_A(x) = \sum_{k=1}^{\infty}\int_{A_k} \log|(S^k)'(x)|d\lambda|_A(x). \tag{5.2.22}$$

Applying the chain rule and log properties to line (5.2.22) yields

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\int_{A_k} \log|S'(S^j(x)|d\lambda|_A(x). \tag{5.2.23}$$

By definition $D_0 = A$ and $D_k = \bigcup_{n>k} A_n$, so line (5.2.23) becomes

$$\sum_{k=0}^{\infty}\int_{D_k} \log|S'(S^k(x)|d\lambda|_A(x) \tag{5.2.24}$$

$$= \underbrace{\int_A \log|S'(x)|d\lambda|_A(x)}_{(I)} + \underbrace{\sum_{k=1}^{\infty}\int_{D_k} \log|S'(S^k(x)|d\lambda|_A(x)}_{(II)}. \tag{5.2.25}$$

We know that $\lambda|_A = \lambda$ on A, so $(I) = \int_A \log|S'(x)|d\lambda(x)$. Thus, in order to complete the proof we must show that $(II) = \int_{A^c} \log|S'(x)|d\lambda(x)$. By the definition of D_k and ψ together with Remark 5.2.8 we know $D_k = \bigcup_{j=1}^{N} \psi_{j,(N+1)[k-1]}(A^c)$. Therefore, we have

$$(II) = \sum_{k=1}^{\infty}\int_{\bigcup_{j=1}^{N} \psi_{j,(N+1)[k-1](A^c)}} \log|S'(S^k(x)|d\lambda|_A(x). \tag{5.2.26}$$

For each k, $\bigcup_{j=1}^{N} \psi_{j,(N+1)[k-1]}(A^c)$ is a disjoint union, so changing the integral over the finite union to a finite sum of integrals yields

$$\sum_{k=1}^{\infty}\sum_{j=1}^{N}\int_{\psi_{j,(N+1)[k-1]}(A^c)} \log|S'(S^k(x)|d\lambda|_A(x). \tag{5.2.27}$$

Let $S^k(x) = y$ and $x = \psi_{j,(N+1)[k-1]}(y)$. Therefore, by change of variables (5.2.27) becomes

$$\sum_{k=1}^{\infty}\sum_{j=1}^{N}\int_{A^c} \log|S'(y)| \cdot J_{\psi_{j,(N+1)[k-1]}}(y)d\lambda(y). \tag{5.2.28}$$

By Lemma 5.2.6 we have that $\log|S'(y)|$ is integrable. Furthermore, by Lemma 5.2.9 we have $\sum_{k=1}^{\infty}\sum_{j=1}^{N} J_{\psi_{j,(N+1)[k-1]}}(y) = 1$. Thus, by the dominated convergence theorem line (5.2.28) becomes

$$\int_{A^c} \log|S'(y)|d\lambda(y) \tag{5.2.29}$$

Finally combining (I) and (II), we have

$$(I)+(II) = \int_A \log|S'(x)|d\lambda(x) + \int_{A^c} \log|S'(x)|d\lambda(x) = \int_{\mathbb{R}} \log|S'(x)|d\lambda(x), \tag{5.2.30}$$

which completes the proof. $\square$

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.2.1. By the definition of Krengel Entropy, $h_{Kr}(S) = h(S_A)$. Also, Theorem 5.2.3 and Lemma 5.2.4 say we can use Rohlin's formula for the entropy of the induced transformation. That is,

$$h(S_A) = \int_A \log|S_A'(x)|d\lambda|_A(x). \tag{5.2.31}$$

By Proposition 5.2.7 we have

$$\int_A \log |S_A'(x)| d\lambda|_A(x) = \int_{\mathbb{R}} \log |S'(x)| d\lambda(x). \tag{5.2.32}$$

Therefore,

$$h_{Kr}(S) = \int_{\mathbb{R}} \log |S'(x)| d\lambda(x). \tag{5.2.33}$$

□

COROLLARY 5.2.10. *If S is a rational R-function of negative type, then*

$$h_{Pa}(S) = h_{Po}(S) = \int_{\mathbb{R}} \log |S'(x)| d\lambda(x). \tag{5.2.34}$$

PROOF. By Theorem 4.6.6 S is quasi-finite with respect to λ. Therefore, by Theorems 5.1.4 and 5.1.6 we have $h_{Kr}(S) = h_{Pa}(S) = h_{Po}(S)$. □

CHAPTER 6

ENTROPY AS AN ISOMORPHISM INVARIANT

6.1. Preliminaries on c-Isomorphisms

Entropy and the notion of uncertainty is a topic of importance in many fields. In ergodic theory, entropy is an isomorphism invariant for measure-preserving transformations. Suppose $(X_1, \mathcal{B}_1, m_1, T_1)$ and $(X_2, \mathcal{B}_2, m_2, T_2)$ are two measure-preserving systems. We first recall from Definition 2.1.9 the definition of an isomorphism between two probability-preserving transformations T_1 and T_2.

We now extend the definition of isomorphism to transformations that preserve an infinite measure. In this case we also have the notion of c-isomorphisms. The definition is similar, but property (2) of the isomorphism ϕ is less restrictive.

Definition 6.1.1 (c-Isomorphic). Let $(X_1, \mathcal{B}_1, m_1, T_1)$ and $(X_2, \mathcal{B}_2, m_2, T_2)$ be two infinite-measure-preserving systems. Suppose we have two sets $M_1 \in \mathcal{B}_1$ and $M_2 \in \mathcal{B}_2$ with $m_1(X_1 \backslash M_1) = 0$ and $m_2(X_2 \backslash M_2) = 0$ such that $T_1(M_1) \subseteq M_1$ and $T_2(M_2) \subseteq M_2$. For $c \in (0, \infty]$ we say $(X_1, \mathcal{B}_1, m_1, T_1)$ is a *c-isomorphic* to $(X_2, \mathcal{B}_2, m_2, T_2)$ if there exists an invertible map $\phi : M_1 \to M_2$ such that for all $A \in \mathcal{B}_2|_{M_2}$,

(1) $\phi^{-1}(A) \in \mathcal{B}_1|_{M_1}$,
(2) $m_1(\phi^{-1}(A)) = c \cdot m_2(A)$, and
(3) $(\phi \circ T_1)(x) = (T_2 \circ \phi)(x)$ for all $x \in M_1$.

We will denote this situation by $\phi : T_1 \to^c T_2$, and ϕ is called a *c-isomorphism.*

As stated above, we are primarily interested in entropy as an isomorphism invariant. For infinite-measure-preserving transformations, however, it is important to be clear about the measure under c-isomorphisms. Our primary measure of interest is still 1-dimensional Lebesgue measure, λ, on $\mathbb{R}$. Given two rational R-functions of negative type we consider only c-isomorphisms $(\mathbb{R}, \mathcal{B}, \lambda, S_1) \to^c (\mathbb{R}, \mathcal{B}, \lambda, S_2)$. If we write $h_{Kr}(S)$, then we assume the measure of interest is λ.

Fixing Lebesgue measure forces the isomorphism $\phi : (\mathbb{R}, \mathcal{B}, \lambda, S_1) \to^c (\mathbb{R}, \mathcal{B}, \lambda, S_2)$ to have the following property

$$\frac{d(\lambda \circ \phi^{-1})}{d\lambda}(x) = c \quad \text{for almost every } x \in \mathbb{R}. \tag{6.1.1}$$

From now on, when we write $\phi : S_1 \to^c S_2$ is a c-isomorphism of rational R-functions of negative type, we mean ϕ is a c-isomorphism such that (6.1.1) holds. In an effort to stay consistent with notation throughout proofs we will often write (6.1.1) as $J_{\phi^{-1}} = c$ almost everywhere. Furthermore, rational R-functions of negative type preserve λ and are piecewise smooth on $\mathbb{R}$, so we often write $J_S = |S'(x)|$.

As stated above, we are interested in studying Krengel entropy as an isomorphism invariant. The following lemma shows that a c-isomorphism multiplies the entropy by a factor of c.

Proposition 6.1.2. *If S_1 and S_2 are rational R-functions of negative type and $\phi : S_1 \to^c S_2$ is a c-isomorphism, then*

$$h_{Kr}(S_1) = c \cdot h_{Kr}(S_2). \tag{6.1.2}$$

Proof. Suppose $\phi : S_1 \to^c S_2$ is a c-isomorphism. By definition, $S_2 = \phi \circ S_1 \circ \phi^{-1}$ and $\lambda \circ \phi^{-1} = c \cdot \lambda$. We begin by noting $J_{\phi^{-1}}(x) = 1/(J_\phi(\phi^{-1}x))$. Then, using Theorem 5.2.1 combined with the chain rule and properties of log yields

$$\begin{aligned} c \int_{\mathbb{R}} \log\left(J_{S_2}(x)\right) d\lambda(x) = c \int_{\mathbb{R}} \log\left(\left|J_\phi(S_1(\phi^{-1}x))\right|\right) d\lambda(x) + c \int_{\mathbb{R}} \log\left(\left|J_{S_1}(\phi^{-1}x)\right|\right) d\lambda(x) \\ - c \int_{\mathbb{R}} \log\left(\left|J_\phi(\phi^{-1}x)\right|\right) d\lambda(x). \end{aligned}$$

After the substitution $u = \phi^{-1}(x)$ the last line becomes

$$\int_{\mathbb{R}} \log\left(|J_\phi(S_1(u))|\right) d\lambda(u) + \int_{\mathbb{R}} \log\left(|J_{S_1}(u)|\right) d\lambda(u) - \int_{\mathbb{R}} \log\left(|J_\phi(u)|\right) d\lambda(u).$$

We note that S_1 preserves Lebesgue measure, so we can simplify the first piece and obtain

$$\int_{\mathbb{R}} \log\left(|J_\phi(u)|\right) d\lambda(u) + \int_{\mathbb{R}} \log\left(|J_{S_1}(u)|\right) d\lambda(u) - \int_{\mathbb{R}} \log\left(|J_\phi(u)|\right) d\lambda(u).$$

Finally, we have a simple cancellation, and the proof is complete. □

COROLLARY 6.1.3. *Krengel entropy is a* 1*-isomorphism invariant for rational* R*-functions of negative type. That is, if* S_1 *and* S_2 *are* 1*-isomorphic, then* $h_{Kr}(S_1) = h_{Kr}(S_2)$.

We note that Krengel entropy is not a c-isomorphism invariant. That is, if $h_{Kr}(S_1) \neq h_{Kr}(S_2)$, then we cannot immediately determine whether or not S_1 and S_2 are c-isomorphic. To illustrate this, consider the following example.

