Dynamical Properties of some non-stationary, non-simple Bratteli-Vershik systems

by
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ABSTRACT

SARAH BAILEY: Dynamical Properties of some non-stationary, non-simple Bratteli-Vershik systems

(Under the direction of Professor Karl Petersen)

Bratteli-Vershik systems, also called adics, are dynamical systems defined on the infinite path space of a Bratteli diagram. We introduce a family adics called limited scope adics for which the number of vertices increases by a constant at each level, and a subfamily determined by positive integer polynomials. We show that the dimension groups of the Bratteli diagrams associated to limited scope adics are intrinsically linked to the dynamics, generalizing a result for Cantor minimal systems, and we explicitly compute them for the subfamily of adics determined by a positive integer polynomials. We show that certain limited scope adics are isomorphic to subshifts. For the systems determined by positive integer polynomials we show that the set of fully-supported invariant ergodic probability measures consists of a one-parameter family of Bernoulli measures. We also show that the systems associated to positive integer polynomials are loosely Bernoulli. A particular limited scope adic system is the Euler adic system, for which the number of paths from the root vertex to a vertex (n,k) is the Eulerian number A(n,k). We show that this system has a unique fully-supported invariant ergodic probability measure and that the system is totally ergodic and loosely Bernoulli.

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CHAPTER 1

Introduction

In this introduction we will give a brief history of Bratteli-Vershik systems and state our main results. Sections 1.1 and 1.2 present necessary background.

Ergodic theory is the study of long-term statistical behavior of evolving systems. Symbolic dynamics is the study of evolving systems using discrete time and space viewpoints. Some systems are studied from a measure-theoretic viewpoint, while others from a strictly topological viewpoint. The basis for the systems of interest in this thesis originated in operator theory, in classification of almost finite (AF) C^*-algebras. In [**5**] Bratteli introduced diagrams which have come to be known as Bratteli diagrams to describe unital almost finite C^*-algebras. Two of these C^*-algebras are isomorphic if and only if their associated diagrams are diagram equivalent (definition given later). These diagrams are infinite graded directed graphs beginning with a single root vertex. The vertices are partitioned into levels, $\mathcal{V}_n$; each level contains finitely many vertices, and edges connect vertices in consecutive levels.

In [**13**] Elliott introduced the notion of a dimension group as an invariant for equivalent Bratteli diagrams and hence for the C^*-algebras that they represent. The dimension group is calculated as the following direct limit:

$$\mathbb{Z}^{|\mathcal{V}_0|=1} \xrightarrow{\phi_1} \mathbb{Z}^{|\mathcal{V}_1|} \xrightarrow{\phi_2} \mathbb{Z}^{|\mathcal{V}_2|} \xrightarrow{\phi_3} \ldots$$

where ϕ_n are the matrices for which $[\phi_n]_{ij}$ = the number of edges connecting vertex $(n-1, j)$ and (n, i), see [**23, 20, 21**]. There is a positive set consisting of the equivalence classes which have a positive vector representative. Elliott introduced the distinguished order unit as the equivalence class of the element 1. Two AF C^*-algebras are isomorphic is their associated dimension groups are order isomorphic (the positive sets are preserved) and the distinguished order units are preserved.

In [**46, 45, 44, 47**], Vershik associated to these diagrams dynamical systems, which we henceforth call Bratteli-Vershik or adic systems. The space X is the set of infinite paths which begin at the root, these paths are given a partial order, and the map T is defined to map a path to its successor (when possible). The maps associated to these systems are commonly referred to as Bratteli-Vershik or adic transformations and will be denoted throughout this thesis as T. Vershik showed that every ergodic, measure-preserving transformation on a Lebesgue space is measure-theoretically isomorphic to a uniquely ergodic adic transformation. These systems provide a concise combinatorial method of representing cutting and stacking transformations of the unit interval.

While's Vershik's original construction dealt with measure-preserving transformations, others took a topological viewpoint. In [**23**], Herman, Putnam, and Skau showed that every Cantor minimal system (the space has countable basis of closed and open sets and there are no proper closed transformation invariant sets) topologically conjugate to a special class of Bratteli-Vershik systems known as properly ordered Bratteli-Vershik systems. This class of Bratteli-Vershik systems has been studied in great detail. Fewer examples of adic transformations that are not properly ordered are found in the literature, and much less is known about them. In this thesis we study a family $\mathcal{S}_{\mathcal{L}}$ of

Bratteli-Vershik systems (of "limited scope")that are not properly ordered but still have structure limited in a particular way, as well as a subfamily of $\mathcal{S}_{\mathcal{L}}$ consisting of systems associated to polynomials and thus preserving a high degree of regularity. We will denote this subfamily by $(\mathcal{S}_{\mathcal{L}})_{p(x)}$. We explore some of the results for Cantor minimal systems and prove analogous results for Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$. For the class of properly ordered Bratteli-Vershik systems the transformations are homeomorphisms. This is not the case for the systems in $\mathcal{S}_{\mathcal{L}}$. In fact, the adic transformations are not even continuous.

In [**20**], Giordano, Putnam and Skau drew connections between the dynamics and the C^*-algebra theory by showing that for a Cantor minimal system the dimension group introduced by Elliott is order isomorphic to the dynamical group of continuous functions from the Cantor set into the integers ($C(X,\mathbb{Z})$) modulo the coboundaries ($\partial_T C(X,\mathbb{Z})$). Since for systems in $\mathcal{S}_{\mathcal{L}}$ the transformation T need not be everywhere continuous, the coboundary $f - f \circ T$ generated by a function $f \in C(X,\mathbb{Z})$ need not be in $C(X,\mathbb{Z})$. But we can make a modification which allows us to extend this result for a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$.

THEOREM 2.2.7. *For $(X,T) \in \mathcal{S}_{\mathcal{L}}$ with underlying Bratteli diagram $(\mathcal{V},\mathcal{E})$, there is an order isomorphism between the dimension group of $(\mathcal{V},\mathcal{E})$ and $C(X,\mathbb{Z})/(\partial_T C(X,\mathbb{Z}) \cap C(X,\mathbb{Z}))$ which maps the distinguished order unit of the dimension group to the equivalence class of the constant function 1.*

Dimension groups are direct limits and in general are difficult to compute. In [**12**] Durand, Host, and Skau compute dimension groups for stationary Bratteli diagrams. In [**29**] Kwiatkowski and Wata give a general method of computing the dimension groups of Cantor minimal systems and do so for some examples. We have been able to compute dimension groups for $(\mathcal{S}_{\mathcal{L}})_{p(x)}$. This subfamily of $\mathcal{S}_{\mathcal{L}}$ consists of Brattel-Vershik systems for which the number of finite paths from the root to a vertex (n, k) is given by the coefficient of x^k in the n'th power of a positive integer polynomial $p(x)$. The most well-known of these is the Pascal adic transformation, where $p(x) = 1 + x$.

Theorem 3.3.1. *The dimension group associated to the Bratteli diagram determined by a positive integer polynomial $p(x)$ is order isomorphic to the ordered group $G_{p(x)}$ of rational functions of the form*

$$\frac{r(x)}{p(x)^m},$$

where $r(x)$ is any polynomial with integer coefficients such that $\deg(r(x)) \leq md$. Addition of two elements is given by

$$\frac{r(x)}{p(x)^m} + \frac{s(x)}{p(x)^l} = \frac{r(x) + s(x)p(x)^{m-l}}{p(x)^m}$$

if $l \leq m$. The positive set $(G_{p(x)})_+$ consists of the elements of $G_{p(x)}$ such that there is an l for which the numerator of

$$\frac{r(x)(p(x))^l}{p(x)^{l+m}}$$

has all positive coefficients. The distinguished order unit of the $G_{p(x)}$ is the constant polynomial 1.

These results are topological in nature, but we have also examined the members of $\mathcal{S}_{\mathcal{L}}$ in a measure-theoretic context. We begin by identifying all the ergodic, invariant, probability measures for the Bratteli-Vershik systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$. For the Pascal adic transformation these have been computed in various places: see [**22, 47, 38, 33, 34**] and the references they contain. In [**32**] Méla determined the ergodic, invariant probability measures for Bratteli-Vershik systems associated to polynomials for which all coefficients are 1. Using similar techniques as [**32**], we have been able to extend this result.

THEOREM 3.2.4. *Let* $p(x) = a_0 + \cdots + a_d x^d$ *and let* $(X_{p(x)}, T_{p(x)})$ *be the Bratteli-Vershik system determined by* $p(x)$. *If* $q \in (0, \frac{1}{a_0})$ *and* t_q *is the unique solution in* $[0,1]$ *of the equation*

$$a_0 q^d + a_1 q^{d-1} t + \ldots + a_d t^d - q^{d-1} = 0,$$

then the invariant, fully supported, ergodic probability measures for the adic transformation $T_{p(x)}$ *are the one-parameter family of Bernoulli measures of the form*

$$\mathcal{B}\left(\underbrace{q, \ldots, q}_{a_0 \text{ times}}, \underbrace{t_q, \ldots, t_q}_{a_1 \text{ times}}, \underbrace{\frac{t_q^2}{q}, \ldots, \frac{t_q^2}{q}}_{a_2 \text{ times}}, \ldots, \underbrace{\frac{t_q^n}{q^{n-1}}, \ldots, \frac{t_q^n}{q^{n-1}}}_{a_n \text{ times}}\right).$$

The question of weak mixing for the Pascal adic was originally posed by Vershik [**45**]. This question is still open, but we can answer the question of weak mixing for other Bratteli-Vershik systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by polynomials of degree 1 with positive integer coefficients.

COROLLARY 3.5.2. *Let* $(X_{p(x)}, T_{p(x)})$ *be the Bratteli-Vershik system determined by the polynomial* $a_0 + a_1 x$ *with a fully-supported,* $T_{p(x)}$*-invariant, ergodic probability measure* μ. *If either* a_0 *or* a_1 *is greater than 1, then* $T_{p(x)}$ *is not weakly mixing.*

In [**17**] it was proved that the set of stationary properly ordered Bratteli-Vershik systems is the disjoint union of the family of the minimal substitution systems and the family of stationary odometers. In [**12**] Durand, Host, and Skau reproved this result in a constructive manner. Their construction is in a topological setting. The path spaces in $\mathcal{S}_{\mathcal{L}}$ are compact metric spaces, but as mentioned earlier, the adic transformations are not continuous. Because of this, the questions of topological weak mixing, topological strong mixing, topological entropy, and complexity cannot be formulated coherently and so we are motivated to find a coding of these systems for which such questions may be posed and determine the implications of such properties on the original Bratteli-Vershik system.

THEOREM 2.3.7. *For a Bratteli-Vershik system $(X, T) \in \mathcal{S}_L$ such that the number of vertices increases by 1 at each level, with a fully-supported invariant ergodic probability measure μ, there are a set $X' \subset X$ of measure 0, and a one-to-one Borel measurable map ϕ from $X \setminus X'$ into a subshift space Σ on a finite alphabet such that $\phi \circ T = \sigma \circ \phi$ on $X \setminus X'$. Furthermore, (Σ, σ) is constructible directly from a coding of the vertices of the Bratteli diagram.*

Another system in $\mathcal{S}_{\mathcal{L}}$ is the Euler adic, for which the number of paths from the root vertex into any vertex (n, k) is the Eulerian number $A(n, k)$, which is the number of permutations of $\{1, 2, \ldots, n+1\}$ with exactly k rises and $n-k$ falls. Besides their obvious combinatorial importance, these numbers are also of interest in connection with the statistics of rankings: see, for example, [**8, 14, 19, 18**], and [**36**]. In studying random permutations, it is often assumed that all permutations are equally likely, each permutation of length $n+1$ occurring with probability $1/(n+1)!$. The following result can be interpreted as saying that if any two permutations of the same length which *have the same number of rises* are equally likely, and if every permutation has positive probability, then in fact *all* permutations of the same length are equally likely. The *symmetric* *measure*, η, is determined by assigning weights $1/(n+2)$ to each edge connecting level n to level $n+1$. Theorem 4.3.4 is joint work with Keane, Petersen, and Salama in [**1, 2**].

THEOREM 4.3.4. *The symmetric measure η is ergodic and is the only T-invariant ergodic Borel probability measure with full support for the Euler adic transformation.*

We also initiate investigation of the dynamical properties of the Euler adic system with its unique fully-supported ergodic invariant measure.

THEOREM 4.4.5. *The Bratteli-Vershik system determined by the Euler graph with the symmetric measure η is totally ergodic.*

The property of loosely Bernoulli was introduced by Feldman in [**15**] as well as by Katok and Sataev in [**26**]. In [**35**] Ornstein, Rudolph, and Weiss systematically studied the property of loosely Bernoulli and Kakutani equivalence for measure-preserving transformations. The systems of limited scope have zero entropy and a zero entropy transformation is loosely Bernoulli if and only if it is isomorphic to a transformation induced by an irrational rotation. In [**25**], Janvresse and de la Rue showed that the Pascal adic is loosely Bernoulli. In [**33**] Méla showed that every Bratteli-Vershik system determined by a polynomial for which all coefficients are 1 is loosely Bernoulli. We extend these results to the family $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ and to the Euler adic.

THEOREM 3.4.3. *The Bratteli-Vershik systems $(X_{p(x)}, T_{p(x)})$ determined by positive integer polynomials are loosely Bernoulli with respect to each of their $T_{p(x)}$-invariant ergodic probability measures.*

THEOREM 4.5.2. *The Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ determined by the Euler graph and with the symmetric measure is loosely Bernoulli.*

1.1. General Concepts, Definitions, and Notation

To establish terminology, notation, and context we list well-known definitions that will be needed in the following. For more background on ergodic theory see [**37, 48**], on probability see [**16, 3**], on C^*-algebras and their dimension groups see [**40, 23, 12, 20**].

Definition 1.1.1. A *dynamical system*, denoted (X, ϕ), consists of a set X and a transformation $\phi : X \to X$.

A *topological dynamical system* consists of a compact metric space and a homeomorphism on the space. A *measure-theoretic dynamical system* consists of a measure space X and a measure-preserving transformation on X. Sometimes the transformation ϕ is not required to be one-to-one, or onto, or defined everywhere. See [**7**] for more background. In this thesis we will be dealing with a measure-preserving transformation that is invertible, but not always continuous.

Definition 1.1.2. Let $(X, \mathcal{B}, \mu)$ be a measure space. A transformation $T : X \to X$ is said to be a *measure-preserving transformation* on X if for every $B \in \mathcal{B}$, $\mu(T^{-1}(B)) = \mu(B)$.

Definition 1.1.3. Let (X, ϕ) be a dynamical system. For $x \in X$, the orbit of x, denoted $\mathcal{O}_\phi(x)$, is $\{\phi^i(x) | i \in \mathbb{Z}\}$. If (X, ϕ) is a topological dynamical system, denote by $\overline{\mathcal{O}_\phi}(x)$ the closure of the orbit of x.

Definition 1.1.4. Let (X, ϕ) be a topological dynamical system. A point $x_0 \in X$ is called a *transitive point* if $\overline{\mathcal{O}_\phi}(x_0) = X$.

Definition 1.1.5. If (X, ϕ) is a topological dynamical system and $x_0 \in X$ is a transitive point, then the triple (X, ϕ, x_0) is called a *pointed topological dynamical system* with *distinguished transitive point* x_0.

Definition 1.1.6. Two pointed dynamical systems (X, ϕ, x_0) and (X', ϕ', x_0') are said to be *pointedly topological conjugate* if they are topologically conjugate with a topological conjugacy $h : X \to X$ that takes x_0 to x_0'.

Definition 1.1.7. Let (X, ϕ) be a dynamical system. Then a set Z in X is said to be *ϕ-invariant* if $\phi(Z) \subseteq Z$.

There are many variants of this definition: $\phi(Z) \subset Z$, $\phi^{-1}(Z) \subset Z$, $\phi^{-1}(Z) = Z$. In this thesis we will assume $\phi(Z) \subseteq Z$.

Definition 1.1.8. Let (X, ϕ) be a topological dynamical system. Then a set Z in X is said to be *minimal* if it is closed, ϕ-invariant, nonempty and minimal among such sets (with respect to set inclusion). A topological dynamical system (X, ϕ) is called a *minimal system* if there are no proper minimal sets.

Definition 1.1.9. A topological dynamical system (X, ϕ) is *essentially minimal* if it has a unique minimal set.

Definition 1.1.10. A *Cantor set* is a set whose topology has a countable basis of sets that are both open and closed. A *Cantor system* is a dynamical system for which the space is a Cantor set. A *Cantor minimal system* is a topological dynamical system for which is both a Cantor system and a minimal system.

DEFINITION 1.1.11. Two topological dynamical systems (X, ϕ) and (X', ϕ') are said to be *topologically conjugate* if there exists a homeomorphism $h : X \to X'$ such that $h \circ \phi = \phi' \circ h$. The homeomorphism h is called a *topological conjugacy*.

DEFINITION 1.1.12. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretic dynamical system with $\mu(X) < \infty$, and let $B \subset \mathcal{B}$ be such that $\mu(B) > 0$. Define the integer $n_B(x) = \inf\{n \geq 1 | T^n x \in B\}$. Then define the *induced transformation* $T_B : B \to B$ by

$$T_B(x) = T^{n_B(x)}(x).$$

DEFINITION 1.1.13. Let $(X, \mathcal{B}, \mu)$ be a probability space. A *finite measurable partition* of X is a collection $\mathcal{P} = \{P_1, \ldots, P_n\}$ of measurable sets such that $\mu(\cup_{i=1}^n P_i) = 1$ and for all $0 \leq i < j \leq n$, $\mu(P_i \cap P_j) = 0$.

DEFINITION 1.1.14. Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be two finite measurable partitions of X. Their *join* is the partition

$$\mathcal{P} \vee \mathcal{Q} = \{P_i \cap Q_j | 1 \leq i \leq n, q \leq j \leq m\}.$$

DEFINITION 1.1.15. Let $(\mathcal{B}_n)$ be a sequence of σ-algebras. $(\mathcal{B}_n)$ is said to increase to the σ-algebra $\mathcal{B}$, denoted $\mathcal{B}_n \nearrow \mathcal{B}$, if $\mathcal{B}$ is the smallest σ-algebra which contains $\cup_{i=1}^\infty \mathcal{B}_i$.

DEFINITION 1.1.16. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretic dynamical system. For any finite measurable partition $\mathcal{P}$ of X, let $\mathcal{F}(\mathcal{P})$ denote the σ-algebra generated by $\mathcal{P}$.

We say the partition P is *generating* if

$$\mathcal{F}\left(\bigvee_{i=-n}^{n} T^i P\right) \nearrow \mathcal{B}.$$

DEFINITION 1.1.17. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretic dynamical system and $\mathcal{P}$ a finite measurable partition of X. Define a function on X, also denoted by $\mathcal{P}$, by $\mathcal{P}(x) = j$ for all $x \in P_j$, $j = 1, \ldots, m$. For each $n = 0, 1, 2, \ldots$, the *$\mathcal{P}$-n-name* of x is the finite block on the symbols $1, \ldots, m$

$$\mathcal{P}_0^n(x) = \mathcal{P}(x)\mathcal{P}(Tx)\ldots\mathcal{P}(T^n x),$$

and the *$\mathcal{P}$-name* of x is the doubly infinite sequence

$$\mathcal{P}_{-\infty}^{\infty}(x) = \ldots\mathcal{P}(T^{-2}x)\mathcal{P}(T^{-1}x).\mathcal{P}(x)\mathcal{P}(Tx)\mathcal{P}(T^2x)\ldots$$

which is defined for almost every x in X.

DEFINITION 1.1.18. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretic dynamical system. A complex number λ is an *eigenvalue* of T if there exists a nonzero function $f \in \mathcal{L}^2(\mu)$ such that $f(Tx) = \lambda f(x)$ μ-almost everywhere. Such a function is called an *eigenfunction* corresponding to λ.

DEFINITION 1.1.19. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretic dynamical system. T is *weakly mixing* if for every $A, B \in \mathcal{B}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0.$$

1.2. Bratteli-Vershik Systems

In this section we give more precise definitions relating to Bratteli-Vershik systems and set up new machinery and notation. We postpone for now the precise definitions and machinery related to the dimension group.

DEFINITION 1.2.1. A *Bratteli diagram*, denoted $(\mathcal{V}, \mathcal{E})$, is an infinite directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ with the following properties:

1. $\mathcal{V}$ and $\mathcal{E}$ are each the union of countably many pairwise disjoint finite sets:

$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2...$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \ldots$, with $|\mathcal{V}_n| < \infty$, $|\mathcal{E}_n| < \infty$ for all n.

2. $\mathcal{V}_0$ consists of a single element v_0, called the *root*.

3. If $r : \mathcal{E} \to \mathcal{V}$ is the associated range map, and $s\ E \to \mathcal{V}$ is the associated source map, then $r(\mathcal{E}_n) = \mathcal{V}_n$ and $s(\mathcal{E}_n) = \mathcal{V}_{n-1}$ for all $n = 1, 2, \ldots$. Furthermore, $s^{-1}\{v\}$ is nonempty for all $v \in \mathcal{V}$ and $r^{-1}\{v\}$ is non-empty for all $v \in \mathcal{V} \setminus \mathcal{V}_0$.

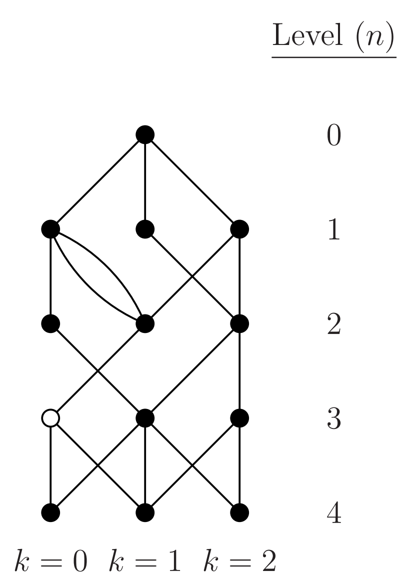


FIGURE 1.1. Levels 0-4 of a Bratteli diagram. The hollow vertex is denoted (3,0).

For notational purposes for each set $\mathcal{V}_n$ the vertices are numbered 0 through $|V_n|-1$. The diagrams are then drawn with the vertices in ascending numerical order from left to right. We will let a specific vertex $v \in \mathcal{V}$ be denoted by (n,k) whenever $v \in \mathcal{V}_n$ and is the k'th vertex in $\mathcal{V}_n$. We will refer to a vertex (n,k) as being on *level* n, see Figure 1.1. With this notation Definition 1.2.1 (3) says that every vertex has at least one edge connecting it to the level below, and every vertex except $(0,0)$ also has at least one edge connecting it to the level above.

Associated to any Bratteli diagram $(\mathcal{V},\mathcal{E})$ is a sequence of *incidence matrices.* For any pair of consecutive levels $n-1$ and n, the incidence matrix D_n describes the range and source of $\mathcal{E}_n$. In particular D_n is a $|\mathcal{V}_n|\times|\mathcal{V}_{n-1}|$ matrix such that $[D_n]_{i,j}$ is the number of edges connecting vertices $(n-1,j)$ and (n,i). For example, for the diagram in Figure 1.1 we have

$$D_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The process of *telescoping* Bratteli diagrams involves condensing the levels in such a way that the total number of paths connecting the remaining levels remains unchanged. The precise definition follows.

Definition 1.2.2. Let $(\mathcal{V},\mathcal{E})$ be a Bratteli diagram. For $k=0,1,\ldots,$ $k<l=1,2,\ldots,$ define $\mathcal{E}_{k,l}$ to be the set of edge paths from $\mathcal{V}_k$ to $\mathcal{V}_l$. Let $m_1, m_2, \ldots$ be an increasing sequence in $\mathbb{N}$ and define another Bratteli diagram $(\mathcal{V}',\mathcal{E}')$ by setting $\mathcal{V}'_0 = \mathcal{V}_0$, and, for $n=1,2,\ldots,$ $\mathcal{V}'_n = \mathcal{V}_{m_n}$ and $\mathcal{E}'_n = \mathcal{E}_{m_{n-1},m_n}$. Then $(\mathcal{V}',\mathcal{E}')$ is called a

telescoping of $(\mathcal{V}, \mathcal{E})$. Then the incidence matrices D'_n for $(\mathcal{V}', \mathcal{E}')$ are given by $D'_n = D_{m_n} D_{m_n - 1} \dots D_{m_{n-1}+1}$.

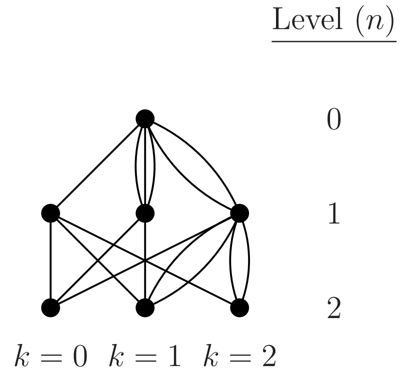


FIGURE 1.2. Telescoping of the Bratteli diagram in Figure 1.1 to the second and fourth levels.

Two Bratteli diagrams $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{V}', \mathcal{E}')$ are said to be *isomorphic* if there are bijections $\rho : \mathcal{V} \to \mathcal{V}'$ and $\alpha : \mathcal{E} \to \mathcal{E}'$ such that $\rho \circ r = r' \circ \alpha$ and $\rho \circ s = s' \circ \alpha$. There is a *diagram equivalence relation* for Bratteli diagrams that is generated by isomorphisms and telescoping. Two Bratteli diagrams being diagram equivalent implies that their associated dimension groups are order isomorphic.

We will now describe the Bratteli-Vershik systems associated with these diagrams. To any Bratteli diagram associate the space $X = X(\mathcal{V}, \mathcal{E})$ of infinite edge paths on $(\mathcal{V}, \mathcal{E})$ beginning at the vertex $v_0 = (0, 0)$. If γ is a path in X, for each $n = 0, 1, \dots$, $k = 0, 1, \dots, |\mathcal{V}_n|$, denote by $(n, k_n(\gamma))$ the vertex through which γ passes on level n. Then denote by γ_i the edge along which γ travels between vertices $(i, k_i(\gamma))$ and $(i+1, k_{i+1}(\gamma))$. X is a compact metric space with the metric given by, $d(\gamma, \xi) = 2^{-j}$, where $j = \inf\{i | \gamma_i \neq \xi_i\}$. A *cylinder set* $C = \{\gamma \in X | \gamma_{i_1} = c_1, \gamma_{i_2} = c_2, \dots \gamma_{i_j} = c_j\}$ fixes a finite number of the coordinates (here, edges) of the paths it contains. A cylinder set of the form

$\{\gamma \in X | \gamma_0 = c_0, \gamma_1 = c_1, \dots \gamma_{j-1} = c_{j-1}$ for all $i = 0, 1, \dots, j-2\}$ will be denoted $[c_0 c_1 \dots c_{j-1}]$ and said to be of *length* j. The cylinder set $[c_0 c_1 \dots c_{j-1}]$ is said to *terminate at vertex* (j, k) if $r(c_{j-1}) = (j, k)$. The cylinder sets are open and closed, and they form a basis for the topology on X.

On any Bratteli diagram one can define a *partial order* on the set $\mathcal{E}$ of edges. Specifically, two edges e and $\tilde{e}$ are said to be *comparable* if $r(e) = r(\tilde{e})$. The edges' sources may be different. We choose and fix a total order on each set of edges with the same range, i.e. on each $r^{-1}\{v\}$, $v \in \mathcal{V}$. The edges that are maximal (respectively minimal) according to this ordering make up the set $\mathcal{E}_{\max}$ (resp. $\mathcal{E}_{\min}$). A Bratteli diagram with such a partial order on its set of edges is called an *ordered Bratteli diagram* and is denoted $(\mathcal{V}, \mathcal{E}, \geq)$. Two ordered Bratteli diagrams, $B = (\mathcal{V}, \mathcal{E}, \geq)$ and $B' = (\mathcal{V}', \mathcal{E}', \geq)$ are *order equivalent* if their underlying Bratteli diagrams are diagram equivalent in a way that preserves the edge orderings.

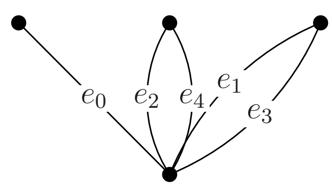


FIGURE 1.3. An edge ordering. $e_0 < e_1 < e_2 < e_3 < e_4$, $e_0 \in \mathcal{E}_{\min}$ and $e_4 \in \mathcal{E}_{\max}$.

On an ordered Bratteli diagram, the partial ordering of edges can be extended to a partial ordering of the entire path space X. Two paths γ and ξ are *comparable* if they agree after some level n ($\gamma_k = \xi_k$ for all $k \geq n$) and $\gamma_{n-1} \neq \xi_{n-1}$; then we define $\gamma < \xi$ if and only if $\gamma_{n-1} < \xi_{n-1}$. The set of maximal paths is denoted by $X_{\max}$. For any path $\gamma \in X_{\max}$, and for all $i \in \{0, 1, \dots\}$, γ_i is a maximal edge according to the partial ordering on edges. Likewise there are minimal paths, which make up the set $X_{\min}$. For every $\gamma \in X_{\min}$, and for all $i \in \{0,1,\dots\}$, γ_i is a minimal edge according the partial ordering on edges.

Definition 1.2.3. Define the *adic transformation* T on $X(\mathcal{V},\mathcal{E})$ as follows:

$$T(\gamma) = \begin{cases} \text{the smallest } \xi > \gamma & \text{if } \gamma \notin X_{\max} \\ \gamma & \text{if } \gamma \in X_{\max}. \end{cases}$$

The pair (X,T) is called the *Bratteli-Vershik system* associated to the ordered Bratteli diagram $(\mathcal{V},\mathcal{E},\geq)$.

We will see after Lemma 1.2.4 and in Section 2.1 that for certain adic transformations a variation of this definition is useful.

T is a Borel map, but it may fail to be continuous, onto, or one-to-one. Here is a sort of algorithm for the action of T on X. Given a non-maximal path γ in X, there is an edge γ_i which is non-maximal. In order to find $T(\gamma)$, let $j = \inf\{i|\gamma_i \text{ is not a maximal edge}\}$. Let ξ_j be the next largest edge in the partial edge ordering of $r^{-1}r(\gamma_j)$. Then let $\xi_0\xi_1\dots\xi_{j-1}$ be the path of all minimal edges connecting vertex $(0,0)$ to $s(\xi_j)$. Then $T(\gamma) = \xi_0\xi_1\dots\xi_j\gamma_{j+1}\gamma_{j+2}\dots$.

Lemma 1.2.4. *$T : X \setminus X_{max} \to X \setminus X_{min}$ is a homeomorphism.*

Proof. Let $\gamma, \gamma' \in X \setminus X_{\max}$, and suppose that $\gamma' \in X \setminus X_{\max}$ agrees with γ to a level n which is past the first non-maximal edge i in γ. From the above description of the action of T on $X \setminus X_{\max}$ we see that $T\gamma$ and $T\gamma'$ agree down to the same level n. Hence T is continuous at the point γ. Similarly for T^{-1} on $X \setminus X_{\min}$. □

In many situations the definition of the adic transformation can be extended to $X_{\max}$.

Definition 1.2.5. Let $A = \{0, 1\}$. Define $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by the following, for each $i \in \mathbb{N}$, $x_i \in A$,

$$\phi(x_1 x_2 \dots) = (x_1 x_2 \dots) + (10000000 \dots)$$

with carry to the right, and $\phi(11111\dots) = (00000\dots)$. Then ϕ is called the *binary odometer*.

There is an equivalent definition of the binary odometer as a Bratteli-Vershik transformation on a Bratteli diagram. Let D be the Bratteli Diagram for which there is one vertex on each level and two edges connecting consecutive levels. Let the edge ordering be so that the left edge is smaller than the right edge. Extend the adic transformation to map the unique maximal path to the unique minimal path. By labeling the left edges 0 and the right edges 1, the topological conjugacy is clear, see Figure 1.4.

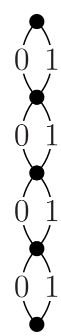


Figure 1.4. The binary odometer represented by a Bratteli-Vershik system.

We extend this idea to a general concept of a Bratteli-Vershik odometer.

Definition 1.2.6. Let D be a Bratteli diagram such that there is one vertex at each level, and let the edge ordering be such that the edges increase from left to right. Let the Bratteli-Vershik transformation associated to this diagram also send the unique maximal path to the unique minimal path. This trasformation is called an *odometer* and the path space together with the odometer is called an *odometer system*. A *stationary odometer* is an odometer for which the number of edges connecting consecutive levels is constant.

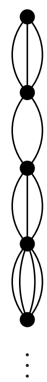


FIGURE 1.5. A general odometer system.

The binary odometer is a stationary odometer, see Figure 1.4.

DEFINITION 1.2.7. An ordered Bratteli diagram $(\mathcal{V}, \mathcal{E}, \geq)$, is *essentially simple* if $X_{\max}$ and $X_{\min}$ consist of one-point sets, $\{\gamma_{\max}\}$ and $\{\gamma_{\min}\}$ respectively.

For an essentially simple Bratteli diagram we extend the adic transformation T to map $\gamma_{\max}$ to $\gamma_{\min}$. Then T is a homeomorphism on the whole space X, and the topological dynamical system (X, T) is essentially minimal. This family of systems has been well studied. Some results appear below.

LEMMA 1.2.8 (Herman, Putnam, and Skau [**23**]). *If $(\mathcal{V}, \mathcal{E}, \geq)$ is an essentially simple ordered Bratteli diagram, then any equivalent ordered Bratteli diagram is also essentially simple.*

THEOREM 1.2.9. [**23**]. *There is a bijective correspondence between equivalence classes of essentially simple ordered Bratteli diagrams and pointed topological conjugacy classes of essentially minimal pointed topological dynamical systems.*

Definition 1.2.10. A Bratteli diagram $(\mathcal{V}, \mathcal{E})$ is said to be *simple* if there exists a telescoping $(\mathcal{V}', \mathcal{E}')$ of $(\mathcal{V}, \mathcal{E})$ such that the incidence matrices of $(\mathcal{V}', \mathcal{E}')$ have all nonzero entries at each level. A *properly ordered Bratteli diagram* is an ordered Bratteli diagram $(\mathcal{V}, \mathcal{E}, \geq)$ such that

1. $(\mathcal{V}, \mathcal{E})$ is simple.
2. $(\mathcal{V}, \mathcal{E}, \geq)$ is essentially simple.

If $X = X(\mathcal{V}, \mathcal{E})$ is not finite, the first condition on a simple ordered Bratteli diagram ensures that associated path space X has no isolated points, and hence in this case X is a Cantor set. The following theorem was alluded to in the introduction and we now give the precise statement.

Theorem 1.2.11 (Herman, Putnam, Skau [**23**]). *Let (X, ϕ, γ_0) be a minimal pointed topological dynamical system, where X is a Cantor set. Then there exists a properly ordered Bratteli diagram $B = (\mathcal{V}, \mathcal{E}, \geq)$ with unique maximal path ξ_{max} so that (X, ϕ, γ_0) is pointedly topologically conjugate to the Bratteli-Vershik system associated to $(\mathcal{V}, \mathcal{E}, \geq)$ with distinguished transitive point $\xi_{\max}$. Moreover, this correspondence establishes a bijection of equivalence classes: if for $i = 1, 2$, B^i are ordered Bratteli diagrams with associated Bratteli-Vershik systems (X_i, ϕ_i) with unique maximal elements γ_i, then (X_1, ϕ_1, γ_1) is pointedly conjugate to (X_2, ϕ_2, γ_2) if and only if B^1 is order equivalent to B^2.*

Definition 1.2.12. A Bratteli diagram $(\mathcal{V}, \mathcal{E})$ is *stationary* if there are an $l \in \mathbb{N}$ and a fixed $l \times l$ nonnegative integer matrix D such that for all $i \geq 1$, $|\mathcal{V}_i| = l$ and $D_i = D$. In other words, $(\mathcal{V}, \mathcal{E})$ repeats itself after the first level. $(\mathcal{V}, \mathcal{E}, \geq)$ is a *stationary ordered Bratteli diagram* if $(\mathcal{V}, \mathcal{E})$ is stationary and for all $n, m = 1, 2, \ldots$ and $k = 0, 1, \ldots l - 1$

the ordering on the set of edges with range (n,k) is the same as the ordering on the set of edges with range (m,k) .

The properly ordered stationary Bratteli-Vershik systems have been classified in [**17, 12**].

Definition 1.2.13. Let $(\mathcal{V},\mathcal{E})$ be a Bratteli diagram. For any $n=0,1,\dots$ and $k=0,1,\dots,|\mathcal{V}_n|-1$ define the *dimension* of the vertex (n,k) to be the number of finite paths from (0,0) to (n,k), and denote it by $\dim(n,k)$. Let $X=X(\mathcal{V},\mathcal{E})$ be the path space associated to $(\mathcal{V},\mathcal{E})$. Given a cylinder set $C\subset X$ and $\gamma\in X$, and $n=0,1,2,\dots,$ define the *dimension of C through* $(n,k_n(\gamma))$ to be the number of paths in C which coincide with γ after the n'th level, and denote it by $\dim(C,(n,k_n(\gamma)))$:

$$\dim(C,(n,k_n(\gamma)))=|\{\xi\in C|\xi_j=\gamma_j \text{ for all } j\geq n\}|$$

The following Lemma will be used throughout this thesis as a method for determining T-invariant, ergodic probability measures. For any adic transformation T and measure μ we will assume that T is defined μ-almost everywhere.

