Perturbative Expansion of the Colored Jones Polynomial

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Abstract

ANDREA OVERBAY: Perturbative Expansion of the Colored Jones Polynomial
(Under the direction of Lev Rozansky)

Both the Alexander polynomial $\Delta_{\mathcal{K}}(t)$ and the colored Jones polynomial $V_\alpha(\mathcal{K}; q)$ are well-known knot invariants. While the Jones polynomial seems similar to the Alexander polynomial, it lacks an interpretation in classical topology. Because the Alexander polynomial has a classical topological definition, exploring a relationship between the two polynomials offers the possibility of interpreting the Jones polynomial topologically.

Melvin and Morton conjectured a relationship between the two through an expansion of the colored Jones polynomial [**18**]. The conjecture was proven by Bar-Natan and Garoufalidis [**4**] and Rozansky extended the result further [**24**]. Rozansky proved the following expansion in $h = q - 1$:

$$V_\alpha(\mathcal{K}; q) = \sum_{n \geq 0} h^n \left(\frac{P^{(n)}(\mathcal{K}; q^{\alpha/2} - q^{-\alpha/2})}{\Delta_{\mathcal{K}}^{2n+1}(q^{\alpha/2} - q^{-\alpha/2})} \right)$$

where $P^{(n)}(\mathcal{K}; q^{\alpha/2} - q^{-\alpha/2}) \in \mathbb{Z}[q^\alpha, q^{-\alpha}]$ are polynomial invariants of the knot $\mathcal{K}$.

In this dissertation, we will describe how we used the quantum group $U_q(\mathfrak{sl}(2))$ and techniques from quantum field theory to calculate the first two of these polynomial invariants for all prime knots of up to nine crossings and present these results. Furthermore, we will provide evidence of the validity of a conjecture from [**23**] by calculating $P^{(1)}(\mathcal{K}; q^{\alpha/2} - q^{-\alpha/2})$ and $P^{(2)}(\mathcal{K}; q^{\alpha/2} - q^{-\alpha/2})$ for all amphicheiral knots of up to ten crossings.

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CHAPTER 1

Introduction

When studying knots or links, one often seeks to find the topological properties of the objects being studied. A key question that one strives to answer when studying knots is, "Are these two knots the same?" This includes seeing if a knot is really the unknot. In the seemingly never-ending quest to answer these questions and others, many different knot invariants have been discovered and explored.

Let $\mathcal{K}$ be a knot in S^3. The Alexander polynomial $\Delta_{\mathcal{K}}(t)$ is a well understood knot invariant that has a topological interpretation. In Chapter 2 we will provide an overview of the Alexander polynomial. Further information about $\Delta_{\mathcal{K}}(t)$ can be found in numerous texts including [**16**]. Another knot polynomial came onto the scene in 1984 with the work of Vaughn Jones [**13**]. The Jones polynomial is a polynomial knot invariant that seems similar to the Alexander polynomial in many ways but is actually a totally different beast. In fact, the Jones polynomial is more powerful than the Alexander polynomial, but it is defined purely combinatorially and lacks any obvious topological interpretation within classical topology. Thus it was particularly exciting when a non-rigorous path integral topological interpretation was suggested by Witten in [**27**]. He suggested that the Jones polynomial is an infinite dimensional integral over all $SU(2)$ connections over S^3 with a certain weight. The colored Jones polynomial, $V_\alpha(\mathcal{K}; q)$, is a generalized version of the Jones polynomial. We will describe a way of calculating the colored Jones polynomial as a quantum trace in Chapter 3.

Melvin and Morton conjectured a relationship between the Jones and Alexander polynomials in [**18**]. This conjecture was proven by Bar-Natan and Garoufalidis in [**4**]. Rozansky extended the result further in [**24**].

Associate an $(\alpha + 1)$-dimensional $U_q(\mathfrak{sl}(2))$-module to a fixed knot, $\mathcal{K} \subset S^3$. Let $J_\alpha(\mathcal{K}; q)$ denote the colored Jones polynomial of $\mathcal{K}$ normalized so that it is multiplicative under disjoint unions and

$$J_\alpha(\text{unknot}; q) = \frac{q^{\alpha/2} - q^{-\alpha/2}}{q^{1/2} - q^{-1/2}}. \tag{1.0.1}$$

We will consider the reduced colored Jones polynomial defined as

$$V_\alpha(\mathcal{K}; q) = \frac{J_\alpha(\mathcal{K}; q)}{J_\alpha(\text{unknot}; q)}. \tag{1.0.2}$$

We have the following theorem due to Rozansky.

THEOREM 1.0.1. *Let $\mathcal{K} \subset S^3$ be a knot. We have the following expansion for the colored Jones polynomial in powers of $h = q - 1$:*

$$V_\alpha(\mathcal{K}; q) = \sum_{n \geq 0} h^n \left(\sum_{0 \leq m \leq n} D_{m,n}(\mathcal{K})(\alpha h)^{2m} \right) \tag{1.0.3}$$

such that the coefficients $D_{m,n+2m}$ have the following property:

$$\sum_{m \geq 0} D_{m,n+2m}(\mathcal{K})(\alpha h)^{2m} = \frac{P^{(n)}(K; q^\alpha)}{\Delta_{\mathcal{K}}^{2n+1}(q^{\alpha/2} - q^{-\alpha/2})}, \tag{1.0.4}$$

where $\Delta_{\mathcal{K}}$ is the Alexander-Conway polynomial normalized so that $\Delta_{unknot} = 1$ and $P^{(n)}(\mathcal{K}; q^\alpha) \in \mathbb{Z}[q^\alpha, q^{-\alpha}]$ are invariants of the knot.

In this theorem, we sum along the diagonals in Figure 1.1.

Our work involves efficiently calculating these invariants, $P^{(n)}(\mathcal{K}; q^\alpha)$, in the hopes of gaining a wealth of experimental data from which we can infer some topological properties. As will be described in Chapter 3, the Jones polynomial can be calculated as a quantum trace. The calculation of $V_\alpha(\mathcal{K}; q)$ is based on representation theory of the quantum group $U_q(\mathfrak{sl}(2))$. The quantum group $U_q(\mathfrak{sl}(2))$ has standard generators

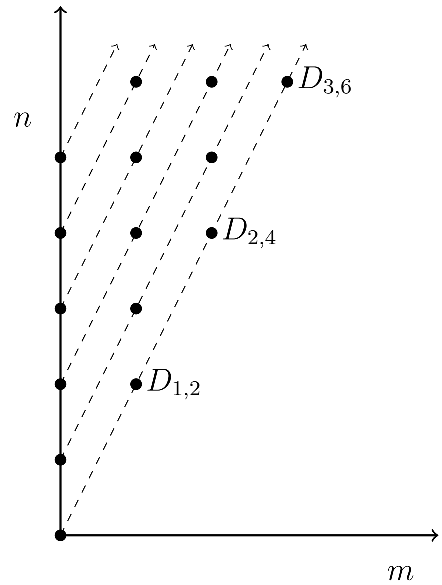


Figure 1.1. Summing along Diagonals

E, F, and H with relations

$$[H,E]=2E, \quad [H,F]=-2F, \quad \text{and} \quad [E,F]=\frac{q^H-q^{-H}}{q-q^{-1}} \tag{1.0.5}$$

where $q \in \mathbb{C}$. Let W_α denote an $(\alpha+1)$-dimensional irreducible $U_q(\mathfrak{sl}(2))$-module. There is an important $U_q(\mathfrak{sl}(2))$-intertwiner

$$\check{\mathcal{R}} : W_\alpha \otimes W_\alpha \to W_\alpha \otimes W_\alpha \tag{1.0.6}$$

that satisfies the braid relations given in Section 2.4. This means there exists a homomorphism

$$f : B_n \to \mathrm{End}_{U_q(\mathfrak{sl}(2))}(W_\alpha^{\otimes n}) \tag{1.0.7}$$

where B_n is the braid group on n generators, such that $f(\sigma_i) = \check{\mathcal{R}}_i$ where σ_i is an elementary braid consisting of a single positive crossing of the i^{th} strand over the $(i+1)^{st}$ strand and

$$\check{\mathcal{R}}_i = \mathbb{1}^{\otimes(i-1)} \otimes \check{\mathcal{R}} \otimes \mathbb{1}^{\otimes(n-i-1)}. \tag{1.0.8}$$

If a knot $\mathcal{K}$ is presented as a circular closure of a braid $\beta \in B_n$, then the Jones polynomial is the "quantum trace"

$$V_\alpha(\mathcal{K};q) = q^{\mathrm{fr}}\, \mathrm{Tr}_{W_\alpha^{\otimes n}}(Q_H f(\beta)) \tag{1.0.9}$$

where $Q_H = (q^{H/2})^{\otimes n}$ and q^{fr} is a framing factor given by the Formula (3.4.1). Importantly, one can compute only a partial trace. Namely, for a linear transformation $A : W_\alpha^{\otimes n} \to W_\alpha^{\otimes n}$ define

$$\mathrm{Tr}^{(1)} : W_\alpha \to W_\alpha \tag{1.0.10}$$

as a trace of A taken over all W_α factors except the first one. The partial quantum trace of $f(\beta)$ with a framing factor is proportional to the identity with the coefficient being the reduced Jones polynomial $V_\alpha(\mathcal{K};q)$, i.e.

$$q^{\mathrm{fr}}\, \mathrm{Tr}^{(1)}_{W_\alpha^{\otimes n}}(Q_H^{(1)} f(\beta)) = V_\alpha(\mathcal{K};q)\, \mathbb{1}_{W_\alpha} \tag{1.0.11}$$

where $Q_H^{(1)} = \mathbb{1} \otimes (q^{H/2})^{\otimes (n-1)}$.

Let us recall the more familiar classical case of $\mathfrak{sl}(2)$ before proceeding to the case of $U_q(\mathfrak{sl}(2))$. The generators E, F, and H of $\mathfrak{sl}(2)$ satisfy the relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad \text{and} \quad [E,F] = H. \tag{1.0.12}$$

In this case, we have a family of homomorphisms $\tilde{f}_\alpha : \mathfrak{sl}(2) \to \mathbb{C}[z,\partial_z]$ where $\alpha \in \mathbb{C}$ and $\mathbb{C}[z,\partial_z]$ is the Heisenberg algebra. Its action on the generators is

$$\tilde{f}_\alpha(E) = z, \;\; \tilde{f}_\alpha(H) = \alpha + 2z\partial_z, \;\; \text{and} \;\; \tilde{f}_\alpha(F) = -\alpha\partial_z - z\partial_z^2. \tag{1.0.13}$$

By deforming $\tilde{f}$, we get a family of algebra homomorphisms $f_\alpha : U_q(\mathfrak{sl}(2)) \to \mathbb{C}[z,\partial_z]$. Before we describe the action of the homomorphisms on the generators of $U_q(\mathfrak{sl}(2))$, let us define a function $g(q,t,x) \in \mathbb{Z}[t][[h,x]]$ as

$$g(q,t,x) = -\frac{(q^x - q^{-x})(t\, q^{x-1/2} - t^{-1} q^{-x+1/2})}{x(q-q^{-1})(q^{1/2}-q^{-1/2})}. \tag{1.0.14}$$

Let $a = \alpha h$, so that $\alpha = \dfrac{a}{h}$ and $q^\alpha = e^a$. Then the action of the family of homomorphisms on the standard generators E, H, and F is:

$$f_\alpha(E) = z, \;\; f_\alpha(H) = \alpha + 2z\partial_z, \;\text{ and }\; f_\alpha(F) = g(q, q^\alpha, z\partial_z + 1). \tag{1.0.15}$$

The Heisenberg algebra acts naturally on the polynomial algebra $\mathbb{C}[z]$, so the homomorphism f_α turns $\mathbb{C}[z]$ into a $U_q(\mathfrak{sl}(2))$-module which we will denote by $\mathbb{C}_\alpha[z]$. If α is a positive integer, then $\mathbb{C}_\alpha[z]$ has a submodule $\widetilde{W}_\alpha$ generated over $\mathbb{C}[z]$ by $z^{\alpha+1}$. That is, it is generated by $\{z^{\alpha+1}, z^{\alpha+2}, ...\}$ over $\mathbb{C}$. The quotient module

$$W_\alpha = \mathbb{C}_\alpha[z]/\widetilde{W}_\alpha \tag{1.0.16}$$

is the $(\alpha{+}1)$-dimensional irreducible representation of $U_q(\mathfrak{sl}(2))$ generated by 1, z, z^2, ..., z^α over $\mathbb{C}$. The action of the $U_q(\mathfrak{sl}(2))$ generators are given by the equations in (1.0.13). Since the universal R-matrix of $U_q(\mathfrak{sl}(2))$ is given in terms of the generators E, F, and H as in Equation (3.3.4), the action of $\check{\mathcal{R}}$, the R-matrix composed with permutation, is determined by substituting the action of the generators into Equation (3.3.4).

For a knot $\mathcal{K}$, it was shown in [**24**] that since W_α is a quotient $\mathbb{C}_\alpha[z]/\widetilde{W}_\alpha$, one can replace the factors W_α in the partial trace (1.0.11) by $\mathbb{C}_\alpha[z]$:

$$q^{\mathrm{fr}}\mathrm{Tr}^{(1)}_{(\mathbb{C}_\alpha[z])^{\otimes n}} Q_H^{(1)} f(\beta) = V_\alpha(\mathcal{K}; q)\mathbb{1}_{\mathbb{C}_\alpha[z]}. \tag{1.0.17}$$

In order to produce the expansion in Theorem 1.0.1 and following [**24**], we study the limit of Equation (1.0.17) when $q \to 1$ and q^α is constant. Let $h = \log(q)$ and $t = q^\alpha = e^a$. Then we can cast

$$\check{\mathcal{R}} : \mathbb{C}[z_1, z_2] \to \mathbb{C}[z_1, z_2] \tag{1.0.18}$$

in the form

$$\check{\mathcal{R}} = Q(z_1, z_2, \partial_{z_1}, \partial_{z_2})\left(1 + \sum_{i=1}^{\infty} h^i R_i(z_1, z_2, \partial_{z_1}, \partial_{z_2})\right) \tag{1.0.19}$$

where $R_i(z_1, z_2, w_1, w_2)$ is a polynomial of its arguments for each i and $R_i(z_1, z_2, \partial_{z_1}, \partial_{z_2})$ is normal ordered while

$$Q(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \exp\left(-z_1\partial_{z_1} + tz_2\partial_{z_1} + tz_1\partial_{z_2} - t^2 z_2\partial_{z_2}\right) \tag{1.0.20}$$

is a $\mathbb{C}[z_1, z_2]$-algebra homomorphism, that is, Q acts on $\mathbb{C}[z_1, z_2]$ by a linear transformation on the generators z_1 and z_2. As a result $f(\beta)$ has a similar form. Suppose $\beta \in B_n$. Let $\underline{z}$ denote $z_1, z_2, ..., z_n$ and similarly let $\underline{\partial}_z$ denote $\partial_{z_1}, \partial_{z_2}, ..., \partial_{z_n}$. Then we can write

$$f(\beta) = Q_\beta(\underline{z}, \underline{\partial}_z)\left(1 + \sum_{i=1}^{\infty} h^i B_{i,\beta}(\underline{z}, \underline{\partial}_z)\right) \tag{1.0.21}$$

where $Q_\beta(z, \partial_z)$ is a $\mathbb{C}[\underline{z}]$-algebra homomorphism whose action on $z_1, z_2, ..., z_n$ coincides with the Burau representation of β while $B_{i,\beta}(\underline{z}, \underline{\partial}_z)$ is a normal ordered polynomial for each i. Recall the special relationship between the Burau representation and the Alexander polynomial. The Alexander polynomial of a knot presented as a braid closure can be computed as a determinant of the Burau representation of the braid. Further details can be found in Section 2.5.

When $h \to 0$ and q^α is kept constant, we are left with only the algebra homomorphism, Q_β. A partial trace of an algebra homomorphism can be computed by the generalized geometric series formula. Let $\tilde{Q}_\beta$ denote the restriction of the action of Q_β to the subspace $\mathbb{C}^n$ of variables $z_2, ..., z_n$. Then we can write

$$\mathrm{Tr}^{(1)}_{\mathbb{C}[\underline{z}]} Q_\beta = \frac{1}{\det_{\mathbb{C}^n}(1 - \tilde{Q}_\beta)}. \tag{1.0.22}$$

Since $\tilde{Q}_\beta$ coincides with the Burau representation, Equation (1.0.22) is precisely the reason the Alexander polynomial appears in the expansion in Theorem 1.0.1.

However, we are interested in the case when h is nonzero. Thus it remains to take the trace of the second portion of (1.0.21). To do this, we will use methods from quantum field theory. We should stress that we are using these methods on a finite number of variables $z_1, z_2, ..., z_n$ while the techniques of quantum field theory are typically performed on a space with infinitely many variables. Hence all of these techniques are completely rigorous in our setting. Our techniques will be explained in Chapters 4 and 5, but here is a brief summary. Suppose that Q_β in the geometric sum formula (1.0.22) depends on an extra parameter ϵ, so Equation (1.0.22) becomes

$$\mathrm{Tr}^{(1)}_{\mathbb{C}[z]} Q_\beta(\epsilon) = \frac{1}{\det_{\mathbb{C}^n}(1-\tilde{Q}_\beta)}. \tag{1.0.23}$$

For a positive integer k, take the k-th derivative over epsilon of both sides and evaluate at $\epsilon = 0$:

$$\mathrm{Tr}^{(1)}_{\mathbb{C}[z]} \left(\frac{d^k Q_\beta(\epsilon)}{d\epsilon^k} \right) \bigg|_{\epsilon=0} = \frac{d^k}{d\epsilon^k} \left(\frac{1}{\det_{\mathbb{C}^n}(1-\tilde{Q}_\beta)} \right) \bigg|_{\epsilon=0}. \tag{1.0.24}$$

Although $\dfrac{d^k Q_\beta(\epsilon)}{d\epsilon^k}$ is no longer an algebra homomorphism, this formula still computes its trace. Equation (1.0.24) is the simplest example of the quantum field theory techniques that we will use to compute the trace of the expanded formula (1.0.21).

We have developed a program in Mathematica [**12**] to efficiently perform these techniques in rigorous finite-dimensional cases. Our program takes in information about the braid representation of a knot and then calculates $P^{(1)}(\mathcal{K}; q^\alpha)$ and $P^{(2)}(\mathcal{K}; q^\alpha)$. We are now able to calculate $P^{(1)}(\mathcal{K}; q^\alpha)$ and $P^{(2)}(\mathcal{K}; q^\alpha)$ for any knot, and we will present these polynomials for all prime knots up to nine crossings in Chapter 6. Also in Chapter 6, we discuss a conjecture about our polynomials for amphicheiral knots from [**23**] and provide evidence to the validity of the conjecture. In Chapter 4 we will expand the R-matrix of $U_q(\mathfrak{sl}(2))$ in powers of h to prove Theorem 4.1.1. We will also provide a similar expansion of $\check{\mathcal{R}}^{-1}$. In Chapter 5 we will discuss our methods of calculation including how to take the trace of $\hat{\beta}$, the action associated to a braid β. We provide a copy of our program in Appendix A. We begin in Chapters 2 and 3 with some preliminaries, the Alexander polynomial, and the Jones polynomial.

CHAPTER 2

The Alexander Polynomial

In this chapter we begin by discussing some preliminary definitions of knot theory. Then we present three different interpretations of the Alexander polynomial $\Delta_{\mathcal{K}}(t)$. These three include a topological interpretation, a skein-relation definition, and a relationship with the Burau representation of the knot presented as a braid closure. The last of these will play a role in our own calculations, but the topological interpretation is the one of most interest. One of the biggest strengths of the Alexander polynomial versus the Jones polynomial is the fact that the Alexander polynomial has a topological interpretation while the Jones polynomial lacks one in classical topology. We proceed with the preliminaries.

2.1. Preliminaries

The study of knots began in 1867 when Lord Kelvin suggested that atoms were knotted up bits of ether. Because of this theory, physicists were interested in tabulating all possible knots. The Scottish physicist Tait started studying when two knots are the same and created a table of knots. The following preliminary definitions are adapted from [**16**].

Definition 2.1.1. A *link* $\mathcal{L}$ of m components is a subset of S^3, or of $\mathbb{R}^3$, that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a *knot* $\mathcal{K}$.

One can envision taking a string, tangling it somehow, and then fusing the ends. What does it mean for two links or knots to be the "same"?

DEFINITION 2.1.2. Links $\mathcal{L}_1$ and $\mathcal{L}_2$ in S^3 are *equivalent* if there is an orientation preserving piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(\mathcal{L}_1) = \mathcal{L}_2$.

Oftentimes instead of studying the knots or links themselves, we study their projections into the plane.

DEFINITION 2.1.3. A *link diagram*, D, of a link, $\mathcal{L}$, is a projection into the plane that keeps crossing information.

Of course, when studying the projections of links into the plane, we must understand when two of these projections represent the same link.

DEFINITION 2.1.4. A *Reidemeister move* refers to one of three local moves on a link diagram. Each move operates on a small region of the diagram and is one of three types:

(1) Twist and untwist in either direction.
(2) Move one loop completely over another.
(3) Move a string completely over or under a crossing.

These moves can be seen in Figure 2.1

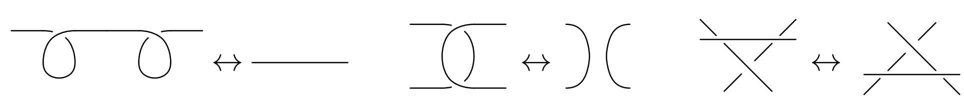


FIGURE 2.1. Reidemeister Moves

Two link diagrams belonging to the same link, up to planar isotopy, can be related by a sequence of the three Reidemeister moves. This fact was shown independently by Reidemeister in 1926 [**21**] and by Alexander and Briggs in 1927 [**3**].

Once we understand what it means for two knots, or knot diagrams, to be equivalent, the next natural step is to try to answer the question, “Are these two the same?”

One way that we can try to answer this question is by discovering and calculating knot invariants. In fact, a variety of topological properties of knots can be investigated by studying knot invariants.

Definition 2.1.5. A *knot invariant* is a quantity (in a broad sense) defined for each knot which is the same for equivalent knots.

Another one of the questions mathematicians seek to answer about knots is, "Is this the unknot?" Currently we know that the Alexander polynomial doesn't detect unknottedness, meaning we have examples of non-trivial knots of 10 crossings for which the Alexander polynomial is equal to 1. It is still an open question whether the Jones polynomial can detect unknottedness. These questions about knots and knot invariants are interesting. Many knot invariants have been discovered and studied, and this process continues today.

2.2. A Topological Definition of the Alexander Polynomial

The Alexander polynomial is perhaps the most well-known invariant of knots. This polynomial was discovered by Alexander in 1928 [**2**]. The Alexander polynomial of a knot can be interpreted topologically using an infinite cyclic cover of the complement of the knot. A convenient way to view and build this infinite cyclic cover is through Seifert surfaces. In this section, we will define these ideas rigorously. A reference for the following discussion is [**16**].

Definition 2.2.1. A *Seifert surface* for an oriented link $\mathcal{L}$ in S^3 is a connected compact surface contained in S^3 that has $\mathcal{L}$ as its oriented boundary.

Of course now that we have this definition, a reasonable question to ask is, "Does every knot have an associated Seifert surface?" The answer is yes for any oriented link thanks to the following theorem.

Theorem 2.2.2. *Any oriented link in S^3 has a Seifert surface.*

This theorem was first proven by Frankl and Pontrjagin in [**11**]. In 1935 Seifert provided another proof which is constructive [**25**]. His proof provided a general method of finding a Seifert surface of a knot which is called the *Seifert algorithm*. However this construction is not unique.

Although a Seifert surface is not necessary to have an infinite cyclic cover, we can use it to build an infinite cyclic cover of the knot complement. As described in [**16**], let F be a Seifert surface of $\mathcal{L}$ and let N be a regular neighborhood of $\mathcal{L}$. Define X to be the closure of $S^3 - N$. Now let Y be X-cut-along-F. This means that Y is homeomorphic to X less the open neighborhood $F \times (-1, 1)$. We can build X_∞, the *infinite cyclic cover*, by stacking countably many copies of Y on top of each other. X_∞ has a natural homeomorphism $t : X_\infty \to X_\infty$ which moves up one stack/copy of Y from the current position.

Definition 2.2.3. The r^{th} *Alexander ideal* of an oriented link L is the r^{th} elementary ideal of the $\mathbb{Z}[t, t^{-1}]$-module $H_1(X_\infty; \mathbb{Z})$. The r^{th} *Alexander polynomial* of $\mathcal{L}$ is the generator of the smallest principal ideal of $\mathbb{Z}[t, t^{-1}]$ that contains the r^{th} Alexander ideal. The first Alexander polynomial is called <u>the</u> *Alexander polynomial* and is written $\Delta_{\mathcal{K}}(t)$.

If we restrict ourselves to considering the Alexander polynomial of a knot $\mathcal{K}$, we have a very satisfying interpretation of $\Delta_{\mathcal{K}}(t)$ as a characteristic polynomial presented in [**16**].

Theorem 2.2.4. *Let $\mathcal{K}$ be a knot in S^3 and let $t : X_\infty \to X_\infty$ be the (covering) translation of X_∞ (the infinite cyclic cover of the exterior of $\mathcal{K}$). Then $H_1(X_\infty; \mathbb{Q})$ is a finite-dimensional vector space over the field $\mathbb{Q}$. The characteristic polynomial of the linear map $t_\star : H_1(X_\infty; \mathbb{Q}) \to H_1(X_\infty; \mathbb{Q})$ is, up to multiplication by a unit, equal to the Alexander polynomial of $\mathcal{K}$.*

Through this set up of the Alexander polynomial and the proceeding theorem, one can see that the polynomial has its basis in topology. There are other descriptions of the Alexander polynomial that are not topological in nature.

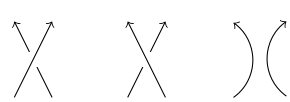
2.3. A Skein Relation Definition of the Alexander Polynomial

In 1970 Conway introduced his polynomial which satisfies certain skein relations [**9**]. In fact, Alexander had shown that his polynomial satisfied the same relations. This result was presented in the miscellaneous section of Alexander's paper and was thus not thoroughly investigated for some time. The Conway polynomial and the Alexander polynomial are related by a simple formula and oftentimes the polynomial knot invariant is simply called the Alexander-Conway polynomial. The Conway polynomial is defined in the following way.

DEFINITION 2.3.1. For oriented links $\mathcal{L}$, the *Conway polynomial* $\nabla_{\mathcal{L}}(z) \in \mathbb{Z}[z, z^{-1}]$ is defined by

(1) $\nabla_{\text{unknot}}(z) = 1$,
(2) whenever three oriented links L_+, L_-, and L_0 are the same except in a neighborhood of a point where they are as shown in Figure 2.2, then

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z). \tag{2.3.1}$$



L_+ L_- L_0

FIGURE 2.2. Conway Triple

The triple in Figure 2.2 is known as the *Conway triple*.

The Alexander polynomial is related to the Conway polynomial as follows:

$$\Delta_{\mathcal{L}}(t^2) = \nabla_{\mathcal{L}}(t - t^{-1}) \tag{2.3.2}$$

for $\Delta_{\mathcal{L}}(t)$ normalized to satisfy the relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t). \tag{2.3.3}$$

Oftentimes this is the method used to calculate the Alexander polynomial, but it lacks the satisfying topological basis of the previous definition of the Alexander polynomial.

2.4. Presenting Knots as Braid Closures

In order to discuss the third interpretation of the Alexander polynomial, we must pause here and discuss braids. Any braid is made up of elementary braids.

DEFINITION 2.4.1. An *elementary braid* σ_i is a braid in which the only crossing is the i^{th}-strand crossing over the $(i+1)^{st}$-strand. The inverse of σ_i, denoted σ_i^{-1}, is the braid consisting of only the crossing in which the $(i+1)^{st}$-strand crosses over the i^{th}-strand.