EXAMPLE 6.1.4. Consider the following family of rational R-functions of negative type

$$T_a(x) = -x - (a+1) - \frac{-(a+1)}{1-x}, \quad \text{where } a < -1. \tag{6.1.3}$$

This family will be studied in detail in Section 6.4, but for now consider the following two rational R-functions of negative type

$$T_{-2}(x) = -x + 1 - \frac{1}{1-x} \quad \text{and} \quad T_{-5}(x) = -x + 4 - \frac{4}{1-x}. \tag{6.1.4}$$

By Theorem 5.2.1 we have $h_{Kr}(T_{-2}) = 2\pi$, but $h_{Kr}(T_{-5}) = 4\pi$. Thus, it is clear that T_{-2} and T_{-5} are not 1-isomorphic, but the entropy alone does not determine whether T_{-2} and T_{-5} are c-isomorphic.

Let $\phi(x) = 3 - 2x$, so $\phi^{-1}(x) = \frac{3-x}{2}$. A simple computation shows

$$(\phi \circ T_{-2} \circ \phi^{-1}) = T_{-5} \quad \text{and} \quad \lambda \circ \phi^{-1} = (1/2) \cdot \lambda. \tag{6.1.5}$$

Thus, $\phi : T_{-2} \to^{1/2} T_{-5}$ is a 1/2-isomorphism, and we have the relationship $2\pi = h_{Kr}(T_{-2}) = (1/2) \cdot h_{Kr}(T_{-5}) = (1/2) \cdot 4\pi$ between the Krengel entropies.

We describe the possible 1-isomorphisms and c-isomorphisms between quadratic rational R-functions of negative type in Sections 6.3.1 and 6.3.2 respectively.

6.2. Rational R-Functions of Negative Type are Not Squashable

Jon Aaronson posed the question of whether or not rational R-functions of negative type are squashable. In this section we give context for Aaronson's question and answer it by proving that rational R-functions of negative type are not squashable. We begin by stating the definition of a squashable transformation as it appears in Section 8.4 of [**Aar4**].

Definition 6.2.1. A conservative, ergodic, infinite-measure-preserving system $(X, \mathcal{B}, m, T)$ is *squashable* if there exists a nonsingular transformation $Q : (X, \mathcal{B}) \to (X, \mathcal{B})$ such that $Q \circ T = T \circ Q$ and $m \circ Q^{-1} = c \cdot m$ for some $c \in (0, \infty]$ and $c \neq 1$.

The following proposition answers Aaronson's question concerning the squashability of rational R-functions of negative type.

Proposition 6.2.2. *If S is a rational R-function of negative type, then S is not squashable.*

The proof of Proposition 6.2.2 requires a few facts about transformations with law of large numbers. We develop the necessary theory here.

DEFINITION 6.2.3. A conservative, ergodic, measure-preserving system, $(X, \mathcal{B}, m, T)$, has *law of large numbers* if there exists $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that

$$L(\mathbb{1}_A(x), \mathbb{1}_A(Tx), \mathbb{1}_A(T^2x), ...) = m(A) \tag{6.2.1}$$

for all $A \in \mathcal{B}$ and almost every $x \in X$.

The following lemma relating rational ergodicity and law of large numbers can be found in [**Aar4**] and is stated here for completeness.

LEMMA 6.2.4. *If T is rationally ergodic, then T has law of large numbers.*

We are now ready to prove Proposition 6.2.2.

PROOF OF PROPOSITION 6.2.2. By Theorem 4.5.1 we have that if S is a rational R-function of negative type, then S is pointwise dual ergodic (and thus rationally ergodic by Theorem 2.4.6). Therefore, by Lemma 6.2.4 S also has law of large numbers. Assume for contradiction that S is squashable. Let Q be the commuting map such that $Q \circ S = S \circ Q$ and $m \circ Q^{-1} = c \cdot m$. Let $Y = \{y \in \mathbb{R} : L(\mathbb{1}_A(y), \mathbb{1}_A(Sy), \mathbb{1}_A(S^2y), ...) = m(A) \text{ for all } A \in \mathcal{B}\}$. By definition of law of large numbers we have $\lambda(\mathbb{R} \setminus Y) = 0$, so by the nonsingularity of Q we also have $\lambda(Q^{-1}(\mathbb{R} \setminus Y)) = 0$. Thus, for λ-almost every $y \in Y$ there exists an $x \in Y$ such that $Qx = y$. Let $A \in \mathcal{B}$ and $y \in Y$ we have

$$\begin{aligned}
\lambda(A) &= L(\mathbb{1}_A(y), \mathbb{1}_A(Sy), \mathbb{1}_A(S^2y), ...) \\
&= L(\mathbb{1}_A(Qx), \mathbb{1}_A(S(Qx)), \mathbb{1}_A(S^2(Qx)), ...) \\
&= L(\mathbb{1}_A(Qx), \mathbb{1}_A(Q(Sx)), \mathbb{1}_A(Q(S^2x)), ...) \\
&= L(\mathbb{1}_{Q^{-1}A}(x), \mathbb{1}_{Q^{-1}A}(Sx), \mathbb{1}_{Q^{-1}A}(S^2x), ...) \\
&= \lambda(Q^{-1}A).
\end{aligned}$$

This implies that Q is measure-preserving, which contradicts the definition of squashable. $\square$

The following lemma shows that if there exists a c-isomorphism between two rational R-functions of negative type, then c is unique.

LEMMA 6.2.5. *If S_1 and S_2 are rational R-functions of negative type, then there is at most one $c \in (0, \infty]$ such that $\phi : S_1 \to^c S_2$ is a c-isomorphism.*

PROOF. Suppose $\phi : S_1 \to^c S_2$ is a c-isomorphism. Now, assume that $\psi : S_1 \to^k S_2$ is a k-isomorphism. We have the following commutative diagram

$$\begin{tikzcd} S_1 \arrow[r, "\phi"] \arrow[d, "\psi"'] & S_2 \\ S_2 \arrow[ur, dashed, "\phi \circ \psi^{-1}"'] & \end{tikzcd}$$

Thus, $\phi \circ \psi^{-1} : S_2 \to S_2$, and for almost every $x \in \mathbb{R}$ we have $J_{\phi\circ\psi^{-1}}(x) = J_\phi(\psi^{-1}(x)) \cdot J_{\psi^{-1}}(x) = (1/c) \cdot k$. Therefore, $\phi \circ \psi^{-1} : S_2 \to^{k/c} S_2$ is a $\frac{k}{c}$-isomorphism. We know, however, that S_2 is not squashable by Proposition 6.2.2. Therefore, $\frac{k}{c} = 1$ and $c = k$. $\square$

6.3. Isomorphism Invariants for Maps of Degree Two

In this section we provide isomorphism invariants for quadratic rational R-functions of negative type. That is, we restrict to maps of degree two. We denote a quadratic rational R-function of negative type by

$$S_{(\beta,p,t)}(x) = -x - \beta - \frac{p}{t - x}, \tag{6.3.1}$$

where $\beta, p, t \in \mathbb{R}$ and $p > 0$.

There are two main questions of interest in this section. First, we note that for all $\beta, \gamma \in \mathbb{R}$

$$J_{S_{(\beta,p,t)}}(x) = 1 + \frac{p}{(t-x)^2} = J_{S_{(\gamma,p,t)}}(x). \tag{6.3.2}$$

That is, the constant does not affect the Jacobian. Therefore, by Theorem 5.2.1 we have $h_{Kr}(S_{(\beta,p,t)}) = h_{Kr}(S_{(\gamma,p,t)})$ for all $\beta, \gamma \in \mathbb{R}$. We know by Lemma 6.1.2 and Remark 6.1.3 that Krengel entropy is a 1-isomorphism invariant. Thus, we are interested in answering:

QUESTION 6.3.1. Is $S_{(\beta,p,t)}$ 1-isomorphic to $S_{(\gamma,p,t)}$ for all $\beta, \gamma \in \mathbb{R}$?

Second, $h_{Kr}(S_{(\beta,p,t)})$ may not equal $h_{Kr}(S_{(\gamma,q,s)})$, but there still may be a c-isomorphism between $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ (as in Example 6.1.4). That is, perhaps $h(S_{(\gamma,q,s)}) = c \cdot h(S_{(\beta,p,t)})$. Thus, we are also interested in answering:

QUESTION 6.3.2. If $h_{Kr}(S_{(\beta,p,t)}) \neq h_{Kr}(S_{(\gamma,q,s)})$, can we determine whether or not there exists a c-isomorphism $\phi : S_{(\beta,p,t)} \to^c S_{(\gamma,q,s)}$?

Before discussing the answers to Questions 6.3.1 and 6.3.2 we prove the following lemma, which says two quadratic rational R-functions of negative type have the same Krengel entropy if and only if the numerators of the linear factors are equal.

LEMMA 6.3.1. *If $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ are two quadratic rational R-functions of negative type, then $h_{Kr}(S_{(\beta,p,t)}) = h_{Kr}(S_{(\gamma,q,s)})$ if and only if $p = q$.*

PROOF. Consider a quadratic rational R-function of negative type, $S_{(\beta,p,t)}$. By Theorem 5.2.1 we have

$$h_{Kr}(S_{(\beta,p,t)}) = \int_{\mathbb{R}} \log(J_{S_{(\beta,p,t)}}(x))d\lambda(x) = \int_{\mathbb{R}} \log\left(1 + \frac{p}{(t-x)^2}\right) d\lambda(x). \tag{6.3.3}$$

We can compute the integral in 6.3.3 using simple integration by parts. We let

$$u = \log\left(1 + \frac{p}{(t-x)^2}\right) \quad \text{and} \quad v = x - t,$$

so

$$du = \frac{2p}{(t-x)^3 + p(t-x)}dx \quad \text{and} \quad dv = dx.$$

Integration by parts yields

$$\begin{aligned}\int \log\left(1+\frac{p}{(t-x)^2}\right)d\lambda(x) &= \log\left(1+\frac{p}{(t-x)^2}\right)(x-t) \\ &\quad - \int (x-t)\frac{2p}{(t-x)^3+p(t-x)}d\lambda(x).\end{aligned}$$

Considering the remaining integral on the right-hand side and using the substitution $w = x - t$, we have

$$\int (x-t)\frac{2p}{(t-x)^3+p(t-x)}d\lambda(x) = -2p\int \frac{1}{p+w^2}d\lambda(w) = -2\sqrt{p}\arctan\left(\frac{t-x}{\sqrt{p}}\right).$$

Thus, we have shown

$$\int \log(J_{S_{(\beta,p,t)}}(x))d\lambda(x) = -2\sqrt{p}\arctan\left(\frac{t-x}{\sqrt{p}}\right) + \log\left(1+\frac{p}{(t-x)^2}\right)(x-t).$$

Evaluating the right-hand side from $-\infty$ to ∞ yields $h_{Kr}(S_{(\beta,p,t)}) = 2\pi\sqrt{p}$. Similarly, $h_{Kr}(S_{(\gamma,q,s)}) = 2\pi\sqrt{q}$. Thus, $h_{Kr}(S_{(\beta,p,t)}) = h_{Kr}(S_{(\gamma,q,s)})$ if and only if $p = q$. $\square$

6.3.1. 1-Isomorphisms Between Maps of Degree Two. We begin by noting that 1-isomorphism is the most natural extension of the concept of isomorphism between probability-preserving transformations. The following lemma shows that for finite-measure-preserving transformations there is only one possible c for c-isomorphisms.

