Are There Common Values in First-Price Auctions? A Tail-Index Nonparametric Test*

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Abstract

We develop a consistent nonparametric test of common values in first-price auctions and apply it to British Columbia Timber Sales data. The test is based on the behavior of the CDF of bids near the reserve price. We show that the curvature of the CDF is drastically different under private values (PV) and common values (CV). We then show that the problem of discriminating between PV and CV is equivalent to estimating the lower tail index of the bid distribution. Our approach admits unobserved auction heterogeneity of an arbitrary form. We develop a Hill (1975)-type tail index estimator and find presence of common values in BC Timber Sales.

Keywords: first-price auctions, common values, private values, tail index, timber auctions

JEL Codes: C14, D44

1 Introduction

Economists have long recognized that auctions with common values (CV), where buyers do not know in advance their willingness to pay, are very different from auctions with private values (PV). For example, a theoretical result of Milgrom and Weber (1982) shows that in the affiliated value model that allows for both common and private components in

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bidder valuations, an open auction is revenue superior to the first-price, sealed-bid auction.¹ Other policy recommendations differ depending on whether the values are private or common.² For example, in a CV environment, the seller may decide to reveal information to mitigate the winner's curse effect. Another concrete example is setting the optimal reserve price. Our application in this paper is to British Columbia Timber Sales (BCTS). In British Columbia, the Ministry of Forests has developed a complex system of reserve prices with the goal of maximizing revenue for the government. As the optimal reserve price setting depends on the employed model, it is important to know whether a private or common value model is more appropriate in this application.

This paper develops a consistent (asymptotic power-one) tail-index nonparametric test for common or private values in first-price auctions and applies it to timber auctions in British Columbia. This is the first paper we are aware of that uses powerful methods from the econometric literature on extremes to shed light on an important problem in empirical industrial organization.

The approach in this paper is inspired by Hendricks, Pinkse, and Porter (2003) who utilize a result in Milgrom and Weber (1982) to show that the behavior of bids around the reserve price is different under PV and CV.³ Specifically, the lower bound of the support of the so-called pseudo-values (see Guerre, Perrigne, and Vuong, 2000) is equal to r under PV but is strictly greater than r under CV. We show that under PV, there must be more clustering of bids near the reserve price than under CV. For *independent* private values, an excessive clustering result was previously obtained by Guerre, Perrigne, and Vuong (2000) in the context of value density estimation. However, the previous literature has not considered a formal statistical test of PV versus CV based on the behavior of bids near the reserve price.

Our innovation is to develop such a test based on the *tail index* of the bid distribution. In our application, the tail index gives a measure of bid clustering around the reserve price. The tail index characterizes the curvature or rate of decay of a CDF in the tail,

¹There is also empirical evidence to the importance of this effect, e.g. in Shneyerov (2006).

²In addition, as Laffont and Vuong (1996) have shown, models with common values are often nonparametrically non-identified, while private-value models are often identified.

³A more detailed discussion appears in Hendricks and Porter (2007).

and is a popular approach to modeling extreme quantiles in economics, finance, insurance and engineering. See Section 2 for a formal definition, and see Embrechts, Kluppleberg, and Mikosch (1997), Gabaix (2008) and Davis (2010) for a review on power tails and their applications. We show that the tail index κ is equal to 1 under CV, and is equal to $1/2$ under PV. Under private values, Guerre, Perrigne, and Vuong (2000) establish this result only for the IPV case. We show that the tail index is $1/2$ whether the private values are independent or affiliated.⁴

To our knowledge, existing tests for common values take the null hypothesis to be that of private values (independent or affiliated) and the alternative to be a model with a common value component. In many applications, however, there is no paradigm (PV or CV) that is generally “accepted”. By switching the null and alternative hypotheses, this paper is the first to propose a test of the null hypothesis that incorporates a CV component versus a PV alternative.

Milgrom and Weber (1982)’s classic model has been fundamental widely in both theoretical and empirical research on auctions, despite the fact that it applies only to the symmetric environment. Empirical applications within a symmetric common-value paradigm include Paarsch (1992) to auctions for tree-planting contracts, Haile (2001) to timber sales, Hong and Shum (2002) to highway procurement, Hendricks, Pinkse, and Porter (2003) to off-shore oil, and Shneyerov (2006) to municipal bonds. Within the symmetric private-value paradigm, applications include Laffont and Vuong (1996) to eggplant auctions in Marmande (France), Krasnokutskaya (2011) to highway procurement, and Li and Perrigne (2003) to timber sales.

However, it is indisputable that bidder asymmetries are important in a number of markets. For example, in a seminal paper, Hendricks and Porter (1988) emphasize bidder asymmetries in sealed-bid auctions for off-shore drainage tracts. Also, within the private-value paradigm, Bajari and Ye (2003) and Jofre-Bonet and Pesendorfer (2003) have studied auctions for highway procurement, Campo, Perrigne, and Vuong (2003) have considered

⁴The distribution of bids near the reserve price satisfies a power-law, hence is regularly varying. This evidently marks a rare case of naturally occurring Paretian tails. In the statistics literature many Markov chains and stochastic recurrence equations are known to exhibit power-law tail decay, including nonlinear GARCH. Consult Embrechts, Kluppleberg, and Mikosch (1997), Gabaix (2008), Hill (2010) and Hill (2011b) for references.

off-shore oil auctions, Flambard and Perrigne (2006) - auctions of snow-removal contracts, while Athey, Levin, and Seira (2011) emphasize asymmetries in timber sales. A question then arises, to what extent is our tail-index approach robust to bidder asymmetries. We consider a class of asymmetric models in which, as we show, the tail index is preserved.

The estimation of the tail index is a well-studied problem in statistics and econometrics, see Hill (1975), Hsing (1993) and more recently Hill (2010) and Hill (2011b). We employ a version of Hill's (1975) celebrated estimator $\hat{\kappa}$ of the tail index. The estimator is consistent and asymptotically normal, and is remarkably easy to implement for a one-sided t-test of PV versus CV. Our test approach works even when there is unobserved auction heterogeneity, and even if the number of potential bidders is unobservable.⁵ Further, since Hill's (1975) estimator $\hat{\kappa}$ exhibits the same stochastic properties under either hypothesis PV and CV, differing only in the asymptotic mean, our test of CV or PV is consistent: it obtains an asymptotic power of one against the chosen alternative.

An early approach to testing for common values was to check if bids increase monotonically with the number of potential bidders; a non-monotonic pattern was believed to provide evidence of common values. This approach was initiated by Gilley and Karels (1981), and applied to second-price sealed-bid and English auctions by Paarsch (1991) and Bajari and Hortaçsu (2004). However, Pinkse and Tan (2005) have shown that in first-price auctions, this pattern can also arise if values are private and affiliated.

The first paper that adopted a structural approach to this problem is Paarsch (1992), where a parametric testing method is developed and applied to the auctions of tree planting contracts in BC. The recent literature has focused on nonparametric approaches. Haile, Hong, and Shum (2003) have proposed a nonparametric test of PV versus CV. Their approach is entirely different from ours and is based on the variation in the number of bidders across auctions. They implement their test on a sample of US Forest Service (USFS) timber auctions and obtain mixed results. Haile, Hong and Shum's approach does not require a binding reserve price.⁶ Recently, Hortaçsu and Kastl (2010) proposed a test

⁵Krasnokutskaya (2011) has argued for the importance of accounting for the unobserved heterogeneity in the estimation of auction markups. See also the discussion in Paarsch and Hong (2006).

⁶In USFS auctions, the reserve price is typically non-binding and the number of potential bidders is observable. See Baldwin, Marshall, and Richard (1997), Haile (2001) and Haile and Tamer (2003).

of common values when some bidders have information about rivals' bids, and applied it to Canadian Treasury Bill auctions.⁷ Their approach is tailored to the environment of Canadian Treasury Bill auctions and is also entirely different from ours.

Our paper also makes a few important econometric contributions by extending the Hill (1975) tail index estimator $\hat{\kappa}$ to imbalanced panels where bids are nonlinearly dependent within auctions of random size. There are only a few applications of tail index estimation for panel data: see, for example, Mikosch and de Vreis (2006) and Jongen, Verschoor, Wolff, and Zwinkels (2006). By exploiting theory developed in Hsing (1991), Hill (2009), and Hill (2010), the estimator $\hat{\kappa}$ is shown to be asymptotically normal where the stochastic nature of bid counts is irrelevant. In particular, $\hat{\kappa}$ has the same asymptotic distribution as though bids were independent. Although this can simplify inference asymptotically, we show by the way of Monte Carlo simulations that for sample sizes that are realistic for auction applications, Hill (2010)'s asymptotic kernel variance estimator provides a better approximation of the small sample dispersion in $\hat{\kappa}$. We recommend using the kernel variance estimator.

We implement our test on a British Columbia Timber Sales [BCTS] dataset that contains all auctions conducted from January 14, 2004 to December 14, 2006. Timber auctions have attracted significant attention in the literature, while most of which focus on US data. The question of which model, PV or CV, is more appropriate for timber auctions has not been fully resolved.⁸ Baldwin (1995) and Athey and Levin (2001) argue for the presence of common values, while Baldwin, Marshall, and Richard (1997), Haile (2001), and Haile and Tamer (2003) adopted a PV paradigm.⁹ Our test rejects private values for BCTS in favor of common values.

⁷In Canadian Treasury bill auctions, bidders naturally fall into two groups — dealers and customers, and the former have an informational advantage over the latter.

⁸When we say CV, we mean a general model with interdependent values, not necessarily pure common values.

⁹We follow Milgrom and Weber (1982) and assume a standard affiliated model of a first-price auction. Some papers that have studied bidding in US timber auctions focused on the aspects specific to that market. Baldwin, Marshall, and Richard (1997) have shown that collusion was a common element in these data. Collusion implies that bids will be close to the reserve. To the extent that there was collusion, the test results based on a competitive model would have little meaning. Athey and Levin (2001) show that bids are multi-dimensional and this gives rise to “bid skewing”. Haile (2001) argues that resale was important which changes the nature of the auction game. We abstract from these possibilities.

2 The Model and Testable Restrictions

The model is the canonical symmetric model first introduced in Wilson (1977). There are $N \geq 2$ potential bidders who consider bidding at a sealed-bid, first-price auction. The rules of the auction specify a minimum bid, or *reserve price*, r . The object is assumed to have value U_i to bidder i , and this value may not be fully known beforehand. Prior to the auction, bidders obtain information concerning the object summarized by a scalar signal S_i . Bidders may have common as well as private components in their valuations. The common component is V . The valuation of the bidder is $U_i = u(V, S_i)$, where u is a nonnegative, continuous and nondecreasing function.¹⁰ As in Milgrom and Weber (1982), we assume that the vector $(V, S_1, \dots, S_N)$ is drawn from some joint distribution F having density f .

The model is symmetric: the function u is the same for all bidders, and the distribution F is symmetric in bidders' signals. We maintain the following assumptions that will allow us to establish the exact rate of decay of the CDF of bids near the reserve price.

Assumption 1 (Symmetry and Affiliation) *The density f is symmetric in the signals $(S_1, \dots, S_N)$ and satisfies the affiliation property.*¹¹

Assumption 2 (Rectangular Support) *The support of F is $[\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}]^N$, where $\underline{v} < \bar{v}$, $\underline{s} < \bar{s}$.*

Assumption 3 (Smoothness) *The function F is smooth, i.e. has continuous partial derivatives of all orders: $F \in C^\infty([\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}]^N)$. The functions $u(v, s)$ is likewise smooth, and $u \in C^\infty([\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}])$.*¹²

Our tail-index results require the following assumption on f , the density of F .

¹⁰Milgrom and Weber (1982) allow more generally $U_i = u(V, S_i, \{S_j\}_{j \neq i})$, i.e. allow the valuation of a given bidder to depend on rivals' signals directly, not only through the common component V . Nothing would change if we adopted this more general specification. We decided to consider a simpler specification because it is easy to interpret empirically in our application.

¹¹Recall affiliation implies $f(\min\{x, y\})f(\max\{x, y\}) \geq f(x)f(y)$, where min and max are understood component-wise here. See Milgrom and Weber (1982).

¹²The infinite differentiability assumption is not required for the derivation of the tail index, but greatly facilitates the derivation of the rate of decay in Proposition 1 by allowing us to invoke the powerful Morse Lemma, a tool from differential topology.

Assumption 4 (Positive Density) *The density f is positive on the support $[\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}]^N$.*

If $u(V, S_i)$ does not depend on V , we have a PV model (an affiliated private values model, or APV, if the signals are strictly affiliated). Otherwise, we have a CV model. Let $Y_i = \max_{j \neq i} S_j$. Milgrom and Weber (1982) have shown that in a symmetric equilibrium, the bidding strategy $B(s)$ satisfies the differential equation

$$[v(s) - B(s)] f_{Y_1|S_1}(s|s) - B'(s) F_{Y_1|S_1}(s|s) = 0, \quad (1)$$

where

$$v(s) \equiv E[U_i | S_i = s, Y_i = s],$$

the value of the object conditional on “just” winning the auction with bid $B(s)$.

Assumption 5 (Smoothness of v) *The function $v \in C^\infty([\underline{s}, \bar{s}])$ is smooth, and its derivative is positive: $v'(s) > 0$ for all $s \in [\underline{s}, \bar{s}]$.*

This differential equation has a unique solution subject to the boundary condition $B(s^*) = r$. Only the bidders with signals $S_i \geq s^*$ can win the auction. *We assume that bidders having $S_i < s^*$ do not bid.* In general the *screening level* $s^* \in [\underline{s}, \bar{s}]$, but we assume that the reserve price is *binding*: $s^* \in (\underline{s}, \bar{s})$. In a symmetric equilibrium, a bidder with signal $S_i = s^*$ can only win if his potential rivals all draw signals below the screening level, $S_j \leq s^*$. His value conditional on winning is $w(s^*)$, where

$$w(s) \equiv E[U_i | S_i = s, Y_i < s].$$

By definition, this bidder is a marginal bidder; i.e., is indifferent between entering or not entering, and makes zero expected profit. The marginal bidder bids r , and the zero expected profit condition together with the fact that the reserve price is binding implies that s^* is determined implicitly from the equation

$$w(s^*) = r.$$

As was first noted in Hendricks, Pinkse and Porter (2003) in their study of wildcat sales in Outer Continental Shelf (OCS) auctions, there is a difference in bidders' behavior around the reserve price under private and common values. Under PV, there is no scope for the winner's curse. Therefore,

$$r = w(s^*) = v(s^*),$$

which implies $B'(s^*) = 0$. Under CV, on the other hand,

$$r = w(s^*) < v(s^*),$$

and the slope is positive, $B'(s^*) > 0$.

The difference in the slope $B'(s^*)$ under PV and CV leads to a difference in the curvature of the distribution of bids around $b = r$. Our testing approach is based on a measure of this curvature, formally captured in the notion of the tail index. It is convenient to normalize the bids

$$B^* \equiv \frac{B}{r} - 1. \tag{2}$$

Let $G^*(b)$ be the equilibrium distribution of the normalized bids,

$$G^*(b) \equiv \Pr[B^*(S_i) \leq b | S_i \geq s^*].$$

The lower bound of the support of G^* is 0.

Our main result shows the bid distribution tail near the reserve price is governed by a power law with constant scale, hence it is Paretian. Recall the definition of Paretian tail decay and the associated tail index (cf. Hall (1982), Embrechts, Kluppleberg, and Mikosch (1997)).

Definition 1 *The (lower) probability tail $G^*(b)$ exhibits Paretian tail decay with tail index $\kappa > 0$ and scale $c > 0$ if*

$$G^*(b) = c \cdot b^\kappa \cdot (1 + o(1)) \quad \text{as } b \downarrow 0. \tag{3}$$

In the proposition below we show the lower normalized bid tail is Paretian, we derive the tail indexes of $G^*(b)$ under CV and PV, and even stronger we characterize the decay scale of $G^*(b)$ as $b \downarrow 0$.

Proposition 1 *Under the Assumptions 1 - 5 we have*

$$G^*(b) = c \cdot b^\kappa \cdot (1 + O(b^\kappa)) \quad (4)$$

as $b \downarrow 0$ for some $c > 0$, where $\kappa = 1$ under CV and $\kappa = 1/2$ under PV.

Proof. We have assumed that the bidders who draw signals below s^* , do not bid. This implies that the bidding strategy $B^*(s)$ is only defined on $[s^*, \bar{s}]$. Our proof will rely on the Inverse Function Theorem and the Morse Lemma.¹³ In order to apply these mathematical results, we need to work on an open domain around $s = s^*$. Therefore, we need to smoothly extend B^* as a function to such an open domain, even though it does not have an economic meaning for $s < s^*$. We first show that such an extension is possible.

Re-arrange the differential equation (1) as

$$\begin{aligned} B^{*'}(s) &= \frac{[v(s) - rB^*(s) - r] f_{Y_1|S_1}(s|s)}{rF_{Y_1|S_1}(s|s)} \\ &= \Phi[s, B(s)]. \end{aligned} \quad (5)$$

Assumptions 3 and 4 imply the smoothness of $f_{Y_1|S_1}(y|s)$ on $(\underline{s}, \bar{s})^2$. This in turn implies that $\Phi(s, b)$ is also smooth in s on $(\underline{s}, \bar{s})$, and therefore is Lipschitz. Theorem 2.3 in Mattheij and Molenaar (2002) then implies that, as the solution to the differential equation (5) the function $B^*(s)$ can be smoothly extended to an open domain $D_\varepsilon \equiv (s^* - \varepsilon, s^* + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small. Assumption 3 implies that the function $F_{S_i|S_i \geq s^*}(s|s)$ can also be smoothly extended to D_ε . *In this proof, we consider both $B^*(s)$ and $F_{S_i|S_i \geq s^*}(s|s)$ smoothly extended to D_ε .*

Under CV, since $B^{*'}(s^*) > 0$, the Inverse Function Theorem implies that for a small

¹³For the statement of the Morse Lemma, see e.g. Guillemin and Pollack (1974), p. 42.

enough $\varepsilon > 0$, B^* is a diffeomorphism on D_ε , so that the inverse bidding strategy B^{*-1} is also smooth (on $B^*(D_\varepsilon)$).¹⁴ Then $F_{S_i|S_i \geq s^*} \circ B^{*-1}$ is smooth on D_ε as a composition of two smooth functions. In particular, it is twice continuously differentiable. Therefore, since B^* is a strictly increasing function on its domain, P

$$\begin{aligned} P[B^*(S_i) \leq b | S_i \geq s^*] &= F_{S_i|S_i \geq s^*}[B^{*-1}(b)] \\ &= cb(1 + O(b)) \end{aligned} \quad (6)$$

as $b \downarrow 0$, where $c > 0$. This proves the tail index representation with $\kappa = 1$ under CV.

