# Multi-Component Diffusion Study and Stability Analysis 

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#### Abstract

The effect of multi-component diffusive convection in a fluid layer is studied, through both an experimental component and a neutral stability analysis. In addition to the temperature field, the analysis looks at a dissolved salt field and the presence of dye. Using salt- and dye-stratified corn syrup and water mixtures, the destabilizing behaviors common to multi-component convection, oscillations and fingering, were observed within a small range of densities. The neutral stability analysis created a comparison between the Rayleigh numbers for salt, dye, and temperature, such that values given by specified relations between parameters correspond to the destabilizing fingering and oscillatory effects. The initial set up of the experiments indicates that the destabilizing features cannot be solely caused by the multi-component convection understood in the theoretical equations, as they lie in a region where all fields are both statically and dynamically stable. Therefore, there must be other factors involved in the observed diffusion.


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## 1 Introduction

Multi-component diffusive convection, where the flow density depends on more than two components, has applications in many natural fields such as oceanography, geophysics, environmental engineering, and astrophysics [9],[14]. In the standard convection problem, instabilities are driven by density differences between the upper and lower planes bounding the fluid [9]. In thermal convection, a fluid with a higher temperature will be less dense, allowing the parcel of fluid to rise, while colder fluid will fall because it is denser than its surroundings. If the fluid layer in question also has some concentration of another substance dissolved within, there are two destabilizing sources. The most common naturally produced scenario, thermohaline convection, involves a temperature field and sodium chloride [3]. When there are two effects, with competing stabilizing and destabilizing forces, the convection phenomenon is called double diffusive convection. In double diffusive flows, the two components often have very different molecular diffusivities, which operate on different time scales [14]. These differences induce interesting flow phenomena such as salt-fingers and oscillating convection cells within the fluid plane. By introducing a third component, the competing behaviors give rise to many combinations of stabilizing and destabilizing phenomena. Applications of this research include modeling geothermal reservoirs, harnessing the sun's energy through solar ponds, and understanding pollution transport [9].

The fingering and oscillation effects usually associated with double diffusion have been recorded in many recent experiments involving the density stratification of corn syrup. To understand if this behavior is connected to multi-component convection or if it is a separate phenomenon, we ran several trials using sharply stratified corn syrup and water mixtures. By analyzing the behavior of these experiments and comparing the results with a stability analysis for multi-component convection, we can better understand the processes at work in these situations.

## 2 Background

Possibly due to the lack of immediate practical applications and the relatively small size of the oceanography community, this field of fluid mechanics developed much slower than other branches of fluid dynamics [3]. A description of the physical phenomenon of double diffusive processes was not presented until Stommel, Arons, and Balnchard wrote "An oceanographical curiosity: the perpetual salt fountain" in 1956 [8]. In their work they imagined a heat-conducting pipe vertically suspended
in an ocean with its upper end in warm, salty water and its lower end in denser, cooler, and fresher water. By pumping fluid from below, through the pipe, the water inside quickly reaches the same temperature as the surrounding ocean outside of the pipe, but this water remains fresh because the salt cannot permeate the pipe walls. This fresh fluid is less dense than the surround water, allowing the water to flow up the pipe. Water from below would continue to move up the pipe until the salinity gradient is gone [3].


Figure 1: A field of salt fingers formed by setting up a stable temperature gradient and pouring a salt and fluorescein solution on top [3], [4].

Stern soon pointed out that the walls of the pipe are not essential to the problem, because salt diffuses much more slowly than heat. Motions similar to those proposed by Stommel et al. within the pipe can occur in a fluid containing temperature and salinity gradients because of the lower transfer of salt compared to heat [8]. If a fluid parcel rises from $z$ to $z+\nabla z$, the much greater thermal diffusivity allows the parcel to absorb heat faster than it absorbs salt. It is then lighter than the surrounding fluid
and continues to rise. Conversely, a fluid parcel that drops by $\nabla z$, quickly becomes heavier than the surrounding fluid as it loses heat, and it continues to sink [4]. These parcels that quickly sink or rise create long and narrow convection cells, called salt fingers. Figure (1) presents a field of salt fingers contrasted with the surrounding fluid by using fluorescein dye [3]. Stern predicted the scale of these fingers using a linear instability calculation [7].


Figure 2: A series of convecting layers and 'diffusive' interfaces, formed by heating a gradient of $\mathrm{K}_{2} \mathrm{CO}_{3}$ solution from below [3].

Stern also found through his stability analysis that the opposite situation, with cold, fresh water over warm, salty water, corresponds to oscillatory instability [3]. Considering the same initial perturbation, the lighter fluid parcel moved up by $\nabla z$ will lose heat quickly, while losing little salt. Therefore, it will fall again, creating oscillations. This non-instantaneous heat transfer leads to the unstable growth of these oscillations, resulting in a mixed layer with a thermal boundary similar to that of
the Rayleigh-Bernard problem [4]. When some critical Rayleigh number is reached, it becomes unstable and a second convecting layer forms above the first [3]. In many cases, the fluid forms several distinct mixed layers as seen in Turner and Stommel's investigations [3]. Such convecting layers can be seen in Figure (2), where a $\mathrm{K}_{2} \mathrm{CO}_{3}$ solution was heated from below.

Veronis [12],[13] realized that the layers and interfaces first studied by Turner and Stommel (1964) could only be explained theoretically by a nonlinear theory [3]. In an extended Rayleigh-Benard problem, Veronis studied the two-dimensional behavior of fluid bounded by two horizontal planes, heated and salted from below [3]. By solving the partial differential equations for conservation of momentum, heat, and salt, researchers found two different solutions, a steady, direct case and an oscillatory case, corresponding to the observed phenomenon. The stability analysis and experimental results revealed that, under certain conditions and perturbations, the resulting destabilization can create movement and density transfers in ways that are very different from the one-component case.

## 3 Diffusion Experiments

To study the impact of multi-component or double diffusion on stratified solutions, we conducted a series of experiments, varying the densities of the fluid and the salt involved. The initial experiments mixed Karo corn syrup with water to create "fresh" fluids varying in densities from 1.20022 to $1.12013 \mathrm{~g} / \mathrm{cm}^{3}$. These corresponded to specified ratios of water $\left(\rho \approx 1 \mathrm{~g} / \mathrm{cm}^{3}\right.$ ) and corn $\operatorname{syrup}\left(\rho \approx 1.36 \mathrm{~g} / \mathrm{cm}^{3}\right.$ ), From 4 parts water and 2 parts corn syrup (4:2) up to five parts corn syrup (4:5). As the phenomena has been observed more often in viscous Karo experiments and less in water, by taking a range of densities we could estimate a critical density threshold for these phenomena. These solutions were created by adding water to the corn syrup using a magnetic stirrer, until the densities were within $0.0005 \mathrm{~g} / \mathrm{cm}^{3}$ of the specified amount.

Each density mixture was then split into two 500 mL containers for the stratification. A calculated amount of salt was added to the bottom layer to increase the density by $0.004 \mathrm{~g} / \mathrm{cm}^{3}$, as well as two drops of dye, which created a visual difference between the layers. The bottom layer was mixed for at least twenty minutes, again, using the magnetic stirrer.

The top layer of un-dyed, fresh corn syrup was stratified over the salty bottom layer
by "floating" the lighter layer on top to maintain an initially sharp stratification. This is done by floating a sponge supported by a highly buoyant material such as styrofoam on the surface of the bottom layer. Filtering the top layer through the sponge disperses the lighter fluid over a broad area, minimizing the mixing between layers. As more fluid is added, the foam allows the sponge to rise with the top layer, allowing the top layer to increase above a sharp stratification.

Once stratified, the four solutions, ranging from the densest at approximately 1.20022 $\mathrm{g} / \mathrm{cm}^{3}$ to the least dense at $1.12013 \mathrm{~g} / \mathrm{cm}^{3}$ were covered with plastic wrap and left to naturally diffuse. The solutions were left at room temperature, susceptible to any fluctuations. Interval shots were taken with a Nikon D3 camera to visually record the diffusion process. Density, conductivity, and viscosity measurements were taken from the top and bottom layer, 5 mm below the surface and 5 mm above the bottom, before and after the experiment. In later experiments, post-experiment measurements of density, conductivity, and viscosity were also taken at the initial point of stratification, 45 mm from the bottom. In the final experiments, continuous temperature measurements in the two density extremes were recorded using thermistors placed 10 mm below and above the initial point of stratification. For visual reference, see Appendix B for all measured values.

Four salts were used to increase the density within the solution, $\mathrm{NaCl}, \mathrm{NaI}, \mathrm{KCl}$, and KI. The combination of salts allowed us to look at the differences between a Chloride and Iodide base when combined with Sodium and Potassium. The length of time each experiment lasted varied slightly for each salt because of the longer time it takes some salts to diffuse over others. The experiments were ended when the dye had visually diffused over most of the solution. One such experiment can be seen in Figure (3).

Since NaCl is the substance behind thermohaline convection and previous experiments by Valchar [11] had recorded the fingering and oscillatory destabilizations with NaCl , more extended experiments were done using this salt. One "zoomed in" experiment looked at a set of four densities ranging between the two most dense solutions that were used in all of the experiments, $1.20022 \mathrm{~g} / \mathrm{cm}^{3}$ and $1.19406 \mathrm{~g} / \mathrm{cm}^{3}$. Using the water to corn syrup ratio language described earlier, these solutions were $\left[(4: 4),\left(4: 4 \frac{1}{3}\right),\left(4: 4 \frac{2}{3}\right),(4: 5)\right]$. To get a better idea of the threshold for the destabilizing phenomena, an "extended" version was also performed, using solutions that were more dense than the original $1.20022 \mathrm{~g} / \mathrm{cm}^{3}$. These were between $1.20901 \mathrm{~g} / \mathrm{cm}^{3}$ and $1.19532 \mathrm{~g} / \mathrm{cm}^{3}$, roughly $\left[\left(4: 4 \frac{3}{4}\right),(4: 5),\left(4: 5 \frac{1}{4}\right),\left(4: 5 \frac{1}{2}\right)\right]$.


Figure 3: Diffusion of corn syrup/ water, stratified by KI and dye, density increasing from right

## 4 Results

### 4.1 Potassium Iodide (KI)

From a visual analysis of the Potassium Chloride or KI experiment, it is immediately clear that no instabilities occur within the given density range. The KI solutions showed no fingering or oscillatory behavior. The least dense solution diffused the fastest. The denser the solution, the longer the dye took to diffuse. Such a correlation can be seen in Figure (3b), where the dye in the least dense (far right) has diffused completely, but the densest has barely changed from the initial stratification.

Comparing the measurements taken before and after the experiment, the density and viscosity changed very little in the top or the bottom for KI. It is clear that some kind of diffusion occurred, though, due to the change in conductivity. The conductivity reading for the top of all four solutions doubled over the course of the experiment, significantly reducing the difference between the top and bottom. Since electrical conductivity is largely influenced by the salt concentration within the solution, this change implies that the salt diffused throughout the mixture over time as one would expect.

### 4.2 Sodium Iodide (NaI)

Sodium Iodide or NaI also exhibited no visible signs of oscillatory or fingering behavior. Similar to KI, the most dense solution diffused the slowest, with speed inversely related to density, as one would expect in a singular diffusion environment. Figure (4) shows how the dye diffused evenly in this experiment, with images taken at the beginning, middle, and end.

Density changed very little in the top and bottom measurements taken before and after diffusion. Unlike KI, viscosity did change significantly over time within the two most dense solutions. In the second most dense (4:4), both the viscosity in the top and the bottom dropped by a similar amount, maintaining the difference between the values. While the measured viscosity in the densest solution (4:5) changed considerably, the thermistor attached to side of this beaker broke loose, disturbing the stratification and diffusion process. This could have caused the changes in the densest solution unrelated to diffusion or any instabilities. As with KI, the conductivity increased in the top of the solutions over time, showing that the salt was most likely diffusing upwards into the fresh corn syrup water mixture.


Figure 4: Diffusion stratified with NaI and dye at $\mathrm{t}=0,95$, and 189 hrs , density increasing from right

Temperature was also measured using four thermistors placed in the two extremes of the densities observed, 10 mm above and below the initial stratification line. Within the first twenty hours there was an initial increase of 1.5 degrees, measured in both the top and the bottom of (4:5) and in the top of (4:2). The bottom of (4:2) measured an initial 1 degree increase. After that, the solutions remained relatively constant, with a small increase of about 0.5 degrees over time in all measured points. This increase did not correspond to any specific visual changes in the dye.

### 4.3 Potassium Chloride ( KCl )

Within minutes of stratifying with Potassium Chloride, or KCl , the denser solutions began to show signs of fingering. First in the most dense $\left(\rho_{4: 5{ }_{\text {Top }}}=1.20005 \mathrm{~g} / \mathrm{cm}^{3}\right)$ and soon in the next solution ( $\rho_{4: 4_{T o p}}=1.18077 \mathrm{~g} / \mathrm{cm}^{3}$ ). In the most dense (4:5) solution, the fingering effect also involved very short oscillations. This caused the dye to diffuse in very visible spurts or steps. After forty hours, the dye hit some kind of barrier halfway up the top part of the stratification and did not diffuse any farther, even though the experiment was left running for another twenty-six hours. In contrast, the second most dense (4:4) quickly formed long fingers which doubled back into oscillation halfway up, overtaking the rest of the solution in the next oscillation or step. This change in the size of the oscillatory layers allowed the dye to be completely uniform throughout within only eleven hours. The next solution (4:3), did show signs a minimal fingering, but mostly looked to diffuse at a normal rate. Within twenty-four hours the dye in the third most dense had diffused uniformly. The results of the least dense solution (4:2) were contaminated due to stratification issues, causing very little convection to take place at all. The final levels of the dye and some of the fingering effect in (4:4) can be seen in Figure (5), where the left most solution is the densest.

As measured, the density in the top and bottom did not change at all. Viscosity in the top increased by $1 \mathrm{mPa}^{*}$ s in the middle two solutions, but the most and least dense saw no change in the top or bottom. The conductivity increased by 500 $\mu S \mathrm{~cm}$ in the top of the three densest solutions that had good initial stratifications. Temperature was not recorded in this experiment.


Figure 5: Diffusion stratified with KCl and dye at $\mathrm{t}=0,5.5,24.4,66.2 \mathrm{hrs}$, density increasing from right

### 4.4 Sodium Chloride ( NaCl )



Figure 6: Fingering effect in NaCl and dye stratified diffusion

As the source of the original observation of this phenomenon, the majority of the experiments were focused on Sodium Chloride's impact on convection. With two trials using the same density range as the other salts, two looking in depth at the two densest values in the original range, and one looking at denser solutions than the original four, we were able to get a better idea of what factors impact the oscillations and fingering observed.

