Essays in Financial Economics

by
Mohammad Reza Jahan-Parvar

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Economics.

Chapel Hill, NC
2006

Approved by:

A. Ronald Gallant, Advisor
Eric Renault, Reader
Evan W. Anderson, Reader
Amarjit Budhiraja, Reader
William R. Parke, Reader
ABSTRACT
MOHAMMAD REZA JAHAN-PARVAR: Essays in Financial Economics.
(Under the direction of A. Ronald Gallant.)

The unifying theme in both chapters of my dissertation is the dynamic inter-temporal investment and consumption decision making of utility maximizing economic agents. In the first chapter, I solve a general equilibrium model where the agents suspect model mis-specification in characterization of the probability distributions of risky asset returns. The agents solve the consumption-investment optimization problem with the possibility of model mis-specification in mind. I model agents’ optimization process using Cox and Huang martingale approach. This is followed by a calibration study of the theoretical results using US and Japan equity investment flows data. The goal is to assess model adequacy and to study how much uncertainty is required to explain the observed home bias of US and Japanese investors. In the second chapter I solve a partial equilibrium model where the agent faces multi-factor and non-linear stochastic volatility in risky asset returns processes. This task is done by solving the stochastic dynamic Hamilton, Jacobi, Bellman equation associated with the discounted expected lifetime utility of the agent following the approach of Kushner and Dupuis. I study numerical solutions to this problem.
To my wonderful parents, my late mother, Ajideh Parsad-Mehr, and my father, Houshang Jahan-Parvar. And to my dear wife, Rocio Lozano-Arcineigas.
I would like to offer my special thanks to my advisor, A. Ronald Gallant for all the help, encouragement, patience, and support through my years as a graduate student. Amarjit Budhiraja and Evan Anderson supported and guided me when I really needed support and guidance. I am grateful for their generosity. Eric Renault graciously tolerated me, even when I was tedious. Garland Durham believed in me and encouraged me to follow my dream and study financial econometrics. My special thanks goes to Ai-Ru Cheng for constant encouragement and faith in me, even when I doubted myself. I owe a great deal to my wife Rocio Lozano-Arcineigas for patiently and generously supporting and believing in me through good times and hard times. Last but not least, I thank my late mother, Ajideh Parsad-Mehr, and my father, Houshang Jahan-Parvar. You sacrificed a lot to give me this opportunity. All I can say is just a simple “thank you”, and hoping that I have made you happy.
## Contents

List of Figures xiii

List of Tables xv

List of Abbreviations xvii

1 Introduction 1

2 Home Bias Puzzle Revisited: A General Equilibrium Solution Based on Model Mis-Specification 3

2.1 Introduction ................................. 6

2.2 The Model ................................. 13

2.2.1 The Real Economy: ......................... 16

2.2.2 The Financial Economy: ...................... 18

2.2.3 The Agents: .............................. 19

2.2.4 The Wealth Process: ....................... 20

2.3 The Equilibrium ............................ 25

2.3.1 The Definition of Radner Equilibrium in Mis-Specification Economy: ........................... 25

2.3.2 State Prices and Equilibrium Allocations: .......................... 27

2.3.3 Equilibrium Asset Price Processes and Portfolio Holdings: ... 30

2.3.4 The Home Bias in this Model: ...................... 31
2.4 Calibration ................................................. 33
   2.4.1 Data: ................................................. 33
   2.4.2 Method of Calibration: ................................. 35
   2.4.3 Calibration Results: ................................... 41
2.5 Conclusion and Future Research: ................................ 44

3 Portfolio Selection in Presence of Multi-Factor Stochastic Volatility 46
   3.1 The Problem: .............................................. 50
   3.2 Theoretical Model: ........................................ 54
      3.2.1 The General Case of Multi Factor Stochastic Volatility: ... 54
      3.2.2 Utility Structure, Consumption, Portfolio Weights, and Wealth Process: ... 56
   3.3 Derivation of Portfolio Weights: ....................... 58
      3.3.1 Merton Problem: ................................... 59
      3.3.2 Finite Horizon case: ................................ 61
   3.4 Numerical Solutions and Analysis: ...................... 76
      3.4.1 Explicit Functional Forms for Log-Linear SV Model: ........ 77
      3.4.2 The Numerical Algorithm: .......................... 79
      3.4.3 Construction of the Approximating Chain: .............. 81
   3.5 Conclusion and Directions for Future Research: .......... 87

Appendix: .................................................. 93

Bibliography ............................................. 117

xii
List of Figures

2.1 Simulated Sample Paths of $\check{\xi}_t$ ........................................ 28
2.2 Simulated Sample Paths of $\check{\xi}_t$ under Mis-Specification and Certainty . 29
2.3 Plots of Data used in Calibration of the Model Mis-Specification Economy: Consumption ........................................ 36
2.4 Plots of Data used in Calibration of the Model Mis-Specification Economy: Relative Prices and Returns ........................................ 36
2.5 Plots of Data used in Calibration of the Model Mis-Specification Economy: Changes in Equity Investment Flows ...................... 37
3.1 Simulated Returns Process, Volatility, and Calibrated Portfolio Weight 88
3.2 Simulated Returns Process, Volatility, and Calibrated Portfolio Weight, Volatility Factors follow an Ornstein-Uhlenbeck Process ................. 89
3.3 Simulated Returns Process, Volatility, and Calibrated Portfolio Weight, Drift Process is Time-Dependent ........................................ 90
3.4 Simulated Returns Process, Volatility, and Calibrated Portfolio Weight, Volatility Factors follow an Ornstein-Uhlenbeck Process and Drift Process is Time-Dependent ........................................ 91
List of Tables

2.1 Home Bias in Equities: ......................................................... 7
2.2 Lintner Portfolio Weights: ....................................................... 9
2.3 Data Summary of Variables Used in Calibration Exercise: ............... 39
2.4 SNP Parameter Estimates ....................................................... 40
2.5 Calibration Results ............................................................... 42
3.1 EMM Parameters Estimates used in Calibration Study ....................... 87
List of Abbreviations

APM: Approximation Policy Space
CAPM: Capital Asset Pricing Model
CHM: Cox-Huang Martingale Solution
CRRA: Constant Relative Risk Aversion
DJIA: Dow Jones Industrial Average
EMM: Efficient Method of Moments
GBM: Geometric Brownian Motion
GE: General Equilibrium
HBP: Home Bias Puzzle
HJB: Hamilton-Jacobi-Bellman Equation
MCA: Markov Chain Approximation
MCMC: Markov Chain, Monte Carlo Method
MCI: Monte Carlo Integration
PDE: Partial Differential Equation
SDE: Stochastic Differential Equation
SNP: Semi Non-Parametric Estimation
SV: Stochastic Volatility

S&P500: Standard and Poor’s 500
Chapter 1

Introduction

In the first chapter of this dissertation, I propose a partial solution to equity home bias puzzle (HBP) based on economic agent’s fear of model mis-specification. Classical rational expectations models assume that the economic agent has perfect knowledge of the probability distribution of all economic variables. Hence, only risk and return trade off matters in portfolio selection. An important strand of research in finance contends that, in addition to risk, uncertainty is also an important factor in portfolio selection and asset pricing.

I propose a two country, two agent, exchange economy, where, due to either limited number of observations or due to the quality of the available data, the agents are not sure about the “correct” model governing the dynamics of the asset returns processes. To hedge against possible mis-specification of their model, the agents choose portfolio weights and consumption policies which appear as if the agents have a pessimistic view about the foreign country and an optimistic view about the home country. This statement is consistent with survey and cross sectional results. The analytical solution follows Cox and Huang’s martingale solution to the dynamic portfolio choice problem. Model mis-specification requires solving the problem under a mis-specification augmented probability measure, such that both risk and model uncertainty carry a
premium.

The semi-closed form analytical results are then used to measure the level of hedging against model mis-specification by agents that is consistent with data on equity investment flows between US and Japan. Using equity investment flow data from US Department of the Treasury’s TIC data bank and S&P 500 and Nikkei 225 market indices as proxies for market portfolios, I match the moments of the analytical results to a selected number of moments from the data, through a calibration procedure. The goal is to study the effect of levels of model uncertainty induced in the portfolio selection through parametrization of the alternatives to the “reference” probability model estimated from the available data on risky asset returns.

In the second chapter of my dissertation, I develop a theoretical model for optimal portfolio choice in the presence of multi factor stochastic volatility by following and extending Fleming and Hernández-Hernández. I rigourously derive portfolio and consumption rules when several pure diffusion factors govern the volatility dynamics of equity return processes. Chernov, Gallant, Ghysels, and Tauchen show that pure diffusion multi factor volatility for asset returns is a reasonable assumption.

This project is geared toward providing theoretically reliable guidelines for short term portfolio management. I propose a numerically efficient algorithm for computation of portfolio and consumption rules. Numerical solutions of the optimal portfolio weights and consumption rule are explored. Attention is focused on providing rigorous proofs for characteristics of the value function, portfolio weights, and optimal consumption rules, as well as efficient methods of numerical computation of the dynamic systems and control policies. A calibration study follows.
Chapter 2

Home Bias Puzzle Revisited: A General Equilibrium Solution Based on Model Mis-Specification
This paper proposes a general equilibrium solution to the “Home Bias Puzzle” based on new findings on the importance of model uncertainty in portfolio selection. The problem is parameterized under a mis-specification augmented probability measure, such that both risk and model uncertainty carry a premium. The analytical results obtained through applying Cox and Huang method generate results demonstrating home bias in consumption and investment decisions. A calibration study is carried out to explore the fit of analytical results to consumption and equity investment flows data of US and Japan.

**Key Words:** Home Bias, Risk, Uncertainty, Model Mis-Specification, Radner Equilibrium, Cox-Huang Martingale Method

**JEL Classification:** C32, C51, C68, E27, F30, F37, G11, G15.
2.1 Introduction

In this paper, I propose a solution for the long standing “Equity Home Bias Puzzle”, the tendency of investors to hold more domestic equities than would be optimal under traditional mean-variance analysis, based on new findings on the role of model uncertainty. In this paper, model uncertainty manifests itself as model mis-specification concerns. It is increasingly clear that uncertainty is an important factor in asset pricing and portfolio selection (see (Anderson et al., 2005), (Maenhout, 2004), (Anderson et al., 2003)). Traditional mean-variance analysis implies a lower share for domestic securities held by investors compared to observed values (see (French and Poterba, 1991); and (Jaske, 2001)). Moreover, a fundamental assumption of the rational expectations model is that economic agents have exact knowledge of the true probability law for the asset returns, as well as other fundamentals of the economy. Besides, the traditional “homogeneous expectations” assumption in Capital Asset Pricing Model (CAPM) and all the related security pricing literature imply that all investors analyze all securities the same way, share the same view of economic fundamentals, and hence have identical estimates of probability distributions of future dividend streams and cash flows of investing in all the available securities.

I consider a setting where “homogeneous expectations” assumption is relaxed and the agent, given available data, formulates a “reference model” of the probability distributions, but recognizes that it is just an approximation to the “true model” (see (Uppal and Wang, 2003), (Anderson et al., 2003), and (Epstein and Miao, 2003)). Such an approximation is by nature subject to model mis-specification. Notice that the probability distributions extracted from the data and used by econometricians is based on ex post information, while agents form their portfolio and consumption policies based on ex ante information.
Table 2.1: Home Bias in Equities:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>93</td>
<td>89</td>
<td>48</td>
</tr>
<tr>
<td>UK</td>
<td>82</td>
<td>78</td>
<td>8</td>
</tr>
<tr>
<td>Japan</td>
<td>96</td>
<td>91</td>
<td>13</td>
</tr>
<tr>
<td>Germany</td>
<td>79</td>
<td>80</td>
<td>4</td>
</tr>
<tr>
<td>France</td>
<td>90</td>
<td>83</td>
<td>4</td>
</tr>
<tr>
<td>Canada</td>
<td>92</td>
<td>88</td>
<td>3</td>
</tr>
</tbody>
</table>


Building on this setting, this paper contributes to the literature in two directions. First, I develop a framework allowing the investor to view long-term growth rates of the asset returns optimistically for his home country and pessimistically for the foreign country. This view is consistent with findings of (Strong and Xu, 2003). This formulation is also consistent with findings of (Bansal and Yaron, 2004) and (Bansal et al., 2003) on predictability of long-run growth rates of an economy’s fundamental variables. Second, by performing a calibration exercise, the performance of the analytical results is studied in more detail in a general equilibrium (GE) setting for consumption and equity investment flows data of US and Japan in 1977-2004 period. The goal is to choose econometric techniques suitable for estimation of the general equilibrium model based on the findings of this calibration study.

This paper proposes that model uncertainty concerns due to sample size or quality of data, lead the agent to demand a premium for model mis-specification. As a result, a lifetime utility maximizing agent will choose investment and consumption rules which differ from those predicted under model certainty. In this paper we show that an investor, acting rationally but with the knowledge that his information regarding dynamics of different asset returns does not allow him to identify the “true model”, chooses an optimal consumption and investment policy that leads to observationally
“biased” outcomes.

Financial theory asserts that non-systematic risk can be eliminated through diversification. At the international level, the benefits of diversification are noted in early works of (Grubel, 1968) and (Solnik, 1974). Using a simple asset allocation model, (Lintner, 1965) proposes that each country should hold portfolios with identical proportions (but of course, not identical sizes). Together with market clearing conditions, this implies that each country should hold a portfolio in which its home shares weight is equal to the weight of its shares in world market capitalization. This prediction is not corroborated by the data (see Table 2.1).

A comprehensive survey of the home bias puzzle literature is given by (Lewis, 1999). According to (Lewis, 1999), equity home bias is the situation where individuals hold too little of their wealth in foreign assets, leading to sub-optimal international risk sharing. Consumption home bias is the inequality of consumption growth rates and marginal utilities of consumption across countries in equilibrium under iso-elastic utilities and complete markets. (French and Poterba, 1991) formally document this phenomenon. (Griever et al., 2001) and (Jaske, 2001) report more diversification in portfolio holdings during 1990s, but the results are far from (Lintner, 1965) predictions (see Table 2.2).

To this date, there is no evidence of a comprehensive theory able to explain this puzzle, either on the financial or on the real side of the economy. In other words, we have no been able to identify a “smoking gun” (Shore and White, 2006). A possible explanation is that home bias may be the result of numerous factors. Hence a single factor may not provide an overwhelmingly convincing treatment of the puzzle. There is an extensive body of literature on this issue. (Black, 1974) and (Stulz, 1981) studied transaction costs as a possible source of home bias. But measurable transaction costs fail to explain observed home bias (see (Ahearne et al., 2004), and (Warnock, 2002)). (Obstfeld and Rogoff, 2000) propose trading and transportation costs as candidates for
<table>
<thead>
<tr>
<th>Year</th>
<th>$\pi_t^{US}$</th>
<th>$\pi_t^{Japan}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988</td>
<td>41.64</td>
<td>58.36</td>
</tr>
<tr>
<td>1989</td>
<td>44.43</td>
<td>55.57</td>
</tr>
<tr>
<td>1990</td>
<td>51.17</td>
<td>48.83</td>
</tr>
<tr>
<td>1991</td>
<td>56.65</td>
<td>43.35</td>
</tr>
<tr>
<td>1992</td>
<td>65.17</td>
<td>34.83</td>
</tr>
<tr>
<td>1993</td>
<td>63.13</td>
<td>36.87</td>
</tr>
<tr>
<td>1994</td>
<td>57.67</td>
<td>42.33</td>
</tr>
<tr>
<td>1995</td>
<td>65.16</td>
<td>34.84</td>
</tr>
<tr>
<td>1996</td>
<td>73.31</td>
<td>26.69</td>
</tr>
<tr>
<td>1997</td>
<td>83.61</td>
<td>16.39</td>
</tr>
<tr>
<td>1998</td>
<td>84.35</td>
<td>15.65</td>
</tr>
<tr>
<td>1999</td>
<td>78.53</td>
<td>21.47</td>
</tr>
<tr>
<td>2000</td>
<td>82.71</td>
<td>17.29</td>
</tr>
<tr>
<td>2001</td>
<td>85.98</td>
<td>14.02</td>
</tr>
<tr>
<td>2002</td>
<td>83.87</td>
<td>16.13</td>
</tr>
<tr>
<td>2003</td>
<td>82.76</td>
<td>17.24</td>
</tr>
</tbody>
</table>

Asset allocation in a two-country world according to (Lintner, 1965). The values reported are calculated based on USA and Japan’s stock market capitalizations in current US Dollars. The data is from World Bank WDI data set. Values are in percentages.
resolving several puzzles in international macroeconomics, among them trade, equity and consumption home bias. The results, both theoretically and empirically when offered, are mixed and not convincing.

Some researchers argue that hedging demand against shocks to non-tradable goods producing sectors can induce home bias. (Dellas and Stockman, 1989) and (Serrat, 2001) provide elaborate and elegant solutions based on this hypothesis, but the results crucially depend on some technical assumptions such as the separability of utility functions of agents with respect to tradable and non tradable goods. Moreover, (Pesenti and van Wincoop, 2002) find evidence that non-tradable hedging can produce only a small amount of home bias in equity holdings. (Cooper and Kaplanis, 1994) test the inflation risk hedging demand and deviations from purchasing power parity. Their empirical findings turn out to be implausibly high.

(French and Poterba, 1991) believe that equity bias may arise due to two sources: institutional factors may reduce the returns from investing abroad or may explicitly limit the ability of investors to hold foreign equity. Examples are different tax treatment, dividend withholding, transaction costs, explicit limits on cross-border investments. These factors seem to be insignificant or unable to explain the bias. The second possible candidate is investor behavior: a. return expectations vary systematically across countries. b. statistical uncertainties associated by estimating expected returns make it difficult for investors to objectively compare domestic and foreign equities. c. perception of risk in equity markets may differ for domestic and foreign markets. Investors may attach “extra” risk to foreign markets due to lack or cost of information. (French and Poterba, 1991) are of the opinion that the second category may do a better job in explaining home bias.

(Kang and Stulz, 1997) study the cross sectional properties of seemingly biased portfolios and find that almost half of equity investment in both US and Japan is in
small, highly levered firms, producing non-tradable goods. Such firms typically do not attract international investors. Size, availability of operational information, and cost of information gathering deter potential international investors. (Coval and Moskowitz, 1999) introduce the idea of “local bias”, where there is high demand for equity of firms in geographical vicinity of the investor. They propose extending this notion to international finance. Their observations are consistent with findings of (Dahlquist and Robertsson, 2001) who report similar results in Swedish firms. (Grinblatt and Keloharju, 2001) study the effects of language and even culture on local bias in Finland, and their results are significant. Findings from these studies confirms the importance of information asymmetry in home bias research.

The other half of the portfolios are invested in large, internationally recognized firms with global operations. These firms are usually listed in more than one market. If it is possible to get the desired exposure to international markets through holding a portfolio of home country’s multinationals, then there is no need for international diversification. (Errunza et al., 1999) show that it seems plausible to get the desired exposure at home. However, (Cai and Warnock, 2006) show that with a more careful treatment of available data, this home grown foreign gains are not as large as what (Errunza et al., 1999) reported. Investors invest abroad as a hedging against domestic market’s cyclical behavior since the dynamic international correlations are weaker than those between domestic assets. Large multinationals constitute a significant part of domestic markets, hence they are not the best hedging candidates against domestic fluctuations. Findings of (Ahearne et al., 2004) and (Edison and Warnock, 2006) show that multiple listings of multinationals represents a desire for more transparency and reduction of information asymmetry. But then again, (Ahearne et al., 2004) report that even if all actively traded foreign shares were listed in NYSE, 50% of home equity bias would remain unexplained.
Model uncertainty about conditional distributions of asset returns may arise either in long-term growth rates, in the volatility of the assets, or in both. We know that estimating the volatility term can be done cheaply and accurately, given data sets of moderate lengths. On the other hand, the same body of knowledge informs us that it is hard to estimate the drift terms, representing long-term growth rates. Moreover, distinguishing different close models requires very long data sets. Hence, we focus on asymmetry in agents’ views of drift terms as the most probable source of model mis-specification.

Several studies support this decision. (French and Poterba, 1991) consider the possibility that investors view their home country asset returns more favorably then foreign asset returns. Based on survey study of fund managers in US, UK, Japan, and continental Europe, (Strong and Xu, 2003) show that fund managers assign an optimistic view of their home equity and discount foreign equity more severely. (Bravo-Ortega, 2003) shows that information asymmetry in a signalling setting may actually worsen the home bias. Thus, we feel more confident about model uncertainty and mis-specification as good candidates to study home bias.

The dominant paradigm in asset pricing, that of perfect information, risk-based models; has well documented empirical failures. Theoretical work by Epstein and co-authors and Hansen and Sargent and their co-authors point to importance of ambiguity or model mis-specification as a factor as important as risk in investment decisions. Ambiguity and demand for robustness give rise to similar results in optimal portfolios. (Maenhout, 2004) and (Epstein and Miao, 2003) this relationship.

The paper proceeds as follows: in section (2.2) we introduce the model and discuss the link between the reference probability measure and the alternative measures. The optimization problems of agents and solutions to the model based on (Cox and Huang, 1989) are discussed. In section (2.3), we study the equilibrium allocations, the behavior
of price processes for both goods and financial assets, and the behavior of equilibrium portfolio weights. Section (2.4) discusses calibration of the model, data, and findings. In this section, we construct a two-country, two-agent world and then study the features of the model, based on data from US and Japan. Section (2.5) concludes the findings and discusses the future research directions.

2.2 The Model

In a two country, two agent, exchange economy, following and modifying (Merton, 1971), (Merton, 1969), and (Lucas, 1978), agents face uncertainty about the choice between close estimates of correct probability distribution of excess returns, as in (Hansen and Sargent, 2004) and (Epstein and Miao, 2003). This section derives a decomposition of excess returns into risk and model uncertainty. The main idea is that the agent makes decisions, *ex ante*, under a probability measure different from what the econometrician extracts from the available data, *ex post*. Similar to (Anderson et al., 2003), (Hansen and Sargent, 2004), and (Hansen et al., 2006), model uncertainty is related to conditional volatility. Agents are more uncertain about the model in periods of high volatility.

Throughout the paper, random variables are defined over probability space \((\Omega, \mathcal{F}, P)\). The random variables are indexed by \(t \in [0, T]\) and \(T < \infty\). Also, assume that there exists an augmented filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) and \(\mathcal{F}_T = \mathcal{F}\). Following (Maenhout, 2004), and (Uppal and Wang, 2003), we assume that the *reference* probability measure \(P\) is the probability measure that agent acknowledges as useful, but possibly mis-specified. The agent believes that at least there is one *alternative measure* \(A^\xi\). Both measures
are defined over the measurable set \((\Omega, \mathcal{F})\). The two measures are related through:

\[
\frac{dA^\xi}{dP} = \xi(\theta, Z_t) = \exp\{-\frac{1}{2} \int_0^t ||\theta(Z_s)||^2 ds - \int_0^t \theta(Z_s) dW_s\}
\]

Where \(Z_t = (Z^1_t, ..., Z^d_t)'\) is a set of state variables, \(\theta \in \Theta\) is a \(\mathbb{R}^d\)-valued process for which the process \(\xi(\theta, Z_t)\) is a \(P\)-martingale, and \(\xi(\theta, z)\) is a density function. One may alternatively consider \(\xi\) as a scalar perturbing \(P\) and view this law as the joint distribution of all state variables.

By assigning a family of possible probability laws to a decision maker, we may distinguish between Knightian uncertainty (uncertainty about the probabilistic model) and risk (uncertainty about probabilistic outcomes). An example of early work on this issue is (Hansen et al., 1999).

The agent needs a very large data set to distinguish the true model from all “close” alternatives. Since portfolio selection is usually done with limited data\(^1\), concerns for model mis-specification are valid. The agent would choose his portfolio with the possibility of mis-specification of the dynamics in mind. (Hansen et al., 2002) present this view in an intuitive fashion. Suppose that there are three model dynamics specified: the unknown true model, an approximating model based on available data, and a constrained worst-case model. Also assume that these three models are related through the dynamics of a class of SDEs as follows:

\[
dZ_t = [\mu(Z_t) - \phi_t]dt + \sigma(Z_t)dW_t
\]

(2.1)

Assume that \(\phi_t = \tilde{\phi}_t\) for the true model, \(\phi_t = \hat{\phi}_t\) for the worst-case model, and

---

\(^1\)Most individual investors invest based on very short data sets. Many fund managers base their decisions on three years of returns data. In some unusual cases, up to five years of data are used. These sample lengths are not long enough to mitigate the fear of model mis-specification.
\( \phi_t = 0 \) for the approximating model. The agent has done some preliminary searching, hence \( \hat{\phi}_t \) is constrained by the method discussed in (Anderson et al., 2003). The agent knows that the true model lies within the closure of approximating and worst-case models (see (Hansen et al., 1999), (Hansen et al., 2002), and (Hansen and Sargent, 2007)). Presence of model mis-specification leads the agent to choose decision rules which work for a set of close models.

