# SATURATION PROBLEM FOR AFFINE KAC-MOODY ALGEBRAS

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## ABSTRACT

Merrick Brown: Saturation problem for affine Kac-Moody algebras
(Under the direction of Shrawan Kumar)

This thesis is a study of the saturated tensor cones of the affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$. We show that the occurrence of certain components in the tensor product of two highest weight integrable representations implies the occurrence of other components. For $A_1^{(1)}$ and $A_2^{(2)}$, we are able to prove the occurrence of enough components to explicitly determine the saturated tensor cone and saturation factors. Moreover, in these two cases, we show that the saturated tensor cone is given by the inequalities conjectured in [2].

To my teachers.

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# INTRODUCTION

For an affine Kac-Moody algebra $\mathfrak{g}$ with Cartan $\mathfrak{h}$, the integrable weight modules with a highest weight are in bijective correspondence with the set of dominant integral weights $P_+ \subset \mathfrak{h}^*$. As such, for a dominant integral $\lambda \in P_+$ we write the integrable, highest weight (irreducible) representation with highest weight $\lambda$ as $L(\lambda)$. Define the *tensor product semigroup* to be the set

$$\overline{\Gamma} := \left\{(\lambda, \mu, \nu) \in P_+^3 \,|\, L(\nu) \subset L(\lambda) \otimes L(\mu)\right\}.$$

Define the *saturated tensor cone* as

$$\Gamma := \left\{(\lambda, \mu, \nu) \in P_+^3 \,|\, \exists N \in \mathbb{Z}_{>0} \text{ such that } (N\lambda, N\mu, N\nu) \in \overline{\Gamma}\right\}.$$

We say that an integer $d > 0$ is a *saturation factor* (for $\mathfrak{g}$) if for any $(\lambda, \mu, \nu) \in \Gamma$ such that $\lambda + \mu - \nu \in Q$, where $Q$ is the root lattice of $\mathfrak{g}$ , then $(d\lambda, d\mu, d\nu) \in \overline{\Gamma}$.

The genesis of this work was the question: Do saturation factors exist for affine Kac-Moody algebras? For semisimple Lie algebras, the answer is affirmative, and for any given simple $\mathfrak{g}$, some saturation factors are known. One part of this thesis is the computation of saturation factors for the Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$. The original question is still open, but these results are the first saturation factors known for any infinite dimensional Kac-Moody algebra.

Our method for determining the saturated tensor cone of an affine Kac-Moody algebra is as follows: For an affine Kac-Moody algebra $\mathfrak{g}$, the decomposition of $L(\lambda) \otimes L(\mu)$ with respect to the derived subalgebra $\mathfrak{g}'$ gives a formally simple answer, although the multiplicity of each $\mathfrak{g}'$-submodule may be infinite. Each of these "multiplicity spaces" - $\mathrm{Hom}_{\mathfrak{g}'}(L(\lambda) \otimes L(\mu), L(\nu))$ - is an unitarizable representation of the Virasoro algebra. Using basic properties of these Virasoro representations, one finds that the occurrence of a $\mathfrak{g}$-submodule implies the occurrence of others. Hence, as we show, determining the saturated tensor cone relies on proving the occurrence of components that we call $\delta$-maximal components.

We then use the Kac-Weyl character formula to write the multiplicity of a component as a complicated alternating sum of power series. Determining the set of $\delta$-maximal components requires a delicate analysis of these sums of power series, in particular, one must determine their lowest degree term. For the affine Kac-Moody algebra $A_1^{(1)}$, this is possible because cancellation in the low degree terms can be controlled (cf 4.1.11). For $A_2^{(2)}$, cancellation in low degree terms can be controlled only when the highest weights of the representations are large enough sums of the fundamental weights. In either case, enough $\delta$-maximal components can be ascertained to fully compute the saturated tensor cone and saturation factors. In higher rank, $A_3^{(1)}$ for instance, cancellation is unavoidable, thus some other method for determining $\delta$-maximal components must be brought to bear.

Finally, in the cases above, we give a geometric interpretation of the results. Let $G$ Kac-Moody group (cf. [7, ch.6]) associated to $\mathfrak{g}$. Let $B$ be the standard positive Borel subgroup of $G$. Write $X := G/B$. For $\mathfrak{g} = A_1^{(1)}$, $A_2^{(2)}$ we show that $\Gamma \subset P_+^3$ is cut out by an infinite collection of linear inequalities, and that these inequalities are indexed by certain products in the singular cohomology ring $H^*(X)$. Moreover, we show that there is a subset of inequalities that suffice to determine $\Gamma$. This smaller set is analogous to those in the finite case, as described in [1].

## CHAPTER 1: AFFINE KAC-MOODY ALGEBRAS

This chapter outlines the definition, structure, and representations of affine Kac-Moody algebras. All algebras are assumed to be over $\mathbb{C}$, unless otherwise noted.

### 1.1 Definition, root space decomposition, and Weyl group

Fix $\overset{\circ}{\mathfrak{g}}$ a simple Lie algebra. Consider the invariant bilinear form $\langle \cdot, \cdot \rangle_{\overset{\circ}{\mathfrak{g}}}$ on $\overset{\circ}{\mathfrak{g}}$ normalized so that $\langle \theta, \theta \rangle_{\overset{\circ}{\mathfrak{g}}} = 2$, for $\theta$ the highest root of $\overset{\circ}{\mathfrak{g}}$. Define the *untwisted affine Kac-Moody algebra associated to* $\overset{\circ}{\mathfrak{g}}$ as

$$\mathfrak{g} := \left( \mathbb{C}[t, t^{-1}] \otimes \overset{\circ}{\mathfrak{g}} \right) \oplus \mathbb{C}d \oplus \mathbb{C}c$$

with the Lie bracket

$$\begin{aligned} \left[ t^m \otimes x + \mu d + zc, t^{m'} \otimes x' + \mu' d + z'c \right] = \\ t^{m+m'}[x, x'] + \mu m' t^{m'} \otimes x' - \mu' m t^m \otimes x + m\delta_{-m,m'} \langle x, x' \rangle_{\overset{\circ}{\mathfrak{g}}} c. \end{aligned} \tag{1.1}$$

We call a Kac-Moody algebra *affine* if its generalized Cartan matrix is positive semi-definite and has corank 1. Fix $\overset{\circ}{\mathfrak{h}} \subset \overset{\circ}{\mathfrak{g}}$ a Cartan subalgebra of $\overset{\circ}{\mathfrak{g}}$. Define $\mathfrak{h} := \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}d \oplus \mathbb{C}c$. Write $\mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$, where $\delta$ is defined by $\delta(d) = 1$, $\delta|_{\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c} = 0$ and $\Lambda_0$ by $\Lambda_0(c) = 1$, $\Lambda_0|_{\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}d} = 0$. If $\{\alpha_i\}$ ($\{\alpha_i^\vee\}$) is the set of simple roots (coroots) of $\overset{\circ}{\mathfrak{g}}$, then $\{\delta - \theta, \alpha_1, \ldots, \alpha_\ell\}$ are the simple roots of $\mathfrak{g}$ and $\{c - \theta^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ are the simple coroots. Write $\alpha_0 := \delta - \theta$ and $\alpha_0^\vee := c - \theta^\vee$.

Write $\overset{\circ}{\Delta}$ to mean the set of roots of $\overset{\circ}{\mathfrak{g}}$. We have that $\mathfrak{g}$ decomposes as a $\mathfrak{h}$-module with respect to the adjoint action as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{m \in \mathbb{Z} \setminus 0} \left( t^m \otimes \overset{\circ}{\mathfrak{h}} \right) \oplus \bigoplus_{m \in \mathbb{Z}, \beta \in \overset{\circ}{\Delta}} \left( t^m \otimes \overset{\circ}{\mathfrak{g}}_\beta \right).$$

Hence, $\Delta$, the set of roots of $\mathfrak{g}$, is

$$\Delta = \{0\} \cup \{m\delta \,|\, m \in \mathbb{Z}\} \cup \{m\delta + \beta \,|\, m \in \mathbb{Z}, \beta \in \overset{\circ}{\Delta}\}.$$

Thus, we have the triangular decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}, \tag{1.2}$$

where

$$\mathfrak{n} := \left(t\mathbb{C}[t]\otimes \overset{\circ}{\mathfrak{g}}\right) \oplus \bigoplus_{\beta\in\overset{\circ}{\Delta}_+} \overset{\circ}{\mathfrak{g}}_\beta$$
$$\mathfrak{n}^- := \left(t^{-1}\mathbb{C}[t^{-1}]\otimes \overset{\circ}{\mathfrak{g}}\right) \oplus \bigoplus_{\beta\in\overset{\circ}{\Delta}_+} \overset{\circ}{\mathfrak{g}}_{-\beta}\ .$$

We write $\mathfrak{b}$ to mean $\mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{b}^- := \mathfrak{h} \oplus \mathfrak{n}^-$. Set the *root lattice* $Q := \sum_{i=0}^{\ell} \mathbb{Z}\alpha_i^\vee$ and $Q_+ := \sum_{i=0}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i$. Fix a partial order $\leq$ on $\mathfrak{h}^*$ by $\mu \leq \lambda$ if $\lambda - \mu \in Q_+$.

**Definition 1.1.1** *The Weyl group of* $\mathfrak{g}$*,* $W \subset GL(\mathfrak{h}^*)$*, is the group generated by* $\{s_i\}_{i=0}^{\ell}$*, where*

$$s_i(\chi) = \chi - \chi(\alpha_i^\vee)\alpha_i. \tag{1.3}$$

*For* $w \in W$*, let* $\ell(w) := \min\{k \mid s_{i_1} \cdots s_{i_k} = w\}$.

Fix a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by:

$$\langle p \otimes x, q \otimes y\rangle = \textstyle\int (t^{-1}pq)\, \langle x, y\rangle_{\overset{\circ}{\mathfrak{g}}}, \qquad \text{for } x, y \in\overset{\circ}{\mathfrak{g}} \text{ and } p, q \in \mathbb{C}[t, t^{-1}]\ , \tag{1.4}$$

$$\langle \mathbb{C}c \oplus \mathbb{C}d, \mathbb{C}[t, t^{-1}]\otimes \overset{\circ}{\mathfrak{g}}\rangle = \langle c, c\rangle = \langle d, d\rangle = 0, \qquad \langle c, d\rangle = 1; \tag{1.5}$$

where $\int : \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ is the $\mathbb{C}$-linear map which sends a Laurent polynomial to its $t^{-1}$ coefficient. By [7, 13.], this is indeed an invariant form on $\mathfrak{g}$. Note that $\langle \cdot, \cdot \rangle|_{\overset{\circ}{\mathfrak{g}}\times\overset{\circ}{\mathfrak{g}}} = \langle \cdot, \cdot \rangle_{\overset{\circ}{\mathfrak{g}}}$, henceforth we will denote both by $\langle \cdot, \cdot \rangle$. In addition, $\langle \cdot, \cdot \rangle$, which by above is clearly nondegenerate on $\mathfrak{h}$, may be carried to a $W$ invariant form on $\mathfrak{h}^*$ via the isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$, $\nu(h) : h' \mapsto \langle h', h\rangle$.

Let $\overset{\circ}{W}$ and $\overset{\circ}{Q}{}^\vee$ be the Weyl group and coroot lattice, respectively, of $\overset{\circ}{\mathfrak{g}}$. Recall [7, 13.1.7], that $W$ is isomorphic to $\overset{\circ}{W} \ltimes \overset{\circ}{Q}{}^\vee$. Moreover, for $\beta \in\overset{\circ}{Q}{}^\vee$, denote its image in $W$ by $T_\beta$, then

$$T_\beta : \chi \mapsto \chi + \chi(c)\nu(\beta) - \left(\chi(\beta) + \frac{1}{2}\langle \beta, \beta\rangle\chi(c)\right) \delta \in GL(\mathfrak{h}^*). \tag{1.6}$$

Fix a real subspace $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ such that the following hold:

1. $\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{h}$,

2. $\{\alpha_0^\vee, \ldots \alpha_\ell^\vee\} \subset \mathfrak{h}_{\mathbb{R}}$, and

3. $\alpha_i(\mathfrak{h}_{\mathbb{R}}) \subset \mathbb{R}$ for $i = 0, \ldots, \ell$.

Define the *dominant chamber* $D_{\mathbb{R}} \subset \mathfrak{h}_{\mathbb{R}}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_b r, \mathbb{R})$ by

$$D_{\mathbb{R}} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \lambda(\alpha_i^\vee) \geq 0 \text{ for all } i\}.$$

The *Tits cone* C, is defined as the union the $W$ translates of $D_{\mathbb{R}}$. By [7, 13.1.E.8.a],

$$C = \mathbb{R}\delta \cup \{\lambda \in \mathfrak{h}^* \mid \lambda(c) > 0\}. \tag{1.7}$$

## 1.2 Integrable highest weight representations

In this section, assume that $\mathfrak{g}$ is affine. For a vector space $V$ and a linear operator $\phi \in \mathrm{End}(V)$, we say that $\phi$ *acts nilpotently on* $v$ if $\exists n \in \mathbb{Z}_{>0}$ such that $\phi^n(v) = 0$ and that $\phi$ *is locally nilpotent* if $\phi$ acts nilpotently on each $v \in V$. We say that a $\mathfrak{g}$-module $V$ is a *weight module* if $V$ decomposes into the sum of finite dimensional $\mathfrak{h}$ weight spaces, that is, $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ as an $\mathfrak{h}$-module, where $V_\mu := \{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$ and $\dim V_\mu < \infty$. $V$ is a *highest weight module of highest weight* $\lambda$ if it is a weight module where there exists $v \in V$ such that $\mathfrak{n}v = 0$, $V = U(\mathfrak{n}^-)v$, and $h \cdot v = \lambda(h)v$. A weight module $V$ is called *integrable* if each $e_i$ and $f_i$ act as locally nilpotent operators on $V$.

For a weight module define the *formal character of* $V$, $\mathrm{ch}\, V := \sum_{\mu\in\mathfrak{h}^*} \dim V_\mu e^\mu$. We think of $\mathrm{ch}\, V$ as an element of a certain unital, commutative, associative algebra over $\mathbb{Z}$ such that $e^{\mu_1} \cdot e^{\mu_2} = e^{\mu_1+\mu_2}$ and $e^0 = 1$.

Define the set of *dominant integral weights* to be $P_+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_+ \text{ for } i = 0, \ldots, \ell\}$. For each $\lambda \in P_+$, there exists a unique integrable, highest weight representation. Moreover, such a representation is irreducible. We denote this representation by $L(\lambda)$.

We recall the Weyl-Kac character formula [7, 2.2.1], which we will need.