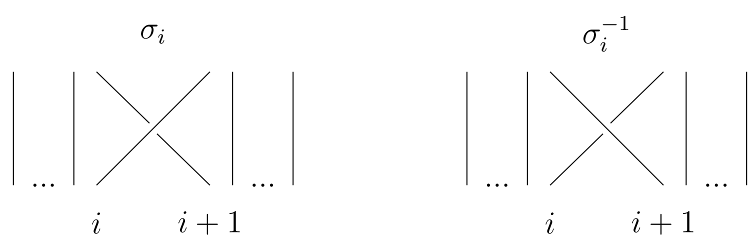


FIGURE 2.3. Elementary Braids

The braid group on n-strands, B_n, is made up of elementary braids with certain relations.

DEFINITION 2.4.2. The *braid group* on n strands, denoted B_n, is generated by the braids σ_i for $i = 1, ..., n-1$ subject to the relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } 1 \le i \le n-2 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| \ge 2 \end{aligned} \tag{2.4.1}$$

Then naturally, a *braid* is an element of B_n for some $n \geq 1$.

The closure of a braid β, denoted $\bar{\beta}$, is created by connecting corresponding ends in pairs. We are talking about braids because there is a clear relationship between braids and knots. We have the following theorem proven by Alexander in 1923 [**1**].

THEOREM 2.4.3. *Every knot can be presented as the closure of a braid.*

Note that this theorem says nothing about the uniqueness of the braid representation of the knot, so we must ask when two different braid closures give rise to the same knot. This question was answered by Markov in [**17**]. There are two actions on a braid, called *Markov moves*, that we consider. The first is conjugation.

DEFINITION 2.4.4. Given two braids α and β in B_n, a *type 1 Markov move*, also called *conjugation*, takes $\alpha\beta \mapsto \beta\alpha$.

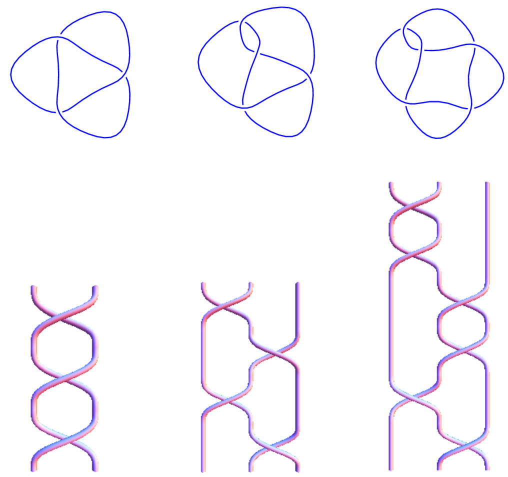
We have another Markov move that is called a stabilization move.

DEFINITION 2.4.5. Given a braid $\beta \in B_n$, a *type 2 Markov move*, also called a *stabilization move*, takes $\beta \mapsto \beta\sigma_n \in B_{n+1}$ or $\beta \mapsto \beta\sigma_n^{-1} \in B_{n+1}$.

We can use these moves to determine when two braids will give rise to the same knot. The following theorem is due to Markov [**17**].

THEOREM 2.4.6. *Given two braids $\alpha \in B_m$ and $\beta \in B_n$, we have that $\bar{\alpha}$ is equivalent to $\bar{\beta}$ if and only if β can be obtained from α through a finite sequence of Markov moves.*

We conclude this section with a table that shows braid representations for the some non-trivial knots. The knot diagrams in this table are from Knot Info [**8**] while the braid representations were created in Mathematica [**12**].



3_1 4_1 5_2

Table 2.1: Braid Representation of Some Knots

2.5. The Alexander Polynomial by the Burau Representation

There is yet another characterization of the Alexander polynomial using the reduced Burau representation of the braid group. The following definition of the reduced Burau representation and Theorem 2.5.2 are presented in [**5**].

Definition 2.5.1. Let $\mathcal{K}$ be the closure of braid on n-strands, β. Let σ_i be the elementary positive braid in which the i^{th} strand crosses over the $(i+1)^{st}$ strand. Then the *reduced Burau representation* associated to σ_i for $i = 2, ..., n-1$ is the $(n-1) \times (n-1)$ matrix of the form

$$\begin{pmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-2} \end{pmatrix} \tag{2.5.1}$$

where I_r denotes the $r \times r$ identity matrix. When $i = 1$, we have a block matrix in the upper left corner of the $(n-1) \times (n-1)$ matrix of the form:

$$\begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \tag{2.5.2}$$

and the identity elsewhere. The *reduced Burau matrix,* $\mathcal{B}$, associated to β is the matrix product of the matrices associated to the sequence of elementary braids in β.

Through the reduced Burau representation, we get the following theorem concerning the Alexander polynomial. This theorem is presented in [**5**] as a modification of a theorem of Burau from [**7**].

THEOREM 2.5.2. *For the Alexander polynomial,* $\Delta_{\mathcal{K}}(t)$*, and the reduced Burau matrix,* $\mathcal{B}$*, we have the following relationship*

$$\Delta_{\mathcal{K}}(t) = \frac{\det(\mathcal{B} - I)}{1 + t + \ldots + t^{n-1}}. \tag{2.5.3}$$

Putting together the information in this section and the discussion from the introduction, we can now understand why the Alexander polynomial appears in Theorem 1.0.1. Since the action of Q_β in the expansion of $f(\beta)$ coincides with the Burau representation and taking the trace of such an expression introduces a factor of $\det(1 - \tilde{Q}_\beta)$, we get the Alexander polynomial in Equation (1.0.4).

CHAPTER 3

The Jones Polynomial as the Quantum Trace of a Braid

3.1. The Bracket Polynomial

The next knot invariant of interest is the Jones polynomial which can also be defined using a skein relation. A reference for the following discussion is [**14**]. Often we will be considering *framed* links.

DEFINITION 3.1.1. A link $\mathcal{L}$ provided with a non-singular normal vector field is said to be *framed*. Two vector fields on $\mathcal{L}$ that are homotopic in the class of non-singular normal vector fields determine the same *framing*.

To compute the Jones polynomial using a skein class of diagrams, we must make the following definitions.

DEFINITION 3.1.2. Fix a nonzero complex number q. Let $E(q)$ be the complex vector space generated by all link diagrams quotiented by

(1) ambient isotopy in the plane;
(2) the relation $D \cup O = -(q + q^{-1})D$, where D is an arbitrary link and O is a simple closed curve bounding a disk in the complement of D;
(3) the identity in Figure 3.1 which is called *Kauffman's skein relation*.

$E(q)$ is a *skein class of diagrams*.

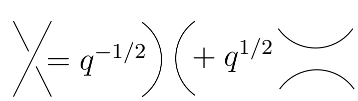


FIGURE 3.1. Kauffman's Skein Relation

Thus given a link diagram, one can use these relations to resolve any crossing of the diagram.

Definition 3.1.3. Every link diagram D represents an element of $E(q)$ which will be denoted by $\langle D\rangle(q)$ or sometimes just $\langle D\rangle$. This is called the *skein class of D.*

For knot diagrams, this skein class makes sense due to the following theorem presented in [**14**].

Theorem 3.1.4. *The skein class of any link diagram is invariant under the Reidemeister moves.*

We should note that introducing a positive or negative curl as in the first Reidemeister move introduces a factor of $-q^{-3/2}$ or $-q^{3/2}$ respectively. However this is considered to be equivalent in the skein class.

Using Theorem 3.1.4, we can define the bracket polynomial of a link $\mathcal{L}$. Present $\mathcal{L}$ by a diagram D and choose $q \in \mathbb{C}$ so that $q + q^{-1} \neq 0$. Then we have the following definition.

Definition 3.1.5. The *bracket polynomial of $\mathcal{L}$* is

$$\langle \mathcal{L}\rangle(q) = -(q + q^{-1})^{-1}\langle D\rangle(q). \tag{3.1.1}$$

Note that the bracket polynomial does depend on the framing of the link $\mathcal{L}$.

3.2. The Jones Polynomial

Using the bracket polynomial we can define the *Jones polynomial*, $V(\mathcal{L}; q)$, of an oriented link $\mathcal{L}$ in $\mathbb{R}^3$. In fact, we can define the Jones polynomial using the diagram D of $\mathcal{L}$. The Jones polynomial is basically the bracket polynomial with a framing correction and renormalization.

Definition 3.2.1. Let $\omega(D) \in \mathbb{Z}$ be the sum of all signs over all crossing points of D. $\omega(D)$ is called the *writhe* of D.

Once we have this definition, we have the following theorem stated in [**14**] about computing the Jones polynomial of a link diagram.

THEOREM 3.2.2. *Let $|D|$ denote the number of crossing points of D. Then the Jones polynomial can be calculated from the diagram D as follows:*

$$V(\mathcal{L};q) = (-1)^{|D|+1} q^{3\omega(D)/2} \frac{\langle D\rangle(q)}{q+q^{-1}}. \tag{3.2.1}$$

While this is a reasonable way to define the Jones polynomial, it can seem overly complicated. There is a characterization like that of Theorem 2.3.1 for the Jones polynomial from [**14**]. Recall that the *Conway Triple* is the relation in Figure 2.2.

THEOREM 3.2.3. *There exists a unique function*

$$V : \{\textit{non-empty oriented links in } \mathbb{R}^3\} \to \mathbb{Z}[q,q^{-1}] \tag{3.2.2}$$

such that

(1) *if $\mathcal{L}$ is isotopic to $\mathcal{L}'$, then $V(\mathcal{L}) = V(\mathcal{L}')$*
(2) $V(\textit{unknot}) = 1$
(3) *for any Conway triple*

$$q^{-2}V(L_+) - q^2 V(L_-) = (q-q^{-1})V(L_0)$$

Furthermore, this unique function is the Jones polynomial.

As was discussed in the introduction, the Jones polynomial is more powerful than the Alexander polynomial, but it lacks any obvious topological interpretation in classical topology. Witten introduced a path integral topological interpretation in [**27**]. The Jones polynomial is an infinite dimensional integral over all $SU(2)$ connections over S^3 with a certain weight. We should note that if you compute this integral with the help of perturbation theory, then you get an expansion of the Jones polynomial in powers of $(q-1)$ or $\log(q)$. In the next chapter, we will be expanding the colored Jones polynomial in $h = \log(q)$.

3.3. The Quantum Group $U_q(\mathfrak{sl}(2))$

The colored Jones polynomial, $V_\alpha(\mathcal{K};q)$, is a generalized version of the Jones polynomial. Here α denotes coloring by an $(\alpha+1)$-dimensional representation of $U_q(\mathfrak{sl}(2))$. Note that $V_1(\mathcal{K};q) = V(\mathcal{K};q)$. The colored Jones polynomial can be calculated using R-matrices of quantum groups. Here we consider $U_q(\mathfrak{sl}(2))$, the Hopf algebra with generators E, F, and H satisfying:

$$[H,E]=2E, \quad [H,F]=-2F, \quad \text{and} \quad [E,F]=\frac{q^H-q^{-H}}{q-q^{-1}} \tag{3.3.1}$$

with comultiplication given by:

$$\begin{aligned} \Delta(E) &= E\otimes q^H + 1\otimes E \\ \Delta(F) &= F\otimes 1 + q^{-H}\otimes F \\ \Delta(H) &= H\otimes 1 + 1\otimes H \end{aligned} \tag{3.3.2}$$

This is a natural definition for the comultiplication of $U_q(\mathfrak{sl}(2))$. For a Lie algebra we usually have that

$$\Delta(x) = x\otimes 1 + 1\otimes x, \tag{3.3.3}$$

so that the comultiplication respects the commutator of the Lie algebra. This is also the case for our quantum group. The comultiplication is chosen in the above way so that it respects the commutator. For example, $[\Delta(H),\Delta(E)] = \Delta([H,E])$.

$U_q(\mathfrak{sl}(2))$ acts on tensor products, but because of the above definition of the comultiplication, permutation $P: V\otimes W \to W\otimes V$ which takes $v\otimes w \mapsto w\otimes v$ is not an $U_q(\mathfrak{sl}(2))$ intertwiner. However we do have many different intertwiners. We consider a special one that satisfies the Yang-Baxter equation and the braid relation. This intertwiner is called an R-matrix and is given as follows:

$$\mathcal{R} = q^{(H\otimes H)/2}\sum_{n=0}^{\infty} R_n(q)(E^n\otimes F^n), \tag{3.3.4}$$

where

$$R_n(h) = q^{n(n-1)/2}(q-q^{-1})^n([n]_q!)^{-1} \quad \text{and} \quad [n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}. \tag{3.3.5}$$

A good reference for this is [**15**].

3.4. Calculating the Colored Jones Polynomial

As previously stated, one can calculate the colored Jones polynomial using this R-matrix structure. This method was developed by Reshetikin and Turaev in [**22**]. First present the knot $\mathcal{K}$ as the closure of a braid β. Using Theorem 2.4.3, we know that any knot can be presented as a braid closure. To any braid β of n-strands, one can associate an action $\hat{\beta}$ on $W_\alpha^{\otimes n}$ where W_α is an $(\alpha+1)$-dimensional representation of $U_q(\mathfrak{sl}(2))$. To the elementary braid σ_i, we associate $\check{\mathcal{R}}$, the R-matrix composed with permutation, acting on $W_\alpha \otimes W_\alpha$ at the i^{th} and $(i+1)^{st}$ positions in $W_\alpha^{\otimes n}$ as in Figure 3.2 and described thoroughly in the introduction.

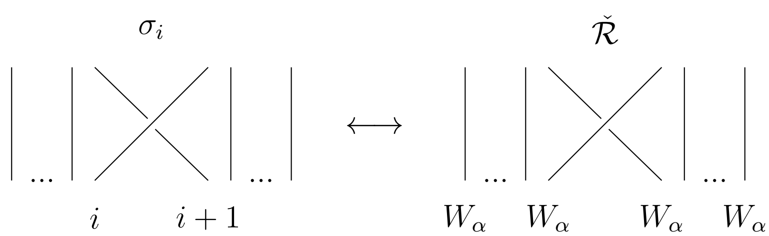


FIGURE 3.2. Associated Action for σ_i

We associate $\check{\mathcal{R}}^{-1}$, $\mathcal{R}^{-1}$ composed with permutation, to the negative elementary braid σ_i^{-1} as in Figure 3.3.

In this construction the action of braids commutes with the global action of $U_q(\mathfrak{sl}(2))$. Now we can compute the colored Jones polynomial using the following theorem.

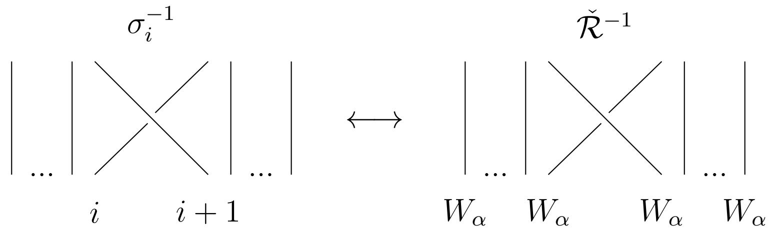


FIGURE 3.3. Associated Action for σ_i^{-1}

THEOREM 3.4.1. *Present a knot $\mathcal{K}$ as the closure of a braid β. Let $\hat{\beta}$ denote the representation of β acting on $W_\alpha^{\otimes n}$. Then the colored Jones polynomial can be calculated as the quantum trace of the braid, i.e.*

$$V_\alpha(\mathcal{K};q) = q^{fr}\, Tr_{W_\alpha^{\otimes n}}(q^{H/2})^{\otimes n}\hat{\beta},$$

where $(q^{H/2})^{\otimes n}$ denotes an operator that acts as $q^{H/2}$ on each module W_α of $W_\alpha^{\otimes n}$. fr is a framing correction:

$$fr = -1/4(\alpha^2 - 1)e(\beta), \tag{3.4.1}$$

where $e(\beta)$ equals the number of positive elementary braids minus the number of negative elementary braids in β.

3.5. The Melvin-Morton-Rozansky Expansion

In the 1990's a relationship between the Jones and Alexander polynomials was explored. Melvin and Morton made the following conjecture concerning the colored Jones polynomial in [**18**].

THEOREM 3.5.1. *Let $V_\alpha(\mathcal{K}) \in \mathbb{Q}(q)$ be the "framing independent colored Jones polynomial" of the knot $\mathcal{K}$, i.e. the framing independent Reshetikhin-Turaev invariant of $\mathcal{K}$ colored by the $(\alpha+1)$-dimensional representation of $\mathfrak{sl}(2)$. Let h be a formal parameter and let $q = e^h$. Then expanding $V_\alpha(\mathcal{K};e^h)$ in powers of α and h,*

$$V_\alpha(\mathcal{K};e^h) = \sum_{j,m\geq 0} a_{jm}(\mathcal{K})\alpha^j h^m,$$

we have:

(1) *"Above diagonal" coefficients vanish:* $a_{jm}(\mathcal{K}) = 0$, *if* $j > m$.

(2) *"On diagonal" coefficients give the inverse of the Alexander-Conway polynomial:*

$$MM(\mathcal{K})(h) \cdot \Delta_{\mathcal{K}}(q)(e^h) = 1, \tag{3.5.1}$$

where $\Delta_{\mathcal{K}}(q)$ *is the Alexander-Conway polynomial and* MM *is defined by*

$$MM(\mathcal{K})(h) = \sum_{m=0}^{\infty} a_{mm}(\mathcal{K}) h^m. \tag{3.5.2}$$

Here we note that this conjecture was proven by Bar-Natan and Garoufalidis in an equivalent form in [**4**]. This idea was studied further by Rozansky in [**24**]. Here we switch to the set up and notation of [**24**]. Let $\mathcal{K}$ be a knot in S^3. Associate an α-dimensional $U_q(\mathfrak{sl}(2))$-module to the knot. Let $V_\alpha(K; q)$ denote the reduced colored Jones polynomial of $\mathcal{K}$ normalized so that it is multiplicative under disconnected sums and $V_\alpha(\text{unknot}; q) = 1$. In this set up, an equivalent statement of Theorem 3.5.1 is as follows.

Theorem 3.5.2. *Let* $\mathcal{K} \subset S^3$ *be a knot and* $h = q - 1$. *We have the following expansion for the colored Jones polynomial:*

$$V_\alpha(\mathcal{K}; q) = \sum_{n \geq 0} h^n \left(\sum_{0 \leq m \leq n} D_{m,n}(K) \alpha^{2m} \right) \tag{3.5.3}$$

such that the coefficients $D_{m,n}$ *have the following properties:*

(1) $D_{m,n}(\mathcal{K}) = 0$ *for* $m > \dfrac{n}{2}$,

(2) $\displaystyle\sum_{m \geq 0} D_{m,2m}(K) a^{2m} = \frac{1}{\Delta_{\mathcal{K}}(t - t^{-1})}$,

where a *is a formal parameter,* $\Delta_{\mathcal{K}}$ *is the Alexander-Conway polynomial normalized so that* $\Delta_{unknot} = 1$, *and* $t = e^{i\pi a}$.

Note that the second part of the theorem says to sum along the first diagonal in Figure 3.4.

In fact, there is another theorem which involves summing along other diagonals which gives powers of the the Alexander polynomial in the denominator. This theorem is due to Rozansky [**24**]. In this theorem, we sum along the other diagonals in Figure 3.4.

THEOREM 3.5.3. *Let $\mathcal{K} \subset S^3$ be a knot and $h = q - 1$. We have the following expansion for the colored Jones polynomial:*

$$V_\alpha(\mathcal{K}; q) = \sum_{n \geq 0} h^n \left(\sum_{0 \leq m \leq n} D_{m,n}(\mathcal{K})(\alpha h)^{2m} \right) \tag{3.5.4}$$

such that the coefficients $D_{m,n+2m}$ have the following property:

$$\sum_{m \geq 0} D_{m,n+2m}(\mathcal{K})(\alpha h)^{2m} = \frac{P^{(n)}(\mathcal{K}; q^{\alpha/2} - q^{-\alpha/2})}{\Delta_{\mathcal{K}}^{2n+1}(q^{\alpha/2} - q^{-\alpha/2})}, \tag{3.5.5}$$

where $\Delta_{\mathcal{K}}$ is the Alexander-Conway polynomial normalized so that $\Delta_{unknot} = 1$and $P^{(n)}(\mathcal{K}; q^\alpha) \in \mathbb{Z}[q^\alpha, q^{-\alpha}]$ are invariants of the knot.

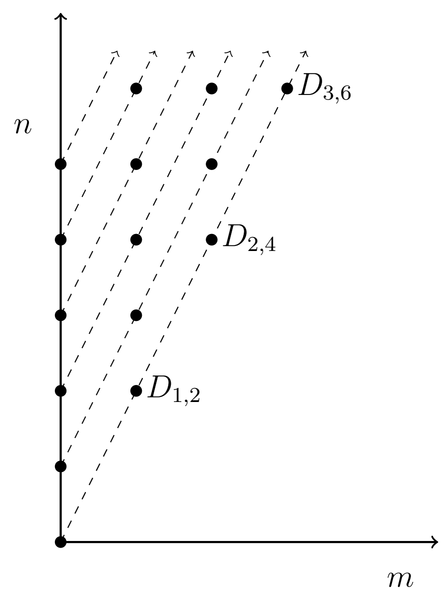


FIGURE 3.4. Summing along Diagonals

In the next chapters, we will describe our methods of calculating $P^{(1)}(\mathcal{K}; q^\alpha)$ and $P^{(2)}(\mathcal{K}; q^\alpha)$ and then present these polynomials for all prime knots up to nine crossings. We will also discuss an interesting property of the polynomials for amphicheiral knots.

An amphicheiral knot is one that is isotopic to its mirror image. The interesting property is given as a conjecture by Rozansky in [**23**] stated by expanding in a new parameter

$$\tilde{h} = (1+h)^{1/2} - (1+h)^{-1/2}. \tag{3.5.6}$$

Expanding $V_\alpha(\mathcal{K};q)$ in $\tilde{h}$, we have

$$V_\alpha(\mathcal{K};q) = \sum_{n=0}^{\infty} \tilde{V}^{(n)}(\mathcal{K};q^{\alpha/2}-q^{-\alpha/2})\tilde{h}^n. \tag{3.5.7}$$

Then the conjecture is as follows:

CONJECTURE 3.5.4. For an amphicheiral knot $\mathcal{K}$,

$$\tilde{V}^{(2n-1)}(\mathcal{K};q^{\alpha/2}-q^{-\alpha/2}) = 0 \tag{3.5.8}$$

and

$$\tilde{V}^{(2n)}(\mathcal{K}) = \frac{\tilde{P}^{(2n)}(K;q^{\alpha/2}-q^{-\alpha/2})}{\Delta_K^{3n+1}(q^{\alpha/2}-q^{-\alpha/2}))}. \tag{3.5.9}$$

for all $n \geq 1$.

Note that this conjecture says two things of importance. The first is that $P^{(2n-1)}(\mathcal{K};q^\alpha)$ vanishes for all $n \geq 1$ for amphicheiral knots. The second is that for $n \geq 1$, $P^{(2n)}(\mathcal{K};q^\alpha)$ is divisible by the powers of Alexander polynomial for amphicheiral knots. When we present our results in Chapter 6, we will provide evidence to the validity of this conjecture. We present the first two polynomials for all amphicheiral knots up to ten crossing and will show that the polynomials have the properties in Conjecture 3.5.4.

CHAPTER 4

Expansion of the R-matrix

4.1. The h-adic Hopf Algebra $U_h(\mathfrak{sl}(2))$

In this chapter we are concerned with expanding the universal R-matrix of $U_q(\mathfrak{sl}(2))$ in powers of $h = \log(q)$. First we discuss the generators of the $U_h(\mathfrak{sl}(2))$ themselves, then we move on to further techniques. We will calculate an expansion of each term in the R-matrix and then talk about how we can put the pieces together to get an expression of the form

$$\check{\mathcal{R}} = Q(z_1, z_2, \partial_{z_1}, \partial_{z_2})(1+h\, R_1(z_1, z_2, \partial_{z_1}, \partial_{z_2})+h^2\, R_2(z_1, z_2, \partial_{z_1}, \partial_{z_2})+\mathcal{O}(h^3)) \tag{4.1.1}$$

First we begin with a description of $U_h(\mathfrak{sl}(2))$.

$U_h(\mathfrak{sl}(2))$ is the h-adic Hopf algebra with generators E, F, and H satisfying:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad \text{and} \quad [E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}} \tag{4.1.2}$$

with comultiplication given by:

$$\begin{aligned} \Delta(E) &= E \otimes e^{hH} + 1 \otimes E \\ \Delta(F) &= F \otimes 1 + e^{-hH} \otimes F \\ \Delta(H) &= H \otimes 1 + 1 \otimes H \end{aligned} \tag{4.1.3}$$

The universal R-matrix is given by

$$\mathcal{R} = e^{h(H\otimes H)/2} \sum_{n=0}^{\infty} R_n(h)(E^n \otimes F^n), \tag{4.1.4}$$

where

$$R_n(h) = e^{hn(n+1)/2}(1-e^{-2h})^n([n]_q!)^{-1} \quad \text{and} \quad [n]_q = \frac{e^{hn}-e^{-hn}}{e^h-e^{-h}}. \tag{4.1.5}$$

Furthermore, $\mathcal{R}$ has an inverse given by

$$\mathcal{R}^{-1} = e^{-h(H\otimes H)/2}\sum_{n=0}^{\infty} e^{-hn}R_n(h)((e^{hH}E)^n \otimes (e^{hH}F)^n). \tag{4.1.6}$$

We denote the R-matrix composed with permutation by $\check{\mathcal{R}}$. Similarly, $\check{\mathcal{R}}^{-1}$ denotes the inverse R-matrix composed with permutation. For further discussion on this, see [**15**] or [**14**].

For the remainder of the chapter, we aim to prove a theorem about the expansion of the R-matrix. As discussed in the introduction, there is a family of algebra homomorphisms $f_\alpha : U_q(\mathfrak{sl}(2)) \mapsto \mathbb{C}[z,\partial_z]$ where $\alpha \in \mathbb{C}$ and $\mathbb{C}[z,\partial_z]$ is the Heisenberg algebra. Recall that its action on the standard generators E, H, and F is:

$$f_\alpha(E) = z, \;\; f_\alpha(H) = \alpha + 2z\partial_z, \;\; \text{and} \;\; f_\alpha(F) = g(q, q^\alpha, z\partial_z + 1), \tag{4.1.7}$$

for

$$g(q,t,x) = -\frac{(q^x - q^{-x})(t\,q^{x-1/2} - t^{-1}q^{-x+1/2})}{x(q-q^{-1})(q^{1/2}-q^{-1/2})} \in \mathbb{Z}[t][[h,x]]. \tag{4.1.8}$$

We will provide a proof of the action of f_α on F in Proposition 4.2.1 using the commutator relations on the generators E, H, and F. Using this family of homomorphisms, we can state our main theorem. As in the introduction, the Heisenberg algebra acts naturally on $\mathbb{C}[z]$, so f_α turns $\mathbb{C}[z]$ into a $U_q(\mathfrak{sl}(2))$-module denoted $\mathbb{C}_\alpha[z]$. If $\widetilde{W}_\alpha$ denotes the submodule over $\mathbb{C}_\alpha[z]$ generated by $z^{\alpha+1}$, then the quotient module

$$W_\alpha = \mathbb{C}_\alpha[z]/\widetilde{W}_\alpha \tag{4.1.9}$$

is the $(\alpha{+}1)$-dimensional irreducible representation of $U_q(\mathfrak{sl}(2))$ generated by $1, z, z^2, \ldots, z^\alpha$ with the action of E, F, and H given by Equation (4.1.7). Then the R-matrix acts on $W_\alpha \otimes W_\alpha$ and can be expanded in $h = \log(q)$ as in the following theorem.

