Can You Borrow From an Already Borrowed From Number?
Insights into Second Graders' Knowledge of Place Value

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A thesis submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Master of Arts in the School of Education.

Chapel Hill
2007

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ABSTRACT

TRACY L. JOHNSON: Can You Borrow From an Already Borrowed From Number?
Insights into Second Graders' Knowledge of Place Value
(Under the direction of Olof Steinthorsdottir)

This purpose of this study is to provide insights about end-of-the year second graders' knowledge of place value and its application in solving two- and three-digit addition and subtraction problems. Twenty-two students in two schools in the same district in rural North Carolina were interviewed using a qualitative, structured, task-based interview. The tasks addressed number combinations, use of ten as a composite unit, conservation of quantity for grouped objects, incrementing on and off the decade by tens with and without physical representations of quantities, two- and three-digit addition and subtraction problems, and understanding of algorithmic procedures. The findings of this study indicate that students' strategy selection is largely algorithm-dependent, and students appear to have stronger procedural knowledge than conceptual understanding of the standard algorithm. Students had more difficulty procedurally and conceptually with subtraction than addition. This study also found that students' highest known number combination may relate to their overall level of base-ten knowledge. Finally, the interview protocol used to assess students' place value knowledge appears to provide comprehensive data for assessing levels of knowledge.

ACKNOWLEDGEMENTS

The project was completed with funding from the UNC University Research Council Research Grant: Pilot Study - Relating Mathematics Curricula, Implementation and Learning Outcomes: What happens at the end of Grade 2? (Dr. Susan N. Friel, Principal Investigator).

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Introduction

The term "place value" refers to many different and complex ideas. Conceptualization of place value is more advanced than labeling columns or digits as "tens" or "ones"; it requires an understanding of the base-ten number system (Richardson, 2002b). However, teaching and learning place value is situated within a context in which "the prevailing view of school mathematics is one of rules and procedures, memorization and practice, and exactness in procedures and in answers" (Lindquist, 1997, p. xiv; see also Boaler, 2003). Place value understanding is a necessary mathematical developmental understanding that must solidify in early to middle elementary school.

Research has established that many children have difficulty understanding place value concepts (Brown & Burton, 1978; Cauley, 1988; Ginsburg, 1989; Kamii, 1986; Miura & Okamoto, 1989; Resnick, 1987; Resnick & Omanson, 1987; Ross, 1989). This is caused by several factors. One challenge is understanding the role of the location of digits. Many students struggle with learning the difference between a number's "face value" (the name of the digit, ranging from 0-9) and its "complete value" (the amount the digit represents, given its position in the numeral) (Becker & Varelas, 1993). For example, many children identify the two in 23 as having a value of 2 instead of 20.

Another challenge to place value understanding is the disparity between physical representations of quantities and the written numeral. Base-ten materials are physical representations of the complete value. Students may be able to manipulate these materials and yet still have difficulty transitioning to written notation in which numbers with the same face value hold different complete values. For example, consider the number 282, where the left 2 is worth 200 and the right 2 is worth 2. When using base-ten materials, students do not need to consider distinctions between the 2 hundreds and the 2 ones because they are represented in their complete value form via two different groupings of blocks (the 200 is represented with (two) hundreds flats and the 2 is represented with two individual units). There is a different physical representation for each 2 based on its place value. However, written notation does not provide distinct representations for the 200 and the 2 as they are both written simply as "2." The fact that a numeral can have the same digit (or face value) in different locations but represent different amounts is not addressed by using base-ten materials and therefore can cause confusion with students when transitioning to written notation (Becker & Varelas, 1993).

Additionally, there is a difference in the verbal and written form of numerals in European languages. Consider how the "teen" numbers, and especially "eleven" and "twelve," do not follow the pattern of higher two-digit numbers in which the tens are spoken first and the ones are spoken second (such as "twenty-one") (Fuson & Smith, 1996; Sharma, 1993). Further, the presence of a place-holding zero is not explicitly verbally communicated. In order to write "one thousand eighty four" students must know to write a zero in the hundreds place. Some errors may include writing 184 or 100084.

Not only is place value challenging to conceptualize, it is also necessary as a foundation to acquire other mathematical concepts. Place value understanding is essential in order for students to have success with addition and subtraction of multi-digit numbers, decimal operations, algebraic expressions and equations, scientific notation, and exponents (Wearne & Hiebert, 2002; Sowder, 2002; Carpenter, Franke, & Levi, 2003; Sharma, 1993). Since number pervades all areas of mathematics, place value understanding radiates throughout additional mathematical concepts as well (National Council of Teachers of Mathematics-NCTM, 2000). "Mathematics in the early years is not just a simpler version of mathematics that children will learn later. Rather, (it) provides foundational concepts that are key to understanding more formal and abstract ideas. To be truly prepared for later mathematics, young children need to develop flexibility in thinking about numbers" (Richardson, 2003, p.2). Developing a strong foundation of place value understanding in early elementary school is a key benchmark that improves students' abilities to conceptualize mathematical operations, processes, and relationships throughout and well beyond elementary school.

Informal experiences with place value and the base-ten number system occur before children enter school (Clements & Sarama, 2007), and formal experiences with place value generally begin in first grade. The National Council of Teachers of Mathematics (NCTM) delineates the importance of place value in its *Principles and Standards for School Mathematics* (2000) and NCTM *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (2006). Students are to develop "beginning ideas of tens and ones" in first grade (NCTM, 2006, p.22). In second grade a more overt emphasis on place value occurs. The North Carolina Standard Course of Study (NCSCOS) indicates that place value is one of the “major concept(s)” in second grade (NCSCOS, 2003). Similarly, two of the three NCTM second grade Focal Points address place value. The first states that students are to “develop an understanding of the base-ten numeration system and place-value concepts” and the second states that students are to “develop fluency with multi-digit addition and subtraction” (NCTM, 2006, p.23). Excerpts from these two second grade focal points follow:

> ***1.*** *Number and Operations:* **Developing an understanding of the base-ten numeration system and place-value concepts:** Children develop an understanding of the base-ten numeration system and place-value concepts (at least to 1000). Their understanding of base-ten numeration includes ideas of counting in units and multiples of hundreds, tens, and ones, as well as a grasp of number relationships…They understand multi-digit numbers in terms of place value, recognizing that place-value notation is a shorthand for the sums of multiples of powers of 10 (e.g., 853 as 8 hundreds + 5 tens + 3 ones).
>
> ***2.*** *Number and Operations and Algebra:* **Developing quick recall of addition facts and related subtraction facts and fluency with multi-digit addition and subtraction:** Children…solve arithmetic problems by applying their understanding of relationships and properties of number (such as place value). Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multi-digit whole numbers…They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.
>
> (NCTM, 2006, p.23)

Both the NCTM *Principles and Standards* (2000) and *Curriculum Focal Points* (2006) indicate that all students nationally should develop an understanding of place value in second grade. This study examined second graders’ mathematical understandings surrounding various aspects of place value. The central question of the exploratory study is this:

> What insights do a series of related interview tasks provide about end-of-year second grade students' knowledge of place value and its application in solving two- and three-digit addition and subtraction problems?

Within this exploratory question, I specifically examined second graders' understanding of some of the big ideas related to place value and the use of ten. These big ideas included what number combinations for numbers 1-10 do students know fluently, what is students' facility with unitizing, and what is students' facility recognizing and using place value patterns? I also explored students' solution strategies for two- and three-digit addition and subtraction problems. Finally, I considered whether there were disconnects between students' procedural knowledge and conceptual understanding of place value when using the standard algorithm at the end of Grade 2.

The first section of this paper (Background) presents the literature on place value. First, the stages of development in children's conceptualization of place value are detailed, and place value is discussed as a key developmental understanding and its composite big ideas are delineated. In the second section (Methodology), the methodology and data analysis of this study are described. The third section (Results) contains the results of this study. The fourth section (Discussion) discusses implications for instruction.

Background

In this section the theoretical framework for place value knowledge is discussed. First, suggested stages of development in children's conceptualization of place value understanding are presented. Second, distinctions between "types" or "levels" of mathematical knowledge and understanding are discussed. Third, place value is explained as a key developmental understanding and several big ideas within place value are summarized. Fourth, what is known about children's solution strategies for solving two-digit addition and subtraction problems is presented. Fifth, the effects of using algorithms on young children's understanding is discussed.

Stages of Development of Place Value Understanding

It is important to know what the existing categorizations of conceptual development are regarding two-digit numbers and place value. This information was used both to select tasks and analyze responses and is presented in the following section.

Wright, Martland, Stafford and Stanger (2006b) outline a framework for students' conceptual place value progression. This learning framework for early number knowledge which contains three levels of development of base-ten arithmetical strategies: Level 1) Initial concept of ten, Level 2) Intermediate concept of ten, and Level 3) Facile concept of ten. The descriptors of each level are presented in the following table.

Table 1

Levels of Base-ten Arithmetic Knowledge

Level	Description
Level 1--Initial Concept of Ten	The child does not see ten as a unit of any kind. The child focuses on the individual items that make up the ten. In addition or subtraction tasks involving tens, children count forward or backward by ones.
Level 2--Intermediate Concept of Ten	Ten is seen as a unit composed of tens and ones. The child is depending on re-presentations (like mental replay or recollection) of units of ten such as hidden ten-strips or open hands of ten fingers. The child can perform addition and subtraction tasks involving tens where these are presented with materials such as covered strips of tens and ones. The child cannot solve addition and subtraction tasks involving tens and ones presented as written number sentences.
Level 3--Facile Concept of Ten	The child can solve addition and subtraction tasks involving tens and ones without using materials or re-presentations of materials. The child can solve written number sentences involving tens and ones by adding or subtracting units of tens and ones.

Note. From *Teaching number: Advancing children's skills and strategies* (p. 10), by R. J. Wright et al., 2006b, Thousand Oaks, CA: Sage. Copyright 2006 by Robert J. Wright, Jim Martland , Ann K. Stafford, and Garry Stanger.

Carpenter, Fennema, Franke, Levi, and Empson (1999) also outline students' procedural and conceptual place value progression in their work on Cognitively Guided Instruction (CGI). The authors document the progression of children's development of number concepts.

Base-ten development begins with the child **Counting by Ones**, and is similar to Level 1—Initial Concept of Ten (Wright et al., 2006b). At this stage the child does not understand that they can count groups of ten directly. Groups of ten appear to carry no significance with regard to the number assigned to the collection of counters. When presented with a collection of objects grouped by tens, a child would count by ones.

The second stage of base-ten development is **Counting by Tens** and is similar to Level 2—Intermediate Concept of Ten (Wright et al., 2006b). . At this stage the child is able to use base-ten number concepts. When presented with a collection of objects grouped by tens, a child would count the grouped objects by tens and then count the ones.

The highest stage of base-ten development is **Direct Place Value** and is similar to Level 3—Facile Concept of Ten (Wright et al., 2006b). This is a more flexible conception of base-ten concepts. When presented with a collection of objects grouped by tens, a child would not count but rather would immediately recognize the total number of objects in the grouped sets (e.g. 5 groups of ten is 50 objects) and add the ones to this number (e.g. three more make 53). This type of thinking is more advance and flexible than that of a child who counts by tens.

The Nature of Mathematical Understanding

Within mathematics education there has been an increased focus on promoting students' understanding. The National Council of Teachers of Mathematics (2000) states that, "students must learn mathematics with understanding" (p.11). Brownell (1947) offers several reasons why learning should be focused on meaning and understanding, including assurance of retention, increased likelihood that ideas and skills will be used, providing a foundation that allows for transferable understandings, reducing the amount of repetitive practice necessary, safeguarding against mathematically absurd answers, and providing versatile and flexible approaches. As Hiebert et al. (1997) state, "understanding is crucial because things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things" (p.1).

Mathematical "understanding" has been defined using various terms and constructs, although it generally describes a distinction between skill and meaning. Piaget (1978) described the difference between successful action and conceptual understanding. Skemp (1978) defined instrumental understanding as knowing what to do or the possession of a rule and the ability to use it and relational understanding as knowing what to do and why. Hiebert and Lefevre (1986) distinguished between conceptual knowledge that is rich in relationships and procedural knowledge that consists of symbolic systems and rules for completing mathematical tasks. Resnick (1982) used the terms syntax and semantics, Gelman and Gallistel (1978) differentiated between principles and skills, and Hatano (1988) distinguished between routine expertise and adaptive expertise.

There is much overlap among these various labels. In this paper the learning of skills will be referred to as "procedural knowledge," and the development of understanding will be referred to as "conceptual understanding."

Place Value as a Key Developmental Understanding

As an underlying support of this study, Simon's (2006) construct of a "key developmental understanding" (KDU) is detailed. A key developmental understanding "identifie(s) critical transitions that are essential for students' mathematical development" (p.360). Further, a KDU "involves a conceptual advance on the part of students…a change in the students' ability to think about and/or perceive particular mathematical relationships" (p.362). Another characteristic of a KDU is that "students without the knowledge do not tend to acquire it as the result of an explanation or demonstration. That is, the transition requires a building up of the understanding through students' activity and reflection and usually comes about over multiple experiences" (p.362).

Place value concepts fit within Simon's construct of a KDU, as place value understanding cannot be imparted simply with telling, showing, explaining, or demonstrating. Furthermore, place value understanding impacts the way students think about mathematical relationships, without which they may suffer from a fragile mathematical foundation that may result in unfulfilled mathematical potential (NCTM, 2006).

This study explores the key developmental understanding of place value in depth during the time in which students are engaged in a "critical transition" that is essential for their mathematical development. End-of-the-year second graders are at this significant mathematical juncture. As Simon (2006) wrote, "A focus is needed on those understandings whose development tends to require more than an explanation or demonstration" (p.362).

Place Value and Big Ideas

A broad and complex concept, “place value” is too intricate to be considered as only one KDU. Rather, place value is comprised of several KDUs, and in order for students to be facile with place value I surmise that they will need to conceptualize several KDUs. This is supported by Fosnot’s (2007) work, which specifies several “big ideas” that relate to place value understanding.

The first big idea that underlies place value, addition, and subtraction is **unitizing**. Fosnot (2007) defines unitizing as counting single objects, groups, and the ability to do both simultaneously. Children require a “shift in perspective” to “treat a group of ten as a unit, and ten groups of ten (100 units) as one unit of a hundred” (Fosnot, 2007, p.7). Unitizing is also referred to in the literature as the use of composite units. Understanding ten as a composite unit means that a student can combine single units into a new, countable unit. As Carpenter et al. (1999) state, “The central principle that children must grasp to understand base-ten numbers is that collections of ten (or one hundred or one thousand) can be counted” (p.59). Students who have conceptualized composite units and unitize can see ten as ten ones and one unit of ten. They also understand that the structure of a number is based on its organization into groups of tens and ones (Richardson, 2002b). Unitizing, or conceptualization of composite units, is “essential for the development of concepts of place value and multiplication—thus, the identification of composite units as a KDU” (Simon, 2006, p.367).

Fosnot’s (2007) second big idea behind place value is that **place determines value**. This refers to children understanding the difference between a numeral’s face value and complete value. “The numeral 2 may represent two units, but the units themselves can change; they can be ones or tens or hundreds or more” (Fosnot, p.7). Where the numeral is placed determines its amount.

Another big idea within place value understanding is **equivalence**. Students need to understand that “amounts can be rearranged and decomposed and still be equivalent” (Fosnot, 2007, p.7). Students with this understanding are able to conceptualize that 3 tens and 4 ones are equivalent to 2 tens and 14 ones as well as 1 ten and 24 ones.

Commutativity and associativity is another big idea related to place value. Fosnot (2007) explains, “Children need many opportunities to compose and decompose numbers before they realize that numbers can be grouped in a variety of ways, even turned around, and the amounts stay the same” (p.7). NCTM (2000) states, “students with number sense naturally decompose number” (p.32). Being able to compose and decompose numbers “contributes to developing part-whole relations, one of the most important accomplishments in arithmetic” (Clements & Sarama, 2007, p.486). “When children learn to subtract without decomposing numbers, they do not have the foundational skills necessary for solving problems with larger and more complex numbers” (Richardson, 2002a, p.26). For example, without being able to decompose numbers a student would not be able to use the strategy of compensation (subsequently described in Background: *Strategies*, p.13). As Kamii (2000) stated, “Our goal in single-digit addition is that children become able to think flexibly about numbers and construct a network of numerical relationships” (p.69).