On the other hand, one may view the same problem as the outcome of model ambiguity discussed in (Epstein and Miao, 2003). In this formulation, after learning all they can about returns that are conditional on the states of the world, agents still have ambiguity regarding the returns. This feature results in adjustments in expected future discounted utility’s volatility. While they are methodologically different, the final results of both approaches are very close.

Unlike Hansen, Sargent or Epstein and their co-authors, in our formulation agents have differing views on long-term growth rates of asset returns. Hence, instead of “worst-case scenario” we call agent’s effective probability measure the alternative model. Since the true model is unknown and falls within the closure of reference and alternative measures, we do not use explicit notation to represent the “true” model.

Denote \( W_t = (W_t^1, \cdots, W_t^d)' \) to be a \( d \)-dimensional Wiener process defined over \((\Omega, \mathcal{F}, \mathbb{P})\), driving the dynamics of the dynamic system studied here. But the driving process is a Wiener process under probability measure \( \mathbb{P} \) and if there are concerns for mis-specification, finding the true driving process is not straightforward. Under the alternative model, \( \mathbb{A}^\xi \), flow of information follows \( W_t^A = (W_t^{1,A}, \cdots W_t^{d,A})' \), a \( d \)-dimensional process. If \( W_t \) and \( W_t^A \) are related through \( W_t^A = W_t + \int_{s=0}^{t} \theta(Z_s)ds \), then \( W_t^A \) is a Wiener process under \( \mathbb{A}^\xi \). Thus ambiguity concerns are limited to the drift process. This is due to Wiener environment and assumption of absolute continuity (see (Epstein and Chen, 2002)).
2.2.1 The Real Economy:

Consider a two country world, one called the domestic country and the other called foreign country. Each country is inhabited by a single representative agent. Call these agents “domestic” and “foreign agent” respectively. We denote domestic agent by (1), and the foreign agent by (2). The agents are heterogeneous due to their different preferences, endowments, and consumption sets. Agents consume/invest continuously over time interval [0, T]. Each country is endowed with a firm producing a perfectly tradable consumption good. We assume that production technology is known and it does not change over the [0, T] time period and we assume that production and transportation of the good is free of cost. Hence, without loss of generality, we may consider this economy as a pure exchange economy with stochastic endowment processes.

Denote the quantities of each good at time \( t \) by \( \varrho_t = (\varrho_1^t, \varrho_2^t)' \). We assume that \( \varrho_t \) follows a multivariate Geometric Brownian Motion (GBM):

\[
d\varrho_t = \text{diag} \varrho_t [\mu_t dt + \sigma_t dW_t] \tag{2.2}
\]

Where \( \mu_t \) is a 2 × 1 vector and \( \sigma_t \) is a full rank, 2 × 2 matrix. Both \( \mu_t \) and \( \sigma_t \) may or may not be functions of other state variables. Under the alternative probability law, \( A^\xi \), each agent views the endowment process following:

\[
\begin{pmatrix}
    d\varrho_1^{1,A}_{t} \\
    d\varrho_2^{2,A}_{t}
\end{pmatrix} = \begin{pmatrix}
    \varrho_1^{1,A}_{0} & 0 \\
    0 & \varrho_2^{2,A}_{t}
\end{pmatrix} \begin{pmatrix}
    \mu_1^t \\
    \mu_2^t
\end{pmatrix} + \begin{pmatrix}
    \sigma_{11} & \sigma_{12} \\
    \sigma_{12} & \sigma_{22}
\end{pmatrix} \begin{pmatrix}
    \nu_1^{1,i} \\
    \nu_2^{2,i}
\end{pmatrix} dt + \begin{pmatrix}
    \sigma_{11} & \sigma_{12} \\
    \sigma_{12} & \sigma_{22}
\end{pmatrix} \begin{pmatrix}
    dW_1^1 \\
    dW_1^2
\end{pmatrix} \tag{2.3}
\]

Where \( \sigma_t \nu_t^i \) is the adjustment for change of measure, as discussed above.

In this representation for alternative model, we follow (Anderson et al., 2003), (Maenhout, 2004), and (Uppal and Wang, 2003). Define this process as \( \nu_t^i = (\nu_1^{i,1}, \nu_2^{i,2})' \).
For simplicity, it may be assumed that $\nu^i_t = \nu^j$. Based on anecdotal evidence from surveys conducted by Survey of Affluent Americans\textsuperscript{2}, Affluent Investor Index\textsuperscript{3}, and World Wealth Report\textsuperscript{4} and empirical evidence from studies of (Kang and Stulz, 1997), (Coval and Moskowitz, 1999), (Grinblatt and Keloharju, 2001), and (Ivković and Weisbenner, 2005) assume symmetric $\nu^i_t$s for investors with respect to distance:

\[
\begin{align*}
\nu_{t}^{11} &= -\nu_{t}^{21} \\
\nu_{t}^{12} &= -\nu_{t}^{22}
\end{align*}
\]

(Epstein and Miao, 2003) assumes that the agent is uncertain about the foreign returns, but certain about home returns, setting $\nu_{t}^{11}$ and $\nu_{t}^{22}$ equal to zero and $\nu_{t}^{12}$ and $\nu_{t}^{21}$ as constants. On the other hand, in this formulation the sign is more important than the size. Meaning that empirically, $\nu_{t}^{12}$ and $\nu_{t}^{21}$ are negative and $\nu_{t}^{11}$ and $\nu_{t}^{22}$ are positive. This implies that on the equilibrium path, the investor is relatively optimistic towards his home country’s assets and is relatively pessimistic towards foreign assets as reported by (Strong and Xu, 2003).

Choose the domestic good as the numeraire. Endowments can be bought and sold at price $p^i_t$ where $i = 1, 2$. Normalize the price of the domestic good to one. Hence the price process for consumption goods (endowments) at any given time $t$ is $p_t = (1, p^2_t)'$. Assume that $\int_0^T p_t dt < \infty$ a.s.

\textsuperscript{3}Spectrum Group Inc. Affluent Investor Index \textcopyright(2002-2004)  
\textsuperscript{4}Merrill Lynch/Cap Gemini Ernst and Young World Wealth Report \textcopyright2002.
2.2.2 The Financial Economy:

Each agent can trade the shares of domestic and foreign firms free of cost. There is a market for instantaneous borrowing/lending with elastic supply. This is equivalent to existence of a risk free asset, paying a (real) risk free rate which is equal across countries. The outcomes of this market are exogenous and actions taken by the agents do not affect these outcomes. Shares of the firms are called equity. Equities are traded, hence they need to be priced such that equities market is cleared continuously. We denote the equity price process as a $2 \times 1$ vector: $P_t = (P^1_t, P^2_t)$. Value of the endowments at each time $t$ is $\varrho'_t \varrho_t = (\varrho^1_t, \varrho^2_t)$'. Denote this process by $\hat{\varrho}_t$, a $2 \times 1$ vector, and call it the dividend process. The agents accumulate wealth from “gains” that they earn from changes in price process of the equities and from accumulated dividend processes. Define the sum of the price process of the equities and the accumulated dividend process as the “gains” process. Since the information flow in this model is Brownian, then the gains process will be an Itô process. Moreover, assume that the process has an absolutely continuous bounded variation component. Thus, the gains process follows:

$$dG_t = dP_t + \hat{\varrho}_t dt = \text{diag} P_t (\mu^G_t dt + \sigma^G_t dW_t)$$ (2.4)

If processes $\mu^G_t$ and $\sigma^G_t$ are adapted and satisfy $\int_{t=0}^{T} (||\mu^G_t|| + ||\sigma_t||) dt < \infty$ a.s., then the above representation of the gains process is allowed. $\sigma_i$ stands for row vectors of $\sigma^G_t$. We also assume that $\sigma_i$ processes are progressively measurable with respect to $\{\mathcal{F}_t\}$, uniformly bounded in $[0, T], \forall t \in [0, T]$ and $\forall \varsigma \in \mathbb{R}^N$ they satisfy $\varsigma'\sigma_t\sigma_t'\varsigma \geq \epsilon ||\varsigma||^2$. Throughout, we assume that $\sigma^G_t = \Upsilon_t' \sigma$. If the $\Upsilon_t$ process follows an explicit stochastic process, then the solution admits stochastic volatility (SV). Constant or deterministic specifications for $\Upsilon_t$ are clearly admissible. Both $\mu^G_t$ and $\sigma^G_t$ are endogenous and are found as part of equilibrium solution. Characterization of $\sigma^G_t$ is equivalent to estimation.
of \( \sigma \) and identification of \( \Upsilon_t \).

Under the alternative model, the gains process follows:

\[
dG_t^i = dP_t + \hat{\nu}_t^A dt = \text{diag}_P[(\mu_t^G + \sigma_t^G \nu_t^i) dt + \sigma_t^G dW_t]
\]  

(2.5)

Values of equity prices are determined on the equilibrium path following standard arguments. Correct specification of \( \sigma_t^G \) process after removing the drift term from the gains process through a change of measure, solves this problem.\(^5\) Assume that risk free asset follow:

\[
dB_t = r_t B_t dt
\]

Without loss of generality, assume that risk free rate itself is not mis-specified and is constant, hence \( r_t = r \).

2.2.3 The Agents:

Denote each agent’s consumption by:

\[
C_t^i = \begin{bmatrix}
c_t^{i,1} \\
c_t^{i,2}
\end{bmatrix}
\]

Such that \( i = 1, 2 \), \( C_t^i \) is adapted and satisfies \( \int_{t=0}^{T} C_t^i dt < \infty \). Each agent chooses consumption policies optimally at each point in time, \( t \). Agents’ preferences are represented through utility functionals:

\[
U^i(C_t^i) = \begin{cases} 
\frac{(c_t^{i,1})^{1-\gamma_i} + (c_t^{i,2})^{1-\gamma_i}}{1-\gamma_i} & : \text{if } \gamma_i \neq 1 \\
\ln(c_t^{i,1}) + \ln(c_t^{i,2}) & : \text{if } \gamma_i = 1
\end{cases}
\]

\(^5\)See (Elliott and Kopp, 2005) for a general discussion.
And discounted lifetime expected utility of each agent is:

$$V^i_t(C^i_t) = \begin{cases} 
E \left[ \int_{t=0}^{T} e^{-\beta_i t} \left( \frac{(c_{i,1}^t)^{1-\gamma_i} + (c_{i,2}^t)^{1-\gamma_i}}{1-\gamma_i} \right) dt \right], & \gamma_i \neq 1; \\
E \left[ \int_{t=0}^{T} e^{-\beta_i t} [\ln(c_{i,1}^t) + \ln(c_{i,2}^t)] dt \right], & \gamma_i = 1.
\end{cases}$$

Where $\beta_i$ is a constant, exogenous discount factor, $\pi^i_t$ is the vector of portfolio weights invested in domestic and foreign risky assets, and $c^i$ is as discussed above. Throughout the paper, without loss of generality and for ease of derivations, assume that $\gamma_1 = \gamma_2 = \gamma$ and $\beta_1 = \beta_2 = \beta$.

### 2.2.4 The Wealth Process:

The initial wealth of each agent is the market value of the firm located in his country at time $t = 0$, denoted by $X^i_0$. This value is assumed to be $X^i_0 \geq 0$. (Epstein and Miao, 2003) allow this value to be equal to the negative of total value of non-tradeable endowment. These assumption are equivalent in binding the value of initial wealth away from negative numbers. Each agent’s portfolio, invested in his home and foreign risky equities, is represented by:

$$\pi^i_t = \begin{bmatrix} \pi_{i,1}^t \\ \pi_{i,2}^t \end{bmatrix} \quad \text{(2.6)}$$

We will derive the semi-closed functional forms of portfolio process as part of the solution to this problem.

**Definition 1:** $(\pi^i, C^i)$ is an admissible strategy if it is a $\mathcal{F}_t$-progressively measurable process, such that $P [ \forall t \in [0, T], |\pi^i_t| \leq A_1, 0 \leq C^i_t \leq A_2 ] = 1$, where there exist $A_1, A_2 \in \mathbb{R}^+$. Denote the set of admissible strategies by $\mathcal{A}$. 

20
Moreover, assume that \( \int_{t=0}^{T} ||\pi_t^i|| dt < \infty \). Given admissible consumption \((C_t^i)\) and portfolio \((\pi_t^i)\) processes, the wealth process of agent \(i\) is:

\[
\frac{dX_t^i}{X_t^i} = \left[ \pi_t^i \left( \mu_t^G + \sigma_t^G \nu_t^i - r_t \mathbf{1} \right) + r_t - \frac{(p_t' C_t^i)'}{X_t^i} \right] dt + \pi_t^i \sigma_t^G dW_t
\]

This process characterizes agents’ dynamic budget constraint\(^6\).

**Definition 2:** If conditions mentioned in Definition 1 hold, \( \int_{t=0}^{T} ||\pi_t^i|| dt < \infty \), and the solution to (2.7), \(X_t^i\), is such that for \(\forall t \in [0, T]\), \(X_t^i \geq 0\)\(^7\) and for arbitrary wealth level \(\hat{X}_t^i\) we have \(X_t^i \geq \hat{X}_t^i\); then we say portfolio process \(\pi_t^i\) finances consumption process \(C_t^i\).

Consider the conditions mentioned above as regularity requirements for existence of solutions to the model. Progressive measurability is an information requirement. Boundedness of portfolio process guarantees the existence of a solution for (2.7). The last condition is a budget constraint.

Under these conditions, the problem of each agent is to choose an adapted and non-negative consumption process \(c_t^i\), given \(X_0^i\) and such that there exists a portfolio process \(\pi_t^i\) that finances \(C_t^i\), and such that \(V^i(C_t^i)\) is maximized.

If we have accurate estimations of gains process’ dynamics, this dynamic optimization problem can be solved either through stochastic optimal control methodology (for example using (Kushner and Dupuis, 2001)), or one may simplify the problem considerably by using (Cox and Huang, 1989) martingale method. In this project the martingale solution is used since this method allows for solving a static problem to identify the optimal consumption and then finds the optimal portfolio financing the optimal consumption bundle. The alternative requires solving a stochastic dynamic control with

\(^6\)Notice that setting \(\nu_t^i = 0\) delivers the familiar wealth process.

\(^7\)This is equivalent to (Epstein and Miao, 2003) no bankruptcy assumption.
two controls and at least two state variables (if we assume constant volatility for the diffusion processes). The reader should bear in mind that solving the static problem is computationally far less intensive than solving the optimal control. For a discussion of computing optimal policies using stochastic control theory, the reader may refer to (Jarvis and Kushner, 1996), and (Kushner, 1998). As long as the assumptions of each method are not violated, the final results are equivalent.

In order to use martingale solutions, following (Cox and Huang, 1989), admissible consumption processes should satisfy the following budget constraints for both domestic and foreign agents.

\[
\mathbb{E}\left( \int_{t=0}^{T} p_t^i \hat{\xi}_t^i \, dt \right) \leq \mathbb{E}\left( \int_{t=0}^{T} \varrho_t^i \hat{\xi}_t^i \, dt \right) \quad (2.8)
\]

\[
\mathbb{E}\left( \int_{t=0}^{T} p_t^i C_t^1 \hat{\xi}_t^i \, dt \right) \leq \mathbb{E}\left( \int_{t=0}^{T} p_t^i C_t^2 \hat{\xi}_t^i \, dt \right)
\]

The processes \( \hat{\xi}_t^i \) represents Arrow-Debreu state price densities under the alternative model. We will derive the exact functional form of these state price densities in our discussion of the equilibrium. This processes can also be represented as a risk free rate augmented Girsanov transformation. Under constant drift and diffusion of the wealth process (or gains process), the market price of risk is constant. Let the set of admissible consumption processes for agent \( i \) be \( \mathcal{C}^i(X_t^i) \). Under the alternative model we have:

\[
\hat{\xi}_t^i = \exp \left( - \int_{s=0}^{t} r_s \, ds - \frac{1}{2} \int_{s=0}^{t} \| \theta_s^i \| \, ds - \int_{s=0}^{t} \theta_s^i \, dW_s \right)
\]

Where \( \theta_t^i = (\sigma_t^G)^{-1}(\mu_t^G + \sigma_t^G \nu_t^i - r_t \mathbf{1}) \). Martingale approach requires that both \( r_t \) and \( \theta_t \) be uniformly bounded. Notice that if \( \nu_t^i = 0 \) (model certainty), then one will have the familiar state-price density formulation. Incorporating the \( \sigma_t^G \nu_t^i \) term in \( \theta_t^i \) assigns a premium to model mis-specification. Hence, we call this new expression “instantaneous
market price of risk and model mis-specification”.

It should be noted that our formulation leads to a conceptually different optimization problem compared to the class of problems studied in (Anderson et al., 2003), (Maenhout, 2004), or (Uppal and Wang, 2003). In robust portfolio literature, the agent is assumed to solve a two-tier problem. The agent plays a “Min-Max” game where he first minimizes a “distance” variable denoted by \( u \) in (Maenhout, 2004) and by \( \hat{g} \) in (Anderson et al., 2003). He then proceeds to maximize his life-long expected discounted utility by choosing optimal consumption and investment policies.

The “penalty” to keep the agent from acting too cautiously and discounting risky assets too steeply is through some entropy measure, denoted in (Maenhout, 2004) by \( \Psi(X, t) \). He gets “penalized” in consumption if he is too pessimistic. We formulate this problem differently. In our formulation, the agent has already done the searching to minimize the distance variable (\( u \) in (Maenhout, 2004)). Instead of directly penalizing the agent through consumption, we offer a more intuitive explanation: since the agent is pessimistic about the foreign country, he knowingly does not fully diversify his consumption and investment. This leads to “unrealized utility” rather than direct lost consumption.

This transformation is done through his pricing kernel or state price of risk and model mis-specification, \( \hat{\xi}_t \). That is, his forecasts for expected returns on the foreign country are lower than what the ex-post data shows. In a sense, our formulation has a slight heterogeneous beliefs flavor. This is the most important difference between this paper and robust portfolio choice literature. It should be noted that no arbitrage requirement implies that both agents should face the same market price for equities.

**Remark 2.2.1** Absence of arbitrage means that \( \hat{\xi}^1_t \) and \( \hat{\xi}^2_t \) should price an equity similarly. In other words, although the functional forms of state-price densities are different,
they should assign the same price on an the same asset. In order to get this condition, agents should have a similar view of perturbed drift terms under the alternative measures. We earlier assumed that $\nu_{11}^t = -\nu_{21}^t$ and $\nu_{12}^t = -\nu_{22}^t$. Under these conditions, our requirement indeed holds. The restriction that we obtain for this condition is:

$$(\sigma_{11} + \sigma_{12}^G)\nu_{11}^t + (\sigma_{22}^G + \sigma_{12}^G)\nu_{12}^t = \mu_2^G - \mu_1^G \tag{2.10}$$

or after rearranging:

$$\nu_{11}^t = \varsigma_1 + \varsigma_2 \nu_{12}^t \tag{2.11}$$

where $\varsigma_1 = (\mu_2^G - \mu_1^G)/(\sigma_{11} + \sigma_{12}^G)$ and $\varsigma_2 = (\sigma_{22}^G + \sigma_{12}^G)/(\sigma_{11} + \sigma_{12}^G)$

Refer to appendix A.1 for a more detailed discussion. Hence, we may assume that state-price densities are equal for both agents almost surely.

The following result is well known.

**Theorem 2.2.2 (Cox-Huang (1989))** We have $C_t^i \in \mathbf{C}^i(X^i_0)$ if and only if there exists an admissible portfolio $\pi_t^i$ financing $C_t^i$. Moreover, this portfolio satisfies:

$$\pi_t^i \sigma_t^G = \dot{\xi}_t^{-1} \kappa_t^i + X_t^i \theta_t \tag{2.12}$$

where $\kappa_t^i$ is the process arising in the martingale representation of the process

$$\zeta_t^i = \mathbb{E}_t^P \left( \int_s^T p_t^i C_s^i \dot{\xi}_s^i ds \right) - \mathbb{E}_t^P \left( \int_s^t p_t^i C_s^i \dot{\xi}_s^i ds \right) \tag{2.13}$$

i.e. an adapted process that satisfies $\zeta_t^i = \int_{s=0}^t \kappa_s^i dW_s$ a.s.

**Definition 3:** Following Serrat (2001), call $\Pi_t^i = \pi_t^i \sigma_t^G$ the “portfolio generating kernel” process of agent $i$. 

24
Now we solve the static problem for finding the optimal consumption rules, $C^i_t$:

$$\sup_{c^i_t \in \mathcal{C}^i(X^i_0)} \mathbb{E} \int_{t=0}^{T} e^{-\beta t} \left[ (c^i_1)^{1-\gamma} + (c^i_2)^{1-\gamma} \right] \frac{dt}{1-\gamma}$$  \hspace{1cm} (2.14)

The next step is to identify the unique portfolio processes that finance the solutions of (2.14).

### 2.3 The Equilibrium

#### 2.3.1 The Definition of Radner Equilibrium in Mis-Specification Economy:

The equilibrium concept here is an adaptation of (Serrat, 2001). We require continuous clearing of consumption goods and risky assets markets, to have a Radner equilibrium implying that agents make their policy decisions for entire time horizon $[0,T]$ at $t = 0$. This definition of equilibrium is consistent with existence of state-price deflator and Martingale methodology used. Walras’ law implies that if all the above mentioned markets clear, in this (Lucas, 1978) exchange economy, the market for instantaneous borrowing and lending clears as well, hence the bond market is in equilibrium.

An equilibrium is defined as an array of stochastic processes $(\{c^1_t\}, \{c^2_t\}, \{\pi^1_t, \pi^2_t\}, P_t, p_t, \hat{\xi}_t)$ such that $c^i_t \in M^i(g, p, \hat{\xi}, X^i_t) \subset \mathcal{C}^i(X^i_t)$ for $i = 1, 2$ and all markets clear:

$$c^{1,j}_t + c^{2,j}_t = g^j_t; \hspace{0.5cm} X^i_t = P^{i,1}_t + P^{i,2}_t \equiv \pi^{i,j}_t . P_t; \hspace{0.5cm} \pi^1_t + \pi^2_t = P_t$$  \hspace{1cm} (2.15)

(Serrat, 2001) shows that if:

i: There exists some adapted process $(c^i_t, P_t, p_t, \hat{\xi}_t)$ such that $c^i_t \in \mathcal{C}^i(X^i_0)$ and markets for consumption goods clear,
ii: Agents finance their consumption through portfolio policies arising from portfolio generating kernels, \( \pi_i \sigma_i^G = \frac{1}{\xi_t} \kappa_i^t + X_i^t \theta_t \).

then, (2.15) holds.

From (Serrat, 2001), we know that clearing of the goods market implies clearing of the financial markets, a remarkable simplification of the problem. In view of this result, we need to prove the existence of equilibrium by showing that a unique price process, \( p_t \) exists that clears the goods markets. Notice that (Kollman, 2006) criticism of (Serrat, 2001) does not affect this result. (Kollman, 2006) has criticisms regarding final derivation of (Serrat, 2001) portfolio weights under assumption of separability of traded and non-traded goods in Cobb-Douglas utility structure.

We take the following steps in solving the problem: first, we compute the equilibrium consumption process of consumption goods, which produces the equilibrium pricing kernel, \( \hat{\xi}_t \), and the relative price process, \( p^2_t \). Second, we use the results from (Serrat, 2001) to identify the portfolio policies on the equilibrium path.

**Consumption Optimization Problem:**

Under the reference model, our dynamic consumption optimization problem transforms into a static optimization problem by (Cox and Huang, 1989) martingale methods:

\[
\sup_{c^1_t \in \mathcal{C}(X^0_t)} \mathbb{E} \int_{t=0}^{T} e^{-\beta t} \left[ \frac{(c^1_t)^{1-\gamma} + (c^2_t)^{1-\gamma}}{1-\gamma} \right] dt \\
\text{s.t.} \quad \mathbb{E} \left( \int_{t=0}^{T} (c^1_t + p^2_t c^2_t) \hat{\xi}_t dt \right) \leq \mathbb{E} \left( \int_{t=0}^{T} p^i_t \xi_t dt \right)
\]

(2.16)

Where \( p^1_t \) is normalized to 1. The first order conditions of this optimization problem
imply that optimal consumption for agent $i$ is:

$$c_i^1 = (e^{\beta t i \lambda_i^*})^{-1/\gamma} \quad (2.17)$$

$$c_i^2 = (e^{\beta t i \lambda_i^2 \hat{\xi}_i})^{-1/\gamma} \quad (2.18)$$

Where $\lambda_i$ is the Lagrange multiplier pertaining to agent $i$’s optimization problem.