Lemma 1.2.14 (Vershik [**46, 45**]). *Let (X,T) be a Bratteli-Vershik system. If μ is a non-atomic probability measure on X which is invariant and ergodic under the adic transformation T, then for every cylinder set $C\subset X$,*

$$\mu(C)=\lim_{n\to\infty}\frac{\dim(C,(n,k_n(\gamma)))}{\dim(n,k_n(\gamma))} \text{ for } \mu\text{-a.e. } \gamma\in X.$$

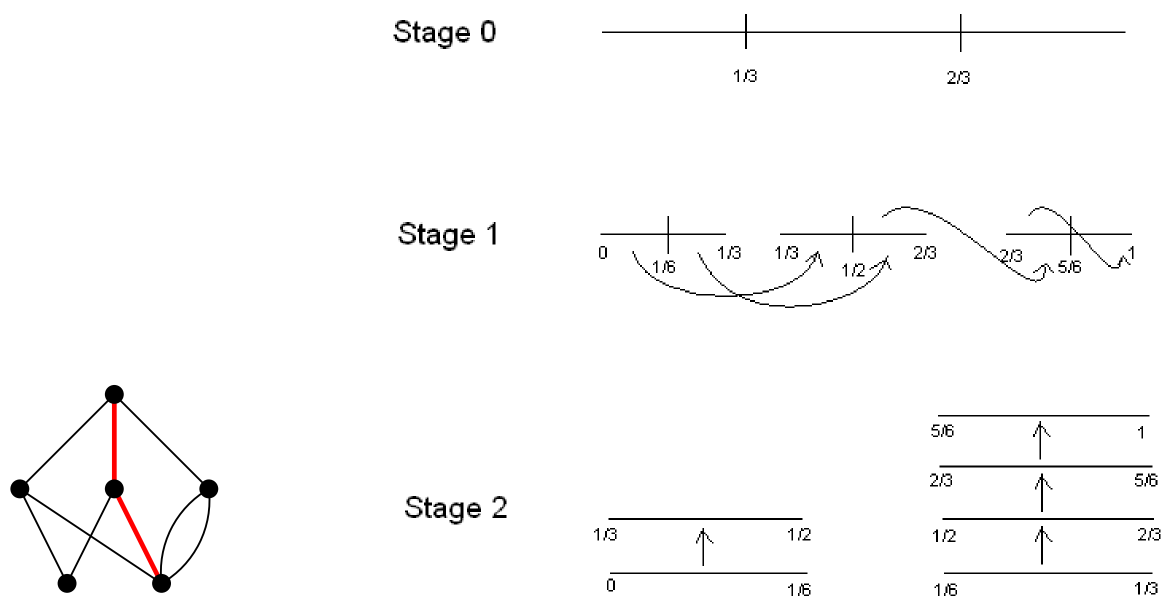
Proof. The proof in [**32**] extends to all Brattel-Vershik systems. □

The following notation will allow a method for discussing particular cylinders in X. Let $(\mathcal{V}, \mathcal{E}, \geq)$ be an ordered Bratteli diagram. For any vertex $(n,k) \in \mathcal{V}$ there is a cylinder determined by the path from the root vertex to (n,k) for which all the edges are contained in $\mathcal{E}_{\min}$ ($\mathcal{E}_{\max}$). We will call this the *minimal (maximal) cylinder terminating at vertex* (n,k). Denote by $Y_n(k,0)$ the minimal cylinder into vertex (n,k), and let $Y_n(k,i) = T^i(Y_n(k,0))$ for $i = 0, 1, \ldots, \dim(n,k) - 1$. For each $n = 0, 1, 2, \ldots$, denote the union of all the minimal cylinders of length n by Y_n, so that

$$Y_n = \bigcup_{0 \leq k \leq |\mathcal{V}_n| - 1} Y_n(k,0).$$

Each Bratteli-Vershik system that is endowed with an adic invariant probability measure is a combinatorial model of a map on the unit interval defined by "cutting and stacking" which preserves Lebesgue measure, m. Each stage of cutting and stacking corresponds to a level in the Bratteli diagram. At each stage $n = 0, 1, 2, \ldots$ there are $|V_n|$ stacks $S_{n,0}, S_{n,1}, \ldots, S_{n,|V_n|-1}$ which correspond to the vertices (n,k), $0 \leq k \leq |V_n| - 1$, of the Bratteli diagram. Stack $S_{n,k}$ consists of $\dim(n,k)$ subintervals of $[0,1]$. Each subinterval corresponds to a cylinder set determined by a path of length n, terminating in vertex (n,k). The transformation $\tilde{T}$ is defined by mapping each level of the stack, except the topmost one, linearly onto the one above it. This corresponds to mapping each non-maximal path of length n to its successor. To proceed to the next stage in the cutting and stacking construction, each stack $S_{n,k}$ is cut into the number of edges leaving the vertex (n,k) substacks with length proportions corresponding to the various weights on the different edges. More precisely, if there are l edges leaving vertex (n,k) with weights $w_1, w_2, \ldots, w_l$, and the width of the stack $S_{n,k}$ is α, then the stack $S_{n,k}$ is cut into l substacks of length $w_1\alpha, w_2\alpha, \ldots, w_3\alpha$. These are recombined into new stacks in the order prescribed by the edge ordering. In this manner, every Bratteli-Vershik system (X, T, μ) is isomorphic to a Lebesgue measure-preserving transformation defined almost everywhere on $[0, 1]$.

Example 1.2.1. The following picture shows the first two levels of a Bratteli-Vershik system for which the edge ordering increases from left to right, all edges connecting level 0 to level 1 are given weight 1/3, all edges connecting level 1 to level 2 are given weight 1/2., and the corresponding first two stages of cutting and stacking. The red cylinder corresponds to the interval (1/2, 2/3).



CHAPTER 2

Limited Scope Adic Transformations

2.1. Description

We study a special family $\mathcal{S}_{\mathcal{L}}$ of Bratteli-Vershik systems whose vertex growth between subsequent levels is bound by a constant; we call them adics of *limited scope* for that reason.

Definition 2.1.1. Let $(\mathcal{V}, \mathcal{E}, \leq)$ be a Bratteli diagram such that for a constant d and all $n = 0, 1, 2, \ldots,$ $|V_n| = nd + 1$ and each vertex (n, k) is connected by some positive number of edges to each vertex in $(n + 1, k + i)$ for all $i \in \{0, 1, 2, \ldots, d\}$, and there are no edges elsewhere. We denote this family of Bratteli diagrams by $\mathcal{D}_{\mathcal{L}}$.

In short, for some constant d and every level n, the incidence matrix D_n is an $(nd + 1) \times ((n-1)d + 1)$ matrix such that $[D_n]_{i,j}$ is nonzero if and only if $j \leq i \leq j + d$.

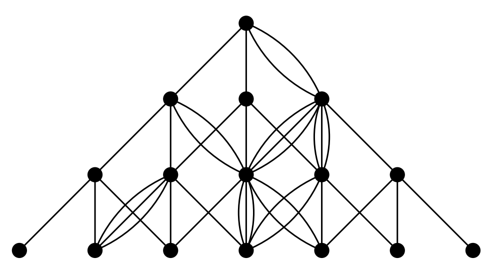


Figure 2.1. An example of a Bratteli diagram in $\mathcal{S}_{\mathcal{L}}$ when $d = 2$.

We draw the diagrams for these systems so that edges with the same range increase in order from left to right, as in Figure 2.2.

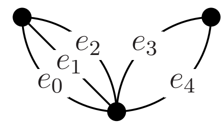


FIGURE 2.2. $e_0 < e_1 < e_2 < e_3 < e_4$

As before, for any diagram $(\mathcal{V}, \mathcal{E}) \in \mathcal{D}_{\mathcal{L}}$, X is the space of infinite edge paths on $(\mathcal{V}, \mathcal{E})$. With the ordering described above, there are a countable number of paths in $X_{\max} \cup X_{\min}$. For every k in the set $\{0, 1, \dots\} \cup \{\infty\}$ there is a unique associated path in $X_{\max}$, denoted $\gamma^k_{\max}$, which is defined as follows. For $k \neq \infty$ $\gamma^k_{\max}$ is the path in X that travels down the far right edge of the graph, following maximal edges, to level $n_0 - 1$, where $n_0 \in \mathbb{N}$ is such that $(n_0 - 1)d < k \leq n_0 d$, and then connects to vertex (n_0, k) along the maximal edge. Then for $n \geq n_0$, $k_n(\gamma^k_{\max}) = k$, and $\gamma^k_{\max}$ follows a maximal edge. The path $\gamma^\infty_{\max}$ is the path which travels through the vertices (n, dn) along maximal edges for all $n \in \mathbb{N}$. Likewise for every k in the set $\{0, 1, \dots\} \cup \{\infty\}$ there is a unique path in $X_{\min}$ denoted $\gamma^k_{\min}$. For $k \neq \infty$ this is the path in X that travels down the left side of the graph along minimal edges to level $n_0 - 1$, where $n_0 \in \mathbb{N}$ is such that $(n_0 - 1)d < k \leq n_0 d$, and then connects to vertex $(n_0, n_0 d - k)$ along the minimal edge. Then for $n \geq n_0$, $k_n(\gamma) = nd - k$. The path $\gamma^\infty_{\min}$ is the path which travels through the vertices $(n, 0)$ along minimal edges for all $n \in N$.

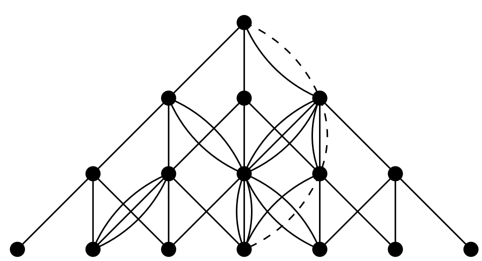


FIGURE 2.3. The dashed path is the first three edges of $\gamma^3_{\max}$.

Let T be the Bratteli-Vershik transformation on the path space X. As with the odometers described in Chapter 1, it is now usefull to redefine T on $X_{\max}$ so that $T(\gamma_{\max}^k) = \gamma_{\min}^k$ for $0 < k < \infty$, $T(\gamma_{\max}^0) = \gamma_{\min}^\infty$, and $T(\gamma_{\max}^\infty) = \gamma_{\min}^0$. In this way T is a bijection on the whole space X; but not continuous on $X_{\max}$. $T(\gamma_{\max}^0)$ and $T(\gamma_{\max}^\infty)$ are defined in this manner to create invariant odometer systems on the far left and far right sides of the diagram.

DEFINITION 2.1.2. Let $(\mathcal{V}, \mathcal{E})$ be a Bratteli diagram in $\mathcal{D}_{\mathcal{L}}$. Let X be the infinite edge path space on $(\mathcal{V}, \mathcal{E})$ and let T be the Bratteli-Vershik transformation on X. The family of such systems is said to have *limited scope* and is denoted $\mathcal{S}_{\mathcal{L}}$.

No member of $\mathcal{D}_{\mathcal{L}}$ is essentially simple since there are countably many paths in $X_{\max}$, and clearly no member of $\mathcal{D}_{\mathcal{L}}$ is stationary. Members of $\mathcal{D}_{\mathcal{L}}$ are also not simple. This is easy to see, since for a vertex (n, k) with $k < dn$ there is no path to vertex (m, dm) for any $m > n$. Hence the theory described in the last chapter is insufficient to understand the workings of the systems based on these diagrams.

DEFINITION 2.1.3. For $(X, T) \in \mathcal{S}_{\mathcal{L}}$, we say a path $\gamma \in X$ is *eventually diagonal to the left* if there exists and $N \geq 0$ such that for $n \geq N$, $k_n(\gamma) = k_N(\gamma)$. We say a path $y \in X$ is *eventually diagonal to the right* if there exists an $M \geq 0$ such that for $m \geq M$, $k_m(\gamma) = dm - k_M(\gamma)$.

REMARK 2.1.4. We will say that a path is *eventually diagonal* if the direction is either clear or unknown. All paths in the orbits of $X_{\max}$ and $X_{\min}$ are eventually diagonal.

PROPOSITION 2.1.5. *For every $\gamma \in X$, exactly one of the following holds.*

1. γ is eventually diagonal to the right.

2. γ *is eventually diagonal to the left.*

3. $\overline{\mathcal{O}(\gamma)} = X$

Proof. Suppose that γ is not eventually diagonal, both $k_n(\gamma)$ and $dn - k_n(\gamma)$ are both unbounded. Then for any $\xi \in X$ and $m \in \mathbb{N}$ there is an $n_0 > m$ such that $k_m(\xi) \leq k_{n_0}(\gamma)$ and $dm - k_m(\xi) \leq dn_0 - k_{n_0}(\gamma)$. Hence, $k_{n_0}(\gamma) - d(n_0 - m) \leq k_m(\xi) \leq k_{n_0}(\gamma)$. Therefore there is a path from $(m, k_m(\xi))$ to $(n_0, k_{n_0}(\gamma))$. Then there is a $j \in \mathbb{Z}$ so that $T^j\gamma$ coincides with ξ along the first m edges, showing that $\mathcal{O}(\gamma)$ is dense in X.

If γ is in eventually diagonal to the right (resp. to the left), we have that for any $\xi \in \mathcal{O}(\gamma)$ and all $n \in \mathbb{N}$, $k_n(\xi)$ (resp. $dn - k_n(\xi)$) is bounded by some number N (resp. M). Now choose $\xi \in X$ and $m \in \mathbb{N}$ for which $k_m(\xi) > N$ (resp. $dm - k_m(\xi) > M$) and let $B_{2^{-m}}(\xi)$ be the ball of radius 2^{-m} around ξ. Then $\mathcal{O}(\gamma) \cap B_{2^{-m}}(\xi) = \emptyset$. Hence, $\mathcal{O}(\gamma)$ is not dense in X. $\square$

2.2. Dimension Groups

Bratteli diagrams first appeared in 1972 in [**5**] with the purpose of studying unital AF-algebras. Elliott [**13**] produced a bijective correspondence between isomorphism classes of unital AF-algebras and order isomorphism classes of dimension groups with distinguished order units. The following discussion focuses on the bijective correspondence between order isomorphism classes of dimension groups with distinguished order units and equivalence classes of Bratteli diagrams. For further references on ordered groups and dimension groups see [**12**],[**4**], and [**23**].

Definition 2.2.1. An *ordered group* is a pair (G, G_+) such that G is a countable abelian group and G_+ is a subset of G containing 0 such that:

1. $G_+ + G_+ = G_+$;
2. $G_+ - G_+ = G$;
3. $G_+ \cap (-G_+) = \{0\}$.

Definition 2.2.2. For each $n = 1, 2, \ldots$ $\mathbb{Z}^n$, with the positive set $\mathbb{Z}^n_+$ of vectors with nonnegative entries, is called a *simplicial group*.

Definition 2.2.3. Let I be a directed set, and for each $i \in I$ let A_i be an abelian group. Suppose that for every pair of indicies i, j with $i \leq j$ there is a group homomorphism $\phi_{ij} : A_i \to A_j$ such that

1. $\phi_{jk} \circ \phi_{ij} = \phi_{ki}$ whenever $i \leq j \leq k$, and
2. $\phi_{ii} = id$ for all $i \in I$.

These maps are called *transition maps*. The family of groups A_i and maps ϕ_{ji} is called a *directed system*. Let B be the disjoint union of all the A_i, and define a relation $\sim$ on B as follows: if $a \in A_i$ and $b \in B_j$, then $a \sim b$ if and only if there is a k with $i, j \leq k$ and $\phi_{ik}(a) = \phi_{jk}(b)$. The set of equivalence classes is called the *direct limit* of the directed system $\{A_i\}$ and is denoted $\varinjlim A_i$. Denote the equivalence class of an element $a \in A_i$ by $\overline{a}$. Define the function $\phi^i : A_i \to \varinjlim A_i$ by $\phi^i(a) = \overline{a}$.

For two elements $\overline{a}, \overline{b} \in \lim_{\to} A_i$, $\overline{a+b}$ is determined by choosing an $i \in \mathbb{N}$ large enough so that there are representatives a_i, b_i of $\overline{a}$ and $\overline{b}$ in A_i. Then $\overline{a+b} = \overline{a_i + b_i}$.

For the rest of this section by an *isomorphism* we will mean a bijective group homomorphism. Two ordered groups (G, G_+) and (G', G'_+) are said to be *order isomorphic* if there is a group isomorphism $\phi : G \to G'$ for which $\phi(G_+) = G'_+$ and $\phi^{-1}(G'_+) = G_+$.

DEFINITION 2.2.4. A *dimension group* is an ordered group which is order isomorphic to a direct limit of simplicial groups.

DEFINITION 2.2.5. $u \in G_+$ is said to be an *order unit* for (G, G_+) if for each $g \in G_+$ there is an $n \in \mathbb{N}$ such that $nu - g \in G_+$.

For every Bratteli diagram $(\mathcal{V}, \mathcal{E})$ there is an associated dimension group, denoted $K_0(\mathcal{V}, \mathcal{E})$, which is the direct limit of the following directed system:

$$\mathbb{Z}^{|V_0|=1} \xrightarrow{\phi_1} \mathbb{Z}^{|V_1|} \xrightarrow{\phi_2} \mathbb{Z}^{|V_2|} \xrightarrow{\phi_3} \ldots$$

where for each $i = 1, 2, \ldots$ ϕ_i is the group homomorphism determined by the incidence matrix between levels $i-1$ and i of the Bratteli diagram. The positive set consists of the equivalence classes for which there is a nonnegative vector representative. The equivalence class of $1 \in \mathbb{Z}$ is called the *distinguished order unit* because it is always an order unit, and in certain situations mappings between dimension groups associated to the Bratteli diagrams that send one distinguished order unit to another have special properties. It is important to note that the dimension group $K_0(\mathcal{V}, \mathcal{E})$ is not dependent on the associated dynamical system but comes to us from C^*-algebra theory. In the case of essentially simple Bratteli-Vershik systems, the relationship between the dynamical system and the dimension group is known, and we describe it now.

If (X, ϕ) is a dynamical system (recall Definition 1.1.1) let $C(X, \mathbb{Z})$ denote the additive group of continuous functions from the space X to $\mathbb{Z}$ and define

$$\partial_\phi C(X, \mathbb{Z}) = \{g \circ \phi - g | g \in C(X, \mathbb{Z})\}.$$

The elements of $\partial_\phi C(X, \mathbb{Z})$ are called the *coboundaries* of (X, ϕ). Note that they may not be continuous. In the case that ϕ is a homeomorphism, $\partial_\phi C(X, \mathbb{Z}) \subset C(X, \mathbb{Z})$, and we define $K^0(X, \phi)$ to be $C(X, \mathbb{Z})/\partial_\phi C(X, \mathbb{Z})$.

THEOREM 2.2.6 ([**23**]). *Let $(\mathcal{V}, \mathcal{E}, \geq)$ be an essentially simple ordered Bratteli diagram and let (X, ϕ) be its associated Bratteli-Vershik system. Then there is an order isomorphism*

$$\theta : K_0(\mathcal{V}, \mathcal{E}) \to K^0(X, \phi)$$

which maps the distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$ to the equivalence class of the constant function 1.

In the case of our family $\mathcal{S}_\mathcal{L}$, which consists of systems based on Bratteli diagrams that are neither stationary nor essentially simple, $\partial_T C(X, \mathbb{Z})$ may not be not contained in $C(X, \mathbb{Z})$ as T is not continuous everywhere. Nevertheless, by slightly adjusting the definition of $K^0(X, T)$ to be $C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}))$ we can achieve a result similar to Theorem 2.2.6.

THEOREM 2.2.7. *For $(X, T) \in \mathcal{S}_\mathcal{L}$, there is an order isomorphism*

$$K_0(\mathcal{V}, \mathcal{E}) \cong K^0(X, T)$$

which maps the distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$ to the equivalence class of the constant function 1.

Proof. This proof is an adaptation of the dynamical proof of Theorem 2.2.6 given by Glasner and Weiss in [**21**]. We will first define a group homomorphism $J : C(X, \mathbb{Z}) \to K_0(\mathcal{V}, \mathcal{E})$. Then we will define a set B and show that it is a subset of $C(X, \mathbb{Z})$. Then we will show $B = \ker(J)$ by first showing $B \subset \ker(J)$ and then $\ker(J) \subset B$. This will induce a one-to-one group homomorphism $\tilde{J} : C(X, \mathbb{Z})/B \to K_0(\mathcal{V}, \mathcal{E})$. We will then show that $\tilde{J}$ is surjective and in fact an order isomorphism. Lastly we will show $B = \partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})$.

Let $f \in C(X, \mathbb{Z})$. Since X is compact, f is bounded and hence takes on only finitely many values. Let $\{l_1, \dots l_j\}$ be the set of these values and let $U_i = f^{-1}\{l_i\}$ for each i. If $i = 1, \dots, j$ and $\gamma \in U_i$, then there is a cylinder set $C_\gamma \subset U_i$ of the form $[c_0 c_1 \dots c_{N_\gamma - 1}]$ which contains γ. From $\{C_\gamma | \gamma \in X\}$ select a finite subcover $\{C_{\gamma^1}, C_{\gamma^2}, \dots, C_{\gamma^r}\}$. Then for some $i \in \{1, 2, \dots, r\}$, C_{γ^i} is of longest length, $N_1(f)$, and f is constant on any cylinder of length $n \geq N_1(f)$. Recall that $Y_n(k, 0)$ is the minimal cylinder into vertex (n, k). For $n \geq N_1(f)$ define an element $\tilde{f}_n \in \mathbb{Z}^{dn+1}$ by letting, for each $0 \leq k \leq dn$ and any $\gamma \in Y_n(k, 0)$,

$$\tilde{f}_n(k) = f(\gamma) + f(T\gamma) + f(T^2\gamma) + \dots + f(T^{\dim(n,k)-1}\gamma).$$

Recall that D_n denotes the adjacency matrix of the edges connecting levels $n-1$ and n. Then

$$\tilde{f}_{n+1}(i) = \sum_{j=0}^{nd} \tilde{f}_n(j)(D_n)_{i,j} = (\tilde{f}_n D_n)(i).$$

Therefore the sequence $\tilde{f}_n$ defines an element $J(f) \in K_0(\mathcal{V}, \mathcal{E})$ (see Definition 2.2.3). Clearly $J : C(X, \mathbb{Z}) \to K_0(\mathcal{V}, \mathcal{E})$ is a group homomorphism.

Recall that Y_n is the union of all the minimal cylinder sets into level n. Let $G = \{g \in C(X, \mathbb{Z})|$ such that there is an $N_2(g)$ and $n \geq N_2(g)$ implies for each $\gamma \in Y_n,\ g(\gamma) = c\}$. In other words, g takes the same value on all the minimal cylinders into level n. We now define $B = \{g \circ T - g | g \in G\}$.

We now show that $B \subset C(X, \mathbb{Z})$. For $f \in B$ with $f = g \circ T - g$, f is continuous on $X \setminus X_{\max}$, since $g \in C(X, \mathbb{Z})$ and T is continuous on $X \setminus X_{\max}$. Hence we only need to check continuity of f on $X_{\max}$. Let $m \geq \max\{N_1(g), N_2(g)\}$ be such that g is constant on each cylinder of length m and g is also constant on Y_m. For $\gamma_{\max} \in X_{\max}$ and $\xi \in X$, $d(\gamma_{\max}, \xi) < 2^{-m}$ implies that $\gamma_{\max}$ and ξ are both in the same maximal cylinder terminating at vertex $(m, k_m(\gamma_{\max}))$, and hence $g(\gamma_{\max}) = g(\xi)$. Since $T(\gamma_{\max})$ and $T(\xi)$ are both in Y_m, we have $(g \circ T)(\gamma) = (g \circ T)(\xi)$. Hence $f(\gamma) = f(\xi)$, and so f is continuous.

We will show that $B = \ker(J)$. If $f = g \circ T - g \in B$, $n \geq \max\{N_1(g) N_2(g)\}$, $0 \leq k \leq dn$, and any $\gamma \in Y_n(k, 0)$, then

$$\tilde{f}_n(k) = f(\gamma) + f(T\gamma) + f(T^2\gamma) + \cdots + f(T^{\dim(n,k)-1}\gamma) = g \circ T^{\dim h(n,k)}(\gamma) - g(\gamma).$$

Since both γ and $T^{\dim(n,k)}(\gamma) \in Y_n$ and g is constant on Y_n, $\tilde{f}_n(k) = 0$. Therefore $J(f) = 0$, which implies $B \subset \ker(J)$.

Conversely, if $f \in C(X, \mathbb{Z})$ and $J(f) = 0$, there is an $n > N_1(f)$ for which $\tilde{f}_n = 0$. We will define a function $g \in C(X, \mathbb{Z})$ so that $f = g \circ T - g$. Let $g = 0$ on Y_n. For $1 \leq l \leq \dim(n, k)$, choose any $\gamma \in Y_n(k, 0)$ and let $g \equiv f(\gamma) + f(T\gamma) + \cdots + f(T^{l-1}\gamma)$ on $Y_n(k, l)$. Now g is everywhere defined, and clearly $f = g \circ T - g$ on every cylinder terminating at vertex (n, k) except maybe on the maximal cylinder. However, for $\gamma \in$

$Y_n(k,0)$, $g(T^{h(n,k)}\gamma)=0$ and $\tilde{f}_n(k)=0$, so we have

$$\begin{aligned}
g(T^{h(n,k)}\gamma)-g(T^{h(n,k)-1}\gamma) &= -g(T^{h(n,k)-1}\gamma)\\
&= -(f(\gamma)+f(T\gamma)+\cdots+f(T^{h(n,k)-2}\gamma))\\
&= -(f(\gamma)+f(T\gamma)+\cdots+f(T^{h(n,k)-1}\gamma))+f(T^{h(n,k)-1}\gamma)\\
&= -\tilde{f}_n(k)+f(T^{h(n,k)-1}\gamma)\\
&= f(T^{h(n,k)-1}\gamma).
\end{aligned}$$

Thus $f=g\circ T-g$ also on the maximal cylinder, and hence $f\in B$. Thus $B=\ker(J)$, and J induces an injective group homomorphism $\tilde{J}: C(X,\mathbb{Z})/B\to K_0(\mathcal{V},\mathcal{E})$.

We now show that $\tilde{J}$ is onto and an order isomorphism. Given $a\in K_0(\mathcal{V},\mathcal{E})$, choose an $n\in\mathbb{Z}_+$ so that the equivalence class a has a representative $a_n\in\mathbb{Z}^{dn+1}$. Define f as follows. For $k=0,1,\ldots,dn$ and $\gamma\in Y_n(k,0)$, let $f(\gamma)=a_n(k)$ and elsewhere put $f=0$. Then $\tilde{f}_n(k)=a_n(k)$, so that $J(f)=a$, and thus $\tilde{J}$ is onto. Clearly $\tilde{J}$ takes positive elements to positive elements, and the preceding argument shows that the unique preimage of every positive element under $\tilde{J}$ is a positive element. Thus $C(X,\mathbb{Z})/B$ is order isomorphic to $K_0(\mathcal{V},\mathcal{E})$ by the map $\tilde{J}$, which maps the equivalence class of the constant function 1 to the distinguished order unit of $K_0(\mathcal{V},\mathcal{E})$.

We now show that $\partial_T C(X,\mathbb{Z})\cap C(X,\mathbb{Z})\subset B$. Let $f\in\partial_T C(X,\mathbb{Z})\cap C(X,\mathbb{Z})$ be given. Then $f=g\circ T-g$ for some $g\in C(X,\mathbb{Z})$, and f is continuous. We have to show that there is an $N_2(g)$ so that for each $n\geq N_2(g)$, g takes the same value on all of Y_n. Since $g\in C(X,\mathbb{Z})$, we can choose $l=N_1(g)$ such that g is constant on cylinder sets of length l. Then for every level $j\geq l$, and every $i\in\{0,1,\ldots,(j-l)d\}$,

$$Y_j(i,0)\subset Y_l(0,0) \tag{2.2.1}$$

(see Figure 2.4). Now consider $k < dl$ and $\gamma^k_{\max} \in X_{\max}$. Then

$$T(\gamma^k_{\max}) = \gamma^k_{\min} \in Y_l(ld-k, 0) \text{ (see Figure 2.5).} \tag{2.2.2}$$

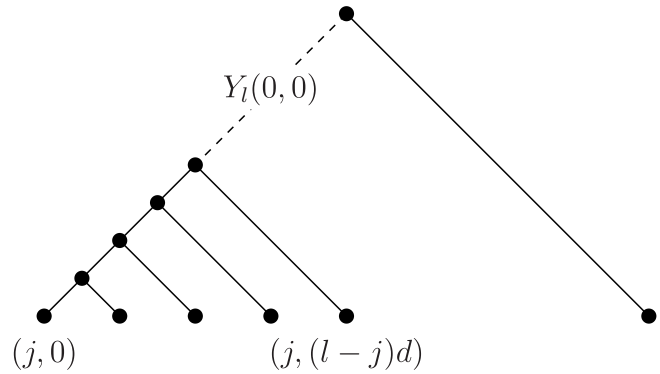


FIGURE 2.4. Connections from level l to j.

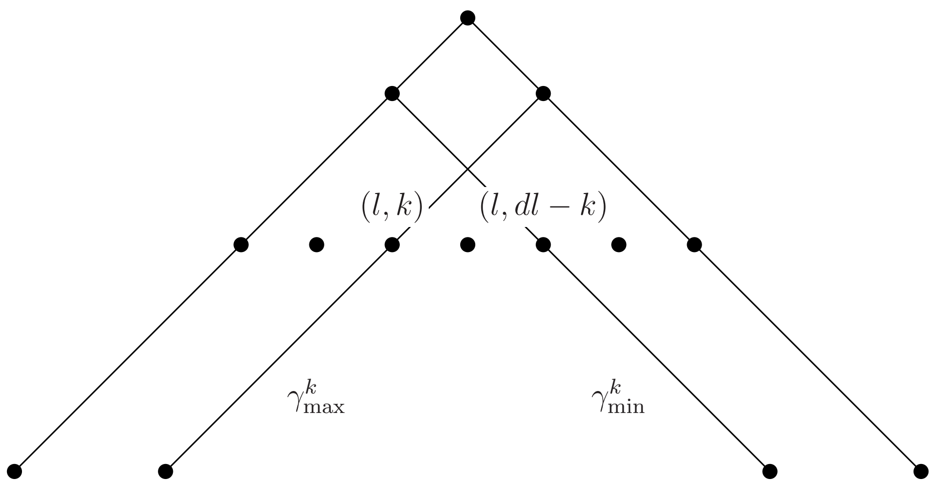


FIGURE 2.5. $T(\gamma^k_{\max}) = \gamma^k_{\min} \in Y_l(ld-k, 0)$.

Since $f = g \circ T - g \in C(X, \mathbb{Z})$, given $\gamma^k_{\max} \in X$ with $k < dl$ there is a $\delta > 0$ such that $d(\gamma^k_{\max}, \xi) < \delta$ implies $f(\gamma^k_{\max}) = f(\xi)$. We will choose a $\xi \in X$ sufficiently close to $\gamma^k_{\max}$ such that $f(\gamma^k_{\max}) = f(\xi)$ and $g(\gamma^k_{\max}) = g(\xi)$ which implies $g \circ T(\gamma^k_{\max} = g \circ T(\xi)$. Choose j so that $2^{-j} < \delta$ and $(j-l)d > k+1$. Now let ξ be a path in X such that $\xi_i = (\gamma^k_{\max})_i$ for each $i = 0, 1, \ldots, j-1$ and $\xi_j \neq (\gamma^k_{\max})_j$. Then $d(\gamma^k_{\max}, \xi) < \delta$, so that $f(\gamma^k_{\max}) = f(\xi)$. Since $j > l$, $\gamma^k_{\max}$ and ξ are in the same maximal cylinder which terminates at (l, k), which implies $g(\gamma^k_{\max}) = g(\xi)$. Thus $f(\gamma^k_{\max}) = f(\xi)$ implies $(g \circ T)(\gamma^k_{\max}) = (g \circ T)(\xi)$. Since $s(\xi_j) = (j, k)$, and ξ_j is the first non-maximal edge of ξ, $T\xi$ is in either $Y_j(k, 0)$ or $Y_j(k+1, 0)$ depending on the source of the successor of ξ_j (see Figure 2.6). Since $k + 1 < (j - l)d$, Equation 2.2.1 implies $T\xi \in Y_l(0, 0)$. Then $(g \circ T)(\gamma^k_{\max}) = (g \circ T)(\xi)$, and g constant on each cylinder of length l implies $g(Y_l(ld - k), 0) = g(Y_l(0, 0))$. Since $k < dl$ was arbitrary, we have shown that g is constant on all $Y_l(k, 0)$ for $k < dl$. It remains only to show that g takes this same value on $Y_l(dl, 0)$. Consider $\gamma^\infty_{\max}$, and choose $j \geq l$ so that $2^{-j} < \delta$. Then $d(\gamma^\infty_{\max}, \gamma^{jd}_{\max}) < \delta$, which implies $\gamma^\infty_{\max}$ and $\gamma^{jd}_{\max}$ are both in the maximal cylinder terminating at vertex (l, dl). Thus $g(\gamma^\infty_{\max}) = g(\gamma^{jd}_{\max})$. Then $f(\gamma^\infty_{\max}) = f(\gamma^{jd}_{\max})$ and $g(\gamma^\infty_{\max}) = g(\gamma^{jd}_{\max})$ implies $(g \circ T)(\gamma^\infty_{\max}) = (g \circ T)(\gamma^{jd}_{\max})$. Thus $T\gamma^\infty_{\max} \in Y_l(dl, 0)$, $T\gamma^{jd}_{\max} \in Y_l(0, 0)$ and g constant on each cylinder of length l implies $g(Y_l(0, 0)) = g(Y_l(dl, 0))$. Hence g is constant on Y_l, as required. $\square$

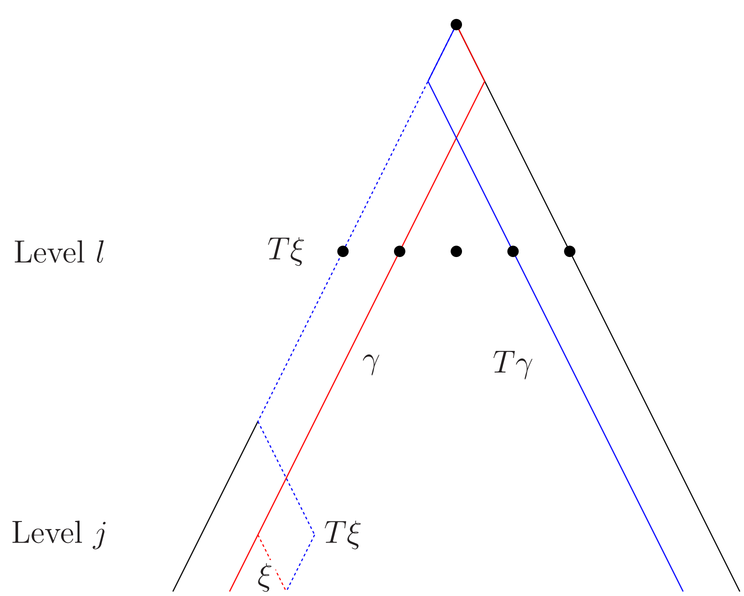


FIGURE 2.6. Tracking $T\gamma$ and $T\xi$.

In the case of Cantor minimal systems, not only does the dimension group have connections with the dynamics through the isomorphism in Theorem 2.2.7, it can also tell us something about orbit equivalence. We discuss this in Chapter 5.

2.3. Systems in $\mathcal{S}_{\mathcal{L}}$ with $d = 1$ and Subshifts

The support of a Borel probability measure μ is the smallest closed subset B such that $\mu(B) = 1$. A measure is *fully supported* on X if $B = X$, hence every cylinder set has positive measure. In this section we will show that each Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ (see Section 2.1) for which $d = 1$, when equipped with a fully-supported ergodic invariant measure, is measure-theoretically isomorphic to a subshift on a finite alphabet. Recall that for systems in $\mathcal{S}_{\mathcal{L}}$ for which $d = 1$, we have that for all $n = 0, 1, \dots$ and $k = 0, 1, \dots n$, $|V_n| = n + 1$ and there are edges between vertices (n, k) and $(n + 1, k)$ as well as (n, k) and $(n + 1, k + 1)$.

In order to show this we will introduce some definitions and lemmas.