The final place value-related big idea Fosnot (2007) defines is **the place value patterns that result from repeatedly adding or subtracting ten, making groups of ten, or**

multiplying by ten. Children must learn the patterns associated with counting on the decade (also referred to as landmark numbers, e.g., 10, 20, 30…) as well as counting off the decade (e.g., 14, 24, 34…). Children also need to learn that these patterns continue past higher landmark numbers, such as 100. The ability to increment by tens is "the forerunner to the development of place value knowledge" (Wright, Martland & Stafford, 2006a, p.93). The ability to count by ten, both on and off the decade, will result in an increased ability of students to engage in mental arithmetic and conceptualize the quantities with which they are working.

Simon (2006) postulated that by observing students engaged in mathematical tasks it is possible to "specify understandings that can account for differences in the actions of different students in response to the same task. A way to explain these observed differences is by postulating a KDU" (p.363). This study used interview tasks designed from place value-related KDUs and big ideas to identify what students are able to do and what students appear to know with respect to place value understanding. By using the constructs of KDUs and big ideas to explore students' responses it is possible to highlight distinctions between students' understandings, given the level of sophistication in their responses and strategies. Student strategies are described in the following section.

Strategies

The NCTM (2000) states "Students should be able to perform computations in different ways" (p.32). Within the CGI framework, children's solution strategies for addition and subtraction move through a progression to become more efficient, sophisticated, and abstract. The stages of strategy development follow.

In the beginning, children solve problems by **direct modeling**. This means students use concrete objects or otherwise represent (e.g., through drawing) quantities in order to solve problems. Direct modeling lays an important foundation for children, as "many children flounder in mathematics when they reach the elementary grades precisely because they haven't had sufficient hands-on experiences with the place-value system in a concrete form" (Ginsberg, Inoue & Seo, 2000, p.139).

Over time, children's direct modeling is replaced by counting strategies. In addition problems (join-result unknown), children may count on from the first number given in the problem (**count on from first**) or they may count on from the larger of the given numbers (**count on from larger**). In subtraction problems (separate-result unknown), they may **count down** to find the answer. Counting strategies are more sophisticated than direct modeling, because "counting strategies indicate a level of understanding of number concepts and an ability to reflect on numbers as abstract entities" (Carpenter et al., 1999, p. 28). Eventually counting strategies are replaced with even more efficient and abstract strategies of using number facts that are **known facts**.

Math Recovery explains additional categorizations of strategies that students may use to solve two-digit addition or subtraction tasks (Wright et al., 2006b). One strategy is called the **jump method,** identified elsewhere in the literature as the N10 method (Beishuizen, 1993) or the sequencing or cumulative method (Thompson, 1999, p.150). A child using this strategy "jumps" from one of the given numbers by the number of tens in the second number, then “jumps” by the number of ones in the second number. For example, to solve 47+ 22 one would begin at 47, add the 20 (or 10 and 10) to get to 67, and then add the 2 to yield a final answer of 69. In order to successfully use this strategy a student must be able to increment by ten off the decade, which is one reason why the big idea of recognizing place value patterns is critical.

Fosnot (2007) outlined a strategy similar to the jump method as **keeping one number whole, using landmark numbers, and adding or removing chunks of tens or hundreds**. To use this strategy a child would begin the same as the jump method, beginning with one of the given numbers and jumping by the number of tens in the second number. Next, the child would use some of the ones from the second number to reach a landmark (or “friendly”) number. Finally, the child would add the remaining ones from the second number. For example, to solve 33 + 29 one would begin at 33, add 10 to get to 43, and add 10 to get to 53. In the next step, one would add 53 + 7 to get to 60 (a landmark or friendly number), then add 2 more to arrive at the answer of 62.

Another strategy that students may use to solve two-digit addition or subtraction tasks is the **split method** (Fosnot, 2007; Wright et al., 2006b), identified elsewhere in the literature as the 1010 method (Beishuizen, 1993) or the partitioning method (Thompson, 1999). A child using this strategy “splits” the numbers by place value position into tens and ones to work with them separately in expanded notation and then recomposes the total quantity to produce the final answer. For example, to solve 47+ 22 one would begin by adding 40 and 20 to get 60, then add 7 and 2 to get 9, and then add 60 and 9 to yield a final answer of 69.

The last strategy that some students may use to solve the two-digit addition and subtraction problems is **compensation** (Chapin & Johnson, 2000). A child using this strategy changes the original numbers to make “friendly” or “nice” numbers that are easy to work with, performs the operation, and then compensates for the change made to the original quantity to arrive at the final answer. For example, to solve 49 + 14 a child may think about adding 50 and 15 to get 65, and then subtract 2 from 65 to yield a final answer of 63.

The Standard Algorithm and Young Children

One specific strand of discussion regarding the distinctions that exist between conceptual understanding versus procedural knowledge focuses on the use of the standard algorithm with young children (i.e., children in kindergarten, first, or second grade). An expanding literature base has focused on the use of the standard algorithm with young children and the effect this may have on their conceptual understanding (Kamii, 2000; McIntosh, 1990; Thompson, 1999).

The standard algorithm was developed to efficiently solve mathematical problems. Standardized symbolic notation has been “streamlined over hundreds of years to contain a maximum amount of information with a minimum amount of writing. This means they are quite dense and students can find it hard to construct meaning for them” (Hiebert et al., 1997, p.57). Although it is efficient for adults to use an algorithm, “an analysis of place value that seems reasonable to an adult in terms of his or her own relatively sophisticated understanding of place value is no substitute for a conceptual analysis of children’s mathematics” (Cobb & Wheatley, 1988). Traditional approaches to teaching place value via two-digit addition and subtraction entail the use of the standard algorithm, in which students "stack" the two numbers vertically and operate in columns, working from right to left and "carrying" or "borrowing" as needed. Typically, this notation is emphasized in second grade.

However, introducing (much less requiring) standardized opaque notation in advance of students' conceptualization of place value can have negative effects. "For many children, the effect of initial instruction on arithmetic symbols is to pry apart conceptual and procedural knowledge and send them in different directions" (Hiebert & Lefevre, 1986, p.20). This is detrimental because, as Hiebert et al. (1997) explain, "if students separate their conceptual understandings from their procedures it means that they cannot solve problems very well" (p. 24). It is very difficult to re-join conceptual understanding and procedural knowledge after they have been separated, and most students are not successful because "it is hard to go back and try to understand a procedure after you have practiced it many times" (Hiebert et al., 1997, p.25). Instead, using the standard algorithm often results in students not understanding the procedures they are using and leads to students making errors because they distort a rule, forget a step in a procedure, over-generalize a rule, or fail to adjust a rule for a different type of problem (see Carpenter, Franke & Levi, 2003; Hiebert et al., 1997; Kamii, 2000). This results in "syntactically plausible but conceptually flawed errors that so often plague children's use of school-taught algorithms" (Sophian, 1999, p.15).

Kamii (2000) explained that after working with second graders she realized that using the standard algorithm is harmful because "they 'unteach' place value, thereby preventing children from developing number sense" (p. 83). Kamii found that the use of standard or school algorithms untaught place value by reinforcing misconceptions (thinking the 2 in 25 is actually worth 2) and resulted in less plausible answers when compared with children who did not use the standard algorithms. School algorithms are detached from the quantities the numbers represent because of the way they operate in independent columns, and therefore children do not necessarily see the link to the quantities represented. Carraher, Carraher, and Schliemann (1987) articulated a similar problem, stating that the use of standard algorithms promotes "manipulation of symbols" as opposed to the "manipulation of quantities." Difficulties arise from the way standardized written procedures are "symbolic and contracted and by their very nature involve pure manipulation of symbols without reference to the particular meanings which the place value system attaches to these individual symbols" (Thompson, 1999, p.173).

"There is little doubt that one of the main reasons for…underperformance in the number tests of international surveys is the very early introduction of formal written calculation method" (Thompson, 1999, p.170). This statement is supported by findings that the early use of standard algorithms can obstruct students' development of mental strategies and arithmetic (Beishuizen & Anghileri, 1998). This is due to a disparity between children's thinking and their strategies versus the way the written algorithm operates. Studies have revealed that children typically manipulate a quantity from left to right, whereas standard algorithms operate from right to left. Children also manipulate the entire quantity rather than isolated digits and show a preference for recording their work horizontally rather than vertically, both of which are in further discord with standard algorithms (Thompson, 1999).

Development of students' strategies is also obstructed by the way standard algorithms encourage "cognitive passivity" (Williams, 1962-3). "The algorithm demands that you not even *try* to think about what the digits actually represent. If you do, you are highly likely to become confused. Instead you are expected to suspend disbelief and follow the recommended steps in the procedure" (Thompson, 1999, p.173). Kamii (2000) also wrote about how standard algorithms "encourage children to give up their own thinking" (p. 83). This idea was further developed by Thompson (1999), who explained that "the decision as to how to set out the calculation, where to start, what value to assign the digits, etc. are all taken out of the individual's hands" (p.173).

Fosnot (2007) emphasized "for students today, a deep understanding of place value and equivalence is critical…to be able to assess the reasonableness of an answer found by using a calculator…to have good mental arithmetic strategies…[and] to know how to calculate efficiently" (p.6). Another reason why it is critical that students understand place value and arithmetic calculations is because they lay a foundation for algebra. If students perceive arithmetic as a series of steps and procedures, then they may not realize the properties and relationships of the numbers that allow for calculations. This, in turn, may mean that they may not recognize it is these same properties and relationships that allow them to simplify expressions and solve equations. "If students genuinely understand arithmetic at a level at which they can explain and justify the properties they are using as they carry out calculations, they have learned some critical foundations for algebra" (Carpenter et al., 2003, p.2).

Methodology

Sampling Procedures

The purpose of this study is to provide insights about end-of-year second graders' knowledge of place value and its application in solving two- and three-digit addition and subtraction problems. This section describes the sample of students who were part of this study. It also presents a detailed discussion of the tasks and data collection procedures used to assess students' knowledge and application. Finally, a description of how data were analyzed is provided.

This study was conducted using a sample of students in a school district located in a rural county in central North Carolina. Qualitative, structured, task-based interviews were carried out in two of the district's seven elementary schools, School A and School B. During the 2006-2007 school year, School A had an enrollment of 504 students, of which 0.4% were Asian, 6.2% Hispanic, 18.8% Black or African American, 72.2% White, 0.6% American Indian, and 1.8% Multi-Racial. At School A, 34% of students received free or reduced lunch. School B had an enrollment of 435 students, of which 0.2% were Asian, 3.0% Hispanic, 13.3% Black or African American, 79.3% White, 0.0% American Indian, and 4.1% Multi-Racial. At School B, 30% of students received free or reduced lunch. School A uses Saxon Math as the basis for their K-2 mathematics curriculum. School B does not use a specific curricular program for their K-2 mathematics instruction. Both schools emphasize the use of the standard algorithm to solve multi-digit addition and subtraction problems in the second grade.

Forty second graders, 23 from School A and 17 from School B, were interviewed during the last week of April and the first week of May 2007. In order to assess students with similar experiences with and access to the curriculum, students who were identified as Exceptional Children (learning disabled or otherwise identified) or English Language Learners were excluded from the potential sample pool from which the students were randomly selected. Ten students were randomly selected from each classroom with the intent of interviewing six of these students once permissions were obtained. In two classrooms, only five permissions were given. In School A, six students were interviewed in each of three of the second grade classes and five students were interviewed in the fourth second grade class, yielding 23 interviews. In School B, six students were interviewed in each of two of the second grade classes and five students were interviewed in the third second grade class, yielding 17 interviews.

The sample selected is representative of the total population of second graders in both schools. Table 2 shows the demographic information of the potential sample pool (all second grade students except for those identified as EC or ELL).

Table 2

Demographic Information of the Potential Sample Pool (All Second Grade Students Except those Identified as EC or ELL)

	Classroom								
	School A				School B			Total	
	1	2	3	4	5	6	7	N	Percent
Total Number of Students Eligible for Study (Non-EC, Non-ELL)	22	20	22	20	16	19	19	138	100%
Gender									
Female	12	9	10	8	9	11	9	68	49.3%
Male	10	11	12	12	7	8	10	70	50.7%
Racial and Ethic Categories									
American Indian or Alaskan Native	0	0	0	0	0	0	0	0	0.00%
Asian	0	1	0	0	0	0	0	1	0.72%
Black or African American	3	4	6	6	2	4	3	28	20.3%
Hispanic or Latino	0	0	2	1	0	0	1	4	2.9%
Native Hawaiian or Other Pacific Islander	0	0	0	0	0	0	0	0	0.00%
White	19	15	13	13	14	14	15	103	74.5%
Multi-Racial	0	0	1	0	0	1	0	2	1.4%

Table 3 shows the demographic information of the students who were randomly selected and interviewed. Note that the percent-representation of each category is consistent with that of the potential sample pool from Table 2.

Table 3

Demographic Information of the Interviewed Students

	Classroom							Total	
	School A				School B				
	1	2	3	4	5	6	7	N	Percent
Total Number of Students Interviewed for Study	6	5	6	6	6	6	5	40	100%
Gender									
Female	4	2	3	3	4	4	3	23	57.5%
Male	2	3	3	3	2	2	2	17	42.5%
Racial and Ethic Categories									
American Indian or Alaskan Native	0	0	0	0	0	0	0	0	0.00%
Asian	0	0	0	0	0	0	0	0	0.00%
Black or African American	1	0	1	3	0	2	0	7	17.5%
Hispanic or Latino	0	0	0	1	0	0	1	2	5.0%
Native Hawaiian or Other Pacific Islander	0	0	0	0	0	0	0	0	0.00%
White	5	5	5	2	6	4	4	31	77.5%
Multi-Racial	0	0	0	0	0	0	0	0	0.00%

For this study, the analyzed sample was further reduced. Of the 40 interviews completed, 22 were randomly selected to be analyzed for this study. The demographic information of the reduced sample set is shown in Table 4 and is consistent with the 138-student potential sample pool and the 40-student interview sample set. Therefore, the demographics of this 22-student reduced sample set are deemed indicative of the population of second grade students at both schools.

Table 4

Demographic Information of Students whose Interviews were Evaluated

	Classroom							Total	
	School A				School B				
	1	2	3	4	5	6	7	N	Percent
Total Number of Student Interviews Evaluated	6	3	1	3	4	2	3	22	100%
Gender									
Female	4	0	0	1	3	1	2	11	50%
Male	2	3	1	2	1	1	1	11	50%
Racial and Ethic Categories									
American Indian or Alaskan Native	0	0	0	0	0	0	0	0	0.00%
Asian	0	0	0	0	0	0	0	0	0.00%
Black or African American	1	0	0	2	0	1	0	4	18.2%
Hispanic or Latino	0	0	0	0	0	0	1	1	4.5%
Native Hawaiian or Other Pacific Islander	0	0	0	0	0	0	0	0	0.00%
White	5	3	1	1	4	1	2	17	77.3%
Multi-Racial	0	0	0	0	0	0	0	0	0.00%

Task Selection and Data Collection Procedures

Each of the 40 students was interviewed individually by one of three researchers using a qualitative, structured, task-based interview (Goldin, 2000). The interviews were videotaped. Students were asked to complete four different place value-related tasks, which were compiled from published well-developed interview protocols that have been in use for quite some time. Each task examines a different big idea (Fosnot, 2007) under the broader umbrella of place value. The four tasks, the big ideas they address, and how these big ideas relate to place value will now be discussed in detail.