Under PV, B^* has a critical point as $s = s^*$, $B^{*'}(s^*) = 0$, and the Inverse Function Theorem does not apply. However, the critical point is non-degenerate. Indeed, re-writing the differentiating equation (1) in terms of B^* and differentiating with respect to s at $s = s^*$ gives

$$[v'(s^*) - rB^{*'}(s^*)] f_{Y_1|S_1}(s^*|s^*) - rB^{*''}(s^*) F_{Y_1|S_1}(s^*|s^*) = 0,$$

which implies

$$B^{*''}(s_*) = \frac{f_{Y_1|S_1}(s^*|s^*) v'(s^*)}{r F_{Y_1|S_1}(s^*|s^*)} > 0.$$

The (one-dimensional) Morse Lemma states that if a smooth function $f : D \rightarrow \mathbb{R}$, defined on an open subset $D \subset \mathbb{R}$, has a *non-degenerate* critical point $a \in D$, then there exists an open neighborhood $D_0 \subset D$ of a , and a diffeomorphism $y : D_0 \rightarrow \mathbb{R}$ such that $\forall x \in D_0$, $f(x) = f(a) + f''(a)y(x)^2$. Since $f''(a) > 0$, $y(x)$ must be equal to $\sqrt{f(x) - f(a)}$ if $x \geq a$ and $-\sqrt{f(x) - f(a)}$ if $x < a$.¹⁵

In our case, the Morse Lemma implies that the function $\beta : D_\varepsilon \rightarrow \mathbb{R}$ defined as

$$\beta(s) = \begin{cases} +B^*(s)^{\frac{1}{2}}, & s \geq s^* \\ -B^*(s)^{\frac{1}{2}}, & s < s^* \end{cases}, \quad (7)$$

¹⁴A smooth map $f : D \rightarrow Y \subset \mathbb{R}$ defined on some interval $D \subset \mathbb{R}$ is called a *diffeomorphism* if it is one to one and onto, and the inverse map f^{-1} is smooth; see Guillemin and Pollack (1974), page 3.

¹⁵The only other possibility would be to have $y(x) = \sqrt{f(x) - f(a)}$ for all $x \in D_0$, but this mapping is not a diffeomorphism because it does not have an inverse in an open neighborhood of $x = a$

is a diffeomorphism on D_ϵ , and therefore has a smooth inverse β^{-1} . Since

$$\begin{aligned} P[B^*(S_i) \leq b | S_i \geq s^*] &= P\left[B^*(S_i)^{\frac{1}{2}} \leq b^{\frac{1}{2}} | S_i \geq s^*\right] \\ &= F_{S_i | S_i \geq s^*}\left[\beta^{-1}(b^{\frac{1}{2}})\right], \end{aligned} \quad (8)$$

and $F_{S_i | S_i \geq s^*} \circ \beta^{-1}$ is smooth as a composition of two smooth functions, we conclude that under PV,

$$P[B^*(S_i) \leq b | S_i \geq s^*] = cb^{\frac{1}{2}}(1 + O(b^{\frac{1}{2}})),$$

as $b \downarrow 0$ for some $c > 0$. *Q.E.D.*

Remark 1 *The result in Proposition 1 relies only on $f_{Y_1|S_1}(s^*|s^*) > 0$, so our Assumption 4 can be relaxed. This latter assumption is, however, indispensable. Otherwise the tail indexes under CV and PV would depend on the rate of convergence of $f_{Y_1|S_1}(s|s)$ to 0 as $s \downarrow s^*$. In particular, if $f_{Y_1|S_1}(s|s) \asymp (s - s^*)^\iota$ for $\iota = 1, 2, \dots$, then it can be shown that $\kappa = 1/(1 + \iota)$ under CV, and $\kappa = 1/(2 + \iota)$ under PV.*

Remark 2 *The second order term $O(b^\kappa)$ in (4) is non-trivial since we have not shown bids near the reserve price are governed by an exact Pareto law $G^*(b) = c \cdot b^\kappa$. We have shown, however, that the tail exhibits power-law decay $G^*(b) = b^\kappa L(b)$ for some $L(b) \rightarrow c$ as $b \downarrow 0$. This in turn has a sharp impact on the precise degree of small sample bias in the Hill (1975) estimator of κ . The bias is a function of the number of bids used to compute the estimator, and therefore bounds how many tail observations can be used to ensure asymptotic unbiasedness. See Section (3.2) for details, and see Haeusler and Teugels (1985), Goldie and Smith (1987), Hsing (1991) and Hill (2010).*

2.1 Robustness to Bidder Asymmetries

Our derivation of the tail index in the preceding section is based on Milgrom and Weber (1982)'s classic model. Although this model has been widely used in both theoretical and empirical work, it is not without limitations. The most important restriction is the assumption that bidders are ex ante symmetric, which limits the scope of application. In this section, we explore robustness of our tail-index results to bidder asymmetries.

Private Values

Most of the papers that have incorporated bidder asymmetries, have done so in a model that allows two bidder groups. Such groups arise naturally in many applications where there are institutional differences among bidders. For example, in US timber sales, the most important bidders can be classified as either mills or loggers, who are likely to have systematic cost differences. See, e.g., Campo, Perrigne, and Vuong (2003) and Athey, Levin, and Seira (2011). Also, Flambard and Perrigne (2006) consider a model where the group classification arises as a result of geographic factors. They look at auctions of snow-removal contracts on the island of Montreal, and explain why the firms on the island may have substantial cost advantages vis-a-viz firms located outside. Capturing the specifics of snow removal on the island of Montreal, they explicitly consider two groups of bidders: strong (those located on the island proper), and weak (those located outside the island).

Within the IPV framework, a relatively complete theoretical characterization of equilibrium is available: see e.g. Maskin and Riley (2000), Lebrun (1999) and Bajari (2001).¹⁶ Another recent paper that also restricts asymmetries to two bidder groups is Athey, Levin, and Seira (2011). The groups they consider are mills and loggers, who are likely to have systematic cost differences.

We consider an asymmetric IPV model with two bidder groups, 1 and 2. Bidders in each group $j \in \{1, 2\}$ are symmetric, and draw their valuations independently from distributions F_j , with common support $[\underline{v}, \bar{v}]$ and with continuously differentiable densities f_j bounded away from 0 on the support. The number of potential bidders in each group is denoted as N_j , and we assume that each group contains at least two bidders. This assumption is made to rule out pooling at the reserve; see the discussion at the end of this section.

Assumption 6 *We have $N_j \geq 2$, $j \in \{1, 2\}$.*

As before, we assume that there is a binding reserve price $r \in (\underline{v}, \bar{v})$.¹⁷ Let $B_j :$

¹⁶Shneyerov and Wong (2011) consider a model with finitely many groups of bidders and show non-parametric identification of model primitives even when the number of potential bidders in each group is unobservable.

¹⁷Our assumption that the support $[\underline{v}, \bar{v}]$ is the same for weak and strong bidders implies that the reserve price is binding for both groups.

$[r, \infty) \rightarrow \mathbb{R}_+$ be the bidding strategy of any bidder from group j . Lebrun (1999) and Maskin and Riley (2000) show that B_j is a continuous, increasing function for each $j \in \{1, 2\}$. Let $\alpha_j \equiv B_j^{-1}$ be the corresponding inverse bidding strategy for bidders in group j , satisfying $B_j(\bar{v}) = \bar{b}$ for some $\bar{b} > 0$. Similar to the previous definition, let the (observable) distribution of bids for group j be denoted as

$$G_j(b) \equiv \frac{F_j[\alpha_j(b)] - F_j(r)}{1 - F_j(r)} \quad (b \geq r). \quad (9)$$

Then the distribution of normalized bids is $G_j^*(b^*) = G_j(r(1 + b^*))$.

Our tail index derivations in this section rely on the differential equations for the inverse bidding strategies. In the IPV case, Theorem 1 of Lebrun (1999) implies that there is a *unique* Bayesian-Nash equilibrium, with differentiable inverse bidding strategies α_j that satisfy the following system of differential equations:

$$\alpha_1'(b) = \frac{F_1[\alpha_1(b)]}{(N-1)f_1[\alpha_1(b)]} \left(\frac{N_2}{\alpha_2(b) - b} - \frac{N_2 - 1}{\alpha_1(b) - b} \right), \quad (10)$$

$$\alpha_2'(b) = \frac{F_2[\alpha_2(b)]}{(N-1)f_2[\alpha_2(b)]} \left(\frac{N_1}{\alpha_1(b) - b} - \frac{N_1 - 1}{\alpha_2(b) - b} \right), \quad (11)$$

where we denoted $N \equiv N_1 + N_2$. Theorem 1 of Lebrun (1999) and the discussion afterward implies that $\alpha_1(r) = \alpha_2(r) = r$. These boundary conditions lead to a *singularity* at the reserve price: the (right) derivatives $\alpha_j'(b)$ are undefined at $b = r$. The proposition below uses this first-order characterization to resolve this singularity, showing that $\lim_{b \downarrow r} \alpha_j'(b) = +\infty$, and, further, that the tail index is equal to $1/2$ for both groups. This implies that, within the IPV framework, *the tail-index of the symmetric IPV model is robust to asymmetries*.

Proposition 2 *We have $\kappa_j = 1/2$, $j \in \{1, 2\}$.*

Common Values

We now present a model that nests the general asymmetric IPV model above, but allows for *interdependent* values. Specifically, we obtain tail indexes for a special case of this

model, the so-called *independent signal model*, that has been studied by e.g. Lizzeri and Persico (2000), Branco (1996) and Bulow and Klemperer (1996).

As before, we assume that there are two bidder groups, 1 and 2. Bidder i from group $j \in \{1, 2\}$ has a *signal* $S_{i,j}$. We assume that the signals are drawn independently, respectively from the cumulative distributions F_1 for group $j = 1$ and F_2 for group $j = 2$. Both distributions are assumed to satisfy the same properties as in the general IPV model considered above. We restrict attention to an additive model: the value $U_{i,j}$ of the object to bidder i from group $j \in \{1, 2\}$ now may also depend on the sum of the signals of other bidders:

$$U_{i,1} = S_{i,1} + \omega_1 \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{N_1} S_{j,1} + \sum_{j=1}^{N_2} S_{j,2} \right\}, \quad U_{i,2} = S_{i,2} + \omega_2 \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{N_2} S_{j,2} + \sum_{j=1}^{N_1} S_{j,1} \right\}. \quad (12)$$

The model nests the general IPV model considered above by setting $\omega_1 = \omega_2 = 0$. From now on, we assume that a common value component is present for both groups.

Assumption 7 (Common Values) *We have $\omega_j > 0$ for $j \in \{1, 2\}$.*

We also continue to make Assumption 6. We consider an equilibrium in which only bidders with signals above the cutoffs $s_j^* \in (\underline{v}, \bar{v})$ submit a bid, and employ increasing bidding strategies $B_j : [s_j^*, \bar{v}] \rightarrow [r, \bar{b}_j]$ with the cutoffs s_j^* found by equating the value of the object, conditional on all rivals staying out, to the reserve price:

$$\begin{aligned} E[U_{i,1} | S_{i,1} = s_1^*, \max_{k \neq i} S_{k,1} \leq s_1^*, \max_k S_{k,2} \leq s_2^*] &= r, \\ E[U_{i,2} | S_{i,2} = s_2^*, \max_{k \neq i} S_{k,2} \leq s_2^*, \max_k S_{k,1} \leq s_1^*] &= r. \end{aligned}$$

The existence of such an equilibrium is implied by Theorem 2.1 of Reny and Zamir (2004).¹⁸

As before, let α_j denote the inverse bidding strategy of a bidder from group $j \in \{1, 2\}$. Let $\alpha_j^* \equiv \alpha_j(r) = s_j^*$. Using (12) and the monotonicity of α_j , this condition can be

¹⁸Even though Reny and Zamir (2004) only show existence in non-decreasing strategies, in our two-group setting with $N_j \geq 2$, the strategies must be increasing. Otherwise, a bidder who is tied with a bidder from the same group with a positive probability will have an incentive to bid higher.

equivalently stated for each group as

$$w_1(\alpha_1^*, \alpha_2^*) \equiv \alpha_1^* + \omega_1(N_1 - 1)h_1(\alpha_1^*) + \omega_2 N_2 h_2(\alpha_2^*) = r, \quad (13)$$

$$w_2(\alpha_1^*, \alpha_2^*) \equiv \alpha_2^* + \omega_2(N_2 - 1)h_2(\alpha_2^*) + \omega_1 N_1 h_1(\alpha_1^*) = r, \quad (14)$$

where we denoted

$$h_j(s) \equiv E_j[S_i | S_i \leq s] = \frac{\int_{\underline{s}}^s \tilde{s} f_j(\tilde{s}) d\tilde{s}}{F_j(s)}.$$

In order to derive the tail index using the same methods as in the proof of Proposition 2, we need the inverse bidding strategies be differentiable in some neighborhood of r . While this property is not implied by Theorem 2.1 of Reny and Zamir (2004), we proceed under this assumption.¹⁹

Assumption 8 (Differentiable Inverse Strategies) *There exists $\eta > 0$ such that the inverse bidding strategies $\alpha_j(b)$ are differentiable on $(r, r + \eta)$.*

We now derive a system of differential equations for the inverse strategies $\alpha_1(b)$ and $\alpha_2(b)$. Consider a bidder from group 1 with a signal realization s_1 who bids b . Assuming all other bidders follow their prescribed equilibrium strategies, this bidder's expected profit is

$$\Pi_1(s_1, b) = \{s_1 + \omega_1(N_1 - 1)h_1(\alpha_1(b)) + \omega_1 N_2 h_2(\alpha_2(b))\} \times F_1^{N_1-1}(\alpha_1(b)) F_2^{N_2}(\alpha_2(b))$$

In parallel, a bidder from group 2 with signal realization s_2 who bids b has expected profit

$$\Pi_2(s_2, b) = \{s_2 + \omega_1 N_1 h_1(\alpha_1(b)) + \omega_1(N_2 - 1)h_2(\alpha_2(b))\} \times F_1^{N_1}(\alpha_1(b)) F_2^{N_2-1}(\alpha_2(b))$$

Tedious, but straightforward algebra shows that the first-order equilibrium conditions

$$\frac{\partial \Pi_1(\alpha_1(b), b)}{\partial b} = 0, \quad \frac{\partial \Pi_2(\alpha_2(b), b)}{\partial b} = 0$$

¹⁹Suffice it to say here that we did not encounter any problems with this in the examples we numerically solved in order to perform Monte-Carlo simulations reported in Section 4.

yield a system of differential equations for the inverse bidding strategies²⁰

$$\alpha'_1(b) = \frac{F_1[\alpha_1(b)]}{(N-1)f_1[\alpha_1(b)]} \left\{ \frac{N_2}{v_2[\alpha_1(b), \alpha_2(b)] - b} - \frac{N_2 - 1}{v_1[\alpha_1(b), \alpha_2(b)] - b} \right\}, \quad (15)$$

$$\alpha'_2(b) = \frac{F_2[\alpha_2(b)]}{(N-1)f_2[\alpha_2(b)]} \left\{ \frac{N_1}{v_1[\alpha_1(b), \alpha_2(b)] - b} - \frac{N_1 - 1}{v_2[\alpha_1(b), \alpha_2(b)] - b} \right\}, \quad (16)$$

where $b \in (r, r + \eta)$ and

$$v_j(\alpha_1, \alpha_2) \equiv w_j(\alpha_1, \alpha_2) + \omega_j[\alpha_j - h_j(\alpha_j)], \quad j \in \{1, 2\}.$$

This system is subject to the boundary conditions given by (13) and (14). Since Assumption 7 implies $h_j(s) < s$ for $j \in \{1, 2\}$, we have

$$v_j[\alpha_1(r), \alpha_2(r)] > r, \quad j \in \{1, 2\}. \quad (17)$$

In contrast to the system of equations (10) and (11) for the inverse bidding strategies in the IPV case, here there is *no* singularity at the reserve: the inverse bidding strategies have non-negative (right) derivatives at $b = r$. Moreover, as the proof of the proposition below shows, $\alpha'_j(r) > 0$ for at least one $j \in \{0, 1\}$. This leads to the following tail-index result. In an (N_1, N_2) auction, the ex ante probability that a randomly chosen bidder is from group j is N_j/N . The *pooled* distribution of (normalized) bids then defined as

$$G^*(b) \equiv \frac{N_1}{N} G_1^*(b) + \frac{N_2}{N} G_2^*(b).$$

The proposition below shows that for $G^*(b)$, the tail index $\kappa = 1$.