**Theorem 1.2.1 (Weyl-Kac character formula)** *Let $\rho \in \mathfrak{h}^*$ satisfy the property that $\rho(\alpha_i^\vee) = 1$ for $i = 0, \ldots, \ell$. For $w \in W$ and $\mu \in \mathfrak{h}^*$ write $w * \mu := w \cdot (\mu + \rho) - \rho$. Let $\lambda \in P_+$,*

$$\operatorname{ch} L(\lambda) = \left( \sum_{w \in W} (-1)^{\ell(w)} e^{w * \lambda} \right) \cdot \prod_{\alpha \in \delta^+} (1 - e^{-\alpha})^{-\dim \mathfrak{g}_\alpha} \tag{1.8}$$

Since $L(0)$ is the trivial 1-dimensional $\mathfrak{g}$-module, we have the following identity:

**Corollary 1.2.2**

$$e^{-\rho} \sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \rho} = \prod_{\alpha \in \delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} \tag{1.9}$$

By (1.3), $\delta$ is $W$ fixed, hence $\operatorname{ch} L(\lambda + n\delta) = e^{n\delta} \operatorname{ch} L(\lambda)$.

**Proposition 1.2.3 (Weights of integrable, highest weight representations)** *For $\lambda \in P_+$, define $P(\lambda)$,* the set of weights of $L(\lambda)$, *by*

$$P(\lambda) := \{\mu \in \mathfrak{h}^* \mid L(\lambda)_\mu \neq 0\}.$$

*Then by [3, Proposition 12.5],*

*a* $P(\lambda) = W \cdot \{\mu \in P_+ \mid \mu \leq \lambda\}$*,*

*b* $P(\lambda) = (\lambda + Q) \cap$ *convex hull of* $W \cdot \lambda$*, and*

*c* $P(\lambda)$ *lies in the paraboloid*

$$\{\mu \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \pi_{\mathbb{R}}(\mu), \pi_{\mathbb{R}}(\mu) \rangle + \lambda(c)\langle \mu, \Lambda_0 \rangle \leq \langle \lambda, \lambda \rangle; \, \lambda(c) = \mu(c)\},$$

*where $\pi_{\mathbb{R}} : \mathfrak{h}_{\mathbb{R}}^* \to \mathring{\mathfrak{h}}^*{}_{\mathbb{R}}$ is dual to the embedding $\mathring{\mathfrak{h}}_{\mathbb{R}} \hookrightarrow \mathfrak{h}_{\mathbb{R}}$. Moreover, $P(\lambda)$ intersects the boundary of the paraboloid precisely at the points $W \cdot \lambda$.*

Define $P^o(\lambda)$ as the set of $\delta$-maximal weights of $L(\Lambda)$, i.e.,

$$P^o(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \in P(\lambda) \text{ but } \mu + n\delta \notin P(\lambda) \text{ for any } n > 0\}. \tag{1.10}$$

Let $\theta = \sum_{i=1}^{\ell} h_i \alpha_i$ be the highest root of $\mathring{\mathfrak{g}}$ (with respect to a choice of the positive roots), written as a linear combination of the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$ of $\mathring{\mathfrak{g}}$. Let

$$S := \{\sum_{i=0}^{\ell} n_i\alpha_i \,|\, n_i \geq 0 \text{ for any } i \text{ and } 0 \leq n_i < h_i \text{ for some } 0 \leq i \leq \ell\},$$

where $h_0 := 1$.

**Proposition 1.2.4** *Let $\mathfrak{g}$ be an untwisted affine Kac-Moody Lie algebra as above. Then, for any $\lambda \in P_+$ with $\lambda(c) > 0$,*

$$P^o(\lambda)_+ = S(\lambda) \cap P_+,$$

*where $P^o(\lambda)_+ := P^o(\lambda) \cap P_+$ and $S(\lambda) = \{\lambda - \beta : \beta \in S\}$.*

**Proof** Take $\mu \in S(\lambda)$. Then, for any $n \geq 1$,

$$\lambda - (\mu + n\delta) = \left(\sum_{i=0}^{\ell} n_i\alpha_i\right) - n\delta = (n_0 - n)\alpha_0 + \sum_{i=1}^{\ell} (n_i - nh_i)\alpha_i,$$

since $\alpha_0 := \delta - \theta$. Now, the coefficient of some $\alpha_i$ in the above sum is negative, for any positive $n$, since $\mu \in S(\lambda)$. Thus, $\mu + n\delta$ could not be a weight of $L(\lambda)$ for any positive $n$. Therefore, if $\mu \in P(\lambda) \cap S(\lambda)$, then it is $\delta$-maximal.

By 1.2.3.a, if $\lambda(c) \neq 0$, then $S(\lambda) \cap P_+ \subset P(\lambda)$. Therefore, $S(\lambda) \cap P_+ \subset P^o(\lambda)_+$.

Conversely, take $\mu \in P^o(\lambda)_+$. Then, $\mu \in P(\lambda) \cap P_+$ and $\mu + \delta \notin P(\lambda)$. Express $\mu = \lambda - n_0\alpha_0 - \sum_{i=1}^{\ell} n_i\alpha_i$, for some $n_i \in \mathbb{Z}_+$. Then,

$$\mu + \delta = \lambda - (n_0 - 1)\alpha_0 - \sum_{i=1}^{\ell} (n_i - h_i)\alpha_i.$$

Again applying 1.2.3.a, $\mu + \delta \notin P(\lambda)$ if and only if $\mu + \delta \not\leq \lambda$, i.e., for some $0 \leq i \leq \ell$, $n_i < h_i$. Thus, $\mu \in S(\lambda)$. This proves the proposition.

# CHAPTER 2: THE VIRASORO ALGEBRA

We recall the definition of the Virasoro algebra and its basic representation theory. We prove some basic facts about the weight spaces of unitarizable representations which will be used crucially later on.

## 2.1 The Virasoro algebra and its unitarizable highest weight representations

The *Virasoro algebra* Vir has a basis $\{C, L_n \,|\, n \in \mathbb{Z}\}$ over $\mathbb{C}$ and the Lie bracket is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}C \text{ and } [\mathrm{Vir}, \mathrm{C}] = 0.$$

Let $\mathrm{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C$. Then, a Vir module $V$ is said to be a *highest weight representation* if there exists a $\mathrm{Vir}_0$-eigenvector $v_o \in V$ such that $L_n v_o = 0$ for $n \in \mathbb{Z}_{>0}$ and $U(\bigoplus_{n<0} \mathbb{C}L_n)v_o = V$. Such a $V$ is said to have *highest weight* $\lambda \in \mathrm{Vir}_0^*$ if $Xv_o = \lambda(X)v_o$, for all $X \in \mathrm{Vir}_0$. (It is easy to see that such a $v_o$ is unique up to a scalar multiple and hence $\lambda$ is unique.) The irreducible highest weight representations of Vir are in 1-1 correspondence with elements of $\mathrm{Vir}_0^*$ given by their highest weight. Denote the basis of $\mathrm{Vir}_0^*$ dual to the basis $\{L_0, C\}$ of $\mathrm{Vir}_0$ as $\{h, z\}$. For any $\mu \in \mathrm{Vir}_0^*$, denote the $\mu$-th weight space of $V$ by $V_\mu$, i.e.,

$$V_\mu := \{v \in V : X \cdot v = \mu(X)v \ \forall X \in \mathrm{Vir}_0\}.$$

Define a Vir module $V$ to be *unitarizable* if there exists a positive definite Hermitian form $(\cdot\,,\,\cdot)$ on $V$ so that $(L_n v\,,\, w) = (v\,,\, L_{-n}w)$ for all $n \in \mathbb{Z}$ and $(Cv\,,\, w) = (v\,,\, Cw)$. It is easy to see that if $M$ is a Vir-submodule of $V$, then $M^\perp$ is also a submodule. Hence, any unitarizable representation of Vir is completely reducible. Note that for a unitarizable highest weight Vir-representation $V$

with highest weight $\lambda$, if $v_o$ is a highest weight vector, then

$$\begin{aligned} 0 &\leq (L_{-n}v_o\,,\, L_{-n}v_o) \\ &= (L_nL_{-n}v_o\,,\, v_o) \\ &= (2n\lambda(L_0) + \frac{1}{12}(n^3 - n)\lambda(C))(v_o\,,\, v_o) \end{aligned} \tag{2.1}$$

for all $n > 0$. Therefore, both $\lambda(L_0)$ and $\lambda(C)$ must be nonnegative real numbers.

**Lemma 2.1.1** *Let $V$ be a unitarizable, highest weight (irreducible) representation of $Vir$ with highest weight $\lambda$.*

*(a) If $\lambda(L_0) \neq 0$, then $V_{\lambda+nh} \neq 0$, for any $n \in \mathbb{Z}_+$.*

*(b) If $\lambda(L_0) = 0$ and $\lambda(C) \neq 0$, then $V_{\lambda+nh} \neq 0$, for any $n \in \mathbb{Z}_{>1}$ and $V_{\lambda+h} = 0$.*

*(c) If $\lambda(L_0) = \lambda(C) = 0$, then $V$ is one dimensional.*

**Proof** If $\lambda(L_0) \neq 0$, then by the equation (2.1) (since both of $\lambda(L_0)$ and $\lambda(C) \in \mathbb{R}_+$), $L_{-n}v_o \neq 0$, for any $n \in \mathbb{Z}_+$.

If $\lambda(L_0) = 0$ and $\lambda(C) \neq 0$, then again by the equation (2.1), $L_{-n}v_o \neq 0$, for any $n \in \mathbb{Z}_{>1}$. Also, $L_{-1}v_o = 0$.

If $\lambda(L_0) = \lambda(C) = 0$, then (by the equation (2.1) again), $L_{-n}v_o = 0$, for any $n \in \mathbb{Z}_{\geq 1}$. This shows that $V$ is one dimensional.

## CHAPTER 3: TENSOR PRODUCT DECOMPOSITION

### 3.1 A general method for tensor product decomposition for affine Kac-Moody algebras

Let $\Lambda \in P_+$ and consider $L(\Lambda)$. For any $\lambda \in P^o(\Lambda)$ (cf. 1.10), define the *$\delta$-character of $L(\Lambda)$ through $\lambda$* by

$$c_{\Lambda,\lambda} = \sum_{n\in\mathbb{Z}} \dim L(\Lambda)_{\lambda+n\delta}\, e^{n\delta}.$$

Since $\delta$ is $W$-invariant,

$$c_{\Lambda,\lambda} = c_{\Lambda,w\lambda}, \text{ for any } w \in W. \tag{3.1}$$

Moreover, $P^o(\Lambda)$ is $W$-stable. It is obvious that

$$ch\, L(\Lambda) = \sum_{\lambda\in P^o(\Lambda)} c_{\Lambda,\lambda} e^{\lambda}. \tag{3.2}$$

By (1.7), for any $\lambda \in P(\Lambda')$ and $\Lambda'' \in P_+$, $\Lambda'' + \lambda + \rho$ belongs to the Tits cone. Hence, there exists $v \in W$ such that $v^{-1}(\Lambda'' + \lambda + \rho) \in P_+$. Moreover, if $\Lambda'' + \lambda + \rho$ has nontrivial $W$-isotropy, then its isotropy group must contain a reflection (cf. [7, 1.4.2.a]). Thus, for such a $\lambda \in P(\Lambda')$, i.e., if $\Lambda'' + \lambda + \rho$ has nontrivial $W$-isotropy,

$$\sum_{w\in W} \varepsilon(w) e^{w(\Lambda''+\lambda+\rho)} = 0. \tag{3.3}$$

Define

$$\bar{P}_+ := \{\Lambda \in P_+ : \Lambda(d) = 0\}.$$

For any $m \in \mathbb{Z}_+$, let

$$P_+^{(m)} := \{\Lambda \in P_+ : \Lambda(c) = m\},$$

and let

$$\bar{P}_+^{(m)} := \bar{P}_+ \cap P_+^{(m)}.$$

Then, $\bar{P}_+^{(m)}$ provides a set of representatives in $P_+^{(m)}$ mod $(P_+ \cap \mathbb{C}\delta)$.

For any $\Lambda, \Lambda', \Lambda'' \in P_+$, define

$$T_\Lambda^{\Lambda',\Lambda''} = \{\lambda \in P^o(\Lambda') : \exists v_{\Lambda,\Lambda'',\lambda} \in W \text{ and } S_{\Lambda,\Lambda'',\lambda} \in \mathbb{Z} \text{ with}$$
$$\lambda + \Lambda'' + \rho = v_{\Lambda,\Lambda'',\lambda}(\Lambda + \rho) + S_{\Lambda,\Lambda'',\lambda}\delta\}.$$

Observe that since $\Lambda + \rho + n\delta \in P_{++}$ for any $n \in \mathbb{Z}$, such a $v_{\Lambda,\Lambda'',\lambda}$ and $S_{\Lambda,\Lambda'',\lambda}$ are unique by [7, 1.4.2.a-b] (if they exist). Also, observe that

$$T_\Lambda^{\Lambda',\Lambda''} = \emptyset, \text{ unless } \Lambda(c) = \Lambda'(c) + \Lambda''(c) \text{ and } \Lambda' + \Lambda'' - \Lambda \in Q. \tag{3.4}$$

**Proposition 3.1.1** *For any $\Lambda'$ and $\Lambda'' \in P_+$,*

$$ch\left(L(\Lambda') \otimes L(\Lambda'')\right) = \sum_{\Lambda \in \bar{P}_+^{(m)}} ch\, L(\Lambda) \sum_{\lambda \in T_\Lambda^{\Lambda',\Lambda''}} \varepsilon(v_{\Lambda,\Lambda'',\lambda}) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda'',\lambda}\delta},$$

*where $m := \Lambda'(c) + \Lambda''(c)$.*

*Moreover, $\sum_{\lambda \in T_\Lambda^{\Lambda',\Lambda''}} \varepsilon(v_{\Lambda,\Lambda'',\lambda}) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda'',\lambda}\delta}$ is the shifted character of a unitary representation (though, in general, not irreducible) of the Virasoro algebra* Vir *with central charge $z_{\mathfrak{g}}^{m'} + z_{\mathfrak{g}}^{m''} - z_{\mathfrak{g}}^{m}$, where $z_{\mathfrak{g}}^m := \frac{m \dim \mathring{\mathfrak{g}}}{m+g}, m' := \Lambda'(c), m'' := \Lambda''(c)$ and $g := \rho(c)$. In fact, the multiplicity space*

$$M(\Lambda; \Lambda', \Lambda'') := \mathrm{Hom}_{\mathfrak{g}'}\left(L(\Lambda') \otimes L(\Lambda''), L(\Lambda)\right)$$

*is a coset-module representation of the Virasoro algebra (cf. [4, Proposition 10.3]), where $\mathfrak{g}'$ is the derived subalgebra of $\mathfrak{g}$*

