VARIATIONAL APPROACHES TO NONLINEAR SCHRÖDINGER AND KLEIN-GORDON EQUATIONS

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ABSTRACT

Mayukh Mukherjee: Variational approaches to nonlinear Schrödinger and Klein-Gordon equations
(Under the direction of Michael Taylor)

This thesis has two chapters. In the first chapter, we investigate *traveling wave solutions* of nonlinear Schrödinger and Klein-Gordon equations. In the compact case, we establish existence of traveling wave solutions via *energy minimization methods* and prove that at least compact isotropic manifolds have genuinely traveling waves. We establish certain sharp estimates on low dimensional spheres that improve results in [T1] and carry out the *subelliptic analysis* for NLKG on spheres of higher dimensions. We also extend the investigation started in [T1] on compact manifolds to complete non-compact manifolds which either have a certain radial symmetry or are weakly homogeneous, using *concentration-compactness* type arguments. In the second chapter of the thesis, we study ground state solutions for these equations on the hyperbolic space $\mathbb{H}^n$ via a study of the *Weinstein functional*, first defined in [W]. The main result is the fact that the supremum value of the Weinstein functional on $\mathbb{H}^n$ is the same as that on $\mathbb{R}^n$ and the related fact that the supremum value of the Weinstein functional is not attained on $\mathbb{H}^n$, when maximization is done in the Sobolev space $H^1(\mathbb{H}^n)$. Lastly, we prove that a corresponding version of the conjecture will hold for the Weinstein functional with the fractional Laplacian as well. The thesis ends with four Appendices and a table of symbols, which are mainly for expository convenience.

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CHAPTER 1: TRAVELING WAVES

In this chapter, we look at traveling wave solutions to nonlinear Schrödinger and Klein-Gordon equations of power-type nonlinearity.

1.1 Introduction, Setting and Notations

Let us consider a complete Riemannian manifold M. Let X be a Killing field (see Definition A.2.1 in Appendix A) on the manifold, which flows by a one-parameter family of isometries $g(t)$ of M. The following is the nonlinear Schrödinger (NLS) equation:

$$i\partial_t v + \Delta v = -K|v|^{p-1}v, \tag{1.1.1}$$

and the following is the nonlinear Klein-Gordon (NLKG) equation:

$$\partial_t^2 v - \Delta v + m^2 v = K|v|^{p-1}v, \tag{1.1.2}$$

where in each case, $K > 0$ is a constant and $m \in \mathbb{R}$.

In this chapter, we will investigate traveling wave solutions (see Definition D.0.1 in Appendix D) to both the NLS and the NLKG. In the past, there has been a lot of investigation on traveling wave solutions to nonlinear Schrödinger, Klein-Gordon and sine-Gordon equations. However, most of the literature focuses on traveling waves in an Euclidean setting $(x, t) \in \mathbb{R} \times [0, \infty)$ and their associated stability analysis. For example, see [JMMP], [MJS]. In the setting $M = \mathbb{R}^n$ and $g(t)x = x + tv$ for $x, v \in \mathbb{R}^n$, such traveling waves have been studied in [St] and [BL].

As far as non-Euclidean settings are concerned, we must also mention recent interest in standing wave solutions (see Definition D.0.2, Appendix D) to (1.1.1) and (1.1.2) in non-Euclidean settings. For example, see [MS], [CM], and [CMMT]. To the best of our knowledge, the study of traveling waves on Riemannian manifolds was initiated in [T1]. Our aim in this chapter is to extend and build on the investigation started in [T1], using variants and modifications of techniques introduced in the aforementioned references, particularly, [T1], [MT] and [CMMT]. We should also mention that the study in [T1] focuses solely on compact manifolds. In this chapter, we extend the investigation to select non-compact manifolds with symmetry, and the results are largely motivated by the investigation in [CMMT] and [MT]. The tools and methods that we use in arriving at these non-compact results will be a hybrid of the latter two papers.

1.2 Setting up the auxiliary equations and standing assumptions

First, to fix notations, we define,

$$F_{\lambda,X}(u) = (-\Delta u - iXu + \lambda u, u), \tag{1.2.1}$$

$$F_{m,\lambda,X}(u) = (-\Delta u + X^2 u + 2i\lambda Xu + (m^2 - \lambda^2)u, u), \tag{1.2.2}$$

$$E_X(u) = \frac{1}{2}(-\Delta u - iXu, u) - \frac{1}{p+1}\int_M |u|^{p+1} dM, \tag{1.2.3}$$

and

$$\mathcal{E}_{\lambda,X}(u) = \frac{1}{2}(-\Delta u + X^2 u + 2i\lambda Xu, u) - \frac{1}{p+1}\int_M |u|^{p+1} dM. \tag{1.2.4}$$

(for an explanation of the volume form dM see Notation D.0.3 in Appendix D). In all of the above, and henceforth, (u, v) denotes the inner product $(u, v) = \int_M u\bar{v}dM$.

In general, if F is an isometry of M and we define $F^*u(x) = u(F(x))$, then it is known that the Laplacian $-\Delta$ commutes with F^* (see [T3], page 155). Since $g(t)x$ flows by isometries, the Laplacian $-\Delta$ commutes with $g(t)^*$ for all t, that is,

$$\Delta(u(g(t)x)) = (\Delta u)(g(t)x).$$

Using this, if we differentiate $v(t,x) = e^{i\lambda t}u(g(t)x)$ with respect to t, we get

$$i\partial_t v = e^{i\lambda t}(-\lambda u(g(t)x) + iXu(g(t)x)),$$

where, as mentioned before, X is the Killing field flowing by $g(t)$. Thus, (1.1.1) holds if and only if

$$-\Delta u + \lambda u - iXu = K|u|^{p-1}u. \tag{1.2.5}$$

Differentiating $v(t,x) = e^{i\lambda t}u(g(t)x)$ twice with respect to t, we get

$$\partial_t^2 v = e^{i\lambda t}(-\lambda^2 u(g(t)x) + 2i\lambda Xu(g(t)x) + X^2 u(g(t)x)).$$

Thus, (1.1.2) holds if and only if

$$-\Delta u + (m^2 - \lambda^2)u + X^2 u + 2i\lambda Xu = K|u|^{p-1}u. \tag{1.2.6}$$

As we mentioned before, we assume that the Killing field X is bounded, that is,

$$\langle X, X\rangle \leq b^2 < \infty,\ \ b \in \mathbb{R}. \tag{1.2.7}$$

On a complete manifold M, the Laplacian $-\Delta$ is essentially self-adjoint when defined on $C_c^\infty(M)$, and still calling $-\Delta$ the self-adjoint extension of the Laplacian (see Section C.1 of Appendix C for a clarification), (1.2.7) means that iX is a small relatively bounded perturbation of Δ on which the Kato-Rellich theorem applies (see Definition B.1.1 of Appendix B and Section C.2 of Appendix C), which in turn means that $-\Delta - iX$ is self-adjoint. This implies

$$\operatorname{Spec}(-\Delta - iX) \subset [\alpha, \infty), \alpha \in \mathbb{R}. \tag{1.2.8}$$

As long as we are concentrating on compact manifolds, (1.2.7) is not a geometric restriction. We will also find the opportunity to say something about non-compact manifolds which have such bounded Killing fields later. Note, however, that all non-compact manifolds do not have to have bounded Killing fields. For example, rotate the parabola $y = x^2, z = 0$ about the x-axis in $\mathbb{R}^3$.

The only Killing fields of the resulting surface of revolution generate rotations about the x-axis and are not bounded.

Remark 1.2.1. Comparing (1.2.5) and (1.2.6) we can now claim and justify a bias in our investigations towards the NLKG, which, as far as traveling waves are concerned, is harder to study because of the presence of the second order operator X^2 in (1.2.6). Depending on the length of X, $-\Delta+X^2$ may be elliptic, subelliptic (see Definition B.2.1 in Appendix B), or even hyperbolic. As an example, consider $\Delta = \partial_{x_1}^2+...+\partial_{x_n}^2$ on the torus $\mathbb{T}^n$ and $X=\sqrt{2}\partial_{x_1}$. Then, $-\Delta+X^2=\partial_{x_1}^2-\partial_{x_2}^2-.....-\partial_{x_n}^2$. This demarcates a point of deviation from the general methodology of [CMMT] and [MT].

By a similar logic as above, we see that $\langle X, X\rangle \leq b^2 < 1$ implies that $-\Delta + X^2$ is a strongly elliptic (see (11.79), Chapter 5 of [T3] for a definition of strong ellipticity) nonnegative semidefinite self-adjoint operator and $2i\lambda X$ is a relatively bounded perturbation of $-\Delta + X^2$ with relative bound $= 0$ (see Section C.2 of Appendix C), meaning that $-\Delta + X^2 + 2i\lambda X$ is self-adjoint, giving

$$\mathrm{Spec}(-\Delta + X^2 + 2i\lambda X) \subset [\beta(\lambda), \infty),\ \beta(\lambda) \in \mathbb{R}. \tag{1.2.9}$$

Now we give a broad outline of the rest of this chapter. In this chapter, we will study traveling wave solutions with bounded Killing fields X (see Definition A.2.1 in Appendix A). [T1] established the existence of such traveling wave solutions for (1.1.1) and (1.1.2) on compact manifolds, by establishing the existence of minimizers of $F_{\lambda,X}(u)$ and $F_{m,\lambda,X}(u)$ respectively in the space $H^1(M)$ keeping the integral $\int_M |u|^{p+1} dM$ constant. In Section 1.3, we establish the existence of constrained energy minimizers, i.e., we minimize the energies $E_X(u)$ and $\mathcal{E}_{\lambda,X}(u)$ subject to the mass (see Definition D.0.4 of Appendix D) being constant and use usual variational arguments to see that these constrained minimizers actually give traveling wave solutions to (1.1.1) and (1.1.2). These are respectively Proposition 1.3.2 and Lemma 1.3.1.

As the whole point of this investigation is to get traveling wave solutions, we must establish that the constrained minimizers u do not always satisfy $Xu \equiv 0$. This is a legitimate concern, as constrained minimizers can even turn out to be constants. This concern is taken up in Section 1.4, where it is shown that on fairly general spaces and for at least a non-empty set of parameters λ and m, we have honest traveling wave solutions to (1.2.6). To be precise, this is Theorem 1.4.2, which generalizes Lemma 2.1 in [T1].

In Section 1.5, we extend the analysis on S^2 done in [T1] to a sphere of arbitrary dimension along similar lines of reasoning. We improve on an estimate on S^2 given in [T1] and show that our estimate is sharp.

Finally, in Section 1.6, we establish our main theorems for this chapter: existence of constrained $F_{m,\lambda,X}$ minimizers for (1.2.5) and constrained $\mathcal{E}_{\lambda,X}$ minimizers for (1.2.6) in the non-compact setting. These are respectively Theorem 1.6.4 and Theorem 1.6.5. Let us note here that among the two, the latter is somewhat more analytically involved and requires the application of the concentration-compactness principle and a stronger symmetry assumption on the manifold to work.

1.3 Existence of constrained Energy minimizers

In [T1], it was proved that on compact M, with α as in (1.2.8) and

$$\lambda > -\alpha, \tag{1.3.1}$$

we have

$$F_{\lambda,X}(u) \cong ||u||^2_{H^1} \ \forall \ u \in H^1(M), \tag{1.3.2}$$

where H^s denotes the usual Sobolev spaces (the above fact comes from elliptic regularity once it is known that λ is above the lowest possible eigenvalue of $-\Delta - iX$; see Appendix C, Section C.3). It was also proved in [T1] that with

$$\langle X, X\rangle \leq b^2 < 1, \ \text{Spec}(-\Delta + X^2 + 2i\lambda X) \subset [\beta(\lambda), \infty), \ \beta(\lambda) \in \mathbb{R} \tag{1.3.3}$$

and

$$m^2 - \lambda^2 > -\beta(\lambda), \tag{1.3.4}$$

we have

$$F_{m,\lambda,X}(u) \cong ||u||^2_{H^1} \ \forall \ u \in H^1(M). \tag{1.3.5}$$

In [T1], (1.3.2) was then used to minimize $F_{\lambda,X}(u)$ over $H^1(M)$, subject to the constraint

$$\int_M |u|^{p+1} dM = \text{constant}. \tag{1.3.6}$$

Similarly, (1.3.5) was used to minimize $F_{m,\lambda,X}(u)$ over $H^1(M)$, subject to the constraint (1.3.6), which would then give a solution to (1.2.6). Here, we find solutions to (1.2.5) and (1.2.6) via constrained energy minimizers, which goes as follows:

For the NLS, we will try to minimize the energy $E_X(u)$ and for the NLKG, we will try to minimize the energy $\mathcal{E}_{\lambda,X}$ subject to

$$Q(u) := \|u\|^2_{L^2} = \beta \text{ (constant)}. \tag{1.3.7}$$

The reason for doing this, as mentioned before, is the following:

Lemma 1.3.1. *(***Energy minimizers imply solutions***) Let M be a compact manifold. Then*

- *If $u \in H^1(M)$ minimizes $E_X(u)$, subject to keeping the mass $||u||^2_{L^2} = \beta$ (constant), then u solves (1.2.5) with $K > 0$ and for some $\lambda \in \mathbb{R}$.*

- *If $u \in H^1(M)$ minimizes $\mathcal{E}_{\lambda,X}(u)$ subject to keeping the mass $||u||^2_{L^2} = \beta$ (constant), then u solves (1.2.6) with $K > 0$ and for some $m \in \mathbb{R}$.*

Proof. On calculation, we can see that with $u, v \in H^1(M)$,

$$\left.\frac{d}{d\tau}\right|_{\tau=0} E_X(u + \tau v) = \text{Re}(-\Delta u - iXu - |u|^{p-1}u, v). \tag{1.3.8}$$

Also,

$$\left.\frac{d}{d\tau}\right|_{\tau=0} Q(u + \tau v) = 2\text{Re}(u, v). \tag{1.3.9}$$

So, for the NLS, if $u \in H^1(M)$ minimizes E_X constrained by $Q(u) =$ constant, then,

$$v \in H^1(M), \mathrm{Re}(u, v) = 0 \implies \mathrm{Re}(-\Delta u - iXu - |u|^{p-1}u, v) = 0. \tag{1.3.10}$$

Since $\mathrm{Re}(.,.)$ is a non-degenerate $\mathbb{R}$-bilinear dual pairing of $H^1(M)$ and $H^{-1}(M)$ (which is the dual of $H^1(M)$ with respect to the L^2 norm on $H^1(M)$), we have that there exists a $\lambda \in \mathbb{R}$ such that a mass-constrained E_X-minimizer u satisfies

$$-\Delta u + \lambda u - iXu = |u|^{p-1}u. \tag{1.3.11}$$

Now, if u solves (1.3.11), then $u_a = au$ solves

$$-\Delta u_a + \lambda u_a - iXu_a = |a|^{1-p}|u_a|^{p-1}u_a, \tag{1.3.12}$$

which finally means that we can solve (1.2.5) for any $K > 0$.

Similarly, for the NLKG, we have

$$\frac{d}{d\tau}\bigg|_{\tau=0} \mathcal{E}_{\lambda,X}(u + \tau v) = \mathrm{Re}(-\Delta u + 2i\lambda Xu + X^2 u - |u|^{p-1}u, v), \tag{1.3.13}$$

and

$$\frac{d}{d\tau}\bigg|_{\tau=0} Q(u + \tau v) = 2\mathrm{Re}(u, v). \tag{1.3.14}$$

As before, since $\mathrm{Re}(.,.)$ is a non-degenerate $\mathbb{R}$-bilinear dual pairing of $H^1(M)$ and $H^{-1}(M)$, we have that there exists a $\sigma \in \mathbb{R}$ such that a mass-constrained $E_{\lambda,X}$-minimizer u satisfies

$$-\Delta u + Xu + 2i\lambda Xu + \sigma u = |u|^{p-1}u. \tag{1.3.15}$$

Clearly, there exists $m \in \mathbb{R}$ be such that $m^2 - \lambda^2 = \sigma$. Finally, using the scaling $u_a = au$, we see that we can produce a solution to (1.2.6) for any constant $K > 0$. $\square$

So far we have argued that mass constrained energy minimizers, if they exist, would indeed give solutions to (1.2.5) and (1.2.6). Now we have to establish the existence of such constrained energy minimizers. Let us label our assumptions

$$\langle X, X\rangle \le b^2,\ \ b \in \mathbb{R}, \tag{1.3.16}$$

and

$$\langle X, X\rangle \le b^2 < 1,\ \ b \in \mathbb{R}. \tag{1.3.17}$$

Proposition 1.3.2. *(***Existence of constrained energy minimizers***) On a compact Riemannian manifold M, if $p \in (1, 1+4/n)$, then we can find, assuming (1.3.16) and (1.3.17) respectively, minimizers for E_X and $\mathcal{E}_{\lambda,X}$ for all $\lambda \in \mathbb{R}$, when the minimization is done in the class of $H^1(M)$ functions having constant L^2-norm.*

Proof. Let us define

$$I_\beta = \inf\{E_X | u \in H^1(M), Q(u) = \beta\}, \tag{1.3.18}$$

$$I'_\beta = \inf\{\mathcal{E}_{\lambda,X} | u \in H^1(M), Q(u) = \beta\}. \tag{1.3.19}$$

Recall the Gagliardo-Nirenberg inequality:

$$||u||_{L^{p+1}} \le C||u||_{L^2}^{1-\gamma}||u||_{H^1}^{\gamma}, \tag{1.3.20}$$

where $\gamma = \frac{n}{2} - \frac{n}{p+1}$, and hence $\gamma(p+1) < 2$.

Choosing λ satisfying (1.3.1), we have,

$$\begin{aligned} F_{\lambda,X}(u) &= (-\Delta u - iXu + \lambda u, u) = (-\Delta u - iXu, u) + (\lambda u, u) \\ &= (-\Delta u - iXu, u) - \frac{2}{p+1}\int_M |u|^{p+1} dM + \frac{2}{p+1}\int_M |u|^{p+1} dM + (\lambda u, u) \\ &= 2E_X(u) + \frac{2}{p+1}\int_M |u|^{p+1} dM + \lambda Q(u). \end{aligned}$$

This gives via (1.3.20),

$$F_{\lambda,X}(u) \leq 2E_X(u) + CQ(u)^{(p+1)\frac{(1-\gamma)}{2}}\|u\|_{H^1}^{\gamma(p+1)} + \lambda Q(u),\ C > 0. \tag{1.3.21}$$

This derivation implies two things:

Since $Q(u) = \beta$ is constant, $I_\beta > -\infty$, since $F_{\lambda,X}(u) \geq 0$. Also, since $\gamma(p+1) < 2$, if u_ν is a sequence in $H^1(M)$ such that $E_X(u_\nu) \to I_\beta$, then (1.3.21) implies that $\|u_\nu\|_{H^1}$ remains bounded. This is because, $F_{\lambda,X}(u) \cong \|u\|_{H^1}^2$.

Similarly, for the NLKG, choosing m such that $m^2 - \lambda^2 > -\beta(\lambda)$, with $\beta(\lambda)$ defined as in (1.3.3), we have

$$\begin{aligned}
\|u\|_{H^1}^2 &\cong F_{m,\lambda,X}(u) \\
&= (-\Delta u + X^2 u + 2i\lambda X u + ((m^2-\lambda^2)u, u) \\
&= (-\Delta u + X^2 u + 2i\lambda X u, u) - \frac{2}{p+1}\int_M |u|^{p+1} dM + \frac{2}{p+1}\int_M |u|^{p+1} dM \\
&+ ((m^2-\lambda^2)u, u) \\
&= 2\mathcal{E}_{\lambda,X}(u) + \frac{2}{p+1}\int_M |u|^{p+1} dM + (m^2-\lambda^2)Q(u).
\end{aligned}$$

This gives

$$\begin{aligned}
||u||_{H^1}^2 &\lesssim 2\mathcal{E}_{\lambda,X}(u) + CQ(u)^{\frac{p+1}{2}(1-\gamma)}||u||_{H^1}^{\gamma(p+1)} + (m^2-\lambda^2)Q(u) && (1.3.22)\\
&= 2\mathcal{E}_{\lambda,X}(u) + K||u||_{H^1}^{\gamma(p+1)} + K', && (1.3.23)
\end{aligned}$$

where $K, K' > 0$ are constants. So, as before, $I'_\beta > -\infty$ and if $u_\nu \in H^1(M)$ is a sequence satisfying $\mathcal{E}_{\lambda,X}(u_\nu) \to I'_\beta$, then $||u_\nu||_{H^1(M)}$ must be bounded.

So, in both the cases, passing to a subsequence if need be, we can assert that there exists $u \in H^1(M)$ such that

$$u_\nu \to u$$

weakly in $H^1(M)$ (for clarification, see Section C.4 in Appendix C).

Now, by the compactness of Sobolev embedding $H^1(M) \hookrightarrow L^2(M)$, u_ν has a convergent subsequence, called u_ν again by abuse of notation, converging in L^2-norm, and the L^2-limit is u. So, by the triangle inequality, $\|u\|_{L^2} = \|u_\nu\|_{L^2}$.

Now to prove that u attains the infimum I_β, that is,

$$E_X(u) = I_\beta.$$

We know that

$$E_X(u) = \frac{1}{2}F_{\lambda,X}(u) - \frac{1}{p+1}\int_M |u|^{p+1} dM - \frac{1}{2}\lambda Q(u).$$

Since $u_\nu \to u$ in L^{p+1}-norm, by the triangle inequality, we have $||u_\nu||_{L^{p+1}} \to ||u||_{L^{p+1}}$. So it suffices to establish that

$$F_{\lambda,X}(u) \leq \liminf F_{\lambda,X}(u_\nu).$$

But this is a consequence of the fact that $u_\nu \to u$ weakly in H^1 and $F_{\lambda,X}(u) \cong ||u||^2_{H^1}$ (for clarification, see Section C.4 in Appendix C). This settles the case for the NLS.

For the NLKG, we have to prove that $\mathcal{E}_{\lambda,X}(u) = I'_\beta$. Now,

$$\mathcal{E}_{\lambda,X}(u) = \frac{1}{2}F_{m,\lambda,X}(u) - \frac{1}{p+1}\int_M |u|^{p+1} dM - \frac{1}{2}(m^2 - \lambda^2)Q(u). \tag{1.3.24}$$

Since $\|u_\nu\|_{L^2} = \|u\|_{L^2}$ and $\|u_\nu\|_{L^{p+1}} \to \|u\|_{L^{p+1}}$, we just have to argue that

$$F_{m,\lambda,X}(u) \leq \liminf F_{m,\lambda,X}(u_\nu).$$

As argued before, this derives from the facts that $u_\nu \to u$ weakly in $H^1(M)$ and $F_{m,\lambda,X}(u) \cong \|u\|^2_{H^1}$. That finishes the proof. $\square$

Remark: Note that the constrained $F_{\lambda,X}$ or $F_{m,\lambda,X}$ minimizers give solutions to (1.2.5) and (1.2.6) respectively for $p \in (1, \frac{n+2}{n-2})$, while the constrained E_X or $\mathcal{E}_{\lambda,X}$ minimizers give solutions to (1.2.5) and (1.2.6) respectively for a smaller range of p; to wit, $p \in (1, 1 + 4/n)$. However, it is not apriori clear that the solutions obtained from the two schemes are the same. Since they are different variational formulations, they can potentially give different solutions.