LEMMA 6.3.2. *If $(X_1, \mathcal{B}_1, m_1, T_1)$ and $(X_2, \mathcal{B}_2, m_2, T_2)$ are such that both $m_1(X_1)$, $m_2(X_2) < \infty$, and $\phi : T_1 \to^c T_2$ is a c-isomorphism, then $c = \frac{m_1(X_1)}{m_2(X_2)}$. In particular, if $m_1(X_1) = m_2(X_2) = 1$, then $c = 1$.*

Proof. Let $\phi : T_1 \to^c T_2$ be a c-isomorphism. We have $m_1(X_1) = m_1(\phi^{-1}(X_2)) = c \cdot m_2(X_2)$. Therefore, $c = \frac{m_1(X_1)}{m_2(X_2)}$. $\square$

Now, motivated by Lemma 6.3.2, we return to infinite-measure-preserving transformations and investigate 1-isomorphism invariants for quadratic rational R-functions of negative type. Recall that Proposition 6.1.2 and Corollary 6.1.3 show that Krengel entropy is a 1-isomorphism invariant for rational R-functions of negative type. That is, if $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ are 1-isomorphic, then $h_{Kr}(S_{(\beta,p,t)}) = h_{Kr}(S_{(\gamma,q,s)})$. We are interested in a complete invariant for 1-isomorphisms of quadratic rational R-functions of negative type. The following theorem is the main result of this section.

Theorem 6.3.3. *The pair* $(|2t+\beta|, h_{Kr}(S_{(\beta,p,t)}))$ *is a complete invariant for* 1-*isomorphisms between quadratic rational R-functions of negative type in the form (6.3.1).*

Before we can prove Theorem 6.3.3 we must develop a few auxiliary results. It is convenient to change each $S_{(\beta,p,t)}$ into a partially normalized form, $S_{(\beta',p,0)}$ (a completely normalized form will be defined and used in Section 6.3.2).

Lemma 6.3.4 (Partially Normalized Form). *Given a quadratic rational R-function of negative type $S_{(\beta,p,t)}$, there exists a* 1-*isomorphism* $\psi_t : S_{(\beta,p,t)} \to^1 S_{(\beta',p,0)}$, *where* $\beta' = 2t + \beta$.

Proof. To partially normalize $S_{(\beta,p,t)}$ to $S_{(\beta',p,0)}$, we move the pole from t to 0 via conjugation by the map

$$\psi_t(x) = x - t \quad \text{and} \quad \psi_t^{-1}(x) = x + t. \tag{6.3.4}$$

Therefore, conjugating $S_{(\beta,p,t)}$ by ψ_t we have

$$\begin{aligned}(\psi_t \circ S_{(\beta,p,t)} \circ \psi_t^{-1})(x) &= -(x+t) - \beta - \frac{p}{t-(x+t)} - t \\ &= -x - (2t+\beta) - \frac{p}{-x} \\ &= S_{(2t+\beta,p,0)}(x) = S_{(\beta',p,0)}(x). \end{aligned} \tag{6.3.5}$$

We have that $J_{\psi_t^{-1}} = 1$, so ψ is a 1-isomorphism, and does not affect the entropy. $\square$

LEMMA 6.3.5. *Fix p and let $S_{(\beta',p,0)}$ and $S_{(\gamma',p,0)}$ be two partially normalized quadratic rational R-functions of negative type. If there exists a c-isomorphism, $\phi : S_{(\beta',p,0)} \to^c S_{(\gamma',p,0)}$, then $\phi(x) = \pm x$ for almost every $x \in \mathbb{R}$, and $c = 1$.*

PROOF. If $\phi : S_{(\beta',p,0)} \to^c S_{(\gamma',p,0)}$ is a c-isomorphism, then by definition we have

$$\phi \circ S_{(\beta',p,0)} = S_{(\gamma',p,0)} \circ \phi, \quad \text{for almost every } x \in \mathbb{R}. \tag{6.3.6}$$

By the chain rule, taking the Jacobian of both sides of (6.3.6) yields

$$J_\phi(S_{(\beta',p,0)}(x)) \cdot J_{S_{(\beta',p,0)}}(x) = J_{S_{(\gamma',p,0)}}(\phi(x)) \cdot J_\phi(x), \quad \text{for almost every } x \in \mathbb{R}. \tag{6.3.7}$$

By definition $J_{\phi^{-1}}(x) = c$ and $J_\phi(x) = \frac{1}{c}$ for almost every $x \in \mathbb{R}$, so (6.3.7) becomes

$$\frac{1}{c} \cdot J_{S_{(\beta',p,0)}}(x) = J_{S_{(\gamma',p,0)}}(\phi(x)) \cdot \frac{1}{c}, \quad \text{for almost every } x \in \mathbb{R}. \tag{6.3.8}$$

Finally, a simple cancellation yields

$$J_{S_{(\beta',p,0)}}(x) = J_{S_{(\gamma',p,0)}}(\phi(x)), \quad \text{for almost every } x \in \mathbb{R}. \tag{6.3.9}$$

Note that the constant doesn't affect the Jacobian (as in (6.3.2)), so we have $J_{S_{(\beta',p,0)}} = J_{S_{(\gamma',p,0)}}$. Thus, (6.3.9) is equivalent to

$$1 + \frac{p}{x^2} = 1 + \frac{p}{(\phi(x))^2}, \quad \text{for almost every } x \in \mathbb{R}. \tag{6.3.10}$$

Therefore, $(\phi(x))^2 = x^2$ for almost every $x \in \mathbb{R}$, so $\phi(x) = \pm x$ for almost every $x \in \mathbb{R}$. $\square$

PROPOSITION 6.3.6. *If $S_{(\beta',p,0)}$ and $S_{(\gamma',p,0)}$ are two partially normalized quadratic rational R-functions of negative type, then $S_{(\beta',p,0)}$ is 1-isomorphic to $S_{(\gamma',p,0)}$ if and only if $|\beta'| = |\gamma'|$. Furthermore, the isomorphism $\phi = \mathcal{I}$ almost everywhere or $\phi = -\mathcal{I}$ almost everywhere, where $\mathcal{I}$ denotes the identity map.*

PROOF. ($\Longrightarrow$) We assume there exists a c-isomorphism $\phi : S_{(\beta',p,0)} \to^c S_{(\gamma',p,0)}$ and first show $|\beta'| = |\gamma'|$. Then we show $\phi = \pm\mathcal{I}$ almost everywhere. By Lemma 6.3.5 $\phi(x) = \pm x$ for almost every $x \in \mathbb{R}$. Let $M \subseteq \mathbb{R}$ be a measurable set such that $\phi \circ S_{(\beta',p,0)} = S_{(\gamma',p,0)} \circ \phi$ and $\lambda(\mathbb{R} \setminus M) = 0$. Note that if $E \in \mathcal{B}$ is a set such that $\lambda(E) > 0$, then $\lambda(M \cap E) > 0$. That is, given any set of positive measure, E, we can find points in E where the conjugation (as in the definition of c-isomorphism) holds.

Define

$$\begin{aligned} A &= \{x \in \mathbb{R} : \phi(x) \neq \pm x\} \cup \{0\} \cup M^c, \\ B &= \{x \in M : \phi(x) = +x\}, \quad \text{and} \\ C &= \{x \in M : \phi(x) = -x\} \end{aligned} \tag{6.3.11}$$

We have $\lambda(\mathbb{R} \setminus (B \cup C)) = 0$ and $\lambda(A) = 0$.

Without loss of generality assume $\beta' \neq 0$ and $\lambda(B) > 0$.

CLAIM 1. $\lambda(S_{(\beta',p,0)}(B) \cap B) > 0$

To prove Claim 1 we show $\lambda(S_{(\beta',p,0)}(B)) > 0$ and both $\lambda(S_{(\beta',p,0)}(B) \cap A) = 0$ and $\lambda(S_{(\beta',p,0)}(B) \cap C) = 0$. Note $S_{(\beta',p,0)}$ is λ-preserving, so we have that $\lambda(B) \leq \lambda(S^{-1}_{(\beta',p,0)}(S_{(\beta',p,0)}(B))) = \lambda(S_{(\beta',p,0)}(B))$. That is, $\lambda(S_{(\beta',p,0)}(B)) > 0$.

By definition of A we have $\lambda(A) = 0$, so $\lambda(S_{(\beta',p,0)}(B) \cap A) = 0$. Now we show $\lambda(S_{(\beta',p,0)}(B) \cap C) = 0$. Suppose $x \in S_{(\beta',p,0)}(B) \cap C$. Then there exists $y \in B$ such that $S_{(\beta',p,0)}(y) = x \in C$. We know $y \in M$ and $S_{(\beta',p,0)}(y) \in M$, so by definition of the c-isomorphism we have

$$(\phi \circ S_{(\beta',p,0)})(y) = (S_{(\gamma',p,0)} \circ \phi)(y). \tag{6.3.12}$$

We have $S_{(\beta',p,0)}(y) \in C$, so

$$(\phi \circ S_{(\beta',p,0)})(y) = S_{(\beta',p,0)}(y) = y + \beta' + \frac{p}{-y}. \tag{6.3.13}$$

We also have $y \in B$, so

$$(S_{(\gamma',p,0)} \circ \phi)(y) = S_{(\gamma',p,0)}(y) = -y - \gamma' - \frac{p}{-y}. \tag{6.3.14}$$

Therefore, (6.3.12) implies the right-hand sides of (6.3.13) and (6.3.14) are equal, which implies

$$2y^2 + (\beta' + \gamma')y - 2p = 0. \tag{6.3.15}$$

There are at most two possible y's for which (6.3.15) is satisfied. Therefore, there are at most two points in $S_{(\beta',p,0)}(B) \cap C$, so $\lambda(S_{(\beta',p,0)}(B) \cap C) = 0$. Combining this with the work above proves Claim 1.

CLAIM 2. If $\lambda(S_{(\beta',p,0)}(B) \cap B) > 0$, then $\beta' = \gamma'$.