Theorem 4.1.1. *Let $t = q^\alpha$ and $h = \log(q)$. Then $\check{\mathcal{R}}$ of $U_h(\mathfrak{sl}(2))$ can be written as*

(4.1.10)

$$\check{\mathcal{R}} = Q(z_1, z_2, \partial_{z_1}, \partial_{z_2})(1 + h\, R_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\, R_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3)),$$

with

$$Q(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \exp\left(-z_1\partial_{z_1} + tz_2\partial_{z_1} + tz_1\partial_{z_2} - t^2 z_2\partial_{z_2}\right), \tag{4.1.11}$$

$$\begin{aligned} R_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = {} & \left(-2t^2 - \frac{1}{t^2} + 3\right) z_1^2\partial_{z_2}^2 + \left(2t - \frac{2}{t}\right) z_1^2\partial_{z_1}\partial_{z_2} \\ & + \left(\frac{1}{t} - 3t\right) z_1 z_2\partial_{z_2}^2 + 2z_1z_2\partial_{z_1}\partial_{z_2}, \end{aligned} \tag{4.1.12}$$

and

(4.1.13)

$$\begin{aligned} R_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = {} & \left(4t^3 - \frac{16t}{3} + \frac{4}{3t}\right) z_1^3\partial_{z_2}^3 + \left(-4t^3 + \frac{2}{t^3} + 10t - \frac{8}{t}\right) z_1^4\partial_{z_2}^3\partial_{z_1} \\ & + \left(6t^3 - \frac{1}{t^3} - 11t + \frac{6}{t}\right) z_2z_1^3\partial_{z_2}^4 + \left(-4t^2 - \frac{1}{t^2} + 5\right) z_1^2\partial_{z_2}^2 \\ & + \left(4 - 4t^2\right) z_1^3\partial_{z_2}^2\partial_{z_1} + \left(2t^2 + \frac{2}{t^2} - 4\right) z_1^4\partial_{z_2}^2\partial_{z_1}^2 \\ & + \left(\frac{4t^2}{3} - \frac{2}{3t^2} + \frac{10}{3}\right) z_2z_1^2\partial_{z_2}^3 \\ & + \left(\frac{9t^2}{2} + \frac{1}{2t^2} - 3\right) z_2^2z_1^2\partial_{z_2}^4 + \left(-10t^2 - \frac{4}{t^2} + 14\right) z_2z_1^3\partial_{z_2}^3\partial_{z_1} \\ & + \left(2t^4 + \frac{1}{2t^4} - 6t^2 - \frac{3}{t^2} + \frac{13}{2}\right) z_1^4\partial_{z_2}^4 + \left(2t - \frac{2}{t}\right) z_1^2\partial_{z_2}\partial_{z_1} \\ & + \left(2t - \frac{2}{t}\right) z_1^3\partial_{z_2}\partial_{z_1}^2 + \left(\frac{1}{t} - 5t\right) z_2z_1\partial_{z_2}^2 + \left(\frac{2}{3t} - \frac{14t}{3}\right) z_2^2z_1\partial_{z_2}^3 \\ & + \left(-2t - \frac{2}{t}\right) z_2z_1^2\partial_{z_2}^2\partial_{z_1} + \left(\frac{2}{t} - 6t\right) z_2^2z_1^2\partial_{z_2}^3\partial_{z_1} + 2z_2z_1\partial_{z_2}\partial_{z_1} \\ & + \left(4t - \frac{4}{t}\right) z_2z_1^3\partial_{z_2}^2\partial_{z_1}^2 + 2z_2^2z_1\partial_{z_2}^2\partial_{z_1} + 2z_2z_1^2\partial_{z_2}\partial_{z_1}^2 + 2z_2^2z_1^2\partial_{z_2}^2\partial_{z_1}^2 \end{aligned}$$

Our key technique will appear over and over in this chapter. For each piece of the R-matrix, we will express it as a higher order term and an exponential of a bilinear form in z_1, z_2, ∂_{z_1} and ∂_{z_2}. When we say higher order term, we mean a term of the form

$$1 + h\, P_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\, P_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3). \tag{4.1.14}$$

For each portion of the R-matrix, we will have an exponential of a bilinear form in z's and ∂_z's and higher order terms. We will normal order each higher order term as we go along. An expression is normal ordered if all partial derivatives appear to the right of any multiplication by z. To complete our theorem, we will have to move a higher order term through the exponential of a bilinear form. The mechanisms of this moving will be discussed in Chapter 5, but it amounts to making linear substitutions in z_1, z_2, ∂_{z_1}, and ∂_{z_2}.

4.2. The Generators E, F, and H

We want to use quantum groups to calculate the colored Jones polynomial of a knot as in Chapter 3. However, we will do it in such a way that we will automatically get the expansion in powers of h in Theorem 3.5.3. The idea is outlined as follows. We will first represent the knot as a braid closure. Next we will look at the associated action of $\hat{\beta}$ on $W_\alpha^{\otimes n}$ as before. As explained in the introduction and the previous section, we consider W_α, which is the $(\alpha+1)$-dimensional irreducible representation of $U_q(\mathfrak{sl}(2))$ generated by $1, z, z^2, ..., z^\alpha$ with the action of E, F, and H as in Equation (4.1.7). Before expounding on this for $U_q(\mathfrak{sl}(2))$, let us recall the more familiar classical case.

Consider $\mathfrak{sl}(2)$ which has generators E, F, and H that satisfy the relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad \text{and} \quad [E, F] = H. \tag{4.2.1}$$

In this case, we have a family of homomorphisms $\tilde{f}_\alpha : \mathfrak{sl}(2) \to \mathbb{C}[z, \partial_z]$. We want to define its action on the generators E, F, and H. It is reasonable to make the following choices for E and H:

$$\tilde{f}_\alpha(E) = z \quad \text{and} \quad \tilde{f}_\alpha(H) = \alpha + 2z\partial_z. \tag{4.2.2}$$

Then

$$[\tilde{f}_\alpha(H), \tilde{f}_\alpha(E)] = [\alpha + 2z\partial_z, z] = 2z = 2\tilde{f}_\alpha(E) \tag{4.2.3}$$

as required. From here we can figure out what $\tilde{f}_\alpha(F)$ should be, using the relation $[E, F] = H$. Let us make an informed guess that $\tilde{f}_\alpha(F) = b\partial_z + c\partial_z^2$. After solving for the coefficients, we find that $b = -\alpha$ and $c = -z$, so

$$\tilde{f}_\alpha(F) = -\alpha\partial_z - z\partial_z^2. \tag{4.2.4}$$

With this choice of $\tilde{f}_\alpha(F)$, we have that $[\tilde{f}_\alpha(H), \tilde{f}_\alpha(F)] = -2\tilde{f}_\alpha(F)$, and $[\tilde{f}_\alpha(E), \tilde{f}_\alpha(F)] = \tilde{f}_\alpha(H)$ as required.

We wish to do a similar process with the generators of $U_q(\mathfrak{sl}(2))$. Now we will let $q = e^h$, $a = \alpha h$, and $t = q^\alpha = e^a$.

Proposition 4.2.1. *Let $t = q^\alpha$ and $h = \log(q)$. If*

$$f_\alpha(E) = z \quad \textit{and} \quad f_\alpha(H) = \alpha + 2z\partial_z, \tag{4.2.5}$$

then

$$\begin{aligned} f_\alpha(F) = &-\frac{t - t^{-1}}{2h}\partial_z - \frac{t + t^{-1}}{2}z\partial_z^2 - \frac{t - t^{-1}}{12}h(4z^2\partial_z^3 + 6z\partial_z^2 - \partial_z) \\ &- \frac{t + t^{-1}}{12}h^2(2z^3\partial_z^4 + 8z^2\partial_z^3 + 3z\partial_z^2) + \mathcal{O}(h^3) \end{aligned} \tag{4.2.6}$$

PROOF. We can expand F in powers of h by considering

$$
\begin{aligned}
FE^n|0\rangle &= \sum_{k=0}^{n-1} E^{n-k-1}[F,E]E^k|0\rangle \\
&= \sum_{k=0}^{n-1} E^{n-k-1}\left(-\frac{q^H - q^{-H}}{q-q^{-1}}\right)E^k|0\rangle \\
&= -\frac{1}{q-q^{-1}}\sum_{k=0}^{n-1}(tq^{2k} - t^{-1}q^{-2k})E^{n-1}|0\rangle \\
&= -\frac{1}{q-q^{-1}}\left(t\,\frac{q^{2n}-1}{q-1} - t^{-1}\,\frac{q^{-2n}-1}{q^{-1}-1}\right)E^{n-1}|0\rangle \\
&= -\frac{(q^n - q^{-n})(t\,q^{n-1/2} - t^{-1}q^{-n+1/2})}{(q-q^{-1})(q^{1/2}-q^{-1/2})}E^{n-1}|0\rangle
\end{aligned}
\tag{4.2.7}
$$

for $t = q^\alpha = e^a$.

Once we have this expression, we can expand

$$
-\frac{(q^n - q^{-n})(t\,q^{n-1/2} - t^{-1}q^{-n+1/2})}{(q-q^{-1})(q^{1/2}-q^{-1/2})}
\tag{4.2.8}
$$

in powers of h. Doing so, we arrive at the following

(4.2.9)

$$
\begin{aligned}
FE^n|0\rangle = \Big\{ &-\frac{t-t^{-1}}{2h}n + \frac{t+t^{-1}}{2}n(n-1) - \frac{t-t^{-1}}{12}\,h(4n(n-1)(n-2) + 6n(n-1) - n) \\
&-\frac{t+t^{-1}}{12}\,h^2(2n(n-1)(n-2)(n-3) + 8n(n-1)(n-2) + 3n(n-1)) + \mathcal{O}(h^3)\Big\}\,E^{n-1}|0\rangle
\end{aligned}
$$

Now we can write $f_\alpha(F)$ in powers of h as follows:

$$
\begin{aligned}
f_\alpha(F) = &-\frac{t-t^{-1}}{2h}\partial_z - \frac{t+t^{-1}}{2}z\partial_z^2 - \frac{t-t^{-1}}{12}\,h(4z^2\partial_z^3 + 6z\partial_z^2 - \partial_z) \\
&- \frac{t+t^{-1}}{12}\,h^2(2z^3\partial_z^4 + 8z^2\partial_z^3 + 3z\partial_z^2) + \mathcal{O}(h^3).
\end{aligned}
\tag{4.2.10}
$$

□

4.3. Expansion of $R_n(h)$

We are now ready to work on expanding the pieces of the R-matrix of $U_h(\mathfrak{sl}(2))$. First we concern ourselves with

$$R_n(h) = q^{n(n+1)/2}(1-q^{-2})^n([n]_q!)^{-1}. \tag{4.3.1}$$

Lemma 4.3.1. *We can expand $R_n(h)$ in powers of $h = \log(q)$ and get the first few terms as follows:*

(4.3.2)

$$\begin{aligned} R_n(h) &= q^{n(n+1)/2}(1-q^{-2})^n([n]_q!)^{-1} \\ &= e^{hn(n+1)/2}(1-e^{-2h})^n([n]_q!)^{-1} \\ &= \frac{(2h)^n}{n!}\Big(1 + \frac{h}{2}n(n-1) + \frac{h^2}{72}(9n(n-1)(n-2)(n-3) + 32n(n-1)(n-2) + 12n) \\ &+\mathcal{O}(h^3)\Big) \end{aligned}$$

Proof. In order to calculate the expansion, we take the natural log of some of the pieces in (4.3.2), work on each piece, then exponentiate to return to what we wish. First we have that

$$\ln(e^{hn(n+1)/2}) = h\left(\frac{1}{2}n(n+1)\right) + \mathcal{O}(h^2). \tag{4.3.3}$$

Next we expand

$$n(\ln(1-e^{-2h}) = n\left(\ln(2) + \ln(h) - h + \frac{h^2}{6} - \frac{h^4}{180} + \mathcal{O}(h^6)\right). \tag{4.3.4}$$

Now we work on an expansion of $[n]_q!$. Recall that

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{e^{hn} - e^{-hn}}{e^h - e^{-h}}. \tag{4.3.5}$$

Consider

$$\frac{e^{hk} - e^{-hk}}{e^h - e^{-h}} = k\left(1 + h^2\,\frac{k^2-1}{6} + h^4\,\frac{3k^4 - 10k^2 + 7}{360} + \mathcal{O}(h^6)\right). \tag{4.3.6}$$

Taking the natural log yields

$$\ln\left(\frac{e^{hk}-e^{-hk}}{e^h-e^{-h}}\right)=\ln(k)+h^2\,\frac{k^2-1}{6}+h^4\,\frac{k^4-1}{180}+\mathcal{O}(h^6). \tag{4.3.7}$$

In order to calculate $[n]_q!$, we sum (4.3.7) with k running from 1 to n. Doing so we obtain

(4.3.8)
$$\sum_{k=1}^{n}\ln\left(\frac{e^{hk}-e^{-hk}}{e^h-e^{-h}}\right)=\ln(n!)+h^2\,\frac{n(2n^2+3n-5)}{36}-h^4\,\frac{n(6n^4+15n^3+10n^2-31)}{5400}.$$

Finally, we exponentiate (4.3.3) + (4.3.4) − (4.3.8) and obtain that

(4.3.9)
$$\begin{aligned}R_n(h)=\frac{(2h)^n}{n!}\Bigg(1&+\frac{h}{2}n(n-1)\\&+\frac{h^2}{72}(9n(n-1)(n-2)(n-3)+32n(n-1)(n-2)+12n)+\mathcal{O}(h^3)\Bigg).\end{aligned}$$

□

4.4. Expansion of $E^n\otimes F^n$

Now that we have expanded $R_n(h)$, we must tackle $E^n\otimes F^n$. First we use the result of Lemma 4.3.9 to write

(4.4.1)
$$\begin{aligned}\sum_{n=0}^{\infty}R_n(h)E^n\otimes F^n&=\left(\sum_{n=0}^{\infty}\frac{(2h)^n}{n!}E^n\otimes F^n\right)\Bigg\{1+\frac{h}{2}(2hE\otimes F)^2+\frac{h^2}{2}\big(9(2hE\otimes F)^4\\&\quad+32(2hE\otimes F)^3+12((2hE\otimes F)\big)+\mathcal{O}(h^3)\Big\}\\&=\exp(2hE\otimes F)\Bigg\{1+\frac{h}{2}(2hE\otimes F)^2+\frac{h^2}{2}\big(9(2hE\otimes F)^4\\&\quad+32(2hE\otimes F)^3+12((2hE\otimes F)\big)+\mathcal{O}(h^3)\Big\}.\end{aligned}$$

Before proceeding further, we pause here and expand

(4.4.2)
$$1+\frac{h}{2}(2hE\otimes F)^2+\frac{h^2}{2}\big(9(2hE\otimes F)^4+32(2hE\otimes F)^3+12((2hE\otimes F)\big)+\mathcal{O}(h^3).$$

Lemma 4.4.1. *Let $T = t - t^{-1}$ and $S = t + t^{-1}$. The expansion in powers of h of Formula* (4.4.2) *is*

$$1 + h\, s_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\, s_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3), \tag{4.4.3}$$

where

$$s_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \frac{1}{2}T^2 z_1^2 \partial_{z_2}^2 \tag{4.4.4}$$

and

$$s_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \frac{1}{2}STz_1^2\partial_{z_2}^2 + STz_1^2 z_2 \partial_{z_2}^3 + \frac{1}{8}T^4 z_1^4 \partial_{z_2}^4 - \frac{4}{9}T^3 z_1^3 \partial_{z_2}^3 - \frac{1}{6}Tz_1\partial_{z_2}. \tag{4.4.5}$$

Proof. There is not much to do here. We simply expand (4.4.2) and normal order the terms based on the commutator relation

$$[\partial_{z_i}, z_j] = \delta_{ij}. \tag{4.4.6}$$

Doing so yields (4.4.3). □

4.5. Expansion of $\exp(2hE \otimes F)$

We are now ready to work on expanding each part of Equation (4.4.1). For each of these pieces, we will have an exponential of a bilinear form and a higher order part which consists of normal ordered polynomials in z_1, z_2, ∂_{z_1} and ∂_{z_2}. First we consider $\exp(2hE \otimes F)$. We wish to express this as

$$\exp(2hE\otimes F) = Q_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) \left(1 + hq_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2 q_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3)\right). \tag{4.5.1}$$

Proposition 4.5.1. *We can expand* $\exp(2hE \otimes F)$ *in powers of h and get that*

$$\exp(2hE\otimes F) = \exp(-Tz_1\partial_{z_2}) \left(1 + h\, q_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\, q_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3)\right), \tag{4.5.2}$$

with

$$Q_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \exp(-(t-t^{-1})z_1\partial_{z_2}), \tag{4.5.3}$$

$$q_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = -\frac{1}{2}STz_1^2\partial_{z_2}^2 - Sz_1z_2\partial_{z_2}^2, \tag{4.5.4}$$

and

(4.5.5)

$$\begin{aligned} q_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = {} & \frac{1}{6}Tz_1\partial_{z_2} - \frac{1}{2}T^2z_1^2\partial_{z_2}^2 \left(\frac{2S^2T}{3} - \frac{2T^3}{9}\right) z_1^3\partial_{z_2}^3 + \frac{1}{8}S^2T^2z_1^4\partial_{z_2}^4 \\ & + \left(S^2 - \frac{2T^2}{3}\right) z_1^2z_2\partial_{z_2}^3 + \frac{1}{2}S^2Tz_1^3z_2\partial_{z_2}^4 + \frac{1}{2}S^2z_1^2z_2^2\partial_{z_2}^4 \\ & - Tz_1z_2\partial_{z_2}^2 - \frac{2}{3}Tz_1z_2^2\partial_{z_2}^3. \end{aligned}$$

Proof. First recall that

$$f_\alpha(E_1) = z_1 \tag{4.5.6}$$

and

$$\begin{aligned} f_\alpha(F_2) = {} & -\frac{t-t^{-1}}{2h}\partial_{z_2} - \frac{t+t^{-1}}{2}z_2\partial_{z_2}^2 - \frac{t-t^{-1}}{12}\, h(4z_2^2\partial_{z_2}^3 + 6z_2\partial_{z_2}^2 - \partial_{z_2}) \\ & - \frac{t+t^{-1}}{12}\, h^2(2z_2^3\partial_{z_2}^4 + 8z_2^2\partial_{z_2}^3 + 3z_2\partial_{z_2}^2) + \mathcal{O}(h^3). \end{aligned} \tag{4.5.7}$$

Thus we can write

(4.5.8)

$$2hE_1\otimes F_2 = -(t-t^{-1})z_1\partial_{z_2} - \frac{t+t^{-1}}{h}\, z_1z_2\partial_{z_2}^2 - \frac{t-t^{-1}}{6}\, h^2z_1(4z_2^2\partial_{z_2}^3 + 6z_2\partial_{z_2}^2 - \partial_{z_2}) + \mathcal{O}(h^3).$$

Before finding the higher order part, let us consider the exponential term. We must get a normal ordered expression of $\exp(-(t-t^{-1})z_1\partial_{z_2})$, denoted $:\exp(-(t-t^{-1})z_1\partial_{z_2}):$. Because z_1 and ∂_{z_2} commute, there is nothing to be done here. We have that

$$\exp(-(t-t^{-1})z_1\partial_{z_2}) =: \exp(-(t-t^{-1})z_1\partial_{z_2}):. \tag{4.5.9}$$

Thus we have

$$Q_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \exp(-(t - t^{-1})z_1\partial_{z_2}). \tag{4.5.10}$$

For convenience, we set $T = t - t^{-1}$ and $S = t + t^{-1}$. Proceeding to the higher order term, we consider

$$\mathcal{U}(s) = e^{-As}e^{s(A+hB)} \tag{4.5.11}$$

where

$$A = -Tz_1\partial_{z_2} \tag{4.5.12}$$

and

$$B = -h(Sz_1z_2\partial_{z_2}^2) - h^2\left(\frac{2T}{3}z_1z_2^2\partial_{z_2}^3 + Tz_1z_2\partial_{z_2}^2 - \frac{T}{6}z_1\partial_{z_2}\right). \tag{4.5.13}$$

Since we are looking for an expansion in h, we write

$$\mathcal{U}(s) = 1 + h\,\mathcal{U}_1(s) + h^2\,\mathcal{U}_2(s) + \mathcal{O}(h^3) \tag{4.5.14}$$

Differentiating both (4.5.11) and (4.5.14) gives us that

$$\mathcal{U}_1'(s) = e^{-s\,ad_A}B, \tag{4.5.15}$$

where ad_A on B is the usual adjoint action $[A, B]$. Thus

$$\begin{aligned} \mathcal{U}_1(s) &= \left(\int_0^s e^{-\tau\,ad_A}d\tau\right)B \\ &= \frac{1 - e^{-s\,ad_A}}{ad_A}B. \end{aligned} \tag{4.5.16}$$

We can expand this in powers of ad_A and conclude that at the order of h we have the following

$$B + \frac{1}{2}[A, B] + \frac{1}{6}[A, [A, B]] - \frac{1}{24}[A[A[A, B]]], \tag{4.5.17}$$

which can be written as a normal ordered operator in terms of z_1, z_2, ∂_{z_1}, and ∂_{z_2} which gives the desired

$$P_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = -\frac{1}{2}STz_1^2\partial_{z_2}^2 - Sz_1z_2\partial_{z_2}^2. \tag{4.5.18}$$

Proceeding in a similar fashion, we see that

$$\mathcal{U}_2'(s) = e^{-s\,ad_{A_1}}\mathcal{U}_1(s) = e^{-s\,ad_{A_1}}\frac{1 - e^{-s\,ad_{A_2}}}{ad_{A_2}}(B, B). \tag{4.5.19}$$

Here we have introduced the notation A_i to mean that this copy of A acts on the i^{th} copy of B in the above expression. Again, we integrate and find that

$$\mathcal{U}_2(s) = \frac{1}{ad_{A_2}}\left(\frac{1 - e^{-s\,ad_{A_1}}}{ad_{A_1}} - \frac{1 - e^{-s(ad_{A_1} + ad_{A_2})}}{ad_{A_1} + ad_{A_2}}\right)(B, B). \tag{4.5.20}$$

Again expanding the above expression in ad_{A_i} and performing the adjoint action on the appropriate copies of B, we get the following,

(4.5.21)
$$\frac{1}{2}B^2 - \frac{1}{3}[A, B]\cdot B + \frac{1}{8}[A, [A, B]]\cdot B - \frac{1}{6}B\cdot[A, B] - \frac{1}{8}[A, B][A, B] + \frac{1}{24}B\cdot[A, [A, B]].$$

Once we compute the commutators and normal order the expression, we get the desired

(4.5.22)
$$\begin{aligned} P_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = {} & \frac{1}{6}Tz_1\partial_{z_2} - \frac{1}{2}T^2z_1^2\partial_{z_2}^2\left(\frac{2S^2T}{3} - \frac{2T^3}{9}\right)z_1^3\partial_{z_2}^3 + \frac{1}{8}S^2T^2z_1^4\partial_{z_2}^4 \\ & + \left(S^2 - \frac{2T^2}{3}\right)z_1^2z_2\partial_{z_2}^3 + \frac{1}{2}S^2Tz_1^3z_2\partial_{z_2}^4 + \frac{1}{2}S^2z_1^2z_2^2\partial_{z_2}^4 \\ & - Tz_1z_2\partial_{z_2}^2 - \frac{2}{3}Tz_1z_2^2\partial_{z_2}^3. \end{aligned}$$

□

4.6. Expansion of $\exp(h(H \otimes H)/2)$

Next we turn our attention to $\exp(h(H \otimes H)/2)$. As before, we wish to write this as an exponential of a bilinear form and a higher order piece. We have the following.

PROPOSITION 4.6.1. *For* $\exp(h(H \otimes H)/2)$ *we can write*

$$e^{h(H\otimes H)/2} = e^{a^2/2h} Q_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) \left(1 + h\, p_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\, p_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3)\right), \tag{4.6.1}$$

with

$$Q_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \exp((t-1)(z_1\partial_{z_1} + z_2\partial_{z_2})), \tag{4.6.2}$$

$$p_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = 2z_1 z_2 \partial_{z_1}\partial_{z_2}, \tag{4.6.3}$$

and

$$p_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = 2z_1 z_2 \partial_{z_1}\partial_{z_2} + z_1^2 z_2 \partial_{z_1}^2 \partial_{z_2} + 2z_1 z_2^2 \partial_{z_1}\partial_{z_2}^2 + 2z_1^2 z_2^2 \partial_{z_1}^2 \partial_{z_2}^2 \tag{4.6.4}$$

PROOF. Finding the expression in Equation (4.6.1) is a simple matter of expanding and normal ordering. The calculation is as follows

$$\begin{aligned} \exp(h(H \otimes H)/2) &= \exp\left(\frac{h}{2}\left(\frac{a}{h} + 2z_1\partial_{z_1}\right)\left(\frac{a}{h} + 2z_2\partial_{z_2}\right)\right) \\ &= \exp(a^2/2h)\exp(a(z_1\partial_{z_1} + z_2\partial_{z_2}))\exp(2hz_1z_2\partial_{z_1}\partial_{z_2}). \end{aligned} \tag{4.6.5}$$

Once we expand the third exponential in powers of h and normal order the coefficients, we arrive at Equations (4.6.3) and (4.6.4). Now we turn our attention to the exponential. For the moment, we will ignore $\exp(a^2/2h)$ as it will be taken care of in our final calculations as a framing factor. Now we must normal order the exponential. Normal ordering we get that

$$\exp(a(z_1\partial_{z_1} + z_2\partial_{z_2}) =: \exp((e^a - 1)(z_1\partial_{z_1} + z_2\partial_{z_2})): . \tag{4.6.6}$$

Thus we conclude that

$$Q_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \exp((t-1)(z_1\partial_{z_1} + z_2\partial_{z_2})). \tag{4.6.7}$$

□

In order to complete the proof of Theorem 4.1.1, we must understand how to compose the exponential terms and how to move a normal ordered operator through an exponential of a bilinear form. For a discussion on the mechanisms of composing and moving through exponentials of bilinear forms, see the next chapter. Now that we have the required expansions, we complete the proof of Theorem 4.1.1 here.