Let $(X, T) \in \mathcal{S}_{\mathcal{L}}$ with $d = 1$. Denote the edges leaving $v_0 = (0, 0)$ by $e_1, e_2, \dots, e_m$, so that $s^{-1}\{v_0\} = \{e_1, \dots, e_m\}$. Define $P_i = \{\gamma \in X | \gamma_0 = e_i\}$. Then $\mathcal{P} = \{P_1, P_2, \dots P_m\}$ is a finite partition of X into pairwise disjoint nonempty clopen cylinder sets. Recall from Definition 1.1.17 that there is a function on X, also denoted by $\mathcal{P}$ such that by $\mathcal{P}(\gamma) = j$ for all $\gamma \in P_j$, $j = 1, \dots, m$. For each $n = 0, 1, 2, \dots$, the $\mathcal{P}$-*n*-*name* of γ is the finite block

$$\mathcal{P}_0^n(\gamma) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma)\dots\mathcal{P}(T^n\gamma),$$

and the $\mathcal{P}$-*name* of γ is the doubly infinite sequence

$$\mathcal{P}_{-\infty}^{\infty}(\gamma) = \dots\mathcal{P}(T^{-2}\gamma)\mathcal{P}(T^{-1}\gamma).\mathcal{P}(\gamma)\mathcal{P}(T\gamma)\mathcal{P}(T^2\gamma)\dots.$$

Recall that for any $(n,k) \in \mathcal{V}$, $\dim(n,k)$ is the number of paths from the root vertex into (n,k) and $Y_n(k,0)$ is the minimal cylinder terminating in vertex (n,k).

For every vertex $(n,k) \in \mathcal{V}$ and $\gamma \in Y_n(k,0)$ define

$$B(n,k) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma)\mathcal{P}(T^2\gamma)\ldots\mathcal{P}(T^{\dim(n,k)-1}\gamma).$$

$B(n,k)$ is called the *basic block at vertex* (n,k).

Let $l(n,k)$ denote the number of edges connecting (n,k) and $(n+1,k)$, and $r(n,k)$ denote the number of edges connecting (n,k) and $(n+1,k+1)$. Then

$$B(n+1,k+1) = B(n,k)^{r(n,k)}B(n,k+1)^{l(n,k+1)}, \tag{2.3.1}$$

where the exponents indicate concatenation, see Figure 2.7.

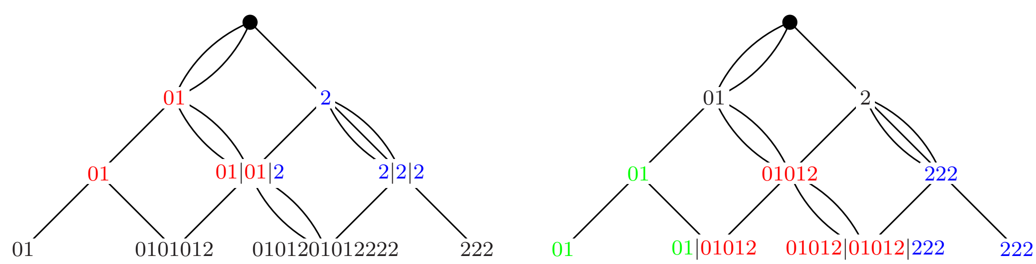


FIGURE 2.7. Relations of $B(n,k)$ seen graphically.

DEFINITION 2.3.1. Let Σ denote the space of bi-infinite sequences on $\{1,2,\ldots,m\}$ for which every finite subsequence appears as a subblock in some $B(n,k)$, and let $\sigma : \Sigma \to \Sigma$ denote the shift map.

Because of the recursion given in Equation 2.3.1, any $B(n,k)$ can be decomposed into a sequence of $B(1,0)$'s and $B(1,1)$'s. For any vertex (n,k), decompose $B(n,k)$ into a sequence of $B(1,0)$'s and $B(1,1)$'s, and let $b_0(n,k)$ be the number of $B(1,0)$ that appear before the first $B(1,1)$ appears or $B(n,k)$ ends, and let $b_1(n,k)$ be the number of $B(1,1)$ that appear after the final $B(1,0)$.

LEMMA 2.3.2. *For* $n = 1, 2, \ldots$ *we have*

1. $b_0(n,k) > b_0(n,k+1)$ *for* $0 < k < n$;
2. $b_0(n,k) > b_0(n-1,k)$ *for* $0 < k \leq n$;
3. $b_1(n,k) > b_1(n,k-1)$ *for* $0 \leq k < n$.

Proof. By definition we have,

$$\begin{aligned}
b_0(1,0) &= 1, \\
b_0(n,n) &= 0, \text{ and, for } n > 1, \\
b_0(n,0) &= l(1,0)l(2,0)\ldots l(n-1,0), \\
b_0(n,1) &= b_0(n-1,0)r(n-1,0) + b_0(n-1,1) \text{ and} \\
&= b_0(n-1,0)r(n-1,0) + \cdots + b_0(1,0)r(1,0).
\end{aligned}$$

Then for $0 < k \leq n$,

$$\begin{aligned}
b_0(n,k) &= b_0(n-1,k-1) = \cdots = b_0(n-k+1,1) \\
&= b_0(n-k,0)r(n-k,0) + b_0(n-k-1,0)r(n-k-1,0) + \cdots + b_0(1,0)r(1,0).
\end{aligned}$$

Since $b_0(n-k,0)r(n-k,0) \geq 1$,

$$b_0(n,k) > b_0(n,k+1) \text{ and}$$

$$b_0(n,k) > b_0(n-1,k).$$

Likewise, we have

$$
\begin{aligned}
b_1(1,1) &= 1, \\
b_1(n,0) &= 0, \text{ and, for } n > 1, \\
b_1(n,n) &= r(1,1)r(2,2)\ldots r(n-1,n-1) \text{ and} \\
b_1(n,n-1) &= b_1(n-1,n-1)l(n-1,n-1) + b_1(n-1,n-2) \\
&= b_1(n-1,n-1)l(n-1,n-1) + \cdots + b_1(1,1)l(1,1).
\end{aligned}
$$

Then for $0 \leq k < n$,

$$
\begin{aligned}
b_1(n,k) &= b_1(n-1,k) = \cdots = b_1(k+1,k) \\
&= b_1(k,k)l(k,k) + \cdots + b_1(1,1)l(1,1).
\end{aligned}
$$

Since $b_1(k,k)l(k,k) \geq 1$,

$$b_1(n,k) > b_1(n,k-1).$$

□

We define X' to consist of the maximal set $X_{\max}$, its orbit, and the set of paths that never leave the far left or far right sides of the diagram:

Lemma 2.3.3. *Define $X' \subset X$ to consist of the following paths:*

1. $\mathcal{O}(X_{\max})$*;*

2. $\{\gamma \in X | k_n(\gamma) = 0 \ \forall n \in \mathbb{N}\}$*;.*

3. $\{\gamma \in X | k_n(\gamma) = dn \ \forall n \in \mathbb{N}\}$*.*

Then for any fully supported, T-invariant, ergodic measure μ, $\mu(X') = 0$.

Proof. Since $X_{\max}$ is countable, the sets of paths that never leave the far left ($k_n \equiv 0$) or far right ($k_n \equiv n$) sides of the diagram are proper closed T-invariant sets, and μ is ergodic, $\mu(X') = 0$. $\square$

LEMMA 2.3.4. *For $\gamma, \xi \in X \backslash X'$ if any of the following occur, then $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$:*

1. $B(n,k)$ starts in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a $B(n,k+1)$ starts in $\mathcal{P}(\xi)_{-\infty}^{\infty}$,

2. There are $m > n$, such that a $B(m,k)$ starts in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a $B(n,k)$ starts in $\mathcal{P}_{-\infty}^{\infty}(\xi)$,

3. There are $m > n$, such that a $B(m,k)$ starts in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a $B(n,k+1)$ starts in $\mathcal{P}_{-\infty}^{\infty}(\xi)$,

4. The end of a $B(n,k)$ in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ lines up with the end of a $B(n,k-1)$ in $\mathcal{P}_{-\infty}^{\infty}(\xi)$.

Proof. $B(1,0)$ consists of only the symbols associated to the edges leaving $(0,0)$ to the left, and $B(1,1)$ consists of only the symbols associated to the edges leaving $(0,0)$ to the right; hence $B(1,0)$ and $B(1,1)$ have no common symbols, and if any of their symbols appear in the same coordinate in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ and $\mathcal{P}_{-\infty}^{\infty}(y)$, then $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$. In case 1, Lemma 2.3.2 says that $b_0(n,k) > b_0(n,k+1)$, and therefore a symbol from $B(1,0)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a symbol from $B(1,1)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\xi)$. Then $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$.

In case 2, Lemma 2.3.2 says that $b_0(m,k) > b_0(n,k)$, since $m > n$; therefore a symbol from $B(1,0)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a symbol from $B(1,1)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\xi)$. Then $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$.

In case 3, Lemma 2.3.2 says that $b_0(m,k) > b_0(n,k) > b_0(n,k+1)$, since $m > n$; therefore a symbol from $B(1,0)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a symbol from $B(1,1)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\xi)$. Then $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$.

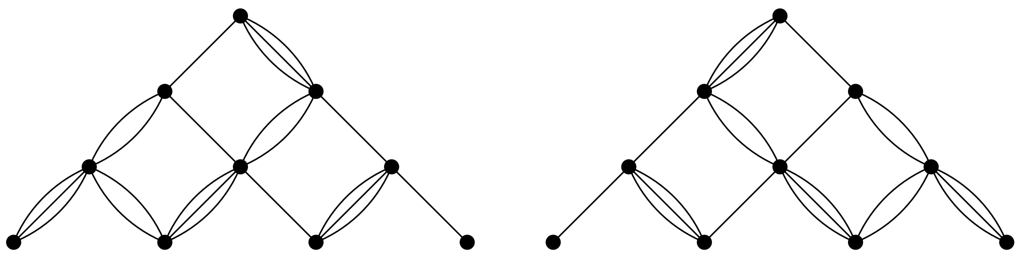


FIGURE 2.8. An example of D and $\overline{D}$

.

In case 4, Lemma 2.3.2 says that $b_1(n,k) > b_1(n,k-1)$; therefore a symbol from $B(1,1)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\gamma)$ in the same coordinate that a symbol from $B(1,0)$ appears in $\mathcal{P}_{-\infty}^{\infty}(\xi)$. Then $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$. □

DEFINITION 2.3.5. Let (X, T_X) be in $\mathcal{S}_{\mathcal{L}}$. For the Bratteli diagram associated to X define the *mirror image of* D to be the Bratteli diagram, $\overline{D}$, such that the number of edges connecting $\overline{(n,k)}$ to vertex $\overline{(n,k+j)}$ in for $j = 0, 1, \ldots, d$ in $\tilde{D}$ is the number of edges connecting the vertices $(n, dn-k)$ and $(n, dn-k+(d-j))$ in D. Define the *mirror image of* X to be the path space, $\overline{X}$, associated to $\overline{D}$. For any $\gamma \in X$ define the *mirror image of* γ to be the path $\overline{\gamma} \in \overline{X}$ such that the following hold:

1. $k_n(\overline{\gamma}) = dn - k_n(\gamma)$ for all n, and

2. If γ_n is the i'th edge connecting (n,k) to $(n+1, k+j)$, $\overline{\gamma_n}$ is the i'th from the last edge connecting vertices $\overline{(n, dn-k)}$ and $\overline{(n+1, dn-k+(d-j))}$.

See Figure 2.8.

LEMMA 2.3.6. *If* $\gamma, \xi \in X$ *with* $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$, *then for* $\tilde{\gamma}, \tilde{\xi} \in \tilde{X}$, $\mathcal{P}_{-\infty}^{\infty}(\tilde{\gamma}) \neq \mathcal{P}_{-\infty}^{\infty}(\tilde{\xi})$.

Proof. $T_X(\xi)$ is the mirror image of $T_{\tilde{X}}^{-1}(\gamma)$. $\mathcal{P}_{-\infty}^{\infty}(\gamma) \neq \mathcal{P}_{-\infty}^{\infty}(\xi)$ implies there is a $j \in \mathbb{Z}$ such that $T_X^j(\gamma)$ and $T_X^j(\xi)$ disagree in the first coordinate. This implies $T_{\tilde{X}}^{-j}(\tilde{\gamma})$ and $T_{\tilde{X}}^{-j}(\tilde{\xi})$ disagree in the first coordinate, hence $\mathcal{P}_{-\infty}^{\infty}(\tilde{\gamma}) \neq \mathcal{P}_{-\infty}^{\infty}(\tilde{\xi})$. $\square$

THEOREM 2.3.7. *Let $(X, T) \in \mathcal{S}_{\mathcal{L}}$ with $d = 1$, and let μ be a fully-supported T-invariant ergodic probability measure on X. Let Σ be the subshift defined above. Then there are a set $X' \subset X$ with $\mu(X') = 0$ and a one-to-one Borel measurable map $\phi : X \setminus X' \to \Sigma$ such that $\phi \circ T = \sigma \circ \phi$ on $X \setminus X'$.*

Proof. For each $\gamma \in X$ define $\phi(\gamma)$ to be the $\mathcal{P}$-name of γ. Then for all $\gamma \in X$,

$$\begin{aligned} \phi \circ T(\gamma) &= \dots \mathcal{P}(T^{-1}\gamma)\mathcal{P}(\gamma).\mathcal{P}(T\gamma)\mathcal{P}(T^2\gamma)\dots \\ &= \sigma(\dots \mathcal{P}(T^{-2}\gamma)\mathcal{P}(T^{-1}\gamma).\mathcal{P}(\gamma)\mathcal{P}(T\gamma)\dots) \\ &= \sigma \circ \phi(\gamma). \end{aligned}$$

It is clear that ϕ^{-1} of any cylinder in Σ is a union of cylinder sets in X, hence ϕ is Borel measurable. Defining X' as above, Lemma 2.3.3 tells us that $\mu(X') = 0$.

The strategy for showing ϕ is one-to-one is to show that for $\gamma, \xi \in X \setminus X'$, $\gamma \neq \xi$, there is a coordinate j such that either $\phi(\gamma)_j$ or $\phi(\xi)_j$ is a symbol from $B(1, 0)$ and the other is a symbol from $B(1, 1)$.

Take $\gamma, \xi \in X \setminus X'$ with $\gamma \neq \xi$. We will consider cases according to the different ways that $\gamma, \xi \in X \setminus X'$ disagree. Begin by assuming they first disagree at edge n and $(n, k_n(\gamma)) = (n, k_n(\xi))$ is an internal vertex; in other words, n is the first place such that $\gamma_n \neq \xi_n$, and $0 < k_n(\gamma) = k_n(\xi) < n$. Since $\phi(T^{-m}\gamma) \neq \phi(T^{-m}\xi)$ implies $\phi(\gamma) \neq \phi(\xi)$, we may assume that γ and ξ follow the minimal path into the vertex $(n, k_n(\gamma))$ by taking the appropriate m.

(I) First consider the case when both γ and ξ connect vertex $(n, k_n(\gamma))$ to vertex $(n+1, k_n(\gamma))$, where $0 < k_n(\gamma) < n$. For ease of notation when it is clear, we will denote $k_n(\gamma)$ by k. Without loss of generality assume $\gamma_n < \xi_n$ according to the edge ordering, say γ_n travels down the i'th edge and ξ_n travels down the j'th edge with $0 \leq i < j < l(n,k)$. Then

$$\phi(\gamma) = \ldots B(n,k-1)^{r(n,k-1)} B(n,k)^i . B(n,k)^{l(n,k)-i} \ldots$$

$$\phi(\xi) = \ldots B(n,k-1)^{r(n,k-1)} B(n,k)^j . B(n,k)^{l(n,k)-j} \ldots$$

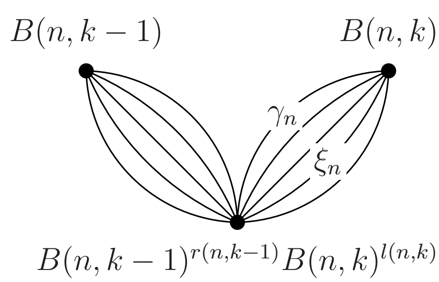


FIGURE 2.9. $r(\gamma_n) = r(\xi_n) = (n+1, k_n(\gamma))$.

Comparing $\phi(\gamma)$ and $\phi(\xi)$, we see that the end of a $B(n,k-1)$ in $\phi(\gamma)$ lines up with the end of a $B(n,k)$ from $\phi(\xi)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

(II) Consider the case when both γ_n and ξ_n connect vertex $(n, k_n(\gamma))$ to vertex $(n+1, k_n(\gamma)+1)$, where $0 < k_n(\gamma) < n$. Without loss of generality assume $\gamma_n < \xi_n$ according to the edge ordering, say γ_n travels down the i'th edge and ξ_n travels down the j'th edge with $0 \leq i < j < r(n,k)$. Then we have

$$\phi(\gamma) = \ldots (B(n,k))^i.(B(n,k))^{r(n,k)-i}B(n,k+1)^{l(n,k+1)}\ldots$$

$$\phi(\xi) = \ldots (B(n,k))^j.(B(n,k))^{r(n,k)-j}B(n,k+1)^{l(n,k+1)}\ldots$$

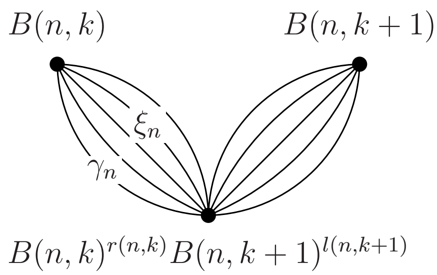


FIGURE 2.10. $r(\gamma_n) = r(\xi_n) = (n+1, k_n(\gamma)+1)$.

Comparing $\phi(\gamma)$ and $\phi(\xi)$, we see that a $B(n,k)$ starts in the same place in $\phi(\gamma)$ that a $B(n,k+1)$ starts in $\phi(\xi)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

(III) Now assume $r(\gamma_n) = (n+1,k+1)$ via the i'th edge and $r(\xi_n) = (n+1,k)$ via the j'th edge, where $0 < k_n(\gamma) = k_n(\xi) < n$; then

$$\phi(\gamma) = \ldots B(n,k)^i.B(n,k)^{r(n,k)-i}B(n,k+1)^{l(n,k+1)}\ldots$$

$$\phi(\xi) = \ldots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}\ldots$$

(A) If $i > j$, then the end of an $B(n,k)$ from $\phi(\gamma)$ lines up with the end of a $B(n,k-1)$ from $\phi(\xi)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

(B) If $l(n,k) - j > r(n,k) - i$, then the beginning of $B(n,k+1)$ in $\phi(\gamma)$ lines up with the beginning of $B(n,k)$ in $\phi(\xi)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

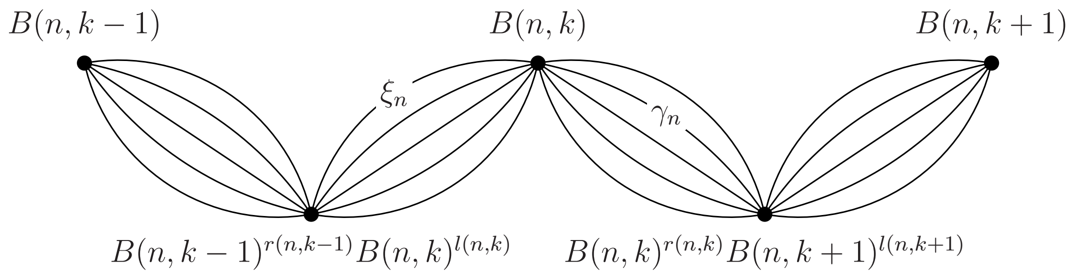


FIGURE 2.11. $r(\gamma_n) = (n+1, k_n(\gamma))$ and $r(\xi_n) = (n+1, k_n(\gamma)+1)$.

(C) Now assume that $i \leq j$ and $l(n,k) - j \leq r(n,k) - i$. Then the information we have about $\phi(\gamma)$ and $\phi(\xi)$ is insufficient to make any conclusions. We will consider the edges ξ_m for $m > n$ to determine more about $\phi(\xi)$ and then make a comparison with $\phi(\gamma)$.

(1) If ξ_{n+1} connects to $(n+2, k_n(\gamma)+1)$ and does not follow the maximal edge between $(n+1, k_n(\gamma))$ and $(n+2, k_n(\gamma)+1)$, then

$$\phi(\gamma) = \ldots B(n,k)^i.B(n,k)^{r(n,k)-i}B(n,k+1)^{l(n,k+1)} \ldots$$

$$\phi(\xi) = \ldots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}B(n+1,k) \ldots$$

If $l(n,k) - j < r(n,k) - i$, a $B(n+1,k)$ from $\phi(\xi)$ lines up with a $B(n,k)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$. If $l(n,k) - j = r(n,k) - i$, a $B(n+1,k)$ from $\phi(\xi)$ lines up with a $B(n,k+1)$ from $\phi(\gamma)$ and then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

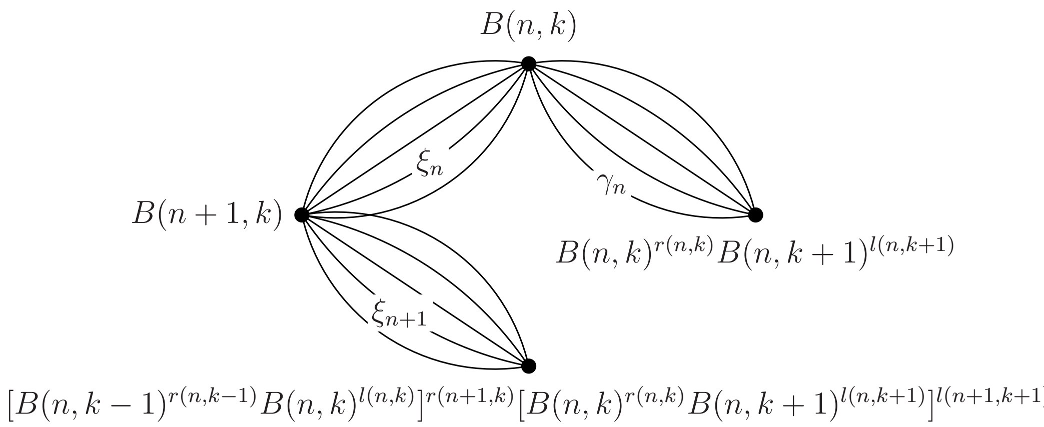


FIGURE 2.12. ξ extended to $(n+2, k_n(\gamma)+1)$ through a non-maximal edge.

(2) If ξ_{n+1} follows the $r(n+1, k_n(\gamma))-1$'st edge from $(n+1, k_n(\gamma))$ into $(n+2, k_n(\gamma)+1)$ we have

$$\begin{aligned}
\phi(\gamma) &= \ldots B(n,k)^i.B(n,k)^{r(n,k)-i}B(n,k+1)^{l(n,k+1)}\ldots \\
\phi(\xi) &= \ldots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}B(n+1,k+1)^{l(n+1,k+1)}\ldots \\
&= \ldots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}B(n,k)^{r(n,k)}B(n,k+1)^{l(n,k+1)}\ldots
\end{aligned}$$

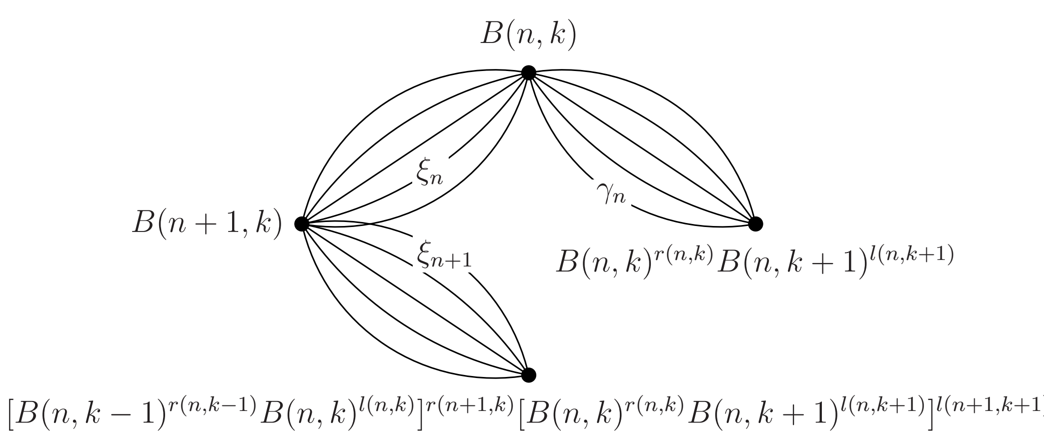


FIGURE 2.13. ξ extended to $(n+2, k_n(\gamma)+1)$ through the maximal edge.

But, $0 \leq j < l(n,k)$ implies $r(n,k) + (l(n,k) - j) > r(n,k) - i$; hence a $B(n,k+1)$ from $\phi(\gamma)$ lines up with a $B(n,k)$ from $\phi(\xi)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

(3) Now we will consider what happens if ξ continues left. In other words $r(\xi_{n+1}) = (n+2, k_n(\gamma))$.

(a) First consider the case when ξ_{n+1} does not follow a maximal edge. Then

$$\phi(\gamma) = \ldots B(n,k)^i.B(n,k)^{r(n,k)-i}B(n,k+1)^{l(n,k+1)}\ldots$$

$$\phi(\xi) = \ldots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}B(n+1,k)\ldots$$

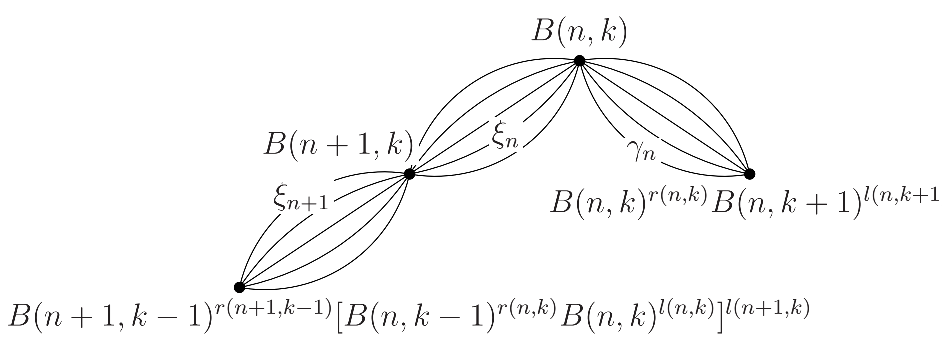


FIGURE 2.14. $r(\xi_{n+1}) = (n+2, k_n(\gamma))$ and ξ_{n+1} a non-maximal edge.

If $l(n,k) - j < r(n,k) - i$, we see that a $B(n+1,k)$ from $\phi(\xi)$ lines up with a $B(n,k)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$. If $l(n,k) - j = r(n,k) - i$, then a $B(n+1,k)$ from $\phi(\xi)$ lines up with a $B(n,k+1)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

(b) If ξ does follow the maximal edge into $(n+2, k_n(\gamma))$, then $\xi \notin X'$ implies that ξ is not in the orbit of $X_{\max}$. Hence there is an $m > n+1$ for which ξ_m is not maximal.

(i) If $r(\xi_m) = (m+1, k_n(\gamma))$, or $r(\xi_m) = (m+1, k_n(\gamma)+1)$ and ξ_m is not the maximal edge between $(m, k_n(\gamma))$ and $(m, k_n(\gamma)+1)$ then

$$\phi(\gamma) = \dots B(n,k)^i.B(n,k)^{r(n,k)-i}B(n,k+1)^{l(n,k+1)}\dots$$

$$\phi(\xi) = \dots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}B(m,k)\dots$$

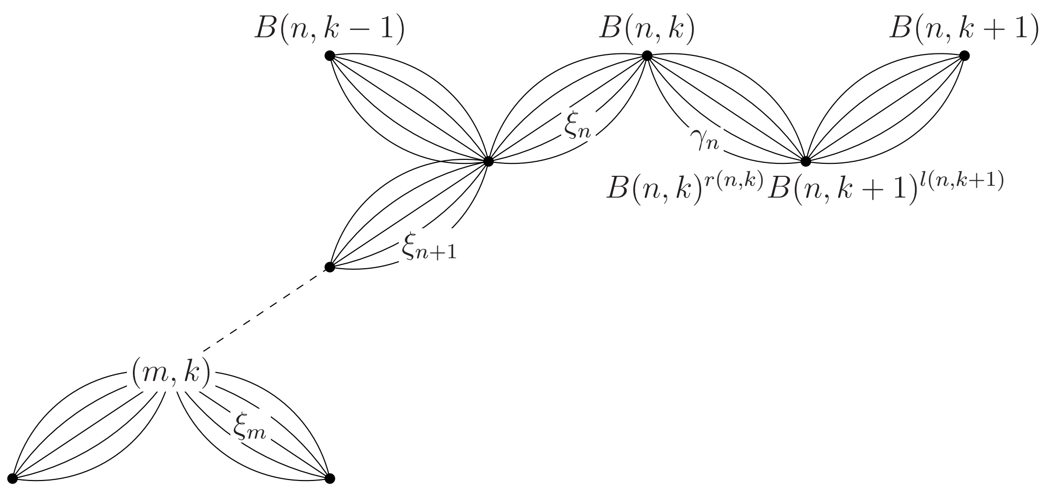


FIGURE 2.15. ξ extended to $(m+1, k_n(\gamma)+1)$ through a non-maximal edge.

If $l(n,k) - j < r(n,k) - i$ we see that a $B(m,k)$ from $\phi(\xi)$ lines up with a $B(n,k)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$. If $l(n,k) - j = r(n,k) - i$, then a $B(m,k)$ from $\phi(\xi)$ lines up with a $B(n,k+1)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

(ii) If ξ_m does follow the maximal edge between $(m, k_n(\gamma))$ and $(m+1, k_n(\gamma)+1)$ (note that while ξ_m is maximal between the vertices $(m, k_n(\gamma))$ and $(m+1, k_n(\gamma)+1)$

this edge is not in $X_{\max}$, see Figure 2.16) we have

$$\phi(\gamma) = \ldots B(n,k)^i.B(n,k)^{r(n,k)-i}B(n,k+1)^{l(n,k+1)}\ldots$$

$$\phi(\xi) = \ldots B(n,k-1)^{r(n,k-1)}B(n,k)^j.B(n,k)^{l(n,k)-j}B(m,k+1)\ldots$$

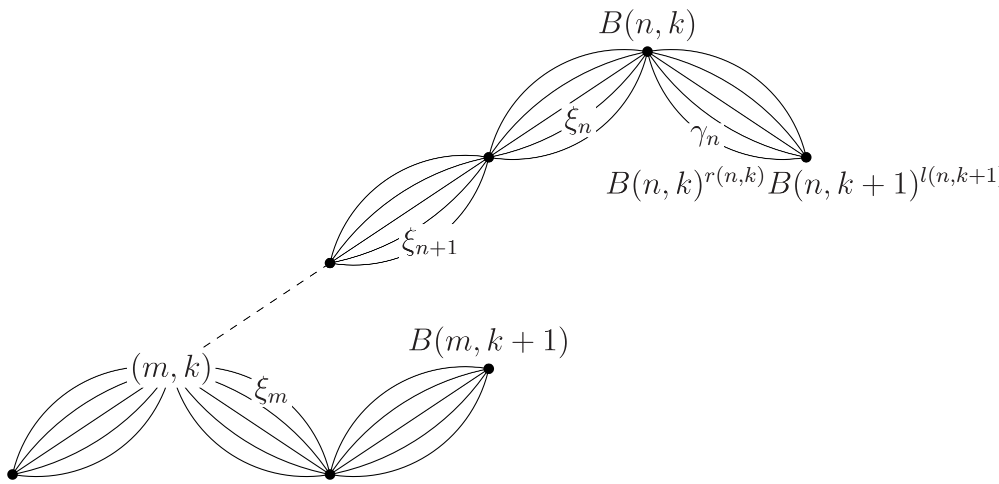


FIGURE 2.16. ξ extended to $(m+1, k_n(\gamma)+1)$ through the maximal edge.

If $l(n,k) - j < r(n,k) - i$ then a $B(m,k+1)$ from $\phi(\xi)$ lines up with a $B(n,k)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$. If $l(n,k) - j = r(n,k) - i$, then a $B(m,k+1)$ from $\phi(\xi)$ lines up with a $B(n,k+1)$ from $\phi(\gamma)$. Then by Lemma 2.3.4 $\phi(\gamma) \neq \phi(\xi)$.

This covers all the cases where the first place that γ and ξ disagree is at an internal vertex. We still need to determine that $\phi(\gamma) \neq \phi(\xi)$ if we assume they first disagree at edge n and $k_n(\gamma) = k_n(\xi) = 0$ or n.

(IV) We will begin with the case when γ and ξ first disagree at level n and $k_n(\gamma) = k_n(\xi) = 0$.

(A) Assume first that γ and ξ leave the far left side of the diagram at different times, m_1 and m_2 respectively down the i'th and j'th edges respectively with $0 \leq i \leq r(m_1, 0) - 1$ and $0 \leq j \leq r(m_2, 0) - 1$. Without loss of generality assume $m_2 > m_1 \geq n$. Since we

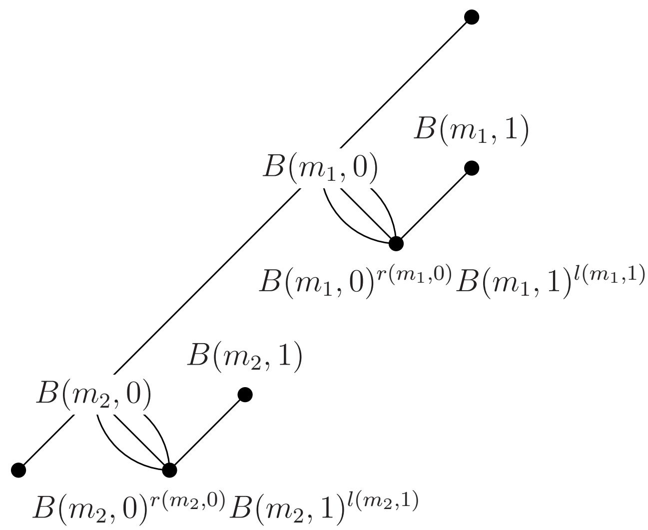


FIGURE 2.17. γ and ξ leave the far left edge at different times

are not necessarily looking at the first place γ and ξ disagree, we can no longer assume they are minimal into the vertices $(m_1, 0)$ and $(m_2, 0)$. Looking at Figure 2.17, we see that there is a nonempty (provided $m_1 \neq 0$ in which case $\phi(\gamma) \neq \phi(\xi)$ is clear) subblock $\gamma_0\gamma_1 \ldots \gamma_{s_\gamma - 1}$ of $B(m_1, 0)$ such that

$$\phi(\gamma) = \ldots . \gamma_0\gamma_1 \ldots \gamma_{s_\gamma-1} B(m_1, 0)^{r(m_1,0)-i-1} B(m_1, 1) \ldots .$$

Likewise there is a suffix $\xi_0\xi_1 \ldots \xi_{s_\xi - 1}$ of $B(m_2, 0)$ such that

$$\phi(\xi) = \ldots . \xi_0\xi_1 \ldots \gamma_{s_\xi-1} B(m_2, 0)^{r(m_2,0)-j-1} B(m_2, 1) \ldots .$$

Now the first $B(1,1)$ after the decimal place appears in $\phi(\gamma)$ in position

$$
\begin{aligned}
&s_\gamma - 1 + |B(m_1,0)|(r(m_1,0) - i - 1) + b_0(m_1,1)|B(1,0)| \\
\leq &|B(m_1,0)|(r(m_1,0) - i) + b_0(m_1,1)|B(1,0)| - 1 \\
= &|B(1,0)|[b_0(m_1,0)(r(m_1,0) - i) + b_0(m_1,1)] - 1 \\
= &|B(1,0)|[b_0(m_1,0)(r(m_1,0) - i) + b_0(m_1 - 1,0)r(m_1 - 1,0) + \cdots + b_0(1,0)r(1,0)] - 1 \\
= &P_1.
\end{aligned}
$$

The first $B(1,1)$ after the decimal place appears in $\phi(\xi)$ in position

$$
\begin{aligned}
&s_\xi - 1 + |B(m_2,0)|(r(m_2,0) - j - 1) + b_0(m_2,1)|B(1,0)| \\
\geq &|B(m_2,0)(r(m_2,0) - j - 1) + b_0(m_2,1)|B(1,0)| \\
= &|B(1,0)|[b_0(m_2,0)(r(m_2,0) - j - 1) + b_0(m_2,1)] \\
\geq &|B(1,0)|(b_0(m_2,1)) \\
= &|B(1,0)|(b_0(m_2 - 1,0)r(m_2 - 1,0) + \cdots + b_0(m_1,0)r(m_1,0) + \cdots + b_0(1,0)r(1,0)). \\
> &P_1.
\end{aligned}
$$

Since $m_2 > m_1$, the first positive position that a symbol from $B(1,1)$ appears in $\phi(\gamma)$ is strictly less than the first positive position a symbol from $B(1,1)$ appears in $\phi(\xi)$. Hence $\phi(\gamma) \neq \phi(\xi)$.