Task One (Richardson, 2002a) assesses knowledge of combinations to ten by determining if students know the missing part of a number without having to figure it out. The interviewer starts with five cubes under her hand and shows the student a predetermined sequence of exposed cubes, with the balance remaining hidden. The students is asked how many cubes are still hiding. If the student knows all combinations with automaticity, another cube is added to the collection and a new sequence begun. Once the student begins using his/her fingers, taking a long time to respond, or makes several errors, the task stops. After determining the highest number the student knows, the student is asked a sequence of questions following the format "What if you had ___ cubes and you gave me ___? How many would you have left?" using a predetermined numeric sequence. This is done to confirm that the student knows the combinations without physical representations. This task relates to place value and multi-digit arithmetic because knowing combinations indicates that a student knows the parts of a number, which is a prerequisite to recognizing "the relationship between composition and decomposition of numbers and addition and subtraction" (Richardson, 2002a, p.26). The first task assesses whether students are able to decompose numbers to ten "so well that when given one part of a number, they automatically know the other part" (Richardson, 2002a, p.26). Task One assessed Fosnot's (2007) "big ideas" of equivalence and commutativity and associativity.

Task Two (Richardson, 2002b) assesses organizing a quantity into tens and ones, conservation of quantity, and counting by groups of tens, fives and twos. The first part of the assessment is a grouping task used to ascertain how the student groups and counts a collection of objects by tens and ones. The student is presented with a pile of 33 cubes and asked to estimate, first, the total number of cubes and, second, the number of tens that could be formed. The student then makes as many groups of tens as possible. The student is asked to report the number of groups of tens and number of leftovers. He/she is then asked if this gives an idea about the total number of cubes. If the student counts the grouped 33 cubes by ones, the number of cubes is reduced to 17 cubes, and the student is asked to estimate the number of cubes and predict how many tens he/she can make. If a student counts the grouped 33 cubes by tens, he/she is asked how many cubes there would be if different adjustments were made (e.g., add 10 more cubes, take away 10 cubes, add 20 more cubes, take away 20 cubes). If the student performs these calculations without counting by ones, he/she is asked how many cubes there would be if given 7 tens and 12 more. The first part of Task Two assesses Fosnot's (2007) big idea of unitizing, students' use of ten as a composite unit, and counting off the decade by tens.

The second part of Task Two assesses students' recognition that counting by different sized groups does not change the total quantity. Although by second grade we expect students have developed conservation of number when counting individual objects, they may not yet have attained conservation of quantity when grouping is involved. In the second part of Task Two, a student considers the pile of 33 cubes and is asked how many cubes there would be if the cubes are counted by fives. The students then counts the cubes by fives. Next, the student again considers the pile of 33 cubes and is asked how many cubes there would be if the cubes are counted by twos. The student then counts the cubes by twos. This portion of Task Two relates to place value understanding because it again addresses to what extent a student unitizes and has conceptualized composite units and how grouping objects relates to the total quantity (Richardson, 2002b).

Task Three (Wright, Martland, and Stafford, 2006a) assesses knowledge of counting by tens on (e.g., 10, 20, 30…) and off the decade (e.g., 4, 14, 24…). First, the student is presented with a strip picturing 10 dots and asked how many dots there are. One more strip is added, and the student is asked how many there are altogether. The interviewer continues to add one strip of 10 at a time until 80 dots were present. Next, the student is shown a strip picturing four dots and asked how many dots there are. One strip of 10 was added to the right of the 4 dots, and the student is asked how many there are altogether. The interviewer continues to add one strip of 10 at a time until 74 dots were present. This task determines whether the student can increment by tens when a representation of the quantity is displayed, and relates to Fosnot's (2007) big idea that children need to recognize place value patterns that result from repeatedly adding 10.

The second part of Task Three uses a large sheet upon which similar dots were pictured in groups of tens or as various numbers of ones. The dot sheet is screened and the student is shown the first quantity and asked the number of dots present, then the dots are screened and the next quantity of dots is uncovered. Subsequently, each student is asked the total number of dots (both the re-screened portion and visible portion). The first dot sheet presents additions of only ones or only tens at a time, whereas the second dot sheet displays additions of both tens and ones at the same time. The second portion of the third task assesses whether students unitize and use ten as a composite unit or as ten single units.

Kamii (2000) explains that although adults can represent “one ten” and “ten ones” *simultaneously* to themselves, “young children…think only *successively* about ‘one ten’ and ‘ten ones’” (p. 31; emphasis in original). A child who thinks about tens and ones successively counts both the tens and the ones in an arrangement by tens, because he/she cannot think about tens and ones simultaneously and therefore once the child starts counting by tens he/she will continue to count by tens even when he/she encounters the single units of an arrangement (Fuson & Smith, 1996; Kamii, 2000). This task presented an activity in which it was possible to see whether students thought about tens and ones simultaneously or successively.

Task Four (adapted from Wright, Martland, and Stafford, 2006a) assesses strategies for solving several different two- and three-digit addition and subtraction problems, with and without regrouping. Problems are presented on cards written as horizontal number sentences. Students are provided marker and paper to solve the number sentences within Task Four if they elect to do so with writing or drawing. The first problem asks students if they had a way to figure out 16+10, and the second asks students if they could use 16+10 to help them do 16+9. There are several subsequent problems; each involves showing the student a card with the number sentence and asking if he/she has a way to figure out the problem. The problems are presented in the same order to each student, as indicated by the ordering of the problems in Table 5.

Table 5

Task Four: Horizontal Number Sentences

Two-digit addition	Two-digit subtraction	Three-digit addition	Three-digit subtraction
1. 16+10	6. 56-23	9. 128+354	11. 267-119
2. 16+9	7. 43-15	10. 168+156	12. 324-133
3. 42+23	8. 73-48		13. 524-239
4. 38+24			
5. 39+53			

This task again assessed the big idea of unitizing and students' use of ten as a composite unit. It also assessed how this understanding related to their work in solving two- and three-digit addition and subtraction number sentences and whether students understood the strategies and/or algorithms they used in terms of rules and procedures (procedural knowledge); in terms of place value, base-ten properties and quantity (conceptual understanding); or whether they demonstrated both procedural knowledge and conceptual understanding. Exploring these areas also assessed the big idea that the place of a digit determines its value and the strategy of regrouping (Fosnot, 2007). Task Four also assessed students' understanding that the structure of written notation is consistent and indicative of base-ten groupings. Students' work with addition and subtraction problems was examined to determine the strategy selected, flexibility, and underlying use of ten as a composite unit.

Within Task Four, two additional questions were asked regarding the two-digit problems 39+53 and 73-48. After a student solved each of these problems, the interviewer posed a scenario in which she had been working with another student, Manny, who solved the problem differently. After solving 39+53, students were told that Manny solved 39+53 by first doing 30+50=80, then doing 9+3=12. The interviewer then said that Manny got confused at this point, and asked the student how he/she would suggest that Manny continue. Questions were asked to ascertain if the student understood why Manny used 30, 50, 9, and 3 in his calculation. After students solved 73-48, the interviewer told students that the same boy, Manny, solved this problem by doing 73- 40=33, then 33-3=30, and lastly 30-5=25. Questions were asked to ascertain if the student could make sense of this strategy and if he/she understood why Manny used 40, 3, and 5 in his calculation. These questions were asked because "understanding can be characterized by the kinds of relationships or connections that have been constructed between ideas, facts, (and) procedures" (Hiebert et al., 1997, p.15). If students think relationally it can facilitate learning arithmetic and provide a foundation that will ease the transition to algebra (Carpenter et al., 2003; Carpenter & Levi, 2000).

The four tasks and sub-tasks were presented in the same order for each student. The interview protocol included the wording for questions on tasks one, two, and three. Task Four questions were presented at a time when they applied to the students' work (e.g., asking about regrouping just after the student used regrouping notation). Some questions for the fourth task were also specific to a student's approach to the problem.

Follow-up questions were used throughout all four tasks to determine if the student understood what he/she was doing and to clarify students' thinking. Students were asked to explain their thinking and strategies as they worked. In some cases, more difficult sub-tasks of the interview were skipped if a student experienced difficulty or frustration on a previous sub-task. Written student work was retained at the end of the interview. Interviews were approximately 35 minutes in length; therefore some students did not finish all of the sub-tasks due to the amount of time.

Prior to beginning this study, the three researchers piloted the interview protocol by using it to interview second graders in a different school and district in central North Carolina. Observations of each other during the interviews and videotaping were done. This helped refine the protocol and promote consistency amongst the interviewers.

Data Analysis

Each video recording of the 22 interviews in the reduced sample set was watched at least twice to code the data in detail. The interviews were not transcribed, except for sections of dialogue that seemed particularly insightful in terms of clarifying a students' thinking, strategy, or conceptualization of place value. Students' written work was also examined in tandem with the videotape and dialogue that transpired as they were writing.

The coding schema utilized the same framework that Richardson (2002a, 2002b) and Wright, Martland, and Stafford (2006a) designed for use with their tasks. Additional schema were generated to capture aspects of place value understanding that the original framework did not directly address. This was done to link the task to the big idea for which it was selected. Strategies and processes were recorded even if the student did not produce the correct numerical answer.

Task One did not require any additional coding beyond that delineated by Richardson (2002a). For Task Two, Richardson's (2002b) coding framework was used as well as two additional codes. The first was to indicate whether students used the quantity ten as ten single ones or as a composite unit. The second was to indicate whether students knew that the total quantity did not change based on grouping arrangement.

For Task Three, the CGI levels of base-ten development were used. The three levels were: counting by ones, counting by tens, and direct place value (Carpenter et al., 1999; previously described in Background: *Stages of Development of Place Value Understanding,* p.6). At the initial level, a student does not realize he/she can count by groups of ten directly and therefore counts by ones. At the intermediate level, a student is able to use base-ten concepts and counts the grouped objects by tens and then counts on the ones. At the most sophisticated level, a student uses direct place value to immediately recognize the total number of objects in the grouped sets (e.g., 5 groups of ten is 50 objects) and then add the ones to this number (e.g., 3 more make 53).

Three additional codes were also introduced for Task Three. The first was to indicate whether students used the quantity ten as ten single ones or as a composite unit. The second was to indicate whether students addressed tens successively or simultaneously (Kamii, 2000).

Students' work on Task Four was perhaps the most difficult to capture because it was a more qualitative task. Another challenge was that although a final coding framework was provided by Wright et al. (2006b), it was necessary to develop intermediate coding schemas to form a bridge from the interview transcripts to Wright et al.'s levels of base-ten conceptualization. First, the student's problem solving strategy was noted, regardless of whether he/she employed the strategy correctly or arrived at a correct answer. The strategy that a student used first and without prompting was the strategy coded, as this was the strategy most indicative of how the student solved the problem when working on his/her own. Therefore the student's strategy for 16+9 was not usually coded, because the interviewer specifically asked the student to solve it using 16+10. However if the student could not solve 16+9 by using 16+10, then the strategy he/she did use was coded since it was considered unprompted and an independent choice. Data were also coded to indicate if the student 1) saw the relationship between 16+10 and 16+9, 2) made sense of the alternative solution strategy for 39+53, and 3) made sense of the alternative solution strategy for 73-48. The following are descriptors used to code a student as making no sense, limited sense, or thorough sense of the two proposed alternative solution strategies.

Table 6

Coding Students' Understanding of Manny's Alternative Solution Strategy

Level of understanding	Description
No understanding	The student does not see how the numbers used relate to the original problem and may even say that the answer is wrong.
Limited understanding	The student understands part of the alternative solution. The student is able to explain where at least some of the numbers "come from" and how the numbers relate to the original problem. The student may not fully understand the alternative strategy. For example, with the problem 73-48, the student may understand that 73-40=33 relates to subtracting the tens quantity in the second number, but for 33-3=30 and 30-5=25 the student may not recognize the subtraction of 8 and that it relates to the ones quantity in the second number.
Thorough understanding	The student fully understands the alternative solution. The student is able to explain where all of the numbers "come from" and how the numbers relate to the original problem.

Data were coded separately for addition and subtraction on whether the student: 1) had procedural fluency, 2) flexibly used multiple strategies, 3) talked in tens, 4) explained regrouping in addition in a way that indicated understanding, 5) explained regrouping in subtraction in a way that indicated understanding, 6) manipulated symbols, and 7) manipulated quantities. Each of these seven aspects of possible evidence of place value understanding was coded as a "yes" or a "no." In a few instances a student's response was coded as "both", although this was avoided whenever possible. When the majority, although perhaps not entirety, of a student's response indicated a "yes" or "no" as a fair assessment, it was appropriately coded. When a student's response seemed evenly representative of a partial understanding(s) with partial misconception(s), "both" was coded. The following are descriptors used to code a student as "yes" for these seven place value-related behaviors, each of which was coded separately for addition and subtraction.

Table 7: *Students' Place Value-Related Behaviors*

Behavior	Definition
Procedural fluency	A student has procedural fluency when he/she consistently performs the standard algorithm correctly.
Flexibility and Multiple strategies	A student demonstrated multiple strategies when he/she used more than one strategy to solve horizontal number problems in Task Four. This means that the student used more than one strategy over the battery of problems solved, not that the student necessarily had multiple ways to solve a single problem. One problem, 16+9, was excluded from analysis for multiple strategies, since each student was asked by the interviewer to use 16+10 to solve 16+9 and, therefore, the student was not making unprompted decisions about strategy use. All other two-and three-digit addition and subtraction problems were analyzed for multiple strategies. This assessed a student's flexibility with selecting a strategy that would be a good match for the type of problem and the numbers in the problem.
Talked in Tens	A student talks in tens if he/she refers to digits by their complete value rather than face value. For example, when asked about the number 25 a student who was talking in tens would talk about the 2 as "twenty", not as "two." Likewise, when explaining 25+43 with an algorithm a student who was talking in tens would say "20 plus 40 equals 60" rather than "2 plus 4 equals 6."
Explained regrouping (carrying in addition) in a way that indicated understanding	A student demonstrated regrouping with understanding when he/she referred to carrying as "carry a ten" versus "carry the one." Students were also asked when carrying the number 12, why the 2 is written in the ones place of the answer and the 1 is carried and not the other way around. Responses were evaluated to determine if students understood carrying based on if they articulated anything in regards to a ten or place value (indicating understanding) or if they talked in ones and procedures (indicating a lack of understanding). When asked about carrying, a student who did not understand may only be able to say, "Put the number down here and the other number up here," whereas a student with understanding may explain that since he/she has ten or more he/she needs to write down the number of ones and write the ten with the other tens.
Explained regrouping (borrowing in subtraction) in a way that indicated understanding	A student demonstrated regrouping with understanding when he/she referred to borrowing as "borrow a ten" versus "borrow a one." Students were also asked when borrowing with the number 43 why the 4 changes to a 3, why the 3 changes to a 13, and why the 3 does not change to a 4 (since 1 is borrowed from the tens and added to the 3 ones). Responses were evaluated to determine if students understood borrowing based on if they articulated anything in regards to a ten or place value (indicating understanding), or if they talked in ones and procedures (indicating a lack of understanding). When asked about borrowing, a student who did not understand may only be able to say, "Cross off the number, write one less, go next door, put the one on the front, then subtract," whereas a student with understanding may explain that since he/she do not have enough ones to subtract he/she needs to move a ten over and then reduce the number of remaining tens by one.
Manipulated symbols	A student who manipulates symbols thinks about the algorithm as a series of steps performed on isolated columns, within which he/she crosses off and moves numbers.
Manipulated quantities	A student who manipulates quantities thinks about the problem and his/her strategy as calculating a total quantity, not working isolated digits.

Table 7: *Students' Place Value-Related Behaviors*

A summary coding was also made to indicate a student's overall level of conceptualization of ten. These levels were inspired by Wright, Martland, and Stafford's (2006a) stages of base-ten arithmetical strategies, which were previously outlined in Table 1, with two significant adjustments. The first change is that the original Level Two/ Intermediate Concept of Ten states that students "cannot solve addition and subtraction involving tens and ones when presented with written number sentences" (Wright et al., 2006a, p.93). However, all of the interviewed second grade students were able to solve the horizontal number sentences. Had the levels not been adjusted, everyone would have been coded at Level Three/Facile Concept of Ten. It was necessary to make finer distinctions between students, especially when considering their calculations in light of their conceptual understanding of quantity versus symbolic manipulation. The second change is that several of the other aspects of the previously delineated coding framework were added to expand the levels from descriptors of base-ten arithmetic levels to descriptors of base-ten knowledge.