Proposition 3 *We have $\kappa = 1$ for the pooled distribution of normalized bids $G^*(b)$.*

Pooling at the reserve. Our Assumption 6 guarantees that there is no pooling (mass point) at the reserve. Otherwise, each bidder would be tied at the reserve with a positive probability. An epsilon deviation to bidding higher will lead to a dramatic increase in

²⁰These calculations are available from the authors on request. See Lizzeri and Persico (2000) for similar derivations in the two-bidder case, and Lebrun (1999) for the IPV case.

his probability of winning, while his value conditional on winning will be at least weakly higher.

In practice, Assumption 6 may not be very restrictive, at least in applications where there is no a priori rationale for restricting the number of bidders in one of the groups to one. Even when the numbers of potential bidders are unobservable, we can condition on the number of *actual* bidders in each group being greater or equal than 2.²¹ Our discussion in Section 3.1 implies that such conditioning will preserve the tail index.

Without Assumption 6, there may be pooling at the reserve. In fact, such pooling has been found in several interesting models. Hendricks and Porter (1988) study a model with one perfectly informed bidder (“neighbor”), and a number of uninformed bidders (“non-neighbors”). They show that, while the distribution of the non-neighbor bids is right-continuous at the reserve, the distribution of neighbor bids has a mass point at the reserve.

Under private values, Haile (2000) shows that pooling at the reserve must arise in English auctions with two bidders and resale. In the asymmetric IPV model considered above, Theorem 1 in Lebrun (1999) allows for a possibility of $\alpha_j(r) > r$, i.e. pooling (a mass point) at the reserve, which then implies that equilibrium inverse bidding strategies α_j must satisfy the boundary condition $\alpha_j(r) = r$. If one of the groups is comprised of a single bidder, the aforementioned theorem allows for a possibility of $\alpha_j(r) > r$, i.e. pooling (a mass point) at the reserve.

The tail index is not even uniquely defined in the mass point case. Our view is that the models that involve pooling at the reserve are tailored to the specific applications, e.g. drainage tract or re-sale. In such applications, the researcher may not even need a test of common values, since the relevant environment is determined by the institutional features. Be as it may, in Appendix B we show that the tail index of this distribution is still equal to 1 for the distribution of the bids *conditional* on $b > r$, i.e. upon elimination of the bids at the reserve.

²¹See also Shneyerov and Wong (2011), who make a related point in their identification results.

3 Estimation and Testing Framework

We consider heterogeneous auctions, each characterized by a reserve price r , characteristics x as well as the number of potential bidders N . The environment (PV or CV) is fixed across auctions. The vector (r, x, N) is assumed to have distribution H with support $[\underline{r}, \bar{r}] \times \mathcal{X} \times [\underline{N}, \bar{N}]$, where $\mathcal{X} \subset \mathbb{R}^d$ is compact, and $2 \leq \underline{N} < \bar{N} < \infty$.

The valuations of the bidders now take the form $u(v, s, r, x, N)$, and the information in the auction is characterized by the conditional density $f(v, s_1, \dots, s_N | r, x, N)$. We make further regularity assumptions on the primitives that mirror Assumptions 1 - 5 in the preceding section.

Assumption 9 *For any*

$$(r, x, N) \in [\underline{r}, \bar{r}] \times \mathcal{X} \times \{\underline{N}, \dots, \bar{N}\} :$$

(i) the density $f(v, s_1, \dots, s_N | r, x, N)$ is symmetric in the signals $(s_1, \dots, s_N)$ and satisfies the affiliation property; (ii) the support of $F(\cdot | r, x, N)$ is $[\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}]^N$; (iii) the distribution function $F(\cdot | r, x, N) \in C^\infty([\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}]^N)$; (iv) the valuation function $u(\cdot, r, x, N)$ is smooth in its first two arguments, $u(\cdot, r, x, N) \in C^\infty([\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}])$; (v) the value conditional on “just” winning is increasing on its domain $[\underline{s}, \bar{s}]$, and also smooth

$$v(\cdot, r, x, N) \equiv E[u(V, S_i, r, x, N) | S_i = \cdot, Y_i = \cdot, r, x, N] \in C^\infty([\underline{s}, \bar{s}]), \quad v'(\cdot, r, x, N) > 0;$$

(vi) the density $f(\cdot | r, x, N)$ is positive on $[\underline{v}, \bar{v}] \times [\underline{s}, \bar{s}]^N$.

The differential equation that characterizes the bidding strategy $B(\cdot | r, x, N)$ now takes the form

$$\begin{aligned} & [v(s, r, x, N) - B(s, r, x, N)] f_{Y_1 | S_1, r, x, N}(s | s, r, x, N) \\ & - \frac{\partial B(s, r, x, N)}{\partial s} F_{Y_1 | S_1, r, x, N}(s | s, r, x, N) = 0 \end{aligned} \tag{18}$$

subject to the boundary condition

$$B(s^*(r, x, N), r, x, N) = r.$$

In addition, we assume that the reserve price is binding for *all* auctions.

Assumption 10 (Binding Reserve Price) *For any $(r, x, N) \in [\underline{r}, \bar{r}] \times \mathcal{X} \times \{\underline{N}, \dots, \bar{N}\}$, the screening function satisfies $s^*(r, x, N) \in (\underline{s}, \bar{s})$.*

3.1 Data Generating Process (DGP)

We assume a sample of L auctions is available, and any information specific to an auction is index by $l = 1, \dots, L$. Thus, each auction is characterized by a reserve price r_l , characteristics x_l and the number of potential bidders N_l , where $2 \leq N_l \leq \bar{N}$. Also, let $n_l \in \{0, 1, \dots, N_l\}$ denote the actual number of bidders, which we discuss below. The data generating process is summarized as follows.

1. Assumption 9 and 10 apply to each auction $l = 1, \dots, L$.
2. The vectors (r_l, x_l, N_l) are drawn independently from distribution H .
3. Conditional on (r_l, x_l, N_l) ($l = 1, \dots, L$), the signals $S_{i,l}$ of potential bidders $i = 1, \dots, N_l$ and the common value components V_l are drawn independently across l from the distribution $F(\cdot | r, x, N)$.
4. Only the bidders whose signals exceed the screening level, $S_{i,l} \geq s^*(r_l, x_l, N_l)$, submit bids. The bids of active bidders are determined according to

$$b_{i,l} = B(S_{i,l}, r_l, x_l, N_l), \quad (i = 1, \dots, n_l),$$

where the bidding strategy B is found as the solution to (18).

Remark 3 *Model symmetry ensures the DGP is stationary, and given a bid is made (i.e. $n_l \geq 1$) the joint distribution of any q -tuple of bids $\{b_{i_1,l}, \dots, b_{i_q,l}\}$, $1 \leq q \leq n_l$, is exchangeable. This follows since the strategy $B(\cdot)$ does not depends on the bidder i*

or auction l , and a bid $b_{i,l}$ depends only on the pre-auction signal $S_{i,l}$ and information common to all bidders $\{r_l, x_l, N_l\}$, hence the order of bids in an active auction ($n_l \geq 1$) is irrelevant.

Denote the j^{th} potential bidder's decision to submit a bid as

$$\mathcal{I}_{j,l} := \mathcal{I}_{j,l}(S_{j,l}, r_l, x_l, N_l) = \begin{cases} 1, & S_{j,l} \geq s^*(r_l, x_l, N_l) \\ 0, & \text{otherwise} \end{cases}$$

The number of bidders n_l in auction l , and the maximum number of bids across auctions, are identically

$$n_l = \sum_{j=1}^{N_l} \mathcal{I}_{j,l}(S_{j,l}, r_l, x_l, N_l) \quad \text{and} \quad \bar{n} := \max_{1 \leq l \leq L} \{n_l\}.$$

We index *active bidders* as $i = 1, \dots, n_l$ for auctions with at least one bidder $n_l \geq 1$, where by assumption $n_l \leq \bar{n} \leq \bar{N} < \infty$. The total number of observations in the sample is $n = \sum_l n_l$.

The data available to the econometrician consists of an (independent and identically distributed across l) sample of observations

$$\left\{ \{b_{i,l}\}_{i=1, \dots, n_l}, r_l, x_l, n_l \right\}_{l=1, \dots, L}.$$

Motivated by practical considerations, we assume that only auctions with a bidder $n_l \geq 1$ are observable.

Consider the reserve-price normalized bids

$$b_{i,l}^* = \frac{b_{i,l}}{r_l} - 1. \tag{19}$$

This normalization has the advantage that the lower bound of the support of $b_{i,l}^*$ is 0 regardless of the covariates, and the tail behavior can be characterized around $b = 0$. Our Assumption 9 allows us to invoke the same arguments as in Proposition 1 to obtain the

following power-law decay property conditional on auction information: as $b \downarrow 0$,

$$P \left[b_{i,l}^* \leq b | r_l, x_l, N_l, \mathcal{I}_{i,l} = 1 \right] = C(r_l, x_l, N_l, \mathcal{I}_{i,l} = 1) \cdot b^\kappa \cdot (1 + O(b^\kappa)) \quad (20)$$

where $C(r_l, x_l, N_l, \mathcal{I}_{i,l} = 1) > 0$ *a.s.* Thus, given an auction has at least one bidder (i.e. some $\mathcal{I}_{i,l} = 1$ hence $n_l \geq 1$), and given observable auction characteristics x_l , normalized bids exhibit the *same* first and second order tail decay rates b^κ and $O(b^\kappa)$ as shown in Proposition 1. If we denote by $\sigma(w)$ the σ -field generated by a random vector w , then since

$$\sigma(r_l, x_l, n_l, \mathcal{I}_{i,l} = 1) \subseteq \sigma(r_l, x_l, N_l, \mathcal{I}_{i,l} = 1),$$

the law of iterated expectations applied to (20) implies that the distribution of $b_{i,l}^*$ conditional on any vector of observable variables z_l governed by our DGP with

$$\sigma(z_l) \subseteq \sigma(r_l, x_l, n_l)$$

also exhibits power-law decay with the same tail index κ :

$$G^*(b|z_l) \equiv P \left[b_{i,l}^* \leq b | z_l, \mathcal{I}_{i,l} = 1 \right] = c(z_l) \cdot b^\kappa \cdot (1 + O(b^\kappa)) \quad \text{as } b \downarrow 0, \quad (21)$$

where $c(z_l) := E[C(r_l, x_l, N_l) | z_l, \mathcal{I}_{i,l} = 1] > 0$ *a.s.* on the support of z_l .

Of particular importance to us will be the cases of conditioning on an auction size $z_l = n_l$, or all observable auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$, or a constant $z_l = c$ since the latter generates the trivial sigma-field corresponding to the unconditional distribution $G^*(b|c) = G^*(b)$. The latter case leads to the following important corollary that shows the invariance of the tail index with respect to auction heterogeneity. This is key to our reduced form method since we do not use any information in our test other than bids.

Corollary 1 *Under Assumption 9, the unconditional power-law (4) for $G^*(b)$ continues to hold even under auction heterogeneity.*

Binding reserve price. Our assumption that the reserve price is binding for all auctions, $s^*(r_l, x_l, N_l) \in (\underline{s}, \bar{s})$, is important for the above derivation, but can be relaxed. For

example, we could alternatively assume that, because of auction heterogeneity, the reserve price is binding some of the time.²² That is,

$$\rho \equiv P[s^*(r_l, x_l, N_l) = \underline{s}] \in (0, 1).$$

Then the marginal (normalized) bid distribution is a mixture of two distributions,

$$G^*(b) = \rho G_1(b) + (1 - \rho) G_2(b)$$

where

$$G_1(b) \equiv P[b_{i,l}^* \leq b | n_l \geq 1, s^*(r_l, x_l, N_l) = \underline{s}]$$

and

$$G_2(b) \equiv P[b_{i,l}^* \leq b | n_l \geq 1, s^*(r_l, x_l, N_l) > \underline{s}].$$

The distribution $G_1(b)$ has tail index $\kappa_1 = 1$ regardless of the environment (PV or CV), while the distribution $G_2(b)$, as before, has tail index $\kappa_2 = \kappa$ that depends on the environment, i.e. $\kappa = 1/2$ under PV and $\kappa = 1$ under CV. It is easy to see that the tail index is preserved under CV. Under PV,

$$\begin{aligned} G^*(b) &= \rho \cdot c_1 b (1 + O(b)) + (1 - \rho) \cdot c_2 b^{\frac{1}{2}} \left(1 + O\left(b^{\frac{1}{2}}\right)\right) \\ &= (1 - \rho) c_2 b^{\frac{1}{2}} \left[\frac{\rho c_1}{(1 - \rho) c_2} b^{\frac{1}{2}} (1 + O(b)) + 1 + O\left(b^{\frac{1}{2}}\right) \right] = (1 - \rho) c_2 b^{\frac{1}{2}} \left(1 + O\left(b^{\frac{1}{2}}\right)\right), \end{aligned}$$

where $c_1, c_2 > 0$. In practice, we would really just need the reserve price to be binding often enough to get the statistical power for our tests.

3.2 The Hill Estimator

In view of Corollary 1 we only require an unconditional estimator of the tail index. The Hill (1975) estimator of the tail index is based on the following fact (see Hsing (1991) and

²²Allowing for a nonbinding reserve, the screening level s^* is defined as in Milgrom and Weber (1982), equation (10) on p.1111: $s^* = \inf\{s \in [\underline{s}, \bar{s}] : w(s) \geq r\}$. With this definition, the reserve price is nonbinding if and only if $s^* = \underline{s}$.

Embrechts, Kluppleberg, and Mikosch (1997)):

$$\kappa^{-1} = E \left\{ \log q - \log b_{i,l}^* | \log b_{i,l}^* \leq \log q \right\} + O(q^\kappa) \text{ as } q \downarrow 0. \quad (22)$$

Thus, the inverse of the tail index κ^{-1} is the mean distance of the log-normalized bid $\log b_{i,l}^*$ below some low threshold $\log q$, as $q \downarrow 0$.

Equation (22) suggests a natural way of estimating κ^{-1} by a sample analog, with an appropriately chosen sequence of threshold bids $q \downarrow 0$ as the sample size goes to infinity. Namely, denote the sample of stacked normalized bids $b_{i,l}^*$ by

$$\{b_t^*\}_{t=1}^n = \left\{ b_{i,l}^* : 1 \leq i \leq n_l \right\}_{l=1}^L,$$

let $b_{(j)}^*$ be the j^{th} sample order statistic of b_t^* , $b_{(1)}^* \leq b_{(2)}^* \leq \dots \leq b_{(n)}^*$, and let m_n be the number of bids near the reserve price in consideration, a function of n . Since auction sizes $\{n_l\}_{l=1}^L$ are random variables, the sample size $n = \sum_l n_l$ and therefore the number of bids near the reserve price m_n are also random. See Section 3.6 for a discussion on the selection of m_n in practice.

Assume $\{m_n\} = \{m_n\}_{n \geq 1}$ is an intermediate order sequence: $1 \leq m_n < n$, $m_n \xrightarrow{a.s.} \infty$, and $m_n/n \xrightarrow{a.s.} 0$ as $L \rightarrow \infty$; cf. Leadbetter, Lindgren, and Rootzen (1983). Then the sequence of threshold bids is chosen as $b_{(m_n+1)}^*$ as in Hall (1982), and the Hill (1975) estimator of κ^{-1} is simply the sample average log-exceedance²³

$$\hat{\kappa}_{m_n}^{-1} = \frac{1}{m_n} \sum_{t=1}^n \left(\log b_{(m_n+1)}^* - \log b_t^* \right)_+ = \frac{1}{m_n} \sum_{i=1}^{m_n} \left(\log b_{(m_n+1)}^* - \log b_{(i)}^* \right). \quad (23)$$

Other estimators exist but none have been shown to be as robust to unknown forms of dependence and heterogeneity. See the literature reviews in Hsing (1991) and Hill (2009, 2010, 2011).

In the case of i.i.d. data with power-law tail decay that satisfies a second order power expansion, and deterministic sample size n , Hall (1982) shows $\hat{\kappa}_{m_n}^{-1}$ is consistent for κ^{-1} and asymptotically normal with the asymptotic variance equal to κ^{-2} . However, our

²³In the following we write $(z)_+$ to denote $\max\{z, 0\}$.

setting is not i.i.d. because bids may be correlated within an auction. Even though the literature contains several results for the non i.i.d. case, these results are for time-series data, and they do not transfer immediately to our setting of an unbalanced panel with stochastic auction sizes (e.g. Resnick and Starica (1998), Hsing (1991), Hill (2010)).

As a first step toward obtaining the limit distribution of $\hat{\kappa}_{m_n}^{-1}$ we need to characterize the intermediate order statistic $b_{(m_n+1)}^*$ in terms of distribution tail properties. Since $b_{(m_n+1)}^*$ is measurable with respect to the sigma-field generated by all bids and auction sizes, it is convenient to denote the sigma-field induced by *observable* auction sizes n_l represented by a sample realization of bids $\{b_t^*\}_{t=1}^n = \{b_{i,l}^* : 1 \leq i \leq n_l\}_{l=1}^L$:

$$F_L := \sigma \left(\{n_l\}_{l=1}^L : n_l \geq 1 \right).$$

We now use F_L to condition on the event of observable auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$, a convenient intermediary step toward unconditional asymptotics. By (20) we may write $P(b_t^* < b | F_L)$, the probability an arbitrary bid b_t^* is below a threshold b , given the bid exists in some active auction in a sample of active auctions $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$. Thus, another contribution of Proposition 1 and therefore the arguments leading to (20) and (21), is they ensure $P(b_t^* < b | F_L)$ is well defined, F_L -measurable, and exhibits power-law decay with the same index $\kappa > 0$,

$$G^*(b | F_L) := P(b_t^* < b | F_L) = c_L \times b^\kappa (1 + O(b^\kappa)), \kappa > 0, \text{ as } b \downarrow 0, \quad (24)$$

where c_L is F_L -measurable and positive with probability one.