4.7. Proof of Theorem 4.1.1

Proof. Using the previous propositions and lemmas, we have expanded the pieces of the R-matrix into exponential and higher order terms. Let us take an inventory of what we have. For ease of notation we will often write p_1 for $p_1(z_1, z_2, \partial_{z_1}, \partial_{z_2})$, Q_1 for $Q_1(z_1, z_2, \partial_{z_1}, \partial_{z_2})$, and so on. From Proposition 4.5.1, we have written

$$\exp(2hE \otimes F) = Q_2 \left(1 + h\, q_1 + h^2\, q_2 + \mathcal{O}(h^3)\right). \tag{4.7.1}$$

From Lemma 4.4.1, we have written Equation (4.4.2) as

$$1 + h\, s_1 + h^2\, s_2 + \mathcal{O}(h^3). \tag{4.7.2}$$

Furthermore, using Proposition 4.6.1, we have expanded

$$e^{h(H \otimes H)/2} = Q_1 \left(1 + h\, p_1 + h^2\, p_2 + \mathcal{O}(h^3)\right). \tag{4.7.3}$$

Now we must appropriately combine these pieces and normal order the resulting higher order term. First let us combine (4.7.1) and (4.7.2) since all we need to do is multiply

$$\left(1 + h\, q_1 + h^2\, q_2 + \mathcal{O}(h^3)\right) \cdot \left(1 + h\, s_1 + h^2\, s_2 + \mathcal{O}(h^3)\right) \tag{4.7.4}$$

and normal order the results. Doing so gives us

$$N(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = 1 + h\, n_1 + h^2\, n_2 + \mathcal{O}(h^3), \tag{4.7.5}$$

where

$$n_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = \left(\frac{1}{t^2} - 1\right) z_1^2 \partial_{z_2}^2 + \left(-t - \frac{1}{t}\right) z_2 z_1 \partial_{z_2}^2 \tag{4.7.6}$$

and

$$\begin{aligned} n_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = {} & \left(\frac{2t^3}{3} - \frac{2}{3t^3} + 2t - \frac{2}{t}\right) z_1^3 \partial_{z_2}^3 + \left(t - \frac{1}{t^3}\right) z_2 z_1^3 \partial_{z_2}^4 \\ & + \left(1 - \frac{1}{t^2}\right) z_1^2 \partial_{z_2}^2 + \left(\frac{4t^2}{3} - \frac{2}{3t^2} + \frac{10}{3}\right) z_2 z_1^2 \partial_{z_2}^3 \\ & + \left(\frac{t^2}{2} + \frac{1}{2t^2} + 1\right) z_2^2 z_1^2 \partial_{z_2}^4 + \left(\frac{1}{2t^4} - \frac{1}{t^2} + \frac{1}{2}\right) z_1^4 \partial_{z_2}^4 \\ & + \left(\frac{1}{t} - t\right) z_2 z_1 \partial_{z_2}^2 + \left(\frac{2}{3t} - \frac{2t}{3}\right) z_2^2 z_1 \partial_{z_2}^3. \end{aligned} \tag{4.7.7}$$

In order to combine the higher order pieces from (4.7.3) and (4.7.5), we must move

$$1 + h\, p_1 + h^2\, p_2 + \mathcal{O}(h^3) \tag{4.7.8}$$

through the exponential Q_2 of (4.7.1). Using the results of Proposition 5.2.1, this amounts to making linear substitutions on z_1, z_2, ∂_{z_1}, and ∂_{z_2} as in equations (5.2.5) and (5.2.9). These substitutions are

$$z_1 \mapsto z_1, \quad z_2 \mapsto z_2 + (t - t^{-1}) z_1, \quad \partial_{z_1} \mapsto \partial_{z_1} - (t - t^{-1}) \partial_{z_2}, \text{ and } \partial_{z_2} \mapsto \partial_{z_2}. \tag{4.7.9}$$

Once we do this moving through and normal order the resulting higher order term, we arrive at

$$1 + h\, \tilde{p}_1 + h^2\, \tilde{p}_2 + \mathcal{O}(h^3), \tag{4.7.10}$$

where

$$\begin{aligned} \tilde{p}_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = {} & \left(-2t^2 - \frac{2}{t^2} + 4\right) z_1^2 \partial_{z_2}^2 + \left(2t - \frac{2}{t}\right) z_1^2 \partial_{z_2} \partial_{z_1} \\ & + \left(\frac{2}{t} - 2t\right) z_2 z_1 \partial_{z_2}^2 + 2 z_2 z_1 \partial_{z_2} \partial_{z_1} \end{aligned} \tag{4.7.11}$$

and

(4.7.12)

$$\begin{aligned}\tilde{p}_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) = &\left(-4t^3 + \frac{4}{t^3} + 12t - \frac{12}{t}\right) z_1^4 \partial_{z_2}^3 \partial_{z_1} + \left(4t^3 - \frac{4}{t^3} - 12t + \frac{12}{t}\right) z_2 z_1^3 \partial_{z_2}^4 \\
&- 2t^2 z_2 z_1^2 \partial_{z_2}^3 + 2t^2 z_2^2 z_1^2 \partial_{z_2}^4 + \left(-2t^2 - \frac{2}{t^2} + 4\right) z_1^2 \partial_{z_2}^2 \\
&+ \left(-2t^2 - \frac{2}{t^2} + 4\right) z_1^3 \partial_{z_2}^2 \partial_{z_1} + \left(2t^2 + \frac{2}{t^2} - 4\right) z_1^4 \partial_{z_2}^2 \partial_{z_1}^2 \\
&+ \left(-8t^2 - \frac{8}{t^2} + 16\right) z_2 z_1^3 \partial_{z_2}^3 \partial_{z_1} - \frac{2 z_2 z_1^2 \partial_{z_2}^3}{t^2} + \frac{2 z_2^2 z_1^2 \partial_{z_2}^4}{t^2} \\
&+ \left(2t^4 + \frac{2}{t^4} - 8t^2 - \frac{8}{t^2} + 12\right) z_1^4 \partial_{z_2}^4 + \left(2t - \frac{2}{t}\right) z_1^2 \partial_{z_2} \partial_{z_1} \\
&+ \left(2t - \frac{2}{t}\right) z_1^3 \partial_{z_2} \partial_{z_1}^2 + \left(\frac{2}{t} - 2t\right) z_2 z_1 \partial_{z_2}^2 + \left(\frac{2}{t} - 2t\right) z_2^2 z_1 \partial_{z_2}^3 \\
&+ \left(\frac{4}{t} - 4t\right) z_2^2 z_1^2 \partial_{z_2}^3 \partial_{z_1} + \left(4t - \frac{4}{t}\right) z_2 z_1^3 \partial_{z_2}^2 \partial_{z_1}^2 + 4 z_2 z_1^2 \partial_{z_2}^3 \\
&- 4 z_2^2 z_1^2 \partial_{z_2}^4 + 2 z_2 z_1 \partial_{z_2} \partial_{z_1} + 2 z_2^2 z_1 \partial_{z_2}^2 \partial_{z_1} + 2 z_2 z_1^2 \partial_{z_2} \partial_{z_1}^2 \\
&+ 2 z_2^2 z_1^2 \partial_{z_2}^2 \partial_{z_1}^2 .\end{aligned}$$

Only one more step remains to get the higher order term associated to $\check{\mathcal{R}}$. We must multiply and normal order

$$\left(1 + h\,\tilde{p}_1 + h^2\,\tilde{p}_2 + \mathcal{O}(h^3)\right) \cdot \left(1 + h\,n_1 + h^2\,n_2 + \mathcal{O}(h^3)\right). \tag{4.7.13}$$

Doing so yields

$$1 + h\,R_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\,R_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3), \tag{4.7.14}$$

with R_1 and R_2 given by Equations (4.1.12) and (4.1.13) respectively.

To conclude, we must prove that $Q(z_1, z_2, \partial_{z_1}, \partial_{z_2})$ is given by Formula (4.1.11). To establish this, we will show that the action on z_1 and z_2 of the composition of Q_1, Q_2, and permutation P sending $z_1 \leftrightarrow z_2$ is the same as the action of Q on z_1 and z_2.

Let us consider

(4.7.15)

$$
\begin{aligned}
(P \circ Q_1 \circ Q_2) \cdot z_1 &= P \circ \exp((t^{-1} - t) z_1 \partial_{z_2}) \circ \exp((t-1)(z_1 \partial_{z_1} + z_2 \partial_{z_2})) \cdot z_1 \\
&= P \circ \exp((t^{-1} - t) z_1 \partial_{z_2}) \cdot (t z_1) \\
&= P \cdot (t z_1) \\
&= t z_2.
\end{aligned}
$$

Similarly the action on z_2 is

(4.7.16)

$$
\begin{aligned}
(P \circ Q_1 \circ Q_2) \cdot z_2 &= P \circ \exp((t^{-1} - t) z_1 \partial_{z_2}) \circ \exp((t-1)(z_1 \partial_{z_1} + z_2 \partial_{z_2})) \cdot z_2 \\
&= P \circ \exp((t^{-1} - t) z_1 \partial_{z_2}) \cdot (t z_2) \\
&= P \cdot (t z_2 + (1 - t^2) z_2) \\
&= t z_1 + (1 - t^2) z_2.
\end{aligned}
$$

Now we consider the action of Q on both z_1 and z_2. We have

$$
Q \cdot z_1 = \exp(-z_1 \partial_{z_1} + t z_2 \partial_{z_1} + t z_1 \partial_{z_2} - t^2 z_2 \partial_{z_2}) \cdot z_1 = t z_2 \tag{4.7.17}
$$

and

$$
Q \cdot z_2 = \exp(-z_1 \partial_{z_1} + t z_2 \partial_{z_1} + t z_1 \partial_{z_2} - t^2 z_2 \partial_{z_2}) \cdot z_1 = t z_1 + (1 - t^2) z_2. \tag{4.7.18}
$$

Since the action of Q is determined by its action on z_1 and z_2, we have established the Q given in Equation (4.1.11) is the exponential bilinear form in the expansion of $\mathcal{R}$. Hence we have completed the proof of Theorem 4.1.1. $\square$

We have a corollary of Theorem 4.1.1 involving the inverse R-matrix composed with permutation, $\check{\mathcal{R}}^{-1}$.

Corollary 4.7.1. *The inverse of the R-matrix can be written as*

(4.7.19)

$$
\check{\mathcal{R}}^{-1} = M(z_1, z_2, \partial_{z_1}, \partial_{z_2})(1 + h\, N_1(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + h^2\, N_2(z_1, z_2, \partial_{z_1}, \partial_{z_2}) + \mathcal{O}(h^3)),
$$

where

$$M(z_1,z_2,\partial_{z_1},\partial_{z_2}) = \exp\left(\frac{1}{t}z_1\partial_{z_2} + \frac{1}{t}z_2\partial_{z_1} - \frac{1}{t^2}z_1\partial_{z_1} - z_2\partial_{z_2}\right), \tag{4.7.20}$$

$$N_1(z_1,z_2,\partial_{z_1},\partial_{z_2}) = \left(1-t^2\right) z_2^2\partial_{z_1}^2 + \left(t+\frac{1}{t}\right) z_2z_1\partial_{z_1}^2 - 2z_2z_1\partial_{z_2}\partial_{z_1}, \tag{4.7.21}$$

and

(4.7.22)

$$\begin{aligned}
N_2(z_1,z_2,\partial_{z_1},\partial_{z_2}) = &\left(\frac{1}{t}-t^3\right) z_2^3z_1\partial_{z_1}^4 + \left(1-t^2\right) z_2^2\partial_{z_1}^2 + \left(4t^2-4\right) z_2^3\partial_{z_2}\partial_{z_1}^2 \\
&+ \left(-\frac{2t^2}{3}+\frac{4}{3t^2}+\frac{10}{3}\right) z_2^2z_1\partial_{z_1}^3 + \left(\frac{t^2}{2}+\frac{1}{2t^2}+1\right) z_2^2z_1^2\partial_{z_1}^4 \\
&+ \left(2t^2-2\right) z_2^3z_1\partial_{z_2}\partial_{z_1}^3 + \left(\frac{t^4}{2}-t^2+\frac{1}{2}\right) z_2^4\partial_{z_1}^4 \\
&+ \left(\frac{8}{3t}-\frac{8t}{3}\right) z_2^3\partial_{z_1}^3 + \left(t-\frac{1}{t}\right) z_2z_1\partial_{z_1}^2 + \left(\frac{2t}{3}-\frac{2}{3t}\right) z_2z_1^2\partial_{z_1}^3 \\
&+ \left(-4t-\frac{4}{t}\right) z_2^2z_1\partial_{z_2}\partial_{z_1}^2 + \left(-2t-\frac{2}{t}\right) z_2^2z_1^2\partial_{z_2}\partial_{z_1}^3 \\
&+ 2z_2z_1\partial_{z_2}\partial_{z_1} + 2z_2^2z_1\partial_{z_2}^2\partial_{z_1} + 2z_2z_1^2\partial_{z_2}\partial_{z_1}^2 + 2z_2^2z_1^2\partial_{z_2}^2\partial_{z_1}^2.
\end{aligned}$$

PROOF. Since we have an expression for $\check{\mathcal{R}}^{-1}$ in terms of the generators E, F, and H, we could prove this corollary in a similar fashion as Theorem 4.1.1. However we will use the expansion of $\check{\mathcal{R}}$ to prove the corollary. As before, we will often write N_1 for $N_1(z_1,z_2,\partial_{z_1},\partial_{z_2})$, M for $M(z_1,z_2,\bar{z}_1,\bar{z}_2)$, and so on. First note that using Theorem 4.1.1

$$\check{\mathcal{R}}^{-1} = \left(\frac{1}{1+h\,R_1+h^2\,R_2+\mathcal{O}(h^3)}\right)Q^{-1}. \tag{4.7.23}$$

Hence to find the higher order part of $\check{\mathcal{R}}^{-1}$, we must expand

$$\left(\frac{1}{1+h\,R_1+h^2\,R_2+\mathcal{O}(h^3)}\right) \tag{4.7.24}$$

and then move it through Q^{-1}. We expand (4.7.24) in powers of h in the usual way,

$$\left(\frac{1}{1+h\,R_1+h^2\,R_2}\right) = 1 - h\,R_1 + h^2(R_1^2 - R_2) + \mathcal{O}(h^3). \tag{4.7.25}$$

The inverse of Q is indeed M as given by Equation (4.7.20). We can establish this by investigating the action of $M \circ Q = Q \circ M$ on z_1 and z_2. We have that

$$(M \circ Q) \cdot z_1 = \exp\left(\frac{1}{t} z_1 \partial_{z_2} + \frac{1}{t} z_2 \partial_{z_1} - \frac{1}{t^2} z_1 \partial_{z_1} - z_2 \partial_{z_2}\right) \cdot (t z_2) = z_1 \tag{4.7.26}$$

and

(4.7.27)

$$(M \circ Q) \cdot z_2 = \exp\left(\frac{1}{t} z_1 \partial_{z_2} + \frac{1}{t} z_2 \partial_{z_1} - \frac{1}{t^2} z_1 \partial_{z_1} - z_2 \partial_{z_2}\right) \cdot (t z_1 + (1-t^2) z_2) = z_2.$$

Similarly,

$$(Q \circ M) \cdot z_1 = \exp(-z_1 \partial_{z_1} + t z_2 \partial_{z_1} + t z_1 \partial_{z_2} - t^2 z_2 \partial_{z_2}) \cdot \left(z_1 + \frac{1}{t} z_2 - \frac{1}{t^2} z_1\right) = z_1 \tag{4.7.28}$$

and

$$(Q \circ M) \cdot z_2 = \exp(-z_1 \partial_{z_1} + t z_2 \partial_{z_1} + t z_1 \partial_{z_2} - t^2 z_2 \partial_{z_2}) \cdot \left(\frac{1}{t} z_1\right) = z_2. \tag{4.7.29}$$

Since $M \circ Q = Q \circ M = \mathbb{1}$, $M = Q^{-1}$ and we have the correct formula for M.

Once we move (4.7.24) through (4.7.20) using Proposition 5.2.1 and normal order the result, we arrive at (4.7.19). $\square$

CHAPTER 5

Composing Operators and Taking the Trace

5.1. Outline of Calculation

In this chapter, we aim to describe the remaining techniques used to calculate the perturbative expansion of the colored Jones polynomial. It will often be important to distinguish when we are discussing an operator acting on $\mathbb{C}[\underline{z}]$ versus when we are discussing variables. To make this distinction, we will use a "hat" to denote an operator, i.e. $\hat{z}$ denotes an operator which acts as multiplication by z, while z simply denotes a variable z.

Recall how we can calculate the colored Jones polynomial. Given a knot, $\mathcal{K}$, we present it as a braid, β. As described in Chapter 2, this braid is made up of elementary braids or crossings. To the positive crossing σ_i, we associate the expansion of $\check{\mathcal{R}}$ acting on the i^{th} and $(i+1)^{st}$ copies of W_α in $W_\alpha^{\otimes n}$. Similarly, to the negative crossing σ_i^{-1}, we associate the expansion of the $\check{\mathcal{R}}^{-1}$ acting on the i^{th} and $(i+1)^{st}$ copies of W_α in $W_\alpha^{\otimes n}$. Hence at each crossing, we have an expansion of the R-matrix of $U_q(\mathfrak{sl}(2))$ given by Theorem 4.1.1. Recall that for each of these expansions, we have an exponential of a bilinear form $\hat{Q}_i$ and a higher order term $\widehat{HO}_i$. So for each σ_i we have an associated

$$\sigma_i \mapsto \hat{Q}_i \circ \widehat{HO}_i \tag{5.1.1}$$

The associated action of the braid β as a whole has the form

$$\hat{\beta} = \widehat{\sigma_{i_k}...\sigma_{i_1}} = \hat{\sigma}_{i_k}...\hat{\sigma}_{i_1} \mapsto \hat{Q}_{i_k} \circ \widehat{HO}_{i_k}...\hat{Q}_{i_1} \circ \widehat{HO}_{i_1} \tag{5.1.2}$$

In order to eventually take the trace of this associated action, we must write $\hat{\beta} = \hat{Q}_\beta \circ \widehat{HO}_\beta$. To do this, we start at the top of the braid and compose the action of the associated R-matrices. At each step of the composition, we have a $\hat{Q}_{i_j}$ and a $\widehat{HO}_{i_j}$. Then we also have an exponential and higher order term from the composition of the previous crossings. Call these $\hat{Q}_\lambda$ and $\widehat{HO}_\lambda$ for $\lambda = \sigma_{i_k} ... \sigma_{i_{j+1}}$. The first step in our composition calculation is to move $\widehat{HO}_\lambda$ through $\hat{Q}_{i_j}$ as in Figure 5.1. This amounts to making linear substitutions as described in Proposition 5.2.1 in Section 5.2. Once we have passed $\widehat{HO}_\lambda$ through $\hat{Q}_{i_j}$, we must compose $\hat{Q}_\lambda$ and $\hat{Q}_{i_j}$. This composition will be described in Proposition 5.3.1 of Section 5.3. We then multiply $\widehat{HO}_\lambda$ (after passing through $\hat{Q}_{i_j}$) and $\widehat{HO}_{i_j}$ together and normal order the result. The method of normal ordering will be discussed in Section 5.4

$$\boxed{\hat{Q}_\lambda} \circ \boxed{\widehat{HO}_\lambda} \qquad \boxed{\hat{Q}_{i_j}} \circ \boxed{\widehat{HO}_{i_j}}$$

FIGURE 5.1. Moving $\widehat{HO}_\lambda$ through $\hat{Q}_{i_j}$

Once we have composed all the associated actions of the crossings, we must place $q^{H/2}$ at the bottom of each strand except for the first as described in Chapter 3. So we must compose $\mathbb{1} \otimes q^{H_2/2} \otimes ... \otimes q^{H_n/2}$ with $\hat{Q}_\beta$ and $\widehat{HO}_\beta$. The mechanisms of this composition are the same as described in the previous paragraph.

To complete our calculation of $P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$, we must take the reduced trace of what we have calculated so far. To accomplish this, we will make use of the holomorphic representation discussed in Section 5.5 and then apply the trace procedure described in Section 5.7. As explained in the introduction, we will close all of the strands except the first and calculate the reduced trace on these. Once the trace is taken, we have the expansion in $h = \log(q)$

$$1 + h\left(\frac{P^{(1)}(\mathcal{K};t)}{\Delta^3_{\mathcal{K}}(t)}\right) + h^2\left(\frac{P^{(2)}(\mathcal{K};t)}{\Delta^5_{\mathcal{K}}(t)}\right) + \mathcal{O}(h^3). \tag{5.1.3}$$

Here we must pause and explain that our h and t must be renormalized for our polynomials to coincide with the polynomials previously calculated for a few torus knots by Rozansky in [**23**]. Let us denote the parameters used in [**23**] by a tilde, i.e. $\tilde{q}$ indicates q used by Rozansky in [**23**]. Rozansky expanded in $\tilde{h} = \tilde{q} - 1$ while we expanded in $h = \log(q)$. Furthermore $\tilde{q} = q^2$, so we have that

$$h = \log(\sqrt{1+\tilde{h}}) = \frac{\tilde{h}}{2} - \frac{h^2}{4} + \mathcal{O}(h^3). \tag{5.1.4}$$

Also we have that $t = \tilde{t}e^h$. Thus as a final step in our calculation, we make the appropriate substitutions and expand in $\tilde{h}$. Then we have $P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$ as given by Theorem 3.5.3.

5.2. Commuting Polynomial Operators and Quadratic Exponentials

Now that we have expressed all the pieces of the R-matrix as exponentials of bilinear forms and normal ordered operators, we must understand how to move an operator through an exponential of a bilinear form and how to compose such exponentials. Here we discuss the former. Please note that for ease of notation, we will often drop the limits of a summation, i.e. we will write $\sum_i$ instead of $\sum_{i=1}^{n}$. We also denote $z_1, ..., z_n$ by $\underline{z}$ and $\partial_{z_1}, ..., \partial_{z_n}$ by $\underline{\partial}_z$. It is well-known (for example, see [**10**]) that an exponential of a bilinear form,

$$\hat{K}_{\underline{z},\underline{w}} = \exp\left(\sum_{i,j} A_{ij} z_i \partial_{w_j}\right) \tag{5.2.1}$$

is an algebra homomorphism $\mathbb{C}[\underline{w}] \to \mathbb{C}[\underline{z}]$. We must consider how to move an operator through $\hat{K}_{\underline{z},\underline{w}}$. Thus we are asking what are $\widehat{p(\underline{z})}$ and $\widehat{r(\underline{\partial}_z)}$ in the equations

$$\hat{K}_{\underline{z},\underline{w}} \circ \hat{w}_k = \widehat{p(\underline{z})} \circ \hat{K}_{\underline{z},\underline{w}} \tag{5.2.2}$$

and

$$\hat{K}_{\underline{z},\underline{w}} \circ \partial w_k = \widehat{r(\underline{\partial}_z)} \circ \hat{K}_{\underline{z},\underline{w}} \tag{5.2.3}$$

Recall one of the reasons we must know how to do this; in Chapter 4 we have expanded the pieces of the $U_q(\mathfrak{sl}(2))$ R-matrix and have an exponential term and higher order term for each. We wish to combine these into a single quadratic kernel and higher order term.

Proposition 5.2.1. *Given an exponential of a bilinear form,*

$$\hat{K}_{\underline{z},\underline{w}} = \exp\left(\sum_{i,j} A_{ij} z_i \partial_{w_j}\right), \tag{5.2.4}$$

we have that

$$\hat{K}_{\underline{z},\underline{w}} \circ \hat{w}_k = \left(\sum_i A_{ik} z_i\right) \hat{K}_{\underline{z},\underline{w}} \tag{5.2.5}$$

Thus when moving $\hat{w}_k$ through the above exponential from right to left,

$$\hat{w}_k \mapsto \sum_i A_{ik} z_i. \tag{5.2.6}$$

If we are moving the operator from left to right, we use A^{-1}.

Proof. This is a consequence of the fact that $\hat{K}_{\underline{z},\underline{w}}$ is an algebra homomorphism. Let $\psi(\underline{w})$ be a test function in $\mathbb{C}[\underline{w}]$ and consider $(\hat{K}_{\underline{z},\underline{w}} \circ \hat{w}_k) \cdot \psi(\underline{w})$. We have that

$$\begin{aligned}
(\hat{K}_{\underline{z},\underline{w}} \circ \hat{w}_k) \cdot \psi(\underline{w}) &= \hat{K}_{\underline{z},\underline{w}}(w_k \psi(\underline{w})) \\
&= \hat{K}_{\underline{z},\underline{w}}(w_k) \circ \hat{K}_{\underline{z},\underline{w}}(\psi(\underline{w})) \\
&= \left(\sum_i A_{ik} z_i\right) \hat{K}_{\underline{z},\underline{w}}(\psi(\underline{w})).
\end{aligned} \tag{5.2.7}$$

So the overall effect when moving multiplication by w_k right to left through $\hat{K}_{\underline{z},\underline{w}}$ is $w_k \mapsto \sum_i A_{ik} z_i$. □

Now we wish to find the same relationship for partial derivatives.

Proposition 5.2.2. *Given an exponential of a bilinear form,*

$$\hat{K}_{\underline{z},\underline{w}} = \exp\left(\sum_{i,j} A_{ij} z_i \partial_{w_j}\right), \tag{5.2.8}$$

we have that

$$\partial_{z_m} \circ \hat{K}_{\underline{z},\underline{w}} = \hat{K}_{\underline{z},\underline{w}} \circ \left(\sum_k A_{mk} \partial_{w_k} \right) \tag{5.2.9}$$

Thus when moving ∂_{z_k} through the above exponential from left to right,

$$\partial_{z_m} \mapsto \sum_k A_{mk} \partial_{w_k}. \tag{5.2.10}$$

Proof. Let $\psi(\underline{w}) \in \mathbb{C}[\underline{w}]$ and consider $(\partial_{z_m} \circ \hat{K}_{\underline{z},\underline{w}}) \cdot \psi(\underline{w})$. We know how $\hat{K}_{\underline{z},\underline{w}}$ acts on $\psi(\underline{w})$; it sends $w_k \mapsto \sum_i A_{ik} z_i$. So we have

$$(\partial_{z_m} \circ \hat{K}_{\underline{z},\underline{w}}) \cdot \psi(\underline{w}) = \partial_{z_k} \psi \left(w_k = \sum_i A_{ik} z_i \right). \tag{5.2.11}$$

Applying the chain rule, we get that

$$\partial_{z_w} \psi \left(w_k = \sum_i A_{ik} z_i \right) = \sum_k \left(\frac{\partial \psi}{\partial w_k} \Big|_{w_k = \sum_i A_{ik}} \frac{\partial w_k}{\partial z_m} \right), \tag{5.2.12}$$

but

$$\frac{\partial w_k}{\partial z_m} = A_{mk} \quad \text{and} \quad \frac{\partial \psi}{\partial w_k} \bigg|_{w_k = \sum_i A_{ik}} = \hat{K}_{\underline{z},\underline{w}} \circ \partial_{w_k}. \tag{5.2.13}$$

So overall we have that

$$(\partial_{z_m} \circ \hat{K}_{\underline{z},\underline{w}}) \cdot \psi(\underline{w}) = \hat{K}_{\underline{z},\underline{w}} \circ \left(\sum_k A_{mk} \partial_{w_k} \right) \psi(\underline{w}). \tag{5.2.14}$$

□

We use these relationships whenever we wish to move an operator through an exponential of a bilinear form. There are two key places that we use these transformations. First, when we were finding an expansion of the R-matrix in terms of an exponential and a higher order operator, and second, whenever we are composing the crossings of a braid representation of a knot.

5.3. Composing Exponentials of Bilinear Forms

Since exponentials of bilinear forms are algebra homomorphisms, it is to be expected that they compose in the natural way. Here we show that this is the case.