In the first trial for NaCl clear fingering was observed within all but the least dense solution. This initial fingering can be seen in Figure(6). Once the fingers extended as far as seen in this figure, they curled over and fell back down, turning into the oscillations depicted in the montage of the most dense solution seen in Figure(7). In (b)-(f) you can see a full oscillation, beginning with fingering. Between (e) and (f) you can see how the dye vertically retreats before rising again to the level seen in (g) after another oscillation. These oscillations quickly create a mixed layer, completely diffusing the dye within the first twenty-four hours. Throughout the experiment, a small part of the top layer remained untouched by the dye in the oscillations as seen in (h) of Figure(7), and no dye ever diffused through this section. The oscillations,
though present in the middle two solutions, were smaller, creating the slower steps up the beaker as layers formed. In both cases, after forty-eight hours the dye reached a point it could go past and did not diffuse in the upper third of the beaker. The least dense solution exhibited no oscillatory or fingering behavior. This trial was purely a visual trial and no measurements were taken at the end of the experiment to make comparisons.


Figure 7: Oscillatory Effect in NaCl Stratified Solution

In the second trial of the usual range, the stratification was not exact in the densest two solutions. Then the thermistor became unattached to the side of most dense solution (4:5), forfeiting any chance of getting useful information out of the solution. Even with a poor stratification the second solution (4:4) exhibited clear oscillatory behavior, going through only two large oscillations before the dye was
completely uniform. This allowed the dye in the denser solution to diffuse within fourteen hours, before the less dense ones. By looking at the conductivity measurements in the top and bottom, it is clear that the salt also diffused upwards, as the conductivity decreased in the bottom and increased in the top. Over the course of the experiment, the bottom of the $(4: 4)$ solution became denser by $0.005 \mathrm{~g} / \mathrm{cm}^{3}$ and more viscous by $1.17 \mathrm{mPa}^{*}$ s, which was the largest increase in both density and viscosity in this experiment. The top also showed increases in viscosity and density. The next solution (4:3) was observed to have some layering creating by oscillations. Though there was very little change in the conductivity, in the conductivity profile, the maximum amount was actually about 30 mm from the bottom, still below original stratification line. There was very little change in density, but viscosity increased in both the top and the bottom, where the top actually surpassed the viscosity in the bottom. The least dense solution (4:2) diffused evenly throughout the experiment with no noticeable effects present. The density in the bottom of this solution experienced the largest increase, going from $1.12436 \mathrm{~g} / \mathrm{cm}^{3}$ to $1.13055 \mathrm{~g} / \mathrm{cm}^{3}$, while the density in the top changed very little. The solution became equally more viscous in the top and the bottom, possibly stemming from some form of evaporation. The conductivity measurements show that the salt did diffuse some, but not enough to shorten the difference between the top and bottom by much. Temperature measurements were taken in the least (4:2) and most dense (4:5) solutions, 10 mm above and below the interface. In the least dense, the bottom temperature increased by 1.5 degrees in the first ten hours. The warmer top solution only increased by 1 degree within that time frame. The temperature difference remain stable after that point for the rest of the experiment. In the most dense experiment, the bottom layer was actually warmer than the top, creating an unstable gradient. Unfortunately, the when the thermistors separated from the side, the two temperatures quickly reached an equilibrium for the rest of the experiment, and we are unable to tell how the temperature in the bottom layer changed.

The "zoomed in" look between the two densest solutions looked at solutions with densities corresponding to the following water and corn syrup ratios: $\left[(4: 4),\left(4: 4 \frac{1}{3}\right)\right.$, $\left.\left(4: 4 \frac{2}{3}\right),(4: 5)\right]$. Unfortunately the pictures on the memory card from the experiment were deleted before being able to analyze them. So we only have the before and after measurements for the first zoomed NaCl experiment. Within the two densest solutions, the density difference between the top and bottom grew over time, as the light upper fluid seemed to get lighter and the bottom heavier over time. In contrast, the less dense (4:4) and ( $4: 4 \frac{1}{3}$ ) grew more dense over time in both the top and the bottom, with the top growing denser. In all the solutions the top grew much less vis-
cous over time while the bottom changed very little. These solutions did experience some diffusion, as all showed decreases in conductivity over time in the bottom.

In the second zoomed experiment, the most dense, corresponding to the regular (4:5), exhibited some fingering and the only oscillatory behavior detected was based on the mixed layers of dye that moved up in steps and spurts rather than a steady diffusion. Yet, the conductivity in this solution did not change significantly. The top grew less dense and much less viscous than the start, a decrease of $1.42 \mathrm{mPa}^{*}$. The loss of density and viscosity in the top suggests that the salt did not diffuse, but the lighter material rose to the top a one would see in a standard convection problem. The middle two experiments that fell between the usual (4:4) and (4:5) exhibited minimal fingering and never fully diffused over the course of the experiment. Their top layers also became considerably less viscous and less dense. The only difference, seen in ( $4: 4 \frac{1}{3}$ ), was that the bottom density decreased on par with the top, maintaining the change in density over time within this solution. Neither solution showed significant change in conductivity, suggesting that the salt did not diffuse over the course of the experiment. The least dense for this experiment corresponds to the second most dense in the other experiments, that of an equal ratio of water and corn syrup (4:4). In this case, the solution $\left(\rho_{\text {Top }_{4: 4}}=1.17987\right)$ did not exhibit any fingering, but immediately started large oscillations, allowing for uniform dye within twenty-four hours. In this solution, you see a similar change in viscosity as in the other solutions, but unlike the others, the density actually increased in both the top and the bottom. The conductivity also increased in the top and decreased in the bottom, showing a good amount of the salt diffused upwards. We were also able to take temperature measurements in this trial, straddling the interface in the least and most dense, (4:4) and (4:5) respectively. In (4:4), there was an intial 0.5 degree difference in the two layers, with warmer fluid on top. Within the first 20 hours, both layers grew warmer by half of a degree, but in the next ten hours the solutions stabilized again, dropping back down to the initial values for the rest of the experiment. In (4:5), the top was only 0.3 degrees warmer than the bottom. This difference remained steady as the temperature increased then decreased by the same 1 degree amount, before reaching a steady temperature.

Finally, we looked at denser, more viscous solutions than the original experiments ranging from $\left(\rho_{4: 5 \frac{1}{2} \text { Top }}=1.20901\right)$ to $\left(\rho_{4: 4 \frac{3}{4}{ }_{\text {Top }}}=1.19532\right)$. None of the solutions showed any sign of oscillations or fingering. The dye in the most dense (4:5.5) and the least dense ( $4: 4.75$ ) solutions diffused a couple of millimeters upwards within the first twenty-four hours, but remained stationary after that. The two middle solu-
tions, diffused farther, about halfway up the top stratification layer, but then they also stopped. Yet, the conductivity profiles show a smooth transition from the top to the bottom, indicating that the salt diffused when the dye did not. The bottom density and viscosity did not change much, but the top grew considerably less dense and less viscous over time. Looking at temperature measurements in ( $4: 5 \frac{1}{2}$ ), the top started off as only .2 degrees warmer than the bottom, but in the first twenty hours, the temperature in the top increased by 1.5 degrees while the bottom increased by only 1 degree, these values remained stable for the rest of the experiment. In $\left(4: 4 \frac{3}{4}\right)$, the bottom was 0.5 degrees warmer than the top, and the solution maintained this difference during the initial 1.5 degree increase and during the stability afterwards.

## 5 Theory

### 5.1 Initial Conditions and Governing Equations

To understand the multi-component phenomenon, we must look at the stability of the diffusion problem. Creating a theoretical framework from which we can see if and when these fingering and oscillation behaviors occur will allow us to conjecture on the correlation between the experimental results and the destabilizing forces at play.

Consider a porous material held in a confined horizontal layer. The bottom layer is at $\mathrm{z}^{\prime}=0$ and the top is at $\mathrm{z}^{\prime}=\mathrm{H}$. The temperature is specified at both boundaries of the solution and the normal component of the velocity, W , is zero. The composition of salt and dye are specified at both boundaries.

$$
\begin{array}{lllll}
T_{H}^{\prime}=T_{H}, & C_{S a}^{\prime}=C_{S a_{H}}^{\prime}, & C_{d}^{\prime}=C_{d_{H}}^{\prime}, & W^{\prime}=0, & \text { at } z^{\prime}=H \\
T_{0}^{\prime}=T_{0}, & C_{S a}^{\prime}=C_{S a_{0}}^{\prime}, & C_{d}^{\prime}=C_{d_{0}}^{\prime}, & W^{\prime}=0, & \text { at } z^{\prime}=0 \tag{2}
\end{array}
$$

The two-dimensional conservation equations governing temperature $T^{\prime}$, composition $C^{\prime}$, volume flux $\boldsymbol{U}^{\prime}=\left(U^{\prime}, 0, W^{\prime}\right)$ and fluid pressure $p^{\prime}$ in the porous layer are

$$
\begin{align*}
& \rho_{0} c_{p}\left[\frac{\partial T^{\prime}}{\partial t^{\prime}}+\boldsymbol{U}^{\prime} \cdot \nabla T^{\prime}\right]=k \nabla^{2} T^{\prime}  \tag{3}\\
& \frac{\partial C_{S a}^{\prime}}{\partial t^{\prime}}+\boldsymbol{U}^{\prime} \cdot \nabla C_{S a}^{\prime}=D_{S a} \nabla^{2} C_{S a}^{\prime}  \tag{4}\\
& \frac{\partial C_{d}^{\prime}}{\partial t^{\prime}}+\boldsymbol{U}^{\prime} \cdot \nabla C_{d}^{\prime}=D_{d} \nabla^{2} C_{d}^{\prime}  \tag{5}\\
& \boldsymbol{U}^{\prime}=-\frac{\Pi_{0}}{\mu}\left(\nabla p^{\prime}+\rho^{\prime}\left(T^{\prime}, C_{S a}^{\prime}, C_{d}^{\prime}\right) g \hat{\mathbf{k}}\right)  \tag{6}\\
& \nabla \cdot \boldsymbol{U}^{\prime}=0 . \tag{7}
\end{align*}
$$

The density of the fluid in the gravity term is given by

$$
\begin{equation*}
\rho^{\prime}\left(T^{\prime}, C_{S a}^{\prime}, C_{d}^{\prime}\right)=\rho_{0}\left[1-\alpha_{S a}\left(C_{S a}^{\prime}-C_{S a_{0}}\right)-\alpha_{d}\left(C_{d}^{\prime}-C_{d_{0}}\right)+\beta\left(T^{\prime}-T_{0}\right)\right] . \tag{8}
\end{equation*}
$$

The composition dependence of the density is only included in the gravity term, and all other appearances are assumed to be the reference density, $\rho_{0}$, at the position $z^{\prime}$ $=0$. Additionally, $c_{p}$ is the specific heat per mass, $k$ is the thermal conductivity, and $D_{i}$ is the solute diffusivity for substance i. $\Pi_{0}$ is the permeability of the porous medium, $\mu$ the fluid viscosity, $g$ is the gravitational acceleration, $\hat{\mathbf{k}}$ the unit vector in the vertical direction, and $\alpha_{i}$ and $\beta$ are solutal and thermal expansion coefficients. Due to the sign convention chosen in the density equation, if $\alpha>0$ and $\beta>0$ the density increases with concentration and decreases with temperature

### 5.2 Non-dimensional Equations

Now, normalizing the above equations, lengths are scaled with the layer thickness $H$, time with $\frac{H^{2}}{\kappa}$, and velocity with $\frac{\kappa}{H}$ (where $\kappa=\frac{k}{\rho_{0} c_{p}}$ is the thermal diffusivity). We can also make temperature dimensionless with $T=\frac{T^{\prime}-T_{H}}{T_{0}-T_{H}}$, such that if $T^{\prime}=T_{H}$, then $\mathrm{T}=0$, and if $T^{\prime}=T_{0}$, then $\mathrm{T}=1$. Similarly for concentration $\mathrm{i}=\mathrm{Sa}, \mathrm{d}$ $C_{i}=\frac{C_{i}^{\prime}-C_{i_{H}}}{C_{i_{0}}-C_{i_{H}}}$.

Dimensionless pressure is given by

$$
\begin{equation*}
p=\frac{\Pi_{0}}{\kappa \mu} p^{\prime} \tag{9}
\end{equation*}
$$

Therefore, the new dimensionless governing equations become

$$
\begin{gather*}
\frac{\partial T}{\partial t}+\boldsymbol{U} \cdot \nabla T=\nabla^{2} T  \tag{10}\\
\frac{\partial C_{S a}}{\partial t}+\boldsymbol{U} \cdot \nabla C_{S a}=\frac{1}{L e_{S a}} \nabla^{2} C_{S a}  \tag{11}\\
\frac{\partial C_{d}}{\partial t}+\boldsymbol{U} \cdot \nabla C_{d}=\frac{1}{L e_{d}} \nabla^{2} C_{d}  \tag{12}\\
\nabla \cdot \boldsymbol{U}=0  \tag{13}\\
\boldsymbol{U}=-\nabla p+\left[-G+R a_{T}(T-1)-R a_{C_{S a}}\left(C_{S a}-1\right)-R a_{C_{d}}\left(C_{d}-1\right)\right] \hat{\mathbf{k}} \tag{14}
\end{gather*}
$$

subject to the boundary conditions:

$$
\begin{array}{llll}
T=0, & C_{S a}=0, & C_{d}=0, & W=0,
\end{array} \begin{array}{ll}
\text { at } z=1 \\
T=1, & C_{S a}=1, \tag{16}
\end{array} C_{d}=1, \quad W=0, \quad \text { at } z=0
$$

The dimensionless parameters in the above equations are

$$
G=\frac{\rho_{0} g H \Pi_{0}}{\kappa \mu}, \quad R a_{T}=\frac{\beta\left(T_{0}-T_{H}\right) \rho_{0} g H \Pi_{0}}{\kappa \mu}, \quad R a_{C_{i}}=\frac{\alpha_{i}\left(C_{i_{0}}-C_{i_{H}}\right) \rho_{0} g H \Pi_{0}}{\kappa \mu}, \quad L e_{i}=\frac{\kappa}{D_{i}},
$$

for $\mathrm{i}=\mathrm{Sa}$ and d .
The Lewis number $L e_{i}$ represents the ratio of thermal $(\kappa)$ to solutal diffusivity $\left(D_{i}\right)$, and is typically much greater than one. G is a dimensionless parameter representing the ratio of hydrostatic pressure $\rho_{0} g H$ to the viscous pressure scale and will only influence the base state pressure. The Rayleigh numbers, $R a_{C_{i}}$ represent solutal buoyancy in the different solutes and $R a_{T}$ represents thermal buoyancy. Assuming positive $\alpha$ and $\beta$, one would expect destabilizing scenarios when $R a_{T}>0$, corresponding to warm fluid under colder fluid $\left(T_{0}>T_{H}\right)$, and $R a_{C_{i}}<0$, corresponding to fresh fluid near the bottom layer and salty, dyed or more sugary fluid at the top $\left(C_{0}<C_{H}\right)$.