To prove Claim 2 we pick $x \in S_{(\beta',p,0)}(B) \cap B$. Then there exists a $y \in B$ such that $S_{(\beta',p,0)}(y) = x \in B$. We know $y \in M$ and $S_{(\beta',p,0)}(y) \in M$, so by definition of c-isomorphism we have

$$(\phi \circ S_{(\beta',p,0)})(y) = (S_{(\gamma',p,0)} \circ \phi)(y). \tag{6.3.16}$$

We have $S_{(\beta',p,0)}(y) \in B$, so

$$(\phi \circ S_{(\beta',p,0)})(y) = S_{(\beta',p,0)}(y) = -y - \beta' - \frac{p}{-y}. \tag{6.3.17}$$

We also have $y \in B$, so

$$(S_{(\gamma',p,0)} \circ \phi)(y) = S_{(\gamma',p,0)}(y) = -y - \gamma' - \frac{p}{-y}. \tag{6.3.18}$$

By (6.3.16) the right-hand sides of (6.3.17) and (6.3.18) are equal. Therefore, $\beta' = \gamma'$.

Claim 3. The isomorphism $\phi = \mathcal{I}$ almost everywhere.

We want to show $\lambda(\mathbb{R} \setminus B) = 0$, which is equivalent to $\lambda(C) = 0$. We show $\lambda(S_{(\beta',p,0)}(C)) = 0$, then use that $S_{(\beta',p,0)}$ is nonsingular with respect to λ. To show $\lambda(S_{(\beta',p,0)}(C)) = 0$ we show $\lambda(S_{(\beta',p,0)}(C) \cap A) = 0$, $\lambda(S_{(\beta',p,0)}(C) \cap B) = 0$, and $\lambda(S_{(\beta',p,0)}(C) \cap C) = 0$.

First, by definition of A we have $\lambda(S_{\beta',p,0}(C) \cap A) = 0$, because $\lambda(A) = 0$.

Now, let $y \in C$, and suppose that $S_{(\beta',p,0)}(y) \in B$. By Claim 2 we have $\beta' = \gamma'$. We also know $y \in M$ and $S_{(\beta',p,0)}(y) \in M$, so by the definition of c-isomorphism we have

$$(\phi \circ S_{(\beta',p,0)})(y) = (S_{(\beta',p,0)} \circ \phi)(y). \tag{6.3.19}$$

By (6.3.19) and an argument similar to that of Claims 1 and 2 we have

$$y - \beta' - \frac{1}{y} = -y - \beta' + \frac{1}{y}, \tag{6.3.20}$$

which implies $2y^2 - 2 = 0$. Thus, there exist at most two points in $S_{(\beta',p,0)}(C) \cap B$, so $\lambda(S_{(\beta',p,0)}(C) \cap B) = 0$.

Finally, let $y \in C$, and suppose that $S_{(\beta',p,0)}(y) \in C$. From Claim 2 we have $\beta' = \gamma'$, so again by an argument similar to that above we have

$$y - \beta' - \frac{1}{y} = y + \beta' - \frac{1}{y}. \tag{6.3.21}$$

Thus, $\beta' = 0$, but we assumed $\beta' \neq 0$ at the beginning. Therefore, $\lambda(S_{(\beta',p,0)}(C) \cap C) = 0$.

Therefore, $\lambda(S_{\beta',p,0}(C)) = 0$, which implies $\lambda(C) = 0$ by the nonsingularity of S with respect to λ. This completes the proof of Claim 3.

Finally, we note that if we had initially assumed $\lambda(C) > 0$ a similar argument would prove $\beta' = -\gamma'$ and $\phi = -\mathcal{I}$ almost everywhere.

($\Longleftarrow$) For the other direction, let $\mathcal{I}$ denote the identity map. If $\beta' = \gamma'$, then $\mathcal{I} : S_{(\beta',p,0)} \to S_{(\gamma',p,0)}$ is a 1-isomorphism. If $\beta' = -\gamma'$, then $-\mathcal{I} : S_{(\beta',p,0)} \to S_{(-\gamma',p,0)}$ is a 1-isomorphism. $\square$

We are now ready to prove Theorem 6.3.3 which provides a complete invariant for 1-isomorphisms of quadratic rational R-functions of negative type.

Proof of Theorem 6.3.3. ($\Longrightarrow$) First, we assume there exists a 1-isomorphism, $\phi : S_{(\beta,p,t)} \to^1 S_{(\gamma,q,s)}$, and we show $(|2t+\beta|, h_{Kr}(S_{(\beta,p,t)})) = (|2s+\gamma|, h_{Kr}(S_{(\gamma,q,s)}))$. By Lemma 6.3.1 we have that $h_{Kr}(S_{(\beta,p,t)}) = 2\pi\sqrt{p} = 2\pi\sqrt{q} = h_{Kr}(S_{(\gamma,q,s)})$, so $p = q$. For ease of notation we replace q with p and write $S_{(\gamma,p,s)} = S_{(\gamma,q,s)}$. By Lemma 6.3.4 we can partially normalize $S_{(\beta,p,t)}$ to $S_{(\beta',p,0)}$ and $S_{(\gamma,p,s)}$ to $S_{(\gamma',p,0)}$ via 1-isomorphisms ψ_t and ψ_s respectively. We have that $S_{(\beta',p,0)}$ is 1-isomorphic to $S_{(\gamma',p,0)}$ via $\phi = \psi_s \circ \phi \circ \psi_t^{-1}$ (see Figure 6.1). Thus, by Lemma 6.3.6 we have $|\beta'| = |\gamma'|$, where $\beta' = 2t + \beta$ and $\gamma' = 2s + \gamma$.

($\Longleftarrow$) For the reverse direction, we assume that $(|2t + \beta|, h_{Kr}(S_{(\beta,p,t)})) = (|2s + \gamma|, h_{Kr}(S_{(\gamma,q,s)}))$ and show that there exists a 1-isomorphism $\xi : S_{(\beta,p,t)} \to^1 S_{(\gamma,q,s)}$. Given $h_{Kr}(S_{(\beta,p,t)}) = h_{Kr}(S_{(\gamma,q,s)})$, by Lemma 6.3.1 we have that $p = q$. Furthermore, by Proposition 6.3.6 if $|2t+\beta| = |2s+\gamma|$, then there exists a 1-isomorphism, ξ, between the partially normalized forms $S_{(\beta',p,0)}$ and $S_{(\gamma',p,0)}$. Therefore, $\xi = \psi_s^{-1} \circ \phi \circ \psi_t : S_{(\beta,p,t)} \to^1 S_{(\gamma,q,s)}$ is a 1-isomorphism (see Figure 6.1).

$\square$

6.3.2. c-Isomorphisms Between Maps of Degree Two. In this section we investigate the possible c-isomorphisms (where $c \neq 1$) between two quadratic rational R-functions of negative type. The approach is similar to that for 1-isomorphisms. In

$$\begin{tikzcd}
S_{(\beta,p,t)} \arrow[r, "\xi"] \arrow[d, "\psi_t"] & S_{(\gamma,q,s)} \arrow[d, "\psi_s"] \\
S_{(\beta',p,0)} \arrow[r, "\phi"] & S_{(\gamma',q,0)}
\end{tikzcd}$$

FIGURE 6.1. A commutative diagram of 1-isomorphisms.

this case, however, there is the extra caveat that there may exist a c-isomorphism $\phi : S_{(\beta,p,t)} \to^c S_{(\gamma,q,s)}$, but $h_{Kr}(S_{(\beta,p,t)}) \neq h_{Kr}(S_{(\gamma,q,s)})$. The following result is the main theorem of this section.

THEOREM 6.3.7. *Two quadratic rational maps $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ are c-isomorphic if and only if*

$$c = \frac{h_{Kr}(S_{(\beta,p,t)})}{h_{Kr}(S_{(\gamma,q,s)})} \quad \text{and} \quad \left|\frac{2t+\beta}{\sqrt{p}}\right| = \left|\frac{2s+\gamma}{\sqrt{q}}\right|. \tag{6.3.22}$$

The proof of Theorem 6.3.7 uses a completely normalized form of quadratic rational R-functions of negative type.

LEMMA 6.3.8 (Completely Normalized Form). *Given a quadratic rational R-function of negative type, $S_{(\beta,p,t)}$, there exists a $\sqrt{p}$-isomorphism to a completely normalized form $S_{(\hat{\beta},1,0)}$, where $\hat{\beta} = \frac{2t+\beta}{\sqrt{p}}$.*

PROOF. To completely normalize $S_{(\beta,p,t)}$ to $S_{(\hat{\beta},1,0)}$ we first conjugate by ψ_t to partially normalize $S_{(\beta,p,t)}$ to $S_{(\beta',p,0)}$. We then change the multiplier p to 1 via conjugation by the following map

$$\zeta_p(x) = \frac{x}{\sqrt{p}} \quad \text{and} \quad \zeta_p^{-1}(x) = \sqrt{p} \cdot x. \tag{6.3.23}$$

That is, conjugating $S_{(\beta',p,0)}$ by ζ_p yields

$$\begin{aligned}
(\zeta_p \circ S_{(\beta',p,0)} \circ \zeta_p^{-1})(x) &= \frac{1}{\sqrt{p}}\left(-\sqrt{p}x - (2t+\beta) - \frac{p}{-\sqrt{p}x}\right) \\
&= -x - \frac{(2t+\beta)}{\sqrt{p}} - \frac{1}{-x} \\
&= -x - \hat{\beta} - \frac{1}{-x} = S_{(\hat{\beta},1,0)}(x). \qquad (6.3.24)
\end{aligned}$$

We note that $\zeta_p : S_{(\beta',p,0)} \to S_{(\hat{\beta},1,0)}$ is a $\sqrt{p}$-isomorphism. Thus, $\zeta_p \circ \psi_t : S_{(\beta,p,t)} \to S_{(\hat{\beta},1,0)}$ is also a $\sqrt{p}$-isomorphism.

$\square$

We are now ready to prove Theorem 6.3.7.