PROPOSITION 5.3.1. *Given two algebra homomorphisms*

$$\hat{K}_{1;\underline{z},\underline{w}} : \mathbb{C}[\underline{z}] \to \mathbb{C}[\underline{w}] \quad \textit{and} \quad \hat{K}_{2;\underline{w},\underline{u}} : \mathbb{C}[\underline{w}] \to \mathbb{C}[\underline{u}] \tag{5.3.1}$$

given by

$$\hat{K}_{1;\underline{z},\underline{w}} = \exp\left(\sum_{i,j} A_{ij} z_i \partial_{w_j}\right) \quad \textit{and} \quad \hat{K}_{2;\underline{w},\underline{u}} = \exp\left(\sum_{j,k} B_{jk} w_j \partial_{u_k}\right) \tag{5.3.2}$$

their composition

$$\hat{K}_{1;\underline{z},\underline{w}} \circ \hat{K}_{2;\underline{w},\underline{u}} : \mathbb{C}[\underline{u}] \to \mathbb{C}[\underline{z}] \tag{5.3.3}$$

is given by

$$\hat{K}_{1;\underline{z},\underline{w}} \circ \hat{K}_{2;\underline{w},\underline{u}} = \exp\left(\sum_{i,j,k} A_{ij} B_{jk} z_i \partial_{u_k}\right). \tag{5.3.4}$$

PROOF. We will establish this by finding the action of $\hat{K}_{1;\underline{z},\underline{w}} \circ \hat{K}_{2;\underline{w},\underline{u}}$ on $u_l \in \mathbb{C}[\underline{u}]$ for a fixed l with $1 \leq l \leq n$. Consider the action of the composition on u_l:

$$\begin{aligned}
(\hat{K}_{1;\underline{z},\underline{w}} \circ \hat{K}_{2;\underline{w},\underline{u}}) \cdot u_k &= \exp\left(\sum_{i,j} A_{ij} z_i \partial_{w_j}\right) \circ \exp\left(\sum_{j,k} B_{jk} w_j \partial_{u_k}\right) \cdot u_l \\
&= \exp\left(\sum_{i,j} A_{ij} z_i \partial_{w_j}\right) \cdot \left(\sum_{j} B_{jl} w_j\right) \\
&= \sum_{i,j} A_{ij} B_{jl} z_i.
\end{aligned} \tag{5.3.5}$$

We compare this to the action of $\hat{K}_{\underline{z},\underline{u}} = \exp\left(\sum_{i,j,k} A_{ij}B_{jk}z_i\partial_{u_k}\right)$ on u_l:

$$\hat{K}_{\underline{z},\underline{u}} \cdot u_l = \exp\left(\sum_{i,j,k} A_{ij}B_{jk}z_i\partial_{u_k}\right) \cdot u_l = \sum_{ij} A_{ij}B_{jl}z_i. \tag{5.3.6}$$

Since the action of the algebra homomorphisms is determined by its action on the subspace of $\mathbb{C}[\underline{u}]$ generated by $\{u_k : 1 \leq u \leq n\}$ and the actions coincide on this subspace, we have that

$$\hat{K}_{1;\underline{z},\underline{w}} \circ \hat{K}_{2;\underline{w},\underline{u}} = \exp\left(\sum_{i,j,k} A_{ij}B_{jk}z_i\partial_{u_k}\right). \tag{5.3.7}$$

$\square$

So now whenever we wish to compose two quadratic exponentials given by matrices A and B, we simply use their matrix product AB. We note that this is indeed what one would expect since $\hat{K}_{1;\underline{z},\underline{\bar{w}}}$ and $\hat{K}_{2;\underline{w},\underline{\bar{u}}}$ are algebra homomorphisms.

5.4. Normal Ordering by Wick's Theorem

In order to take the trace using the holomorphic representation, all of the operators that we encounter must be normal ordered. A differential operator is normal ordered when all of the multiplication by any z_i appears to the left of any partial derivatives. We use $\hat{z}_i$ to indicate the operator that acts as multiplication by z_i:

$$\hat{z}_i\psi(\underline{z}) = z_i\psi(\underline{z}). \tag{5.4.1}$$

Recall that we are working with polynomial operators in $\underline{z}$ and $\underline{\partial}_z$ and they satisfy the relation

$$[\partial_{z_i}, \hat{z}_j] = \delta_{ij}. \tag{5.4.2}$$

In addition to this relationship, all of the partial derivatives commute with each other as do each of the $\hat{z}_i$'s:

$$\begin{aligned} \left[\partial_{z_i}, \partial_{z_j}\right] &= 0, \\ \left[\hat{z}_i, \hat{z}_j\right] &= 0. \end{aligned} \tag{5.4.3}$$

If we encounter $\hat{z}_i$ to the right of ∂_{z_i}, we can use the relation in (5.4.2) to write

$$\partial_{z_i}\hat{z}_i = 1 + \hat{z}_i\partial_{z_i}. \tag{5.4.4}$$

We aim to be able to use this relationship to efficiently normal order our operators in the Mathematica program. To do this, we use Wick's Theorem of normal ordering. A discussion of this theorem can be found in many texts including [**6**] and [**19**]. For a historical perspective, this theorem was proven by Wick in 1950 in [**26**] as a way to normal order creation and annihilation operators in quantum field theory. In this language, we call $\hat{z}_i$ creation and ∂_{z_i} annihilation. In order to state the theorem, we must discuss the contraction.

Let $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, and $\hat{\mathcal{O}}$ denote operators containing creation and annihilation operators. Then we make the following definitions.

Definition 5.4.1. *An operator $\hat{\mathcal{O}}$ is said to be normal ordered when all of the annihilation operators appear to the right of the creation operators. We denote a normal ordered operator by* $:\hat{\mathcal{O}}:$

Definition 5.4.2. *For two operators, $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, we define their contraction, $\overbracket{\hat{\mathcal{A}}\hat{\mathcal{B}}}$, as*

$$\overbracket{\hat{\mathcal{A}}\hat{\mathcal{B}}} = \hat{\mathcal{A}}\hat{\mathcal{B}} - :\hat{\mathcal{A}}\hat{\mathcal{B}}: \tag{5.4.5}$$

Using our new definition of contraction, we get the following concerning the contractions of creation and annihilation operators.

$$
\begin{aligned}
\overbracket{\partial_{z_i}\hat{z}_j} &= \partial_{z_i}\hat{z}_j - :\partial_{z_i}\hat{z}_j: \ = \delta_{ij}, \\
\overbracket{\hat{z}_i\partial_{z_j}} &= \hat{z}_i\partial_{z_j} - :\hat{z}_i\partial_{z_j}: \ = 0, \\
\overbracket{\hat{z}_i\hat{z}_j} &= \hat{z}_i\hat{z}_j - :\hat{z}_i\hat{z}_j: \ = 0, \\
\overbracket{\partial_{z_i}\partial_{z_j}} &= \partial_{z_i}\partial_{z_j} - :\partial_{z_i}\partial_{z_j}: \ = 0.
\end{aligned}
\tag{5.4.6}
$$

Now that we have the necessary definitions, we are ready to state Wick's theorem regarding normal ordering operators. The content of the theorem is the following: given a string of creation and annihilation operators, the string can be rewritten as the normal-ordered product of the string plus the normal-ordered sum of all possible contractions.

Theorem 5.4.3. *The ordinary product of linear operators $\hat{\mathcal{A}}_1\ldots\hat{\mathcal{A}}_n$ can be written as*

(5.4.7)

$$\hat{\mathcal{A}}_1\ldots\hat{\mathcal{A}}_n. =: \hat{\mathcal{A}}_1\ldots\hat{\mathcal{A}}_n: + \sum_{i\neq j} : \hat{\mathcal{A}}_1\ldots\overbracket{\hat{\mathcal{A}}_i\ldots\hat{\mathcal{A}}_j}\ldots\hat{\mathcal{A}}_n: + \sum_{i,j,k,l} : \hat{\mathcal{A}}_1\ldots\wick{\c1{\hat{\mathcal{A}}}_i\ldots\c2{\hat{\mathcal{A}}}_j\ldots\c1{\hat{\mathcal{A}}}_k\ldots\c2{\hat{\mathcal{A}}}_l}\ldots\hat{\mathcal{A}}_n: + \ldots$$

Whenever we need to normal order a higher order term or a product of such terms, we apply Wick's theorem to efficiently perform this action.

5.5. The Heisenberg Algebra and the Holomorphic Representation

Throughout the remainder of this chapter, we will be using the holomorphic representation to facilitate the trace and other calculations. A good reference on this subject is [**10**] and much of the following discussion is based on that text. We consider the Heisenberg algebra denoted $\mathcal{H}_{\underline{\hat{z}},\underline{\partial}_z}$ which is

$$\mathcal{H}_{\underline{\hat{z}},\underline{\partial}_z} = \mathbb{C}[\underline{\hat{z}}, \underline{\partial}_z]/([\partial_{z_j}, \hat{z}_i] = \delta_{ij}) \tag{5.5.1}$$

and a module $\mathbb{C}[\underline{z}]$ over it. Recall that an operator is said to be *normal ordered* if all partial derivatives appear to the right of any $\hat{z}_i$. We can construct a linear map

$$\mathcal{F} : \mathcal{H}_{\underline{\hat{z}},\underline{\partial}_z} \to \mathbb{C}[[\underline{z}, \underline{\bar{z}}]] \tag{5.5.2}$$

such that for every $\psi(\underline{z}) \in \mathbb{C}[\underline{z}]$ and normal ordered operator $\hat{\mathcal{O}} \in \mathcal{H}_{\underline{\hat{z}},\underline{\partial}_z}$

$$\hat{\mathcal{O}}(\psi(\underline{z})) = \int \mathcal{F}[\hat{\mathcal{O}}](\underline{z}, \underline{\bar{w}}) \exp\left(-\sum_i w_i \bar{w}_i\right) \psi(\underline{w}) \frac{d^2\underline{w}}{2\pi i}. \tag{5.5.3}$$

We will refer to $K(z, \bar{z}) = \mathcal{F}[\mathcal{O}](z, \bar{z})$ as *kernels* and $\mathcal{O}$ as *operators*. As described in [**10**], we have the following theorem concerning our linear map $\mathcal{F}$ which allows us to move freely between kernels and operators:

Theorem 5.5.1. *Let*

$$\hat{\mathcal{O}} = \sum_{m,n} A_{mn} z^m \partial_z^n \tag{5.5.4}$$

be a normal ordered operator in $\mathcal{H}_{\underline{\hat{z}},\underline{\partial}_z}$. *Then* $\mathcal{F}[\hat{\mathcal{O}}](\underline{z}, \underline{\bar{z}})$ *is given by the formula*

$$K(\underline{z}, \underline{\bar{z}}) = \mathcal{F}[\hat{\mathcal{O}}](\underline{z}, \underline{\bar{z}}) = \exp\left(\sum_i z_i \bar{z}_i\right) \sum_{m,n} A_{mn} z^m \bar{z}^n. \tag{5.5.5}$$

We will make use of this theorem repeatedly throughout the remainder of this chapter.

For us, the kernels we will be dealing with will involve exponential functions of bilinear forms. We will call these *quadratic kernels*. When integrating a kernel of this form, we are integrating what is called a *Gaussian integral*. In each of our calculations, we will make use of a well-known property of integrals of this form: the integral is equal to the value of the integrand at the extremum point of the exponent of the exponential function, up to a constant that is included in the definition of the integration measure [**10**].

5.6. The Kernel Associated to $\hat{\beta}$

Once we have an associated action $\hat{\beta}$ to a braid β as discussed in the previous chapters, we must take the trace of this action. Recall that we will have two pieces after composing the crossings of the braid representation of a knot and placing $q^{H/2}$ at the bottom of each strand except the first. We will have an exponential term and a higher order term. We must first get the kernel of the composition of these two, and then we will be ready to take the trace of this action using the holomorphic representation.

For the matrix A associated to the exponential piece of the braid calculation, the kernel is given by

$$K_Q(\underline{z}, \underline{\bar{z}}) = \exp\left(\sum_{i,m} A_{im} z_i \bar{z}_m\right). \tag{5.6.1}$$

The higher order piece of the braid calculation is given by a normal ordered expression $\widehat{HO}_\beta = P(\underline{z}, \underline{\partial}_z)$. Using Theorem 5.5.1, the kernel associated to this is given by

$$K_{HO}(\underline{z}, \underline{\bar{z}}) = \mathcal{F}[\widehat{HO}_\beta](\underline{z}, \underline{\bar{z}}) = \exp\left(\sum_{i} z_i \bar{z}_i\right) P(\underline{z}, \underline{\bar{z}}). \tag{5.6.2}$$

Now we must compose these and encode this action into our program.

Proposition 5.6.1. *For the associated action of β together with $\mathbb{1} \otimes (q^{H/2})^{\otimes(n-1)}$, we have an exponential of a bilinear form, $\hat{Q}_\beta$, and higher order terms, $\widehat{HO}_\beta = P(\underline{z}, \underline{\partial}_z)$. In the holomorphic representation, the kernel associated to their composition is:*

$$\mathcal{F}[\hat{Q}_\beta \circ \widehat{HO}_\beta] = \exp\left(\sum_{j,l} A_{lj} w_l \bar{u}_j\right) P\left(\sum_{l} \underline{A_{li} w_l}, \underline{\bar{u}}\right). \tag{5.6.3}$$

Proof. We must calculate

(5.6.4)

$$\mathcal{F}[\hat{Q}_\beta \circ \widehat{HO}_\beta] = \int \exp\left(\sum_i z_i \bar{u}_i - \sum_j z_j \bar{z}_j + \sum_{k,l} A_{lk} w_l \bar{z}_k\right) P(\underline{z}, \underline{\bar{u}}) d^2\underline{z}$$

$$= \int \exp\left(-\sum_j (\bar{z}_j - \bar{u}_j)(z_j - \sum_l A_{lj} w_l) + \sum_{j,l} A_{lj} w_l \bar{u}_j\right) P(\underline{z}, \underline{\bar{u}}) d^2\underline{z}.$$

Using the property of the Gaussian integral this calculation yields

$$\exp\left(\sum_{j,l} A_{lj} w_l \bar{u}_j\right) P\left(\sum_l \underline{A_{li} w_l}, \underline{\bar{u}}\right). \tag{5.6.5}$$

$\square$

5.7. Taking the Trace

In order to complete our calculation, we must take the reduced trace of the $\hat{Q}_\beta \circ \widehat{HO}_\beta$. Given any $\hat{K}$, we can take its trace using the holomorphic representation:

$$\mathrm{Tr}^{(1)} \hat{K} = \int \exp\left(-\sum_{i=2}^{n} z_i \bar{z}_i\right) \mathcal{F}[\hat{K}](\underline{z}, \underline{\bar{z}})\, d^2\underline{z}^{(1)}, \tag{5.7.1}$$

where $d^2\underline{z}^{(1)} = d^2 z_2 ... d^2 z_n$ normalized appropriately.

We wish to take the reduced trace of an exponential of a bilinear form $\hat{Q}_\beta$ together with a normal ordered operator $\widehat{HO}_\beta$. We will describe how to take the reduced trace of $\hat{Q}_\beta$ together with a general monomial from $\widehat{HO}_\beta$ after the substitution described in Proposition 5.6.1. This description can be extended linearly to all of $\widehat{HO}_\beta$.

Consider $\hat{\mathcal{O}}_{\underline{m},\underline{r}} =: \prod_{i=1}^n z_i^{m_i} \prod_{j=1}^n \partial_{z_j}^{r_j} :$. We know that the kernel associated to this operator is

$$\mathcal{O}_{\underline{m},\underline{r}} = \mathcal{F}[\hat{\mathcal{O}}_{\underline{m},\underline{r}}] = \exp\left(\sum_i z_i \bar{w}_i\right) \prod_{i,j=1}^{n} z_i^{m_i} \bar{w}_j^{r_j}. \tag{5.7.2}$$

In order to calculate the reduced trace of $\hat{Q}_\beta \circ \hat{\mathcal{O}}_{\underline{m},\underline{r}}$, we need to calculate

(5.7.3)
$$\int \exp\left(\sum_{i,j=2}^{n}(A_{ij}z_i\bar{z}_j - \delta_{ij}) + A_{11}z_1\bar{z}_1 + \sum_{k=2}^{n} A_{1k}z_1\bar{z}_k + \sum_{l=2}^{n} z_l\bar{z}_1\right)\prod_{i,j=1}^{n} z_i^{m_1}\bar{z}_j^{r_j}\, d^2\underline{z}^{(1)}.$$

We can calculate this by introducing $Q_{ij} = \delta_{ij} - A_{ij}$ for $2 \le i, j \le n$. Then we have that the above integral (5.7.3) is

(5.7.4)
$$e^{A_{11}z_1\bar{z}_1}\int \exp\left(\sum_{i,j=2}^{n} -Q_{ij}\left(z_i - \sum_{k=2}^{n} Q_{ki}^{-1}A_{1k}z_1\right)\left(\bar{z}_j - \sum_{l=2}^{n} Q_{jl}^{-1}A_{l1}\bar{z}_1\right) + \sum_{k,l=2}^{n} z_1A_{1k}Q_{kl}^{-1}A_{l1}\bar{z}_1\right)$$
$$\prod_{i,j=1}^{n} z_i^{m_1}\bar{z}_j^{r_j}\, d^2\underline{z}^{(1)}.$$

By defining $z_i' = z_i - \sum_{k=2}^{n} Q_{ki}^{-1}A_{1k}z_1$ and $\bar{z}_j' = \bar{z}_j - \sum_{l=2}^{n} Q_{jl}^{-1}A_{l1}\bar{z}_1$, the integral becomes

(5.7.5)
$$D\int \exp\left(-\sum_{i,j=2}^{n} Q_{ij}z_i'\bar{z}_j'\right)\prod_{i,j=2}^{n}\left(z_i' + \sum_{k=2}^{n} Q_{ki}^{-1}A_{1k}z_1\right)^{m_i}\left(\bar{z}_j' + \sum_{l=2}^{n} Q_{jl}^{-1}A_{l1}\bar{z}_1\right)^{r_j} d^2\underline{z}'^{(1)},$$

where $D = \exp\left(A_{11}z_1\bar{z}_1 + \sum_{k,l=2}^{n} z_1A_{1k}Q_{kl}^{-1}A_{l1}\bar{z}_1\right)z_1\bar{z}_1$. So now to complete the calculation of $\mathrm{Tr}^{(1)}(\hat{Q}_\beta \circ \hat{\mathcal{O}}_{\underline{m},\underline{r}})$, we are concerned with calculating the integral

$$\int \exp\left(-\sum_{i,j=2}^{n} Q_{ij}z_i'\bar{z}_j'\right)\prod_{i,j} z_i'^{m_i}\bar{z}_j'^{r_j}\, d^2\underline{z}'^{(1)}, \tag{5.7.6}$$

where $Q_{ij} = \delta_{ij} - A_{ij}$ for $2 \le i, j \le n$. To calculate this integral, we use a technique that is common in quantum field theory. For a reference, see [**20**]. As stressed in the introduction, we are performing this technique in the rigorous case with a finite number of variables. We introduce parameters

$$\underline{\epsilon} = (\epsilon_2, ..., \epsilon_n) \quad \text{and} \quad \underline{\bar{\epsilon}} = (\bar{\epsilon}_2, ..., \bar{\epsilon}_n) \tag{5.7.7}$$

and then we note that

$$\prod_{i,j} z_i'^{m_i} \bar{z}_j'^{r_j} = \prod_{i,j} \partial_{\epsilon_i}^{m_i} \partial_{\epsilon_j}^{r_j} \left(\exp\left(\sum_{i=2}^{n} \epsilon_i z_i' + \sum_{j=2}^{n} \bar{\epsilon}_j \bar{z}_j' \right) \right) \Big|_{\underline{\epsilon}=\bar{\underline{\epsilon}}=0}. \tag{5.7.8}$$

Hence if we define $Z(\underline{\epsilon}, \bar{\underline{\epsilon}})$ as

$$Z(\underline{\epsilon}, \bar{\underline{\epsilon}}) = \int \exp\left(-\sum_{i,j=2}^{n} Q_{ij} z_i' \bar{z}_j' + \sum_{i=2}^{n} \epsilon_i z_i' + \sum_{j=2}^{n} \bar{\epsilon}_j \bar{z}_j' \right) d^2 \underline{z}'^{(1)}, \tag{5.7.9}$$

then the integral (5.7.6) is equal to

$$\prod_{i,j} \partial_{\epsilon_i}^{m_i} \partial_{\bar{\epsilon}_j}^{r_j} \left(Z(\underline{\epsilon}, \bar{\underline{\epsilon}}) \right) \Big|_{\underline{\epsilon}=\bar{\underline{\epsilon}}=0}. \tag{5.7.10}$$

We can use our usual technique to calculate the integral $Z(\underline{\epsilon}, \bar{\underline{\epsilon}})$:

$$\begin{aligned} Z(\underline{\epsilon}, \bar{\underline{\epsilon}}) &= \int \exp\left(-\sum_{i,j=2}^{n} Q_{ij} z_i' \bar{z}_j' + \sum_{i=2}^{n} \epsilon_i z_i' + \sum_{j=2}^{n} \bar{\epsilon}_j \bar{z}_j' \right) d^2 \underline{z}'^{(1)} \\ &= \int \exp\left(-\sum_{i,j=2}^{n} Q_{ij} \left(z_i' - \sum_k Q_{ki}^{-1} \bar{\epsilon}_k \right) \left(\bar{z}_j' - \sum_l Q_{jl}^{-1} \epsilon_l \right) + \sum_{i,j,k,l=2}^{n} Q_{ij} Q_{ki}^{-1} Q_{jl}^{-1} \bar{\epsilon}_k \epsilon_l \right) d^2 \underline{z}'^{(1)} \\ &= \exp\left(\sum_{k,l=2}^{n} Q_{kl}^{-1} \bar{\epsilon}_k \epsilon_l \right). \end{aligned} \tag{5.7.11}$$

So now in order to calculate the reduced trace, we must find

$$\prod_{i,j} \partial_{\epsilon_i}^{m_i} \partial_{\bar{\epsilon}_j}^{r_j} \left(\exp\left(\sum_{k,l=2}^{n} Q_{kl}^{-1} \bar{\epsilon}_k \epsilon_l \right) \right) \Big|_{\underline{\epsilon}=\bar{\underline{\epsilon}}=0}. \tag{5.7.12}$$

To calculate Formula (5.7.12), we consider the N^{th} term of the formal power series expansion of $Z(\underline{\epsilon}, \bar{\underline{\epsilon}})$

$$T_N = \frac{1}{N!} \left(\sum_{k,l=2}^{n} Q_{kl}^{-1} \bar{\epsilon}_k \epsilon_l \right)^N, \tag{5.7.13}$$

where $N = \sum_{i=2}^{n} m_i = \sum_{j=2}^{n} r_j$. Furthermore, we will only need to consider the monomials in T_N of the form

$$\epsilon_2^{m_2}\,\epsilon_3^{m_3}\,...\,\epsilon_n^{m_n}\,\bar{\epsilon}_2^{r_2}\,\bar{\epsilon}_3^{r_3}\,...\,\bar{\epsilon}_n^{r_n} \tag{5.7.14}$$

because these are the only ones that will survive the partial derivatives and then setting $\underline{\epsilon} = \underline{\bar{\epsilon}} = 0$. Indeed in the one-dimensional case

$$\partial_\epsilon^m \epsilon^r\Big|_{\epsilon=0} = \begin{cases} m! & \text{if } m = r \\ 0 & \text{if } m \neq r \end{cases} . \tag{5.7.15}$$

So we see for the monomial $\hat{\mathcal{O}}_{\underline{m},\underline{r}} =: \prod_i z_i^{m_i} \prod_j \partial_{z_j}^{r_j} :$, the contributing term to the reduced trace is

$$\frac{m_2!m_3!...m_n!r_2!...r_n!}{N!}\, C_{\underline{m},\underline{r}}, \tag{5.7.16}$$

where $C_{\underline{m},\underline{r}}$ is the coefficient of $\epsilon_2^{m_2}\epsilon_3^{m_3}...\epsilon_n^{m_n}\bar{\epsilon}_2^{r_2}\bar{\epsilon}_3^{r_3}...\bar{\epsilon}_n^{r_n}$ in the formal series expansion of $Z(\underline{\epsilon}, \underline{\bar{\epsilon}}) = \exp\left(\sum_{k,l=2}^{n} Q_{kl}^{-1}\bar{\epsilon}_k\epsilon_l\right)$ and $N = \sum_i m_i = \sum_j r_j$. We perform this procedure for each monomial in $\widehat{HO}_\beta$ and sum the results. This gives us the reduced trace.

Now that we have taken the reduced trace, we have arrived at the perturbative expansion of the colored Jones polynomial. From this we get $P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$. In the next and final chapter, we present the results of this work. We have calculated the first two polynomials for all prime knots of up to nine crossings and for all amphicheiral knots of up to ten crossings.

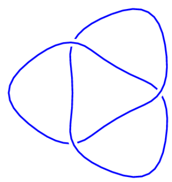
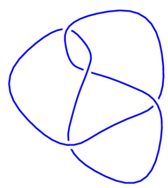
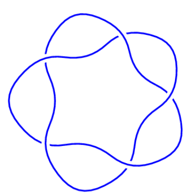
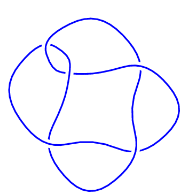
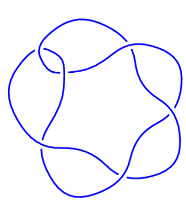
CHAPTER 6

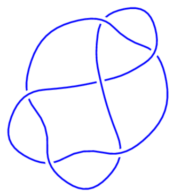
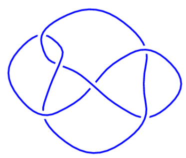
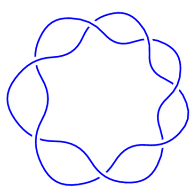
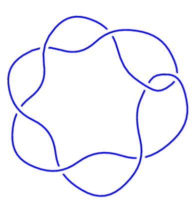
Results and Evidence Supporting the Conjecture

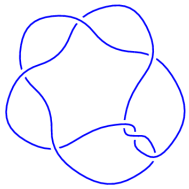
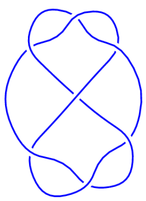
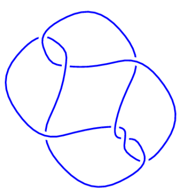
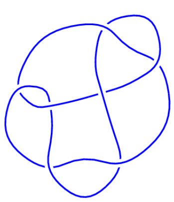
6.1. Results

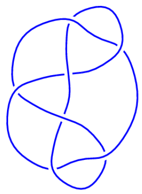
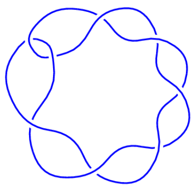
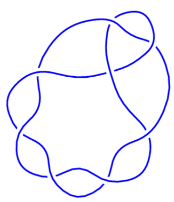
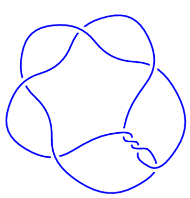
Now that we have explained the mechanisms by which we calculate these polynomials, we are ready to demonstrate and discuss our results. So far, we have calculated $P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$ for all prime knots up to nine crossings. In addition, we have calculated the polynomials for all amphicheiral knots of up to 10 crossings. Our calculations provide evidence to the validity of a conjecture concerning these polynomials for amphicheiral knots. As we continue to run our program, we will be able to calculate and study these polynomials for knots of ten crossings and so on. After we present these polynomials, we will discuss how our results verify the conjecture from [**23**] about amphicheiral knots. For our table, we have used the knot diagrams from Knot Info [**8**].