### 5.3 Steady Base State

A steady, motionless base state solution of the above equations would portray zero flux and a linear progression of composition for sugar, salt and dye with respect to vertical position. This is given by

$$
\begin{equation*}
\boldsymbol{U}=\mathbf{0}, \quad T_{B}(z)=1-z, \quad C_{S a_{B}}=1-z, \quad C_{d_{B}}=1-z \tag{17}
\end{equation*}
$$

The pressure in the base state is also only a function of z , and $\frac{d p_{B}}{d z}=-G-R a_{T} z+$ $R a_{C_{S a}} z+R a_{C_{d}} z$. In the dimensional form the density gradient has the form

$$
\begin{equation*}
\frac{d \rho_{B}}{d z}=\rho_{0}\left[-\alpha_{S a}\left(C_{S a_{0}}-C_{S a_{H}}\right)-\alpha_{d}\left(C_{d_{0}}-C_{d_{H}}\right)+\beta\left(T_{0}-T_{H}\right)\right] \tag{18}
\end{equation*}
$$

Static stability can be defines as when density decreases with vertical position $\left(\frac{d \rho_{B}}{d z}<\right.$ 0 ), ie. heavier fluid over light fluid. The combination of thermal and solutal fields determine this base state density gradient. A specific base state is statically stable with respect to temperature if $\beta\left(T_{0}-T_{H}\right)<0$, meaning, if taken alone, the temperature field would indicate stability if $T_{0}<T_{H}$. Similarly, a specific base state is statically stable with respect to the concentration of a solute if $\alpha\left(C_{0}-C_{H}\right)>0$. In this system there are 14 basic combinations of the solutal effects.

1. All solutal fields are stabilizing ( warm, fresh fluid over cold, salty, dyed fluid )
2. All solutal fields are destabilizing ( cold, salty, dyed fluid over warm fresh fluid)
3. Some combination of stabilizing and destabilizing agents

For (1) and (2), all agents are promoting the same outcome, making the stability prediction simple, but case (3) lead to competing agendas and an unknown stability outcome, studied in the following section.

### 5.4 Disturbance Equations

Consider two-dimensional perturbations to the base state solution of the form

$$
\begin{align*}
T & =T_{B}(z)+\tilde{T}(x, z, t)  \tag{19}\\
C_{S a} & =C_{S a_{B}}(z)+\tilde{C}_{S a}(x, z, t),  \tag{20}\\
C_{d} & =C_{d_{B}}(z)+\tilde{C}_{d}(x, z, t),  \tag{21}\\
\boldsymbol{U} & =\mathbf{0}+\tilde{\boldsymbol{U}}(x, z, t)  \tag{22}\\
p & =p_{B}(z)+\tilde{p}(x, z, t), \tag{23}
\end{align*}
$$

where $\tilde{\boldsymbol{U}}=(\tilde{U}, \tilde{W})$.
Substituting these into the disturbance equations, consider for concentrations $\mathrm{i}=\mathrm{Sa}$ and d

$$
\begin{aligned}
\nabla C_{i} & =\nabla\left(C_{i_{B}}+\tilde{C}_{i}\right) \\
& =\nabla\left(1-z+\tilde{C}_{i}\right) \\
& =\left\langle\frac{\partial \tilde{C}_{i}}{\partial x}, \frac{\partial \tilde{C}_{i}}{\partial z}-1\right\rangle
\end{aligned}
$$

Then,

$$
\begin{aligned}
\boldsymbol{U} \cdot \nabla C_{i} & =\langle\tilde{U}, \tilde{W}\rangle \cdot\left\langle\frac{\partial \tilde{C}_{i}}{\partial x}, \frac{\partial \tilde{C}_{i}}{\partial z}-1\right\rangle \\
& =\tilde{U} \cdot \nabla C_{i}-\tilde{W}
\end{aligned}
$$

Finally, since $\nabla^{2} C_{i_{B}}=0$,

$$
\nabla^{2} C_{i}=\nabla^{2} \tilde{C}_{i}
$$

Therefore,

$$
\frac{\partial C_{i}}{\partial t}+\boldsymbol{U} \cdot \nabla C_{i}=\frac{1}{L e_{i}} \nabla^{2} C_{n}
$$

becomes

$$
\frac{\partial \tilde{C}_{i}}{\partial t}+\tilde{U} \cdot \nabla C_{i}-\tilde{W}=\frac{1}{L e_{i}} \nabla^{2} \tilde{C}_{i}
$$

For temperature,

$$
\begin{aligned}
\nabla T & =\nabla\left(T_{B}+\tilde{T}\right) \\
& =\nabla(1-z+\tilde{T}) \\
& =\left\langle\frac{\partial \tilde{T}}{\partial x}, \frac{\partial \tilde{T}}{\partial z}-1\right\rangle
\end{aligned}
$$

Then,

$$
\begin{aligned}
\boldsymbol{U} \cdot \nabla T & =\langle\tilde{U}, \tilde{W}\rangle \cdot\left\langle\frac{\partial \tilde{T}}{\partial x}, \frac{\partial \tilde{T}}{\partial z}-1\right\rangle \\
& =\tilde{U} \cdot \nabla T-\tilde{W}
\end{aligned}
$$

Therefore,

$$
\frac{\partial T}{\partial t}+\boldsymbol{U} \cdot \nabla T=\nabla^{2} T
$$

becomes

$$
\frac{\partial \tilde{T}}{\partial t}+\tilde{U} \cdot \nabla T-\tilde{W}=\nabla^{2} \tilde{T}
$$

Since $\frac{d p_{B}}{d z}=-G-R a_{T} z+R a_{C_{S}} z+R a_{C_{d}} z$,

$$
\begin{aligned}
\boldsymbol{U} & =-\nabla p+\left[-G+R a_{T}(T-1)-R a_{C_{S a}}\left(C_{S a}-1\right)-R a_{C_{d}}\left(C_{d}-1\right)\right] \hat{\boldsymbol{k}} \\
& =-\nabla \tilde{p}+\left[-\frac{d p_{B}}{d z}+\left(-G+R a_{T}(\tilde{T}-z)-R a_{C_{S a}}\left(\tilde{C}_{S a}-z\right)-R a_{C_{d}}\left(\tilde{C}_{d}-z\right)\right)\right] \hat{\boldsymbol{k}} \\
& =-\nabla \tilde{p}+\left[R a_{T} \tilde{T}-R a_{C_{S a}} \tilde{C}_{S a}-R a_{C_{d}} \tilde{C}_{d}\right] \hat{\boldsymbol{k}}
\end{aligned}
$$

So, the perturbations satisfy the equations

$$
\begin{align*}
\frac{\partial \tilde{T}}{\partial t}+\tilde{U} \cdot \nabla T-\tilde{W} & =\nabla^{2} \tilde{T},  \tag{24}\\
\frac{\partial \tilde{C_{S a}}}{\partial t}+\tilde{U} \cdot \nabla C_{S a}-\tilde{W} & =\frac{1}{L e_{S a}} \nabla^{2} \tilde{C}_{S a},  \tag{25}\\
\frac{\partial \tilde{C}_{d}}{\partial t}+\tilde{U} \cdot \nabla C_{d}-\tilde{W} & =\frac{1}{L e_{d}} \nabla^{2} \tilde{C}_{d},  \tag{26}\\
-\nabla \tilde{p}+\left[R a_{T} \tilde{T}-R a_{C_{S a}} \tilde{C}_{S a}-R a_{C_{d}} \tilde{C}_{d}\right] \hat{\boldsymbol{k}} & =\tilde{\boldsymbol{U}}  \tag{27}\\
\nabla \cdot \tilde{\boldsymbol{U}} & =0 . \tag{28}
\end{align*}
$$

To get ride of the pressure variable within Darcy's Equation (27), we can take the
curl twice and use the Laplacian relationship, $\nabla \times \nabla \times \tilde{\boldsymbol{U}}=\nabla(\nabla \cdot \tilde{\boldsymbol{U}})-\nabla^{2} \tilde{\boldsymbol{U}}$. Due to equation (28), we have $\nabla \times \nabla \times \tilde{\boldsymbol{U}}=-\nabla^{2} \tilde{\boldsymbol{U}}$.

Then,

$$
\begin{aligned}
-\nabla^{2} \boldsymbol{U}= & \nabla \times \nabla \times\left(-\nabla \tilde{p}+\left[R a_{T} \tilde{T}-R a_{C_{S a}} \tilde{C}_{S a}-R a_{C_{d}} \tilde{C}_{d}\right] \hat{\boldsymbol{k}}\right) \\
= & \nabla \times \nabla \times\left\langle-\frac{\partial \tilde{p}}{\partial x},-\frac{\partial \tilde{p}}{\partial z}+R a_{T} \tilde{T}-R a_{C_{S a}} \tilde{C}_{S a}-R a_{C_{d}} \tilde{C}_{d}\right\rangle \\
= & \nabla \times\left(\frac{\partial}{\partial z}\left[-\frac{\partial \tilde{p}}{\partial x}\right]-\frac{\partial}{\partial x}\left[-\frac{\partial \tilde{p}}{\partial z}+R a_{T} \tilde{T}-R a_{C_{S a}} \tilde{C}_{S a}-R a_{C_{d}} \tilde{C}_{d}\right]\right) \hat{\boldsymbol{j}} \\
= & \nabla \times\left(-\frac{\partial^{2} \tilde{p}}{\partial x \partial z}+\frac{\partial^{2} \tilde{p}}{\partial x \partial z}-R a_{T} \frac{\partial \tilde{T}}{\partial x}+R a_{C_{S a}} \frac{\partial \tilde{C}_{S a}}{\partial x}+R a_{C_{d}} \frac{\partial \tilde{C}_{d}}{\partial x}\right) \hat{\boldsymbol{j}} \\
= & \left\langle R a_{T} \frac{\partial^{2} \tilde{T}}{\partial x \partial z}-R a_{C_{S a}} \frac{\partial^{2} \tilde{C}_{S a}}{\partial x \partial z}-R a_{C_{d}} \frac{\partial^{2} \tilde{C}_{d}}{\partial x \partial z},\right. \\
& \left.\quad-R a_{T} \frac{\partial^{2} \tilde{T}}{\partial x^{2}}+R a_{C_{S a}} \frac{\partial^{2} \tilde{C}_{S a}}{\partial x^{2}}+R a_{C_{d}} \frac{\partial^{2} \tilde{C}_{d}}{\partial x^{2}}\right\rangle .
\end{aligned}
$$

So

$$
\begin{align*}
\nabla^{2} \tilde{U} & =-R a_{T} \frac{\partial^{2} \tilde{T}}{\partial x \partial z}+R a_{C_{S a}} \frac{\partial^{2} \tilde{C}_{S a}}{\partial x \partial z}+R a_{C_{d}} \frac{\partial^{2} \tilde{C}_{d}}{\partial x \partial z}  \tag{29}\\
\nabla^{2} \tilde{W} & =R a_{T} \frac{\partial^{2} \tilde{T}}{\partial x^{2}}-R a_{C_{S a}} \frac{\partial^{2} \tilde{C}_{S a}}{\partial x^{2}}-R a_{C_{d}} \frac{\partial^{2} \tilde{C}_{d}}{\partial x^{2}} \tag{30}
\end{align*}
$$

If we take the disturbance quantities to be small $(\boldsymbol{U}=\mathbf{0}+\tilde{\boldsymbol{U}} \approx \mathbf{0})$ and linearize the equations, the problem can be decoupled such that,

$$
\begin{align*}
\frac{\partial \tilde{T}}{\partial t}-\tilde{W} & =\nabla^{2} \tilde{T}  \tag{31}\\
\frac{\partial \tilde{C_{S a}}}{\partial t}-\tilde{W} & =\frac{1}{L e_{S a}} \nabla^{2} \tilde{C}_{S a}  \tag{32}\\
\frac{\partial \tilde{C}_{d}}{\partial t}-\tilde{W} & =\frac{1}{L e_{d}} \nabla^{2} \tilde{C}_{d}  \tag{33}\\
\nabla^{2} \tilde{W} & =R a_{T} \frac{\partial^{2} \tilde{T}}{\partial x^{2}}-R a_{C_{S a}} \frac{\partial^{2} \tilde{C}_{S a}}{\partial x^{2}}-R a_{C_{d}} \frac{\partial^{2} \tilde{C}_{d}}{\partial x^{2}} \tag{34}
\end{align*}
$$

where $\tilde{T}=\tilde{C_{S a}}=\tilde{C}_{d}=\tilde{W}=0$ on $\mathrm{z}=0,1$.
We want solutions to the disturbance equations in the form of

$$
\begin{align*}
\tilde{T} & =\hat{T}(z) e^{\sigma t+i a x}+c . c .  \tag{35}\\
\tilde{C}_{S a} & =\hat{C}_{S a}(z) e^{\sigma t+i a x}+c . c .  \tag{36}\\
\tilde{C}_{d} & =\hat{C}_{d}(z) e^{\sigma t+i a x}+c . c .  \tag{37}\\
\tilde{W} & =\hat{W}(z) e^{\sigma t+i a x}+c . c . . \tag{38}
\end{align*}
$$

where $\sigma$ is the growth rate and $a$ is horizontal wave number. We can then substitute these into the disturbance equations. Using

$$
\frac{\partial \tilde{C}}{\partial t}=\sigma \hat{C}(z) e^{\sigma t+i a x}, \quad \nabla^{2} \tilde{C} \quad=\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{C}(z) e^{\sigma t+i a x}, \quad \frac{\partial^{2} \tilde{C}}{\partial x^{2}}=-a^{2} \hat{C}(z) e^{\sigma t+i a x}
$$

(32) and (33) can be reduced to

$$
\begin{aligned}
\left(\sigma \hat{C}(z) e^{\sigma t+i a x}\right)-\hat{W}(z) e^{\sigma t+i a x} & =\frac{1}{L e}\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{C}(z) e^{\sigma t+i a x} \\
\sigma \hat{C}(z)-\hat{W}(z) & =\frac{1}{L e}\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{C}(z) \\
\sigma \hat{C} & =\frac{1}{L e}\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{C}+\hat{W}
\end{aligned}
$$

(31) to

$$
\begin{aligned}
\frac{\partial \tilde{T}}{\partial t}-\tilde{W} & =\nabla^{2} \tilde{T} \\
\sigma \hat{T}(z) e^{\sigma t+i a x}-\hat{W}(z) e^{\sigma t+i a x} & =\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{T}(z) e^{\sigma t+i a x} \\
\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{T}+\hat{W} & =\sigma \hat{T}
\end{aligned}
$$

and (34) to

$$
\begin{aligned}
\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{W}(z) e^{\sigma t+i a x} & =\left[R a_{T}\left(-a^{2} \hat{T}(z)\right)-R a_{C_{S a}}\left(-a^{2} \hat{C}_{S a}(z)\right)-R a_{C_{d}}\left(-a^{2} \hat{C}_{d}(z)\right)\right] e^{\sigma t+i a x} \\
\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{W} & =a^{2}\left(-R a_{T} \hat{T}+R a_{C_{S a}} \hat{C}_{S a}+R a_{C_{d}} \hat{C}_{d}\right)
\end{aligned}
$$