(B) Now assume that both γ and ξ leave the far left side of the diagram from the same vertex, $(m,0)$, with γ_m and ξ_m through the i'th and j'th edges respectively with

$0 \leq i < j < r(m,0)$. There are $1 \leq s_\gamma, s_\xi \leq |B(m,0)|$, such that

$$\begin{aligned}
\phi(\gamma) &= \ldots .\gamma_0\gamma_1 \ldots \gamma_{s_\gamma - 1} B(m,0)^{r(m,0)-i-1} B(m,1)\ldots \\
\phi(\xi) &= \ldots .\xi_0\xi_1 \ldots \xi_{s_\xi - 1} B(m,0)^{r(m,0)-j-1} B(m,1)\ldots
\end{aligned}$$

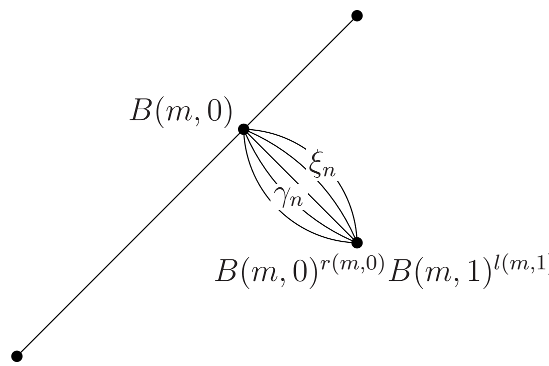


FIGURE 2.18. $\gamma_m \neq \xi_m$ leave the far left side of the diagram from the same vertex

Since $i < j$ the position of the first $B(1,1)$ in $\phi(\gamma)$ to appear after the decimal is

$$\begin{aligned}
& s_\gamma - 1 + |B(m,0)|(r(m,0) - i - 1) + b_0(m,1)|B(1,0)| \\
\geq & |B(m,0)|(r(m,0) - i - 1) + b_0(m,1)|B(1,0)| \\
\geq & |B(m,0)|(r(m,0) - j) + b_0(m,1)|B(1,0)| \\
> & |B(m,0)|(r(m,0) - j) + b_0(m,1)|B(1,0)| - 1 \\
\geq & s_\xi - 1 + |B(m,0)|(r(m,0) - j - 1) + b_0(m,1)|B(1,0)|,
\end{aligned}$$

which is the position of the first $B(1,1)$ in $\phi(\xi)$ to appear after the decimal. Hence $\phi(\gamma) \neq \phi(\xi)$.

(C) Assume γ and ξ leave the far left side of the diagram at the same time, along the same i'th edge connecting to vertex $(m+1, 1)$. Then we have

$$\phi(\gamma) = \ldots .\gamma_0\gamma_1 \ldots \gamma_{s_\gamma - 1} B(m,0)^{r(m,0)-i-1} B(m,1) \text{ and}$$
$$\phi(\xi) = \ldots .\xi_0\xi_1 \ldots \xi_{s_\xi - 1} B(m,0)^{r(m,0)-i-1} B(m,1).$$

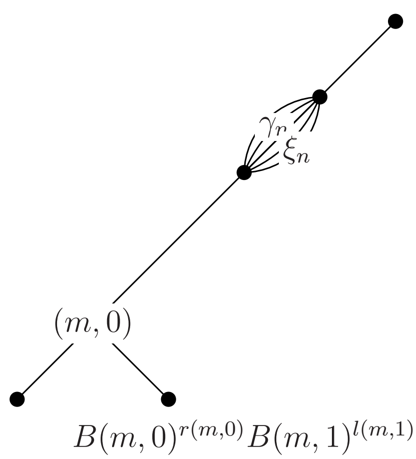


FIGURE 2.19. $\gamma_m = \xi_m$ leave the far left side of the diagram

Since this is the case for which γ and ξ disagree first on the far left side of the diagram, we know that at some point earlier, γ and ξ disagreed. Hence they take different paths into vertex $(m, 0)$, and therefore $s_\gamma \neq s_\xi$. This implies that the first $B(1,1)$ in $\phi(\gamma)$ appears in a different position than the first $B(1,1)$ in $\phi(\xi)$. Hence, $\phi(\gamma) \neq \phi(\xi)$.

(V) Similarly, the edges could first disagree on the far right side of the diagram. Using the above three arguments on $\tilde{X}$, the cases in which $k_n(\gamma) = k_n(\xi) = n$ are taken care of by Lemma 2.8. Hence ϕ is one-to-one.

$\square$

Corollary 2.3.8. *If (X, T) is a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ such that $d = 1$, with fully supported, T-invariant, ergodic measure μ and Borel sets $\mathcal{B}$, the partition $\mathcal{P}$ is a generating partition.*

While we only know that this partition is generating on systems in $\mathcal{S}_{\mathcal{L}}$ with $d = 1$, we can define such a partition by the first edge for any system in $\mathcal{S}_{\mathcal{L}}$, and define (Σ, σ) as in Definition 2.3.1.

Lemma 2.3.9. *For large n the number of words of length n appearing in Σ is bounded above by a polynomial in n (and hence has topological entropy 0).*

Proof. Recall that for each vertex (n, k) in V_n and a path $\gamma \in Y_n(k, 0)$, $B(n, k) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma)\ldots\mathcal{P}(T^{\dim(n,k)-1}\gamma)$.

At each level l, we determine the maximum possible number of new words of length n formed by concatenating two words $B(l, k_1)$ and $B(l, k_2)$. The concatenation of $B(l, k_1)$ and $B(l, k_2)$ can form at most $n - 1$ new words. Since there are $dl + 1$ vertices, there are $(dl + 1)^2$ possible distinct concatenations. Hence there are at most $(dl + 1)^2(n - 1)$ new words formed by concatenation.

At level n all blocks except possibly $B(n, 0)$ and $B(n, dn + 1)$ have length at least n. Concatenating $B(n, 0)$ and $B(n, 1)$ creates the word $B(1, 0)^n$. For all levels $m \geq 0$, the edges of the diagrams dictate that $B(m, 0)$ only joins with $B(m, 1)$, hence the concatenation of $B(m, 0)$ and $B(m, 1)$ at levels $m \geq n$ will only create $B(1, 0)$ across their juncture and hence no longer create words that have not been seen before. Likewise for $B(n, dn - 1)$, $B(n, dn)$, and $B(1, d)^n$. All other blocks at level n are of length at least

n. Since all words in subsequent levels are created by some concatenations of entries on level n, no more new words are formed.

Therefore the number of words of length n is bounded above by

$$\sum_{l=1}^{n}(dl+1)^2(n-1) \leq n^2(dn+1)^2$$
$$\leq d^2n^4+2dn^3+n^2.$$

□

DEFINITION 2.3.10. The *entropy of a partition* $P = \{A_1, \ldots, A_n\}$ with respect to a measure μ is

$$H_\mu(P) = -\sum_{i=1}^{l}\mu(A_i)\log\mu(A_i).$$

The *entropy of a system* (X, T, μ) with respect to the partition P is

$$h_\mu(P,T) = \lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{i=0}^{n-1}T^iP\right).$$

The *entropy of the system* (X, T, μ) is

$$h_\mu(T) = \sup\{h_\mu(P,T)|P \text{ is a finite partition}\}.$$

See [**37**] for further development of the theory of entropy.

THEOREM 2.3.11. *Let (X, T) be a Bratteli-Vershik system in $\mathcal{S}_\mathcal{L}$ with a T-invariant measure μ. Then (X, T, μ) has entropy 0.*

Proof. One may replace the partition in the proof of Theorem 2.3.9 by the partition by the first l edges, $\mathcal{P}_l$, and use the subshift (Σ_l, σ) corresponding to the partition $\mathcal{P}_l$. A similar counting argument will yield that the number of n-blocks in Σ_l is again bounded by a polynomial in n. Now for each $n = 1, 2, \ldots,$

$$H_\mu\left(\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l\right) = -\sum_{A\in\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l} \mu(A)\log(\mu(A))$$
$$\leq H_{ud}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l\right),$$

where ud is the measure on $\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l$ that gives each element equal measure. If p_n is the cardinality of $\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l$, then

$$H_{ud}\left(\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l\right) = -\sum_{i=1}^{p_n} \frac{1}{p_n}\log\left(\frac{1}{p_n}\right)$$
$$= \log(p_n).$$

Since there is a constant c_l such that $p_n \leq n^{c_l}$ for all large n, the entropy of the system with respect to the partition $\mathcal{P}_l$ is

$$h_\mu(\mathcal{P}_l, T) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^i\mathcal{P}_l\right) \leq \lim_{n\to\infty} \frac{c_l}{n}\log(n) = 0.$$

Now let $\mathcal{B}_l$ be the σ-algebra generated by $\mathcal{P}_l$ and $\mathcal{B}$ the σ-algebra of (X, T). Since $\mathcal{B}_l \nearrow \mathcal{B}$, $h_\mu(T)$ is the limit of $h_\mu(\mathcal{P}_l, T)$. Hence the entropy of (X, T, μ) is 0. $\square$

CHAPTER 3

Polynomial Systems

3.1. Description

In this section we will describe how every positive integer polynomial of degree d determines a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$; we will denote this subfamily of systems by $(\mathcal{S}_{\mathcal{L}})_{p(x)}$. We will then label edges with an alphabet and denote an infinite edge path by its corresponding infinite edge labeling.

Definition 3.1.1. Let $a_0, a_1, ..., a_d \in \mathbb{N}$ and $p(x) = a_0 + a_1 x + ... + a_d x^d$. Define the *Bratteli diagram associated to* $p(x)$, $(\mathcal{V}, \mathcal{E})_{p(x)}$, as follows:

1. $|\mathcal{V}_0| = 1$ and $|\mathcal{V}_n| = |\mathcal{V}_{n-1}| + d = nd + 1$ for all $k > 0$.
2. The number of edges from (n, k) to $(n+1, k+j)$ is a_j, with a_j=0 for $j > d$ and $j < 0$.

We will denote the path space by $X_{p(x)}$ and the transformation by $T_{p(x)}$. Recall that for a cylinder set C in $X_{p(x)}$, $\dim(C, (n, k_n(\gamma)))$ is the number of paths from the terminal vertex (m, l) of C, to the vertex $(n, k_n(\gamma))$. Define a function $\text{coeff}_{p(x)} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $\text{coeff}_{p(x)}(n, k) =$ the coefficient of x^k in the polynomial $(p(x))^n$. Because of the self-similarity of this class of Bratteli diagrams, if C terminates at vertex (m, l),

$$\dim(C, (n, k_n(\gamma))) = \text{coeff}_{p(x)}(n - m, k_n(\gamma) - l) \text{ and}$$

$$\dim(n, k) = \text{coeff}_{p(x)}(n, k).$$

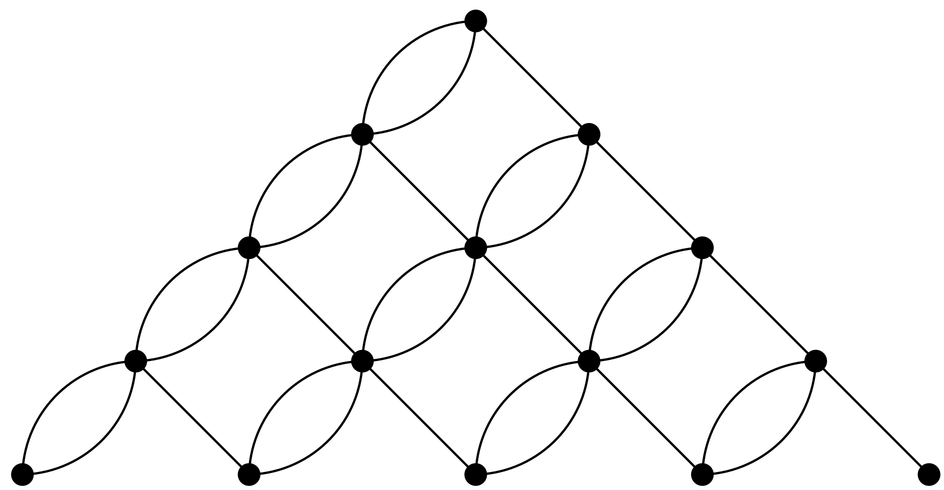


FIGURE 3.1. The first four levels of $(\mathcal{V}, \mathcal{E})_{2+x}$

For $n = 0, 1, \ldots$ and $d \leq k \leq d(n-1)$, the number of edges into vertex (n, k) is exactly $a_0 + a_1 + \cdots + a_d$. In addition, for every vertex (n, k) the number of edges leaving (n, k) is exactly $a_0 + \cdots + a_d$. Because of this it is convenient to use an alphabet to label the edges of paths in $X_{p(x)}$. The alphabet associated to $X_{p(x)}$ will be $A = \{0, 1, \ldots, a_0 + a_1 + \cdots + a_d - 1\}$. Begin with the leftmost edge connecting $(0, 0)$ with $(1, d)$, labeling it 0. Then label the next edge directly to the right with 1. Continue in this manner until all a_d edges from $(0, 0)$ to $(1, d)$ have been labeled in order from 0 to $a_d - 1$. Now move to the edges connecting $(0, 0)$ to $(1, d-1)$, and label the leftmost edge a_d, and as before move to the right, labeling up through $a_d + a_{d-1} - 1$. Repeat this process until all edges leaving $(0, 0)$ are labeled. In the same manner, label edges leaving each vertex (n, k); see Figure 3.2. By labeling in this manner, the lexicographic ordering on comparable edges is consistent with the edge ordering given for the general family $\mathcal{S}_{\mathcal{L}}$. Then any path in $X_{p(x)}$ is uniquely determined by the labeling of its edges, and because of this, we use both X and a one sided infinite sequence in $A^{\mathbb{N}}$ to denote the infinite edge paths on a Bratteli diagram.

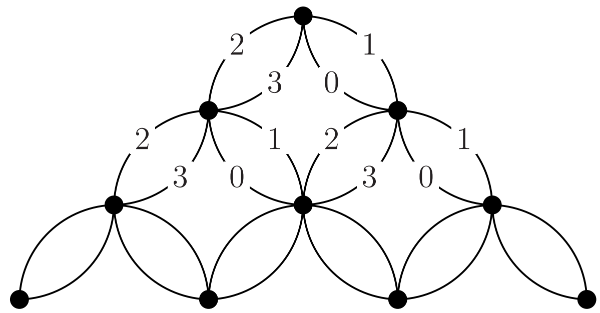


FIGURE 3.2. Labeling of the Bratteli-Vershik system $(\mathcal{V}, \mathcal{E})_{2+2x}$

For ease of notation we will refer to a path by the infinite labeling of its edges, and when the context is clear, we will refer to an edge by its label. With this labeling the sets $(X_{p(x)})_{\max}$ and $(X_{p(x)})_{\min}$ can be described in terms of the alphabet. Let l be the edge label $a_d + a_{d-1} + \cdots + a_0 - 1$. For any vertex (n, k) the maximal edge connecting (n, k) to $(n+1, k)$ is labeled l. Define $M = \{a_d - 1, a_d + a_{d-1} - 1, \ldots, a_d + a_{d-1} + \cdots + a_1 - 1\}$; this is the set of remaining labels on the maximal edges with range in level 1. Then the set of maximal paths is

$$(X_{p(x)})_{\max} = \{(a_d - 1)^\infty, (a_d - 1)^j m l^\infty, l^\infty | m \in M, \quad j = 0, 1, 2, \ldots\}.$$

Likewise let s be the edge label $a_d + \cdots + a_1$. Any minimal edge connecting a vertex (n, k) to vertex $(n+1, k)$ is labeled s. Then define $R = \{0, a_d, a_d + a_{d-1}, \ldots, a_d + \cdots + a_2\}$; this is the set of remaining labels on the minimal edges with range in level 1. Then the set of minimal paths is

$$(X_{p(x)})_{\min} = \{0^\infty, \ s^j r 0^\infty, s^\infty | r \in S, \ j = 0, 1, 2, \ldots\}.$$

With this new description of $(X_{p(x)})_{\max}$ and $(X_{p(x)})_{\min}$ in terms of the alphabet, we can also describe the adic transformation on the maximal paths as follows.

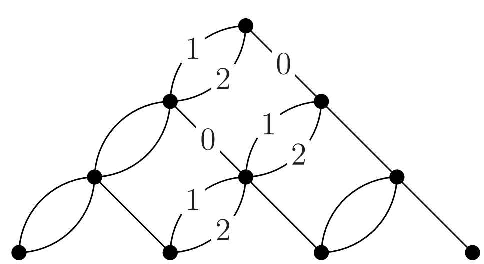
Remark 3.1.2. The adic transformation $T_{p(x)} : X_{p(x)} \to X_{p(x)}$ is described by the following:

$$T_{p(x)}(\gamma) = \begin{cases} \text{the smallest } \xi > \gamma & \text{if } \gamma \notin (X_{p(x)})_{\max} \\ (0)^\infty & \text{if } \gamma = (a_d - 1)^\infty \\ s^\infty & \text{if } \gamma = p^\infty \\ s^j r 0^\infty & \text{if } (a_d - 1)mp^\infty \end{cases}$$

This description of the adic transformation is the same as the definition given on general systems in $\mathcal{S}_{\mathcal{L}}$.

Example 3.1.1. For T_{2+x} acting on $(\mathcal{V}, \mathcal{E})_{2+x}$ we have,

$$\begin{aligned} T_{2+x}(101\ldots) &= 201\ldots \\ T_{2+x}(201\ldots) &= 011\ldots \\ T_{2+x}(011\ldots) &= 021\ldots \\ T_{2+x}(021\ldots) &= 102\ldots \\ &\vdots \end{aligned}$$



Remark 3.1.3. A path $\gamma \in (X_{p(x)}, T_{p(x)})$ is *eventually diagonal to the left* (recall Definition 2.1.3) if the tail of γ is a string on the alphabet $\{s, s+1, \ldots, s+a_0-1\}$. In other words, there exists an $N \geq 0$ such that for all $n \geq N$, $\gamma_n \in \{s, s+1, \ldots, s+a_0-1\}$. We say a path ξ is *eventually diagonal to the right* (recall Definition 2.1.3) if the tail of

ξ is a string on the alphabet $\{0, 1, \ldots, a_d - 1\}$. In other words, there exists and $M \geq 0$ such that for $m \geq M$, $\xi_m \in \{0, 1, \ldots, a_d - 1\}$.

3.2. Invariant Ergodic Probability Measures

DEFINITION 3.2.1. Let A be a finite alphabet. Define a function $p : A \to [0, 1]$ such that $\sum_{a \in A} p(a) = 1$. A measure μ on $A^{\mathbb{N}}$ is said to be *Bernoulli* if for any cylinder $C = [a_0 a_1 \ldots a_n]$, $\mu(C) = p(a_0)p(a_1)\ldots p(a_n)$.

DEFINITION 3.2.2. Consider some cylinder set $C = [c_0 c_1 ... c_{n-1}] \in X_{p(x)}$ and any $T_{p(x)}-$invariant Borel probability measure μ on $X_{p(x)}$. Define the *weight* w_{c_0} on the edge c_0 to be $\mu([c_0])$. For $n > 0$ and $\mu([c_0 c_1 \ldots c_{n-1}]) = 0$ define the weight w_{c_n} on c_n to be 0. For $n > 0$ and $\mu([c_0 \ldots c_{n-1}]) > 0$ define w_{c_n} on c_n to be $\mu([c_0 ... c_n])/\mu([c_0 ... c_{n-1}])$. Then $\mu([c_0 ... c_n]) = w_{c_0} ... w_{c_n}$.

These weights are well defined because as we will see in Lemma 3.2.6, all cylinders with the same terminal vertex have the same measure.

REMARK 3.2.3. Recall that if $p(x) = a_0 + a_1 x + \ldots a_d x^d$, the alphabet determined by $(X_{p(x)}, T_{p(x)})$ is $A = \{0, 1, \ldots, a_0 + \cdots + a_d - 1\}$. In this section we discuss measures, $T_{p(x)}$-invariant Borel probability measures for which edges with the same label have the same weight. Then for the probability space $(X_{p(x)}, \mathcal{B}, \mu)$ there are at most $a_0 + a_1 + \cdots + a_d$ different weights. For each $j \in A$ we will denote by w_j the weight associated to each edge labeled j. Since $(X_{p(x)}, \mathcal{B}, \mu)$ is a probability space, $\sum_{i=0}^{a_0 + \cdots + a_d - 1} a_i w_i = 1$. In view of the labeling of paths by the alphabet A, these measures are Bernoulli and we denote such a measure by $\mathcal{B}(w_{a_0 + \cdots + a_d - 1}, \ldots, w_0)$.

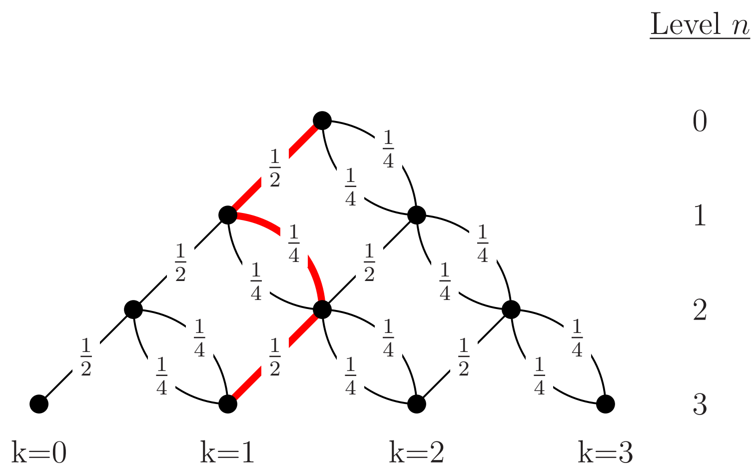


FIGURE 3.3. The measure of the red cylinder is 1/16.

In [**32**], Méla showed that when all the coefficients of $p(x)$ are 1, the invariant ergodic probability measures for $T_{p(x)}$ are the Bernoulli measure $\mathcal{B}(0, ..., 0, 1)$ and the one-parameter family $\mathcal{B}(q, t_q, \frac{t_q^2}{q}, \frac{t_q^3}{q^2}, .., \frac{t_q^n}{q^{n-1}})$, where t_q is the unique solution in [0,1] to the equation

$$q^n - q^{n-1} + q^{n-1}t + q^{n-2}t^2 + ... + qt^{n-1} + t^n = 0.$$

Using similar techniques we have extended this result to the following:

THEOREM 3.2.4. *Let $p(x) = a_0 + \dots a_d x^d$ and let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by $p(x)$. If $q \in (0, \frac{1}{a_0})$, and t_q is the unique solution in $[0, 1]$ to the equation*

$$a_0 q^d + a_1 q^{d-1} t + ... + a_d t^d - q^{d-1} = 0, \tag{3.2.1}$$

then the invariant, fully supported, ergodic probability measures for the adic transformation $T_{p(x)}$ are the one-parameter family of Bernoulli measures

$$\mathcal{B}\left(\underbrace{q, ..., q}_{a_0\ times}, \underbrace{t_q, ..., t_q}_{a_1\ times}, \underbrace{\frac{t_q^2}{q}, ..., \frac{t_q^2}{q}}_{a_2\ times}, \dots, \underbrace{\frac{t_q^n}{q^{n-1}}, ..., \frac{t_q^n}{q^{n-1}}}_{a_n\ times}\right).$$

Proposition 3.2.5. *Let* $p(x) = a_0 + \dots a_d x^d$ *and* $(X_{p(x)}, T_{p(x)})$ *be the Bratteli-Vershik system determined by* $p(x)$. *The only* $T_{p(x)}$ *invariant, ergodic probability measures that are not fully supported are the Bernoulli measures*

$$\mathcal{B}\left(\underbrace{\frac{1}{a_0}, \dots, \frac{1}{a_0}}_{a_0\ times}, 0, \dots, 0\right) \text{ and } \mathcal{B}\left(0, \dots, 0, \underbrace{\frac{1}{a_n}, \dots, \frac{1}{a_n}}_{a_n\ times}\right).$$

The proofs of Theorem 3.2.4 and Proposition 3.2.5 use many other results and definitions, which are presented below. Proposition 3.2.10 shows that every invariant fully supported ergodic probability measure for $(X_{p(x)}, T_{p(x)})$ must be Bernoulli. Proposition 3.2.12 says that the Bernoulli measures that are $T_{p(x)}$-invariant are in fact ergodic. Proposition 3.2.13 shows which Bernoulli measures are $T_{p(x)}$-invariant. This will prove Theorem 3.2.4. We will then conclude with the proof of Proposition 3.2.5.

Lemma 3.2.6. *Any non-atomic measure on* $X_{p(x)}$ *is* $T_{p(x)}$*-invariant if and only if all cylinders with the same terminal vertex have the same measure.*

Proof. ($\Rightarrow$) Let $(n,k) \neq (0,0)$ be a vertex of $(\mathcal{V}, \mathcal{E})_{p(x)}$. Consider the maximal path from (0,0) to (n,k) and the cylinder set, $C_{\max}$, defined by this path. There exists an i such that $T^{-i}_{p(x)}(C_{\max})$ is the minimal cylinder $Y_n(k,0)$ determined by the minimal path from $(0,0)$ to (n,k). Since the measure is $T_{p(x)}$-invariant, the elements of the set $\{T^{-j}_{p(x)}(C_{\max})\}_{j=0}^{i}$ all have the same measure. Because all the cylinders with terminal vertex (n,k) are contained in the above set, all cylinders with terminal vertex (n,k) have the same measure.

($\Leftarrow$) It is enough to show that for each cylinder set C, $T^{-1}_{p(x)}C$ has the same measure as C. Let C be any cylinder set, with terminal vertex (n,k).

Case 1: Suppose that C is not minimal. Since C is not minimal, $T_{p(x)}^{-1}(C)$ has terminal vertex (n, k), and hence the same measure as C.

Case 2: Suppose that $C = [0^j]$. Let A be the alphabet of edge labels, $\{0, 1, ..., \sum_{m=0}^{d} a_m - 1\}$, then $C = \cup_{i \in A} C^i$, where C^i is the cylinder set extended by i. Then $C = C^0 \cup (\cup_{i \in A, i \neq 0} C^i)$. Decompose $C^0 = C^{00} \cup (\cup_{i \in A, i \neq 0} C^{0i})$, with C^{0i} non-minimal. Repeating the process we can see that C is a disjoint union of non-minimal cylinders and a unique minimal path. Since the measure is non-atomic, by Case 1, the measure of C equals the measure of $T_{p(x)}^{-1}(C)$.

Case 3: Suppose that (as above) $s = a_d + \cdots + a_1$, $r \in R = \{0, a_d, a_d + a_{d-1}, ..., a_d + a_{d-1} + ... + a_2\}$, and C is the minimal cylinder $[s^j r]$. As before, write $C = C^0 \cup (\cup_{i \in A, i \neq 0} C^i)$, with C^i non-minimal. Decompose $C^0 = C^{00} \cup (\cup_{i \in A, i \neq 0} C^{0i})$, with C^{0i} non-minimal. Repeating this process we see that C is a disjoint union of non-minimal cylinders and a unique minimal path. Since the measure is non-atomic, the measure of $C =$ measure $T_{p(x)}^{-1}(C)$.

Case 4: Let $C = [s^j] = [s^{j+1}] \cup (\cup_{r \in R} C^r) \cup (\cup_{i \in A \setminus R, i \neq s} C^i)$. Then for each $i \in A \setminus R$ $(i \neq s)$, C^i is non-minimal; and for each $r \in R$, C^r is as in Case 3. Repeating this process on $[s^{j+1}]$, C is a disjoint union of non-minimal cylinders and a countable number of minimal paths. Since the measure is non-atomic, the measure of $C =$ measure $T_{p(x)}^{-1}(C)$.

$\square$

Recall the following Lemma from Section 1.2.

LEMMA 3.2.7 (Vershik [**46, 45**]). *If μ is an invariant non-atomic ergodic probability measure for the adic transformation $T_{p(x)}$, then for every cylinder set C,*

$$\mu(C) = \lim_{n \to \infty} \frac{\dim(C, (n, k_n(\gamma))}{\dim(n, k_n(\gamma))} \text{ for } \mu\text{-a.e. } \gamma \in X$$

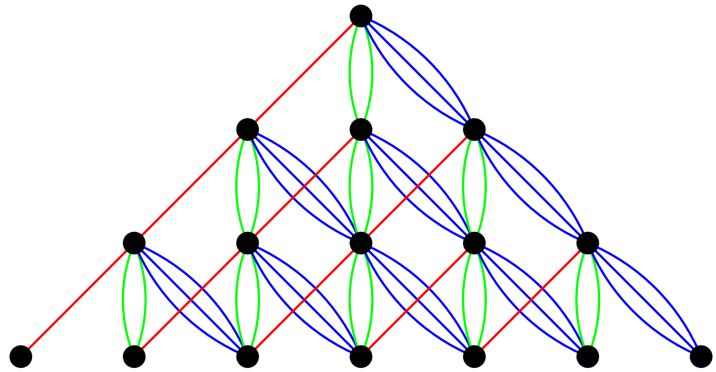


FIGURE 3.4. g(red)=0, g(green)=1, and g(blue)=2

The following lemma shows that the invariant and ergodic probability measures for $T_{p(x)}$ are also invariant for the one-sided shift σ on $A^{\mathbb{N}}$.

LEMMA 3.2.8. *For each $\gamma \in X_{p(x)}$ and $j \in A$, let $\sigma_j\gamma = j\gamma_0\gamma_1....$ If μ is invariant and ergodic for $T_{p(x)}$, then for any cylinder set C,*

$$\mu(C) = \mu(\sigma_0 C) + \mu(\sigma_1 C) + ... + \mu(\sigma_{a_d+...+a_0-1}C) = \mu(\sigma^{-1}C).$$

Proof. Define $g : A \to \{0, 1, ..., d\}$ as follows:

$$g(j) = \begin{cases} d & if\ 0 \leq j \leq a_d - 1 \\ d-1 & if\ a_d \leq j \leq a_d + a_{d-1} - 1 \\ \vdots & \\ 0 & if\ a_{d-1} + ... + a_1 \leq j \leq a_d + a_{d-1} + ... + a_0 - 1 \end{cases} \tag{3.2.2}$$

In other words, if the letter $j \in A$ is the label of an edge which connects vertex (n, k) to $(n, k+i)$, then $g(j) = i$ (see Figure 3.4). Then for any cylinder set $C = [c_0...c_{n-1}]$ which terminates at vertex (n, k), we have $\sum_{i=0}^{n-1} g(c_i) = k$.

Let C be a cylinder set with terminal vertex (m, l). For almost every γ in X and each $j \in A$ we have

$$\mu(\sigma_j C) = \lim_{n\to\infty} \frac{\dim(\sigma_j C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}.$$

The terminal vertex of C^j is $(m+1, l+g(j))$, and since the first $m+1$ edges of $\sigma_j C$ are a permutation of those of C^j, the terminal vertex of $\sigma_j C$ is also $(m+1, l+g(j))$. Hence for all $n > m$, $\dim(C^j, (n, k_n(\gamma))) = \dim(\sigma_j C, (n, k_n(\gamma)))$.

The set of finite paths starting from (m, l) and ending at $(n, k_n(\gamma))$ can be divided into $a_d + ... + a_0$ groups, according to whether the edge is labeled $0, 1, \dots (\sum_{i=0}^{d} a_i) - 1$. Then we have:

$$\begin{aligned}
\dim(C, (n, k_n(\gamma))) &= \dim(C^0, (n, k_n(\gamma))) + ... + \dim(C^{(\sum_{i=0}^d a_i)-1}, (n, k_n(\gamma))) \\
&= \dim(\sigma_0 C, (n, k_n(\gamma))) + ... + \dim(\sigma_{(\sum_{i=0}^d a_i)-1} C, (n, k_n(\gamma))). \\
\text{Hence } \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} &= \frac{\dim(\sigma_0 C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} + ... + \frac{\dim(\sigma_{(\sum_{i=0}^d a_i)-1} C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}.
\end{aligned}$$

Taking limits as $n \to \infty$,

$$\mu(C) = \mu(\sigma_0 C) + ... + \mu(\sigma_{a_d+...+a_0-1} C).$$

$\square$

Lemma 3.2.9. *For $j_0, j_1 \in A$,* $\dim(C^{j_0}, (n, k_n(\gamma))) = \dim(C^{j_1 j_0}, (n, k_{n+1}(\sigma_{j_1}\gamma)))$.

Proof. Assume that C terminates at vertex (m, k). Then C^{j_0} terminates at $(m+1, l+g(j_0))$, where $g(j)$ is as in the proof of Lemma 3.2.8. Hence, $\dim(C^{j_0}, (n, k_n(\gamma))) = \operatorname{coeff}_{p(x)}(n-(m+1), k_n(\gamma) - (l+g(j_0))$. Also, $k_{n+1}(\sigma_{j_1}\gamma) = k_n(\gamma) + g(j_1)$, and $C^{j_1 j_0}$ terminates at vertex $(m+2, l+g(j_1)+g(j_0))$. Hence

$$\dim(C^{j_1 j_0}, (n, k_{n+1}(\sigma_{j_1}\gamma))) = \operatorname{coeff}_{p(x)}(n+1-(m+2), k_n(\gamma)+g(j_1)-(l+g(j_1)+g(j_0)))$$

$$= \operatorname{coeff}_{p(x)}(n-(m+1), k_n(\gamma)-(l+g(j_0)))$$

$$= \dim(C^{j_0}, (n, k_n(\gamma))).$$

$\square$

PROPOSITION 3.2.10. *Every $T_{p(x)}$-invariant fully supported ergodic probability measure for $(X_{p(x)}, T_{p(x)})$ is Bernoulli.*

Proof. Let μ be a $T_{p(x)}$-invariant fully supported ergodic probability mesure for $(X_{p(x)}, T_{p(x)})$. To prove that μ is a Bernoulli measure, it is enough to show that for each $i \in A$ there exists a number p_i such that for every cylinder set C, $\dfrac{\mu(C^i)}{\mu(C)} = p_i$. Now for any $\gamma \in X_{p(x)}$,

$$\frac{\dim(C^i, (n, k_n(\gamma)))}{\dim(C, (n, k_n(\gamma)))} = \frac{\dim(C^{ji}, (n, k_{n+1}(\sigma_j\gamma)))}{\dim(C^j, (n, k_{n+1}(\sigma_j\gamma)))}.$$

By Lemma 3.2.7, there exists a set E of full measure such that for all $\gamma \in E$ and all $i \in A$

$$\frac{\mu(C^i)}{\mu(C)} = \lim_{n\to\infty} \frac{\dim(C^i, (n, k_n(\gamma)))}{\dim(C, (n, k_n(\gamma)))}.$$

If there is $j \in A$ such that $E \cap \sigma_j E = \emptyset$, then $\mu(E \cap \sigma_j E) = 0$, and, since E has full measure, $\mu(\sigma_j E) = 0$. Denote by $[r]$ the cylinder set $\{\gamma \in X : \gamma_0 = r\}$.

Then by Lemma 3.2.8:

$$1 = \mu(E) = \sum_{r\neq j} \mu(\sigma_r E) \leq \sum_{r\neq j} \mu[r] \leq 1 \text{ which implies } \mu([j]) = 0,$$

contradicting our earlier assumption that μ has full support. Hence there exists $\gamma \in E \cap \sigma_j E$. Let ξ be the path in E such that $\sigma_j \xi = \gamma$; then

$$\lim_{n\to\infty} \frac{\dim(C^i, (n, k_n(\xi)))}{\dim(C, (n, k_n(\xi)))} = \lim_{n\to\infty} \frac{\dim(C^{ji}, (n, k_{n+1}(\sigma_j\xi)))}{\dim(C^j, (n, k_{n+1}(\sigma_j\xi)))},$$

$$\text{showing that } \frac{\mu(C^i)}{\mu(C)} = \frac{\mu(C^{ji})}{\mu(C^j)}.$$

Then for any cylinder set $C = [c_0 c_1 ... c_{m-1}]$ we have:

$$\frac{\mu(C^i)}{\mu(C)} = \frac{\mu([c_0...c_{m-2}]^{c_{m-1}i})}{\mu([c_0...c_{m-2}]^{c_{m-1}})} = \frac{\mu([c_0...c_{m-2}]^i)}{\mu([c_0...c_{m-2}])} = ... = \frac{\mu([c_0]^i)}{\mu([c_0])}.$$

Also, for all $j, k, l \in A$ we have:

$\mu([jli]) = \mu([lji])$, since $[jli]$ and $[lji]$ have the same terminal vertex. Then

$$\frac{\mu([j]^{li})}{\mu([j]^l)} = \frac{\mu([l]^{ji})}{\mu([l]^j)} \text{ so that } \frac{\mu([j]^i)}{\mu([j])} = \frac{\mu([l]^i)}{\mu([l])}.$$

This shows that $\dfrac{\mu(C^i)}{\mu(C)}$ is independent of C, and hence equal to $\mu([i])$. Therefore μ is a Bernoulli. □

To see that these measures are in fact ergodic we need a result which can be found in [**3**].