1.4 Nontriviality of solutions and a few other remarks

As mentioned in the outline, we must note that the mere existence of minimizers will not guarantee waves that are *actually* traveling. For example, on a compact manifold M

$$u = [(m^2 - \lambda^2)/K]^{\frac{1}{p-1}}$$

solves (1.2.6) and it is natural to ask if this is an $F_{m,\lambda,X}$ minimizer subject to (1.3.6). In general, it is possible to have constrained minimizers u such that $Xu = 0$; such waves will definitely not be traveling.

1.4.1 Nontriviality on the sphere: discussion

This problem is discussed for the NLKG on S^n with $\lambda = 0$ and $m > 0$ in [T1]. Let us first sketch the main lines of argument as appear there:

Step 1. Let, as before, $u \in H^1(S^n)$ minimize $F_{m,0,X}(u)$, subject to (1.3.6), so u solves

$$-\Delta u + X^2 u + m^2 u = K|u|^{p-1}u.$$

Firstly, it is proved that if u is constant on each orbit of X, or equivalently, $Xu = 0$, then u is actually constant.

Step 2. The metric on S^n is then scaled, with S^n_r denoting the sphere with distance magnified by a factor of r. Picking a "north pole" o on S^n_r, and using exponential coordinates around o, it is observed that as $r \to \infty$, S^n_r approaches flat Euclidean space $\mathbb{R}^n$, whilst the Laplacian approaches the flat Laplacian. Now, if $u^r \in H^1(S^n_r)$ denotes a minimizer of

$$F^r_{m,0,X}(u) = ((-\Delta_r + X_r^2 + m^2)u, u)_{L^2(S^n_r)}$$

subject to the constraint

$$I^r_p(u) = \int_{S^n_r} |u|^{p+1} dM = A \text{ (independent of } r), \tag{1.4.1}$$

and u^r is a constant on each orbit of X_r, then u^r is constant on S^n_r. That means,

$$\begin{aligned} F^r_{m,0,X}(u^r) &= \int_{S^n_r} m^2 |u^r|^2 dM \\ &\simeq r^{\frac{n(p-1)}{p+1}}, \end{aligned}$$

which is also the infimum of $F^r_{m,0,0}$, as $X^r u^r = 0$.

Step 3. A contradiction was then derived with the help of the fact that we know that for $n \geq 2$, there is a minimizer $u^0 \in H^1(\mathbb{R}^n)$ to $F^\infty_{m,0,0}(u) = ((-\Delta u + m^2)u, u)_{L^2(\mathbb{R}^n)}$ subject to (1.3.6) (see Lemma 1.4.1 below). However, in the above calculation, as $r \to \infty$, $F^r_{m,0,0}(u^r)$ blows up.

To complete the above discussion, we quote the following

Lemma 1.4.1. *(***Global constrained minimizer of** $((-\Delta + m^2)u, u)_{L^2(\mathbb{R}^n)}$*)*
Given

$$n \geq 2,\ p \in (1, \frac{n+2}{n-2}),\ A \in (0, \infty), \tag{1.4.2}$$

there is a minimizer $u^0 \in H^1(\mathbb{R}^n)$ *to* $F_m(u) = ((-\Delta + m^2)u, u)_{L^2(\mathbb{R}^n)}$ *subject to the constraint* $\int_{\mathbb{R}^n} |u|^{p+1} d\mathbb{R}^n = A$.

Proof. For the proof, refer to Lemma 2.2 of [T1] and also [BL]. □

We just want to point out the following important fact about the above lemma: the proof, as stated in [T1] (which in turn cites [BL]), also establishes that we can arrange so that the constrained minimizer u^0 is a radial function. We will use this fact in the sequel.

1.4.2 General case

We have the following

Theorem 1.4.2. *(***Traveling waves on isotropic manifolds***) Given a compact isotropic manifold* M *of dimension* $n \geq 2, p \in (1, (n+2)/(n-2))$, $m > 0, K > 0$ *and a Killing field* X *such that* $\langle X, X \rangle \leq b^2 < 1$, *there exists* $\delta > 0$ *such that for* $\varepsilon \in (0, \delta]$, *the constrained* $F_{m,0,X}$*-minimizing* *process produces a solution to*

$$-\Delta u + X^2 u + \frac{1}{\varepsilon^2} m^2 u = \frac{1}{\varepsilon^2} K|u|^{p-1} u$$

with $Xu \neq 0$.

Proof. We have

$$F_{m,0,0}(u) = ((-\Delta + m^2)u, u) = ||\nabla u||_{L^2}^2 + m^2 ||u||_{L^2}^2.$$

Now, as pointed out in Section C.2 of Appendix C, iX is a densely defined symmetric operator on $L^2(M)$. Then, for all $u \in H^1(M)$, we have

$$F_{m,0,X}(u) = \|\nabla u\|_{L^2}^2 + (X^2 u, u) + m^2 \|u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \|Xu\|_{L^2}^2 + m^2 \|u\|_{L^2}^2 \leq F_{m,0,0}(u).$$

Now, if u is not traveling, that is, $Xu = 0$, then $F_{m,0,X}(u) = F_{m,0,0}(u)$, which means that if $u \in H^1(M)$ minimizes $F_{m,0,X}$ subject to (1.3.6), then u also minimizes $F_{m,0,0}$ subject to (1.3.6). Now let us consider the function $v(x) = u(\phi(x))$, where $\phi \in \mathrm{Isom}(M)$. We have $F_{m,0,0}(v) = F_{m,0,0}(u)$. Also,

$$\begin{aligned} F_{m,0,X}(v) &\geq F_{m,0,X}(u) \text{ (since } u \text{ is a } F_{m,0,X} - \text{minimizer)} \\ &= F_{m,0,0}(u) \text{ (since } Xu = 0). \end{aligned}$$

Now,

$$\begin{aligned} F_{m,0,0}(v) &= (-\Delta u(\phi(x)) + m^2 u(\phi(x)), u(\phi(x))) \\ &= \int_M (-\Delta u(\phi(x)) + m^2 u(\phi(x)) \overline{u(\phi(x))} dM \\ &= \int_M (-\Delta u(y) + m^2 u(y)) \overline{u(y)} dM \\ &= F_{m,0,0}(u), \end{aligned}$$

where in the third step above we have used the fact that $y = \phi(x)$ is an isometry and hence the Jacobian determinant of this transformation is 1.

Now, we have

$$F_{m,0,X}(v) = F_{m,0,0}(v) \implies Xv = 0.$$

This happens for all $\phi \in \mathrm{Isom}(M)$. Now, choose a point $p \in M$ and let Y be a smooth vector field on M such that X_p and Y_p are linearly independent and X_p and Y_p have the same length. Consider the isometry $\phi \in \mathrm{Isom}(M)$ such that $\phi(p) = p$ and $d\phi(X_p) = Y_p$. Then, if $v(x) = u(\phi(x))$, we have $Xv|_p = Yu|_p = 0$. Since this happens for all vector fields Y, we can see that u is locally constant. Also, since p can be any point on M (and M is connected), we finally have that u is globally constant.

Now, let us scale the metric on (M, g) to $M_r = (M, g^r_{ij})$ by $g^r_{ij} = r^2 g_{ij}$. Consider a metric ball U of radius k on M which is small enough so that U is diffeomorphic to the open Euclidean 1-ball in $\mathbb{R}^n$. Let U^r be the dilated image of U under the scaling. On M_r, consider the vector field $X_r = \frac{1}{r}X$. Let u_r denote the minimizer of $F^r_{m,0,X_r}(u)$, subject to $\int_{M_r} |u|^{p+1} dM_r = A$. If u_r is constant, on calculation,

$$u_r = (\frac{A}{V})^{\frac{1}{p+1}} r^{-\frac{n}{p+1}},$$

where V is the volume of (M, g).

That gives,

$$\begin{aligned} F^r_{m,0,X_r}(u_r) &= m^2 (\frac{A}{V})^{\frac{2}{p+1}} V r^n r^{-\frac{2n}{p+1}} \\ &= C r^{\frac{n(p-1)}{p+1}}, \end{aligned}$$

where C is a constant. Since $X_r u_r = 0$, this is also the infimum of $F^r_{m,0,0}(u)$, subject to $\int_{M_r} |u|^{p+1} dM_r = A$.

Now,

$$\inf_{u \in H^1(M_r), \mathrm{supp}\ u \subset U^r} F^r_{m,0,0}(u) \geq \inf_{u \in H^1(M_r)} F^r_{m,0,0}(u).$$

So, as $r \to \infty$,

$$\inf_{u \in H^1_0(U^r)} F^r_{m,0,0}(u) \to \infty. \tag{1.4.3}$$

But, as $r \to \infty$, U^r approaches the flat Euclidean space $\mathbb{R}^n$.

Let p be the centre of the balls U^r, which have radius rk. Using the radial minimizer u^0 of Lemma 1.4.1, define

$$v_r(x) = \chi(x)u^0(\mathrm{dist}_r(p,x)), x \in U^r, \tag{1.4.4}$$

where dist_r is the metric distance in (M, g^r_{ij}) and $\chi(x)$ is a smooth radial cut-off function such that $\chi \equiv 1$ on $B^r_p(rk - \frac{1}{r})$, where the superscript r on the ball denotes the ball in the g^r_{ij} metric.

We have

$$\int_{M_r} |v_r|^{p+1} \to A$$

and

$$F^r_{m,0,0}(v_r) \to F^0_{m,0,0}(u^0) = ((-\Delta_{\mathbb{R}^n} + m^2)u, u)_{L^2(\mathbb{R}^n)} < \infty,$$

thereby contradicting (1.4.3).

So, for r large, there exists a constrained minimizer u_r such that $X_r u_r \neq 0$, which solves

$$-\Delta_r u_r + X_r^2 u_r + m^2 u_r = K|u_r|^{p-1}u_r, \tag{1.4.5}$$

where $K > 0$ is arbitrary, as we can scale $u_r \mapsto a u_r$. Seeing that $-\Delta_r = -\frac{1}{r^2}\Delta$ and $X_r = \frac{1}{r}X$, and scaling back (1.4.5), we finally have our result.

$\square$

1.5 $\langle X, X\rangle \leq 1$: subelliptic phenomenon on S^n, $n \geq 3$.

In this section, let us relax (for the traveling waves of the NLKG) the previously held restriction that $\langle X, X\rangle \leq b^2 < 1$. Now, if we allow the length of X to equal 1 at some points of M, then $-\Delta + X^2$ is not elliptic there anymore, which somewhat restricts the techniques we have at our disposal. To balance for that, we will carry out the investigation on a much more restricted geometric setting, namely, the sphere S^n. [T1] has a detailed investigation of this on the sphere S^2. We will now extend the analysis done for S^2 in [T1] to a sphere of dimension n.

1.5.1 Setting up the problem

Let $X_{ij}, i < j$, denote the vector field on S^n, which is the restriction of the vector field $x_i\partial_j - x_j\partial_i$ on $\mathbb{R}^{n+1}$ onto S^n. It is known that the Laplacian on S^n is given by $\Delta = \Sigma_{i<j}X_{ij}^2$ (actually, much more general statements can be made; for a survey article, see [C], Appendix B.4).

So pick one of these X_{ij}'s, say without loss of generality, X_{12}, henceforth called just X. It is to be noted that $\langle X, X\rangle < 1$ does not hold here always. So $L_0 = \Delta - X^2$ is not globally elliptic, but it satisfies Hörmander's condition for hypoellipticity (see Section B.3 in Appendix B, also see [Ho]). Let us justify this: from the above sum of squares, we see that L_0 is elliptic on $S^n \setminus \{(\pm 1, 0, ..., 0), (0, \pm 1, 0, ..., 0)\}$. Pick, without loss of generality, the point $(1, 0, ..., 0)$. On calculation,

$$[X_{23}, X_{13}] = (x_2\partial_1 - x_1\partial_2)|_{S^n} = X_{12}.$$

Together, at $(1, 0, .., 0)$, X_{1j}, $j = 2, 3, ..., n+1$ generate the full tangent space of S^n.

Also, by results in [T2] (Chapter XV, Theorem 1.4 and Theorem 1.8), L_0 is hypoelliptic with loss of a single derivative, which means the following:

$$L_0\phi \in H^s_{\text{loc}} \Rightarrow u \in H^{s+1}_{\text{loc}}. \tag{1.5.1}$$

This gives that

$$\mathcal{D}(L_0) \subseteq H^1(S^n),$$

which in turn implies, by interpolation (see [T3], Chapter 4, Section 2; for a definition, see Appendix B, Section B.4),

$$\mathcal{D}((-L_0)^{1/2}) = [L^2(S^n), \mathcal{D}(-L_0)]_{1/2} \subset [L^2(S^n), H^1(S^n)]_{1/2} = H^{1/2}(S^n). \tag{1.5.2}$$

Now, if we let

$$L_\alpha = L_0 - i\alpha X, \tag{1.5.3}$$

we see that L_α is self-adjoint for all $\alpha \in \mathbb{R}$ (refer to Section C.2 of Appendix C). However, to work with $(-L_\alpha)^{1/2}$, or even to define it via the spectral theorem (Theorem B.6.1), we need to establish the negative semidefiniteness of L_α for a certain range of α and establish what the range is. We actually have

Lemma 1.5.1. $L_\alpha = \Delta - X^2 - i\alpha X$ *is negative semidefinite for* $|\alpha| < n-1$.

Proof. To start, we can do an eigenvector decomposition of $L^2(S^n)$ with respect to the self-adjoint Δ. Since X is Killing, it commutes with Δ (see Chapter 2, Proposition 4.2 of [T3]) and respects the eigenspace decomposition. This means, X maps any eigenspace of Δ into itself. Let V_k denote the space of degree k harmonic homogeneous polynomials, defined on $\mathbb{R}^{n+1}$ and then restricted to S^n. It is known that all the eigenfunctions of the Laplacian on S^n are given by the members of V_k (see [T4], Chapter 8, Section 4). The eigenvalue corresponding to V_k is $k(k+n-1)$. It is also known that V_k is generated by polynomials of the form
$P_C(x) = (c_1x_1 + ... + c_{n+1}x_{n+1})^k$, where $x_i \in \mathbb{R}^{n+1}$, $c_i \in \mathbb{C}$ and $\Sigma c_i^2 = 0$. Now,

$$X(P_C(x)\Big|_{S^n}) = [(x_1\partial_2 - x_2\partial_1)P_C(x)]\Big|_{S^n}.$$

But,

$$\begin{aligned} [(x_1\partial_2 - x_2\partial_1)P_C(x)]\Big|_{S^n} &= [(x_1\partial_2 - x_2\partial_1)(c_1x_1 + ... + c_{n+1}x_{n+1})^k]\Big|_{S^n} \\ &= k(x_1c_2 - x_2c_1)(c_1x_1 + ... + c_{n+1}x_{n+1})^{k-1}\Big|_{S^n}. \end{aligned}$$

If $P_C(x)\Big|_{S^n}$ is an eigenfunction of X, then we must have $\gamma(x_1c_2 - x_2c_1)\Big|_{S^n} = (c_1x_1 + ... + c_{n+1}x_{n+1})\Big|_{S^n}, \gamma \in \mathbb{C}$. That gives, $c_3 = ... = c_{n+1} = 0$. Also, using that $c_1^2 + c_2^2 = 0$, we see that $\gamma = \pm i$.

So, we see that $P_C(x)\Big|_{S^n}$ is an eigenfunction of X if and only if $P_C(x)\Big|_{S^n} = (c_1x_1 + c_2x_2)^k\Big|_{S^n}$ and then it has eigenvalue ik or $-ik$, depending on the coefficients c_i.

Now, on the finite dimensional vector space V_k, the operator iX is Hermitian, allowing it to have a basis of eigenfunctions, say, $v_1, v_2, ..., v_{m_k}$, where $m_k = \dim V_k$. Choose any of these basis eigenfunctions, and call it u. Then,

$$\begin{aligned}(-L_\alpha u, u) &= k(k+n-1)\|u\|_{L^2}^2 + (X^2 u + i\alpha X u, u)\\ &\geq k(k+n-1)\|u\|_{L^2}^2 - k^2\|u\|_{L^2}^2 - |\alpha| k\|u\|_{L^2}^2\\ &= (k(n-1) - |\alpha| k)\|u\|_{L^2}^2.\end{aligned}$$

This finally implies that L_α is negative semidefinite with one dimensional kernel (containing only the constants) when $|\alpha| < n-1$. $\square$

Clearly, $\mathcal{D}((-L_{2\lambda})^{1/2}) = \mathcal{D}((-L_0)^{1/2})$, and by (1.5.2), both lie inside $H^{1/2}(S^n)$, and by Sobolev embedding, $H^{1/2}(S^n) \hookrightarrow L^{\frac{2n}{n-1}}(S^n)$.

Now, let q_* be the optimal (greatest) number such that

$$\mathcal{D}((-L_0)^{1/2}) \subset L^q(S^n), \forall\, q \in [2, q_*], \tag{1.5.4}$$

or

$$\mathcal{D}((-L_0)^{1/2}) \subset L^q(S^n), \forall\, q \in [2, q_*). \tag{1.5.5}$$

Whichever be the case, we can see that the inclusions (1.5.4) and (1.5.5) are continuous via the closed graph theorem applied to the inclusion operator. For the norm on $\mathcal{D}((-L_0)^{1/2})$, we use the graph norm given by

$$\|u\|^2_{\mathcal{D}((-L_0)^{1/2})} = ((-L_0)^{1/2}u, (-L_0)^{1/2}u) + (u, u),$$

which turns $\mathcal{D}((-L_0)^{1/2})$ into a Hilbert space (see Proposition 1.4 of [Sc]). Let us argue the applicability of the closed graph theorem here. It suffices to demonstrate the impossibility of the following scenario: u_n is a sequence in $\mathcal{D}((-L_0)^{1/2})$, such that $u_n \to u$ in $\mathcal{D}((-L_0)^{1/2})$-norm, $u_n \to v$ in L^q norm, and $u \neq v$. Observe that $u_n \to u$ in $\mathcal{D}((-L_0)^{1/2})$-norm implies that $u_n \to u$ in L^2-norm. Also, being in a compact setting, $u_n \to v$ in L^q-norm means $u_n \to v$ in L^2-norm, meaning $u = v$.

We also note that the continuity of the inclusion in (1.5.4) or (1.5.5) will actually guarantee that (1.5.5) is compact. Let us argue this first: by interpolation (see [T3], Chapter 4, Section 2), for all $q \in [2, q^*)$, we can produce $s \in (0,1)$ such that $\mathcal{D}((-L_0)^{s/2}) \subset L^q(S^n)$ is a continuous inclusion.

$$\mathcal{D}((-L_0)^{s/2}) = [L^2(S^n), \mathcal{D}((-L_0)^{1/2})]_s \subset [L^2(S^n), L^{q'}(S^n)]_s,$$

where $q' < q_*$ is chosen such that $[L^2(S^n), L^{q'}(S^n)]_s = L^q$. We can then compose the continuous inclusion $\mathcal{D}((-L_0)^{s/2}) \subset L^q(S^n)$ with the compact inclusion $\mathcal{D}((-L_0)^{1/2}) \hookrightarrow \mathcal{D}((-L_0)^{s/2})$ (the fact that this last inclusion is compact is not trivial; for a proof, see Theorem A.38 of [MK]). Since the composition of a bounded and a compact operator is compact, we have our claim that continuity of the inclusion (1.5.4) or (1.5.5) would imply compactness of (1.5.5).

Now, we have our existence result:

Proposition 1.5.2. *(***Existence result on** S^n*) With X as above, assume*

$$2 < p + 1 < q_*. \tag{1.5.6}$$

Also assume

$$|\lambda| < \frac{n-1}{2}, \; m^2 > \lambda^2. \tag{1.5.7}$$

Then, given $K > 0$, the equation

$$-L_{2\lambda} u + (m^2 - \lambda^2) u = K|u|^{p-1} u \tag{1.5.8}$$

has a nonzero solution $u \in \mathcal{D}((-L_{2\lambda})^{1/2})$.

Proof. As we have shown above, $-L_{2\lambda}$ is positive semidefinite when $|\lambda| < \frac{n-1}{2}$. So, the spectral theorem (Theorem B.6.1 in Appendix B) gives a definition of $(-L_{2\lambda})^{1/2}$. Then we use the fact that

$$F_{m,\lambda,X}(u) = (-L_{2\lambda} u, u) + (m^2 - \lambda^2)(u, u) \cong \|u\|^2_{\mathcal{D}((-L_{2\lambda})^{1/2})}, \tag{1.5.9}$$

where $\|u\|^2_{\mathcal{D}((-L_{2\lambda})^{1/2})}$ is the graph norm given by

$$\|u\|^2_{\mathcal{D}((-L_{2\lambda})^{1/2})} = ((-L_{2\lambda})^{1/2}u, (-L_{2\lambda})^{1/2}u) + (u, u),$$

which turns $\mathcal{D}((-L_{2\lambda})^{1/2})$ into a Hilbert space (see Proposition 1.4 of [Sc]).

Let

$$I_\beta = \inf\{F_{m,\lambda,X}(u) : u \in \mathcal{D}((-L_{2\lambda})^{1/2})\},$$

under the constraint (1.3.6). Now, take a sequence of functions $u_\nu \in \mathcal{D}((-L_{2\lambda})^{1/2})$ such that $F_{m,\lambda,X}(u_\nu) \to I_\beta$. Then, (1.5.9) implies that $\|u_\nu\|_{\mathcal{D}((-L_{2\lambda})^{1/2})}$ is uniformly bounded, which in turn means (a subsequence of) u_ν weakly converges to $u \in \mathcal{D}((-L_{2\lambda})^{1/2})$. By virtue of the compactness of (1.5.5), u_ν has a subsequence, still called u_ν with mild abuse of notation, that is strongly L^{p+1} convergent to u, where $p + 1 < q_*$, meaning that $\|u\|^{p+1}_{L^{p+1}} = \beta$, as in (1.3.6). Also (see arguments in Section C.4, Appendix C),

$$F_{m,\lambda,X}(u) \leq \liminf F_{m,\lambda,X}(u_\nu).$$

So, $u \in \mathcal{D}((-L_{2\lambda})^{1/2}))$ gives a constrained minimizer to $F_{m,\lambda,X}(u)$ subject to (1.3.6). The constrained minimizer will give a solution to (1.2.6), as wanted. □

Remark: Arguing as before with the closed graph theorem applied to the identity map, we can establish that

$$\|.\|_{\mathcal{D}((-L_{2\lambda})^{1/2}))} \cong \|.\|_{\mathcal{D}((-L_0)^{1/2}))}. \tag{1.5.10}$$

This is because $u_n \to u$ in $\|.\|_{\mathcal{D}((-L_{2\lambda})^{1/2}))}$-norm implies $u_n \to u$ in L^2-norm, and if $u_n \to v$ in $\|.\|_{\mathcal{D}((-L_0)^{1/2}))}$-norm, then $u_n \to v$ in L^2-norm, whence $u = v$.