Therefore the reduced system with the solutions in the form of equations (35) - (38) is,

$$
\begin{align*}
\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{T}+\hat{W} & =\sigma \hat{T}  \tag{39}\\
\frac{1}{L e_{S a}}\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{C}_{S a}+\hat{W} & =\sigma \hat{C}_{S a}  \tag{40}\\
\frac{1}{L e_{d}}\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{C}_{d}+\hat{W} & =\sigma \hat{C}_{d} \tag{41}
\end{align*}
$$

and

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial z^{2}}-a^{2}\right) \hat{W}-a^{2}\left(-R a_{T} \hat{T}+R a_{C_{S a}} \hat{C}_{S a}+R a_{C_{d}} \hat{C}_{d}\right)=0 \tag{42}
\end{equation*}
$$

subject to the following conditions at $z=0,1$ :

$$
\begin{equation*}
\hat{T}=\hat{C}_{S a}=\hat{C}_{d}=\hat{W}=0 \tag{43}
\end{equation*}
$$

The system above can be viewed as an eigenvalue problem in which the growth rate $\sigma$ can be interpreted as the eigenvalue. We want eigenfunction solutions to the disturbance equations in the form

$$
\begin{align*}
\hat{T} & =\sin (n \pi z)  \tag{44}\\
\hat{C}_{S a} & =\bar{C}_{S a} \sin (n \pi z)  \tag{45}\\
\hat{C}_{d} & =\bar{C}_{d} \sin (n \pi z)  \tag{46}\\
\hat{W} & =\bar{W} \sin (n \pi z) \tag{47}
\end{align*}
$$

where $n=1,2,3, \ldots$. These forms satisfy the boundary conditions, and inserting these into equations (39) - (42) gives us an eigensystem:

$$
\left[\begin{array}{cccc}
-J_{n}-\sigma & 0 & 0 & 1  \tag{48}\\
0 & \frac{-J_{n}}{L e_{S a}}-\sigma & 0 & 1 \\
0 & 0 & \frac{-J_{n}}{L e_{d}}-\sigma & 1 \\
a^{2} R a_{T} & -a^{2} R a_{C_{S a}} & a^{2} R a_{C_{d}} & -J_{n}
\end{array}\right]\left[\begin{array}{c}
\bar{T} \\
\bar{C}_{S a} \\
\bar{C}_{d} \\
\bar{W}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right] .
$$

where $J_{n}=n^{2} \pi^{2}+a^{2}$.
Taking the determinant of this matrix and setting it equal to zero, we get,

$$
\begin{equation*}
-J_{n}\left(\sigma+J_{n}\right)+a^{2} R a_{T}-a^{2} R a_{C_{S a}} \frac{\left(J_{n}+\sigma\right)}{\left(\frac{J_{n}}{L e_{S a}}+\sigma\right)\left(1-\phi_{S a}\right)}-a^{2} R a_{C_{d}} \frac{\left(J_{n}+\sigma\right)}{\left(\frac{J_{n}}{L e_{d}}+\sigma\right)\left(1-\phi_{d}\right)}=0 \tag{49}
\end{equation*}
$$

Rewritten as,

$$
\begin{align*}
&-J_{n}\left(\sigma+J_{n}\right)\left(\frac{J_{n}}{L e_{S a}}+\sigma\right)\left(\frac{J_{n}}{L e_{d}}+\sigma\right)+a^{2} R a_{T}\left(\frac{J_{n}}{L e_{S a}}+\sigma\right)\left(\frac{J_{n}}{L e_{d}}+\sigma\right) \\
&-a^{2} R a_{C_{S a}}\left(J_{n}+\sigma\right)\left(\frac{J_{n}}{L e_{d}}+\sigma\right)-a^{2} R a_{C_{d}}\left(J_{n}+\sigma\right)\left(\frac{J_{n}}{L e_{S a}}+\sigma\right)=0 \tag{50}
\end{align*}
$$

### 5.5 Stability Analysis

### 5.5.1 Two components

Taking (50) and reducing it to the two component case, we can set one of the $R a_{C}$ terms to zero. Then $\left(J_{n}+L e_{d}\right)$ factors out such that you are left with an equation in terms of only temperature and one other component.

$$
\begin{equation*}
-a^{2} L e R a_{C}\left(J_{n}+\sigma\right)-J_{n}\left(\sigma+J_{n}\right)\left(J_{n}+L e \sigma\right)+a^{2} R a_{T}\left(J_{n}+L e \sigma\right)=0 \tag{51}
\end{equation*}
$$

Let $\sigma=\alpha+i \beta$, then you can split the above equation into its real and imaginary parts.

$$
\begin{align*}
-J_{n}^{3}-a^{2} J_{n} L e R a_{C} & +a^{2} J_{n} R a_{T}+J_{n} L e \beta^{2} \\
& +\left(-J_{n}^{2}(1+L e)-a^{2} L e R a_{C}+a^{2} L e R a_{T}\right) \alpha-J_{n} L e \alpha^{2}=0 \tag{52}
\end{align*}
$$

and

$$
\begin{equation*}
\beta\left(-J_{n}^{2}(1+L e)-a^{2} L e\left(R a_{C}-R a_{T}\right)-2 J L e \alpha\right)=0 \tag{53}
\end{equation*}
$$

Case 1: $\beta=0$.
From equation (53) we find that the factor $\beta$ gives one zero of the solution. Then the real equation becomes,

$$
\begin{align*}
-J_{n}^{3}-a^{2} J_{n} L e R a_{C} & +a^{2} J_{n} R a_{T} \\
& +\left(-J_{n}^{2}(1+L e)-a^{2} L e R a_{C}+a^{2} L e R a_{T}\right) \alpha-J_{n} L e \alpha^{2}=0 \tag{54}
\end{align*}
$$

This is a quadratic in terms of $\alpha:\left(R a_{T}, R a_{C}\right)+B\left(R a_{T}, R a_{C}\right) \alpha+C \alpha^{2}=0$. From the quadratic formula, we know that

$$
\alpha=\frac{-B \pm \sqrt{B^{2}-4 A C}}{2 C}
$$

Since $\alpha$ is the real component of the eigenvalue, $\sigma, \sqrt{B^{2}-4 A C}$ must be real for it to be an eigenvalue. Looking at the discriminant, $B^{2}-4 A C$, we can determine the regions of validity.

$$
\begin{equation*}
B^{2}-4 A C=J_{n}^{4}(L e-1)^{2}+a^{4} L e^{2}\left(R a_{C}-R a_{T}\right)^{2}-2 a^{2} J_{n}^{2}(L e-1) L e\left(R a_{C}+R a_{T}\right) \tag{55}
\end{equation*}
$$

This is a quadratic in terms of $R a_{C}$, such that when its discriminant is less than zero, the quadratic is non-zero.

$$
\begin{equation*}
16 a^{6} J_{n}^{2} L e^{3}(L e-1) R a_{T}=\left|16 a^{6} J_{n}^{2} L e^{3}(L e-1)\right| R a_{T}<0 \tag{56}
\end{equation*}
$$

Case 1a: $R a_{T}<0$ Therefore, when $R a_{T}<0$, the discriminant is non-zero. By picking a point where $R a_{T}<0,\left(R a_{T}, R a_{C}\right)=(-1,1)$ we can check the sign of the discriminant in this case.

$$
\begin{equation*}
\left(B^{2}-4 A C\right)_{(-1,1)}=J_{n}^{4}(L e-1)^{2}+4 a^{4} L e^{2}>0 \tag{57}
\end{equation*}
$$

Therefore when $R a_{T}<0$ the discriminant is $>0$ and all roots of $\alpha$ are allowed eigenvalues. The stability of this situation depends on the sign of $\alpha$.

Therefore, for $\left(B^{2}-4 A C\right)>0$, the solution is unstable for $\alpha=\frac{-B \pm \sqrt{B^{2}-4 A C}}{2 C}>0$.
From (54), we also know that $C<0$, reducing the inequality to two subcases, $B>0$ and $B<0$.

For $B>0$,
$\frac{-B}{2 C}>0$, and $\sqrt{B^{2}-4 A C}>0$ implies $B^{2}>4 A C$. So, $B^{2}>B^{2}-4 A C$.
Therefore, when $B>0, \alpha$ is always unstable. This can be written as,

$$
\begin{gather*}
B=-J_{n}^{2}(1+L e)-a^{2} L e R a_{C}+a^{2} L e R a_{T}>0 \\
R a_{T}-R a_{C}>\frac{J_{n}^{2}}{a^{2}}\left(\frac{1}{L e}+1\right) \tag{58}
\end{gather*}
$$

For $B<0$,
$\frac{-B}{2 C}<0$, and the negative square root will correspond to stable solutions. Looking at the positive square root, $\frac{-B+\sqrt{B^{2}-4 A C}}{2 C}>0$ corresponds to unstable situations.

Then,

$$
\begin{aligned}
-B+\sqrt{B^{2}-4 A C} & \geq 0 \\
\sqrt{B^{2}-4 A C} & \geq B \\
B^{2}-4 A C & \geq B^{2} \\
-4 A C & \geq 0
\end{aligned}
$$

since $C<0$ and $B<0$, A must be $>0$ for instabilities. This can be written as

$$
\begin{align*}
A=-J_{n}^{3}-a^{2} J_{n} L e R a_{C}+a^{2} J_{n} R a_{T} & >0  \tag{59}\\
R a_{T}-L e R a_{C} & >\frac{J_{n}^{2}}{a^{2}} \tag{60}
\end{align*}
$$

Case 1b: $R a_{T}>0$ If $R a_{T}>0$, then we can solve for where the discriminant is equal
to zero.

$$
\begin{align*}
0 & =J_{n}^{4}(L e-1)^{2}+a^{4} L e^{2}\left(R a_{C}-R a_{T}\right)^{2}-2 a^{2} J_{n}^{2}(L e-1) L e\left(R a_{C}+R a_{T}\right)  \tag{61}\\
R a_{C} & =\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T} \pm 2 \sqrt{a^{6} J_{n}^{2}(L e-1) L^{3} R a_{T}}\right) \tag{62}
\end{align*}
$$

Off of these curves, either the discriminant is strictly greater than or less than zero. Choosing a point inside these curves, let $\left(R a_{T}, R a_{C}\right)=(140,60)$, then the discriminant $<0$. Choosing a point outside these curves, let $\left(R a_{T}, R a_{C}\right)=(175,70)$, then the discriminant $>0$.
Therefore, when $R a_{T}>0$ eigenvalues are allowed everywhere except inside the curves give in equation (62).
So the solutions found in Case 1a extend to when when $R a_{T}>0$, except where equation (62) interferes. So direct instabilities, corresponding to a purely real eigenvalue are given by

$$
\begin{equation*}
R a_{T}-L e R a_{C}>\frac{J_{n}^{2}}{a^{2}} \tag{63}
\end{equation*}
$$

and

$$
\begin{equation*}
R a_{T}-R a_{C}>\frac{J_{n}^{2}}{a^{2}}\left(\frac{1}{L e}+1\right), \tag{64}
\end{equation*}
$$

and when $R a_{T}>\frac{J_{n}^{2}}{\left(1-\frac{1}{L e}\right) a^{2}}$,

$$
\begin{equation*}
R a_{C}<\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T}-2 \sqrt{a^{6} J_{n}^{2}(L e-1) L e^{3} R a_{T}}\right) \tag{65}
\end{equation*}
$$

These curves intersect at the point $\left(\frac{J_{n}^{2}}{\left(1-\frac{1}{L e}\right) a^{2}}, \frac{J_{n}^{2}}{L e(1-L e) a^{2}}\right)$. Before this point, the boundary given by (64) is less than that of (63). After this point, (63) and (64) are less than (65). Therefore, we can reduce these instability requirements to

$$
\begin{equation*}
R a_{T}-L e R a_{C}>\frac{J_{n}^{2}}{a^{2}} \tag{66}
\end{equation*}
$$

and when $R a_{T}>\frac{J_{n}^{2}}{\left(1-\frac{1}{L e}\right) a^{2}}$,

$$
\begin{equation*}
R a_{C}<\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T}-2 \sqrt{a^{6} J_{n}^{2}(L e-1) L e^{3} R a_{T}}\right) \tag{67}
\end{equation*}
$$

Case 2: $\beta \neq 0$.
When the eigenvalue has an imaginary component, then the zeros of the imaginary equation depend on,

$$
\begin{equation*}
-J_{n}^{2}(1+L e)-a^{2} L e\left(R a_{C}-R a_{T}\right)-2 J L e \alpha=0 \tag{68}
\end{equation*}
$$

By solving for $\alpha$ you get,

$$
\begin{equation*}
\alpha=\frac{-J_{n}^{2}(1+L e)+a^{2} L e\left(R a_{T}-R a_{C}\right)}{2 J_{n} L e} \tag{69}
\end{equation*}
$$

As before, unstable solution correspond to where $\alpha>0$ which can be written as

$$
\begin{equation*}
R a_{T}-R a_{C}>\frac{J_{n}^{2}}{a^{2}}\left(1+\frac{1}{L e}\right) \tag{70}
\end{equation*}
$$

This solution is the same as (58) and represents an oscillatory mode since the eigenvalue is complex. Furthermore, by plugging (69) into the real equation, we get a
$0=\frac{1}{4 J_{n} L e}\left(J_{n}^{4}(L e-1)^{2}+a^{4} L e^{2}\left(R a_{C}-R a_{T}\right)^{2}+2 J_{n}^{2} L e\left(2 L e \beta^{2}-a^{2}(L e-1)\left(R a_{C}+R a_{T}\right)\right)\right)$
By solving this for $\beta^{2}$ we get a boundary on where we can have oscillations.
$0<\beta^{2}=-\frac{1}{4 J_{n}^{2} L e^{2}}\left(J_{n}^{4}(L e-1)^{2}+a^{4} L e^{2}\left(R a_{C}-R a_{T}\right)^{2}-2 a^{2} J_{n}^{2}(L e-1) L e\left(R a_{C}+R a_{T}\right)\right)$
As $\beta$ approaches zero, we can find the curves under which there is no oscillatory mode.

$$
\begin{equation*}
R a_{C}=\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T} \pm 2 \sqrt{a^{6} J_{n}^{2}(L e-1) L^{3} R a_{T}}\right) \tag{73}
\end{equation*}
$$

Therefore the final solutions for oscillatory instabilities are,

$$
\begin{equation*}
R a_{T}-R a_{C}>\frac{J_{n}^{2}}{a^{2}}\left(\frac{1}{L e}+1\right) \tag{74}
\end{equation*}
$$

where $R a_{T}>\frac{J_{n}^{2}}{\left(1-\frac{1}{L e}\right) a^{2}}$ and

$$
\begin{equation*}
R a_{C}>\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T}-2 \sqrt{a^{6} J_{n}^{2}(L e-1) L e^{3} R a_{T}}\right) \tag{75}
\end{equation*}
$$