On the equilibrium path, $\lambda_i^*$ is of the form:

$$\lambda_i^* = \left[ \frac{E_t \int_{s=t}^{T} g_{s}^1 \hat{\xi}^1_s ds}{E_t \int_{s=t}^{T} \exp \left( -\frac{\beta s}{\gamma} \right) \hat{\xi}^1_s \left[ 1 + (p_{s}^2)^{-(1-\gamma)} \right] ds} \right]^{\gamma} \quad (2.19)$$

$$\lambda_i^* = \left[ \frac{E_t \int_{s=t}^{T} g_{s}^2 \hat{\xi}^2_s ds}{E_t \int_{s=t}^{T} \exp \left( -\frac{\beta s}{\gamma} \right) \hat{\xi}^2_s \left[ 1 + (p_{s}^2)^{-(1-\gamma)} \right] ds} \right]^{\gamma} \quad (2.20)$$

### 2.3.2 State Prices and Equilibrium Allocations:

From first order conditions of consumption optimization problem and market clearing conditions, we derive the equilibrium Arrow-Debreu state-price densities (in terms of the numeraire) under reference and alternative models:

$$\hat{\xi}_i^1 = (g_{i}^1)^{-\gamma} \left[ \exp \left( -\frac{t}{\gamma} \beta \right) \left( \lambda_i^* - \frac{1}{\gamma} \lambda_i^2 \right) \right]^{\gamma} \quad (2.21)$$

$$\hat{\xi}_i^2 = \frac{(g_{i}^2)}{p_{i}^2} \left[ \exp \left( -\frac{t}{\gamma} \beta \right) \left( \lambda_i^* - \frac{1}{\gamma} \lambda_i^2 \right) \right]^{\gamma} \quad (2.22)$$

From Figure (2.1) it is clear that while state-price densities demonstrate similar behavior, they do not price a dividend stream in precisely the same manner, as required in the model. Moreover, from Figure (2.2) it can be seen that there is a significant and persistent difference between the behavior of state-price densities under $P$ and $A$ probability measures.
The equilibrium real price process, under all measures, follows:

\[ p_t^2 = \left( \frac{\varrho_t^2}{\varrho_t^1} \right)^\gamma \quad (2.23) \]

We expect this result since relative prices depend on the endowment processes in the real economy, which are unaffected by agents’ perception of risk and uncertainty. It is of interest for us to know how the dividend process is divided by the agents. Equilibrium consumption processes along with market clearing conditions and price processes allow us to find the sharing rule in this economy.

On the equilibrium path, agents divide the endowment processes of the economy and their equilibrium market clearing consumption satisfies the following under the reference model.

\[ c_t^1 = \alpha_t \varrho_t \quad (2.24) \]

\[ c_t^2 = (1 - \alpha_t) \varrho_t \]
Where $\alpha_i^t$ is:

$$
\alpha_i^t = \frac{\left( \left[ \left( \frac{\lambda_i^2}{\lambda_i^1} \right) \left( \frac{\hat{\xi}_1^i}{\hat{\xi}_2^i} \right) \right]^{-1/\gamma} \right)}{1 + \left( \left[ \left( \frac{\lambda_i^2}{\lambda_i^1} \right) \left( \frac{\hat{\xi}_1^i}{\hat{\xi}_2^i} \right) \right]^{-1/\gamma} \right)} \tag{2.25}
$$

or equivalently,

$$
\alpha_i^t = \frac{c_{i,j}^i}{\varrho_t^j} \tag{2.26}
$$

and $i, j = 1, 2$.

**Proposition 2.3.1** An equilibrium exists up to a nominal scaling, regardless of absence or presence of model mis-specification.

**Proof:** The proof in both cases closely follows the results reported in (Serrat, 2001). A detailed discussion is reported in appendix A.2.

This proposition establishes the existence of a unique equilibrium path price processes which clears the goods market in this economy. Since (Serrat, 2001) establishes that clearing of the goods market implies clearing of the equities market, then there exists a unique equity price process which clears the equities market. Call this process $P_t = [P_t^1, P_t^2]^\prime$. 

29
At this stage, given the well known properties of pricing under different probability measures\(^8\) it suffices to correctly specify the diffusion matrix of the gains process in order to have closed form values for asset prices.

### 2.3.3 Equilibrium Asset Price Processes and Portfolio Holdings:

Following (Serrat, 2001), we find the equilibrium path values for the portfolio weights under the reference model. The reader should note that as a check, for constant volatility case, these results are easily obtained from first order conditions of the Hamilton, Jacobi, Bellman equation associated with gains process defined earlier. Through martingale solution approach, we find an alternative solution in terms of conditional expectations, which is more suitable for our calibration study.

\[
P^1_t = \mathbb{E}_t \left[ \int_{s=t}^{T} \varrho_s \xi^i_s \xi^j_t ds \right] \quad (2.27)\\
P^2_t = \mathbb{E}_t \left[ \int_{s=t}^{T} p^2_s \varrho^2_s \xi^i_s \xi^j_t ds \right] \quad (2.28)
\]

\[
\pi^1_t = \begin{bmatrix} \pi^{1,1}_{t,1} \\ \pi^{1,1}_{t,2} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_t \left[ \int_{s=t}^{T} \alpha_s \varrho_s \xi^i_s \xi^j_t ds \right] \\ \mathbb{E}_t \left[ \int_{s=t}^{T} \varrho^2_s \xi^i_s \xi^j_t ds \right] \end{bmatrix} \quad (2.29)\\
\pi^2_t = \begin{bmatrix} \pi^{2,1}_{t,1} \\ \pi^{2,2}_{t,2} \end{bmatrix} = \begin{bmatrix} \mathbb{E}_t \left[ \int_{s=t}^{T} (1 - \alpha_s) \varrho_s \xi^i_s \xi^j_t ds \right] \\ \mathbb{E}_t \left[ \int_{s=t}^{T} (1 - \alpha_s) \varrho^2_s \xi^i_s \xi^j_t ds \right] \end{bmatrix} \quad (2.30)
\]

The following lemma verifies the results. It is based on the Radner equilibrium conditions introduced in (2.15) and given (2.2.1) restriction.

**Lemma 2.3.2** ((Serrat, 2001)) Equilibrium path portfolio weights are (2.29) and (2.30).

\(^8\)For example, refer to (Elliott and Kopp, 2005)
Proof: In Appendix.

Since portfolio weights are expressed as conditional expectations of state variables \((\varrho_1^t, \varrho_2^t)\) and fundamental parameters of the economy \((\gamma, \beta, \mu^G, \sigma^G)\), we can solve for these expectations. The question will be ease of computation and choice of the method.

There have been some attempts to estimate similar policies. (Brandt, 1999) proposes a conditional Euler equation estimation technique for finding optimal consumption and portfolio weight policies. His approach involves a non-parametric estimation of optimal policies. We compute optimal portfolio weights through a non-linear and parametric framework.

2.3.4 The Home Bias in this Model:

Home bias in a mis-specification economy arises from uncertainty about model specification. If the agent regards the reference model as informative but misspecified, and proceeds to form expectations based on the alternative (worst case scenario model), as we observed in derivation of portfolio weights, the equilibrium path consumption and portfolio rules under reference and alternative model are not the same.

As mentioned earlier, \(P\) and \(A\) refer to the reference and alternative probability measures respectively. Notice that the utility maximizing agent under measure \(A\) is dividing realized and unambiguous \(\varrho_t\) by using \(\alpha_t\) which is constructed under the alternative probability measure \(A\). In other words, any equilibrium achieved under measure \(A\), implies choice of consumption bundles which are by construction different from those chosen under measure \(P\). Since choice under probability measure \(A\) by construction views the foreign dividend and returns processes more pessimistically, consumption and investment rules derived under this probability measure tend to favor the home country compared to certainty case.
Consumption home bias is defined as positive correlation between country specific shocks and consumption growth rates. Abusing the terminology, we may represent this statement as \( \text{Cov}_t(d\left(\frac{c_i^t}{\tilde{\xi}_t^i}\right), dW^i_t) > 0 \).

In order to have consumption home bias, it is sufficient to have:

\[
\lambda_1^P \neq \lambda_1^A \\
\lambda_2^P \neq \lambda_2^A \\
\mathbb{E}[\hat{\xi}_t^P - \hat{\xi}_t^A] \neq 0, \text{a.s.}
\]

These conditions guarantee that \( c_{i,t}^{i,A} \neq c_{i,t}^{i,P} \) with \( c_{i,t}^{ii,A} > c_{i,t}^{ii,P} \) and \( c_{i,t}^{ij,A} < c_{i,t}^{ij,P} \). Since we are in a (Cox and Huang, 1989) setting and portfolios are constructed to finance consumption, equity weights demonstrate the characteristics of consumption process. These results are intuitively obvious, given the formulation of the problem. But they need to be checked for their ability to match real-world data. To do so, the following need to be addressed. First, what values of \( \lambda_i^v \) and \( \hat{\xi}_t \), as derived in the previous sections, are necessary to match the moments of the observed data? To address this question, we need to find values of \( \nu_i \), given (2.2.1) constraint, such that our theoretical results match the data. But the sign of these values is also of interest. More precisely, we need to check and see if whether \( \nu_{ii} = -\nu_{ij} \) where \( i, j = \{1, 2\} \) and \( i \neq j \), holds. If both requirements are satisfied, then we clearly have home bias in our portfolio weights. Notice that if the above holds, on the equilibrium path, \( \mathbb{E}[\pi_t^i|_P - \pi_t^i|_A] \neq 0, \text{a.s.} \) We base our verification on calibration study that follows.


2.4 Calibration

2.4.1 Data:

To measure the degree of home bias in actual data, we need a measure of equity investment flows between the US and Japan. US Treasury Department keeps a record of all fixed income and equity flows between US and any country where the residents (or institutions) have engaged in financial transactions with the US residents (or institutions) worth more than a million US dollars per fiscal year. The data is in public domain under Treasury International Capital System (TIC). This system tracks mutual financial investments on a monthly basis, in millions of US Dollars, from January 1977 to present. We use monthly data from January 1977 to December of 2004. This data set includes transactions in fixed income securities (bonds) as well as equities. We do not model the transactions in bonds. The demand for bonds, especially between Japan and the US, reflects institutional and governmental concerns which are beyond the scope of this study. We concentrate on equity investments between US residents and Japanese residents.

Unfortunately, this data is not available at the firm level. TIC system reports aggregates at the national level. There are some concerns about using this data to study home bias, as reported in (Cai and Warnock, 2006). But assuming that we are modeling the behavior of the representative agents, who invest in a broad market index for the US and Japan, we may use this data at per capita level.

We use S&P 500 index and Nikkei 225 as broad, market based indices used by the representative agents. Thus, we need a measure of market performance. Daily and monthly returns for S&P 500 from CRSP/COMPUSTAT address this issue for US data. We use monthly returns for calibrating the model. CRSP/COMUSTAT also reports aggregate market capitalization of equities reported in S&P 500 Index. For Nikkei
225 returns, data is extracted from Thompson DataStream. Market capitalizations are from Thompson DataStream, International Federation of Stock Exchanges, and World Banks’s World Development Indicators. To have a measure of relative size of the US and Japanese financial markets, we use annual data from International Federation of Stock Exchanges. This step provides us with limited observations since first, the data is annually sampled and second, it covers the 1991-2003 period. World Bank reports a competing series in the World Development Indicators (WDI) data bank. This source reports annual data on market capitalization of Japan and US stock exchanges starting in 1988. Table (2.2) is compiled using these series.

Since we model both portfolio investments and consumption decisions, we need several macroeconomic time series. Consumption data for the US is from FRED data bank at Federal Reserve Bank of St. Louis. We use real personal consumption data, seasonally adjusted and reported for each month between 1977 to 2004. For the Japanese consumption, we use values reported in International Statistical Yearbook, based on International Monetary Fund data. Japanese personal consumption data is reported on a quarterly basis in Yen. The data is in real prices and seasonally adjusted. To convert this quarterly data to monthly basis required in this study, we fill in the months between two observations using an cubic spline scheme. Since consumption is a slow moving process compared to market returns, this scheme does not affect the calibration procedure.

We use exports of US to Japan and US imports from Japan as proxies for consumption of foreign good in each country. This data is compiled by U.S. Department of Commerce, Bureau of Economic Analysis and reported in FRED II data bank at St. Louis FED. In this fashion, we construct four time series for consumption of home and foreign goods\(^9\). Exchange rates are from FRED II data bank at Federal Reserve Bank

\(^9\)Optimally, we would like to limit these series to consumption of non-durable goods. One may
of St. Louis. Since the model is estimated on per capita basis, we use US monthly population data from FRED II. Japanese population data are from Japan Bureau of Statistics, also reported monthly.

We construct the world risk free rate as the ratio of real Japanese three months bank rate and real US three month Treasury Bill rate. T-Bill rates are from FRED II and Japanese three-month bank rates are from IFS data bank. Notice that we do not model the formation of this value in this paper. Debt markets are heavily influenced by actions of governments and central banks. Since we use a variation of (Lucas, 1978) economy, we may assume without loss of generality that this risk free rate is exogenously determined. The market for risk free rate clears due to Walras’ law. All monetary values are scaled by respective US and Japan CPI to neutralize the inflation effects.

Plots of the data used in this calibration exercise are reported in Tables (2.3), (2.4), and (2.5).

2.4.2 Method of Calibration:

The calibration study here follows a scheme to match the first two moments of the simulated variables to the first two moments of observed data. The observed variables market returns for US and Japan, represented by S&P 500 and Nikkei 225 respectively, relative price of goods process, represented by ration of inflation rates in the two countries, investment positions in home and foreign country (US in US, Japan in Japan, and cross holdings), consumption of home goods, and consumption of foreign goods. Proxy for consumption of foreign goods are respective imports of goods and services between the two countries. we constructed the share holding positions based on “Report on US find the time series of non-durable consumption for the US. But IFS data bank only reports aggregate private consumption for Japan.
Figure 2.3: Plots of Data used in Calibration of the Model Mis-Specification Economy: Consumption

Figure 2.4: Plots of Data used in Calibration of the Model Mis-Specification Economy: Relative Prices and Returns
Figure 2.5: Plots of Data used in Calibration of the Model Mis-Specification Economy: Changes in Equity Investment Flows

Portfolio Holdings of Foreign Securities” and “Report on Foreign Portfolio Holdings of U.S. Securities” compiled by the US Department of the Treasury, Federal Reserve Bank of New York, nd the Board of Governors of the Federal Reserve System in various years, index and stock market capitalizations, index returns, population data, and flow of equity investment funds data from US Department of Treasury’s TIC system.

The goal here to match the sample means, sample variances, and a select number of sample covariances of the observed data by simulated values in the model. The theoretical model of mis-specification economy developed in the previous sections is in continues time. We use an Euler discretization scheme to discretize the model for numerical applications. This step will introduce an error component to the calibration exercise, which is acknowledged but is inevitable for such a highly non-linear model. To compute the integration steps required for generating model-based sample paths of (2.29), (2.30, (2.19), (2.20), (2.9), a Markov Chain Integration (MCI) scheme is used. The alternative is numerical integration. If implemented carefully, MCI is as efficient...
as numerical integration schmoes and is easier to code. The results of these integration procedures are crucial for solving the problem and accuracy of the calibration exercise. Control variables, $\pi^{ij}$ and $c^{ij}$ depend on $\alpha^i$ (2.25). In turn, $\alpha^i$ depends on $\lambda^1$, $\lambda^2$, and state-price densities.

After discretization, we have the following formulations:

$$\hat{\xi}_j = \exp \left\{ \sum_{k=1}^{j} -r_h - \frac{1}{2} \sum_{k=1}^{j} \|\hat{\theta}^i\|_h - \sum_{k=1}^{j} (\hat{\theta}_i^i)^\top \sqrt{h}.Z^i_k \right\}$$

(2.31)

$$\lambda^{i*} = \left[ \frac{\frac{1}{N} \sum_{j=1}^{N} \mu_j^i \sigma_j^2 \xi^1_j}{\frac{1}{N} \sum_{j=1}^{N} \exp \left( -\frac{\beta h}{\gamma} \xi_j^i \right) (1 + (\mu_j^2)^{(1-\gamma)^{-1}}) \right]^{-\gamma}$$

(2.32)

where $h$ is the time step of the simulation, $n \in [1, N]$, $N$ is the length of the simulation, $Z^i_j$ is a $d$-dimensional standard Gaussian innovation process, and $\hat{\theta}^i = (\sigma^G)^{-1}(\mu^G + \sigma^G \nu^i)$ is the above mentioned term for market price of risk and uncertainty. Use these results to construct $\alpha^i$, by (2.26) and the compute $c^{i*}$. Use $\alpha^i$, (2.31), and (2.32) to construct the portfolio weights:

$$\pi^1_j = \left[ \begin{array}{c} \pi^{1,1}_j \\
\pi^{1,2}_j \end{array} \right] = \left[ \begin{array}{c} \frac{1}{N} \left[ \sum_{j=n}^{N} \alpha^1_j g_j^i \xi^1_j \right] \\
\frac{1}{N} \left[ \sum_{j=n}^{N} \alpha^2_j g_j^i \xi^1_j \right] \end{array} \right]$$

(2.33)

$$\pi^2_j = \left[ \begin{array}{c} \pi^{2,1}_j \\
\pi^{2,2}_j \end{array} \right] = \left[ \begin{array}{c} \frac{1}{N} \left[ \sum_{j=n}^{N} (1 - \alpha^1_j) g_j^i \xi^1_j \right] \\
\frac{1}{N} \left[ \sum_{j=n}^{N} (1 - \alpha^2_j) g_j^i \xi^1_j \right] \end{array} \right]$$

(2.34)

Since data is expressed in terms of logarithmic growth rates of the observed variables, the same transformation is carried out for values generated at the level in simulation runs.

Simulation length, $N$, is fixed at 10,000. Since we are simulating monthly data,
Table 2.3: Data Summary of Variables Used in Calibration Exercise:

<table>
<thead>
<tr>
<th></th>
<th>sample mean</th>
<th>sample variances</th>
<th>sample covariances</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta p_t$</td>
<td>-0.00207815476</td>
<td>4.5629e-004</td>
<td></td>
</tr>
<tr>
<td>$\Delta G^1_t$</td>
<td>0.01375776786</td>
<td>0.00241777</td>
<td>$\text{Cov}(\Delta G^1, \Delta G^2)$</td>
</tr>
<tr>
<td>$\Delta G^2_t$</td>
<td>0.00417321429</td>
<td>0.00313270</td>
<td></td>
</tr>
<tr>
<td>$\Delta c^{1,1}_t$</td>
<td>0.00182483036</td>
<td>1.6741e-005</td>
<td>$\text{Cov}(\Delta c^{1,1}, \Delta c^{1,2})$</td>
</tr>
<tr>
<td>$\Delta c^{1,2}_t$</td>
<td>0.01159551786</td>
<td>0.013107497</td>
<td>$\text{Cov}(\Delta c^{1,2}, \Delta \pi^{1,1})$</td>
</tr>
<tr>
<td>$\Delta c^{2,1}_t$</td>
<td>0.0050389881</td>
<td>8.9207e-004</td>
<td>$\text{Cov}(\Delta c^{2,1}, \Delta \pi^{1,2})$</td>
</tr>
<tr>
<td>$\Delta c^{2,2}_t$</td>
<td>0.00849010714</td>
<td>0.0024599698</td>
<td>$\text{Cov}(\Delta c^{2,2}, \Delta \pi^{2,2})$</td>
</tr>
<tr>
<td>$\Delta \pi^{1,1}_t$</td>
<td>0.00880429167</td>
<td>0.0017816285</td>
<td></td>
</tr>
<tr>
<td>$\Delta \pi^{1,2}_t$</td>
<td>0.0016529167</td>
<td>0.042543728</td>
<td>$\text{Cov}(\Delta \pi^{1,1}, \Delta \pi^{1,2})$</td>
</tr>
<tr>
<td>$\Delta \pi^{2,1}_t$</td>
<td>-0.00016529167</td>
<td>0.0039412441</td>
<td></td>
</tr>
<tr>
<td>$\Delta \pi^{2,2}_t$</td>
<td>0.00911314583</td>
<td>0.0039412441</td>
<td>$\text{Cov}(\Delta \pi^{2,1}, \Delta \pi^{2,2})$</td>
</tr>
</tbody>
</table>

This number roughly translated into 833 years of data. There are two integration loops in this calibration. The first loop integrates simulated data points to the monthly level, constituting the outer loop. There are several choices available for the length of this loop. One may choose to integrate at five minute, hourly, daily, or even weekly frequencies. Our choice in this exercise is hourly frequency. The assumption is trading during the working hours, hence length of this loop is equal to $M = 180$ hours per month, corresponding to an average working month of 20 days. The inner loop is used to integrate out latent processes and to simulate the forecasts of agents, at any time $s \in [t, T], t \leq T$. For example, simulation of portfolio weights, $\pi^j_i$, requires computation of $\tilde{\xi}^n_i$ at each step. This task is done within this inner loop. The length of this loop corresponds to five minute sampling for each working day, and is set to $S = 2,160$ for the period of one month. This choice implies a monthly re-balancing of portfolios held by the agent, which is more frequent than is observed, but is not unreasonable.

Raw moments of the data are reported in Table (2.3). To start the process, the available series are fit through a flexible, semi-nonparametric estimation methodology.
Parameter estimates. G-VAR, SP-VAR, and NL-NP represent Gaussian VAR, Semi-Parametric VAR, and nonlinear-non-parametric SNP.

to obtain parameter values for series statistics, joint density of the variables, and study the dynamic behavior of covariance functions. We use the SNP software included in (EMM 2.3) source code. For more information on SNP software and its technology, see (Gallant and Tauchen, 2006), (Gallant and Long, 1997), (Gallant and Tauchen, 1989), and (Gallant, 1987). We fit the data via a Gaussian Vector Auto-Regressive (VAR) formulation. In SNP methodology, we use the following procedures to fit the data:

\[
L_u = 1, L_g = 0, L_r = 0, L_p = 1, K_z = 0, K_x = 0 : \text{Gaussian VAR}
\]

\[
L_u = 1, L_g = 0, L_r = 0, L_p = 1, K_z = 1, K_x = 0 : \text{semi-parametric VAR}
\]

\[
L_u = 1, L_g = 1, L_r = 1, L_p = 1, K_z = 1, K_x = 1 : \text{nonparametric nonlinear}
\]

Results of the SNP estimates are reported in Table (2.4). Simple Gaussian VAR results are used in this calibration exercise, but the results of the other two fits are reported for comparison.

Calibration exercise is carried out for an exhaustive set of parameter values. We fix \(\{\mu_1^G, \mu_2^G, \sigma_{11}^G, \sigma_{12}^G, \sigma_{22}^G\}\) by setting them equal to SNP parameter estimates. The case for \(\{\mu_1, \mu_2, \sigma_{11}, \sigma_{12}, \sigma_{22}\}\) set is slightly more complicated. These are parameters of latent processes. The main criteria for recovering them was whether the first two moments of growth rates in \(p_t\) can be matched with some level of success, without sacrificing matching of moments in other series. After this criteria was met at least to some
level of satisfaction, \( \{\mu_1, \mu_2, \sigma_{11}, \sigma_{12}, \sigma_{22}\} \) were fixed for the next step. We start the main thrust of calibration after the preliminary steps mentioned so far, by moving the uncertainty parameters, \( \nu^i \), for a given value of coefficient of relative risk aversion, \( \gamma \). Here, \( \gamma \) is assumed to be fixed, known, and chosen from a set of real numbers. It turns out that choice of \( \gamma \) has an effect on stability of results, since they seem not to be robust to the choice of \( \gamma \). Through several trials it became clear that this parameter should be chosen from \( \gamma \in [2, 10] \) range. For \( \gamma \in [0.05, 1.5] \) range, the results did not converge to reasonable numbers, while for \( \gamma > 12 \), the exercise generated very large peaks in one or more processes and failed to match the data moments. Discount factor, \( \beta \), turns out to be unimportant for reasonable values (between 0.6 to 0.975 or even 0.999). It is assumed to be fixed and known.

