Stability of noncharacteristic boundary-layers for the compressible nonisentropic Navier-Stokes equations

Indrani Rao

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Approved by

Advisor: Professor Mark Williams
Reader: Professor Roberto Camassa
Reader: Professor Christopher Jones
Reader: Professor Jason Metcalfe
Reader: Professor Michael Taylor

ABSTRACT

INDRANI RAO: Stability of noncharacteristic boundary-layers for the compressible nonisentropic Navier-Stokes equations

(Under the direction of Professor Mark Williams)

In this dissertation, we prove the stability of noncharacteristic viscous boundary layers for the compressible nonisentropic Navier-Stokes equations subject to no-slip suction-type boundary conditions.

These boundary conditions correspond to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic suface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G. I. Taylor as a means to reduce drag by stablizing laminar flow. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds.

In [**8**], existence and stability of noncharacterisitic viscous boundary layers for the compressible Navier-Stokes equations has been proved for pure Dirichlet and pure Neumann boundary conditions.

In this dissertation, our boundary conditions are mixed Dirichlet-Neumann and we establish stability in this case.

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Chapter

CHAPTER 1

Introduction

Our goal in this thesis is to establish the stability of noncharacteristic boundary layers for the following compressible, nonisentropic Navier-Stokes equations:

$$A_0(U)U_t + \sum_{j=1}^{2} A_j(U)\partial_j(U) - \epsilon \sum_{j,k=1}^{2} \partial_j(B_{jk}(U)\partial_k U) = 0, \tag{1.0.1}$$

where $U = (\rho, u, v, T)$, where ρ is density, u and v are velocities in the x and y directions and T is the temperature,

subject to the no-slip suction-type boundary conditions:

$$\begin{aligned} u_{|y=0} &= g_1(t,x) \\ v_{|y=0} &= g_2(t,x) < 0(\text{outflow}) \\ \partial T_{|y=0} &= 0 \end{aligned}$$

and separately to the boundary conditions

$$\begin{aligned} \rho_{|y=0} &= h_1(t,x) \\ u_{|y=0} &= h_2(t,x) \\ v_{|y=0} &= h_3(t,x) > 0(\text{inflow}) \\ \partial T_{|y=0} &= 0 \end{aligned}$$

converging to the hyperbolic problem:

$$A_0(U)U_t + \sum_{j=1}^{2} A_j(U)\partial_j(U) = 0, \tag{1.0.2}$$

as viscosity goes to 0, with boundary conditions to be determined.

It turns out that these boundary conditions are given in terms of $\mathcal{C}$-manifolds which exist if profiles are transversal.

These boundary conditions correspond to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic suface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G. I. Taylor as a means to reduce drag by stablizing laminar flow. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds.

Here v corresponds to the component of the velocity of the air that is being sucked into the microscopic holes normal to the boundary of the airfoil and u is the component tangential to the boundary.

It has been verified in that the NS equations satisfy the hypotheses (H1) - (H5) (Assumption 3.0.1) provided the normal velocity of the fluid is nonvanishing on $\mathfrak{U}$ and the normal characteristic speeds (eigenvalues of $\bar{A}(u,\nu)$) are nonvanishing on $\mathfrak{U}$ as in Chapter 2.

These hypotheses are required in the following main results from [**7**]:

(i) Assuming the uniform Lopatinski condition and transversality of layer profiles(Definitions 1 and 11), arbitrarily high-order approximate boundary-layer solutions matching an inner boundary-layer profile to an outer hyperbolic solution have been constructed.

(ii) Assuming uniform Evans stability and using results of [**2**] [**10**] the existence and stability of exact boundary-layer solutions close to the approximate solutions has been shown and consequently the convergence of viscous solutions to solutions of the residual hyperbolic problem in the small viscosity limit has been shown.

(iii) Uniform Evans stability of small-amplitude boundary layers is equivalent to uniform Evans stability of the associated limiting constant layer.

Chapters 2 through 5 review these results in detail.

The spectral stability condition on layer profiles is expressible in terms of an Evans function(Definition 5.0.13). In this thesis we focus on determining the Evans stability condition for the following four cases:

(a) Subsonic, outflow, (b) Subsonic, inflow, (c) Supersonic, outflow and (d) Supersonic, inflow.

We obtain stability in cases (a) and (c) as has been verified experimentally. Stability fails in (d) and is still undetermined in (b).

Before we verify the stability of boundary layers we need to make sure that $\mathcal{C}$-manifolds exist and that the inviscid problem is well-posed. The first follows from [**8**] by checking that the profiles are transversal and the second follows from Lopatinski condition.

We explicitly do the calculations for (a). For this purpose, we divide frequencies into three ranges: Low frequency

Medium frequency and

High frequency.

The low frequency Evans condition is obtained using Rousset's theorem.

In [**8**], it has been proved that high frequency Evans condition holds for NS equations under hypotheses* either for full Dirichlet or full Neumann conditions. Since our boundary conditions are mixed Dirichlet-Neumann conditions, we modified the proof by noting the following:

By [**8**], we know that the uniform Evans condition holds for profiles of (1.0.1) if and only if they hold for profiles of

$$\lambda U - \tilde{B}^{22}_{22} U'' - i(\tilde{B}^{22}_{21} + \tilde{B}^{22}_{12})\eta_1 U' + \eta_1^2 \tilde{B}^{22}_{11} U = 0 \tag{1.0.3}$$

We then observe that these equations can be decoupled as follows:

(a) $\lambda u - \frac{\mu u''}{\rho} - \frac{i\eta_1(\mu+\eta)v'}{\rho} + \frac{\eta_1^2(2\mu+\eta)u}{\rho} = 0$

(b) $\lambda v - \frac{(2\mu+\eta)v''}{\rho} - \frac{i\eta_1(\mu+\eta)u'}{\rho} + \frac{\eta_1^2\mu v}{\rho} = 0$

(c) $\lambda T - \frac{\kappa T''}{\rho c_v} + \frac{\eta_1^2 \kappa T}{\rho c_v} = 0$

We then apply the method for pure Dirichlet conditions in [**8**] separately to (a) and (b) and that for pure Neumann conditions to (c).

For the medium frequencies we can't do any such decoupling as we cannot use (6.3.7) here. We also observe that our proof for the Evans condition works for low frequencies as well and thus we don't really need to use Rousset's theorem here.

CHAPTER 2

Overview

The aim in this thesis is to establish the stability of non-characteristic boundary layers of the full Navier-Stokes equation.

2.1. The Navier Stokes equations

Consider the equation

(a) $\rho_t + (\rho u)_x + (\rho v)_y = 0$

(b) $(\rho u)_t + (\rho u^2)_x + (\rho uv)_y + p_x = (2\mu + \eta)u_{xx} + \mu u_{yy} + (\mu + \eta)v_{xy}$

(c) $(\rho v)_t + (\rho uv)_x + (\rho v^2)_y + p_y = \mu v_{xx} + (2\mu + \eta)v_{yy} + (\mu + \eta)u_{yx}$

(d) $(\rho E)_t + (u\rho E)_x + (v\rho E)_y + (pu)_x + (pv)_y = \kappa T_{xx} + \kappa T_{yy} +$

$((2\mu + \eta)uu_x + \mu v(v_x + u_y) + \eta uv_y)_x +$

$((2\mu + \eta)vv_y + \mu u(v_x + u_y) + \eta vu_x)_y$

on $[-T, T] \times \Omega$ where $\Omega = \{(x, y)|y \geq 0\}$,

subject to the boundary conditions

$$u_{|y=0} = g_1(t, x)$$

$$v_{|y=0} = g_2(t, x) < 0(\text{outflow})$$

$$\partial T_{|y=0} = 0$$

and separately to the boundary conditions

$$
\begin{aligned}
\rho_{|y=0} &= h_1(t,x)\\
u_{|y=0} &= h_2(t,x)\\
v_{|y=0} &= h_3(t,x) > 0(\text{inflow})\\
\partial T_{|y=0} &= 0
\end{aligned}
$$

where ρ is density, u and v are velocities in the x and y directions, p is pressure, and e and $E = e + \frac{u^2}{2} + \frac{v^2}{2}$ are specific internal and total energy respectively. The constants $\mu > |\eta| \geq 0$ and $\kappa > 0$ are coefficients of first and second viscosity and heat conductivity. More specifically, $\exists$ constants, $\tilde{\mu}$, $\tilde{\eta}$ and $\tilde{\kappa}$ such that $\mu = \epsilon\tilde{\mu}$, $\eta = \epsilon\tilde{\eta}$ and $\kappa = \epsilon\tilde{\kappa}$.
Finally T is the temperature and we assume that the internal energy e and the pressure p are known functions of density and temperature: $p = p(\rho, T)$, $e = e(\rho, T)$. Also here, $u_{xy} = \partial_y\left(\partial_x(u)\right)$.
The boundary conditions on the velocities correspond to an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the condition on the temperature corresponds to an insulative condition.

The above equations can be written in the form:

$$
(2.1.1) \qquad A_0(U)U_t + \sum_{j=1}^{2} A_j(U)\partial_j(U) - \epsilon \sum_{j,k=1}^{2} \partial_j(B_{jk}(U)\partial_k U) = 0,
$$

where $U = (\rho, u, v, T)$. We assume the splitting $U = (U^1, U^2) \in \mathbb{R} \times \mathbb{R}^3$ and as observed in Chapter 5, we obtain the block structure:

$$A_0(U) = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix} \tag{2.1.2}$$

$$B_{jk}(U) = \begin{pmatrix} 0 & 0 \\ 0 & B_{jk}^{22} \end{pmatrix} \tag{2.1.3}$$

where we have suppressed U in the entries on the right hand side for convenience.

Later we will be using the following notation:

We set,

$$\bar{A}_j = A_0^{-1} A_j, \qquad \bar{B}_{jk} = A_0^{-1} B_{jk}, \tag{2.1.4}$$

$$\bar{A}(u, \xi) = \sum_{j=1}^{2} \xi_j \bar{A}_j(U) \text{ and } \bar{B}(u, \xi) = \sum_{j,k=1}^{2} \xi_j \xi_k \bar{B}_{jk}(U). \tag{2.1.5}$$

Let

$$\begin{aligned} \mathcal{H}(U) &:= A_0(U) U_t + \sum_{j=1}^{2} A_j(U) \partial_j(U) \text{ and} \\ \mathcal{E}(U) &:= \sum_{j,k=1}^{2} \partial_j (B_{jk}(U) \partial_k U) = 0 \end{aligned} \tag{2.1.6}$$

We want to find exact solutions U^ϵ of $\mathcal{H}(U) - \epsilon \mathcal{E}(U) = 0$ which: (i) converge to solutions of the hyperbolic problem $\mathcal{H}(U) = 0$ with suitable boundary conditions (to be determined) as $\epsilon \to 0$ and (ii) exhibit boundary layers satisfying (i). The first step in constructing such exact solutions is to construct high order approximate solutions with the same property.

2.2. Approximate solutions

In particular, we look for an approximate solution of the form

$$u_a(t,x,y) = \sum_{0 \le j \le M} \epsilon^j \mathfrak{U}^j(t,x,\frac{y}{\epsilon}) + \epsilon^{M+1} U^{M+1}(t,x,y). \tag{2.2.1}$$

where

$$\mathfrak{U}^j(t,x,\frac{y}{\epsilon}) = V^j(t,x,\frac{y}{\epsilon}) + U^j(t,x,y) \tag{2.2.2}$$

and the $V^j(t,x,z)$ are boundary layer profiles constructed to be exponentially decreasing to 0 as $z \to \infty$, where $z = \frac{y}{\epsilon}$.

We will from now on until later in Chapter 5 focus only on the outflow case.

Define $W(t,x,z) = U^0(t,x,0) + V^0(t,x,z)$. Denoting $W = (\rho_W, u_W, v_W, T_W)$, we then expect the boundary condition on W to be:

$$(u_W)_{|y=0} = g_1(t,x)$$

$$(v_W)|y = 0 = g_2(t,x) < 0(\text{outflow})$$

$$\partial(T_W)_{|y=0} = 0$$

and observe that $W(t,x,z) \to U^0(t,x,0)$ as $z \to \infty$.

Substituting u_a in equation (2.1.1), we obtain the left hand side as:

$$\sum_{j=-1}^{M} \epsilon^j \mathcal{F}^j(t,x,z) + \epsilon^M R^{\epsilon,M}(t,x) \tag{2.2.3}$$

where we separate $\mathcal{F}^j$ into slow and fast parts

$$\mathcal{F}^j(t,x,z) = F^j(t,x) + G^j(t,x,z) \tag{2.2.4}$$

and the G^j decrease exponentially to 0 as $z \to \infty$.

We then set F^j and G^j to 0. In particular, setting G^{-1} and F^0 to 0 gives

$$A_2(W)W_z - \frac{d}{dz}(B_{22}(W)W_z) = 0 \tag{2.2.5}$$

and

$$\mathcal{H}(u^0) = 0 \tag{2.2.6}$$

respectively.

2.3. Profiles and $\mathcal{C}$ - manifolds

The equation (2.2.5) is referred to as a profile equation.

This motivates the following definition.

Definition 2.3.1. A solution of (2.2.5) satisfying

$$\lim_{z\to\infty} W(z) = \underline{u} \tag{2.3.1}$$

is called a layer profile.

Let

$$\mathcal{C}_g(t,x) = \{\underline{u} \mid \exists \text{ a layer profile W satisfying 2.2.5 and 2.3.1}\}. \tag{2.3.2}$$

In certain cases, $\mathcal{C}_g(t,x)$ turns out to be a manifold.

This gives the boundary condition for the aforementioned inviscid problem which can now be stated as:

$$\begin{aligned} &\mathcal{H}(U) = 0 \text{ on } [-T_0, T_0] \times \Omega \\ &U(t,x,0) \in \mathcal{C}(t,x) \text{ on } [-T_0, T_0] \times \partial\Omega \end{aligned} \tag{2.3.3}$$

For $(t,x,0) \in [-T_0, T_0] \times \partial\Omega$, we freeze a state $p := U^0(t,x,0)$ and define

$$H(p,\zeta) := -A_2(p)^{-1}\left((i\tau + \gamma)A_0(p) + i\eta_1 A_1(p)\right) \tag{2.3.4}$$

We assume here that $\mathcal{C}(t,x)$ is a manifold. So let

$$\psi : \mathbb{R}^4 \to \mathbb{R}^{N_+} \tag{2.3.5}$$

be a defining function for $\mathcal{C}(t,x)$ near p, i.e., $\mathcal{C}(t,x) = \{u : \psi(U) = 0\}$, with $\nabla\psi$ of full rank N_+. Then the residual boundary condition may be expressed, locally to p, as $\Upsilon_{res}(U) := \psi(U)$, hence the linearized residual boundary condition at p takes the form

$$\Gamma_{res}(p)\dot{U} = 0 \Leftrightarrow \psi'(p)\dot{U} = 0 \Leftrightarrow \dot{U} \in T_p\mathcal{C}(t,x). \tag{2.3.6}$$

REMARK 2.3.2. Suppose $W(z)$ is a solution of the profile (2.2.5) converging to $p = U^0(t,x,0) \in \mathcal{C}(t,x)$ as $z \to \infty$. Let us write the linearized equation of (2.2.5) around $W(z)$ as

$$\mathbb{L}(t,x,z,\partial_z)\dot{W} = 0, \Gamma_2(\dot{W}, \dot{W}^2) = 0. \tag{2.3.7}$$

The fact that the tangent space $T_p\mathcal{C}(t,x)$ may be characterized as the set of limits at $z \to \infty$ of solutions to (2.3.7) follows readily from the definition of $\mathcal{C}(t,x)$; see [**15**], Prop. 5.5.5.