The following are the new descriptors that were used to determine and code a student's overall level of conceptualization of base-ten knowledge. Level One uses all of Wright et al.'s (2006b) material as well as additional descriptors. Level Two is a completely new definition that was piloted with this study. Level Three uses all of Wright et al.'s material as well as additional descriptors.

Table 7: *Students' Place Value-Related Behaviors*

Table 8

Level of Base-Ten Knowledge

Level	Description
Level 1--Initial Concept of Ten	"The child does not see ten as a unit of any kind. The child focuses on the individual items that make up the ten. In addition or subtraction tasks involving tens, children count forward or backward by ones" (Wright et al., 2006a, p.93). The child may still rely on direct modeling. When using an algorithm the child thinks about procedures and manipulates symbols, not quantities. The child talks in ones, even to describe a two-digit algorithm. The child explains carrying and regrouping in terms of ones. The child might be able to identify the "tens" and "ones", but does not conceptualize their meaning as anything beyond labels. In summary, "ten" is neither mentioned (beyond a label) nor used.
Level 2--Intermediate Concept of Ten	The child is able to perceive ten as a unit composed of tens and ones. The child may use tens as a composite unit, although this may not be consistent (e.g., may use ten as a composite unit to count groups but as ten single units when asked to add or subtract, especially when numbers get larger). The child may address tens successively, not simultaneously. The child identifies the "tens" and "ones" and may begin to talk in tens but does not do so consistently (e.g., can explain that the 2 in 26 is twenty, but when explaining the algorithm may refer to it as "add 2"). Conceptualization of ten has moved beyond labels; there is some notion of ten, but it is not consistent or solid. When adding or subtracting begins to manipulate quantities, at other times still manipulates symbols and remains focused on procedures rather than their meaning. In summary, "ten" is inconsistently mentioned and used.
Level 3--Facile Concept of Ten	"The child can solve addition and subtraction tasks involving tens and ones without using materials or re-presentations of materials. The child can solve written number sentences involving tens and ones by adding or subtracting units of tens and ones" (Wright et al., 2006a, p.93). Addresses tens simultaneously, not successively. The child consistently talks in tens, both to identify parts of a numeral as well as when explaining the algorithm. The child explains regrouping in terms of tens. Consistently manipulates quantities, not just symbols. Understands underpinnings of procedures and symbolic notation associated with the algorithm. In summary, "ten" is consistently mentioned and used.

Table 7: *Students' Place Value-Related Behaviors*

These coding schema were developed and used in order to tie the interview back to the big ideas associated with place value. This allowed for analysis to be based on how students conceptualized and used ten for counting, grouping, incrementing, and in solution strategies.

Table 7: *Students' Place Value-Related Behaviors*

Results

This section examines the overall findings from the interview data. First, what students were able to do and seemed to know in regards to place value-related big ideas is presented. Second, students' solution strategies are reported. Third, disconnects between procedural knowledge and conceptual understanding within the standard algorithm are discussed. Given that this study is based on a qualitative interview, descriptive statistics and frequency distributions were generated by compiling the coded data from each student's interview. These statistics were used to generate the tables in this section.

Students' Place Value-Related Understandings of Big Ideas

The first area this study explored was students' use and understanding of some of the big ideas related to place value. The first big idea this study assessed was number combinations. This was done by determining students' highest known number combination from Task One. Another big idea was students' ability to unitize, which was analyzed based on their work in Task Two and Task Three. A third big idea that this study explored was students' ability to perceive place value patterns, which was evaluated by analyzing their work on Task Two and Task Three.

Number Combinations

This study assessed students' number combinations through Task One: Hiding Assessment. Students' automaticity with number combinations reveals that they are able to

Table 7: *Students' Place Value-Related Behaviors*

decompose a number into parts. That is, when given one part of a number they automatically knew the other part. This supports the development of a part/whole understanding of number.

Task One was used to determine if a student was fluent with number combinations to 10, and if not then what was the highest number with which the student demonstrated fluency. The following table shows the highest number that each of the 22 students was able to decompose with automaticity.

Table 9

Task One, Hiding Assessment: Highest Number for which Combinations are Known with Automaticity

	Highest Known Number							
	3	4	5	6	7	8	9	10
Number of Students	1	2	6	4	3	3	0	3

Note. N=22

The median highest known number combination is six. Just over forty percent of students' highest known numbers was 5 or less, 59.1% was 6 or less, and 72.7% was 7 or less. This means that only 27.3% of students had a highest known number of 8 or more, and only 13.6% demonstrated fluency with decomposing the number 10.

Unitizing

Another big idea this study explored was unitizing. Unitizing involves counting single objects as units, groups of objects as units, and the ability to do both simultaneously.

Table 7: *Students' Place Value-Related Behaviors*
Children require a "shift in perspective" to "treat a group of ten as a unit, and ten groups of ten (100 units) as one unit of a hundred" (Fosnot, 2007, p.7). Unitizing is also referred to as the use of composite units. Understanding ten as a composite unit means that a student can combine single units into a new, countable unit. Students who have conceptualized composite units and unitize can simultaneously see ten as ten ones and one unit of ten. Task Two and Task Three were analyzed to investigate students' use of ten as a composite unit, whether they addressed tens successively or simultaneously, and whether they conserved quantity regardless of the grouping arrangement. The findings are presented in the following table.

Table 10

Unitizing

	Big Ideas							
	Uses ten as 10 single units	Uses ten both as 10 single units and as a composite unit	Uses 10 as a composite unit	Addresses tens successively	Addresses tens successively and simultaneously	Addresses tens simultaneously	Tens-to-ones shift errors	Conserves total quantity despite grouping arrangement[a]
Number of students (n=22)	2	6	14	5	4	13	10	3
Percent	9.1%	27.3%	63.6%	22.7%	18.2%	59.1%	45.5%	14.3%

[a]n=21 because one student did not complete Task 2

Of the 21 students, 9.1% of students were unable to unitize and used ten in the form of ten single units. When presented with groupings of objects, these students counted by ones and not by tens. These students counted objects grouped by tens as ones, and also

Table 7: *Students' Place Value-Related Behaviors*
counted by ones to increment by ten. Ninety- one percent of students were able to use ten as a composite unit at least some of the time, and 63.6% consistently used ten as a composite unit. Two differences were found between students who were able to use ten consistently and those who reverted to using single units at times. While both groups of students were able to count groups of objects by ten, a difference was found in their ability to a) continue to use ten as a composite unit with larger numbers, and b) use ten as a composite unit when mentally adding or subtracting ten from an off-decade number when presented with a physical representation of the original quantity. Sixty-four percent of students were able to use ten consistently to count groups of objects, mentally add or subtract ten from an off-decade number, and increment with larger numbers.

Data were examined to determine whether students addressed tens successively or simultaneously. Fifty-five percent of students addressed tens simultaneously, meaning they were able to conceptualize ten as both one unit of ten and ten units of one at the same time. These students were able to successfully "switch" from counting the tens portion to counting the ones portion of a number. In contrast, 45% of students addressed tens successively at least some of the time, and were not able to conceptualize ten existing both as one unit of ten and ten units of one at the same time.

The majority of students appear not to know that a quantity remains constant regardless of how it is grouped or arranged. Students were given 33 cubes and asked to predict how many cubes there would be if the cubes were a) grouped by fives, and b) grouped by twos. Only 14.3% of students responded that there would continue to be 33 cubes. Seven students (33.3%) responded that there would be 30 or 35 cubes when grouped by fives and five students (23.8%) responded that there would be 32 or 34 cubes when

Table 7: *Students' Place Value-Related Behaviors*
grouped by twos, suggesting some students were thinking about counting by groups of fives and twos.

The majority of students (11 students or 52.4% for grouping by fives, and 12 students or 57.1% for grouping by twos) responded with an entirely different quantity of cubes (responses included 82, 90 and 120). Several students had interesting explanations for this misconception. One girl thought there would be 43 cubes because "there would be more because we're counting by fives." Another student thought that there would be more cubes when grouping by two than by five, because "I'm counting by a less number which makes it more cubes when you put them together." When asked if she meant more number of cubes or more groups, she confirmed "more number of cubes."

The results of this study also indicate that there may also be a relationship between students' highest known number combination and their use of ten as a composite unit. The following table presents these findings.

Table 11

Number Combinations and Use of Ten as a Composite Unit

Use of ten	Highest known number	N	Median highest known number
Uses Ten 10 single units	5, 6	2	5.5
Uses Ten Both as 10 single units and as a Composite Unit	3, 4, 5, 5, 6, 7	6	5
Uses Ten as a Composite Unit	4, 5, 5, 5, 6, 6, 7, 7, 8, 8, 8, 10, 10, 10	14	7

Note. n=22.

Table 7: *Students' Place Value-Related Behaviors*

Although the group sizes are different and only two students used ten only in the form of single units, it seems there may be some relationship between the highest number combination a student knows and his/her understanding of how to unitize and use ten as a composite unit. Students who consistently used ten as a composite unit also had the highest median of known number combinations. Students who used ten single units and students who used ten both as 10 single units and as a composite unit appeared to have similar knowledge of number combinations. The numbers of students in the three groups are small; this finding suggests further direction for analysis.

Students' overall level of base-ten knowledge was also compared to their use of ten. As previously described (Methodology: *Data Analysis*, p.31), students use of ten was coded to indicate whether they worked exclusively in ones, whether they worked at times in ones and at times in tens, or whether they worked exclusively in tens. The results are presented in the following table, which identifies students' use of ten as sorted by level of base-ten knowledge (previously defined in Table 8, p. 37).

Table 7: *Students' Place Value-Related Behaviors*

Table 12

Use of Ten by Conceptual Level of Base-Ten Knowledge

Category and level	Use of Ten[a]	N
Two-Digit Addition		
Level 1: Initial Concept of Ten	2 ones, 3 both, 4 tens	9
Level 2: Intermediate Concept of Ten	2 both, 9 tens	11
Level 3: Facile Concept of Ten	2 tens	2
Two-Digit Subtraction		
Level 1: Initial Concept of Ten	2 ones, 2 both, 9 tens	13
Level 2: Intermediate Concept of Ten	2 both, 4 tens	6
Level 3: Facile Concept of Ten	2 tens	2
Three-Digit Addition		
Level 1: Initial Concept of Ten	1 ones, 3 tens	4
Level 2: Intermediate Concept of Ten	1 both, 2 tens	3
Level 3: Facile Concept of Ten	3 tens	3
Three-Digit Subtraction		
Level 1: Initial Concept of Ten	1 ones, 1 both, 6 tens	8
Level 2: Intermediate Concept of Ten	NA	0
Level 3: Facile Concept of Ten	1 ten	1

[a]Ones: student worked exclusively in ones
Both: student sometimes worked in ones and other times worked in tens
Tens: student worked exclusively in tens
Note. Total number of students changes across categories because some students did not complete all of the tasks in the 35 minutes allotted for the interview.

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The results in the previous table suggest that students who exclusively worked in ones were operating at Level One for base-ten knowledge (only 2 students). Students who were at Level Three for base-ten knowledge appear to be thinking exclusively in tens. However, this does not mean that all students who exclusively worked in tens also were evaluated to be at Level Three, because, in fact, students who consistently worked in tens were found at all three levels. Students who worked in both ones and tens were at Level One or Level Two, but not at Level Three.

Place-value Patterns

Another big idea related to place value that this study investigated is students' ability to perceive place value patterns. Fosnot (2007) defines this big idea as the place value patterns that result from repeatedly adding or subtracting ten, making groups of ten, or multiplying by ten. Children must learn the patterns associated with counting on the decade (also referred to as landmark numbers, e.g., 10, 20, 30…) as well as counting off the decade (e.g., 14, 24, 34…). Children also need to learn that these patterns continue past higher landmark numbers, such as 100. The ability to increment by tens is "the forerunner to the development of place value knowledge" (Wright, Martland & Stafford, 2006a, p.93). The ability to count by ten, both on and off the decade, will result in an increased ability of students to engage in mental arithmetic and conceptualize the quantities with which they are working.

This knowledge was evaluated by analyzing students' work on Task Two and Task Three. Task Two asked students to mentally increment by ten to calculate 33+10, 33+20, 33-10, 33-20. Task Three asked students to increment by tens on the decade, by tens off the

Table 7: *Students' Place Value-Related Behaviors*
decade, by tens or ones separately, and by tens and ones simultaneously. These tasks assessed students' ability to recognize place value patterns when adding or subtracting groups of ten. The following table indicates students' facility with such tasks.

Table 13

Strategies for Incrementing by Ten and Incrementing by Tens and Ones

	Tens strips: Visible		Screened dots: Not Visible	
Strategy	Counting by ten on the decade	Incrementing by ten off the decade	Incrementing by only tens or only ones	Incrementing by both tens and ones[a]
Counting by Ones	0	0	6	3
Counting by Tens	2	12	7	14
Direct Place Value	20	10	9	4

Note. N=22.
[a]n=21, because one student had considerable difficulty with Sheet One and therefore was not asked to complete Sheet Two.

Students were very successful counting by ten on the decade (e.g. 10, 20, 30…) when presented with a strip of paper showing ten dots. Twenty students (90.9%) knew the number instantly by direct place value (strategies previously defined in Background: *Stages of Development of Place Value Understanding*, p.6). All students were able to increment off the decade (e.g. 4, 14, 24…) by using ten as a composite unit. Whereas the majority of students were able to use direct place value to count on the decade, 12 students (54.5%) counted by tens to determine the off-the-decade quantity. Of the 12 students who counted by tens, eight of these students (36.4% overall) began by counting by ones, but were eventually

Table 7: *Students' Place Value-Related Behaviors*

able to switch over to counting by tens. After counting the ten dot strips by one for 4-14, 14-24, and 24-34, one student switched to counting by tens to figure out 44 and up. When asked how he was able to figure it out faster he said, "They all keep ending in fours."

When the task switched from using visible tens strips to screened sheets so that students could not see the quantity from which they were incrementing, strategies became less sophisticated. The sheet was uncovered one section at a time and at this point presented a picture of only tens or only ones to add to the previous screened quantity. Six students (27.3%) needed to count on by ones, and seven students (31.8%) counted on by tens. Nine students (40.9%) were able to use direct place value to immediately know the new total quantity.

When the task again increased in difficulty, the number of students using the most advanced strategy of direct place value decreased. On the second sheet, students were presented with a picture of both tens and ones simultaneously to add to the previous screened quantity. Now only four students (19.0%) were able to use direct place value to determine the new total quantity. Two-thirds of students (66.7%) counted by tens and then continued counting on by ones. Three students (14.3%) counted everything, including the tens, by ones. One boy who was trying to work with large numbers in ones exclaimed, "I keep losing count!"

Several students demonstrated using less sophisticated strategies as the tasks increased in difficulty. An example of one such student was a girl who used direct place value to recognize the number of dots on the tens strips to increment by ten on the decade. When asked to increment by ten off the decade from 4 she began counting by ones but was

Table 7: *Students' Place Value-Related Behaviors*
able to switch to counting by tens once she reached 24. For all of the screened tasks in which the original quantity was not visible she counted on by ones.

Strategy Use

The second area of interest of this study was to identify students' strategies for solving two-and three-digit horizontal number sentences. Students were presented with a maximum of thirteen number sentences, although only nine students completed all thirteen problems. Since interviews were kept to about thirty-five minutes in length, most students did not complete all of the problems. The remaining thirteen students completed between five and twelve of the problems. The following table displays the problems in the order they were presented, the number of students who attempted each problem, and the number and percent of problems that were solved correctly.

Table 7: *Students' Place Value-Related Behaviors*

Table 14

Correct Answers for Horizontal Number Sentences

Problem	Number of students who attempted	Total number correct	Percent correct
Two-Digit Addition			
16+10	22	21	95.5%
16+9	22	22	100.0%
42+23	22	21	95.5%
38+24	22	22	100.0%
39+53	22	21	95.5%
Two-Digit Subtraction			
56-23	21	21	100.0%
43-15	21	11	52.4%
73-48	19	14	73.7%
Three-Digit Addition			
128+354	9	8	88.9%
168+156	11	11	100.0%
Three-Digit Subtraction			
267-119	9	4	44.4%
324-133	5	3	60.0%
524-239	7	2	28.6%

Note. Total number of students who attempted each problem changed because some students did not complete all of the problems in the 35 minutes allotted for the interview.