It is not apriori clear that the constrained $F_{m,\lambda,X}$ minimizer obtained above is non-constant always. However, when $\lambda = 0$, the arguments of Theorem 1.4.2 go through, giving the following:

Lemma 1.5.3. *Given $p + 1 \in (2, q_*)$, $m > 0, K > 0$ and the Killing field X as mentioned at the beginning of Section 1.5.1, there exists $\delta > 0$ such that for $\varepsilon \in (0, \delta]$, the constrained*

$F_{m,0,X}$*-minimizing process on* S^n *produces a solution to*

$$-\Delta u + X^2 u + \frac{1}{\varepsilon^2} m^2 u = \frac{1}{\varepsilon^2} K|u|^{p-1} u$$

with $Xu \neq 0$.

Proof. We have

$$F_{m,0,0}(u) = ((-\Delta + m^2)u, u) = \|\nabla u\|_{L^2}^2 + m^2 \|u\|_{L^2}^2.$$

Now, as pointed out in Section C.2 of Appendix C, iX is a densely defined symmetric operator on $L^2(M)$. Then, we have

$$F_{m,0,X}(u) = \|\nabla u\|_{L^2}^2 + (X^2 u, u) + m^2 \|u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \|Xu\|_{L^2}^2 + m^2 \|u\|_{L^2}^2 \leq F_{m,0,0}(u).$$

Now, if u is not traveling, that is, $Xu = 0$, then $F_{m,0,X}(u) = F_{m,0,0}(u)$, which means that if $u \in \mathcal{D}((-L_0)^{1/2})$ minimizes $F_{m,0,X}$ subject to (1.3.6), then u also minimizes $F_{m,0,0}$ subject to (1.3.6). Now let us consider the function $v(x) = u(\phi(x))$, where $\phi \in \mathrm{Isom}(S^n)$. We have $F_{m,0,0}(v) = F_{m,0,0}(u)$. Also,

$$\begin{aligned} F_{m,0,X}(v) &\geq F_{m,0,X}(u) \text{ (since } u \text{ is a } F_{m,0,X} - \text{minimizer)} \\ &= F_{m,0,0}(u) \text{ (since } Xu = 0). \end{aligned}$$

Now,

$$\begin{aligned} F_{m,0,0}(v) &= (-\Delta u(\phi(x)) + m^2 u(\phi(x)), u(\phi(x))) \\ &= \int_{S^n} (-\Delta u(\phi(x)) + m^2 u(\phi(x)) \overline{u(\phi(x))} dM \\ &= \int_{S^n} (-\Delta u(y) + m^2 u(y)) \overline{u(y)} dM \\ &= F_{m,0,0}(u), \end{aligned}$$

where in the third step above we have used the fact that $y = \phi(x)$ is an isometry and hence the Jacobian determinant of this transformation is 1.

Now, we have

$$F_{m,0,X}(v) = F_{m,0,0}(v) \implies Xv = 0.$$

This happens for all $\phi \in \mathrm{Isom}(S^n)$. Now, choose a point $p \in S^n$ and let Y be a smooth vector field on M such that X_p and Y_p are linearly independent and X_p and Y_p have the same length. Consider the isometry $\phi \in \mathrm{Isom}(S^n)$ such that $d\phi(X_p) = Y_p$. Then, $Xv|_p = Yu|_p = 0$. Since this happens for all vector fields Y, we can see that u is locally constant. Also, since p can be any point on S^n (and S^n is connected), we finally have that u is globally constant.

Now, let us scale the metric on (S^n, g) to $S^n_r = (S^n, g^r_{ij})$ by $g^r_{ij} = r^2 g_{ij}$. Consider a metric ball U of radius k on S^n which is small enough so that U is diffeomorphic to the open Euclidean 1-ball in $\mathbb{R}^n$ and U does not intersect the points where $\langle X, X\rangle = 1$. Let U^r be the dilated image of U under the scaling. On S^n_r, consider the vector field $X_r = \frac{1}{r}X$. Let u_r denote the minimizer of $F^r_{m,0,X_r}(u)$, subject to $\int_{S^n_r} |u|^{p+1} dS^n_r = A$. If u_r is constant, on calculation,

$$u_r = (\frac{A}{V})^{\frac{1}{p+1}} r^{-\frac{n}{p+1}},$$

where V is the volume of (S^n, g).

That gives,

$$\begin{aligned} F^r_{m,0,X_r}(u_r) &= m^2 (\frac{A}{V})^{\frac{2}{p+1}} V r^n r^{-\frac{2n}{p+1}} \\ &= C r^{\frac{n(p-1)}{p+1}}, \end{aligned}$$

where C is a constant. Since $X_r u_r = 0$, this is also the infimum of $F^r_{m,0,0}(u)$, subject to $\int_{S^n_r} |u|^{p+1} dS^n_r = A$.

Now,

$$\inf_{u \in H^1(S^n_r), \text{supp } u \subset U^r} F^r_{m,0,0}(u) \geq \inf_{u \in H^1(S^n_r)} F^r_{m,0,0}(u).$$

So, as $r \to \infty$,

$$\inf_{u \in H^1_0(U^r)} F^r_{m,0,0}(u) \to \infty. \tag{1.5.11}$$

But, as $r \to \infty$, U^r approaches the flat Euclidean space $\mathbb{R}^n$.

Let p be the centre of the balls U^r, which have radius rk. Using the radial minimizer u^0 of Lemma 1.4.1, define

$$v_r(x) = \chi(x)u^0(\mathrm{dist}_r(p,x)), x \in U^r, \tag{1.5.12}$$

where dist_r is the metric distance in (S^n, g^r_{ij}) and $\chi(x)$ is a smooth radial cut-off function such that $\chi \equiv 1$ on $B^r_p(rk - \frac{1}{r})$, where the superscript r on the ball denotes the ball in the g^r_{ij} metric.

We have

$$\int_{S^n_r} |v_r|^{p+1} \to A$$

and

$$F^r_{m,0,0}(v_r) \to F^0_{m,0,0}(u^0) = ((-\Delta_{\mathbb{R}^n} + m^2)u, u)_{L^2(\mathbb{R}^n)} < \infty,$$

thereby contradicting (1.4.3).

So, for r large, there exists a constrained minimizer u_r such that $X_r u_r \neq 0$, which solves

$$-\Delta_r u_r + X_r^2 u_r + m^2 u_r = K|u_r|^{p-1}u_r, \tag{1.5.13}$$

where $K > 0$ is arbitrary, as we can scale $u_r \mapsto au_r$. Seeing that $-\Delta_r = -\frac{1}{r^2}\Delta$ and $X_r = \frac{1}{r}X$, and scaling back (1.5.13), we finally have our result. $\square$

1.5.2 What is the optimal q_*?

On S^n, $H^{1/2}$ Sobolev embeds in $L^{\frac{2n}{n-1}}$. By mimicking the calculations in [T1], we now try to see if this can be improved. For any vector field $X_{ij} \neq X$, the points where X_{ij} vanish, will lie on, say, U_{ij} where U_{ij} is isometric to S^{n-2}. By using the ellipticity of L_0 away from points which have coordinates $(\pm 1, 0, ..., 0)$ and $(0, \pm 1, .., 0))$, or the "poles", we have,

$$u \in \mathcal{D}((-L_0)^{1/2}) \Rightarrow \phi u \in H^1(S^n) \subseteq L^{\frac{2n}{n-2}}(S^n) \tag{1.5.14}$$

by Sobolev embedding, where $\phi \in C^\infty_c(S)$ and $S = S^n \setminus \{(\pm 1, 0, ..., 0), (0, \pm 1, .., 0)\}$.

Before proceeding, let us prove the following

Lemma 1.5.4.

$$\mathcal{D}((-L_0)^{1/2}) = \{u \in L^2(S^n) : X_{ij}u \in L^2(S^n), X_{ij} \neq X\}. \tag{1.5.15}$$

Proof. We start by referring to Proposition 1.10, Chapter 8 of [T4], which gives a characterization of $\mathcal{D}(A^{1/2})$, where A is a non-negative, unbounded self-adjoint operator on a Hilbert space H constructed by the Friedrichs method (see Section 10.4 of [Sc]).

In the notation of the said proposition, here $H = L^2(S^n)$. Also, let

$$H_1 = \{u \in L^2(S^n) : X_{ij}u \in L^2(S^n), X_{ij} \neq X\}.$$

$$(u,v)_{H_1} = (u,v) + \sum_{X_{ij} \neq X} (X_{ij}u, X_{ij}v).$$

J is the natural inclusion $H_1 \to L^2(S^n)$. Then we have that

$$\mathcal{D}(-L_0) = \{u \in H_1 : v \mapsto (u,v) + \sum_{X_{ij} \neq X} (X_{ij}u, X_{ij}v) \text{ is continuous in } v$$
$$\forall v \in H_1 \text{ in the } L^2\text{-norm}\}.$$

Now, if we can prove that H_1 is a Hilbert space with inner product $(.,)_{H_1}$, then the conclusion of Proposition 1.10 (Chapter 8 of [T4]) gives that $\mathcal{D}((-L_0)^{1/2}) = H_1$.

Now, call $H_{ij} = \mathcal{D}(X_{ij}) = \{u \in L^2 : X_{ij}u \in L^2\}$, which becomes a Hilbert space with graph inner product $(u,v)_{ij} = (u,v) + (X_{ij}u, X_{ij}v)$. Then,

$$H_1 = \bigcap_{i<j} H_{ij}$$

will become a Hilbert space with the norm $(.,.)_{H_1}$. This is because, given a Cauchy sequence in H_1, it becomes a Cauchy sequence in each H_{ij}, and since the above intersection is finite, we can select a subsequence which is convergent in every H_{ij}. Also, the limit of this subsequence must be the same in every H_{ij}, because of the shared component $(.,.)$ (L^2 inner product) in each $(.,.)_{ij}$. The limit then lies in the intersection H_1, which proves that H_1 is a Hilbert space with inner product $(.,.)_{H_1}$. $\square$

With that in place, we take a function $u \in \mathcal{D}((-L_0)^{1/2}))$ having small support in a neighborhood around any of above poles, say, without loss of generality, $(1, 0, ..., 0)$, and project it down to $\mathbb{R}^n$. This produces a compactly supported projected function on $\mathbb{R}^n$, still called u with mild abuse of notation, such that

$$u \in H^{1/2}(\mathbb{R}^n),\ \partial_{x_i} u \in L^2(\mathbb{R}^n),\ \forall i \in \{2, 3, ..., n\}. \tag{1.5.16}$$

Now, observe that (1.5.16) implies, after Fourier transforming, $(\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^{1/4}\hat{u} \in L^2(\mathbb{R}^n)$ and $\xi_i \hat{u} \in L^2(\mathbb{R}^n)$ for all $i \in \{2, 3, ..., n\}$. That means

$$\hat{f} = (\xi_1^2 + \xi_2^4 + ... + \xi_n^4)^{1/4}\hat{u} \in L^2(\mathbb{R}^n).$$

We label $u = k * f$, where

$$\hat{k} = (\xi_1^2 + \xi_2^4 + ... + \xi_n^4)^{-1/4}.$$

This means that

$$k \in C^\infty(\mathbb{R}^n \setminus \{0\}).$$

For a justification of this, see Section C.5 of Appendix C.

Also, k satisfies the anisotropic homogeneity

$$k(\delta^2 x_1, \delta x_2, ..., \delta x_n) = \delta^{-n} k(x_1, x_2, ..., x_n). \tag{1.5.17}$$

Define $\Omega_0 = \{(x_1, ..., x_n) : 1/2 \leq |x|^2 < 1\}$ and define Ω_j for $j \in \mathbb{Z}$ as the image of Ω_{j-1} under the map

$$(x_1, x_2, ..., x_n) \mapsto (2^{-1}x_1, 2^{-1/2}x_2, ..., 2^{-1/2}x_n).$$

Using (1.5.17), we have

$$|k| \leq C 2^{nj/2} \text{ on } \Omega_j, \tag{1.5.18}$$

where

$$|\Omega_j| = 2^{-(n+1)/2}|\Omega_{j-1}| = C2^{-((n+1)/2)j}.$$

Set

$$k_1 = k \text{ on } \bigcup_{j\geq 0}\Omega_j, \ 0 \text{ elsewhere}$$

and

$$k_2 = k \text{ on } \bigcup_{j<0}\Omega_j, \ 0 \text{ elsewhere},$$

so that $k = k_1 + k_2$. Also, let $u_l = k_l * f, l = 1, 2$.

By (1.5.18), we have

$$\int |k_1|^r d\mathbb{R}^n \leq C \sum_{j\geq 0} 2^{njr/2-((n+1)/2)j} < \infty \tag{1.5.19}$$

when $r < \frac{n+1}{n}$. Also,

$$\int |k_2|^r d\mathbb{R}^n \leq C \sum_{j<0} 2^{njr/2-((n+1)/2)j} < \infty \tag{1.5.20}$$

when $r > \frac{n+1}{n}$. Now by using Young's inequality for convolutions, we have, $u_1 \in L^q$, where $q \in [2, \frac{2(n+1)}{(n-1)})$ and $u_2 \in L^q$, where $q \in (\frac{2(n+1)}{(n-1)}, \infty)$. But $u_2 = u - u_1 \Rightarrow u_2 \in L^2$, and by interpolation, $u_2 \in L^q, q \in [2, \infty)$. So, finally, $u \in L^q$, where $q \in [2, \frac{2(n+1)}{(n-1)})$. So, in our previously introduced notation, $q_* = \frac{2(n+1)}{(n-1)}$.

1.5.3 The endpoint case $q = \frac{2(n+1)}{(n-1)} = 6$ for $n = 2$.

In the special case of $n = 2$, our setting is now the sphere S^2. We already have (also c.f. [T1])

$$\mathcal{D}((-L_0)^{1/2}) \subset L^q(S^2), \ \forall q \in [2, 6). \tag{1.5.21}$$

Here we extend the above inclusion up to $q = 6$ and also argue that this is sharp. We have

Lemma 1.5.5. *(***Optimal embedding and sharpness***)*

$$\mathcal{D}((-L_0)^{1/2}) \subset L^6(S^2).$$

Also, this embedding is sharp. That is,

$$\mathcal{D}((-L_0)^{1/2}) \subset L^q(S^2) \Longrightarrow q \leq 6.$$

Proof. We start by observing that, similar to (1.5.15) above,

$$\mathcal{D}((-L_0)^{1/2}) = \{u \in L^2(S^2) : Yu, Zu \in L^2(S^2)\},$$

where Y, Z are respectively the restrictions on S^2 of the vector fields that generate rotations about the y-axis and the z-axis in $\mathbb{R}^3$.

Ellipticity of $-L_0$ away from the poles $(0, \pm 1, 0), (0, 0, \pm 1)$ implies

$$u \in \mathcal{D}((-L_0)^{1/2}) \Rightarrow \varphi u \in H^1(S^2),$$

where $S = S^2 \setminus \{(0, \pm 1, 0), (0, 0, \pm 1)\}$ and $\varphi \in C_c^\infty(S)$.

With that in place, we take a function $u \in \mathcal{D}((-L_0)^{1/2}))$ having small support in a neighborhood around any of the points in S, say, without loss of generality, $(0, 1, 0)$ and project it to $\mathbb{R}^2$ in the following way: let γ_{xy}, γ_{yz} and γ_{zx} denote the great circles on S^2 lying on the xy, yz and zx-planes respectively. Then the projection takes a neighborhood of $(0, 1, 0)$ in γ_{xy} onto the y-axis and a neighborhood of $(0, 1, 0)$ in γ_{yz} onto the x-axis. This produces a compactly supported projected function on $\mathbb{R}^2$, still called u with mild abuse of notation, such that

$$u \in H^{1/2}(\mathbb{R}^2), \partial_y u \in L^2(\mathbb{R}^2). \tag{1.5.22}$$

Also, since we have already asserted that $Zu \in L^2(S^2)$, this will give

$$(x\partial_y - y\partial_x)u \in L^2(\mathbb{R}^2). \tag{1.5.23}$$

Since u is compactly supported, $\partial_y u \in L^2(\mathbb{R}^2) \implies x\partial_y u \in L^2(\mathbb{R}^2)$, which, coupled with the last fact, implies, $y\partial_x u \in L^2(\mathbb{R}^2)$.

We will use the first two pieces of data, namely, $u \in H^{1/2}(\mathbb{R}^2)$ and $\partial_y u \in L^2(\mathbb{R}^2)$. We observe that this means

$$u \in H_x^{1/2}(L_y^2) \cap L_x^2(H_y^1) \tag{1.5.24}$$

(see Notation D.0.5 in Appendix D). Let us first justify this.
$u \in L_x^2(H_y^1)$ actually means $||||u||_{H_y^1}||_{L_x^2} < \infty \Leftrightarrow ||||(1+\eta^2)^{\frac{1}{2}}\hat{u}^y||_{L_\eta^2}||_{L_x^2} < \infty$, where $\hat{u}^y$ represents Fourier transform with respect to y, that is, $\hat{u}^y$ is now a function of x and η.

Now,

$$\begin{aligned}
\|\|u\|_{H_y^1}\|_{L_x^2} &= \|\|(1+\eta^2)^{\frac{1}{2}}\hat{u}^y\|_{L_x^2}\|_{L_\eta^2} \\
&= \|(1+\eta^2)^{\frac{1}{2}}\|\hat{u}^y\|_{L_x^2}\|_{L_\eta^2} = \|(1+\eta^2)^{\frac{1}{2}}\|\hat{u}||_{L_\xi^2}\|_{L_\eta^2} \\
&= \|\|(1+\eta^2)^{\frac{1}{2}}\hat{u}\|_{L_\xi^2}\|_{L_\eta^2} < \infty,
\end{aligned}$$

since $(1+\eta^2)^{1/2}\hat{u} \in L^2(\mathbb{R}^2)$.

Similarly, $u \in H_x^{1/2}(L_y^2) \Leftrightarrow \|(1+\xi^2)^{1/4}\hat{u}^x(\xi)\|_{L_y^2} \in L_\xi^2$, where $\hat{u}^x$ means Fourier transform with respect to x only. This holds iff $\|\|((1+\xi^2)^{1/4}\hat{u}^x(\xi)\|_{L_y^2}\|_{L_\xi^2} < \infty$, which follows from $u \in H^{1/2}(\mathbb{R}^2)$.

This implies (1.5.24). Now we propose to use interpolation ([LM], Chapter 4 has a detailed treatment of these sorts of spaces and allied results). By interpolation, we can say that for $\theta \in [0,1]$

$$u \in H^{\frac{1}{2}\theta}(H_y^{1-\theta}), \tag{1.5.25}$$

where $H_x^r(H_y^s)$ denotes H_y^s-valued H^r-functions of x. This is because,

$$\begin{aligned}
u \in H^{\frac{1}{2}\theta}(H_y^{1-\theta}) &\Leftrightarrow (1+\xi^2)^{\theta/4}\hat{u}(\xi, y) \in L_\xi^2(H_y^{1-\theta}) \\
&\Leftrightarrow (1+\xi^2)^{\theta/4}(1+\eta^2)^{(1-\theta)/2}\hat{u}(\xi,\eta) \in L_\xi^2(L_\eta^2).
\end{aligned}$$

But this follows by interpolation from $(1+\xi^2)^{1/4}\hat{u}(\xi,\eta) \in L_\xi^2(L_\eta^2)$ and $(1+\eta^2)^{1/2}\hat{u}(\xi,\eta) \in L_\xi^2(L_\eta^2)$.

Now, from (1.5.25), particularly for $\theta = 2/3$, we have

$$u \in H_x^{1/3}(H_y^{1/3}).$$

Now, when we use Sobolev embedding in one dimension, we know that $H^{1/3}$ embeds in L^6. That means, $u \in L_x^6(L_y^6)$, which implies, $u \in L^6(\mathbb{R}^2)$.

We will now prove the next part of the lemma: that the estimate of $u \in L^6$ as obtained above is sharp. To do this, let us emulate the scaling technique as appears in Appendix A.2 of [CMMT]. Since u has compact support, $\partial_y u \in L^2(\mathbb{R}^2) \Rightarrow x\partial_y u \in L^2(\mathbb{R}^2)$, which, coupled with (1.5.23) implies $y\partial_x u \in L^2(\mathbb{R}^2)$. We also have (1.5.22).

Let us define

$$u(r, \sigma, a, b, x, y) = r^\sigma u(r^a x, r^b y).$$

In the ensuing calculations, we will write, when convenient, $u(r, \sigma, a, b)$ for $u(r, \sigma, a, b, x, y)$ for ease of handling symbols.

We have,

$$\begin{aligned}
||\partial_y u(r, \sigma, a, b)||_{L^2}^2 &= \int_{\mathbb{R}^2} |\partial_y u(r, \sigma, a, b, x, y)|^2 dxdy = \int_{\mathbb{R}^2} |\partial_y r^\sigma u(r^a x, r^b y)|^2 dxdy \\
&= \int_{\mathbb{R}^2} |r^b \partial_{z_2} r^\sigma u(z_1, z_2)|^2 r^{-a} r^{-b} dz_1 dz_2 \\
&= r^{b+2\sigma-a} ||\partial_y u||_{L^2}^2.
\end{aligned}$$

Similarly, we can calculate,

$$||y\partial_x u(r, \sigma, a, b)||_{L^2}^2 = r^{2\sigma+a-3b} ||y\partial_x u||_{L^2}^2.$$

Now,

$$\begin{aligned}
\hat{u}(r,\sigma,a,b,\xi,\eta) &\simeq \int_{\mathbb{R}^2} u(r,\sigma,a,b,x,y)e^{-i(\xi x+\eta y)}dxdy \\
&= \int_{\mathbb{R}^2} r^{\sigma}u(r^a x, r^b y)e^{-i(\xi x+\eta y)}dxdy \\
&= \int_{\mathbb{R}^2} r^{\sigma}u(z_1,z_2)e^{-i(\frac{\xi}{r^a}z_1+\frac{\eta}{r^b}z_2)}r^{-a}r^{-b}dz_1dz_2 \\
&= r^{\sigma-a-b}\hat{u}(r^{-a}\xi, r^{-b}\eta).
\end{aligned}$$

So,

$$\begin{aligned}
\|u(r,\sigma,a,b)\|^2_{H^{1/2}} &= \int_{\mathbb{R}^2}(1+\xi^2+\eta^2)^{1/2}r^{2\sigma-2a-2b}|\hat{u}(r^{-a}\xi,r^{-b}\eta)|^2 d\xi d\eta \\
&= r^{2\sigma-2a-2b}\int_{\mathbb{R}^2}(1+r^{2a}\theta^2+r^{2b}\phi^2)^{1/2}|\hat{u}(\theta,\phi)|^2 r^a r^b d\theta d\phi, \theta = r^{-a}\xi, \phi = r^{-b}\eta \\
&= \int_{\mathbb{R}^2}(r^{2(2\sigma-a-b)}+r^{4\sigma-2b}\xi^2+r^{4\sigma-2a}\eta^2)^{1/2}|\hat{u}(\xi,\eta)|^2 d\xi d\eta.
\end{aligned}$$

We will want to compare this estimate with $\|u\|^2_{H^{1/2}} = \int_{\mathbb{R}^2}(1+\xi^2+\eta^2)^{1/2}|\hat{u}(\xi,\eta)|^2 d\xi d\eta$.