### 2.4.3 Calibration Results:

The key interest is finding plausible values for \( \nu^i \) which support matching of select moments of observed data.

Parameter space is \( \Theta = \{\beta, \gamma, \mu_1, \mu_2, \sigma_{11}, \sigma_{12}, \sigma_{22}, \mu_1^G, \mu_2^G, \sigma_{11}^G, \sigma_{12}^G, \sigma_{22}^G, \nu^{11}, \nu^{12}, \nu^{21}, \nu^{22}\} \).

Two parameters are fixed for each calibration run: \( \beta, \gamma \). Five parameters, \( \mu_1^G, \mu_2^G, \sigma_{11}^G, \sigma_{12}^G, \sigma_{22}^G \), and \( \sigma_{22}^G \), are initialized by setting them equal to the SNP estimates of sample moments.

Table (2.5) reports the calibration results. For ease of comparison, moments of the sample (Table (2.3)) are reported on the top panel in this table. Throughout this exercise, matching the second moments did not cause much trouble. The values of calibrated second moments tend to be smaller, in some cases significantly lower, then second moments of the data. The values computed for covariances are all well below those observed in the data. We slightly more success in fitting the variances. The model does not produce enough variability to match the second moments, but on the
Table 2.5: Calibration Results

<table>
<thead>
<tr>
<th>Sample Moments</th>
<th>Sample Mean</th>
<th>Sample Variances</th>
<th>Sample Covariances</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p_t )</td>
<td>-0.00207815476</td>
<td>4.5629e-004</td>
<td>0.00156778461</td>
</tr>
<tr>
<td>( \Delta G^1_t )</td>
<td>0.0137577686</td>
<td>0.00241777</td>
<td>0.00451942</td>
</tr>
<tr>
<td>( \Delta G^2_t )</td>
<td>0.00417324129</td>
<td>0.00313270</td>
<td></td>
</tr>
<tr>
<td>( \Delta c_{1,1} )</td>
<td>0.00182483063</td>
<td>0.16741e-004</td>
<td>0.002783935</td>
</tr>
<tr>
<td>( \Delta c_{1,2} )</td>
<td>0.01159551786</td>
<td>0.013107497</td>
<td>0.015278853</td>
</tr>
<tr>
<td>( \Delta c_{2,1} )</td>
<td>0.0102268125</td>
<td>0.013499109</td>
<td>0.01280691</td>
</tr>
<tr>
<td>( \Delta c_{2,2} )</td>
<td>0.0050838881</td>
<td>8.9207e-004</td>
<td>0.013589032</td>
</tr>
<tr>
<td>( \Delta \pi_{1,1} )</td>
<td>0.00084010714</td>
<td>0.002459698</td>
<td>0.005472841</td>
</tr>
<tr>
<td>( \Delta \pi_{1,2} )</td>
<td>0.00098429167</td>
<td>0.0017816285</td>
<td>0.02037543</td>
</tr>
<tr>
<td>( \Delta \pi_{2,1} )</td>
<td>-0.00016529167</td>
<td>0.004253728</td>
<td>0.058913539</td>
</tr>
<tr>
<td>( \Delta \pi_{2,2} )</td>
<td>0.00911314583</td>
<td>0.0039412441</td>
<td>0.003450451</td>
</tr>
</tbody>
</table>

Calibration Results, Panel No. 1

<table>
<thead>
<tr>
<th>( \gamma = 4 )</th>
<th>( \nu^{\pi,+} = 0.25 )</th>
<th>( \nu^{\pi,-} = 0.0539 )</th>
<th>( \beta = 0.975 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Variances</td>
<td>Covariances</td>
<td></td>
</tr>
<tr>
<td>( \Delta p_t )</td>
<td>-1.5913e-009</td>
<td>2.4762e-014</td>
<td>0.0144e-004</td>
</tr>
<tr>
<td>( \Delta G^1_t )</td>
<td>0.0044</td>
<td>0.1226e-004</td>
<td>0.0015e-003</td>
</tr>
<tr>
<td>( \Delta G^2_t )</td>
<td>0.0043</td>
<td>0.1442e-004</td>
<td>0.0026e-003</td>
</tr>
<tr>
<td>( \Delta c_{1,1} )</td>
<td>0.0020</td>
<td>0.0057e-003</td>
<td>0.0051e-003</td>
</tr>
<tr>
<td>( \Delta c_{1,2} )</td>
<td>0.089</td>
<td>0.0075e-003</td>
<td>0.0081e-003</td>
</tr>
<tr>
<td>( \Delta c_{2,1} )</td>
<td>0.057</td>
<td>0.0667e-003</td>
<td>0.0246e-003</td>
</tr>
<tr>
<td>( \Delta c_{2,2} )</td>
<td>0.0064</td>
<td>0.0266e-003</td>
<td>0.0370e-003</td>
</tr>
<tr>
<td>( \Delta \pi_{1,1} )</td>
<td>5.4440e-003</td>
<td>0.1487e-003</td>
<td>0.0025e-003</td>
</tr>
<tr>
<td>( \Delta \pi_{1,2} )</td>
<td>4.5200e-003</td>
<td>0.874e-003</td>
<td>0.2348e-003</td>
</tr>
<tr>
<td>( \Delta \pi_{2,1} )</td>
<td>6.3459e-003</td>
<td>0.2770e-003</td>
<td>0.3077e-003</td>
</tr>
<tr>
<td>( \Delta \pi_{2,2} )</td>
<td>8.5943e-003</td>
<td>0.3707e-003</td>
<td>0.598e-003</td>
</tr>
</tbody>
</table>

Calibration Results, Panel No. 2

<table>
<thead>
<tr>
<th>( \gamma = 6 )</th>
<th>( \nu^{\pi,+} = 0.05 )</th>
<th>( \nu^{\pi,-} = -0.12248 )</th>
<th>( \beta = 0.955 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Variances</td>
<td>Covariances</td>
<td></td>
</tr>
<tr>
<td>( \Delta p_t )</td>
<td>9.9349e-011</td>
<td>3.0883e-018</td>
<td>0.0162e-004</td>
</tr>
<tr>
<td>( \Delta G^1_t )</td>
<td>0.0044</td>
<td>0.1165e-004</td>
<td>0.0011e-004</td>
</tr>
<tr>
<td>( \Delta G^2_t )</td>
<td>0.0043</td>
<td>0.1625e-004</td>
<td>0.0011e-003</td>
</tr>
<tr>
<td>( \Delta c_{1,1} )</td>
<td>0.0020</td>
<td>0.0589e-004</td>
<td>0.0152e-004</td>
</tr>
<tr>
<td>( \Delta c_{1,2} )</td>
<td>0.0045</td>
<td>0.0832e-004</td>
<td>0.0142e-004</td>
</tr>
<tr>
<td>( \Delta c_{2,1} )</td>
<td>0.0089</td>
<td>0.0938e-004</td>
<td>0.0142e-004</td>
</tr>
<tr>
<td>( \Delta c_{2,2} )</td>
<td>0.0063</td>
<td>0.2480e-004</td>
<td>0.0142e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{1,1} )</td>
<td>2.4828e-005</td>
<td>0.0175e-004</td>
<td>0.0115e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{1,2} )</td>
<td>2.2801e-005</td>
<td>0.0243e-004</td>
<td>0.0162e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{2,1} )</td>
<td>3.0125e-005</td>
<td>0.1287e-004</td>
<td>0.0253e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{2,2} )</td>
<td>4.2809e-005</td>
<td>0.2055e-004</td>
<td>0.001e-003</td>
</tr>
</tbody>
</table>

Calibration Results, Panel No. 3

<table>
<thead>
<tr>
<th>( \gamma = 8 )</th>
<th>( \nu^{\pi,+} = 0.30 )</th>
<th>( \nu^{\pi,-} = 0.097075 )</th>
<th>( \beta = 0.975 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Variances</td>
<td>Covariances</td>
<td></td>
</tr>
<tr>
<td>( \Delta p_t )</td>
<td>-0.0012</td>
<td>7.8904e-004</td>
<td>0.9958e-005</td>
</tr>
<tr>
<td>( \Delta G^1_t )</td>
<td>0.0042</td>
<td>9.985e-005</td>
<td>0.0523e-004</td>
</tr>
<tr>
<td>( \Delta G^2_t )</td>
<td>0.0042</td>
<td>8526e-005</td>
<td>0.4071e-004</td>
</tr>
<tr>
<td>( \Delta c_{1,1} )</td>
<td>0.0022</td>
<td>0.0625e-004</td>
<td>0.2715e-004</td>
</tr>
<tr>
<td>( \Delta c_{1,2} )</td>
<td>0.0027</td>
<td>0.0526e-004</td>
<td>0.5721e-004</td>
</tr>
<tr>
<td>( \Delta c_{2,1} )</td>
<td>0.0082</td>
<td>0.0984e-004</td>
<td>0.0582e-004</td>
</tr>
<tr>
<td>( \Delta c_{2,2} )</td>
<td>0.0064</td>
<td>0.0548e-004</td>
<td>0.0052e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{1,1} )</td>
<td>0.35250e-004</td>
<td>0.2715e-004</td>
<td>0.0032e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{1,2} )</td>
<td>0.5259e-004</td>
<td>0.2341e-004</td>
<td>0.0032e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{2,1} )</td>
<td>6.5207e-004</td>
<td>0.4071e-004</td>
<td>0.0023e-004</td>
</tr>
<tr>
<td>( \Delta \pi_{2,2} )</td>
<td>0.0702e-004</td>
<td>0.8040e-004</td>
<td>0.0023e-004</td>
</tr>
</tbody>
</table>
other hand there is no discernible trade off in matching moments either.

On the other hand, matching the first moments appears to be more complicated. There seems to be a trade off between a good match for $\Delta c_{t}^{11}$ and $\Delta c_{t}^{22}$ versus those for $\Delta c_{t}^{12}$ and $\Delta c_{t}^{21}$. Similarly, to match the first moment of $\Delta \pi_{t}^{11}$ and $\Delta \pi_{t}^{22}$, there is a sacrifice of a good match for $\Delta \pi_{t}^{12}$ and $\Delta \pi_{t}^{21}$. Matching the first moment of $\Delta G_{t}^{1}$ and $\Delta G_{t}^{2}$ does not generate a trade off with those for consumption or portfolio weights, but there seems to be some negative relationship between these two processes’ match and that of relative prices. Notice that variances reported in these tables are variances for the process and not those of statistics. The reason is rather large size of simulation length, $N = 10,000$. In this case, $\sqrt{N}$ is still 100 which would make the statistic variance very small and uninformative.

Over all, this calibration study had reasonable success in matching the first moments, and moderate success in matching the second moments with high values for $\nu_{1,1}^{1}$ and positive but low values for $\nu_{2,2}^{2}$. Low values for $\nu_{1,1}^{1}$ and negative values for $\nu_{2,2}^{2}$ led to far less success in matching of moments.

One significant trend which emerges from this calibration study is the size of $\nu$’s needed to successfully match the empirical moments. Calibration results indicate that in order to reasonably match the empirical moments and complying to (2.2.1) at the same time, either US investors should have very high values of $\nu^{1,1}$s as in Panels No. 1 and No. 3 in Table (2.5), or the Japanese investors should view their home market very skeptically as in panel No. 2, Table (2.5). There are explanations for both results. How realistic they are is a rather different question.

Take the case of pessimistic Japanese investors. During 1990s, Japanese financial markets were bearish, while US equity markets were booming. It is conceivable that a relatively informed Japanese investor could reasonably infer this difference and hence form a pessimistic view towards home equities. But the same market was booming in
1980s. You expect to see a level of “averaging out” of this boom and bust effect that
we have not been able to detect.

On the other hand, high values for \( \nu^{i,i} \) seem to reinforce the view held by (Anderson et al., 2003) and (Maenhout, 2004) that demand for robustness in choice of investment and consumption translates into higher levels of risk aversion. That is, the agent instead of \( \gamma \), has a risk aversion of \( \gamma + \theta \), where \( \theta \) is the level of demand for robustness. On the other hand, this result broadly supports some empirical and calibration studies in “Habit Formation” literature, for example (Shore and White, 2006), where calibrating or fitting the data requires very high levels of relative risk aversion.

### 2.5 Conclusion and Future Research:

In this paper we propose a unified approach for solving consumption and equity home bias puzzles. We use ideas from model uncertainty literature and martingale methods to solve the dynamic consumption and portfolio choice problem. Specifically, we solve for the case of model mis-specification concerns.

The calibration results indicate two directions for explaining home bias in equities. One direction points out towards pessimistic views held by Japanese investors towards their home equity market. This result while plausible for 1990s and early 2000s, does not explain the boom years of late 1970s and 1980s.

The other direction broadly agrees with findings of both “Robust Control” approach pioneered by Hansen and Sargent, as seen in (Anderson et al., 2003), (Uppal and Wang, 2003), and (Maenhout, 2004), and also with contributions of “Habit Formation” literature (see (Shore and White, 2006)). This direction supports the idea that fear of model mis-specification forces the agents to act more risk averse when choosing foreign equities for their portfolios.
So far home bias is studied under constant volatility and power utility assumptions. It is natural to extend the paper to the case of stochastic volatility. The case of log utility, where due to myopic consumer behavior hedging demand for volatility factors does not exist is rather straightforward. The case of general power utility and recursive utility is more challenging, since study of these cases would introduce hedging behavior on the part of the agent against both model mis-specification and against volatility factors resulting in a challenging and rewarding problem to tackle. To this date a study of home bias when agents have hedging demand for several state variables is not present in the theoretical literature.

Case of home bias against emerging market equities may be viewed in the light of catastrophic events in such markets. Market crashes, government and corporate defaults, natural disasters, fear of appropriation, and other similar events may induce extreme outcomes in the joint probability distribution of the returns. It is reasonable to draw on the two branches of finance and statistics, portfolio choice and extreme value theory, to explain the home bias resulting from extreme outcomes.

An interesting issue worth further study is whether portfolio generating kernels studied here have a functional relationship with portfolio generating functions discussed in (Fernholz, 2002).
Chapter 3

Portfolio Selection in Presence of Multi-Factor Stochastic Volatility
ABSTRACT

MOHAMMAD REZA JAHAN-PARVAR: Essays in Financial Economics.
(Under the direction of A. Ronald Gallant.)

We develop a theoretical model for optimal portfolio choice in the presence of multifactor stochastic volatility, following and extending Fleming and Hernández-Hernández (2003). The project is geared toward providing guidelines for short term portfolio managers. Possibilities of numerically computing the optimal portfolio weights and consumption rule are explored. Our attention is focused on providing rigorous proofs for characteristics of value function, portfolio weights, and optimal consumption rules, as well as efficient methods to numerically compute the dynamical systems and control policies.

**Key Words:** Optimal Portfolio Choice, Log-linear Multifactor Stochastic Volatility, Portfolio Weights, Optimal Consumption Rules, Mean Reversion, Kushner-Dupuis Markov Chain Approximation, Efficient Method of Moments.

**JEL Classification:** C63, E27, and G11.
3.1 The Problem:

Traditional portfolios rules such as “buy and hold” policy or (Merton, 1969) fixed weight portfolios are not suitable for those investors who need to balance their portfolios frequently, such as fund managers. While all forms of CAPM imply a passive investment strategy, fund managers often need to balance their portfolios with respect to the current state of the economy. Moreover, while small investors may ignore volatility behavior for their investment horizon (Brandt, 2006), fund managers and other active investors should pay close attention to time-varying behavior of the equity returns and volatilities. Notice that passive policies (fixed weight and buy and hold) produce rather contrary results. More precisely, to maintain fixed weights, the investor should sell the winning assets and buy losing ones. On the other hand, buy and hold strategy is divorced from varying states of the economy.

In this paper, we develop a framework that is sensitive to states of the economy through a multi-factor stochastic volatility formulation for the asset returns. Investors’ time horizon is assumed to be infinite in the model, but our results are shown to be robust for any time horizon $t \in [0, T]$ such that $T > 0$. The derived portfolio weight policies explicitly depend on investor’s level of risk aversion. Although the results in this paper are derived for a two-asset case, extension to a multi asset scenario is trivial as long as the flow of information to different assets is from the same random variable generating process.

Time varying volatility of financial time series is a well documented. There are two main methodologies for modeling this feature of financial data. One method is the ARCH/GARCH formulations of (Engle, 1982) and (Bollerslev, 1986). This method

---

1Through the stochastic volatility formulation, sensitivity to past and current states of the economy is guaranteed. Given a Markovian structure for the volatility factors, future states are also incorporated based on Martingale properties. Relaxing the Markovian structure alters this feature.
is very popular and well researched. On the other hand, (Hull and White, 1987) and (Scott, 1987) introduced Stochastic Volatility (hence forth SV) as an alternative to ARCH/GARCH method. Since initial introduction of the SV models, they have proved to be versatile and accurate in capturing dynamic dependencies in financial markets, providing a powerful alternative to ARCH/GARCH. Among competing SV models, logarithmic specification, built on findings of (Clark, 1973), (Taylor, 1982), and (Gallant et al., 1997) has proved to be the most successful in capturing the dynamics of market time series.

While many researchers have explored the price dynamics of SV models (under various specifications) and their derivative pricing implications, there are very few papers exploring portfolio choice for price processes with stochastic volatility dynamics. Specifically, we can not recall any papers to date which explore portfolio choice implications of multi-factor SV price dynamics. In this paper, we try to address this shortcoming in the literature. We provide rigorous proofs for our derivations and a numerical solution to the problem. We provide proofs for the general case and numerically compute the results based on a successful estimation of multi-factor SV dynamics of Dow Jones Industrial Average (DJIA).

(Merton, 1969) introduced optimal portfolio choice problem. His formulation, similar to existing literature of the time, assumes constant volatility for a price (or returns) process. In recent years, some authors have tried to expand the original portfolio selection model and incorporate stochastic volatility into the optimization problem, including theoretical works by (Chacko and Viceira, 2006),(Fleming and Hernández-Hernández, 2003), and (Grön et al., 2004). Other researchers have produced simulation based solutions or have tried to attack the problem using numerical techniques. Among them, we note (Brandt and Santa-Clara, 2005), (Detemple et al., 2003), and (Das and Uppal, 2004).
The unifying feature of all these papers is the use of single factor or single factor with jumps formulation for stochastic volatility. It is a well documented fact that single factor stochastic volatility models do not distinguish between fast and slow mean reversion in financial time series. In the literature, this shortcoming of single factor models is addressed by adding factors or by incorporating jumps in the model. Since derivation and computation of optimal policies with jumps is well researched (see (Das and Uppal, 2004), or (Kushner and Dupuis, 2001)), we focus on multi factor models.

This paper extends the literature in the following ways: First, it extends (Fleming and Hernández-Hernández, 2003) to incorporate multi factor stochastic volatility models. We know that two factor stochastic volatility decisively outperform the single factor models (refer to (Chernov et al., 2003)). Thus, when we discuss portfolio selection, it is desirable to use policies that explicitly allow for multi factor formulation. The second innovation in the paper is semi-closed form derivation of the portfolio and consumption decision rules (as functions of the life time discounted expected utility of the investor), assuming feedback from factors to the price or returns process. This formulation is accurate and fits the US data well, as documented in (Chernov et al., 2003). Third, it is of practical interest to consider formulations that easily lend themselves to simulation or numerical solutions. We try to provide rigorous but tractable treatment of our extension, to ease the process of simulation and numerical computation. Fourth, we develop an algorithm based on (Kushner and Dupuis, 2001) and (Kushner, 1998) Markov chain approximation of dynamic models, and proceed to implement the algorithm using a code based on (Jarvis and Kushner, 1996).

We provide an efficient methodology for computation of the policy rules, consistent with our theoretical findings. In this paper we examine a three dimensional optimal control problem. There are two sets of controls: portfolio weights ($\pi_t$) and optimal consumption rule ($c_t$). We initially assume that both controls belong to bounded sets.
The reader will notice later that relaxing this assumption does not affect our findings. As a result, we will have a straightforward generalization of the bounded case, as will be shown. The usual economic budget constraint for utility maximization is satisfied through the wealth process, \( X_t \). The objective function is a discounted utility function, denoted by \( J(\cdot; \pi, c) \). Defining the derived utility, or value function, as \( W(Y^2_t; \pi, c) \), we proceed to prove that \( W \) is the unique positive classical solution to the Hamilton-Bellman-Jacobi Equation (HJB) for our control problem. To compute the optimal portfolio weights and consumption rules and to show their optimality, we need to know about bounds of \( W \) and its partial derivatives with respect to volatility factors as well as the inverse function, \( W^{-1} \).

The paper proceeds as follows: in section 3.2, we introduce the theoretical model. Rigorous proofs for optimality of portfolio weights, consumption rule, and uniqueness of value function \( W(Y^1_t, Y^2_t) \) as a solution to dynamic programming equation are presented in section 3.3. In section 3.4, we proceed by outlining and introducing numerical methods used to estimate and compute the decision rules. We use the estimated parameters for Dow Jones Industrial Average (DJIA) index in (Chernov et al., 2003) and numerically compute the decision rules for a risk averse investor who wants to optimize a two-asset portfolio (DJIA index and a risk free asset) by (Kushner and Dupuis, 2001) MCA method following (Jarvis and Kushner, 1996). (Chernov et al., 2003) use Efficient Method of Moments (EMM) of (Gallant and Tauchen, 2001) for estimation of price dynamics in DJIA index. Section 3.5 concludes and suggests directions for future research.
3.2 Theoretical Model:

Throughout the paper, terms are defined over a probability space \((\Omega, \mathcal{F}, P)\). We assume that there exists filtration \(\{\mathcal{F}_t\}_{t=0}^\infty\) such that \(\mathcal{F}_t \subset \mathcal{F}\).

Assume a stochastic process for asset price \(P_t\), representing either a market index, an equity, or a mutual fund. This asset is risky. Volatility factors are presented as \(Y^1_t\) and \(Y^2_t\) representing fast and slow mean-reverting volatility in the market, respectively. Their dynamic behavior is discussed in detail.

3.2.1 The General Case of Multi Factor Stochastic Volatility:

\[
\begin{align*}
\text{d}P_t &= \mu P_t \text{d}t + \sigma_p(Y^1_t, Y^2_t)P_t \left[ \sqrt{1 - \rho_1^2 - \rho_2^2} \text{d}W^P_t + \rho_1 \text{d}W^1_t + \rho_2 \text{d}W^2_t \right] \\
\text{d}Y^1_t &= g_1(Y^1_t) \text{d}t + \sigma_1(Y^1_t) \text{d}W^1_t \\
\text{d}Y^2_t &= g_2(Y^2_t) \text{d}t + \sigma_2(Y^2_t) \text{d}W^2_t 
\end{align*}
\]

Where \(W^i_t, i = \{1, 2, P\}\) are standard Brownian motions (or Wiener processes). While we assume independence between \(W^1_t\) and \(W^2_t\), both factor processes are correlated with price process. The respective coefficients of correlation are \(\rho_i\), where \(i = 1, 2\). This feature provides feedback to the price process. Correlation between price process and volatility factor is also known as “instantaneous leverage effect”. This formulation rules out correlation between factors. However, we are interested in implications of existence of such a correlation. To explore this possibility, an equivalent formulation is
proposed. Consider the following processes:

\[
W^*_t = \sqrt{1 - \rho^2_1 - \rho^2_2}W^P_t + \rho_1 W^*_t + \rho_2 W^*_t
\]

\[
W^*_{t,1} = \frac{W^*_t - \rho_1 W^*_t}{\sqrt{1 - \rho^2_1}}
\]

\[
W^*_{t,2} = \frac{W^*_t - \rho_2 W^*_t}{\sqrt{1 - \rho^2_2}}
\]

This formulation is useful for another reason, (Chernov et al., 2003) use the former formulation for price dynamics (3.2), while (Fleming and Hernández-Hernández, 2003) use a formulation similar to (3.2). It is easy to show that the quadratic variations of these transformed processes satisfy the desired qualities. There is correlation between the two factor processes and the factors and the price process are independent:

\[
\langle W^*_t, W^*_{t,1} \rangle = \langle W^*_t, W^*_{t,2} \rangle = 0
\]

\[
\langle W^*_{t,1}, W^*_{t,2} \rangle = \frac{-\rho_1 \rho_2 t}{\sqrt{(1 - \rho^2_1)(1 - \rho^2_2)}} = \delta t
\]

The latter result is not generally equal to zero, hence these two processes show the desired correlation. Rewrite the model as:

\[
dP_t = \mu P_t dt + \sigma_p(Y^1_t, Y^2_t)P_t dW^*_t
\]

\[
dY^1_t = g_1(Y^1_t)dt + \sigma_1(Y^1_t)\sqrt{1 - \rho^2_1}dW^*_t + \rho_1 W^*_{t,1}
\]

\[
dY^2_t = g_2(Y^2_t)dt + \sigma_2(Y^2_t)\sqrt{1 - \rho^2_2}dW^*_t + \rho_2 W^*_{t,2}
\]

(3.2)

This model provides feedback in the price process, correlation between volatility factors, as we desire, and lends itself more easily for derivation than the previous formulation. Denote initial values of \(Y^1_t\) and \(Y^2_t\) by \(Y^1_0 = y_1\) and \(Y^2_0 = y_2\).
3.2.2 Utility Structure, Consumption, Portfolio Weights, and Wealth Process:

Denote the wealth process by $X_t$, consumption process by $C_t$, optimal fraction of wealth consumed by $c_t$, portfolio weights for risky asset(s) by $\pi_t$, and risk free rate by $r$. Without loss of generality, we assume that the risk free rate, $r$ is constant.