DEFINITION 2.3.3. 1) The inviscid problem (2.3.3) satisfies the uniform Lopatinski condition at $p = u(t,x,0)$ provided there exists $C > 0$ such that for all ζ with $\gamma > 0$

$$|D_{Lop}(p,\zeta)| := |\det(\mathbb{E}^-(H(p,\zeta)), \ker \Gamma_{res}(p))| \geq C. \tag{2.3.8}$$

where for a matrix A, $\mathbb{E}^-(A)$ is the stable space of A, that is the generalized eigenspace of A corresponding to eigenvalues with negative real part.

2) The inviscid problem (2.3.3) satisfies the uniform Lopatinski condition provided (2.3.8) holds with a constant that can be chosen independently of $(t,x,0) \in [-T_0, T_0] \times \partial\Omega$.

Here by a determinant of subspaces we mean the determinant of the matrix with subspaces replaced by smoothly chosen bases of column vectors, specifying D_{Lop} up to a smooth nonvanishing factor.

The following theorem ([**8**], Theorem 1.17) gives well-posedness of the inviscid problem.

Theorem 2.3.4. *Given a smooth manifold* $\mathcal{C}$ *as in Assumption 12, consider the hyperbolic problem* (2.3.3)

under hypotheses (H1) - (H5).

Let $s > \frac{3}{2} + 1$ *and suppose that we are given initial data* $v^0(x) \in H^{s+1}(\Omega)$ *at* $t = 0$ *satisfying corner compatibility conditions to order* $s-1$ *for* (2.3.3)*. Suppose also that the uniform Lopatinski condition is satified at all points* $x_0 \in \partial\Omega$, $t = 0$*. Then there exists a* $T_0 > 0$ *and a function* $U^0(t, x, 0) \in H^s([0, T_0] \times \Omega)$ *satisfying* (2.3.3) *with*

$$U^0_{|t=0} = v^0, \tag{2.3.9}$$

and so that the uniform Lopatinski condition holds on $[0, T_0] \times \partial\Omega$*.*

The hypotheses (Hn) are given in the next Chapter.

CHAPTER 3

Layer profiles and transversality

The results from the GMWZ literature used throughout this treatise uses the following assumptions in their hypotheses. By inspection, we know that our system of equations actually satisfies these assumptions.

We first define the high-frequency principal part of (2.1.1) by

$$\begin{aligned} &\partial_t U^1 + \bar{A}^{11}(U,\partial)U^1 = 0, \\ &\partial_t U^2 - \epsilon \bar{B}^{22}(U,\partial)U^2 = 0 \end{aligned} \tag{3.0.1}$$

We assume the existence of an open set $\mathfrak{U}$ in the state space such that the following hypotheses hold for all $U \in \mathfrak{U}$.

ASSUMPTION 3.0.1. (H1) The matrices A_j and B_{jk} are $C^\infty 4 \times 4$ real matrix-valued functions of the variable $U \in \mathfrak{U}$. Moreover, for all $U \in \mathfrak{U}$, $\det A_0(U) \neq 0$.

(H2) There is $c > 0$ such that for all $U \in \mathfrak{U}$ and $\xi \in \mathbb{R}^2$, the eigenvalues of $\bar{B}^{22}(U,\xi)$ satisfy $\mathrm{Re}\mu \geq c|\xi|^2$.

(H3) For all $U \in \mathfrak{U}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$, the eigenvalues of $\bar{A}^{11}(U,\xi)$ are real, semi-simple and of constant multiplicity. Moreover, for all $U \in \mathfrak{U}$, $\det \bar{A}^{11}(u,\nu) \neq 0$, all positive (inflow) or all negative (outflow), where $\nu = (0,1)$.

(H4) For all $U \in \mathfrak{U}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$, the eigenvalues of $\bar{A}(U,\xi)$ are real, semisimple, and of constant multiplicity. Moreover, for $U \in \mathfrak{U}$, $\det \bar{A}(u,\nu) \neq 0$, with number of positive (negative) eigenvalues of $\bar{A}(u,\nu)$ independent of $U \in \mathfrak{U}$.

(H5) There is $c > 0$ such that for $U \in \mathfrak{U}$ and $\xi \in \mathbb{R}^2$, the eigenvalues of $i\bar{A}(U,\xi) + \bar{B}(U,\xi)$ satisfy $\mathrm{Re}\mu \geq c\frac{|\xi|^2}{1+|\xi|^2}$.

Remark 3.0.2. Hypothesis (H4) is a hyperbolicity condition on the inviscid equation $\mathcal{L}_0(U) = 0$, while (H2), (H4) implies hyperbolic-parabolicity of the viscous equation $\mathcal{L}_\epsilon(U) = 0$ when $\epsilon > 0$. (H3) is a hyperbolicity condition on the first equation in (3.0.1). The conditions on the normal matrices in (H3)-(H4) mean that the boundary is noncharacteristic for both the inviscid and the viscous equations. Hypothesis (H5) is a dissipativity condition reflecting genuine coupling of hyperbolic and parabolic parts for $U \in \mathfrak{U}$.

Definition 3.0.3. The system (2.1.1) is said to be symmetric dissipative if there exists a real matrix $S(U)$, which depends smoothly on $U \in \mathfrak{U}$, such that for all $U \in \mathfrak{U}$ and all $\xi \in \mathbb{R}^2 \setminus \{0\}$, the matrix $S(U)A_0(U)$ is symmetric definite positive and block-diagonal, $S(U)A(U,\xi)$ is symmetric, and the symmetric matrix $\mathrm{Re}S(U)B(U,\xi)$ is nonnegative with kernel of dimension 1.

Definition 3.0.4. A symmetric-dissipative system satisfies the genuine coupling condition if for all $U \in \mathfrak{U}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$, no eigenvector of $\sum_j \bar{A}_j\xi_j$ lies in the kernel of $\sum_{j,k} \bar{B}_{jk}\xi_j\xi_k$.

Hypothesis $H4'$. For all $\xi \in \mathbb{R}^2 \setminus \{0\}$, the eigenvalues of $\bar{A}(U,\xi)$are real and are either semisimple and of constant multiplicity or are totally nonglancing in the sense of [**10**], Definition 4.3. Moreover, for all $U \in \mathfrak{U}$ we have $\det \bar{A}(U) \neq 0$, with the number of positive (negative) eigenvalues of $\bar{A}(U)$ independent of $U \in \mathfrak{U}$.

Notations. With assumptions as above, N_+ (constant) denotes the number of positive eigenvalues of $\bar{A}_2(U) := \bar{A}(U)$ and N_+^1 the number of positive eigenvalues of

$\bar{A}_2^{11}(U) := \bar{A}^{11}(U)$. We also set $N_b = 3 + N_+^1$ is the correct number of boundary conditions for the well posedness of (2.1.1).

Assumption 3.0.5. (H6) Υ_1, Υ_2 and Υ_3 are smooth functions of their arguments with values in $\mathbb{R}^{N_+^1}$, $\mathbb{R}^{3-N''}$ and $\mathbb{R}^{N''}$ respectively, where $N'' \in \{0, 1, 2, 3\}$. Moreover

the equation for a layer profile w reads

$$
\begin{aligned}
&A_2(w)\partial_z w - \partial_z(B_{22}(w)\partial_z w) = 0, z \geq 0, \\
&\Upsilon(w, 0, \partial_z w^2)_{|z=0} = (g_1(t,x), g_2(t,x), 0).
\end{aligned}
\tag{3.0.2}
$$

The profile equation (3.0.2) can be written as a first order system for $U = (w, \partial_z w^2)$, which is nonsingular if and only if A_2^{11} is invertible, $(H3)$:

$$
\begin{aligned}
&\partial_z w^1 = -(A_2^{11})^{-1} A_2^{12} w^3, \\
&\partial_z w^2 = w^3, \\
&\partial_z (B_2 2^{22} w^3) = (A_2^{22} - A_2^{21}(A_2^{11})^{-1} A_2^{12}) w^3,
\end{aligned}
\tag{3.0.3}
$$

and the matrices are evaluated at $w = (w^1, w^2)$.

Consider now the linearized equation of (3.0.2) about $w(z)$, written as a first-order system

$$
\partial_Z \dot{W} - \mathcal{G}_2(z)\dot{W} = 0, z \geq 0, \tag{3.0.4}
$$

$$
\Gamma_2 \dot{W}_{|z=0} = 0 \tag{3.0.5}
$$

in $\dot{W} = (\dot{w}^1, \dot{w}^2, \dot{w}^3)$, where

$$\mathcal{G}_2(\infty) := \lim_{z\to\infty} \mathcal{G}_2(z) = \begin{pmatrix} 0 & 0 & -(A_2^{11})^{-1}A_2^{12} \\ 0 & 0 & I \\ 0 & 0 & (B_{22}^{22})^{-6}(A_2^{22} - A_2^{21}(A_2^{11})^{-1}A_2^{12}) \end{pmatrix} (\underline{U}) \tag{3.0.6}$$

Lemma 3.0.6. *(*[**10**], [**14**]*). Let N_-^2 denote the number of stable eigenvalues $Re\mu < 0$ of $\mathcal{G}_2(\infty)$, N_+^2 the number of unstable eigenvalues $Re\mu > 0$, $\mathcal{S}$ the subspace of solutions of (*3.0.4*) that approach finite limits as $z \to \infty$, and $\mathcal{S}_0$ the subspace of solutions of (*3.0.4*) that decay to 0. Then, (i) $N_-^2 + N_+^2 = N'$ and*

$$N_+ + N_-^2 = N_b := N' + N_+^1, \tag{3.0.7}$$

(ii) profile $w(.)$ decays exponentially to its limit $\underline{U}$ as $z \to \infty$ in all derivatives, and

(iii) $\dim \mathcal{S} = N + N_+^2$ *and* $\dim \mathcal{S} = N_-^2$*.*

Definition 3.0.7. The profile w is said to be transversal if

i) there is no nontrivial solution $\dot{w} \in \mathcal{S}_0$ which satisfies the boundary conditions

$\Gamma_2(\dot{w}, \partial_z \dot{w}^2)_{|z=0} = 0$,

ii) the mapping $\dot{w} \mapsto \Gamma_2(\dot{w}, \partial_z \dot{w}^2)_{|z=0}$ from $\mathcal{S}$ to $\mathbb{C}^{N_b}$ has rank N_b.

The following assumption is the starting point for our construction of exact boundary layer solutions to (2.1.1).

Assumption 3.0.8. Fix a choice of (g_1, g_2) as in the boundary conditions. We are given a smooth manifold $\mathcal{C}$ defined as the graph

$$\mathcal{C} = \{(t, x, \mathcal{C}(t, x)) : (t, x, 0) \in [-T, T] \times \partial\Omega\}, \tag{3.0.8}$$

where each $\mathcal{C}(t,x)$ defined in Chapter 1, is now assumed to be a smooth manifold of dimension $N - N_+$. In addition we are given a smooth function

$$W : [0,\infty) \times \mathcal{C} \to \mathbb{R}^4 \tag{3.0.9}$$

such that for all $(t,x,q) \in \mathcal{C}$, W(z, t, x, q) is a transversal layer profile satisfying (3.0.2) and converging to q as $z \to \infty$ at an exponential rate that can be taken uniform on compact subsets of $\mathcal{C}$.

This assumption is hard to check in general. We will later use a proposition which gives necessary and sufficient conditions on boundary operator in order for the above assumption to hold in the small-amplitude case.

CHAPTER 4

Construction of Approximate Solutions

In this chapter, we give the construction of approximate solutions, following the method used in [**8**]. Here we obtain approximate solutions on the half space $\Omega = \{(x,y) \in \mathbb{R}^2 | y \geq 0\}$.

We seek high-order approximate solutions to

$$\mathcal{L}_\epsilon(U) := A_0(U)U_t + \sum_{j=1}^{2} A_j(U)\partial_j(U) - \epsilon \sum_{j,k=1}^{2} \partial_j(B_{jk}(U)\partial_k U) = 0, \tag{4.0.1}$$

subject to the boundary conditions

$$u_{|y=0} = g_1$$

$$v_{|y=0} = g_2$$

$$T'_{|y=0} = 0$$

as mentioned in the previous chapter,

which converge to a given solution $U^0(t,x,y)$ of the inviscid hyperbolic problem:

(4.0.2)

$$\mathcal{H}(U) = 0 \text{ on } [-T_0, T_0] \times \Omega U(t,x,0) \in \mathcal{C}(t,x) \text{ on } [-T_0,T_0] \times \partial\Omega$$

where $\mathcal{C}(t,x)$ is the endstate manifold defined in Chapter 1.

As mentioned in Chapter 1, we look for an approximate solution of the form

$$u_a(t,x,y) = \sum_{0 \leq j \leq M} \epsilon^j \mathfrak{U}^j(t,x,\frac{y}{\epsilon}) + \epsilon^{M+1} U^{M+1}(t,x,y). \tag{4.0.3}$$

where

$$\mathfrak{U}^j(t,x,\frac{y}{\epsilon}) = V^j(t,x,\frac{y}{\epsilon}) + U^j(t,x,y) \tag{4.0.4}$$

Here U^0 satisfies (4.0.2) and V^0 is given by

$$V^0(t,x,z) = W(z,t,x,U^0(t,x,0)) - U^0(t,x,0), \tag{4.0.5}$$

for a profile $W(z,t,x,U^0(t,x_0))$ as in Assumption 11.

The $V^j(z,x,t)$ are boundary layer profiles constructed to be exponentially decreasing to 0 as $z \to \infty$. For the moment we just assume enough regularity so that all the operations involved in the construction make sense. A precise statement is given in Proposition 13 below.