All 22 students completed the 5 two-digit addition problem tasks. Students were successful in responding to these problems; correct responses ranged from 95% to 100%. This suggests a high level of fluency with two-digit addition. Two-digit subtraction problems without regrouping also were answered correctly 95% of the time. However, the two problems that involved two-digit subtraction with regrouping were answered correctly 52% and 73% of the times attempted, respectively. Students appear more fluent with two-digit operations with and without regrouping in addition and without regrouping in subtraction more so than when regrouping in subtraction.

Table 7: *Students' Place Value-Related Behaviors*

Each of the three-digit problems required regrouping (carrying or borrowing) at least once. Students were able to complete the first three-digit addition problem (which required regrouping from the ones to the tens) with 88.9% success and the second three-digit addition problem (which required regrouping from the ones to the tens and from the tens to the hundreds) with 100% success. Students were not as successful at calculating the correct answer for three-digit subtraction problems. Students completed the first three-digit subtraction problem (which required regrouping from the tens to the ones) with 44.4% success, the second three-digit subtraction problem (which required regrouping from the hundreds to the tens) with 60% success, and the third three-digit subtraction problem (which required regrouping from the tens to the ones and the hundreds to the tens) with 28.6% success. Given that work with three-digit numbers using addition and subtraction for most students is introduced late in the year in second grade, if at all, it is not surprising that performance appears to be less successful than work with two-digit number problems.

Although this is a helpful starting point, a goal of this study to move beyond assessment based on analysis of correct and incorrect responses. The literature indicates that attention to strategy and understanding is needed in mathematics education. The strategies used for all of the horizontal number sentences that students completed are presented in the following table.

~~Table 7: *Students' Place Value-Related Behaviors*~~
Table 15

Strategy Use for Horizontal Number Sentences

Strategy	Number of Students who Used Strategy (n=22)	Number of Times Strategy was Used (n=196)
Direct Modeling	1	3
Counts on from First	5	6
Counts on from Larger	0	0
Counts Down	0	0
Split Strategy	3	5
Jump Strategy	0	0
Compensation	0	0
Known Fact	8	8
Incorrect Strategy	2	3
Standard Algorithm	22	172

Students' independently selected strategies relied heavily on standard algorithms. All twenty-two students used standard algorithms at least three times. Eight students made use of the known fact strategy, all of which were for solving 16+10. Five students (22.7%) counted on from the first number in the problem, three students (13.6%) used the split strategy, two students (9.1%) used an incorrect strategy (both students added the digits in the problem together as their first step), and one student (4.5%) used direct modeling (she drew tally marks and circles).

When considering the overall number of times each strategy was used, the results reveal that students use standard algorithms significantly more than any other strategy. It should be noted that the curriculum and instructional methods students experienced focused attention on the use of the standard algorithm. Students use standard algorithms 172 times (87.8%) to solve the horizontal number sentences, whereas the next-most used strategy was that of using a known fact and occurred only seven times (3.6%). Twelve students (54.5%)

Table 7: *Students' Place Value-Related Behaviors*
used standard algorithms as their only strategy, and all 22 students (100%) used standard algorithms at least two-thirds of the time, with a range of 66.6% to 100% algorithm use. Other seldom-used strategies included counting on from first, split strategy, direct modeling, and incorrect strategy. Counting on from larger, counting down, jump strategy, compensation and using landmark numbers were never used by any of the students.

This study sought to examine not only students' selection of strategies but also their ability to think relationally. This study analyzed students' relational understanding by asking students to make use of 16+10 to solve 16+9, presenting students with two alternative solution strategies. Students' flexibility was also analyzed by considering whether students used multiple strategies to solve the horizontal number sentences or if they used the same strategy regardless of the problem-type and numbers involved. The results are presented in Table 16.

Table 16

Relational Understanding and Flexibility in Students' Solution Strategies

	Relational understanding					Flexibility
	Saw relationship between 16+10 and 16+9	Limited sense of alternative solution strategy for 39+53	Thorough sense of alternative solution strategy for 39+53	Limited sense of alternative solution strategy for 73-48	Thorough sense of alternative solution strategy for 73-48	Multiple strategies
Number of students who demonstrated/ Total number of students asked	17/22	0/17	9/17	5/11	1/11	6/22
Percent	77.3%	0%	52.9%	45.5%	9.1%	27.3%

Note. Total number of students asked changed for some tasks because some students did not complete all of the tasks in the 35 minutes allotted for the interview.

Table 7: *Students' Place Value-Related Behaviors*

Just over 77% of students used 16+10=26 to solve 16+9. Many students said that the answer was 25 "because it's just one less." Twenty-three percent of students chose to use a written strategy, most often the standard algorithm, to solve 16+9 despite knowing 16+10=26.

Students tended to have more difficulty with the "Manny" alternative solution strategy scenarios that were presented. 53% of students could thoroughly explain the alternative solution strategy for 39+53 (previously detailed in Methodology: *Task Selection and Data Collection Procedures*, p.24). Since no students demonstrated a limited understanding of the strategy, this means that approximately 47% of students had no conception of what was happening with the alternative strategy, where the numbers "came from," how the numbers related to the original problem, or if the strategy even resulted in a correct answer.

For the alternative solution strategy for 73-48 (previously detailed in Methodology: *Task Selection and Data Collection Procedures*, p.24), one student (9.1%) appeared to have a thorough understanding of the alternative strategy, and five students (45.5%) provided limited understanding. This means that the remaining five students (45.4%) who were asked about this strategy were not able to relate the alternative strategy to the original problem. Two of these students (18.2%) thought that the answer was actually wrong. One boy insisted, "He obviously didn't look at the problem. You're supposed to copy it off like this. He probably should do it over…he should do it how it says on the card." Some students thought that the alternative strategy was using any numbers that combined to 92, which they already knew was the correct answer having first solved the problem themselves. For example, one girl explained the alternative strategy by saying "He just picked random numbers to see if

Table 7: *Students' Place Value-Related Behaviors*
they added up to 12 and then to see if they would add up to 80… That gets you to 92."
Similarly, a boy thought that to solve the alternative solution for 73-48 "you could use an addition problem…24+1 'cuz that equals 25" even though these numbers had no relation to the problem other than arriving at the correct answer.

Six students (27.3%) had multiple strategies for solving the horizontal number sentences. Since all students used standard algorithms (see Table 15), this means that only six students were able to use a strategy in addition to standard algorithms, excluding problems 16+10 and 16+9 (their exclusion was previously explained in Methodology: *Data Analysis*, p.31). Also, the strategy of a "known fact" was not considered, as the aim was to discover how children engage in problem solving and those students who used known fact to answer 16+10 were not engaging in problem solving since they already knew the answer. When one student solved a problem using the standard algorithm for addition and was asked if she could solve it a different way, she responded "That's always how our teacher does it so that's how we do it on paper."

Three of the six students who used multiple strategies used strategies that could be categorized as inefficient: one student drew tally marks and two students counted on from the first number using their fingers. This means that three students (13.6%) made use of efficient, alternative strategies at some point during the battery of number sentences. All three of these students used the split strategy to calculate an answer, two of them doing so mentally and one in writing.

Procedural Knowledge and Conceptual Understanding

The third area of interest of this study focuses on students' disparity between procedural knowledge and conceptual understanding of two-digit operations and place value

Table 7: *Students' Place Value-Related Behaviors*
knowledge while using standard algorithms. This was evaluated based on students' procedural fluency, manipulation of symbols versus quantities, talking in tens, and understanding of regrouping. The results are presented in the Table 17.

Table 17

Procedural Knowledge and Conceptual Understanding

	Number of students who demonstrated /Total number of students			
	Addition		Subtraction	
	Two-digit	Three-digit	Two-digit	Three-digit
Procedural Knowledge				
Procedural Fluency	20/22	10/10	11/21	1/8
Manipulated Symbols	18/22	6/10	19/21	7/8
Initial Conceptual Understanding				
Manipulated Symbols and Quantities	3/22	1/10	1/21	0/8
Conceptual Understanding				
Manipulated Quantities	1/22	3/10	1/21	1/8
Talked in Tens	3/22		2/21	
Understands Carrying in Addition	4/22			
Understands Borrowing in Subtraction			4/21	

Note. Total number of students asked changed for some categories because some students did not complete all of the problems in the 35 minutes allotted for the interview.

Students' procedural fluency, evaluated based on their correct use of the standard algorithm, was found to be very high for addition. Twenty students (91%) demonstrated procedural fluency for two-digit addition and all ten students who completed the three-digit addition problems (100%) demonstrated procedural fluency. There were only two students

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who did not have procedural fluency with two-digit addition. One of these students had difficulty regrouping: When doing 39+53 she arrived at an incorrect answer 82. She explained, "3+5=8, 9+3=2." When asked about 9+3=2 she said "9+3=12. The one goes away, the two stays." Instead of regrouping (or "carrying) the ten she dropped it out of the answer. The second student who had trouble with addition procedures was working on 42+23. He wrote out the algorithm, then got stuck. The dialogue that followed is transcribed below:

> Student: "I forgot how to carry."
> TJ: "What's that mean?"
> Student: "It means when you carry a number to the top number."
> TJ: "Why do you do that?"
> Student: "So you can add."
> TJ: "Do you always have to carry every time you add?"
> Student: "When this number is bigger than this number you have to carry" (when the bottom ones digit is larger than the top ones digit you have to carry).
> Student rewrites algorithm, this time with 23 on top and 42 on bottom
> TJ: "You did that because now the 3 is bigger than the 2 now you can add them?"
> Student: Nods
> TJ: "But when it was 2 + 3 you couldn't add them?"
> Student: Nods

This student appeared to think that to add he had to have the larger ones digit "on top" and the smaller digit on the bottom. He seemed not to realize the commutative nature of addition and that it does not matter which number is written first or second. Perhaps he was thinking about subtraction and starting with the larger number from which to subtract the smaller number.

Students' procedural fluency with subtraction was notably less well-developed, with eleven students (52.4%) demonstrating procedural fluency with two-digit subtraction and only one out of the eight students (12.5%) who completed the three-digit subtraction

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problems demonstrating procedural fluency. When subtracting, eleven students did not regroup when necessary. Ten of these students subtracted "up." One girl explained this strategy by saying, "I always start with the higher number, even if it's on the bottom." One student who did not borrow subtracted to zero, and for the problem 73-48 explained that the answer was 30 because, "It has to be zero…3-8=0 because 8 is a greater number than 3."

Some students' difficulties and mistakes were unique to operations with three-digit numbers. One student who was trying to calculate 524-239 said, "I don't really get borrowing from the hundreds that much….I'm just guessing." He could not explain how he arrived at his incorrect answer of 105. Another student, who eventually did calculate the correct answer, said while working on 524-239 in regards to the tens (which you first borrow *from* and then borrow *for*), "This part hurts my head a little. Can you borrow from already borrowed numbers?…I think so…as long as it's a number you can mostly borrow from it." He realized that as long as the number is not zero you could continue to use it to regroup.

Manipulation of Symbols vs. Manipulation of Quantities

While analyzing students' verbal explanations of their written work, it was found that the majority of students appear to have manipulated symbols rather than quantities. This was true in all four categories of problems, with 60% symbolic manipulation for three-digit addition, 81.8% for two-digit addition, 87.5% for three-digit subtraction, and 90.5% symbolic manipulation for two-digit subtraction. Students demonstrated symbol manipulation when their explanations were limited to procedures and markings but failed to touch on place determining value. For example, when one student was solving 43-15 he said, "You can't do that (3-5) so you mark out the 4 and put a 3 and then put a 1." Similarly, when

Table 7: *Students' Place Value-Related Behaviors*
explaining 42+23, one student said "I add this one up (2+3) and this one up (4+2) and it gave me 65."

Many fewer students demonstrated that at times they were manipulating symbols and at times quantities, with 4.8% for two-digit subtraction, 10% for three-digit addition, and 13.6% for two-digit addition. For example, the one student who used tally marks manipulated quantity in this strategy, but when asked if she could solve the problem with numbers, she used the standard algorithm and talked in ones, indicating that she was manipulating symbols. Another student manipulated symbols and quantities for two-digit subtraction. When solving 56-23 he used the standard algorithm and said "on the 5 and 2 I subtracted. 5, 4, 3." This indicated symbol manipulation. When solving 43-15 he was able to talk about borrowing a ten and even demonstrated borrowing using cubes. This demonstrated quantity manipulation.

A low number of students consistently manipulated quantities, with 4.5% for two-digit addition, 4.8% for two-digit subtraction, 12.5% for three-digit subtraction, and 30.0% quantity manipulation for three-digit addition. For example, one student solved 39+53 mentally by using the split strategy. She did her calculation out loud, saying "50, 60, 70, 80, 89, 90, 91, 92." She started at the 50 (decomposed from 53), counted on 30 (decomposed from 39) to get to 80, "jumped" up by 9 (from the 39), then counted on 3 (from the 53). When asked if she had a way to write this down, she wrote 50+10+10+10+9+3. This is an example of a student who is able to manipulate quantities, not just symbols. Her strategy and explanation show that she is able to decompose and recompose numbers, understands some properties of addition (including the big ideas of commutativity and associativity), and

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understands that place determines value and the associated complete values of two-digit numbers.

Another student who was trying to calculate 43-15 first wrote 40-10=30. Then he wrote 3-5, at which point he crossed off the 40 and replaced it with a 30 and changed the answer of 30-10 to 20. Then he wrote 13 instead of 3. Next he did 13-5=8. His final step was 20+8=28. This student also was able to decompose and recompose numbers and thought in terms of quantities and complete values in order to perform this split strategy.

Procedural Knowledge and Conceptual Understanding of Regrouping: Addition

In addition to analyzing students' procedural abilities, students' conceptual understandings were considered. Most students appeared to demonstrate a procedural knowledge of why one regroups in addition, which all students referred to as "carrying". Four students (18.2%) explained carrying with some understanding that it was a ten that was being carried, whereas eighteen students (81.8%) referred to it as a one.

Many students appear to have some notion that one needs to carry because "there isn't enough room" and "you can't put two numbers in one space." One girl said that she carried when "I know I don't have enough fingers for that" (for adding two digits in the ones place whose total was more than ten). When students were asked to explain carrying, most students' talked about rules or procedures, not numeric relationships.

The most common wording was "carry a one." When describing their work, the majority of students used language such as the following for explaining 38+24:

> Student begins by writing standard algorithm
> Student: 8+4 is 12. Put the 2 down here, put the one above the 3.
> Interviewer: That's a 1?

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Student: Yes

Interviewer: And it's worth 1?

Student: Yes. Then it's 1+3+2=6, so the answer is 62.

The majority of students described carrying in this way. This indicates the presence of a procedural knowledge but also suggests a lack of conceptual understanding as to Fosnot's (2007) big idea of place determines value within the workings of the standard algorithm.

In order to further gauge students' conceptual understanding, they were asked how they knew which of the two digits to carry and which to write in the ones place of the answer. Most students explained this in terms of rules or procedures, not numeric relationships. One student said, "Carrying means putting the second number down here and the first number up here." Five students similarly explained that they "carry the first part." Two students said "if it's higher than nine we carry our one and put it at the top."

Two students thought "you put the biggest number down there and the smallest up here" (these students thought that the larger number was written as part of the ones answer and the smaller number was carried to the tens, regardless of their order). This misconception could result from the fact that it is always a one that is carried, and usually the number written in the ones answer is going to be larger than one. However, this reasoning undermines the role of place value in carrying and is an example of a misconception coming from the standard algorithm (Kamii, 2000; Sophian, 1996).

As expected student responses indicated a range of conceptualization about regrouping in addition (carrying). I will present portions of student responses in order from less to more developed conceptual understanding.