**Assumption 1:** Adjustment of portfolios or trading of securities do not incur costs. This assumption is consistent with the literature.

**Assumption 2:** This problem is solved from a representative investor’s point of view. (Assume that we are solving the problem from US investor’s point of view.) The investor maximizes his expected utility either over an infinite horizon or over a finite horizon (we will show that our results are robust for both cases). In case of finite period, we assume that the investor does not leave a bequest. This implies that at time $T$, he will not invest at all and will consume all his wealth.

**Assumption 3:** $\sigma_p(Y^1_t, Y^2_t)$ and $g_i(Y^i_t) \in C^1(\mathbb{R})$, $(i = 1, 2)$

i) $\sigma_p(Y^1_t, Y^2_t)$ is bounded by $\sigma_l$ and $\sigma_u$ for some $0 < \sigma_l \leq \sigma_u < \infty$.

ii) $g_{y,i}(Y^i_t)$ is bounded such that there exists $k_i > 0$ and $g_{y,i}(Y^i_t) \leq -k_i$; $i = 1, 2$. Regarding drift terms, if we assume that volatility factors follow Ornstein-Uhlenbeck processes, then $g_{y,i}(Y^i_t) = \alpha_{ii}$ where $\alpha_{ii}$ is the coefficient of mean reversion. We know from empirical evidence and stability studies that we should have $\alpha_{ii} < 0$ for stability of the volatility processes. Then there exists arbitrary $k_i > 0$ such that $\alpha_{ii} \leq -k_i$, as required.

\[\text{Refer to the Appendix for a discussion of upper and lower bounds of volatilities and drift terms presented here.}\]
Consumption:

We assume that consumption is a fraction of financial wealth, at each point in time. But this fraction is optimally and continuously chosen by the consumer. There is no other source for financing consumption but the returns from the optimally invested wealth (we are ignoring salary, inheritance, or other sources of income). That is, for wealth process $X_t$ and optimal consumption rule $c_t$, we have consumption $C_t$:

$$C_t = c_t X_t$$

Utility Structure:

Throughout the paper, we use the usual constant relative risk aversion (CRRA) power utility class:

$$U(C_t) = \begin{cases} 
\frac{C_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\
\ln C_t & \text{if } \gamma = 1 
\end{cases}$$

As a result:

$$U(C_t) = \begin{cases} 
\left[ c_t X_t \right]^{1-\gamma} \frac{1-\gamma}{1-\gamma} & \text{if } \gamma \neq 1 \\
\ln(c_t X_t) & \text{if } \gamma = 1 
\end{cases}$$

This utility structure exhibits constant relative risk aversion, time separability, and admits the no bequest assumption mentioned earlier. The investor derives utility from consuming part of his wealth. At the infinite horizon, discounted expected utility of wealth is:

$$J(x, y_1, y_2; \pi, c) = \mathbb{E} \int_{t=0}^{\infty} \frac{1}{1-\gamma} \exp(-\alpha t) \left[ c_t X_t \right]^{1-\gamma} dt \quad (3.3)$$

In this formulation, discount factor for future consumption is $\alpha$ and is assumed to be constant. Discounted expected utility is our usual and familiar “discounted cost function” of optimal control theory. We proceed to use this function as the main vehicle
of approximation of optimal policies in subsequent sections.

**Wealth Process:**

The wealth process faces the same sources of shocks as the price process discussed above. It can be readily modified for both functional forms presented in section 4. The investor will invest his wealth in risky asset as long as $\mu - r > 0$, or else invest it in risk free asset (usually in short term bonds, for example T-Bills). Consumption decisions are made optimally, given any time $t \leq T$ or $t < \infty$.

$$
\begin{align*}
  dX_t &= X_t \left[ (\mu - r)\pi_t + r - c_t \right] dt + \pi_t\sigma_p(Y^1_t, Y^2_t)dW^*_t \\
  X_0 &= x \gg 0 \\
  \Rightarrow X_t &= x \exp \left[ rt + \int_0^t \left( (\mu - r)\pi_s - c_s - \frac{1}{2}\pi^2_s\sigma^2_p(Y^1_s, Y^2_s) \right) ds + \int_0^t \pi_s\sigma_p(Y^1_s, Y^2_s)dW^*_s \right]
\end{align*}
$$

Wealth process is the familiar budget constraint of the economic theory in our utility maximization problem. We characterize admissible strategies (feedback functions) for this dynamic utility maximization problem:

**Definition 1:** $(\pi, c)$ is an admissible strategy if it is a $\mathcal{F}_t$-progressively measurable process, such that $P\left[ \forall t > 0, |\pi_t| \leq A_1, 0 \leq c_t \leq A_2 \right] = 1$, where there exist $A_1, A_2 \in \mathbb{R}^+$. Denote the set of admissible strategies by $A$.

### 3.3 Derivation of Portfolio Weights:

Following (Fleming and Hernández-Hernández, 2003), we proceed in constructing the portfolio weights in presence of general multi factor stochastic volatility. Then, we provide explicit functional forms to compute the optimal policies. In this section we
ignore the explicit forms and provide all the statements in a general fashion, since at this point, we are concerned with theoretical rigor and consistency of our treatment of the problem.

Since $\pi_t$ and $c_t$ are optimal controls and feedback functions of $(Y^1_t, Y^2_t)$, the particular choice of probability space and $\{\mathcal{F}_t\}$ is not important, although the probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ once chosen, is assumed to be fixed.

We postulate that the value function $V(x, y_1, y_2)$ exists. We define this value function as follows:

\[
V(x, y_1, y_2) = \inf_{\pi, c \in A} J(x, y_1, y_2; \pi, c); \gamma < 1 \quad (3.4)
\]

\[
V(x, y_1, y_2) = \sup_{\pi, c \in A} J(x, y_1, y_2; \pi, c); \gamma > 1 \quad (3.5)
\]

\[
V(x, y_1, y_2) = x^{1-\gamma} \bar{W}(y_1, y_2) \quad (3.6)
\]

Where $\bar{W}(y_1, y_2)$ is characterized as the value function from solving another optimal control problem, as will be discussed. The equality of the first two definitions, (3.4), and (3.5), and the last definition, (3.6), is a central issue of our research and we provide suitable proofs in subsequent sections. As usual, we need to start our work by introducing a suitable benchmark for the model. The usual benchmark in optimal portfolio choice literature is (Merton, 1971) problem.

**3.3.1 Merton Problem:**

Consider the classical Merton problem (Merton, 1971). We assume that price process follows a Geometric Brownian Motion with constant volatility ($\sigma$ is a constant value), and constant risk free rate, $r$. The assumptions regarding CRRA power utility class
and wealth process still hold. The formulation of the problem is as follows:

\[
\begin{align*}
\frac{dP_t}{P_t} &= \mu P_t dt + \sigma P_t dW_t^p \\
\frac{dX_t}{X_t} &= X_t \left[ (\mu - r) \pi_t + r - c_t \right] dt + \pi_t \sigma dW_t^p 
\end{align*}
\]

In a similar fashion and given constant volatility, we proceed to construct discounted expected utility, \( J(x; \pi, c) \) and solve for value function \( V(x; \pi, c) = \frac{x^{1-\gamma}}{1-\gamma} \bar{W} \). Notice that since \( \sigma \) is a constant, \( \bar{W} \) is a constant as well. This problem has a HJB equation of the form:

\[
\sup_{\pi, c \in A} \mathcal{L}V(x) - \alpha V(x) + v(x, \alpha) = 0
\]

Where \( \mathcal{L} \) is the Itô operator. One can verify that \( V(x) \) is the unique positive classical solution to the equation above. The reader may refer to (Merton, 1971) or (Duffie, 1996) for the solution and verification. Value function follows:

\[
V(x) = \bar{W} \frac{x^{1-\gamma}}{1-\gamma}
\]

Since \( \bar{W} \) is constant, \( V \) is just a function of \( x \). If \( \pi \in \Pi \) and \( \Pi = \mathbb{R} \), then optimal portfolio weight and optimal consumption rule, \((\pi^*, c^*)\), are:

\[
\begin{align*}
\pi^* &= \frac{\mu - r}{\gamma \sigma^2} \\
c^* &= \frac{\alpha - (1 - \gamma) \left[ \frac{(\mu - r)^2 + r}{2 \gamma \sigma^2} \right]}{\gamma}
\end{align*}
\]

Moreover, after performing some calculations we will notice that \( \bar{W} = (c^*)^{1/\gamma} \).

\[
^3\text{These results are true for } \gamma \in (0, 1)
\]
### 3.3.2 Finite Horizon case:

Starting with this section, we introduce the tools needed for theoretical proofs of our results. Going back to the original problem, we are trying to motivate the functional characteristics of $\tilde{W}$. Control policies are present in front of the diffusion term of our wealth process. We need to reformulate the problem such that we do not have any controls associated with a Wiener process, to simplify derivation and computation. It is possible to approximate a process with controls in the diffusion process, but the computational burden will be much heavier. Notice that one may have actual control policies in the diffusion process. We simply do not deal with this case here, since it does not arise in our problem. The reader may refer to (Kushner, 1998) for computational issues associated with this case. Let $T > 0$ be an arbitrary time, fixed in the future. Then, for each $\pi, c \in A$ define the finite horizon functional, discounted expected utility, $J(\cdot)$. We define a new measure $\tilde{P}$ through a Radon-Nikodym change of measure as follows:

$$
\left. \frac{d\tilde{P}}{dP} \right|_{\mathcal{F}_T} = \exp \left[ (1 - \gamma) \int_{s=0}^{T} \pi_s \sigma_p(Y_{s1}^1, Y_{s2}^2) dW_s^* - \frac{1}{2}(1 - \gamma)^2 \int_{s=0}^{T} \pi_s^2 \sigma_p^2(Y_{s1}^1, Y_{s2}^2) ds \right]
$$

Under this new measure, we define the finite horizon discounted expected utility functional $\tilde{J}(x, y_1, y_2; \pi, c, T)$ as follows:

$$
\tilde{J}(x, y_1, y_2; \pi, c, T) = \mathbb{E} \int_{t=0}^{T} \exp \left( -\alpha t \right) \frac{(c_t X_t)^{1-\gamma}}{1 - \gamma} dt
$$

$$
= \mathbb{E} \int_{t=0}^{T} \frac{x^{1-\gamma}}{1 - \gamma} c_t^{1-\gamma} e \left[ -\alpha t + \mu t + (1 - \gamma) \int_{s=0}^{t} \left( \mu - \frac{1}{2} \pi_s^2 \sigma_p^2(Y_{s1}^1, Y_{s2}^2) \right) ds + (1 - \gamma) \int_{s=0}^{t} \pi_s \sigma_p(Y_{s1}^1, Y_{s2}^2) dW_s^* \right] dt
$$

$$
= \frac{x^{1-\gamma}}{1 - \gamma} \tilde{E} \int_{t=0}^{T} c_t^{1-\gamma} e \left[ -\alpha t + \mu t + (1 - \gamma) \int_{s=0}^{t} \left( \mu - \frac{1}{2} \pi_s^2 \sigma_p^2(Y_{s1}^1, Y_{s2}^2) \right) ds + \frac{(1-\gamma)^2}{2} \int_{s=0}^{t} \pi_s^2 \sigma_p^2(Y_{s1}^1, Y_{s2}^2) ds \right] dt
$$

61
\[ J(x, y_1, y_2; \pi, c, T) = \frac{x^{1-\gamma}}{1-\gamma} \tilde{J}(y_1, y_2; \pi, c, T) \]  \hspace{1cm} (3.7)

Given admissible controls, \( (\pi_t, c_t) \in \mathcal{A} \), we simplify the results above by defining:

\[ z_t = \bar{\alpha} t + \bar{r} (1-\gamma) t + (1-\gamma) \int_{s=0}^{t} [(\mu - r)\pi_s - c_s] \, ds + \frac{\gamma (\gamma - 1)}{2} \int_{s=0}^{t} \pi_s^2 \sigma_p^2 (Y^1_s, Y^2_s) \, ds \]  \hspace{1cm} (3.8)

Rewrite the indirect utility function under the new measure as:

\[ \tilde{J}_t(y_1, y_2; \pi, c, T) = \tilde{E} \int_{t=0}^{T} c_t^{1-\gamma} \exp(z_t) \, dt \]  \hspace{1cm} (3.9)

**Case 1 - \( \gamma < 0 \):**

This assumption regarding risk tolerance, \( 1 - \gamma < 0 \), is the usual assumption in finance theory. As the reader recalls from our discussion of Merton’s problem, optimal policies depend on value function, \( V(\cdot) \). In that particular problem, value function just depends on \( x \), since we have constant volatility. But if we have time variable volatility, the value function will depend on both initial wealth and the volatility factors, \( Y^i_t \). The effects of volatility factors enter the derivation through the value function. Recall that the value function follows equation (3.6), hence we need to characterize \( \tilde{W} \) to correctly specify these factor related effects. In this section we characterize \( \tilde{W} \) in consecutive steps, starting with \( W \) and fixed time \( T \) and then we set out to find bounds on value functions and their respective inverse and derivatives.

Define the value function as:

\[ W(y_1, y_2, T) = \inf_{\pi, c} \tilde{J}(y_1, y_2; \pi, c, T) \]

Notice that we take the infimum of this process since \( 1 - \gamma < 0 \). Since the value function is defined under measure \( \tilde{P} \), we need to make appropriate changes for our
volatility factor processes. Under the new measure \( \tilde{P} \), we define:

\[
B^i_t = W^*_{t} - (1 - \gamma) \int_{s=0}^{t} \pi_s \sqrt{1 - \rho_i \sigma_p(Y^1_t, Y^2_t)} ds
\]

\( B^i_t \) is a Brownian motion adapted to \( \{F_t\} \) and specific to the factor \( i \). These results are valid since \( \pi \)'s are bounded.

\[
dY^i_t = \left[ g_i(Y^i_t) + \sigma_i(Y^i_t)(1 - \gamma) \sqrt{1 - \rho_i \pi_t \sigma_p(Y^1_t, Y^2_t)} \right] dt + \sigma_i(Y^i_t) dB^i_t
\]

This change of measure does not change the correlation between factor specific Brownian motions. That is, \( \langle B^1_t, B^2_t \rangle = \langle W^*_{t}^{1}, W^*_{t}^{2} \rangle = \delta t \). Define \( W(y_1, y_2) \), value function corresponding to \( \tilde{J}(x, y_1, y_2; \pi, c) \) as \( T \uparrow \infty \), as follows:

\[
W(y_1, y_2) \overset{\pi, c}{=} \inf \tilde{E} \int_{t=0}^{\infty} c^{1 - \gamma} \exp(z_t) dt
\] (3.10)

Recall that \( \tilde{W} \) corresponds to the value function, as discussed above and by equation (3.6). This should coincide with \( \tilde{W}(y_1, y_2) \overset{\pi, c}{=} \lim_{T \to \infty} W(y_1, y_2, T) \). Notice that:

\[
J(x, y_1, y_2; \pi, c, T) = \frac{x^{1 - \gamma}}{1 - \gamma} \tilde{J}(y_1, y_2; \pi, c, T) \leq J(x, y_1, y_2; \pi, c)
\]

To illustrate the results more clearly, consider the following equalities:

\[
\tilde{W}(y_1, y_2; \pi, c) = \tilde{E} \int_{t=0}^{\infty} c^{1 - \gamma} \exp(z_t) dt
\]

\[
W(y_1, y_2) = \inf_{(\pi, c) \in A} \tilde{E} \int_{t=0}^{\infty} c^{1 - \gamma} \exp(z_t) dt
\]

\[
\bar{W}(y_1, y_2) = \liminf_{T \to \infty} \inf_{(\pi, c) \in A} \tilde{E} \int_{t=0}^{T} c^{1 - \gamma} \exp(z_t) dt
\]

63
This implies, for each \(y_1, y_2 \in \mathbb{R}\):

i. \(\bar{W}(y_1, y_2) \geq W(y_1, y_2, T) \Rightarrow \bar{W}(y_1, y_2) \geq \bar{W}(y_1, y_2)\)

ii. \(W(y_1, y_2) \geq \bar{W}(y_1, y_2)\)

In subsequent sections, it is shown that equality holds for both \(i\) and \(ii\). Notice that we desire specific characteristics in our results. We need our finite horizon results to converge to infinite horizon results, since once we start computing the optimal policies, we always obtain a finite horizon approximation to the infinite horizon problem. Hence, convergence is a minimal requirement. Second, we need to show that our policy space is “feasible” and “reasonable” from both economic and control theoretic point of view. Thus, we need to show that our policy space is dense and general enough to admit all reasonable policy choices while being meaningful in the economic sense, for example, we should show that our optimal consumption can not assume negative values. Thus we start our proofs under a restricted policy space, and then extend the proofs to the unrestricted case.

**Properties of the Value Function:**

Optimal policies depend on values of \(W\). Thus, our main objective is to correctly characterize \(W\), and find regular analytical solutions for \(W\), its inverse, and partial derivatives. As demonstrated later, our policies are functions of the above mentioned values. Hence, in order to obtain well behaved solutions, we need to show that first these values exist, second we need to show that they are unique, and third that they show regular behavior. This goals goes beyond the usual proof for existence of a strong or weak solution to SDEs. We start the proofs for the bounded policy space.

**Assumption 4:** Control sets are bounded intervals:

i. \(c_t \in C\) where \(C = [c_l, c_u]\) such that \(0 < c_l < c_u < \infty\).
ii. \( \pi_t \in \Pi \) where \( \Pi = [-L, L] \), for \( L > 0 \)

In subsequent sections we relax these bounds on controls. We want to put the properties of the value functions \( W(y_1, y_2, T) \) and \( W(y_1, y_2) \) in rigorous basis. Dynamic programming equation associated with \( W(y_1, y_2, T) \) is easy to obtain. Following (Kushner and Dupuis, 2001), we apply the Itô operator, \( \mathcal{L} \), to get the following equality:

\[
-\alpha W + \mathcal{L} W + v(x, y_1, y_2) = 0
\]  

(3.11)

Applying the operator, and gathering the terms, we get:

\[
\alpha W + W_T = g_1(y_1)W_{y_1} + g_2(y_2)W_{y_2} + \frac{\sigma_1^2(y_1)}{2} W_{y_1 y_1} + \frac{\sigma_2^2(y_2)}{2} W_{y_2 y_2} + \delta \sigma_1(y_1) \sigma_2(y_2) W_{y_1 y_2} + \inf_{c_t \in C} [-c_t (1 - \gamma) W + c_t^{1-\gamma}] \\
+ \inf_{\pi_t \in \Pi_L} [(1 - \gamma) r + (\mu - r) \pi_t - \frac{\gamma}{2} \pi_t \sigma_p(y_1, y_2)] W \]  

(3.12)

What we are interested in is \( W(y_1, y_2) \), that is value function where \( T \to \infty \). Notice that if \( W_T = 0 \) for \( T \to \infty \) in the previous equation, we get:

\[
\alpha W = g_1(y_1)W_{y_1} + g_2(y_2)W_{y_2} + \frac{\sigma_1^2(y_1)}{2} W_{y_1 y_1} + \frac{\sigma_2^2(y_2)}{2} W_{y_2 y_2} + \delta \sigma_1(y_1) \sigma_2(y_2) W_{y_1 y_2} + \inf_{c_t \in C} [-c_t \gamma W + c_t^\gamma] + \inf_{\pi_t \in \Pi_L} [(1 - \gamma) r + (\mu - r) \pi_t - \frac{\gamma}{2} \pi_t \sigma_p(y_1, y_2)] W
\]  

(3.13)

Since the value function should be bounded to be useful, we start deriving the bounds for the \( W \) process. To get the positive upper bound on \( W \), simplify the process by setting \( \pi_t = 0 \) and choose admissible \( c_t \geq 0 \), and then minimize \( \tilde{J} \). After setting
\[ \pi = 0, \bar{J}(x, y_1, y_2; \pi, c) = \bar{E} \int_0^\infty c_t^{1-\gamma} \exp[-\alpha t + (1-\gamma)rt] dt \] We minimize \( \bar{J} \) with respect to suitable \( c_t \). Gather and rearrange the terms to obtain the upper bound for \( W \):

\[
W^+ = \left\{ \frac{\alpha - (1-\gamma)r}{\gamma} \right\}^{-\gamma} = K_1
\] (3.14)

Notice that for optimal \( c_t \geq 0 \), \( W^+ \) chosen as above solves:

\[
\alpha W^+ = g_1(y_1)W_{y_1}^+ + g_2(y_2)W_{y_2}^+ + \frac{\sigma_1^2(y_1)}{2}W_{y_1y_1}^+ + \frac{\sigma_2^2(y_2)}{2}W_{y_2y_2}^+ + \delta \sigma_1(y_1)\sigma_2(y_2)W_{y_1y_2}^+ \\
- c_t(1-\gamma)W^+ + c_t^{1-\gamma} + (1-\gamma)r W^+
\]

Based on results presented so far, notice that \( W(y_1, y_2) \leq W^+(y_1, y_2) \).

Now we get the positive lower bound. Since:

\[
r + (\mu - r)\pi_t - \frac{\gamma}{2} \sigma^2_p(y_1, y_2)\pi_t^2 \geq r + (\mu - r)\pi_t - \frac{\gamma}{2} \sigma^2_t \pi_t^2 \geq r - \frac{(\mu - r)^2}{2\gamma \sigma^2_t} \geq W^- \] (3.15)

We define \( W^- = r - (\mu - r)^2[2\gamma \sigma^2_t]^{-1} \) as positive lower bound of \( W \).

Following from the previous equality, \( \forall c_t \geq 0 \) and \( c_t \in C \), given \( T > 0 \) define:

\[
\bar{\alpha} = \alpha - (1-\gamma)W^- \\
z_t \geq -\bar{\alpha}t - (1-\gamma) \int_{s=0}^{t} c_s ds
\]

Thus:

\[
\bar{J}(y_1, y_2; c, \pi, T) \geq \bar{E} \int_{t=0}^{T} \exp(-\bar{\alpha}t) c_t^{1-\gamma} \zeta_t^{1-\gamma} dt
\] (3.16)

Where \( \frac{d\zeta}{dt} = -c_t dt \) and \( \zeta_0 = 1 \).

Define the auxiliary pure consumption minimization problem using \( \zeta_t \)'s ODE, for
all \( c_t \geq 0 \):

\[
\nu(\zeta, t) = \min_{c} \int_{t=0}^{T} \exp(-\bar{\alpha}t)c_t^{1-\gamma} \zeta^{-\gamma} dt
\]

The corresponding dynamic programming equation, based on (Kushner and Dupuis, 2001) MCA method, is:

\[
\nu_T + \bar{\alpha} \nu = \min_{c \geq 0} \left[ - c_t \zeta \nu + (c_t \zeta)^{1-\gamma} \right]
\]

Based on similar ideas to solve the Merton problem, it can be seen that \( \nu(\zeta, T) = \zeta^{1-\gamma} \omega(T) \), where \( \omega \) solves this ODE:

\[
\omega_T + \bar{\alpha} \omega = \gamma \omega^{\frac{1}{\gamma - 1}}
\]

Moreover, as \( T \uparrow \infty \), from (Fleming and Hernández-Hernández, 2003) we know that \( \omega(T) \rightarrow \{ \frac{\bar{\alpha}}{\gamma} \} \). Since \( W(y_1, y_2, T) \geq \nu(1, 1, T) = \omega(T) \) and since we know that there exists a positive lower bound on \( W \), then we know that there exists \( T_1 \) and an arbitrary \( K_2 \geq 0 \) and \( K_2 < W^- \) such that for \( T > T_1, W(y_1, y_2, T) \geq K_2 \), and so we have the uniform bounds on \( W \):

\[
K_2 \leq W(y_1, y_2, T), W(y_1, y_2) \leq K_1
\]  \hspace{1cm} (3.17)

Note that if \( c_t \leq K_1^{\gamma - 1} \) and \( K_2^{\gamma - 1} \leq c_u \), then, \( \inf_{c} [-c\gamma W + c^\gamma] = W(y_1, y_2)^{\frac{1}{\gamma - 1}} = c^* \in [c_l, c_u] \). Thus we may let \( C = [0, \infty) \) or \( C = [c_l, c_u] \) without any loss of generality. This satisfies one important requirement in our results, positivity of optimal consumption.