**Proof** By (1.8) and the identity (3.2), for any $\Lambda', \Lambda'' \in P_+$,

$$\begin{aligned}
&\left(\sum_{w\in W}\varepsilon(w)e^{w\rho}\right)\cdot ch\, L(\Lambda')\cdot ch\, L(\Lambda'')\\
&=\left(\sum_{\lambda\in P^o(\Lambda')}c_{\Lambda',\lambda}e^{\lambda}\right)\cdot\left(\sum_{w\in W}\varepsilon(w)e^{w(\Lambda''+\rho)}\right)\\
&=\sum_{\lambda\in P^o(\Lambda')}c_{\Lambda',\lambda}\sum_{w\in W}\varepsilon(w)e^{w(\Lambda''+\lambda+\rho)},\ \text{by (3.1)}\\
&=\sum_{\Lambda\in\bar{P}_+^{(m)}}\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}}c_{\Lambda',\lambda}\sum_{w\in W}\varepsilon(w)e^{w(v_{\Lambda,\Lambda'',\lambda}(\Lambda+\rho))+S_{\Lambda,\Lambda'',\lambda}\delta},\ \text{by (3.3)}\\
&=\sum_{\Lambda\in\bar{P}_+^{(m)}}\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}}c_{\Lambda',\lambda}\sum_{w\in W}\varepsilon(w)\varepsilon(v_{\Lambda,\Lambda'',\lambda})e^{w(\Lambda+\rho)}e^{S_{\Lambda,\Lambda'',\lambda}\delta}\\
&=\sum_{\Lambda\in\bar{P}_+^{(m)}}\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}}\varepsilon(v_{\Lambda,\Lambda'',\lambda})c_{\Lambda',\lambda}e^{S_{\Lambda,\Lambda'',\lambda}\delta}.
\end{aligned}$$

Thus,

$$ch\left(L(\Lambda')\otimes L(\Lambda'')\right)=\sum_{\Lambda\in\bar{P}_+^{(m)}}ch\, L(\Lambda)\sum_{\lambda\in T_\Lambda^{\Lambda',\Lambda''}}\varepsilon(v_{\Lambda,\Lambda'',\lambda})c_{\Lambda',\lambda}e^{S_{\Lambda,\Lambda'',\lambda}\delta}.$$

To prove the second part of the proposition, we apply [4, Proposition 10.3]:

$$\sum_{n\in\mathbb{Z}}e^{-n\delta}\dim\mathrm{Hom}_{\mathfrak{g}}(L(\Lambda')\otimes L(\Lambda''),L(\Lambda-n\delta))=e^{h_\Lambda^{\Lambda',\Lambda''}\delta}\sum_{\xi\in\mathbb{C}}e^{-\xi\delta}\dim M(\Lambda;\Lambda',\Lambda'')_{\xi h},\tag{3.5}$$

where

$$h_\Lambda^{\Lambda',\Lambda''}:=\frac{\langle\Lambda'+2\rho,\Lambda'\rangle}{2(m'+g)}+\frac{\langle\Lambda''+2\rho,\Lambda''\rangle}{2(m''+g)}-\frac{\langle\Lambda+2\rho,\Lambda\rangle}{2(m+g)}.$$

This proves the proposition.

If we combine this theorem with Lemma 2.1.1 we get the following useful fact:

**Corollary 3.1.2** *Suppose $L(\nu)$ is a submodule of $L(\lambda)\otimes L(\mu)$, then for any $n\in\mathbb{Z}_{>1}$, $L(\nu-n\delta)$ is also a submodule of $L(\lambda)\otimes L(\mu)$.*

**Remark** By [5], $\mathrm{Hom}_{\mathfrak{g}'}\left(L(\Lambda')\otimes L(\Lambda''),L(\Lambda)\right)\neq 0$ if and only if $\Lambda\in(\Lambda'+\Lambda''+\mathring{Q}+\mathbb{C}\delta)\cap P_+$, where $\mathring{Q}$ is the root lattice of $\mathring{\mathfrak{g}}$.

# CHAPTER 4: SATURATED TENSOR CONE FOR $A_1^{(1)}$

## 4.1 Computation of $\delta$-maximal components for $A_1^{(1)}$

In this section, we consider $\mathfrak{g} = A_1^{(1)} = \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{C}t^n \otimes \mathfrak{sl}_2\right) \oplus \mathbb{C}c \oplus \mathbb{C}d$. In this case $\mathfrak{h}^* = \mathbb{C}\alpha \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$, where $\alpha$ is the simple root of $\mathfrak{sl}_2$ and $\Lambda_0{}_{|\overset{\circ}{\mathfrak{h}}\oplus\mathbb{C}d} \equiv 0$ and $\Lambda_0(c) = 1$. Then, $\Lambda_0$ is a zeroth fundamental weight. The simple roots of $\widehat{\mathfrak{sl}_2}$ are $\alpha_0 := \delta - \alpha$ and $\alpha_1 := \alpha$. The simple coroots are $\alpha_0^\vee := c - \alpha^\vee$ and $\alpha_1^\vee := \alpha^\vee$. It is easy to see that an element of $\mathfrak{h}^*$ of the form $m\Lambda_0 + \frac{j}{2}\alpha$ belongs to $P_+$ if and only if $m, j \in \mathbb{Z}_+$ and $m \geq j$.

Specializing Proposition 1.2.4 to the case of $\mathfrak{g} = A_1^{(1)}$, we get the following.

**Corollary 4.1.1** *For $\mathfrak{g} = A_1^{(1)}$ and $\Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+$,*

$$P^o(\Lambda)_+ = \left\{ \Lambda - k\alpha,\ \Lambda - l(\delta - \alpha) \;\middle|\; \begin{array}{c} k, l \in \mathbb{Z}_+ \text{ and} \\ k \leq \frac{j}{2},\ l \leq \frac{m-j}{2} \end{array} \right\}. \tag{4.1}$$

**Proof** The corollary follows from Proposition 1.2.4 since $m_1\Lambda_0 + \frac{m_2}{2}\alpha + m_3\delta$ belongs to $P_+$ if and only if $m_1, m_2 \in \mathbb{Z}_+$ and $m_1 \geq m_2$.

Let $\pi$ be the projection $\mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta \to \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha$.

**Lemma 4.1.2** *For $\Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+$ (i.e., $m, j \in \mathbb{Z}_+$ and $m \geq j$) such that $m > 0$,*

$$\pi(P^o(\Lambda)) = \{\Lambda + k\alpha : k \in \mathbb{Z}\}. \tag{4.2}$$

*Moreover, for any $k \in \mathbb{Z}$, let $n_k$ be the unique integer such that $\Lambda + k\alpha + n_k\delta \in P^o(\Lambda)$. Then, writing $k = qm + r, 0 \leq r < m$, we have:*

$$n_k = n_r - q(k + r + j). \tag{4.3}$$

**Proof** The assertion (4.2) follows from the identity (4.1) together with the action of the affine Weyl group $W \simeq \overset{\circ}{W} \times (\mathbb{Z}\alpha^\vee)$ on $\mathfrak{h}^*$, where $\overset{\circ}{W}$ is the Weyl group of $\mathfrak{sl}_2$ and $\mathbb{Z}\alpha^\vee$ acts on $\mathfrak{h}^*$ via:

$$T_{n\alpha^\vee}(\mu) = \mu + n\mu(c)\alpha - [n\mu(\alpha^\vee) + n^2\mu(c)]\delta, \ \ \text{for}\, n \in \mathbb{Z}, \mu \in \mathfrak{h}^*. \tag{4.4}$$

Since $P^o(\Lambda)$ is $W$-stable, the identity (4.3) can be established from the action of the affine Weyl group element $T_{-q\alpha^\vee}$ on $\Lambda + k\alpha + n_k\delta$.

The value of $n_r$ for $0 \le r < m$ can be determined from the identity (4.1) by applying $T_{\alpha^\vee}, T_{\alpha^\vee} \cdot s_1$ to $\Lambda - k\alpha$ and applying $1, T_{\alpha^\vee} \cdot s_1$ to $\Lambda - l(\delta - \alpha)$, where $s_1$ is the nontrivial element of $\overset{\circ}{W}$. We record the result in the following lemma.

**Lemma 4.1.3** *With the notation as in the above lemma, the value of $n_r$ for any integer $0 \le r < m$ is given by*

$$n_r = \begin{cases} -r\,, & \text{for}\, 0 \le r \le m - j \\ m - j - 2r & \text{for}\, m - j \le r < m. \end{cases}$$

**Lemma 4.1.4** *Take the following elements in $P_+$:*

$$\Lambda = m\Lambda_0 + \frac{j}{2}\alpha,\ \Lambda' = m'\Lambda_0 + \frac{j'}{2}\alpha,\ \Lambda'' = m''\Lambda_0 + \frac{j''}{2}\alpha,$$

*where $m := m' + m''$ and we assume that $m' > 0$. Then,*

$$\pi\left(T_\Lambda^{\Lambda',\Lambda''}\right) = \left\{ \Lambda' + k\alpha \;\middle|\; k \in \mathbb{Z},\ \begin{array}{ll} k \equiv \frac{1}{2}(j - j' - j'') & \text{mod}\ M \text{ or} \\ k \equiv -\frac{1}{2}(j + j' + j'') - 1 & \text{mod}\ M \end{array} \right\},$$

*where $M := m + 2$. In particular, by the equation (3.4), $T_\Lambda^{\Lambda',\Lambda''}$ is nonempty if and only if $\frac{j-j'-j''}{2} \in \mathbb{Z}$.*

*Moreover, for $\lambda = \Lambda' + k\alpha + n_k\delta \in T_\Lambda^{\Lambda',\Lambda''}$,*

$$v_{\Lambda,\Lambda'',\lambda} = \begin{cases} T_{\frac{k-\frac{1}{2}(j-j'-j'')}{M}\alpha^\vee}, & \text{if}\, k \equiv \frac{1}{2}(j - j' - j'') \quad \text{mod}\ M \\ s_1 T_{-\frac{k+\frac{1}{2}(j+j'+j'')+1}{M}\alpha^\vee}, & \text{if}\, k \equiv -\frac{1}{2}(j + j' + j'') - 1 \quad \text{mod}\ M, \end{cases}$$

*where $T_{n\alpha^\vee}$ is defined by the equation (4.4). Further,*

$$S_{\Lambda,\Lambda'',\lambda} = n_k + \frac{\left(k - \frac{1}{2}(j - j' - j'')\right)\left(k + \frac{1}{2}(j + j' + j'') + 1\right)}{M}.$$

**Proof** Follows from the fact that $W = \overset{\circ}{W} \rtimes \mathbb{Z}\alpha^\vee$ and that $\rho = 2\Lambda_0 + \frac{1}{2}\alpha$.

We have the following very crucial result.

**Proposition 4.1.5** *Fix $\Lambda, \Lambda'$ and $\Lambda''$ as in Lemma 4.1.4 and asume that $\frac{j-j'-j''}{2} \in \mathbb{Z}$ and both of $m', m'' > 0$. Then, the maximum of*

$$\left\{S_{\Lambda,\Lambda'',\lambda} : \ \lambda \in T_\Lambda^{\Lambda',\Lambda''} \ \textit{and}\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\right\}$$

*is achieved precisely when $\pi(\lambda) = \Lambda' + \frac{1}{2}(j - j' - j'')\alpha$.*

**Proof** By Lemma 4.1.4 and the fact that $ell(T_{n\alpha^\vee}) = 2|n|$,

$$\pi\{\lambda \in T_\Lambda^{\Lambda',\Lambda''} \,|\, \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\} = \{\Lambda' + k_l\alpha \,|\, l \in \mathbb{Z}\},$$

where $M := m + 2$ and $k_l := \frac{j-j'-j''}{2} + lM$. Take $\lambda = \Lambda' + k_l\alpha \in \pi\left(T_\Lambda^{\Lambda',\Lambda''}\right)$ for $l \in \mathbb{Z}$. Write $k_l = q_l m' + r_l$ for $q_l \in \mathbb{Z}$ and $0 \le r_l < m'$. Then, by Lemmas 4.1.2, 4.1.3 and 4.1.4, for $\lambda = \Lambda' + k_l\alpha$ (setting $J := \frac{j-j'-j''}{2}$),

$$\begin{aligned}
S_{\Lambda,\Lambda'',\lambda} =& n_{r_l} - \frac{(J + j' + lM + r_l)(J + lM - r_l)}{m'} + l(lM + 1 + j) \\
=& l^2 M \left(1 - \frac{M}{m'}\right) + l\left(1 + j - \frac{M(j - j'')}{m'}\right) - \frac{(j - j'')^2 - j'^2}{4m'} \\
& + \frac{r_l^2}{m'} + \frac{r_l j'}{m'} + n_{r_l} \\
=& l^2 M \left(1 - \frac{M}{m'}\right) + l\left(1 + j - \frac{M}{m'}(j - j'')\right) - \frac{(j - j'')^2 - j'^2}{4m'} \\
& + p(k_l),
\end{aligned}$$

where

$$p(k_l) := \frac{r_l^2}{m'} + \frac{r_l}{m'}j' + n_{k_l}.$$

Let $P = P_{m',j'} : \mathbb{R} \to \mathbb{R}$ be the following function:

$$P(s) \quad := \quad \begin{cases} \frac{(s-\frac{m'}{2}k)^2}{m'} - \frac{(j')^2}{4m'}, & \text{if } \left|s - \frac{m'}{2}k\right| \leq \frac{j'}{2} \text{ for some } k \in 2\mathbb{Z} \\ \frac{(s-\frac{m'}{2}k)^2}{m'} - \frac{(m'-j')^2}{4m'}, & \text{if } \left|s - \frac{m'}{2}k\right| \leq \frac{m'-j'}{2} \text{ for some } k \in 2\mathbb{Z}+1. \end{cases}$$

Let $k_s \in \mathbb{Z}$ be such a $k$. (Of course, $k_s$ depends upon $m'$ and $j'$.)

**Claim 4.1.6** $P(s) = p(s - \frac{j'}{2})$ *for* $s \in \frac{j'}{2} + \mathbb{Z}$.

**Proof** Clearly, both of $P$ and $p$ are periodic with period $m'$. So, it is enough to show that $P(s) = p(s - \frac{j'}{2})$, for $s - \frac{j'}{2}$ equal to any of the integral points of the interval $[-j', m' - j']$. By Lemma 4.1.3 and the identity (4.3), for any integer $-j' \leq r \leq 0$,

$$p(r) = \frac{1}{m'} r(r + j'),$$

and for any integer $0 \leq r \leq m' - j'$,

$$p(r) = \frac{r(r + j')}{m'} - r.$$

From this, the claim follows immediately.