The following dialogue is a transcription of how one student explained 38+24:

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> Student: "I learned this in class…Miss X said to just write down the last number and take the first number over there, and if you have twenty something you just put 2 ones up there (carry the 2 "ones")."
> Interviewer: "Why do you put the two down and not the one? Could I put the 2 up and the one down?"
> Student: "No, because you'd count that as two and its not supposed to be and you'd get the answer wrong."
> Interviewer: "Why don't you leave the one at the bottom?"
> Student: "Because you would get the answer wrong."

This student has a procedurally-based conception of carrying. There was no mention of "ten," no labels for columns, and no indication that place determines value. Five students responded similarly to the interviewer's question about why you couldn't put the one "down" and the two "up" when the ones column added to 12 by saying that "you would get the answer wrong." They could give no additional explanation as to why. One student responded, "It's math's nature." This again indicates math is more about getting correct answers than making sense.

One boy who correctly solved 38+24 was asked to explain his carrying. The subsequent dialogue is a transcription of the conversation:

> Student: "There's an extra because the number is too big and you can't put two numbers cuz it won't be really equal, so you put it up here."
> TJ: "How do you know to put the 2 here (in the ones of the answer) and the 1 up there (carried to the top of the tens)?"
> Student: "Because I did it a lot in class"
> TJ: "Why don't you put the 1 down there and put the 2 up there?"
> Student: "Only the ones works up there…This is the tens (points to the ones) and this is the ones (points to the tens)."
> TJ: "Oh so that's why the 1 goes up there because that side is the ones (referring to the tens side)?"
> Student: "Yep."
> TJ: "So if that side is the tens side, what are they worth? What's that 2 worth, it's worth just 2?"
> Student: "Um hm."
> TJ: "Even though it's on the tens side?"
> Student: "Um hm."

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This student's response is telling in several ways. First, he has some notion that he has to carry because "the number is too big," but he is not yet able to fully explain why. Second, he is willing to put his faith in the algorithm but he does not know how it works, only that he "did it a lot in class" and it gets him the correct answer. Third, this student, unlike the previous student, is aware that there are labels (although he has not made a connection to their values) for the digits. Finally, this student thinks the tens side is called the ones because that is where you carry the "one" (where you write a one to indicate a ten). This is another good example of a misconception coming from the confusing nature of the standard algorithm (Kamii, 2000; Sophian, 1996). It seems this student was trying to make sense of the algorithm, and it certainly makes sense to call the ones where the ones "go," but unfortunately he is incorrect in his thinking and again doesn't realize the role of place determines value.

The next example is of a student with slightly more developed procedural knowledge and demonstrates an initial conceptual understanding of carrying. This student explained how he solved 38+24:

> Student: "8+4 is 12 so I carry my 1, put my 2, 3 plus 2 is 5 plus the one is 6. 65.
> Interviewer: So you said you put your 2 and carry your 1, what were you talking about?
> Student: In my class…If it's higher than 9 we carry our 1 and put it at the top and then put our 2 down here. Then 3+2 is 5, plus 1 is 6."
> Interviewer: "Why didn't you carry your 2 and put your 1 there?"
> Student: "Because we haven't learned to carry your two yet."
> Interviewer: Why does the 1 go over here?"
> Student: "Because it comes first."
> Interviewer: "What does it mean when it come first?"
> Student: "I think I remember I think she said the first number goes to the top."
> Interviewer: "To the top of what?"
> Student: "To the top of the tens."
> Interviewer: "So what does the one represent? What does it mean?"
> Student: "It represents the 12…the 2."

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Interviewer: "Well the 2 is there, what does this 1 represent?"
Student: "The tens."

This student is able to state that the "1" represents ten, but only after a series of questions. He thinks of carrying in terms of moving the "first part" before he thinks about tens and how place determines value. At this point, he is essentially using "ten" as a label more than as a conceptualization of place value.

The next two examples show students who have an initial sense as to why one carries. One boy who solved 38+24 with the standard algorithm explained it the following way:

Student: "8+4 is 12. You put the 1 right here (above the 3)."
TJ: "Why didn't you put the 12 down here?"
Student: "Because it would be the wrong answer."
TJ: Why is it wrong?
Student: "It wouldn't make sense…..because if you add these two numbers you don't usually get 612….the 1 floats….so I put it up top…because that's where the 1 goes…you put the ones column down and the tens column here."
TJ: "So this is a 1?"
Student: Nods
TJ: "Is it worth 1?"
Student: Nods
TJ: "You add 1 + 3 + 2?"
Student: Nods

Another student similarly explained the same problem (38+24):

Student: "8 + 4….I know I don't have enough fingers for that…it's too high a number…there's not enough room….it would be 612 which is really big for just 38+24. So you drag a one over."
TJ: "That's a one?"
Student: "Yeah."

Both students have an encouraging notion that you carry so that your answer "makes sense." The first student is also using "ten" and "one" as a label, but has connected the label with their place value. Other students conveyed similar explanations that the answer would be "too big" without carrying.

Table 7: *Students' Place Value-Related Behaviors*

The final example of a student's explanation of carrying indicates a move to a further developed conceptual understanding. This student explained carrying within the standard algorithm using the problem 38+24:

> Student: "Eight plus four is twelve. You can't put a ten in the ones column, so you put it up here."
> TJ: "So if you have 12, how do you have 10?"
> Student: "Twe-lve" (points to 1-points to 2)
> TJ: "How do you know to put the 1 up there? Why don't you put the 1 down here and put the 2 up there?"
> Student: "Because the 1 is not really a one."
> TJ: "What is it?"
> Student: "It's a 10."
> TJ: "What's the 2?"
> Student: "A one."

This student was able to not only label the columns and digits, but recognized it was the associated place value that is the basis for what digits represent and that you carry a ten, not the "first part."

Procedural Knowledge and Conceptual Understanding of Regrouping: Subtraction

Most students also demonstrated procedural knowledge of why one regroups in subtraction, which all students called "borrowing". Four students (19.0%) explained borrowing with an understanding that a ten was being borrowed, while seventeen students (81.0%) said that a one was borrowed. Many had some notion that one needs to borrow and said things including "you can't do 3-5" and "you don't have enough." One boy said, "You borrow a 1 because the lower number is up there (meaning the smaller number is "on top" of a larger number in the algorithm)." Another student said, "You need a bigger number to be able to subtract." A few students said, "It (the answer) would be negative."

Table 7: *Students' Place Value-Related Behaviors*

When students were asked to explain borrowing, most students explained this in terms of rules or procedures, not numeric relationships. The most common wording was that numbers "turned into" or "became" a number one more or one less than they were originally. For example, when doing the problem 43-15 the most used wording to explain regrouping was to say the four "turns into" or "becomes" a 3, and the 3 "turns into" or "becomes" a 13. However, students rarely could explain how they were able to change the numbers, again falling back on comments such as "this is how we do it in my class" and that they borrow "to get the right answer." This type of language indicates that students are aware of the procedural outcomes (a 4 becomes a 3) but not the processes involved that allow this to happen.

Naturally there was a range of student responses indicating a range of conceptualization of regrouping. I will present portions of student responses in order from less to more developed conceptual understanding. At the most procedural level, students described borrowing in terms of ones, that is, "borrow a one." For example, with the problem 43-15 the most common explanation was as follows:

> Student writes standard algorithm
> Student: "You can't do 3-5, so you cross off the 4 and make it a 3, add 1 to the 3 and it's 13. 13-5=8. 3-1=2. The answer is 28."

To see if students conceptually understood regrouping they were asked why, in the problem 43-15, the 3 "turned into" a 13 and not a 4. One boy said the following:

> Student: "Take away the 4, make that a 3, carry the 1, make that a 13."
> Interviewer: "1 and 3 is 4 so how is it 13?"
> Student: "It's not like you add them. You put it in front."
> Interviewer: "Is it really a 1?"
> Student: "Yeah…(pause)…I don't know that's how I just do it."

Table 7: *Students' Place Value-Related Behaviors*
A girl with a similar procedural knowledge about regrouping had a comparable explanation:

> Student: " You don't have enough numbers so I borrow from the 4. Cross off the 4, make it a 3. Drag the 1 over and make it into 13. I borrowed 1."
> TJ: "Why doesn't the 3 turn into a 4 instead of a 13?"
> Student: "Because I'm not adding I'm taking away (pause)…Because it still wouldn't be enough…I don't know, we just learned it this way."

When asked why 3 turns into 13 not 4, several students responded that "it still wouldn't be enough" (meaning 4-5 was still not possible). At this level of conception there is a notion of "getting a big enough number" rather than understanding where that number comes from or how it relates to the overall quantity.

Students who had a slightly more conceptual understanding of regrouping responded in ways similar to the following boy:

> Student: "5 is bigger than 3, so we cross it out and get a bigger number, and the 3 asks the 1, the 3 asks the 5, the 3 asks the 4 if he can borrow a ten, so it gets a 1."
> Interviewer: "So it gives up a 1? It sends a 1 next door?"
> Student: "Yes."

A girl explained 73-48 in a similar way:

> Student: "3 asked the 7 to borrow a ten."
> TJ: "You said the 7 gives a 10 to the 3. How does it have a 10 to give if it's only a 7?"
> Student: "I don't know."

Even though these students originally identified that they were borrowing a ten, it seems to instantly turn into a one in their subsequent explanations. When asked again about the one, they do not say that it is a ten but rather that it is a one. Several students had similar explanations, that is, when they initially used language (most likely similar to that used by their teacher) they would say "borrow a ten" but before they finished their explanation it had turned into a one, and when asked what it was worth they usually said it was a one. This is

Table 7: *Students' Place Value-Related Behaviors*

important to realize because without further questioning it may seem these students understand borrowing because they initially say the phrase "borrow a ten." However, upon further questioning it becomes clear that they have not conceptualized what this means and the ten turns into a one. Developing this concept could be delayed by the written notation of the standard algorithm, in which a student simply writes a "1" in front of the original ones digit. They literally do not see a ten but only a one. Also, several students used wording such as "you knock on the tens and borrow one." Here again, adults understand this wording to mean you borrow one *ten*, but children only hear that you "borrow one." These are further examples of how using the standard algorithm may, as Kamii (2000) said, "unteach place value" (p.83).

Students who had a more developed conceptual understanding of the big idea that place determines value responded in ways similar to the following boy, who explained 43-15:

> Student: "Since the 5 is bigger than the 3 I took 1 from the 4 and I have 3 and I took 10 over."
> TJ: "How did the 3 turn into a 13?"
> Student: "10 plus 3 equals 13."
> TJ: "Where did 10 come from?"
> Student: "The 4."
> TJ: "How did 10 become 4?"
> Student: "The 3 borrowed 1 ten from 4. I had to cross it out and put 3."
> TJ: "Why did 4 turn into a 3?"
> Student: "Because it's subtraction and when you run into problems like this you gotta take 1 away and you put 10 over here."
> TJ: "So how does 1 from here turn into 10 over here?"
> Student: "Because you only want 1 ten."

Another student explained the same problem (43-15) this way: "Take a 10 from the 4 to make the 3 a 13, which leaves you with 3 tens. But you still have 43, just in a different way. Thirty plus 13 equals 43."

Table 7: *Students' Place Value-Related Behaviors*

These students were able to not only explain the procedure and initially state that a ten was borrowed, but were able to thoroughly explain what was happening within the mechanics of the procedures. They were able to explain borrowing in terms of place determines value rather than procedures.

Talked in Tens

Three students (13.6%) talked in tens in regards to addition, and two students did so for subtraction (9.5%). Two of these were the same student talking in tens for both addition and subtraction, and the third student talked in tens for addition but not subtraction. The majority of students (86.4% for addition and 90.5% for subtraction) explained their strategies and calculations by talking in ones. That is, if they used standard algorithms, even the tens column was referred to as if it were ones and there was no indication that the numerals represented tens. When asked what a digit in the tens column was worth, students responded with the digit's face value rather than its complete value. For example (in regards to the problem 38+24):

> Student begins by writing standard algorithm.
> Student: "8+4 is 12. Put the 2 down here, put the one above the 3."
> Interviewer: "That's a 1?"
> Student: "Yes."
> Interviewer: "And it's worth 1?"
> Student: "Yes. Then it's 1+3+2=6, so the answer is 62."

Some students inconsistently referred to tens, such as this student who was adding 168+156:

> Interviewer: "So you said 1+1+1=3 (in regards to the hundreds)?"
> Student: "Yes…it's 3 hundreds."
> TJ: "What is this (pointing to the 1 in 156)?"
> Student: "It's a 1."
> TJ: "It's worth 1?"

Table 7: *Students' Place Value-Related Behaviors*
Student: "Yes…well actually it's 100."

Three students consistently talked in tens, evident in explanations such as a student explaining the tens in 42+23 as "40 plus 20 equals 60."

Level of Base-Ten Knowledge

This study also examined students' overall level of base-ten knowledge, which summarized their use of ten and overall conceptual understanding of place value. The results for the three levels (previously defined in Table 8, p.37) are presented in the following table.

Table 18

Conceptual Level of Base-Ten Knowledge

	Addition		Subtraction	
Level	Two-Digit	Three-Digit	Two-Digit	Three-Digit
Level 1: Initial Concept of Ten	9	4	13	8
Level 2: Intermediate Concept of Ten	11	3	6	0
Level 3: Facile Concept of Ten	2	3	2	1

Note. Total number of students asked changed is different across categories because some students did not complete all of the problems in the 35 minutes allotted for the interview.

Overall the majority of students were found to have an initial concept of ten. For two-digit addition nine students (40.9%) had an initial concept of ten, eleven students (50%) had an intermediate concept of ten, and two students (9.1%) had a facile concept of ten. Of the ten students who completed three-digit addition tasks, four (40%) had an initial concept

Table 7: *Students' Place Value-Related Behaviors*
of ten, three students (30%) had an intermediate concept of ten, and three students (30%) had a facile concept of ten.

When presented with subtraction tasks, students' level of base-ten knowledge generally decreased from what it was with addition. With two-digit subtraction, the majority of students (61.9%) had an initial concept of ten, six students (28.6%) had an intermediate concept of ten, and two students (9.5%) had a facile concept of ten. Of the nine students who completed three-digit subtraction tasks, eight students (88.9%) had an initial concept of ten and one student (11.1%) had a facile concept of ten.

Eight students (36.4%) had an initial concept of ten in all categories they attempted and one student (4.5%) had a facile concept of ten in all four categories. Twelve students (54.5%) had the same levels of base-ten arithmetic knowledge for both addition and subtraction. Seven students (33.3%) had a higher level of base-ten knowledge in addition than subtraction, and one student (4.8%) had a higher level of base-ten knowledge in subtraction than addition. Five students (50.0%) had the same level of base-ten knowledge for both two-digit and three-digit operations. Five students (50.0%) had a higher level of base-ten knowledge for two-digit operations than three-digit operations.

Students' overall level of base-ten knowledge was also considered along side their highest known number combination. The results are presented in the following table, which identifies each student's highest known number combination as sorted by level of base-ten knowledge (defined in Table 8, p.37).

Table 7: *Students' Place Value-Related Behaviors*

Table 19

Highest Known Number Combination by Conceptual Level of Base-Ten Knowledge

Category and level	Highest known number	N	Median highest known number
<u>Two-Digit Addition</u>			
Level 1: Initial Concept of Ten	3, 4, 4, 5, 5, 5, 6, 7, 10	9	5
Level 2: Intermediate Concept of Ten	5, 5, 5, 6, 6, 6, 7, 7, 8, 8, 10	11	6
Level 3: Facile Concept of Ten	8, 10	2	9
<u>Two-Digit Subtraction</u>			
Level 1: Initial Concept of Ten	3, 4, 4, 5, 5, 5, 5, 6, 7, 7, 8, 8, 10	13	5
Level 2: Intermediate Concept of Ten	5, 6, 6, 6, 8, 10	6	6
Level 3: Facile Concept of Ten	7, 10	2	8.5
<u>Three-Digit Addition</u>			
Level 1: Initial Concept of Ten	6, 6, 8, 10	4	7.5
Level 2: Intermediate Concept of Ten	5, 6, 7	3	6
Level 3: Facile Concept of Ten	8, 10, 10	3	10
<u>Three-Digit Subtraction</u>			
Level 1: Initial Concept of Ten	5, 6, 6, 6, 7, 8, 10, 10	8	6.5
Level 2: Intermediate Concept of Ten	NA	0	NA
Level 3: Facile Concept of Ten	10	1	10

Note. Total number of students changes across categories because some students did not complete all of the tasks in the 35 minutes allotted for the interview.