We have proved that relaxing the bounds on consumption rules do not change our results. In other words, as long as \( c_t \in \mathbb{R}^+ \), our results are valid. Thus, one of the two restrictions on optimal policies is relaxed. Now we set out to explore the effects of relaxing the other restriction. Notice that bounds \( K_1 \) and \( K_2 \) are independent of \( L \),

67
the bound on $|\pi_t|$. This is an important result and we will use this independence in subsequent sections.

**Case $1 - \gamma > 0$**

This case is unnecessary from finance theoretic point of view. It will be meaningful where $1 > 1 - \gamma > 0$, an empirically unlikely event, but it is meaningless for $1 - \gamma > 1$. Thus this case is presented just for mathematical thoroughness of the proofs. Since $1 - \gamma > 0$, corresponding value functions $W(y_1, y_2, T)$ and $W(y_1, y_2)$ are defined as:

$$W(y_1, y_2, T) = \sup_{\pi, c} \tilde{J}(y_1, y_2, \pi, c, T)$$  \hspace{1cm} (3.18)

$$W(y_1, y_2) = \sup_{\pi, c} \tilde{E} \int_{t=0}^{\infty} c_1^{1-\gamma} \exp(z_t) dt$$  \hspace{1cm} (3.19)

Notice that if $\bar{\alpha} > 0$ then we have the following uniform bounds on value functions:

$$K_1 \leq W(y_1, y_2, T), W(y_1, y_2) \leq K_2$$  \hspace{1cm} (3.20)

$W(y_1, y_2, T)$ is a classical solution to the following HJB equation:

$$\alpha W + W_T = g_1(y_1)W_{y_1} + g_2(y_2)W_{y_2} + \frac{\sigma_1^2(y_1)}{2} W_{y_1 y_1} + \frac{\sigma_2^2(y_2)}{2} W_{y_2 y_2}$$

$$+ \delta \sigma_1(y_1) \sigma_2(y_2) W_{y_1 y_2} + \sup_{c \in C} [-c(1 - \gamma)W + c^{1-\gamma}]$$

$$+ \sup_{\pi \in \Pi_L} [(1 - \gamma)[r + (\mu - r)\pi - \frac{\gamma}{2} \pi^2 \sigma_p^2(y_1, y_2)]W$$

$$+ \sigma_1(y_1)(1 - \gamma) \sqrt{1 - \rho_1^2 \pi \sigma_p(y_1, y_2)} W_{Y_1} + \sigma_2(y_2)(1 - \gamma) \sqrt{1 - \rho_2^2 \pi \sigma_p(y_1, y_2)} W_{Y_2}]$$  \hspace{1cm} (3.21)

It can be shown that regardless of the sign of $1 - \gamma$, $W(\cdot, \cdot, T)$ is Lipschitz for fixed $T$.\footnote{Refer to (Fleming and Hernández-Hernández, 2003) and the appendix for further discussion.} We may summarize the findings as follows:
Lemma 3.3.1 If $K_i - L|\sigma_i(y_i)\sqrt{1 - \rho_i^2(1 - \gamma)}|\sigma_{y_i,p}(y_1,y_2)| > 0$ for $i = 1, 2$, then $W(y_1,y_2,T)$ and $\bar{W}(y_1,y_2)$ are Lipschitz and $W(y_1,y_2,T) \to \bar{W}(y_1,y_2)$ as $T \uparrow \infty$ uniformly on compact sets.

Proof: Refer to Appendix.

The result holds for $\rho_i = 1$ trivially. For the alternative case, $\rho_i \neq 1$, the results are not satisfactory and we need to develop some new arguments to obtain the bounds.

Correlated Case, with Positive $1 - \gamma$:

Consistent with discussion regarding positive $1 - \gamma$, assume that $1 - \gamma \in (0,1)$ throughout this section. The goal is the study of value function $W(y_1,y_2,T)$ and it’s asymptotic limit when $T \uparrow \infty$ and $L \uparrow \infty$. In this section, the reader will notice that relaxing the bounds on portfolio weights will not alter our findings. Thus, we may find optimal portfolios in both infinite time horizon and for all values in $\mathbb{R}$. This step allows us to allow for short selling in our optimization problem formally.

(i) Constrained Case:

Throughout this section, we assume where $\pi \in \Pi_L = [-L,L]$ and $L \in (0,\infty)$.

Theorem 3.3.1 $W(y_1,y_2,T)$ is the unique and bounded classical solution to (??) with initial condition $W(y_1,y_2,0) = 0$.

Proof: Refer to (Fleming and Hernández-Hernández, 2003), p.253 for a sketch of the proof and reference to original sources.

In the next step, we formally characterize the properties of $\bar{W}$, the limit function of $W$, along with smooth solutions to steady state PDE. This step allows us, if the conditions for the theorem hold, to specify the characteristics of the optimal control
policies. This characterization defines these policies as functions of the unique solution to the steady state PDE. We show that this solution is also the limit function of $W$. Eventually, we need to characterize the optimal policies as functions of $W$. That will follow in the subsequent steps.

**Theorem 3.3.2** Let $W^-$ be defined as:

$$W^- \doteq r + \frac{(\mu - r)^2}{2\gamma \sigma^2_t}$$

and assume that $\tilde{\alpha} = \alpha - (1 - \gamma)W^-$

Then:

\begin{enumerate}
  \item $\bar{W}$ is a classical solution to steady state PDE:

$$\alpha \omega = g_1(y_1)\omega_{y_1} + g_2(y_2)\omega_{y_2} + \frac{\sigma^2_1(y_1)}{2} \omega_{y_1,y_1} + \frac{\sigma^2_2(y_2)}{2} \omega_{y_2,y_2}$$

$$+ \delta \sigma_1(y_1)\sigma_2(y_2)\omega_{y_1,y_2} + (1 - \gamma)\omega^{-\gamma - 1} + \mathcal{H}(y_1, y_2, \omega, \omega_{y_1}, \omega_{y_2})$$

such that:

$$\mathcal{H}(y_1, y_2, l, p, q) = \sup_{\pi \in \Pi_l} \left[(1 - \gamma)\left[r + (\mu - r)\pi - \frac{\gamma}{2}\pi^2\sigma^2_p(y_1, y_2)\right]l + \left[(1 - \gamma)\sigma_1(y_1)\sqrt{1 - \rho^2_1\pi}\sigma_p(y_1, y_2)p + (1 - \gamma)\sigma_2(y_2)\sqrt{1 - \rho^2_2\pi\sigma_p(y_1, y_2)}q]\right]$$

\item $\bar{W}(y_1, y_2) = W(y_1, y_2)$ for each $y_i \in \mathbb{R}$ and $i = 1, 2$ where $W$ is:

$$W(y_1, y_2) \doteq \sup_{\pi,c} \mathbb{E} \int_{t=0}^{\infty} c_t^{1-\gamma} \exp(z_t)dt$$

\end{enumerate}

**Proof:** In Appendix
From the above theorem, we know that the following feedback policies are bounded, optimal, and locally Lipschitz.

\[
\tilde{\pi}(y_1, y_2) = \arg\max_{\pi \in \mathcal{P}} \left\{ (1 - \gamma) \left[ r + (\mu - r)\pi - \frac{\gamma}{2} \pi^2 \sigma_p^2(y_1, y_2) \right] \tilde{W}(y_1, y_2) + \sigma_1(y_1) \sqrt{1 - \rho_1^2(1 - \gamma)} \pi \sigma_p(y_1, y_2) \tilde{W}_{y_1}(y_1, y_2) + \sigma_2(y_2) \sqrt{1 - \rho_2^2(1 - \gamma)} \pi \sigma_p(y_1, y_2) \tilde{W}_{y_2}(y_1, y_2) \right\} \tag{3.22}
\]

\[
\tilde{c}(y_1, y_2) = \tilde{W}(y_1, y_2) \frac{1}{1 - \gamma} \tag{3.23}
\]

Given the definition of \( z_t \) and the suitable functional forms proposed for \( \sigma_p(Y_t^1, Y_t^2) \), we may find analytical expressions for these policies. The problem is that due to presence of \( \arg\max \) in the expressions, the result is not practically useful. For computationally useful solutions, we need to find out the analytical solutions, after applying the \( \arg\max \) to the policies.

We need to notice and keep in mind the following facts. Routine calculations show that \( \tilde{V}(x, y_1, y_2) = \frac{x^{1-\gamma}}{1-\gamma} W(y_1, y_2) \), is the classical solution to the following HJB equation:

\[
\alpha \nu = g_1(y_1) \nu_{y_1} + g_2(y_2) \nu_{y_2} + \frac{\sigma_1^2(y_1)}{2} \nu_{y_1 y_1} + \frac{\sigma_2^2(y_2)}{2} \nu_{y_2 y_2} + \delta \sigma_1(y_1) \sigma_2(y_2) \nu_{y_1 y_2} + \\
+ \sup_{\pi \in \mathcal{P}} \left\{ \left[ r + (\mu - r)\pi - c \right] x \nu_{x} + \frac{1}{2} \pi^2 \sigma_p^2(y_1, y_2) x^2 \nu_{xx} + \sigma_1(y_1)(1 - \gamma) \sqrt{1 - \rho_1^2} \pi \sigma_p(y_1, y_2) \nu_{y_1 x} + \sigma_2(y_2)(1 - \gamma) \sqrt{1 - \rho_2^2} \pi \sigma_p(y_1, y_2) \nu_{y_2 x} + \frac{c^{1-\gamma} x^{1-\gamma}}{1 - \gamma} \right\} \tag{3.24}
\]

Also, the reader will notice that using the proof for the previous theorem, it is intuitively obvious that \( \tilde{V} = V \) and the more important result is that by this argument, it follows that \( \tilde{W} = W \). Notice that our policies depend on values of \( W \). Thus, our main objective is to correctly characterize \( W \), and find analytical solutions based on
W, its inverse and partial derivatives. We have to establish existence, and then show that our analytical results are unique and bounded.

(ii) **Unconstrained Case:**

Let \( W^L \), denote the value function \( W \), constrained by limits to portfolio weight control, \( \Pi = \Pi_L = [-L, L] \). Our goal in this section is a study of asymptotic properties, when \( L \to \infty \). That is, we want to study the effects of relaxing boundedness condition on admissible portfolio set. The reader will notice that while by allowing \( -L \leq 0 \) our formulation is sufficiently flexible to allow short selling, we are still interested in the study of asymptotic properties of relaxing the bounds.

Define \( W(y_1, y_2) = \lim_{L \to \infty} W^L(y_1, y_2) \). This limit exists, since \( L \to W^L(y_1, y_2) \) is increasing. We also note that (??) holds for \( W \).

**Theorem 3.3.3** Suppose \( \bar{\alpha} > 0 \), then \( W \) is a positive classical solution to:

\[
\begin{align*}
\alpha W &= g_1(y_1)W_{y_1} + g_2(y_2)W_{y_2} + \frac{\sigma_1^2(y_1)}{2} W_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2} W_{y_2,y_2} \\
&+ \delta \sigma_1(y_1)\sigma_2(y_2) W_{y_1,y_2} + (1 - \gamma)W_{\sigma_1,\sigma_2}(y_1, y_2) + (1 - \gamma)r W \\
&+ (1 - \gamma)W \left[ \mu - \sigma_1(y_1)\sqrt{1 - \rho_1^2}\sigma_p(y_1, y_2) \frac{W_{y_1}}{W} + \sigma_2(y_2)\sqrt{1 - \rho_2^2}\sigma_p(y_1, y_2) \frac{W_{y_2}}{W} \right] \\
&\quad \frac{2\gamma\sigma_p^2(y_1, y_2)}{2}
\end{align*}
\]

**Proof:** Refer to Appendix.

So far we have shown existence. To characterize \( W \) as the unique solution of (3.24), we need to obtain some estimates on \( W_{y_i} \). This step also provides us with infinite horizon optimal strategies, introduced in Section (3.2.2). In the first step, define \( \tilde{g}_i(y_i) \) as:

\[
\begin{align*}
\tilde{g}_i(y_i) &= g_i(y_i) - \frac{(1 - \gamma)\sqrt{1 - \rho_i^2}\sigma_i(y_i)(\mu - r)}{\gamma\sigma_p(y_1, y_2)} \\
\tilde{g}_{i,y_i}(y_i) &= g_{i,y_i}(y_i) - \frac{(1 - \gamma)\sqrt{1 - \rho_i^2}(\mu - r)\left[ \sigma_{i,y_i}(y_i)\sigma_p(y_1, y_2) - \sigma_i(y_i)\sigma_{p,y_i}(y_1, y_2) \right]}{\gamma\sigma_p^2(y_1, y_2)}
\end{align*}
\]

72
If $\sigma_{i,y_i}(y_i) = 0$, as is the case with log-linear multi factor SV, then:

$$
\tilde{g}_{i,y_i}(y_i) = g_{i,y_i}(y_i) + (1 - \gamma) \sqrt{1 - \rho_i^2} \frac{(\mu - r)\sigma_i(y_i)\sigma_{p,y_i}(y_1, y_2)}{\gamma \sigma_p^2(y_1, y_2)}
$$

By construction, there exists $\tilde{k}_i$ such that $\tilde{g}_{y_i,i}(y_i) \leq -\tilde{k}_i$. The following lemma shows boundedness for $W_{y_i}$, another important result for solving the problem.

**Lemma 3.3.4** Suppose that conditions of the previous theorem hold, and $\tilde{k} > 0$. If $W$ is a positive classical solution of 3.24 with $W$ and $W^{-1}$ are bounded, then $W_{y_i}$'s are also bounded.

**Proof:** Refer to Appendix.

**Remark:** Under Assumption 3, the results of this lemma impose growth conditions on $\sigma_p(Y_t^1, Y_t^2)$. Notice that the second term of $\tilde{g}$ and $\tilde{g}_{y_i}$ is in terms of $\sigma_p(y_1, y_2)$ and its derivatives. Its size influences the size and sign of $\tilde{k}$. Also, notice that the lemma remains true under weaker assumption that $\tilde{k} + b > 0$.

In the next step we need to show that the value function, $W$, is the unique classical solution to (3.24). The following theorem provides a verification.

**Theorem 3.3.5** Under the conditions of Theorem 3.3.3 and Lemma 3.3.4, the value function

$$
W(y_1, y_2) = \sup_{\pi, c} \mathbb{E} \int_0^\infty c_t^{1-\gamma} \exp(z_t) dt
$$

with $\pi \in \Pi = \mathbb{R}$, is the unique classical solution to (3.24) with $W, W^{-1}$, and $W_{y_i}$ bounded. Moreover,

$$
\pi^*(y_1, y_2) = \frac{\mu - r}{\gamma \sigma_p^2(y_1, y_2)} + \frac{\sigma_1(y_1) \sqrt{1 - \rho_1^2} W_{y_1}(y_1, y_2) + \sigma_2(y_2) \sqrt{1 - \rho_2^2} W_{y_2}(y_1, y_2)}{\gamma \sigma_p(y_1, y_2) W(y_1, y_2)} \quad (3.25)
$$

$$
c^*(y_1, y_2) = W(y_1, y_2)^{\frac{1}{1-\gamma}} \quad (3.26)
$$
are optimal portfolio weight and optimal consumption policy.

**Proof:** Refer to Appendix.

Now we have our optimal policies in closed form. Moreover, this result recasts the portfolio weights as the sum of Merton weights and two hedging demands, one for each factor. These demands mean that if identification of the factors is possible, rational investor would hedge them. This result is important since mis-specification or failure in identification would lead to under diversification.

Our results to this point characterize the optimal policies for correlated volatility factors where relative risk aversion is positive and $1 - \gamma \in (0, 1)$. For completeness of our findings, we need to show that these results are also true when $1 - \gamma < 0$. The next section provides the required proofs.

**Correlated Case, with $1 - \gamma \leq 0$:**

(i) **Constrained Case:**

Now we provide some additional proofs for the case where $1 - \gamma < 0$. These proofs are very similar to what the reader has encountered so far. We start from constrained case, and then relax the constraints in subsequent steps. The objective is to provide proofs for the most general case. We need to account for effects of the correlation. Given these findings, our optimal policies are universally applicable. We start by letting $\pi \in \Pi^L = [-L, L]$, and $L > 0$.

**Theorem 3.3.6** Assume that $K_i - L|\sigma_i(Y_i)|\sqrt{1 - \rho_i^2(1 - \gamma)}|\sigma_p(y_1, y_2)| > 0$ holds. Then
\( \bar{W}(y_1, y_2) \) is a classical solution to:

\[
\begin{align*}
\alpha \omega &= g_1(y_1) \omega_{y_1} + g_2(y_2) \omega_{y_2} + \frac{\sigma_1^2(y_1)}{2} \omega_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2} \omega_{y_2,y_2} + \delta \sigma_1(y_1) \sigma_2(y_2) \omega_{y_1,y_2} \\
&+ (1 - \gamma) \omega^{\gamma - 1} + \mathcal{H}(y_1, y_2, \omega, \omega_{y_1}, \omega_{y_2}) \\
\text{s.t.} \\
\mathcal{H}(y_1, y_2, l, p, q) &= \inf_{\pi \in \Pi} \left[ (1 - \gamma) \left[ r + (\mu - r) \pi - \frac{\gamma}{2} \pi^2 \sigma_p^2(y_1, y_2) \right] l \\
&+ \left[ (1 - \gamma) \sigma_1(y_1) \sqrt{1 - \rho_1^2} \pi \sigma_p(y_1, y_2) + (1 - \gamma) \sigma_2(y_2) \sqrt{1 - \rho_2^2} \pi \sigma_p(y_1, y_2) \right] \right]
\end{align*}
\]

**Proof:** A sketch is provided here. This proof follows standard arguments for parabolic PDEs. From Lemma 3.3.1 we know that when \( T \to \infty, \bar{W}(y_1, y_2, T) \) converges uniformly to \( \bar{W}(y_1, y_2, T) \) on compact sets. Furthermore, \( \bar{W}(y_1, y_2, T) \) is Lipschitz. Since \( \mathcal{H}(y_1, y_2, \omega, p, q) \) is locally Lipschitz, standard Holder estimates for parabolic PDEs (refer to (Fleming and Hernández-Hernández, 2003), page 258 or (Fleming and Rishel, 1975) for further discussion), then there exists a sequence \( T_n \to \infty \) such that \( W_{y_1}(y_1, y_2, T) \to \bar{W}_{y_1}(y_1, y_2, T) \) and \( W_{y_1,y_1}(y_1, y_2, T) \to \bar{W}_{y_1,y_1}(y_1, y_2, T) \). This means that the limit function \( \bar{W} \in C^2(\mathbb{R}) \) and is a solution to equation (3.27).

\[ \square \]

(ii) **Unconstrained Case:**

For small enough \( |\rho_i| \), and bounded \( \bar{W}_{y_i} \), \( \bar{W} \) is a classical solution to (3.24). Moreover, in studying the unconstrained case, \( \pi \in \Pi = \mathbb{R} \), for each \( y_i \in \mathbb{R} \), \( \mathcal{H} \) attains a minimum at:

\[
\pi^*(y_1, y_2) = \frac{\mu - r}{\gamma \sigma_p^2(y_1, y_2)} + \frac{\sigma_1(y_1) \sqrt{1 - \rho_1^2} \bar{W}_{y_1}(y_1, y_2) + \sigma_2(y_2) \sqrt{1 - \rho_2^2} \bar{W}_{y_2}(y_1, y_2)}{\gamma \sigma_p(y_1, y_2) \bar{W}(y_1, y_2)}
\]

(3.28)
The next theorem recasts the arguments in a more precise fashion. Notice that HJB equation for unconstrained case is invariant to the sign of $\gamma$.

**Theorem 3.3.7** There exists $L_o > 0$ and $\varrho > 0$ such that $\bar{W}$ is a classical solution to (3.24) for $L > L_o$ and $|\rho_1| \leq \varrho$.

**Proof:** Refer to Appendix.

Thus, optimal consumption rule and optimal portfolio weights in the multi factor stochastic volatility have the general form of:

$$
\begin{align*}
\pi^*(y_1, y_2) &= \frac{\mu - r}{\gamma \sigma^2_p(y_1, y_2)} + \frac{\sigma_1(y_1)\sqrt{1 - \rho_1^2}W_{y_1}(y_1, y_2) + \sigma_2(y_2)\sqrt{1 - \rho_2^2}W_{y_2}(y_1, y_2)}{\gamma \sigma_p(y_1, y_2) W(y_1, y_2)} \\
c^*(y_1, y_2) &= W(y_1, y_2)^{\frac{1}{1 - \gamma}}
\end{align*}
$$

independent from bounds on consumption, portfolio weights, time, and the sign of risk aversion coefficient. Notice that our results are not meaningful from an economic point of view if $1 - \gamma > 1$. This formulation recasts the portfolio weight as a hedging demand against risk from volatility factors, as we discussed before. The results are in semi-closed form, allowing us to use computational methods to obtain optimal policies.

### 3.4 Numerical Solutions and Analysis:

Our goal is to numerically compute the optimal portfolio weights and consumption rules derived in the previous section for a special class of multi factor SV models. We also want to introduce an algorithm which is efficient, cheap to compute, and accurate given the state of knowledge regarding stochastic control theory.

There are many methods available to solve for optimal rules. One possibility is to solve the PDEs numerically. This exercise is computationally intensive and expensive. (Brandt, 1999) proposes a non-parametric estimation approach for estimating the
portfolio weights. However, his proposal, and an extension developed in (Brandt and Santa-Clara, 2005), is not suitable for estimation of our derived portfolio weights and consumption rule, since we obtained these rules as functionals of the value function and partial derivatives.

Our solution is an implementation of (Kushner and Dupuis, 2001) method, which is well known in stochastic control literature. In this method, one solves the control problem through MCA. The details, modified for our problem, follow.

Since (Chernov et al., 2003) provide us with the most accurate estimation of parameters for dynamics of Dow-Jones Industrial Average (DJIA) for 1953-1999 period, we simply adopt their estimated parameter values for our computations. These estimated parameter values are reported in Table 3.1. (Chernov et al., 2003) use (Gallant and Tauchen, 96 a; Gallant and Tauchen, 2001; Gallant and Tauchen, 2006) “Efficient Method of Moments” or EMM for estimation.

In computing our optimal portfolio and consumption rules, we need to infer the return and volatility dynamics of the risky asset (here assumed to be the Dow-Jones Index) and a suitable risk free asset. We consider average three months T-Bill rate for the same period to be our risk free return. Data for risk free asset is from FRED II Data set at Federal Reserve Bank of St. Louis.

3.4.1 Explicit Functional Forms for Log-Linear SV Model:

As we proceed, we need to simulate the processes and compute the decision rules. In anticipation of that step, we introduce two explicit examples of log-linear stochastic volatility: two factor stochastic volatility model with constant volatility and two factor diffusion log-linear stochastic volatility model. Both models are presented below as an illustration. Since exponential class of stochastic volatility models violate Itô growth
conditions, following (Chernov et al., 2003), we splice the process to avoid instability\(^5\). These models are designed to capture volatility spikes observed in the financial time series.