Also, on calculation, $\|u(r,\sigma,a,b)\|^p_{L^p} = r^{\sigma p-a-b}\|u\|^p_{L^p}$.

Now, suppose that 6 is not a sharp exponent. We begin by choosing a $u \in \mathcal{D}((-L_0)^{1/2})$ satisfying $u \in L^{6+\varepsilon}$, where $\varepsilon > 0$. In the above equations, we let $\sigma = 1$ by observation. Then we see that for $a = 4$ and $b = 2$ (and calling $u(r,1,4,2) = u_r$), we see that $\|\partial_y u_r\|_{L^2} = \|\partial_y u\|_{L^2}, \|y\partial_x u_r\|_{L^2} = \|y\partial_x u\|_{L^2}$ and $\|u_r\|_{H^{1/2}} \leq \|u\|_{H^{1/2}}$ when $r \geq 1$.

On calculation, $\|u_r\|^{6+\varepsilon}_{L^{6+\varepsilon}} = r^{\varepsilon}\|u\|^{6+\varepsilon}_{L^{6+\varepsilon}}$. Clearly, as we let r increase, the left hand side increases, with a fast decreasing support, since the support of u was compact to begin with.

Finally, to get a contradiction, we just have to take a sequence of u_r for fast increasing r, with disjoint supports, and sum them up. To be precise, we already have $\|u_r\|_{L^{6+\varepsilon}} = Kr^{\theta}$, where $K = \|u\|_{L^{6+\varepsilon}}$ is a constant and $\theta = \frac{\varepsilon}{6+\varepsilon} > 0$.

Define a new function u^* by $u^* = \Sigma\frac{1}{2^n}v_{r_n}$, where r_n is chosen such that $2^{n-1} \leq r_n^{\theta} < 2^n$ and v_{r_n} is obtained by a translate of u_{r_n} parallel to the x-axis, in such a way that all the v_{r_n} have disjoint support. That way, we still preserve control over $\|\partial_y u^*\|_{L^2}, \|y\partial_x u^*\|_{L^2}$ and $\|u^*\|_{H^{1/2}}$, but the $L^{6+\varepsilon}$-norm of u^* blows up, contrary to our assumption. $\square$

1.5.4 Higher regularity in case of nonsmooth nonlinearity on S^2

It has already been shown that

$$u \in \mathcal{D}((-L_0)^{1/2}) \Rightarrow u \in L^6.$$

Now if u solves (1.2.6), then we can do better. A specific case ($p = 3$) has been worked out in [T1] and it has been shown (using an elliptic bootstrapping argument) that u is then smooth. Now, if p is not an odd integer, we cannot expect a similar smoothness, because the nonlinearity of (1.2.6) itself is then not smooth. However, we can expect higher Sobolev spaces and, in turn, higher L^r spaces (by Sobolev embedding) for u when p is not an odd integer. Here, we calculate one explicit case, namely, $p = 4$. A word is in order regarding this choice. Firstly, let $P = -L_{2\lambda} + (m^2 - \lambda^2)$ and $F(u) = K|u|^{p-1}u$ whence $Pu = F(u)$. Let $3 < p < 6$.

Then, $F(u) \in L^{6/p}$. On calculation, $(L^{6/p})^* = L^{\frac{6}{6-p}}$. By using the Sobolev embedding theorem, we can find a $\delta > 0$ such that $H^\delta \subset L^{\frac{6}{6-p}}$. On calculation, this happens when $\delta > p/3 - 1$. So, by duality,

$$F(u) \in L^{6/p} \subseteq \bigcap_{\delta > p/3-1} H^{-\delta}, \tag{1.5.26}$$

which means

$$u = P^{-1}(F(u)) \subseteq \bigcap_{\delta > p/3-1} H^{1-\delta}. \tag{1.5.27}$$

The above claim comes from the fact that P is hypoelliptic from Hörmander's condition. From Section B.3, we see that P is hypoelliptic if $L_{2\lambda}$ is. Since $-\Delta + X^2 = Y^2 + Z^2$, and $[Y, Z] = X$ (see the more general demonstration on page 16), $L_{2\lambda}$ is hypoelliptic. Also, from Theorem 1.8, Chapter XV of [T2], P is hypoelliptic with the loss of a single derivative.

Note that we already know that $u \in H^{1/2}$. So this bootstrapping process yields something better than what we started with only when $\delta < 1/2$, or equivalently, $3 < p < 9/2$. So for an explicit demonstration we have chosen $p = 4$.

When $p = 4$, according to previous calculation,

$$u = P^{-1}(F(u)) \subseteq \bigcap_{\delta > 1/3} H^{1-\delta} = \bigcap_{\varepsilon > 0} H^{2/3-\varepsilon}. \tag{1.5.28}$$

As argued before, $u \in H^1$ when u is supported away from the poles. So, choose neighborhoods around the "north pole" of the 2-sphere in the following manner: U, V and W are open neighborhoods such that $V \subset \overline{V} \subset W \subset \overline{W} \subset U$. Also choose a smooth bump function ϕ such that $\operatorname{supp} \phi \subset \overline{W}$ and $\phi \equiv 1$ on $\overline{V}$. Note that ϕu satisfies (1.2.6) inside V, so with a suitably chosen ϕ we can ensure that $\phi u \in \bigcap_{\varepsilon>0} H^{2/3-\varepsilon}(U)$.

Now we are going to determine if ϕu belongs in a higher Sobolev space. Surely, ϕu will not solve (1.2.6) on U, but that is fine. All we want to investigate is the behavior of u around the pole, which can be tracked by the behavior of ϕu inside V. Now, projecting down U on the plane, we see that the projection of ϕu, called v, satisfies $\partial_y v \in L^2$ and v has compact support. This implies, by the interpolation procedure on mixed Sobolev spaces carried out in the proof of Lemma 1.5.5, that,

$$v \in L^r, r < 10. \tag{1.5.29}$$

Let us argue how this goes. We know $v \in \bigcap_{\varepsilon>0} H^{2/3-\varepsilon}$ and $\partial_y v \in L^2$. By arguments outlined in the proof of Lemma 1.5.5, that means

$$v \in H_x^{2/3-\varepsilon}(L_y^2) \cap L_x^2(H_y^1), \forall \varepsilon > 0. \tag{1.5.30}$$

By interpolation,

$$v \in H_x^{(2/3-\varepsilon)\theta}(H_y^{1-\theta}), \forall \varepsilon > 0, \theta \in [0,1]. \tag{1.5.31}$$

Choosing $\theta = \frac{1}{5/3-\varepsilon}$, we finally get that

$$v \in H_x^{(2/3-\varepsilon)(\frac{1}{5/3-\varepsilon})}(H_y^{(2/3-\varepsilon)(\frac{1}{5/3-\varepsilon})}), \forall \varepsilon > 0. \tag{1.5.32}$$

Sobolev embedding then gives (1.5.29).

(1.5.29), in turn, through the bootstrapping procedure given by (1.5.26), (1.5.27) and (1.5.28), implies that $v \in \bigcap_{\varepsilon>0} H^{4/5-\varepsilon}$, which means, $u \in \bigcap_{\varepsilon>0} H^{4/5-\varepsilon}$. This is a gain in regularity.

1.6 Results in the non-compact setting: Main theorems

In this section, we enter into our main theorems, which deal with constrained minimizing solutions of (1.2.5) and (1.2.6) on non-compact manifolds. As mentioned at the outset of this chapter, we extend results from [T1] and also results proved earlier in this chapter to a non-compact setting. More precisely, we will try to repeat the analysis of Section 1 of [T1] and Section 1.3 of this thesis in the case of non-compact manifolds M. Before we begin, here is a heuristic story. What will cost us most dearly in the non-compact setting is the failure of compact Sobolev embedding. That means, we have to find out means of exercising some control over functions outside a large compact set. We do this in two different ways: one is to consider constrained minimization in a subclass of $H^1(M)$ functions and impose appropriate geometric restrictions on the manifold M that will make elements of the said subclass vanish at infinity, and this will make the failure of the Sobolev embedding a manageable problem. This we do in the case of $F_{m,\lambda,X}$ minimizers in Section 1.6.1. Another way to proceed is to use concentration compactness arguments in the presence of a certain geometric homogeneity of the space M. In that case, once we prove concentration, we can use the geometric homogeneity of the space to bring all the zones of concentration within a compact region and get compact Sobolev embedding into action. This we implement in the case of constrained $\mathcal{E}_{\lambda,X}$ minimizers in Section 1.6.2.

1.6.1 $F_{m,\lambda,X}$ minimizers

Here we consider non-compact manifolds M with C^∞ bounded geometry which are of the form $[0,\infty) \times N$. Here N is assumed to be compact and $(n-1)$ dimensional, and M has the product metric $g = dr^2 + \phi(r)g_N$, where ϕ is smooth and positive with $\phi(1) = 1$, and g_N denotes the metric on N. Also, we assume that M is complete, and all the points $(0,x), x \in N$ are identified to a single point (which could be thought of as the origin). The last two assumptions respectively mean that we do not have to worry about cone points and boundaries.

If X is a Killing field on (N, g_N), consider the push-forward vector field X_r on $(N, \phi(r)g_N)$, that is, $X_r = i_{r*}X$, $i_r : (N, g_N) \to (N, \phi(r)g_N)$ being the identity map. This induces a Killing field on M, still called X by abuse of notation. Note that, for $r \in [0,\infty), x \in N$, $X_{(r,x)} = \sqrt{\phi(r)}X_{(1,x)}$. We will consider only those M which have bounded geometry (see Definition A.4.1 in Appendix

A) and those functions ϕ such that $\langle X, X\rangle \leq b^2 < 1$. As an example of the kind of space we are talking about, consider the cylinder $[1,\infty)\times S^1$ fitted with a hemispherical cap (diffeomorphic to the closed 2-disc) to make it complete. Here X is given by (slow) rotation about the axis of the cylinder.

In general (see [CMMT], Section 2.3, for example), we should not expect minimizers of $F_{m,\lambda,X}$ on $H^1(M)$ when M is complete and non-compact, even if it has rotational symmetry. However, we can minimize $F_{m,\lambda,X}$ on the class of radial functions which are in $H^1(M)$; that is, we will try to minimize $F_{m,\lambda,X}$ over

$$H^1_r(M) = \{u \in H^1(M) : u \text{ is a radial function}\}.$$

A word is in order regarding what is meant by a radial function. Here it means those functions which are dependent only on the variable r running over $[0,\infty)$ of the space $M = [0,\infty)\times N$, i.e., we are considering only those functions f for which $f(r,x) = \varphi(r)$. Also, if $A(r)drdN$ represents the volume form of M, then by calculation, we have $A(r) = (\phi(r))^{\frac{n-1}{2}}$.

To work out constrained $F_{m,\lambda,X}$ minimizers, we first need a lemma:

Lemma 1.6.1. *Consider a non-compact complete manifold M of dimension n satisfying the properties described at the beginning of this section. Also, assume a positive lower bound on $\phi(r)$ outside a compact set, say, when $r > 1$. Then, if $f \in H^1_r(M)$, f vanishes at infinity.*

Proof. We start by justifying that $f \in H^1_r(M) \Rightarrow f \in C(M\setminus U)$, where U is a neighborhood of the origin, let us say, without loss of generality, a ball of radius 1. The argument behind this is essentially local. Choose $(r',x') \in M$ and a small open ball $B = (r'-\delta, r'+\delta)\times V$ around (r',x'), where V is open in N. We can see that $u \in H^1(B) \Rightarrow u(r,x') \in H^1((r'-\delta, r'+\delta))$, and since all the components of the metric tensor g are uniformly bounded on B, one-dimensional Sobolev embedding gives $u(r,x') \in C((r'-\delta,r'+\delta)) \Rightarrow u \in C(B)$. Also, since functions in $C(M\setminus U)$ can be uniformly approximated by functions in $C^\infty(M\setminus U)$, we can assert that it is enough to prove the lemma for $f \in C^1(M\setminus U)\cap H^1_r(M)$.

Now, if f does not vanish at infinity, then, there exists an $\varepsilon > 0$ such that no matter what compact set in M we select, f attains a value greater than ε outside this compact set. By scaling the function if necessary, we can use $\varepsilon = 1$.

Let $q_k \in M$ be a sequence of points satisfying the following:

(a) q_k has coordinates (r_k, x), $r_k \in (0, \infty)$, $x \in N$ (fixed), such that r_k is a strictly increasing sequence in k,

(b) $\mathrm{dist}(q_k, q_{k+1}) > 2$ for all k,

(c) $f(q_k) > 1$ for all k,

(d) there exist annuli $D_k = (r_k - s_k, r_k + s'_k) \times N$, $s_k, s'_k > 0$ such that f falls below $1/2$ somewhere inside each D_k and the D_k's do not intersect each other, and

(e) $|D_k|$ is bounded above by a positive constant.

Clearly,

$$\begin{aligned}
\int_{D_k} |\nabla f|^2 A(r) dr dN &\geq C_k \left(\int_{D_k} |\nabla f| A(r) dr dN\right)^2 \text{ (using (e))} \\
&\gtrsim C_k \left(\int_{r_k - s_k}^{r_k + s'_k} |\nabla f| A(r) dr\right)^2 \gtrsim \left(\int_{r_k - s_k}^{r_k + s'_k} |\nabla f| dr\right)^2 \\
&\gtrsim 1/4 \text{ (using (c) and (d))} .
\end{aligned}$$

where $C_k = \frac{1}{|D_k|}$ is bounded below, since $|D_k|$'s are bounded above. Since this is happening for all k, this will contradict the fact that $f \in H^1_r(M)$.

We must point out that (d) and (e) above hold necessarily, as otherwise, we will have a sequence of annuli B_k such that $|f| > 1/2$ on B_k and $|B_k| \to \infty$. That will imply $f \notin H^1_r(M)$. $\square$

Here, we have assumed a lower bound on the function $A(r)$. To give some alternative criteria under which we can force f to vanish at infinity, we refer to Lemma 2.1.1 from [MT], which says the following:

Lemma 1.6.2. *Assume that $A(r)$ satisfies either*

$$\int_{|r| \geq 1} \frac{dr}{A(r)} < \infty$$

or

$$\lim_{|r| \to \infty} A(r) = \infty, \text{ and } \sup_{|r| \geq 1} \left| \frac{A'(r)}{A(r)} \right| < \infty.$$

Then

$$f \in H^1_r(M) \Rightarrow f|_{M_1} \in C(M_1) \text{ and}$$

$$\lim_{|r|\to\infty} |f(r)| = 0,$$

where M_1 consists of all the points of M having r-coordinates ≥ 1.

Let us also prove the following

Lemma 1.6.3. *Consider a non-compact manifold M as described in the statement of Lemma 1.6.1. Given $m, \lambda \in \mathbb{R}$, we assume the following bounds on b:*

$$b^2 + 2|\lambda|b < 1, \text{ and also } 2|\lambda|b < m^2 - \lambda^2 \text{ if } m^2 - \lambda^2 > 0. \tag{1.6.1}$$

Now, under (1.3.3), and if (1.3.4) holds, then we have,

$$F_{m,\lambda,X}(u) \cong \|u\|^2_{H^1}.$$

Proof. We have, $\mathrm{Spec}(-\Delta + X^2 + 2i\lambda X) \subset [\beta(\lambda), \infty)$ and $m^2 - \lambda^2 > -\beta(\lambda)$. Assume first that $m^2 - \lambda^2 > 0$. We have,

$$F_{m,\lambda,X}(u) \leq (-\Delta u, u) + |(Xu, Xu)| + 2|\lambda||(Xu, u)| + ((m^2 - \lambda^2)u, u).$$

Using $\langle X, X\rangle \leq b^2$, on calculation this gives, $F_{m,\lambda,X}(u) \lesssim \|u\|^2_{H^1}$. Also,

$$(-\Delta u, u) - |(Xu, Xu)| - 2|\lambda||(Xu, u)| + ((m^2 - \lambda^2)u, u) \leq F_{m,\lambda,X}(u).$$

We want to show that

$$C\|u\|^2_{H^1} \leq (-\Delta u, u) - |(Xu, Xu)| - 2|\lambda||(Xu, u)| + ((m^2 - \lambda^2)u, u), \tag{1.6.2}$$

where $C > 0$ is independent of u. This will hold if and only if we can find a constant C such that

$$|(Xu, Xu)| + 2|\lambda||(Xu, u)| \leq (1 - C)(-\Delta u, u) + (m^2 - \lambda^2 - C)\|u\|^2_{L^2}.$$

On calculation, using $\langle X, X\rangle \le b^2$, we get

$$|(Xu, Xu)| + 2|\lambda||(Xu, u)| \le (b^2 + 2|\lambda|b)(-\Delta u, u) + 2|\lambda|b\|u\|_{L^2}^2.$$

which finally proves (1.6.2).

Now let us consider the case $0 \ge m^2 - \lambda^2 > -\beta(\lambda)$. As before, letting $-L_{2\lambda} = -\Delta + X^2 + 2i\lambda X$, we have

$$F_{m,\lambda,X}(u) \le (-L_{2\lambda}u, u) + (u, u) \lesssim \|u\|_{H^1}^2 \text{ (using } \langle X, X\rangle \le b^2).$$

Also, the calculation for $\|u\|_{H^1}^2 \lesssim (-L_{2\lambda}u, u) + (u, u)$ is similar to the proof of (1.6.2). So, we are done if we can prove that

$$(-L_{2\lambda}u, u) + (u, u) \lesssim F_{m,\lambda,X}(u). \tag{1.6.3}$$

We see that $(-L_{2\lambda}u, u) \ge \beta(\lambda)(u, u)$. When $\alpha > -\beta(\lambda)$, we have

$$\begin{aligned}(-L_{2\lambda}u, u) + \alpha(u, u) \ge C(-L_{2\lambda}u, u) &\ge \frac{C}{2}(-L_{2\lambda}u, u) + \frac{\beta(\lambda)}{2}C(u, u)\\ &\gtrsim (-L_{2\lambda}u, u) + (u, u),\end{aligned}$$

where $C = 1 + \frac{\alpha}{\beta(\lambda)}$. $\square$

Now, we have our first main theorem of this chapter:

Theorem 1.6.4. (Main theorem I) *Consider a non-compact manifold M as described in the statement of Lemma 1.6.1. Given $m, \lambda \in \mathbb{R}$, we assume (1.6.1) and (1.3.3). Now, if (1.3.4) is satisfied, then we can minimize $F_{m,\lambda,X}(u)$ in the class of functions $H_r^1(M)$ subject to (1.3.6). Here we keep p in the range $(1, \frac{n+2}{n-2})$.*

Proof. We already know, under our assumptions,

$$F_{m,\lambda,X}(u) \cong \|u\|_{H^1(M)}^2. \tag{1.6.4}$$

We also have, $H^1(M) \hookrightarrow L^q(U)$ compactly, $q \in [2, \frac{2n}{n-2})$, where $\overline{U}$ is compact in M. Also, by

Lemma 1.6.1,

$$u \in H^1_r(M) \Rightarrow u \text{ vanishes at infinity.}$$

So,

$$u \in H^1_r(M) \Rightarrow u \in L^\infty(M \setminus U).$$

Also, $u \in L^2(M)$. This means, by interpolation,

$$u \in L^q(M \setminus U) \text{ for all } q \in [2, \infty].$$

We also have,

$$\begin{aligned} \int_{M\setminus U} |u|^q dM &\leq \|u\|^{q-2}_{L^\infty(M\setminus U)} \int_{M\setminus U} |u|^2 dM \\ &\leq \|u\|^{q-2}_{L^\infty(M\setminus U)} \|u\|^2_{H^1(M)}, \end{aligned} \tag{1.6.5}$$

and this gives,

$$\begin{aligned} u \in H^1_r(M) &\Rightarrow u \in L^q(M) \ \forall \ q \in [2, \frac{2n}{n-2}) \\ &\Rightarrow u \in L^{p+1}(M) \ \forall \ p \in (1, \frac{n+2}{n-2}) \ (p = 1 \text{ is not in our range}). \end{aligned}$$

As usual, let

$$I_\beta = \inf\{F_{m,\lambda,X}(u) : u \in H^1_r(M), \text{ subject to (1.3.6)}\}.$$

Clearly, $I_\beta > 0$, because of (1.6.4), (1.6.5) and the constraint (1.3.6). Now, take a sequence $u_\nu \in H^1_r(M)$ such that $||u_\nu||^{p+1}_{L^{p+1}} = \beta$, and $F_{m,\lambda,X}(u_\nu) \leq I_\beta + 1/\nu$.

Passing to a subsequence if necessary and without changing the notation, $u_\nu \to u \in H^1_r(M)$ weakly, which implies, by compact Sobolev embedding,

$$u_\nu \longrightarrow u \text{ in } L^{p+1}(U)\text{-norm for all relatively compact } U. \tag{1.6.6}$$

Also, using (1.6.6) with very large U's and the fact that u_ν, u vanish at infinity, we have from

(1.6.5),

$$\|u_\nu - u\|_{L^{p+1}(M\setminus U)} \longrightarrow 0, \tag{1.6.7}$$

meaning finally that

$$||u||^{p+1}_{L^{p+1}} = \beta.$$

Also, we have to prove that $F_{m,\lambda,X}(u) = I_\beta$. This comes from the fact that

$$F_{m,\lambda,X}(u) \leq \liminf F_{m,\lambda,X}(u_\nu).$$

So finally a constrained $F_{m,\lambda,X}$ minimizer is obtained. $\square$

1.6.2 Constrained energy minimizers

We now write about constrained energy minimizers in a non-compact setting. To be precise, we assume that our non-compact manifold M is weakly homogeneous (see Definition D.0.6 in Appendix D). On such spaces, we are trying to minimize the energy

$$\mathcal{E}_{\lambda,X}(u) = \frac{1}{2}(-\Delta u + X^2 u + 2i\lambda X u, u) - \frac{1}{p+1}\int_M |u|^{p+1} dM$$

subject to $||u||^2_{L^2} = \beta$ (constant) and (1.3.3), the minimization being done over $H^1(M)$, and $p \in (1, 1 + 4/n)$.

Let

$$I_\beta = \inf\{\mathcal{E}_{\lambda,X}(u) : u \in H^1(M), \|u\|^2_{L^2} = \beta\}.$$

We will make the following technical assumption:

$$I_\beta < -\frac{(m^2 - \lambda^2)}{2}\beta, \tag{1.6.8}$$

where m is selected such that $m^2 - \lambda^2 >$ max $\{-\beta(\lambda), 0\}$, with $\beta(\lambda)$ defined as in (1.3.3). We also assume that (1.6.1) is satisfied.

With that in place, we state the second main theorem of this chapter:

Theorem 1.6.5. (Main Theorem II) *If M is a non-compact weakly homogeneous manifold, under the technical assumption (1.6.8), we can minimize $\mathcal{E}_{\lambda,X}(u)$ in the class of functions $H^1(M)$ subject to $\|u\|_{L^2} = \beta$ and (1.3.3). Here we want p in the range $(1, 1 + \frac{4}{n})$.*

Arguing with the Gagliardo-Nirenberg inequality as in Proposition 1.3, we can reach equation (1.3.22), which lets us conclude that $I_\beta > -\infty$, and if u_ν is a sequence in $H^1(M)$ satisfying $\mathcal{E}_{\lambda,X}(u_\nu) < I_\beta + \frac{1}{\nu}$, then (a subsequence) u_ν is weakly convergent to $u \in H^1(M)$.