Fix $m' > 0$. Let

$$I := \left\{ (t, j', m'', j'', j) \in \mathbb{R}^5 \;\middle|\; \begin{array}{l} 0 \leq j' \leq m',\, 1 \leq m'', \\ 0 \leq j'' \leq m'',\, 0 \leq j \leq m' + m'' \end{array} \right\}.$$

Define $F : I \to \mathbb{R}$ by

$$\begin{aligned} F : (t, j', m'', j'', j) \quad \mapsto \quad & t^2 M \left(1 - \frac{M}{m'}\right) + t \left[ j \left(1 - \frac{M}{m'}\right) + 1 + \frac{M}{m'} j'' \right] \\ & + \frac{(j')^2 - (j - j'')^2}{4m'} + P\left(\frac{1}{2}\left(j - j''\right) + tM\right). \end{aligned}$$

Thus, $F$ is a continuous, piecewise smooth function with failure of differentiability along the set

$$\{(t, j', m'', j'', j) \in I \,:\, \frac{1}{2}(j \pm j' - j'') + tM \in m'\mathbb{Z}\}.$$

**Claim 4.1.7** *Let $\Delta(t) = \Delta(t, j', m'', j'', j) := F(t+1, j', m'', j'', j) - F(t, j', m'', j'', j)$. Then, on $I$,*

1. *$\Delta$ is a nonincreasing function of $t$*

2. *$\Delta$ is increasing with respect to $j''$*

3. *$\Delta$ is nonincreasing in $j$*

4. *$\Delta(0)$ is decreasing in $m''$*

5. *$\Delta(-1)$ is nondecreasing in $m''$.*

**Proof** We compute and give bounds for the partial derivatives of $\Delta$, where they exist.

$$\begin{aligned}\Delta(t) &= 2tM\left(1-\frac{M}{m'}\right)+\left((j+M)\left(1-\frac{M}{m'}\right)+1+\frac{M}{m'}j''\right)\\ &\quad +P\left(tM+M+\frac{1}{2}(j-j'')\right)-P\left(tM+\frac{1}{2}(j-j'')\right).\end{aligned}$$

Hence,

$$\begin{aligned}\partial_t\Delta(t) &= 2M\left(1-\frac{M}{m'}\right)+M\cdot P'\left(tM+M+\frac{1}{2}(j-j'')\right)\\ &\quad -M\cdot P'\left(tM+\frac{1}{2}(j-j'')\right)\\ &= 2M\left(1-\frac{M}{m'}\right)+2\frac{M}{m'}\left(M-\frac{m'}{2}k_1+\frac{m'}{2}k_0\right)\\ &= 2M\left(1-\frac{k_1-k_0}{2}\right),\end{aligned}$$

where $k_1 := k_{(t+1)M+\frac{1}{2}(j-j'')}$ and $k_0 := k_{tM+\frac{1}{2}(j-j'')}$. Since $2 \le k_1 - k_0$, we see that $\partial_t\Delta \le 0$, wherever $\partial_t\Delta$ exists. Since $\Delta$ is continuous everywhere and differentiable on all but a discrete set, $\Delta$ is nonincreasing in $t$.

$$\partial_{j''}\Delta(t) = \frac{M}{m'} - \frac{1}{2}\left[P'\left(tM+M+\frac{1}{2}(j-j'')\right)-P'\left(tM+\frac{1}{2}(j-j'')\right)\right].$$

Now, $|P'| \le 1$, so $\frac{M}{m'}+1 \ge \partial_{j''}\Delta \ge \frac{M}{m'}-1 = \frac{m''+2}{m'} > 0$.

For (3):

$$\begin{aligned}
\partial_j \Delta(t) &= 1 - \frac{M}{m'} + \frac{1}{2}\left[P'\left(tM + M + \frac{1}{2}(j - j'')\right) - P'\left(tM + \frac{1}{2}(j - j'')\right)\right] \\
&= 1 - \frac{M}{m'} + \frac{1}{m'}\left(M - \frac{m'}{2}k_1 + \frac{m'}{2}k_0\right) \\
&= 1 - \frac{k_1 - k_0}{2} \leq 0.
\end{aligned}$$

(4) and (5) follow from the following calculation:

$$\begin{aligned}
\partial_{m''}\Delta = 2t\left(1 - 2\frac{M}{m'}\right) &+ \left(1 - 2\frac{M}{m'} + \frac{1}{m'}(j'' - j)\right) \\
&+ (t+1)P'\left(tM + M + \frac{1}{2}(j - j'')\right) \\
&- tP'\left(tM + \frac{1}{2}(j - j'')\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\partial_{m''}\Delta(0) &= 1 - 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) + P'\left(M + \frac{1}{2}(j - j'')\right) \\
&\leq 1 - 2\frac{M}{m'} + \frac{m''}{m'} + 1 \\
&= \frac{-m'' - 4}{m'} < 0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_{m''}\Delta(-1) &= -2\left(1 - 2\frac{M}{m'}\right) + \left(1 - 2\frac{M}{m'} + \frac{1}{m'}(j'' - j)\right) \\
&\quad + P'\left(-M + \frac{1}{2}(j - j'')\right) \\
&= -1 + 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) + P'\left(-M + \frac{1}{2}(j - j'')\right) \\
&= -1 + 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) - 2\frac{M}{m'} + \frac{1}{m'}(j - j'') - k_0 \\
&= -1 - k_0.
\end{aligned}$$

Note that $k_0 \leq -1$ since $-\frac{(j-j'')}{2} - M < -\frac{m'}{2}$. Thus, $\partial_{m''}\Delta(-1) \geq 0$.

**Claim 4.1.8** *The maximum of $F = F(-, j', m'', j'', j) : \mathbb{Z} \to \mathbb{R}$ occurs at 0.*

**Proof** We show that $\Delta(-1) > 0 > \Delta(0)$. Since $\Delta$ is nonincreasing in $t$, it would follow that $F(0) > F(t)$ for all $t \in \mathbb{Z}_{\neq 0}$.

Let us begin with $\Delta(-1)$. By the previous claim 4.1.7, $\Delta(-1)$ is as small as possible when $m'' = 1$, $j'' = 0$, and $j = m' + 1$. So, let us compute with these values:

$$\begin{aligned}
\Delta(-1) &\geq \frac{6}{m'} + 1 + P\left(\frac{1}{2}m' + \frac{1}{2}\right) - P\left(-2 - \frac{1}{2}m' - \frac{1}{2}\right) \\
&= \frac{6}{m'} + 1 + \frac{\left(\frac{1}{2}m' + \frac{1}{2} - \frac{1}{2}m'k_1\right)^2}{m'} - \frac{\left(2 + \frac{1}{2}m' + \frac{1}{2} + \frac{1}{2}m'k_0\right)^2}{m'} \\
&+ \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\ 0 & \text{if } k_1 - k_0 \text{ even} \\ \frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases}
\end{aligned}$$

Note that for $m' \geq 5$, the possible values of $(k_1, k_0)$ are $(1, -1)$; $(1, -2)$; or $(2, -2)$. So, the result, that $\Delta(-1) > 0$, is established by considering such pairs directly and by cases for smaller $m'$.

For $\Delta(0)$, we take $m'' = 1$, $j'' = 1$, and $j = 0$.

$$\begin{aligned}
\Delta(0) &= \left(\frac{-3(3 + m')}{m'} + 1 + \frac{3 + m'}{m'}\right) + P(\frac{1}{2} + 2 + m') - P(-\frac{1}{2}) \\
&= 1 - \frac{2(3 + m')}{m'} + P\left(\frac{1}{2} + 2 + m'\right) - P\left(-\frac{1}{2}\right) \\
&= 1 - \frac{2(3 + m')}{m'} + \frac{\left(\frac{1}{2} + 2 + m' - \frac{1}{2}m'k_1\right)^2}{m'} - \frac{\left(\frac{1}{2} + \frac{1}{2}m'k_0\right)^2}{m'} \\
&+ \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\ 0 & \text{if } k_1 - k_0 \text{ even} \\ \frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases}
\end{aligned}$$

For $m' \geq 5$, the possible values of $(k_1, k_0)$ are $(3, -1)$; $(3, 0)$; or $(2, 0)$. So, again the result, that $\Delta(0) < 0$, is established by considering such pairs directly and by cases for smaller $m'$.

This completes the proof of the proposition.

**Remark** We have shown that $F(l, j', m'', j'', j) = S_{\Lambda,\Lambda'',\lambda}$ for integral values of $l$. If $l$ is not an integer, then $\lambda_l := \Lambda' + (lM + J)\alpha$ may not be in $\pi(T_\Lambda^{\Lambda',\Lambda''})$, in which case $S_{\Lambda,\Lambda'',\lambda_l}$ is not defined. On the other hand, if $\lambda_l \in \pi(T_\Lambda^{\Lambda',\Lambda''})$, we note that the equality $F(l, j', m'', j'', j) = S_{\Lambda,\Lambda'',\lambda_l}$ holds, as can be seen by letting $k_l = lM - \frac{1}{2}(j + j' + j'') - 1$ in the above proof.

Now, let us apply the same analysis to the case that $\varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1$. By Lemma 4.1.4, this corresponds to $k_l = -\frac{1}{2}(j + j' + j'') - 1 + lM$. For $\lambda = \Lambda' + k_l\alpha$, let us denote the function $S_{\Lambda,\Lambda'',\lambda}$ by $G_{\mathbb{Z}}(l) = G_{\mathbb{Z}}(l, j', m'', j'', j)$. Thus, $G_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$.

**Lemma 4.1.9** *Define the function $G = G(-, j', m'', j'', j) : \mathbb{R} \to \mathbb{R}$ by*

$$G\left(t, j', m'', j'', j\right) = F\left(t - \frac{j+1}{M}, j', m'', j'', j\right).$$

*Then, $G_{|\mathbb{Z}} = G_{\mathbb{Z}}$.*

*Hence, $S_{\Lambda,\Lambda'',\lambda}$ has a maximum when $l = 0$ or $l = 1$.*

**Proof** By the proof of Proposition 4.1.5 and Remark 4.1, $S_{\Lambda,\Lambda'',\lambda+(j+1)\alpha} = F(l)$, for $\lambda = \Lambda' + k_l\alpha$. Since $\lambda = \Lambda' + (-\frac{1}{2}(j + j' + j'') - 1 + lM)\alpha$, by Proposition 4.1.5, $S_{\Lambda,\Lambda'',\lambda} = F(l - \frac{j+1}{M})$. This proves the lemma.

**Lemma 4.1.10** *Suppose*

$$\Lambda' - (\frac{1}{2}\left(j + j' + j''\right) + 1)\alpha + n_1\delta$$

*and*

$$\Lambda' + \frac{1}{2}\left(j - j' - j''\right)\alpha + n_2\delta$$

*are $\delta$-maximal weights of $L(\Lambda')$. Then the following are equivalent: $n_1 = n_2$, $n_1 = n_2 = 0$, and*

$$j'' + 2 \leq j' - j. \tag{4.5}$$

**Proof** Fix an integer $n$ and consider the set $P_n = \{\nu \in P(\Lambda') \,|\, \nu - \Lambda' = k\alpha + n\delta,\ k \in \mathbb{Z}\}$. We give a description of $P_n \cap P^o(\Lambda')$. Clearly, $P_n = \{\lambda, \lambda - \alpha, \ldots, \lambda - \langle\lambda, \alpha^\vee\rangle\alpha\}$ for some $\lambda = \lambda_n$ and that $\lambda$ is uniquely determined by $n$. Suppose that some $\mu \in P_n$ is not $\delta$-maximal, then none of $\{\mu, \ldots, \mu - \langle\mu, \alpha^\vee\rangle\alpha\}$ are $\delta$-maximal, since if $\mu + k\delta \in P(\Lambda')$, then the whole string $\{\mu + k\delta, \ldots, \mu + k\delta - \langle\mu, \alpha^\vee\rangle\alpha\} \subset P(\Lambda')$. In particular, if $\lambda - \alpha$ is $\delta$-maximal, then so is $\lambda$. Hence,

$\mathfrak{g}_{\delta-\alpha}L(\Lambda')_\lambda = 0$ and $\mathfrak{g}_\alpha L(\Lambda')_\lambda = 0$. Therefore, $\lambda$ is the highest weight $\Lambda'$. Thus, $P_n \cap P^o(\Lambda')$ is either empty, the set $\{\lambda, s_1\lambda\}$, or $\lambda = \Lambda'$ (in the case that $n = 0$).

If $P_n \cap P^o(\Lambda')$ contains $\Lambda' - (\frac{1}{2}(j+j'+j'')+1)\alpha + n\delta$ and $\Lambda' + \frac{1}{2}(j-j'-j'')\alpha + n\delta$, the first two possibilities are impossible. From this and 4.1.1 the lemma follows easily.

From Lemma 4.1.9 and the definition of $F$, it is easy to see that

$$\begin{aligned} &G\left(t, j', m'', j'', j\right) \\ &= G\left(1-t, m'-j', m'', m''-j'', m'+m''-j\right) + \frac{1}{2}(j'+j''-j), \end{aligned} \tag{4.6}$$

for any $t \in \mathbb{R}$. Hence, if the maximum of $G_{\mathbb{Z}}$ occurs at 1, it is equal to

$$G\left(0, m'-j', m'', m''-j'', m'+m''-j\right) + \frac{1}{2}(j'+j''-j). \tag{4.7}$$

We also record the following identity, which is easy to prove from the definition of $F$.

$$\begin{aligned} &F\left(0, j', m'', j'', j\right) \\ &= F\left(0, m'-j', m'', m''-j'', m'+m''-j\right) + \frac{1}{2}(j'+j''-j). \end{aligned} \tag{4.8}$$

As a corollary of 4.1.5 and 4.1.9, we get the following 'Non-Cancellation Lemma'.

**Corollary 4.1.11** *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 4.1.5 and let*

$$\begin{aligned} \mu_\Lambda^{\Lambda',\Lambda''} &:= \max\left\{S_{\Lambda,\Lambda'',\lambda} : \ \lambda \in T_\Lambda^{\Lambda',\Lambda''} \ \ \textit{and}\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\right\}, \\ \bar{\mu}_\Lambda^{\Lambda',\Lambda''} &:= \max\left\{S_{\Lambda,\Lambda'',\lambda} : \ \lambda \in T_\Lambda^{\Lambda',\Lambda''} \ \ \textit{and}\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\}. \end{aligned}$$

*Assume that $\mu_\Lambda^{\Lambda',\Lambda''} = \bar{\mu}_\Lambda^{\Lambda',\Lambda''}$. Then,*

$$\mu_\Lambda^{\Lambda'',\Lambda'} \neq \bar{\mu}_\Lambda^{\Lambda'',\Lambda'}.$$

**Proof** We proceed in two cases:

*Case I.* Suppose the maximum $\bar{\mu}_\Lambda^{\Lambda',\Lambda''}$ occurs when
$\pi(\lambda) = \Lambda' - (\frac{1}{2}(j+j'+j'')+1)\alpha$, (cf. Lemma 4.1.9). This means that the $\delta$-maximal weights of $L(\Lambda')$ through $\Lambda' - (\frac{1}{2}(j+j'+j'')+1)\alpha$ and through $\Lambda' + \frac{1}{2}(j-j'-j'')\alpha$ have the same $\delta$

coordinate (cf. Proposition 4.1.5). By Lemma 4.1.10, we know that this occurs if and only if $\frac{1}{2}(j+j'')+1 \leq \frac{j'}{2}$.