There is no oscillatory mode below this value.
Then the total solution set for instabilities is given by

$$
\begin{equation*}
R a_{T}-L e R a_{C}>\frac{J_{n}^{2}}{a^{2}} \tag{76}
\end{equation*}
$$

and

$$
\begin{equation*}
R a_{C}<\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T}-2 \sqrt{a^{6} J_{n}^{2}(L e-1) L e^{3} R a_{T}}\right) \tag{77}
\end{equation*}
$$

OR

$$
\begin{equation*}
R a_{T}-R a_{C}>\frac{J_{n}^{2}}{a^{2}}\left(\frac{1}{L e}+1\right) \tag{78}
\end{equation*}
$$

and

$$
\begin{equation*}
R a_{C}>\frac{1}{a^{4} L e^{2}}\left(a^{2} J_{n}^{2}(L e-1) L e+a^{4} L e^{2} R a_{T}-2 \sqrt{a^{6} J_{n}^{2}(L e-1) L e^{3} R a_{T}}\right) \tag{79}
\end{equation*}
$$

Looking at the first node, $\mathrm{n}=1$, the minimum value of right hand side of (78) and (76) with respect to the wave number, a, is $\pi$, such that $\frac{J_{n}^{2}}{a^{2}}=4 \pi^{2}$. Substituting these values into our set of equations, we get a set of inequalities outlining where the system is unstable.

$$
\begin{equation*}
R a_{T}-L e R a_{C}>4 \pi^{2} \tag{80}
\end{equation*}
$$

and

$$
\begin{equation*}
R a_{C}<2 \pi^{2}-2 \sqrt{2} \pi \sqrt{R a_{T}}+R a_{T} \tag{81}
\end{equation*}
$$

OR

$$
\begin{equation*}
R a_{T}-R a_{C}>4 \pi^{2}\left(\frac{1}{L e}+1\right) \tag{82}
\end{equation*}
$$

and

$$
\begin{equation*}
R a_{C}>2 \pi^{2}-2 \sqrt{2} \pi \sqrt{R a_{T}}+R a_{T} \tag{83}
\end{equation*}
$$

The neutral stability for this case, given by $\alpha=0$, can be graphed as lines in the $R a_{C}, R a T$ plane, below which gives positive eigenvalues corresponding to instability. If we let $L e=2$ we get the following equation for direct instability,

$$
\begin{equation*}
R a_{T}-2 R a_{C}>4 \pi^{2} \tag{84}
\end{equation*}
$$

Plotting the case where $\beta=0$, one can see where the system is directly unstable, without oscillations.


Figure 8: Stability of Two-Component System, no oscillations $(\beta=0)$.

The black line in Figure (8) refers to where the system is neutrally stable, as defined by the density relationship before perturbations. Below this line, the system is statically unstable. The green line refers to equation (80), below which, in the gray region, the system is unstable with respect to perturbations. The orange line refers to equation (81), below which is also in gray, representing direct instability. Equation (83) is outlined in red, but does not impact direct instability because of its location above the orange curve and below the green line. If we check a point above the green line and below the orange curve before their intersection such as $\left(R a_{T}, R a_{C}\right)=(20,-5)$, shown by the purple dot in Figure (8, we see that the roots are given by $\sigma \approx-13.69$ and -3.42 , which are both stable. This ensures that the directly unstable region is given by the bounds seen above.


Figure 9: Full Instability Plot of Two-Component System.

When $\beta>0$ you can also have oscillatory instabilities within the range,

$$
\begin{equation*}
2 \pi^{2}-2 \sqrt{2} \pi \sqrt{R a_{T}}+R a_{T}<R a_{C}<R a_{T}-6 \pi^{2} \tag{85}
\end{equation*}
$$

as $\beta$ approaches zero, $R a_{T^{*}}=8 \pi^{2}$ and $R a_{C} *=2 \pi^{2}$ correspond to the point on the neutral stability line where this unstable region is no longer oscillatory. Along with the previous graph showing direct and static instability, when $\beta>0$, Figure (9) shows this oscillatory region. This occurs below the red dashed line and above the orange line, in the blue region. Oscillations only occur when equation (85) holds, so before the red point $\left(2 \pi^{2}, 8 \pi^{2}\right)$, there are no oscillations. It is also interesting to note that the orange lower boundary is asymptotically parallel to the upper boundary red line, both with a slope of one.

In previous work [1][4][5], $\alpha$ was set to zero from the beginning and the unstable regions were found intuitively by the individual stability of the Rayleigh numbers. Instead of a band of frequencies where $\beta^{2}$ holds in the oscillatory case, other studies found a bound on a single variable, $R a_{T}$, from plugging in equation(78) into the real
equation (52) and solving for $\beta^{2}$. From this they get,

$$
\begin{equation*}
\beta^{2}=\pi^{2}\left(R a_{T}\left(1-\frac{1}{L e}\right)-2\right)>0 \Longrightarrow R a_{T}>\frac{4 \pi^{2}}{1-\frac{1}{L e}} \tag{86}
\end{equation*}
$$



Figure 10: Showing point $(175,70)$ in purple between the oscillatory and direct instability regions.

This corresponds to the initial $R a_{T}$ value along the neutral stability curve, but does not take into account a minimum value for oscillations. The assumption is that oscillations occur above the green line and below the red line in Figure (9). By including this complete band of frequencies, one can see that there are values of $R a_{T}$ and $R a_{C}$, such that the region can be directly unstable, yet remain obove the green line. For example, take the point $\left(R a_{T}, R a_{C}\right)=(175,70)$, plotted in purple in Figure (10). From equation (92) we get that $R a_{T}-2 R a_{C}=35<4 \pi^{2}$, and from equation (85), $R a_{C}-R a_{T}=105>6 \pi^{2}$. Therefore, this point lies between the red and green lines, so by previous rational [1][4][5] this should be an oscillatory mode. Yet, using
equation (69) for $\alpha$, derived directly from the imaginary equation,

$$
\begin{equation*}
\alpha=\frac{-J_{n}^{2}(1+L e)+a^{2} L e\left(R a_{T}-R a_{C}\right)}{2 J_{n} L e} \approx 11.45 \tag{87}
\end{equation*}
$$

and plugging this into the real equation (52) we get $\beta^{2} \approx-108.9$, which is not an allowable value for eigenvalues, since $\beta$ is real by definition. By using our second boundary on oscillatory behavior from equation (85), we see that

$$
\begin{equation*}
2 \pi^{2}-2 \sqrt{2} \pi \sqrt{R a_{T}}+R a_{T}=2 \pi^{2}-2 \sqrt{2} \pi \sqrt{175}+175 \approx 77.19>75=R a_{C} \tag{88}
\end{equation*}
$$

This implies that it is below the allowable oscillatory region, given by the orange curve in Figure (10) but within the directly unstable regime. Therefore, the complete two-dimensional stability analysis needs all restrictions given above to give a true picture of what happens in double diffusion. This complete solution can be extended into the third dimension when you add a third component.

### 5.5.2 Interpretation of Two-Dimensional Stability Plot

Understanding the physical interpretation of the two dimensional case, we can expand the results to see what occurs when another component is added.

Case (1): In the case where hot, salty fluid is on top of cold, fresh fluid, the experiment lies within the bottom left quadrant. If the solution is stably stratified from a static stand point, then it would lie above the black line in Figure (9). Since the thermal diffusivity is much larger than the solute diffusivity, as a fluid parcel is displaced upwards, it would quickly equilibrate to the surrounding temperature of the stable thermal field, yet still remain fresh, allowing it to continue upwards towards even warmer fluid. This is the direct mode of instability between the green and black lines.

Case (2): Conversely, when the hot, salty fluid is below cold, fresh fluid (Quadrant 1 in Figure (9)), the salt field is stabilizing and the thermal field is destabilizing the layer. The theoretical results show that the observed behavior greatly depends upon the exact location of the layer. Below a certain threshold value of $R a_{C}$, the stabilizing field is not strong enough, and the instability is controlled by the thermal field. The oscillations, shown by the region shaded in blue, occur when a parcel rises and loses heat quickly due to the strong thermal destabilization, but does not lose salt because of the stable salt environment. This causes the heavy parcel to fall again. When the stability of the solute field reaches some minimum amount, corresponding to
the orange line, the solution becomes directly unstable. This suggests that there is a small region where the stability of the salt is not strong enough to create oscillations, allowing the thermal field to control the unstable behavior.

Case(3): For most of the statically stable situations, especially those that lie within Quadrant 2, the solution is both thermally and solutably stable. Therefore there are no oscillation or fingering behaviors. This would be the case where cold, salty fluid is under warmer fresh fluid.

### 5.5.3 Three components

Returning to the three-component case with equation (50), let $\sigma=\alpha+i \beta$. Then (50) can be split into real and imaginary parts. Where the real equation is

$$
\begin{align*}
& J_{n} \alpha a^{2}\left(L e_{d}\left(R a_{C_{d}}-R a_{T}\right)+L e_{S a}\left(R a_{C_{S a}}-R a_{T}\right)+L e S a L e_{d}\left(R a_{C_{S} a}+R a_{C_{d}}\right)\right) \\
& \quad+J_{n}^{2}\left(a^{2}\left(L e_{S a} R a_{C_{S a}}+L e_{d} R a_{C_{d}}-R a_{T}\right)+\left(L e_{S a}+L e_{d}+L e_{S a} L e_{d}\right)\left(\alpha^{2}-\beta^{2}\right)\right) \\
& \quad+J_{n} \alpha L e_{S a} L e_{d}\left(\alpha^{2}-3 \beta^{2}\right)+J_{n}^{4}+J_{n}^{3}\left(1+L e_{S a}+L e_{d}\right) \alpha \\
& \quad+a^{2} L e_{S a} L e_{d}\left(R a_{C_{S a}}+R a_{C_{d}}-R a_{T}\right)\left(\alpha^{2}-\beta^{2}\right)=0 \tag{89}
\end{align*}
$$

and the imaginary

$$
\begin{align*}
& \quad 0=\beta\left[J_{n}^{3}\left(1+L e_{S a}+L e_{d}\right)+2 J_{n}^{2}\left(L e_{S a}+L e_{d}+L e_{S a} L e_{d}\right) \alpha+2 a^{2} L e_{S a} L e_{d}\left(R a_{C_{S a}}+R a_{C_{d}}-R a_{T}\right) \alpha\right. \\
& +  \tag{90}\\
& \left.J_{n} \alpha\left(a^{2}\left(L e_{d}\left(R a_{C_{d}}-R a_{T}\right)+L e_{S a}\left(R a_{C_{S a}}-R a_{T}\right)+L e S a L e_{d}\left(R a_{C_{S} a}+R a_{C_{d}}\right)\right)+L e_{S a} L e_{d}\left(3 \alpha^{2}-\beta^{2}\right)\right)\right]
\end{align*}
$$

Solving these equations, gives us direct and oscillatory regions in the three-dimensional $\left(R a_{T}, R a_{C_{S a}}, R a_{C_{d}}\right)$.

Case 1: $\beta=0$ The factor $\beta$ gives one zero of the solution. Then the equation for neutral stability becomes

$$
\begin{equation*}
R a_{T}-R a_{C_{S a}} L e_{S a}-R a_{C_{d}} L e_{d}=\frac{J_{n}^{2}}{a^{2}} \tag{91}
\end{equation*}
$$

This solution reduces to the two-dimensional case discussed in the previous section, but understanding the stability of the rest of the system, requires solving for the cubic roots of $\alpha$. We can find these using the analytical formula for cubic roots created by Tartaglia and Cardano. By finding where these roots are positive and real with respect to the parameter space, we can find the regions of direct instability. In order to simplify this calculation, the horizontal wavenumber was set to $\pi$, as was derived in the two-component case. We will not show that this is the minimum wave number for the three-component case, but will use it as a placeholder for farther analysis. By setting a to $\pi$ and therefore, $J_{n}$ to $2 \pi^{2}$ in the first node, we can reduce the equations considerably. We also chose values for the Lewis numbers, such that $L e_{S a}=2$ (the same as used in the two-component case) and $L e_{d}=3$. Then we are only solving,

$$
\begin{align*}
& 12 \pi^{2} \alpha^{3}+\left(44 \pi^{4}+6 \pi^{2}\left(R a_{C_{S a}}+R a_{C_{d}}-R a_{T}\right)\right) \alpha^{2}+16 \pi^{8}+8 \pi^{6} R a_{C_{S a}} \\
& +12 \pi^{6} R a_{C_{d}}-4 \pi^{6} R a_{T}+\left(48 \pi^{6}+16 \pi^{4} R a_{C_{S a}}+18 \pi^{4} R a_{C_{d}}-10 \pi^{4} R a_{T}\right) \alpha=0 \tag{92}
\end{align*}
$$



Figure 11: Algorithm 1 for three-components reduced to two-component case for direct instability regions.

Once we have the three roots in terms of the parameters, we can check to see when they are real and positive by looping through values in a specified range. If the imaginary part of the root is less than machine error and the real part is greater than machine error, then the parameter point is directly unstable. This was done using a Mathematica script following Algorithm 1, as shown in Appendix C. These parameters can then be plotted in three-dimensions. The script was run on the two dimensional case by setting $R a_{C_{d}}=0$ and the two-dimensional case discussed above was rediscovered. Using Algorithm 1 we can look at perturbed values of $R a_{C_{d}}$ ranging from -50 to 50 by setting the values of R 2 as a constant and plotting a 2 dimensional plot of (RT, R1). We can see how the addition of a third component changes the oscillatory region. The curve of the line seems to stay the same as you perturb the third component. With positive values of $R a_{C_{d}}$, corresponding to a stable dye concentration, the unstable region drops lower along the negative $R a_{C_{S a}}$ axis. This makes sense, because as you increase the stability of the solution, an unstable regime would require a greater destabilizing force with respect to the other component. In comparison, as $R a_{C_{d}}$ decreases along the negative axis, corresponding to an unstable dye concentration, almost all values of $R a_{C_{S a}}<0$ within the range will give destabilizing behavior because the strong unstable fields of both salt and dye. The unstable region continues to extend upwards into the thermally and solutally stable quadrant as the instability in the third component increases.

```
Algorithm 1 Algorithm for Direct Instabilities
    1: Set \(\beta=0\) in Equation (92)).
    2: Set parameters values for \(J_{n}, a, L e_{S a}\), and \(L e_{d}\). (Values used for this analysis
    shown below.)
\[
\begin{aligned}
J_{n} & =2 \pi^{2}(\text { minimum value }) \\
a & =\pi \quad(\text { minimum value }) \\
L e_{S a} & =2 \\
L e_{d} & =3
\end{aligned}
\]
3: Set precision, \(\varepsilon\), for finding roots. (Value used for this analysis shown below.)
\[
\varepsilon=10^{-20}
\]
: Solve Equation (92) for \(3 \alpha\) roots: Root1[RT, R1, R2], Root2[RT, R1, R2], Root3[RT, R1, R2]
for (RT, R1, R2) in Range [ \(-\mathrm{n}, \mathrm{n}, \mathrm{h}]\) do if \(\operatorname{Abs}(\operatorname{Im}(\operatorname{Root}[R T, R 1, R 2]))<\varepsilon\) and \(\operatorname{Re}(\operatorname{Root}[R T, R 1, R 2])>\varepsilon\) then
Point is unstable \(\Longrightarrow\) Append (RT, R1, R2) to list else
Point is stable \(\Longrightarrow\) Continue end if
end for
```



Figure 12: Perturbed $R a_{C_{d}}$ values using Three-Component Direct Instability Algorithm.