Now, we can see that establishing u as the constrained energy minimizer amounts to establishing two things:

- $u_\nu \longrightarrow u$ in L^2-norm, so that $\|u\|^2_{L^2} = \beta$,
- $\mathcal{E}_{\lambda,X}(u) = I_\beta$.

Now, in view of (1.3.24), the second bullet point will be established if we can prove that

$$\|u_\nu\|_{L^{p+1}} \to \|u\|_{L^{p+1}}, \tag{1.6.9}$$

and

$$F_{m,\lambda,X}(u) \leq \liminf F_{m,\lambda,X}(u_\nu). \tag{1.6.10}$$

Since $\|u_\nu\|_{H^1}$ is uniformly bounded, (1.6.9) will be established via the Gagliardo-Nirenberg inequality (applied to $u - u_\nu$), in conjunction with the first bullet point above. Also, (1.6.10) is a consequence of weak convergence, spectral assumption (1.3.3) and $m^2 - \lambda^2 > -\beta(\lambda)$.

So now, our entire task hinges on proving the first bullet point, namely

$$u_\nu \to u \text{ in } L^2\text{-norm}. \tag{1.6.11}$$

To accomplish this, we use the techniques of concentration-compactness, as laid out in [L]. Below we give a formal statement of this. The statement was originally made in the setting of the Euclidean space, but as noted in [CMMT], the concentration-compactness principle and most of the subsidiary results generalize to manifolds of bounded geometry with essentially no changes at all. We will state the reformulated version as appears in [CMMT].

Proposition 1.6.6. *Let M be a Riemannian manifold with C^∞ bounded geometry. Fix $\beta \in (0, \infty)$. Let $\{u_\nu\} \in L^{p+1}(M)$ be a sequence satisfying $\int_M |u_\nu|^{p+1} dM = \beta$. Then, after extracting a subsequence, one of the following three cases holds:*

(i) Vanishing: If $B_R(y) = \{x \in M : d(x, y) \leq R\}$ is the closed R-ball around y, then for all $R \in (0, \infty)$,

$$\lim_{\nu \to \infty} \sup_{y \in M} \int_{B_R(y)} |u_\nu|^{p+1} dM = 0.$$

(ii) Concentration: There exists a sequence of points $\{y_\nu\} \subset M$ with the property that for each $\varepsilon > 0$, there exists $R(\varepsilon) < \infty$ such that

$$\int_{B_{R(\varepsilon)}(y_\nu)} |u_\nu|^{p+1} dM > \beta - \varepsilon.$$

(iii) Splitting: There exists $\alpha \in (0, \beta)$ with the following properties: For each $\varepsilon > 0$, there exists $\nu_0 \geq 1$ and sets $E_\nu^\#, E_\nu^b \subset M$ such that

$$d(E_\nu^\#, E_\nu^b) \to \infty \text{ as } \nu \to \infty \tag{1.6.12}$$

and

$$\left| \int_{E_\nu^\#} |u_\nu|^{p+1} dM - \alpha \right| < \varepsilon, \left| \int_{E_\nu^b} |u_\nu|^{p+1} dM - (\beta - \alpha) \right| < \varepsilon, \nu > \nu_0. \tag{1.6.13}$$

For a statement of the above fact in the even more general setting of measure metric spaces, see Appendix A.1 of [CMMT]. A couple of lines about the heuristics of the concentration-compactness principle: when we have a sequence of elements in a Banach space with fixed norm, or, in other words, lying on a sphere in the Banach space, we cannot necessarily pick a norm convergent subsequence unless the Banach space itself is finite dimensional. But, we can give an exhaustive list of the possible behaviors of subsequences, at least in the context of the L^p spaces. That is what the concentration-compactness principle gives. In our case, the only handle we have on the sequence u_ν is that all of them have the same L^2-norm. This should make the application of the concentration-compactness argument seem natural. In applications such as ours, the usual line of attack is to rule out vanishing and splitting phenomena, so we are left with concentration phenomenon as the only possibility. From there, we will show how to go to compactness, i.e., convergence of the subsequence, $\|u_\nu - u\|_{L^2} \to 0$, which has been the goal of the first bullet point.

Ruling out vanishing and splitting

Following closely the corresponding analyses of [CMMT] and [MT], to rule out vanishing, one has to make the technical assumption mentioned before:

$$I_\beta < -\frac{(m^2 - \lambda^2)}{2}\beta, \tag{1.6.14}$$

where m is selected such that $m^2 - \lambda^2 > -\beta(\lambda)$, with $\beta(\lambda)$ defined as in (1.3.3) and also $m^2 - \lambda^2 > 0$.

It is not clear that we can always have (1.6.8) regardless of the manifold type. Some discussion about the assumption $I_\beta < 0$ is found in (3.0.10) and (3.0.11) of [CMMT].

Step I: Ruling out vanishing.

Assume vanishing occurs, that is, $\forall\ R \in (0, \infty)$,

$$\lim_{\nu\to\infty} \sup_{y\in M} \int_{B_R(y)} |u_\nu|^2 dM = 0.$$

We already know that u_ν's satisfy $\mathcal{E}_{\lambda,X}(u_\nu) < I_\beta + 1/\nu$ and that, $\{u_\nu\}$ is bounded in $H^1(M)$.

Then, we have, by Lemma 2.1.2 of [CMMT]

$$2 < r < \frac{2n}{n-2} \Longrightarrow \|u_\nu\|_{L^r(M)} \to 0.$$

That means,

$$\|u\|_{H^1}^2 \cong F_{m,\lambda,X}(u) = 2\mathcal{E}_{\lambda,X}(u) + \frac{2}{p+1}\int_M |u|^{p+1} dM + (m^2 - \lambda^2)\beta$$

implies in conjunction with (1.6.8) that

$$||u||_{H^1}^2 \le \liminf \|u_\nu\|_{H^1}^2 \le \frac{2}{C^*} I_\beta + \frac{1}{C^*}(m^2 - \lambda^2)\beta < 0,$$

which gives a contradiction. Here C^* is a constant such that $C^*||f||_{H^1}^2 \le F_{m,\lambda,X}(f)$ for all $f \in H^1(M)$.

Step II: Ruling out splitting.

To rule out the splitting phenomenon, we first need a technical lemma, which is a special case of Propositions 3.1.2 and 3.1.3 of [CMMT].

Lemma 1.6.7. *(i) If* $\beta > 0, I_\beta < -\frac{m^2-\lambda^2}{2}\beta, \sigma > 1$, *then*

$$I_{\sigma\beta} < \sigma I_\beta. \tag{1.6.15}$$

(ii) If $0 < \eta < \beta$ *and* $I_\beta < -\frac{m^2-\lambda^2}{2}\beta$, *we have*

$$I_\beta < I_{\beta-\eta} + I_\eta. \tag{1.6.16}$$

Finally, we work to rule out splitting phenomena. We have

Proposition 1.6.8. *If* $\{u_\nu\} \in H^1(M)$ *is a* $\mathcal{E}_{\lambda,X}$*-minimizing sequence with* $\|u_\nu\|_{L^2}^2 =$ *constant, then splitting ((1.6.12) and (1.6.13)) cannot occur.*

Proof. Begin by choosing $\varepsilon > 0$ sufficiently small such that

$$I_\beta < I_\alpha + I_{\beta-\alpha} - C_1\varepsilon, \tag{1.6.17}$$

where C_1 is a constant that will be chosen later.

Suppose now that splitting happens. We have already argued that $||u_\nu||_{H^1}$ is uniformly bounded. Also, seeing that $\|u_\nu\|_{L^2} =$ constant and $||u_\nu||_{L^{p+1}}$ is uniformly bounded by an application of the Gagliardo-Nirenberg inequality, it follows from (1.6.12) and (1.6.13) that there exists ν_1 such that when $\nu \geq \nu_1$, we have

$$\int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |\nabla u_\nu|^2 dM + \int_{S_\nu} |u_\nu|^{p+1} dM < \varepsilon, \tag{1.6.18}$$

where S_ν is a set of the form

$$S_\nu = \{x \in M : d_\nu < d(x, E_\nu^\#) \leq d_\nu + 2\} \subset M \setminus (E_\nu^\# \cup E_\nu^b)$$

for some $d_\nu > 0$. Call

$$\tilde{E}_\nu(r) = \{x \in M : d(x, E_\nu^{\#}) \leq r\}.$$

Now define functions $\chi_\nu^{\#}$ and χ_ν^b by

$$\chi_\nu^{\#}(x) = \begin{cases} 1, & \text{if } x \in \tilde{E}_\nu(d_\nu) \\ 1 - d(x, \tilde{E}_\nu(d_\nu)), & \text{if } x \in \tilde{E}_\nu(d_\nu + 1) \\ 0, & \text{if } x \notin \tilde{E}_\nu(d_\nu + 1) \end{cases}$$

and

$$\chi_\nu^b(x) = \begin{cases} 0, & \text{if } x \in \tilde{E}_\nu(d_\nu + 1) \\ d(x, \tilde{E}_\nu(d_\nu + 1)), & \text{if } x \in \tilde{E}_\nu(d_\nu + 2) \\ 1, & \text{if } x \notin \tilde{E}_\nu(d_\nu + 2). \end{cases}$$

Observe that both $\chi_\nu^{\#}(x)$ and $\chi_\nu^b(x)$ are Lipschitz with Lipschitz constant 1 and the intersection of their supports has measure zero. Also set

$$u_\nu^{\#} = \chi_\nu^{\#} u_\nu, u_\nu^b = \chi_\nu^b u_\nu.$$

Just to motivate what we are doing, we want a control on the term $|\mathcal{E}_{\lambda,X}(u_\nu) - \mathcal{E}_{\lambda,X}(u_\nu^{\#} + u_\nu^b)|$ i.e., show that

$$|\mathcal{E}_{\lambda,X}(u_\nu) - \mathcal{E}_{\lambda,X}(u_\nu^{\#} + u_\nu^b)| = |\mathcal{E}_{\lambda,X}(u_\nu) - [\mathcal{E}_{\lambda,X}(u_\nu^{\#}) + \mathcal{E}_{\lambda,X}(u_\nu^b)]| \lesssim \varepsilon, \tag{1.6.19}$$

and get a contradiction from the fact that $|I_\beta - I_\alpha - I_{\beta-\alpha}| > C_1\varepsilon$ which comes from (1.6.17). Choosing m such that $m^2 - \lambda^2 > -\beta(\lambda)$, with $\beta(\lambda)$ as in (1.3.3), we know that

$$2\mathcal{E}_{\lambda,X}(u_\nu) = F_{m,\lambda,X}(u_\nu) - \frac{2}{p+1}||u_\nu||_{L^{p+1}}^{p+1} - (m^2 - \lambda^2)\|u_\nu\|_{L^2}^2,$$

and hence we see by triangle inequality that controlling each of the terms

$$\int_M \Big(|u_\nu|^{p+1} - (|u_\nu^\#|^{p+1} + |u_\nu|^{p+1}) \Big) dM, \tag{1.6.20}$$

$$\int_M \Big(||u_\nu||_{H^1} - (||u_\nu^\#||_{H^1} + ||u_\nu^b||_{H^1}) \Big) dM, \tag{1.6.21}$$

$$|F_{m,\lambda,X}(u_\nu) - (F_{m,\lambda,X}(u_\nu^\#) + F_{m,\lambda,X}(u_\nu^b))|, \tag{1.6.22}$$

would be sufficient. To that end, we first note that when $\nu \geq \nu_1$,

$$||u_\nu^\#||_{L^2}^2 = \alpha_\nu, \text{ where } |\alpha - \alpha_\nu| < 2\varepsilon$$

and

$$||u_\nu^b||_{L^2}^2 = \beta_\nu - \alpha_\nu, \text{ where } |(\beta - \alpha) - (\beta_\nu - \alpha_\nu)| < 2\varepsilon.$$

Now, we have

$$\int_M \Big(|u_\nu|^{p+1} - (|u_\nu^\#|^{p+1} + |u_\nu|^{p+1}) \Big) dM \leq \int_{S_\nu} |u_\nu|^{p+1} dM < \varepsilon$$

and

$$\int_M \Big(|u_\nu|^2 - (|u_\nu^\#|^2 + |u_\nu|^2) \Big) dM \leq \int_{S_\nu} |u_\nu|^2 dM < \varepsilon. \tag{1.6.23}$$

Using $\nabla u_\nu^\# = \chi_\nu^\# \nabla u_\nu + (\nabla \chi_\nu^\#) u_\nu$, the corresponding identity for ∇u_ν^b and the fact that both $\chi_\nu^\#(x)$ and $\chi_\nu^b(x)$ have Lipschitz constant 1, we see that

$$\int_M \Big(|\nabla u_\nu|^2 - (|\nabla u_\nu^\#|^2 + |\nabla u_\nu|^2) \Big) dM \leq \int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |\nabla u_\nu|^2 dM \lesssim \varepsilon. \tag{1.6.24}$$

(1.6.23) and (1.6.24) together give (1.6.21). Now we are left with (1.6.22).

From the definition of $F_{m,\lambda,X}(u)$ and what has gone before, we see that it suffices to control

$$|(X^2u_\nu, u_\nu) - (X^2u_\nu^\#, u_\nu^\#) - (X^2u_\nu^b, u_\nu^b)|$$

or, equivalently,

$$\left| \int_M (|Xu_\nu|^2 - |Xu_\nu^\#|^2 - |Xu_\nu^b|^2) dM \right|.$$

and also

$$\left| (iXu_\nu, u_\nu) - (iXu_\nu^\#, u_\nu^\#) - (iXu_\nu^b, u_\nu^b) \right|.$$

Now, as before, $Xu_\nu^\# = \chi_\nu^\# Xu_\nu^\# + X(\chi_\nu^\#)u_\nu^\#$, so

$$\left| \int_M (|Xu_\nu|^2 - |Xu_\nu^\#|^2 - |Xu_\nu^b|^2) dM \right| \leq \int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |Xu_\nu|^2 dM \tag{1.6.25}$$

$$\lesssim \int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |\nabla u_\nu|^2 dM \lesssim \varepsilon, \tag{1.6.26}$$

the last observation coming from the fact that X is bounded, which means that $|Xu_\nu| = |X.\nabla u_\nu| \lesssim |\nabla u_\nu|$. Lastly, we can also control

$$\left| (iXu_\nu, u_\nu) - (iXu_\nu^\#, u_\nu^\#) - (iXu_\nu^b, u_\nu^b) \right|$$

by using the Cauchy-Schwarz inequality. This is because

$$\begin{aligned} |(iXu_\nu, u_\nu) - (iXu_\nu^\#, u_\nu^\#) - (iXu_\nu^b, u_\nu^b)| &= |(Xu_\nu, u_\nu) - (\chi_\nu^\# Xu_\nu^\# + X(\chi_\nu^\#)u_\nu^\#, u_\nu^\#) \\ &\quad -(\chi_\nu^b Xu_\nu^b + X(\chi_\nu^b)u_\nu^b, u_\nu^b)| \\ = |(Xu_\nu, u_\nu) - (Xu_\nu, \chi_\nu^\# u_\nu) - (Xu_\nu, \chi_\nu^b u_\nu) &- (X(\chi_\nu^\#)u_\nu, u_\nu^\#) + (X(\chi_\nu^b)u_\nu, u_\nu^b)| \end{aligned}$$

$$\leq |(Xu_\nu, u_\nu) - (Xu_\nu, \chi_\nu^{\#} u_\nu) - (Xu_\nu, \chi_\nu^{b} u_\nu)| + |(X(\chi_\nu^{\#})u_\nu, u_\nu^{\#})| + |(X(\chi_\nu^{b})u_\nu, u_\nu^{b})|$$

$$\leq |\int_{S_\nu} Xu_\nu \overline{u_\nu}| + |\int_{S_\nu} u_\nu \overline{u_\nu^{\#}}| + |\int_{S_\nu} u_\nu \overline{u_\nu^{b}}| \leq |\int_{S_\nu} Xu_\nu \overline{u_\nu}| + 2\int_{S_\nu} |u_\nu|^2.$$

Now,

$$|\int_{S_\nu} Xu_\nu \overline{u_\nu}| \leq \|Xu_\nu\|_{L^2(S_\nu)} \|u_\nu\|_{L^2(S_\nu)} \leq \|X.\nabla u_\nu\|_{L^2(S_\nu)} \|u_\nu\|_{L^2(S_\nu)} \lesssim \|\nabla u_\nu\|_{L^2(S_\nu)} \|u_\nu\|_{L^2(S_\nu)} \leq \varepsilon$$

the last step coming from (1.6.18). That completes the proof. □

Now that we have ruled out the alternatives, we can say that the minimizing sequence u_ν will concentrate. Recall that this means

Corollary 1.6.9. *Under the setting of Lemma 1.6.7, there is a sequence of points $y_\nu \in M$ such that for all $\varepsilon > 0$, there exists $R(\varepsilon) < \infty$ (independent of ν) such that*

$$\int_{M \setminus B_{R(\varepsilon)}(y_\nu)} |u_\nu|^2 dM < \varepsilon. \tag{1.6.27}$$

This allows us to invoke the assumption of weak homogeneity at last. Using weak homogeneity, we can map the sequence y_ν into a compact region $K \subset M$ and we still call the translates of u_ν as u_ν. Now, any subsequence which concentrates will have compact Sobolev embedding, i.e., we use the compact embedding $H^1(M) \hookrightarrow L^2(K)$. Then, weak H^1 convergence of u_ν allows us to find a subsequence, still called u_ν, such that

$$\|u_\nu - u\|_{L^2(K)} \to 0. \tag{1.6.28}$$

We see that (1.6.28) holds for all bounded $K \subset M$. Hence, from (1.6.27), we see that $\|u\|_{L^2}^2 = \beta$ and $u_\nu \to u$ in $L^2(M)$-norm. This is what we intended to prove.

CHAPTER 2: GROUND STATES AND THE WEINSTEIN FUNCTIONAL

2.1 Introduction

The Weinstein functional on a manifold M for a function u is defined by

$$W(u) = \frac{||u||_{L^{p+1}}^{p+1}}{||u||_{L^2}^{\alpha}||\nabla u||_{L^2}^{\beta}} \tag{2.1.1}$$

with $\alpha = 2-(n-2)(p-1)/2, \beta = n(p-1)/2, n = \dim(M)$. We also keep p in the range $(1, \frac{n+2}{n-2})$ unless otherwise mentioned. We are interested in whether $W(u)$ *attains* a maximum over $H^1(M)$. It is clear that if the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{p+1}}^{p+1} \leq C\|u\|_{L^2}^{\alpha}\|\nabla u\|_{L^2}^{\beta} \tag{2.1.2}$$

holds, then $W(u)$ is bounded above, and moreover, the best constant in the Gagliardo-Nirenberg inequality will also be the supremum of the Weinstein functional over $H^1(M)$, denoted by $W_M^{\sup}$. As a notational convenience, we will sometimes drop the subscript M when there is no cause for confusion.

The functional was first introduced in [W] to study the bound states for nonlinear Schrödinger equations. Now why is it important? Consider the nonlinear Schrödinger equation

$$iv_t + \Delta v + |v|^{p-1}v = 0, x \in M,\ v(0,x) = v_0(x). \tag{2.1.3}$$

A nonlinear bound state/standing wave solution of (2.1.3) is a choice of an initial condition $u_\lambda(x)$ such that

$$v(t,x) = e^{i\lambda t}u_\lambda(x).$$

Plugging in this ansatz in (2.1.3) yields the following auxiliary elliptic equation

$$-\Delta u_\lambda + \lambda u_\lambda - |u_\lambda|^{p-1} u_\lambda = 0. \tag{2.1.4}$$

We also note that seeking a standing wave solution to the nonlinear Klein-Gordon equation

$$v_{tt} - \Delta v + m^2 v - |v|^{p-1} v = 0,\ v(t,x) = e^{i\mu t} u(x) \tag{2.1.5}$$

will lead to (2.1.4) with $\lambda = m^2 - \mu^2$ (from the point of view of standing waves, there is no essential difference in the analyses of the NLS and the NLKG; this is markedly different than the situation we encountered in Chapter I).

Now, with $u, v \in H^1(M)$, we calculate that,

$$\frac{d}{d\tau} W(u + \tau v)\Big|_{\tau=0} = \frac{\mathrm{Re}(N(u), v)}{||u||_{L^2}^{2\alpha} ||\nabla u||_{L^2}^{2\beta}}, \tag{2.1.6}$$

where

$$N(u) = (p+1)\|u\|_{L^2}^{\alpha} \|\nabla u\|_{L^2}^{\beta} |u|^{p-1} u - \beta \|u\|_{L^{p+1}}^{p+1} \|u\|_{L^2}^{\alpha} \|\nabla u\|_{L^2}^{\beta-2} (-\Delta u) - \alpha \|u\|_{L^2}^{\alpha-2} \|\nabla u\|_{L^2}^{\beta} u \|u\|_{L^{p+1}}^{p+1}.$$

Let u be a maximizer of the Weinstein functional, and let

$$\lambda = \frac{\alpha}{\beta} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2},\ K = \frac{p+1}{\beta} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{p+1}}^{p+1}}. \tag{2.1.7}$$

Then, (2.1.6) shows that u will give a solution to

$$-\Delta v + \lambda v = K|v|^{p-1} v. \tag{2.1.8}$$

Now, if u solves (2.1.8), then $u_a = au$ solves

$$-\Delta v + \lambda v = |a|^{1-p} |v|^{p-1} v, \tag{2.1.9}$$

which finally means that we can solve (2.1.4) for any $K > 0$. This allows us to pass in between

(2.1.8) and (2.1.4).

Theorem B of [W] establishes the existence of a maximizer of the Weinstein functional inside $H^1(\mathbb{R}^n)$. The main objective of Weinstein's work was to establish a sharp criterion for the existence of global solutions to the focusing nonlinear Schrödinger equation on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$iv_t + \Delta v = -\frac{1}{2}|v|^{p-1}v,\ v(x,0) = v_0(x), \tag{2.1.10}$$

in the energy critical case $p = 1 + 4/n$. Before his work, (2.1.10) was already known to have global solutions for any $v_0 \in H^1(\mathbb{R}^n)$ with $\|v_0\|_{L^2}$ sufficiently small. The question was: exactly how small? This was answered in the energy critical case by

Theorem 2.1.1. *(Weinstein [W]) Let $v_0 \in H^1(\mathbb{R}^n)$. For $p = 1 + 4/n$, a sufficient condition for global existence in the initial-value problem (2.1.10) is*

$$\|v_0\|_{L^2} < \|\psi\|_{L^2},$$

where ψ is a positive solution of the equation

$$-\Delta u + u = u^{1+4/n}$$

of minimal L^2 norm.

Such solutions of minimal L^2 norm are also known as ground states. Theorem B of [W] shows that the Weinstein functional maximizer exists in $H^1(\mathbb{R}^n)$ and also that it gives a ground state solution to (2.1.10).