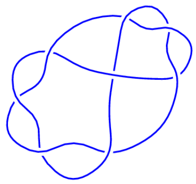
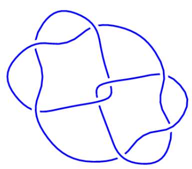
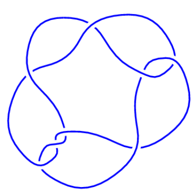
Table 6.1: $P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$ for 3_1 through 9_{49}

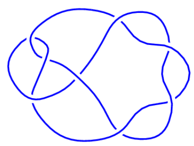
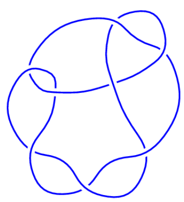
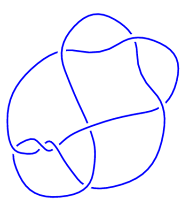
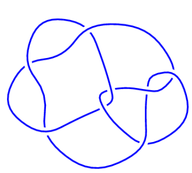
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
3_1		$P^{(1)}(\mathcal{K};t) = t^{-4} - 2t^{-2} + 2 - 2t^2 + t^4$ $P^{(2)}(\mathcal{K};t) = -t^{-4} + t^{-2} + 1 + t^2 - t^4$
4_1		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = t^{-4} - 4t^{-2} + 5 - 4t^2 + t^4$
5_1		$P^{(1)}(\mathcal{K};t) = 2t^{-8} - 4t^{-6} + 5t^{-4} - 6t^{-2} + 6 - 6t^2 + 5t^4$ $- 4t^6 + 2t^8$ $P^{(2)}(\mathcal{K};t) = t^{-16} - 4t^{-14} + 8t^{-12} - 12t^{-10} + 13t^{-8} - 18t^{-6}$ $+ 32t^{-4} - 49t^{-2} + 61 - 49t^2 + 32t^4 - 18t^6 + 13t^8$ $- 12t^{10} + 8t^{12} - 4t^{14} + t^{16}$
5_2		$P^{(1)}(\mathcal{K};t) = -5t^{-4} + 14t^{-2} - 18 + 14t^2 - 5t^4$ $P^{(2)}(\mathcal{K};t) = 23t^{-8} - 130t^{-6} + 356t^{-4} - 629t^{-2} + 762 - 629t^2$ $+ 356t^4 - 130t^6 + 23t^8$
6_1		$P^{(1)}(\mathcal{K};t) = t^{-4} - 6t^{-2} + 10 - 6t^2 + t^4$ $P^{(2)}(\mathcal{K};t) = -t^{-8} + 10t^{-6} - 32t^{-4} + 57t^{-2} - 70 + 57t^2 - 32t^4$ $+ 10t^6 - t^8$

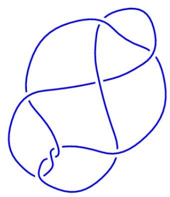
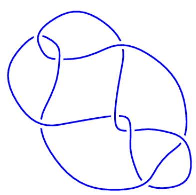
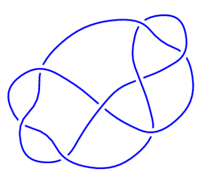
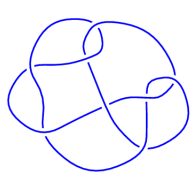
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
6_2		$P^{(1)}(\mathcal{K};t) = t^{-8} - 6t^{-6} + 13t^{-4} - 16t^{-2} + 16 - 16t^{2} + 13t^{4}$ $- 6t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-12} + 14t^{-10} - 63t^{-8} + 135t^{-6} - 131t^{-4} + 2t^{-2}$ $+ 87 + 2t^{2} - 131t^{4} + 135t^{6} - 63t^{8} + 14t^{10} - t^{12}$
6_3		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = -t^{-12} + 4t^{-10} + 2t^{-8} - 52t^{-6} + 177t^{-4} - 332t^{-2}$ $+ 405 - 332t^{2} + 177t^{4} - 52t^{6} + 2t^{8} + 4t^{10} - t^{12}$
7_1		$P^{(1)}(\mathcal{K};t) = 3t^{-12} - 6t^{-10} + 8t^{-8} - 10t^{-6} + 11t^{-4} - 12t^{-2}$ $- 12 - 12t^{2} + 11t^{4} - 10t^{6} + 8t^{8} - 6t^{10} + 3t^{12}$ $P^{(2)}(\mathcal{K};t) = 3t^{-24} - 12t^{-22} + 27t^{-20} - 48t^{-18} + 72t^{-16}$ $- 9t^{-14} + 114t^{-12} - 141t^{-10} + 189t^{-8} - 255t^{-6}$ $+ 333t^{-4} - 402t^{-2} + 438 - 402t^{2} + 333t^{4}$ $- 255t^{6} + 189t^{8} - 141t^{10} + 114t^{12} - 96t^{14}$ $+ 72t^{16} - 48t^{18} + 27t^{20} - 12t^{22} + 3t^{24}$
7_2		$P^{(1)}(\mathcal{K};t) = 14t^{-4} - 44t^{-2} + 60 - 44t^{2} + 14t^{4}$ $P^{(2)}(\mathcal{K};t) = 42t^{-8} - 248t^{-6} + 705t^{-4} - 1294t^{-2} + 1593$ $- 1294t^{2} + 705t^{4} - 248t^{6} + 42t^{8}$

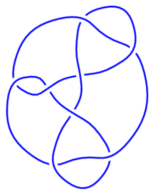
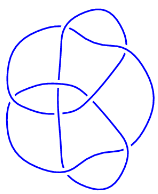
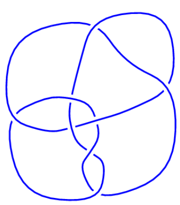
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
7_3		$P^{(1)}(\mathcal{K};t) = 9t^{-8} - 26t^{-6} + 41t^{-4} - 52t^{-2} + 56 - 52t^{2} + 41t^{4} - 26t^{6} + 9t^{8}$ $P^{(2)}(\mathcal{K};t) = 23t^{-16} - 130t^{-14} + 372t^{-12} - 740t^{-10} + 1193t^{-8} - 1793t^{-6} + 2615t^{-4} - 3462t^{-2} + 3849 - 3462t^{2} + 2615t^{4} - 1793t^{6} + 1193t^{8} - 740t^{10} + 372t^{12} - 130t^{14} + 23t^{16}$
7_4		$P^{(1)}(\mathcal{K};t) = 24t^{-4} - 80t^{-2} - 80t^{2} + 112 + 24t^{4}$ $P^{(2)}(\mathcal{K};t) = 112t^{-8} - 720t^{-6} + 2177t^{-4} - 4064t^{-2} + 4994 - 4064t^{2} + 2177t^{4} - 720t^{6} + 112t^{8}$
7_5		$P^{(1)}(\mathcal{K};t) = 9t^{-8} - 34t^{-6} + 70t^{-4} - 102t^{-2} + 114 - 102t^{2} + 70t^{4} - 34t^{6} + 9t^{8}$ $P^{(2)}(\mathcal{K};t) = 23t^{-16} - 168t^{-14} + 632t^{-12} - 1652t^{-10} + 3389t^{-8} - 5847t^{-6} + 8697t^{-4} - 11117t^{-2} + 12090 - 11117t^{2} + 8697t^{4} - 5847t^{6} + 3389t^{8} - 1652t^{10} + 632t^{12} - 168t^{14} + 23t^{16}$
7_6		$P^{(1)}(\mathcal{K};t) = t^{-8} - 10t^{-6} + 36t^{-4} - 66t^{-2} + 78 - 66t^{2} + 36t^{4} - 10t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = 8t^{-10} - 60t^{-8} + 155t^{-6} - 99t^{-4} - 253t^{-2} + 499 - 253t^{2} - 99t^{4} + 155t^{6} - 60t^{8} + 8t^{10}$

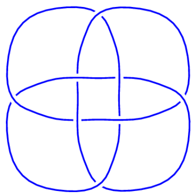
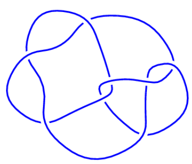
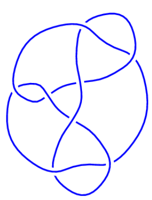
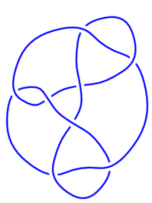
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
7_7		$P^{(1)}(\mathcal{K};t) = 3t^{-4} - 14t^{-2} + 22 - 14t^2 + 3t^4$ $P^{(2)}(\mathcal{K};t) = -t^{-12} + 12t^{-10} - 46t^{-8} + 36t^{-6} + 207t^{-4}$ $- 673t^{-2} + 929 - 673t^2 + 207t^4 + 36t^6 - 46t^8$ $+ 12t^{10} - t^{12}$
8_1		$P^{(1)}(\mathcal{K};t) = 5t^{-4} - 26t^{-2} + 42 - 26t^2 + 5t^4$ $P^{(2)}(\mathcal{K};t) = -3t^{-8} + 20t^{-6} - 14t^{-4} - 63t^{-2} + 117 - 63t^2$ $- 14t^4 + 20t^6 - 3t^8$
8_2		$P^{(1)}(\mathcal{K};t) = 2t^{-12} - 12t^{-10} + 28t^{-8} - 40t^{-6} + 47t^{-4} - 50t^{-2}$ $+ 50 - 50t^2 + 47t^4 - 40t^6 + 28t^8 - 12t^{10} + 2t^{12}$ $P^{(2)}(\mathcal{K};t) = t^{-24} - 12t^{-22} + 63t^{-20} - 194t^{-18} + 407t^{-16}$ $- 628t^{-14} + 738t^{-12} - 742t^{-10} + 962t^{-8} - 1826t^{-6}$ $+ 3361t^{-4} - 4979t^{-2} + 5698 - 4979t^2 + 3361t^4$ $- 1826t^6 + 962t^8 - 742t^{10} + 738t^{12} - 628t^{14}$ $+ 407t^{16} - 194t^{18} + 63t^{20} - 12t^{22} + t^{24}$
8_3		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = 16t^{-8} - 176t^{-6} + 795t^{-4} - 1848t^{-2} + 2422$ $- 1848t^2 + 795t^4 - 176t^6 + 16t^8$

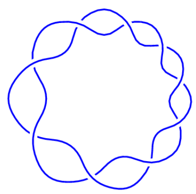
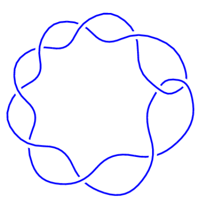
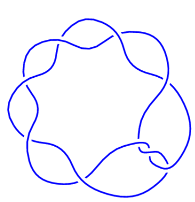
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
8_4		$P^{(1)}(\mathcal{K};t) = 3t^{-8} - 14t^{-6} + 25t^{-4} - 24t^{-2} + 20 - 24t^{2} + 25t^{4} - 14t^{6} + 3t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 10t^{-14} - 60t^{-12} + 268t^{-10} - 761t^{-8} + 1205t^{-6} - 697t^{-4} - 852t^{-2} + 1773 - 852t^{2} - 697t^{4} + 1205t^{6} - 761t^{8} + 268t^{10} - 60t^{12} + 10t^{14} - t^{16}$
8_5		$P^{(1)}(\mathcal{K};t) = 2t^{-12} - 12t^{-10} + 31t^{-8} - 54t^{-6} + 75t^{-4} - 88t^{-2} + 92 - 88t^{2} + 75t^{4} - 54t^{6} + 31t^{8} - 12t^{10} + 2t^{12}$ $P^{(2)}(\mathcal{K};t) = t^{-24} - 12t^{-22} + 65t^{-20} - 214t^{-18} + 501t^{-16} - 927t^{-14} + 1492t^{-12} - 2298t^{-10} + 3596t^{-8} - 5579t^{-6} + 8009t^{-4} - 10117t^{-2} + 10965 - 10117t^{2} + 8009t^{4} - 5579t^{6} + 3596t^{8} - 2298t^{10} + 1492t^{12} - 927t^{14} + 501t^{16} - 214t^{18} + 65t^{20} - 12t^{22} + t^{24}$
8_6		$P^{(1)}(\mathcal{K};t) = 5t^{-8} - 30t^{-6} + 73t^{-4} - 108t^{-2} + 120 - 108t^{2} + 73t^{4} - 30t^{6} + 5t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-16} - 36t^{-14} + 182t^{-12} - 498t^{-10} + 915t^{-8} - 1593t^{-6} + 3179t^{-4} - 5544t^{-2} + 6782 - 5544t^{2} + 3179t^{4} - 1593t^{6} + 915t^{8} - 498t^{10} + 182t^{12} - 36t^{14} + 3t^{16}$

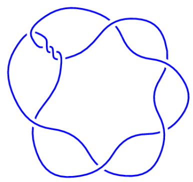
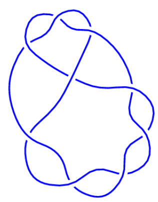
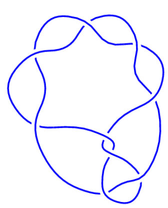
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
8_7		$P^{(1)}(\mathcal{K};t) = t^{-12} - 6t^{-10} + 19t^{-8} - 36t^{-6} + 47t^{-4} - 50t^{-2}$ $+ 50 - 50t^2 + 47t^4 - 36t^6 + 19t^8 - 6t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -t^{-20} + 10t^{-18} - 58t^{-16} + 238t^{-14} - 692t^{-12}$ $+ 1401t^{-10} - 1908t^{-8} + 1464t^{-6} + 226t^{-4} - 2319t^{-2}$ $+ 3280 - 2319t^2 + 226t^4 + 1464t^6 - 1908t^8 +$ $1401t^{10} - 692t^{12} + 238t^{14} - 58t^{16} + 10t^{18} - t^{20}$
8_8		$P^{(1)}(\mathcal{K};t) = t^{-8} - 6t^{-6} + 21t^{-4} - 44t^{-2} + 56 - 44t^2 + 21t^4$ $- 6t^6 + t^8$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 12t^{-14} - 82t^{-12} + 326t^{-10} - 790t^{-8}$ $+ 1179t^{-6} - 996t^{-4} + 296t^{-2} + 114 + 296t^2 - 996t^4$ $+ 1179t^6 - 790t^8 + 326t^{10} - 82t^{12} + 12t^{14} - t^{16}$
8_9		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = -t^{-20} + 12t^{-18} - 58t^{-16} + 156t^{-14} - 236t^{-12}$ $+ 60t^{-10} + 735t^{-8} - 2372t^{-6} + 4578t^{-4} - 6544t^{-2}$ $+ 7338 - 6544t^2 + 4578t^4 - 2372t^6 + 735t^8 + 60t^{10}$ $- 236t^{12} + 156t^{14} - 58t^{16} + 12t^{18} - t^{20}$
8_{10}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 6t^{-10} + 20t^{-8} - 42t^{-6} + 64t^{-4} - 78t^{-2}$ $+ 82 - 78t^2 + 64t^4 - 42t^6 + 20t^8 - 6t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -2t^{-20} + 19t^{-18} - 99t^{-16} + 351t^{-14} - 892t^{-12}$ $+ 1634t^{-10} - 2074t^{-8} + 1502t^{-6} + 270t^{-4} - 2351t^{-2}$ $+ 3287 - 2351t^2 + 270t^4 + 1502t^6 - 2074t^8$ $+ 1634t^{10} - 892t^{12} + 351t^{14} - 99t^{16} + 19t^{18} - 2t^{20}$

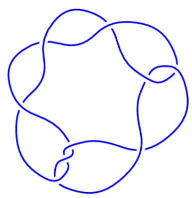
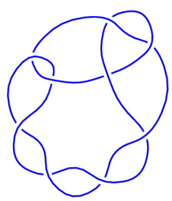
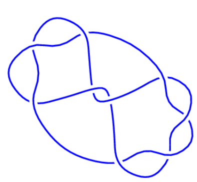
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
8_{11}		$P^{(1)}(\mathcal{K};t) = 5t^{-8} - 34t^{-6} + 92t^{-4} - 146t^{-2} + 166 - 146t^{2} + 92t^{4} - 34t^{6} + 5t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-16} - 38t^{-14} + 199t^{-12} - 552t^{-10} + 1029t^{-8} - 1949t^{-6} + 4339t^{-4} - 7971t^{-2} + 9879 - 7971t^{2} + 4339t^{4} - 1949t^{6} + 1029t^{8} - 552t^{10} + 199t^{12} - 38t^{14} + 3t^{16}$
8_{12}		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = 4t^{-12} - 72t^{-10} + 557t^{-8} - 2436t^{-6} + 6669t^{-4} - 12000t^{-2} + 14553 - 12000t^{2} + 6669t^{4} - 2436t^{6} + 557t^{8} - 72t^{10} + 4t^{12}$
8_{13}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 6t^{-6} + 23t^{-4} - 52t^{-2} + 68 - 52t^{2} + 23t^{4} - 6t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 14t^{-14} - 98t^{-12} + 410t^{-10} - 1057t^{-8} + 1647t^{-6} - 1373t^{-4} + 242t^{-2} + 433 + 242t^{2} - 1373t^{4} + 1647t^{6} - 1057t^{8} + 410t^{10} - 98t^{12} + 14t^{14} - t^{16}$
8_{14}		$P^{(1)}(\mathcal{K};t) = 5t^{-8} - 38t^{-6} + 118t^{-4} - 210t^{-2} + 250 - 210t^{2} + 118t^{4} - 38t^{6} + 5t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-16} - 40t^{-14} + 248t^{-12} - 948t^{-10} + 2618t^{-8} - 5829t^{-6} + 10838t^{-4} - 16247t^{-2} + 18714 - 16247t^{2} + 10838t^{4} - 5829t^{6} + 2618t^{8} - 948t^{10} + 248t^{12} - 40t^{14} + 3t^{16}$

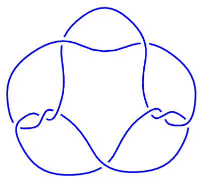
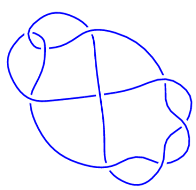
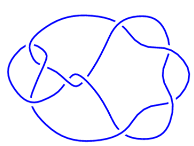
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
8_{15}		$P^{(1)}(\mathcal{K};t) = 21t^{-8} - 106t^{-6} + 269t^{-4} - 444t^{-2} + 520 - 444t^{2}$ $+ 269t^{4} - 106t^{6} + 21t^{8}$ $P^{(2)}(\mathcal{K};t) = 129t^{-16} - 1271t^{-14} + 6263t^{-12} - 20644t^{-10}$ $+ 50993t^{-8} - 99815t^{-6} + 159284t^{-4} - 210051t^{-2}$ $+ 230228 - 210051t^{2} + 159284t^{4} - 99815t^{6}$ $+ 50993t^{8} - 20644t^{10} + 6263t^{12} - 1271t^{14} + 129t^{16}$
8_{16}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 8t^{-10} + 30t^{-8} - 68t^{-6} + 108t^{-4} - 134t^{-2}$ $+ 142 - 134t^{2} + 108t^{4} - 68t^{6} + 30t^{8} - 8t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -2t^{-20} + 25t^{-18} - 156t^{-16} + 621t^{-14} - 1709t^{-12}$ $+ 3325t^{-10} - 4416t^{-8} + 3259t^{-6} + 820t^{-4} - 5821t^{-2}$ $+ 8109 - 5821t^{2} + 820t^{4} + 3259t^{6} - 4416t^{8}$ $+ 3325t^{10} - 1709t^{12} + 621t^{14} - 156t^{16} + 25t^{18} - 2t^{20}$
8_{17}		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = -t^{-20} + 12t^{-18} - 56t^{-16} + 108t^{-14} + 135t^{-12}$ $- 1564t^{-10} + 5571t^{-8} - 13056t^{-6} + 22931t^{-4}$ $- 31704t^{-2} + 35247 - 31704t^{2} + 22931t^{4} - 13056t^{6}$ $+ 5571t^{8} - 1564t^{10} + 135t^{12} + 108t^{14} - 56t^{16}$ $+ 12t^{18} - t^{20}$

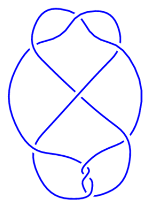
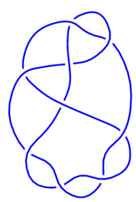
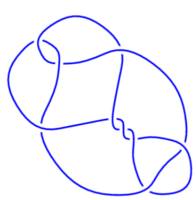
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
8_{18}		$P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = 13t^{-16} - 180t^{-14} + 1168t^{-12} - 4760t^{-10} + 13742t^{-8} - 29964t^{-6} + 51208t^{-4} - 70092t^{-2} + 77731 - 70092t^{2} + 51208t^{4} - 29964t^{6} + 13742t^{8} - 4760t^{10} + 1168t^{12} - 180t^{14} + 13t^{16}$
8_{19}		$P^{(1)}(\mathcal{K};t) = 3t^{-12} - 6t^{-10} + 3t^{-8} + 4t^{-6} - 5t^{-4} - 2t^{-2} + 6 - 2t^{2} - 5t^{4} + 4t^{6} + 3t^{8} - 6t^{10} + 3t^{12}$ $P^{(2)}(\mathcal{K};t) = 3t^{-24} - 12t^{-22} + 18t^{-20} - 7t^{-18} - 17t^{-16} + 30t^{-14} - 25t^{-12} + 20t^{-10} - 40t^{-8} + 65t^{-6} - 10t^{-4} - 130t^{-2} + 215 - 130t^{2} - 10t^{4} + 65t^{6} - 40t^{8} + 20t^{10} - 25t^{12} + 30t^{14} - 17t^{16} - 7t^{18} + 18t^{20} - 12t^{22} + 3t^{24}$
8_{20}		$P^{(1)}(\mathcal{K};t) = 4t^{-4} - 12t^{-2} + 16 - 12t^{2} + 4t^{4}$ $P^{(2)}(\mathcal{K};t) = -7t^{-12} + 46t^{-10} - 120t^{-8} + 138t^{-6} + 14t^{-4} - 278t^{-2} + 416 - 278t^{2} + 14t^{4} + 138t^{6} - 120t^{8} + 46t^{10} - 7t^{12}$
8_{21}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 10t^{-6} + 33t^{-4} - 60t^{-2} + 72 - 60t^{2} + 33t^{4} - 10t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = t^{-14} + 4t^{-12} - 60t^{-10} + 276t^{-8} - 775t^{-6} + 1550t^{-4} - 2331t^{-2} + 2670 - 2331t^{2} + 1550t^{4} - 775t^{6} + 276t^{8} - 60t^{10} + 4t^{12} + t^{14}$

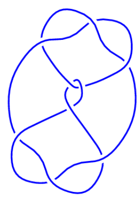
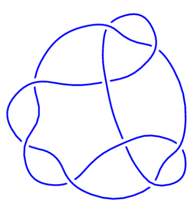
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_1		$P^{(1)}(\mathcal{K};t) = 4t^{-16} - 8t^{-14} + 11t^{-12} - 14t^{-10} + 16t^{-8} - 18t^{-6} + 19t^{-4} - 20t^{-2} + 20 - 20t^{2} + 19t^{4} - 18t^{6} + 16t^{8} - 14t^{10} + 11t^{12} - 8t^{14} + 4t^{16}$ $P^{(2)}(\mathcal{K};t) = 6t^{-32} - 24t^{-30} + 56t^{-28} - 104t^{-26} + 166t^{-24} - 240t^{-22} + 320t^{-20} - 400t^{-18} + 470t^{-16} - 556t^{-14} + 680t^{-12} - 845t^{-10} + 1050t^{-8} - 1275t^{-6} + 1496t^{-4} - 1670t^{-2} + 1750 - 1670t^{2} + 1496t^{4} - 1275t^{6} + 1050t^{8} - 845t^{10} + 680t^{12} - 556t^{14} + 470t^{16} - 400t^{18} + 320t^{20} - 240t^{22} + 166t^{24} - 104t^{26} + 56t^{28} - 24t^{30} + 6t^{32}$
9_2		$P^{(1)}(\mathcal{K};t) = 30t^{-4} - 100t^{-2} + 140 - 100t^{2} + 30t^{4}$ $P^{(2)}(\mathcal{K};t) = 252t^{-8} - 1628t^{-6} + 4934t^{-4} - 9186t^{-2} + 11260 - 9186t^{2} + 4934t^{4} - 1628t^{6} + 252t^{8}$
9_3		$P^{(1)}(\mathcal{K};t) = 13t^{-12} - 38t^{-10} + 62t^{-8} - 82t^{-6} + 97t^{-4} - 108t^{-2} + 112 - 108t^{2} + 97t^{4} - 82t^{6} + 62t^{8} - 38t^{10} + 13t^{12}$ $P^{(2)}(\mathcal{K};t) = 59t^{-24} - 342t^{-22} + 1037t^{-20} - 2256t^{-18} + 3989t^{-16} - 6102t^{-14} + 8457t^{-12} - 11235t^{-10} + 14817t^{-8} - 19267t^{-6} + 24063t^{-4} - 28032t^{-2} + 29633 - 28032t^{2} + 24063t^{4} - 19267t^{6} + 14817t^{8} - 11235t^{10} + 8457t^{12} - 6102t^{14} + 3989t^{16} - 2256t^{18} + 1037t^{20} - 342t^{22} + 59t^{24}$

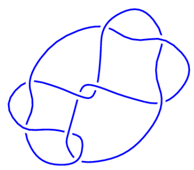
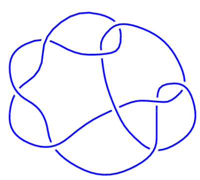
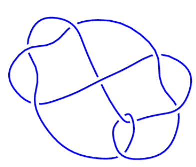
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_4		$P^{(1)}(\mathcal{K};t) = 23t^{-8} - 74t^{-6} + 125t^{-4} - 164t^{-2} + 180 - 164t^{2}$ $+ 125t^{4} - 74t^{6} + 23t^{8}$ $P^{(2)}(\mathcal{K};t) = 168t^{-16} - 1064t^{-14} + 3397t^{-12} - 7494t^{-10}$ $+ 13245t^{-8} - 20655t^{-6} + 29463t^{-4} - 37560t^{-2}$ $+ 41007 - 37560t^{2} + 29463t^{4} - 20655t^{6} + 13245t^{8}$ $- 7494t^{10} + 3397t^{12} - 1064t^{14} + 168t^{16}$
9_5		$P^{(1)}(\mathcal{K};t) = 65t^{-4} - 230t^{-2} + 330 - 230t^{2} + 65t^{4}$ $P^{(2)}(\mathcal{K};t) = 1095t^{-8} - 7654t^{-6} + 24458t^{-4} - 46567t^{-2}$ $+ 57342 - 46567t^{2} + 24458t^{4} - 7654t^{6} + 1095t^{8}$
9_6		$P^{(1)}(\mathcal{K};t) = 13t^{-12} - 50t^{-10} + 106t^{-8} - 166t^{-6} + 217t^{-4}$ $- 252t^{-2} + 264 - 252t^{2} + 217t^{4} - 166t^{6} + 106t^{8}$ $- 50t^{10} + 13t^{12}$ $P^{(2)}(\mathcal{K};t) = 59t^{-24} - 448t^{-22} + 1780t^{-20} - 4946t^{-18}$ $+ 10794t^{-16} - 19788t^{-14} + 31980t^{-12} - 47371t^{-10}$ $+ 65965t^{-8} - 86985t^{-6} + 107770t^{-4} - 123616t^{-2}$ $+ 129619 - 123616t^{2} + 107770t^{4} - 86985t^{6}$ $+ 65965t^{8} - 47371t^{10} + 31980t^{12} - 19788t^{14}$ $+ 10794t^{16} - 4946t^{18} + 1780t^{20} - 448t^{22} + 59t^{24}$

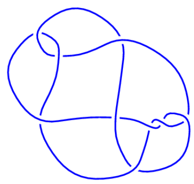
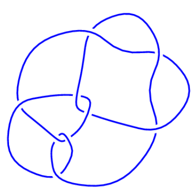
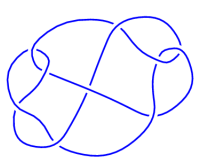
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_7		$P^{(1)}(\mathcal{K};t) = 23t^{-8} - 102t^{-6} + 234t^{-4} - 362t^{-2} + 414 - 362t^2 + 234t^4 - 102t^6 + 23t^8$ $P^{(2)}(\mathcal{K};t) = 168t^{-16} - 1456t^{-14} + 6392t^{-12} - 18984t^{-10} + 42948t^{-8} - 78779t^{-6} + 120667t^{-4} - 155813t^{-2} + 169719 - 155813t^2 + 120667t^4 - 78779t^6 + 42948t^8 - 18984t^{10} + 6392t^{12} - 1456t^{14} + 168t^{16}$
9_8		$P^{(1)}(\mathcal{K};t) = 3t^{-8} - 22t^{-6} + 64t^{-4} - 102t^{-2} + 114 - 102t^2 + 64t^4 - 22t^6 + 3t^8$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 16t^{-14} - 120t^{-12} + 556t^{-10} - 1613t^{-8} + 2695t^{-6} - 1855t^{-4} - 1345t^{-2} + 3334 - 1345t^2 - 1855t^4 + 2695t^6 - 1613t^8 + 556t^{10} - 120t^{12} + 16t^{14} - t^{16}$
9_9		$P^{(1)}(\mathcal{K};t) = 13t^{-12} - 50t^{-10} + 116t^{-8} - 206t^{-6} + 297t^{-4} - 364t^{-2} + 388 - 364t^2 + 297t^4 - 206t^6 + 116t^8 - 50t^{10} + 13t^{12}$ $P^{(2)}(\mathcal{K};t) = 59t^{-24} - 448t^{-22} + 1856t^{-20} - 5554t^{-18} + 13358t^{-16} - 27310t^{-14} + 49099t^{-12} - 79427t^{-10} + 117166t^{-8} - 158525t^{-6} + 196982t^{-4} - 224610t^{-2} + 234716 - 224610t^2 + 196982t^4 - 158525t^6 + 117166t^8 - 79427t^{10} + 49099t^{12} - 27310t^{14} + 13358t^{16} - 5554t^{18} + 1856t^{20} - 448t^{22} + 59t^{24}$