Table 7: *Students' Place Value-Related Behaviors*

These results indicate there may be a relationship between students' highest known number combination and their overall level of base-ten knowledge. This is evident by the fact that the median highest known number is always highest in all four categories for students who are also in Level 3. The only place this trend does not hold is in levels one and two for three-digit addition, in which Level 1 students have a median of 7.5 and Level 2 students have a median of 6. In two-digit addition, two-digit subtraction, and three-digit subtraction, the trend is consistent for all three levels with Level One students having the lowest median number combination, Level 2 students having the middle median, and Level 3 students having the highest median. Again, the numbers of students in the groups are small. These findings suggest further directions for research.

Table 7: *Students' Place Value-Related Behaviors*

Discussion

The purpose of this study is to provide insights about end-of-year second graders' knowledge of place value and its application in solving two- and three- digit addition and subtraction problems. Data were gathered through the use of a qualitative, task-based structured interview that occurred individually with each of the 22 students and was video taped. Data analysis reveals several interesting results, which are discussed in the following section.

Key Developmental Understandings, Big Ideas, and Place Value

It was found that 63.6% of students consistently unitized and used ten as a composite unit, 27.3% of students used ten as a composite unit at times, and 9.1% of students used ten as ten single units. These results give specific insights into the KDU of composite units. As Simon (2006) explained, "One way to identify KDUs is to observe students engaged in mathematical tasks to specify understandings that can account for differences in the actions of different students in response to the same task" (p.363). Two such differences between students who unitized and used ten as a composite unit consistently as compared to those who did so inconsistently were a) continuing to use ten as a composite unit with larger numbers, and b) using ten as a composite unit when mentally adding or subtracting ten from an off-decade number when presented with a physical representation of the original quantity. Therefore, this study indicates that such a difference exists in students who are at varying stages of development with the KDU of composite units and big idea of unitizing.

These results indicate that students develop an ability to unitize and use ten as a composite unit first with lower numbers, and they continue to use ones until they have developed a sense of composite units with higher numbers. Furthermore, students develop a concept of composite units to count groups of physical objects by tens earlier than they develop an ability to apply composite units to mental addition and subtraction. This was evidenced by the fact that all of the students who used ten inconsistently could count groups of objects by ten, but had to mentally add and subtract in ones. Although not all children will learn concepts in the same order, this study identified some of the stages within the development of the conception and use of ten as a composite unit.

This study also found that students who consistently used ten as a composite unit had a higher median of known number combinations than students who used ten as a composite unit inconsistently or not at all. This suggests that students who have more facility with composing and decomposing numbers are also more likely to have more facility working in groups of ten rather than single units. This indicates that students may need a solid sense about numbers before being able to manipulate groups of numbers. Students who are able to compose and decompose numbers have "construct(ed) a network of numerical relationships" (Kamii, 2000, p.69) which allows them to use ten as a composite unit instead of as single ones.

One of the more skewed results of this study was finding that a majority of students (85.7%) did not know that a quantity remains constant regardless of how it is grouped or arranged. This indicates that conservation of quantity when grouping develops after conservation of individual objects. Most students who did not conserve the original quantity of 33 thought that there would be more cubes when counting by groups.

Students’ strategies generally became less sophisticated when tasks involving incrementing by ten increased in difficulty by counting off the decade, removing a visual representation, and asking students to increment simultaneously by both tens and ones. The increased demand on students’ working memory was evident as they computed. Whereas to count by tens students generally responded before or as the next tens strip was placed on the table, students were slightly slower incrementing by tens off the decade and considerably slower when working with a screened quantity. Many students mentally calculated for twenty seconds or so, and then asked for the screened quantity again because they forgot it in their mental work or had otherwise floundered and needed to start over.

Students who were able to continue working in tens did so with either the split strategy or the jump strategy. Five students used the jump strategy, four students used the split strategy, and five students used both the jump and split strategy (it remained unclear as to what strategy two students used). Everyone who used the split strategy explained that they grouped the tens first, then the ones. When using the jump strategy, students first “jumped” up by the new number of tens, then by the new number of ones. These findings are consistent with research that reveals children typically manipulate a quantity from left to right (Thompson, 1999, p.170). This is important to consider because the standard algorithm, which was the primary means of calculation that all of these students were taught, operates from right to left. Although all of the students used standard algorithms in their written work, when presented with a high-cognitive demand task that they had to solve mentally, students who were able to work in tens used the split strategy and/or jump strategy and moved from left to right. This indicates an ability for using a alternative strategy in mental work.

Another finding was that all five students who used both the split strategy and the jump strategy began by using the split strategy and later switched to the jump strategy. This indicates an initial preference and ability on the part of these students to use the split strategy. However, as the task proceeded the numbers grew larger and the screened arrangements of dots were increasingly difficult to picture. These five students most likely switched from visualizing the arrangements of dots, which they used with smaller numbers to first group the tens together and then the ones. Instead they began to use the jump strategy which did not require a visualization of the screened dots (although it did not preclude it, either) but rather only required retaining the total number of screened dots and then counting on first by the new number of tens and then by the new number of ones. This is consistent with Wright et al.'s (2006b) statement that students had more difficulty using the split strategy with more difficult problems than they did using the jump strategy.

Strategy Use

The second area of interest of this study was to examine students' strategy use on two-and three-digit horizontal number sentences. This study found that students' independently selected strategies relied heavily on standard algorithms. All twenty-two students used standard algorithms for at least two-thirds of their strategies, and twelve students used standard algorithms as their only strategy. Standard algorithms accounted for approximately 88% of all of the strategies used. Although these results favored standard algorithms for written work, as highlighted previously, students' appeared to be able to conceptualize mental incrementing and addition by using the jump strategy or split strategy, rather than the standard algorithm.

These results confirm findings that the early use of standard algorithms can obstruct students' development of other strategies (Beishuizen & Anghileri, 1998). Although NCTM (2000, p.32) states "Students should be able to perform computations in different ways," several other viable strategies were hardly used (known fact, counting on from first, split strategy, direct modeling), and others were never used (counting on from larger, counting down, jump strategy, and compensation). This confirms Kamii's (2000) findings that early use of standard algorithms "encourage(s) children to give up their own thinking" (p. 83). Indeed, very little original thinking was evident during the series of horizontal number sentences, as most students were focused on the steps and procedures of the standard algorithms.

Likewise, students' flexibility was fairly limited as only six students used a strategy in addition to the standard algorithm, and only three of these students made use of an efficient, alternative strategy. NCTM (2000) states that students in grades pre-Kindergarten through grade two need to "develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers" (p.78). Students generally did not adapt their strategy to the problem type or numbers involved, but rather continued using standard algorithms regardless of whether an alternative strategy would have been easier or more efficient to calculate. Students learning the many procedures in the standard algorithms may have decreased their range of available strategies, as they have not been afforded time to develop or be exposed to alternative strategies. As Fosnot (2007) states, an objective for addition and subtraction "is for children to look to the numbers first before deciding on a strategy. Mathematicians do not use the same strategy for every problem; their strategies vary depending on the numbers" (pp.8-9).

Although roughly three-quarters of students were able to use a relational strategy to solve 16+9 from 16+10, just under half had difficulty understanding alternative solution strategies that made use of the split strategy. These students had several common responses to the problem. They tended to say that the problem was written wrong, it was too many steps, it added or subtracted too many times, that you cannot change the numbers from the original problem, and that you cannot change the steps from those of the algorithm. Some students thought that the alternative strategy was using any numbers that combined to 92, which they already knew was the correct answer having first solved the problem themselves. This shows a lack of understanding that the numbers used are directly related to the original problem, even if they are in a different form (in this case decomposed into tens and ones). Some students said you could use other numbers even though they had no relation to the problem other than arriving at the correct answer.

This indicates that for some students, math is about generating the correct answer instead of about making sense. "If from an early age, children are taught to approach problem solving as a way of making sense out of problem situations, they may come to believe that learning and doing mathematics involves the solution of problems in ways that are always meaningful" (Carpenter et al., 1999, p. 57). Early algorithm use may have resulted in some students having a limited and rigid conception of how calculations, and mathematics in general, work. At this point they are thinking in terms of steps in algorithms, but not numeric relationships and about half have a hard time understanding alternative strategies when asked.

Procedural Knowledge versus Conceptual Understanding

This leads into the results for the third area of interest of this study, which was to examine whether students had disconnects between procedural knowledge and conceptual understanding within the standard algorithm. Regrouping is a major concept housed within the procedures of the standard algorithms. Students referred to regrouping in addition as "carrying" and regrouping in subtraction as "borrowing." Students were generally procedurally fluent with two-digit addition with regrouping. There were only two students who did not have procedural fluency with two-digit addition.

Students had the most procedural difficulty with subtraction with regrouping. The most common mistakes associated with two-digit subtraction procedures included: a) not borrowing, b) subtracting the smaller number from the larger number even if the larger number was in the second quantity (thereby effectively subtracting "up" the algorithm), c) disregarding that there "wasn't enough" to subtract and subtracting to zero, d) adding all of the digits in the original problem together, e) subtracting the tens but adding the ones, and f) unnecessarily borrowing. Eleven students did not borrow or regroup when necessary. Ten of these students subtracted "up," which indicates a lack of understanding that subtraction is not commutative. Some students subtracted to zero. Although these students did not know how to address having a smaller number minus a larger number in the ones column, they did understand that subtraction is not commutative because they did not "flip" the problem to subtract "up" like many other students. Two students added all of the digits in the original digits in the problem together and tried to manipulate them to calculate an answer. Two students subtracted the tens but then added the ones instead of subtracting. One student borrowed when it was unnecessary.

Students were procedurally fluent with three-digit addition procedures (100% correct), but again had more difficulty with three-digit subtraction with regrouping (12.5% correct). Some students had similar procedural difficulties with three-digit subtraction as with two-digit subtraction. Two students subtracted "up," one subtracted to zero, and one added all of the digits together. Some students had procedurally difficulties unique to three-digit subtraction. For example, one student began 267-119 by borrowing from the hundreds, but then did not know how to use this to help him with 7-9 in the ones. It may be that this student has a misconception that when borrowing you always start with the digit furthest to the left.

In summary, students' were generally procedurally strong with both two- and three-digit addition. Students made more procedural errors with subtraction, some of which reveal misconceptions associated with subtraction and others reveal misconceptions about the role of place value within the standard algorithm.

Procedural Knowledge and Conceptual Understanding of Regrouping: Summary

Students' responses indicate a range of conceptual understanding of regrouping both for addition and subtraction. In both operations, students seem to begin with procedural knowledge and little conceptual understanding. They explain regrouping in terms of procedures that one follows so that one does not get the answer wrong. Next students begin to label tens and ones and talk about regrouping in terms of "carrying/borrowing a one." Later students understand that regrouping helps you get an answer that makes sense. Finally, students merge the labels and procedures with the big idea that place determines value to consistently recognize the role of ten in regrouping, and students realize that they are regrouping a ten. This study found that the students at the end of second grade were generally still in the procedural stages of understanding regrouping.

Talking in Tens

In this study, the majority of students (86.4% for addition and 90.5% for subtraction) were talking in ones not tens. Examples of talking in ones can be seen throughout students' verbal explanations of regrouping in both addition and subtraction, which were previously detailed and therefore will not be repeated here. The fact that the majority of students talked in ones is significant because it indicates students' procedural comprehension of the standard algorithm. Talking in ones reveals that students think about independent columns of numbers rather than the overall quantity, and may not understand that there is a difference between face value and complete value. These results are consistent with research that indicates students' perceive the standard algorithm as independent columns and in terms of face value (Carraher et al., 1987; Kamii, 2000; McIntosh, 1990; Thompson, 1999). Some students inconsistently referred to tens, showing an initial conception of complete value. Only three students consistently talked in tens.

After analyzing all 22 student responses, it is evident that as a student moves from a procedural knowledge to a conceptual understanding of regrouping the way he/she verbally address tens also develops. Students begin without mentioning tens at all. All procedures and why they are performed are in terms of ones, for example "the 4 turns into a 3, the 3 turns into a 13." Next, a student uses tens and ones as labels, but continues talking in ones when asked what a digit in the tens column is worth by responding with its face value rather than complete value. A third step comes when students begin to not only label but also inconsistently refer to tens as part of their explanations. For example, a student may say that he/she borrows a ten initially but later may refer to it as a one. Finally, students consistently talk in tens in meaningful ways and connect the big idea that place determines value to procedures.

Manipulation of Symbols versus Quantities

This study found that the majority of students manipulated symbols not quantities. This is consistent with Carraher et al.'s (1987) finding that standard algorithm use can encourage symbol manipulation rather than quantity manipulation. Students demonstrated symbol manipulation when their explanations were limited to procedures and markings but failed to touch on place determining value.

Going back to a previously mentioned misconception reveals a student's manipulation of symbols rather than quantities. To solve 43-15 a student subtracted to zero (resulting in an incorrect answer of 30). In the context of symbol manipulation a student who subtracts to zero does not perceive the digits as related to the same overall quantity. This means that when there are not enough ones to subtract, the answer in the ones equals zero because the ones are viewed as a stand-alone quantity. There is no connection of any given digit to the overall quantity and no realization that there is "more" available to permit full subtraction of the ones and the tens. Multi-digit numbers are viewed as a series of independent, single-digit quantities.

Students' manipulations of symbols were further evident in their language. When students talked in ones, this indicated a focus on symbols and face values rather than quantities and complete values. In addition to talking in ones, using phrases such as a digit “turns into” or “becomes” a different number, or that you “make” a 3 into a 4 indicates attention to symbols not quantities.

A few students did manipulate quantities (ranging from 4.5% to 30% depending on problem type). Three students did not use an algorithm but rather used the split strategy. These students were able to manipulate quantities, not just symbols. Their strategy and explanation show that they are able to decompose and recompose numbers, understand some properties of addition (including the big idea of commutativity and associativity), and understand that place determines value and the associated complete values of two-digit numbers. One student who used the split strategy also solved the same problem with the standard algorithm. When asked which strategy was easier for her to think about, she said “this one (standard algorithm) is easier because you don’t have to add in your head.” This seems to confirm findings that students using the split strategy find it difficult to adapt this strategy (Wright et al., 2006b). However, Wright et al. also indicate that using the split strategy makes it more likely that a student will make errors, but she did not make any errors.

Level of Base-Ten Knowledge

The majority of students were found to have an initial concept of ten (levels previously defined in Table 8, p.37). In particular more students had an initial concept of ten for subtraction than for addition. For two-digit subtraction nearly 62% of students had an initial concept of ten and for three-digit subtraction almost 88% of students had an initial concept of ten. These numbers were higher for subtraction than addition. One-third of students scored higher in addition than subtraction.

These findings support previously discussed results that more students manipulated symbols (rather than quantities) in subtraction than addition and more students made procedural errors with subtraction than addition. This indicates that subtraction may be harder and/or take longer to develop a quantity-based conceptual understanding as opposed to addition. A further exploration of the literature would be informative in this regard. One possible explanation is that the standard subtraction algorithm has a higher number of symbols that require manipulation, and the increased quantity demands more time for which to develop an understanding. Or perhaps there is something inherent in the actual operation of subtraction that children have a harder time conceptualizing. Upon considering the steps of both addition and subtraction with objects (how children are often taught early operations), the increased complexity of subtraction becomes apparent. With addition children can make groups for both numbers, add both groups of objects, count every object, and do so in any order. Subtraction is a slightly more complex concept for children to learn given that they only make a group for the first number, they never make a group for the second number, only one group of objects is the correct one with which to begin, the second amount needs to be removed from the original amount, and only the remaining sub-set of the original number is counted for the final answer. When considering the algorithm for two-digit subtraction, numbers are crossed off, new amounts written, and "ones" are placed in front of digits. Perhaps the convergence of the subtraction operation and the standard algorithm require more time and experience in order for students to develop an understanding. Until students develop this understanding and make sense of both the operation and the algorithm, they will manipulate symbols not quantities, talk in ones, and have only an initial concept of ten. Again, an exploration of the literature in regards to subtraction would be informative.