In the setting of the hyperbolic space, consider the focusing nonlinear Schrödinger equation

$$iv_t + \Delta_{\mathbb{H}^n} v = -|v|^{p-1}v,\ v(0,x) = v_0 \in H^1(\mathbb{H}^n). \tag{2.1.11}$$

We know that the Gagliardo-Nirenberg inequality holds on $\mathbb{H}^n$ (see, for example, [Ba], Section 6.1). Let C represent the best constant of the Gagliardo-Nirenberg inequality or $W_{\mathbb{H}^n}^{\sup}$ in the energy critical case $p = 1 + 4/n$. Then, as stated in [Ba], (2.1.11) has global solution if

$$\|v_0\|_{L^2} < (\frac{2 + 4/n}{2C})^{n/4}.$$

Now, we can raise the following question: how do the best constants of the Gagliardo-Nirenberg inequality on $\mathbb{R}^n$ and $\mathbb{H}^n$ compare? It is known that the best constant in the Gagliardo-Nirenberg inequality on $\mathbb{H}^n$ is greater than or equal to the one on $\mathbb{R}^n$ (see Proposition 2.1.3 below), but not obviously equal to it (see [Ba], Remark 6.1). This motivates us to investigate this natural question in Section 2.2 below. In this regard, also refer to Section 4.3 of [CMMT].

Applications to Schrödinger equations apart, the Weinstein functional is an interesting nonlinear functional in its own right, and establishing where it can be maximized (that is, there exists a function which attains the maximum) is intrinsically related to the geometry of the manifold M and can be quite tricky. The functional is not at all well-behaved with respect to conformal changes of the metric, which adds to the difficulty. To the best of our knowledge, the question of existence of Weinstein functional maximizers is largely unexplored in the compact setting, for example, on compact manifolds with boundary with Dirichlet boundary conditions.

In the setting of non-compact Riemannian manifolds, it is not even clear when the Gagliardo-Nirenberg inequality (2.1.2) holds, let alone existence of Weinstein functional maximizers. For the sake of completeness, we recall that the Gagliardo-Nirenberg inequality is implied by any of the following equivalent statements (we will prove a more general version of this implication later on):

- the heat kernel $p(t, x, y)$ of the manifold M satisfying

$$p(t, x, y) \leq Ct^{-n/2}, t > 0, x, y \in M, \tag{2.1.12}$$

  where C is a constant independent of x, y and t.

- Existence of Sobolev embeddings of the form

$$\Big(\int_M |u|^{2n/(n-2)} dM\Big)^{(n-2)/n} \lesssim \int_M |\nabla u|^2 dM, \ \forall\ u \in C_0^\infty(M). \tag{2.1.13}$$

In fact, the above two statements are equivalent. For details on the proofs, see [N] and [V]. To be specific, [N] establishes the heat kernel bounds starting from the Sobolev embeddings given by (2.1.13). [V] has the converse.

In particular, among other things, it is known that non-negative lower bounds on the Ricci curvature (which means, $\text{Ric} \geq \lambda g, \lambda > 0$, as bilinear forms) implies any of the above (actually the lower bound on the Ricci curvature is a much stronger condition; it can even imply Gaussian bounds on the heat kernel, see [SY]). The heat kernel bounds (2.1.12) are known separately for the hyperbolic space and many other nice rank 1 symmetric spaces (see Definition A.5.1, Appendix A, and also see [HS]). In any case, we know that W_M^{sup} exists at least when $M = \mathbb{R}^n, \mathbb{H}^n$ as well as on compact manifolds with boundary with Dirichlet boundary conditions. We must also state the obvious at this point: the Weinstein functional maximization problem does not make sense on a compact manifold without boundary, as the constants would make the $\|\nabla u\|_{L^2}^{\beta}$ term on the denominator vanish. One of the better ways to make sense of the problem on a compact manifold M with boundary is to use Dirichlet boundary conditions; it disallows one from plugging in nonzero constant functions u into $W(u)$. A Weinstein functional maximizer in $H_0^1(M)$ will give a solution to (2.1.4) with Dirichlet boundary conditions.

Before we state our main theorems for this chapter, let us begin with a few preliminary lemmas. The first thing we want to point out is the following

Lemma 2.1.2. *Scaling the metric has no effect on the Weinstein functional. In other words, consider a manifold (M, g) and the same (smooth) manifold with a scaled metric (M, rg) $(r > 0)$. Let $W(u)$ and $W_r(u)$ represent the Weinstein functionals of u with respect to the metrics g and rg respectively. Then*

$$W_r(u) = W(u),$$

which implies that

$$W_{(M,g)}^{\text{sup}} = W_{(M,rg)}^{\text{sup}}.$$

Proof. Let $g_r = rg$ be the scaled metric and ∇_r denote the gradient of u with respect to g_r. Then,

$$\int_M |u|^{p+1}\sqrt{g_r}dx = r^{n/2}\int_M |u|^{p+1}\sqrt{g}dx, \tag{2.1.14}$$

$$(\int_M |u|^2\sqrt{g_r}dx)^{\alpha/2} = r^{\alpha n/4}(\int_M |u|^2\sqrt{g}dx)^{\alpha/2}, \tag{2.1.15}$$

Also, $|\nabla_r u|^2 = \frac{1}{r}|\nabla u|^2$, which means

$$||\nabla_r u||^{\beta}_{L^2} = r^{\beta n/4-\beta/2}||\nabla u||^{\beta}_{L^2}. \tag{2.1.16}$$

Finally, from (2.1.14), (2.1.15) and (2.1.16), we have that $W_r(u) = W(u)$. □

So let us talk about one consequence of this lemma. Consider any manifold M of dimension n. Then (also c.f. [CMMT], (4.3.18)), we have

Proposition 2.1.3.

$$W_M^{sup} \geq W_{\mathbb{R}^n}^{sup}. \tag{2.1.17}$$

Proof. Start by selecting an open ball $U \subset M$ small enough so that it is diffeomorphic to the Euclidean 1-ball. When we scale the metric $g \mapsto g_r = rg$, as $r \to \infty$, let U_r denote the dilated ball obtained from U. We see that U_r approaches $\mathbb{R}^n$ as $r \to \infty$. Then, using the scaling independence of $W(u)$, we have,

$$W_{\mathbb{R}^n}^{\sup} = \lim W_{U_r}^{\sup} = \lim W_U^{\sup} = W_U^{\sup},$$

where $W_{U_r}^{\sup}$ is taken over all $u \in H_0^1(U_r)$. Also, since $U \subset M$,

$$W_M^{\sup} \geq W_U^{\sup}. \tag{2.1.18}$$

□

We will describe in a later section how to construct compact manifolds with boundary $\overline{M}$ with the Dirichlet boundary condition for which we have

$$W_{\overline{M}}^{\sup} > W_{\mathbb{R}^n}^{\sup},$$

which will demonstrate that equality does not always hold in (2.1.18).

2.2 Comparing $W_{\mathbb{H}^n}^{\mathbf{sup}}$ with $W_{\mathbb{R}^n}^{\mathbf{sup}}$

Since the Gagliardo-Nirenberg inequality holds on $\mathbb{H}^n$, $W_{\mathbb{H}^n}^{\sup}$ does exist, and as proven in Proposition 2.1.3, $W_{\mathbb{H}^n}^{\sup} \geq W_{\mathbb{R}^n}^{\sup}$. Now we investigate the question whether $W_{\mathbb{H}^n}^{\sup}$ is attained, or, in other words, whether there exists a Weinstein functional maximizer in $H^1(\mathbb{H}^n)$. To attack this question, it seems convenient to use the following model of $\mathbb{H}^n$:

$$\mathbb{H}^n = \{v = (v_0, v') \in \mathbb{R}^{n+1} : \langle v, v\rangle = 1, v_0 > 0\},$$

and the metric on $\mathbb{H}^n$ is given by the restriction of the Lorentzian metric on $\mathbb{R}^{n+1}$

$$g = -d_{x_1}^2 + d_{x_2}^2 + ... + d_{x_{n+1}}^2$$

to $\mathbb{H}^n$. Let us parametrize $\mathbb{H}^n$ using the following "polar" model:

$$\mathbb{H}^n = \{(t, x) \in \mathbb{R}^{1+n} : t = \cosh\, r, x = \sinh\, r\omega, r \geq 0, \omega \in S^{n-1}\}. \tag{2.2.1}$$

We note that the "polar metric" of $\mathbb{H}^n$ is given by

$$ds^2 = dr^2 + \sinh^2 r d\omega^2 \tag{2.2.2}$$

as compared to the corresponding "polar" metric on $\mathbb{R}^n$, given by

$$ds^2 = dr^2 + r^2 d\omega^2. \tag{2.2.3}$$

Comparing these two, we define the following map $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{H}^n)$ by

$$T(u) = \phi u, \tag{2.2.4}$$

where

$$\phi(r) = (\frac{r}{\sinh\, r})^{\frac{n-1}{2}}. \tag{2.2.5}$$

It is clear that T is an isometry, since

$$\int_{\mathbb{H}^n} |\phi u|^2 d\mathbb{H}^n = \int_{r=0}^{\infty}\int_{S^{n-1}} |u|^2 (\frac{r}{\sinh\, r})^{n-1} \sinh^{n-1} r dr d\omega$$
$$= \int_{r=0}^{\infty}\int_{S^{n-1}} |u|^2 r^{n-1} dr d\omega = \int_{\mathbb{R}^n} |u|^2 d\mathbb{R}^n. \tag{2.2.6}$$

Now we can state our first main theorem of this chapter:

Theorem 2.2.1. (Main Theorem I)

$$W^{sup}_{\mathbb{H}^n} = W^{sup}_{\mathbb{R}^n}. \tag{2.2.7}$$

Proof. The following is the scheme of our proof: we show that, given a function $v \in H^1(\mathbb{H}^n)$, we can find a corresponding function $u \in H^1(\mathbb{R}^n)$ such that

$$W_{\mathbb{H}^n}(v) < W_{\mathbb{R}^n}(u).$$

So, if we can use a map that preserves the L^2 norm (we have the map T as defined above in mind), that is, $\|u\|_{L^2(\mathbb{H}^n)} = \|v\|_{L^2(\mathbb{R}^n)}$, the major issue to address is how to compare their L^{p+1} and gradient-L^2 norms. That is, we are done if we can show that, with ϕ as in (2.2.4) and (2.2.5),

- $\|\nabla(\phi u)\|_{L^2(\mathbb{H}^n)} > \|\nabla u\|_{L^2(\mathbb{R}^n)}$ and
- $\|\phi u\|_{L^{p+1}(\mathbb{H}^n)} < \|u\|_{L^{p+1}(\mathbb{R}^n)}$.

To that end, we quote the following calculation from [CM]:

$$\partial_r(\phi) = \frac{n-1}{2}\left(\frac{r}{\sinh\, r}\right)^{\frac{n-3}{2}} \left(\frac{\sinh\, r - r\cosh\, r}{\sinh^2(r)}\right)$$

and

$$\partial_r^2(\phi) = \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{r}{\sinh\, r}\right)^{\frac{n-5}{2}} \left(\frac{\sinh\, r - r\cosh\, r}{\sinh^2(r)}\right)^2$$
$$+ \frac{n-1}{2}\left(\frac{r}{\sinh\, r}\right)^{\frac{n-3}{2}} \left(\frac{2r\sinh\, r\cosh^2(r) - 2\sinh^2(r)\cosh\, r - r\sinh^3(r)}{\sinh^4(r)}\right).$$

Then we have

$$
\begin{aligned}
\phi^{-1}(-\Delta_{\mathbb{H}^n})(\phi u) &= \phi^{-1}(-\partial_r^2 - (n-1)\frac{\cosh r}{\sinh r}\partial_r - \frac{1}{\sinh^2(r)}\Delta_{S^{n-1}})(\phi u) \\
&= -\partial_r^2 u - 2\phi^{-1}(\partial_r\phi)(\partial_r u) - \phi^{-1}u\partial_r^2\phi \\
&\quad - (n-1)\frac{\cosh r}{\sinh r}\partial_r u - (n-1)\frac{\cosh r}{\sinh r}\phi^{-1}u\partial_r\phi - \frac{1}{\sinh^2(r)}\Delta_{S^{n-1}}u \\
&= -\partial_r^2 u + V_0(r)\partial_r u + \left[V_n(r) + \left(\frac{n-1}{2}\right)^2\right]u - \frac{1}{\sinh^2(r)}\Delta_{S^{n-1}}u \\
&= -\Delta' u + \left[V_n(r) + \left(\frac{n-1}{2}\right)^2\right]u,
\end{aligned}
\tag{2.2.8}
$$

where

$$
\begin{aligned}
V_0(r) &= \frac{1-n}{r} \\
V_n(r) &= \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\frac{1}{\sinh^2 r} - \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\frac{1}{r^2} \\
&= \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)V(r) \\
-\Delta' &= -\Delta_{\mathbb{R}^n} + \frac{\sinh^2 r - r^2}{r^2 \sinh^2 r}\Delta_{S^{n-1}},
\end{aligned}
\tag{2.2.9}
$$

where $V(r) = \frac{1}{\sinh^2 r} - \frac{1}{r^2}$.

Now, start by selecting a radial function $u \in H^1(\mathbb{R}^n)$. By the preceding calculation, using the fact that ϕ is an isometry and $-\Delta_{S^{n-1}}u = 0$ (since u is radial), we have

$$
(-\Delta_{\mathbb{H}^n}\phi u, \phi u) = (-\Delta_{\mathbb{R}^n}u, u) + \epsilon||u||_{L^2}^2 \tag{2.2.10}
$$

for some $\epsilon > 0$, because we have for all r (see justification below),

$$
(n-3)(\frac{1}{r^2} - \frac{1}{\sinh^2 r}) < n-1 \tag{2.2.11}
$$

when $n \neq 2$ and

$$
(n-1)(n-3)(\frac{1}{r^2} - \frac{1}{\sinh^2 r}) < 0 < (n-1)^2 \tag{2.2.12}
$$

when $n = 2$.

Together (2.2.11) and (2.2.12) give us that for all $r > 0$,

$$V_n(r) + \left(\frac{n-1}{2}\right)^2 > 0,$$

which in turn implies that $\epsilon > 0$.

Let us justify (2.2.11): this can be seen by observing that

$$\lim_{r \to 0+} V(r) = -1/3$$

and the fact that $V_n(r)$ does not attain an extremum for any $r > 0$. In fact $V_n'(r) = 0$ only when $r = 0$. This is because, we see that

$$\begin{aligned} V'(r) = 0 &\Longrightarrow \frac{\sinh^3 r - r^3 \cosh r}{r^3 \sinh^3 r} = 0 \\ &\Longrightarrow \frac{\sinh^3 r}{\cosh r} = r^3. \end{aligned}$$

If we let

$$h(r) = \frac{\sinh r}{\cosh^{1/3} r},$$

then proving that $h'(r) > 1$ for all $r > 0$ will suffice, as then $h(r)$ can never equal r. Now,

$$h'(r) = \frac{3\cosh^2 r - \sinh^2 r}{3\cosh^{4/3} r} = \frac{2\cosh^2 r + 1}{3\cosh^{4/3} r}.$$

Now, writing $\cosh^2 r = z$, we have that

$$\begin{aligned} \frac{2\cosh^2 r + 1}{3\cosh^{4/3} r} \leq 1 &\Longrightarrow 8z^3 - 15z^2 + 6z + 1 \leq 0 \\ &\Longrightarrow (z-1)^2(8z+1) \leq 0 \Longrightarrow z = 1, \end{aligned}$$

which can only happen if $r = 0$. So everywhere else, we have $h'(r) > 1$.

When $r \to \infty$, $V(r) \to 0-$. Also, the fact that $V(r)$ does not attain an extremum means that $V(r) > -1/3 > -\frac{n-1}{n-3}$ always.

So, finally, from (2.2.10) we have that

$$||\nabla(\phi u)||^2_{L^2(\mathbb{H}^n)} > ||\nabla u||^2_{L^2(\mathbb{R}^n)}. \tag{2.2.13}$$

Also, when $p > 1$, we have

$$\begin{aligned}
\int_{\mathbb{H}^n} |\phi u|^{p+1} d\mathbb{H}^n &= \int_0^\infty \int_{S^{n-1}} |u|^{p+1} \frac{r^{(n-1)(p+1)/2}}{\sinh^{(n-1)(p+1)/2} r} \sinh^{n-1}(r) dr d\omega \\
&= \int_0^\infty \int_{S^{n-1}} |u|^{p+1} \frac{r^{(n-1)(p-1)/2}}{\sinh^{(n-1)(p-1)/2} r} r^{n-1} dr d\omega \\
&< \int_0^\infty \int_{S^{n-1}} |u|^{p+1} r^{n-1} dr d\omega = \int_{\mathbb{R}^n} |u|^{p+1} d\mathbb{R}^n.
\end{aligned}$$

So, ultimately, we have,

$$W_{\mathbb{H}^n}(\phi u) < W_{\mathbb{R}^n}(u). \tag{2.2.14}$$

However, it is known that

$$W^{\sup}_{\mathbb{H}^n} = \sup\{W_{\mathbb{H}^n}(u) : u \text{ is a radial function } \in H^1(\mathbb{H}^n)\}. \tag{2.2.15}$$

For details on this, see [CM]. Note that, by a radial function in this context, we mean a function whose value at a point depends solely on the distance of the point from a pre-chosen fixed point, which can be called the origin. Heuristically, the basic argument is that we start with an arbitrary function u and then consider its symmetric decreasing rearrangement u^* (see Definition D.0.7 of Appendix D), and make use of the fact that symmetric decreasing rearrangements keep the same L^s-norms for all s, that is,

$$\|u^*\|_{L^s(\mathbb{H}^n)} = \|u\|_{L^s(\mathbb{H}^n)},\ s \in [1, \infty],$$

but they decrease gradient norms, that is,

$$\|\nabla u^*\|_{L^s(\mathbb{H}^n)} \leq \|\nabla u\|_{L^s(\mathbb{H}^n)},\ s \in [1, \infty]. \tag{2.2.16}$$

To prove (2.2.16), [CM] writes

$$||\nabla f||_{L^2(\mathbb{H}^n)} = \lim_{t\to 0} I^t(f),$$

where

$$I^t(f) = t^{-1}[(f,f)_{\mathbb{H}^n} - (f, e^{t\Delta_{\mathbb{H}^n}} f)_{\mathbb{H}^n}],$$

$(.,.)_{\mathbb{H}^n}$ denoting the usual inner product in $L^2(\mathbb{H}^n)$.

Since the symmetric decreasing rearrangement keeps same L^2-norm, now one just needs to see

$$(f^*, e^{t\Delta_{\mathbb{H}^n}} f^*)_{\mathbb{H}^n} \geq (f, e^{t\Delta_{\mathbb{H}^n}} f)_{\mathbb{H}^n}. \tag{2.2.17}$$

Lemma 3.3 of [CM] proves (2.2.17) with the help of a rearrangement inequality from [D] (which we reproduce below).

Also, the statement for $\mathbb{R}^n$ corresponding to (2.2.16) is given by:

$$\|\nabla u^*\|_{L^s(\mathbb{R}^n)} \leq \|\nabla u\|_{L^s(\mathbb{R}^n)}. \tag{2.2.18}$$

For a proof of (2.2.18), see [LL].

Finally, from what has gone,

$$W_{\mathbb{H}^n}(u^*) \geq W_{\mathbb{H}^n}(u) \ \forall \ u \in H^1(\mathbb{H}^n),$$

which establishes (2.2.15).

Lastly, we mention the fact that it does not matter where the radial functions are centered in the respective spaces, that is, if φ is a radial function in $H^1(M)$ ($M = \mathbb{H}^n$ or $\mathbb{R}^n$), centered at $P \in M$, and ψ is a translate of φ centered at another point $Q \in M$, then $W_M(\varphi) = W_M(\psi)$. Towards that end, let $(1, \bar{0}) \in \mathbb{H}^n$ be the point $t = 1, x = \bar{0} = (0,..,0)$, as per the notation of (2.2.1). Using the homogeneity of $\mathbb{H}^n$, we can infer that

$$\sup\{W_{\mathbb{H}^n}(u) : u \text{ is a radial function}\} = \sup\{W_{\mathbb{H}^n}(u) : u \text{ is a radial function centered at } (1,\bar{0})\}.$$

Also, using (2.2.18) and the homogeneity of $\mathbb{R}^n$, we have,

$$\begin{aligned} W_{\mathbb{R}^n}^{\sup} &= \sup\{W_{\mathbb{R}^n}(u) : u \text{ is a radial function}\} \\ &= \sup\{W_{\mathbb{R}^n}(u) : u \text{ is a radial function centered at } 0\}. \end{aligned}$$

So, using (2.2.14), and the conclusion of Proposition 2.1.3, we ultimately have our result. $\square$

We include here the aforementioned rearrangement result from [D], as quoted in [CM].

Theorem 2.2.2. *(Draghici [D]) Let* $X = \mathbb{H}^n$, $f_i = X \to R_+$ *be* m *nonnegative functions,* $\Psi \in AL_2(R_+^m)$ *be continuous and* $K_{ij} : [0,\infty) \to [0,\infty)$, $i < j$, $j \in \{1, ..., m\}$ *be decreasing functions. We define*

$$I[f_1, ..., f_m] = \int_{X^m} \Psi(f_1(\Omega_1), ..., f_m(\Omega_m)) \Pi_{i<j} K_{ij}(d(\Omega_i, \Omega_j)) d\Omega_1 ... d\Omega_m.$$

Then the following inequality holds:

$$I[f_1, ..., f_m] \le I[f_1^*, ..., f_m^*].$$

Theorem 2.2.1 was conjectured in [CMMT] (also see [Ba]). Harris ([Ha]) had collected some numerical evidence of this phenomenon in the special case $p = n = 2$.

Note that we have also proved another related conjecture in [CMMT], which says in effect that for all $u \in H^1(\mathbb{H}^n)$, $W(u) < W_{\mathbb{H}^n}^{\sup}$, which means that there is no Weinstein functional maximizer in $H^1(\mathbb{H}^n)$. Let us justify this: in case there exists $v \in H^1(\mathbb{H}^n)$ such that $W_{\mathbb{H}^n}(v) = W_{\mathbb{H}^n}^{\sup}$, then the spherical decreasing rearrangement $v^* \in H^1(\mathbb{H}^n)$ of $|v|$ also satisfies $W_{\mathbb{H}^n}(v^*) = W_{\mathbb{H}^n}^{\sup}$. But then, $u^* = \phi^{-1}v^* \in H^1(\mathbb{R}^n)$ will satisfy $W_{\mathbb{R}^n}(u^*) > W_{\mathbb{H}^n}(v^*)$. By Theorem 2.2.1, this is a contradiction.