Now let $\{q_n\}$ denote any sequence of positive random numbers that satisfies

$$\lim_{L \rightarrow \infty} P \left(\left| \frac{n}{m_n} G^*(q_n | F_L) - 1 \right| > \varepsilon | F_L \right) = 0 \quad \forall \varepsilon > 0 \text{ on } F_L - a.e.,$$

which we write compactly as

$$\frac{n}{m_n} G^*(q_n | F_L) \xrightarrow{p} 1 \text{ on } F_L - a.e., \quad (25)$$

and *a.e.* denotes *almost everywhere*. Note that “on $F_L - a.e.$ ” implies convergence is in probability, *almost surely* with respect to an asymptotic draw of observed auction sizes $\{\{n_l\}_{l=1}^\infty : n_l \geq 1\}$.²⁴ By (24) we can always choose $q_n = (m_n/n)^{1/\kappa} c_L^{-1/\kappa}$, but any sequence $\{q_n\}$ that satisfies $q_n = (m_n/n)^{1/\kappa} c_L^{-1/\kappa} (1 + o_p(1))$ is also valid.

By construction $q_n \xrightarrow{p} 0$ follows from $\{m_n\}$ being an intermediate order sequence hence $m_n/n \xrightarrow{a.s.} 0$, and since $b_{(m_n+1)}^*$ is the sample m_n/n^{th} quantile for a given sequence of observed auction sizes $\{\{n_l\}_{l=1}^\infty : n_l \geq 1\}$, it follows $b_{(m_n+1)}^*$ estimates q_n . Notice the bid threshold q_n is random because it is a function of random auction sizes $n = \sum_{l=1}^L n_l$.

The Hill (1975) estimator $\hat{\kappa}_{m_n}^{-1}$ for panels of stochastic sample size still approximates the mean distance of the log-normalized bid $\log b_{i,l}^*$ below a low threshold, a la (22). See Appendix A for omitted proofs.

Lemma 4 *Under the bid DGP*

$$\kappa^{-1} = \lim_{L \rightarrow \infty} E \left\{ \frac{n}{m_n} (\log q_n - \log b_t^*)_+ \right\}.$$

Conditioned on observable auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$, bids $b_{i,l}^*$ exhibit only finite dependence due to bounded auction sizes and cross-auction independence, so asymptotic normality follows straightforwardly. Since *unconditional* asymptotic normality is then trivial, we dedicate the majority of the following discussion to framing how arguments developed in Hsing (1991), Hill (2009) and Hill (2010) lead to normality when we condition on auction sizes.

Define conditionally mean centered bid tail log-exceedances and events

$$U_{m_n,t}^{(L)} := (\log(q_n/b_t^*))_+ - E \left[(\log(q_n/b_t^*))_+ | F_L \right]$$

$$I_{m_n,t}^{(L)}(u) := I(b_t^* < q_n e^{-u}) - G^*(q_n e^{-u} | F_L), \quad u \geq 0,$$

²⁴See Leadbetter et al (1983: Theorem 1.7.13) for classic treatments of intermediate order sequences. We impose (25) to ensure the stochastic sequences $\{m_n\}$ and $\{q_n\}$ have the same asymptotic relationship *irrespective* of the draw of observable auction sizes asymptotically $\{\{n_l\}_{l=1}^\infty : n_l \geq 1\}$. By dominated convergence (25) implies unconditional probability convergence $\lim_{L \rightarrow \infty} P(|(n/m_n)G^*(q_n|F_L) - 1| > \varepsilon) \rightarrow 0 \forall \varepsilon > 0$. Technically the latter limit also holds on $F_L - a.e.$ because $(n/m_n)G^*(q_n|F_L)$ is F_L -measurable, and convergence is in probability to the constant 1 for any draw $\{\{n_l\}_{l=1}^\infty : n_l \geq 1\}$. We simply use $\xrightarrow{p}$ without writing “on $F_L - a.e.$ ” to denote unconditional convergence, e.g. $(n/m_n)G^*(q_n|F_L) \xrightarrow{p} 1$, to avoid confusion with conditional probability convergence (25).

construct a conditionally mean-zero tail array

$$\left\{ T_{m_n,t}^{(L)}(u) : 1 \leq t \leq n \right\}_{n \geq 1} = \left\{ T_{m_n,t}^{(L)}(u) \right\} := \left\{ U_{m_n,t}^{(L)} - \kappa^{-1} I_{m_n,t}^{(L)}(u) \right\},$$

and compactly write

$$I_{m_n,t}^{(L)} := I_{m_n,t}^{(L)} \left(u/m_n^{1/2} \right) \quad \text{and} \quad T_{m_n,t}^{(L)} := T_{m_n,t}^{(L)} \left(u/m_n^{1/2} \right).$$

As usual we write interchangeably the stacked array $\{U_{m_n,t}^{(L)}, I_{m_n,t}^{(L)}, T_{m_n,t}^{(L)}\}$ or the auction/bid array $\{U_{m_n,i,l}^{(L)}, I_{m_n,i,l}^{(L)}, T_{m_n,i,l}^{(L)}\}$.

The array $\{T_{m_n,t}^{(L)}\}$ plays an elegant role in Hill (1975) estimator asymptotics. In the case of non-random n and for a broad class of second order regularly varying tails that encompasses (24), $\hat{\kappa}_{m_n}^{-1}$ expands into a sample mean of $\{T_{m_n,t}^{(L)}\}$ (Hsing (1991), Hill (2009, 2010)), and $\{T_{m_n,t}^{(L)}\}$ satisfies a Gaussian central limit theorem for a large class of dependent and heterogeneous processes (Hill (2009, 2010, 2011)). We summarize the argument below.

In the following we operate conditionally on F_L . Define $X_n := m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})$ and $Y_n := 1/m_n^{1/2} \sum_{t=1}^n T_{m_n,t}^{(L)}$. Given power law tail decay (24) and finite dependence, as long as there are sufficiently many bids drawn close to the reserve price asymptotically in the sense $m_n \xrightarrow{a.s.} \infty$ and $m_n/n^{2/3} \xrightarrow{a.s.} 0$, then $|X_n - Y_n| \xrightarrow{P} 0$ by Lemma A.7 in Hill (2009)²⁵, where the limit holds *almost surely* with respect to F_L . We may therefore write

$$m_n^{1/2} \left(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1} \right) = \frac{1}{m_n^{1/2}} \sum_{t=1}^n T_{m_n,t}^{(L)} + o_p(1), \text{ almost surely with respect to } F_L. \quad (26)$$

The next step is to characterize the distribution limit of $1/m_n^{1/2} \sum_{t=1}^n T_{m_n,t}^{(L)}$. Define its conditional variance:

$$\sigma_{m_n|L}^2 := E \left[\left(\frac{1}{m_n^{1/2}} \sum_{t=1}^n T_{m_n,t}^{(L)} \right)^2 \middle| F_L \right].$$

Power law tail decay and finite dependence ensure Corollary 3.3 of Hill (2009) applies to

²⁵See Hsing (1991) for seminal theory contributions along these lines.

deduce a Gaussian limit,

$$\frac{1}{m_n^{1/2} \sigma_{m_n|L}} \sum_{t=1}^n T_{m_n,t}^{(L)} \xrightarrow{d} N(0,1) \text{ almost surely with respect to } F_L. \quad (27)$$

Further, in Section 3.3 and in Appendix A.1 we show conditionally $\sigma_{m_n|L}^2 \xrightarrow{p} \kappa^{-2}$ *almost surely* with respect to F_L . That is, even under bid dependence the classic asymptotic result for i.i.d. data holds. See especially Section 3.3 for discussion. In the present setting the important finding is non-degeneracy: $\liminf_{L \rightarrow \infty} \sigma_{m_n|L}^2 > 0$ with probability one with respect to F_L . In view of (26) it therefore follows if $m_n \xrightarrow{a.s.} \infty$ and $m_n/n^{2/3} \xrightarrow{a.s.} 0$ then

$$\frac{m_n^{1/2}}{\sigma_{m_n|L}} (\hat{\kappa}_{m_n}^{-1} - \kappa^{-1}) = \frac{1}{m_n^{1/2} \sigma_{m_n|L}} \sum_{t=1}^n T_{m_n,t}^{(L)} + o_p(1), \quad (28)$$

where the $o_p(1)$ term converges to zero in probability *almost surely* with respect to F_L . By Theorem 3.1 in Billingsley (1999) it therefore follows $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\sigma_{m_n|L}$ has the same conditional distribution limit as $m_n^{-1/2} \sigma_{m_n|L}^{-1} \sum_{t=1}^n T_{m_n,t}^{(L)}$. Now combine (27) and (28) to deduce as long as $m_n \xrightarrow{a.s.} \infty$ and $m_n/n^{2/3} \xrightarrow{a.s.} 0$, that $\hat{\kappa}_{m_n}^{-1}$ is asymptotically normal

$$\frac{m_n^{1/2}}{\sigma_{m_n|L}} (\hat{\kappa}_{m_n}^{-1} - \kappa^{-1}) \xrightarrow{d} N(0,1) \text{ almost surely with respect to } F_L. \quad (29)$$

Limit (29) is an extension of Theorem 3 in Hill (2010) to imbalanced panels with known sample sizes. Since the limiting standard normal distribution in (29) is the same for any asymptotic draw $\{\{n_l\}_{l=1}^\infty : n_l \geq 1\}$, apply Lebesgue's dominated convergence theorem to prove $\hat{\kappa}_{m_n}^{-1}$ is unconditionally asymptotically normal.

Theorem 5 *Under the bid DGP, for any $\sigma(\cup_{L \geq 1} F_L)$ -measurable intermediate order sequence $\{m_n\}$, $m_n \xrightarrow{a.s.} \infty$ and $m_n/n^{2/3} \xrightarrow{a.s.} 0$, unconditionally $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\sigma_{m_n|L} \xrightarrow{d} N(0,1)$.*

Remark 4 *Define $v_{m_n|L}^2 := E[m_n(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})^2 | F_L]$, the mean-squared-error of $\hat{\kappa}_{m_n}^{-1}$ conditioned on a random draw of auction sizes.²⁶ Although (29) holds with scale $\sigma_{m_n|L}$*

²⁶Note that in general $v_{m_n|L}^2$ is not the conditional variance of $\hat{\kappa}_{m_n}^{-1}$ since the Hill (1975) estimator is biased (e.g. Hall (1982), Embrechts, Kluppleberg, and Mikosch (1997), Segers (2002)).

and implies convergence to a law with unit variance, this does not suffice to deduce $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/v_{m_n|L} \xrightarrow{d} N(0, 1)$. This is irrelevant for practical purposes since $\sigma_{m_n|L}^2$ is easily estimated. See Section 3.3.

Hill (1975)'s estimator is perfectly designed for a distribution with an exact Pareto tail, e.g. $G^*(b|F_L) = c_L \cdot b^\kappa \forall b \leq \bar{b}$ and some $\bar{b} > 0$, in which case the bias term $O(q^\kappa)$ in (22) is identically zero. Otherwise, power-law tails with a second order regular variation property like (24) imply a mean log-exceedance with bias like (22). The bias identically arises in Hill's (1975) estimator $E[\hat{\kappa}_{m_n}^{-1}] \neq \kappa^{-1}$ since $\hat{\kappa}_{m_n}^{-1}$ is nothing more than a sample conjugate of the expectation in (22). Since the second order term vanishes as $b \downarrow 0$, the bias can be corrected asymptotically as long as bids used to compute κ^{-1} are obtained sufficiently near the reserve price, while maintaining $m_n \xrightarrow{a.s.} \infty$ for asymptotic normality. This translates to setting $m_n \xrightarrow{a.s.} \infty$ sufficiently *slowly*. In order for $\hat{\kappa}_{m_n}^{-1}$ to have zero asymptotic bias for the power-law class $G^*(b|F_L) = c_L \cdot b^\kappa(1 + O(b^\theta))$, $\theta \geq 0$, as $b \downarrow 0$ when n is non-random it is well known $m_n = o(n^{2\theta/(2\theta+\kappa)})$ suffices. See Hall (1982) and Haeusler and Teugels (1985). Since by Proposition 1 we have $\theta = \kappa$, and n is random by our DGP, we must have $m_n/n^{2/3} \xrightarrow{a.s.} 0$.

Even though the unconditional Hill estimator of κ is consistent, an estimator obtained by first conditioning on observable auction characteristics, and then averaging the conditional estimators, could be more efficient. However, the normalization of bids (19) likely removes the effect of covariates on bids, at least to some extent.²⁷ For example, if we assume a multiplicative structure $V_l = a(x_l)\tilde{V}_l$, $S_{i,l} = a(x_l)\tilde{S}_{i,l}$ and $r_l = a(x_l)\tilde{r}_l$, where $(\tilde{V}_l, \tilde{S}_{i,l}, \tilde{r}_l)$ and x_l are independent, then $b_{i,l} = a(x_l)\tilde{b}_{i,l}$, where $\tilde{b}_{i,l}$ and x_l are also independent.²⁸ This implies that $b_{i,l}^* = (b_{i,l}/r_l) - 1$ and x_l are likewise independent, and conditioning on x_l is not necessary. However, even in this specification, the dependence between $b_{i,l}^*$ and n_l generally remains, so in principle one could benefit from conditioning on n_l . But our Monte-Carlo simulations have shown that in samples of typical size, even though the reduction in the variance of the estimator can be substantial, it is potentially

²⁷See Roberts (2008) who proposes a general argument.

²⁸A variant of this specification was also considered in Krasnokutskaya (2011).

overwhelmed by the small sample bias.²⁹

3.3 Variance Characterization for CV and PV Auctions

As a consequence of Theorem 5 we require an estimate of $\sigma_{m_n|L}^2$ for our test of CV or PV. The first task is to decompose $\sigma_{m_n|L}^2$ into manageable components. In Appendix A we show cross-auction independence implies the decomposition

$$\sigma_{m_n|L}^2 = \kappa^{-2} + \frac{n}{m_n} E \left[T_{m_n,1,l}^{(L)} \times T_{m_n,2,l}^{(L)} | F_L \right] \times \left\{ \frac{\sum_{l=1}^L n_l (n_l - 1)}{\sum_{l=1}^L n_l} \right\} + o_p(1), \quad (30)$$

where $o_p(1)$ reflects convergence to zero in probability as $L \rightarrow \infty$. There are three properties to take note. First, $\sum_{l=1}^L n_l (n_l - 1) / \sum_{l=1}^L n_l \in (0, \infty)$ *a.s.* as $L \rightarrow \infty$ since under our DGP auction sizes are finite $n_l \leq \bar{N}$ and infinitely many auctions have more than one bidder $n_l > 1$. Second, symmetry and exchangeability imply dependence across any two bidders is the same, so we need only look at bidders 1 and 2. Therefore, third for our DGP $\sigma_{m_n|L}^2$ asymptotically reduces to the i.i.d. case κ^{-2} plus a correction term that captures total within-auction bid dependence across different bidders.

Under our DGP, however, $E[T_{m_n,1,l}^{(L)} T_{m_n,2,l}^{(L)} | F_L] = O_p((m_n/n)^2) = o_p(m_n/n)$ hence from (30) we obtain $\sigma_{m_n|L}^2 \xrightarrow{p} \kappa^{-2}$, the classic result for i.i.d. data (cf. Hall (1982)). The property $E[T_{m_n,1,l}^{(L)} T_{m_n,2,l}^{(L)} | F_L] = O_p((m_n/n)^2)$ follows from the fact that signal distribution smoothness Assumption 9(iii) and boundedness from below Assumption 9(vi) imply the joint bid density is bounded by the marginals, while tail variables $(\log(q_n/b_t^*))_+$ and $I(b_t^* < q_n e^{-u})$ have conditional means of order $O_p(m_n/n)$.³⁰ Thus, although bids may be dependent, and dependent near the reserve price in the sense $E[T_{m_n,1,l}^{(L)} T_{m_n,2,l}^{(L)} | F_L] \neq 0$ with positive probability, dependence is negligible enough relative to the size of the vanishing bid tail portion $m_n/n \xrightarrow{a.s.} 0$ that it does not impact Hill (1975) estimator asymptotics. See Appendix A for a proof, and see Hill and Shneyerov (2010) for a deeper treatment on interpreting $E[T_{m_n,1,l}^{(L)} T_{m_n,2,l}^{(L)} | F_L]$. We summarize the above conclusion with the following result.

²⁹The simulations are available on request.

³⁰We thank a referee for pointing out the joint density may be bounded by the marginal product under our DGP, and suggesting this may lead to the class asymptotic variance κ^{-2} .

Theorem 6 *Under the bid DGP for either CV or PV bidding strategies $\sigma_{m_n|L}^2 \xrightarrow{p} \kappa^{-2}$. Therefore $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\hat{\kappa}_{m_n}^{-1} \xrightarrow{d} N(0, 1)$.*

The outcome that a nonparametric estimator with dependent data has an asymptotic variance equivalent to the i.i.d. case is certainly not unique to tail index estimation for auction bids. Hill (2011a), for example, shows $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\kappa^{-1} \xrightarrow{d} N(0, 1)$ for a stochastic volatility process that is dependent in general, and tail dependent, yet exhibits tail array uncorrelatedness similar to the present case. In general the issue is well known in the nonparametric econometrics literature. Consider Rosenblatt (1971) and Robinson (1983) for classic treatments, and see Section 4 in Ichimura and Todd (2007) for further details and references.

Nevertheless, while ignoring bid dependence is legitimate under the first order asymptotics that justify $\hat{\kappa}_{m_n}^{-2}$ as a variance estimator, Monte Carlo evidence in Hill and Shneyerov (2010) suggests that taking dependence into account is important in finite samples.³¹ In the following subsection we present a consistent non-parametric estimator of $\sigma_{m_n|L}^2$ due to Hill (2010) that leads to sharp inference in practice.