Case 2: $\beta \neq 0$ When $\beta$ is not zero, the rest of the imaginary equation can be
solved for the $\beta^{2}$ term within. Using the same variables as above,

$$
\begin{align*}
& \beta^{2}=48 \pi^{6}+16 \pi^{4} R a_{C_{S a}}+18 \pi^{4} R a_{C_{d}}-10 \pi^{4} R a_{T} \\
&+\left(88 \pi^{4}+12 \pi^{2}\left(R a_{C_{S a}}+R a_{C_{d}}-R a_{T}\right)\right) \alpha+36 \pi^{2} \alpha^{2}>0 \tag{93}
\end{align*}
$$

This gives a quadratic in terms of $\alpha$ that must be greater than zero for oscillatory modes of instability. Solving this with the quadratic formula, we get two roots. Let $R a=\left(R a_{C_{S a}}+R a_{C_{d}}-R a_{T}\right)$, then,

$$
\begin{equation*}
\alpha=-72 \pi^{4}\left(22 \pi^{2}+3 R a \pm \sqrt{52 \pi^{4}+9 R a^{2}-6 \pi^{2}\left(2 R a_{C_{S a}}+5 R a_{C_{d}}-7 R a_{T}\right)}\right) \tag{94}
\end{equation*}
$$

If you plug in the equation for $\beta^{2}$ into the real equation (89), you get a cubic $n$ terms of $\alpha$. Solving these roots as we did for the direct case, we can again check to see where they are real valued and positive. Because of the second condition, though, oscillations only occur when $\beta^{2} \dot{i} 0$. Since these are both equations in terms of the parameters, we can loop through within a given range to find the oscillatory range.Either the discriminant is positive or if negative other conditions must be applied. In equation (93), the leading term is positive, which indicates that the quadratic is concave up. Therefore, when the discriminant is positive, the two roots given by equation (94) are ordered such that the positive square root is always larger than the negative. Then we can check to see if the cubic root of $\alpha$ is greater than the larger $\beta$ root or smaller than the smaller one. This along with ensuring that $\alpha$ and the discriminant are real-valued and positive, gives a set of Rayleigh values where oscillatory behavior occurs.

For example, consider the point $\left(R a_{T}, R a_{C_{S a}}, R a_{C_{d}}\right)=(130,50,0)$, which, from the two-component case, we know lies in the blue oscillatory region. By graphing $\beta^{2}$ (blue curve) and the real equation (orange curve), we can see where the roots lie. In Figure (13a) we can see that, as expected the positive real root given by where the orange line crosses the horizontal axes exists where $\beta^{2}>0$ as it is larger than the right-most root in blue. Conversely, as we increase the $R a_{T}$ value, we leave the oscillatory region. Looking at the parameter point, (135, 50,0), in Figure (13b) we can see that the root lies inside of the $\beta^{2}$ curve, where eigenvalues cannot exist. As you increase the third component, extending into the third-dimension, you see that this point becomes oscillatory again in Figure (13c).

(a) Oscillatory: $\left(R a_{T}, R a_{C_{S a}}, R a_{C_{d}}\right)=(130,50,0)$.

(b) Not Oscillatory: $\left(R a_{T}, R a_{C_{S a}}, R a_{C_{d}}\right)=(135,50,0)$.

(c) Oscillatory: $\left(R a_{T}, R a_{C_{S a}}, R a_{C_{d}}\right)=(135,50,3)$.

Figure 13: Checking to see where roots give oscillatory region values


Figure 14: Three-component script reduced to two-component case for oscillatory instability.

This was done using a Mathematica script following Algorithm 2, as shown in Appendix C. These parameters can then be plotted in three-dimensions. If we reduce to the two dimensional case by setting $R a_{C_{d}}=0$, we rediscover the two-component oscillatory instability region. Using Algorithm 2 we can look at perturbed values of $R a_{C_{d}}$ ranging from -50 to 50 by setting the values of R 2 as a constant and plotting a 2 dimensional plot of (RT, R1). We can see how the addition of a third component changes the oscillatory region. Like the direct mode of instability, adding positive values of the third component causes the oscillatory region to shift downwards, while negative values shift the region up. As $R a_{C_{d}}$ becomes more and more negative, corresponding to a larger unstable field in the third component, the oscillatory region is drawn away from the origin into a more thermally unstable and solutally stable area. This could be because the large In addition, as $R a_{C_{d}}$ reaches 50, the region elongates to the point where part of the oscillatory region extends into Quadrant 2. Unlike the stable region, there seems to be another boundary curve controlling the oscillatory region of the three-component case as $R a_{C_{d}}$ increases.

```
Algorithm 2 Algorithm for Oscillatory Instabilities
    : Create function for Equation (93) and (89)) shown below.)
\[
\begin{aligned}
J_{n} & =2 \pi^{2} \\
a & =\pi \\
L e_{S a} & =2 \\
L e_{d} & =3
\end{aligned}
\]
```

2: Set parameters values for $J_{n}, a, L e_{S a}$, and $L e_{d}$. (Values used for this analysis
: Set precision, $\varepsilon$, for finding roots. (Value used for this analysis shown below.)

$$
\varepsilon=10^{-20}
$$

Solve Equation (89) for 3 a roots: Root1[RT, R1, R2], Root2[RT, R1, R2], $\operatorname{Root} 3[R T, R 1, R 2]$
: Solve Equation (93) for $2 \alpha$ roots: $\beta$ Root $_{+}[R T, R 1, R 2], \beta$ Root_ $_{-}[R T, R 1, R 2]$
: Find discriminant of Equation (93): disc[RT, R1, R2]
for (RT, R1, R2) in Range[ $-\mathrm{n}, \mathrm{n}, \mathrm{h}$ ] do
if $\operatorname{Abs}(\operatorname{Im}(\operatorname{Root}[R T, R 1, R 2]))<\varepsilon$ and $\operatorname{Re}(\operatorname{Root}[R T, R 1, R 2])>\varepsilon$ then Point is unstable $\Longrightarrow$ Continue if $\operatorname{disc}[R T, R 1, R 2]<\varepsilon$ then Point is Oscillatory $\Longrightarrow$ Append (RT, R1, R2) to list else if disc $[R T, R 1, R 2]<\varepsilon$ then if $\beta$ Root $_{+}[R T, R 1, R 2]<\operatorname{Re}(\operatorname{Root}[R T, R 1, R 2])$ then Point is Oscillatory $\Longrightarrow$ Append (RT, R1, R2) to list else if $\beta$ Root_ $[R T, R 1, R 2]>\operatorname{Re}(\operatorname{Root}[R T, R 1, R 2])$ then Point is Oscillatory $\Longrightarrow$ Append (RT, R1, R2) to list else Point is not Oscillatory $\Longrightarrow$ Continue end if end if
else
Point is Stable $\Longrightarrow$ Continue
end if
end for

## 6 Discussion

In diffusion experiments set up as these experiments were, with fresh fluid stratified over dyed, salty fluid, one would expect the salt and dye to diffuse upwards to the lower concentration gradient in the fresh fluid. This would cause the density and viscosity to increase in the top and decrease in the bottom. The experiments measuring temperature over time, show that the temperature consistently rose by at least a degree in the top and the bottom over time. A temperature increase causes the density to drop much more than the slower, upward diffusion of salt causes it to increase. This is seen in both the extended and the zoomed in NaCl experiments. It is also a factor in the NaI experiments, except that the salt did not diffuse. While most experiments had a stable system of warm fluid over cold fluid, there were a few cases where the opposite held, but there was not enough data to show if this unstable regime changed the outcome of the stability. In any case, one can see that temperature is a strong factor in determining the outcome of the diffusion.

The second component of the diffusion, the dye, can visually be seen diffusing quickly throughout the solution. The dye is what we use to visualize the fingering and oscillations. While the dye does appear with these instabilities, a study by Camassa, McLaughlin, and Valchar found this fingering effect to happen without dye, and when dye was added to either the top or the bottom layer [11]. This suggests that this component is not crucial to the instability regime. The dye must be impacted by some other diffusing factor affecting the flux within the solution and creating the fingering and oscillatory behavior.

The third component, salt, seems to have the largest impact on the magnitude and frequency of oscillations and fingering. Camassa, et al. also determined that there was an order to how strong of an effect was created by different salts created in stratified fluids. The salts with an iodide anion, like KI, were not observed to have any instability effect. These solutions stratified by Iodide-based salts also took much longer to diffuse, on the order of months instead of weeks. Camassa et al. did observe instability behavior for several Chloride based salts, such as $\mathrm{CaCl}, \mathrm{NaCl}$, and KCl . The strongest and most prevalent fingering in our experiments was observed in NaCl , followed by KCl , and the stratification with KI and NaI did not create any fingering or oscillatory instabilities. The difference in salts could be a matter of density, since


Sodium and Potassium Iodide have densities of 3.67 and $3.12 \mathrm{~g} / \mathrm{cm}^{3}$ respectively, compared to the Chloride based salts which range from 1.98 to $2.16 \mathrm{~g} / \mathrm{cm}^{3}$. This might indicate it is hard for small parcels of the denser salts to break away from the whole to allow the fingering or oscillatory behavior. Since Iodide based salts are denser, it takes less salt mass added to create the same density stratification. Fewer salt particles diffused within the bottom layer could change the way that the diffusion occurs.

Therefore, though this was considered a three-component set up, there seems to only be two active components within the diffusion-convection problem. In the experiments, with salt and dye on the bottom, the stratifications were solutably stable with respect to both. In all of the experiments where temperature was recorded, the thermistors show that the experiments were also thermally stable, with warmer fluid over colder fluid. Considered stable in every regime, the experiments would be within the second quadrant, in the two dimensional case. This set up does not correspond to any instability behavior. The fact that the fingers and oscillations occur suggest that there might be another underlying cause.


Figure 16: Periodic Table with Studied Elements in Red [6].
One such alternative could be the impact of Chlorine on dye. Chlorine is often used to strip color from things such as hair or clothing. It is possible that there is some reaction between the dye and the Chlorine that causes the instabilities we see. Bendig, Maier, and Vetter [2] explain that Bromine, directly below Chlorine on the periodic table (see Figure (16)), is used to make brominated vegetable oil, an emulsifying agent in many soft drinks. With a density of $1.33 \mathrm{~g} / \mathrm{cm}^{3}$, it mixes with less dense dye or flavor oil to produce an oil with the same density as the environment, in this case the water in soda. As Bromine is the Halogen directly between Chlorine and Iodine, it stand to reason that some of these chemical qualities that allow the Bromine to react with the dye could also be present in Chlorine based salts. On the other hand, it could be Iodine which is similar to Bromine, explaining why the fingering of the dye does not occur with KI, or why the diffusion process takes so much longer than with NaCl and KCl .

One of the purposes of these experiments was to see if there was some range of density where fingering and oscillations occurred. Simply looking at the NaCl experiments, where we had a larger range of information and data available, there does not seem to be much correlation. Sometimes instabilities would present themselves in the less dense solutions, but not in the more dense ones where they had previously
been recorded. While more experimentation would have to be done to show any true range, very rarely were oscillatory and fingering behavior observed in solutions where the top density was less than $1.18 \mathrm{~g} / \mathrm{cm}^{3}$. Since very little oscillatory and fingering behavior was observed in the extended NaCl trial, we can also conjecture an upper bound around $1.20 \mathrm{~g} / \mathrm{cm}^{3}$. Within this range, instabilities were not guaranteed, but most often occurred with varying intensity.

## 7 Conclusion

The stability analysis shows that while there are regions where oscillations and fingering occur, the absence of an adverse gradient implies that there must be another factor involved in the salt-stratified experiments. For future work, it would be best to study the other cases of diffusion outlined in Section 6.5.3 to see if they align more closely with the theoretical results, and explore other salts such as Bromine. There was no clear correlation within the studied density ranges corresponding to greater intensity or frequency of unstable behavior, but the solutions exhibiting oscillations or fingering largely fell between $\rho_{\text {Top }}=1.18 \mathrm{~g} / \mathrm{cm}^{3}$ and $\rho_{\text {Top }}=1.20 \mathrm{~g} / \mathrm{cm}^{3}$. More experiments could better understand the presence and occurrence of the observed behavior over time. There also needs to be a more in depth study into the stability of the three-dimensional case as well as looking into the math involved in the true fluid regime. Most interestingly, we found results that were slightly different from those presented in previous work, including a larger region for direct instabilities and a smaller region for oscillatory instabilities. The results in this analysis and set of experiments show that there may be a larger problem behind the observed instabilities, but without farther research we can only conclude that multi-component diffusive convection cannot be the only root of the problem.

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## Appendices

A Diffusion Experiments by Camassa et al. [11]


Figure 17: Corn Syrup Stratified with NaCl (Left) and KI (Right) on the Bottom, Alternating the Placement of the Dye, Top or Bottom.