Let $\{Z_i\}_{i=1}^{\infty}$ be a set of random variables on a probability space $(\Omega, \mathcal{B}, \mu)$ (recall that random variables are measurable functions from Ω into $\mathbb{R}$). A set B is said to *depend symmetrically* on $Z_1, \ldots, Z_n$ if whenever ξ is in B and π is a permutation of $\{1, 2, \ldots, n\}$, the set

$$\{\omega \in X | (Z_1(\omega), \ldots, Z_n(\omega), \ldots) = (Z_{\pi 1}(\xi), Z_{\pi 2}(\omega), \ldots, Z_{\pi n}(\omega), Z_{n+1}(\omega), \ldots)\}$$

is also in B. Define $\mathcal{S}_n$ to be σ-algebra generated by all such sets. Then $\mathcal{S} = \cap_{i=1}^{\infty} \mathcal{S}_n$ is the σ-algebra of sets invariant under all finite permutations of the coordinates of $(Z_1(\omega), Z_2(\omega), \ldots)$ in the following sense: $\mathcal{S}$ is generated by the sets B for which if $\omega \in B$ then all $\xi \in X$ for which there exists a number M such that $m > M$ implies that $Z_m(\omega) = Z_m(\xi)$ and the first M coordinates, $(Z_1(\xi), Z_2(\xi), \ldots, Z_M(\xi))$ are a permutation of $(Z_1(\omega), Z_2(\omega), \ldots, Z_M(\omega))$, are also in B. $\mathcal{S}$ is called the *symmetric σ-algebra generated by* $\{Z_i\}_{i=1}^{\infty}$.

A sequence $\{Z_i\}_{i=1}^{\infty}$ of random variables is said to be an *independent, identically distributed (i.i.d.) process* if all Z_i have the same probability distribution and are mutually independent.

THEOREM 3.2.11 (Hewitt-Savage). *If $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables under the probability measure μ, then for every set B in the symmetric σ-algebra $\mathcal{S}$ generated by $\{Z_i\}_{i=1}^{\infty}$, $\mu(B) = 0$ or 1.*

PROPOSITION 3.2.12. *The $T_{p(x)}$-invariant Bernoulli measures on $X_{p(x)}$ are ergodic.*

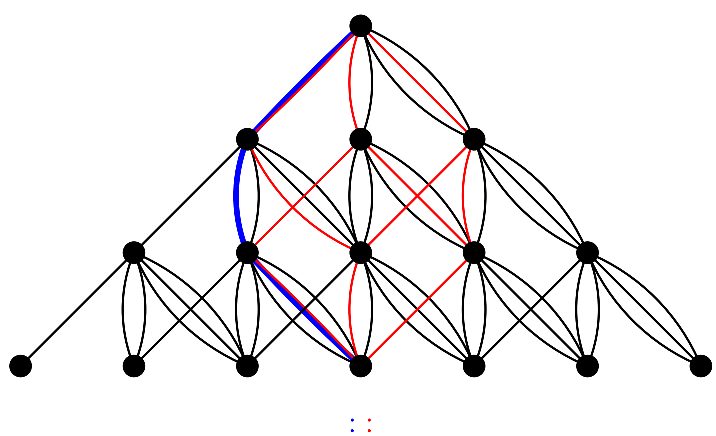


FIGURE 3.5. If the blue path is in $B \in \mathcal{S}_3$, then all the red paths (which agree with the blue path after level 3) are also in B.

Proof. Define the random variable Z_i on $X_{p(x)}$ by letting $Z_i(\gamma)$ be the label on the $i-1$'th edge of γ, for $i = 1, 2, \ldots$, $Z_i(\gamma) = \gamma_{i-1}$. Since the probability measure is Bernoulli, the Z_i are independent and identically distributed. If B is a set that depends symmetrically on $Z_1, \ldots, Z_n$, then $\gamma \in B$ implies that $\{\xi \in X | \xi_0\xi_1 \ldots \xi_{n-1}$ is a permutation of $\gamma_0\gamma_1 \ldots \gamma_{n-1}$ and for $m \geq$

$n,\ \xi_m = \gamma_m\}$ is also in B, see Figure 3.5. If $\mathcal{S}_n$ is σ-algebra set generated by such B, the Hewitt-Savage theorem implies that $\mathcal{S} = \cap_{n=1}^{\infty} \mathcal{S}_n$ is trivial.

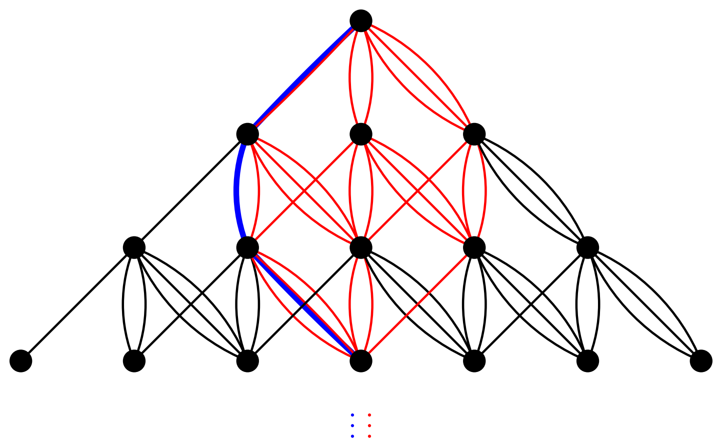


FIGURE 3.6. If the blue path is in $B' \in \mathcal{T}_3$, then all the red paths (which agree with the blue path after level 3) are also in B'.

Let $\mathcal{T}_n$ be the σ-algebra generated by sets B' such that if $\gamma \in B'$, then $\{\xi \in X |$ for $m \geq n,\ \xi_m = \gamma_m\}$ is also in B', see Figure 3.6. Then for each generator B' of $\mathcal{T}_n$, there are a finite number of generators B_i of $\mathcal{S}_n$ such that $\cup_{i=1}^{m} B_i = B'$. Hence $B' \subset \mathcal{S}_n$. Then $\mathcal{T}_n \subset \mathcal{S}_n$, and $\cap_{i=1}^{\infty} \mathcal{T}_n = \mathcal{T} \subset \mathcal{S}$. Since $\mathcal{S}$ is trivial, so is $\mathcal{T}$. But $\mathcal{T}$ is the σ-algebra of $T_{p(x)}$-invariant sets. Therefore the invariant Bernoulli measures for $T_{p(x)}$ are ergodic. $\square$

It remains to determine which Bernoulli measures are invariant.

PROPOSITION 3.2.13. *The Bernoulli measures invariant for the adic transformation $T_{p(x)}$ are the fully supported ones described in Theorem 3.2.4, along with*

$$\mathcal{B}\left(\underbrace{\frac{1}{a_0},\ldots,\frac{1}{a_0}}_{a_0\ times},0,\ldots,0\right) \ and\ \mathcal{B}\left(0,\ldots,0,\underbrace{\frac{1}{a_d},\ldots,\frac{1}{a_d}}_{a_d\ times}\right).$$

Proof. Recall from Remark 3.2.3 that any edge label j has weight $w(j)$. Recall the definition of $g : A \to \{0, 1, \dots, d\}$ as given in Equation 3.2.2. By Lemma 3.2.6, $g(j_1) = g(j_2)$ implies $w(j_1) = w(j_2)$. For $0 \le t \le d$, define $p_t = w(j)$ whenever $g(j) = t$. Then

$$a_0 p_0 + ... + a_d p_d = 1. \tag{3.2.3}$$

For $s \in \mathbb{N}$ and $i_k, j_k \in \{0, 1, \dots, d\}$ for all $k = 0, 1, \dots, s$, Lemma 3.2.6 implies that a Bernoulli measure is $T_{p(x)}$ invariant if and only if whenever

$$\sum_{k=0}^{s} i_k = \sum_{k=0}^{s} j_k, \text{ we have } \prod_{k=0}^{s} p_{i_k} = \prod_{k=0}^{s} p_{j_k}. \tag{3.2.4}$$

Assume for now that $p_0, p_1 > 0$. Claim: Equation 3.2.4 is satisfied if and only if $p_0 p_j = p_1 p_{j-1}$ for $1 \le j \le d$.

Clearly Equation 3.2.4 implies $p_0 p_j = p_1 p_{j-1}$. It remains to be shown that $p_0 p_j = p_1 p_{j-1}$ implies Equation 3.2.4.

For $1 \le j \le d$ we will assume

$$p_0 p_j = p_1 p_{j-1}. \tag{3.2.5}$$

We will use induction to prove to prove our claim. The hypothesis is that for $i_0, i_1 \dots, i_{s-1}, j_0, j_1, \dots, j_{s-1}$ in $\{0, 1, \dots, d\}$, whenever

$$i_0 \cdots + i_{s-1} = j_0 + \dots j_{s-1}, \text{ we have } \prod_{k=0}^{s-1} p_{i_k} = \prod_{k=0}^{s-1} p_{j_k}. \tag{3.2.6}$$

We will show that for $i_0, i_1 \dots, i_s, j_0, j_1, \dots, j_s$ in $\{0, 1, \dots, d\}$, whenever

$$i_0 \cdots + i_s = j_0 + \dots j_s, \text{ we have } \prod_{k=0}^{s} p_{i_k} = \prod_{k=0}^{s} p_{j_k}.$$

We now show the base case. For $1 \leq i \leq d$ and $0 \leq k \leq d-1$, Equation 3.2.5 implies

$$p_i = \frac{p_1}{p_0} p_{i-1} \text{ and } p_k = \frac{p_0}{p_1} p_{k+1}.$$

Hence

$$p_i p_k = \frac{p_1}{p_0} p_{i-1} \frac{p_0}{p_1} p_{k+1} = p_{i-1} p_{k+1}.$$

For $i, k, l, m \in \{0, 1, \dots, d\}$ we then have that whenever $i+k = l+m$, $p_i p_k = p_l p_m$, hence we have shown the base case.

Now consider $i_0, i_1, \dots, i_s, j_0, j_1, \dots, j_s$ in $\{0, 1, \dots, d\}$ such that

$$i_0 + \cdots + i_s = j_0 + \cdots + j_s.$$

Then $i_0 + \cdots + i_s - i_s - j_s = j_0 + \cdots + j_s - j_s - i_s$, hence

$$i_0 + \cdots + i_{s-1} - j_s = j_0 + \cdots + j_{s-1} - i_s.$$

There also exist $l_0, l_1, \dots, l_{s-2}$ in $\{0, 1, \dots, d\}$ such that

$$l_0 + \cdots + l_{s-2} = i_0 + \cdots + i_{s-1} - j_s = j_0 + \cdots + j_{s-1} - i_s.$$

Adding j_s to both sides, we see that $l_0 + \cdots + l_{s-2} + j_s = i_0 + \cdots + i_{s-1}$, hence the induction hypothesis implies

$$p_{j_s} \prod_{k=0}^{s-2} p_{l_k} = \prod_{k=0}^{s-1} p_{i_k}. \tag{3.2.7}$$

Likewise, $l_0 + \cdots + l_{s-2} + i_s = j_0 + \cdots + j_{s-1}$, and the induction hypothesis implies

$$p_{i_s} \prod_{k=0}^{s-2} p_{l_k} = \prod_{k=0}^{s-1} p_{j_k}. \tag{3.2.8}$$

Combining Equation 3.2.7 and Equation 3.2.8 we see that

$$\prod_{k=0}^{s} p_{i_k} = p_{i_s} p_{j_s} \prod_{k=0}^{s-2} p_{l_k} = p_{j_s} p_{i_s} \prod_{k=0}^{s-2} p_{l_k} = \prod_{k=0}^{s} p_{j_k}.$$

Therefore we have proved the claim and the Bernoulli measures are $T_{p(x)}$ invariant if and only if for $1 \leq j \leq d$,

$$p_0 p_j = p_1 p_{j-1}.$$

For simplicity of notation define $p_0 = q$ and $p_1 = t$. For $1 \leq j \leq d$, $p_j = \dfrac{t}{q} p_{j-1}$. Hence every p_j can be defined inductively by t and q. In particular, for $1 \leq j \leq d$,

$$p_j = \frac{t^j}{q^{j-1}}.$$

By Equation 3.2.3

$$a_0 q + a_1 t + a_2 \frac{t^2}{q} + \cdots + a_d \frac{t^d}{q^{d-1}} = 1.$$

Multiplying through by q^{d-1} and simplifying, we see that

$$a_0 q^d + a_1 q^{d-1} t + ... + a_d t^d - q^{d-1} = 0. \tag{3.2.9}$$

To conclude that $p_1, ..., p_d$ are completely determined by the choice of $p_0 = q$, it remains only to show that for each $q \in (0, 1/a_0)$, Equation 3.2.9 has a unique solution in [0,1].

Consider $m(t) = a_0 q^d + a_1 q^{d-1} t + ... + a_d t^d - q^{d-1}$. Then $m(0) = a_0 q^d - q^{d-1} = q^{d-1}(a_0 q - 1) \leq 0$, since $a_0 q \leq 1$. Also, $m(1) = a_0 q^d + a_1 q^{d-1} + ... + a_d - q^{d-1} > 0$, since $a_1 \geq 1$ implies $a_1 q^{d-1} - q^{d-1} \geq 0$. By the intermediate value theorem, there exists a root in $[0, 1]$. Now, $m'(t) = a_1 q^{d-1} + ... + d a_d t^{d-1}$ is strictly greater than 0 on $[0, 1]$, so that $m(t)$ is strictly increasing on $[0, 1]$; therefore there is a unique solution t_q to $m(t) = 0$ in the interval $[0, 1]$.

If $p_0 = 1/a_0$, $a_0 p_0 = 1$ and all other $p_i = 0$, hence the $T_{p(x)}$-invariant measure μ is supported on the paths for which $k_n(\gamma) = 0$ for all $n \geq 0$. Finally, if $i \leq d-2$ and $p_i = 0$, then $i+2 \leq d$, and $p_i p_{i+2} = p_{i+1} p_{i+1} = 0$. Hence $p_{i+1} = 0$. If $p_0 = 0$ then $p_i = 0$ for all $0 \leq i < d$. This implies that the only nonzero probability is p_d and $a_d p_d = 1$, hence $p_d = 1/a_d$ and the $T_{p(x)}$-invariant measure μ is supported on the set of paths for which $k_n(\gamma) = dn - 1$ for all $n \geq 0$.

□

We now have enough tools to prove Theorem 3.2.4.

Proof of Theorem 3.2.4. Direct from Proposition 3.2.13, Proposition 3.2.12, and Proposition 3.2.10. □

Proof of Proposition 3.2.5. According to Proposition 3.2.13 and Proposition 3.2.12, the Bernoulli measures

$$\mathcal{B}\left(\underbrace{\frac{1}{a_0}, \ldots, \frac{1}{a_0}}_{a_0 \ times}, 0, \ldots, 0\right) \text{ and } \mathcal{B}\left(0, \ldots, 0, \underbrace{\frac{1}{a_n}, \ldots, \frac{1}{a_n}}_{a_n \ times}\right)$$

are ergodic and invariant for $(X_{p(x)}, T_{p(x)})$. It remains to show that these are the only invariant, ergodic probability measures that are not fully supported on $X_{p(x)}$.

By Proposition 2.1.5, the only proper closed invariant sets are those for which the tails are eventually diagonal (recall Definition 2.1.3), see Figure 3.7.

Define A_k to be the closed invariant set

$$A_k = \{\gamma \in X | \text{ either } k_n(\gamma) \leq k \text{ for all } n \text{ or } dn - k_n(\gamma) \leq k \text{ for all } n\}.$$

In Figure 3.7, $k = 2$. Define C to be the maximal cylinder from the root vertex to vertex (l, k), where l is the first level for which $dl - k > k$. In other words, C is the maximal

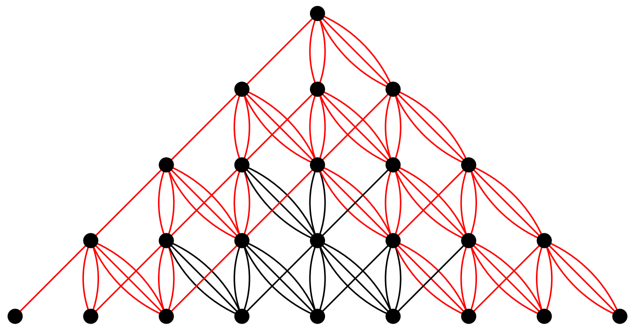


FIGURE 3.7. The infinite red paths form a proper closed invariant set.

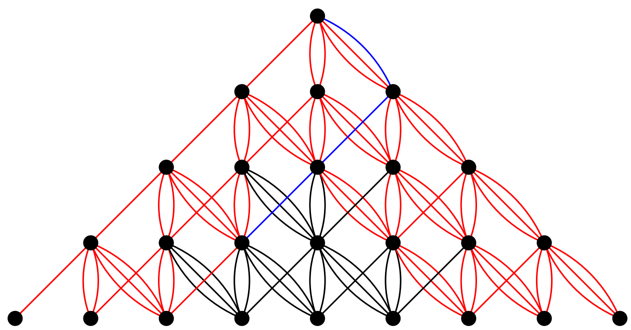


FIGURE 3.8. C is denoted by the blue cylinder.

cylinder of shortest length such that for each $\gamma \in C$ and $n \geq l$, $k_n(\gamma) = k$. In Figure 3.8, $l = 3$.

If μ is a $T_{p(x)}$-invariant ergodic probability measure that is not fully supported on $X_{p(x)}$, we will compute $\mu(C)$ using Lemma 3.2.7. If there is a set of positive measure $B_k \subset A_k$ for which every $\gamma \in B_k$ and $n \geq l$ has $k_n(\gamma) \neq k$, then for all $n \geq l$, $\dim(C, (n, k_n(\gamma))) = 0$, hence $\mu(C) = 0$. Therefore we will assume that almost every γ in A_k has $k_n(\gamma) \equiv k$ for sufficiently large n. This implies that the measure μ is supported on the paths for which $k_n(\gamma) \leq k$ (see Figure 3.9).

Then

$$\mu(C) = \lim_{n \to \infty} \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}.$$

It is clear that $\dim(C, (n, k_n(\gamma))) = a_0^{n-l}$. We will now find a lower bound for $\dim(n, k_n(\gamma))$. Let $m \in \mathbb{Z}_+$ such that $k - md > 0$ and $k - (m+1)d \leq 0$. Then $dm + i = k$ for some

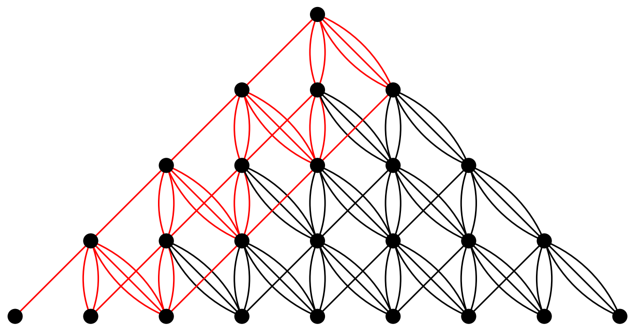


FIGURE 3.9. μ is supported on the infinite red paths.

$k \in \{1, 2, \dots, d\}$. Recall the function $g(j)$ defined in the proof of Lemma 3.2.8. We will only count the paths $\xi \in A_k$ for which $g(\xi_0) = g(\xi_1) = \cdots = g(\xi_{j-1}) = 0$, $g(\xi_j) = i$, $g(\xi_{j+1}) = g(\xi_{j+2}) = \cdots = g(\xi_{j+m}) = d$, and $g(\xi_{j+m+1}) = g(\xi_{j+m+2} = \cdots = g(\xi_{n-1}) = 0$, where $0 \leq j \leq n-(m+1)$ (see Figure 3.10). The range of ξ_{n-1} is $(n, k_n(\gamma))$. These paths form a subset of the paths which are counted when computing $\dim(n, k_n(\gamma))$. For a fixed j, the number of such paths is

$$a_0^j a_i a_d^m a_0^{n-m-j-1} = a_0^{n-m-1} a_i a_d^m.$$

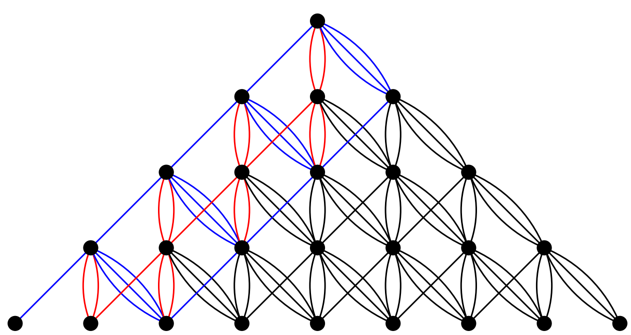


FIGURE 3.10. Only the blue paths will be counted. In this case $i = d = 2$ and $m = 0$.

Letting j range over 0 to $n-(m+1)$ we see that $\dim(n, k_n(\gamma)) \geq (n-m)a_0^{n-m-1}a_i a_d^m$. Hence

$$\mu(C) = \lim_{n\to\infty} \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} \leq \lim_{n\to\infty} \frac{a_0^{n-l}}{(n-m)a_0^{n-m-1}a_i a_d^m} = \lim_{n\to\infty} \frac{a_0^{m+1-l}}{(n-m)a_i a_d^m} = 0.$$

By the invariance of $T_{p(x)}$, no cylinder whose terminal vertex is (j, k), where $j \geq l$, has positive measure. Using this same argument for vertices (j, i) where $i \leq k$, we see that the only edges on which μ is supported are the edges on the far left of the diagram. In this case we have an odometer, and the measure is as stated in the proposition. A symmetric argument shows that the only measure supported on paths for which $dn - k_n(\gamma) \leq k$ for all $n \geq 0$ is supported on the paths for which $k_n(\gamma) = dn$ for all $n \geq 0$. Hence μ is as stated in the proposition. $\square$

We now describe the cutting and stacking equivalent for Bratteli-Vershik systems determined by positive integer polynomials. At each stage $n = 0, 1, 2, \ldots$ we have $dn+1$ stacks $S_{n,0}, S_{n,1}, \ldots, S_{n,dn}$ (corresponding to the vertices (n, k), $0 \leq k \leq dn$, of the Bratteli diagram). Stack $S_{n,k}$ consists of $\dim(n, k)$ subintervals of $[0, 1]$. Each subinterval corresponds to a cylinder set determined by a path of length n, terminating in vertex (n, k). The transformation $\tilde{T}$ is defined by mapping each level of the stack, except the topmost one, linearly onto the one above it. This corresponds to mapping each non-maximal path of length n to its successor. To proceed to the next stage in the cutting and stacking construction, each stack $S_{n,k}$ is cut into $a_0 + a_1 + \cdots + a_d$ substacks with length proportions corresponding to the various weights on the different edges. These are recombined into new stacks in the order prescribed by the way $T_{p(x)}$ maps their corresponding cylinder sets.

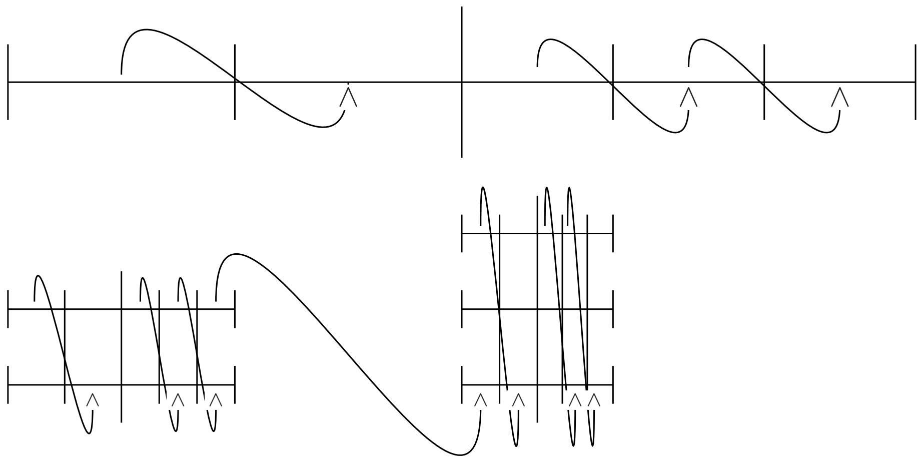


FIGURE 3.11. Cutting and Stacking for $p(x) = 2 + 3x$ and $\mu = \mathcal{B}(1/4, 1/4, 1/6, 1/6, 1/6)$.

3.3. Dimension Group Computation

In Section 2.2 the dimension group of a Bratteli diagram was described and we proved Theorem 2.2.7, which described the general relationship between the group $C(X, \mathbb{Z})/(C(X, \mathbb{Z}) \cap \partial_T C(X, \mathbb{Z})$ of a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ and the dimension group of the associated diagram. In the case of the Bratteli diagrams in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$, the dimension groups are directly computable, and we compute them in this section.

If $\varinjlim A_i$ is a direct limit (recall Definition 2.2.3) with directed set I, transition functions $\phi_{ij} : A_i \to A_j$, and the equivalence class of $a \in A_i$ is denoted $\overline{a}$, then $\varinjlim A_i$ has the following universal mapping property, which can be found in [**11**]. If G is any abelian group such that for each $i \in I$ there is a homomorphism $\rho_i : A_i \to G$ with $\rho_i = \rho_j \circ \phi_{ij}$ whenever $i \leq j$, then there is a unique homomorphism $\rho : \varinjlim A_i \to G$ such that $\rho \circ \phi^i = \rho_i$ for all $i \in I$, where $\phi^i(a) = \overline{a}$ for $a \in A_i$.

Theorem 3.3.1. *The dimension group $K_0(\mathcal{V},\mathcal{E})_{p(x)}$ associated to $(\mathcal{V},\mathcal{E})_{p(x)}$ is order isomorphic to the ordered group $G_{p(x)}$ of rational functions of the form*

$$\frac{r(x)}{p(x)^m},$$

where $r(x)$ is any polynomial with integer coefficients such that $\deg(r(x)) \leq md$. Addition of two elements is given by

$$\frac{r(x)}{p(x)^m} + \frac{s(x)}{p(x)^l} = \frac{r(x) + s(x)p(x)^{m-l}}{p(x)^m}$$

if $l \leq m$. The positive set $(G_{p(x)})_+$ consists of the elements of $G_{p(x)}$ such that there is an l for which the numerator of

$$\frac{r(x)(p(x))^l}{p(x)^{l+m}}$$

has all positive coefficients. The distinguished order unit of $K_0(\mathcal{V},\mathcal{E})_{p(x)}$ is the constant polynomial 1.

Proof. We will construct an order isomorphism from $K_0(\mathcal{V},\mathcal{E})_{p(x)}$ into G. The transposes of the incidence matrices will be used for typographical reasons in the computation in order to make the computations on row vectors. If $p(x) = a_0 + \ldots a_d x^d$, the transposes of the incidence matrices associated to $(\mathcal{V},\mathcal{E})_{p(x)}$ will take the following form:

$$\phi_1 = \begin{bmatrix} a_0 & a_1 & \ldots & a_d \end{bmatrix},$$

$$\phi_2 = \begin{bmatrix} a_0 & a_1 & \ldots & a_d & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & a_{d-1} & a_d & \ldots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & \ldots & 0 & a_0 & a_1 & \ldots & a_d \end{bmatrix}.$$

More generally, ϕ_k will be a $((k-1)d+1) \times (kd+1)$ matrix with

$$(\phi_k)_{ij} = \begin{cases} a_{(j-i)} & \text{if } 0 \leq j-i \leq d \\ 0 & \text{otherwise} \end{cases}$$

For $l \leq m$, define $\phi_{lm} : \mathbb{Z}^{d(l-1)+1} \to \mathbb{Z}^{dm+1}$ by $\phi_l \phi_{l+1} \ldots \phi_m$. We will identify $\mathbb{Z}^i$ with the additive group of polynomials of degree at most $i-1$, $\mathbb{Z}_{i-1}[x]$ in the following manner. For $v = [v_0\ v_1 \ldots v_{i-1}] \in \mathbb{Z}^i$, define $v(x) \in \mathbb{Z}_{i-1}[x]$ by $v(x) = \sum_{j=0}^{i-1} v_j x^j$. For example if $v = [1\ 2\ 3] \in \mathbb{Z}^3$, then $v(x) = 1 + 2x + 3x^2 \in \mathbb{Z}_2[x]$. Now if $v \in \mathbb{Z}^{dm+1}$, we have $(v\phi_m)(x) = v(x)p(x)$. Under the above correspondence, ϕ_l becomes multiplication by $p(x)$ for all l, and ϕ_{lm} becomes multiplication by $(p(x))^{m-l}$.

Define $\rho_m : \mathbb{Z}_{md}[x] \to G$ by $\rho_m(r(x)) = \dfrac{r(x)}{(p(x))^m}$. In order to satisfy the hypothesis of the universal mapping property of direct limits, it needs to be shown that for $l \leq m$, $\rho_l = \rho_m \circ \phi_{lm}$:

$$\begin{aligned} \rho_m \circ \phi_{lm}(r(x)) &= \rho_m(r(x)(p(x))^{m-l}) \\ &= \frac{r(x)(p(x))^{m-l}}{(p(x))^m} \\ &= \frac{r(x)}{(p(x))^l} \\ &= \rho_l(r(x)). \end{aligned}$$

Hence the hypothesis for the universal mapping property of direct limits is satisfied, and the ρ_l are constant on equivalence classes. It follows that there is a unique homomorphism $\rho : K_0(\mathcal{V}, \mathcal{E})_{p(x)} \to G_{p(x)}$, which can be defined on an equivalence class by taking any representative in $\mathbb{Z}^{di+1}$ and applying ρ_i to it. This is well defined because ρ_i is constant on equivalence classes, and there is only one element of each equivalence class in each $\mathbb{Z}^{id+1}$. We claim that ρ is an isomorphism.

1. ρ is a homomorphism. For without loss of generality, assume $l \leq m$, $r(x) \in \mathbb{Z}_{md}[x]$, and $s(x) \in \mathbb{Z}_{ld}[x]$. Then

$$\begin{aligned}
\rho\left(\overline{r(x)} + \overline{s(x)}\right) &= \rho\left(\overline{r(x) + (p(x))^{m-l}s(x)}\right) \\
&= \frac{r(x) + (p(x))^{m-l}s(x)}{(p(x))^m} \\
&= \frac{r(x)}{(p(x))^m} + \frac{s(x)}{(p(x))^l} \\
&= \rho(\overline{r(x)}) + \rho(\overline{s(x)}).
\end{aligned}$$

2. ρ is onto. Given $\dfrac{r(x)}{(p(x))^m} \in G_{p(x)}$, then for $r(x) \in \mathbb{Z}_{md}[x]$, $\rho(\overline{r(x)}) = \dfrac{r(x)}{(p(x))^m}$.

3. ρ is injective. If $r(x) \in \mathbb{Z}_{md}[x]$, $\rho(\overline{r(x)}) = 0$, then

$$\frac{r(x)}{(p(x))^m} = 0 \text{ therefore } r(x) = 0 \text{ and } \overline{r(x)} = \overline{0}.$$

Hence ρ is an isomorphism, and $G_{p(x)}$ is isomorphic to $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$. In addition, $G_{p(x)}$ is order isomorphic to $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ because

$$(G_{p(x)})_+ = \left\{ \frac{r(x)}{(p(x))^m} \mid r(x)(p(x))^l \text{ has all positive coefficients for some } l \geq 0 \right\}$$

is exactly the image of the positive set of $\varinjlim \mathbb{Z}^{dk+1}$ under ρ. Finally, the image of 1 is $\dfrac{1}{(p(x))^0} = 1$. $\square$

Corollary 3.3.2. *For $(X_{p(x)}, T_{p(x)}) \in (\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by $p(x)$, there is an order isomorphism*

$$G_{p(x)} \cong C(X_{p(x)}, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}))$$

which maps the constant function 1 in $G_{p(x)}$ to the equivalence class of the constant function 1 in $C(X_{p(x)}, \mathbb{Z})/(\partial_{T_{p(x)}} C(X_{p(x)}, \mathbb{Z}) \cap C(X_{p(x)}, \mathbb{Z}))$.

Proof. This is a direct consequence of Theorem 2.2.7 and Proposition 3.3.1. ☐

3.4. Loosely Bernoulli

The property of loosely Bernoulli was introduced by Feldman in [**15**] as well as by Katok and Sataev in [**26**]. A transformation that has zero entropy (see [**37**]) is loosely Bernoulli if and only if it is isomorphic to an induced map (recall Definition 1.1.12) of an irrational rotation on the circle ($\gamma \in S^1, \lambda \in \mathbb{R} \setminus \mathbb{Q}, T\gamma = \gamma + \lambda \pmod 1$).

Definition 3.4.1. The $\overline{f}$ distance between two words $v = v_1 \dots v_l$ and $w = w_1 \dots w_l$ of the same length $l > 0$ on the same alphabet is

$$\overline{f}(v, w) = \frac{l - s}{l},$$

where s is the greatest integer in $\{0, 1, \dots, l\}$ such that there are $1 \leq i_1 < i_2 < \cdots < i_s \leq l$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq l$ with $v_{i_r} = w_{j_r}$ for $r = 1, \dots, s$.

Example 3.4.1.

$$\overline{f}(abababab, babababa) = 1/8$$

Recall that for a measure-theoretic dynamical system (X, T), a partition $\mathcal{P} = \{P_1, \dots, P_j\}$, and the corresponding function $\mathcal{P} : X \to \{1, 2, \dots, j\}$, the $\mathcal{P}$-n-name of $\gamma \in X$ is defined to be

$$\mathcal{P}_0^n(\gamma) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma) \dots \mathcal{P}(T^n\gamma).$$

Definition 3.4.2. Let T be a zero-entropy measure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$, and let $\mathcal{P}$ be a finite measurable partition of X. Then the process $(\mathcal{P}, T)$ is said to be *loosely Bernoulli* (LB) if for all $\varepsilon > 0$ and for all sufficiently large l we can find $A \subset X$ with $\mu(A) > 1 - \varepsilon$ such that for all $\gamma, \xi \in A$,

$$\overline{f}(\mathcal{P}_0^l(\gamma), \mathcal{P}_0^l(\xi)) < \varepsilon.$$

T is said to be *loosely Bernoulli* if $(\mathcal{P}, T)$ is loosely Bernoulli for all partitions $\mathcal{P}$.

T is LB if for a generating partition $\mathcal{P}$, $(\mathcal{P}, T)$ is LB. Some of the Bratteli-Vershik systems determined by positive integer polynomials have already been shown to be loosely Bernoulli. Janvresse and de la Rue proved it for the Pascal adic (the Bratteli-Vershik system associated to $p(x) = 1 + x$) in [**25**], and in [**33**], Méla showed it for polynomials of arbitrary degree where all the coefficients are 1. We have established this property for Bratteli-Vershik systems determined by arbitrary positive integer polynomials.

THEOREM 3.4.3. *The Bratteli-Vershik systems $(X_{p(x)}, T_{p(x)})$ in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by positive integer polynomials are loosely Bernoulli with respect to each of their $T_{p(x)}$-invariant ergodic probability measures.*

The proof of Theorem 3.4.3 will follow the ideas of Janvresse and de la Rue. There are two cases, depending on whether or not the ergodic measure has full support. The following lemma gives a seemingly weaker sufficient condition for the loosely Bernoulli property to hold.

LEMMA 3.4.4 (Janvresse, de la Rue [**25**]). *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system with entropy 0. Suppose that for every $\varepsilon > 0$ and for $\mu \times \mu$-almost every $(\gamma, \xi) \in X \times X$ we can find an integer $l(\gamma, \xi) \geq 1$ such that*

$$\overline{f}\left(\mathcal{P}_0^{l(\gamma,\xi)}(\gamma), \mathcal{P}_0^{l(\gamma,\xi)}(\xi)\right) < \varepsilon.$$

Then the process $(\mathcal{P}, T)$ is LB.

DEFINITION 3.4.5. Let $\{Z_i\}_{i=i}^{\infty}$ be a sequence of independent, identically distributed discrete random variables. For each positive integer n, we let S_n denote the sum $Z_1 + Z_2 + \cdots + Z_n$. The sequence $\{S_n\}_{n=1}^{\infty}$ is called a *random walk*. A random walk for which the expectation of each S_n is 0, is called a *symmetric random walk*.