(i) Two Factor Stochastic Volatility Model with Spliced Constant Volatilities:

\[
\begin{align*}
    dP_t &= \mu P_t dt + \left\{ \exp \left[ \eta_{10} + \eta_{11} Y_{1t}^1 + \eta_{12} Y_{2t}^2 \right] \wedge u \right\} P_t dW_t^* \\
    dY_{1t}^1 &= \alpha_{11} Y_{1t}^1 dt + \sigma_1 \left[ \sqrt{1 - \rho_{12}^2} dW_t^* + \rho_{11} W_t^{*1} \right] \\
    dY_{2t}^2 &= \alpha_{22} Y_{2t}^2 dt + \sigma_2 \left[ \sqrt{1 - \rho_{22}^2} dW_t^* + \rho_{21} W_t^{*2} \right]
\end{align*}
\]

(ii) Two Factor Stochastic Volatility Model with Spliced Diffusions:

\[
\begin{align*}
    dP_t &= \mu P_t dt + \left\{ \exp \left[ \eta_{10} + \eta_{11} Y_{1t}^1 + \eta_{12} Y_{2t}^2 \right] \wedge u \right\} P_t dW_t^* \\
    dY_{1t}^1 &= \alpha_{11} Y_{1t}^1 dt + (1 + \beta_{11} Y_{1t}^1) \left[ \sqrt{1 - \rho_{12}^2} dW_t^* + \rho_{11} W_t^{*1} \right] \\
    dY_{2t}^2 &= \alpha_{22} Y_{2t}^2 dt + \left[ \sqrt{1 - \rho_{22}^2} dW_t^* + \rho_{21} W_t^{*2} \right]
\end{align*}
\]

Where \( \left\{ \exp \left[ \eta_{11} Y_{1t}^1 + \eta_{12} Y_{2t}^2 \right] \wedge u \right\} \) denotes spliced volatility as discussed above. Volatility fluctuates up to level \( u \), and then is truncated at that level. Thus, between \( u \) and \( \exp \left[ \eta_{11} Y_{1t}^1 + \eta_{12} Y_{2t}^2 \right] \) the one that is smaller is chosen at each point in time.

For simplicity, assume that both models generate a wealth process with the following dynamics:

\[
    dX_t = b(Y; c, \pi) X_t dt + a(Y) X_t dW_t^*
\]

Both explicit functional models presented, log-linear constant volatility two factor model and log-linear two factor diffusion model, are considered here. We proceed to compute their respective implied optimal consumption and portfolio policies.

\(^5\text{Refer to the Appendix}\)
in the following steps: First, both models are discretized, using an Euler discretization scheme. Then, both models are simulated given estimated coefficients of (Chernov et al., 2003). The third step is computing the policy rules, given (Kushner and Dupuis, 2001) “Markov Chain Approximation” (MCA) method.

3.4.2 The Numerical Algorithm:

For numerical purposes we need to work in a bounded domain. We chose the upper and lower boundaries of returns process and the risk free rate given the available data, as absorbing boundaries. We let \( t \in [0, T] \) where \( T \) is a suitably large number. For the sake of simplicity, let our state-space be a regular \( h \)-grid (\( h \) denotes the size of our grid), denoted by \( G_h \). We denote transition probabilities by \( p^h(x, y_1, y_2|\pi, c) \).

One should notice that our HJB equation for our control problem is highly non-linear, even without boundary conditions, in order to solve the model we need to approximate the model using Markov chain approximation. One important feature of the (Kushner and Dupuis, 2001) MCA method is that under minimal requirements (with Lipschitz conditions uniform in \( (x, y_1, y_2; c, \pi, t) \) on the drift and the diffusion terms of the processes in question) we know that our discretized controls converge to their continuous counterparts. We do not discuss the convergence theorems since they are extensively discussed in (Kushner and Dupuis, 2001), (Fleming and Rishel, 1975), and (Fleming and Soner, 2006). Also, it is important to note that this method leads to approximating problems that have a close resemblance to the original control models and hence the approximating models have physical interpretations as well. This fact should increase our confidence in the method.

There are, however, several steps in construction of the approximating Markov chains that we need to discuss. The reader should notice that this exercise provides
a verification for closed form solutions provided above. We need to demonstrate local consistency properties for the approximating chain. This condition guarantees that numerical noise is very small. In fact, we can show that this noise goes to zero under certain conditions (Kushner and Dupuis, 2001). The objective of this exercise is to find a controlled Markov chain \( \{\xi^h_n\} \), and an adaptation of the cost (discounted expected utility) function \( \tilde{J}(x, y_1, y_2; \pi, c, T) \) such that the associated optimal control problem is conveniently solvable and the results of this solution conveniently converge to the original as \( h \to 0 \).

The conditions required for our approximating chain to converge to the original are **Local Consistency**. This condition imposes (or rather states) some requirements for conditional expectation and covariances of the approximating Markov chains. Let expectation be denoted by \( E_{h, \pi, c}^n \) at state \( Y^* = (y_1^*, y_2^*)' \) of the chain at step \( n \). Let **interpolation interval** be denoted by \( \Delta t^h(Y, \pi, c) > 0 \) and \( \Delta t^h(Y, \pi, c) \to 0 \) uniformly on \( (Y; \pi, c) \) as \( h \to 0 \).

Given state \( Y^* \), then the following relationships should hold:

\[
E_{Y^*, n}^{h, \pi, c}[\xi^h_{n+1} - Y^*] = b(Y^*, \pi, c)\Delta t^h(Y^*, \pi, c) + o(\Delta t^h(Y^*, \pi, c)) \tag{3.32}
\]

\[
Cov_{Y^*, n}^{h, \pi, c}[\xi^h_{n+1} - Y^*] = a(Y^*)'a(Y^*)\Delta t^h(Y^*, \pi, c) + o(\Delta t^h(Y^*, \pi, c)) \tag{3.33}
\]

Where \( b(Y^*, \pi, c) \) is the drift term in the controlled process, and as \( |\xi^h_{n+1} - \xi^h_n| \to 0 \). Then if the controls on this chain are denoted by \( \pi^h(\cdot) \) and \( c^h(\cdot) \), we may define the interpolation interval as \( \Delta t^h_n = \Delta t^h(\xi^h_n, \pi^h_n, c^h_n) \).

We may have several candidates for the cost function (discounted expected utility).
One of them is:

\[ \tilde{J}^h(y_1, y_2; \pi^h, c^h, T^h) = \mathbb{E}^{(\pi^h, c^h)}_{(y_1, y_2)} \sum_{n=0}^{N^h-1} \exp(-\alpha t^h_n)k(\xi^h_n, \pi^h_n, c^h_n)\Delta t^h_n + \mathbb{E}^{(\pi, c)}_{(y_1, y_2)} \exp(-\alpha T^h)g(\xi^h_{N^h}) \]  

(3.34)

Where \( N^h \) is an admissible stopping time for the chain and selected by the programmer. We define \( V^h = \inf_{(\pi^h, c^h, T^h)} \tilde{J}^h(x, y_1, y_2, \pi^h, c^h, T^h) \). Under conditions discussed in (Kushner and Dupuis, 2001):

\[ V^h(x, y_1, y_2) \to \inf_{(\pi, c, T)} \frac{x^{1-\gamma}}{1-\gamma} \tilde{J}(y_1, y_2; \pi, c, T) = V(x, y_1, y_2) \]  

(3.35)

Hence, our dynamic programming equation is:

\[
V^h(x, y_1, y_2) = \min \left\{ \min_{(\pi, c) \in \mathcal{A}} \left[ e^{(-\alpha \Delta t^h(y_1, y_2; \pi, c))} \sum_{Y'} p^h(Y^*, Y' | \pi, c)V^h(x, Y') \right] + k(x, y_1, y_2; \pi, c)\Delta t^h(y_1, y_2; \pi, c) \right\} 
\]  

(3.36)

Thus, we face two strategies: control and continue, or stop. Note that it is computationally expedient to choose the most simple possible expressions for transition probabilities and interpolation intervals. It can be shown (Kushner, 1998) that \( \Delta t^h(y_1, y_2; \pi, c) \) need not depend on \( (y_1, y_2; \pi, c) \). This is a very useful result, to be discussed in more detail.

### 3.4.3 Construction of the Approximating Chain:

Now we can start constructing the approximating chain. We construct this chain for a finite time horizon case on the classical Approximation Policy Space (henceforth APS). A simple and intuitive method for approaching the problem is the finite difference based approximation. When a carefully chosen finite difference approximation is applied to
the differential operator (Itô operator) of the (un-reflected) system process, the coefficients of the resulting discrete equation are transition probabilities which satisfy the local consistency requirements. Moreover, if we choose the finite difference approximation with reasonable care, the method should work in any dimensions, provided that the covariance matrix is diagonally dominant. Notice that the validity of this approach does not depend on the validity of finite difference solution techniques for PDEs. Finite difference approximations are purely used as guides in construction of locally consistent approximating Markov chains. The rest, conditional on availability of locally consistent approximating Markov chains, is just dealt with with probabilistic methods rooted in weak convergence theory (Kushner, 1998; Kushner and Dupuis, 2001). The resulting transition probabilities can be altered in many ways without sacrificing local consistency.

Suppose that for small enough \( h > 0 \), \( a(Y^*)'a(y^*) > h |b(Y^*, \pi, c)| \). This condition may fail at some \((\tilde{Y}, \tilde{\pi}, \tilde{c})\). If this happens, we use the “upwind” approximation. Following the usual finite difference method, we define numerical first and second derivatives through:

\[
f_{y_1}(y_1, y_2) \rightarrow \frac{[f(y_1 + h, y_2) - f(y_1 - h, y_2)]}{2h} \quad (3.37)
\]

\[
f_{y_2}(y_1, y_2) \rightarrow \frac{[f(y_1, y_2 + h) - f(y_1, y_2 - h)]}{2h} \quad (3.38)
\]

\[
f_{y_1y_1}(y_1, y_2) \rightarrow \frac{[f(y_1 + h, y_2) + f(y_1 - h, y_2) - 2f(y_1, y_2)]}{h^2} \quad (3.39)
\]

\[
f_{y_2y_2}(y_1, y_2) \rightarrow \frac{[f(y_1, y_2 + h) + f(y_1, y_2 - h) - 2f(y_1, y_2)]}{h^2} \quad (3.40)
\]

\[
f_{y_1y_2}(y_1, y_2) \rightarrow \frac{[f(y_1 + h, y_2 + h) - f(y_1 + h, y_2) - f(y_1, y_2 + h) - f(y_1, y_2)]}{h^2} \quad (3.41)
\]
and the “upwind approximation”, also known as “one sided approximation” as:

\[ f_{y_1}(y_1, y_2) \to \frac{[f(y_1 + h, y_2) - f(y_1, y_2)]}{h}; \quad b(\tilde{Y}, \tilde{\pi}, \tilde{c}) \geq 0, \quad (3.42) \]

\[ f_{y_2}(y_1, y_2) \to \frac{[f(y_1, y_2 + h) - f(y_1, y_2)]}{h}; \quad b(\tilde{Y}, \tilde{\pi}, \tilde{c}) < 0 \quad (3.43) \]

Apply the derived finite difference definitions for first and second order derivatives to the model. If we choose an arbitrary function of state variables, \( f(Y) \), satisfying what we discussed so far, then we may write an approximation for this function given arbitrary \( k(\cdot) \) as:

\[
\mathcal{L}^{\pi,c} f(y_1, y_2) + k(y_1, y_2; c, \pi) = b_1(y_1, y_2; c, \pi)f_{y_1}(Y) + b_2(y_1, y_2; c, \pi)f_{y_2}(Y) \\
+ \frac{1}{2}a(Y)'a(Y)f_{y_1y_1}(Y) + \frac{1}{2}a(Y)'a(Y)f_{y_2y_2}(Y) \\
+ \delta a(Y)'a(Y)f_{y_1y_2}(Y) + k(Y; c, \pi) \\
= \alpha f(y_1, y_2)
\]

As we recall, this formula closely resembles the PDE associated with \( W(y_1, y_2) \) as \( T \to \infty \). Since we are interested in writing an HBJ equation solving a closely related Markov approximation, we apply the MCA algorithm to this PDE. Some algebra and
substitution results in this equation:

\[
0 = \mathcal{L}^{c,\pi}W - \alpha W + v(x, y_1, y_2)
\]

\[
\alpha W = g_1(y_1) \frac{W(y_1 + h, y_2) - W(y_1 - h, y_2)}{2h} + g_2(y_2) \frac{W(y_1, y_2 + h) - W(y_1, y_2 - h)}{2h}
\]

\[
+ \frac{\sigma_y^2(y_1)}{2} \left[ \frac{W(y_1 + h, y_2) + W(y_1 - h, y_2) - 2W(y_1, y_2)}{h^2} \right]
\]

\[
+ \frac{\sigma_y^2(y_2)}{2} \left[ \frac{W(y_1, y_2 + h) + W(y_1, y_2 - h) - 2W(y_1, y_2)}{h^2} \right] + \delta \sigma_1^2(y_1) \sigma_2^2(y_2) \left[ \frac{W(y_1 + h, y_2 + h) - W(y_1 - h, y_2) - W(y_1, y_2 - h) - W(y_1, y_2)}{h^2} \right]
\]

\[
+ v(x, y_1, y_2)
\]

(3.44)

After gathering and reordering of the terms we have the transition probabilities:

\[
\alpha W^h = \left[ \frac{\sigma_y^2(y_1) + hg_1(y_1)}{\sigma_y^2(y_1) + \sigma_y^2(y_2) + 2\delta \sigma_1(y_1) \sigma_2(y_2)} \right] W(y_1 + h, y_2)
\]

\[
+ \left[ \frac{\sigma_y^2(y_1) + hg_2(y_2)}{\sigma_y^2(y_1) + \sigma_y^2(y_2) + 2\delta \sigma_1(y_1) \sigma_2(y_2)} \right] W(y_1, y_2 + h)
\]

(3.45)

\[
+ \left[ \frac{\sigma_y^2(y_1) - hg_1(y_1)}{\sigma_y^2(y_1) + \sigma_y^2(y_2) + 2\delta \sigma_1(y_1) \sigma_2(y_2)} \right] W(y_1 - h, y_2)
\]

\[
+ \left[ \frac{\sigma_y^2(y_1) - hg_2(y_2)}{\sigma_y^2(y_1) + \sigma_y^2(y_2) + 2\delta \sigma_1(y_1) \sigma_2(y_2)} \right] W(y_1, y_2 - h)
\]

\[
+ \frac{v(x, y_1, y_2) h^2}{\sigma_y^2(y_1) + \sigma_y^2(y_2) + 2\delta \sigma_1(y_1) \sigma_2(y_2)}
\]

\[
\alpha W^h = p^h(y_1 + h, y_2|c, \pi) W(y_1 + h, y_2)
\]

\[
+ p^h(y_1, y_2 + h|c, \pi) W(y_1, y_2 + h)
\]

(3.46)

\[
+ p^h(y_1 - h, y_2|c, \pi) W(y_1 - h, y_2)
\]

\[
+ p^h(y_1, y_2 - h|c, \pi) W(y_1, y_2 - h)
\]

\[
+ v(x, y_1, y_2)\Delta t^h(Y, c, \pi)
\]

84
The \( p^h(Y, Y \pm h|c, \pi) \) are non-negative and if for \( \tilde{y}_i \neq y_i \pm h \) we define \( p^h(Y, Y \pm h|c, \pi) = 0 \), the \( \sum_Y p^h(Y, Y \pm h|c, \pi) = 1 \). As a result, interpolation interval and transition probabilities are locally consistent as required. If we use one sided approximations for first and second order derivatives, then transition probabilities and interpolation intervals are:

\[
p^h(y_i, y_i \pm h|c, \pi) = \frac{\sigma_i^2(y_i)/2 + hg_i^\pm(y_i)}{\sigma_i^2(y_i)/2 + h|g_i(y_i)|}
\]

\[
\Delta t^h(Y; c, \pi) = \frac{h^2}{\sigma_i^2(y_i)/2 + h|g_i(y_i)|}
\]

Simplify the computation step further by fixing the interpolation intervals. This simplification is particularly important for the finite time case. Without loss of efficiency in the numerical algorism (Kushner, 1998) define \( Q^h = \sigma_1^2(y_1) + \sigma_2^2(y_2) + 2\delta \sigma_1(y_1)\sigma_2(y_2) \) and define \( \bar{Q}^h = \max Q^h \). Then the following are locally consistent:

\[
\Delta t^h = h^2/\bar{Q}^h
\]

\[
\bar{p}^h(y_1, y_2, y_i \pm h|c, \pi) = \frac{\sigma_i^2(y_i) \pm hg_i(y_i)}{Q^h}
\]

\[
\bar{p}^h(y_1, y_2, y_i = \tilde{y}_i|c, \pi) = [\bar{Q}^h - Q^h]/\bar{Q}^h
\]

Notice that in this formulation, the interpolation interval is “uniformized” by allowing each state \( y_i \) to communicate with itself, hence reducing complexity in the computation of the minima in the APS algorithm.

With interpolation interval and transition probabilities available, we can implement the Markov chain approximation. The method used here generates a sequence of policies that converge to the optimal. By defining an absorbing state (chosen by the controller), the stopping time and the stopping policies can be incorporated into the rest of the problem, with entry cost (discounted expected utility) \( g(x, y_1, y_2) \) and a set of unique
control actions which will take a set of states to the absorbing state.

To start the procedure, choose a set of admissible feedback controls, \( \{c_o(y_1, y_2), \pi_o(y_1, y_2)\} \). Then compute the discounted expected utility \( \tilde{J}^h(y_1, y_2; c_o, \pi_o) \) which satisfies:

\[
\tilde{J}^h(y_1, y_2; c_o, \pi_o) = \exp(-\alpha \bar{\Delta}t^h) \sum_{y_i \pm h} \tilde{p}^h(y_1, y_2; y_i \pm h | c, \pi) \tilde{J}^h(Y \pm h; c_o, \pi_o) + k(y_1, y_2; c_o(\cdot), \pi_o(\cdot)) \bar{\Delta}t^h
\]

(3.52)

Then by projecting forward and rolling back, one may “approximately” solve (3.52) for arbitrary \( k^{th} \) step. If we have \( (c_{k-1}(\cdot), \pi_{k-1}(\cdot)), \tilde{J}^h(y_1, y_2; c_{k-1}, \pi_{k-1}) \), then we may obtain the next candidate from:

\[
\begin{pmatrix}
  c_k(y_1, y_2) \\
  \pi_k(y_1, y_2)
\end{pmatrix} = \arg \min_{c, \pi \in A} \left[ e^{-\alpha \Delta t^h} \sum_{y_i \pm h} \tilde{p}^h(y_1, y_2; y_i \pm h | \pi, c) \tilde{J}^h(y_1, y_2; c_{k-1}, \pi_{k-1}) + k(y_1, y_2; \pi, c) \bar{\Delta}t^h \right]
\]

(3.53)

In this paper, a first attempt series of results based on logarithmic utility are reported in Figures (3.1 to 3.4). (Chernov et al., 2003) parameter estimates are reported in Table (3.1). Notice that due to the absence of hedging demands under logarithmic utility, it is much easier to compute the optimal policies compared to iterations based on power utility specification.

In this exercise, we let the simulated processes run for 12,000 observations, then truncate the first 2,000 observations, repeat the results 200 times, and take the mean for each simulated observation. The resulting 10,000 simulated returns are then used to compute the optimal portfolio weights. Since we assume an infinitely lived investor, we do not report an “end of the time” policy. Figures (3.1) and (3.2) use drift terms for return processes generated outside of the loop. Figures (3.3) and (3.4) use drift terms
Table 3.1: EMM Parameters Estimates used in Calibration Study

<table>
<thead>
<tr>
<th>Parameter</th>
<th>est.</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{11}$</td>
<td>-0.0512</td>
<td>0.0410</td>
</tr>
<tr>
<td>$\alpha_{22}$</td>
<td>-0.6673</td>
<td>3.1107</td>
</tr>
<tr>
<td>$\eta_{10}$</td>
<td>-2.1969</td>
<td>0.0414</td>
</tr>
<tr>
<td>$\eta_{11}$</td>
<td>0.0863</td>
<td>0.0400</td>
</tr>
<tr>
<td>$\eta_{12}$</td>
<td>2.7688</td>
<td>0.2597</td>
</tr>
<tr>
<td>$\beta_{11}$</td>
<td>1.9228</td>
<td>0.2260</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>0.0000</td>
<td>-</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>-0.2966</td>
<td>0.0240</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>-0.2915</td>
<td>0.0408</td>
</tr>
</tbody>
</table>

Estimated parameters for DJIA (1953-1999) from (Chernov et al., 2003). These parameters are estimated using EMM, for two-factor log-linear SV models. For stability of the solutions, we calibrated the model with $\alpha_{22} = -0.6673$. The original value is $\alpha_{22} = -52.6673$.

Since ease of computation is one of our main concerns, codes are developed with MATLAB 6.5 (Release 13). At this point, with extensive use of MATLAB’s control, optimization, ordinary and partial differential equation tool boxes, we can easily compute the policies under log-utility assumption. In the near future, when we need to compute the power utility based policies, we need to call codes in other computational libraries. This code may need to be developed using a higher power language such as C++. We are currently working on this section.

3.5 Conclusion and Directions for Future Research:  

This paper develops a rigorous treatment of the theoretical model for optimal portfolio choice in the presence of multi factor stochastic volatility. Since reliable empirical benchmarks are not available, it is important to address the theoretical derivations as rigorously as possible. Otherwise, numerical solutions are highly susceptible to severe
mis-specification. The project is geared toward providing guidelines for active portfolio managers. Possibilities of numerically computing the optimal portfolio weights and consumption rule are extensively discussed. We provided rigorous proofs for characteristics of value function, portfolio weights, and optimal consumption rules, as well as efficient methods to numerically compute the dynamical systems and control policies. This treatment covers one of cases studied by (Chernov et al., 2003). (Fleming and Hernández-Hernández, 2003) study another case, that of one factor stochastic volatility. (Das and Uppal, 2004) study jump-in-returns case. It is possible to include jump terms in volatility factor process or both in returns and volatility processes. Contribution of these factors to portfolio choice of small, individual investors are insignificant, as pointed out by (Brandt, 2006). But for a professional investor who actively manages
a large portfolio, these small differences may translate to significant gains, potential
unrealized gains, or losses. It is interesting to numerically compare the size and timing
implications of portfolio choice rules derived for all the models estimated in (Chernov
et al., 2003), find a benchmark to compare their performance, and study the implica-
tions on the management style of an active fund manager who re-balances his portfolio
often.

On the other hand, research on relationship between this class of problems and the
“Stochastic Portfolio Theory” introduced in (Fernholz, 2002) might be fruitful and of
interest.
Figure 3.3: Simulated Returns Process, Volatility, and Calibrated Portfolio Weight, Drift Process is Time-Dependent

Figure No. 5
Two-Factor SV Model with Constant VoV and Log-Utility

Computed Portfolio Weights, $\pi_t$

$\mu = -1.3994 \times 10^{-4}$

$\pi_{Merton} = -0.0091$
Figure 3.4: Simulated Returns Process, Volatility, and Calibrated Portfolio Weight. Volatility Factors follow an Ornstein-Uhlenbeck Process and Drift Process is Time-Dependent.
Appendix:

Proofs for Chapter 2:

Proof of Proposition 2.1:

If we have the following relationships for Arrow-Debreu state-price densities under the reference and alternative models:

\[
\hat{\xi}_t = \exp \left( - \int_{s=0}^{t} r_s ds - \frac{1}{2} \int_{s=0}^{t} ||\theta_s||^2 ds - \int_{s=0}^{t} \theta_s dW_s \right)
\]

\[
\hat{\xi}_1^t = \exp \left( - \int_{s=0}^{t} r_s ds - \frac{1}{2} \int_{s=0}^{t} ||\theta_1^s||^2 ds - \int_{s=0}^{t} \theta_1^t dW_s \right)
\]

\[
\hat{\xi}_2^t = \exp \left( - \int_{s=0}^{t} r_s ds - \frac{1}{2} \int_{s=0}^{t} ||\theta_2^s||^2 ds - \int_{s=0}^{t} \theta_2^t dW_s \right)
\]

Where \( \theta_t = (\sigma_t^G)^{-1}(\mu_t^G - r_t 1) \) and \( \theta^i_t = (\sigma_t^G)^{-1}(\mu_t^G + \sigma_t^G \nu_t^i - r_t 1) \) then, given our assumption that \( \nu_t^{11} = -\nu_t^{21} \) and \( \nu_t^{12} = -\nu_t^{22} \).

State-price densities should assign a unique price to each asset to fulfil no arbitrage condition. This requires that each agent should view the returns of their respective country similarly. Since both agents observe the state variables without distortion (\([W_1^t, W_2^t]^{t}\) is the common state vector for both agents), agents need to have similar
view of the drift terms in their home country’s asset:

\[
\mu_1^G + [\sigma_{11}^G, \sigma_{12}^G] \nu_1^t = \mu_2^G + [\sigma_{12}^G, \sigma_{22}^G] \nu_2^t
\]

\[
\sigma_{11}^G \nu_1^{11} - \sigma_{12}^G \nu_1^{21} + [\sigma_{12}^G, \sigma_{22}^G] \nu_2^{12} = \mu_2^G - \mu_1^G
\]

and in the foreign country’s asset, which yields a similar result.