4.1. Profile equations

We substitute (4.0.3) into (4.0.1) and write the result as

$$\sum_{-1}^{M} \epsilon^j \mathcal{F}^j(t,x,z)|_{z=\frac{y}{\epsilon}} + \epsilon^M R^{\epsilon,M}(t,x), \tag{4.1.1}$$

where we separate $\mathcal{F}^j$ into slow and fast parts

$$\mathcal{F}^j(t,x,z) = F^j(t,x) + G^j(t,x,z), \tag{4.1.2}$$

and the G^j decrease exponentially to 0 as $z \to \pm\infty$.

The interior profile equations are obtained by setting the F^j, G^j equal to zero. In the following expressions for $G^j(t,x,z)$, the functions $U^j(t,x)$ and their derivatives are evaluated at (t,x). With $W = (W(z,t,x,U^0(t,x,0)))$ set

$$\begin{aligned}\mathbb{L}(t,x,z,\partial_z)v := A_2(W)v_z + (d_u A_2(W)\cdot v)W_z - \frac{d}{dz}(B_{22}(W)v_z)\\ -\frac{d}{dz}((v\cdot d_u B_{22}(W))W_z),\end{aligned} \tag{4.1.3}$$

the operator determined by linearizing the profile equations about W, and

$$\mathcal{L}_0 v := A_0(U^0)v_t + \sum_{j=1}^{2} A_j(U^0)\partial_j v. \tag{4.1.4}$$

We have

$$\begin{aligned}F^{-1}(t,x) &= 0\\ G^{-1}(t,x,z) &= A_2(W)W_z - \frac{d}{dz}(B_{22}(W)W_z),\end{aligned} \tag{4.1.5}$$

$$\begin{aligned}F^{0}(t,x) &= \mathcal{H}U^0\\ G^{0}(t,x,z) &= \mathbb{L}(t,x,z,\partial_z)\mathcal{U}^1 - Q^0(t,x,z),\end{aligned} \tag{4.1.6}$$

where Q^0 decays exponentially as $z\to+\infty$ and depends only on (U^0,V^0). For $j\geq 1$ we have

$$\begin{aligned}F^{j}(t,x) &= \mathcal{L}U^j - P^{j-1}(t,x)\\ G^{j}(t,x,z) &= \mathbb{L}(t,x,z,\partial_z)\mathcal{U}^{j+1} - Q^j(t,x,z),\end{aligned} \tag{4.1.7}$$

where Q^j decays exponentially as $z \to +\infty$ and P^j, Q^j depend only on (U^k, V^k) for $k \leq j$.

In writing out the boundary profile equations, we note first that the boundary conditions are equivalent for $\epsilon > 0$ to

$$u_{|y=0} = g_1$$

$$v_{|y=0} = g_2$$

$$\epsilon T'_{|y=0} = 0$$

With $\mathcal{U}^j(t, x, y, z) = (\rho^j, u^j, v^j, T^j)$ and their derivatives always evaluated at $(t, x, 0)$, the boundary profile equations at order ϵ^j take the form:

$$u^0_{|y=0} = g_1$$

$$v^0_{|y=0} = g_2$$

$$\partial_z T^0_{|y=0} = 0 \qquad \text{(order } \epsilon^0\text{)}$$

$$u^1_{|y=0} = 0$$

$$v^1_{|y=0} = 0$$

$$\partial_z T^1_{|y=0} = 0 \qquad \text{(order } \epsilon^1\text{)}$$

$$u^1_{|y=0} = 0$$

$$v^1_{|y=0} = 0$$

$$\partial_z T^1_{|y=0} = 0 \qquad (\text{order } \epsilon^j, j \geq 2),$$

4.2. Solution of the profile equations

The solution of the profile equations given below assumes transversality of $W(z, U^0(t, x, 0))$ and the uniform Lopatinski condition, as well as the existence of a K-family of smooth inviscid symmetrizers.

1. The interior equations $G^{-1} = 0$ and $F^0 = 0$ and the boundary equation for order ϵ^0 are satisfied because of our assumptions about U^0 and
$W(z, t, x, U^0(t, x, 0))$.

2. Construction of $(\mathcal{U}^1, U^1)$. We construct the functions $\mathcal{U}^1(t, x, z)$ and $U^1(t, x)$ from the equations $G^0 = 0$, $F^1 = 0$, and the boundary equation for order ϵ^1. $\mathcal{U}^1$ will be a sum of three parts

$$\mathcal{U}^1(t, x, z) = \mathcal{U}^1_a + \mathcal{U}^1_b + \mathcal{U}^1_c, \text{ where } \mathcal{U}^1_k(t, x, z) = U^1_k(t, x) + V^1_k(t, x, z), k = a, b, c. \tag{4.2.1}$$

First use the exponential decay of Q^0 to find an exponentially decaying solution $V^1_a(t, x, z)$ to

$$\begin{gathered} \mathbb{L}(t, x, z, \partial_z) V^1_a = Q^0(t, x, z) \text{ on } \pm z \geq 0 \\ V^1_a \to 0 \text{ as } z \to +\infty, \end{gathered} \tag{4.2.2}$$

and define $U_a^1(t,x) \equiv 0$. This problem is easily solved after first conjugating to a constant coefficient ODE using the operators $\mathcal{W}$ defined in Lemma 14.

Next, for $\mathcal{U}_a^1$ fixed as above, use part (ii) of the definition of transversality (Definition 10) to see that we can solve for $\mathcal{U}_b^1(t,x,z) \in \mathcal{S}$ satisfying

$$\begin{aligned} &\mathbb{L}(t,x,z,\partial_z)\mathcal{U}_b^1 = 0 \text{ on } z \geq 0 \\ &(u_a^1 + u_b^1)_{|y=0} = 0 \\ &(v_a^1 + v_b^1)_{|y=0} = 0 \\ &\partial_z((T_a^1 + T_b^1))_{|y=0} = 0(\text{order } \epsilon^j, j \geq 2), \end{aligned} \tag{4.2.3}$$

Recalling the definition of $\mathcal{S}$ from Lemma 9, we see that $\mathcal{U}_b^1$ has limits as $z \to \infty$. Define

$$\begin{aligned} &U_b^1(t,x,0) := \lim_{z\to\infty} \mathcal{U}_b^1(t,x,z), \\ &V_b^1(t,x,z) := \mathcal{U}_b^1(t,x,z) - U_b^1(t,x,0), \end{aligned} \tag{4.2.4}$$

and let $U_b^1(t,x)$ be any smooth extension of $U_b^1(t,x,0)$ to $[-T_0,T_0] \times \Omega$.

Finally, for an appropriate choice of $U_c^1(t,x,0)$ we need $\mathcal{U}_c^1(t,x,z)$ to satisfy

$$
\begin{aligned}
&\mathbb{L}(t,x,z,\partial_z)\mathcal{U}_c^1 = 0\\
&(u_c^1)_{|y=0} = 0\\
(4.2.5)\qquad &(v_c^1)_{|y=0} = 0\\
&\partial_z(T_c^1)_{|y=0} = 0\\
&\lim_{z\to+\infty}\mathcal{U}_c^1(t,x,z) = U_c^1(t,x,0).
\end{aligned}
$$

According to the characterization of $T_q\mathcal{C}(t,x)$ given in Remark 2, this is possible if and only if $U_c^1(t,x,0) \in T_{U^0(t,x,0)}\mathcal{C}(t,x)$. Thus, we first solve for $U_c^1(t,x)$ satisfying the linearized inviscid problem

$$
(4.2.6)\qquad
\begin{aligned}
&\mathcal{L}_0 U_c^1 = P^0 - \mathcal{L}_0 U_b^1\\
&U_c^1(t,x) \in T_{U^0(t,x,0)}\mathcal{C}(t,x).
\end{aligned}
$$

This problem requires an initial condition in order to be well-posed. The right side in the interior equation of (4.2.6) is initially defined just for $t \in [-T_0, T_0]$. With a C^∞ cutoff that is identically one in $t \geq -T_0/2$, we can modify the right side to be zero in $t \leq -T_0 + \delta$, say. Requiring U_c^1 to be identically zero in $t \leq -T_0 + \delta$, we thereby obtain a problem for U_c^1 that is forward well-posed since U^0 satisfies the uniform Lopatinski condition. Thus, there exists a solution to (4.2.6) on $[-\frac{T_0}{2}, T_0]$. This allows us to obtain $\mathcal{U}_c^1(t,x,z)$ satisfying (4.2.5) and to define

$$
(4.2.7)\qquad V_c^1(t,x,z) := \mathcal{U}_c^1(t,x,z) - U_c^1(t,x,0).
$$

By construction, the functions $(\mathcal{U}^1, U^1)$ satisfy the equations $G^0 = 0$, $F^1 = 0$, and the boundary conditions for order ϵ^1.

3. Construction of $(\mathcal{U}^j, U^j)$, $j \geq 2$**.** In the same way, for $j \geq 2$, we use the equations $G^{j-1} = 0$, $F^j = 0$, and the boundary conditions for the order ϵ^j to determine the functions $(\mathcal{U}^j, U^j)$. The corrector $\epsilon^{M+1} U^{M+1}$ is chosen simply to solve away an $O(\epsilon^{M+1})$ error that remains in the boundary conditions after the construction of $\mathcal{U}^M$.

In the next Proposition we formulate a precise statement summarizing the construction of this section. The regularity assertions in the Proposition are justified as in [**2**], Prop. 5.7. Regularity is expressed in terms of the following spaces:

DEFINITION 4.2.1. 1. Let H^s (resp. H^s_b) be the standard Sobolev space on $[-T_0, T_0] \times \Omega$ (resp. $[-T_0, T_0] \times \partial\Omega$).

2. Let $\tilde{H}^s$ be the set of functions $V(t, x, z)$ on $[-T_0, T_0] \times \partial\Omega \times \bar{\mathbb{R}}_+$ such that $V \in \mathcal{C}^\infty(\bar{\mathbb{R}}_+, H^s([-T_0, T_0] \times \partial\Omega))$ and satisfies

$$|\partial_z^k V(t, x, z)|_{H^s_b} \leq C_{k,x} e^{-\delta|z|} \text{ for all } k \tag{4.2.8}$$

for some $\delta > 0$.

PROPOSITION 4.2.2. *(**Approximate solutions.**) Assume (H1)-(H6) (with (H4') replacing (H4) in the symmetric-dissipative case). for given integers* $m \geq 0$ *and* $M \geq 1$ *let*

$$s_0 > m + \frac{7}{2} + 2M + \frac{d+1}{2}. \tag{4.2.9}$$

Suppose that the inviscid solution U^0 as in (4.0.2) satisfies the uniform Lopatinski condition and that the profiles $W(z, U^0(t,x,0))$ are transversal. Assume $U^0 \in H^{s_0}$ and $U^0|_{\partial\Omega} \in H_b^{s_0}$.

Then one can construct u_a as in (4.0.3) satisfying:

$$\mathcal{L}_\epsilon u_a = \epsilon^M R^M(t,x) \text{ on } [-\frac{T_0}{2}, T_0] \times \Omega \tag{4.2.10}$$

We have

$$U^j(t,x) \in H^{s_0-2j}, V^j(t,x,z) \in \tilde{H}^{s_0-2j}, \tag{4.2.11}$$

and $R^M(t,x)$ satisfies

(a) $|(\partial_t, \partial_x, \epsilon\partial_z)^\alpha R^M|_{L^2} \leq C_\alpha$ *for* $|\alpha| \leq m + \frac{d+1}{2}$

(b) $|(\partial_t, \partial_x, \epsilon\partial_z)^\alpha R^M|_{L^\infty} \leq C_\alpha$ *for* $|\alpha| \leq m.$

CHAPTER 5

Evans function

Fix a point (t,x) and consider again the viscous problem (4.0.1). Consider a planar layer profile

$$U^\epsilon(t,x,y) = w(y/\epsilon) \tag{5.0.1}$$

as in Definition 1, which is an exact solution to (2.1.1) on $y \geq 0$ when the coefficients and boundary data $(g_1, g_2, 0)$ are frozen at $(t,x,0)$.

Taking $z = \frac{y}{\epsilon}$, we see that $\partial_y = \frac{1}{\epsilon}\partial_z$. So equation (2.1.1) becomes

$$\begin{aligned}
&\epsilon A_0(U)\partial_t U + \epsilon A_1(U)\partial_x U + A_2(U)\partial_y U - \epsilon^2\partial_x(B_{11}(U)\partial_x U) \\
&-\epsilon\partial_x(B_{12}(U)\partial_z U) - \epsilon\partial_z(B_{21}(U)\partial_x U) - \partial_z(B_{22}(U)\partial_z U) = 0.
\end{aligned} \tag{5.0.2}$$

Write the linearized equations of (2.1.1) about w:

$$\mathcal{L}'_w(\dot{U}) = \dot{f}, \qquad \Upsilon'(\dot{U}, \partial_x \dot{U}^2, \partial_z \dot{U}^2)_{|y=0} = \dot{g}. \tag{5.0.3}$$

Performing a Fourier-Laplace transform of (5.0.3) in (t,x), with frequency variables denoted by $\gamma + i\tau$ and η respectively, yields the family of ordinary differential systems

$$L(z, \gamma + i\tau, i\eta, \partial_z)U = f, \begin{pmatrix} u \\ v \\ \partial_Z T \end{pmatrix}_{|z=0} = g \tag{5.0.4}$$

where,

$$L = -\mathcal{B}(z)\partial_z^2 + \mathcal{A}(z,\zeta)\partial_z + \mathcal{M}(z,\zeta), \tag{5.0.5}$$

with coefficients given by

$$\mathcal{B}(z) = B_{22}(w(z)) \tag{5.0.6}$$

$$\mathcal{A}(z,\zeta) = A_2(w(z)) - i\eta(B_{12} + B_{21})(w(z)) + E_2(z) \tag{5.0.7}$$

$$\mathcal{M}(z,\zeta) = (i\tau + \gamma)A_0(w(z)) + i\eta(A_1(w(z)) + E_1(z)) + \eta^2 B_{11}(w(z)) + E_0(z). \tag{5.0.8}$$

The E_k are functions independent of ζ which involve derivatives of w and thus converge to 0 at an exponential rate when z tends to infinity. Moreover, we note that

$$E_k^{11} = 0, \qquad E_k^{12} = 0 \text{ for } k > 0. \tag{5.0.9}$$

From the given NS equations, we also remark that $\mathcal{M}^{12}$ does not depend on τ and γ.