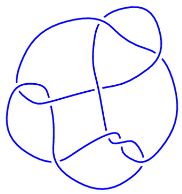
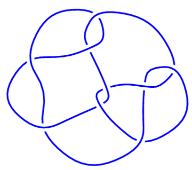
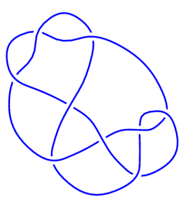
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{10}		$P^{(1)}(\mathcal{K};t) = 40t^{-8} - 152t^{-6} + 298t^{-4} - 420t^{-2} + 468 - 420t^{2} + 298t^{4} - 152t^{6} + 40t^{8}$ $P^{(2)}(\mathcal{K};t) = 496t^{-16} - 3680t^{-14} + 13704t^{-12} - 34792t^{-10} + 69151t^{-8} - 116396t^{-6} + 171083t^{-4} - 218294t^{-2} + 237464 - 218294t^{2} + 171083t^{4} - 116396t^{6} + 69151t^{8} - 34792t^{10} + 13704t^{12} - 3680t^{14} + 496t^{16}$
9_{11}		$P^{(1)}(\mathcal{K};t) = 2t^{-12} - 20t^{-10} + 75t^{-8} - 150t^{-6} + 211t^{-4} - 248t^{-2} + 260 - 248t^{2} + 211t^{4} - 150t^{6} + 75t^{8} - 20t^{10} + 2t^{12}$ $P^{(2)}(\mathcal{K};t) = t^{-24} - 20t^{-22} + 175t^{-20} - 886t^{-18} + 2949t^{-16} - 7028t^{-14} + 12910t^{-12} - 19946t^{-10} + 29077t^{-8} - 43383t^{-6} + 63831t^{-4} - 84100t^{-2} + 92844 - 84100t^{2} + 63831t^{4} - 43383t^{6} + 29077t^{8} - 19946t^{10} + 12910t^{12} - 7028t^{14} + 2949t^{16} - 886t^{18} + 175t^{20} - 20t^{22} + t^{24}$
9_{12}		$P^{(1)}(\mathcal{K};t) = 5t^{-8} - 46t^{-6} + 161t^{-4} - 304t^{-2} + 368 - 304t^{2} + 161t^{4} - 46t^{6} + 5t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-16} - 58t^{-14} + 485t^{-12} - 2290t^{-10} + 7255t^{-8} - 17421t^{-6} + 33493t^{-4} - 50852t^{-2} + 58771 - 50852t^{2} + 33493t^{4} - 17421t^{6} + 7255t^{8} - 2290t^{10} + 485t^{12} - 58t^{14} + 3t^{16}$

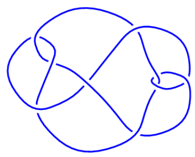
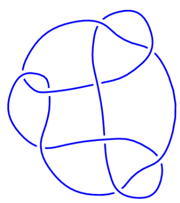
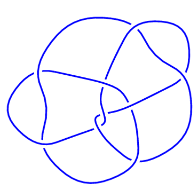
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{13}		$P^{(1)}(\mathcal{K};t) = 40t^{-8} - 172t^{-6} + 378t^{-4} - 568t^{-2} + 644 - 568t^{2}$ $+ 378t^{4} - 172t^{6} + 40t^{8}$ $P^{(2)}(\mathcal{K};t) = 496t^{-16} - 4192t^{-14} + 17765t^{-12} - 50726t^{-10}$ $+ 110764t^{-8} - 198080t^{-6} + 299319t^{-4} - 384612t^{-2}$ $+ 418539 - 384612t^{2} + 299319t^{4} - 198080t^{6}$ $+ 110764t^{8} - 50726t^{10} + 17765t^{12} - 4192t^{14} + 496t^{16}$
9_{14}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 10t^{-6} + 52t^{-4} - 138t^{-2} + 190 - 138t^{2}$ $+ 52t^{4} - 10t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 18t^{-14} - 160t^{-12} + 844t^{-10} - 2640t^{-8}$ $+ 4723t^{-6} - 4160t^{-4} + 233t^{-2} + 2285 + 233t^{2}$ $- 4160t^{4} + 4723t^{6} - 2640t^{8} + 844t^{10} - 160t^{12}$ $+ 18t^{14} - t^{16}$
9_{15}		$P^{(1)}(\mathcal{K};t) = 5t^{-8} - 50t^{-6} + 193t^{-4} - 392t^{-2} + 488 - 392t^{2}$ $+ 193t^{4} - 50t^{6} + 5t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-16} - 60t^{-14} + 550t^{-12} - 2970t^{-10} + 10685t^{-8}$ $- 27763t^{-6} + 54520t^{-4} - 82014t^{-2} + 94100$ $- 82014t^{2} + 54520t^{4} - 27763t^{6} + 10685t^{8} - 2970t^{10}$ $+ 550t^{12} - 60t^{14} + 3t^{16}$

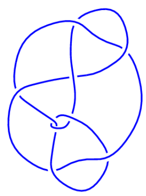
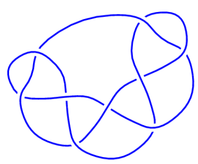
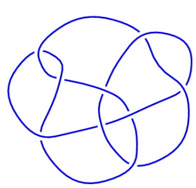
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{16}		$P^{(1)}(\mathcal{K};t) = 13t^{-12} - 62t^{-10} + 165t^{-8} - 316t^{-6} + 481t^{-4}$ $- 610t^{-2} + 658 + -610t^{2} + 481t^{4} - 316t^{6} + 165t^{8}$ $- 62t^{10} + 13t^{12}$ $P^{(2)}(\mathcal{K};t) = 59t^{-24} - 554t^{-22} + 2735t^{-20} - 9433t^{-18}$ $+ 25475t^{-16} - 57266t^{-14} + 111164t^{-12} - 190776t^{-10}$ $+ 293705t^{-8} - 409021t^{-6} + 517431t^{-4} - 595632t^{-2}$ $+ 624232 - 595632t^{2} + 517431t^{4} - 409021t^{6}$ $+ 293705t^{8} - 190776t^{10} + 111164t^{12} - 57266t^{14}$ $+ 25475t^{16} - 9433t^{18} + 2735t^{20} - 554t^{22} + 59t^{24}$
9_{17}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 10t^{-10} + 40t^{-8} - 86t^{-6} + 115t^{-4} - 112t^{-2}$ $+ 104 - 112t^{2} + 115t^{4} - 86t^{6} + 40t^{8} - 10t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -t^{-20} + 24t^{-18} - 234t^{-16} + 1300t^{-14} - 4658t^{-12}$ $+ 11261t^{-10} - 18129t^{-8} + 16995t^{-6} - 1646t^{-4}$ $- 20708t^{-2} + 31590 - 20708t^{2} - 1646t^{4} + 16995t^{6}$ $- 18129t^{8} + 11261t^{10} - 4658t^{12} + 1300t^{14} - 234t^{16}$ $+ 24t^{18} - t^{20}$

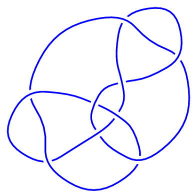
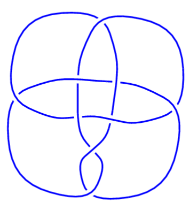
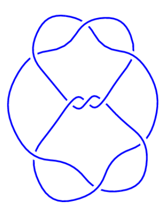
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{18}		$P^{(1)}(\mathcal{K};t) = 40t^{-8} - 188t^{-6} + 449t^{-4} - 714t^{-2} + 826 - 714t^{2} + 449t^{4} - 188t^{6} + 40t^{8}$ $P^{(2)}(\mathcal{K};t) = 496t^{-16} - 4528t^{-14} + 20876t^{-12} - 64904t^{-10} + 152656t^{-8} - 287776t^{-6} + 447556t^{-4} - 581629t^{-2} + 634512 - 581629t^{2} + 447556t^{4} - 287776t^{6} + 152656t^{8} - 64904t^{10} + 20876t^{12} - 4528t^{14} + 496t^{16}$
9_{19}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 10t^{-6} + 37t^{-4} - 72t^{-2} + 88 - 72t^{2} + 37t^{4} - 10t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 20t^{-14} - 154t^{-12} + 590t^{-10} - 992t^{-8} - 767t^{-6} + 7539t^{-4} - 17494t^{-2} + 22516 - 17494t^{2} + 7539t^{4} - 767t^{6} - 992t^{8} + 590t^{10} - 154t^{12} + 20t^{14} - t^{16}$
9_{20}		$P^{(1)}(\mathcal{K};t) = 2t^{-12} - 20t^{-10} + 81t^{-8} - 194t^{-6} + 332t^{-4} - 442t^{-2} + 482 - 442t^{2} + 332t^{4} - 194t^{6} + 81t^{8} - 20t^{10} + 2t^{12}$ $P^{(2)}(\mathcal{K};t) = t^{-24} - 20t^{-22} + 179t^{-20} - 962t^{-18} + 3582t^{-16} - 10116t^{-14} + 23136t^{-12} - 45110t^{-10} + 77842t^{-8} - 120895t^{-6} + 168287t^{-4} - 207005t^{-2} + 222164 - 207005t^{2} + 168287t^{4} - 120895t^{6} + 77842t^{8} - 45110t^{10} + 23136t^{12} - 10116t^{14} + 3582t^{16} - 962t^{18} + 179t^{20} - 20t^{22} + t^{24}$

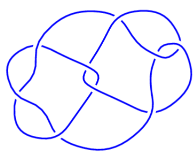
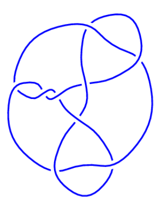
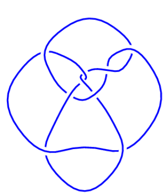
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{21}		$P^{(1)}(\mathcal{K};t) = 5t^{-8} - 54t^{-6} + 220t^{-4} - 462t^{-2} + 582 - 462t^{2} + 220t^{4} - 54t^{6} + 5t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-16} - 62t^{-14} + 587t^{-12} - 3294t^{-10} + 12394t^{-8} - 33645t^{-6} + 68358t^{-4} - 104909t^{-2} + 121139 - 104909t^{2} + 68358t^{4} - 33645t^{6} + 12394t^{8} - 3294t^{10} + 587t^{12} - 62t^{14} + 3t^{16}$
9_{22}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 10t^{-10} + 41t^{-8} - 94t^{-6} + 140t^{-4} - 154t^{-2} + 152 - 154t^{2} + 140t^{4} - 94t^{6} + 41t^{8} - 10t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -2t^{-20} + 38t^{-18} - 315t^{-16} + 1554t^{-14} - 5114t^{-12} + 11635t^{-10} - 17839t^{-8} + 15665t^{-6} + 422t^{-4} - 22883t^{-2} + 33677 - 22883t^{2} + 422t^{4} + 15665t^{6} - 17839t^{8} + 11635t^{10} - 5114t^{12} + 1554t^{14} - 315t^{16} + 38t^{18} - 2t^{20}$
9_{23}		$P^{(1)}(\mathcal{K};t) = 40t^{-8} - 208t^{-6} + 539t^{-4} - 902t^{-2} + 1062 - 902t^{2} + 539t^{4} - 208t^{6} + 40t^{8}$ $P^{(2)}(\mathcal{K};t) = 496t^{-16} - 5040t^{-14} + 25633t^{-12} - 86784t^{-10} + 218571t^{-8} - 432908t^{-6} + 694782t^{-4} - 918113t^{-2} + 1006731 - 918113t^{2} + 694782t^{4} - 432908t^{6} + 218571t^{8} - 86784t^{10} + 25633t^{12} - 5040t^{14} + 496t^{16}$

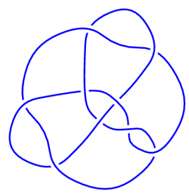
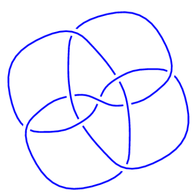
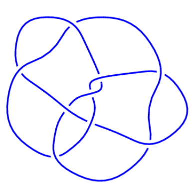
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{24}		$P^{(1)}(\mathcal{K};t) = -4t^{-6} + 24t^{-4} - 56t^{-2} + 72 - 56t^{2} + 24t^{4} - 4t^{6}$ $P^{(2)}(\mathcal{K};t) = 6t^{-18} - 67t^{-16} + 334t^{-14} - 922t^{-12} + 1164t^{-10}$ $+ 1418t^{-8} - 10332t^{-6} + 26064t^{-4} - 42438t^{-2}$ $+ 49547 - 42438t^{2} + 26064t^{4} - 10332t^{6} + 1418t^{8}$ $+ 1164t^{10} - 922t^{12} + 334t^{14} - 67t^{16} + 6t^{18}$
9_{25}		$P^{(1)}(\mathcal{K};t) = 12t^{-8} - 94t^{-6} + 305t^{-4} - 564t^{-2} + 682 - 564t^{2}$ $+ 305t^{4} - 94t^{6} + 12t^{8}$ $P^{(2)}(\mathcal{K};t) = 21t^{-16} - 327t^{-14} + 2327t^{-12} - 10128t^{-10}$ $+ 30936t^{-8} - 71781t^{-6} + 131152t^{-4} - 189471t^{-2}$ $+ 214542 - 189471t^{2} + 131152t^{4} - 71781t^{6}$ $+ 30936t^{8} - 10128t^{10} + 2327t^{12} - 327t^{14} + 21t^{16}$
9_{26}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 10t^{-10} + 48t^{-8} - 134t^{-6} + 244t^{-4} - 326t^{-2}$ $+ 354 - 326t^{2} + 244t^{4} - 134t^{6} + 48t^{8} - 10t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = 2t^{-18} - 27t^{-16} + 210t^{-14} - 964t^{-12} + 2465t^{-10}$ $- 2545t^{-8} - 4459t^{-6} + 22456t^{-4} - 44607t^{-2} + 54938$ $- 44607t^{2} + 22456t^{4} - 4459t^{6} - 2545t^{8} + 2465t^{10}$ $- 964t^{12} + 210t^{14} - 27t^{16} + 2t^{18}$

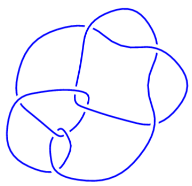
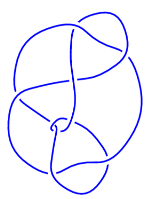
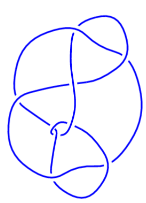
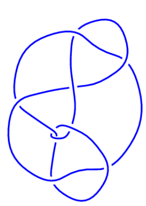
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{27}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 10t^{-6} + 41t^{-4} - 88t^{-2} + 112 - 88t^{2} + 41t^{4} - 10t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-20} + 20t^{-18} - 163t^{-16} + 756t^{-14} - 2212t^{-12} + 3982t^{-10} - 2931t^{-8} - 5979t^{-6} + 24262t^{-4} - 44398t^{-2} + 53328 - 44398t^{2} + 24262t^{4} - 5979t^{6} - 2931t^{8} + 3982t^{10} - 2212t^{12} + 756t^{14} - 163t^{16} + 20t^{18} - t^{20}$
9_{28}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 10t^{-10} + 47t^{-8} - 136t^{-6} + 271t^{-4} - 398t^{-2} + 450 - 398t^{2} + 271t^{4} - 136t^{6} + 47t^{8} - 10t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -2t^{-20} + 27t^{-18} - 167t^{-16} + 612t^{-14} - 1375t^{-12} + 1464t^{-10} + 1842t^{-8} - 11767t^{-6} + 28198t^{-4} - 44740t^{-2} + 51817 - 44740t^{2} + 28198t^{4} - 11767t^{6} + 1842t^{8} + 1464t^{10} - 1375t^{12} + 612t^{14} - 167t^{16} + 27t^{18} - 2t^{20}$
9_{29}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 10t^{-10} + 43t^{-8} - 108t^{-6} + 181t^{-4} - 222t^{-2} + 230 - 222t^{2} + 181t^{4} - 108t^{6} + 43t^{8} - 10t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -4t^{-20} + 69t^{-18} - 532t^{-16} + 2466t^{-14} - 7662t^{-12} + 16466t^{-10} - 23513t^{-8} + 17305t^{-6} + 9140t^{-4} - 43892t^{-2} + 60315 - 43892t^{2} + 9140t^{4} + 17305t^{6} - 23513t^{8} + 16466t^{10} - 7662t^{12} + 2466t^{14} - 532t^{16} + 69t^{18} - 4t^{20}$

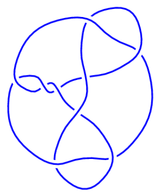
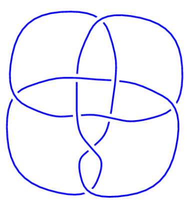
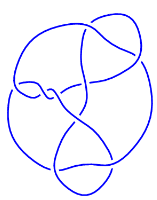
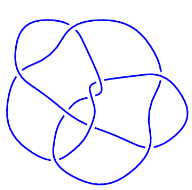
Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{30}		$P^{(1)}(\mathcal{K};t) = 2t^{-8} - 14t^{-6} + 47t^{-4} - 92t^{-2} + 114 - 92t^{2} + 47t^{4} - 14t^{6} + 2t^{8}$ $P^{(2)}(\mathcal{K};t) = -2t^{-20} + 31t^{-18} - 210t^{-16} + 822t^{-14} - 1944t^{-12} + 2181t^{-10} + 2558t^{-8} - 17256t^{-6} + 41804t^{-4} - 66537t^{-2} + 77105 - 66537t^{2} + 41804t^{4} - 17256t^{6} + 2558t^{8} + 2181t^{10} - 1944t^{12} + 822t^{14} - 210t^{16} + 31t^{18} - 2t^{20}$
9_{31}		$P^{(1)}(\mathcal{K};t) = -13t^{-10} + 115t^{-8} - 463t^{-6} + 1126t^{-4} - 1857t^{-2} + 2185 - 1857t^{2} + 1126t^{4} - 463t^{6} + 115t^{8} - 13t^{10}$ $P^{(2)}(\mathcal{K};t) = -13t^{-16} + 180t^{-14} - 1207t^{-12} + 5157t^{-10} - 15630t^{-8} + 35539t^{-6} - 62672t^{-4} + 87453t^{-2} - 97613 + 87453t^{2} - 62672t^{4} + 35539t^{6} - 15630t^{8} + 5157t^{10} - 1207t^{12} + 180t^{14} - 13t^{16}$
9_{32}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 12t^{-10} + 63t^{-8} - 188t^{-6} + 363t^{-4} - 508t^{-2} + 562 - 508t^{2} + 363t^{4} - 188t^{6} + 63t^{8} - 12t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = 3t^{-18} - 49t^{-16} + 347t^{-14} - 1297t^{-12} + 2213t^{-10} + 1982t^{-8} - 21345t^{-6} + 60831t^{-4} - 105664t^{-2} + 125957 - 105664t^{2} + 60831t^{4} - 21345t^{6} + 1982t^{8} + 2213t^{10} - 1297t^{12} + 347t^{14} - 49t^{16} + 3t^{18}$

Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{33}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 12t^{-6} + 51t^{-4} - 110t^{-2} + 140 - 110t^{2}$ $+ 51t^{4} - 12t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = 4t^{-18} - 48t^{-16} + 210t^{-14} - 133t^{-12} - 2812t^{-10}$ $+ 15721t^{-8} - 47484t^{-6} + 97856t^{-4} - 148061t^{-2}$ $+ 169495 - 148061t^{2} + 97856t^{4} - 47484t^{6} + 15721t^{8}$ $- 2812t^{10} - 133t^{12} + 210t^{14} - 48t^{16} + 4t^{18}$
9_{34}		$P^{(1)}(\mathcal{K};t) = 3t^{-8} - 24t^{-6} + 82t^{-4} - 160t^{-2} + 198 - 160t^{2}$ $+ 82t^{4} - 24t^{6} + 3t^{8}$ $P^{(2)}(\mathcal{K};t) = -2t^{-20} + 34t^{-18} - 243t^{-16} + 899t^{-14} - 1343t^{-12}$ $- 3320t^{-10} + 25323t^{-8} - 78832t^{-6} + 162282t^{-4}$ $- 244513t^{-2} + 279429 - 244513t^{2} + 162282t^{4}$ $- 78832t^{6} + 25323t^{8} - 3320t^{10} - 1343t^{12} + 899t^{14}$ $- 243t^{16} + 34t^{18} - 2t^{20}$
9_{35}		$P^{(1)}(\mathcal{K};t) = 90t^{-4} - 324t^{-2} + 468 - 324t^{2} + 90t^{4}$ $P^{(2)}(\mathcal{K};t) = 2173t^{-8} - 15532t^{-6} + 50362t^{-4} - 96550t^{-2}$ $+ 119101 - 96550t^{2} + 50362t^{4} - 15532t^{6} + 2173t^{8}$

Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{36}		$P^{(1)}(\mathcal{K};t) = 2t^{-12} - 20t^{-10} + 78t^{-8} - 170t^{-6} + 263t^{-4}$ $- 328t^{-2} + 350 - 328t^{2} + 263t^{4} - 170t^{6} + 78t^{8}$ $- 20t^{10} + 2t^{12}$ $P^{(2)}(\mathcal{K};t) = t^{-24} - 20t^{-22} + 177t^{-20} - 919t^{-18} + 3197t^{-16}$ $- 8176t^{-14} + 16646t^{-12} - 29163t^{-10} + 47115t^{-8}$ $- 72304t^{-6} + 102929t^{-4} - 130019t^{-2} + 141075$ $- 130019t^{2} + 102929t^{4} - 72304t^{6} + 47115t^{8}$ $- 29163t^{10} + 16646t^{12} - 8176t^{14} + 3197t^{16} - 919t^{18}$ $+ 177t^{20} - 20t^{22} + t^{24}$
9_{37}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 10t^{-6} + 39t^{-4} - 80t^{-2} + 100 - 80t^{2} + 39t^{4}$ $- 10t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = -t^{-16} + 22t^{-14} - 174t^{-12} + 616t^{-10} - 530t^{-8}$ $- 3765t^{-6} + 16716t^{-4} - 34640t^{-2} + 43509 - 34640t^{2}$ $+ 16716t^{4} - 3765t^{6} - 530t^{8} + 616t^{10} - 174t^{12}$ $+ 22t^{14} - t^{16}$
9_{38}		$P^{(1)}(\mathcal{K};t) = 62t^{-8} - 328t^{-6} + 852t^{-4} - 1420t^{-2} + 1668$ $- 1420t^{2} + 852t^{4} - 328t^{6} + 62t^{8}$ $P^{(2)}(\mathcal{K};t) = 1182t^{-16} - 12246t^{-14} + 62904t^{-12} - 213694t^{-10}$ $+ 538946t^{-8} - 1069918t^{-6} + 1723338t^{-4}$ $- 2285080t^{-2} + 2509142 - 2285080t^{2} + 1723338t^{4}$ $- 1069918t^{6} + 538946t^{8} - 213694t^{10} + 62904t^{12}$ $- 12246t^{14} + 1182t^{16}$

Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{39}		$P^{(1)}(\mathcal{K};t) = 12t^{-8} - 108t^{-6} + 390t^{-4} - 772t^{-2} + 956 - 772t^{2}$ $+ 390t^{4} - 108t^{6} + 12t^{8}$ $P^{(2)}(\mathcal{K};t) = 21t^{-16} - 368t^{-14} + 2958t^{-12} - 14510t^{-10}$ $+ 49295t^{-8} - 124072t^{-6} + 238666t^{-4} - 353822t^{-2}$ $+ 403666 - 353822t^{2} + 238666t^{4} - 124072t^{6}$ $+ 49295t^{8} - 14510t^{10} + 2958t^{12} - 368t^{14} + 21t^{16}$
9_{40}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 14t^{-10} + 82t^{-8} - 270t^{-6} + 574t^{-4} - 866t^{-2}$ $+ 986 - 866t^{2} + 574t^{4} - 270t^{6} + 82t^{8} - 14t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = -t^{-18} + 27t^{-16} - 329t^{-14} + 2398t^{-12} - 11706t^{-10}$ $+ 40644t^{-8} - 104066t^{-6} + 200988t^{-4} - 296867t^{-2}$ $+ 337823 - 296867t^{2} + 200988t^{4} - 104066t^{6}$ $+ 40644t^{8} - 11706t^{10} + 2398t^{12} - 329t^{14} + 27t^{16}$ $- t^{18}$
9_{41}		$P^{(1)}(\mathcal{K};t) = 3t^{-8} - 26t^{-6} + 113t^{-4} - 268t^{-2} + 356 - 268t^{2}$ $+ 113t^{4} - 26t^{6} + 3t^{8}$ $P^{(2)}(\mathcal{K};t) = -6t^{-16} + 87t^{-14} - 620t^{-12} + 2604t^{-10} - 6336t^{-8}$ $+ 7289t^{-6} + 2780t^{-4} - 21985t^{-2} + 32374 - 21985t^{2}$ $+ 2780t^{4} + 7289t^{6} - 6336t^{8} + 2604t^{10} - 620t^{12}$ $+ 87t^{14} - 6t^{16}$

Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{42}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 4t^{-6} + 4t^{-4} + 4t^{-2} - 10 + 4t^{2} + 4t^{4} - 4t^{6}$ $+ t^{8}$ $P^{(2)}(\mathcal{K};t) = -5t^{-12} + 50t^{-10} - 187t^{-8} + 300t^{-6} - 16t^{-4}$ $- 718t^{-2} + 1150 - 718t^{2} - 16t^{4} + 300t^{6} - 187t^{8}$ $+ 50t^{10} - 5t^{12}$
9_{43}		$P^{(1)}(\mathcal{K};t) = 2t^{-12} - 12t^{-10} + 25t^{-8} - 24t^{-6} + 16t^{-4} - 16t^{-2}$ $+ 18 - 16t^{2} + 16t^{4} - 24t^{6} + 25t^{8} - 12t^{10} + 2t^{12}$ $P^{(2)}(\mathcal{K};t) = t^{-24} - 12t^{-22} + 61t^{-20} - 169t^{-18} + 272t^{-16}$ $- 209t^{-14} - 148t^{-12} + 693t^{-10} - 950t^{-8} + 319t^{-6}$ $+ 1351t^{-4} - 3368t^{-2} + 4319 - 3368t^{2} + 1351t^{4}$ $+ 319t^{6} - 950t^{8} + 693t^{10} - 148t^{12} - 209t^{14} + 272t^{16}$ $- 169t^{18} + 61t^{20} - 12t^{22} + t^{24}$
9_{44}		$P^{(1)}(\mathcal{K};t) = -2t^{-6} + 13t^{-4} - 32t^{-2} + 42 - 32t^{2} + 13t^{4} - 2t^{6}$ $P^{(2)}(\mathcal{K};t) = 3t^{-14} - 31t^{-12} + 126t^{-10} - 245t^{-8} + 147t^{-6}$ $+ 390t^{-4} - 1143t^{-2} + 1506 - 1143t^{2} + 390t^{4}$ $+ 147t^{6} - 245t^{8} + 126t^{10} - 31t^{12} + 3t^{14}$
9_{45}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 16t^{-6} + 76t^{-4} - 168t^{-2} + 214 - 168t^{2}$ $+ 76t^{4} - 16t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = 2t^{-14} + 16t^{-12} - 316t^{-10} + 1841t^{-8} - 6088t^{-6}$ $+ 13476t^{-4} - 21370t^{-2} + 24880 - 21370t^{2} + 13476t^{4}$ $- 6088t^{6} + 1841t^{8} - 316t^{10} + 16t^{12} + 2t^{14}$

Knot	Knot Diagram	$P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$
9_{46}		$P^{(1)}(\mathcal{K};t) = 3t^{-4} - 18t^{-2} + 30 - 18t^{2} + 3t^{4}$ $P^{(2)}(\mathcal{K};t) = 3t^{-8} - 34t^{-6} + 191t^{-4} - 493t^{-2} + 664 - 493t^{2}$ $+ 191t^{4} - 34t^{6} + 3t^{8}$
9_{47}		$P^{(1)}(\mathcal{K};t) = t^{-12} - 8t^{-10} + 28t^{-8} - 52t^{-6} + 57t^{-4} - 48t^{-2}$ $+ 44 - 48t^{2} + 57t^{4} - 52t^{6} + 28t^{8} - 8t^{10} + t^{12}$ $P^{(2)}(\mathcal{K};t) = 5t^{-18} - 72t^{-16} + 466t^{-14} - 1743t^{-12} + 4097t^{-10}$ $- 6037t^{-8} + 4413t^{-6} + 2575t^{-4} - 11715t^{-2} + 16021$ $- 11715t^{2} + 2575t^{4} + 4413t^{6} - 6037t^{8} + 4097t^{10}$ $- 1743t^{12} + 466t^{14} - 72t^{16} + 5t^{18}$
9_{48}		$P^{(1)}(\mathcal{K};t) = t^{-8} - 14t^{-6} + 67t^{-4} - 148t^{-2} + 188 - 148t^{2}$ $+ 67t^{4} - 14t^{6} + t^{8}$ $P^{(2)}(\mathcal{K};t) = 3t^{-12} - 42t^{-10} + 279t^{-8} - 1149t^{-6} + 3177t^{-4}$ $- 5916t^{-2} + 7299 - 5916t^{2} + 3177t^{4} - 1149t^{6}$ $+ 279t^{8} - 42t^{10} + 3t^{12}$
9_{49}		$P^{(1)}(\mathcal{K};t) = 21t^{-8} - 80t^{-6} + 158t^{-4} - 220t^{-2} + 242 - 220t^{2}$ $+ 158t^{4} - 80t^{6} + 21t^{8}$ $P^{(2)}(\mathcal{K};t) = 129t^{-16} - 960t^{-14} + 3562t^{-12} - 8910t^{-10}$ $+ 17424t^{-8} - 29382t^{-6} + 44204t^{-4} - 57934t^{-2}$ $+ 63740 - 57934t^{2} + 44204t^{4} - 29382t^{6} + 17424t^{8}$ $- 8910t^{10} + 3562t^{12} - 960t^{14} + 129t^{16}$

6.2. Amphicheiral Knots

Recall the following definitions.