Remark 2.2.3. For a generic manifold M, we do not have $W_M^{\sup} = W_{\mathbb{R}^n}^{\sup}$. In fact, consider the following counterexample:

Let M_k be the sphere S^n with a tiny open ball (homeomorphic to $B_1(0) \subset \mathbb{R}^n$) of radius r_k removed. As we make the radius of the removed ball $r_k \to 0$, we see that the first eigenvalue $\lambda_1^{(k)}$ of the Laplacian $-\Delta_k$ of M_k goes to 0, because M_k approaches the sphere S^n, whose first eigenvalue is 0. Now, consider a sequence of functions u_l^k such that when k is fixed, $W_{M_k}(u_l^k) \to W_{M_k}^{\sup}$. Since all the M_k's are compact with uniformly bounded volume, we can find a constant C (independent of k) such that $\|u_l^k\|_{L^2} \leq C\|u_l^k\|_{L^{p+1}}$. Now,

$$\frac{\|u_l^k\|^{p+1}_{L^{p+1}(M_k)}}{\|u_l^k\|^{\alpha}_{L^2(M_k)}\|\nabla u_l^k\|^{\beta}_{L^2(M_k)}} = \frac{\|u_l^k\|^{p+1}_{L^{p+1}(M_k)}\|u_l^k\|^{\beta}_{L^2(M_k)}}{\|u_l^k\|^{p+1}_{L^2(M_k)}\|\nabla u_l^k\|^{\beta}_{L^2(M_k)}} \geq \frac{\|u_l^k\|^{\beta}_{L^2(M_k)}}{C^{p+1}\|\nabla u_l^k\|^{\beta}_{L^2(M_k)}}, \ .$$

So,

$$\sup \frac{\|u_l^k\|^{p+1}_{L^{p+1}(M_k)}}{\|u_l^k\|^{\alpha}_{L^2(M_k)}\|\nabla u_l^k\|^{\beta}_{L^2(M_k)}} \geq \frac{1}{C^{p+1}\left(\lambda_1^{(k)}\right)^{\beta}}.$$

This means that we have $W^{\sup}_{M_k} \to \infty$.

On a compact domain inside $\mathbb{R}^n$ with Dirichlet boundary condition, it is known via a Harnack inequality argument (see Proposition 4.3.1 of [CMMT]) that there is no optimal constant for the Gagliardo-Nirenberg inequality. It is however, an interesting (and largely unanswered) question as to what happens in the case of generic compact manifolds with boundary (with Dirichlet boundary condition).

2.3 Weinstein functional and fractional Laplacian

We know that $\mathrm{Spec}(-\Delta_{\mathbb{H}^n}) \subset [\frac{(n-1)^2}{4}, \infty)$ (see Chapter 8, Proposition 5.1 of [T4]). So the spectral theorem (Theorem B.6.1 of Appendix B) can be applied to define the fractional Laplacian $(-\Delta)^{\alpha}$, $\alpha \in (0, 1)$. Now we investigate the corresponding Weinstein functional maximization problem for the fractional Laplacian $(-\Delta)^{\alpha}$. In other words, we try to investigate what we can say about the maximization problem for

$$W_{\alpha}(u) = \frac{\|u\|^{p+1}_{L^{p+1}}}{\|u\|^{\gamma}_{L^2}\|(-\Delta)^{\frac{\alpha}{2}}u\|^{\rho}_{L^2}},$$

where $\gamma = 2 - (n-2\alpha)(p-1)/(2\alpha), \rho = n(p-1)/(2\alpha)$. We will want $p \in (1, \frac{n+2\alpha}{n-2\alpha})$. The reason for our interest in this is the following: if we consider the fractional NLS of the form

$$iv_t - (-\Delta)^{\alpha}v + |v|^{p-1}v = 0, x \in M$$

$$v(0, x) = v_0(x),$$

and plug in

$$v(t,x) = e^{i\lambda t} u_\lambda(x),$$

we get the the following auxiliary elliptic equation

$$(-\Delta)^\alpha u_\lambda + \lambda u_\lambda - |u_\lambda|^{p-1} u_\lambda = 0. \tag{2.3.1}$$

By a similar calculation as before, a maximizer u for the fractional Weinstein functional will solve

$$(-\Delta)^\alpha v + \lambda v = K|v|^{p-1} v, \tag{2.3.2}$$

where

$$\lambda = \frac{\gamma}{\rho} \frac{\|(-\Delta)^{\alpha/2} u\|_{L^2}^2}{\|u\|_{L^2}^2}, \; K = \frac{p+1}{\rho} \frac{\|(-\Delta)^{\alpha/2} u\|_{L^2}^2}{\|u\|_{L^{p+1}}^{p+1}}. \tag{2.3.3}$$

Also, let us mention here that there has been some recent interest in nonlocal equations of the type (2.3.1). For example, see [FL] and references therein.

Now, the fractional Gagliardo-Nirenberg inequality (the fact that it actually holds is the content of Proposition 2.3.1 below) implies that $W_\alpha(u)$ is actually bounded from above on both $\mathbb{R}^n$ and $\mathbb{H}^n$, when u is chosen from the natural domain of $(-\Delta)^{\alpha/2}$, which is

$$\mathcal{D}((-\Delta)^{\frac{\alpha}{2}}) = H^\alpha(M) \subset L^q(M), \; \forall q \; \in \left[2, \frac{2n}{n-2\alpha}\right], M = \mathbb{R}^n, \mathbb{H}^n.$$

Let us discuss when the fractional Gagliardo-Nirenberg inequality holds. We want to justify (our tacit claim above) that it holds on the hyperbolic space $\mathbb{H}^n$ and the Euclidean space $\mathbb{R}^n$. Actually we have, more generally:

Proposition 2.3.1. *Let M be a complete Riemannian manifold on which the heat kernel satisfies the following pointwise bounds:*

$$|p(t,x,y)| \le Ct^{-n/2}, \; t > 0, \; x, y \in M, \tag{2.3.4}$$

where C is constant independent of t, x and y. Then the fractional Gagliardo-Nirenberg inequality

$$\|u\|_{L^{p+1}}^{p+1} \leq C\|(-\Delta)^{\alpha/2}u\|_{L^2}^{\rho}\|u\|_{L^2}^{\gamma}$$

holds on M, where $\gamma = 2-(n-2\alpha)(p-1)/(2\alpha)$, and $\rho = n(p-1)/(2\alpha)$.

Proof. We have,

$$\begin{aligned}\int_M |u|^{p+1}dM &= \int_M |u|^{(p+1)\theta}|u|^{(p+1)(1-\theta)}dM \\ &\leq \||u|^{(p+1)\theta}\|_{L^{r'}}\||u|^{(p+1)(1-\theta)}\|_{L^{s'}} \\ &= \|u\|_{L^{r'(p+1)\theta}}^{(p+1)\theta}\|u\|_{L^{s'(p+1)(1-\theta)}}^{(p+1)(1-\theta)}\end{aligned}$$

where $\frac{1}{r'}+\frac{1}{s'}=1$.

That means,

$$\|u\|_{L^{p+1}} \leq \|u\|_{L^{r'(p+1)\theta}}^{\theta}\|u\|_{L^{s'(p+1)(1-\theta)}}^{1-\theta}.$$

Let $r'(p+1)\theta = r$ and $s'(p+1)(1-\theta) = s$. So

$$\|u\|_{L^{p+1}} \leq \|u\|_{L^r}^{\theta}\|u\|_{L^s}^{1-\theta},$$

where

$$\frac{\theta}{r}+\frac{1-\theta}{s}=\frac{1}{p+1}.$$

Now, we can assert that the Hardy-Littlewood-Sobolev estimates

$$\|u\|_{L^r} \lesssim \|(-\Delta)^{\alpha/2}u\|_{L^m}$$

where $r = \frac{nm}{n-\alpha m}$, $0<\alpha<1$, $1<m<\frac{n}{\alpha}$, will follow from the heat kernel bounds (see [VSC], Chapter II, Theorem II.2.4 and the following discussion; also see [Bau]). Given that, we now have

$$\|u\|_{L^{p+1}}^{p+1} \lesssim \|(-\Delta)^{\alpha/2}u\|_{L^m}^{\theta(p+1)}\|u\|_{L^s}^{(1-\theta)(p+1)}$$

with

$$\theta(\frac{1}{m}-\frac{\alpha}{n})+\frac{1-\theta}{s}=\frac{1}{p+1}.$$

In the special case of $m = s = 2$, we retrieve the Gagliardo-Nirenberg inequality in the form that we use here. $\square$

Remark 2.3.2. By [DGM], it is known that the heat kernel bounds (2.3.4) hold on complete simply connected manifolds of dimension n and sectional curvature less than or equal to 0. This is also true on compact manifolds with the Dirichlet Laplacian. As regards symmetric spaces, a similar heat kernel bound holds on spaces of the form $G_{\mathbb{C}}/G$, where G is a compact Lie group and $G_{\mathbb{C}}$ is the complexification of G (for details see [Ga]).

Now we have the second main theorem of this chapter

Theorem 2.3.3. *(***Main Theorem II***)*

$$W^{sup}_{\alpha,\mathbb{R}^n}=W^{sup}_{\alpha,\mathbb{H}^n}.$$

Proof. Morally, as in the proof of Theorem 2.2.1, we want to compare $W_{\alpha,\mathbb{R}^n}(u)$ with $W_{\alpha,\mathbb{H}^n}(v)$ for functions $u\in H^\alpha(\mathbb{R}^n), v\in H^\alpha(\mathbb{H}^n)$. As usual, we use the isometric isomorphism T defined before that keeps L^2-norms same and lowers the L^{p+1}-norm on the hyperbolic side, that is, if $v = Tu$, then

$$\|u\|_{L^2(\mathbb{R}^n)}=\|v\|_{L^2(\mathbb{H}^n)}, \|u\|_{L^{p+1}(\mathbb{R}^n)}>\|v\|_{L^{p+1}(\mathbb{H}^n)}. \tag{2.3.5}$$

Seeing what has gone before, comparing the supremum values of the fractional Weinstein functionals just amounts to comparing $\|(-\Delta_{\mathbb{R}^n})^{\frac{\alpha}{2}}u\|_{L^2(\mathbb{R}^n)}$ with $\|(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}}\phi u\|_{L^2(\mathbb{H}^n)}$. Now we use the following functional calculus (see [B]; also see Proposition 3.1.12 of [H])

$$A^\alpha u=\frac{\sin\alpha\pi}{\pi}\int_0^\infty t^{\alpha-1}(t+A)^{-1}Au dt,\ \forall u\in\mathcal{D}(A),$$

where A is a sectorial operator (see Appendix B, Section B.11) on a Banach space X and $0<\alpha<1$. Now, it is known that on a Hilbert space H, a non-negative self-adjoint operator $A:\mathcal{D}(A)\subset H\longrightarrow H$ is sectorial with $\omega=0$ (see Chapter 2, Section 2.1.1 of [H]).

So then, writing $(.,.)_M$ for the inner product in $L^2(M)$, where $M = \mathbb{R}^n, \mathbb{H}^n$, we get,

$$((-\Delta_{\mathbb{H}^n})^{\alpha}\phi u, \phi u)_{\mathbb{H}^n} = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\int_{\mathbb{H}^n} t^{\alpha-1}(t-\Delta_{\mathbb{H}^n})^{-1}(-\Delta_{\mathbb{H}^n})(\phi u)\overline{\phi u}d\mathbb{H}^n dt$$

and

$$((-\Delta_{\mathbb{R}^n})^{\alpha}u, u)_{\mathbb{R}^n} = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\int_{\mathbb{R}^n} t^{\alpha-1}(t-\Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})u\overline{u}d\mathbb{R}^n dt.$$

So we have reduced the problem to comparing

$$\int_{\mathbb{H}^n}(t-\Delta_{\mathbb{H}^n})^{-1}(-\Delta_{\mathbb{H}^n})(\phi u)\overline{\phi u}d\mathbb{H}^n$$

with

$$\int_{\mathbb{R}^n}(t-\Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})u\overline{u}d\mathbb{R}^n.$$

Now, if we let $u = u_1 + iu_2$, we will see that for the above comparison it is enough to consider real-valued u (see Section C.6 of Appendix C). So we have reduced the problem to the comparison of

$$A = \int_{\mathbb{H}^n}(t-\Delta_{\mathbb{H}^n})^{-1}(-\Delta_{\mathbb{H}^n})(\phi u)(\phi u)d\mathbb{H}^n$$

with

$$B = \int_{\mathbb{R}^n}(t-\Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})(u)(u)d\mathbb{R}^n,$$

where u is real-valued. So, let us call

$$\begin{aligned}
F(t) &= ((t-\Delta_{\mathbb{H}^n})^{-1}(-\Delta_{\mathbb{H}^n})\phi u, \phi u)_{\mathbb{H}^n} - ((t-\Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})u, u)_{\mathbb{R}^n} \\
&= ((t-\phi^{-1}\Delta_{\mathbb{H}^n}\phi)^{-1}(-\phi^{-1}\Delta_{\mathbb{H}^n}\phi)u, u)_{\mathbb{R}^n} - ((t-\Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})u, u)_{\mathbb{R}^n} \\
&= (((t-\overline{\Delta})^{-1}(-\overline{\Delta}) - (t-\Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n}))u, u)_{\mathbb{R}^n},
\end{aligned}$$

where $\overline{\Delta} = \phi^{-1}\Delta_{\mathbb{H}^n}\phi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$. Writing $(t-\overline{\Delta})^{-1}u = u_1, (t-\Delta_{\mathbb{R}^n})^{-1}u = u_2$, we get

$$\begin{aligned} F(t) &= (-\overline{\Delta}u_1, u)_{\mathbb{R}^n} - (-\Delta_{\mathbb{R}^n}u_2, u)_{\mathbb{R}^n} \\ &= (-\overline{\Delta}u_1, (t-\Delta_{\mathbb{R}^n})u_2)_{\mathbb{R}^n} - (-\Delta_{\mathbb{R}^n}u_2, (t-\overline{\Delta})u_1)_{\mathbb{R}^n} \\ &= t[(-\overline{\Delta}u_1, u_2)_{\mathbb{R}^n} - (-\Delta_{\mathbb{R}^n}u_2, u_1)_{\mathbb{R}^n}]. \end{aligned}$$

Writing $V(r) = V, K_1 = (\frac{n-1}{2})(\frac{n-3}{2}), K_2 = (\frac{n-1}{2})^2$, we get from (2.2.8) and (2.2.9),

$$\begin{aligned} F(t)/t &= ((-\overline{\Delta}u_1 - (-\Delta_{\mathbb{R}^n}))u_1, u_2)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, u_2)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, (t-\Delta_{\mathbb{R}^n})^{-1}(t-\overline{\Delta})u_1)_{\mathbb{R}^n}. \end{aligned}$$

Seeing that

$$(t-\Delta_{\mathbb{R}^n})^{-1}(t-\overline{\Delta}) = (t-\Delta_{\mathbb{R}^n})^{-1}(t-\Delta_{\mathbb{R}^n} + (-\overline{\Delta} - (-\Delta_{\mathbb{R}^n}))) = I + (t-\Delta_{\mathbb{R}^n})^{-1}(-\overline{\Delta} - (-\Delta_{\mathbb{R}^n}))$$

we have

$$\begin{aligned} F(t)/t &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, (I + (t-\Delta_{\mathbb{R}^n})^{-1}(-V\Delta_{S^{n-1}} + K_1V + K_2))u_1)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, u_1)_{\mathbb{R}^n} + ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, (t-\Delta_{\mathbb{R}^n})^{-1} \\ &\quad (-V\Delta_{S^{n-1}} + K_1V + K_2)u_1)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, u_1)_{\mathbb{R}^n} + ((t-\Delta_{\mathbb{R}^n})w, w)_{\mathbb{R}^n} \\ &> (V(-\Delta_{S^{n-1}})u_1, u_1)_{\mathbb{R}^n}, \end{aligned}$$

where $w = (-V\Delta_{S^{n-1}} + K_1V + K_2)u_1$. If we now assume that u_1 is radial, then

$$(V(-\Delta_{S^{n-1}})u_1, u_1)_{\mathbb{R}^n} = 0.$$

This means that $F(t)/t > 0$.

Now, the reason that we can just choose u_1 radial in the above calculation is because we have

$$W^{\text{sup}}_{\alpha,\mathbb{H}^n} = \sup\{W_{\alpha,\mathbb{H}^n}(u) : u \text{ is a radial function } \in H^\alpha(\mathbb{H}^n)\}, \tag{2.3.6}$$

and

$$W^{\text{sup}}_{\alpha,\mathbb{R}^n} = \sup\{W_{\alpha,\mathbb{R}^n}(u) : u \text{ is a radial function } \in H^\alpha(\mathbb{R}^n)\}. \tag{2.3.7}$$

(2.3.7) follows from (5.0.3) and (5.0.4) of [CMMT1].

To show (2.3.6), we need to verify that, replacing u by the radial decreasing rearrangement u^* of $|u|$ lowers the kinetic energy term, that is,

$$\|(-\Delta_{\mathbb{H}^n})^{\alpha/2}u^*\|^2_{L^2(\mathbb{H}^n)} \leq \|(-\Delta_{\mathbb{H}^n})^{\alpha/2}u\|^2_{L^2(\mathbb{H}^n)}.$$

This can be realized by the methods used in [CM] as mentioned in the proof of Theorem 2.2.1, in conjunction with the functional calculus used above. A proof more or less along such lines appears as Lemma 4.0.2 in [CMMT1], which we reproduce below. Taking this for granted, we have established that it is enough to compare the Weinstein functional values for radial functions in $H^\alpha(\mathbb{R}^n)$ and $H^\alpha(\mathbb{H}^n)$.

Finally, we see that

$$W^{sup}_{\alpha,\mathbb{R}^n} = W^{sup}_{\alpha,\mathbb{H}^n},$$

and the corresponding fact that $W^{sup}_{\alpha,\mathbb{H}^n}$ is not attained in $H^\alpha(\mathbb{H}^n)$. □

The following lemma finishes the proof (for notational convenience, in the lemma below, $-\Delta$ refers to $-\Delta_{\mathbb{H}^n}$):

Lemma 2.3.4. *Replacing $u \in H^\alpha(\mathbb{H}^n)$ by the radial, decreasing rearrangement u^* of $|u|$ lowers the term $\|(-\Delta)^{\frac{\alpha}{2}}u\|^2_{L^2(\mathbb{H}^n)}$.*

Proof. For $u \in H^\alpha(\mathbb{H}^n)$, we have

$$\begin{aligned}\|(-\Delta)^{\frac{\alpha}{2}}u\|^2_{\mathbb{H}^n} &= ((-\Delta)^\alpha u, u)_{\mathbb{H}^n} \\ &= \lim_{t\to 0}\frac{1}{t}((I - e^{-t(-\Delta)^\alpha})u, u)_{\mathbb{H}^n}.\end{aligned}$$

To prove our lemma, it suffices to demonstrate that

$$(e^{-t(-\Delta)^\alpha}u, u)_{\mathbb{H}^n} \leq (e^{-t(-\Delta)^\alpha}u^*, u^*)_{\mathbb{H}^n}.$$

Now,

$$(e^{-t(-\Delta)^\alpha}u, u)_{\mathbb{H}^n} = \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} p_\alpha(t, \mathrm{dist}(x,y))u(x)u(y)dxdy,$$

where $p_\alpha(t, \mathrm{dist}(x,y))$ represents the integral kernel of the semigroup $e^{-t(-\Delta)^\alpha}$. We observe that

$$e^{-t(-\Delta)^\alpha} = \int_0^\infty f_{t,\alpha}(s)e^{s\Delta}ds,\ t > 0,$$

with $f_{t,\alpha}(s) \geq 0$ (see [Y], pp. 260-261). So,

$$e^{-t(-\Delta)^\alpha}u(x) = \int \left(\int_0^\infty f_{t,\alpha}(s)p(t, \mathrm{dist}(x,y))ds\right)u(y)dy,$$

which gives,

$$p_\alpha(t, \mathrm{dist}(x,y)) = \int_0^\infty f_{t,\alpha}(s)p(t, \mathrm{dist}(x,y))ds.$$

Hence, given $\alpha \in (0,1)$, $t > 0$, and writing $r = \mathrm{dist}(x,y)$, we have that $p_\alpha(t,r)$ is monotonically decreasing in r (since we have from [CM] that $p(t,r)$ is monotonically decreasing in r), and

$$p_\alpha(t,r) \geq 0.$$

This gives,

$$(e^{-t(-\Delta)^\alpha}u, u)_{\mathbb{H}^n} \leq (e^{-t(-\Delta)^\alpha}|u|, |u|)_{\mathbb{H}^n}.$$

Now, we want to demonstrate that

$$(e^{-t(-\Delta)^\alpha}|u|, |u|)_{\mathbb{H}^n} \leq (e^{-t(-\Delta)^\alpha}u^*, u^*)_{\mathbb{H}^n}.$$

But this follows from Theorem 2.2.2, by using $\Psi(f_1, f_2) = f_1 f_2$ and $K_{12} = p_\alpha(r,t)$. $\square$

APPENDIX A: DIFFERENTIAL GEOMETRIC STATEMENTS AND FACTS

Here we collect some results about differential geometry used in the main body of the thesis, for the purpose of reference.

A.1 Laplacian/Laplace-Beltrami operator

To us, the Laplacian, or, the Laplace-Beltrami operator on a manifold, denoted by Δ, is the negative of the Hodge Laplacian $d\delta + \delta d$, acting on functions, or, 0-forms, which makes $\Delta = \delta d$, where d is the exterior differential operator and δ is its adjoint; see [T3].

Naturally, we use the analyst's convention, which makes the Laplacian negative semidefinite. We also point out that in local coordinates, the Laplacian is given by

$$\Delta f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j f),$$

where g_{ij} denote the metric tensor in local coordinates.

A.2 Killing fields

Definition A.2.1. A vector field X on a Riemannian manifold M is said to be a Killing field if the Lie derivative of the metric g with respect to X vanishes, that is, $\mathcal{L}_X g = 0$.

It can be proved that Killing fields are infinitesimal generators of isometries (see [T3]).

A Killing field X on a manifold M is said to be bounded if $\langle X, X\rangle_x, x \in M$ is bounded, where $\langle ., .\rangle_x$ is the square of the length of X at x given by the Riemannian metric of M.

A.3 Isotropic manifolds

Definition A.3.1. Isotropic manifolds are defined as those Riemannian manifolds such that, given any $p \in M$ and unit vectors $v, w \in T_p(M)$, there exists $\varphi \in \mathrm{Isom}(M)$ such that $\varphi(p) = p$ and $d\varphi_p(v) = w$.

Intuitively, these are the manifolds in which "the geometry is same in every direction", or, which "look the same in every direction".

A.4 Manifolds of bounded geometry

Definition A.4.1. A Riemannian manifold M is said to have C^k-bounded geometry provided that for all $x \in M$, there is a geodesic ball $B_r(x)$ of radius r (independent of x) such that $\exp_x : T_xM \to M$ is a diffeomorphism of $B_r(0) \subset T_xM$ onto $B_r(x)$ so that the following are satisfied:

1. the metric g_{ij} on $B_r(x)$, pulled back to $B_r(0)$ by the exp map, is bounded in C^k-norm for T_xM, and
2. the inverse matrix g^{ij} is bounded in the sup norm.

For more details on this, see [CGT]. Heuristically, manifolds of bounded geometry are essential in keeping control on volume growth of balls outside a compact set and also hindering neckpinch type pathologies by keeping a control on the injectivity radius.