The problem (5.0.4) may be written as a first order system

$$\partial_z \tilde{U} - \mathcal{G}(z,\zeta)\tilde{U} = F, \qquad \Gamma(\zeta)\tilde{U}_{|z=0} = G, \tag{5.0.10}$$

where $\tilde{U} = (U, \partial_z U^2) = (U^1, U^2, U^3) \in \mathbb{C}^7$ and $\zeta = (\tau, \gamma, \eta)$. The components of $\mathcal{G}(z,\zeta)$ are given below.

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & \mathcal{G}^{13} \\ 0 & 0 & \mathrm{Id} \\ \mathcal{G}^{31} & \mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix}, \tag{5.0.11}$$

where,

$$\begin{aligned} \mathcal{G}^{11} &= -(\mathcal{A}^{11})^{-1}\mathcal{M}^{11}, \qquad & \mathcal{G}^{31} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{11} + \mathcal{M}^{21}), \\ \mathcal{G}^{12} &= -(\mathcal{A}^{11})^{-1}\mathcal{M}^{12}, \qquad & \mathcal{G}^{32} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{12} + \mathcal{M}^{22}), \\ \mathcal{G}^{13} &= -(\mathcal{A}^{11})^{-1}\mathcal{A}^{12}, \qquad & \mathcal{G}^{33} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{13} + \mathcal{A}^{22}). \end{aligned} \tag{5.0.12}$$

A necessary condition for stability of the inhomogeneous equations (5.0.10) is stability of the homogeneous case $F = 0$, $G = 0$, i.e., nonexistence for $\gamma \geq 0$, $\zeta \neq 0$ of solutions U decaying as $z \to \infty$ and satisfying $\Gamma(\zeta)U(0) = 0$. These may be detected by vanishing of the *Evans function*

$$D(\zeta) := \det_{7\times 7}(\mathbb{E}^{-}(\zeta), \ker \Gamma(\zeta)), \tag{5.0.13}$$

where $\mathbb{E} = \{U(0) | \partial_z \tilde{U} - \mathcal{G}(z,\zeta)\tilde{U} = 0 \text{ and } U \text{ decays at } z = \infty\}$.

Let $\mathcal{G}(z, \zeta, p)$ be as in (5.0.10), a frequency-dependent matrix arising from linearization around a profile $w(z)$ such that for some positive constants C, β, uniform with respect to model parameters p,

$$|w(z) - w(\infty)| \leq Ce^{-\beta z}, \tag{5.0.14}$$

and also $p \mapsto (w, \partial_z w_2)(., p)$ is continuous as a function from p to $L^\infty(0,\infty)$. Thus also,

$$|\mathcal{G}(z,.) - \mathcal{G}(\infty,.)| \leq Ce^{-\beta z}, \tag{5.0.15}$$

and $\mathcal{G}(.,\zeta,p)$ is continuous as a function from p to $L^\infty(0,\infty)$.

The following lemma which is called the conjugation lemma is useful in converting a first-order system like (5.0.10) to one where $\mathcal{G}(z)$ is replaced by the constant coefficient matrix $\mathcal{G}(\infty)$.

LEMMA 5.0.1. [**13**]***, Lemma 2.6****. For all $\underline{\zeta} \in \mathbb{R}^3$ with $\underline{\gamma} \geq 0$, there is a neighborhood ω of $(\underline{p},\underline{\zeta})$ and there is a matrix $\mathcal{W}$ defined and C^∞ on $[0,\infty) \times \omega$ such that*

i) $\mathcal{W}^{-1}$ is uniformly bounded and there is $\theta > 0$ such that

$$|\mathcal{W}(z,p,\zeta) - Id| \leq Ce^{-\theta z} \tag{5.0.16}$$

ii) $\mathcal{W}$ satisfies

$$\partial_z \mathcal{W} = \mathcal{G}(z)\mathcal{W}(z) - \mathcal{W}(z)\mathcal{G}^\infty. \tag{5.0.17}$$

Observe that U satisfies (5.0.10) on $z \geq 0$ if and only if V defined by $U = PV$ satisfies

$$\partial_z V = \mathcal{G}(\infty)V + P^{-1}F, \Gamma P(0)V_{|z=0} = G \tag{5.0.18}$$

This implies that the decaying space $\mathbb{E}^-(\zeta,p)$ as in (5.0.13) is exactly the image under $P(0,\zeta,p)$ of the stable subspace of $\mathcal{G}(\infty,\zeta,p)$, denoted $\mathbb{E}^-_\infty(\zeta,p)$. Thus, by the calculation of [**10**], Lemma 2.12, $\mathbb{E}^-(\zeta,p)$ has dimension $N_b = rank\Gamma$ for $\gamma \geq 0$, $\zeta \neq 0$. The Evans determinant (5.0.13)

$$D_p(\zeta) = \det(\mathbb{E}^-(\zeta, p), \ker \Gamma(\zeta, p)), \tag{5.0.19}$$

now denoted with additional dependence on model parameters p, is then well-defined on $\gamma \geq 0$, $\zeta \neq 0$ and depends smoothly on ζ and continuously (in all ζ derivatives) on p. For high frequencies $|\zeta| \geq R > 0$, we also define a rescaled Evans function $D^{sc}(\zeta)$. By [**10**], the maximal stability estimate for (5.0.10) for high frequencies is

$$\begin{aligned}(1+\gamma)\|U^1\|_{L^2(\mathbb{R}_+)} + \Lambda\|U^2\|_{L^2(\mathbb{R}_+)} + \|\partial_z U^2\|_{L^2(\mathbb{R}_+)} + (1+\gamma)^{\frac{1}{2}}|U^1(0)|\\ +\Lambda^{\frac{1}{2}}|U^2(0)| + \Lambda^{-\frac{1}{2}}|\partial_z U^2(0)| \leq C(\|f^1\|_{L^2(\mathbb{R}_+)} + \Lambda^{-1}\|f^2\|_{L^2(\mathbb{R}_+)})\\ +C((1+\gamma)^{\frac{1}{2}}|g^1| + \Lambda^{\frac{1}{2}}|g^2| + \Lambda^{-\frac{1}{2}}|g^3|),\end{aligned} \tag{5.0.20}$$

where C is an independent constant and Λ is the natural parabolic weight

$$\Lambda(\zeta) = (\tau^2 + \gamma^2 + |\eta|^4)^{1/4}. \tag{5.0.21}$$

Taking $f = 0$ in (5.0.20) yields the necessary condition

$$\begin{aligned}(1+\gamma)^{\frac{1}{2}}|u^1| + \Lambda^{\frac{1}{2}}|u^2| + \Lambda^{-\frac{1}{2}}|u^3| \leq C((1+\gamma)^{\frac{1}{2}}|\Gamma_1 u^1|\\ +\Lambda^{\frac{1}{2}}|\Gamma_2 u^2| + \Lambda^{-\frac{1}{2}}|\Gamma_3(\zeta)(u^2, u^3)|)\end{aligned} \tag{5.0.22}$$

$\forall \zeta \in \bar{\mathbb{R}}^{d+1}_+$, $|\zeta| \geq R$, $\forall U = (u^1, u^2, u^3) \in \mathbb{E}_-(\zeta)$. This can be reformulated in terms of a *rescaled Evans function.* Introduce maps defined on $\mathbb{C}^7$ and $\mathbb{C}^3$ respectively by

$$J_\zeta(u^1,u^2,u^3) := ((1+\gamma)^{\frac{1}{2}}u^1, \Lambda^{\frac{1}{2}}u^2, \Lambda^{-\frac{1}{2}}u^3)$$
$$J_\zeta(g^1,g^2,g^3) := ((1+\gamma)^{\frac{1}{2}}g^1, \Lambda^{\frac{1}{2}}g^2, \Lambda^{-\frac{1}{2}}g^3) \tag{5.0.23}$$

Note that $J_\zeta\Gamma(\zeta)U = \Gamma^{sc}J_\zeta U$ with

$$\Gamma^{sc}U = (\Gamma_1 u^1, \Gamma_2 u^2, K_2 u^3). \tag{5.0.24}$$

Thus (5.0.22) reads

$$\forall U \in J_\zeta\mathbb{E}^-(\zeta): \qquad |U| \le C|J_{zeta}\Gamma(\zeta)J_\zeta^{-1}U| = C|\Gamma^{sc}U|. \tag{5.0.25}$$

Introducing the *rescaled Evans function*

$$D^{sc}(\zeta) := |\det(J_\zeta\mathbb{E}^-(\zeta), J_\zeta \ker\Gamma(\zeta))| = |\det(J_\zeta\mathbb{E}^-(\zeta), \ker\Gamma^{sc}(\zeta))|, \tag{5.0.26}$$

and using Lemma 15 below, we see that this stability condition is equivalent to the following definition.

LEMMA 5.0.2. ***([10], Lemma 2.19).*** *Suppose that $\mathbb{E} \subset \mathbb{C}^n$ and $\Gamma : \mathbb{C}^n \to \mathbb{C}^m$, with $rank\Gamma = \dim\mathbb{E} = m$. If $|\det(\mathbb{E}, \ker\Gamma)| \ge c > 0$, then there is C, which depends only on c and $|\Gamma^*(\Gamma\Gamma^*)^{-1}|$ such that*

$$|U| \le C|\Gamma U| \textit{ for all } U \in \mathbb{E}. \tag{5.0.27}$$

Conversely, if this estimate is satisfied then $|\det(\mathbb{E}, \ker\Gamma)| \geq c > 0$, *where* $c > 0$ *depends only on* C *and* $|\Gamma|$.

REMARK 5.0.3. By Lemma 15, the uniform Evans condition $|D(\zeta)| \geq C > 0$ on some subset S of frequencies is equivalent to

$$|U| \leq C|\Gamma U| \text{ for all } U \in \mathbb{E}^{-}(\zeta) \tag{5.0.28}$$

for some constant $C > 0$ independent of $\zeta \in \mathcal{S}$.

DEFINITION 5.0.4. (a) Given a profile w, the linearized equation (5.0.3) satisfies the uniform Evans condition for high frequencies when there are $c > 0$ and $R > 0$ such that $|D^{sc}(\zeta)| \geq c$ for all $\zeta \in \bar{\mathbb{R}}^{d+1}_{+} \geq R$.

(b) The linearized equation (5.0.3) satisfies the uniform Evans condition when there are $c > 0$ and $R > 0$ such that

$$|D(\zeta)| \geq c \text{ for } |\zeta| \leq R \text{ and } |D^{sc}(\zeta)| \geq c \text{ for } |\zeta| \geq R. \tag{5.0.29}$$

We use the Evans condition to establish a version of Theorem 1.30 from [**8**]

CHAPTER 6

Stability of the non-characteristic boundary layers

Our aim in this chapter is to establish the stability of non-characteristic boundary layers of the full Navier-Stokes equation.

6.1. The Navier Stokes equations

Consider the equation

$$\begin{aligned}
&\text{(a)}\ \rho_t + (\rho u)_x + (\rho v)_y = 0\\
&\text{(b)}\ (\rho u)_t + (\rho u^2)_x + (\rho uv)_y + p_x = (2\mu + \eta)u_{xx} + \mu u_{yy} + (\mu + \eta)v_{xy}\\
&\text{(c)}\ (\rho v)_t + (\rho uv)_x + (\rho v^2)_y + p_y = \mu v_{xx} + (2\mu + \eta)v_{yy} + (\mu + \eta)u_{yx}\\
&\text{(d)}\ (\rho E)_t + (u\rho E)_x + (v\rho E)_y + (pu)_x + (pv)_y = \kappa T_{xx} + \kappa T_{yy} +\\
&\qquad ((2\mu + \eta)uu_x + \mu v(v_x + u_y) + \eta uv_y)_x +\\
&\qquad ((2\mu + \eta)vv_y + \mu u(v_x + u_y) + \eta vu_x)_y
\end{aligned}$$

subject to the boundary conditions

$$u_{|y=0} = g_1$$

$$v_{|y=0} = g_2$$

$$T'_{|}y = 0 = 0$$

where ρ is density, u and v are velocities in the x and y directions, p is pressure, and e and $E = e + \frac{u^2}{2} + \frac{v^2}{2}$ are specific internal and total energy respectively. The constants $\mu > |\eta| \geq 0$ and $\kappa > 0$ are coefficients of first and second viscosity and heat conductivity. Finally T is the temperature and we assume that the internal energy e and the pressure p are known functions of density and temperature: $p = p(\rho, T)$, $e = e(\rho, T)$. Also here, $u_{xy} = \partial_y (\partial_x(u))$.

We want to write it in a matrix format to assist in our calculations. The above equations can be written in the form:

$$A_0(U)U_t + \sum_{j=1}^{2} A_j(U)\partial_j(U) - \epsilon \sum_{j,k=1}^{2} \partial_j(B_{jk}(U)\partial_k U) = 0, \tag{6.1.1}$$

where $U = (\rho, u, v, T)$. We follow the notations established in the first two chapters. Below, we evaluate all the matrices.

$$A_0(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ E & \rho u & \rho v & \rho c_v \end{pmatrix}$$

$$A_1(U) = \begin{pmatrix} u & \rho & 0 & 0 \\ u^2 + p_\rho & 2\rho u & 0 & p_T \\ uv & \rho v & \rho u & 0 \\ uE + up_\rho & \rho E + u^2\rho + p & u\rho v & u\rho c_v + up_T \end{pmatrix}$$

$$A_2(U) = \begin{pmatrix} v & 0 & \rho & 0 \\ uv & \rho v & \rho u & 0 \\ v^2 + p_\rho & 0 & 2\rho v & p_T \\ vE + p_\rho v & v\rho u & \rho E + v^2\rho + p & v\rho c_v + p_T v \end{pmatrix}$$

$$B_{11}(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\mu + \eta & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & (2\mu + \eta)u & \mu v & \kappa \end{pmatrix}$$

$$B_{21}(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mu+\eta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \eta v & \mu u & 0 \end{pmatrix}$$

$$B_{12}(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu+\eta & 0 & 0 \\ 0 & \mu v & \eta u & 0 \end{pmatrix}$$

$$B_{22}(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 2\mu+\eta & 0 \\ 0 & \mu u & (2\mu+\eta)v & \kappa \end{pmatrix}$$

The following result from [**7**] reduces the problem of proving existence and nonlinear stability of boundary-layer solutions to verification of the uniform evans condition.

We rewrite our NS equations with our boundary conditions in a compact form below:

$$A_0(U)U_t + \sum_{j=1}^{2} A_j(U)\partial_j(U) - \epsilon \sum_{j,k=1}^{2} \partial_j(B_{jk}(U)\partial_k U) = 0, \tag{6.1.2}$$

$$\Upsilon(U, \partial_x, \partial_z) = (g_1, g_2, 0) \text{ on } [0, T_0] \times \partial\Omega$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$.

It turns out that our system is symmetric dissipative.