3.4 Non-Parametric Variance Estimation

By exploiting the construction of q_n in (25) and the property $E[(\ln(q_n/b_t^*))_+ | F_L] = (m_n/n)\kappa^{-1} \times (1 + o_p(1))$ verified in the proof of Lemma 8 in Appendix A.2, a natural estimator of $\sigma_{m_n|L}^2$ appears to be $1/m_n \sum_{s,t=1}^n \hat{T}_{m_n,s}^{(L)} \hat{T}_{m_n,t}^{(L)}$, where $\hat{T}_{m_n,t}^{(L)} = \hat{U}_{m_n,t}^{(L)} - \hat{\kappa}_{m_n}^{-1} \hat{I}_{m_n,t}^{(L)}$ with $\hat{U}_{m_n,t}^{(L)} = (\log(b_{(m_n+1)}^*/b_t^*))_+ - (m_n/n)\hat{\kappa}_{m_n}^{-1}$ and $\hat{I}_{m_n,t}^{(L)} = I(b_t^* < b_{(m_n+1)}^*) - m_n/n$. There are two issues to take note here. First, it is easy to verify that $1/m_n \sum_{s,t=1}^n \hat{T}_{m_n,s}^{(L)} \hat{T}_{m_n,t}^{(L)}$ reduces to $1/m_n \sum_{s,t=1}^n \hat{U}_{m_n,s}^{(L)} \hat{U}_{m_n,t}^{(L)}$ by exploiting the indicator $I(b_t^* < b_{(m_n+1)}^*)$, the two appearances of $\hat{\kappa}_{m_n}^{-1}(m_n/n)$ in $\hat{T}_{m_n,t}^{(L)}$, and the definition $\hat{\kappa}_{m_n}^{-1} = 1/m_n \sum_{t=1}^n (\log(b_{(m_n+1)}^*/b_t^*))_+ = 1/m_n \sum_{i=1}^{m_n} \log(b_{(m_n+1)}^*/b_{(i)}^*)$. Second, $1/m_n \sum_{s,t=1}^n \hat{U}_{m_n,s}^{(L)} \hat{U}_{m_n,t}^{(L)}$ is not guaranteed to be positive (Newey and West (1987)).

In view of the latter problem, Hill (2010) exploits a now classic nonparametric ap-

³¹CV bids are positively associated due to affiliation, even when there is no unobserved heterogeneity. In practice, unobserved heterogeneity is likely to make bids positively associated within the auction even under PV. See Krasnokutskaya (2011).

proach for estimating the variance $\sigma_{m_n|L}^2$ when n is non-random. In the present setting this instantly extends to $\sigma_{m_n|L}^2$ by kernel weighting cross-products $\hat{U}_{m_n,s}^{(L)}\hat{U}_{m_n,t}^{(L)}$ (cf. Newey and West (1987), Andrews (1991), de Jong and Davidson (2000)):

$$\hat{\sigma}_{m_n|L}^2 := \frac{1}{m_n} \sum_{s,t=1}^n k((s-t)/\gamma_n) \hat{U}_{m_n,s}^{(L)} \hat{U}_{m_n,t}^{(L)}, \quad (31)$$

where k denotes a kernel function, $k(0) = 1$, with bandwidth γ_n . The kernel $k((s-t)/\gamma_n)$ asymptotically negligibly weighs or trims cross-products $\hat{U}_{m_n,s}^{(L)}\hat{U}_{m_n,t}^{(L)}$ at large displacements $|s-t|$ so that $\hat{\sigma}_{m_n|L}^2 > 0$ a.s. for all $n \geq 1$, while ostensibly retaining consistency (Newey and West (1987)). As a bonus, a fully nonparametric approach allows the analyst to have only a vague idea about cross auction dependence and heterogeneity.

A large class of kernel functions $k(z)$ can be considered in the following arguments, including Parzen, Tukey-Hanning, and Quadratic-Spectral (cf. Hill (2010)), but by far the Bartlett kernel $k(z) = (1 - |z|)_+$ is the most popular in the economics literature. See de Jong and Davidson (2000) for references. Since the construction $(\log(b_{(m_n+1)}^*/b_t^*))_+$ implies at most m_n^2 tail bids enters into $\hat{\sigma}_{m_n|L}^2$, and Bartlett $k((s-t)/\gamma_n)$ trims a number of $\hat{U}_{m_n,s}^{(L)}\hat{U}_{m_n,t}^{(L)}$ that is gauged by bandwidth γ_n , some care for choosing γ_n must be taken. Further, if we take γ_n to be a function of n then bandwidth is a random variable.

Theorem 7 *Let $k(z) = (1 - |z|)_+$, $\gamma_n \xrightarrow{a.s.} \infty$, $(n^{1/2}/m_n)\gamma_n \xrightarrow{a.s.} 0$, $m_n/n \xrightarrow{a.s.} 0$ and $m_n/n^{1/2} \xrightarrow{a.s.} \infty$. Under the bid DGP, $\hat{\sigma}_{m_n|L}^2 > 0$ a.s. and $\hat{\sigma}_{m_n|L}^2/\sigma_{m_n|L}^2 \xrightarrow{P} 1$.*

Remark 5 *The bounds on the fractile m_n and bandwidth γ_n together imply $\gamma_n/n^{1/2} \xrightarrow{a.s.} 0$. We require sufficiently many tail observations $m_n/n^{1/2} \xrightarrow{a.s.} \infty$ and sufficiently few cross-products $\gamma_n/n^{1/2} \xrightarrow{a.s.} 0$ to ensure the estimators $b_{(m_n+1)}^*$ and $\hat{\kappa}_{m_n}^{-1}$ that occur in every $\hat{U}_{m_n,t}$ do not affect the limit of $\hat{\sigma}_{m_n|L}^2$. Intuitively, a larger m_n for each n implies with increased probability less error $b_{(m_n+1)}^*/q_n \approx 1$ and $\hat{\kappa}_{m_n}^{-1} \approx \kappa^{-1}$, and smaller γ_n for each n limits the amassed error over all included $\hat{U}_{m_n,s}^{(L)}\hat{U}_{m_n,t}^{(L)}$. In many regression contexts the former is not an option since plug-ins typically have a fixed rate of convergence.³² Here we have the*

³²Estimation of tail parameters, and population parameters by tail-trimming, may both involve variable rates of convergence. In both cases selecting a tail fractile sequence $\{m_n\}$ is synonymous with selecting a rate of convergence. See Hill (2011b) and his references.

option to manipulate the $m_n^{1/2}$ -rate of convergence of $b_{(m_n+1)}^*$ and $\hat{\kappa}_{m_n}^{-1}$ simply by choice of $\{m_n\}$. In practice $m_n/n^{1/2} \xrightarrow{a.s.} \infty$ and $(n^{1/2}/m_n)\gamma_n \xrightarrow{a.s.} 0$ are easily satisfied. Consider, for example, a fractile $m_n = \lfloor \lambda n^{1/2+\delta} \rfloor$ for any $\lambda \in (0, 1]$ and $\delta \in (0, 1/2)$, and a bandwidth $\gamma_n = \lfloor n^\delta / \ln(n) \rfloor$.

By Theorem 7 $\hat{\sigma}_{m_n|L}^2$ consistently estimates the Hill (1975) estimator scale $\sigma_{m_n|L}^2$, but by simplicification (31) $\hat{\sigma}_{m_n|L}^2$ also estimates the small sample mean-squared-error $v_{m_n|L}^2 = E[m_n(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})^2 | F_L]$. This follows from an easily verified expansion $v_{m_n|L}^2 = 1/m_n \sum_{s,t=1}^n E[U_{m_n,s} U_{m_n,t} | F_L]$ where $U_{m_n,t} := (\log(b_{(m_n+1)}^*/b_t^*))_+ - (m_n/n)\kappa^{-1}$. Clearly $\hat{U}_{m_n,t}^{(L)} = (\log(b_{(m_n+1)}^*/b_t^*))_+ - (m_n/n)\hat{\kappa}_{m_n}^{-1}$ estimates $U_{m_n,t}$. Thus, although we do not prove $v_{m_n|L}^2/\sigma_{m_n|L}^2 \xrightarrow{P} 1$ it is nevertheless true that $\hat{\sigma}_{m_n|L}^2$ directly approximates the small sample dispersion of $\hat{\kappa}_{m_n}^{-1}$ by estimating $v_{m_n|L}^2$. Combined with $\hat{\sigma}_{m_n|L}^2 > 0$ a.s. for any sample of bids $\{b_{i,l}^* : 1 \leq i \leq n_l\}_{l=1}^L$, these are the most promising properties of $\hat{\sigma}_{m_n|L}^2$. Since $\hat{\sigma}_{m_n|L}^2$ directly approximates the second moment of $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})$, it follows the ratio $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\hat{\sigma}_{m_n|L}$ is close to a law with zero mean and unit variance. Further, even for moderately sized samples of auction bids $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})$ is close to normally distributed, so the ratio $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\hat{\sigma}_{m_n|L}$ is close to a $N(0, 1)$ distribution. This is fundamentally important for our test since it is based on a t-ratio with critical values taken from the asymptotic $N(0, 1)$ distribution (see below). By contrast, although $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})/\hat{\kappa}_{m_n}^{-1}$ is valid by Theorem 6, it leads to non-negligible over-rejection of CV or PV hypotheses since $\hat{\kappa}_{m_n}^{-2}$ does not well approximate the mean-squared-error of $\hat{\kappa}_{m_n}^{-1}$. See Section 4 and see Hill and Shneyerov (2010).

3.5 Consistent Test of PV against CV

Consider a test of PV against CV for the sake of discussion, and define the t -ratio

$$\tau_{m_n} = m_n^{1/2} (\hat{\kappa}_{m_n}^{-1} - 2) / s_L$$

for any consistent estimator s_L^2 of $\sigma_{m_n|L}^2$ (e.g. $\hat{\kappa}_{m_n}^{-2}$ or $\hat{\sigma}_{m_n|L}^2$). Let Z_θ be the upper θ -quantile of a standard normal distribution: $P(Z > Z_\theta) = \theta$ where Z is distributed $N(0, 1)$. An asymptotic test at nominal significance level $\theta \in [0, 1]$ reduces to rejecting H_0

: $\kappa = 1/2$ in favor of $H_1 : \kappa > 1/2$ when $\tau_{m_n} \leq -Z_\theta$ since by Theorem 5 and Cramér's theorem under PV the ratio satisfies $\tau_{m_n} \xrightarrow{d} N(0,1)$. Conversely, under CV ($\kappa = 1$) it follows $|\tau_{m_n}| \xrightarrow{p} \infty$ hence τ_{m_n} promotes a consistent test.³³

3.6 Fractile Selection

In the literature the only theory offered for selecting an optimal fractile $m_n \in \{1, 2, \dots, n-1\}$ for any finite sample is for i.i.d. data with a non-random sample size n . Examples include minimizing the unconditional mean-squared-error $E[m_n(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1})^2]$ (e.g. Huisman, Koedijk, Kool, and Palm (2001)), working with variants of Hill-plots, like plots of $\hat{\kappa}_{[n^\lambda]}^{-1}$ over λ (Resnick and Starica (1997)), or measuring the occupation time of $\hat{\kappa}_{m_n}^{-1}$ in a neighborhood of κ^{-1} (e.g. Drees, de Haan, and Resnick (2000)).

It is unknown if any of the above methods are appropriate for panels with a random sample size and finite dependent data since a supporting theory is not available. Consider a general class of cadlag fractile functions $m_n(\lambda)$, e.g. $m_n(\lambda) = \lfloor \lambda n / \ln(n) \rfloor$ or $m_n(\lambda) = \lfloor n^\lambda \rfloor$, where λ is restricted to some compact subset $\Lambda \subset [0, 1]$. In principle we may minimize an estimator of $\sigma_{m_n(\lambda)|L}^2$ like $\hat{\kappa}_{m_n(\lambda)}^{-2}$ or $\hat{\sigma}_{m_n(\lambda)|L}^2$ over Λ , but $\hat{\kappa}_{m_n(\lambda)}^{-2}$ is a poor approximation of $\sigma_{m_n(\lambda)|L}^2$ for auction bids in small samples, uniform consistency of $\hat{\sigma}_{m_n(\lambda)|L}^2$ is unknown, and of course Λ must somehow be chosen.^{34,35}

In our simulation study and timber auction application we therefore must resort to conventional Hill-plots of $\hat{\kappa}_{m_n}^{-1}$ over m_n discussed in Embrechts, Kluppelberg, and Mikosch (1997), Drees, de Haan, and Resnick (2000) and elsewhere. If the data are truly generated by an exact i.i.d. Pareto law then $\hat{\kappa}_{m_n}^{-1}$ will hover fairly close to κ^{-1} over a broad range

³³By an extension of the Neyman-Pearson Lemma and Theorem 5 our test is Asymptotically Most Powerful within the class of statistics based on the estimator $\hat{\kappa}_{m_n}^{-1}$, and obtains an asymptotic power of one against *any* arbitrary global deviation from the null (e.g. $H_0 : \kappa = 1/2$ against $H_1 : \kappa \neq 1/2$). Whether a different class of tests exists and obtains more small sample power than ours is not considered here.

³⁴In the case of non-random sample sizes, Hill (2010) proves for dependent heterogeneous data uniform consistency $\sup_{\lambda \in \Lambda} |\hat{\kappa}_{m_n(\lambda)}^{-1} - \kappa^{-1}| \xrightarrow{p} 0$ for a class of cadlag fractiles $m_n(\lambda)$, e.g. $m_n(\lambda) = \lfloor \lambda n / \ln(n) \rfloor$, but only proves consistency of $\hat{\sigma}_{m_n|L}^2$. Similarly Hill (2009) derives the Gaussian weak limit of $\{m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1}) : \lambda \in [0, 1]\}$ on $\mathcal{D}[0, 1]$ for dependent and heterogeneous data, where $\hat{\kappa}_{m_n}^{-1}(\lambda) := 1/m_n \sum_{t=1}^{\lfloor \lambda n \rfloor} (\log b_{(\lfloor \lambda m_n \rfloor + 1)}^* - \log b_t^*)_+$. Uniform consistency of $\hat{\sigma}_{m_n(\lambda)|L}^2$, however, requires a stochastic equicontinuity condition (e.g. Newey (1991)) that appears in reach but has not been shown.

³⁵In simulations not reported here a fractile $\hat{m}_n := m_n(\hat{\lambda}_n)$ computed with $\hat{\lambda}_n := \operatorname{argmin}_{\lambda \in \Lambda} \{\hat{\kappa}_{m_n(\lambda)}^{-2}\}$ or $\operatorname{argmin}_{\lambda \in \Lambda} \{\hat{\sigma}_{m_n(\lambda)|L}^2\}$ does not work well for auction bids. Inference in the form of t-tests based on $\hat{m}_n^{1/2}(\hat{\kappa}_{\hat{m}_n}^{-1} - \kappa^{-1})/\hat{\kappa}_{\hat{m}_n}^{-1}$ or $\hat{m}_n^{1/2}(\hat{\kappa}_{\hat{m}_n}^{-1} - \kappa^{-1})/\hat{\sigma}_{\hat{m}_n|L}$ is not sharp.

of m_n . See the references above for simulation evidence. In a non-i.i.d. setting this does not hold as a general rule of thumb³⁶, although we find strong evidence it applies to CV and PV data based on homogeneous or heterogeneous bidders with or without signal symmetry. In particular, by increasing L in controlled experiments below we are indeed able to portray an optimal range of m_n for CV or PV data, at least for our chosen Monte Carlo design.

4 A Monte Carlo Study

4.1 Data Generating Processes

We study two types of DGP's for bids: the first imposes symmetry and allows unobserved heterogeneity; while the second allows for asymmetry under homogeneity.

In the first experiment we consider a model where bidders may have a common as well as private value components in their valuations as in Wilson (1977, 1998).³⁷ Suppose the log of bidder i 's true valuation u_i is a sum of a common value component v and an idiosyncratic component a_i : $u_i = v + a_i$, where v is normally distributed with mean μ_v and variance σ_v^2 , while a_i is normally distributed with mean 0 and variance σ_a^2 . Generally, the bidders do not observe their valuations, but observe signals s_i that are informative about the valuations: $s_i = u_i + \varepsilon_i$, where the "noise" term ε_i is also mean zero normally distributed, with variance σ_ε^2 . This model nests naturally a private values environment within a common values one. If $\sigma_\varepsilon = 0$, then the environment is PV, and the private values are correlated to the extent that $\sigma_v > 0$. (If also $\sigma_v = 0$, then the environment is independent private values, IPV.) But if $\sigma_\varepsilon > 0$, then the true valuations are unobservable, and we have a model with a common value component.

Figure 1 shows numerically computed bidding strategies $B(s)$ and bid densities $g(b)$ for two examples of the above model.³⁸ In the first example (the CV example, on the

³⁶We find in numerous experiments not presented here that if the data exhibit random volatility, or are governed by a non-finite lag process with short- or long-range dependence, then Hill-plots can be essentially non-informative.

³⁷See also Hong and Shum (2002) for an application to highway procurement auctions.

³⁸A Mathematica notebook used to compute these examples is available at <http://artyom239.googlepages.com>

left panel), we set $\sigma_v = \sigma_a = \sigma_\varepsilon = 0.3$. In the second example (the PV example, on the right panel) we set $\sigma_v = \sigma_a = 0.3$ and $\sigma_\varepsilon = 0$. In both examples, $N = 6$, the mean log valuation $\mu_v = \log 100$ and the reserve price is \$80. A barely noticeable, but important difference between the graphs in the top panel is that under PV the bidding strategy has zero slope at $s = s^*$ (the right graph), while it has a positive slope there under CV (the left graph). This behavior of the bidding strategy translates into a profoundly different behavior of the density of bids $g(b)$ around the reserve price, illustrating the power of our Proposition 1 (see the graphs on the lower panel). Under CV, $g(b)$ is continuous around r . But under PV, the density around the reserve price has a "spike". (The fact that the slope of $B(s)$ is zero at $s = s^*$ implies that the density is unbounded.)