*Case II.* Suppose the maximum $\bar{\mu}_{\Lambda}^{\Lambda',\Lambda''}$ occurs when $\pi(\lambda) = \Lambda' - (\frac{1}{2}(j+j'+j'')+1-M)\alpha$. Then, by the identities (4.7) and (4.8), we get

$$G(0, m'-j', m'', m''-j'', m'+m''-j) = F(0, m'-j', m'', m''-j'', m'+m''-j). \qquad (4.9)$$

So, from the case I, we get in case II, $\mu_{\Lambda}^{\Lambda',\Lambda''} = \bar{\mu}_{\Lambda}^{\Lambda',\Lambda''}$ if and only if

$$\frac{1}{2}\big((m'+m''-j)+(m''-j'')\big)+1 \leq \frac{1}{2}(m'-j'). \qquad (4.10)$$

So, if either of the inequalities (4.5) or (4.10) is satisfied, then none of them can be satisfied for the triple $(\Lambda, \Lambda', \Lambda'')$ replaced by $(\Lambda, \Lambda'', \Lambda')$. This proves the corollary.

**Definition 4.1.12** *Let $\Lambda' \in P_+^{(m')}, \Lambda'' \in P_+^{(m'')}$ and $\Lambda \in P_+^{(m'+m'')}$. Then, we call $L(\Lambda+n\delta)$ the $\delta$-maximal component of $L(\Lambda') \otimes L(\Lambda'')$ through $\Lambda$ if $L(\Lambda+n\delta)$ is a submodule of $L(\Lambda')\otimes L(\Lambda'')$ but $L(\Lambda+m\delta)$ is not a component for any $m > n$.*

**Theorem 4.1.13** *Let $\Lambda', \Lambda'', \Lambda$ be as in Proposition 4.1.5 . Then, $L(\Lambda+n\delta)$ is a $\delta$-maximal component of $L(\Lambda')\otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where $n_1$ is such that $\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda')$ and $n_2$ is such that $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$.*

**Proof** This follows immediately by combining Propositions 3.1.1, 4.1.5 and Lemma 4.1.4.

## 4.2 Saturation factor for $A_1^{(1)}$

**Lemma 4.2.1** *Fix a positive integer $N$. Let $\Lambda \in \bar{P}_+$ and let $\lambda \in \Lambda + Q$, where $Q$ is the root lattice $\mathbb{Z}\alpha \oplus \mathbb{Z}\delta$ of $A_1^{(1)}$. Then, $N\lambda \in P^o(N\Lambda)$ if and only if $\lambda \in P^o(\Lambda)$.*

**Proof** The validity of the lemma is clear for $\lambda \in P^o(\Lambda)_+$ from Corollary 4.1.1. But since $P^o(\Lambda) = W \cdot (P^o(\Lambda)_+)$, and the action of $W$ on $\mathfrak{h}^*$ is linear, the lemma follows for any $\lambda \in P^o(\Lambda)$.

**Definition 4.2.2** *A positive integer $d_o$ is called a* saturation factor *for $\mathfrak{g}$ if for any $\Lambda$, $\Lambda'$, $\Lambda'' \in P_+$ such that $\Lambda - \Lambda' - \Lambda'' \in Q$ and $L(N\Lambda)$ is a submodule of $L(N\Lambda')\otimes L(N\Lambda'')$, for some $N \in \mathbb{Z}_{>0}$, then $L(d_o\Lambda)$ is a submodule of $L(d_o\Lambda')\otimes L(d_o\Lambda'')$.*

**Corollary 4.2.3** *Any $d_o \in \mathbb{Z}_{>1}$ is a saturation factor for $A_1^{(1)}$.*

**Proof** If $\Lambda'(c) = 0$ or $\Lambda''(c) = 0$, then

$$L(N\Lambda') \otimes L(N\Lambda'') \simeq L(N(\Lambda' + \Lambda'')),$$

for any $N \geq 1$. Thus, the corollary is clearly true in this case. So, let us assume that both of $\Lambda'(c) > 0$ and $\Lambda''(c) > 0$. Let $L(N\Lambda + n\delta)$ be the $\delta$-maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ through $L(N\Lambda)$, for some $n \geq 0$. For any $\Psi \in P_+$, let $\bar{\Psi} \in \bar{P}_+$ be the projection $\pi(\Psi)$ defined just before Lemma 4.1.2. Applying 4.1.13 to $\bar{\Lambda}', \bar{\Lambda}'', \bar{\Lambda}$, and observing that

$$L(\bar{\Psi} + k\delta) \simeq L(\bar{\Psi}) \otimes L(k\delta) \tag{4.11}$$

and $L(k\delta)$ is one dimensional, we get that there is a $\delta$-maximal component $L(\Lambda + \widetilde{n}\delta)$ of $L(\Lambda') \otimes L(\Lambda'')$ through $L(\Lambda)$, for some (unique) $\widetilde{n} \in \mathbb{Z}$.

Again applying Theorem 4.1.13 to $N\bar{\Lambda}', N\bar{\Lambda}'', N\bar{\Lambda}$, and observing (using Corollary 4.1.1) that

$$P^o(N\bar{\Psi}) \supset NP^o(\bar{\Psi}), \tag{4.12}$$

we get that $L(N\Lambda + N\widetilde{n}\delta)$ is the $\delta$-maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ through $L(N\Lambda)$. Thus, $n = N\widetilde{n}$. In particular,

$$\widetilde{n} \geq 0. \tag{4.13}$$

Let

$$\sum_{\lambda \in T_{\bar{\Lambda}}^{\Lambda', \Lambda''}} \varepsilon(v_{\bar{\Lambda}, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\bar{\Lambda}, \Lambda'', \lambda}\delta} = \sum_{k \in \mathbb{Z}_+} c_k e^{(\Lambda(d) + \widetilde{n} - k)\delta}, \tag{4.14}$$

for some $c_k \in \mathbb{Z}_+$ with $c_0$ nonzero. By Proposition 3.1.1, this is the character of a unitarizable Virasoro representation with each irreducible component having the same nonzero central charge. Thus, by Lemma 2.1.1, for any $k > 1$, we get $c_k \neq 0$.

By the above argument, $L(d_o\Lambda + d_o\widetilde{n}\delta)$ is the $\delta$-maximal component of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$

through $L(d_o\Lambda)$. If $\widetilde{n} = 0$, we get that

$$L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'').$$

If $\widetilde{n} > 0$, then $d_o\widetilde{n}$ being $> 1$, by the analogue of (4.14) for $d_o\Lambda', d_o\Lambda''$ and $d_o\Lambda$, $L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'')$. This proves the corollary.

**Remark** We note that $L(2\Lambda_0 - \delta)$ is not a component of $L(\Lambda_0) \otimes L(\Lambda_0)$ (cf. [3, Exercise 12.16]). But, of course, $L(2\Lambda_0)$ is a $\delta$-maximal component. By the identity (4.14), we know that $L(2d_o\Lambda_0 - d_o\delta)$ must be a component of $L(d_o\Lambda_0) \otimes L(d_o\Lambda_0)$, for any $d_o > 1$. So $d_o$ can not be taken to be 1 in Corollary 4.2.3.

## 4.3 Saturated tensor cone for $A_1^{(1)}$

**Theorem 4.3.1** *Let $\mathfrak{g} = A_1^{(1)}$. Let $\Lambda', \Lambda'', \Lambda \in P_+$ be such that $\Lambda' + \Lambda'' - \Lambda \in Q$ and both of $\Lambda'(c)$ and $\Lambda''(c)$ are nonzero. Then, the following are equivalent:*

*(a)* $(\Lambda', \Lambda'', \Lambda) \in \Gamma$.

*(b) The following set of inequalities is satisfied for all $w \in W$ and $x_i \in \mathfrak{h}$, such that $\alpha_j(x_i) = \delta_{i,j}$ for $i, j = 0, 1$:*

$$\Lambda'(x_i) + \Lambda''(wx_i) - \Lambda(wx_i) \geq 0, \quad \textit{and}$$
$$\Lambda'(wx_i) + \Lambda''(x_i) - \Lambda(wx_i) \geq 0.$$

**Proof** By Lemma 4.1.2, there exist (unique) $n_1, n_2 \in \mathbb{Z}$ such that

$$\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda'), \quad \text{and} \quad \Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'').$$

Let $n := \min\,(n_1, n_2)$. By our description of the $\delta$-maximal components as in 4.1.13 applied to $\bar{\Lambda}', \bar{\Lambda}'', \bar{\Lambda}$ and using the identity (4.11), we see that $L(\Lambda + n\delta)$ is a $\delta$-maximal component of $L(\Lambda') \otimes L(\Lambda'')$. Thus, by the (4.12), for any $N \geq 1$, $L(N\Lambda + Nn\delta)$ is a $\delta$-maximal component of

$L(N\Lambda') \otimes L(N\Lambda'')$. In particular, by 3.1.1 and 2.1.1,

$$L(N\Lambda) \subset L(N\Lambda') \otimes L(N\Lambda'') \quad \text{for some } N > 1 \quad \text{if and only if } n \geq 0. \tag{4.15}$$

By 1.2.3, if a weight $\gamma + k\delta \in P(\Lambda')$ (for some $k \in \mathbb{Z}_+$), then $\gamma \in P(\Lambda')$. Thus,

$$n \geq 0 \quad \text{if and only if } \Lambda \in \left(P(\Lambda') + \Lambda''\right) \cap \left(P(\Lambda'') + \Lambda'\right). \tag{4.16}$$

We next show that

$$P(\Lambda') = (\Lambda' + Q) \cap C'_\Lambda, \tag{4.17}$$

where $C'_\Lambda := \{\gamma \in \mathfrak{h}^* : \Lambda'(x_i) - \gamma(wx_i) \geq 0 \text{ for all } w \in W \text{ and all } x_i\}$. Clearly,

$$P(\Lambda') \subset (\Lambda' + Q) \cap C'_\Lambda.$$

Since $\Lambda' + Q$ and $C'_\Lambda$ are $W$-stable, and $\Lambda' + Q$ is contained in the Tits cone (by [7, 13.1.E.8.a]), $(\Lambda' + Q) \cap C'_\Lambda = W \cdot ((\Lambda' + Q) \cap C'_\Lambda \cap P_+)$.

Conversely, take $\gamma \in (\Lambda' + Q) \cap C'_\Lambda \cap P_+$. Then, $(\Lambda' - \gamma)(x_i) \geq 0$ and $(\Lambda' - \gamma)(c) = 0$ and hence $\Lambda' - \gamma \in \oplus_i \mathbb{Z}_+ \alpha_i$, i.e., $\Lambda' \geq \gamma$. Thus, by 1.2.3, $\gamma \in P(\Lambda')$. This proves (4.17). Now, combining (4.15), (4.16) and (4.17), we get $L(N\Lambda) \subset L(N\Lambda') \otimes L(N\Lambda'')$ for some $N > 1$ if and only if for all $w \in W$ and $i = 0, 1$,

$$\Lambda'(x_i) + (\Lambda'' - \Lambda)(wx_i) \geq 0, \ \text{and} \Lambda''(x_i) - (\Lambda - \Lambda')(x_i) \geq 0.$$

This proves the equivalence of (a) and (b) in the theorem.

# CHAPTER 5: SATURATED TENSOR CONE FOR $A_2^{(2)}$

## 5.1 The algebra $A_2^{(2)}$

Let $A_2^{(2)}$ be the Kac-Moody algebra with generalized Cartan matrix [7, ch.1]

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Fix a realization of $A_2^{(2)}$: $\mathfrak{h} := \mathbb{C}c \oplus \mathbb{C}\alpha^\vee \oplus \mathbb{C}d$ and $\mathfrak{h}^* = \mathbb{C}\omega_1 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta$ where $\alpha(\alpha^\vee) = 2$, $\delta(d) = 1$, $\omega_0(c) = 1$, and all other combinations 0. We have simple roots $\{\alpha_0 := \delta - 2\alpha, \alpha_1 := \alpha\}$ and simple coroots $\{\alpha_0^\vee := c - \frac{1}{2}\alpha^\vee, \alpha_1^\vee := \alpha^\vee\}$. Equivalently, by [3, ch.8], $A_2^{(2)}$ is isomorphic to the subalgebra of the untwisted affine Kac-Moody algebra associated to $\mathfrak{sl}_3$ which is fixed by the order 2 automorphism

$$t^j \otimes x + \mu d + zc \mapsto (-1)^j t^j \otimes \tau(x) + \mu d + zc,$$

where $\tau \in \mathrm{Aut}_{\mathrm{Lie}}\, \mathfrak{sl}_3$ is the nontrivial diagram automorphism. Write the fundamental weights of $A_2^{(2)}$ as $\omega_0$ and $\omega_1 = \frac{1}{2}\omega_0 + \frac{1}{2}\alpha$ and therefore the dominant weights of a fixed level $m$ are precisely the weights with $\alpha$ coordinate between 0 and $m$ (inclusive) so that the level and $\alpha$ coordinate are equivalent modulo 1. This easily allows one to compute the dominant $\delta$-maximal weights. If $\lambda = m_0\omega_0 + m_1\omega_1$ is the highest weight of an integrable representation, by 1.2.3,

$$P^o(\lambda) \cap P_+ = \{\lambda - j\alpha,\ \lambda + k(2\alpha - \delta),\ \lambda + \alpha - \delta + l(2\alpha - \delta) \mid j,k,l \in \mathbb{Z}_{\geq 0}\} \cap P_+$$

and $P^o(\lambda) = W(P^o(\lambda) \cap P_+)$.