Results also suggest that there is a relationship between students' highest known number combination and their overall level of base-ten knowledge. It is important to emphasize that while results imply a relationship between highest known number combination and overall base-ten knowledge, it does not mean that knowing higher number combinations *causes* a student to have a higher level of base-ten knowledge. Given that both the ability to compose and decompose numbers and conceptualize place value are key developmental understandings, it is more likely that high levels in both areas are developed over time and as a result of numerous experiences. Therefore, isolated drilling of number facts will be unlikely to raise a students' base-ten knowledge.

The relationship between higher known number combinations and higher levels of base-ten conceptualization suggests that there is more involved with knowing number facts than may appear on the surface. This result contributes to previous findings that knowing number combinations indicates the development of complex networks of numeric relationships (Richardson, 2002a). It may be that it is the development of a network of numeric relationships that allows students to conceptualize and use ten and students who have less developed numeric relationships are less likely to conceptualize ten.

It may also be that it is the ability to compose and decompose numbers that allows students to know number combinations and allows them to perceive the complete values within multi-digit numbers. For example, a student who is not able to decompose numbers may be less successful realizing 24 is decomposed into 20+4, and thereby may remain focused on face value, talking in ones, and manipulating symbols. A student who can decompose 24 into 20+4 may be more likely to recognize complete value, talk in tens, and manipulate quantities.

Summary

This study provided insights into end-of-year second graders' knowledge of place value and its application in solving two-and three-digit addition and subtraction problems. The interview protocol provided ways to look at and understand end-of-year second grade students' thinking about tens and ones.

Within this exploratory study, I examined second graders' understanding of some of the big ideas related to place value and the use of ten using a carefully designed set of assessment tasks. These big ideas included what number combinations for numbers 1-10 students know fluently, students' facility with unitizing, and students' facility recognizing and using place value patterns. The median highest known number combination was six. The majority of students' (59.1%) highest known number combination was 6 or lower, with 13.6% demonstrating fluency with combinations of the number 10.

In regards to unitizing, nearly all students (90.9%) were able to use ten as a composite unit at least some of the time. Two distinguishing abilities between students who used ten as a composite unit consistently as compared to those who did so inconsistently were a) continuing to use ten as a composite unit with larger numbers, and b) using ten as a composite unit when mentally adding or subtracting ten from an off-decade number. Fifty-nine percent of students addressed tens simultaneously, and 41% addressed tens successively all or part of the time. The majority of students (85.7%) did not know that a quantity remains constant regardless of how it is grouped or arranged. In regards to the big idea of place value patterns, the level of strategy sophistication decreased when tasks involving incrementing by ten increased in difficulty by counting off the decade, removing a visual representation, and asking students to increment simultaneously by both tens and ones.

This study also examined students' strategy use on two-and three-digit horizontal number sentences. Students appear to be highly fluent with two-digit operations with and without regrouping in addition and without regrouping in subtraction (ranging from 95 to 100% correct), and had some difficulty with regrouping in subtraction (ranging from 52 to 74% correct). Students were fluent with three-digit addition procedures (ranging from 89 to 100% correct), but had more difficulty with three-digit subtraction (ranging from 29 to 60% correct). Students' independently selected strategies relied heavily on the standard algorithms. All twenty-two students used the standard algorithms for at least two-thirds of their strategies and twelve students used the standard algorithms as their only strategy. Only six students used a strategy in addition to the standard algorithms, and only three of these students made use of an efficient, alternative strategy. Students also used known fact, counting on from first, split strategy, direct modeling, and incorrect strategy. Counting on from larger, counting down, jump strategy, and compensation were never used by any of the students. The majority of students (77%) were able to use a relational strategy to solve 16+9 from 16+10, but had difficulty understanding alternative solution strategies. Approximately half of students could not explain the alternative addition strategy, and only one student had a thorough understanding of the alternative subtraction strategy.

The third area of interest for this study was whether students had disconnects between procedural knowledge and conceptual understanding of two-digit operations and place value within the standard algorithm. Nearly all of the students demonstrated procedural fluency with two-digit addition, but only slightly more than half of the students were procedurally sound with two-digit subtraction. Similar results were found with three-digit operations, with high levels of fluency for addition (ranging from 91 to 100%) and much lower levels for subtraction (ranging from 12.5 to 52%). The majority of students were found to manipulate symbols not quantities, and talked in ones not tens. Most students appear not to understand regrouping in addition (82%) or regrouping in subtraction (81%) beyond procedures. The majority of students were also found to have an initial concept of base-ten. Although some students had an intermediate or facile concept of ten for addition (59.1% for two-digit and 60% for three-digit), these numbers again decreased for subtraction for which fewer students demonstrated an intermediate or facile concept of ten (38.1% for two-digit and 12.5% for three-digit). Results also indicate that there may be a relationship between students' highest known number combination and their overall level of base-ten knowledge.

Limitations of Study

One limitation of this study was that not all students completed all of the horizontal number sentences in Task Four. Interviews were kept to similar lengths of time, aiming for around 35 minutes so that students could return to their classrooms in reasonable amounts of time. However, this meant many students did not finish all of the number sentences. Consequently, it was more likely that students who were naturally slower workers or who worked slower due to difficulties on earlier tasks were the ones who did not complete all of the problems. This means it tended to be students who were more efficient workers or had an easier time with the interview tasks who completed all of the horizontal number sentences. Therefore there was a smaller sample size for evaluating three-digit operations, since those were the very last problems asked. This also resulted in a smaller sample size to evaluate three-digit procedural fluency, symbol versus quantity manipulation, and base-ten knowledge.

A second limitation was that while this interview provided a detailed snap-shot, the study did not follow teaching throughout the school year to know what was taught and how it was taught. This study did conduct a focus group with the classroom teachers in both schools that discussed their teaching of place value, but this could be enhanced by classroom observations throughout the year.

A third limitation of this study related to determining students' strategy use in Task Four. For some of the two-digit addition problems it was difficult to determine if a student "counted on from first" or "counted on from larger" because four of the five horizontal number sentences were written with the larger number first. This made it difficult to know if students began with the number because it was first or because it was larger. This could be remedied by writing some of the horizontal number sentences in the opposite order, with the larger number located second.

Questions for Further Research

The set of tasks used provided rich and useful insights into students' place value knowledge. The results of data analysis suggest a number of findings that are worthy of further investigation.

An issue that emerged from this study and warrants further research pertains to the relationship between higher known number combinations and higher level of base-ten knowledge. Although the results from this study indicate a relationship, it remains unclear as to why this is the case.

Another issue generated by this current study provokes the question, what is each student's stage of two-digit number conceptualization? Fuson and Smith (1996) provide stages for two-digit number conceptualization that were considered when analyzing the data in an attempt to determine each student's level. Although it was possible to determine some student's levels, it remained unclear what many student's levels were. Usually some of the six levels were confidently eliminated as possibilities, but at times two or three potential levels remained and it was not possible to reliably select one level for each student. If these levels were of particular interest, a new sub-task could be added to the interview protocol to better assess a student's conceptualization of a two-digit number. Specifically, asking students to draw a given two-digit quantity would help to see how a student pictures the quantity and allow for finer distinctions to be made than verbal descriptions and written work allowed.

The next step to extend this research would be to use the interview data from this study to form student profiles. For example, a student profile could be made for a student who demonstrated a Level One conception of base-ten knowledge, a Level Two conception, and a Level Three conception. The development of these learning profiles may help teachers and researchers get a sense of responses that are typical of a student who is operating at Level One.

Implications

This study has implications both for the use of this interview protocol to assess student knowledge and for teaching and learning about key developmental understandings and big ideas, strategies, and procedures related to place value and the use of ten. This final section highlights some ways that teachers in the early grades (K-2) can support students' place value learning and conceptualization.

Students' highest known number combination appears to be related to higher levels of base-ten knowledge. One area that early grades teachers can all help to develop is students' ability to compose and decompose numbers to ten with automaticity. NCTM (2000) states that students in grades PreK-2 need to "develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers" (p.78). An emphasis within most kindergarten and first grade curriculums is the development of number sense. Engaging in activities in which students learn the various ways each number can be composed will help lay a foundation and begin building numeric networks to promote students' base-ten knowledge. Rather than drill-and-practice, this study's evidence appears to support a process of manipulative-based experiences in which students "make" numbers with various combinations and "break" numbers into component parts.

One of the findings of this study was that students are more accurate using ten to count groups of objects and increment on the decade. Six students reverted to using ones for mentally incrementing from an off-decade number, for higher numbers, when there was no visual representation of the final quantity, and to increment simultaneously by both tens and ones. Having taught first grade it seems that the curriculum was focused more on counting groups and counting by tens on the decade than these other skills. As a result, most of my K-2 colleagues spent more time using tens to count on the decade and groups of objects than they did these other tasks. By presenting activities in which students can use ten to count off the decade, count higher quantities, increment without physical representations, increment mentally from an off-decade number, and increment simultaneously by both tens and ones will support students' progress on the big idea of composite units.

A third area that curriculum and teachers could expand time and focus on is conservation of quantity while grouping. One of the most skewed results of the entire study was that only three students knew that a quantity remains constant regardless of how it is grouped or arranged. Curriculum often addresses conservation of number, but may need to also consider that it is a separate conceptual understanding for a student to realize quantities are conserved when grouped. It would also be helpful for first and second grade teachers to be aware of this so that they can offer additional experiences to students in which they engage in activities that involve a number of manipulatives which students group in various ways and count to build their understanding that the quantity is constant regardless of if they have ones, groups of twos, groups of fives, or groups of tens.

Another implication of this study on teaching and learning is that students need to have multiple ways to compute problems and should have flexible approaches so that they can chose a strategy that best fits the given problem type and numbers. As Fosnot (2007) explained, "The justification for using different strategies is based on the big idea of the commutative and associative properties and a good sense of place value and landmark numbers" (p.9). Similarly, Kilpatrick, Swafford, and Findell (2001) stated, "Flexibility of approach is the major cognitive requirement for solving non-routine problems" (p. 127). Delaying the introduction of the standard algorithm until the second half of second grade or the beginning of third grade would allow students to develop their own meaningful solution strategies. Even if the standard algorithm is to be used earlier, students can still be exposed to alternative strategies, such as the split strategy and jump strategy. As Fosnot (2007) explained, the standard algorithms should not be "seen as the ultimate strategies for computation, only as other (albeit generalizable) strategies in a toolbox for computation based on a deep sense of number and operation" (p.6). Exposure to and conceptualization of multiple strategies may help students develop relational thinking, increase flexibility, understand quantity manipulation, and make sense of the algorithm. Alternative strategies will also help students with mental calculations. Some of the students who were interviewed tried to mentally calculate a multi-digit addition problems by visualizing the algorithm, but either arrived at an incorrect answer or had to resort to writing the algorithm out on paper. While the algorithm is efficient when written, the split or jump strategy are more efficient for mental calculations.

Given that students had more procedural and conceptual difficulties with subtraction than addition, students need to be afforded more time to develop their understanding of this operation. This will likely require more time and more experiences so that students can make sense of both the procedures and the overall operation. When planning these experiences it is important to remember "it is not children's manipulations of materials that is important; it is their understanding of the principles involved in the manipulations" (Carpenter et al., 1999, p. 68). Teachers can help students' development of subtraction understanding by engaging in more conceptually-based experiences with single-digit subtraction in both kindergarten and first grade. This would mean that students would enter second grade with a stronger understanding of properties of subtraction. Then second grade teachers could build off of this and focus on students' understandings of multi-digit subtraction.

Students who accurately calculate with the standard algorithms still need experiences to develop a conceptual understanding of the underlying processes and role of place value. NCTM (2000) stated, "computational fluency should develop in tandem with understanding" (p.32). Students need to be able to manipulate quantities not symbols, talk in tens not ones, and understand regrouping conceptually rather than procedurally. Most likely, this will require more time to allow teachers and students to work on these key developmental understandings before demanding computational mastery. This study suggests that just because a student is procedurally fluent does not mean he/she has a conceptual understanding beyond rules and procedures. In regards to using the standard algorithms, NCTM (2006) states that students need to "understand why the procedures work (on the basis of place value and properties of operations)" (p.23). Given that quantity manipulation, talking in tens, and conceptual understanding of regrouping are all key developmental understandings, it will not be possible for students to develop in these areas simply by increasing teacher-directed explanations or demonstrations because as Fosnot (2007) explained, "When regrouping methods are taught as rote procedures, children often latch on to them without understanding deeply why they work" (p.8). When working with addition and subtraction of multi-digit numbers, Fosnot (2007) endorsed "the important underlying ideas to emphasize…are place value and equivalence, not computational procedures" (p.6). Teachers need to spend time introducing two-digit operations and regrouping with direct modeling so that students can actually see what is happening and why. This should be done before introducing notation and transitioning to paper and pencil. Linndquist (1997) explained the need for direct modeling: "As you talk, do not expect the students to learn without observing the actions on the objects. Then help students symbolize the actions, and later generalize to larger numbers for which the actions on objects become awkward" (p.xi). This process needs to be thoughtfully monitored and not rushed so that students have time to learn through observing before moving on to the abstraction of the algorithm.

A related area that teachers can specifically reflect on and improve is in their own use of language when explaining procedures in algorithms. Again, conceptual understanding will not be developed by explanations alone, but it does seem that a teacher's language can promote conceptual understanding as opposed to procedural knowledge. For example, telling students to "carry the one", "borrow a one," have a number "turn into" another number, and "make" a 3 into a 4 most likely contribute to students' similar explanations of multi-digit operations and detract from conceptual understanding. Common phrases such as borrowing "from next door" or "from your neighbor" or "putting the one next door" or "carrying a ten to your neighbor" may increase students' perception that numbers are comprised of isolated digits rather than considered as one overall quantity. Similarly, teaching "tricks" to know when to carry or borrow undermines learning about place value and numeric relationships and elevates rules above sense-making. Teachers can help students by using language that consistently talks in tens instead of ones and emphasizes quantity rather than symbol manipulation. The last contribution of this study to teachers is the compilation of helpful questions that teachers can ask their students to informally assess their strategy use and understandings about multi-digit operations and regrouping. The following questions were taken from the interviews and tended to result in telling responses from students. These questions could be used to illicit responses that indicate students' procedural versus conceptual understandings and may reveal some misconceptions. Classroom teachers could use these questions to informally assess students' understandings as they work on multi-digit addition or subtraction problems.

Table 20

Questions for Teachers to ask Students for Multi-Digit Addition or Subtraction

Problem Type	Questions
General Questions	Do you have any other ways to solve this problem? Can you draw/show what this problem means?
Questions about Regrouping in Addition	Why do you do that (in reference to carrying a ten/writing a one)? What does it mean to carry? When do you carry? (E.g. 38 + 24) Why don't you write the 12 down there (under the equal sign as in 512)? (E.g. 38 + 24) Why do you put the 1 on top and the 2 down below? Can you put the 2 on top and the 1 down below? Why don't/can't you put the 2 on top and the 1 down below? Can you draw/show what carrying means?
Questions about Regrouping in Subtraction	Why do you do that (in reference to borrowing a ten/writing a one)? What does it mean to regroup/borrow? When do you regroup/borrow? (E.g. 43-15) When you take a 1 from the 4, why/how does the 3t become 13, why doesn't it become a 4 (since 3 + 1 = 4)? If student talks about borrowing a ten: (E.g. 43-15) How can you borrow a ten if it's only a 4? Can you draw/show what borrowing means?
Three-digit Subtraction	Can you borrow for a number that you already borrowed from?

To evaluate student responses a teacher can consider whether a student talks in ones or tens, whether he/she talks about manipulating symbols or quantities, and whether he/she has any misconceptions.

Taking into consideration the results of this study, there are evidenced-based strategies that teachers can apply to help students in their progress on key developmental understandings that support place value and the use of ten. By integrating these suggestions into students' mathematical experiences in kindergarten through second grade, young children are afforded specific support for place value-related key developmental understandings, big ideas, and strategies.

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