PROOF OF THEOREM 6.3.7. ($\Longrightarrow$) First, we assume there exists a c-isomorphism, $\xi : S_{(\beta,p,t)} \to^{c} S_{(\gamma,q,s)}$, and we show $c = \frac{h_{Kr}(S_{(\beta,p,t)})}{h_{Kr}(S_{(\gamma,q,s)})}$ and $\left|\frac{2t+\beta}{\sqrt{p}}\right| = \left|\frac{2s+\gamma}{\sqrt{q}}\right|$. By Proposition 6.1.2 we have $h_{Kr}(S_{(\beta,p,t)}) = c \cdot h_{Kr}(S_{(\gamma,q,s)})$. Therefore, $c = \frac{h_{Kr}(S_{(\beta,p,t)})}{h_{Kr}(S_{(\gamma,q,s)})}$. To show $\left|\frac{2t+\beta}{\sqrt{p}}\right| = \left|\frac{2s+\gamma}{\sqrt{q}}\right|$ we transform $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ to their completely normalized forms. That is,

$$\zeta_p \circ \psi_t : S_{(\beta,p,t)} \to^{\sqrt{p}} S_{(\hat{\beta},1,0)} \quad \text{and} \quad \zeta_q \circ \psi_s : S_{(\gamma,q,s)} \to^{\sqrt{q}} S_{(\hat{\gamma},1,0)}, \tag{6.3.25}$$

where $\hat{\beta} = \frac{2t+\beta}{\sqrt{p}}$ and $\hat{\gamma} = \frac{2s+\gamma}{\sqrt{q}}$ Thus, there exists a map $\phi = \zeta_q \circ \psi_s \circ \xi \circ \psi_t^{-1} \circ \zeta_p^{-1} : S_{(\hat{\beta},1,0)} \to^{\hat{c}} S_{(\hat{\gamma},1,0)}$ (see Figure 6.2). By Lemma 6.3.5 $\hat{c} = 1$, so by Theorem 6.3.3 $|\hat{\beta}| = |\hat{\gamma}|$.

($\Longleftarrow$) We now assume that $\left|\frac{2t+\beta}{\sqrt{p}}\right| = \left|\frac{2s+\gamma}{\sqrt{q}}\right|$ and show there exists a c-isomorphism, $\xi : S_{(\beta,p,t)} \to^c S_{(\gamma,q,s)}$, where $c = \frac{h_{Kr}(S_{(\beta,p,t)})}{h_{Kr}(S_{(\gamma,q,s)})}$. We transform $S_{(\beta,p,t)}$ and $S_{(\gamma,q,s)}$ to their completely normalized forms. Thus, we are now interested in $\hat{c}$-isomorphisms between $S_{(\hat{\beta},1,0)}$ and $S_{(\hat{\gamma},1,0)}$. By Theorem 6.3.3 there exists a 1-isomorphism $\phi : S_{(\hat{\beta},1,0)} \to^1 S_{(\hat{\gamma},1,0)}$, because $|\hat{\beta}| = |\hat{\gamma}|$. Therefore, $\xi = \psi_s^{-1} \circ \zeta_q^{-1} \circ \phi \circ \zeta_p \circ \psi_t : S_{(\beta,p,t)} \to^c S_{(\gamma,q,s)}$ and $c = \frac{\sqrt{p}}{\sqrt{q}} = \frac{h_{Kr}(S_{(\beta,p,t)})}{h_{Kr}(S_{(\gamma,q,s)})}$ (see Figure 6.2).

$$
\begin{tikzcd}
S_{(\beta,p,t)} \arrow[r, "\xi"] \arrow[d, "\psi_t"] & S_{(\gamma,q,s)} \arrow[d, "\psi_s"'] \\
S_{(\beta',p,0)} \arrow[d, "\zeta_p"] & S_{(\gamma',q,0)} \arrow[d, "\zeta_q"'] \\
S_{(\hat{\beta},1,0)} \arrow[r, "\phi"] & S_{(\hat{\gamma},1,0)}
\end{tikzcd}
$$

FIGURE 6.2. A commutative diagram of c-isomorphisms.

$\square$

6.4. Examples from Complex Dynamics

Let R denote a rational map on the Riemann sphere, $\mathbb{C}_\infty$. The following elementary theorem can be found in [**Bea**].

THEOREM 6.4.1. *Every rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ has infinitely many periodic points.*

Some rational maps, however, may lack periodic points of certain periods. The following theorem due to Baker describes the the possible combinations for degree of the rational map and the missing periodic points.

THEOREM 6.4.2 ([**Bak**]). *Let R be a rational map of degree d, where $d \geq 2$, and suppose that R has no periodic points of period n. Then (d, n) is one of the pairs*

$$(2, 2), \quad (2, 3), \quad (3, 2), \quad (4, 2).$$

Moreover, there exists an R corresponding to each pair.

In [**Hag**] Hagihara classified the rational maps (up to conformal conjugacy) which correspond to the pairs in the previous theorem and proved the following result for the $(2, 2)$ case.

THEOREM 6.4.3. *A rational map, R, of degree 2 lacks period 2 orbits if and only if R is conformally conjugate to a member of the one-parameter family*

$$R_a(z) = \frac{z^2 - z}{1 + az}, \quad \text{where } a \in \mathbb{C} \setminus \{1\}.$$

DEFINITION 6.4.4. The *Fatou set* of a rational map, R, is the maximal open subset of the Riemann sphere on which the iterates, $\{R^n\}$, are equicontinuous. We denote the Fatou set of R by $F(R)$. The *Julia set*, $J(R)$, is the complement of the Fatou set.

The following proposition can be found in [**Hag**].

Proposition 6.4.5. *For $a < -1$, $J(R_a) = \mathbb{R} \cup \{\infty\}$.*

We are primarily interested in one-dimensional maps, so we study the dynamics of $R_a|_{J(R_a)}$ when $a < -1$. That is, from now on we consider $R_a : \mathbb{R} \to \mathbb{R}$.

Lemma 6.4.6. *For $a < -1$, R_a preserves the measure ν, where $d\nu(x) = \frac{1}{x^2} d\lambda(x)$.*

Proof. By Lemma 2.3.3 we know $\mathcal{L}_{R_a}(f(y)) = f(y)$ if and only if ν defined by $d\nu = f d\lambda$ is invariant for R_a. Let $f(x) = \frac{1}{x^2}$. If $x_+(y)$ and $x_-(y)$ are solutions to $R_a(x) = y$, then

$$\mathcal{L}_{R_a} f(y) = f(x_+(y)) \cdot |x'_+(y)| + f(x_-(y)) \cdot |x'_-(y)| \tag{6.4.1}$$

We compute both $x_+(y)$ and $x_-(y)$ and obtain

$$\begin{aligned} x_+(y) &= \frac{1}{2}\left(1 + ay + \sqrt{1 + 4y + 2ay + a^2y^2}\right), \quad \text{and} \\ x_-(y) &= \frac{1}{2}\left(1 + ay - \sqrt{1 + 4y + 2ay + a^2y^2}\right). \end{aligned} \tag{6.4.2}$$

Computing derivatives we have

$$\begin{aligned} x'_+(y) &= \frac{1}{2}\left(a + \frac{4 + 2a + 2a^2y}{2\sqrt{1 + 4y + 2ay + a^2y^2}}\right), \quad \text{and} \\ x'_-(y) &= \frac{1}{2}\left(a - \frac{4 + 2a + 2a^2y}{2\sqrt{1 + 4y + 2ay + a^2y^2}}\right). \end{aligned} \tag{6.4.3}$$

Finally, a routine calculation shows $\mathcal{L}_{R_a}(f(y)) = \frac{1}{y^2}$. $\square$

6.4.1. Connection to Rational R-Functions of Negative Type. Define

$$T_a(x) = -x - (a+1) - \frac{-(a+1)}{1-x}. \tag{6.4.4}$$

We note that T_a is a quadratic rational R-function of negative type.

Lemma 6.4.7. *The system $(\mathbb{R}, \mathcal{B}, \nu, R_a)$ is isomorphic to $(\mathbb{R}, \mathcal{B}, \lambda, T_a)$.*

PROOF. Let $\phi(x) = \frac{1}{x}$ and $\phi^{-1}(x) = \frac{1}{x}$. We have

$$(\phi \circ R_a \circ \phi^{-1})(x) = \frac{ax + x^2}{1 - x} = -x - (a+1) - \frac{-(a+1)}{1-x} = T_a(x). \tag{6.4.5}$$

Now, we need to show that $\nu \circ \phi^{-1} = \lambda$. Computing the Radon-Nikodym derivative using the chain rule yields

$$\frac{d\nu}{d\lambda}(\phi^{-1}(x)) \cdot J_{\phi^{-1}}(x) = \frac{1}{1/x^2} \cdot \frac{1}{x^2} = 1 = \frac{d\lambda}{d\lambda}. \tag{6.4.6}$$

□

DEFINITION 6.4.8. Let R be a rational map on $\mathbb{C}$. To each fixed point, ω, of R, we associate a *multiplier*, $\mathfrak{m}$, defined by

$$\mathfrak{m}(R, \omega) = \begin{cases} R'(\omega) & \text{if } \omega \neq \infty \\ 1/R'(\infty) & \text{if } \omega = \infty, \end{cases} \tag{6.4.7}$$

where $R'(\infty) = \lim_{z \to \infty} R'(z)$.

The following theorem due to Milnor also gives necessary and sufficient conditions for a quadratic rational map to lack period 2 orbits.

THEOREM 6.4.9 ([**Mil**]). *A rational map of degree 2 lacks period 2 orbits if and only if one of its fixed points has multiplier -1.*

We know ∞ is a fixed point with multiplier -1 for all rational R-functions of negative type. Therefore, as a consequence of Theorem 6.4.9 we have that any quadratic rational R-function of negative type is conformally conjugate to an R_a and therefore a T_a when viewed as a map of the Riemann sphere. Furthermore, the following lemma produces an explicit c-isomorphism between the boundary function associated to a quadratic rational R-function of negative type and the boundary function of a T_a.

Before stating the lemma, we set some notation. Let $b = -(1+a)$, then

$$T_a = H_b = -x + b - \frac{b}{1-x}, \quad \text{where } b \in (0,2]. \tag{6.4.8}$$

Without loss of generality from now on we consider the family $\{H_b\}$ where $b \in [0,2]$.