THEOREM 6.1.1. ***([7], Theorem 1.25).*** *Consider the viscous problem (6.1.2) under assumptions (H1) - (H6)(with (H4') replacing (H4) in the symmetric-dissipative case). Given an inviscid solution $U^0 \in H^{s_0}([0,T_0] \times \Omega)$ as in Theorem 4, suppose that the uniform Evans condition holds on $[0, T_0 \times \partial\Omega]$. Suppose the constants k, M, and s_0 satisfy*

$$k > \frac{3}{2} + 4, M > k + 2, s_0 > k + \frac{7}{2} + 2M + \frac{d+1}{2}. \tag{6.1.3}$$

Then there exists $\epsilon_0 > 0$, an approximate solution u_a^ϵ as in Chapter 3 satisfying

$$\begin{gathered} \|\mathcal{L}_\epsilon(u_a^\epsilon)\|_{H^s([0,T_0]\times\Omega)} \le C\epsilon^M \\ \Upsilon(u_a^\epsilon, \partial_x u_a^\epsilon, \partial_z u_a^\epsilon) = (g_1, g_2, 0) \text{ on } [0,T_0] \times \partial\Omega, \end{gathered} \tag{6.1.4}$$

and an exact solution U^ϵ of (6.1.2) such that for $0 < \epsilon \le \epsilon_0$:

$$\begin{gathered} \|U^\epsilon - u_a^\epsilon\|_{W^{1,\infty}([0,T_0]\times\Omega)} \le C\epsilon^{M-k}, \\ \|U - U^0\|_{L^2([0,T_0]\times\Omega)} \le C\epsilon^{1/2}, \\ U^\epsilon - U^0 = O(\epsilon) \text{ in } L^\infty_{loc}([0,T_0] \times \Omega^0) \end{gathered} \tag{6.1.5}$$

where Ω^0 denotes the interior of Ω. Moreover, the linearized equations about either u_a^ϵ or u^ϵ satisfy maximal stability estimates.

6.2. The four cases of the problem

The following result from [7] reduces the problem of proving existence and nonlinear stability of boundary-layer solutions to verification of the uniform Evans condition.

We divide the problem into the following four cases and attempt to verify the Evans condition in each of the cases:

(1) Subsonic, outflow,

(2) Subsonic, inflow,

(3) Supersonic, outflow,

(4) Supersonic, inflow.

DEFINITION 6.2.1. **(Small amplitude profiles).** Let $\mathfrak{U}$ be as defined in Chapter 1. For $\epsilon > 0$ and any compact set $D \subset \mathfrak{U}$, the set of ϵ-amplitude profiles associated to D is the set of functions $w(z) = w(z, u)$ for which there exist $\underline{u} \in D$ such that:

a) $A_2(w)\partial_z w - \partial_z(B_22(w)\partial_z w) = 0$ on $z \geq 0$,

b) $w(z, u) \to u$ as $z \to \infty$,

c) $\|(w, w_z^2) - (\underline{u}, 0)\|_{L^\infty(0,\infty)} \leq \epsilon$.

When ϵ is small we refer to such profiles as small amplitude profiles.

Thanks to the following theorem from [**7**], our job of verification of Evans condition is greatly simplified.

THEOREM 6.2.2. *([**7**], **Theorem 1.28**). For any compact subset $D \subset \mathfrak{U}$, there exists an $\epsilon > 0$ such that the uniform Evans condition is satisfied for the set of ϵ-amplitude profiles associated to D if and only if it is satisfied for the set of constant layers $\{w(z, u) : w = u$ for all $z \geq 0$ and $u \in D\}$.*

First we fix some notation :

$$A_2^{11} = (v)$$

$$A_2^{12} = \begin{pmatrix} 0 & \rho & 0 \end{pmatrix}$$

$$A_2^{21} = \begin{pmatrix} uv \\ v^2 + p_\rho \\ vE + v\rho e_\rho + p_\rho v \end{pmatrix}$$

$$A_2^{22} = \begin{pmatrix} \rho v & \rho u & 0 \\ 0 & 2\rho v & p_T \\ \rho uv & \rho E + \rho v^2 + p & \rho v c_v + v p_T \end{pmatrix}$$

$$B_{22}^{22} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & 2\mu + \eta & 0 \\ \mu u & (2\mu + \eta)v & \kappa \end{pmatrix}$$

By our assumption on μ, η and κ, we see that B_{22}^{22} is invertible and

$$(B_{22}^{22})^{-1} = \begin{pmatrix} \frac{1}{\mu} & 0 & 0 \\ 0 & \frac{1}{2\mu+\eta} & 0 \\ \frac{-u}{\kappa} & \frac{-v}{\kappa} & \frac{1}{\kappa} \end{pmatrix}$$

$$A_2^{22} - A_2^{21}(A_2^{11})^{-1}A_2^{12} = \begin{pmatrix} \rho v & 0 & 0 \\ 0 & \rho v - \frac{\rho p_\rho}{v} & p_T \\ \rho uv & \rho v^2 + p - \rho p_\rho & \rho v c_v + v p_T \end{pmatrix}$$

We now turn our attention to verifying the uniform Evans condition for constant profiles for all frequencies which we divide into:

i) Small frequencies: i.e. ζ such that $|\zeta| \leq r$ for a sufficiently small $r > 0$

ii) Medium frequencies: i.e. ζ such that $r \leq |\zeta| \leq R$ for r as above and $R > 0$

iii) Large frequencies: i.e. ζ such that $|\zeta| \geq R$ for $R > 0$.

6.3. Subsonic, outflow

Subsonic means $|v| <$ speed of sound through the respective fluid. Outflow means $v < 0$. For our boundary conditions $N'' = 1$

We will first establish the uniform Evans condition for low frequencies. For this purpose, we use Rousset's theorem from [**10**].

LEMMA 6.3.1. *(*[**10**] ***Theorem. 2.28).*** *Assume (H1) - (H6) (with (H4') replacing (H4) in the symmetric dissipative case), and consider a layer profile $w(z) \to \underline{p}$ as $z \to \infty$. The uniform Evans condition holds for low frequencies, that is, there exist positive constants r, c such that*

$$|D(\zeta)| \geq c \text{ for } |\zeta| \leq r, \tag{6.3.1}$$

if and only if w is transversal and the uniform Lopatinski condition holds at $\underline{p}$ for the residual hyperbolic problem.

Since we intend to verify the uniform Evans condition at constant profiles, in the above lemma, we have $w = \underline{p}$ and we have to verify transversality and the uniform Lopatinski condition at $\underline{p}$.

6.3.1. Low frequency.

6.3.1.1. *Transversality at $\underline{p}$.* Recalling that N_+^1 is the number of positive eigenvalues of A_2^{11} and that $A_2^{11} = (v)$, we see in this case that $N_+^1 = 0$.

So $N_b = 3$ is the correct number of boundary conditions in this case.

Let $G_2 = (B_{22}^{22})^{-1}(A_2^{22} - A_2^{21}(A_2^{11})^{-1}A_2^{12})$.

We use the following proposition which gives an equivalent condition for transverality of a constant layer $\underline{p}$.

PROPOSITION 6.3.2. ***([7], Proposition 2.4.(a)).*** *The constant layer $\underline{p}$ is transversal if and only if (i) the 3×3 matrix* $\begin{pmatrix} \Upsilon_2' G_2^{-1} \\ K_2 \end{pmatrix}$ *is injective on $\mathbb{E}_-(G_2(\underline{p}))$, and (ii) K_2 is of full rank = 1 on $\mathbb{E}_-(G_2(\underline{p}))$.*

$$G_2 = \begin{pmatrix} \frac{v\rho}{\mu} & 0 & 0 \\ 0 & \frac{\rho v^2 - \rho p_\rho}{(2\mu+\eta)v} & \frac{p_T}{2\mu+\eta} \\ 0 & \frac{p}{\kappa} & \frac{v\rho c_v}{\kappa} \end{pmatrix}$$

We first verify that G_2 is indeed invertible.

Suppose we have a 4×4 matrix A, of the form

$$\begin{pmatrix} \lambda & X \\ a_1 & Y \\ a_2 & Z \\ a_3 & W \end{pmatrix}$$

where X, Y, Z, W each are row vectors in $\mathbf{R}^3$ and $\lambda \neq 0, a_1, a_2, a_3$ are scalars.

Claim 6.3.3. If we assume that A is invertible then the 3×3 matrix, say

$$B = \begin{pmatrix} Y - \frac{a_1}{\lambda}X \\ Z - \frac{a_2}{\lambda}X \\ W - \frac{a_3}{\lambda}X \end{pmatrix}$$

is also invertible.

Proof. Assume the contrary. Then we have three scalars, a, b, c not all zero s.t. $a(Y - \frac{a_1}{\lambda}X) + b(Z - \frac{a_2}{\lambda}X) + c(W - \frac{a_3}{\lambda}X) = 0$. This implies that the 1×4 matrix

$$a\begin{pmatrix} a_1 - \lambda\frac{a_1}{\lambda} & Y - \frac{a_1}{\lambda}X \end{pmatrix} + b\begin{pmatrix} a_2 - \lambda\frac{a_2}{\lambda} & Z - \frac{a_2}{\lambda}X \end{pmatrix} + c\begin{pmatrix} a_3 - \lambda\frac{a_3}{\lambda} & W - \frac{a_3}{\lambda}X \end{pmatrix} = 0$$

i.e.

$$a\begin{pmatrix} a_1 & Y \end{pmatrix} + b\begin{pmatrix} a_2 & Z \end{pmatrix} + c\begin{pmatrix} a_3 & W \end{pmatrix} - (\frac{aa_1 + ba_2 + ca_3}{\lambda})\begin{pmatrix} \lambda & X \end{pmatrix} = 0.$$

But

$$a\begin{pmatrix} a_1 & Y \end{pmatrix} + b\begin{pmatrix} a_2 & Z \end{pmatrix} + c\begin{pmatrix} a_3 & W \end{pmatrix} - (\frac{aa_1 + ba_2 + ca_3}{\lambda})\begin{pmatrix} \lambda & X \end{pmatrix}$$

is a linear combination of the four rows of A which we have assumed to be linearly independent (A is assumed to be invertible). Therefore, $a = 0$, $b = 0$, $c = 0$. This proves the claim. $\square$

Since $A_2^{22} - A_2^{21}(A_2^{11})^{-1}A_2^{12}$ is of the same form as B above, and

$$G_2 = (B_{22}^{22})^{-1}(A_2^{22} - A_2^{21}(A_2^{11})^{-1}A_2^{12}),$$

we conclude that G_2 is invertible.

Next we verify (i) of Proposition 23.

Since G_2 is invertible, $\det(G_2) \neq 0$. We observe that

$$(G_2)^{-1} = \frac{1}{det(G_2)} \begin{pmatrix} \frac{\mu}{v\rho} & 0 & 0 \\ 0 & *_1 & *_2 \\ 0 & *_3 & *_4 \end{pmatrix}$$

for some values of $*_1, *_2, *_3, *_4$.

In our case $\Upsilon_2(u, v, T) = (u, v)$. So

$$\Upsilon_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$K_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Let $\alpha = \det(G_2)$. Now note that the 3×3 matrix

$$\begin{pmatrix} \Upsilon_2' G_2^{-1} \\ K_2 \end{pmatrix} = \begin{pmatrix} \frac{\mu}{v\rho\alpha} & 0 & 0 \\ 0 & \frac{*_1}{\alpha} & \frac{*_2}{\alpha} \\ 0 & 0 & 1 \end{pmatrix}.$$

Also observe that $*_1 = \frac{v^2\rho^2 c_v}{\mu\kappa}$ which is clearly nonzero owing to the assumption that v, ρ, c_v are all nonzero.

Thus

$$\begin{pmatrix} \Upsilon_2' G_2^{-1} \\ K_2 \end{pmatrix}$$

is injective as an operator on $\mathbf{R}^3$ and hence is injective on $\mathbf{E}_-(G_2)$. This shows that our profile satisfies (i) of Proposition 23.

$$A_0^{-1}(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ -\frac{E}{\rho c_v} + \frac{u^2}{\rho c_v} + \frac{v^2}{\rho c_v} & -\frac{u}{\rho c_v} & -\frac{v}{\rho c_v} & \frac{1}{\rho c_v} \end{pmatrix}$$

Thus,

$$\bar{A}_1(U) = \begin{pmatrix} u & \rho & 0 & 0 \\ \frac{p_\rho}{\rho} & u & 0 & \frac{p_T}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \frac{p}{\rho c_v} & 0 & u \end{pmatrix}$$

$$\bar{A}_2(U) = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ \frac{p_\rho}{\rho} & 0 & v & \frac{p_T}{\rho} \\ 0 & 0 & \frac{p}{\rho c_v} & v \end{pmatrix}$$

The characteristic polynomial $f(\lambda)$ of the matrix $\bar{A}_2$ is given by

$$\det(\lambda I - \bar{A}_2) = \begin{pmatrix} \lambda - v & 0 & -\rho & 0 \\ 0 & \lambda - v & 0 & 0 \\ -\frac{p_\rho}{\rho} & 0 & \lambda - v & -\frac{p_T}{\rho} \\ 0 & 0 & -\frac{p}{\rho c_v} & \lambda - v \end{pmatrix}$$

$$= (\lambda - v)^2[(\lambda - v)^2 - \frac{pp_T}{\rho^2 c_v} - p_\rho]$$

Thus the eigenvalues of $\bar{A}_2$, counted with multiplicities are: v, v, $v + \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$ and $v - \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$.

Since we are assuming that $v < 0$ and by subsonicity we have that $|v| < \sqrt{\frac{pp_t}{\rho^2 c_v} + p_\rho}$, we see that $\bar{A}_2$ has only one positive eigenvalue, viz., $v + \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$ and three negative eigenvalues.$\therefore N_+ = 1$. Also we have $\bar{A}_2{}^{11}$ has no positive eigenvalues $\Rightarrow N_+^1 = 0$.

$\therefore N_-^2 = N_b - N_+ = N' + N_+^1 - N_+ = 3 + 0 - 1 = 2$. Thus $\dim \mathbb{E}_-(G_2) = 2$.

Next we verify (ii) of Proposition 23.

Claim 6.3.4. K_2 is injective on $\mathbb{E}_-(G_2)$.

Proof. Suppose not. That would mean that every element in $\mathbb{E}_-(G_2) \subset \mathbb{R}^3$ is of the form $(x, y, 0)$. Now for $(x, y, 0) \in \mathbb{E}_-(G_2)$, $G_2 \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} (\frac{v\rho}{\mu})x \\ (\frac{\rho v^2 - \rho p_\rho}{(2\mu+\eta)v})y \\ (\frac{p}{\kappa})y \end{bmatrix}$.