In the Monte Carlo simulation for this model, we incorporate unobserved heterogeneity by assuming that in each auction, μ takes a value in the set $\{\log 80, \log 100, \log 120\}$, each with equal probability.

In the second experiment we consider a variant of the asymmetric model with interdependent values from Section 2. For this model, we employ uniform $[0, 1]$ distributions for bidder signals, $F_1(s) = F_2(s) = s$, and fix the number of potential bidders from groups 1 and 2 to $N_1 = N_2 = 2$. The reserve price is $r = 0.2$. The bidders' valuations are specified as follows:

$$\begin{aligned} u_1 &= s_1 + 0.1 \times s_2 = 0.1 \times (s_1 + s_2) + 0.9 \times s_1, \\ u_2 &= s_2 + 0.2s_1 = 0.2 \times (s_1 + s_2) + 0.8 \times s_2. \end{aligned}$$

This corresponds to relatively small common value components for both groups. The bidding strategies, which we computed by solving the system numerically, are shown in Figure 2.

The randomly generated samples of bids are $\{b_{i,l} : i = 1, \dots, n_l\}_{l=1}^L$; and $R = 250$ samples of $L = 250, 500$, or 1000 auctions are drawn. As always we use the centered bids $b_{i,l}^* = b_{i,l}/r - 1$ where $r = 80$ with symmetric bidders, or $r = 0.2$ with asymmetric bidders. We use $L = 1000$ to gauge the "asymptotic" performance of our test statistic, and $L = 250$ or 500 to gauge small sample performance for the type of sample sizes encountered in

practice. For each PV and CV sample the Hill estimator $\hat{\kappa}_{m_n}^{-1}$ and corresponding kernel variance estimator $\hat{\sigma}_{m_n|L}^2$ are computed over the fractile window $m_n \in \{5, \dots, 200\}$ with a Bartlett kernel and bandwidth $\gamma_n = n^{.225}$.³⁹ We then use CV and PV test performance under null and alternative hypotheses to deduce an optimal range of m_n . In order to conserve space we relegate simulation plots with $L = 500$ or 1000 to Hill and Shneyerov (2010), and make remarks below where helpful.

In view of the fact that $\hat{\kappa}_{m_n}^{-2}$ as an estimator of the asymptotic variance leads to non-negligible over rejection of the null hypotheses, we relegate associated simulation results and comments to Hill and Shneyerov (2010).

4.2 Tail Inference - Symmetric Bids with Unobserved Heterogeneity

Consider the first experiment with symmetric bidders and unobserved heterogeneity. We plot the average $\hat{\kappa}_{m_n}^{-1}$ and 90% asymptotic confidence bands $\hat{\kappa}_{m_n}^{-1} \pm 1.64 \times \hat{\sigma}_{m_n|L} / \sqrt{m_n}$ over the $R = 250$ simulated samples, along with the simulation average 5% and 95% empirical quantiles. We also plot t-test rejection frequencies for the test of H_0 : PV $\kappa = 1/2$ against H_1 : $\kappa > 1/2$, and test of H_0 : CV $\kappa = 1$ against H_1 : $\kappa < 1$, each at the 5% level. See Figure 3.

Figure 3(a)-3(c) presents results for PV data with $L = 250$ auctions. The figure displays $\hat{\kappa}_{m_n}^{-1} \pm 1.64 \times \hat{\sigma}_{m_n|L} / \sqrt{m_n}$ and the 5% and 95% quantiles of $\hat{\kappa}_{m_n}^{-1}$ in panel (a); rejection frequencies over the $R = 250$ samples in panel (b) are for the one-sided tests of PV $\kappa = 1/2$ based on the t-ratio $\sqrt{m_n}(\hat{\kappa}_{m_n}^{-1} - 2) / \hat{\sigma}_{m_n|L}$; and rejection frequencies for tests of CV $\kappa = 1$ are in panel (c). The Hill estimator uniformly hovers near $\hat{\kappa}_{m_n}^{-1} = 2$ in panel (a) where the 90% bands never contain $\kappa^{-1} = 1$, providing strong support for Proposition 1.

The tests of PV $\kappa = 1/2$ against $\kappa > 1/2$ in Figure 3(b) result in t -ratio rejection frequencies near 5% for all fractiles $m_n \geq 20$. Thus, when using asymptotic Gaussian critical values, the kernel estimator $\hat{\sigma}_{m_n|L}^2$ with Bartlett kernel and bandwidth $\gamma_n = n^{.225}$ results in a sharp approximation of the 5% nominal test size. If we flip the hypotheses

³⁹The bandwidth must satisfy $\gamma_n/n^{1/2} \xrightarrow{a.s.} 0$ to ensure consistency $\hat{\sigma}_{m_n|L}^2/v_{m_n|L}^2 \xrightarrow{P} 1$ by Theorem 7. We repeatedly find $\gamma_n = [n^\delta]$ for $\delta \in [.2, .25]$ to be superior across a large variety of data generating processes, hence we use the mid-point $\delta = .225$. Any δ in the range $[.2, .25]$ leads to qualitatively similar results.

and test CV $\kappa = 1$ against $\kappa < 1$ then in panel (c) rejection frequencies are above 90% for $m_n \geq 40$ and above 95% for $m_n \geq 45$.

Now consider Figure 3(d)-3(f) for CV data with $L = 250$ auctions. The figure displays $\hat{\kappa}_{m_n}^{-1} \pm 1.64 \times \hat{\sigma}_{m_n|L}/\sqrt{m_n}$ and the 5% and 95% quantiles of $\hat{\kappa}_{m_n}^{-1}$ in panel (d); panel (e) contains rejection frequencies for the one-sided test of CV $\kappa = 1$ based on t-ratios $\sqrt{m_n}(\hat{\kappa}_{m_n}^{-1} - 1)/\hat{\sigma}_{m_n|L}$; and rejection frequencies for tests of PV $\kappa = 1/2$ are in panel (f). The Hill estimator hovers near $\hat{\kappa}_{m_n}^{-1} = 1$, where the 90% bands never contain $\kappa^{-1} = 2$, again supporting Proposition 1.

Tests of CV $\kappa = 1$ against $\kappa < 1$ result in rejection frequencies roughly monotonically increasing over m_n , breaking 5% at $m_n = 65, 90, 150$ respectively for $L = 250, 500, 1000$. By comparison, if we test PV $\kappa = 1/2$ against $\kappa > 1/2$ then rejection frequencies are above 90% for $m_n \geq 40$ and above 95% for $m_n \geq 45$ cross $L \in \{250, 500, 1000\}$.

In practice a policy for selecting the number of bids near the reserve price m_n must be entertained. A reasonably clear picture emerges from our simulation study. In both cases of PV and CV data, and across the number of auctions $L \in \{250, 500, 1000\}$, if the number m_n is chosen from the range $\{40, \dots, 65\}$ then empirical size is close to or below 5% and empirical power is above 90%.

4.3 Tail Inference - Bidder Asymmetries

Now consider the second experiment based on an asymmetric model with interdependent values, resulting in bids under CV. By Proposition 3 the index $\kappa = 1$ for pooled bids, and $\kappa_j > 1$ for at most one group $j \in \{1, 2\}$. See Figure 4 for the case $L = 250$ and Hill and Shneyerov (2010) for $L = 500$ or 1000.

The Hill-plots in panel (a) reveal the well known small sample bias of the Hill (1975) estimator: all $\hat{\kappa}_{m_n}^{-1} > 1$ where theory predicts $\kappa^{-1} < 1$. Nevertheless, the 90% confidence bands contain $\kappa^{-1} = 2$ only at the smallest fractiles values $m_n \in \{5, 6, 7\}$ when $L = 250$ and only at $m_n = 5$ when $L = 500$ or 1000. Similarly, the tests of CV $\kappa = 1$ against $\kappa < 1$ generate rejection frequencies above 5% for fractiles m_n starting at 30, 34, 52 respectively for $L = 250, 500, 1000$, while tests of PV $\kappa = 1/2$ against $\kappa > 1/2$ lead to frequencies above 70% for all $m_n \geq 20$ across all L , and above 80% for all $m_n \geq 20$ and $L = 500, 1000$.

A tighter optimal fractile range is therefore revealed to be safely $m_n \in \{20, \dots, 30\}$. Thus, even with known estimator bias, under CV we obtain the same qualitative test results with symmetric or asymmetric bidders.

5 Empirical Application: BCTS

Prior to 2003, the Ministry of Forests⁴⁰ sold some timber rights through auctions under the Small Business Forest Enterprise Program (SBFEP) first studied by Paarsch (1997). In 2003, this program was transformed into BC Timber Sales (BCTS), an independent organization whose aim is to generate the best possible financial return from publicly-owned timber, provide timber harvesting opportunities, and set a credible reference point for the price of harvested timber.

The BCTS sells timber through first-price, sealed-bid auctions, and it has developed a complex system of reserve prices, also known in the industry as upset rates. The reserve prices are set in dollars per cubic meter of timber. Bids must be submitted as bonus payments, i.e. the dollar amounts per cubic meter over and above the reserve prices. Negative bonus bids are not allowed. Moreover, there are auctions that end up being canceled due to no bids submitted. In fact, over the period covered by our dataset, there were 33 such auctions, or about 5.4% of all auctions conducted. Paarsch (1997) also argues convincingly that non-participation in SFBEP is explained by binding reserve prices. His summary statistics show that the mean number of potential bidders (92.39) exceeds by a large margin the mean number of actual bidders (3.29).⁴¹ So it is very likely that the reserve prices are strictly binding in BCTS, which is a necessary condition for the applicability of our testing approach.

As we have already mentioned our approach is theoretically valid in an unconditional form (see Corollary 1). However, it might be desirable to incorporate relevant auction characteristics in order to increase the power of the test. In BCTS, the reserve prices are set using a pricing model that accounts for timber tract heterogeneity and uses the relevant information from comparable past sales. The variables used by the BCTS for this

⁴⁰Now known as the Ministry of Forests and Range.

⁴¹See Table 4 on p. 353 in Paarsch (1997).

purpose include average log selling prices in the region, net cruise volume, estimated species composition on the tract, and measures of distance to the closest major location (e.g. Vancouver, Nanaimo, Prince Rupert, etc.). The list of variables is not long, but probably detailed enough to capture the main aspects of auction heterogeneity, so normalizing the bids by the reserve price is likely to homogenize the bids to a large extent. For this reason, we do not incorporate any covariates in the empirical implementation of our test beyond the normalization of bids by the reserve prices.

Our data were collected from the BCTS website.⁴² The raw data is in the form of PDF files for each sale. We have all bonus bids, the reserve price, the identities of the bidders and also data on the characteristics of the sale; e.g., the location, the species of trees present, slope, and so forth. The reserve prices and bids are quoted per 1m^3 of timber. Our dataset comprises of all auctions conducted from January 14, 2004 to December 14, 2006. The sample of auctions that have received bids contains 611 auctions and a total of 1874 bids. For each auction, the reserve price as well as the bonus bids are available for all bidders. The summary statistics of the sample are shown in Table 1. It is evident that there is substantial heterogeneity in timber tract values, as reflected by the variances of both bids and reserve prices. The heterogeneity persists even when bids are normalized by the reserve prices.

Why Should We Expect Common Values in BCTS? Paarsch (1997) assumed a private-values model in his study of SBFEP, the predecessor of BCTS. However, common value uncertainty is likely to be present in BCTS at least to some extent. First, the auction design has changed. In BCTS, even though bidders pay for the actual volume of timber on the tract, their bids must specify a single price for all timber, whereas in SBFEP, bidders were allowed to enter separate bids for each species. To the extent that the composition of timber on the tract is uncertain at the time of bidding, this new design increased the common value uncertainty for the bidders. Another factor at play over the period covered by our data was the epidemic of Mountain Pine Beetle in BC. The epidemic affected the predominant merchantable timber species, the Lodgepole Pine. Timber quality was

⁴²<http://www.for.gov.bc.ca/bcts/>, accessed in January 2007.

affected in a manner that was likely to be tract-specific, and this contributed to the common value uncertainty in the auction.⁴³

Can Bidders Be Treated Symmetrically? Our testing approach has been developed under the assumption that bidders are symmetric. Paarsch (1997) also treated bidders symmetrically in his study of SBFEP. But how well is symmetry supported in BCTS? Since one of the goals of BCTS is to obtain market value for the timber by promoting entry, the eligibility requirements in BCTS were somewhat relaxed. In particular, the eligibility is no longer restricted to small logging firms. In principle, any entity in BC that (i) is an individual 19 years of age or older or (ii) a corporation registered in British Columbia; and, has one year logging experience or own a timber processing facility, can participate for a two-year period. by paying a small registration fee (250 Canadian dollars). But many features of the SBFEP have been preserved in BCTS; e.g., the tracts are still quite small and, as important, no firm is allowed to hold more than five licenses at the same time. This alone is likely to limit the participation of large mills and timber processing companies.

In our data, we find no evidence of major players. Figure 5(a) depicts the frequencies of participation by firms in the auctions over the period covered by our data. About half of the bidders participated in only one auction. Even the maximum of the participation rate is very small, 25 out of 611 auctions, or the rate of about 0.02. This also implies that information asymmetries, for example due to superior information about the beetle infestation, are unlikely in this market, and bidders can still be treated symmetrically.

5.1 Test Results

Figure 5(b) depicts the histogram of normalized bonus bids. The histogram exhibits an overall declining pattern consistent with our findings in the numerical examples (Figure 1), for both PV and CV environments.⁴⁴ There is some evidence of bid clustering around the

⁴³According to an estimate by the BC Ministry of Forests and Range, since late 1990s beetles have killed over 620 million cubic meters of timber, covering over 130,000 square kilometers. Some information about the Mountain Pine Beetle epidemic in BC can be found in the Mountain Pine Beetle Action Plan, available at the BC Ministry of Forests and Range website, at http://www.for.gov.bc.ca/hfp/mountain_pine_beetle/MPB_ActionPlan_ProgressReport.pdf (accessed on 28 April, 2009).

⁴⁴Roise (2005) presents a similar pattern in his Figure 8, without normalizing bids by the reserve price.

reserve price. This could be evidence of a PV environment, but could also be a statistical artifact, especially since our numerical examples have also identified some clustering under CV.

The tail index test however provides robust and conclusive evidence. The results of the estimation of the tail index κ^{-1} and the t-test are reported in Figure 6. We plot the asymptotic 90% confidence bands $\hat{\kappa}_{m_n}^{-1} \pm 1.64 \times \hat{\sigma}_{m_n|L}/\sqrt{m_n}$ over the same fractile window $m_n \in \{5, \dots, 200\}$ used in the simulation study, and plot p-values from the t-tests of PV $\kappa = 1/2$ against $\kappa > 1/2$, and CV $\kappa = 1$ against $\kappa < 1$. Related plots based on the variance estimator $\hat{\kappa}_{m_n}^{-2}$ are presented in Hill and Shneyerov (2010).

The CV value $\kappa = 1$ lies within the confidence band for over 70% of the fractiles $m_n \in \{5, \dots, 200\}$, while the PV value $\kappa = 1/2$ never lies in the band. In particular, there is no evidence at any fractile m_n for PV: p-values for the t-test of PV are no larger than .0045. Conversely, when CV is the null hypothesis p-values are predominantly above 5%-10%.

Recall when m_n is restricted to the range $\{40, \dots, 65\}$ simulation evidence for symmetric heterogeneous bidders shows the kernel estimator $\hat{\sigma}_{m_n|L}^2$ leads to empirical sizes at or below 5%, and empirical power at or above 90%. Similarly, the optimal range under asymmetry and homogeneity is $\{20, \dots, 30\}$. In the former range the CV value $\kappa = 1$ lies within the 90% confidence band for over 88% of the m_n . In particular, for the test of PV $\kappa = 1/2$ against $\kappa > 1/2$ the average p-value is .0004, and the percent of m_n on which we reject the null of PV in favor of CV is 100%.⁴⁵ Similarly, for the one-sided test of CV $\kappa = 1$ against $\kappa < 1$ the average p-value is 15.5%, and the percent of m_n on which we reject the null of CV in favor of PV is 7.7%. In the restricted range $\{20, \dots, 30\}$ the test results are even more stark: if the null is CV the average p-value is 75.7% and we do not reject the null for any m_n , and if the null is PV the p-value average is 2.6E-07 and we reject the null for every m_n .

Although the t-ratio τ_{m_n} works exceptionally well within the broad range $m_n \in \{5, \dots, 200\}$, if we take into consideration (i) the superlative performance of τ_{m_n} on the restricted range $m_n \in \{40, \dots, 65\}$ and the strong performance on $\{20, \dots, 30\}$, and (ii) the

⁴⁵A more appropriate test statistic may be $\sup_{\lambda \in \Lambda} |\tau_{m_n(\lambda)}|$ where, e.g., $m_n(\lambda) = \lfloor \lambda n / \ln(n) \rfloor$ with λ restricted to a compact subset Λ of $[0, 1]$. A Gaussian weak limit theory exists for $\hat{\kappa}_{m_n(\lambda)}^{-1}$ but a weak limit for $\hat{\sigma}_{m_n(\lambda)|L}^2$ is not yet available (Hill (2009, 2010)). See the discussion in Footnote 34.

empirical evidence within those ranges, then we conclude the evidence overwhelmingly points to a CV strategy within BTCS data.

6 Concluding Remarks

In this paper, we have developed a new tail-index nonparametric, asymptotically most powerful test of common values in first-price auctions and applied it to BC Timber Sales, an important institution both in the economy of British Columbia and in the larger context of the United States-Canada soft lumber dispute.