LEMMA 6.4.10. *A quadratic rational R-function of negative type, $S_{(\beta,p,t)} = -x - \beta - \frac{p}{t-x}$, is c-isomorphic to an H_b for some $b \in (0,2]$. The isomorphism $\phi : S_{(\beta,p,t)} \to^c H_b$ is given by*

$$\chi_K(x) = \frac{K-t+x}{K}, \quad \textit{where } K = \frac{1}{4}\left(\beta + 2t \pm \sqrt{(\beta+2t)^2 + 8p}\right). \tag{6.4.9}$$

Letting $K_+ = \frac{1}{4}\left(\beta + 2t + \sqrt{(\beta+2t)^2+8p}\right)$ and $K_- = \frac{1}{4}\left(\beta + 2t - \sqrt{(\beta+2t)^2+8p}\right)$ yields the following two cases for the isomorphism ϕ:

(1) *If $\beta \geq -2t$, then $\chi_{K_+}(x) = \frac{K_+ - t + x}{K_+}$.*
(2) *If $\beta < -2t$, then $\chi_{K_-}(x) = \frac{K_- - t + x}{K_-}$.*

PROOF. Define $\zeta_{K^2}(x) = \frac{x}{K}$ and $\zeta_{K^2}^{-1}(x) = Kx$ as in the proof of Lemma 6.3.8. We have that $S_{(\beta,p,t)}$ is K-isomorphic to

$$S_{(\beta/K, p/K^2, t/K)} = -x - \frac{\beta}{K} - \frac{p/K^2}{t/K - x} \tag{6.4.10}$$

via $\zeta_K(x)$. Now, define $\psi_{t/K-1}(x) = x - (t/K - 1)$ and $\psi_{t/K-1}^{-1}(x) = x + (t/K-1)$ as in the proof of Lemma 6.3.4. We have $S_{(\beta/K,p/K^2,t/K)}$ is 1-isomorphic to

$$S_{(2(t/K-1)+\beta/K, p/K^2, 1)} = -x - (2(t/K-1) + \beta/K) - \frac{p/K^2}{1-x} \tag{6.4.11}$$

via the isomorphism $\psi_{t/K-1}$. We want to show $S_{(2(t/K-1)+\beta/K,p/K^2,1)} = H_b$ for some $b \in (0,2]$. Thus, we solve for K, by setting

$$-(2(t/K-1) + \beta/K) = p/K^2. \tag{6.4.12}$$

A calculation shows

$$K = \frac{1}{4}\left(\beta + 2t \pm \sqrt{(\beta+2t)^2 + 8p}\right). \tag{6.4.13}$$

Now, we need to check that $-(2(t/K-1)+\beta/K) = p/K^2 \in (0,2]$. Note,

$$\frac{p}{K^2} = \frac{16p}{(\beta+2t \pm \sqrt{(\beta+2t)^2+8p})^2}, \tag{6.4.14}$$

so $p/K^2 > 0$. Suppose $\beta \geq -2t$, then $\beta + 2t \geq 0$. Therefore,

$$\frac{p}{K_+^2} = \frac{16p}{(\beta+2t + \sqrt{(\beta+2t)^2+8p})^2} \leq \frac{16p}{(\sqrt{8p})^2} = 2. \tag{6.4.15}$$

Now, suppose $\beta < -2t$, then $\beta + 2t < 0$. Therefore,

$$\frac{p}{K_-^2} = \frac{16p}{(\beta+2t - \sqrt{(\beta+2t)^2+8p})^2} < \frac{16p}{(\sqrt{8p})^2} = 2. \tag{6.4.16}$$

Thus, we have shown $S_{(2(t/K-1)+\beta/K, p/K^2, 1)} = H_b$ where $b \in (0,2]$, so $\chi_K = \psi_{t/K-1} \circ \zeta_{K^2} : S_{(\beta,p,t)} \to^K H_b$ is a K-isomorphism. □

Lemma 6.4.11. *Two transformations H_b and $H_{b'}$ of the form (6.4.8) where $b, b' \in (0,2]$ are c-isomorphic if and only if $b = b'$.*

Proof. By Theorem 6.3.7 we know H_b is c-isomorphic to $H_{b'}$ if and only if $\left|\frac{2-b}{\sqrt{b}}\right| = \left|\frac{2-b'}{\sqrt{b'}}\right|$. Therefore, $b = b'$ or $b = \frac{4}{b'}$. By assumption $b, b' \in (0,2]$, so $b = b'$. □

Let $[S_{(\beta,p,t)}]$ denote the c-isomorphism class of $S_{(\beta,p,t)}$. That is,

$$[S_{(\beta,p,t)}] = \{S_{(\gamma,q,s)} : S_{(\gamma,q,s)} \text{ is } c\text{-isomorphic to } S_{(\beta,p,t)}\}. \tag{6.4.17}$$

Theorem 6.4.12. *The set of of c-isomorphism classes of quadratic rational R-functions of negative type are in one-to-one correspondence with the interval* $(0,2]$.

Proof. Let $S_{(\beta,p,t)}$ be a representative of the c-isomorphism class $[S_{(\beta,p,t)}]$. By Lemma 6.4.10 we have $S_{(\beta,p,t)}$ is c-isomorphic to an H_b where $b \in (0,2]$. Furthermore,

b is unique, because if there exists a c'-isomorphism from $S_{(\beta,p,t)}$ to $H_{b'}$ with $b' \in (0,2]$, then H_b is $\frac{c'}{c}$-isomorphic to $H_{b'}$. Therefore, by Lemma 6.4.11 we have $b = b'$. $\square$

6.5. Isomorphism Invariants for Maps of Degree Three

In this section we consider cubic rational R-functions of negative type, that is,

$$S(x) = -x - \beta - \frac{p_1}{t_1 - x} - \frac{p_2}{t_2 - x}, \tag{6.5.1}$$

where $\beta, p_i, t_i \in \mathbb{R}$ and $p_i > 0$ for $i = 1, 2$.

We present results for the subset of cubic rational R-functions of negative type where $p_1 = p_2$ and $t_1 = -t_2$.

Lemma 6.5.1. *If $S(x) = -x - \beta - \frac{p}{(t-x)} - \frac{p}{-t-x}$, then*

$$h_{Kr}(S) = 2\pi\left(\sqrt{p - t^2 - \sqrt{p^2 - 4pt^2}} + \sqrt{p - t^2 + \sqrt{p^2 - 4pt^2}}\right). \tag{6.5.2}$$

Proof. By Theorem 5.2.1 we know

$$h_{Kr}(S) = \int_{\mathbb{R}} \log(J_S(x))d\lambda(x) = \int_{\mathbb{R}} \log\left(1 + \frac{p}{(t-x)^2} + \frac{p}{(-t-x)^2}\right) d\lambda(x). \tag{6.5.3}$$

We first get a closed form for the indefinite integral $\int \log\left(1 + \frac{p}{(t-x)^2} + \frac{p}{(-t-x)^2}\right) dx$. We integrate by parts letting $u = \log\left(1 + \frac{p}{(t-x)^2} + \frac{p}{(-t-x)^2}\right)$ and $v = x$, so (6.5.3) becomes

(6.5.4)
$$x\log\left(1 + \frac{p}{(t-x)^2} + \frac{p}{(-t-x)^2}\right) - \int \frac{4px^3(3t^2 + x^2)}{(t-x)(t+x)((t^2-x^2)^2 + 2p(t^2+x^2))}dx.$$

Considering the integral in (6.5.4), we use partial fractions to obtain

$$\int \left[\frac{-2t}{-t+x} + \frac{2t}{t+x} - \frac{4(2pt^2 + t^4 + px^2 - t^2x^2)}{2pt^2 + t^4 + 2px^2 - 2t^2x^2 + x^4}\right] dx. \tag{6.5.5}$$

Considering the integral of the first two pieces of (6.5.5) we have

$$\int \left[\frac{-2t}{-t+x} + \frac{2t}{t+x}\right] dx = -2t\log(-t+x) + 2t\log(t+x). \tag{6.5.6}$$

Now, we consider the third term of (6.5.5). A calculation shows

$$\frac{4(2pt^2+t^4+px^2-t^2x^2)}{2pt^2+t^4+2px^2-2t^2x^2+x^4} = \frac{2(p-t^2+\sqrt{p^2-4pt^2})}{p-t^2+\sqrt{p^2-4pt^2}+x^2} + \frac{2(p-t^2-\sqrt{p^2-4pt^2})}{p-t^2-\sqrt{p^2-4pt^2}+x^2}. \tag{6.5.7}$$

Using the fact that $\int \frac{a^2}{a^2+x^2} dx = a\arctan\left(\frac{x}{a}\right)$, we have

$$\begin{aligned} &\int \frac{2(p-t^2+\sqrt{p^2-4pt^2})}{p-t^2+\sqrt{p^2-4pt^2}+x^2} dx \\ &= 2\sqrt{p-t^2+\sqrt{p^2-4pt^2}}\arctan\left(\frac{x}{\sqrt{p-t^2+\sqrt{p^2-4pt^2}}}\right) \end{aligned} \tag{6.5.8}$$

and

$$\begin{aligned} &\int \frac{2(p-t^2-\sqrt{p^2-4pt^2})}{p-t^2-\sqrt{p^2-4pt^2}+x^2} dx \\ &= 2\sqrt{p-t^2-\sqrt{p^2-4pt^2}}\arctan\left(\frac{x}{\sqrt{p-t^2-\sqrt{p^2-4pt^2}}}\right). \end{aligned} \tag{6.5.9}$$

Combining the work in (6.5.4), (6.5.6), (6.5.8), and (6.5.9) gives

$$\begin{aligned} \int \log|J_S(x)|dx &= x\log\left(1+\frac{p}{(t-x)^2}+\frac{p}{(t+x)^2}\right) + 2t\log(-t+x) - 2t\log(t+x) \\ &+ 2\sqrt{p-t^2-\sqrt{p^2-4pt^2}}\arctan\left(\frac{x}{\sqrt{p-t^2-\sqrt{p^2-4pt^2}}}\right) \\ &+ 2\sqrt{p-t^2+\sqrt{p^2-4pt^2}}\arctan\left(\frac{x}{\sqrt{p-t^2+\sqrt{p^2-4pt^2}}}\right). \end{aligned} \tag{6.5.10}$$

Evaluating the right-hand side of (6.5.10) from $-\infty$ to ∞ yields

(6.5.11)
$$\int_{\mathbb{R}} \log|J_S(x)|d\lambda(x) = 2\pi\left(\sqrt{p-t^2-\sqrt{p^2-4pt^2}}+\sqrt{p-t^2+\sqrt{p^2-4pt^2}}\right),$$

which proves the result. □

LEMMA 6.5.2. *Any cubic rational R-function of negative type, $S(x) = -x - \beta - \frac{p}{r-x} - \frac{p}{s-x}$, where $\beta, p, r, s \in \mathbb{R}$ and $p > 0$ is 1-isomorphic to a cubic rational R-function of negative type of the form $\tilde{S}(x) = -x - \beta' - \frac{p}{t-x} - \frac{p}{-t-x}$, where $\beta, p, t \in \mathbb{R}$ and $p > 0$.*