But $\mathbb{E}_-(G_2)$ is invariant under G_2. Hence $(\frac{p}{\kappa})y = 0 \implies y = 0 \implies \mathbb{E}_-(G_2)$ is at most one dimensional which is a contradiction. This proves the claim. $\square$

This shows that our profile satisfies condition (ii) of Proposition 23. Thus constant profiles in this case are transversal.

6.3.1.2. *Lopatinski condition.* Next we want to establish the Lopatinski condition. We use the result from [**7**] that says that maximal dissipativity of Γ_{res} implies the Lopatinski condition. Thus in order to verify Lopatinski condition, we first check for maximal dissipativity of Γ_{res} defined later.

We first note that our system is symmetric dissipative with the corresponding symmetrizer S given by

$$S = \begin{pmatrix} \frac{p_\rho}{\rho^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{p_T c_v}{p} \end{pmatrix}$$

Definition 6.3.5. Γ_{res} is said to be maximally dissipative, if $S\bar{A}_2$ is negative definite on the kernel of Γ_{res}.

By Remark 2, the tangent space to the C-manifold of states q near a constant layer $\underline{p}$, $C_{\underline{p}}$ is $\dot{C}_{\underline{p}} = \{(\dot{q}^1, \dot{q}^2) : \dot{W}(z, (\dot{q}^1, \dot{q}^2))$ is a solution of the linearized profile problem (3.0.4), (3.0.5) with $(\dot{w}_1(z), \dot{w}_2(z)) \to (\dot{q}^1, \dot{q}^2)$ as $z \to \infty\}$.

By Prop. 5.5.5 of [**15**], we know that $C_{\underline{p}} = \ker \Gamma_{res}$.

Consider the linearized profile equation (3.0.4) at $\underline{p}$ with

$\dot{W} = (\dot{w}_1, \dot{w}_2, \dot{w}_3)$.

By part 2. of Lemma 4.6 of [**7**], for any $(q^1, q^2) \in \mathbb{R}^4$, equation 3.0.4 is integrated to yield a solution with $(\dot{w}_1(z), \dot{w}_2(z)) \to (q^1, q^2)$ as $z \to \infty$:

$$\dot{w}_1(z) = -(A_2^{11})^{-1} A_2^{12} e^{zG_2(\underline{p})} (G_2(\underline{p}))^{-1} r^2 + \dot{q}^1$$

$$\dot{w}_2(z) = e^{zG_2(\underline{p})} (G_2(\underline{p}))^{-1} r^2 + \dot{q}^2$$

$$\dot{w}_3(z) = e^{zG_2(\underline{p})} r^2, \text{ where } r^2 \in \mathbb{E}_-(G_2(\underline{p})).$$

In particular take an arbitrary $(\dot{q}^1, \dot{q}^2)$

Also we have $\Upsilon_1(\dot{\rho}) = 0$ and $\Upsilon_2(\dot{u}, \dot{v}, \dot{T}) = (\dot{u}, \dot{v})$.

Setting $\Upsilon_1'(p^1)\dot{w}_1(0) = 0$, $\Upsilon_2'(p^2)\dot{w}_2(0) = 0$ and $\Upsilon_3'(p^2)\dot{w}_3(0) = 0$, we get the following.

By the definition of Υ_1 and Υ_2, we see that $\dot{q}^1$ and the last coordinate of $\dot{q}^2$ could be arbitrary whereas the first two components of $(G_2(p)^{-1})r^2 + \dot{q}^2$ should be 0 where $r^2 \in \mathbf{E}_-(G_2(p))$. This implies that the first two components of $\dot{q}^2$ should be equal to $-\Upsilon_2'(p^2)(G_2(p))^{-1}r^2$.

We know that $\mathbf{E}_-(G_2(p))$ has a basis of the form $\{(1,0,0),(0,y,z)\}$ where $z \neq 0$. So for a typical $r^2 \in \mathbf{E}_-(G_2(p))$, we would have $r^2 = (a, by, bz)$ for some real constants a and b.

Since $\Upsilon_3'(p^2)\dot{w}_3(0) = 0$, we have $\Upsilon_3'(p^2)r^2 = 0$.

$\implies bz = 0$. But $z \neq 0 \implies b = 0$.

$\implies r^2 = (a, 0, 0)$.

$\implies$ that the second coordinate of $\dot{q}^2$ is 0.

In conclusion, we have

$\dot{C}_{\underline{p}} = \{(x, y, 0, z) : x, y, z \in \mathbb{R}\}$.

Thus, $\ker \Gamma_{res} = \{(x, y, 0, z) : x, y, z \in \mathbb{R}\}$

$$S\bar{A}_2 = \begin{pmatrix} \frac{vp_\rho}{\rho^2} & 0 & \frac{p_\rho}{\rho} & 0 \\ 0 & v & 0 & 0 \\ \frac{p_\rho}{\rho} & 0 & v & \frac{p_T}{\rho} \\ 0 & 0 & \frac{p_T}{\rho} & \frac{vp_Tc_v}{p} \end{pmatrix}$$

Let $(a, b, 0, c) \in C_{\underline{p}}$.

Then,

$$S\bar{A}_2 \begin{pmatrix} a \\ b \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} \frac{avp_\rho}{\rho^2} \\ bv \\ \frac{ap_\rho}{\rho} + \frac{zp_T}{\rho} \\ \frac{cvp_Tc_v}{p} \end{pmatrix} \tag{6.3.2}$$

Then $\left\langle S\bar{A}_2(\rho, u, 0, T), (\rho, u, 0, T)\right\rangle = a^2\frac{vp_\rho}{\rho^2} + b^2v + c^2\frac{vp_Tc_v}{p}$.

Since $v < 0$, the coefficients of a^2, b^2 and c^2 are all negative. Let $C = \max\{\frac{vp_\rho}{\rho^2}, v, \frac{vp_Tc_v}{p}\}$.

So $C < 0$.

Thus $\left\langle S\bar{A}_2(a, b, 0, c), (a, b, 0, c)\right\rangle \leq C\left\langle (a, b, 0, c), (a, b, 0, c)\right\rangle$ for all $(a, b, 0, c) \in C_{\underline{p}}$. Thus Γ_{res} is maximally dissipative.

This establishes the uniform Evans condition for low frequencies.

6.3.2. High frequencies.

$$S\bar{A}_1 = \begin{pmatrix} \frac{up_\rho}{\rho^2} & \frac{p_\rho}{\rho} & 0 & 0 \\ \frac{p_\rho}{\rho} & u & 0 & \frac{p_T}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \frac{p_T}{\rho} & 0 & \frac{up_Tc_v}{p} \end{pmatrix}$$

With $\lambda = i\tau + \gamma$ we consider the Fourier-Laplace transformed problem with coefficients evaluated at the constant layer p:

$$\lambda A_0 u + A_2 u' + i A_1 \eta_1 u - B_{22} u'' - i(B_{21} + B_{12})\eta_1 u' + \eta_1^2 B_{11} u = 0 \tag{6.3.3}$$

To make the system (6.3.3) symmetric-dissipative, we shall multiply (6.3.3) by the symmetrizer SA_0^{-1}. We'll first compute $\bar{B}_{jk}$'s.

$$\bar{B}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\mu+\eta}{\rho} & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\kappa}{\rho c_v} \end{pmatrix}$$

$$\bar{B}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu+\eta}{\rho} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\eta v}{\rho c_v} & -\frac{\eta u}{\rho c_v} & 0 \end{pmatrix}$$

$$\bar{B}_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mu+\eta}{\rho} & 0 & 0 \\ 0 & -\frac{\eta v}{\rho c_V} & \frac{\eta u}{\rho c_V} & 0 \end{pmatrix}$$

$$\bar{B}_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & \frac{2\mu+\eta}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\kappa}{\rho c_v} \end{pmatrix}$$

So

$$S\bar{B}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\mu+\eta}{\rho} & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & \frac{p_T\kappa}{\rho^2} \end{pmatrix}$$

$$S\bar{B}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu+\eta}{\rho} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{p_T\eta v}{\rho^2} & -\frac{p_T\eta u}{\rho^2} & 0 \end{pmatrix}$$

$$S\bar{B}_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mu+\eta}{\rho} & 0 & 0 \\ 0 & -\frac{p_T\eta v}{\rho^2} & \frac{p_T\eta u}{\rho^2} & 0 \end{pmatrix}$$

$$S\bar{B}_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & \frac{2\mu+\eta}{\rho} & 0 \\ 0 & 0 & 0 & \frac{p_T\kappa}{\rho^2} \end{pmatrix}$$

$$S^{\frac{1}{2}} = \begin{pmatrix} \frac{\sqrt{p_\rho}}{\rho} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{p_T c_v}}{\sqrt{p}} \end{pmatrix}$$

$$S^{-\frac{1}{2}} = \begin{pmatrix} \frac{\rho}{\sqrt{p_\rho}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{p}}{\sqrt{p_T c_v}} \end{pmatrix}$$

Thus multiplying (6.3.3) by SA_0^{-1} we get

$$\text{(6.3.4)} \qquad \lambda S u + S\bar{A}_2 u' + iS\bar{A}_1 \eta_1 u - S\bar{B}_{22} u'' - i(S\bar{B}_{21} + S\bar{B}_{12})\eta_1 u' + \eta_1^2 S\bar{B}_{11} u = 0$$

where now S is symmetric, positive definite and block diagonal and $S\bar{A}_1$ and $S\bar{A}_2$ are symmetric and the $S\bar{B}_{jk}$'s are dissipative.

We now want to keep all the properties of the coefficient matrices intact except that we want to be able to assume that S is the identity matrix. So we multiply each of the coefficient matrices on the left and right by $S^{-\frac{1}{2}}$ to obtain

(6.3.5)

$$\lambda u + S^{\frac{1}{2}}\bar{A}_2 S^{-\frac{1}{2}} u' + iS^{\frac{1}{2}}\bar{A}_1 S^{-\frac{1}{2}}\eta_1 u - S^{\frac{1}{2}}\bar{B}_{22} S^{-\frac{1}{2}} u'' - iS^{\frac{1}{2}}\bar{B}_{21} S^{-\frac{1}{2}} + S^{\frac{1}{2}}\bar{B}_{12} S^{-\frac{1}{2}}\eta_1 u'$$

$$+\eta_1^2 S^{\frac{1}{2}}\bar{B}_{11} S^{-\frac{1}{2}} u = 0$$

For convenience, let us denote $S^{\frac{1}{2}} P S^{-\frac{1}{2}}$ by $\tilde{P}$ for any 4×4 matrix P.

We can now rewrite (6.3.5) as

$$\lambda U + \tilde{A}_2 U' + i\tilde{A}_1 \eta_1 U - \tilde{B}_{22} U'' - i(\tilde{B}_{21} + \tilde{B}_{12})\eta_1 U' + \eta_1^2 \tilde{B}_{11} U = 0 \tag{6.3.6}$$

$$\tilde{A}_1 = \begin{pmatrix} u & \sqrt{p_\rho} & 0 & 0 \\ \sqrt{p_\rho} & u & 0 & \frac{\sqrt{p}p_T}{\rho\sqrt{c_v}} \\ 0 & 0 & u & 0 \\ 0 & \frac{\sqrt{p}p_T}{\rho\sqrt{c_v}} & 0 & u \end{pmatrix}$$

$$\tilde{A}_2 = \begin{pmatrix} v & 0 & \sqrt{p_\rho} & 0 \\ 0 & v & 0 & 0 \\ \sqrt{p_\rho} & 0 & v & \frac{\sqrt{p}p_T}{\rho\sqrt{c_v}} \\ 0 & 0 & \frac{\sqrt{p}p_T}{\rho\sqrt{c_v}} & v \end{pmatrix}$$

$$\tilde{B}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\mu+\eta}{\rho} & 0 & 0 \\ 0 & 0 & \frac{\mu}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\kappa}{\rho c_v} \end{pmatrix}$$

$$\tilde{B}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu+\eta}{\rho} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\eta v\sqrt{p_T}}{\rho\sqrt{p c_v}} & \frac{-\eta u\sqrt{p_T}}{\rho\sqrt{p c_v}} & 0 \end{pmatrix}$$

$$\tilde{B}_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mu+\eta}{\rho} & 0 & 0 \\ 0 & -\frac{\eta v\sqrt{p_T}}{\rho\sqrt{pc_v}} & \frac{\eta u\sqrt{p_T}}{\rho\sqrt{pc_v}} & 0 \end{pmatrix}$$

$$\tilde{B}_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho} & 0 & 0 \\ 0 & 0 & \frac{2\mu+\eta}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\kappa}{\rho c_v} \end{pmatrix}$$

Given $(\xi_1, \xi_2) \in \mathbb{R}$ such that $(\xi_1, \xi_2) \neq (0, 0)$,

$$\sum_{j,k=1}^{2} \xi_j \xi_k \tilde{B}_{jk}^{22} = \begin{pmatrix} \xi_1^2(\frac{2\mu+\eta}{\rho}) + \xi_2^2\frac{\mu}{\rho} & \xi_1\xi_2(\frac{\mu+\eta}{\rho}) & 0 \\ \xi_1\xi_2(\frac{\mu+\eta}{\rho}) & \xi_1^2\frac{\mu}{\rho} + \xi_2^2(\frac{2\mu+\eta}{\rho}) & 0 \\ 0 & 0 & (\xi_1^2 + \xi_2^2)\frac{\kappa}{\rho c_v} \end{pmatrix}$$

For later reference we record the first and second components of (6.3.6):

$$(a) \quad (i\tau + \gamma)\rho + v\rho' + \sqrt{p_\rho}v' + i\eta_1 u(\rho + \sqrt{p_\rho}) = 0$$

$$(b) \quad (i\tau + \gamma)\begin{pmatrix} u \\ v \\ \rho \end{pmatrix} + i\eta_1\rho\begin{pmatrix} \sqrt{p_\rho} \\ 0 \\ 0 \end{pmatrix} + i\eta_1\begin{pmatrix} u^2 + \frac{(\sqrt{pp_T})T}{\rho\sqrt{c_v}} \\ uv \\ \frac{u\sqrt{pp_T}}{\rho\sqrt{c_v}} + uT \end{pmatrix} + \rho'\begin{pmatrix} 0 \\ \sqrt{p_\rho} \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} vu' \\ vv' + \frac{(\sqrt{pp_T})T}{\rho c_v} \\ \frac{\sqrt{pp_T}v'}{\rho\sqrt{c_v}} + vT' \end{pmatrix} - \begin{pmatrix} \frac{\mu u''}{\rho} \\ \frac{(2\mu+\eta)v''}{\rho} \\ \frac{\kappa T''}{\rho c_v} \end{pmatrix} - i\eta_1\begin{pmatrix} \frac{(\mu+\eta)v'}{\rho} \\ \frac{(\mu+\eta)u'}{\rho} \\ 0 \end{pmatrix} + \eta^2\begin{pmatrix} \frac{(2\mu+\eta)u}{\rho} \\ \frac{\mu v}{\rho} \\ \frac{\kappa T}{\rho c_v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For now we will consider the part of 6.3.6 that corresponds to (3.17)(b) from [**8**]. So we consider:

the equation

$$\lambda U - \tilde{B}^{22}_{22} U'' - i(\tilde{B}^{22}_{21} + \tilde{B}^{22}_{12})\eta_1 U' + \eta_1^2 \tilde{B}^{22}_{11} U = 0 \tag{6.3.7}$$

Now let's write (6.3.7) as three separate equations.