Let $T_k := (s_0 s_1)^k$, then by a simple computation:

$$\begin{aligned} T_k(m\omega_0 + j\alpha + n\delta) = & \quad (s_0s_1)^k(m\omega_0 + j\alpha + n\delta) \\ = & \quad m\omega_0 + (2km + j)\alpha + (n - mk^2 - jk)\delta \\ = & \quad m\omega_0 + j\alpha + n\delta + 2km\alpha - k(km + j)\delta. \end{aligned}$$

## 5.2 Computation of some $\delta$-maximal components for $A_2^{(2)}$

We proceed in a very similar manner to the $A_1^{(1)}$ case. First we must compute

$$T_\Lambda^{\Lambda',\Lambda''} = \left\{\lambda \in P^o(\Lambda') \mid \lambda + \Lambda'' + \rho \in W(\Lambda + \rho) \bmod \mathbb{C}\delta\right\}.$$

Note that $\rho = \frac{3}{2}\omega_1 + \frac{1}{2}\alpha$. We decompose $W$ as $T_\mathbb{Z} \sqcup s_1 T_\mathbb{Z}$. Hence, let

$$T_{\Lambda,+}^{\Lambda',\Lambda''} = \left\{\lambda \in P^o(\Lambda') \mid \lambda + \Lambda'' + \rho \in T_\mathbb{Z}(\Lambda + \rho) \bmod \mathbb{C}\delta\right\},$$

and

$$T_{\Lambda,-}^{\Lambda',\Lambda''} = \left\{\lambda \in P^o(\Lambda') \mid \lambda + \Lambda'' + \rho \in T_\mathbb{Z} s_1(\Lambda + \rho) \bmod \mathbb{C}\delta\right\}.$$

We have that

$$\begin{aligned} T_{\Lambda,+}^{\Lambda',\Lambda''} &= \left\{\lambda \in P^o(\Lambda') \mid \lambda = \Lambda' + k\alpha + n_{\Lambda',k}\delta,\ k \in -\frac{1}{2}(m_1' + m_1'' - m_1) + (2m+3)\mathbb{Z}\right\} \\ T_{\Lambda,-}^{\Lambda',\Lambda''} &= \left\{\lambda \in P^o(\Lambda') \mid \lambda = \Lambda' + k\alpha + n_{\Lambda',k}\delta,\ k \in -\frac{1}{2}(m_1' + m_1'' + m_1) - 1 + (2m+3)\mathbb{Z}\right\}. \end{aligned}$$

**Claim 5.2.1** *Assume $\lambda \in T_\Lambda^{\Lambda',\Lambda''}$. Write $\lambda = \Lambda' + k\alpha + n\delta$. Then*

$$v_{\Lambda,\Lambda'',\lambda} = \begin{cases} T_{\frac{k+\frac{1}{2}(m_1'+m_1''-m_1)}{2m+3}} & \text{if } k \equiv -\frac{1}{2}(m_1' + m_1'' - m_1) \mod (2m+3) \\ T_{\frac{k+\frac{1}{2}(m_1'+m_1''+m_1)+1}{2m+3}} s_1 & \text{if } k \equiv -\frac{1}{2}(m_1' + m_1'' + m_1) - 1 \mod (2m+3) \end{cases}.$$

Now we can compute $S_{\Lambda,\Lambda'',\lambda}$, for $\delta S_{\Lambda,\Lambda'',\lambda} = \Lambda'' + \lambda + \rho - v_{\Lambda,\Lambda'',\lambda}(\Lambda + \rho)$, keeping the same notation as above we get:

$$\delta S_{\Lambda,\Lambda'',\lambda} = \Lambda'' + \lambda + \rho - v_{\Lambda,\Lambda'',\lambda}(\Lambda+\rho) = n + \frac{(k+\frac{1}{2}(m_1'+m_1''-m_1))(k+1+\frac{1}{2}(m_1'+m_1''+m_1))}{2(2m+3)}$$

Using the fact that $P^o(\Lambda') = WP^o_{\Lambda'+}$, let us compute the $n$ above. Let $\lambda \in P^{\delta\,\max}_{\Lambda'}$. Write $\lambda = m'\omega_1 + (2m'q+r)\alpha + n_{qm'+r}\delta$, $0 \le r < 2m'$. Now, $n_r$ is known for $0 \le r \le m'$. $P(\Lambda')$ is $W$ invariant, so by 1.2.3.b, we can write $n_k$ in the following way:

$$n_k = -\frac{k}{2m'}(\frac{k+m_1'}{2}) + P(k)$$

where $P(k)$ is a periodic function with period $2m'$. Moreover, we have essentially computed $P(k)$ when we determined the dominant $\delta$- maximal weights, since the dominant weights along with their images under $s_1$ gives a fundamental domain for the action of the translational part of the Weyl group on the set of weights of $L(\Lambda')$.

First, consider the case that $\lambda \in T^{\Lambda',\Lambda''}_{\Lambda,+}$.

Let $k = lM - \frac{1}{2}(m_1'+m_1''-m_1) = 2qm' + r - \frac{m_1'}{2}$ where $M = 2(m'+m'')+3$, then

$$\begin{aligned} S_{\Lambda,\Lambda'',\lambda} = & -\frac{lM-\frac{1}{2}(m_1''+m_1'-m_1)}{2m'}(\frac{lM-\frac{1}{2}(m_1''-m_1'-m_1)}{2}) + \frac{l(lM+1+m_1)}{2} \\ & +P(lM-\frac{1}{2}(m_1'+m_1''-m_1)) \\ = & \frac{l^2}{2}M(1-\frac{M}{2m'}) + \frac{l}{2}(1+m_1-\frac{M(m_1-m_1'')}{2m'}) - \frac{(m_1-m_1'')^2-(m_1')^2}{16m'} \\ & +P(lM-\frac{1}{2}(m_1'+m_1''-m_1)) \end{aligned}$$

To use the method applied to the $A_1^{(1)}$ case, a suitable piecewise smooth function must be introduced. The simplest way to do this in the case at hand is to use 2 piecewise quadratic functions. The upper function $P^+$ is given by

$$P^+(s) = \begin{cases} \frac{(s+\frac{1}{2}m'-m'x)^2}{4m'} - \frac{m_1'^2}{16m'} & \text{if } |s+\frac{1}{2}m'-m'x| \le \frac{m_1'}{2} \text{ for some } x \in 2\mathbb{Z} \\ \frac{(s+\frac{1}{2}m'-m'x)^2}{4m'} - \frac{(m_1'-2m')^2}{16m'} & \text{if } |s+\frac{1}{2}m'-m'x| \le \frac{2m'-m_1'}{2} \text{ for some } x \in 2\mathbb{Z}+1 \end{cases}$$

the lower function is given by

$$P^-(s) = \begin{cases} \frac{(s+\frac{1}{2}m'-m'x)^2}{4m'} - \frac{m_1'^2}{16m'} & \text{if } |s+\frac{1}{2}m'-m'x| \leq \frac{m_1'}{2} - 1 \text{ for some } x \in 2\mathbb{Z} \\ \frac{(s+\frac{1}{2}m'-m'x)^2}{4m'} - \frac{(m_1'-2m')^2}{16m'} - \frac{1}{2} & \text{if } |s+\frac{1}{2}m'-m'x| \leq \frac{2m'-m_1'}{2} + 1 \text{ for some } x \in 2\mathbb{Z}+1 \end{cases}$$

Let

$$\begin{aligned} F^+(t) = &\frac{t^2}{2}M(1-\frac{M}{2m'}) + \frac{t}{2}(1+m_1-\frac{M(m_1-m_1'')}{2m'}) - \frac{(m_1-m_1'')^2-(m_1')^2}{16m'} \\ &+ P^+(tM-\frac{1}{2}(m_1'+m_1''-m_1)) \end{aligned}$$

and

$$\begin{aligned} F^-(t) = &\frac{t^2}{2}M(1-\frac{M}{2m'}) + \frac{t}{2}(1+m_1-\frac{M(m_1-m_1'')}{2m'}) - \frac{(m_1-m_1'')^2-(m_1')^2}{16m'} \\ &+ P^-(tM-\frac{1}{2}(m_1'+m_1''-m_1)) \end{aligned}$$

Then

$$\begin{aligned} \Delta^+(t) =&F^+(t+1) - F^+(t) \\ =&tM(1-\frac{M}{2m'}) - \frac{M}{4m'}(M+m_1-m_1'') + \frac{1+M+m_1}{2} \\ &+ P^+(tM+M-\frac{1}{2}(m_1'+m_1''-m_1)) - P^+(tM-\frac{1}{2}(m_1'+m_1''-m_1)) \end{aligned}$$

Hence the derivative of $\Delta^+(t)$, where it exists, is equal to

$$\begin{aligned} {\Delta^+}'(t) =&M(1-\frac{M}{2m'}) + M({P^+}'(tM+M-\frac{1}{2}(m_1'+m_1''-m_1)) - {P^+}'(tM-\frac{1}{2}(m_1'+m_1''-m_1))) \\ =&M(1-\frac{M}{2m'}) + M(\frac{M+m'k_0-m'k_1}{2m'}) \\ =&M(1+\frac{1}{2}(k_0-k_1)) \end{aligned}$$

The final quantity must be non-positive since $M > 2m'$ and $k_1 - k_0 \geq 2$. The identical computation holds for ${\Delta^-}'(t)$.

Extend the values of some of the parameters: $m'' \in [\frac{1}{2}, \infty)$, $m_1'' \in [0, 2m'']$, and $m_1 \in [0, 2m' +$

$2m'']$. We shall denote the set of these parameter values as $I \subset \mathbb{R}^3$, therefore we may write $\Delta^\pm : \mathbb{R} \times I \to \mathbb{R}$ and $\Delta^\pm(t) = \Delta^\pm\big|_{\{t\}\times I}$. Thus $\Delta^\pm$ are continuous and piecewise smooth on $\mathbb{R} \times I$.

**Claim 5.2.2** $\Delta^\pm(0)$ *achieves its maximum on* $I$ *when* $m_2'' = 1, m_1 = 0, m'' = \frac{1}{2}$. $\Delta^\pm(-1)$ *achieves it's minimum when* $m_1'' = 0, m'' = \frac{1}{2}, m_1 = 2m' + 1$.

**Proof** We compute and give bounds for derivatives, where they exist.

$$\partial_{m_1''}\Delta^\pm = \frac{M}{4m'} - \frac{1}{2}P^{\pm\prime}(tM + M - \frac{1}{2}(m_1' + m_1'' - m_1)) + \frac{1}{2}P^{\pm\prime}(tM - \frac{1}{2}(m_1' + m_1'' - m_1))$$

Now, $|P^{\pm\prime}| \le \frac{1}{2}$, so $\frac{M}{4m'} + \frac{1}{2} \ge \partial_{m_1''}\Delta \ge \frac{M}{4m'} - \frac{1}{2} = \frac{2m''+3}{4m'} > 0$.

For (2):

$$\begin{aligned}
\partial_{m_1}\Delta^\pm &= -\frac{M}{4m'} + \frac{1}{2} + \frac{1}{2}P^+(tM + M - \frac{1}{2}(m_1' + m_1'' - m_1)) - \frac{1}{2}P^+(tM - \frac{1}{2}(m_1' + m_1'' - m_1)) \\
&= \frac{1}{2} + \frac{1}{2}\left(\frac{m'k_0 - m'k_1}{2m'}\right) \\
&= \frac{1}{2}(1 + \frac{k_0 - k_1}{2}) \le 0
\end{aligned}$$

Now let us specialize to when $m_1'' = 2m''$ and $m_1 = 0$ and $t = 0$.

$$\begin{aligned}
\Delta^\pm(0) = &-\frac{M}{4m'}(M - 2m'') + \frac{1+M}{2} \\
&+ P^+(M - \frac{1}{2}(m_1' + 2m'')) - P^+(-\frac{1}{2}(m_1' + 2m'')) \\
= &-\frac{2m' + 2m'' + 3}{4m'}(2m' + 3) + m' + m'' + 2 \\
&+ P^\pm(2m' + m'' + 3 - \frac{1}{2}m_1') - P^\pm(-\frac{1}{2}m_1' - m'')
\end{aligned}$$

Now take the derivative of the above expression with respect to $m''$, we get

$$\begin{aligned}
&-\frac{3}{2m'} \\
&+ P^{\pm\prime}(2m' + m'' + 3 - \frac{1}{2}m_1') + P^{\pm\prime}(-\frac{1}{2}m_1' - m'') \\
=&-\frac{3}{2m'} + \frac{2m' + 3 - k_1 m' - k_0 m'}{2m'} \\
=&\frac{2m' - k_1 m' - k_0 m'}{2m'} = 1 - \frac{k_1 + k_0}{2} \leq 0
\end{aligned}$$

Indeed, $k_1 + k_0 \geq 2$ and $0 \leq m_1' \leq 2m'$.

For $\Delta^{\pm}(-1)$ let $\mathrm{m}_2'' = 0, m_1 = 2m' + 2m''$.

$$\begin{aligned}
\Delta^{+}(-1) =& -M(1 - \frac{M}{2m'}) - \frac{M}{4m'}(2M - 3) + M - 1 \\
&+ P^{+}(-\frac{1}{2}m_1' + m' + m'')) - P^{+}(-\frac{1}{2}m_1' - m' - m'' - 3)
\end{aligned}$$

Taking derivative with respect to $m''$

$$\begin{aligned}
&\frac{3}{2m'} \\
&+ P^{+\prime}(-\frac{1}{2}m_1' + m' + m'') + P^{+\prime}(-\frac{1}{2}m_1' - m' - m'' - 3) \\
=&\frac{1}{2m'}\left(-k_1 m' - k_0 m'\right) = -\frac{k_1 + k_0}{2} \geq 0
\end{aligned}$$

Since $k_1 + k_0 \leq 0$.

So let us consider the case when $\Delta^{\pm}(-1)$ is a small as possible, that is when $m_1'' = 0$, $m_1 = 2m' + 1$, and $m'' = \frac{1}{2}$.

$$
\begin{aligned}
\Delta^+(-1) \geq & \frac{1}{2} + \frac{3}{m'} \\
& + P^+(\frac{1}{2} + m' - \frac{1}{2}m'_1) - P^+(-\frac{7}{2} - m' - \frac{1}{2}m'_1) \\
= & -1 + \frac{(k_1 + k_0)}{4}(m'k_1 - m'k_0 - 1 - 2m') - \frac{3}{2}k_0 \\
& + \begin{cases} 0 & k_1 - k_0 \text{is even} \\ \frac{m' - m'_1}{4} & k_1 \text{even and } k_0 \text{odd} \\ \frac{m'_1 - m'}{4} & k_0 \text{even and } k_1 \text{odd} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\Delta^-(-1) \geq & \frac{1}{2} + \frac{3}{m'} \\
& + P^-(\frac{1}{2} + m' - \frac{1}{2}m'_1) - P^-(-\frac{7}{2} - m' - \frac{1}{2}m'_1) \\
= & -1 + \frac{(k_1 + k_0)}{4}(m'k_1 - m'k_0 - 1 - 2m') - \frac{3}{2}k_0 \\
& + \begin{cases} 0 & k_1 - k_0 \text{is even} \\ \frac{m' - m'_1 + 2}{4} & k_1 \text{even and } k_0 \text{odd} \\ \frac{m'_1 - m' - 2}{4} & k_0 \text{even and } k_1 \text{odd} \end{cases}
\end{aligned}
$$

For $\Delta^+$ we have $k_1 = 1$ provided $2m' \neq m'_1$and $k_0 = -1$ when $m'_1 \leq 2m' - 7$ and $k_0 = -2$ otherwise. If $2m' = m'_1$ then $k_1 = 2$. For$\Delta^-$, $k_1$ is always 1.