The test is based on auction theory, but is reduced-form in the sense that it is based on the properties of bids distribution, which is directly observable. The test exploits the difference in the clustering of bids near the reserve price. A measure of such clustering is provided by the tail index. We have shown that the tail index of the bid distribution is equal to one-half under private values, but is equal to one if there is a common-value component in bidders' valuations. The estimation of the tail index is a well-studied problem in econometrics, where the Hill (1975) estimator is by far the most widely-used method. Nevertheless, the available asymptotic results do not cover our setting of imbalanced panels with stochastic dimension and conditional sub-sample heterogeneity, and few results treat inference in the non-i.i.d. case. Our econometric contributions are to develop a nonparametric asymptotic inference framework for the Hill (1975) estimator in this setting, show arbitrary within-auction dependence does not alter the asymptotic distribution relative to the i.i.d. case, and show kernel estimator methods greatly improve small sample inference. These methods are potentially useful in other contexts where dependent heterogeneous data and a stochastic sample size are encountered. In this regard applicable fields are quite diverse, including marked point process in finance, and spatial point processes at the heart of stochastic geometry.⁴⁶

Our nonparametric tail-index test has uncovered the presence of common values in

⁴⁶A marked point process in finance may denote the date and value (e.g. price, volume) of a traded asset jointly within a deterministic window of time. The number of marks (i.e. trades) observed is therefore random. This generalizes the setting of a first-price auction, while the theory developed here easily extends to a general class of dependent heterogeneous processes (cf. Hill (2010, 2011b)). Forestry growth and earthquake magnitudes in space and time are further examples. See, e.g., Stoyan, Kendall, and Mecke (1987) and Jacobsen (2006).

BCTS. Simply put, the bids do not cluster around the reserve price as much as they should if values were purely private. Why is this important? Recall that BCTS pursues a goal of obtaining the maximal revenue for the Crown, and currently uses first-price, sealed-bid auction mechanism. To the extent that common values are present, theory (Milgrom and Weber (1982)) suggests that BCTS would obtain a higher expected revenue if it adopted an ascending-bid auction. Shneyerov (2006) proposes nonparametric methods for estimation and bounding of such counter-factual revenues. In addition, Tang (2011) has recently developed nonparametric bounds for expected revenues under counter-factual reserve prices. An application of these methods to BC Timber Sales would be an interesting topic for future research.

Appendix A: Variance and Omitted Proofs

In this appendix we characterize the Hill (1975) estimator scale $\sigma_{m_n|L}^2$ required for our test, and present omitted proofs. We first summarize the major properties of bids governed by our DGP for the sake of reference. Recall $\{b_t^*\}_{t=1}^n$ is the sample of stacked bids $b_{i,l}^*$, and the sigma-field induced by observable auction sizes:

$$F_L := \sigma \left(\{n_l\}_{l=1}^L : n_l \geq 1 \right).$$

The following is implied by Corollary 1 and therefore (24).

A1 (conditional tail) *The left-tail behavior of b_t^* satisfies for every t*

$$G^*(b|F_L) := P(b_t^* < b|F_L) = c_L \times b^\kappa (1 + O(b^\kappa)), \kappa > 0, \text{ as } b \downarrow 0 \quad (32)$$

where the F_L -measurable random variable $c_L > 0$ a.s.

Bids across auctions are independent.

A2 (cross-auction independence) *$b_{i,l}^*$ is possibly dependent over i within auction l , and independent across auctions l .*

Finally, signal symmetry and the DGP of Section 3.1 ensure stationarity and exchangeability of bids. This implies the bivariate distribution of $\{b_{i,l}^*, b_{j,l}^*\}$, conditional on observable auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$, depends only on F_L and on whether the bidders are the same $i = j$ or different $i \neq j$: in particular, $P(b_{i,l}^* \leq b_1, b_{j,l}^* \leq b_2 | F_L) = P(b_{r,l}^* \leq b_1, b_{s,l}^* \leq b_2 | F_L)$ for every pair $i \neq j$ and $r \neq s$.

A3 (stationarity and exchangeability) *Bids $b_{i,l}^*$ within auction l are stationary and exchangeable, conditional on observable auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$. Unconditionally, bids are stationary.*

A.1 Variance

By the definition of the variance $\sigma_{m_n|L}^2 := E(1/m_n^{1/2} \sum_{l=1}^L \sum_{i=1}^{n_l} T_{m_n,i,l}^{(L)} | F_L)^2$, by the construction $E[T_{m_n,i,l}^{(L)} | F_L] = 0$, and by the fact that bids $b_{i,l}^*$ are independent across auctions under A2, it follows

$$\sigma_{m_n|L}^2 = E \left[\left(\frac{1}{m_n^{1/2}} \sum_{l=1}^L \sum_{i=1}^{n_l} T_{m_n,i,l}^{(L)} \right)^2 | F_L \right] = \frac{1}{m_n} \sum_{l=1}^L \sum_{i,j=1}^{n_l} E \left[T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L \right]. \quad (33)$$

Stationarity implies $E[T_{m_n,i,l}^{(L)} T_{m_n,j,l}^{(L)} | F_L]$ depends only on the auction size n_l and whether the bids come from the same bidder $i = j$ or different bidders $i \neq j$. Further, in auctions of size $n_l \geq 2$ exchangeability implies bid dependence between any two bidders is the same, hence we need only consider the first two bidders. Specifically, define

$$\begin{aligned} C_{m_n,l}^{(L)} &= \frac{n}{m_n} T_{m_n,1,l}^{(L)} T_{m_n,2,l}^{(L)} \left\{ \frac{\sum_{l=1}^L n_l (n_l - 1)}{\sum_{l=1}^L n_l} \right\} \text{ if } n_l \geq 2 \text{ for some } l = 1, \dots, L \\ &= 0 \text{ if } n_l = 1 \text{ for all } l = 1, \dots, L. \end{aligned} \quad (34)$$

Then by (33), (34), stationarity and exchangeability A3, and $n = \sum_l n_l$, we have

$$\begin{aligned} \sigma_{m_n|L}^2 &= \frac{1}{m_n} \sum_{l=1}^L \sum_{i=1}^{n_l} E \left[\left(T_{m_n,i,l}^{(L)} \right)^2 | F_L \right] + \frac{1}{m_n} \sum_{l=1}^L \sum_{i \neq j}^{n_l} E \left[T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L \right] \\ &= \frac{n}{m_n} E \left[\left(T_{m_n,1,l}^{(L)} \right)^2 | F_L \right] + E \left[C_{m_n,l}^{(L)} | F_L \right]. \end{aligned} \quad (35)$$

We use the following result to characterize the first conditional expectation in (35). Notice each limit is in unconditional probability: in Appendix A.2 we prove each claim by first conditioning on auction sizes, and then by exploiting dominated convergence.

Lemma 8 $(n/m_n)E[(\ln(q_n/b_t^*))_+|F_L] \xrightarrow{p} \kappa^{-1}$, $(n/m_n)E[(U_{m_n,t}^{(L)})^2|F_L] \xrightarrow{p} 2\kappa^{-2}$, $(n/m_n)E[(I_{m_n,t}^{(L)})^2|F_L] \xrightarrow{p} 1$ and $(n/m_n)E[(T_{m_n,t}^{(L)})^2|F_L] \xrightarrow{p} \kappa^{-2}$.

Combine decomposition (35) and Lemma 8 to characterize the unconditional probability limit of $\sigma_{m_n|L}^2$.

Lemma 9 *The conditional variance satisfies unconditionally $\sigma_{m_n|L}^2 \xrightarrow{p} \kappa^{-2} + \text{plim}_{L \rightarrow \infty} \{E[C_{m_n,t}^{(L)}|F_L]\}$ hence $\sigma_{m_n|L}^2 = \kappa^{-2} + E[C_{m_n,t}^{(L)}|F_L] + o_p(1)$.*

The conditional variance asymptotically reduces to the asymptotic variance κ^{-2} in the i.i.d. case plus a term capturing total within auction bid dependence near the reserve price. The next result suffices to prove $\text{plim}_{L \rightarrow \infty} \{E[C_{m_n,t}^{(L)}|F_L]\} = 0$ under our DGP (see the proof of Theorem 6).

Lemma 10 *The joint bid density $g(b_1, b_2|F_L)$ satisfies $\limsup_{0 \leq b_1, b_2 \leq q_n} \{g(b_1, b_2|F_L) / (g(b_1|F_L)g(b_2|F_L))\} \leq K$ on $F_L - a.e.$*

A.2 Omitted Proofs

Proof of Lemma 4. A variation on Hsing's (1991: p.1548) argument suffices. Under the DGP conditional tail decay (24) applies by an application of Proposition 1 and its consequence (21). Under (24) b_t^* has a regularly varying conditional tail with index $\kappa > 0$, and by (25) the thresholds q_n and fractiles m_n satisfy $(n/m_n)G^*(q_n|F_L) \xrightarrow{p} 1$ on $F_L - a.e.$ Therefore, by dominated convergence and Fubini's Theorem as $L \rightarrow \infty$

$$\begin{aligned} E \left[\frac{n}{m_n} \left(\ln \left(\frac{q_n}{b_t^*} \right)_+ \right) \right] &= \int_0^\infty P(b_t^* < q_n e^{-um_n/n}) du = \int_0^\infty E \left[P(b_t^* < q_n e^{-um_n/n} | F_L) \right] du \\ &= \int_0^\infty E \left[P(b_t^* < q_n | F_L) \frac{P(b_t^* < q_n e^{-um_n/n} | F_L)}{P(b_t^* < q_n | F_L)} \right] du \\ &\sim \int_0^\infty E \left[\frac{m_n}{n} e^{-\kappa um_n/n} \right] du = \kappa^{-1} E \left[\frac{m_n}{n} \frac{n}{m_n} \right] = \kappa^{-1}. \end{aligned}$$

Q.E.D.

Proof of Theorem 6. By Lemma 10 we have $\limsup_{0 \leq b_1, b_2 \leq q_n} \{g(b_1, b_2|F_L) / (g(b_1|F_L)g(b_2|F_L))\} \leq K$ on $F_L - a.e.$ Now apply dominated convergence and tail-array moment bounds verified in Lemma 8 to deduce

$$E \left[\left(\ln \left(q_n / b_{i,l}^* \right) \right)_+ \left(\ln \left(q_n / b_{j,l}^* \right) \right)_+ | F_L \right] \leq O_p \left(\left(E \left[\left(\ln \left(q_n / b_{i,l}^* \right) \right)_+ | F_L \right] \right)^2 \right) \quad (36)$$

and

$$E \left[I \left(b_{i,l}^* < q_n e^{-\frac{u}{m_n^{1/2}}} \right) I \left(b_{j,l}^* < q_n e^{-\frac{u}{m_n^{1/2}}} \right) | F_L \right] \leq O_p \left(\left(E \left[I \left(b_{i,l}^* < q_n e^{-\frac{u}{m_n^{1/2}}} \right) | F_L \right] \right)^2 \right) \quad (37)$$

are $O_p((m_n/n)^2)$. Together (36) and (37) prove $E[T_{m_n,1,l}^{(L)} T_{m_n,2,l}^{(L)} | F_L] = O_p((m_n/n)^2)$. Since $\sum_{l=1}^L n_l(n_l - 1) / \sum_{l=1}^L n_l \in (0, \infty)$ a.s. as $L \rightarrow \infty$ we have shown $\text{plim}_{L \rightarrow \infty} \{E[C_{m_n,l}^{(L)} | F_L]\} = 0$ for $C_{m_n,l}^{(L)}$ defined in (34). In conjunction with Lemma 9 this proves $\hat{\sigma}_{m_n|L}^2 \xrightarrow{p} \kappa^{-2}$. The claim $m_n^{1/2}(\hat{\kappa}_{m_n}^{-1} - \kappa^{-1}) / \hat{\kappa}_{m_n}^{-1} \xrightarrow{d} N(0, 1)$ now follows from Theorem 5 and Cramér's Theorem. *Q.E.D.*

Proof of Theorem 7. Under the stated assumptions and the arguments leading to Theorem 5, all conditions of Hill (2010)'s Lemmas B.8 and B.9 hold for bids b_i^* conditioned on observable auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$. Therefore $P(\hat{\sigma}_{m_n|L}^2 > 0 | F_L) = 1$ and $\lim_{L \rightarrow \infty} P(|\hat{\sigma}_{m_n|L}^2 / \sigma_{m_n|L}^2 - 1| > \varepsilon | F_L) = 0 \forall \varepsilon > 0$ on $F_L - a.e.$ See especially the proof of Theorem 3 in Hill (2010). Hence, by dominated convergence unconditionally $\hat{\sigma}_{m_n|L}^2 > 0$ a.s. and $\hat{\sigma}_{m_n|L}^2 / \sigma_{m_n|L}^2 \xrightarrow{p} 1$ as claimed. *Q.E.D.*

Proof of Lemma 8. It can be shown by multiplying out all parts of $T_{m_n,i,l}^{(L)}$ and canceling

opposing terms that

$$\begin{aligned}
E \left[\left(T_{m_n, t}^{(L)} \right)^2 | F_L \right] &= \left\{ E \left[\left(\ln (q_n / b_t^*) \right)_+^2 | F_L \right] + \left(E \left[\left(\ln (q_n / b_t^*) \right)_+ | F_L \right] \right)^2 \right\} \\
&\quad + \left\{ \kappa^{-2} P \left(b_t^* < q_n e^{-u/m_n^{1/2}} | F_L \right) - \kappa^{-2} P \left(b_t^* < q_n e^{-u/m_n^{1/2}} | F_L \right)^2 \right\} \\
&\quad - 2\kappa^{-1} E \left[\left(\ln (q_n / b_t^*) \right)_+ \times I \left(b_t^* < q_n e^{-u/m_n^{1/2}} \right) | F_L \right] \\
&\quad + 2\kappa^{-1} E \left[\left(\ln (q_n / b_t^*) \right)_+ | F_L \right] \times P \left(b_t^* < q_n e^{-u/m_n^{1/2}} | F_L \right).
\end{aligned}$$

Use conditional power law A1 with properties of regularly varying functions (e.g. Leadbetter, Lindgren, and Rootzen (1983)), and the threshold construction $(n/m_n)G^*(q_n|F_L) \xrightarrow{p} 1$ on $F_L - a.e.$ by (25) to deduce three properties. First, $P(b_t^* < q_n e^{-u/m_n^{1/2}} | F_L) \sim (m_n/n)e^{-\kappa u/m_n^{1/2}} = (m_n/n) \times (1 + o_p(1))$ on $F_L - a.e.$ Second, choose any $p \in \mathbb{N}$. By Fubini's Theorem and bounded convergence, $E[(\ln(q_n/b_t^*))_+^p | F_L]$ is on $F_L - a.e.$

$$\begin{aligned}
\int_0^\infty P \left(b_t^* < q_n e^{-u/p} | F_L \right) du &= P \left(b_t^* < q_n | F_L \right) \times \int_0^\infty e^{-\kappa u/p} du \sim (m_n/n) p! \kappa^{-p} \\
&\sim (m_n/n) \times \int_0^\infty e^{-\kappa u/p} du \sim (m_n/n) p! \kappa^{-p}.
\end{aligned}$$

Hence $(n/m_n)E[(\ln(q_n/b_t^*))_+^p | F_L] \xrightarrow{p} p! \kappa^{-p}$ on $F_L - a.e.$

Similarly, on $F_L - a.e.$

$$\begin{aligned}
E \left[\left(\ln (q_n / b_t^*) \right)_+^p \times I \left(b_t^* < q_n e^{-u/m_n^{1/2}} \right) | F_L \right] &= E \left[\int_{u/m_n^{1/2}}^\infty P \left(b_t^* < q_n e^{-v/p} | F_L \right) dv | F_L \right] \\
&\sim (m_n/n) p! \kappa^{-p}.
\end{aligned}$$

Combine the three properties with $m_n/n \xrightarrow{a.s.} 0$ to prove each claim. For example, $(n/m_n) \times E[(U_{m_n, t}^{(L)})^2 | F_L] = (n/m_n)E[(\log(q_n/b_t^*))_+^2 | F_L] - (n/m_n)(E[(\log(q_n/b_t^*))_+ | F_L])^2 \sim 2\kappa^{-2} - (n/m_n)(m_n/n)^2 \kappa^{-2} \xrightarrow{p} 2\kappa^{-2}$ on $F_L - a.e.$, hence by dominated convergence $(n/m_n) \times E[(U_{m_n, t}^{(L)})^2 | F_L] \xrightarrow{p} 2\kappa^{-2}$.

Similarly, together we obtain $(n/m_n)E[(T_{m_n, t}^{(L)})^2 | F_L] \sim \{\kappa^{-2} + \kappa^{-2}\} + \{\kappa^{-2} - o_p(1)\} - 2\kappa^{-2} + 2\kappa^{-2}(m_n/n) \xrightarrow{p} \kappa^{-2}$ on $F_L - a.e.$ hence $(n/m_n)E[(T_{m_n, t}^{(L)})^2 | F_L] \xrightarrow{p} \kappa^{-2}$. *Q.E.D.*

Proof of Lemma 10. By the arguments leading to Corollary 1, under our DGP we may replicate the proof of Proposition 1 for bids conditional on auction sizes $\{\{n_l\}_{l=1}^L : n_l \geq 1\}$.