(a) $\lambda u - \frac{\mu u''}{\rho} - \frac{i\eta_1(\mu+\eta)v'}{\rho} + \frac{\eta_1^2(2\mu+\eta)u}{\rho} = 0$

(b) $\lambda v - \frac{(2\mu+\eta)v''}{\rho} - \frac{i\eta_1(\mu+\eta)u'}{\rho} + \frac{\eta_1^2 \mu v}{\rho} = 0$

(c) $\lambda T - \frac{\kappa T''}{\rho c_v} + \frac{\eta_1^2 \kappa T}{\rho c_v} = 0$

Observe that (a) and (b) are decoupled from (c). By a modification of Proposition 3.8 from [**7**] we get that the uniform Evans condition holds for high frequencies.

Proposition 6.3.6. *Consider a layer profile as in (*5.0.1*) and the linearized equations about $w(z)$ given by(*5.0.10*). The uniform high-frequency Evans condition is satisfied for our Navier-Stokes equations with the boundary conditions:*

$$u_{|y=0} = g_1$$

$$v_{|y=0} = g_2$$

$$T'_{|} y = 0 = 0$$

Proof. By Corollary 3.7 from [**5**] and Remark 17, the uniform high-frequency Evans condition in the case of the given boundary conditions is equivalent to the estimate

$$|u'(0)| + |v'(0)| + |T(0)| \leq C(|u(0)| + |v(0)| + |T'(0)|) \tag{6.3.8}$$

for decaying solutions of (6.3.7) where the coefficients are evaluated at $w(0)$ with the boundary conditions

$$u_{|y=0} = g_1$$

$$v_{|y=0} = g_2$$

$$T'_{|}y = 0 = 0$$

where the constant C in (6.3.8) is independent of (λ, η) in the positive parabolic unit sphere, $\gamma \geq 0$, $|\lambda| + \eta_1^2 = 1$.

Taking the real part of the inner product of (u, v) with equations (a) and (b), we obtain after integrating w.r.t z from 0 to ∞:

$$(\gamma + \eta_1^2)(\|u\|^2 + \|v\|^2) + \|u'\|^2 + \|v'\|^2 \leq C(\eta_1 |u(0)||v(0)| + |u(0)||u'(0)| + |v(0)||v'(0)|) \tag{6.3.9}$$

Similarly, taking the real part of the inner product of $-(u'', v'')$ with (a) and (b), we obtain:

$$(\gamma + \eta_1^2)(\|u'\|^2 + \|v'\|^2) + \|u''\|^2 + \|v''\|^2 \leq C((|\lambda| + \eta_1^2)(|u'(0)||u(0)| + |v'(0)||v(0)|) + |\eta_1||u'(0)||v'(0)|) \tag{6.3.10}$$

Using the Sobolev bound, we get

$$
\begin{aligned}
|u'(0)|^2 + |v'(0)|^2 \le \|u'\|\|u''\| + \|v'\|\|v''\| &\le C_\delta(\|u'\|^2 + \|v'\|^2) \\
&\quad +\delta(\|u''\|^2 + \|v''\|^2) \\
&\le C(C_\delta(\eta_1|u(0)||v(0)| + |u(0)||u'(0)| + |v(0)||v'(0)|) \\
+\delta((|\lambda| + \eta_1^2)(|u'(0)||u(0)| &+ |v'(0)||v(0)|) + |\eta_1||u'(0)||v'(0)|)
\end{aligned} \tag{6.3.11}
$$

Since (λ, η_1) lie on the positive unit parabolic sphere, we can conclude that there exists $C > 0$ such that

$$|u'(0)| + |v'(0)| \le C(|u(0)| + |v(0)|). \tag{6.3.12}$$

Taking inner product of (c) with T and integrating by parts as above, but now taking both real parts and imaginary parts separately and then combining them, we get a $C > 0$ such that

$$(|\lambda| + \eta_1^2)\|T\|^2 + \|T'\|^2 \le C|T(0)||T'(0)| \tag{6.3.13}$$

Similarly, pairing $-T''$ with (c), we get,

$$(|\lambda| + \eta_1^2)\|T'\|^2 + \|T''\|^2 \le C|T(0)||T'(0)|. \tag{6.3.14}$$

As above, using the Sobolev embedding,

$$|T(0)|^2 \le \|T\|\|T'\| \tag{6.3.15}$$

we get a $C > 0$ such that

$$|T(0)| \leq C|T'(0)|. \tag{6.3.16}$$

Combining (6.3.12) and (6.3.16), we get that there exists $C > 0$ such that (6.3.8) holds.

□

6.3.3. Medium frequencies. By Remark 17, the uniform Evans condition is:

$$|\rho(0)| + |T(0)| + |u'(0)| + |v'(0)| \leq C(|u(0)| + |v(0)| + |T'(0)|) \tag{6.3.17}$$

Pairing (6.3.6) with U, we get,

(6.3.18)

$$\begin{aligned}
&\rho\bar{\lambda}\bar{\rho} + \rho v_0\bar{\rho}' + \rho\sqrt{p^0_\rho}\bar{v}' - i\rho\eta_1 u_0\bar{\rho} - i\rho\eta_1\sqrt{p^0_\rho}\bar{u} + \bar{\lambda}u\bar{u} + v_0 u\bar{u}' - i\eta_1\sqrt{p^0_\rho}u\bar{\rho} - i\eta_1 u_0 u\bar{u} - \\
&i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}u\bar{T} - \frac{\mu}{\rho_0}u\bar{u}'' + i\eta_1\frac{\mu+\eta}{\rho_0}u\bar{v}' + \eta_1^2\frac{2\mu+\eta}{\rho_0}u\bar{u} + \bar{\lambda}v\bar{v} + \sqrt{p^0_\rho}v\bar{\rho}' + v_0 v\bar{v}' + \\
&\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}v\bar{T}' - i\eta_1 u_0 v\bar{v} - \frac{2\mu+\eta}{\rho_0}v\bar{v}'' + i\eta_1\frac{\mu+\eta}{\rho_0}v\bar{u}' + \eta_1^2\frac{\mu}{\rho_0}v\bar{v} + \bar{\lambda}T\bar{T} + \\
&\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}T\bar{v}' + v_0 T\bar{T}' - i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}T\bar{u} - i\eta_1 u_0 T\bar{T} - \frac{\kappa}{\rho_0 c_{v_0}}T\bar{T}'' + \eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}T\bar{T} = 0
\end{aligned}$$

Rewriting (6.3.18) we get,

(6.3.19)

$$\bar{\lambda}\rho\bar{\rho} + v_0\rho\bar{\rho}' + \sqrt{p^0_\rho}\rho\bar{v}' - i\eta_1 u_0\rho\bar{\rho} - i\eta_1\sqrt{p^0_\rho}\rho\bar{u} + \bar{\lambda}u\bar{u} + v_0 u\bar{u}' - i\eta_1\sqrt{p^0_\rho}u\bar{\rho} - i\eta_1 u_0 u\bar{u} -$$
$$i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}u\bar{T} - \frac{\mu}{\rho_0}u\bar{u}'' + i\eta_1\frac{\mu+\eta}{\rho_0}u\bar{v}' + \eta_1^2\frac{2\mu+\eta}{\rho_0}u\bar{u} + \bar{\lambda}v\bar{v} + \sqrt{p^0_\rho}v\bar{\rho}' + v_0 v\bar{v}' +$$
$$\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}v\bar{T}' - i\eta_1 u_0 v\bar{v} - \frac{2\mu+\eta}{\rho_0}v\bar{v}'' + i\eta_1\frac{\mu+\eta}{\rho_0}v\bar{u}' + \eta_1^2\frac{\mu}{\rho_0}v\bar{v} + \bar{\lambda}T\bar{T} +$$
$$\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}T\bar{v}' + v_0 T\bar{T}' - i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}T\bar{u} - i\eta_1 u_0 T\bar{T} - \frac{\kappa}{\rho_0 c_{v_0}}T\bar{T}'' + \eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}T\bar{T} = 0$$

That is,

(6.3.20)

$$\bar{\lambda}|\rho|^2 + v_0\rho\bar{\rho}' + \sqrt{p^0_\rho}\rho\bar{v}' - i\eta_1 u_0|\rho|^2 - i\eta_1\sqrt{p^0_\rho}\rho\bar{u} + \bar{\lambda}|u|^2 + v_0 u\bar{u}' - i\eta_1\sqrt{p^0_\rho}u\bar{\rho} -$$
$$i\eta_1 u_0|u|^2 - i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}u\bar{T} - \frac{\mu}{\rho_0}u\bar{u}'' + i\eta_1\frac{\mu+\eta}{\rho_0}u\bar{v}' + \eta_1^2\frac{2\mu+\eta}{\rho_0}|u|^2 + \bar{\lambda}|v|^2 + \sqrt{p^0_\rho}v\bar{\rho}' +$$
$$v_0 v\bar{v}' + \frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}v\bar{T}' - i\eta_1 u_0|v|^2 - \frac{2\mu+\eta}{\rho_0}v\bar{v}'' + i\eta_1\frac{\mu+\eta}{\rho_0}v\bar{u}' + \eta_1^2\frac{\mu}{\rho_0}|v|^2 + \bar{\lambda}|T|^2 +$$
$$\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}T\bar{v}' + v_0 T\bar{T}' - i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}T\bar{u} - i\eta_1 u_0|T|^2 - \frac{\kappa}{\rho_0 c_{v_0}}T\bar{T}'' + \eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}|T|^2 = 0$$

That is,

$$\bar{\lambda}(|\rho|^2 + |u|^2 + |v|^2 + |T|^2) + v_0(\rho\bar{\rho}' + u\bar{u}' + v\bar{v}' + T\bar{T}') + \sqrt{p^0_\rho}(\rho\bar{v}' + v\bar{\rho}') -$$
$$i\eta_1 u_0(|\rho|^2 + |u|^2 + |v|^2 + |T|^2) - i\eta_1\sqrt{p^0_\rho}(\rho\bar{u} + u\bar{\rho}) - i\eta_1\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}(u\bar{T} + T\bar{u}) -$$
$$\frac{\mu}{\rho_0}u\bar{u}'' - \frac{2\mu+\eta}{\rho_0}v\bar{v}'' - \frac{\kappa}{\rho_0 c_{v_0}}T\bar{T}'' + i\eta_1\frac{\mu+\eta}{\rho_0}(u\bar{v}' + v\bar{u}') + \eta_1^2\frac{2\mu+\eta}{\rho_0}|u|^2 + \quad (6.3.21)$$
$$\frac{\sqrt{p^0 p^0_T}}{\rho_0\sqrt{c_{v_0}}}(v\bar{T}' + T\bar{v}') + \eta_1^2\frac{\mu}{\rho_0}|v|^2 + \eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}|T|^2 = 0$$

Taking the real part of (6.3.21), we get,

$$\gamma(|\rho|^2+|u|^2+|v|^2+|T|^2)+\eta_1^2\frac{2\mu+\eta}{\rho_0}|u|^2+\eta_1^2\frac{\mu}{\rho_0}|v|^2+\eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}|T|^2+$$

$$v_0\Re(\rho\bar{\rho'}+u\bar{u'}+v\bar{v'}+T\bar{T'})+\sqrt{p_\rho^0}\Re(\rho\bar{v'}+v\bar{\rho'})+\Re(i\eta_1\frac{\mu+\eta}{\rho_0}(u\bar{v'}+v\bar{u'}))+ \tag{6.3.22}$$

$$\frac{\sqrt{p^0p_T^0}}{\rho_0\sqrt{c_{v_0}}}\Re(v\bar{T'}+T\bar{v'})-\frac{\mu}{\rho_0}\Re(u\bar{u''})-\frac{2\mu+\eta}{\rho_0}\Re(v\bar{v''})-\frac{\kappa}{\rho_0 c_{v_0}}\Re(T\bar{T''})=0$$

Taking integral of (6.3.22) w.r.t. z from 0 to ∞, we get,

(6.3.23)

$$\gamma(\|\rho\|_{L^2}^2+\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|T\|_{L^2}^2)+\eta_1^2\frac{2\mu+\eta}{\rho_0}\|u\|_{L^2}^2+\eta_1^2\frac{\mu}{\rho_0}\|v\|_{L^2}^2+\eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}\|T\|_{L^2}^2+$$

$$v_0\Re\int_0^\infty(\rho\bar{\rho'}+u\bar{u'}+v\bar{v'}+T\bar{T'})+\sqrt{p_\rho^0}\Re\int_0^\infty(\rho\bar{v'}+v\bar{\rho'})+\Re(i\eta_1\frac{\mu+\eta}{\rho_0}\int_0^\infty(u\bar{v'}+v\bar{u'}))+$$

$$\frac{\sqrt{p^0p_T^0}}{\rho_0\sqrt{c_{v_0}}}\Re\int_0^\infty(v\bar{T'}+T\bar{v'})-\frac{\mu}{\rho_0}\Re\int_0^\infty(u\bar{u''})-\frac{2\mu+\eta}{\rho_0}\Re\int_0^\infty(v\bar{v''})-$$

$$\frac{\kappa}{\rho_0 c_{v_0}}\Re\int_0^\infty(T\bar{T''})=0$$

That is,

(6.3.24)

$$\gamma(\|\rho\|_{L^2}^2+\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|T\|_{L^2}^2)+\eta_1^2\frac{2\mu+\eta}{\rho_0}\|u\|_{L^2}^2+\eta_1^2\frac{\mu}{\rho_0}\|v\|_{L^2}^2+$$

$$\eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}\|T\|_{L^2}^2+\|u'\|_{L^2}^2+\|v'\|_{L^2}^2+\|T'\|_{L^2}^2-\frac{1}{2}v_0(|\rho(0)|^2+|u(0)|^2+|v(0)|^2+|T(0)|^2)-$$

$$\sqrt{p_\rho^0}\Re(\rho(0)\overline{v(0)})-\Re(i\eta_1\frac{\mu+\eta}{\rho_0}u(0)\overline{v(0)})-\frac{\sqrt{p^0p_T^0}}{\rho_0\sqrt{c_{v_0}}}\Re(v(0)\overline{T(0)})+\frac{\mu}{\rho_0}\Re(u(0)\overline{u'(0)})+$$

$$\frac{2\mu+\eta}{\rho_0}\Re(v(0)\overline{v'(0)})+\frac{\kappa}{\rho_0 c_{v_0}}\Re(T(0)\overline{T'(0)})=0$$