THEOREM 3.4.6 ([**30**] page 384). *A symmetric random walk is recurrent.*

LEMMA 3.4.7. *Let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system determined by the positive integer polynomial $p(x)$ of degree d, with ergodic, $T_{p(x)}$-invariant probability measure μ. For $\mu \times \mu$-almost every (γ, ξ) in $X_{p(x)} \times X_{p(x)}$, we can find arbitrarily large n such that $k_n(\gamma) = k_n(\xi)$.*

Proof. Define the random variables $\{Z_i\}_{i=1}^{n}$ from $X_{p(x)}$ to $\{0, 1, \ldots, d\}$ by letting $Z_i(\gamma) = k_i(\gamma) - k_{i-1}(\gamma)$. This is an i.i.d. process. Define $S_n : X_{p(x)} \times X_{p(x)} \to \{-d, -d + 1, \ldots, 0, 1, \ldots, d\}$ by $S_n(\gamma, \xi) = \sum_{i=1}^{n} [Z_i(\gamma) - Z_i(\xi)]$. (S_n) is a symmetric random walk, and hence recurrent. Thus for $\mu \times \mu$-almost every (γ, ξ) in $X_{p(x)} \times X_{p(x)}$, there are infinitely many n such that $k_n(\gamma) = k_n(\xi)$. $\square$

PROPOSITION 3.4.8. *Let $(X_{p(x)}, T_{p(x)})$ be a Bratteli-Vershik system determined by a positive integer polynomial of degree d, with a fully-supported, $T_{p(x)}$-invariant, ergodic probability measure μ. Let $\mathcal{P}$ be the partition determined by the first edge. Then the process $(\mathcal{P}, T_{p(x)})$ is loosely Bernoulli.*

Proof. The partition $\mathcal{P}$ was described in detail in Section 2.3 for polynomials of degree 1. The same notations will be used here. In particular, recall that for each vertex (n, k), and a path $\gamma \in Y_n(k,0)$,

$$B(n,k) = \mathcal{P}(\gamma)\mathcal{P}(T_{p(x)}\gamma)\dots\mathcal{P}(T_{p(x)}^{\dim(n,k)-1}\gamma).$$

From Lemma 3.4.7, we know that for $\mu \times \mu$-almost every $(\gamma, \xi) \in X_{p(x)} \times X_{p(x)}$, γ and ξ meet infinitely often. Also, for μ-almost every γ, for each cylinder C,

$$\frac{\dim(C,(n,k_n(\gamma)))}{\dim(n,k_n(\gamma))} \to \mu(C).$$

Let $p_0 = \mu([0])$ denote the weight associated to each edge labeled 0. For each $r \geq 1$, with probability $p_0^{2r} > 0$, both γ and ξ continue along edges labeled 0 for the next r edges.

Given $\varepsilon > 0$, choose r so that $p_0^r < \varepsilon/2$. Let C be a cylinder with terminal vertex (r, dr). Then for $(\mu \times \mu)$-almost every (γ, ξ) there are infinitely many n for which $k_n(\gamma) = k_n(\xi) = k$, $\left|\frac{\dim(C,(n,k_n(\gamma)))}{\dim(n,k_n(\gamma))} - \mu(C)\right| < \frac{\varepsilon}{2}$, and$\gamma_n = \xi_n = \gamma_{n+1} = \xi_{n+1} = \cdots = \gamma_{n+r-1} = \xi_{n+r-1} = 0$, see Figure 3.12.

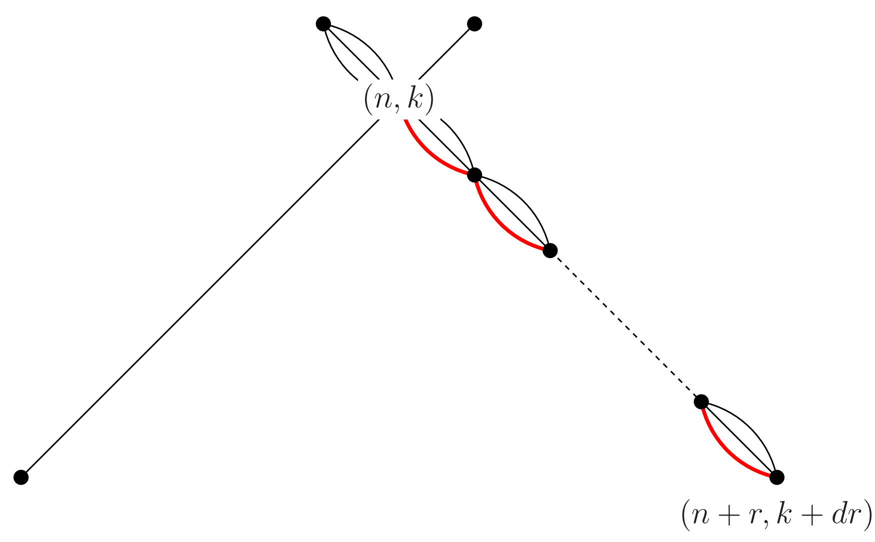


FIGURE 3.12. Both γ and ξ pass through vertex (n,k) and continue along the red path to vertex $(n+r,k+dr)$

Then the $\mathcal{P}$-names of both γ and ξ have long central block $B(n+r, k+dr)$. If we decompose $B(n+r, k+dr)$ into blocks from level n, we see that the first block to appear is $B(n, k)$. Both γ and ξ have their decimal point in this first block of $B(n, k)$.

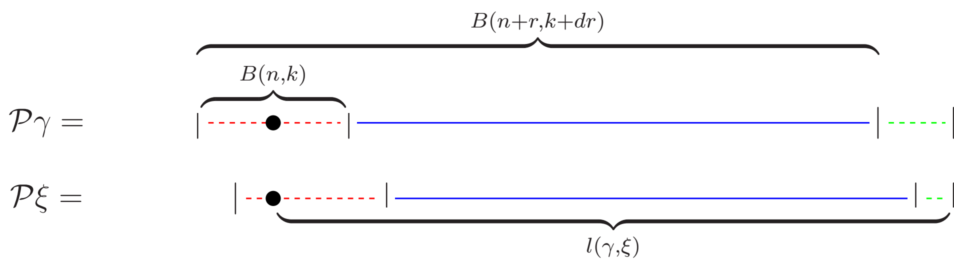


FIGURE 3.13. $\mathcal{P}\gamma$ and $\mathcal{P}\xi$ agree on the blue line, which is the rest of $B(n+r, k+dr)$ after the end of the initial $B(n, k)$.

If we let $l(\gamma, \xi) = |B(n+r, k+dr)|$ we have

$$\overline{f}(\mathcal{P}_0^{l(\gamma,\xi)}\gamma, \mathcal{P}_0^{l(\gamma,\xi)}\xi) \leq \frac{|B(n,k)|}{|B(n+r,k+dr)|} = \frac{\dim(n,k)}{\dim(n+r,k+dr)}.$$

By the isotropic nature of the diagram, $\dim(n, k) = \dim(C, (n+r, k+dr))$ (see Figure 3.14). Hence,

$$\frac{|B(n,k)|}{|B(n+r,k+dr)|} = \frac{\dim(n,k)}{\dim(n+r,k+dr)} = \frac{\dim(C,(n+r,k+dr))}{\dim(n+r,k+dr)}.$$

By Lemma 3.2.7,

$$\frac{\dim(C,(n+r,k+dr))}{\dim(n+r,k+dr)} \to \mu(C) = p_0^r.$$

We can now take n large enough so that

$$\frac{|B(n,k)|}{|B(n+r,k+dr)|} < p_0^r + \frac{\varepsilon}{2} < \varepsilon.$$

Hence $(\mathcal{P}, T_{p(x)})$ is loosely Bernoulli.

$\square$

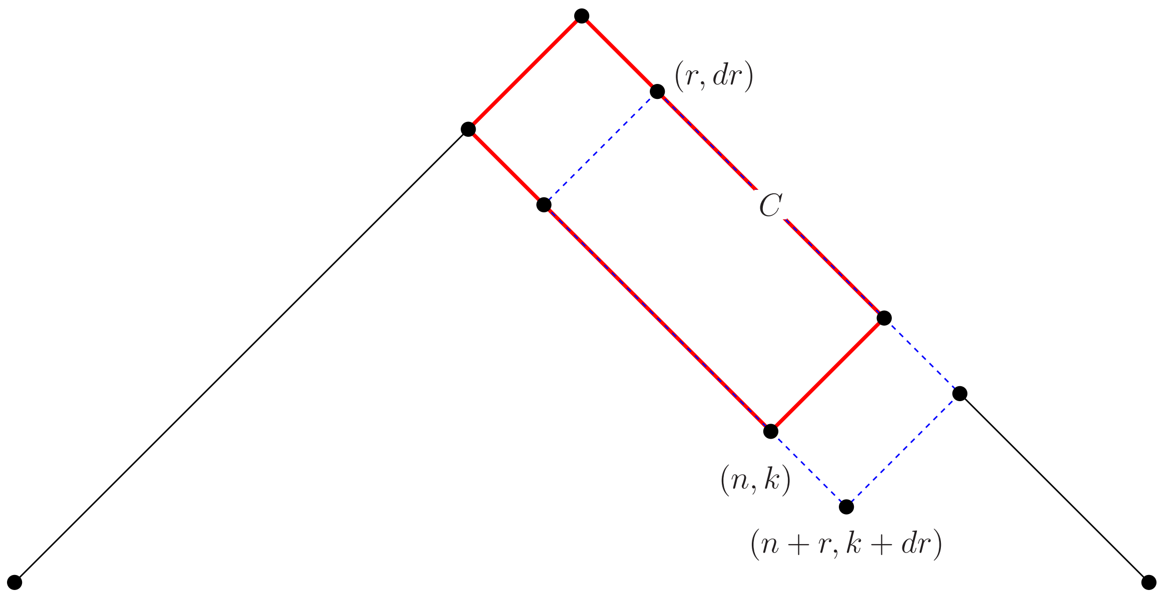


FIGURE 3.14. The red area and the dashed blue area contain the same number of paths.

If the partition by the first edge is a generating partition, then Proposition 3.4.8 would be enough to say that all Bratteli-Vershik systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ are loosely Bernoulli. We conjecture that this is the case in Conjecture 5. As it stands, we are able to prove that all the systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ are loosely Bernoulli without Conjecture 5.

COROLLARY 3.4.9. *Let $(X_{p(x)}, T_{p(x)})$ be a Bratteli-Vershik system determined by a positive integer polynomial $p(x)$ of degree d with fully-supported, $T_{p(x)}$-invariant, ergodic probability measure μ. Let $\mathcal{P}_l$ be the partition determined by the first l edges. Then the process $(\mathcal{P}_l, T_{p(x)})$ is loosely Bernoulli.*

Proof. By telescoping to the levels which are multiples of l, the Bratteli-Vershik system becomes the system determined by the polynomial $q(x) = (p(x))^l$, and $\mathcal{P}_l$ becomes the partition on the first edge. By Proposition 3.4.8, this system is LB.

Alternatively, one may replace $\mathcal{P}$ in Proposition 3.4.8 with $\mathcal{P}_l$. Provided $n \geq l$, the blocks $B(n, k)$ will remain the same length although over a different alphabet. Taking extra care to choose $n \geq l$, the same proof also shows that the process $(\mathcal{P}_l, T_{p(x)})$ is LB.

□

Theorem 3.4.10 (Ornstein, Rudolph, Weiss [**35**]). *If G is a compact group and $\phi : G \to G$ is rotation by ρ with $\mathbb{Z}_\rho$ dense in G, then for any partition $\mathcal{P}$, $(\mathcal{P}, \phi)$ is LB.*

Theorem 3.4.11 (Onrstein, Rudolph, Weiss [**35**]). *If $\mathcal{B}_n \nearrow \mathcal{B}$ and $(X, \mathcal{B}_n, T)$ is LB for each n, so is $(X, \mathcal{B}, T)$.*

Proof of Theorem 3.4.3. If μ does not have full support, by Proposition 3.2.5, the Bratteli-Vershik system is a stationary odometer. Every odometer is a compact group rotation, and hence by Theorem 3.4.10 is LB.

For a fully-supported measure μ, Corollary 3.4.9 says that for each l, the process $(\mathcal{P}_l, T_{p(x)})$ is LB. Let $\mathcal{B}_l$ be the σ-algebra generated by $\mathcal{P}_l$. Then $(X_{p(x)}, \mathcal{B}_l, \mu, T_{p(x)})$ is LB and $\mathcal{B}_l \nearrow \mathcal{B}$. Hence Theorem 3.4.11 tells us that $(X_{p(x)}, \mathcal{B}, \mu, T_{p(x)})$ is LB. □

3.5. Eigenvalues

In this section we will show that every Bratteli-Vershik system determined by a positive integer polynomial of degree 1 for which either of the coefficients is greater than 1, when endowed with a $T_{p(x)}$-invariant ergodic measure, is not weakly mixing.

Theorem 3.5.1. *Let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by $p(x) = a_0 + a_1 x$, with fully-supported, $T_{p(x)}$-invariant, ergodic probability measure μ. Then $e^{2\pi i/(a_0 a_1)}$, $e^{2\pi i/a_0}$, and $e^{2\pi i/a_1}$ are eigenvalues of $T_{p(x)}$.*

Proof. The sets $\{\gamma \in X_{p(x)} | k_n(\gamma) = 0 \text{ for all } n = 0, 1, \ldots\}$ and $\{\gamma \in X_{p(x)} | k_n(\gamma) = n \text{ for all } n = 0, 1, \ldots\}$ are both $T_{p(x)}$-invariant sets. Since μ is ergodic and has full support these are sets of measure 0. Recall from Section 1.2 that the minimal cylinder into vertex (n,k), is denoted $Y_n(k,0)$ and that $Y_n(k,i) = T^i(Y_n(k,0))$ for $i = 0,1,\dots,\dim(n,k)-1$. For μ-almost every $\gamma \in X_{p(x)}$ there exist $n \geq 0$, $0 < k < n$, and $0 \leq j \leq \dim(n,k)-1$ for which $\gamma \in Y_n(k,j)$.

Define the function $f : X_{p(x)} \to \mathbb{C}$ by the following: for $n > 1$, $0 < k < n$, and $0 \leq j < \dim(n,k)$,

$$f(Y_n(k,j)) = (e^{2\pi i/(a_0a_1)})^{j+1},$$

and $f = 0$ elsewhere.

In the cutting and stacking construction, this amounts to defining f on the bottom levels of the stacks as $(e^{2\pi i/(a_0a_1)})$ and multiplying by $(e^{2\pi i/(a_0a_1)})$ as one progresses up the stacks.

In order to show that f is well defined it is enough to show that for a minimal cylinder C and an extension C^j of C

$$f(C^j) = e^{2\pi i/(a_0a_1)}$$

(see the proof of Lemma 3.2.6). We will divide the argument into two cases, the case when C is extended by an edge $j \in \{0,1,\dots,a_1-1\}$ (C extended to the right) and the case when C is extended by an edge $j \in \{a_1, a_1+1\dots, a_1+a_0-1\}$ (C extended to the left). First assume that C terminates at vertex $0 < k < n$, and let C^j be the extension of the cylinder C by the edge $j \in \{0,1,\dots,a_1-1\}$, which has terminal vertex $(n+1,k+1)$. Then $C^j = Y_{n+1}(k+1, j\dim(n,k))$. Since $0 < k < n$,

$$\dim(n,k) = \binom{n}{k} a_0^{n-k} a_1^k = 0 \bmod a_0a_1, \tag{3.5.1}$$

and therefore

$$f(C^j) = (e^{2\pi i/(a_0a_1)})^{j\dim(n,k)+1} = (e^{2\pi i/(a_0a_1)})^{j\dim(n,k)}e^{2\pi i/(a_0a_1)} = e^{2\pi i/(a_0a_1)}.$$

Now let C^j be the extension of the cylinder C by an edge $j \in \{a_1, a_1+1, \ldots, a_1+a_0-1\}$. Then $C^j = Y_{n+1}(k, a_1\dim(n,k-1) + (j-a_1)\dim(n,k))$. Since $0 < k < n$,

$$a_1\dim(n,k-1) = \binom{n}{k-1}a_0^{n-k+1}a_1^k = 0 \bmod a_0a_1,$$

and from Equation 3.5.1 we see that,

$$\begin{aligned} f(C^j) &= (e^{2\pi i/(a_0a_1)})^{a_1\dim(n,k-1)+(j-a_1)\dim(n,k)+1} \\ &= (e^{2\pi i/(a_0a_1)})^{a_1\dim(n,k-1)}(e^{2\pi i/(a_0a_1)})^{(j-a_1)\dim(n,k)}e^{2\pi i/(a_0a_1)} \\ &= e^{2\pi i/(a_0a_1)}. \end{aligned}$$

Hence f is well defined μ-almost everywhere.

For $n > 1$, $0 < k < n$, $0 \leq j < \dim(n,k) - 1$ and $\gamma \in Y_n(k,j)$, it is clear that $f(T_{p(x)}\gamma) = e^{2\pi i/(a_0a_1)}f(\gamma)$. For $n \geq 0$, $0 < k < n$, and $\gamma \in Y_n(k, \dim(n,k)-1)$, there are$m \geq 0$ and $0 < l < m$ such that $T_{p(x)}\gamma \in Y_m(k,0)$. Then $f(\gamma) = 1$ and $f(T_{p(x)}\gamma) = e^{2\pi i/(a_0a_1)}$. Hence for μ-almost every $\gamma \in X$, $f(T_{p(x)}\gamma) = e^{2\pi i/(a_0a_1)}f(\gamma)$, and $e^{2\pi i/(a_0a_1)}$ is an eigenvalue of $T_{p(x)}$.

The same argument can be repeated using the eigenvalues $e^{2\pi i/a_0}$ and $e^{2\pi i/a_1}$. $\square$

COROLLARY 3.5.2. *Let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system determined by the polynomial $a_0 + a_1x$ with a fully-supported, $T_{p(x)}$-invariant, ergodic probability measure μ. If either a_0 or a_1 is greater than 1, then $T_{p(x)}$ is not weakly mixing.*

Remark 3.5.3. The main result of this section is possible because for a degree one polynomial, $p(x) = a_0 + a_1x$, all the coefficients of $(p(x))^n$ except the coefficients of x^0 and x^{dn} are divisible by a_0a_1, and thus in the cutting and stacking representation all but maybe the end stacks have heights divisible by a_0a_1. For a polynomial $p(x)$ of degree higher than 1, the coefficients (and stack heights) do not necessarily have a common factor, see Figure 3.15; therefore this argument is not sufficient for polynomials of higher degree.

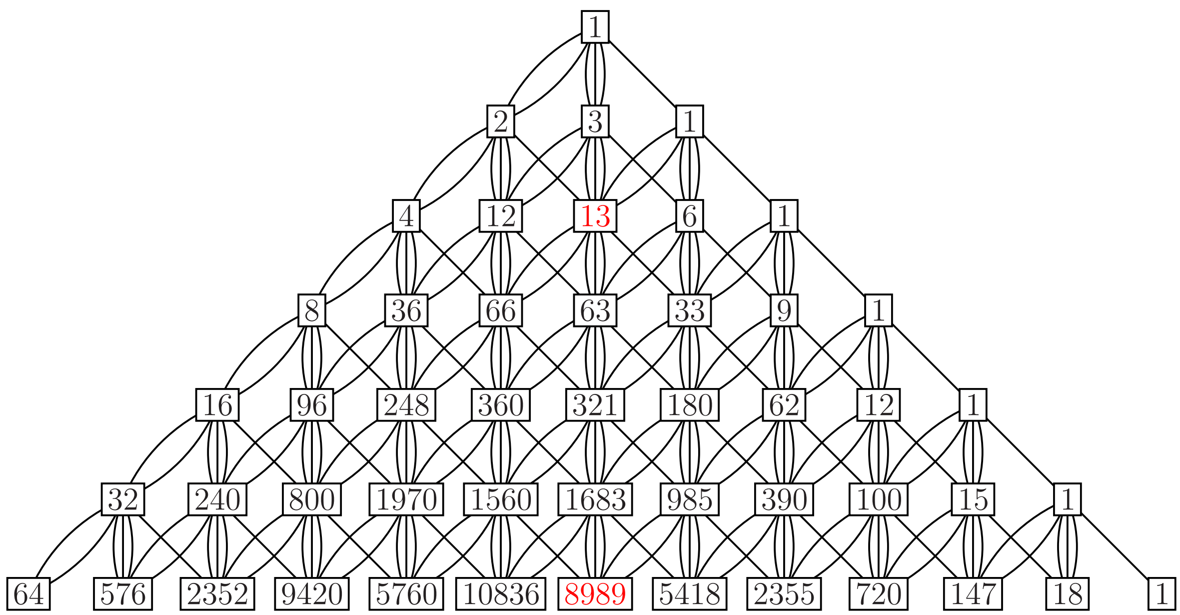


Figure 3.15. The coefficients of the powers of $p(x) = 2 + 3x + x^2$.

CHAPTER 4

The Euler Adic

Much of this chapter is taken from two papers that have been submitted for publication. Section 4.2 is joint work Micheal Keane, Karl Petersen, and Ibraham Salama. Section 4.3 is joint work with Karl Petersen.

4.1. Description

For each $n = 0, 1, 2, \ldots, k = 0, 1, \ldots n$, the *Eulerian number* $A(n, k)$ is the number of permutations $i_1 i_2 \ldots i_{n+1}$ of $\{1, \ldots, n+1\}$ with exactly k *rises* (indices $j = 1, 2, \ldots, n$ with $i_j < i_{j+1}$) and $n - k$ *falls* (indices $j = 1, 2, \ldots, n$ with $i_j > i_{j+1}$).

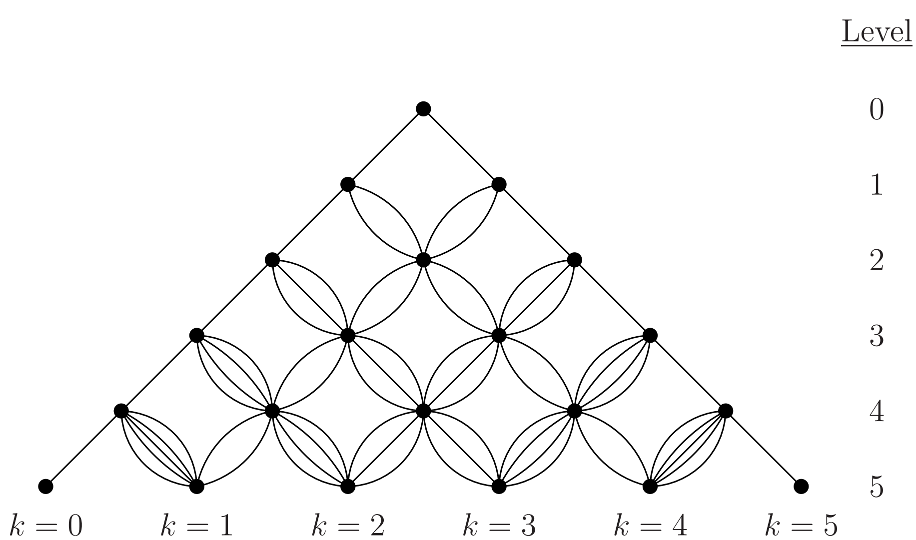


FIGURE 4.1. The Euler graph

The *Euler graph* is a Bratteli diagram in $\mathcal{D}_{\mathcal{L}}$ (see Section 2.1) for which $d = 1$, and for $n = 0, 1, \ldots,$ and $k = 0, 1, \ldots, n$, the vertex (n, k) has $n + 2$ total edges leaving it, with $k + 1$ edges connecting it to vertex $(n + 1, k)$ and $n - k + 1$ edges connecting it to vertex $(n + 1, k + 1)$. We say that an edge is a *left turn* if it connects vertices (n, k) and $(n + 1, k)$ and a *right turn* if it connects vertices (n, k) and $(n + 1, k + 1)$. The number of paths from the root vertex (0,0) to the vertex (n, k) is the Eulerian number, $A(n, k)$.

These numbers satisfy the recursion

$$A(n + 1, k) = (n - k + 2)A(n, k - 1) + (k + 1)A(n, k), \tag{4.1.1}$$

which can be visualized in Figure 4.2.

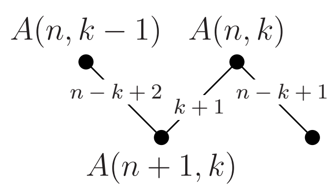


FIGURE 4.2. The Euler Graph gives rise to Equation 4.1.1.

DEFINITION 4.1.1. The Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ associated to the Euler graph will be denoted (X, T), and the transformation T is called the *Euler adic*.

The *symmetric measure*, η, on the infinite path space X is the Borel probability measure that for each n gives every cylinder of length n starting at the root vertex the same measure. A similar argument to Lemma 3.2.6 tells us that a non-atomic probability measure on X is T-invariant if and only if all cylinders with the same terminal vertex have the same measure. Clearly η is T-invariant. The measure of any cylinder set can be computed by multiplying *weights* on the edges, each weight on an edge connecting

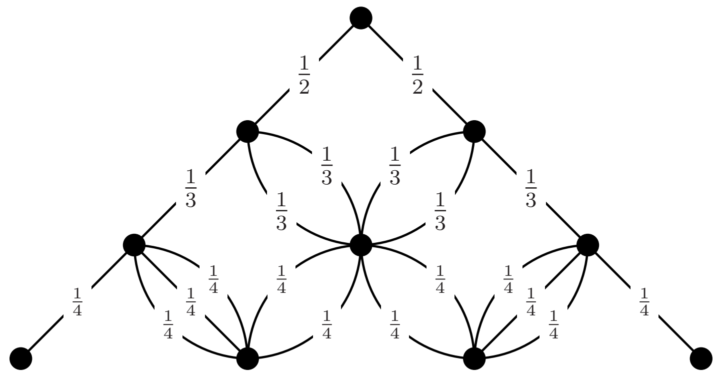


FIGURE 4.3. The Symmetric Measure

level n to level $n+1$ being $1/(n+2)$. We can think of the weights as assigning equal probabilities to all the allowed steps for a random walker who starts at the root and descends step by step to form an infinite path $\gamma \in X$. See Figure 4.3.

We can also view the transformation T as a map on the unit interval defined by "cutting and stacking" as in Section 1.2. At each stage $n = 0, 1, 2, \dots$ we have $n+1$ stacks $S_{n,0}, S_{n,1}, \dots, S_{n,n}$ (corresponding to the vertices (n,k), $0 \leq k \leq n$, of the Euler graph). Stack $S_{n,k}$ consists of $A(n,k)$ subintervals of $[0,1]$. In the case of the Euler adic, in order to proceed to the next stage in the cutting and stacking construction, each stack $S_{n,k}$ is cut into $n+2$ equal substacks. These are recombined into new stacks in the order prescribed by the way T maps their corresponding cylinder sets. Hence, (X, η) is isomorphic to $([0,1], m)$ where m is Lebesgue measure.

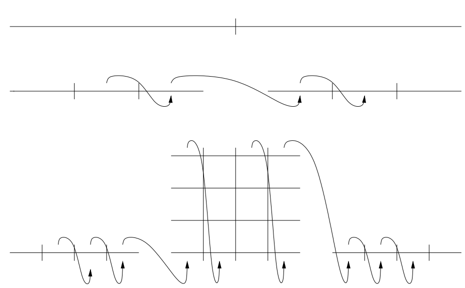


FIGURE 4.4. The Euler adic as a cutting and stacking transformation.

4.2. The Symmetric Measure is Ergodic

The main result of this section is joint work with Michael Keane, Karl Petersen, and Ibrahim Salama and will appear in [**1**]. In order to prove that the Euler adic T is ergodic with respect to the symmetric measure η, we adapt the proof in [**28**] of ergodicity of the $\mathcal{B}(1/2, 1/2)$ measure for the Pascal adic. For previous proofs of the ergodicity of Bernoulli measures for the Pascal adic, see [**22**],[**47**], [**38**],[**33**],[**34**] and the references that they contain.

Proposition 4.2.1. *For each $\gamma \in X$, denote by $I_n(\gamma)$ the cylinder set determined by $\gamma_0\gamma_1...\gamma_{n-1}$. Then for each measurable $A \subseteq X$,*

$$\frac{\eta(A \cap I_n(\gamma))}{\eta(I_n(\gamma))} \to \chi_A(\gamma) \text{ almost everywhere.}$$

Proof. In view of the isomorphism of (X, η) and $([0, 1], m)$, this is just the Lebesgue Density Theorem. □

Denote by ρ the measure $\eta \times \eta$ on $X \times X$.

Proposition 4.2.2. *For ρ-almost every $(\gamma, \xi) \in X \times X$, there are infinitely many n such that $I_n(\gamma)$ and $I_n(\xi)$ have the same terminal vertex in the Euler graph, equivalently $(n, k_n(\gamma)) = (n, k_n(\xi))$.*

This is equivalent to saying that for infinitely many n the number of left turns in $\gamma_0...\gamma_{n-1}$ equals the number of left turns in $\xi_0...\xi_{n-1}$, or that in the cutting and stacking representation the subintervals of [0,1] corresponding to $I_n(\gamma)$ and $I_n(\xi)$ are in the same stack. This happens because the symmetric measure has a central tendency: if a path is not near the center of the graph at level n, there is a greater probability that at level

$n+1$ it will be closer to the center than before (and the farther from the center, the greater the probability). We defer momentarily the proof of Proposition 4.2.2 in order to show how it immediately implies the main result.

THEOREM 4.2.3. *The Euler adic T is ergodic with respect to the symmetric measure, η.*

Proof. Suppose that $A \subseteq X$ is measurable and T-invariant and that $0 < \eta(A) < 1$. By Proposition 4.2.1,

$$\frac{\eta(A \cap I_n(\gamma))}{\eta(I_n(\gamma))} \to 1 \text{ and } \frac{\eta(A^c \cap I_n(\xi))}{\eta(I_n(\xi))} \to 1 \text{ for } \rho\text{-almost every } (\gamma, \xi) \in A \times A^c.$$

Hence for almost every $(\gamma, \xi) \in A \times A^c$ we can pick an $n_0 = n_0(\gamma, \xi)$ such that for all $n \geq n_0$,

$$\frac{\eta(A \cap I_n(\gamma))}{\eta(I_n(\gamma))} > \frac{1}{2} \qquad \text{and} \qquad \frac{\eta(A^c \cap I_n(\xi))}{\eta(I_n(\xi))} > \frac{1}{2}. \tag{4.2.1}$$

Then, by Proposition 4.2.2, we can choose $n \geq n_0$ such that $I_n(\gamma)$ and $I_n(\xi)$ end in the same vertex, and hence there is $j \in \mathbb{Z}$ such that $T^j(I_n(\gamma)) = I_n(\xi)$. Since A is T-invariant, this contradicts Equation (4.2.1). Then we must have $\eta(A) = 0$ or $\eta(A) = 1$, and so T is ergodic with respect to η. □

It remains to prove Proposition 4.2.2.

For a random variable Z_n on the probability space $(\Omega, \mathcal{B}, \mu)$ denote the expected value of Z_n by $\mathbb{E}(Z_i)$ and the conditional expected value of Z_n with respect to the sub-σ-algebra generated by $\mathcal{A}$ by $\mathbb{E}_\eta(Z_n|\mathcal{A})$. Denote by $V(Z_n)$ the variance of Z_n.

DEFINITION 4.2.4. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$ an increasing sequence of sub-σ-algebras of $\mathcal{B}$. A sequence $Z_1, Z_2, \dots$ of functions in $\mathcal{L}^1(\Omega)$

such that for each $n = 1, 2, \dots$, the random variable Z_n is measurable with respect to $\mathcal{B}_n$ is called a *supermartingale* if

$$\mathbb{E}_\eta(Z_{n+1}|\mathcal{B}_n) \le Z_n \quad \eta\text{-almost everywhere.}$$

DEFINITION 4.2.5. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and $\{\mathcal{B}_n\}$ an increasing sequence of sub-σ-algebras of $\mathcal{B}$. Then $\tau : \Omega \to \{1, 2, \dots\}$ is said to be a *stopping time* (with respect to $\{\mathcal{B}_n\}$) if for all $n = 1, 2, \dots$, $\{\omega : \tau(\omega) \le n\} \in \mathcal{B}_n$.

For references on general probability theory see [**3**] and [**16**].

LEMMA 4.2.6. *On $(X \times X, \rho)$, for each $n = 1, 2, \dots$ let $D_n(\gamma, \gamma') = |k_n(\gamma) - k_n(\gamma')|$, and let $\mathcal{F} = \mathcal{B}((\gamma_0, \gamma'_0), \dots, (\gamma_{n-1}, \gamma'_{n-1}))$ denote the σ-algebra generated by $(\gamma_0, \gamma'_0), \dots, (\gamma_{n-1}, \gamma'_{n-1})$. Let $\sigma(\gamma, \gamma')$ be a stopping time with respect to $(\mathcal{F}_n)$ such that $D_{\sigma(\gamma,\gamma')}(\gamma, \gamma') > 0$. Fix $M > 0$ and let*

$$\tau(\gamma, \gamma') = \inf\{n > \sigma(\gamma, \gamma') : D_n \in \{0, M\}\}.$$

For $n = 0, 1, 2, \dots$, let

$$Y_n(\gamma, \gamma') = \begin{cases} D_{\sigma(\gamma,\gamma')}(\gamma, \gamma') & \text{if} \quad 0 \le n \le \sigma(\gamma, \gamma') \\ D_n(\gamma, \gamma') & \text{if} \quad \sigma(\gamma, \gamma') < n \le \tau(\gamma, \gamma') \\ D_{\tau(\gamma,\gamma')}(\gamma, \gamma') & \text{if} \quad n \ge \tau(\gamma, \gamma') \end{cases}$$

Then $(Y_n(\gamma, \gamma') : n = 0, 1, 2, \dots)$ is a supermartingale with respect to $(\mathcal{F}_n)$.

Proof. We have to check the defining inequality for supermartingales only for the range of n where $Y_n = D_n$, since otherwise $Y_n(\gamma, \gamma')$ is constant in n.

If γ turns to the left at stage n, then $k_{n+1}(\gamma) = k_n(\gamma)$, but if γ turns to the right $k_{n+1}(\gamma) = k_n(\gamma) + 1$. From Figure 4.2 we see that

$$\eta\{k_{n+1}(\gamma) = k_n(\gamma) | \gamma_0 ... \gamma_{n-1}\} = \frac{k_n(\gamma) + 1}{n + 2} \tag{4.2.2}$$

and

$$\eta\{k_{n+1}(\gamma) = k_n(\gamma) + 1 | \gamma_0 ... \gamma_{n-1}\} = \frac{n - k_n(\gamma) + 1}{n + 2}. \tag{4.2.3}$$

Without loss of generality assume that $k_n(\gamma') > k_n(\gamma)$. Note that

$$D_{n+1} = \begin{cases} D_n & \text{on the set} \quad A = \{k_{n+1}(\gamma) = k_n(\gamma), k_{n+1}(\gamma') = k_n(\gamma')\} \cup \\ & \qquad\qquad\quad \{k_{n+1}(\gamma) = k_n(\gamma) + 1, k_{n+1}(\gamma') = k_n(\gamma') + 1\} \\ D_n + 1 & \text{on the set} \quad B = \{k_{n+1}(\gamma) = k_n(\gamma), k_{n+1}(\gamma') = k_n(\gamma) + 1\} \\ D_n - 1 & \text{on the set} \quad C = \{k_{n+1}(\gamma) = k_n(\gamma) + 1, k_{n+1}(\gamma') = k_n(\gamma')\}. \end{cases}$$

From (4.2.2) and (4.2.3),

$$\mathbb{E}_\rho(D_{n+1} - D_n | \mathcal{F}_n) = 0 \cdot \rho(A|\mathcal{F}_n) + 1 \cdot \rho(B|\mathcal{F}_n) - 1 \cdot \rho(C|\mathcal{F}_n)$$

$$= \frac{1}{n+2}[(k_n(\gamma) + 1)(n - k_n(\gamma') + 1) - (k_n(\gamma') + 1)(n - k_n(\gamma) - 1)] \leq 0.$$

Hence $\mathbb{E}_\rho(D_{n+1} | \mathcal{F}_n) \leq D_n$. $\square$

LEMMA 4.2.7. *$\frac{k_n(\gamma)}{n} \to \frac{1}{2}$ in measure.*

Proof. Let $u_n(\gamma) = 2k_n(\gamma) - n$ for all n. We will show that $u_n/n \to 0$ in measure. We begin by computing the variance of u_n. Note that if $k_{n+1}(\gamma) = k_n(\gamma)$ then $u_{n+1} = u_n - 1$, and if $k_{n+1}(\gamma) = k_n(\gamma) + 1$ then $u_{n+1} = u_n + 1$. Following the calculations in [**42**], and using (4.2.2) and (4.2.3),

$$\mathbb{E}_\eta(u_{n+1}|u_n) = \left(\frac{n+1}{n+2}\right) u_n;$$

so, since $u_0 = 0$, $\mathbb{E}_\eta(u_n) = 0$ for all $n = 1, 2, \dots$. Similarly,

$$\mathbb{E}_\eta(u_{n+1}^2|u_n) = (u_n - 1)^2 \left(\frac{k_n(\gamma)+1}{n+2}\right) + (u_n+1)^2 \left(\frac{n - k_n(\gamma) + 1}{n+2}\right)$$

$$= (u_n - 1)^2 \left(\frac{u_n + n + 2}{2(n+2)}\right) + (u_n+1)^2 \left(\frac{n - u_n + 2}{2(n+2)}\right)$$

$$= \frac{nu_n^2}{n+2} + 1.$$

Then

$$\mathbb{E}_\eta(u_{n+1}^2|u_{n-1}) = \frac{n}{n+2}\left(\frac{(n-1)u_{n-1}^2}{n+1} + 1\right) + 1,$$

and continuing this recursively we see that the variance of u_{n+1} is

$$V(u_{n+1}) = \mathbb{E}_\eta(u_{n+1}^2) = \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} (i+1)(i+2)$$

$$= \frac{n+3}{3}.$$

Then by Chebyshev's Inequality,

$$\eta\left\{\left|\frac{u_n}{n}\right| \geq \varepsilon\right\} \leq \frac{c}{n\varepsilon^2} \to 0 \text{ as } n \to \infty,$$

so that $\dfrac{u_n}{n} \to 0$ in measure, i.e. $\dfrac{k_n(\gamma)}{n} \to \dfrac{1}{2}$ in measure. $\square$

Proof of Proposition 4.2.2. From Lemma 4.2.6, (D_n) is a supermartingale with respect to $\mathcal{F}_n = (\mathcal{B}((\gamma_0, \gamma_0'), \ldots, (\gamma_{n-1}, \gamma_{n-1}')))$. Fix $M > 0$ and define stopping times $\sigma(\gamma, \gamma') = \inf\{n | k_n(\gamma) \neq k_n(\gamma')\}$ and $\tau(\gamma, \gamma') = \inf\{n > \sigma(\gamma, \gamma') | D_n \in \{0, M\}\}$. Then $\mathbb{E}_\rho(D_\tau) \leq \mathbb{E}_\rho(D_\sigma) = 1$. If τ is finite almost everywhere, then

$$\mathbb{E}_\rho(D_\tau) = M(\rho\{D_\tau = M\}) + 0(\rho\{D_\tau = 0\}),$$

so that

$$\rho\{D_n \neq 0 \text{ for any } n > \sigma(\gamma, \gamma')\} \leq \rho\{D_\tau = M\} \leq 1/M$$

for all M. Letting $M \to \infty$ implies that

$$\rho\{D_n \neq 0 \text{ for any } n > \sigma(\gamma, \gamma')\} = 0.$$

Hence with ρ-probability 1 there is an n_0 for which $k_{n_0}(\gamma) = k_{n_0}(\gamma')$. Repeat this process with $\sigma(\gamma, \gamma') = \inf\{n > n_0(\gamma, \gamma') | k_n(\gamma) \neq k_n(\gamma')\}$ to see that with ρ-probability 1, $k_n(\gamma) = k_n(\gamma')$ infinitely many times. It remains to show that τ is finite almost everywhere.