Definition 6.2.1. *For a given knot, K, its mirror image, K', is the knot obtained by reflecting K in some plane.*

This definition leads us to the idea of an ampheichiral knot.

Definition 6.2.2. *A knot is ampheichiral if it is isotopic to its mirror image.*

Something very interesting is happening with our polynomials for amphicheiral knots. For knots up to eight crossings, the following are amphicheiral: 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, and 8_{18}. There are several more amphicheiral knots of 10 crossing which we include in our table of amphicheiral knots as well. Notice that our results support Conjecture 3.5.4 involving amphichieral knots which was first conjectured by Rozansky in [**23**].

Knot	$\Delta_{\mathcal{K}}(t),\ P^{(1)}(\mathcal{K};t)\,\&\,P^{(2)}(\mathcal{K};t)$
4_1	$\Delta_{\mathcal{K}}(t) = -t^{-2}+3-t^2$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-t^{-2}+1-t^2)$
6_3	$\Delta_{\mathcal{K}}(t) = t^{-4}-3t^{-2}+5-3t^2+t^4$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-t^{-8}+t^{-6}+10t^{-4}-30t^{-2}+41-30t^2+10t^4+t^6-t^8)$
8_3	$\Delta_{\mathcal{K}}(t) = -4t^{-2}+9-4t^2$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-4t^{-6}+35t^{-4}-116t^{-2}+166-116t^2+35t^4-4t^6)$
8_9	$\Delta_{\mathcal{K}}(t) = -t^{-6}+3t^{-4}-5t^{-2}+7-5t^2+3t^4-t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(t^{-14}-9t^{-12}+26t^{-10}-26t^{-8}-40t^{-6}+180t^{-4}-335t^{-2}$ $+404-335t^2+180t^4-40t^6-26t^8+26t^{10}-9t^{12}+t^{14})$
8_{12}	$\Delta_{\mathcal{K}}(t) = t^{-4}-7t^{-2}+13-7t^2+t^4$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(4t^{-8}-44t^{-6}+197t^{-4}-457t^{-2}+597-457t^2+197t^4$ $-44t^6+4t^8)$

Table 6.2: $P^{(1)}(\mathcal{K};t)$ and $P^{(2)}(\mathcal{K};t)$ for Amphicheiral Knots

Knot	$\Delta_{\mathcal{K}}(t),\ P^{(1)}(\mathcal{K};t)\,\&\,P^{(2)}(\mathcal{K};t)$
8_{17}	$\Delta_{\mathcal{K}}(t) = -t^{-6}+4t^{-4}-8t^{-2}+11-8t^2+4t^4-t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(t^{-14}-8t^{-12}+16t^{-10}+31t^{-8}-235t^{-6}+620t^{-4}$ $-1031t^{-2}+1211-1031t^2+620t^4-235t^6+31t^8+16t^{10}-8t^{12}$ $+t^{14})$
8_{18}	$\Delta_{\mathcal{K}}(t) = -t^{-6}+5t^{-4}-10t^{-2}+13-10t^2+5t^4-t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-13t^{-10}+115t^{-8}-463t^{-6}+1126t^{-4}-185t^{-2}+2185$ $-1857t^2+1126t^4-463t^6+115t^8-13t^{10})$
10_{17}	$\Delta_{\mathcal{K}}(t) = t^{-8}-3t^{-6}+5t^{-4}-7t^{-2}+9-7t^2+5t^4-3t^6+t^8$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(t^{-20}+9t^{-18}-44t^{-16}+124t^{-14}-212t^{-12}+192t^{-10}+$ $78t^{-8}-638t^{-6}+1366t^{-4}-2003t^{-2}+2260-2003t^2+1366t^4$ $-638t^6+78t^8+192t^{10}-212t^{12}+124t^{14}-44t^{16}+9t^{18}-t^{20})$
10_{33}	$\Delta_{\mathcal{K}}(t) = 4t^{-4}-16t^{-2}+25-16t^2+4t^4$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(4t^{-12}-40t^{-10}+265t^{-8}-1232t^{-6}+3666t^{-4}-6944t^{-2}$ $+8562-6944t^2+3666t^4-1232t^6+265t^8-40t^{10}+4t^{12})$

Knot	$\Delta_{\mathcal{K}}(t),\ P^{(1)}(\mathcal{K};t)\,\&\,P^{(2)}(\mathcal{K};t)$
10_{37}	$\Delta_{\mathcal{K}}(t) = 4t^{-4} - 13t^{-2} + 19 - 13t^2 + 4t^4$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(16t^{-16} - 208t^{-14} + 1347t^{-12} - 6164t^{-10} + 21438t^{-8}$ $-\ 56528t^{-6} + 112519t^{-4} - 169544t^{-2} + 194251 - 169544t^2$ $+ 112519t^4 - 56528t^6 + 21438t^8 - 6164t^{10} + 1347t^{12} - 208t^{14}$ $+ 16t^{16})$
10_{43}	$\Delta_{\mathcal{K}}(t) = -t^{-6} + 7t^{-4} - 17t^{-2} + 23 - 17t^2 + 7t^4 - t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-2t^{-14} + 18t^{-12} - 110t^{-10} + 566t^{-8} - 2096t^{-6} + 5232t^{-4}$ $-\ 8924t^{-2} + 10634 - 8924t^2 + 5232t^4 - 2096t^6 + 566t^8 - 110t^{10}$ $+ 18t^{12} - 2t^{14})$
10_{45}	$\Delta_{\mathcal{K}}(t) = -t^{-6} + 7t^{-4} - 21t^{-2} + 31 - 21t^2 + 7t^4 - t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(8t^{-12} - 133t^{-10} + 929t^{-8} - 3689t^{-6} + 9361t^{-4} - 16070t^{-2}$ $+ 19186\ - 16070t^2 + 9361t^4 - 3689t^6 + 929t^8 - 133t^{10} + 8t^{12})$

Knot	$\Delta_{\mathcal{K}}(t),\ P^{(1)}(\mathcal{K};t)\,\&\,P^{(2)}(\mathcal{K};t)$
10_{79}	$\Delta_{\mathcal{K}}(t) = t^{-8} - 3t^{-6} + 7t^{-4} - 12t^{-2} + 15 - 12t^2 + 7t^4 - 3t^6 + t^8$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-2t^{-20} + 14t^{-18} - 54t^{-16} + 128t^{-14} - 171t^{-12} - 15t^{-10}$ $+\ 722t^{-8} - 2105t^{-6} + 3931t^{-4} - 5539t^{-2} + 6187 - 5539t^2 + 3931t^4$ $- 2105t^6 + 722t^8 - 15t^{10} - 171t^{12} + 128t^{14} - 54t^{16} + 14t^{18} - 2t^{20})$
10_{81}	$\Delta_{\mathcal{K}}(t) = -t^{-6} + 8t^{-4} - 20t^{-2} + 27 - 20t^2 + 8t^4 - t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-4t^{-14} + 40t^{-12} - 243t^{-10} + 1136t^{-8} - 3847t^{-6}$ $+ 9079t^{-4} - 15043t^{-2} + 17767 - 15043t^2 + 9079t^4 - 3847t^6$ $+ 1136t^8 - 243t^{10} + 40t^{12} - 4t^{14})$
10_{88}	$\Delta_{\mathcal{K}}(t) = -t^{-6} + 8t^{-4} - 24t^{-2} + 35 - 24t^2 + 8t^4 - t^6$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-2t^{-14} + 40t^{-12} - 362t^{-10} + 1930t^{-8} - 6689t^{-6}$ $+ 15822t^{-4} - 26252t^{-2} + 31025 - 26252t^2 + 15822t^4 - 6689t^6$ $+ 1930t^8 - 362t^{10} + 40t^{12} - 2t^{14})$

Knot	$\Delta_{\mathcal{K}}(t),\ P^{(1)}(\mathcal{K};t)\,\&\,P^{(2)}(\mathcal{K};t)$
10_{99}	$\Delta_{\mathcal{K}}(t) = t^{-8} - 4t^{-6} + 10t^{-4} - 16t^{-2} + 19 - 16t^{2} + 10t^{4} - 4t^{6} + t^{8}$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-2t^{-20} + 16t^{-18} - 64t^{-16} + 144t^{-14} - 118t^{-12} - 416t^{-10}$ $+\ 2056t^{-8} - 5152t^{-6} + 9222t^{-4} - 12816t^{-2} + 14264 - 12816t^{2}$ $+ 9222t^{4} - 5152t^{6} + 2056t^{8} - 416t^{10} - 118t^{12} + 144t^{14} - 64t^{16}$ $+ 16t^{18} - 2t^{20})$
10_{109}	$\Delta_{\mathcal{K}}(t) = t^{-8} - 4t^{-6} + 10t^{-4} - 17t^{-2} + 21 - 17t^{2} + 10t^{4} - 4t^{6} + t^{8}$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-2t^{-20} + 16t^{-18} - 63t^{-16} + 134t^{-14} - 68t^{-12} - 581t^{-10}$ $+\ 2462t^{-8} - 5935t^{-6} + 10446t^{-4} - 14403t^{-2} + 15991 - 14403t^{2}$ $+ 10446t^{4} - 5935t^{6} + 2462t^{8} - 581t^{10} - 68t^{12} + 134t^{14} - 63t^{16}$ $+ 16t^{18} - 2t^{20})$
10_{115}	$\Delta_{\mathcal{K}}(t) = -t^{-6} + 9t^{-4} - 26t^{-2} + 37 - 26t^{2} + 9t^{4} - t^{6}$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-5t^{-14} + 75t^{-12} - 573t^{-10} + 2804t^{-8} - 9339t^{-6}$ $+ 21685t^{-4} - 35672t^{-2} + 42051 - 35672t^{2} + 21685t^{4} - 9339t^{6}$ $+ 2804t^{8} - 573t^{10} + 75t^{12} - 5t^{14})$

Knot	$\Delta_{\mathcal{K}}(t),\ P^{(1)}(\mathcal{K};t)\,\&\,P^{(2)}(\mathcal{K};t)$
10_{118}	$\Delta_{\mathcal{K}}(t) = t^{-8} - 5t^{-6} + 12t^{-4} - 19t^{-2} + 23 - 19t^{2} + 12t^{4} - 5t^{6} + t^{8}$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(-t^{-20} + 11t^{-18} - 53t^{-16} + 124t^{-14} - 18t^{-12} - 917t^{-10}$ $+\ 3703t^{-8} - 8971t^{-6} + 15915t^{-4} - 22063t^{-2} + 24540 - 22063t^{2}$ $+ 15915t^{4} - 8971t^{6} + 3703t^{8} - 917t^{10} - 18t^{12} + 124t^{14} - 53t^{16}$ $+ 11t^{18} - t^{20})$
10_{123}	$\Delta_{\mathcal{K}}(t) = t^{-8} - 6t^{-6} + 15t^{-4} - 24t^{-2} + 29 - 24t^{2} + 15t^{4} - 6t^{6} + t^{8}$ $P^{(1)}(\mathcal{K};t) = 0$ $P^{(2)}(\mathcal{K};t) = \Delta_{\mathcal{K}}(t)(5t^{-16} - 86t^{-14} + 628t^{-12} - 2718t^{-10} + 8058t^{-8}$ $-\ 17716t^{-6} + 30279t^{-4} - 41372t^{-2} + 45842 - 41372t^{2} + 30279t^{4}$ $- 17716t^{6} + 8058t^{8} - 2718t^{10} + 628t^{12} - 86t^{14} + 5t^{16})$

One can see that our data does support the conjecture, and it will be an interesting thing to explore in the future. By exploring this further, one can hope to get a more topological understanding of the Jones polynomial.

6.3. Future Work

In the near future, we will continue working on increasing the efficiency of the Mathematica program and distribute it for use by others. We will also calculate more terms in the expansion of the colored Jones polynomial in Equation (3.5.4). This will give more polynomial invariants, first starting with $P^{(3)}(\mathcal{K};t)$ from Equation (3.5.5) and moving on to $P^{(4)}(\mathcal{K};t)$, $P^{(5)}(\mathcal{K};t)$ and so on.

The overarching and long term goal of this research is to gain a topological understanding of the Jones polynomial. There are many things that we may need to do in order to reach that goal. Once the polynomials that arise in the expansion from the $U_q(\mathfrak{sl}(2))$ case have been explored, we will study similar polynomials that arise from considering $U_q(\mathfrak{sl}(3))$. This is a quantum group built from $\mathfrak{sl}(3)$. Like $U_q(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(3))$ also has an R-matrix and similar, although more complicated, calculations can be made using this.

Mathematica Program

```
knot =;
initialcrossing = firstcrossing[];
initialmatrix = firstmatrix[];
braidword = Sequence[];
n = KnotData[knot, "BraidIndex"];
firstcrossing[x_, y_] := If[x < y, HO_{x,y}, INV_{y,x}] ;
firstmatrix[x_, y_] := If[x < y, Matrix_{x,y}, Imatrix_{y,x}];
alex = KnotData[knot, "AlexanderPolynomial"][t^2];
amp = KnotData[knot, "Amphichiral"];
rep2[i_, j_] :=
 Which[i == j == 0, 1, True, ((i! j! Coefficient[exp, d_2^i z_2^j ]) /. {z_2 -> 0, d_2 -> 0})]
rep3[i_, j_, k_, l_] := Which[i == j == k == l == 0, 1, True,
  ((i! j! k! l! Coefficient[exp, d_2^i z_2^j d_3 ^k z_3^l]) /. {z_2 -> 0, d_2 -> 0, z_3 -> 0, d_3 -> 0})]
rep4[i_, j_, k_, l_, m_, n_] := Which[i == j == k == l == m == n == 0, 1, True,
  ((i! j! k! l! m! n! Coefficient[exp, d_2^i z_2^j d_3 ^k z_3^l d_4 ^m z_4^n]) /.
    {z_2 -> 0, d_2 -> 0, z_3 -> 0, d_3 -> 0, z_4 -> 0, d_4 -> 0})]
rep5[i_, j_, k_, l_, m_, n_, o_, p_] := Which[i == j == k == l == m == n == o == p == 0, 1, True,
  ((i! j! k! l! m! n! o! p! Coefficient[exp, d_2^i z_2^j d_3 ^k z_3^l d_4 ^m z_4^n d_5 ^o z_5^p]) /.
    {z_2 -> 0, d_2 -> 0, z_3 -> 0, d_3 -> 0, z_4 -> 0, d_4 -> 0, z_5 -> 0, d_5 -> 0})]
rep6[i_, j_, k_, l_, m_, n_, o_, p_, q_, r_] := Which[i == j == k == l == m == n == o == p == q == r == 0,
  1, True, ((i! j! k! l! m! n! o! p! q! r!
      Coefficient[exp, d_2^i z_2^j d_3 ^k z_3^l d_4 ^m z_4^n d_5 ^o z_5^p d_6^q z_6^r]) /.
    {z_2 -> 0, d_2 -> 0, z_3 -> 0, d_3 -> 0, z_4 -> 0, d_4 -> 0, z_5 -> 0, d_5 -> 0, z_6 -> 0, d_6 -> 0})]
trace_2[s_] := Expand[Plus @@ (s /. Rule[{a_, b_}, c_] :> rep2[a, b] c)]
trace_3[s_] := Expand[Plus @@ (s /. Rule[{a_, b_, e_, f_}, c_] :> rep3[a, b, e, f] c)]
trace_4[s_] :=
 Expand[Plus @@ (s /. Rule[{a_, b_, e_, f_, g_, h_}, c_] :> rep4[a, b, e, f, g, h] c)]
trace_5[s_] := Expand[
  Plus @@ (s /. Rule[{a_, b_, e_, f_, g_, h_, i_, j_}, c_] :> rep5[a, b, e, f, g, h, i, j] c)]
trace_6[s_] := Expand[Plus @@ (s /.
     Rule[{a_, b_, e_, f_, g_, h_, i_, j_, k_, l_}, c_] :> rep6[a, b, e, f, g, h, i, j, k, l] c)]
qh[n_] := (1 + h Coefficient[Product[1 + 2 h z_i d_i + 2 h^2 (z_i d_i + z_i^2 d_i^2), {i, 2, n}], h, 1] +
   h^2 Coefficient[Product[1 + 2 h z_i d_i + 2 h^2 (z_i d_i + z_i^2 d_i^2), {i, 2, n}], h, 2])
p_{l_,k_}[i_, j_] := Which[i == k && j == l, t, i == l && j == k, t, i == j == l,
  0, i == j == k, 1 - t^2, i == j, 1, True, 0]
Matrix_{l_,k_} := Array[p_{l,k}, {n, n}]
Imatrix_{l_,k_} := Inverse[Matrix_{l,k}]
Sz_{l_,k_}[z_{j_}] := Sum[z_i Imatrix_{l,k}[[i, j]], {i, 1, n}];
Sder_{l_,k_}[d_{j_}] := Sum[Matrix_{l,k}[[i, j]] d_i, {i, 1, n}];
ISz_{l_,k_}[z_{j_}] := Sum[z_i Matrix_{l,k}[[i, j]], {i, 1, n}];
ISder_{l_,k_}[d_{j_}] := Sum[Imatrix_{l,k}[[i, j]] d_i, {i, 1, n}];
```

```
Higher[a_, b_, w_, x_] :=
  1 + h ((3 - 1/t^2 - 2 t^2) a^2 * x^2 + (-2/t + 2 t) a^2 * x * w + (1/t - 3 t) b * a * x^2 + 2 b * a * x * w) +
   h^2 ((5 - 1/t^2 - 4 t^2) a^2 * x^2 + (4/(3 t) - 16 t/3 + 4 t^3) a^3 * x^3 + (13/2 + 1/(2 t^4) - 3/t^2 - 6 t^2 + 2 t^4) a^4 * x^4 +
      (-2/t + 2 t) a^2 * x * w + (4 - 4 t^2) a^3 * x^2 * w + (2/t^3 - 8/t + 10 t - 4 t^3) a^4 * x^3 * w +
      (-2/t + 2 t) a^3 * x * w^2 + (-4 + 2/t^2 + 2 t^2) a^4 * x^2 * w^2 + (1/t - 5 t) b * a * x^2 + (2/(3 t) - 14 t/3) b^2 * a * x^3 +
      (10/3 - 2/(3 t^2) + 4 t^2/3) b * a^2 * x^3 + (-3 + 1/(2 t^2) + 9 t^2/2) b^2 * a^2 * x^4 + (-1/t^3 + 6/t - 11 t + 6 t^3) b * a^3 * x^4 +
      2 b * a * x * w + 2 b^2 * a * x^2 * w + (-2/t - 2 t) b * a^2 * x^2 * w + (2/t - 6 t) b^2 * a^2 * x^3 * w +
      (14 - 4/t^2 - 10 t^2) b * a^3 * x^3 * w + 2 b * a^2 * x * w^2 + 2 b^2 * a^2 * x^2 * w^2 + (-4/t + 4 t) b * a^3 * x^2 * w^2)
Inv[a_, b_, x_, y_] := 1 + h ((1 - t^2) b^2 * x^2 + (1/t + t) b * a * x^2 - 2 b * a * y * x) +
  h^2 ((1 - t^2) b^2 * x^2 + (8/(3 t) - 8 t/3) b^3 * x^3 + (1/2 - t^2 + t^4/2) b^4 * x^4 + (-1/t + t) b * a * x^2 + (-4 + 4 t^2)
      b^3 * y * x^2 + (10/3 + 4/(3 t^2) - 2 t^2/3) b^2 * a * x^3 + (-2/(3 t) + 2 t/3) b * a^2 * x^3 + (1/t - t^3) b^3 * a * x^4 +
     (1 + 1/(2 t^2) + t^2/2) b^2 * a^2 * x^4 + 2 b * a * y * x + 2 b^2 * a * y^2 * x + (-4/t - 4 t) b^2 * a * y * x^2 +
     2 b * a^2 * y * x^2 + 2 b^2 * a^2 * y^2 * x^2 + (-2 + 2 t^2) b^3 * a * y * x^3 + (-2/t - 2 t) b^2 * a^2 * y * x^3)
HO_{i_,j_} := Higher[z_i, z_j, d_i, d_j]
INV_{i_,j_} := Inv[z_i, z_j, d_i, d_j]
Der_{i_}[f_] := f + D[f, w_i, x_i] + (1 / 2) D[f, {w_i, 2}, {x_i, 2}] + (1 / 6) D[f, {w_i, 3}, {x_i, 3}] +
   (1 / 24) D[f, {w_i, 4}, {x_i, 4}] + 1 / (5 !) D[f, {w_i, 5}, {x_i, 5}] + (1 / 6 !) D[f, {w_i, 6}, {x_i, 6}]
NO[l_, {i_, j_}] := If[i < j,
   (Der_j[Der_i[Expand[((1 /. Flatten[Table[{z_k → Sz_{i,j}[z_k], d_k → Sder_{i,j}[d_k]}, {k, 1, n}]]) /.
           {d_i → w_i, d_j → w_j}) + h Coefficient[Higher[x_i, x_j, d_i, d_j], h, 1] +
         h^2 (Coefficient[Higher[x_i, x_j, d_i, d_j], h, 2] + Coefficient[
              ((1 /. Flatten[Table[{z_k → Sz_{i,j}[z_k], d_k → Sder_{i,j}[d_k]}, {k, 1, n}]]) /. {d_i → w_i,
                  d_j → w_j}), h, 1] * Coefficient[Higher[x_i, x_j, d_i, d_j], h, 1])]]]) /.
    {w_i → d_i, w_j → d_j, x_i → z_i, x_j → z_j}, (Der_j[Der_i[Expand[((1 /. Flatten[
              Table[{z_k → ISz_{j,i}[z_k], d_k → ISder_{j,i}[d_k]}, {k, 1, n}]]) /. {d_i → w_i, d_j → w_j}) +
         h Coefficient[Inv[x_j, x_i, d_j, d_i], h, 1] + h^2 (Coefficient[Inv[x_j, x_i, d_j, d_i], h, 2] +
            Coefficient[((1 /. Flatten[Table[{z_k → ISz_{j,i}[z_k], d_k → ISder_{j,i}[d_k]},
                    {k, 1, n}]]) /. {d_i → w_i, d_j → w_j}), h, 1] * Coefficient[
              Inv[x_j, x_i, d_j, d_i], h, 1])]]]) /. {w_i → d_i, w_j → d_j, x_i → z_i, x_j → z_j}]
```

```
g[l_, {x_ , y_}] := If[x < y, l.(Matrix_{x,y}), l.(Imatrix_{y,x})]
qhend[f_] := Expand[(f /. Flatten[Table[d_i → w_i, {i, 2, n}]]) +
    h (Coefficient[qh[n], h, 1] /. Flatten[Table[z_i → x_i, {i, 2, n}]]) +
    h^2 (Coefficient[qh[n], h, 2] /. Flatten[Table[z_i → x_i, {i, 2, n}]]) +
    h^2 ((Coefficient[f, h, 1] /. Flatten[Table[d_i → w_i, {i, 2, n}]]) *
        (Coefficient[qh[n], h, 1] /. Flatten[Table[z_i → x_i, {i, 2, n}]]))]
Quad = Expand[Fold[g, initialmatrix, {braidword}]];
a_{i_,j_} := If[i == j, Quad[[i, j]] - 1, Quad[[i, j]]]
A = Table[a_{i,j}, {i, 2, n}, {j, 2, n}];
F = Inverse[-A];
HigherOrder = Fold[NO, initialcrossing, {braidword}];
kerbeforesub = ((l = qhend[HigherOrder]; Do[l = Der_i[l], {i, 2, n, 1}]; l) /.
     Flatten[Table[{x_i → z_i, w_i → d_i}, {i, 2, n}]]);
keraftersub =
   Expand[kerbeforesub /. Flatten[Table[z_k → Sum[Quad[[i, k]] z_i, {i, 1, n}], {k, 1, n}]]];
poly = Expand[
    keraftersub /. Flatten[Table[{z_i → z_i + Sum[F[[k, i - 1]] Quad[[1, k + 1]] z_1, {k, 1, n - 1}],
         d_i -> d_i + Sum[F[[i - 1, l]] Quad[[l + 1, 1]] d_1, {l, 1, n - 1}]}, {i, 2, n}]]];
exp =
   Expand[1 + Sum[(Expand[Sum[F[[i - 1, j - 1]] d_i z_j, {j, 2, n}, {i, 2, n}]])^i/i!, {i, 1, 6}]];
poly2 = CoefficientRules[poly, Flatten[Table[{d_i, z_i}, {i, 2, n}]]]
tr = trace_n[poly2];
test1 = Together[Coefficient[tr, h, 1]];
test2 = Together[Coefficient[tr, h, 2]];
p = If[amp == True, Cancel[Denominator[test1] / alex], Cancel[Denominator[test1] / alex^2]];
q =
  If[amp == True, Cancel[Denominator[test2] / alex^3], Cancel[Denominator[test2] / alex^4]];
term1 = Expand[Numerator[test1] / p] / Expand[Denominator[test1] / p];
term2 = Expand[Numerator[test2] / q] / Expand[Denominator[test2] / q];
series = Series[((((1 + h term1 + h^2 term2) (1 / alex)) /. {t → t Exp[h]}) alex) /.
     {h → h / 2 - h^2 / 4}, {h, 0, 2}];
p1 = Expand[Numerator[Coefficient[series, h, 1]] / p]
p2 = Expand[Numerator[Coefficient[series, h, 2]] / q]
```

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