## B Experimental Measurements and Plots

B. 1 KI


Figure 18: Measurements Taken on KI Stratified Solutions

## B. 2 NaI



Figure 19: Measurements Taken on NaI Stratified Solutions

## B. 3 KCl



Figure 20: Measurements Taken on KCl Stratified Solutions

## B. 4 NaCl



Figure 21: Measurements Taken on NaCl Stratified Solutions (Second Trial)


Figure 22: Measurements Taken on "Zoomed" NaCl Stratified Solutions (First Trial)


Figure 23: Measurements Taken on "Zoomed" NaCl Stratified Solutions (Second Trial)


Figure 24: Measurements Taken on "Extended" NaCl Stratified Solutions

## C Mathematica Script for Instabilities in ThreeComponent Case

## Three Components

$$
\begin{aligned}
& \text { rel }=J^{4}+a^{2} J^{2} \text { L1R1 }+a^{2} J^{2} \text { L2 R2 }-a^{2} J^{2} R T+J^{3} \alpha+J^{3} L 1 \alpha+J^{3} L 2 \alpha+a^{2} J \text { L1R1 } \alpha+ \\
& a^{2} \text { J L1 L2 R1 } \alpha+a^{2} \text { J L2 R2 } \alpha+a^{2} \text { J L1 L2R2 } \alpha-a^{2} \text { J L1RT } \alpha-a^{2} \text { JL2RT } \alpha+J^{2} \text { L1 } \alpha^{2}+ \\
& J^{2} \text { L2 } \alpha^{2}+J^{2} \text { L1 L2 } \alpha^{2}+a^{2} \text { L1 L2 R1 } \alpha^{2}+a^{2} \text { L1 L2 R2 } \alpha^{2}-a^{2} \text { L1 L2 RT } \alpha^{2}+\text { JL1 L2 } \alpha^{3}-J^{2} \text { L1 } \beta^{2}- \\
& J^{2} \text { L2 } \beta^{2}-J^{2} \text { L1 L2 } \beta^{2}-a^{2} \text { L1 L2 R1 } \beta^{2}-a^{2} \text { L1 L2 R2 } \beta^{2}+a^{2} \text { L1 L2 RT } \beta^{2}-3 \text { J L1 L2 } \alpha \beta^{2} \\
& J^{4}+\mathrm{a}^{2} \mathrm{~J}^{2} \mathrm{~L} 1 \mathrm{R} 1+\mathrm{a}^{2} \mathrm{~J}^{2} \mathrm{~L} 2 \mathrm{R} 2-\mathrm{a}^{2} \mathrm{~J}^{2} \mathrm{RT}+\mathrm{J}^{3} \alpha+\mathrm{J}^{3} \mathrm{~L} 1 \alpha+\mathrm{J}^{3} \mathrm{~L} 2 \alpha+\mathrm{a}^{2} \mathrm{JL} 1 \mathrm{R} 1 \alpha+\mathrm{a}^{2} \mathrm{JL} \mathrm{~L} \mathrm{~L} 2 R 1 \alpha+ \\
& a^{2} \text { J L2 R2 } \alpha+a^{2} \text { J L1 L2 R2 } \alpha-a^{2} \text { J L1RT } \alpha-a^{2} \text { JL2RT } \alpha+J^{2} \text { L1 } \alpha^{2}+J^{2} L 2 \alpha^{2}+ \\
& J^{2} L 1 L 2 \alpha^{2}+a^{2} L 1 L 2 R 1 \alpha^{2}+a^{2} L 1 L 2 R 2 \alpha^{2}-a^{2} L 1 L 2 R T \alpha^{2}+J L 1 L 2 \alpha^{3}-J^{2} L 1 \beta^{2}- \\
& J^{2} \text { L2 } \beta^{2}-J^{2} \text { L1 L2 } \beta^{2}-a^{2} \text { L1 L2 R1 } \beta^{2}-a^{2} \text { L1 L2 R2 } \beta^{2}+a^{2} \text { L1 L2 RT } \beta^{2}-3 \text { JL1 L2 } \alpha \beta^{2} \\
& \text { img }=\left(J^{3} \beta+J^{3} L 1 \beta+J^{3} L 2 \beta+a^{2} \text { JL1R1 } \beta+a^{2} J L 1 L 2 R 1 \beta+a^{2} \text { JL2R2 } \beta+\right. \\
& a^{2} \text { J L1 L2 R2 } \beta-a^{2} \text { JL1RT } \beta-\mathrm{a}^{2} \text { JL2RT } \beta+2 \mathrm{~J}^{2} \mathrm{~L} 1 \alpha \beta+2 \mathrm{~J}^{2} \mathrm{~L} 2 \alpha \beta+2 \mathrm{~J}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \alpha \beta+ \\
& 2 \mathrm{a}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \mathrm{R} 1 \alpha \beta+2 \mathrm{a}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \mathrm{R} 2 \alpha \beta-2 \mathrm{a}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \mathrm{RT} \alpha \beta+3 \mathrm{JL} 1 \mathrm{~L} 2 \alpha^{2} \beta-\mathrm{JL} 1 \mathrm{~L} 2 \beta^{3} \text { ) } / \beta \\
& \frac{1}{\beta}\left(J^{3} \beta+J^{3} L 1 \beta+J^{3} L 2 \beta+a^{2} J L 1 R 1 \beta+a^{2} J L 1 L 2 R 1 \beta+a^{2} J L 2 R 2 \beta+\right. \\
& a^{2} \text { J L1 L2 R2 } \beta-a^{2} \text { J L1 RT } \beta-a^{2} \text { J L2 RT } \beta+2 \mathrm{~J}^{2} \mathrm{~L} 1 \alpha \beta+2 \mathrm{~J}^{2} \mathrm{~L} 2 \alpha \beta+2 \mathrm{~J}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \alpha \beta+ \\
& 2 \mathrm{a}^{2} \text { L1 L2 R1 } \alpha \beta+2 \mathrm{a}^{2} \text { L1 L2 R2 } \alpha \beta-2 \mathrm{a}^{2} \text { L1 L2 RT } \alpha \beta+3 \text { J L1 L2 } \alpha^{2} \beta-\text { J L1 L2 } \beta^{3} \text { ) }
\end{aligned}
$$

Finding Unstable Values for Direct Mode

```
Collect[rel/. { \beta -> 0, R2 }->0},\alpha
```




```
16 \mp@subsup{\pi}{}{8}+8 \mp@subsup{\pi}{}{6}R1+12 \mp@subsup{\pi}{}{6}R2-4 \mp@subsup{\pi}{}{6}RT+(48 \mp@subsup{\pi}{}{6}+16\mp@subsup{\pi}{}{4}\textrm{R}1+18 \mp@subsup{\pi}{}{4}\textrm{R}2-10 \mp@subsup{\pi}{}{4}\textrm{RT})\alpha+
    (44 \pi}\mp@subsup{\pi}{}{4}+6\mp@subsup{\pi}{}{2}R1+6\mp@subsup{\pi}{}{2}R2-6\mp@subsup{\pi}{}{2}RT) \mp@subsup{\alpha}{}{2}+12\mp@subsup{\pi}{}{2}\mp@subsup{\alpha}{}{3
```

$\alpha$ Roots of direct mode

```
dirRoot1[RT_, R1_, R2_] = \alpha /. Solve[dirR[RT, R1, R2] == 0, \alpha][[1]];
```

dirRoot2[RT_, R1_, R2_] = $\alpha /$. Solve[dirR[RT, R1, R2] == 0, $\alpha$ ][[2]];
dirRoot3[RT_, R1_, R2_] = $\alpha$ /. Solve[dirR[RT, R1, R2] == 0, $\alpha$ ][[3]];

When the $\alpha$ roots are positive and real, the system is directly unstable:
$\mathrm{n}=200$; (*checking values in the range of this $\mathrm{n}^{3}$ space*)
$h=1$; (*stepping by $h *$ )
$r=\operatorname{Range}[-n, n, h]$;
r1vals2D = \{\};
Do[rt = r[[j]];
$r 1=r[k]]$;
$\operatorname{If}\left[A b s[\operatorname{Im}[N[d i r \operatorname{Root} 1[r t, r 1,0]]]]<10^{\wedge}-16 \& \& \operatorname{Re}[N[d i r \operatorname{Root} 1[r t, r 1,0]]]>10^{\wedge}-16\right.$, AppendTo[r1vals2D, \{rt, r1\}], Continue], $\{j, 1, \operatorname{Length}[r], 1\},\{k, 1$, Length[r], 1\}]

Checking 2D case for each root

```
n=200; (*checking values in the range of this n }\mp@subsup{}{}{3}\mathrm{ space*)
h = 1; (*stepping by h*)
r = Range[-n, n, h];
r1vals2D = {};
Do[rt = r[[j]];
    r1 = r[[k]]; If[Abs[Im[N[dirRoot1[rt, r1, 0], 40]]] < 10^-20 &
        Re[N[dirRoot1[rt, r1, 0], 40]] > 10^-20, AppendTo[r1vals2D, {rt, r1}],
    Continue], {j, 1, Length[r], 1}, {k, 1, Length[r], 1}]
n=200; (*checking values in the range of this n }\mp@subsup{}{}{3}\mathrm{ space*)
h = 1; (*stepping by h*)
r = Range[-n, n, h];
Do[rt = r[[j]];
    r1 = r[[k]]; If[Abs[Im[N[dirRoot2[rt, r1, 0], 40]]]< <10^-20&&
        Re[N[dirRoot2[rt, r1, 0], 40]] > 10^-20, AppendTo[r1vals2D, {rt, r1}],
    Continue], {j, 1, Length[r], 1}, {k, 1, Length[r], 1}]
n=200; (*checking values in the range of this n }\mp@subsup{}{}{3}\mathrm{ space*)
h = 1; (*stepping by h*)
r = Range[-n, n, h];
Do[rt = r[[j]];
    r1 = r[[k]]; If[Abs[Im[N[dirRoot3[rt, r1, 0], 40]]] < 10^-20 &
        Re[N[dirRoot3[rt, r1, 0], 40]] > 10^-20, AppendTo[r1vals2D, {rt, r1}],
        Continue], {j, 1, Length[r], 1}, {k, 1, Length[r], 1}]
```

3D Case for direct roots

```
n=200; (*checking values in the range of this n3 space*)
h = 5; (*stepping by h*)
r = Range[-n, n, h];
rlvals3D = {};
Do[rt = r[[j]];
    r1 = r[[k]];
    r2 = r[[l]];
    If[Abs[Im[N[dirRoot1[rt, r1, r2], 40]]] < 10^-20&&
        Re[N[dirRoot1[rt, r1, r2], 40]] > 10^-20, AppendTo[r1vals3D, {rt, r1, r2}],
    Continue], {j, 1, Length[r], 1}, {k, 1, Length[r], 1}, {l, 1, Length[r], 1}]
Do[rt = r[[j]];
    r1 = r[[k]];
    r2 = r[[l]];
    If[Abs[Im[N[dirRoot2[rt, r1, r2], 40]]]< 10^-20&&
        Re[N[dirRoot2[rt, r1, r2], 40]] > 10^-20, AppendTo[r1vals3D, {rt, r1, r2}],
        Continue], {j, 1, Length[r], 1}, {k, 1, Length[r], 1}, {l, 1, Length[r], 1}]
Do[rt = r[[j]];
    r1 = r[[k]];
    r2 = r[[l]];
    If[Abs[Im[N[dirRoot3[rt, r1, r2], 40]]] < 10^-20 &&
        Re[N[dirRoot3[rt, r1, r2], 40]] > 10^-20, AppendTo[r1vals3D, {rt, r1, r2}],
    Continue], {j, 1, Length[r], 1}, {k, 1, Length[r], 1}, {l, 1, Length[r], 1}]
```


## Oscillatory

Solve for $\beta^{2}$ in imaginary
$\beta 2$ /.

$$
\begin{aligned}
& \text { Solve }\left[J^{3}+J^{3} L 1+J^{3} L 2+a^{2} J L 1 R 1+a^{2} J L 1 L 2 R 1-a^{2} J L 1 R T-a^{2} J L 2 R T+2 J^{2} L 1 \alpha+2 J^{2} L 2 \alpha+\right. \\
& \quad 2 J^{2} L 1 L 2 \alpha+2 a^{2} L 1 L 2 R 1 \alpha-2 a^{2} L 1 L 2 R T \alpha+3 J L 1 L 2 \alpha^{2}+ \\
& \left.\quad R 2\left(a^{2} J L 2+a^{2} J L 1 L 2+2 a^{2} L 1 L 2 \alpha\right)-J L 1 L 2 \beta 2=0, \beta 2\right][[1]] \\
& \frac{1}{J L 1 L 2}\left(J^{3}+J^{3} L 1+J^{3} L 2+a^{2} J L 1 R 1+a^{2} J L 1 L 2 R 1+\right. \\
& \quad a^{2} J L 2 R 2+a^{2} J L 1 L 2 R 2-a^{2} J L 1 R T-a^{2} J L 2 R T+2 J^{2} L 1 \alpha+2 J^{2} L 2 \alpha+ \\
& \left.2 J^{2} L 1 L 2 \alpha+2 a^{2} L 1 L 2 R 1 \alpha+2 a^{2} L 1 L 2 R 2 \alpha-2 a^{2} L 1 L 2 R T \alpha+3 J L 1 L 2 \alpha^{2}\right)
\end{aligned}
$$

```
Collect \(\left[\frac{1}{J L 1 L 2}\left(J^{3}+J^{3} L 1+J^{3} L 2+a^{2} J L 1 R 1+a^{2} J L 1 L 2 R 1+\right.\right.\)
    \(a^{2} J L 2 R 2+a^{2} J L 1 L 2 R 2-a^{2} J L 1 R T-a^{2} J L 2 R T+2 J^{2} L 1 \alpha+2 J^{2} L 2 \alpha+\)
    \(2 \mathrm{~J}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \alpha+2 \mathrm{a}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \mathrm{R} 1 \alpha+2 \mathrm{a}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \mathrm{R} 2 \alpha-2 \mathrm{a}^{2} \mathrm{~L} 1 \mathrm{~L} 2 \mathrm{RT} \alpha+3 \mathrm{~J}\) L1 L2 \(\alpha^{2}\) ) , \(\left.\alpha\right]\)
\(\frac{1}{J L 1 L 2}\)
    \(\left(J^{3}+J^{3} L 1+J^{3} L 2+a^{2} J L 1 R 1+a^{2} J L 1 L 2 R 1+a^{2} J L 2 R 2+a^{2} J L 1 L 2 R 2-a^{2} J L 1 R T-a^{2} J L 2 R T\right)+\)
    \(\frac{1}{J L 1 L 2}\left(2 J^{2} L 1+2 J^{2} L 2+2 J^{2} L 1 L 2+2 a^{2} L 1 L 2 R 1+2 a^{2} L 1 L 2 R 2-2 a^{2} L 1 L 2 R T\right) \alpha+3 \alpha^{2}\)
```