PROOF. Define $\psi_{\frac{r+s}{2}}(x) = x - \frac{r+s}{2}$, so $\psi^{-1}_{\frac{r+s}{2}}(x) = x + \frac{r+s}{2}$. We have

$$\begin{aligned}
(\psi_{\frac{r+s}{2}} \circ S \circ \psi^{-1}_{\frac{r+s}{2}})(x) &= -x - \frac{r+s}{2} - \beta - \frac{p}{\left(s-\frac{(r+s)}{2}\right)-x} - \frac{p}{\left(r-\frac{(r+s)}{2}\right)-x} - \frac{r+s}{2} \\
&= -x - (r+s+\beta) - \frac{p}{\frac{s-r}{2}-x} - \frac{p}{\frac{r-s}{2}-x} \\
&= -x - \beta' - \frac{p}{t-x} - \frac{p}{-t-x}, \qquad (6.5.12)
\end{aligned}$$

where $\beta' = (r+s+\beta)$ and $t = \frac{s-r}{2}$. □

LEMMA 6.5.3. *Let $S_1(x) = -x-\beta-\frac{p}{(t-x)}-\frac{p}{-t-x}$ and $S_2(x) = -x-\gamma-\frac{q}{(s-x)}-\frac{q}{-s-x}$. Suppose $h_{Kr}(S_1) = h_{Kr}(S_2)$. We have the following two results:*

(1) *If $q = p$, then $s = \pm t$.*
(2) *If $s = t$, then $q = p$ or $q = p + 2t^2 + 2\sqrt{2pt^2 + t^4}$.*

PROOF. We begin by proving (1). Suppose $p = q$. That is, $S_1(x) = -x - \beta - \frac{p}{(t-x)} - \frac{p}{-t-x}$ and $S_2(x) = -x - \gamma - \frac{p}{(s-x)} - \frac{p}{-s-x}$. Assuming $h_{Kr}(S_1) = h_{Kr}(S_2)$, then by Lemma 6.5.1 we have

$$\begin{aligned}
&\sqrt{p-t^2-\sqrt{p^2-4pt^2}}+\sqrt{p-t^2+\sqrt{p^2-4pt^2}} \\
&= \sqrt{p-s^2-\sqrt{p^2-4ps^2}}+\sqrt{p-s^2+\sqrt{p^2-4ps^2}}. \qquad (6.5.13)
\end{aligned}$$

Squaring both sides of (6.5.13) and simplifying yields

$$s^2 - t^2 + \sqrt{2pt^2 + t^4} = \sqrt{2ps^2 + s^4}. \tag{6.5.14}$$

Squaring both sides of (6.5.14) and simplifying yields

$$2pt^2 - 2s^2t^2 + 2t^4 + 2s^2\sqrt{2pt^2 + t^4} - 2t^2\sqrt{2pt^2 + t^4} - 2ps^2s^2 = 0. \tag{6.5.15}$$

We have that (6.5.15) factors as

$$-2(p + t^2 - \sqrt{2pt^2 + t^4})(s^2 - t^2) = 0. \tag{6.5.16}$$

Then, solving for s shows that $s = \pm t$.

We now show (2) by a similar argument. Suppose $t = s$. That is, $S_1(x) = -x - \beta - \frac{p}{(t-x)} - \frac{p}{-t-x}$ and $S_2(x) = -x - \gamma - \frac{q}{(t-x)} - \frac{q}{-t-x}$. Assuming $h_{Kr}(S_1) = h_{Kr}(S_2)$, then again by Lemma 6.5.1 we have

$$\begin{aligned} &\sqrt{p - t^2 - \sqrt{p^2 - 4pt^2}} + \sqrt{p - t^2 + \sqrt{p^2 - 4pt^2}} \\ &= \sqrt{q - t^2 - \sqrt{q^2 - 4pt^2}} + \sqrt{q - t^2 + \sqrt{q^2 - 4qt^2}}. \end{aligned} \tag{6.5.17}$$

Squaring both sides of (6.5.17) and simplifying yields

$$p - q + \sqrt{2pt^2 + t^4} = \sqrt{2pt^2 + t^4}. \tag{6.5.18}$$

Squaring both sides of (6.5.18) and simplifying yields

$$p^2 - 2pq + q^2 + 2pt^2 + 2p\sqrt{2pt^2 + t^4} - 2q\sqrt{2pt^2 + t^4} - 2qt^2 = 0. \tag{6.5.19}$$

We have that (6.5.19) factors as

$$(q - p)\left(q - (p + 2t^2 + 2\sqrt{2pt^2 + t^4})\right) = 0. \tag{6.5.20}$$

Then, solving for q shows $q = p$ or $q = p + 2t^2 + 2\sqrt{2pt^2 + t^4}$. $\square$

Corollary 6.5.4. *Let* $S_1(x) = -x - \beta - \frac{p}{(t-x)} - \frac{p}{-t-x}$ *and* $S_2(x) = -x - \gamma - \frac{q}{(s-x)} - \frac{q}{-s-x}$.

(1) *If* S_1 *and* S_2 *are* 1*-isomorphic and* $p = q$*, then* $t = \pm s$.

(2) *If* S_1 *and* S_2 *are* 1*-isomorphic and* $t = s$*, then* $q = p$ *or* $q = p + 2t^2 + 2t\sqrt{2p + t^2}$.

CHAPTER 7

FUTURE WORK

7.1. Exactness of Negative R-Functions

In this section lay the framework for studying transformations which are not necessarily rational. Let F be the boundary function associated to an R-function of positive type such that

$$F(x) = x + \beta + \int_R \frac{d\nu(t)}{t-x}, \tag{7.1.1}$$

where $\beta \in \mathbb{R}$ and ν is finite, singular, and compactly supported on $\mathbb{R}$. Recall from Chapter 3 that Aaronson proved Theorem 3.4.3 which says F is exact.

We study the negatives of such maps. That is, we consider transformations

$$H(x) = -x - \beta - \int_R \frac{d\nu(t)}{t-x}, \tag{7.1.2}$$

where $\beta \in \mathbb{R}$ and ν is finite, singular, and compactly supported on $\mathbb{R}$. Recall that Letac proved that maps of the form (7.1.2) preserve λ ([**Let**]). We are interested in studying the measure-theoretic properties of these transformations.

We first prove the following lemma which relates maps of the form (7.1.2) to boundary functions associated to an R-functions of positive type.

LEMMA 7.1.1. *If H is a transformation of the form (7.1.2), then H^2 is the boundary function associated to an R-function of positive type.*

PROOF. Suppose $H(x) = -x - \beta - \int_{\mathbb{R}} \frac{d\nu(t)}{t-x}$. Let $F_{\beta,\nu}$ be the boundary function associated to an R-function of positive type such that $F_{\beta,\nu}(x) = x + \beta + \int_{\mathbb{R}} \frac{d\nu(t)}{t-x}$. That is, $F_{\beta,\nu} = -H$. Let $\hat{\nu}$ be a measure on $\mathbb{R}$ such that $\hat{\nu}(A) = \nu(-A)$. We have $F_{-\beta,\hat{\nu}}$ is an R-function, and $F_{-\beta,\hat{\nu}}(x) = x - \beta + \int_{\mathbb{R}} \frac{d\hat{\nu}(t)}{t-x}$. We can write H^2 as the composition of two boundary functions associated to R-functions of positive type. That is,

$$H^2 = F_{-\beta,\hat{\nu}} \circ F_{\beta,\nu},$$

so H^2 is the boundary function associated to an R-function of positive type. $\square$

Now that we have H^2 is the boundary function associated to an R-function of positive type, our goal is to show that H^2 satisfies the conditions of Theorem 3.4.3 which would prove the following conjecture.

CONJECTURE 7.1.1. *If $H(x) = -x - \beta - \int_{\mathbb{R}} \frac{d\nu(t)}{t-x}$, where $\beta \in \mathbb{R}$ and ν is finite, singular, and compactly supported on $\mathbb{R}$, then H is exact with respect to λ on $\mathbb{R}$.*

In the previous sections we our work focused on the case when H is rational. The results did not use the motivation and theory coming from harmonic analysis, so there was no need for a notational distinction between the map on $\mathbb{R}$ as opposed to $\mathbb{C}$. In this section, however, we will use the theory developed in Chapter 3. From now on, we will write H^2 is the boundary function of an R-function of positive type, $h^2 : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$. From Theorem 3.3.9 we have

$$h^2(z) = \beta + \alpha z + \int_{\mathbb{R}} \frac{1+tz}{t-z} d\mu(t), \tag{7.1.3}$$

where $\beta, \alpha \in \mathbb{R}$ and μ is finite, singular, and compactly supported on $\mathbb{R}$.

We prove the following lemma, which allows us to apply Lemma 3.3.6 to h^2 to obtain $h^2 - \mathcal{I} - \beta \in R_0$ where $\mathcal{I}$ denotes the idenity function.

LEMMA 7.1.2. *If $g(z) = h^2(z) - z$, then $\lim_{y\to\infty} iyg(iy) = c < \infty$.*

PROOF. Given $h(z) = -z - \beta - \int_{\mathbb{R}} \frac{d\nu(t)}{t-z}$ we have

$$h^2(z) = z + \int_{\mathbb{R}} \frac{d\nu(t)}{t-z} - \int_{\mathbb{R}} \frac{d\nu(t)}{t-h(z)}. \tag{7.1.4}$$

Letting $g(z) = h^2(z) - z$ we have

$$g(z) = \int_{\mathbb{R}} \frac{d\nu(t)}{t-z} - \int_{\mathbb{R}} \frac{d\nu(t)}{t-h(z)}. \tag{7.1.5}$$

We evaluate g at iy to obtain

$$g(iy) = \int_{\mathbb{R}} \frac{d\nu(t)}{t-iy} - \int_{\mathbb{R}} \frac{d\nu(t)}{t+iy+\beta+\int_{\mathbb{R}} \frac{d\nu(t)}{t-iy}}. \tag{7.1.6}$$

Rationalizing the denominator of $\int_{\mathbb{R}} \frac{d\nu(t)}{t-iy}$ yields $\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2} + i\int_{\mathbb{R}} \frac{yd\nu(t)}{t^2+y^2}$, so (7.1.6) becomes

$$\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2} + i\int_{\mathbb{R}} \frac{yd\nu(t)}{t^2+y^2} + \int_{\mathbb{R}} \frac{d\nu(t)}{t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2} + iy\left(\int_{\mathbb{R}} \frac{yd\nu(t)}{t^2+y^2}\right)}. \tag{7.1.7}$$

Rationalizing the denominator of the third integral in (7.1.6) yields

(7.1.8)

$$\int_{\mathbb{R}} \frac{d\nu(t)}{t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2} + iy\left(\int_{\mathbb{R}} \frac{yd\nu(t)}{t^2+y^2}\right)} = \int_{\mathbb{R}} \frac{\left(\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right) - iy\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)d\nu(t)}{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)^2 + \left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2}.$$

Therefore, (7.1.7) equals

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2} + \int_{\mathbb{R}} \frac{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)d\nu(t)}{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)^2 + \left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2} \\
&+ i\left(\int_{\mathbb{R}} \frac{yd\nu(t)}{t^2+y^2} - \int_{\mathbb{R}} \frac{y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2y^2}\right)d\nu(t)}{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)^2 + \left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2}\right).
\end{aligned} \tag{7.1.9}$$