In particular, first assume PV and imitate the proof of Proposition 1 leading to (8) to deduce bidder signal distribution smoothness Assumption 9(iii) and density boundedness from below Assumption 9(vi) implies

$$P\left(b_{1,l}^* \leq b_1, b_{2,l}^* \leq b_2 | F_L\right) = P\left(S_1 \leq \beta_l^{-1}(b_1^{1/2}), S_2 \leq \beta_l^{-1}(b_2^{1/2}) | F_L\right) \quad (38)$$

where $\beta_l : D_\varepsilon \rightarrow R$, $\beta_l(b) := \beta(b; r_l, x_l, N_l)$, with β defined by (7). Note that $\beta_l(\cdot)$ is a diffeomorphism, i.e. is smooth with a smooth inverse β^{-1} . Also denote, respectively as $f_l(s)$ and $f_l(s_1, s_2)$, the density of the signal $S_{1,l}$ and the joint density of signals $(S_{1,l}, S_{2,l})$, conditional on (r_l, x_l, N_l) . Then the joint bid density $g(b_1, b_2 | F_L)$ satisfies by iterated expectations

$$\begin{aligned} g(b_1, b_2 | F_L) &= \frac{\partial}{\partial b_1} \frac{\partial}{\partial b_2} P\left(b_{1,l}^* \leq b_1, b_{2,l}^* \leq b_2 | F_L\right) \\ &= E \left[f_l\left(\beta_l^{-1}(b_1), \beta_l^{-1}(b_2)\right) \prod_{i=1}^2 \left(\frac{d\beta_l^{-1}(b_i)}{db} \frac{1}{2\sqrt{b_i}} \right) | F_L \right] \\ &= E \left[f_l\left(\beta_l^{-1}(b_1), \beta_l^{-1}(b_2)\right) \prod_{i=1}^2 \frac{d\beta_l^{-1}(b_i)}{db} | F_L \right] \times \prod_{i=1}^2 \frac{1}{2\sqrt{b_i}}. \end{aligned} \quad (39)$$

Similarly, the marginal bid density is

$$g(b | F_L) = E \left[f_l(\beta_l^{-1}(b)) \frac{d\beta_l^{-1}(b)}{db} | F_L \right] \times \frac{1}{2\sqrt{b}}.$$

Assumption 9(iii) implies that a joint signal density $f_l(s_1, s_2)$ is bounded from above $\sup_{s_1, s_2 \in [\underline{s}, \bar{s}]} \{f_l(s_1, s_2)\} \leq K$ a.s., and density positiveness Assumption 9(vi) ensures boundedness from below, hence $\inf_{s \in [\underline{s}, \bar{s}]} \{f_l(s)\} > 0$ a.s. Moreover, being a diffeomorphism, the function $\beta_l(\cdot)$ has a bounded derivative, which is also bounded away from 0. Therefore,

as claimed

$$\sup_{0 \leq b_1, b_2 \leq q_n} \left\{ \frac{g(b_1, b_2 | F_L)}{g(b_1 | F_L) g(b_2 | F_L)} \right\} \leq \sup_{s_1, s_2 \in [\underline{s}, \bar{s}]} \left\{ \frac{E \left[f_l \left(\beta_l^{-1}(b_1), \beta_l^{-1}(b_2) \right) \prod_{i=1}^2 \frac{d\beta_l^{-1}(b_i)}{db} | F_L \right]}{\prod_{i=1}^2 E \left[f_l(\beta_l^{-1}(b_i)) \frac{d\beta_l^{-1}(b_i)}{db} | F_L \right]} \right\} \\ \leq K \text{ on } : F_L - a.e.$$

The proof for CV is essentially identical by replicating arguments leading to (6). In equations (38) and (39) replace $\beta(s)$ with the smooth bidding strategy $B^*(s)$ from the proof of Proposition 1, and replace $\sqrt{b_i}$ with b_i . *Q.E.D.*

Appendix B: Robustness Checks

Proof of Proposition 2. A change of variables $\alpha_i(b) - b \equiv \phi_i(\tau)$, $\tau \equiv \sqrt{b - r}$, and a re-arrangement transforms system (10)-(11) into

$$\phi'_1(\tau) = \frac{2F_1[r + \tau^2 + \phi_1(\tau)]}{(N-1)f_1[r + \tau^2 + \phi_1(\tau)]} \left[N_2 \frac{\tau}{\phi_2(\tau)} - (N_2 - 1) \frac{\tau}{\phi_1(\tau)} \right] - 2\tau, \quad (40)$$

$$\phi'_2(\tau) = \frac{2F_2[r + \tau^2 + \phi_2(\tau)]}{(N-1)f_2[r + \tau^2 + \phi_2(\tau)]} \left[N_1 \frac{\tau}{\phi_1(\tau)} - (N_1 - 1) \frac{\tau}{\phi_2(\tau)} \right] - 2\tau. \quad (41)$$

We first show that we cannot have as $\tau \downarrow 0$

$$\phi'_j(\tau) \rightarrow \infty \quad (42)$$

or

$$\phi'_j(\tau) \rightarrow 0. \quad (43)$$

Fix one one of the groups, say $j = 1$. (The argument is symmetric for $j = 2$.) *To rule out (42)*, note first that L'Hopital's rule then implies $\frac{\tau}{\phi_1(\tau)} \rightarrow 0$. Also $\tau/\phi_2(\tau) \rightarrow 0$, for otherwise the r.h.s. of (41) would be negative for all sufficiently small $\tau > 0$, and the slope $\phi'_2(\tau)$ would also be negative. In view of $\phi_2(0) = 0$, this would imply that $\alpha_2(b) < b$ for some $b > r$ sufficiently close to r , which is impossible. But if both $\tau/\phi_1(\tau) \rightarrow 0$ and $\tau/\phi_2(\tau) \rightarrow 0$, then from (40) we have $\phi'_1(\tau) \rightarrow 0$, contrary to what we assumed in (42).

To rule out (43), first rewrite the system (40)-(41) as

$$\phi_1'(\tau) = \frac{2F_1[r + \tau^2 + \phi_1(\tau)]}{(N-1)f_1[r + \tau^2 + \phi_1(\tau)]} \frac{\tau}{\phi_1(\tau)} \left[N_2 \frac{\phi_1(\tau)}{\phi_2(\tau)} - (N_2 - 1) \right] - 2\tau, \quad (44)$$

$$\phi_2'(\tau) = \frac{2F_2[r + \tau^2 + \phi_2(\tau)]}{(N-1)f_2[r + \tau^2 + \phi_2(\tau)]} \frac{\tau}{\phi_2(\tau)} \left[N_1 \frac{\phi_2(\tau)}{\phi_1(\tau)} - (N_1 - 1) \right] - 2\tau. \quad (45)$$

Since $\phi_1'(\tau) \rightarrow 0$, we obtain $\frac{\tau}{\phi_1(\tau)} \rightarrow \infty$ by L'Hopital's rule. Then our assumption (43) and equation (44) above together imply

$$N_2 \frac{\phi_1(\tau)}{\phi_2(\tau)} - (N_2 - 1) \rightarrow 0, \quad \frac{\phi_1(\tau)}{\phi_2(\tau)} \rightarrow \frac{N_2 - 1}{N_2}. \quad (46)$$

Also, we must have $\tau/\phi_2(\tau) \rightarrow \infty$, for otherwise we would have from (40) $\phi_1'(\tau) < 0$ for all sufficiently small $\tau > 0$, which is impossible. But then (45) imply

$$N_1 \frac{\phi_2(\tau)}{\phi_1(\tau)} - (N_1 - 1) \rightarrow 0, \quad \frac{\phi_2(\tau)}{\phi_1(\tau)} \rightarrow \frac{N_1 - 1}{N_1}. \quad (47)$$

Now observe that the product of the left-hand sides of (46) and (47) is equal to 1, while the limit of the product of their right-hand sides is $\frac{(N_1-1)(N_2-1)}{N_1 N_2} < 1$, a contradiction.

With the possibilities in (42) and (43) eliminated, it follows that $\phi_j'(\tau)$ converge to finite, positive limits: $\lim_{\tau \rightarrow 0} \phi_j'(\tau) \in (0, \infty)$.⁴⁷ Therefore, the derivatives $\phi_j'(\tau)$ are right-continuous and positive at $\tau = 0$. As $\alpha_j(b) = b + \phi_j(\sqrt{b-r})$, and the bid distributions are given by (9), we obtain $G_j(b) = \frac{F_j[\alpha_j(b)] - F_j(r)}{1 - F_j(r)} = \frac{H_j(\sqrt{b-r})}{1 - F_j(r)}$, where $H_j(x) \equiv F_j[r + x^2 + \phi_j(x)] - F_j(r)$ is a twice continuously differentiable around 0 function, with $H_j(0) = 0, H_j'(0) > 0$. The result in the proposition follows from the mean value theorem for H_j . For some $\zeta \in [0, \sqrt{b-r}]$, we have $H_j(\sqrt{b-r}) = H_j'(\zeta)\sqrt{b-r} = [H_j'(0) + o(1)]\sqrt{b-r}$. *Q.E.D.*

Proof of Proposition 3. Evaluating the r.h.s. of (15) and (16) at $b = r$ and taking into account (17) yields existence of the (right) derivatives $\alpha_j'(r) \in [0, \infty)$. Moreover,

⁴⁷These limits can be computed explicitly by applying L'Hopital's rule to the system (44) and (45) as $\tau \rightarrow 0$.

$\alpha'_j(r) > 0$ for at least one of the groups. Indeed, if e.g. $\alpha'_1(r) = 0$, then

$$\begin{aligned} \frac{N_2}{v_2[\alpha_1(r), \alpha_2(r)] - r} - \frac{N_2 - 1}{v_1[\alpha_1(r), \alpha_2(r)] - r} &= 0, \\ \Rightarrow \frac{N_1}{v_1[\alpha_1(r), \alpha_2(r)] - r} - \frac{N_1 - 1}{v_2[\alpha_1(r), \alpha_2(r)] - r} &= \frac{1}{v_1[\alpha_1(r), \alpha_2(r)] - r} \\ &\times \left[N_1 \frac{N_2}{N_2 - 1} - (N_1 - 1) \right] > 0 \end{aligned}$$

By the mean value theorem, we have for the pooled distribution

$$G(b) \equiv \sum_{j=1}^2 \frac{N_j}{N} G_j(b) = \sum_{j=1}^2 \beta_j f_j(\tilde{\alpha}_j) (\alpha_j(b) - \alpha_j(r)) = (1 + o(1)) \times \sum_{j=1}^2 \beta_j f_j(r) \alpha'_j(r) \times (b - r), \quad (48)$$

where $\beta_j \equiv \frac{N_j}{N} \frac{1}{1 - F_j(r)} > 0$. Because $\alpha'_j(r) \geq 0$, with a strict inequality for some $j \in \{1, 2\}$, it follows that in (48), $\sum_{j=1}^2 \beta_j f_j(r) \alpha'_j(r) > 0$, and therefore the tail index of the corresponding normalized bid distribution $G^*(b)$ is $\kappa = 1$. *Q.E.D.*

Drainage Tract Model. This is the model considered in the classic study by Hendricks and Porter (1988). There is an informed bidder and n uninformed bidders. The informed bidder knows the value of the object V , a random variable drawn from CDF F with support $[\underline{v}, \bar{v}]$, and with continuously differentiable density f bounded away from 0 on the support. The uninformed bidder doesn't have any information concerning V . The payoff to the informed from winning the auction is V , while the payoff to an uninformed is $V - c$, where $c \geq 0$ may reflect production cost differences between the firms. There is a binding reserve price $r \in (\underline{v}, \bar{v})$.

For simplicity, we assume that there is one uninformed bidder ($n = 1$); however, the analysis can be generalized. Hendricks and Porter (1988) characterize Bayesian-Nash equilibrium bidding strategies as follows. The informed bidder adopts a pure bidding strategy given by

$$\sigma^*(v) = \begin{cases} E[V|V \leq v] - c, & v > \hat{v} \\ r, & v \in [r, \hat{v}] \\ 0, & v < r \end{cases} \quad (49)$$

(Here we follow Hendricks and Porter (1988) and assume that the bidder with valuation below the reserve bids 0.) The threshold $\hat{v}$ is found from the break-even condition of the uninformed bidder at the reserve, $E[V|V \leq \hat{v}] - c = r$. The uninformed bidder adopts a mixed strategy. Accordingly, the distribution of the informed and uninformed bids are

$$G_I^*(b) = \begin{cases} 1, & b > E[V] - c \\ F[\sigma^{*-1}(b)], & b \in (r, E[V] - c] \\ F(r), & b \in [0, r] \end{cases}, \quad G_U^*(b) = \begin{cases} 1, & b > \bar{v} - c \\ \varphi[\sigma^{*-1}(b)], & b \in (r, \bar{v} - c] \\ \varphi(r), & b \in [0, r] \end{cases}$$

where $\varphi(h) \equiv \exp \left\{ - \int_h^\infty \frac{f(s) \int_h^s F(u) du ds}{cF(s)^2 + F(s) \int_h^s F(u) du} \right\}$. (If $c = 0$, it can be verified that $\varphi(h) = F(h)$.)

Note that the distribution G_I^* has a probability mass at the reserve, equal to $F(\hat{v}) - F(r)$. On the contrary, the distribution G_U^* is continuous at the reserve. *We therefore focus on the tail index of both distributions with the bids at (and below) reserve price removed, i.e. conditional on $b > r$:* $G_I^{**}(b) = \frac{G_I^*(b)}{1 - F(\hat{v})}$ and $G_U^{**}(b) = \frac{G_U^*(b)}{1 - \varphi(r)}$. We now show that both of these distributions have tail index equal to 1: $\kappa_I = \kappa_U = 1$. First, the slope of the informed bidder's strategy at the threshold $v = \hat{v}$ can be evaluated as $\frac{d\sigma^*(\hat{v})}{db} = \frac{f(\hat{v})}{F(\hat{v})} \int_{\underline{v}}^{\hat{v}} F(s) ds \in (0, \infty)$. Therefore the right-derivative of the inverse bidding strategy at $b = r$ is also positive, $\frac{d\sigma^{*-1}(r+0)}{db} \in (0, \infty)$. Since we have assumed that F has a positive density, it follows that the cumulative distribution G_I^{**} has a right-continuous density at $b = r$, and therefore its tail index at $b = r$ is $\kappa_I = 1$.

Turning to the uninformed bidders, we first verify that, for $c > 0$,

$$\varphi'(\hat{v}) = \varphi(\hat{v}) \int_{\hat{v}}^\infty \frac{cF(s)^2}{(cF(s)^2 + F(s) \int_{\hat{v}}^s F(u) du)^2} f(s) F(\hat{v}) ds,$$

while for $c = 0$, $\varphi'(\hat{v}) = f(\hat{v})$. In both cases, we can see that $\varphi'(\hat{v}) \in (0, \infty)$. We can see that the right-density $g_U^{**}(r+0) = \varphi'(\hat{v}) \frac{d\sigma^{*-1}(r+0)}{db} \in (0, \infty)$, and therefore $\kappa_U = 1$.

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Figure 1: Bidding strategies and bid densities for CV (left panel) and PV (right panel) models.

Figure 2: Bidding strategies in the asymmetric example. The lower curve corresponds to $B_1(s)$, the upper curve - to $B_2(s)$.

(a) The tail index estimator $\hat{\kappa}_{m_n}^{-1}$. $b(\cdot)$ denotes the asymptotic 90% confidence band based on the variance estimator (kern) or the sample quantiles (quant).

(d) The tail index estimator $\hat{\kappa}_{m_n}^{-1}$. $b(\cdot)$ denotes the asymptotic 90% confidence band based on the variance estimator (kern) or the sample quantiles (quant).

(b) Test rejection frequencies: PV against CV. $r(\cdot)$ is the rejection frequency of the null of PV at the 5% level.

(e) Test rejection frequencies: CV against PV. $r(\cdot)$ is the rejection frequency of the null of PV at the 5% level.

(c) Test rejection frequencies: CV against PV. $r(\cdot)$ is the rejection frequency of the null of CV at the 5% level.

(f) Test rejection frequencies: PV against CV. $r(\cdot)$ is the rejection frequency of the null of CV at the 5% level.

Figure 3: Simulated PV and CV data ($L = 250$). PV = panels (a)-(c), CV = panels (d)-(f).

(a) The tail index estimator $\hat{\kappa}_{m_n}^{-1}$. $b(\cdot)$ denotes the asymptotic 90% confidence band based on the variance estimator (kern) or the sample quantiles (quant).

(b) Test rejection frequencies: CV against PV. $r(\cdot)$ is the rejection frequency of the null of CV at the 5% level.

(c) Test rejection frequencies: CV against PV. $r(\cdot)$ is the rejection frequency of the null of PV at the 5% level

Figure 4: Simulated CV data under Asymmetry ($L = 250$)

(a) The histogram of firms' participation rates.

(b) The histogram of normalized bonus bids.

Figure 5: BCTS participation and bidding

(a) The tail index estimator $\hat{\kappa}_{m_n}^{-1}$ for BCTS data and 90% confidence bands $b(kern)$ computed from the kernel asymptotic variance estimator $\hat{\sigma}_{m_n|L}^2$.

(b) P-values for the t-test of PV $\kappa = 1/2$ on BCTS data.

(c) P-values for the t-test of CV $\kappa = 1$ on BCTS data.

Figure 6: BCTS: tail index test

Table 1: Summary statistics

Variable	Description	Mean	St. Dev.	Min	Max
Auction date				14-Jan-05	14-Dec-06
Bonus bid	Bonus bid in \$ per m ³	10.277	9.947	0.000	63.550
Price	Maximum bid in \$ per m ³	12.749	12.271	0.010	63.550
Reserve price	Upset rate per m ³	28.655	10.858	0.250	50.220
Normalized bid	Bid divided by upset rate	0.457	0.599	0.000	5.278
Normalized price	Price divided by upset rate	0.581	0.740	0.000	411.364
Distance	Distance from bidder's to site location	44.898	52.278	0.000	411.364
Volume	Estimated volume of timber (m ³)	26,372	20,944	200	122,113
Number of bidders		4.560	2.230	1	10
Species composition	Percentage of wood species				
Lodgepole		48.068	38.508	0.000	100.000
Spruce		12.696	15.609	0.000	93.000
Douglas-fir		9.382	20.618	0.000	99.000
Other		29.867	37.449	0.000	100.000
Number of observations		1835			
Number of auctions		611			