This is more complicated than ergodicity for the $\mathcal{B}(1/2, 1/2)$ measure because the process of going left or right is not a genuine symmetric random walk, as the probabilities depend on the location in the diagram. Nevertheless, the increments are Markovian, and near the center of the diagram the probability of going left or right is near 1/2. The general idea of the following is to see that for η-almost every $\gamma \in X$, the path γ is eventually bounded within a region such that the probabilities of going left or right are near 1/2.

We have a fixed M; fix also a large L. Fix a small enough ε so that if $k_n(\gamma)/n,\ k_n(\gamma')/n$ are in the interval $(1/2-\varepsilon, 1/2+\varepsilon)$, then

$$\frac{k_{n+i}(\gamma)}{n+i}, \frac{n-k_{n+i}(\gamma)}{n+i}, \frac{k_{n+i}(\gamma')}{n+i}, \frac{n-k_{n+i}(\gamma')}{n+i} \geq \frac{1}{4} \text{ for } i = 0, 1, \ldots, ML.$$

In other words, starting from $(n, k_n(\gamma))$ all the probabilities of going left or right for both γ and γ' are at least $1/4$ for ML steps. Let

$$A_n = \{(\gamma, \gamma') \in X \times X | k_n(\gamma)/n, k_n(\gamma')/n \in (1/2-\varepsilon, 1/2+\varepsilon)\},$$

and note that $\rho(A_n) \to 1$ as $n \to \infty$, by the convergence in measure. Let

$$B_n = \{(\gamma, \gamma') \in X \times X | k_{n+i}(\gamma) = k_n(\gamma), k_{n+i}(\gamma') = k_n(\gamma') + i \text{ for all } i = 0, 1 \ldots, M\}.$$

For every n, $\{\gamma | \tau(\gamma) = \infty\} \cap A_n \subset A_n \cap B_n^c \cap B_{n+M}^c \cap \cdots \cap B_{n+(L-1)M}^c = G_n$, since (γ, γ') in B_n implies $D_{n+i}(\gamma, \gamma')$ is either 0 or M for some $i \leq M$. Conditioned on the set A_n, the sets $B_n, B_{n+M}, \ldots, B_{n+(L-1)M}$ are not independent, because at each step the probabilities of going left or right, given by sums of the weights on the edges, are changing. But since the probabilities of going left or right at each step are all near $1/2$, so that the probability of each event we are considering is near the probability that it would be assigned by a genuine symmetric random walk, we can estimate the measure of G_n.

For each $j = 0, 1, \ldots, L-1$, abbreviate $E_j = B_{n+jM}^c$. Then for each pair of vertices $v = ((jM-1, k), (jM-1, k'))$,we have $\rho(E_j | v) \leq (1 - 1/4^{2M})$. Thus

$$\rho(E_j | E_{j-1} \cap \cdots \cap E_0 \cap A_n) = \sum_{\substack{\text{vertices } v \text{ at} \\ \text{level } jM-1}} \rho(E_j | v) \rho(v | E_{j-1} \cap \cdots \cap E_0 \cap A_n) \leq (1 - 1/4^{2M}),$$

and iterating gives $\rho(E_{L-1} \cap \cdots \cap E_0 | A_n) \leq \left(1 - 1/4^{2M}\right)^L$.

Therefore $\rho(\tau = \infty | A_n) \leq (1 - 1/4^{2M})^L$ for all L. Letting $n \to \infty$ and then $L \to \infty$, we conclude that $\rho\{\tau = \infty\} = 0$. $\square$

Remark 4.2.8. In fact $k_n(\gamma)/n \to 1/2$ almost everywhere. We can see this as follows. Continue to let $u_n(\gamma) = 2k_n(\gamma) - n$ as in Lemma 4.2.7. Since $E((n+2)u_{n+1}|(n+1)u_n) = (n+1)u_n,\ S_n = (n+1)u_n$ forms a mean-0 martingale. If $X_n = S_n - S_{n-1}$, then the X_n are a martingale difference sequence in L^2, thus mean 0 and orthogonal. The variance of X_n is

$$\mathbb{E}(X_n^2) = \mathbb{E}(S_n^2) - \mathbb{E}(S_{n-1}^2) = \frac{3n^2 + 5n}{3}.$$

Recall Kolmogrov's Criterion for the Strong Law of Large Numbers [**16**, p.238]: Let $\{X_k\}$ be random variables such that $\mathbb{E}(X_n | X_1, \ldots, X_{n-1}) = 0$, $S_n = \sum_{i=1}^n X_i$, $b_1 < b_2 < \cdots \to \infty$, and $\sum \mathbb{E}(X_k^2)/b_n^2 < \infty$. Then with probability one, $b_n^{-1} S_n \to 0$.

If we let $b_n = n^2$, then $\sum \mathbb{E}(X_n^2)/b_n^2 < \infty$, so $S_n/b_n \to 0$ almost everywhere, that is to say, $u_n/n \to 0$ almost everywhere.

4.3. The Uniqueness of the Symmetric Measure

In this section we give a new proof of the main result of the last section and [**1**], in fact we prove a stronger result by a different method; namely, we show that not only is the symmetric measure η ergodic, but that it is the only T-invariant, ergodic measure with full support. This section has been submitted as a paper jointly with Karl Petersen.

For the Euler graph and associated adic transformation defined in Section 4.1 there is a bijective correspondence between paths (or cylinders) of length n_0 starting at the root vertex and terminating at vertex (n_0, k_0) and permutations of $\{1, 2, \ldots, n_0 + 1\}$ with k_0

rises. Consider the cylinder set defined by the single edge connecting the vertex $(0,0)$ to the vertex $(1,0)$. This cylinder set is of length 1 with 1 left turn, and we assign to it the permutation 21, which has one fall. Likewise, the cylinder set defined by the single edge connecting the vertex $(0,0)$ to the vertex $(1,1)$ is of length 1 with one right turn, and we assign to it the permutation 12, which has one rise. When a cylinder F of length n, corresponding to the permutation $\pi(F)$ of $\{1,2,\dots,n+1\}$, is extended by an edge from level n to level $n+1$, we extend $\pi(F)$ in a unique way to a permutation of $\{1,2,\dots,n+2\}$, as follows. If F is extended by a left turn down the i'th edge connecting (n,k) to $(n+1,k)$, insert $n+2$ into $\pi(F)$ in the i'th place that adds an additional fall to $\pi(F)$. Likewise, if F is extended by a right turn down the i'th edge connecting (n,k) to $(n+1,k+1)$, insert $n+2$ into $\pi(F)$ in the i'th place that adds an additional rise to $\pi(F)$. This correspondence produces a labeling of infinite paths in the Euler graph starting at the root; then the path space X corresponds to the set of all linear orderings of $\mathbb{N}=\{1,2,3,\dots\}$, and the adic transformation T can be thought of as moving from an ordering to its successor.

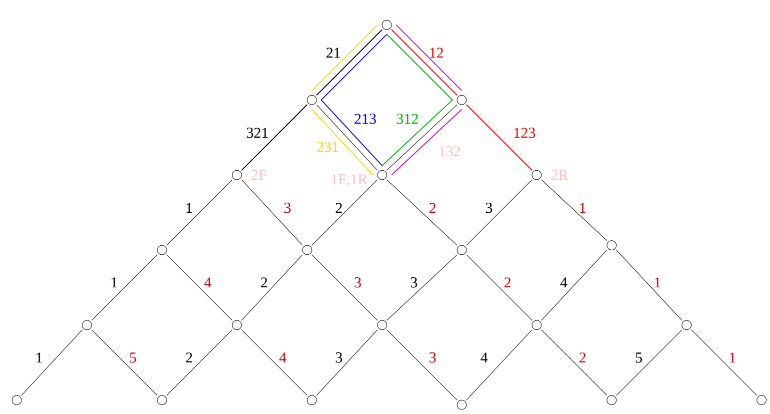


FIGURE 4.5. Some cylinders and their corresponding permutations

Denote by $\mathcal{I}$ the σ-algebra of T-invariant Borel measurable subsets of X and recall the following result.

Lemma 4.3.1 (Vershik [**46, 45**]). *Let μ be an invariant probability measure for the Euler adic transformation. Then for every cylinder set F and μ-almost every $\gamma \in X$,*

$$\lim_{n \to \infty} \frac{\dim(F, (n, k_n(\gamma)))}{\dim((n, k_n(\gamma)))} = \mathbb{E}_\mu(\chi_F | \mathcal{I}).$$

It is clear from the Markov property and the patterns of weights on the edges that with respect to η almost every path $\gamma \in X$ has infinitely many left and right turns. We note now that this property is shared by all fully-supported ergodic measures for the Euler adic.

Lemma 4.3.2. *Let μ be a T-invariant ergodic probability measure with full support for the Euler adic transformation. For μ-almost every $\gamma \in X$, there are infinitely many left and right turns (i.e., $k_n(\gamma)$ and $n - k_n(\gamma)$ are unbounded a.e.).*

Proof. For each $K = 1, 2, \dots,$ let

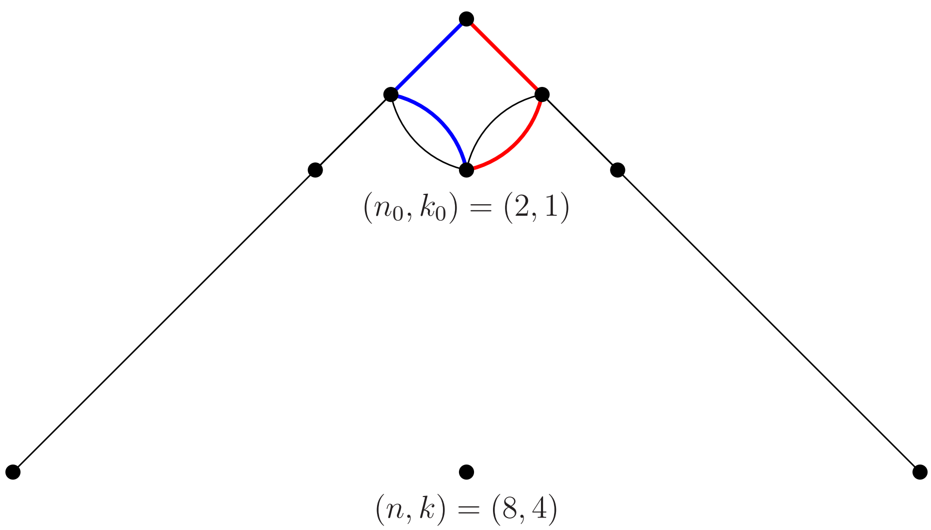
$$A_K = \{\gamma \in X | \text{either } k_n(\gamma) \leq K \text{ for } n = 0, 1, \dots, \text{ or } n - k_n(\gamma) \leq K \text{ for } n = 0, 1, 2, \dots\}.$$

In other words, each path γ in X has either a finite number of left turns or a finite number of right turns. A_K is a proper closed T-invariant set. Since μ is ergodic and has full support, $\mu(A_K) = 0$. $\square$

Proposition 4.3.3. *If μ is an invariant probability measure for the Euler adic transformation such that $k_n(\gamma)$ and $n - k_n(\gamma)$ are unbounded a.e., then $\mu = \eta$.*

Proof. For any string w on an ordered alphabet denote by $r(w)$ the number of rises in w and by $f(w)$ the number of falls in w (defined as above). Let F and F' be cylinder sets in X specified by fixing the first n_0 edges, and let $\pi(F)$ and $\pi(F')$ be the permutations assigned to them by the correspondence described in the preceding section. Suppose that the paths corresponding to F and F' terminate in the vertices (n_0, k_0) and (n_0, k_0') respectively. Fix $n \gg n_0$ and $k \gg k_0$. We aim to show that $\mu(F) = \mu(F')$.

EXAMPLE 4.3.1. Let F be the blue cylinder, and F' the red. Then $\pi(F) = 213$ and $\pi(F') = 132$. For purposes of following an example through, we will let $(n, k) = (8, 4)$.



Each path s in the Euler graph from (n_0, k_0) to (n, k) corresponds to a permutation σ_s of $\{1, 2, \dots, n+1\}$ with k rises in which $1, 2, \dots, n_0 + 1$ appear in the order $\pi(F)$. Counting $\dim(F, (n, k))$ is equivalent to counting the number of distinct such σ_s. Each such permutation σ_s has associated to it a permutation $t(\sigma_s)$ of $\{n_0 + 2, \dots, n + 1\}$ obtained by deleting $1, 2, \dots, n_0 + 1$ from σ_s, see Example 4.3.2. Taking a reverse view, one obtains σ_s from $\rho = t(\sigma_s)$ by inserting $1, 2, \dots, n_0 + 1$ from left to right, in the order prescribed by $\pi(F)$, into ρ.

We define a *cluster* in σ_s to be a subset of $\{1, 2, \ldots, n_0 + 1\}$ whose members are found consecutively in σ_s, with no elements of $\{n_0 + 2, \ldots, n + 1\}$ separating them, in the order prescribed by $\pi(F)$. The set M_s of clusters in σ_s is an ordered partition of the permutation $\pi(F)$, and we define

$$r(M_s) = \sum_{c \in M_s} r(c).$$

In general, $1 \leq |M_s| \leq n_0 + 1$ and $0 \leq r(M_s) \leq k_0$.

EXAMPLE 4.3.2. $\pi(F) = 213$

$\sigma_{s_1} = 297146385$	$\sigma_{s_2} = 962471358$
$t(\sigma_{s_1}) = 974685$	$t(\sigma_{s_2}) = 964758$
$M_{s_1} = \{2, 1, 3\}$	$M_{s_2} = \{2, 13\}$
$r(M_{s_1}) = 0$	$r(M_{s_2}) = 1$

Given a permutation ρ of $\{n_0+2, \ldots, n+1\}$, $1 \leq m \leq n_0+1$, and an ordered partition M of $\pi(F)$ with $|M| = m$, there are $C(n - n_0 + 1, m)$ (the binomial coefficient) choices for how to insert the members of M as clusters into the permutation ρ in order to form a permutation σ_s. But not all of these choices yield valid permutations σ_s, which have exactly k rises. Looking more closely, we see that placing a cluster $c \in M$ at the tail end of ρ or into a rise in ρ produces a permutation $\overline{\rho}$ whose number of rises is $r(\overline{\rho}) = r(\rho) + r(c)$, while placing c at the beginning or into a fall produces $\overline{\rho}$ with $r(\overline{\rho}) = r(\rho) + r(c) + 1$. So we must have

$$k = r(\sigma_s) = r(\rho) + r(M) +$$

$$\#\{c \in M | c \text{ is placed into a fall or at the beginning of } \rho\}.$$

In order to count the number of ways to place the members of M, we will first count the number of places where we could insert $\{c \in M | c$ is placed into a fall or at the beginning of $\rho\}$. There are $n - n_0 - (k - k_0) + 1$ possible places and we must choose $k - r(\rho) - r(M)$ of them. For each of these possibilities we must then choose places to place the remaining clusters of M. There are $k - k_0 + 1$ places, and $m - (k - r(\rho) - r(M))$ remaining clusters. Therefore the number of ways to place the members of M into ρ in such a way as to form a valid permutation σ_s, with k rises and $n - k$ falls, is

$$C(n - n_0 - (k - k_0) + 1, k - r(\rho) - r(M))\, C(k - k_0 + 1, m - (k - r(\rho) - r(M))).$$

For each $m = 1, 2, \ldots, n_0 + 1$ denote by $P_m(F)$ the set of ordered partitions M of $\pi(F)$ such that $|M| = m$. For each $r = 0, 1, \ldots, n - n_0 - 1$ denote by $Q(n, n_0, r)$ the set of permutations of $\{n_0 + 2, \ldots, n + 1\}$ with exactly r rises, so that

$$|Q(n, n_0, r)| = A(n - n_0 - 1, r).$$

Example 4.3.3. $\pi(F) = 213$, $(n, k) = (8, 4)$.

$$P_1(F) = \{\{213\}\}$$

$$P_2(F) = \{\{2, 13\}, \{21, 3\}\}$$

$$P_3(F) = \{\{2, 1, 3\}\}$$

$$Q(8, 2, 0) = \{987654\} \qquad |Q(8, 2, 0)| = A(5, 0) = 1$$

$$Q(8, 2, 5) = \{456789\} \qquad |Q(8, 2, 5)| = A(5, 5) = 1$$

In order to compute $\dim(F(n, k))$ we will partition the set of permutations σ_s of $\{1, 2, \ldots, n + 1\}$ which have exactly k rises and a subpermutation $\pi(F)$ in the following manner. First, partition the set of all such permutations according to the cardinality $m \in \{1, \ldots, n_0 + 1\}$ of the set of clusters. Now partition further by grouping the σ_s by their corresponding $M_s \in P_m(F)$. Partition yet again by grouping the σ_s by the number of rises, r, in their corresponding permutations $t(\sigma_s)$ of $\{n_0+2,\ldots,n+1\}$. Since the total number of rises in each σ_s must be k, and there are $r(M)$ rises in M, the minimum such r is $k-r(M)-m$ and the maximum is $k-r(M)$. Finally, partition these groups by the distinct permutations $t(\sigma_s)$ of $\{n_0+2,\ldots,n+1\}$ which meet the foregoing criteria. Then for each r there are $|Q(n,n_0,r)| = A(n-n_0-1,r)$ of these groups, and each group has cardinality

$$C(n-n_0-(k-k_0)+1, k-r(\rho)-r(M))\, C(k-k_0+1, m-(k-r(\rho)-r(M))).$$

Example 4.3.4. Fix $\pi(F) = 213$ and $(n,k) = (8,4)$.

1. Partition by m.

$$\begin{aligned} m=1 &\ :\ \sigma_s = \ldots 213 \ldots \\ m=2 &\ :\ \sigma_s = \ldots 2 \ldots 13 \ldots \text{ or } \sigma_s = \ldots 23 \ldots 1 \ldots \\ m=3 &\ :\ \sigma_s = \ldots 2 \ldots 1 \ldots 3 \ldots \end{aligned}$$

2. Fix $m=2$, partition by M.

$$\begin{aligned} &\ldots 2 \ldots 13 \ldots \quad \text{appears in each } \sigma_s \\ &\ldots 21 \ldots 3 \ldots \quad \text{appears in each } \sigma_s \end{aligned}$$

3. Fix $m=2$, $M=\{2,13\}$, then $r(M)=1$, partition over $r(t(\sigma_s))$.

$$r(t(\sigma_s)) = 1$$

$$r(t(\sigma_s)) = 2$$

$$r(t(\sigma_s)) = 3$$

4. Fix $m = 2$, $M = \{2, 13\}$, $r(M) = 1$, $r(t(\sigma_s)) = 3$, then partition over $t(\sigma_s)$ in $Q(8, 2, 3)$. There are $A(5, 3) = 302$ sets in this partition, and each one contains $C(4, 0)C(4, 2) = 6$ elements.

For ease of notation, for each r, let $l = k - r - r(M)$. Then $\dim(F, (n, k)) =$

$$\sum_{m=1}^{n_0+1} \sum_{M \in P_m(F)} \sum_{r=l}^{k-r(M)} A(n - n_0 - 1, r)C(n - n_0 - (k - k_0) + 1, l)\, C(k - k_0 + 1, m - l).$$

The numerator of the first binomial coefficient simplifies to having l factors chosen from $[n - n_0 - (k - k_0) - m, n - n_0 - (k - k_0)]$, and the numerator of the second has $m - l$ terms chosen from $[k - k_0 - m, k]$. Combining the two, the numerator of the binomial factors in each individual term in this sum consists of m factors chosen from $[n - k - m, n - k] \cup [k - m, k]$, and the product of the denominators is bounded above by $m!$. Since $m \leq n_0 + 1$, each of the m factors is comparable to (i.e., between two constant multiples of) either k or $n - k$.

Regrouping the terms, rewrite the sum as

$$\dim(F, (n, k)) = \sum_{r=k-(n_0+1)}^{k} \left(\sum_{m=1}^{n_0+1} \alpha(F, r, m) \right) A(n - n_0 - 1, r),$$

in which each of the coefficients

$$\beta(F, r) = \sum_{m=1}^{n_0+1} \alpha(F, r, m)$$

is a polynomial in k and $n - k$, with constant coefficients.

Clearly the dominant term within $\beta(F, r)$ when k and $n - k$ are large is the one of maximum degree in the variables k and $n - k$, which occurs when $m = n_0 + 1$. $P_{n_0+1}(F)$ has only the partition M of $\pi(F)$ into singletons, and $r(M) = 0$. The *main point* is that this term is the same for $\pi(F)$ as for any other permutation $\pi(F')$ of $\{1, 2, \ldots, n_0 + 1\}$:

$$\alpha(F, r, n_0 + 1) = \alpha(F', r, n_0 + 1)$$

for all r and all F'. When $k-r$ of $\{1, \ldots, n_0+1\}$ are put (as singleton clusters) into falls in ρ or at the beginning, and the rest are put into rises or at the end, no matter which elements are placed in which slots we always produce a permutation σ_s with exactly k rises.

Let us now consider the ratio $\dim(F, (n, k))/\dim(F', (n, k))$ when n and k are both very large. Divide top and bottom by the sum on r of the dominant terms (taking maximum degree coefficients in k and $n-k$ for each r),

$$\sum_{r=k-(n_0+1)}^{k} \alpha(F, r, n_0 + 1) A(n - n_0 - 1, r),$$

which is the *same* for F and F'. This shows that the ratio is very near 1 when k and $n-k$ are both very large.

Thus if $k_n(\gamma)$, $n - k_n(\gamma) \to \infty$ a.e. $d\mu$, we have for any two cylinders F and F' starting at the root vertex and of the same length that

$$\mathbb{E}_\mu(X_F|\mathcal{I})(\gamma) = \mathbb{E}_\mu(X_{F'}|\mathcal{I})(\gamma) \text{ a.e. } d\mu.$$

Integrating gives $\mu(F) = \mu(F')$, so that $\mu = \eta$. $\square$

THEOREM 4.3.4. *The symmetric measure η is ergodic and is the only T-invariant ergodic Borel probability measure with full support for the Euler adic transformation.*

Proof. If we show that there is an ergodic measure μ which has $k_n(\gamma)$ and $n - k_n(\gamma)$ unbounded a.e., then it will follow from Proposition 4.3.3 that $\mu = \eta$, and hence η is ergodic and is the only T-invariant ergodic measure on X with full support.

If an ergodic measure has, say, $k_n(\gamma)$ bounded on a set of positive measure, then $k_n(\gamma)$ is bounded a.e., since each set $\{\gamma | k_n(\gamma) \leq K\}$ is T-invariant. Let $\mathcal{E}_0 = \emptyset$, and for each $K = 1, 2, \ldots$ let $\mathcal{E}_K$ be the set of ergodic measures supported on either $\{\gamma \in X | k_n(\gamma) \leq K \text{ for all } n\}$ or $\{\gamma \in X | n - k_n(\gamma) \leq K \text{ for all } n\}$. If no ergodic measure has $k_n(\gamma)$ and $n - k_n(\gamma)$ unbounded a.e., then the set of ergodic measures is

$$\mathcal{E} = \bigcup_K \mathcal{E}_K.$$

Form the ergodic decomposition of η:

$$\eta = \int_{\mathcal{E}} e \, dP_\eta(e) = \sum_{K=1}^{\infty} \int_{\mathcal{E}_K \setminus \mathcal{E}_{K-1}} e \, dP_\eta(e).$$

If S is the set of paths $\gamma \in X$ for which both $k_n(\gamma)$ and $n - k_n(\gamma)$ are unbounded, then, from the remark before Lemma 4.3.2, $\eta(S) = 1$; but, for each K, $e(S) = 0$ for all e in $\mathcal{E}_K$. Hence there is an ergodic measure for which $k_n(\gamma)$ and $n - k_n(\gamma)$ are unbounded a.e.. $\square$

4.4. Total Ergodicity

Definition 4.4.1. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretic dynamical system. T is said to be *totally ergodic* if T^m is ergodic for all $m \in \mathbb{N}$.

Thus T is totally ergodic if and only if T is ergodic and has no roots of unity (other than 1) as eigenvalues.

Lemma 4.4.2. *Let (X, T) be the Bratteli-Vershik system determined by the Euler graph. Let $\gamma \in X$ be a path such that there is an $n \in \mathbb{N}$ for which $n - k_n(\gamma) + 1 > 1$, $k_{n+1}(\gamma) = k_n(\gamma) + 1$, and γ_n is not the largest edge (with respect to the edge ordering) connecting $(n, k_n(\gamma))$ and $(n+1, k_n(\gamma)+1)$. Then $d(T^{A(n,k_n(\gamma))}\gamma, \gamma) = 2^{-n}$.*

Proof. For the remainder of the proof let $k = k_n(\gamma)$. Recall that $Y_n(k,0)$ and $Y_n(k, A(n,k) - 1)$ are respectively the minimal and maximal cylinders into vertex (n,k). There are $l, m \in \mathbb{N}$ such that $T^l\gamma \in Y_n(k, A(n,k)-1)$, $\gamma \in T^m(Y_n(k,0))$, and $l+m+1 = A(n,k)$.

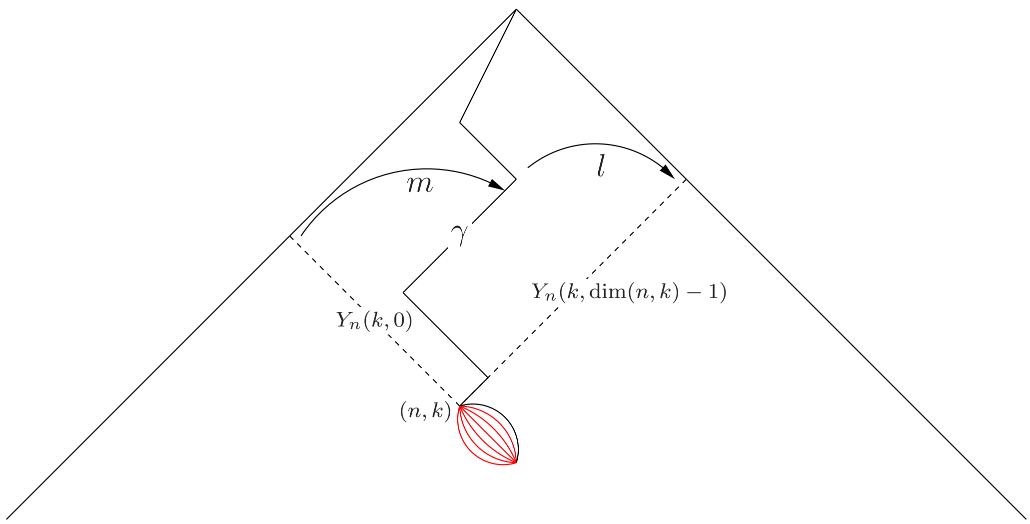


FIGURE 4.6. γ_n is one of the red paths, and $m+l+1 = \dim(n, k_n(\gamma))$.

From the description of the transformation given in Section 2.1, $T(T^l\gamma) \in Y_n(k,0)$ and $(T^{l+1}\gamma)_n$ is the successor of γ_n with respect to the edge ordering. Then $T^m(T^{l+1}\gamma) = T^{A(n,k)}\gamma \in T^m(Y_n(k),0)$. Hence

$$(T^{A(n,k)}\gamma)_0 = \gamma_0,\ (T^{A(n,k)}\gamma)_1 = \gamma_1, \ldots, (T^{A(n,k)}\gamma)_{n-1} = \gamma_{n-1}$$

and $(T^{A(n,k)}\gamma)_n \neq \gamma_n$; therefore $d(T^{A(n,k)}\gamma, \gamma) = 2^{-n}$. $\square$

Key elements of the proof of the following Lemma are already in [**9**], and a similar result has been known for a long time for substitution and related systems, see [**9, 24, 41, 43, 39, 34, 33, 6**].

LEMMA 4.4.3. *Let (X, T) be the Bratteli-Vershik system determined by the Euler graph with the symmetric measure η. If λ is an eigenvalue for T, then $\lambda^{A(n,k_n(\gamma))} \to 1$ η-almost everywhere.*

Proof. Let f be an eigenfunction for T with corresponding eigenvalue λ. Let $\varepsilon_j = 2^{-j}$. By Lusin's Theorem, there exists a compact set $E_j \subset X$ such that $\mu(E_j) > 1 - \varepsilon_j$ and $f|_{E_j}$ is continuous.

Since E_j is compact, so is $f(E_j)$. For every point $z \in f(E_j)$ let U_z be the ball of radius $\frac{1}{2}\varepsilon_j$ around z. These balls cover $f(E_j)$, hence there is a finite subcover $U=\{U_1, U_2, \ldots, U_r\}$ of $f(E_j)$. Then $f^{-1}(U_1), \ldots, f^{-1}(U_r)$ cover E_j.

For each $i \in \{1, 2, \ldots, r\}$ and each $\gamma \in f^{-1}(U_i) \cap E_j$, there is a cylinder set $C_\gamma \subset f^{-1}(U_i)$ which begins at the root vertex and contains γ. Letting i range over $1, \ldots, r$, these C_γ form a cover of E_j, hence there is a finite cover of E_j by cylinders $C_1, \ldots, C_l$. For each cylinder $C_m \in \{C_1, \ldots, C_l\}$, there is a k for which $C_m \subset f^{-1}(U_k)$; therefore for any two paths $\gamma, \xi \in C_m$, $|f(\gamma), f(\xi)| < \varepsilon_j$.

There is a cylinder in $\{C_1, \ldots, C_l\}$ of maximal length, N_j. Let F be a cylinder of length $n \geq N_j$ for which $F \cap E_j \neq \emptyset$. Then there is a cylinder $C_i \in \{C_1, \ldots, C_l\}$ for which $F \subset C_i$. Then for any two paths $\gamma, \xi \in F$, $|f(\gamma), f(\xi)| < \varepsilon_j$.

Let $E = \cap_{i=1}^{\infty} \cup_{j=i}^{\infty} E_j$. Then $\eta(E) = 1$, and for η-almost every $\gamma \in X$ there exist infinitely many E_j for which $\gamma \in E_j$.

Let $\varepsilon > 0$ be given. Let $\gamma \in E$. Choose $j > 0$ so that $\varepsilon_j < \varepsilon$ and $\gamma \in E_j$. Choose $n \geq N_j$ and define F to be the cylinder $[\gamma_0\gamma_1 \ldots \gamma_{n-1}]$. Let e_n be the minimal edge connecting vertices $(n, k_n(\gamma))$ and $(n, k_n(\gamma) + 1)$. Let F^{e_n} be the cylinder F extended by the edge e_n. Since $\eta(F^{e_n}) > 0$, there is a path $\xi \in F^{e_n}$ such that for $i = 0, 1, \ldots, A(n, k_n(\gamma))$,

$f(T^i\xi) = \lambda^i f(\xi)$. By Lemma 4.4.2, $d(T^{A(n,k_n(\gamma))}\xi, \xi) = 2^{-n}$. Also, $T^{A(n,k_n(\gamma))}\xi \in F$. Then $|f(T^{A(n,k_n(\gamma))}\xi), f(\xi)| < \frac{1}{2}\varepsilon$, hence $|\lambda^{A(n,k_n(\gamma))} f(\xi), f(\xi)| < \frac{1}{2}\varepsilon_j < \varepsilon$. Therefore for η-almost every $\gamma \in X$, $\lambda^{A(n,k_n(\gamma))} \to 1$.

$\square$

There is a closed-form formula for $A(n,k)$ which can be found in [**42**]:

$$A(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+2}{j} (k+1-j)^{n+1}.$$

THEOREM 4.4.4 (E. Lucas [**31**]). *Let p be a prime number and $j \leq n$. Consider the base p decompositions of n and j:*

$$n = n_0 + n_1 p + \cdots + n_s p^s$$

$$j = j_0 + j_1 p + \cdots + j_s p^s$$

where $0 \leq j_i, n_i < p$ for all i. Then

$$\binom{n}{j} \equiv_p \binom{n_0}{j_0} \cdots \binom{n_s}{j_s},$$

with the convention that $\binom{n_i}{j_i} = 0$ if $j_i > n_i$.

THEOREM 4.4.5. *The Bratteli-Vershik system determined by the Euler graph with the symmetric measure η is totally ergodic.*

Proof. It is enough to show that for any prime p, $e^{2\pi i/p}$ is not an eigenvalue for T.

If $\lambda = e^{2\pi i/p}$ is an eigenvalue of T, by Lemma 4.4.3 we know that $\lambda^{A(n,k_n(\gamma))} \to 1$ for η-almost every $\gamma \in X$. Since λ is a root of unity, for η-almost every γ in X, there must be an N such that $n \geq N$ implies $\lambda^{A(n,k_n(\gamma))} = 1$. Therefore for η-almost every $\gamma \in X$ there is an N such that $n \geq N$ implies

$$A(n, k_n(\gamma)) = 0 \mod p.$$

We will show that for every $\gamma \in X$, there are infinitely many n for which $A(n, k_n(\gamma)) \equiv_p 1$. In particular, for every $l = 0, 1, \ldots,$ and $0 \leq k \leq p^l - 1$, $A(p^l - 1, k) \equiv_p 1$. Recall that for $k \geq 1$,

$$A(p^l - 1, k) = \sum_{j=0}^{k} (-1)^j \binom{p^l + 1}{j} (k + 1 - j)^{p^l}.$$

We will examine this sum by computing each term in the sum mod p.

For $j = 0$ we have

$$\binom{p^l + 1}{0} (k+1)^{p^l} = (k+1)^{p^l} \equiv_p k + 1 \text{ by Fermat's Little Theorem.} \tag{4.4.1}$$

For $j = 1$, we have

$$(-1)\binom{p^l + 1}{1} k^{p^l} = -(p^l + 1)k^{p^l} \equiv_p -k. \tag{4.4.2}$$

For $2 \leq j \leq p^l - 1$ we have

$$(-1)^j \binom{p^l + 1}{j} (k - j)^{p^l}.$$

By Theorem 4.4.4,

$$(-1)^j \binom{p^l + 1}{j} (k - j)^{p^l} \equiv_p (-1)^j \binom{1}{j_0}\binom{0}{j_1} \cdots \binom{0}{j_{l-1}}\binom{1}{0} (k - j)^{p^l}.$$

Since $2 \leq j \leq p^l - 1$, at least one of $j_1, j_2, \ldots, j_{l-1}$ must be positive. Therefore

$$(-1)^j \binom{p^l + 1}{j} (k - j)^{p^l} \equiv_p 0. \tag{4.4.3}$$

For fixed p and $0 \leq k \leq p^l - 1$, we will now compute $A(p^l - 1, k)$.

$$A(p^l - 1, 0) = (-1)^0 \binom{p^l + 1}{0} (1)^{p^l} = 1.$$

Combining Equations 4.4.1, 4.4.2, and 4.4.3 we see that for $k > 0$,

$$A(p^l - 1, k) = \sum_{j=0}^{k} (-1)^j \binom{p^l + 1}{j} (k + 1 - j)^{p^l} \equiv_p (k + 1) - k \equiv_p 1.$$

Hence $\lambda^{A(n,k_n(\gamma))}$ does not converge to 1, and therefore T has no roots of unity as eigenvalues. $\square$

4.5. Loosely Bernoulli

Recall the following result from Section 3.4

Lemma 4.5.1 (Janvresse, de la Rue [**25**]). *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system with entropy 0. Suppose that for every $\varepsilon > 0$ and for $\mu \times \mu$-almost every $(\gamma, \xi) \in X \times X$ we can find an integer $l(\gamma, \xi) \geq 1$ such that*

$$\overline{f}\left(\mathcal{P}_0^{l(\gamma,\xi)}(\gamma), \mathcal{P}_0^{l(\gamma,\xi)}(\xi)\right) < \varepsilon.$$

Then the process $(\mathcal{P}, T)$ is LB.