That is,

(6.3.25)
$$\gamma\|U\|_{L^2}^2+\eta_1^2\frac{2\mu+\eta}{\rho_0}\|u\|_{L^2}^2+\eta_1^2\frac{\mu}{\rho_0}\|v\|_{L^2}^2+\eta_1^2\frac{\kappa}{\rho_0 c_{v_0}}\|T\|_{L^2}^2+\|U_2'\|_{L^2}^2-\frac{1}{2}v_0|U(0)|^2-$$
$$\sqrt{p_\rho^0}\Re(\rho(0)\overline{v(0)})-\Re(i\eta_1\frac{\mu+\eta}{\rho_0}u(0)\overline{v(0)})-\frac{\sqrt{p^0p_T^0}}{\rho_0\sqrt{c_{v_0}}}\Re(v(0)\overline{T(0)})+\frac{\mu}{\rho_0}\Re(u(0)\overline{u'(0)})+$$
$$\frac{2\mu+\eta}{\rho_0}\Re(v(0)\overline{v'(0)})+\frac{\kappa}{\rho_0 c_{v_0}}\Re(T(0)\overline{T'(0)})=0$$

Thus, $\exists$ a constant $C>0$, such that

$$\gamma\|U\|_{L^2}^2+\eta_1^2\|U_2\|_{L^2}^2+\|U_2'\|_{L^2}^2+|U(0)|^2$$
$$\le C(\sqrt{p_\rho^0}\Re(\rho(0)\overline{v(0)})+\Re(i\eta_1\frac{\mu+\eta}{\rho_0}u(0)\overline{v(0)})+\frac{\sqrt{p^0p_T^0}}{\rho_0\sqrt{c_{v_0}}}\Re(v(0)\overline{T(0)})-$$
(6.3.26)
$$\frac{\mu}{\rho_0}\Re(u(0)\overline{u'(0)})-\frac{2\mu+\eta}{\rho_0}\Re(v(0)\overline{v'(0)})-\frac{\kappa}{\rho_0 c_{v_0}}\Re(T(0)\overline{T'(0)}))$$
$$\le C(|\rho(0)||v(0)|+|u(0)||v(0)|+|v(0)||T(0)|+|u(0)||u'(0)|+|v(0)||v'(0)|+$$
$$|T(0)||T'(0)|)$$

Thus we have,

$$\gamma\|U\|_{L^2}^2+\eta_1^2\|U_2\|_{L^2}^2+\|U_2'\|_{L^2}^2+|U(0)|^2\le C(|\rho(0)||v(0)|+|u(0)||v(0)|$$
(6.3.27)
$$+|v(0)||T(0)|$$
$$+|u(0)||u'(0)|+|v(0)||v'(0)|+|T(0)||T'(0)|)$$

Thus we get,

$$
\begin{aligned}
\gamma\|U\|_{L^2}^2+\eta_1^2\|U_2\|_{L^2}^2+\|U_2'\|_{L^2}^2+|U(0)|^2 \le C(|u(0)|^2+|v(0)|^2+|T'(0)|^2)+ \\
\delta(|u'(0)|^2+|v'(0)|^2)
\end{aligned} \tag{6.3.28}
$$

for $\delta > 0$ sufficiently small.

So we have,

$$
\begin{aligned}
|\rho(0)|^2+|u(0)|^2+|v(0)|^2+|T(0)|^2 \le C(|u(0)|^2+|v(0)|^2+|T'(0)|^2) \\
+\delta(|u'(0)|^2+|v'(0)|^2)
\end{aligned} \tag{6.3.29}
$$

Also by (4.25) from [**8**], we know

$$
|\rho(0)|^2+|u'(0)|^2+|v'(0)|^2+|T'(0)|^2 \le C(|u(0)|^2+|v(0)|^2+|T(0)|^2) \tag{6.3.30}
$$

From (6.3.30), we get

$$
|u'(0)|^2+|v'(0)|^2 \le C(|u(0)|^2+|v(0)|^2+|T(0)|^2) \tag{6.3.31}
$$

Combining (6.3.29) and (6.3.31) we have,

$$
|\rho(0)|^2+|T(0)|^2 \le C(|u(0)|^2+|v(0)|^2+|T'(0)|^2) \tag{6.3.32}
$$

Adding (6.3.29) and (6.3.31),

(6.3.33)

$$|\rho(0)|^2 + |u(0)|^2 + |v(0)|^2 + |T(0)|^2 + |u'(0)|^2 + |v'(0)|^2 \leq C(|u(0)|^2 + |v(0)|^2 + |T'(0)|^2) + \delta(|u'(0)|^2 + |v'(0)|^2)$$

Combining (6.3.31) and (6.3.33),

(6.3.34)

$$|\rho(0)|^2 + |u(0)|^2 + |v(0)|^2 + |T(0)|^2 + |u'(0)|^2 + |v'(0)|^2 \leq C(|u(0)|^2 + |v(0)|^2 + |T'(0)|^2) + \delta(|u(0)|^2 + |v(0)|^2 + |T(0)|^2)$$

Thus,

$$|\rho(0)|^2 + |u(0)|^2 + |v(0)|^2 + |T(0)|^2 + |u'(0)|^2 + |v'(0)|^2 \leq C(|u(0)|^2 + |v(0)|^2 + |T'(0)|^2) \tag{6.3.35}$$

Thus,

$$|\rho(0)|^2 + |u(0)|^2 + |v(0)|^2 + |T(0)|^2 + |u'(0)|^2 + |v'(0)|^2 \leq C(|u(0)|^2 + |v(0)|^2 + |T'(0)|^2) \tag{6.3.36}$$

Thus,

$$|\rho(0)|^2 + |u'(0)|^2 + |v'(0)|^2 + |T(0)|^2 \leq C(|u(0)|^2 + |v(0)|^2 + |T'(0)|^2) \tag{6.3.37}$$

This establishes the Evans condition for all frequencies.

Thus we have also proved the following corollary:

Corollary 6.3.7. *For our system of Navier Stokes equations with the given boundary conditions, under hypotheses (H1) - (H6), but with (H4') replacing (H4), we have the following:*

Given a smooth global assignment of states $p(t,x)$*, there exists a* $\mathcal{C}$ *manifold satisfying Assumption 12 with* $p(t,x) \in \mathcal{C}(t,x) \subset \mathfrak{U}$ *for all* (t,x)*, and associated small amplitude profiles* $W(z,t,x,q)$ *satisfying the uniform Evans condition on* $[-T,T] \times \partial\Omega$*. The manifold* $\mathcal{C}$ *defines a residual hyperbolic boundary condition.*

Given initial data v^0 *satisfying appropriate corner compatibility conditions for the hyperbolic problem, there exists an inviscid solution* U^0*, an approximate solution* u_a^ϵ*, and as exact boundary layer solution* u^ϵ *satisfying all the conclusions of Theorem 19 for constants* s_0*,* k *and* M *as described there.*

6.4. Subsonic, inflow

Just like in the previous case, we will check for Uniform Evans condition for all frequencies for constant profiles.

Consider G_2 as before. We need to determine if in this case, (i) and (ii) of Proposition 23 hold in order to establish transversality.

When we showed that $\begin{pmatrix} \gamma_2' G_2^{-1} \\ K_2 \end{pmatrix}$ is injective on $\mathbb{E}^-(G_2)$, we didn't use subsonicity or outflow in the proof. Thus in this case too, we can conclude that (i) of Proposition 23 holds.

We know that the eigenvalues of $\bar{A}_2$, counted with multiplicities are: v, v, $v + \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$ and $v - \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$.

Now inflow implies that $v > 0$ and subsonic implies that $|v| < \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$. This means that $N_+ = 3$. Also $N_+^1 = 1$.

$\therefore N_-^2 = N_b - N_+ = N' + N_+^1 - N_+ = 3 + 1 - 3 = 1$. Thus $dim \mathbb{E}_-(G_2) = 1$.

By the third equation of (4.54) of [**7**] we know that whichever r_2 works should have its last coordinate $= 0$. Since v is positive in this case and $\mathbb{E}^-(G_2)$ is one dimensional and invariant under G_2, we have that $\mathbb{E}_-(G_2)$ should be generated by a vector $(0, y, z)$.

CLAIM 6.4.1. $(0, 1, 0) \notin \mathbb{E}_-(G_2)$.

Proof. If not, then since $\dim \mathbb{E}_-(G_2) = 1$ and $\mathbb{E}_-(G_2)$ is invariant under G_2, we would get the second column of G_2 to be a multiple of $(0, 1, 0)$ which is a contradiction since the $(3, 2)$th entry of G_2 is $\frac{p}{\kappa}$ which is known to be non-zero. This proves the claim. □

Thus $z \neq 0$. This proves that K_2 is of full rank on $\mathbb{E}_-(G_2)$. Thus (ii) of Proposition 23 holds and we have that constant profiles are transversal in this case as well.

Also, we get that the only r_2 that works in (4.54) in [**7**] is $(0, 0, 0)$.

Now $N_+^1 = 1 \implies N_b = 4$ and the boundary conditions are

$$\rho(0) = g_1$$

$$u(0) = g_2$$

$$v(0) = g_3$$

$$T'(0) = 0 \tag{6.4.1}$$

Thus by the definition of v_1 and v_2 in this case, we get that $q_1 = 0$ and the first two coordinates of q_2 are both 0. But since there is no restriction on the last coordinate of q_2, we get that $\dot{C}_{\nu,p} = \mathrm{Span}_{\mathbb{R}}\{(0,0,0,1)\}$.

To verify maximal dissipativity of residual boundary conditions for constant profiles, we need to check for the negative definiteness of $S\bar{A}_2$ on $\dot{C}_{\nu,p}$.

For $(0,0,0,T) \in \dot{C}_{\nu,p}$, $\left\langle S\bar{A}_2(0,0,0,T),(0,0,0,T)\right\rangle = \frac{vp_Tc_v}{p}T^2 =$

$C\langle(0,0,0,T),(0,0,0,T)\rangle$ where $C = \frac{vp_Tc_v}{p}T^2$ which is positive due to our assumption that $v > 0$. Thus we see that in this case, the residual boundary condition is not maximally dissipative.

Since maximal dissipativity is stronger than the Uniform Lopatinski condition, we still don't know if the Uniform Lopatinski condition fails.

Thus we need to find a $C > 0$ independent of p such that for all ζ with $\gamma > 0$,

$$|\det(\mathbb{E}_-(H(p,\zeta)), \ker\Gamma_{res}(p))| \geq C.$$

We attempt to check whether Uniform Lopatinski condition fails at $\zeta = (\lambda, 0)$. Fix such a ζ, then the Lopatinski determinant for this ζ is

$|\det(\mathbb{E}_-(-\lambda\bar{A}_2^{-1}), \ker\Gamma_{res}(p))|$.

Since $\gamma > 0$, $\mathbb{E}_-(-\lambda\bar{A}_2^{-1}) = \mathbb{E}_+(\lambda\bar{A}_2^{-1})$.

Since $\bar{A}_2$ is invertible and $\gamma > 0$, $\mathbb{E}_+(\lambda\bar{A}_2^{-1}) = \mathbb{E}_+(\bar{A}_2)$.

By inspection, we see that $(0,1,0,0)$, $(1/p_\rho, 1, 0, -1/p_T)$ and

$(\rho, 0, c, p/(\rho c_v))$ form a basis of $\mathbb{E}_+(\bar{A}_2)$. We also note that $(0,0,0,1) \notin \mathbb{E}_+(\bar{A}_2)$.

By continuity of the Lopatinski determinant,$\exists C > 0$ and $\exists\delta > 0$ such that for $|\eta_1| < \delta$,

$$|\det(\mathbb{E}_-(H(p,\zeta)), \ker\Gamma_{res}(p))| \geq C.$$

But $H(p,\zeta)$ being linear in ζ, we see that for any $\alpha > 0$,

$|\det(\mathbb{E}_-(H(p,\alpha\zeta)), \ker\Gamma_{res}(p))| \geq C$ for the above C and ζ with $|\eta_1| < \delta$.

Thus the Lopatinski condition holds for some ζ. But we still don't know if it does for all the ζ required in the definition of the Lopatinski condition.

6.5. Supersonic, outflow

As we saw in the last section we already know that (i) of Proposition 23 holds in this case as well. So in order to determine the transversality of a constant profile, we only need to establish (ii).

In this case, outflow implies that $v < 0$ and supersonic implies that $|v| > \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$. This means that $N_+ = 0 = N_+^1$.

$\therefore N_-^2 = N_b - N_+ = N' + N_+^1 - N_+ = 3 + 0 - 0 = 3$. Thus $\dim \mathbb{E}_-(G_2) = 3$.

Now since $(1,0,0) \in \mathbb{E}_-(G_2)$ we know that K_2 must be of full rank on $\mathbb{E}_-(G_2)$. This shows that constant profiles are transversal in this case.

In order to determine the uniform Evans condition for low frequencies we need to determine maximum dissipativity which in turn would imply that the uniform Lopatinski condition holds.

In this case we get that the tangent space to the C-manifold is 4 dimensional and hence is $\mathbb{R}^4$. But maximal dissipativity holds just as it does for the subsonic outflow case. Thus low frequency Evans condition holds in this case as well.

The proof for verification of Evans condition is similar to that in the subsonic case.

6.6. Supersonic, inflow

As we saw in the last section we already know that (i) of Proposition 23 holds in this case as well. So in order to determine the transversality of a constant profile, we only need to establish (ii).

In this case, inflow implies that $v > 0$ and supersonic implies that $|v| > \sqrt{\frac{pp_T}{\rho^2 c_v} + p_\rho}$. This means that $N_+ = 4$ and that $N_+^1 = 1$.

$\therefore N_-^2 = N_b - N_+ = N' + N_+^1 - N_+ = 3 + 1 - 4 = 0$. Thus $\dim \mathbb{E}_-(G_2) = 0$. Thus K_2 is not of full rank on $\mathbb{E}_-(G_2)$. Thus transversality fails in this case and thus uniform Evans condition does not hold.

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