- $k_1 = 1, k_0 = -1$

$$
\Delta^{\pm}(-1) \geq \qquad \frac{1}{2}
$$

- $k_1 = 1, k_0 = -2$

$$
\Delta^+(-1) \geq \qquad 2 + \frac{1}{4}(m'_1 - 2m' + 1) \geq \frac{1}{2}
$$

- $k_1 = 2, k_0 = -2$

$$\Delta^+(-1) \geq \qquad 2$$

An upper bound for $\Delta^+(0)$ is computed as follows:$m''_2 = 1, m_1 = 0, m'' = \frac{1}{2}$

$$\begin{aligned}
\Delta^{\pm}(0) =& -\frac{M}{4m'}(M + m_1 - m''_1) + \frac{1 + M + m_1}{2} \\
&+ P^+(M - \frac{1}{2}(m'_1 + m''_1 - m_1)) - P^+(-\frac{1}{2}(m'_1 + m''_1 - m_1)) \\
\leq& -1 - \frac{3}{m'} \\
&+ P^+(2m' + \frac{7}{2} - \frac{m'_1}{2}) - P^+(-\frac{1}{2} - \frac{m'_1}{2}) \\
=&\frac{5}{2} - (k_0 + k_1)\frac{m'(k_1 - k_0) + 1}{4} - \frac{3}{2}k_1 - (k_1 - 1)m' \\
&+ \begin{cases} 0 & k_1 - k_0 \text{is even} \\ \frac{m' - m'_1}{4} & k_1 \text{even and } k_0 \text{odd} \\ \frac{m'_1 - m'}{4} & k_0 \text{even and } k_1 \text{odd} \end{cases} \\
&< 0
\end{aligned}$$

Indeed, $k_0 + k_1 \geq 0$, $k_1 - k_0 \geq 0$, and $k_1 \geq 2$.

This tells us that the

Now, suppose that $\lambda \in T^{\Lambda', \Lambda''}_{\Lambda, -}$. We have

$$\begin{aligned}
S_{\Lambda, \Lambda'', \lambda} = \quad & -\frac{lM - \frac{1}{2}(m'_1 + m''_1 + m_1) - 1}{2m'}(\frac{lM + \frac{1}{2}(m'_1 - m''_1 - m_1) - 1}{2}) + \frac{(lM - m_1 - 1)l}{2} \\
& + P(lM - \frac{1}{2}(m'_1 + m''_1 + m_1) - 1)
\end{aligned}$$

So let

$$\begin{aligned}
G^{\pm}(l) = \quad & -\frac{lM - \frac{1}{2}(m'_1 + m''_1 + m_1) - 1}{2m'}(\frac{lM + \frac{1}{2}(m'_1 - m''_1 - m_1) - 1}{2}) + \frac{(lM - m_1 - 1)l}{2} \\
& + P^{\pm}(lM - \frac{1}{2}(m'_1 + m''_1 + m_1) - 1)
\end{aligned}$$

It is easy to see that $F^{\pm}(l) = G^{\pm}(l + \frac{m_1+1}{M})$. This means that $S_{\Lambda,\Lambda'',\lambda}$ is maximized when $\lambda \equiv \Lambda' - (\frac{1}{2}(m_1' + m_1'' + m_1) + 1)\alpha$ or when $\lambda \equiv \Lambda' - (\frac{1}{2}(m_1' + m_1'' + m_1) + 1 - M)\alpha$

**Lemma 5.2.3** *Non-Cancellation Lemma: Suppose that $\Lambda' + \Lambda'' + \Lambda \in Q$ such that that $m_1', m_1'' \neq 1$ and $m', m'' \geq 2$, then if cancellation occurs at the head of $\sum_{\lambda \in T_\Lambda^{\Lambda',\Lambda''}} \varepsilon(v_{\Lambda,\Lambda'',\lambda}) c_{\Lambda',\lambda} e^{S_{\Lambda,\Lambda'',\lambda}\delta}$, then cancellation does not occur at the head of $\sum_{\lambda \in T_\Lambda^{\Lambda'',\Lambda'}} \varepsilon(v_{\Lambda,\Lambda',\lambda}) c_{\Lambda'',\lambda} e^{S_{\Lambda,\Lambda',\lambda}\delta}$.*

**Proof** Suppose

$$\max\left\{S_{\Lambda,\Lambda'',\lambda} \middle|\, \lambda \in T_\Lambda^{\Lambda',\Lambda''},\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\} = \max\left\{S_{\Lambda,\Lambda'',\lambda} \middle|\, \lambda \in T_\Lambda^{\Lambda',\Lambda''},\ \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = 1\right\}$$

. We proceed in cases:

- Suppose the maximum of $\left\{S_{\Lambda,\Lambda'',\lambda} \middle|\, \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\}$ occurs when

$$\lambda \equiv \Lambda' - (\frac{1}{2}\left(m_1 + m_1' + m_1''\right) + 1)\alpha \mod \mathbb{C}\delta.$$

  This means that the $\delta$-maximal weights of $L(\Lambda')$ through $\Lambda' - (\frac{1}{2}\left(m_1 + m_1' + m_1''\right) + 1)\alpha$ and $\Lambda' + \frac{1}{2}\left(m_1 - m_1' - m_1''\right)\alpha$ have the same $\delta$ coordinate. By our knowledge of the $\delta$-maximal weights of $L(\Lambda')$, we know that this occurs only if

$$\begin{gathered} -\frac{m_1'}{2} \leq \frac{1}{2}m_1 - \frac{1}{2}m_1'' \leq \frac{m_1'}{2} \\ -\frac{m_1'}{2} \leq -\frac{1}{2}m_1 - \frac{1}{2}m_1'' - 1 \leq \frac{m_1'}{2} \end{gathered}$$

  or

$$\frac{1}{2}\left(m_1 + m_1''\right) + 1 = \frac{1}{2}\left(m_1 - m_1''\right) + C,$$

  where $C$ can be 0,1, or $-1$. The latter is impossible unless $C = 1$, $m_1 = 0$, $m_1' = 1$, which is ruled out by the hypotheses of the Lemma. Note that if the two inequalities hold, then they do not if $(m', m_1')$ is interchanged with $(m'', m_1'')$. To see this, simply add one inequality to the other with $(m', m_1')$ and $(m'', m_1'')$ swapped.

- Suppose the maximum of $\left\{S_{\Lambda,\Lambda'',\lambda} |\, \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\}$ occurs when

$$\lambda \equiv \Lambda' - (\frac{1}{2}\left(m_1 + m_1' + m_1''\right) + 1 - M)\alpha \mod \mathbb{C}\delta.$$

  We describe a necessary condition for cancellation as follows: the line joining the $\delta$-maximal weights through $\Lambda' - (\frac{1}{2}(m_1' + m_1'' + m_1) + 1 - M)\alpha$ and $\Lambda' + \frac{1}{2}\left(m_1 - m_1' - m_1''\right)\alpha$ must have direction $2\alpha - \delta$. Since we know the $\delta$-maximal weights of $L(\Lambda')$, we can write this in terms of the inequalities:

$$0 \leq -\frac{1}{2}(-m_1 + m_1'' + m_1') \leq 2m' - m_1'$$
$$0 \leq 2m' + 2m'' + 2 - \frac{1}{2}(m_1 + m_1'' + m_1') \leq 2m' - m_1'$$

  or, in the case that $-m_1'' + m_1 = m_1' - 2$, $2m'' - 2 - \frac{1}{2}m_1 - \frac{1}{2}m_1'' \leq -\frac{1}{2}m_1' + 1$ (Note that this case implies that $m'' \leq 1$, which is explicitly ruled out by the hypotheses). There may be cancellation if $m' - \frac{1}{2}m_1 + \frac{1}{2}m_1'' = m' + 2m'' + 2 - \frac{1}{2}m_1 - \frac{1}{2}m_1''$, but this is clearly impossible, since $m_1'' \leq 2m''$.

- This is the case that covers the possibility that $\left\{S_{\Lambda,\Lambda'',\lambda} |\, \varepsilon(v_{\Lambda,\Lambda'',\lambda}) = -1\right\}$ occurs when $\lambda \equiv \Lambda' - (\frac{1}{2}\left(m_1 + m_1' + m_1''\right) + 1)\alpha \mod \mathbb{C}\delta$ but that $\left\{S_{\Lambda,\Lambda',\lambda} |\, \varepsilon(v_{\Lambda,\Lambda',\lambda}) = -1\right\}$ occurs when $\lambda \equiv \Lambda'' - (\frac{1}{2}\left(m_1 + m_1' + m_1''\right) + 1 - M)\alpha \mod \mathbb{C}\delta$. For this to happen, it is necessary that

$$0 \leq \frac{1}{2}m_1 - \frac{1}{2}m_1'' + \frac{1}{2}m_1' \leq m_1'$$
$$0 \leq -\frac{1}{2}m_1 - \frac{1}{2}m_1'' - 1 + \frac{1}{2}m_1' \leq m_1'$$

  and

$$0 \leq \frac{1}{2}m_1 - \frac{1}{2}m_1'' - \frac{1}{2}m_1' \leq 2m'' - m_1''$$
$$0 \leq 2m' + 2m'' + 2 - \frac{1}{2}m_1 - \frac{1}{2}m_1'' - \frac{1}{2}m_1' \leq 2m'' - m_1''$$

  These inequalities are incompatible, since $\frac{1}{2}m_1 + \frac{1}{2}m_1'' + 1 - \frac{1}{2}m_1' \leq 0$, adding this inequality to the last inequality yields $2m' + 3 - m_1' \leq -m_1''$, which is impossible because $2m' - m_1' \geq 0$

and $m_1'' \geq 0$.

**Theorem 5.2.4** *Let $\Lambda', \Lambda'', \Lambda$ satisfy the hypotheses of 5.2.3 . Then, $L(\Lambda + n\delta)$ is a $\delta$-maximal component of $L(\Lambda') \otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where $n_1$ is such that $\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda')$ and $n_2$ is such that $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$.*

**Proof** This follows immediately by combining 5.2.3 and 3.1.1.

## 5.3 Saturation factor for $A_2^{(2)}$

**Lemma 5.3.1** *Fix a positive integer $N > 1$. Let $\Lambda \in \bar{P}_+$ and let $\lambda \in \Lambda + Q$, where $Q$ is the root lattice $\mathbb{Z}\alpha \oplus \mathbb{Z}\delta$ of $A_2^{(2)}$. Then, $N\lambda \in P^o(N\Lambda)$ if and only if $\lambda \in P^o(\Lambda)$.*

**Proof** The validity of the lemma is clear for $\lambda \in P^o(\Lambda)_+$ from (5.1). But since $P^o(\Lambda) = W \cdot (P^o(\Lambda)_+)$, and the action of $W$ on $\mathfrak{h}^*$ is linear, the lemma follows for any $\lambda \in P^o(\Lambda)$.

**Corollary 5.3.2** 4 *is a saturation factor for $A_2^{(2)}$.*

**Proof** If $\Lambda'(c) = 0$ or $\Lambda''(c) = 0$, then

$$L(N\Lambda') \otimes L(N\Lambda'') \simeq L(N(\Lambda' + \Lambda'')),$$

for any $N \geq 1$. Thus, the corollary is clearly true in this case. So, let us assume that both of $\Lambda'(c) > 0$ and $\Lambda''(c) > 0$.

Suppose that $L(N\Lambda + n\delta)$ is a $\delta$-maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ and that $\Lambda - \Lambda' - \Lambda'' \in Q$. By 5.2.3, provided that $N \geq 4$, $n = \min(n_1, n_2)$, where $n_1$ is such that $N\Lambda - N\Lambda'' + n_1\delta \in P^o(N\Lambda')$ and $n_2$ satisfies $N\Lambda - N\Lambda' + n_2\delta \in P^o(N\Lambda'')$. Since $\Lambda - \Lambda' \in \Lambda'' + Q$, any $\Lambda - \Lambda'' + \widetilde{n}_1\delta \in \Lambda' + Q$. Thus, by 5.3.1, $\Lambda - \Lambda'' + \widetilde{n}_1\delta \in P^o(\Lambda')$ if and only if $N\Lambda - N\Lambda'' + N\widetilde{n}_1\delta \in P^o(N\Lambda')$. Clearly then $N\widetilde{n}_1 = n_1$, and likewise $N\widetilde{n}_2 = n_2$. Applying 5.3.1 again, $\Lambda - \Lambda'' + \widetilde{n}_1\delta \in P^o(\Lambda')$ if and only if $4\Lambda - 4\Lambda'' + 4\widetilde{n}_1\delta \in P^o(4\Lambda')$ and $\Lambda - \Lambda' + \widetilde{n}_2\delta \in P^o(\Lambda'')$ if and only if $4\Lambda - 4\Lambda' + 4\widetilde{n}_2\delta \in P^o(4\Lambda'')$.

Clearly, $4\Lambda$, $4\Lambda'$, and $4\Lambda''$ satisfy the conditions of 5.2.3, so by 5.2.4 if $N \geq 4$ and $\Lambda - \Lambda' - \Lambda'' \in Q$, then $L(4\Lambda + 4\widetilde{n}\delta)$ is a $\delta$-maximal component of $L(4\Lambda') \otimes L(4\Lambda'')$ if $L(N\Lambda + N\widetilde{n}\delta)$ is a $\delta$-maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$.

Let

$$\sum_{\lambda\in T_{4\bar{\Lambda}}^{4\Lambda',4\Lambda''}} \varepsilon(v_{4\bar{\Lambda},4\Lambda'',\lambda})c_{4\Lambda',\lambda}e^{S_{4\bar{\Lambda},4\Lambda'',\lambda}\delta} = \sum_{k\in\mathbb{Z}_+} c_k e^{(4\Lambda(d)+4\widetilde{n}-k)\delta}, \tag{5.1}$$

for some $c_k \in \mathbb{Z}_+$ with $c_0$ nonzero. By 3.5, this is the character of a unitarizable Virasoro representation with each irreducible component having the same nonzero central charge. Thus, by Lemma 2.1.1, for any $k > 1$, we get $c_k \neq 0$.

By the above argument, $L(4\Lambda + 4\widetilde{n}\delta)$ is the $\delta$-maximal component of $L(4\Lambda') \otimes L(4\Lambda'')$ through $L(4\Lambda)$. If $\widetilde{n} = 0$, we get that

$$L(4\Lambda) \subset L(4\Lambda') \otimes L(4\Lambda'').$$

If $\widetilde{n} > 0$, then $4\widetilde{n}$ being $> 1$, by the analogue of (5.1) for $4\Lambda', 4\Lambda''$ and $4\Lambda$, $L(4\Lambda) \subset L(4\Lambda') \otimes L(4\Lambda'')$. For the case that $N < 4$, we use the fact that $\overline{\Gamma}$ is a semigroup. This proves the corollary.

By virtue of 5.2.4, the proof of the following is identical to that for the $A_1^{(1)}$.