Theorem 4.5.2. *Let (X, T) be the Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ determined by the Euler graph and with the symmetric measure η. Then T is loosely Bernoulli.*

Proof. Let $\mathcal{P}$ be the partition according to the first edge, described in Section 2.3 for Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$ for which $d = 1$. By Corollary 2.3.8 this is a generating partition; therefore it is sufficient to show that the process $(\mathcal{P}, T)$ is LB.

Proposition 4.2.2 tells us that for $\eta\times\eta$-almost every $(\gamma,\xi)\in X\times X$, $k_n(\gamma)=k_n(\xi)=k$ infinitely many times. For such an n, with conditional probability

$$\left(\frac{n-k+1}{2n+2}\right)^2,$$

$k_{n+1}(\gamma)=k_n(\gamma)+1=k_{n+1}(\xi)$ and both γ_n and ξ_n are one of the first $(n-k+1)/2$ edges into $(n+1,k+1)$. By Lemma 4.2.7, $k_n(\gamma)/n\to 1/2$ η-almost everywhere. Therefore, for η-almost every $\gamma\in X$,

$$\frac{n-k_n(\gamma)+1}{2n+2}\to\frac{1}{4}.$$

Then for η-almost every $\gamma\in X$ we can take n large enough so that

$$\left(\frac{n-k_n(\gamma)+1}{2n+2}\right)^2>\frac{1}{64}.$$

Hence the set of (γ,ξ) for which there are infinitely many n such that $k_n(\gamma)=k_n(\xi)=k$, $k_{n+1}(\gamma)=k_{n+1}(\xi)=k+1$, and each of γ_n and ξ_n are one of the first $(n-k+1)/2$ edges connecting (n,k) and $(n+1,k+1)$ has full measure.

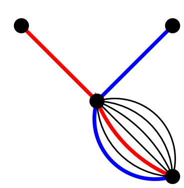


FIGURE 4.7. γ and ξ meet at vertex (n,k) and continue along one of the first $(n-k+1)/2$ edges to vertex $(n+1,k+1)$.

Then the $\mathcal{P}$-names of both γ and ξ have long central block $B(n+1,k+1)=B(n,k)^{n-k+1}B(n,k+1)^{k+2}$. Both γ and ξ have their decimal point in this first block of $B(n,k)$. For some subblocks $w_0w_1\dots w_{j_1}$ and $w_0'w_1'\dots w_{j_2}'$ of $B(n,k)$ and m_1,m_2 with

$(n-k+1)/2 \le m_1, m_2 \le n-k+1$,

$$\mathcal{P}_0^\infty \gamma = \ldots_\bullet w_0 w_1 \ldots w_{j_1} (B(n,k))^{m_1} (B(n,k+1))^{k+2} \ldots$$

$$\mathcal{P}_0^\infty \xi = \ldots_\bullet w_0' w_1' \ldots w_{j_2} (B(n,k))^{m_2} (B(n,k+1))^{k+2} \ldots$$

Then $\mathcal{P}_0^\infty \gamma$ and $\mathcal{P}_0^\infty \xi$ agree on $\min\{m_1, m_2\}$ consecutive blocks $B(n,k)$. Let $l(\gamma, \xi) = \min\{m_1, m_2\} A(n,k) + \max\{j_1, j_2\}$. Then

(4.5.1)
$$\overline{f}(\mathcal{P}_0^{l(\gamma,\xi)}(\gamma), \mathcal{P}_0^{l(\gamma,\xi)}(\xi)) \le \frac{\max\{j_1, j_2\}}{\min\{m_1, m_2\}|B(n,k)|} = \frac{A(n,k)}{1/2(n-k+1)A(n,k)} = \frac{2}{n-k+1}.$$

Lemma 4.2.7 says that $k_n(\gamma)/n \equiv d_n(\xi)/n \to 1/2$ as $n \to \infty$. Thus given $\varepsilon > 0$, we can let n be large enough so that

$$\frac{2}{n-k+1} < \varepsilon.$$

Then T is LB. $\square$

CHAPTER 5

Partial Results and Open Questions

5.1. Orbit Equivalence and the Dimension Group

DEFINITION 5.1.1. Two topological dynamical systems (X_1, ϕ_1) and (X_2, ϕ_2) are *(topologically) orbit equivalent* if there exists a homeomorphism $h : X_1 \to X_2$ so that $h(\mathcal{O}_{\phi_1}(x)) = \mathcal{O}_{\phi_2}(h(x))$ for all $x \in X_1$. The homeomorphism h is called an *orbit map*.

DEFINITION 5.1.2. Let (X_1, ϕ_1) and (X_2, ϕ_2) be two orbit-equivalent dynamical systems and let h be an orbit map. Then for each $x \in X$ there is an integer $n(x)$ so that $h \circ \phi_1(x) = \phi_2^{n(x)} \circ h(x)$ and an integer $m(x)$ so that $h \circ \phi_1^{m(x)}(x) = \phi_2 \circ h(x)$. We call m and n the *orbit cocyles* associated to the orbit map h.

DEFINITION 5.1.3. Let (X_1, ϕ_1) and (X_2, ϕ_2) be two orbit-equivalent dynamical systems. Then (X_1, ϕ_1) and (X_2, ϕ_2) are said to be *strongly (topologically) orbit equivalent* if there is an orbit map $h : X_1 \to X_2$ so that the associated orbit cocyles each have at most one point of discontinuity.

THEOREM 5.1.4 ([**20, 21**]). *If (X, T) and (Y, S) are Cantor minimal systems, the following are equivalent:*

1. There exist ordered, simple ordered Bratteli diagrams $(\mathcal{V}, \mathcal{E}, \geq)$ and $(\mathcal{V}', \mathcal{E}', \geq)$ such that the associated Bratteli-Vershik systems (X', T') and (Y', T') are topologically conjugate to (X, T) and (Y, S) respectively and an unordered simple Bratteli diagram B of which *both $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{V}', \mathcal{E}')$ are contractions.*

2. (X, T) and (Y, S) are strongly orbit equivalent.

3. $K_0(X', T') \cong K_0(Y', S')$ as ordered dimension groups with order units.

At present it is unclear how Theorem 5.1.4 might extend to the Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$. Because we are dealing with transformations that are not continuous on the set of maximal paths, the definition of strong orbit equivalence will need to be adjusted.

Definition 5.1.5. Let (X_1, T_1) and (X_2, T_2) be two orbit-equivalent Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$. Then (X_1, T_1) and (X_2, T_2) are said to be $\mathcal{S}_{\mathcal{L}}$*-strongly orbit equivalent* if there is an orbit map $h : X_1 \to X_2$ so that the associated orbit cocylces are continuous on $X_1 \setminus (X_1)_{\max}$ and $X_2 \setminus (X_2)_{\max}$ respectively.

Other variations of this definition may include the ability to throw out any first category set or dropping the requirement that the orbit map be a homeomorphism from all of X_1 to all of X_2.

Conjecture 1. *If (X, T) and (Y, S) are Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$, then the following are equivalent:*

1. (X, T) and (Y, T) are $\mathcal{S}_{\mathcal{L}}$-strong orbit equivalent.

2. The dimension groups of the Bratteli diagrams associated to (X, T) and (Y, S) are order isomorphic by a map sending distinguished order unit to distinguished order unit.

We have some preliminary results showing that dimension groups are preserved in some explicit examples. On the other hand, it is possible to construct a system $(X, T) \in$

$\mathcal{S}_{\mathcal{L}}$ and a map $S : X \to X$ such that

1. S is continuous except on $X_{\max}$,
2. S is orbit equivalent to T using the identity map, but
3. $\partial_S C(X, \mathbb{Z}) \neq \partial_T C(X, \mathbb{Z})$.

PROPOSITION 5.1.6. *Let (X, T) be a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ with Bratteli diagram $(\mathcal{V}, \mathcal{E})$, and let (X', T') be a Bratteli-Vershik system obtained from a reordering of the edges in $(\mathcal{V}, \mathcal{E})$ such that $X'_{\max} = X_{\max}$, $X'_{\min} = X_{\min}$, and $T' = T$ on $X_{\max}$. Then (X, T) and (X', T') are $\mathcal{S}_{\mathcal{L}}$-strongly orbit equivalent and*

$$C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})) \text{ and } C(X', \mathbb{Z})/(\partial_{T'} C(X', \mathbb{Z}) \cap C(X', \mathbb{Z}))$$

are order isomorphic with corresponding distinguished order units.

Proof. (X, T) and (X', T') are clearly $\mathcal{S}_{\mathcal{L}}$-strongly orbit equivalent by letting the orbit map be the identity map since both S and T are both constant on nonmaximal cylinder sets. Since (X, T) and (X', T') share the same underlying diagram $(\mathcal{V}, \mathcal{E})$, they have the same dimension group, $K_0(\mathcal{V}, \mathcal{E})$. If $X'_{\max} = X_{\max}$, $X'_{\min} = X_{\min}$, and $T' = T$ on $X_{\max}$, the proof of Theorem 2.2.7 (which shows that for $(X, T) \in \mathcal{S}_{\mathcal{L}}$ with associated diagram $(\mathcal{V}, \mathcal{E})$, $K_0(\mathcal{V}, \mathcal{E}) \cong C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}))$) holds for (X', T') as well. Hence

$$C(X', \mathbb{Z})/(\partial_{T'} C(X', \mathbb{Z}) \cap C(X', \mathbb{Z})) \cong K_0(\mathcal{V}, \mathcal{E}) \cong C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})).$$

$\square$

While it is clear that all reorderings of the edges give $\mathcal{S}_{\mathcal{L}}$-strong orbit equivalences by using the identity as the orbit map, some of these reorderings might produce topologically conjugate systems. This may be true if the edge reordering is done in such a way that it scrambles only the ordering of edges that have both the same range and source.

CONJECTURE 2. *Let* (X, T) *be a Bratteli-Vershik system in* $\mathcal{S}_{\mathcal{L}}$ *with Bratteli diagram* $(\mathcal{V}, \mathcal{E})$*, and let* (X', T') *be a Bratteli-Vershik system obtained by reodering the edges in* $(\mathcal{V}, \mathcal{E})$ *in such a way that* $X'_{\max} = X_{\max}$*,* $X'_{\min} = X_{\min}$ *and if two edges* e_1 *and* e_2 *have terminal vertex* (n, k) *and source vertices* $(n-1, k-i_1)$ *and* $(n-1, k-i_2)$ *respectively, then* $i_1 < i_2$ *implies that* $e_2 < e_1$*. Then* (X, T) *and* (X', T') *are topologically conjugate.*

PROPOSITION 5.1.7. *Let* $(X_{p(x)}, T_{p(x)})$ *and* $(X_{q(x)}, T_{q(x)})$ *be Bratteli-Vershik systems determined by positive integer polynomials* $p(x) = a_0 + a_1x + \cdots + a_dx^d$ *and* $q(x) = a_d + a_{d-1}x + \ldots a_0x^d$ *respectively. Then there exists a homeomorphism* $F : X_{p(x)} \to X_{q(x)}$ *such that* $F \circ T_{p(x)} = T_{q(x)}^{-1} \circ F$ *and an order isomorphism* $\sigma : K_0(\mathcal{V}, \mathcal{E})_{p(x)} \to K_0(\mathcal{V}, \mathcal{E})_{q(x)}$ *that sends the distinguished order unit of* $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ *to the distinguished order unit of* $K_0(\mathcal{V}, \mathcal{E})_{q(x)}$*.*

Proof. The diagrams $(\mathcal{V}, \mathcal{E})_{p(x)}$ and $(\mathcal{V}, \mathcal{E})_q(x)$ are mirror images of each other (recall Definition 2.8). The flip conjugacy F is defined by sending $\gamma \in X_{p(x)}$ to $\overline{\gamma} \in (\mathcal{V}, \mathcal{E})_q(x)$. Then clearly $F \circ T_{p(x)} = T_{q(x)}^{-1} \circ F$.

Let $r(x)/(p(x))^l \in K^0(\mathcal{V}, \mathcal{E})_{p(x)}$ with $r(x) = r_0 + r_1x + \cdots + r_{dl}x^{dl}$, where $r_i \in \mathbb{Z}$ for each $i = 0, 1, \ldots, dl$. Then the order isomorphism σ is defined by

$$\sigma\left(\frac{r(x)}{(p(x))^l}\right) = \frac{r_{dl} + r_{dl-1}x + \cdots + r_0x^{dl}}{(q(x))^l}.$$

The map σ is clearly well defined, sends the positive set of $K^0(\mathcal{V}, \mathcal{E})_{p(x)}$ to the positive set of $K^0(\mathcal{V}, \mathcal{E})_{q(x)}$, and sends the constant function 1 to the constant function 1. $\square$

PROPOSITION 5.1.8. *Let* $(X_{p(x)}, T_{p(x)})$ *and* $(X_{q(x)}, T_{q(x)})$ *be the Bratteli-Vershik systems in* $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ *determined by positive integer polynomials* $p(x)$ *and* $q(x)$ *such that there exists* $k \geq 1$ *such that* $q(x) = (p(x))^k$. *Then there exists a homeomorphism* $F : X_{p(x)} \to X_{q(x)}$ *such that* $F \circ T_{p(x)} = T_{q(x)} \circ F$ *and an order isomorphism* $\sigma : K_0(\mathcal{V}, \mathcal{E})_{p(x)} \to K_0(\mathcal{V}, \mathcal{E})_{q(x)}$ *that sends the distinguished order unit of* $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ *to the distinguished order unit of* $K_0(\mathcal{V}, \mathcal{E})_{q(x)}$.

Proof. $(X_{q(x)}, T_{q(x)})$ is a telescoping of $(X_{p(x)}, T_{p(x)})$ to every k'th level, and telescoping preserves the system and dimension group. □

The two previous propositions have given examples in which the dimension groups are order isomorphic. The following proposition shows explicitly that not all dimension groups associated to systems in $\mathcal{S}_{\mathcal{L}}$ are order isomorphic.

PROPOSITION 5.1.9. *Let* $p(x) = a_0 + a_1 x$ *with* a_0, a_1 *integers larger than 1. Then* $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ *is not order isomorphic to* $K_0(\mathcal{V}, \mathcal{E})_{x+1}$ *by any map that preserves the distinguished order units.*

Proof. We will attempt to construct such a map $\phi : K_0(\mathcal{V}, \mathcal{E})_{p(x)} \to K_0(\mathcal{V}, \mathcal{E})_{x+1}$ and arrive at a contradiction. The constant function 1 is the distinguished order unit of both $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ and $K_0(\mathcal{V}, \mathcal{E})_{x+1}$, so $\phi(1) = 1$. We will attempt to define ϕ on some generators of $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$. Suppose that

$$\phi\left(\frac{1}{a_0 + a_1 x}\right) = \frac{r(x)}{(x+1)^m}.$$

Since $1/(a_0+a_1x) \in K_0(\mathcal{V},\mathcal{E})^+_{p(x)}$, $r(x)/(x+1)^m \in K_0(\mathcal{V},\mathcal{E})^+_{x+1}$. Thus $r(x)$ has a positive leading coefficient. Since ϕ is a group homomorphism,

$$\phi\left(\frac{a_0}{a_0+a_1x}\right) = \frac{a_0 r(x)}{(x+1)^m}.$$

We also have

$$\phi\left(\frac{a_1x}{a_0+a_1x}\right) = \phi\left(1-\frac{a_0}{a_0+a_1x}\right) = \frac{(x+1)^m - a_0r(x)}{(x+1)^m}. \tag{5.1.1}$$

Since $(a_1x)/(a_0+a_1x) \in K_0(\mathcal{V},\mathcal{E})^+_{p(x)}$,

$$\frac{(x+1)^m - a_0r(x)}{(x+1)^m} \in K_0(\mathcal{V},\mathcal{E})^+_{x+1}.$$

Since $a_0 > 1$, $\deg(r(x)) < m$. Hence the leading coefficient of $(x+1)^m - a_0r(x)$ is 1. Since ϕ is a group homomorphism, and by Equation 5.1.1,

$$\phi\left(\frac{a_1x}{a_0+a_1x}\right) = \underbrace{\phi\left(\frac{x}{a_0+a_1x}\right) + \cdots + \phi\left(\frac{x}{a_0+a_1x}\right)}_{a_1 \text{ times}} = \frac{(x+1)^m - a_0r(x)}{(x+1)^m}$$

which implies there is an $s(x)/(x+1)^l \in K_0(\mathcal{V},\mathcal{E})_{x+1}$ such that

$$\phi\left(\frac{x}{a_0+a_1x}\right) = \frac{s(x)}{(x+1)^l} \text{ and so } \frac{a_1s(x)}{(x+1)^l} = \frac{(x+1)^m - a_0r(x)}{(x+1)^m}.$$

Since the leading coefficient of $a_1s(x) > 1$, this is not possible. Hence ϕ is not an order isomorphism. $\square$

5.2. Order Ideals

Definition 5.2.1. Let (G, G^+) be a dimension group. An *order ideal* is a subgroup J so that $J = J^+ - J^+$, where $J^+ = J \cap G^+$, and $0 \le a \le b \in J$ implies $a \in J$. We say that (G, G^+) is a *simple* dimension group if it contains no nontrivial ($J \neq \{0\}$ or G) order ideal.

The dimension groups computed in Proposition 3.3.1 for Bratteli-Vershik systems determined by positive integer polynomials are not simple. For instance, for the Bratteli-Vershik system determined by the d'th degree polynomial $p(x)$,

$$J = \left\{ \frac{r(x)}{(p(x))^l} \in K_0(\mathcal{V}, \mathcal{E})_{p(x)} | \deg(r(x)) \leq dl - 1 \right\}$$

is an order ideal. A question is to determine all order ideals for $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$.

5.3. Infinitesimals

Definition 5.3.1. Let G be a dimension group, with fixed order unit $u \in G^+ \setminus \{0\}$. A homomorphism $p : G \to \mathbb{R}$ is a *state* if $p(G^+) \geq 0$ and $p(u)$=1. Denote the collection of all states by $S_u(G)$.

Definition 5.3.2. We say that $a \in G \setminus \{0\}$ is *infinitesimal* if $-\varepsilon u \leq a \leq \varepsilon u$ for all $0 < \varepsilon \in \mathbb{Q}$, i.e. $-p\varepsilon \leq qa \leq p\varepsilon$ for all $p, q \in \mathbb{N}$.

Equivalently, $a \in G \setminus \{0\}$ is infinitesimal if $p(a) = 0$ for all $p \in S_u(G)$. The set of infinitesimals forms a subgroup of (G, G^+) called the *infinitesimal subgroup of* G, denoted by $\mathrm{Inf}(G)$.

For a Cantor minimal system, there is a bijective correspondence between the states of a dimension group and the invariant probability measures. This correspondence allows us to characterize the infinitesimal subgroup in terms of continuous functions from the path space into $\mathbb{Z}$, and their integrals over the path space. We conjecture that this formulation of the infinitesimal subgroup also holds for the Bratteli-Vershik systems determined by positive integer coefficients.

Conjecture 3. *Let* $(X_{p(x)}, T_{p(x)})$ *be the Bratteli-Vershik system in* $\mathcal{S}_{\mathcal{L}}$ *determined by the positive integer polynomial* $p(x)$. *Then*

(i) Every T*-invariant probability measure* μ *on* $X_{p(x)}$ *induces a state* $S(\mu)$ *on* $C(X_{p(x)}, \mathbb{Z})/(\partial_T C(X_{p(x)}, \mathbb{Z}) \cap C(X_{p(x)}, \mathbb{Z})$ *by sending each* $f \in C(X_{p(x)}, \mathbb{Z})$ *to*

$$\int_X f \, d\mu.$$

(ii) Let

$$Z_T = \left\{ f \in C(X_{p(x)}, \mathbb{Z}) | \int_X f \, d\mu = 0 \text{ for all } T\text{-invariant probability measures } \mu \right\}.$$

Then

$$\mathit{Inf}(K_0(\mathcal{V}, \mathcal{E})_{p(x)}) = Z_T/(\partial_T C(X_{p(x)}, \mathbb{Z}) \cap C(X_{p(x)}, \mathbb{Z})),$$

and $K_0(\mathcal{V}, \mathcal{E})_{p(x)}/\mathit{Inf}(K_0(\mathcal{V}, \mathcal{E})_{p(x)})$ *is naturally isomorphic to* $C(X_{p(x)}, \mathbb{Z})/Z_T$ *by an order isomorphism preserving the distinguished order units.*

For Cantor minimal systems there is a connection between the infinitesimal subgroup and regular orbit equivalence.

Theorem 5.3.3. [**20**]. *Let* (X_1, ϕ_1) *and* (X_2, ϕ_2) *be two minimal Cantor systems. Then* (X_1, ϕ_1) *and* (X_2, ϕ_2) *are orbit equivalent if and only if the dimension groups*

$$K^0(X_1, \phi_1)/\mathit{Inf}(K^0(X_1, \phi_1)) \text{ and } K^0(X_1, \phi_2)/\mathit{Inf}(K^0(X_1, \phi_2))$$

are order isomorphic by a map preserving the distinguished order units.

We can show that for the special class of systems determined by positive integer polynomials, there are no infinitesimals. This may have implications for the kinds of orbit equivalence that are possible between pairs of such systems. Maybe for these systems orbit equivalence implies strong orbit equivalence.

PROPOSITION 5.3.4. *Let* $(X_{p(x)}, T_{p(x)})$ *be the Bratteli-Vershik system determined by the positive integer polynomial* $p(x)$*. Then the dimension group* $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ *has no infinitesimals.*

Proof. Let $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ be the dimension group computed in Proposition 3.3.1. Recall that

$$K_0(\mathcal{V}, \mathcal{E})^+_{p(x)} = \left\{ \frac{r(x)}{p(x)^l} | \text{ there is a } k \geq 0 \text{ for which } r(x)p(x)^k \text{ has all positive coefficients} \right\}.$$

If $s(x)/p(x)^l \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)}$, then $s(x)$ can have no positive real roots. For if it did, there would be an $x_0 \in \mathbb{R}^+$ and $k \geq 0$ for which $s(x_0)p(x_0)^k = 0$ and $s(x)p(x)^l$ has all positive coefficients, which is impossible. Hence for all $x \in \mathbb{R}^+$, $s(x) > 0$.

Assume that $r(x)/p(x)^l \in K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ is an infinitesimal. For every rational $\varepsilon > 0$, $-\varepsilon p(x)^l/p(x)^l \leq r(x)/p(x)^l \leq \varepsilon p(x)^l/p(x)^l$. Since $r(x)/p(x)^l$ is nonzero there is an $x_0 > 0$ for which $r(x_0) \neq 0$. We will assume for now that $r(x_0) = c > 0$. Let $\varepsilon = c/(2p(x_0)^l)$. Then $r(x) \leq \varepsilon p(x)^l$ implies $2p(x_0)^l r(x) < cp(x)^l$. Let $q(x) = cp(x)^l - 2p(x_0)^l r(x)$, then

$$\frac{q(x)}{p(x)^l} = \frac{cp(x)^l - 2p(x_0)^l r(x)}{p(x)^l} \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)}.$$

But $q(x_0) = cp(x_0)^l - 2cp(x_0)^l = -cp(x_0)^l$, which contradicts $q(x)/p(x)^l \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)}$. Hence, $r(x)/p(x)^l$ is not an infinitesimal.

If $r(x_0) = c < 0$, let $\varepsilon = -c/(2p(x_0)^l)$. Then $-\varepsilon p(x)^l < r(x)$ implies $cp(x)^l < 2p(x_0)^l r(x)$. In other words,

$$\frac{q(x)}{p(x)^l} = \frac{2p(x_0)^l r(x) - cp(x)^l}{p(x)^l} \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)}.$$

But $q(x_0) = 2cp(x_0)^l - cp(x_0)^l = cp(x_0)^l < 0$, which contradicts, $q(x)/p(x)^l \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)}$. Hence, $r(x)/p(x)^l$ is not an infinitesimal. $\square$

In the proof of Proposition 5.3.4 we showed that if

$$\frac{r(x)}{p(x)^k} \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)},$$

then $r(x)$ must have no real positive roots. If the converse also holds, we would have an especially effective characterization of the order relation in these dimension groups.

Conjecture 4. *Let $r(x)/(p(x))^l \in K_0(\mathcal{V}, \mathcal{E})_{p(x)}$. If $r(x)$ has no positive real roots, then $r(x)/(p(x))^l \in K_0(\mathcal{V}, \mathcal{E})^+_{p(x)}$.*

5.4. Coding

The class $\mathcal{S}_{\mathcal{L}}$ contains many different Bratteli-Vershik systems. While Theorem 2.2.7, which describes the relationship between the dimension group and the continuous functions from the path space into the integers modulo the coboundaries, applies to each Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$, the remaining results in this thesis are more specific. It would be interesting to determine other properties that hold for all Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$. For example, Theorem 2.3.7 describing a measure-theoretic isomorphism that codes a system as a subshift is specific to Bratteli-Vershik systems in $\mathcal{S}_{\mathcal{L}}$ with $d = 1$. For each Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ with a fully-supported T-invariant, ergodic probability measure, let $\mathcal{P}$ be the partition of X according to the first edge. For each vertex (n, k), let $\gamma \in Y_n(k, 0)$ and define $B(n, k) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma)\mathcal{P}(T^2\gamma) \ldots \mathcal{P}(T^{\dim(n,k)-1}\gamma)$. Let $\Sigma(X, T)$ be the subshift such that for each $x \in \Sigma(X, T)$, each finite subblock of x is contained in some $B(n, k)$.

Conjecture 5. *Let $(X,T) \in \mathcal{S}_{\mathcal{L}}$, and let μ be a fully-supported T-invariant ergodic probability measure on X. Then there are a set $X' \subset X$ with $\mu(X') = 0$ and a one-to-one Borel measurable map $\phi : X \setminus X' \to \Sigma(X,T)$ such that $\phi \circ T = \sigma \circ \phi$ on $X \setminus X'$.*

A topological dynamical system (X,T) is called *topologically weakly mixing* if and only if for every two non-empty open sets $U, V \subset X$, the set $\{n \in \mathbb{Z} : T^n U \cap V \neq \emptyset\}$ contains arbitrarily long intervals. In [**32**] Méla showed that the subshift $(\Sigma(X_{x+1}, T_{x+1}), \sigma)$ is topologically weak mixing. Whether this is also true for other $(\Sigma(X_{p(x)}, T_{p(x)}), \sigma)$ is open.

A similar question involves determining if any of these subshifts are topologically strongly mixing. A topological dynamical system (X,T) is *topologically strongly mixing* if given two nonempty open subsets $U, V \subset X$ there is an N such that $n \geq N$ implies $T^n U \cap V \neq \emptyset$.

In [**32**], Méla investigated the complexity of the $(\Sigma(X_{x+1}, T_{x+1}), \sigma)$. It would be of interest to compare this with the complexity of $(\Sigma(X_{p(x)}, T_{p(x)}), \sigma)$ for another positive integer polynomial $p(x)$.

5.5. Eigenvalues and Weak Mixing

In Section 3.5 we found eigenvalues for Bratteli-Vershik systems determined by degree 1 positive integer polynomials depending on the divisibility properties of the coefficients of powers of the given polynomial. A remaining question is to investigate the divisibility of the coefficents of positive integer polynomials of higher degree in order to determine whether the adic transformations have any roots of unity as eigenvalues. Although we were able to show that for $p(x) = a_0 + a_1 x$ the Bratteli-Vershik system determined by

$p(x)$ is not weakly mixing if either a_0 or a_1 is greater than 1, the long-standing question of weak mixing for the Pascal adic remains open. It was shown in Proposition 4.4.5 that the Euler adic has no roots of unity as eigenvalues. Whether the Euler adic is weak mixing remains open.

On the other hand conceivably some of these systems, when endowed with fully-supported ergodic measures, actually have discrete spectrum and could even be odometers in disguise. For example, consider the Bratteli-Vershik system (X_{2+2x}, T_{2+2x}) determined by the polynomial $p(x) = 2 + 2x$ with a fully-supported, $T_{p(x)}$-invariant, ergodic, probability measure μ. The total number of paths into vertex (n, k) is $2^n\binom{n}{k}$. The cylinders of length n partition the space X. For each cylinder C of length n, there are $0 \leq k \leq n$ and $0 \leq j \leq \dim(n, k) - 1$ for which $C = Y_n(k, j)$ (see the notation in Section 1.2). Define the function f_n on C by

$$f_n(Y_n(k, j)) = (e^{2\pi i/2^n})^j.$$

Then for each $n \geq 0$, f_n is a continuous eigenfunction of $T_{p(x)}$ with corresponding eigenvalue $e^{2\pi i/(2^n)}$.

Conjecture 6. *There exist fully-supported, ergodic, T_{2+2x}-invariant probability measures μ on (X_{2+2x}, T_{2+2x}) for which the Bratteli-Vershik system (X_{2+2x}, T_{2+2x}) has discrete spectrum.*

This conjecture may hold for all of the ergodic, T-invariant, probability measures computed in Theorem 3.2.4 or possibly only for $\mathcal{B}(1/4, 1/4, 1/4, 1/4)$. Using the same construction, we can ask similar questions for each Bratteli-Vershik system $(X_{p(x)}, T_{p(x)})$ determined by the polynomial $p(x) = a + ax$.

It seems even more difficult to determine whether the spectral measures of the systems in $\mathcal{S}_{\mathcal{L}}$ are always singular with respect to Lebesgue measure.

5.6. Other Invariant Measures on the Euler Adic

The symmetric measure for the Euler adic was shown in Theorem 4.3.4 to be the only T-invariant, ergodic probability measure having full support. An open question is to determine the ergodic probability measures without full support. These measures will be supported on the sets

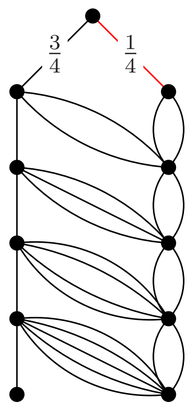
$$A_K = \{\gamma \in X | \text{ either } k_n(x) \leq K \text{ for } n = 0, 1, \ldots, \text{ or } n - k_n(x) < K \text{ for } n = 1, 2, \ldots\}.$$

This is more complicated than the polynomial case (Proposition 3.2.5), as the number of edges leaving a vertex varies. We do know that unlike the measures for systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$, the Euler adic has measures that do not have full support but are also supported on more than the far left or far right sides of the diagram.

Example 5.6.1. There is a measure μ_1 on A_1 that has full support on either the following graph or its mirror image. The measures of the initial cylinders are seen on the graph. Using the same methods as in the proof of Proposition 3.2.5, we can determine the measure of the red cylinder, C. In fact, if μ_1 is ergodic,

$$\mu(C) = \lim_{n\to\infty} \frac{\dim(C, (n, 1))}{\dim(n, 1)} = \lim_{n\to\infty} \frac{2^{n-1}}{A(n, 1)} = \lim_{n\to\infty} \frac{2^{n-1}}{2^{n+1} - (n + 2)} = \frac{1}{4}.$$

Since μ_1 is an invariant probability measure, the rest of the weights are completely determined as follows. If the edge e connects a vertex $(n - 1, 0)$ and $(n, 0)$, then $w(e) = (n+2)/(2n+2)$. If the edge e connects vertices $(n-1, 0)$ and $(n, 1)$, then $w(x) = 1/(2n+2)$. Lastly, if the edge e connects vertices $(n - 1, 1)$ to $(n, 1)$, then $w(e) = 1/2$.



The formula for $A(n,k)$ becomes increasingly more complicated as k increases, as does the above computation. It is an open question to determine all nonfully-supported, ergodic, probability measures for the Euler system.

5.7. Reinforced Random Walks

Bratteli-Vershik systems with invariant measures, given by weights on the edges, naturally correspond to (probably nonstationary) random walks.

For example, *the reversed Euler graph* is the Bratteli diagram associated to a system in $\mathcal{S}_{\mathcal{L}}$ with $d = 1$ such that for each vertex (n,k) the number of edges connecting vertices (n,k) and $(n+1,k)$ is $n-k+1$, while the number of edges connecting vertices (n,k) and $(n+1,k+1)$ is $k+1$. Then the number of paths from the root vertex to (n,k) is $n!$. Define the reverse symmetric measure by giving each cylinder of length n measure $1/(n+2)!$.

We now describe a particular reinforced random walk between two points A and B. Initially (time 0) the probabilities of visiting points A and B are equal. Having visited the points n times (time n), the conditional probability of visiting A in the next step is

$$\frac{1 + \#\text{ of previous visits to } A}{n+2},$$

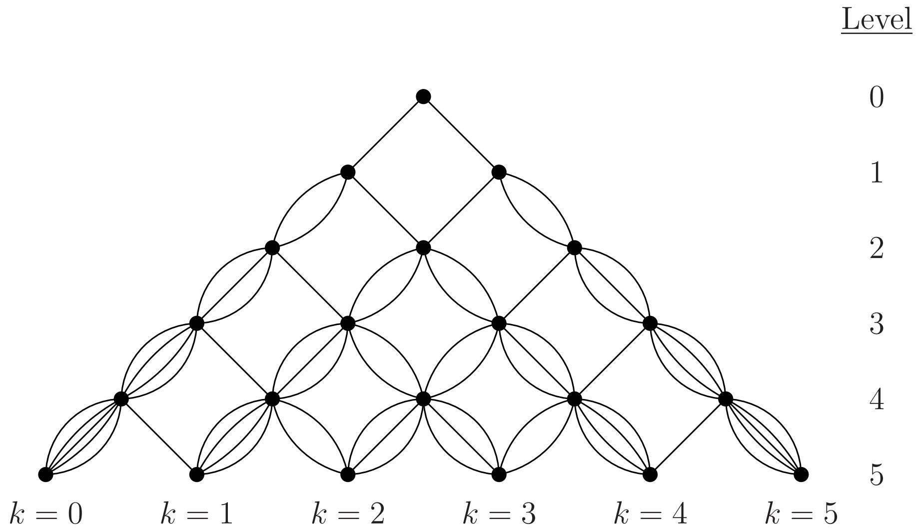


FIGURE 5.1. The Reverse Euler graph

while the conditional probability of visiting B is

$$\frac{1 + \#\text{ of previous visits to } B}{n+2}.$$

The reverse Euler graph endowed with the reverse symmetric measure models this process. Consider a visit to A at time n as a right turn at level n, and a visit to B at time n as a left turn at time n. Each vertex (n, k) in the reverse Euler graph corresponds to paths for which the total number of right turns is k and the total number of left turns is $n - k$. From vertex (n, k) the conditional probability that a path makes a left turn is $(n - k + 1)/(n + 2)$, and the conditional probability that a path makes a right turn is $(k + 1)/(n + 2)$. These conditional probabilities for left and right turns are exactly the conditional probabilities of a visit to the points B and A respectively.

The above random walk has been studied by Diaconis in [**10**] and by Keane and Rolles in [**27**]. Keane and Rolles study the joint distribution of the time spent visiting A and B with the total number of visits. The distributions are shown to converge weakly to an absolutely continuous distribution. The reverse Euler graph allows us to study this system from another viewpoint. We have identified the adic-invariant ergodic measures for this system and hope to use this information to gain new insight into such results.

It remains to be seen what other systems in $\mathcal{S}_{\mathcal{L}}$ correspond to reinforced random walks, and it would be interesting to explore these connections more thoroughly.

5.8. Rank

A Bratteli-Vershik system is said to have *finite rank* if it can be represented (in the sense of measure-theoretical isomorphism of systems) by a diagram for which there are finitely many vertices at each level. Each system in $\mathcal{S}_{\mathcal{L}}$ appears to be of infinite rank, but whether this is true is still an open question, even for the Pascal adic.

5.9. Rigidity

A measure-preserving transformation T is said to be *rigid* if there is a sequence $n_1 < n_2 < \cdots \to \infty$ such that T^{n_k} converges to the identity in the strong operator topology, in other words $T^{n_k} f \to f$ in L^2 for all $f \in L^2$. A probably deep and difficult question involves determining which systems in $\mathcal{S}_{\mathcal{L}}$ are rigid.

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