As examples, $\mathbb{R}^n$ trivially has bounded geometry, but $\mathbb{H}^n$ does not have bounded geometry.

A.5 Symmetric space of rank 1

Definition A.5.1. A connected Riemannian manifold M is called a symmetric space if for each point $p \in M$ and each geodesic γ passing through p, there exists an isometry φ of M fixing p and reversing the geodesic, that is,

$$\varphi(\gamma(t)) = \gamma(-t).$$

The rank of a symmetric space is defined as the dimension of the maximal flat submanifold of the symmetric space.

If the rank is 1, then the maximal flat submanifolds are the geodesics.

Remark: Here we have used the geometric definition of a symmetric space. More generally, a non-compact symmetric space can be defined as G/K, where G is a semisimple real Lie group, and K is a maximal compact subgroup of G.

APPENDIX B: FUNCTIONAL ANALYTIC DEFINITIONS AND FACTS

Here we collect some definitions and results about functional analysis used in the main body of the thesis, for the purpose of reference.

B.1 Relatively bounded perturbation

Definition B.1.1. Let X be a Banach space and T_1 and T_2 be linear operators such that $\mathcal{D}(T_1) \subset \mathcal{D}(T_2)$ and

$$\|T_2 u\| \leq a\|u\| + b\|T_1 u\|, u \in \mathcal{D}(T_1), \tag{B.1.1}$$

where a and b are nonnegative constants. Then T_2 is called a relatively bounded perturbation of T_1. The infimum of all b such that (B.1.1) holds is called the relative bound of T_2 with respect to T_1. For more details, see [RS] Chapter X.

B.2 Subelliptic operator

Definition B.2.1. A self-adjoint second order differential operator $\mathcal{L}$ is called subelliptic of order ε $(0 < \varepsilon < 1)$, at $x \in M$ if there is a neighborhood U of x such that

$$\|u\|^2_{H^\varepsilon} \leq C(|(\mathcal{L}u, u)| + \|u\|^2) \ \forall \ u \in C_c^\infty(U).$$

For a nice reference, see [F]).

B.3 Hypoelliptic operators and Hörmander's condition

Definition B.3.1. A (pseudo)differential operator D of order 2 defined on an open set U is said to be hypoelliptic when for all distributions ϕ,

$$D\phi \in C_c^\infty(U) \Rightarrow \phi \in C_c^\infty(U).$$

We have the following

Theorem B.3.2. *Let $X_i, i = 0, 1, .., r$ be vector fields on a compact manifold M. Consider the differential operator $P = \sum_1^r X_j^2 + X_0 + \varphi$ where $\varphi \in C^\infty(M)$. Then P is hypoelliptic if the tangent* *space at every point in M is generated by the commutators*

$$X_{j_1}, [X_{j_2}, X_{j_3}], [X_{j_1}, [X_{j_2}, X_{j_3}]], ..., [X_{j_1}, [X_{j_2}, ...X_{j_k}]]]...$$

where $j_i = 0, 1, ..., r$.

B.4 Complex Interpolation spaces

Here we define complex interpolation spaces. Consider two Banach spaces E and F with continuous inclusion $F \hookrightarrow E$. Let $\Omega = \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$.

Define

$$\mathcal{H}_{E,F}(\Omega) = \{u(z) \text{ bounded and continuous on } \overline{\Omega} \text{ with values in } E,$$
$$\text{holomorphic on } \Omega : \|u(1+iy)\|_F \text{ is bounded, for } y \in \mathbb{R}\}.$$

Define interpolation spaces $[E, F]_\theta$ by

$$[E, F]_\theta = \{u(\theta) : u \in \mathcal{H}_{E,F}(\Omega)\},\ \theta \in [0, 1]\}.$$

We give $[E, F]_\theta$ the Banach space topology, making it isomorphic to the quotient

$$\mathcal{H}_{E,F}(\Omega)/\{u : u(\theta) = 0\}.$$

For more details on complex interpolation, see [T3], Chapter 4, Section 2.

B.5 Pseudodifferential operators on $\mathbb{R}^n$ and compact manifolds

Definition B.5.1. By $S^m_{\rho,\delta}(\mathbb{R}^n)$, we mean all C^∞ functions $p(x, \xi)$ such that

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{B.5.1}$$

where $\rho, \delta \in [0, 1], m \in \mathbb{R}$.

Now, the pseudodifferential operator associated to $p(x,\xi)$ is given by

$$p(x,D)f(x) = \int p(x,\xi)\hat{f}(\xi)e^{ix.\xi}d\xi.$$

When $p(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $p(x,D)$ is said to lie in $OPS^m_{\rho,\delta}$.

Now, let M be a compact manifold. Then, $P : C_c^\infty(M) \to \mathcal{D}'(M) \in OPS^m_{\rho,\delta}(M)$ if

(a) its Schwartz kernel (see Section B.9 below) is smooth off the diagonal in $M \times M$,

(b) There exists an open cover U_j of M, a subordinate partition of unity φ_j, and diffeomorphisms $F_j : U_j \to \Omega_j \subset \mathbb{R}^n$ such that $\varphi_k P \varphi_j : C^\infty(U_j) \to \mathcal{E}'(U_k)$ are pseudodifferential operators lying in $OPS^m_{\rho,\delta}$, as defined above.

A reference on pseudodifferential operators is [T2].

B.6 Spectral theorem for non-negative self-adjoint operators

We state the spectral theorem in slightly less than full generality, as this is the version we are mainly using in this thesis (a more general version appears as Theorem 1.7 in Chapter 8 of [T4]).

Theorem B.6.1. *Let A be a nonnegative self-adjoint operator on a separable Hilbert space H. Then there exist a measure space (Ω,μ), a unitary map $U : L^2(\Omega,\mathbb{R}) \to H$ and a nonnegative real valued measurable function λ on Ω such that*

$$U^{-1}AUf(x) = \lambda(x)f(x), x \in \Omega, Uf \in \mathcal{D}(A).$$

Also

$$\mathcal{D}(A) = \{Uf : f \in L^2(\Omega,\mathbb{R}), \int_\Omega \lambda^2 f^2 d\mu < \infty\}.$$

For a Borel function $f : \mathbb{R} \to \mathbb{C}$, define $f(A)$ by

$$U^{-1}f(A)Ug(x) = f(\lambda(x))g(x).$$

If f is a bounded Borel function, this is defined for all $g \in L^2(\Omega, d\mu)$, and provides a bounded operator $f(A)$ on H. More generally,

$$\mathcal{D}(f(A)) = \{Ug \in H : g \in L^2(\Omega, d\mu) \text{ and } f(\lambda(x))g \in L^2(\Omega, d\mu)\}.$$

B.7 Heat semigroup and heat kernel

Definition B.7.1. Let M be a complete Riemannian manifold. The self-adjoint operator $e^{t\Delta}$, defined by the spectral theorem (Theorem B.6.1) on $L^2(M)$ is called the heat semigroup.

We also have the following

Theorem B.7.2. *There exists a function $p(t,x,y)$ such that*

1. *$p(t,x,y)$ is a smooth real-valued function on $\mathbb{R}^+ \times M \times M$,*

2. *$p(t,x,y) = p(t,y,x)$, and*

3. *$\int_M |p(t,x,y)|dy \leq 1$ for all x and $t > 0$, such that*

$$e^{t\Delta}u(x) = \int_M p(t,x,y)u(y)dy.$$

The above function $p(t,x,y)$ is defined as the heat kernel.

A reference for the above is [Str].

B.8 Heat semigroup on L^p spaces

We have the following

Theorem B.8.1. *Let $e^{t\Delta}$ denote the heat semigroup on a complete Riemannian manifold M, defined by the spectral theorem (Theorem B.6.1) as a self-adjoint operator on L^2. Let $1 \leq p \leq \infty$. Then, on $L^p(M)$, there exists a unique contraction semigroup, denoted by $e^{t\Delta}_{(p)}$, $t \geq 0$, such that for $f \in L^p \cap L^2$, $e^{t\Delta}_{(p)}f = e^{t\Delta}f$.*

By abuse of notation, we will drop the subscript (p) and refer to $e^{t\Delta}_{(p)}$ as $e^{t\Delta}f$. See Theorem 3.5 of [Str].

B.9 Integral kernel or Schwartz kernel

Let us consider a compact Riemannian manifold and a continuous operator $T : C_c^\infty(M) \to \mathcal{D}'(M)$, the space of all distributions of M. Then the Schwartz kernel theorem states

Theorem B.9.1. *Each such operator T has a unique integral kernel or Schwartz kernel $K \in \mathcal{D}'(M \times M)$ such that*

$$(Tu, v) = \int_M K(x, y)u(y)dy.$$

For more details, see [T3], Section 6, Chapter 4.

B.10 Riesz-Thorin interpolation theorem

Theorem B.10.1. *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and $\theta \in (0,1)$. Define $1 \leq p, q \leq \infty$ by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. If T is a linear map such that $T : L^{p_0} \to L^{q_0}$, $T : L^{p_1} \to L^{q_1}$ are bounded, and $\|T\|_{\mathcal{L}(L^{p_0}, L^{q_0})} = M_0$, $\|T\|_{\mathcal{L}(L^{p_1}, L^{q_1})} = M_1$, then for every $f \in L^{p_0} \cap L^{p_1}$, $\|Tf\|_{L^q} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p}$. Hence T extends uniquely as a bounded map from L^p to L^q with $\|T\|_{\mathcal{L}(L^p, L^q)} = M_0^{1-\theta} M_1^\theta$.*

For more on this theorem, see Theorem 21 of [Tao], where a proof is also given.

B.11 Sectorial operators

Let X be a Banach space and A a linear operator on X. For $0 \leq \omega \leq \pi$ let

$$S_\omega = \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \omega\} & \text{if } \omega \in (0, \pi], \\ (0, \infty) & \text{if } \omega = 0. \end{cases}$$

An operator A on X is called sectorial of angle $\omega \in [0, \pi)$ if

1. $\mathrm{Spec}(A) \subset \overline{S_\omega}$,
2. $\sup\{\|\lambda(\lambda - A)^{-1}\| : \lambda \in \mathbb{C} \setminus \overline{S_{\omega'}}\} < \infty$ for all $\omega' \in (\omega, \pi)$.

For more details, see [H].

APPENDIX C: CLARIFICATION OF ASSORTED STATEMENTS

C.1 Self-adjointness of the Laplacian and other perturbations

Since the Laplacian $-\Delta_M$ on a complete Riemannian manifold M satisfies $(-\Delta u, u) \geq 0$ for all $u \in C_c^\infty(M)$, it is a densely defined non-negative linear operator on $L^2(M)$. Also, the Laplacian is symmetric with respect to the L^2 inner product. So, by the Friedrichs extension procedure (see Section 10.4 of [Sc]), the Laplacian has a non-negative self-adjoint extension, which we still call $-\Delta_M$ by mild abuse of notation. If M is compact, one can show that the domain of the self-adjoint extension is $H^2(M)$ (see Theorem 57 of [Ca]).

Also, with the restriction $\langle X, X\rangle \leq b^2 \leq 1$, we have that

$$\begin{aligned}|(X^2u, u)| &= |(Xu, Xu)| = |(X.\nabla u, X.\nabla u)| \\ &\leq b^2(\nabla u, \nabla u) = b^2(-\Delta u, u).\end{aligned}$$

This, on calculation implies that $((-\Delta + X^2)u, u) \geq (1 - b^2)(-\Delta u, u)$.

Also, we see that the operator iX is symmetric if X is Killing (this follows from Proposition 2.5, Chapter 2 of [T3], and Proposition 16.33 of [Lee]). So, by the Friedrichs extension procedure, we see that $-\Delta + X^2$ has a self-adjoint extension, which we still call $-\Delta + X^2$ by mild abuse of notation.

C.2 Relatively bounded perturbations

Let α be any real number. First we prove that $i\alpha X$ is a relatively bounded perturbation of $-\Delta$. We have

$$\begin{aligned}\|i\alpha Xu\|^2 &= \alpha^2\|Xu\|^2 = \alpha^2\|X.\nabla u\|^2 \leq \alpha^2 b^2\|\nabla u\|^2 \\ &= \alpha^2 b^2(-\Delta u, u) \leq \alpha^2 b^2\|\Delta u\|\|u\| \\ &\leq \alpha^2 b^2 C\|\Delta u\|^2 + \frac{1}{C}\alpha^2 b^2\|u\|^2 + 2\alpha^2 b^2\|\Delta u\|\|u\|,\end{aligned}$$

which implies

$$\|i\alpha Xu\| \leq \alpha b\sqrt{C}\|\Delta u\| + \alpha b\frac{1}{\sqrt{C}}\|u\|,$$

where $C \in \mathbb{R}$, and C can be chosen as small (positive) as wanted.

The operator $i\alpha X$ is densely defined on $L^2(M)$, $\mathcal{D}(i\alpha X) \supset \mathcal{D}(\Delta)$, and, as mentioned above, $i\alpha X$ is symmetric if X is Killing. By applying the Kato-Rellich theorem ([RS], Theorem X.12), we see that $-\Delta - i\alpha X$ is self-adjoint.

Now we prove that $i\alpha X$ is a relatively bounded perturbation of $-\Delta + X^2$ with relative bound 0. We have

$$\begin{aligned}
(i\alpha Xu, i\alpha Xu) &= 2\alpha^2(iXu, iXu) - \alpha^2(iXu, iXu) = 2\alpha^2(X.\nabla u, X.\nabla u) + \alpha^2(X^2u, u) \\
&\leq 2\alpha^2 b^2(-\Delta u, u) + \alpha^2(X^2u, u) \leq \alpha^2 C(-\Delta u, u) + \alpha^2 C(X^2u, u) \\
&= \alpha^2 C((-\Delta + X^2)u, u) \leq \alpha^2 C\|(-\Delta + X^2)u\|\|u\| \\
&\leq \alpha^2 CC'\|(-\Delta + X^2)u\|^2 + \alpha^2 C\frac{1}{C'}\|u\|^2.
\end{aligned}$$

Similarly as above, we can choose $C' \in \mathbb{R}$ as small as we want, proving our contention. Also, by the arguments given above, the Kato-Rellich theorem applies, giving that $-\Delta + X^2 + i\alpha X$ is self-adjoint.

C.3 $F_{\lambda,X} \cong \|u\|^2_{H^1}$ (from (1.3.2))

More generally, let us consider a self-adjoint elliptic pseudodifferential operator $P \in OPS^2$ on a compact manifold M such that $\mathrm{Spec}(P) \subset (0, \infty)$. We can prove that the norm $(Pu, u)^{1/2}$ on $H^1(M)$ is equivalent to $||u||_{H^1(M)}$, the usual Sobolev norm.

Take the functional square root $Q = \sqrt{P}$, as defined by the spectral theorem (Theorem B.6.1). It is a 1st order elliptic, selfadjoint positive pseudodifferential operator on M (see [**?**] for a vast generalization of this statement). Then $(Pu, u)^{1/2} = \|Qu\|_{L^2}$, $\forall u \in C^\infty(M)$. The operator Q induces a continuous linear map

$$Q : H^1(M) \to L^2(M).$$

In fact, the above statement has vast generalizations. See, for example, Theorem 1.3, Chapter XII and Theorem 2.5, Chapter XI of [T2].

Using injectivity of P and the open mapping theorem, we deduce that

$$\|u\|_{H^1} \cong \|Qu\|_{L^2}.$$

C.4 Consequence of weak convergence

In general, when applied to a Banach space X, the Banach-Alaoglu theorem gives weak* compactness of a ball in X^*. So, applying the Banach-Alaoglu theorem to X^* instead of X, and seeing that X is reflexive if it is a Hilbert space, we can say that a ball in a Hilbert space X is weakly compact.

As in the proof of Proposition 1.3.2, let a sequence of functions f_n converge weakly to f in $H^1(M)$, and $\|.\|'$ be a norm on $H^1(M)$ which is comparable/equivalent to the usual Sobolev norm $\|.\|_{H^1}$ on $H^1(M)$. We necessarily have

$$\|f\|' \leq \liminf_n \|f_n\|'. \tag{C.4.1}$$

This is because, $\|.\|_{H^1}$ and $\|.\|'$ generate the same topology, which means f_n converge weakly to f with respect to the topology on $H^1(M)$ generated by $\|.\|'$. It can be shown that (C.4.1) is a consequence of weak convergence.

More generally, let x_n be a sequence in a reflexive Banach space X weakly converging to x. We can assume that $x \neq 0$, otherwise the claim

$$\|x\| \leq \liminf_n \|x_n\| \tag{C.4.2}$$

is trivial. By the Hahn Banach theorem, we can pick $x^* \in X^*$, such that $\|x^*\| = 1, x^*(x) = \|x\|$. Then we have, $\lim |x^*(x_n)| = |x^*(x)| = \|x\|$. Since $\|x^*\| = 1$, we have

$$|x^*(x_n)| \leq \|x_n\|, \ \forall\ n.$$

This gives (C.4.2).

C.5 Proof of $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

We know that $\hat{k}(\xi_1, \xi_2, ..., \xi_n) = \frac{1}{(\xi_1^2+\xi_2^4+...+\xi_n^4)^{1/4}}$. Choose $\psi(\xi)$, a bump function that is identically equal to 1 around the origin. Then $\psi\hat{k}$ is compactly supported, which means $\widehat{(\psi\hat{k})}$ is smooth. So it suffices to prove that $(\widehat{(1-\psi)}\hat{k}) \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

Calling $g = (1-\psi)\hat{k}$, we see that g is smooth. Now, choose $x \neq 0$, and let φ be a smooth bump function around x that is zero at the origin. We need to prove that $\varphi\hat{g}$ is smooth, or equivalently, $\hat{\varphi} * g$ vanishes faster than powers of $|\xi|$.

Define $\eta(x) = |x|^{-2m}\varphi(x)$. We see that $\eta(x) \in C_c^\infty(\mathbb{R}^n)$. Then we have, $\hat{\varphi} = (-\Delta)^m\hat{\eta}$, and $\hat{\varphi} * g = ((-\Delta)^m\hat{\eta}) * g = \hat{\eta} * ((-\Delta)^m g)$. Now, no matter how high m is, $\hat{\eta}$ is always Schwartz. Since we can choose m as large as we want, we can make $(-\Delta)^m g$ decay as fast as we want, which gives that $\hat{\varphi} * g$ decays faster than powers of $|\xi|$ as infinity, which proves our contention.

C.6 Reduction to real-valued functions

Generally, consider a linear self-adjoint operator L on $L^2(M)$ and a function $v = v_1 + iv_2$. Then,

$$\begin{aligned}\int_M Lv\bar{v}dM &= \int_M L(v_1+iv_2)(v_1-iv_2)dM = \int_M (Lv_1+iLv_2)(v_1-iv_2)dM \\ &= \int_M Lv_1v_1dM + \int_M Lv_2v_2dM + i\int_M(-Lv_1v_2+v_1Lv_2)dM.\end{aligned}$$

Consider the operator $L = (\lambda - \Delta_M)^{-1}(-\Delta_M)$, where $M = \mathbb{R}^n$ or $\mathbb{H}^n$. If we can prove that for any real valued $\varphi \in \mathcal{D}(L)$, $L\varphi$ is real-valued, then the symmetry of L will imply that $\int_M(-Lv_1v_2+v_1Lv_2)dM = 0$.

If φ is real-valued, then so is $-\Delta\varphi = \psi$. For real-valued $f, g \in L^2(M)$, if $(\lambda - \Delta_M)(f+ig) = \psi$, then that would imply $-\Delta_M g = -\lambda g$ for $\lambda \geq 0$, which is impossible for $\lambda > 0$. Also, Spec $(-\Delta_{\mathbb{H}^n}) \subset [\frac{(n-1)^2}{4}, \infty)$, and since there are no L^2 harmonic functions on $\mathbb{R}^n$, we can rule out $\lambda = 0$. That settles the question.

APPENDIX D: DEFINITIONS AND NOTATIONS

Definition D.0.1. Let M be a complete Riemannian manifold. A traveling wave solution to the NLS ((1.1.1)) or the NLKG ((1.1.2)) is defined as a solution of the form

$$v(t,x) = e^{i\lambda t}u(g(t)x), \tag{D.0.1}$$

where $\lambda \in \mathbb{R}$ and $g(t)$ is a one-parameter family of isometries on M.

Understandably, such solutions are called traveling waves because of the dependence of u on the traveling component $g(t)$.

Definition D.0.2. A standing wave solution to the NLS or the NLKG is defined as a solution of the form

$$v(t,x) = e^{i\lambda t}u(x),$$

where $\lambda \in \mathbb{R}$.

Notation D.0.3. By dM, we mean the volume form $\sqrt{g}dx_1dx_2...dx_n$, where $(x_1, x_2, ..., x_n)$ are the local coordinates on a Riemannian manifold M of dimension n. As a notational convention, throughout the thesis, we use dM to indicate volume form for integration on a manifold M. For example, when integrating on the hyperbolic space, we will use the notation $d\mathbb{H}^n$ for the volume form. However, in situations where the variable of integration is important, we will use dx, dy, dz etc., for integration on a manifold also. This is particularly the case in Part II of the thesis, where integral kernels are ubiquitous and the variable with respect to which integration is being done needs to be displayed explicitly.

Definition D.0.4. We define $||u||^2_{L^2(M)}$ as the "mass" of $u \in L^2(M)$. Also, we call

$$E_X(u) = \frac{1}{2}(-\Delta u - iXu, u) - \frac{1}{p+1}\int_M |u|^{p+1} dM$$

and

$$\mathcal{E}_{\lambda,X}(u) = \frac{1}{2}(-\Delta u + X^2 u + 2i\lambda X u, u) - \frac{1}{p+1}\int_M |u|^{p+1} dM$$

the “energy” of u for the NLS and the NLKG respectively, where $\lambda \in \mathbb{R}$, X is a Killing field of the manifold M.

Notation D.0.5. $H_x^{1/2}(L_y^2)$ means L_y^2-valued functions of x which lie in the Sobolev space $H_x^{1/2}(\mathbb{R})$. Similarly, $L_x^2(H_y^1)$ means H_y^1-valued functions of x lying in $L_x^2(\mathbb{R})$.

For a good reference on these sorts of spaces and allied results, see [LM], Chapter 4, particularly Section 2.1.

Definition D.0.6. A manifold M is said to be weakly homogeneous if there is a group G of isometries of M and a number $D > 0$ such that for every $x, y \in M$, there exists a $g \in G$ such that $\mathrm{dist}(x, g(y)) \leq D$.

As examples, compact and/or homogeneous spaces are clearly weakly homogeneous. As stated in [CMMT] (see page 39), any covering space of a compact manifold is also weakly homogeneous.

Definition D.0.7. The symmetric decreasing rearrangement of a function u on $\mathbb{H}^n$ is given by

$$u^*(x) = \inf\{t : \lambda_u(t) \leq \mu(B_{\mathrm{dist}\ (x,0)}(0))\}, \tag{D.0.2}$$

where μ is the natural measure on $\mathbb{H}^n$, dist is the hyperbolic distance on $\mathbb{H}^n$, 0 is a fixed point called the origin and

$$\lambda_u(t) = \mu(\{|u| > t\}).$$

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