Plug in $\beta^{2}$ into real

$$
\begin{aligned}
& p p=J^{4}+a^{2} J^{2} L 1 R 1+a^{2} J^{2} L 2 R 2-a^{2} J^{2} R T+J^{3} \alpha+J^{3} L 1 \alpha+J^{3} L 2 \alpha+a^{2} J \operatorname{LIR1} \alpha+ \\
& a^{2} \text { J L1 L2 R1 } \alpha+a^{2} \text { J L2 R2 } \alpha+a^{2} \text { J L1 L2R2 } \alpha-a^{2} \text { JL1RT } \alpha-a^{2} \text { JL2RT } \alpha+J^{2} \text { L1 } \alpha^{2}+ \\
& J^{2} L 2 \alpha^{2}+J^{2} L 1 L 2 \alpha^{2}+a^{2} \text { L1 L2 R1 } \alpha^{2}+a^{2} \text { L1 L2R2 } \alpha^{2}-a^{2} \text { L1 L2RT } \alpha^{2}+J \text { L1 L2 } \alpha^{3}+ \\
& \left(-J^{2} L 1-J^{2} L 2-J^{2} L 1 L 2-a^{2} L 1 L 2 R 1-a^{2} L 1 L 2 R 2+a^{2} L 1 L 2 R T-3 \text { LL1L2 } \alpha \text { ) } \beta\right. \text { /. } \\
& \beta \rightarrow \frac{1}{J L 1 L 2}\left(J^{3}+J^{3} L 1+J^{3} L 2+a^{2} J L 1 R 1+a^{2} J L 1 L 2 R 1+a^{2} J L 2 R 2+\right. \\
& a^{2} \text { J L1 L2 R2 - } a^{2} \text { J L1 RT - } a^{2} J L 2 R T+2 J^{2} L 1 \alpha+2 J^{2} L 2 \alpha+2 J^{2} L 1 L 2 \alpha+ \\
& 2 a^{2} \text { L1 L2 R1 } \alpha+2 a^{2} \text { L1 L2 R2 } \alpha-2 a^{2} \text { L1 L2RT } \alpha+3 \mathrm{JL} 1 \text { L2 } \alpha^{2} \text { ) } \\
& J^{4}+a^{2} J^{2} L 1 R 1+a^{2} J^{2} L 2 R 2-a^{2} J^{2} R T+J^{3} \alpha+J^{3} L 1 \alpha+J^{3} L 2 \alpha+a^{2} J L 1 R 1 \alpha+ \\
& a^{2} \text { J L1 L2 R1 } \alpha+a^{2} \text { J L2 R2 } \alpha+a^{2} \text { J L1 L2 R2 } \alpha-a^{2} \text { JL1RT } \alpha-a^{2} \text { JL2RT } \alpha+J^{2} \text { L1 } \alpha^{2}+ \\
& J^{2} L 2 \alpha^{2}+J^{2} L 1 L 2 \alpha^{2}+a^{2} L 1 L 2 R 1 \alpha^{2}+a^{2} L 1 L 2 R 2 \alpha^{2}-a^{2} L 1 L 2 R T \alpha^{2}+J L 1 L 2 \alpha^{3}+ \\
& \frac{1}{J L 1 L 2}\left(-J^{2} L 1-J^{2} L 2-J^{2} L 1 L 2-a^{2} L 1 L 2 R 1-a^{2} L 1 L 2 R 2+a^{2} L 1 L 2 R T-3 J L 1 L 2 \alpha\right) \\
& \left(J^{3}+J^{3} L 1+J^{3} L 2+a^{2} J L 1 R 1+a^{2} J L 1 L 2 R 1+a^{2} J L 2 R 2+\right. \\
& a^{2} J L 1 L 2 R 2-a^{2} J L 1 R T-a^{2} J L 2 R T+2 J^{2} L 1 \alpha+2 J^{2} L 2 \alpha+2 J^{2} L 1 L 2 \alpha+ \\
& 2 a^{2} \text { L1 L2 R1 } \alpha+2 a^{2} \text { L1 L2 R2 } \alpha-2 a^{2} \text { L1 L2RT } \alpha+3 \mathrm{JL} 1 \text { L2 } \alpha^{2} \text { ) }
\end{aligned}
$$

Letting $\mathrm{a}=\pi$ and $\mathrm{J}=2 \pi^{2}$

```
\beta2[RT_, R1_, R2_] =
```



```
        a}\mp@subsup{}{2}{J
        2 a'L1 L2 RT \alpha + 3 J L1 L2 \alpha
```



```
    2 L1 L2 \pi}\mp@subsup{\pi}{}{4}\textrm{R}2-2\textrm{L}1\mp@subsup{\pi}{}{4}\textrm{RT}-2\textrm{L}2\mp@subsup{\pi}{}{4}\textrm{RT}+8\textrm{L}1\mp@subsup{\pi}{}{4}\alpha+8\textrm{L}2\mp@subsup{\pi}{}{4}\alpha+8\textrm{L}1\textrm{L}2\mp@subsup{\pi}{}{4}\alpha
    2 L1 L2 \pi}\mp@subsup{\pi}{}{2}\textrm{R}1\alpha+2\textrm{L}1\textrm{L}2\mp@subsup{\pi}{}{2}\textrm{R}2\alpha-2\textrm{L}1\textrm{L}2\mp@subsup{\pi}{}{2}\textrm{RT}\alpha+6\textrm{L}1\textrm{L}2\mp@subsup{\pi}{}{2}\mp@subsup{\alpha}{}{2
```

```
\(R 1\left[R T_{-}, R 1_{-}, R 2_{-}\right]=p p / .\{J \rightarrow 2 * P i \wedge 2, a \rightarrow P i\}\)
\(16 \pi^{8}+4 \mathrm{~L} 1 \pi^{6} \mathrm{R} 1+4 \mathrm{~L} 2 \pi^{6} \mathrm{R} 2-4 \pi^{6} \mathrm{RT}+8 \pi^{6} \alpha+8 \mathrm{~L} 1 \pi^{6} \alpha+8 \mathrm{~L} 2 \pi^{6} \alpha+2 \mathrm{~L} 1 \pi^{4} \mathrm{R} 1 \alpha+\)
    \(2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4} \mathrm{R} 1 \alpha+2 \mathrm{~L} 2 \pi^{4} \mathrm{R} 2 \alpha+2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4} \mathrm{R} 2 \alpha-2 \mathrm{~L} 1 \pi^{4} \mathrm{RT} \alpha-2 \mathrm{~L} 2 \pi^{4} \mathrm{RT} \alpha+4 \mathrm{~L} 1 \pi^{4} \alpha^{2}+\)
    \(4 \mathrm{~L} 2 \pi^{4} \alpha^{2}+4 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4} \alpha^{2}+\mathrm{L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} 1 \alpha^{2}+\mathrm{L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} 2 \alpha^{2}-\mathrm{L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{RT} \alpha^{2}+2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2} \alpha^{3}+\)
    \(\frac{1}{2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2}}\left(-4 \mathrm{~L} 1 \pi^{4}-4 \mathrm{~L} 2 \pi^{4}-4 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4}-\mathrm{L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} 1-\mathrm{L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} 2+\mathrm{L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{RT}-6 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2} \alpha\right)\)
        \(\left(8 \pi^{6}+8 \mathrm{~L} 1 \pi^{6}+8 \mathrm{~L} 2 \pi^{6}+2 \mathrm{~L} 1 \pi^{4} \mathrm{R} 1+2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4} \mathrm{R} 1+2 \mathrm{~L} 2 \pi^{4} \mathrm{R} 2+\right.\)
            \(2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4} \mathrm{R} 2-2 \mathrm{~L} 1 \pi^{4} \mathrm{RT}-2 \mathrm{~L} 2 \pi^{4} \mathrm{RT}+8 \mathrm{~L} 1 \pi^{4} \alpha+8 \mathrm{~L} 2 \pi^{4} \alpha+8 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{4} \alpha+\)
            \(\left.2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} 1 \alpha+2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} 2 \alpha-2 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2} \mathrm{R} T \alpha+6 \mathrm{~L} 1 \mathrm{~L} 2 \pi^{2} \alpha^{2}\right)\)
```

Assigning Lewis Numbers Le1 = 2 and Le2 = 3
L1 = 2;
L2 $=3$;
Simplify[ $\beta 2$ [RT, R1, R2]]
Simplify[Rl[RT, R1, R2]]
$2 \pi^{2}\left(24 \pi^{4}+6 \alpha(R 1+R 2-R T+3 \alpha)+\pi^{2}(8 R 1+9 R 2-5 R T+44 \alpha)\right)$
$-\frac{1}{3} \pi^{2}\left(480 \pi^{6}+18 \alpha(R 1+R 2-R T+4 \alpha)^{2}+2 \pi^{4}(112 R 1+117 R 2-85 R T+628 \alpha)+\right.$
$\left.3 \pi^{2}\left(8 R 1^{2}+9 R 2^{2}-14 R 2 R T+5 R T^{2}+124 R 2 \alpha-108 R T \alpha+352 \alpha^{2}+R 1(17 R 2-13 R T+120 \alpha)\right)\right)$

## Roots of Oscillatory Mode

OscRoot1[RT_, R1_, R2_] = $\alpha / . \operatorname{Solve[Rl[RT,~R1,~R2]~==~0,~} \alpha][[1]] ;$
OscRoot2[RT_, R1_, R2_] = $\alpha /$. Solve[Rl[RT, R1, R2] == 0, $\alpha$ ][[2]];
OscRoot3[RT_, R1_, R2_] = $\alpha /$. Solve[Rl[RT, R1, R2] == 0, $\alpha$ ][[3]];
Discriminant of $\beta^{2}$ equation for Quadratic $\alpha$ Solution

```
Collect[\beta2[RT, R1, R2], 人]
48 \mp@subsup{\pi}{}{6}+16\mp@subsup{\pi}{}{4}R1+18\mp@subsup{\pi}{}{4}R2-10 \mp@subsup{\pi}{}{4}RT+(88\mp@subsup{\pi}{}{4}+12\mp@subsup{\pi}{}{2}R1+12 \mp@subsup{\pi}{}{2}R2-12 \mp@subsup{\pi}{}{2}RT})\alpha+36\mp@subsup{\pi}{}{2}\mp@subsup{\alpha}{}{2
Aa = 48 \mp@subsup{\pi}{}{6}+16 \mp@subsup{\pi}{}{4}R1+18\mp@subsup{\pi}{}{4}\textrm{R}2-10 \mp@subsup{\pi}{}{4}RT;
```



```
Cc = 36 告;
disc[RT_, R1_, R2_] = Bb^2 - 4* Aa*Cc;
Roots of \(\beta^{2}\)
Simplify[\beta2Rootp[RT_, R1_, R2_] = (-b + Sqrt[b^2-4*a*c])/2c]
Simplify[\beta2Rootn[RT_, R1_, R2_] = (-b-Sqrt[b^2-4*a* c])/2c]
-72 \mp@subsup{\pi}{}{4}(22\mp@subsup{\pi}{}{2}+3R1+3R2-3RT-\sqrt{}{52\mp@subsup{\pi}{}{4}+9(R1+R2-RT\mp@subsup{)}{}{2}-6\mp@subsup{\pi}{}{2}(2R1+5R2+7RT})}
-72 \mp@subsup{\pi}{}{4}(22\mp@subsup{\pi}{}{2}+3R1+3R2-3RT+\sqrt{}{52\mp@subsup{\pi}{}{4}+9(R1+R2-RT\mp@subsup{)}{}{2}-6\mp@subsup{\pi}{}{2}(2R1+5R2+7RT)})
```

Finding unstable oscillatory values in 2D case

```
n= 200; (*checking values in the range of this n}\mp@subsup{}{}{3}\mathrm{ space*)
h = 1; (*stepping by h*)
r = Range[-n, n, h];
Osc0 = {};
Do[rt = r[[j]];
    r1 = r[[k]];
    If[Abs[Im[N[OscRoot1[rt, r1, 0], 40]]] < 10^-20
            (*real valued*) &&Re[N[OscRoot1[rt, r1, 0], 40]] > 10^^20
            (*positive*) && (N[disc[rt, r1, 0], 40] < 10^- 20(*real valued \beta*)||
                    (N[disc[rt, r1, 0], 40] > 10^-20(*real valued \beta*) &&
                    (Re[N[\beta2Rootp[rt, r1, 0], 40]] < Re[N[OscRoot1[rt, r1, 0], 40]]
                            (*\alpha greater than largest root*) ||
                            Re[N[\beta2Rootn[rt, r1, 0], 40]] > Re[N[OscRoot1[rt, r1, 0], 40]])))
            (* or \alpha less than smallest root*), AppendTo[Osc0, {rt, r1}],
        Continue]
    If[Abs[Im[N[OscRoot2[rt, r1, 0], 40]]]< 10^-20 &&
            Re[N[OscRoot2[rt, r1, 0], 40]] > 10^-20 &&
            (N[disc[rt, r1, 0], 40] < 10^-20|| (N[disc[rt, r1, 0], 40] > 10^-20 &&
                    (Re[N[\beta2Rootp[rt, r1, 0], 40]] < Re[N[OscRoot2[rt, r1, 0], 40]] ||
                            Re[N[\beta2Rootn[rt, r1, 0], 40]] > Re[N[OscRoot2[rt, r1, 0], 40]]))),
        AppendTo[Osc0, {rt, r1}], Continue] If[Abs[Im[N[OscRoot3[rt, r1, 0], 40]]] <
            10^-20 &&Re[N[OscRoot3[rt, r1, 0], 40]] > 10^-20 &&
            (N[disc[rt, r1, 0], 40] < 10^-20||(N[disc[rt, r1, 0], 40] > 10^-20&&
                    (Re[N[\beta2Rootp[rt, r1, 0], 40]] < Re[N[OscRoot3[rt, r1, 0], 40]]||
                            Re[N[\beta2Rootn[rt, r1, 0], 40]] > Re[N[OscRoot3[rt, r1, 0], 40]]))),
        AppendTo[Osc0, {rt, r1}], Continue], {j, 1, Length[r], 1}, {k, 1, Length[
        r], 1}]
```

```
3 D case
n = 200; (*checking values in the range of this n}\mp@subsup{}{}{3}\mathrm{ space*)
h = 1; (*stepping by h*)
r = Range[-n, n, h];
Osc0 = {};
Do[rt = r[[j]];
    r1 = r[[k]];
    r2 = r[[l]]
        If[Abs[Im[N[OscRoot1[rt, r1, r2], 40]]] < 10^-20
            (*real valued*) &&Re[N[OscRoot1[rt, r1, r2], 40]] > 10^-20
            (*positive*) && (N[disc[rt, r1, r2], 40] < 10^- 20(*real valued \beta*) ||
                (N[disc[rt, r1, r2], 40] > 10^-20(*real valued \beta*) &&
                    (Re[N[\beta2Rootp[rt, r1, r2], 40]] < Re[N[OscRoot1[rt, r1, 0], 40]]
                            (*\alpha greater than largest root*) ||
                    Re[N[\beta2Rootn[rt, r1, r2], 40]] > Re[N[OscRoot1[rt, r1, r2], 40]])))
                    (* or \alpha less than smallest root*), AppendTo[Osc0, {rt, r1, r2}],
        Continue]
        If[Abs[Im[N[OscRoot2[rt, r1, r2], 40]]] < 10^-20 &&
            Re[N[OscRoot2[rt, r1, r2], 40]] > 10^-20 &&
            (N[disc[rt, r1, r2], 40] < 10^- 20|| (N[disc[rt, r1, r2], 40] > 10^-20 &&
                    (Re[N[\beta2Rootp[rt, r1, r2], 40]] < Re[N[OscRoot2[rt, r1, 0], 40]] ||
                            Re[N[\beta2Rootn[rt, r1, r2], 40]] > Re[N[OscRoot2[rt, r1, 0], 40]]))),
        AppendTo[Osc0, {rt, r1, r2}], Continue] If[Abs[Im[N[OscRoot3[rt, r1, r2], 40]]]<
            10^-20 &&Re[N[OscRoot3[rt, r1, r2], 40]] > 10^-20 &&
            (N[disc[rt, r1, r2], 40] < 10^- 20|| (N[disc[rt, r1, r2], 40] > 10^-20 &&
                    (Re[N[\beta2Rootp[rt, r1, r2], 40]] < Re[N[OscRoot3[rt, r1, r2], 40]] ||
                            Re[N[\beta2Rootn[rt, r1, r2], 40]] > Re[N[OscRoot3[rt, r1, r2], 40]]))),
        AppendTo[Osc0, {rt, r1, r2}], Continue], {j, 1, Length[r],
        1}, {k, 1, Length[r], 1}, {l,
    1,
        Length[
        r], 1}]
```