Therefore,

$$\begin{aligned}
iyg(iy) = iy&\left(\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2} + \int_{\mathbb{R}} \frac{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)d\nu(t)}{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)^2 + \left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2}\right) \\
- y^2&\left(\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2} - \int_{\mathbb{R}} \frac{\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2y^2}\right)d\nu(t)}{\left(t+\beta+\int_{\mathbb{R}} \frac{td\nu(t)}{t^2+y^2}\right)^2 + \left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2}\right).
\end{aligned} \tag{7.1.10}$$

It is clear that the imaginary part of (7.1.10) tends to 0 as $y \to \infty$. For the real part, we have

$$\int_{\mathbb{R}} \frac{y^2 d\nu(t)}{t^2+y^2} \leq \int_{\mathbb{R}} d\nu(t) = C, \tag{7.1.11}$$

and

$$\begin{aligned} y^2 \int_{\mathbb{R}} & \frac{\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2y^2}\right) d\nu(t)}{\left(t+\beta+\int_{\mathbb{R}} \frac{t d\nu(t)}{t^2+y^2}\right)^2 + \left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2} \\ & \leq y^2 \int_{\mathbb{R}} \frac{\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2y^2}\right) d\nu(t)}{\left(y\left(1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}\right)\right)^2} = \int_{\mathbb{R}} \frac{d\nu(t)}{1+\int_{\mathbb{R}} \frac{d\nu(t)}{t^2+y^2}} \\ & \leq \int_{\mathbb{R}} d\nu(t) = C. \end{aligned} \tag{7.1.12}$$

□

Outline for Proof of Conjecture 7.1.1. The idea is to show H^2 is exact, then appeal to Lemma 4.2.2 to show that H is also exact. Let $g(z) = h^2(z) - z$. By Lemma 7.1.2 we know $\lim_{y\to\infty} iyg(iy) = c < \infty$. This convergence allows us to appeal to Lemma 3.3.6 to show that $g \in R_0$ as in Definition 3.3.5. That is,

$$g(z) = h(z) - z = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}, \tag{7.1.13}$$

so

$$H^2(x) = x + \int_{\mathbb{R}} \frac{d\mu(t)}{t-x}, \tag{7.1.14}$$

where μ is a finite measure on $\mathbb{R}$. The goal is to apply Theorem 3.4.3 to show that H^2 is exact. The question still remains whether or not the new measure μ is singular and compactly supported on $\mathbb{R}$.

7.2. Other Work

7.2.1. Direct Computation of Parry Entropy. We know that all rational R-functions of negative type are quasi-finite (Theorem 4.6.6), so the Parry entropy is equal to the Krengel entropy. By Theorem 5.2.1 the Krengel (and therefore Parry) entropy can be computed using Rohlin's formula. There are, however, no known examples where the Parry entropy has been computed directly.

QUESTION 7.2.1. Can the Parry entropy of a boundary function associated to a rational R-function of negative type be computed without passing through the Krengel entropy?

7.2.2. Higher-Dimensional Transformations. In [**PBGP**] it was shown that a two-dimensional Boole mapping of the form $F(x, y) = (x - \frac{1}{y}, y - \frac{1}{x})$ is ergodic with respect to the invariant product measure $d\mu(x, y) = dxdy$. This result motivates the following questions along these lines.

QUESTION 7.2.2. Are two-dimensional Boole mappings of the form $F(x, y) = (x - \frac{1}{y}, y - \frac{1}{x})$ also conservative and exact?

QUESTION 7.2.3. Can we obtain the same result as [**PBGP**] for two-dimensional negative Boole mappings? In other words, are maps of the form $S(x, y) = (-x + \frac{1}{y}, -y + \frac{1}{x})$ ergodic with respect to the invariant product measure $d\mu(x, y) = dxdy$? Does a similar statement hold for two-dimensional negative generalized Boole transformations (i.e. increasing the number of poles)? If so, are they also exact and conservative?

The following conjecture on higher-dimensional maps was also made in [**PBGP**], which raises the question about whether the above questions can be answered in an n-dimensional setting.

CONJECTURE 7.2.1. *If σ is any element of the permutation group Σ_n, $n \in \mathbb{Z}_+$, then a generalized Boole transformation of the form*

$$F(x_1, x_2, ..., x_n) = \left(x_1 - \frac{1}{x_{\sigma(1)}}, x_2 - \frac{1}{x_{\sigma(2)}}, ..., x_n - \frac{1}{x_{\sigma(n)}} \right)$$

is ergodic with respect to the measure $d\mu(x_1, x_2, ..., x_n) = dx_1 dx_2 ... dx_n$.

References

[Aar1] Jon Aaronson. Rational ergodicity and a metric invariant for Markov shifts. *Israel J. Math.*, 27(2):93–123, 1977.

[Aar2] Jon Aaronson. Ergodic theory for inner functions of the upper half plane. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 14(3):233–253, 1978.

[Aar3] Jon Aaronson. A remark on the exactness of inner functions. *J. London Math. Soc. (2)*, 23(3):469–474, 1981.

[Aar4] Jon Aaronson. *An Introduction to Infinite Ergodic Theory*. American Mathematical Society, 1997.

[AP] Jon Aaronson and Kyewon Koh Park. Predictability, entropy and information of infinite transformations. *Fund. Math.*, 206:1–21, 2009.

[AW] Roy L. Adler and Benjamin Weiss. The ergodic infinite measure preserving transformation of Boole. *Israel J. Math.*, 16:263–278, 1973.

[Bak] I. N. Baker. Fixpoints of polynomials and rational functions. *J. London Math. Soc.*, 39:615–622, 1964.

[Bea] Alan F. Beardon. *Iteration of Rational Functions*. Springer-Verlag, 175 Fifth Avenue, New York, NY 10010, 1991.

[BGT] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

[Con] John B. Conway. *Functions of One Complex Vkreiariable. II*, volume 159 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[EH] Stanley Eigen and Jane Hawkins. Examples and properties of nonexact ergodic shift measures. *Indag. Math. (N.S.)*, 10(1):25–44, 1999.

[GG] M. L. Gorbachuk and V. I. Gorbachuk. *M. G. Krein's Lectures on Entire Operators*, volume 97 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[Hag] Rika Hagihara. *Rational maps lacking certain periodic orbits*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–The University of North Carolina at Chapel Hill.

[Haw] J. M. Hawkins. Amenable relations for endomorphisms. *Trans. Amer. Math. Soc.*, 343(1):169–191, 1994.

[HT] Olga Holtz and Mikhail Tyaglov. Structured matrices, continued fractions, and root localization of polynomials. *SIAM Review*, 54(3):421–509, 2012.

[Hur] Witold Hurewicz. Ergodic theorem without invariant measure. *Ann. of Math. (2)*, 45:192–206, 1944.

[JMRdlR] Élise Janvresse, Tom Meyerovitch, Emmanuel Roy, and Thierry de la Rue. Poisson suspensions and entropy for infinite transformations. *Trans. Amer. Math. Soc.*, 362(6):3069–3094, 2010.

[Kar] J. Karamata. Sur un mode de croissance régulière. Théorèmes fondamentaux. *Bull. Soc. Math. France*, 61:55–62, 1933.

[KK] I. S. Kac and M. G. Kreĭn. Criteria for the discreteness of the spectrum of a singular string. *Izv. Vysš. Učebn. Zaved. Matematika*, 1958(2 (3)):136–153, 1958.

[Kre] Ulrich Krengel. Entropy of conservative transformations. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:161–181, 1967.

[Let] Gérard Letac. Which functions preserve Cauchy laws? *Proc. Amer. Math. Soc.*, 67(2):277–286, 1977.

[LS] Tien Yien Li and Fritz Schweiger. The generalized Boole's transformation is ergodic. *Manuscripta Math.*, 25(2):161–167, 1978.

[Mah] D. Maharam. Incompressible transformations. *Fund. Math.*, 56:35–50, 1964.

[Mil] John Milnor. Geometry and dynamics of quadratic rational maps. *Experiment. Math.*, 2(1):37–83, 1993. With an appendix by the author and Lei Tan.

[Par] William Parry. *Entropy and Generators in Ergodic Theory.* W.A. Benjamin, Inc., New York, NY 10016, 1969.

[PBGP] Yarema Prykarpatsky, Denis Blackmore, Jolanta Golenia, and Anatoliy Prykarpatsky. Invariant measures for discrete dynamical systems and ergodic properties of generalized Boole type transformations. *arXiv:1210.1746v1 [math.DS]*, Oct. 4, 2012.

[Pet] Karl E. Petersen. *Ergodic Theory.* Cambridge University Press, 40 West 20th St, New York, NY 10011, 1983.

[PS] George Pólya and Gabor Szegő. *Problems and Theorems in Analysis. I.* Classics in Mathematics. Springer-Verlag, Berlin, 1998. Series, integral calculus, theory of functions, Translated from the German by Dorothee Aeppli, Reprint of the 1978 English translation.

[PY] Mark Pollicott and Michiko Yuri. *Dynamical Systems and Ergodic Theory*, volume 40 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1998.

[Rem] Reinhold Remmert. *Theory of complex functions*, volume 122 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the second German edition by Robert B. Burckel, Readings in Mathematics.

[Rén] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar*, 8:477–493, 1957.

[Roh1] V. A. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Translation*, 1952(71):55, 1952.

[Roh2] V. A. Rohlin. Exact endomorphisms of a lebesgue space. *Amer. Math. Soc. Trans.*, 36(1):1–36, 1964.

[Roy] Emmanuel Roy. Poisson suspensions and infinite ergodic theory. *Ergodic Theory Dynam. Systems*, 29(2):667–683, 2009.

[RR] Marvin Rosenblum and James Rovnyak. *Topics in Hardy Classes and Univalent Functions*. Birkhauser Advanced Texts, P.O. Box 133, CH - 4010 Basel, Switzerland, 1994.

[Tha] Maximilian Thaler. Transformations on [0, 1] with infinite invariant measures. *Israel J. Math.*, 46(1-2):67–96, 1983.

[Tsu] M. Tsuji. *Potential Theory in Modern Function Theory*. Chelsea Publishing Co., New York, 1975. Reprinting of the 1959 original.

[Wal] Peter Walters. *An Introduction to Ergodic Theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[Wei] Joachim Weidmann. *Linear Operators in Hilbert Spaces*. Springer-Verlag, 175 Fifth Avenue, New York, NY 10010, 1980.

[Yur] Michiko Yuri. Multi-dimensional maps with infinite invariant measures and countable state sofic shifts. *Indag. Math. (N.S.)*, 6(3):355–383, 1995.