**Theorem 5.3.3** *Let $\mathfrak{g} = A_2^{(2)}$. Let $\Lambda', \Lambda'', \Lambda \in P_+$ be such that $\Lambda' + \Lambda'' - \Lambda \in Q$ and both of $\Lambda'(c)$ and $\Lambda''(c)$ are nonzero. Then, the following are equivalent:*

*(a) $(\Lambda', \Lambda'', \Lambda) \in \Gamma$.*

*(b) The following set of inequalities is satisfied for all $w \in W$ and $x_i \in \mathfrak{h}$, such that $\alpha_j(x_i) = \delta_{i,j}$ for $i, j = 0, 1$:*

$$\Lambda'(x_i) + \Lambda''(wx_i) - \Lambda(wx_i) \geq 0, \quad \textit{and}$$

$$\Lambda'(wx_i) + \Lambda''(x_i) - \Lambda(wx_i) \geq 0.$$

# CHAPTER 6: A GEOMETRIC INTERPRETATION

In [2], we show that the saturated tensor cone (for an arbitrary symmetrizable Kac-Moody algebra) is contained in a set cut out by inequalities indexed by certain products in the cohomology ring of the corresponding full flag variety. In this chapter, we will show that in the $A_1^{(1)}$ and $A_2^{(2)}$ cases, this set is equal to the saturated tensor cone. Moreover, we show that a much smaller set of inequalities suffice to determine $\Gamma$, and we directly show that they are analogous to certain sets of inequalities in the case that $\mathfrak{g}$ is finite dimensional.

## 6.1 Necessary inequalities for $\Gamma$

To state the result of [2], we must explain some notation. Let $G$ be the Kac-Moody group associated to $\mathfrak{g}$, as defined in [7, ch.6], let $B \subset G$ be the standard positive Borel subgroup and $P \supseteq B$ a standard parabolic subgroup of $G$. Write $X^P := G/P$ ($X := G/B$, and further, if $B = P$, we omit $P$ in the notation). For $w \in W^P$, let $C_w^P := BwP/P$ and $X_w^P := \overline{C_w^P} \subset X^P$. By [7, 11.3.2], the singular homology of $X^P$ with integer coefficients $H_*(X^P) = \bigoplus_{w \in W^P} \mathbb{Z} \cdot [X_w^P]$, where $[X_w^P]$ is the homology class of $X_w^P$. In other words, $\{[X_w^P]\}_{w \in W^P}$ forms a $\mathbb{Z}$-basis for $H_*(X^P)$. Let $\{\varepsilon_w^P\}_{w \in W^P}$ be the corresponding dual basis of the singular homology of $X^P$, $H^*(X^P)$, with respect to the standard pairing between homology and cohomology. We refer to $\{\varepsilon_w^P\}_{w \in W^P}$ as the *Schubert basis of* $H^*(X^P)$.

**Theorem 6.1.1** *[2] Suppose that $\mathfrak{g}$ is any symmetrizable Kac-Moody algebra. Let $(\Lambda', \Lambda'', \Lambda) \in \Gamma$. Then, using the notation above, for any $u_1, \ldots, u_s, v \in W$ such that $n_{u_1,u_2}^v \neq 0$, where*

$$\varepsilon^{u_1} \cdot \varepsilon^{u_2} = \sum_w n_{u_1,u_2}^w \, \varepsilon^w,$$

*we have*

$$\Lambda'(u_1 x_i) + \Lambda'(u_2 x_i) - \Lambda(v x_i) \geq 0, \ \textit{for any } x_i,$$

*where $x_i \in \mathfrak{h}$ is dual to the simple roots of $\mathfrak{g}$.*

If $\mathfrak{g} = A_1^{(1)}, A_2^{(2)}$, the sufficiency of the above inequalities is readily apparent from Theorems 4.3.1 and 5.3.3 along with the triviality that for all $w \in W$, $n^w_{e,w} = n^w_{w,e} = 1$.

## 6.2 Calculation of $H^*(X)$ and $H^*(X^P)$ for $A_1^{(1)}$ and $A_2^{(2)}$

We begin by computing $H^*(X)$: Recall [6] to determine the structure coefficients $n^w_{u_1,u_2}$. All rank 2 infinite type Kac-Moody algebras have isomorphic Weyl groups - they are all the infinite dihedral group by [7, 1.3.11 and 1.3.21] - and moreover, all have the same Coxeter presentation. Thus we will parametrize the Schubert bases for $A_1^{(1)}$ and $A_2^{(2)}$ by the same formulas. For $n \in \mathbb{Z}_{\geq 0}$,

$$\delta_{2n} := \varepsilon_{(s_1 s_0)^n} \qquad \delta_{2n+1} := \varepsilon_{s_0(s_1 s_0)^n} \tag{6.1}$$

$$\sigma_{2n} := \varepsilon_{(s_0 s_1)^n} \qquad \sigma_{2n+1} := \varepsilon_{s_1(s_0 s_1)^n}. \tag{6.2}$$

**Lemma 6.2.1** *Let $\mathfrak{g} = A_1^{(1)}$. The structure constants for $H^*(X)$ are as follows:*

$$\sigma_n \cdot \sigma_m = \binom{n+m}{n} \sigma_{n+m} \tag{6.3}$$

$$\delta_n \cdot \delta_m = \binom{n+m}{n} \delta_{n+m} \tag{6.4}$$

$$\sigma_n \cdot \delta_m = \binom{m+n-1}{m} \sigma_{n+m} + \binom{n+m-1}{n} \delta_{n+m}. \tag{6.5}$$

*For $\mathfrak{g} = A_2^{(2)}$, the structure constants are*

$$\sigma_n \cdot \sigma_m = e_{n,m}^{-1} \binom{n+m}{n} \sigma_{n+m} \tag{6.6}$$

$$\delta_n \cdot \delta_m = e_{n,m} \binom{n+m}{n} \delta_{n+m} \tag{6.7}$$

$$\sigma_n \cdot \delta_m = e_{n-1,m}^{-1} \binom{m+n-1}{m} \sigma_{n+m} + e_{n,m-1} \binom{n+m-1}{n} \delta_{n+m}. \tag{6.8}$$

*where $e_{n,m} = 1$ if either $n$ or $m$ is even and $e_{n,m} = 2$ if both $n$ and $m$ are odd.*

**Proof** Let $\left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ be the generalized Cartan matrix of a (rank 2) Kac-Moody algebra. The corresponding algebra is infinite dimensional if and only if $ab \geq 4$ (cf [7]). The structure constants in the rank 2 infinite type case are determined by a pair of integer sequences, $c_j$ and $d_j$, given by the following rule:

$$c_0 = d_0 = 0, \quad c_1 = d_1 = 1, \tag{6.9}$$

$$c_{j+1} = ad_j - c_{j-1}, \tag{6.10}$$

$$d_{j+1} = bc_j - d_{j-1}. \tag{6.11}$$

By [6], the cup product satisfies the following identities:

$$\delta_1 \cdot \delta_m = d_{m+1}\delta_{m+1}, \quad \delta_1 \cdot \sigma_m = \delta_{m+1} + d_m\sigma_m, \tag{6.12}$$

$$\sigma_1 \cdot \sigma_m = c_{m+1}\sigma_{m+1}, \quad \sigma_1 \cdot \delta_m = \sigma_{m+1} + c_m\delta_{m+1}. \tag{6.13}$$

Consider the “generalized binomial coefficients” for the the sequences $d_i$ and $c_i$, that is, for $n, m \in \mathbb{Z}_{\geq 0}$,

$$D(n,m) := \frac{d_{n+m}d_{n+m-1}\dots d_1}{d_nd_{n-1}\dots d_1d_md_{m-1}\dots d_1} \quad C(n,m) := \frac{c_{n+m}c_{n+m-1}\dots c_1}{c_nc_{n-1}\dots c_1c_mc_{m-1}\dots c_1}. \tag{6.14}$$

Then, by repeated application of (6.12),

$$\sigma_n \cdot \sigma_m = C(n,m)\sigma_{n+m} \tag{6.15}$$

$$\delta_n \cdot \delta_m = D(n,m)\delta_{n+m} \tag{6.16}$$

$$\sigma_n \cdot \delta_m = D(n-1,m)\sigma_{n+m} + C(m-1,n)\delta_{n+m}. \tag{6.17}$$

For the $A_1^{(1)}$ case, $a = b = 2$, so the sequences $c_n = d_n = n$ satisfy (6.9), and the result follows.

For $A_2^{(2)}$, $a = 1$and $b = 4$. The sequences $c_{2n} = n$, $c_{2k+1} = 2k+1$; $d_{2k} = 4k$, $d_{2k+1} = 2k+1$ solve the recurrence.

Let $\pi_P^* : H^*(X^P) \to H^*(X)$ be the injective homomorphism on cohomology induced from the canonical projection $\pi_P : X \to X^P$. Let $P_i$ be the maximal parabolic such that $\alpha_i$ is not a root of the Levi of $P_i$. Then, by [7, 11.3.3], $\pi_{P_0}^*(H^*(X^{P_0})) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Z}\delta_n$ and $\pi_{P_1}^*(H^*(X^{P_1})) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Z}\sigma_n$. Let us compare our results for $A_1^{(1)}$ and $A_2^{(2)}$ with the following theorem, valid in the finite case. This result, as part of a survey of the general theory in the finite dimensional case, can be found in

[8].

**Theorem 6.2.2** *Let $\mathfrak{g}$ be a simple Lie algebra and $G$ the corresponding complex algebraic group. Then the following are equivalent:*

*(a)* $(\lambda, \mu, \nu) \in \Gamma$

*(b) For each maximal parabolic $P \subset G$, and every $(u_1, u_2, v) \in (W^P)^3$ such that $n^v_{u_1,u_2} = 1$, the following inequality holds:*

$$\lambda(u_1 x_P) + \mu(u_2 x_P) - \nu(v x_P) \geq 0,$$

*where $x_P$ is dual to the root not in the Levi of $P$.*

For $A_1^{(1)}$, we apply 6.3 to see that $n^v_{u_1,u_2} = 1$ implies $u_1 = e$ and $u_2 = v$ or $u_1 = v$ and $u_2 = e$. For $A_2^{(2)}$, applying 6.6, $n^v_{u_1,u_2} = 1$ implies $u_1 = e$ and $u_2 = v$, $u_1 = v$ and $u_2 = e$, or $u_1 = s_1$, $u_2 = s_1$ and $v = s_0 s_1$. The first two cases give us all the inequalities in 5.3.3. The last case yields $\lambda(s_1 x_1) + \mu(s_1 x_1) - \nu(s_0 s_1 x_1) \geq 0$. This inequality is, in fact, redundant:

$$\begin{aligned} \lambda(s_1 x_1) + \mu(s_1 x_1) - \nu(s_0 s_1 x_1) &= \lambda(x_1) + \mu(x_1) - \nu(x_1) \\ &\qquad - \left(\lambda(\alpha_1^\vee) + \mu(\alpha_1^\vee) - \nu(\alpha_1^\vee) - 4\nu(\alpha_1^\vee)\right), \end{aligned}$$

so it is enough to show that another inequality insures that $\lambda(\alpha_1^\vee) + \mu(\alpha_1^\vee) - \nu(\alpha_1^\vee) - 4\nu(\alpha_0^\vee) \leq 0$. Indeed, any other inequality will give that $\lambda(c) + \mu(c) = \nu(c)$, by varying $x_i$. Hence

$$\begin{aligned} &\lambda(\alpha_1^\vee) + \mu(\alpha_1^\vee) - \nu(\alpha_1^\vee) - 4\nu(\alpha_0^\vee) \\ &= -2\left(\lambda(c - \frac{1}{2}\alpha_1^\vee) + \mu(c - \frac{1}{2}\alpha_1^\vee) - \nu(c - \frac{1}{2}\alpha_1^\vee) + 2\nu(\alpha_0^\vee)\right) \\ &= -2\left(\lambda(\alpha_0^\vee) + \mu(\alpha_0^\vee) - \nu(\alpha_0^\vee) + 2\nu(\alpha_0^\vee)\right) = -2\left(\lambda(\alpha_0^\vee) + \mu(\alpha_0^\vee) + \nu(\alpha_0^\vee)\right) \leq 0, \end{aligned}$$

Since $(\lambda, \mu, \nu) \in P_+^3$. Thus in these cases, (a) and (b) are equivalent, since the corresponding inequalities are precisely what we compute for $\Gamma$.

In the finite dimensional case, Belkale-Kumar [1] show that using a deformed product $\odot_0$ for $H^*(X^P)$ reduces the number of required inequalities.

**Definition 6.2.3 (Deformed product)** *Let $G$ be a connect simple algebraic group. Let $P$ be a standard maximal parabolic. Define a product $\odot_0$ on $H^*(X^P)$ by fixing structure constants with respect to the Schubert basis:*

$$\varepsilon^P_{u_1} \odot_0 \varepsilon^P_{u_2} := \sum_{w \in W^P} d^w_{u_1,u_2} \varepsilon^P_w, \tag{6.18}$$

*where*

$$d^w_{u_1,u_2} := \begin{cases} 0 \text{ if } \left(u_1^{-1}\rho + u_2^{-1}\rho - w^{-1}\rho - \rho\right)(x_P) \neq 0 \\ n^w_{u_1,u_2} \text{ otherwise.} \end{cases} \tag{6.19}$$

Belkale-Kumar [1] prove the following theorem for finite type $\mathfrak{g}$:

**Theorem 6.2.4** *Let $\mathfrak{g}$ be a simple Lie algebra and $G$ the corresponding complex algebraic group. Then the following are equivalent:*

*(a)* $(\lambda, \mu, \nu) \in \Gamma$

*(b) For each maximal parabolic $P \subset G$, and every $(u_1, u_2, v) \in (W^P)^3$ such that $d^v_{u_1,u_2} = 1$, the following inequality holds:*

$$\lambda(u_1 x_P) + \mu(u_2 x_P) - \nu(v x_P) \geq 0,$$

*where $x_P$ is dual to the root not in the Levi of $P$. Moreover, as proved by Ressayre, this set of inequalities is irredundant.*

We will "formally" apply this theorem when $\mathfrak{g} = A_1^{(1)}, A_2^{(2)}$. In the $A_1^{(1)}$ case, $n^v_{u_1,u_2} = 1$ if and only if $d^v_{u_1,u_2} = 1$, so we get no fewer inequalities. For $A_2^{(2)}$, the triples $(w, e, w)$ and $(e, w, w)$ are relevant for the deformed product as well. Let us compute $d^{s_0 s_1}_{s_1,s_1}$. A simple computation yields

$$s_1\rho + s_1\rho - s_1 s_0 \rho - \rho = -\alpha_0 + 3\alpha_1,$$

so $d^{s_0 s_1}_{s_1,s_1} = 0$. In both cases, selecting only those $(u_1, u_2, v)$ such that $d^v_{u_1,u_2} = 1$ yields an irredundant set of inequalities.

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