In a related fashion, we may require that \( \mathbb{E}(\hat{\xi}_1^t - \hat{\xi}_2^t) = 0 \), a.s. We may rearrange the terms in \( \hat{\xi}_t^i \) to get:

\[
\mathbb{E}[\hat{\xi}_t^1 - \hat{\xi}_t^2] = 0, \text{a.s.}
\]

\[
\mathbb{E}\left[\exp\left(-\int_{s=0}^{t} (\theta_s + \nu_s^1)(\theta_s + \nu_s^1)ds - \int_{s=0}^{t} \nu_s^1 dW_s\right) - \exp\left(-\int_{s=0}^{t} (\theta_s + \nu_s^2)(\theta_s + \nu_s^2)ds - \int_{s=0}^{t} \nu_s^2 dW_s\right)\right] = 0, \text{a.s.}
\]

Choose \( \nu_t^i \) such that they are orthogonal with respect to \( \theta_t \). Since we assume that \( \nu_t^1 = -\nu_t^2 \), after substitution we have:

\[
\mathbb{E}\left[\exp\left(-\int_{s=0}^{t} \nu_s^1 dW_s\right) - \exp\left(\int_{s=0}^{t} \nu_s^1 dW_s\right)\right] = 0, \text{a.s.}
\]

Since the terms specified above are both martingales and indistinguishable, the desired result simply follows.
Proof of Proposition 2.2:

Proof: This proof establishes the existence of market clearing prices on the equilibrium path, hence the existence of the Radner equilibrium specified in (2.15). Regardless of absence or presence of model mis-specification, we are interested in a fixed point approach to existence of Lagrange multipliers, \( \lambda_1 \) and \( \lambda_2 \). This proof simplifies and adapts results of (Serrat, 2001) for our formulation of the problem.

First we prove the existence of the equilibrium: we need to show that \( \lambda_1 \) and \( \lambda_2 \) solving

\[
\mathbb{E} \int_{s=0}^{T} c_1 p_s \xi_1^1 ds = \mathbb{E} \int_{s=0}^{T} \varrho_1^1 \xi_1^1 ds
\]

\[
\mathbb{E} \int_{s=0}^{T} c_2 p_s \xi_2^2 ds = \mathbb{E} \int_{s=0}^{T} \varrho_2^2 \xi_2^2 ds
\]

exits and are unique.

For any \((x, y) \in \mathbb{R}^2\), define

\[
\Delta_t(x, y) = \exp\left(-\frac{\beta t}{\gamma}\right)x + \exp\left(-\frac{\beta t}{\gamma}\right)y
\]

and the map \( \Lambda = (\Lambda^1, \Lambda^2) : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \) where:

\[
\Lambda^1_{(x, y)} = \frac{\mathbb{E} \left( \int_{t=0}^{T} (\varrho_1^1)^{1-\gamma} \Delta_t(x, y)^\gamma dt \right)}{\mathbb{E} \left( \int_{t=0}^{T} e^{\frac{\beta t}{\gamma}} [(\varrho_1^1)^{1-\gamma} + (\varrho_2^2)^{1-\gamma}] \Delta_t(x, y)^{(1-\gamma)} dt \right)}
\]

\[
\Lambda^2_{(x, y)} = \frac{\mathbb{E} \left( \int_{t=0}^{T} (\varrho_2^2)^{1-\gamma} \Delta_t(x, y)^\gamma dt \right)}{\mathbb{E} \left( \int_{t=0}^{T} e^{\frac{\beta t}{\gamma}} [(\varrho_1^1)^{1-\gamma} + (\varrho_2^2)^{1-\gamma}] \Delta_t(x, y)^{(1-\gamma)} dt \right)}
\] (3.55)

Let \( x = \lambda_1^\frac{1}{\gamma} \) and \( y = \lambda_2^\frac{1}{\gamma} \). Then from (2.15), (2.21), (2.23), and (2.25), it is clear that we are looking for a fixed point of map \( \Lambda \). Thus, \( \forall (x, y) \in \mathbb{R}^2_+ \) such that \( x = \Lambda^1_{(x, y)} \),
there must be \( y = \Lambda^2_{(x,y)} \). To show this, look at \( x = \Lambda^1_{(x,y)} \) then

\[
\mathbb{E}\left( \int_{t=0}^{T} \alpha_t \left[ (g_t^{2})^{1-\gamma} \Delta_t(x,y)^{\gamma} \right] dt \right) = \mathbb{E}\left( \int_{t=0}^{T} (g_t^{1})^{1-\gamma} \Delta_t(x,y)^{\gamma} dt \right)
\]  

(3.56)

where \( \alpha_t = \frac{\exp(-\frac{\beta t}{\gamma})}{\Lambda_t(x,y)} \). Thus, from (3.56) we are allowed to claim that \( \Lambda^2_{(x,y)} = y \).

In the next step, we should show that there exists \( y \), such that \( \Lambda^1_{(1,y)} = 1 \). Such a number exists, if and only if for \( g : \mathbb{R} \to \mathbb{R} \) we have:

\[
g(z) = \mathbb{E}\left( \int_{t=0}^{T} \exp\left(-\frac{\beta t}{\gamma}\right) \left[ (g_t^{1})^{1-\gamma} + (g_t^{2})^{1-\gamma} \right] \Delta_t(1,z)^{\gamma} dt \right) - \mathbb{E}\left( \int_{t=0}^{T} (g_t^{2})^{1-\gamma} \Delta_t(1,z)^{\gamma} dt \right)
\]

then from (Cox and Huang, 1989) we know that \( g(y) = 0 \).

Note that \( g \) is continuous, \( g' < 0 \), and \( \lim_{z \to \infty} g(z) = -\infty \). Thus we have \( g(0) = \mathbb{E}\left( \int_{t=0}^{T} (g_t^{1})^{1-\gamma} \exp\left(-\frac{\beta t}{\gamma}\right) dt \right) > 0 \). Thus, there exists \( y > 0 \) such that \( g(y) = 0 \). This establishes existence.

To show uniqueness up to a scaling, consider the following:

If an equilibrium exists, then budget constraint must be met as an equality. This fact is due to non-satiability of utility schedule of consumers. For brevity, denote an equilibrium just with \( (\lambda_1, \lambda_2, \hat{\xi}_t) \). Note that if \( (\lambda_1, \lambda_2, \hat{\xi}_t) \) characterizes an equilibrium, then by setting \( \phi_1 = \lambda_1^{1/\gamma} \) and \( \phi_2 = \lambda_2^{1/\gamma} \) we may correctly claim that \( (\phi_1, \phi_2) \) is a fixed point of \( \Lambda \). To verify, we substitute \( (k^{1/\gamma} \phi_1, k^{1/\gamma} \phi_2) \) and we will see that it is also a fixed point of \( \Lambda \) for all positive values of \( k \). So we may claim that \( (k \lambda_1, k \lambda_2, k^{-1} \hat{\xi}_t) \) characterizes an equilibrium as well. Assume that \( \Phi = (\phi_1, \phi_2) \) and \( \bar{\Phi} = (\bar{\phi}_1, \bar{\phi}_2) \) are both fixed points of \( \Lambda \), hence both \( (\Phi^\gamma, \hat{\xi}_t^{(\Phi)}) \) and \( (\bar{\Phi}^\gamma, \hat{\xi}_t^{(\Phi)}) \) are solutions for the equilibrium conditions, where \( \hat{\xi}_t^{(x)} = (\hat{g}_t^{1})^{1-\gamma} \left[ \exp(-\beta t/\gamma)x_1 + \exp(-\beta t/\gamma)x_2 \right]^\gamma \) for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). If \( \Phi = k \bar{\Phi} \), for some positive valued \( k \), then we may conclude that \( \Phi \) and \( \bar{\Phi} \) are on the same line. So we define \( \psi = \max \left[ \frac{\phi_1}{\bar{\phi}_1}, \frac{\phi_2}{\bar{\phi}_2} \right] \) and let \( \Psi = \psi \Phi = (\psi_1, \psi_2) \).
We have $\Psi \geq \Phi$ with strict inequality for one component and equality for the other component. Without loss of generality, we assume that $\psi_1 = \phi_1$ and $\psi_2 > \phi_2$. Since $\Psi$ lies on the line through $\Phi$, then $(\Psi^{-\gamma}, \hat{\xi}_t^{(\Psi)})$ characterizes an equilibrium and we have:

$$
\mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} (c_t^{1(\Psi)} + p_t^2 c_t^{2(\Psi)}) dt \right) = \mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} dt \right) \quad (3.57)
$$

where $c_t^{1(\Psi)}$ and $c_t^{2(\Psi)}$ are optimal consumption choices of the domestic agent associated with $\hat{\xi}_t^{(\Psi)}$. The reader should notice that $\hat{\xi}_t^{(\Psi)} > \hat{\xi}_t^{(\Phi)}$, a.s. due to our choice of $\Phi$. As a result:

$$
\int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} dt > \int_{t=0}^{T} \hat{\xi}_t^{(\Phi)} dt \quad (3.58)
$$

moreover, by first order conditions one can see that (3.57) implies

$$
\mathbb{E}\left( \int_{t=0}^{T} \left( e^{-\beta_t} \lambda_1 \frac{1}{\gamma} + e^{-\beta_t} \lambda_1 \frac{1}{\gamma} (p_t^2)^{\frac{2-1}{\gamma}} \right) (\hat{\xi}_t^{(\Psi)} \frac{2-1}{\gamma} \right) dt \right) = \mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} dt \right) \quad (3.59)
$$

Then from (3.57) and (3.59) we may claim that

$$
\mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} (c_t^{1(\Psi)} + p_t^2 c_t^{2(\Psi)}) dt \right) = \mathbb{E}\left( \int_{t=0}^{T} e^{-\beta_t} (\frac{\psi_1}{\gamma} + \psi_2 (\frac{p_t^2}{\gamma})) e^{-\beta_t/\gamma} dt \right) < \mathbb{E}\left( \int_{t=0}^{T} e^{-\beta_t} (\frac{\phi_1}{\gamma} + \phi_2 (\frac{p_t^2}{\gamma})) e^{-\beta_t/\gamma} dt \right) = \mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} (c_t^{1(\Psi)} + p_t^2 c_t^{2(\Psi)}) dt \right) \quad (3.60)
$$

since we assumed $\phi_1 = \psi_1$ and $\phi_2 < \psi_2$. From (3.58) and (3.60) we have:

$$
\mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} (c_t^{1(\Psi)} + p_t^2 c_t^{2(\Psi)}) dt \right) - \mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Phi)} dt \right) < \mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Psi)} (c_t^{1(\Psi)} + p_t^2 c_t^{2(\Psi)}) dt \right) - \mathbb{E}\left( \int_{t=0}^{T} \hat{\xi}_t^{(\Phi)} dt \right) = 0 \quad (3.61)
$$
This result contradicts (3.57). Setting $\psi_1 > \phi_1$ and $\psi_2 = \phi_2$ we obtain a similar contradiction. Hence, we prove that $\Phi$ and $\bar{\Phi}$ lie on the same line.

□

Proof for Lemma 2.3:

Proof: Standard arguments, see (Elliott and Kopp, 2005) for a textbook treatment, yield that to price a risky asset following pure diffusion dynamics such as GBM, it is sufficient to remove the drift term and substitute by the risk free rate, using a Radon-Nykodim change of measure and then correctly characterize the diffusion matrix.

The diffusion matrix $\sigma_t^G$ is characterized previously (proof available upon request). Let $\sigma_t^G \equiv \Upsilon'_t \sigma$. From the definition of portfolio generating kernels (Serrat, 2001) we have: $\Pi_t^i = \pi_t^i \sigma_t^G$ and $\Pi_t^{i,Q} = \pi_t^{i,Q} \sigma_t^G$. It obvious that $\Pi_t^i' = \sigma' \Upsilon_t \pi_t^i'$ and $\Pi_t^{i,Q}' = \sigma' \Upsilon_t \pi_t^{i,Q}'$.

Since $\sigma$ is full rank by assumption, then we should have $\Upsilon_t \pi_t'^{i} = \hat{\Pi}_t'^{i}$ and $\Upsilon_t \pi_t'^{i,Q} = \hat{\Pi}_t'^{i,Q}$ where $\hat{\Pi}_t'$ and $\hat{\Pi}_t'^{i,Q}$ solve $\Pi_t^i = \hat{\Pi}_t'^i \sigma_t$ and $\Pi_t^{i,Q} = \hat{\Pi}_t'^{i,Q} \sigma_t$, respectively.

Let

$$\Upsilon_t = \begin{bmatrix} \Upsilon_1^t \\ \Upsilon_2^t \end{bmatrix}$$

where $\Upsilon_1^t$ and $\Upsilon_2^t$, two $2 \times 1$ vectors, denote the first and second rows of $\Upsilon_t$ respectively.

Let $\tilde{0}$ denote a $2 \times 1$ vector of zeros and let $I_2$ denote a $2 \times 2$ identity matrix.

By (Cox and Huang, 1989) and Radner equilibrium specified in (2.15), we know that $\pi_t^{1'} + \pi_t^{2'} = P_t$, hence the following must hold:

$$\begin{bmatrix} \Upsilon_1^t & \tilde{0} \\ \tilde{0} & \Upsilon_2^t \\ I_2 & I_2 \end{bmatrix} \begin{bmatrix} \pi_t^1 \\ \pi_t^2 \end{bmatrix} = \begin{bmatrix} \hat{\Pi}_t^{1'} \\ \hat{\Pi}_t^{2'} \\ P_t^1 \\ P_t^2 \end{bmatrix}$$

(3.62)

98
Since the first two rows of $\Upsilon_t$ are never collinear, the system above is exactly identified for vector $(\pi^1_t, \pi^2_t)'$. Using the identification of $\sigma^G_t$ in previous lemma, one can verify that the solution for the above system is indeed (2.29) and (2.30).
Proofs for Chapter 3:

Discussion of Bounds for Volatility and Drift Terms:

In the most general formulation, we can model the volatility factors as:

\[ dY^i_t = (\alpha_{i1} + \alpha_{i2} Y^i_t)dt + (\eta_{i1} + \eta_{i2} Y^i_t)^{\phi_i}dW^i_t; i = 1, 2 \]

The necessary condition for boundedness is:

\[ (\eta_{i1} + \eta_{i2} Y^i_t) \geq 0 \Rightarrow Y^i_t \geq \frac{-\eta_{i1}}{\eta_{i2}}, \forall t \in [0, \infty) \]

Substituting in explicit functional forms, for exponential volatility case we have:

\[ \left[ \kappa_{10} + \kappa_{11}Y_1 + \kappa_{12}Y_2 \right] > 0 \]

\[ \exp \left[ \kappa_{01} - \left( \frac{\eta_{11} \kappa_{11}}{\eta_{12}} \right) \right] = \sigma_i \]

Notice that if volatility of volatility is a constant coefficient, \( \sigma_i \), then:

\[ \sigma_i = \exp \left[ \kappa_{01} + (\kappa_{11} \sigma_1(Y_1)) + (\kappa_{12} \sigma_2(Y_2)) \right] \]

Since both models are subject to splicing as shown earlier in the paper, the upper limit of volatility bound is met, and Itô growth conditions are satisfied. The upper bound for volatility process is fixed at \( u \). For simulation and estimation purposes, this upper limit is chosen from distribution of data available.

For further discussion refer to (Chernov et al., 2003).
Proof for \(W(Y_1, Y_2, T)\) is Lipschitz:

Proof: Let \(Y^1_t, \hat{Y}^1_t\) and \(Y^2_t, \hat{Y}^2_t\) be solutions to the following SDE system, given \((\pi, c) \in \mathcal{A}\) and \(y_1, \hat{y}_1\) and \(y_2, \hat{y}_2\) initial conditions.

\[
\begin{align*}
\frac{dY^2_t}{Y^2_t} &= \left[ g_1(Y^1_t) + \sigma_1(y_1)(1 - \gamma) \sqrt{1 - \rho^2 \pi \sigma_p(Y^1_t, Y^2_t)} \right] dt + \sigma_1(Y^1_t) dB^1_t \\
\frac{dY^1_t}{Y^1_t} &= \left[ g_1(\hat{Y}^1_t) + \sigma_1(y_1)(1 - \gamma) \sqrt{1 - \rho^2 \pi \sigma_p(\hat{Y}^1_t, \hat{Y}^2_t)} \right] dt + \sigma_1(\hat{Y}^1_t) dB^1_t \\
\frac{dY^2_t}{Y^2_t} &= \left[ g_2(Y^2_t) + \sigma_2(Y^2_t)(1 - \gamma) \sqrt{1 - \rho^2 \pi \sigma_p(Y^1_t, Y^2_t)} \right] dt + \sigma_2(Y^2_t) dB^2_t \\
\frac{dY^1_t}{Y^1_t} &= \left[ g_2(\hat{Y}^2_t) + \sigma_2(\hat{Y}^2_t)(1 - \gamma) \sqrt{1 - \rho^2 \pi \sigma_p(\hat{Y}^1_t, \hat{Y}^2_t)} \right] dt + \sigma_2(\hat{Y}^2_t) dB^2_t
\end{align*}
\]

Case 1: Let \(\sigma_1(Y^1_t) = \sigma_1(\hat{Y}^1_t) = \sigma_1\) and \(\sigma_2(Y^2_t) = \sigma_2(\hat{Y}^2_t) = \sigma_2\) such that:

\[
\tilde{J}(\hat{y}_1, \hat{y}_2; c, \pi, T) - \tilde{J}(y_1, y_2; c, \pi, T) = \mathbb{E} \int_{t=0}^{T} c_t^{1-\gamma} \exp(\tilde{z}_t - z_t) dt \\
\leq \mathbb{E} \int_{t=0}^{T} c_t^{1-\gamma} \exp(\tilde{z}_t)(1 - e^{z_t - \tilde{z}_t}) dt \\
\leq \mathbb{E} \int_{t=0}^{T} c_t^{1-\gamma} \exp(\tilde{z}_t)(z_t - \tilde{z}_t) dt
\]

Notice that since \(|\pi_t| \leq L\), we have\(^6\):

\[
\tilde{z}_t - z_t = \frac{\gamma(\gamma - 1)}{2} \int_{s=0}^{t} \pi^2 s \left[ \sigma^2_p(\hat{Y}^1_s, \hat{Y}^2_s) - \sigma^2_p(Y^1_s, Y^2_s) \right] ds \\
\leq |\gamma(\gamma - 1)| L^2 u |\sigma_{p,y_1}(Y^1_t, Y^2_t) - \sigma_{p,y_2}(Y^1_t, Y^2_t)| \int_{s=0}^{t} |\hat{Y}^1_s - Y^1_s + \hat{Y}^2_s - Y^2_s| ds
\]

The reader will notice that since sample paths of both \(\hat{Y}^1_t - Y^1_t\) and \(\hat{Y}^2_t - Y^2_t\) are

---

\(^6\)Since \(e^y = e^x + e^x(y - x)\), then \(|e^y - e^x| \leq |e^x||y - x|\)
continuously differentiable, then for each $|\hat{Y}_i^i - Y_i^i|$ and $i = 1, 2$ we have:

$$d|\hat{Y}_i^i - Y_i^i| = 2(\hat{Y}_i^i - Y_i^i)[g_i(\hat{Y}_i^i) - g_i(Y_i^i) + \sigma_i(1 - \gamma)\sqrt{1 - \rho_i^2}\pi_i[\sigma_p(\hat{Y}_i^i, \hat{Y}_i^i) - \sigma_p(Y_i^i, Y_i^i)]]$$

Implying that (We discuss squared processes for the sake of continuity):

$$|\hat{Y}_i^i - Y_i^i|^2 \leq |\hat{y}_i - y_i|^2 - 2k_i \int_0^t |\hat{Y}_s^i - Y_s^i|^2 ds + 2Lk_i(1 - \gamma)\sqrt{1 - \rho_i^2}[\sigma_{p,1}(Y_i^1, Y_i^2) + \sigma_{p,2}(Y_i^1, Y_i^2)] \int_0^t |\hat{Y}_s^i - Y_s^i|^2 ds$$

From Gronwall inequality we get:

$$|\hat{Y}_i^i - Y_i^i| \leq |\hat{y}_i - y_i| \exp[-2(K_i - Lk_i(1 - \gamma)\sqrt{1 - \rho_i^2}[\sigma_{p,1}(Y_i^1, Y_i^2) + \sigma_{p,2}(Y_i^1, Y_i^2)])t]$$

And then:

$$\hat{z}_t - z_t \leq \frac{|\gamma(\gamma - 1)|L^2u|\sigma_{p,1}(Y_i^1, Y_i^2) + \sigma_{p,2}(Y_i^1, Y_i^2)|}{2\sum_{i=1}^2[K_i - \frac{L}{2}\sqrt{1 - \rho_i^2}(1 - \gamma)][\sigma_{p,1}(Y_i^1, Y_i^2) + \sigma_{p,2}(Y_i^1, Y_i^2)]}$$

$$\times \left[1 - \exp[-2\sum_{i=1}^2(K_i - \frac{L}{2}\sqrt{1 - \rho_i^2}(1 - \gamma))|\sigma_{p,1}(Y_i^1, Y_i^2) + \sigma_{p,2}(Y_i^1, Y_i^2)|t]|\hat{y}_i - y_i|]\right]$$

Notice that this result and the fact that

$$\tilde{\mathbb{E}} \int_0^T c_t^{1-\gamma}\exp(\hat{z}_t)(z_t - \hat{z}_t)dt \leq K|\hat{y}_1 - y_1 + \hat{y}_2 - y_2|\tilde{J}(\hat{y}_1, \hat{y}_2; \pi, c, T)$$

for some constant $K$ dependent on $T$ and $L$, imply that:

$$|W(\hat{y}_1, \hat{y}_2, T) - W(y_1, y_2, T)| \leq K'K[|\hat{y}_1 - y_1 + \hat{y}_2 - y_2|]$$

102
where:

\[
K' = \begin{cases} 
K_1 & \text{if } 1 - \gamma < 0 \\
K_2 & \text{if } 1 - \gamma > 0 
\end{cases}
\]

Simple algebra shows that \(K\) is independent of \(L\) when

\[
K_i - L|\sigma_i(y_i)\sqrt{1 - \rho_i^2(1 - \gamma)}||\sigma_p,y_1(y_1, y_2) + \sigma_p,y_2(Y^1_t, Y^2_t)| > 0
\]

**Case 2:** (Note: This proof is not complete) Let \(\sigma_1(Y^1_t), \sigma_1(\hat{Y}^1_t), \sigma_2(Y^2_t),\) and \(\sigma_2(\hat{Y}^2_t)\) follow a diffusion process defined in (3.30), as discussed before. Notice that we need to proceed on a case by case basis on Lipschitz property proofs. There are these facts that we need to pay attention to:

1. \(\sigma_p(Y^1_t, Y^2_t)\) is defined as \(\{\exp(\eta_{10} + \eta_{11}Y^1_t + \eta_{12}Y^2_t) \wedge u\}\). Thus, for values of \(\exp(\eta_{10} + \eta_{11}Y^1_t + \eta_{12}Y^2_t) > u\), \(\sigma_p(Y^1_t, Y^2_t)\) is a constant. Hence, for all such values the discussion of the previous case is valid. Also, to simplify the analysis, without loss of generality we assume that \(\eta_{10} = 0\) and \(\eta_{11} = \eta_{12} = 1\).

2. Since \(\sigma_i(Y^i_t)\)s are diffusions as defined in (3.30), \(dY^i_t\) and \(d\hat{Y}^i_t\) processes are not bounded, although they follow the regularity conditions for existence of a solution. To simplify the analysis again we assume, without loss of generality, that \(\beta_i = 1\).

Thus, we are concerned with values of \(\sigma_p(Y^1_t, Y^2_t)\) between the lower bound and upper bound \(u\).

\[
\hat{z}_t - z_t = \frac{\gamma (\gamma - 1)}{2} \int_{s=0}^{t} \pi^2_s[\sigma^2_p(Y^1_s, \hat{Y}^2_s) - \sigma^2_p(Y^1_s, Y^2_s)] \, ds
\]

\[
= \frac{\gamma (\gamma - 1)}{2} \int_{s=0}^{t} \pi^2_s[\exp(\hat{Y}^1_s + \hat{Y}^2_s) - \exp(Y^1_s + Y^2_s)] \, ds
\]

\[
|\exp(\hat{Y}^1_t + \hat{Y}^2_t) - \exp(Y^1_t + Y^2_t)| \leq |u||\hat{Y}^1_t - Y^1_t + \hat{Y}^2_t - Y^2_t|
\]
Writing down the inequality, we have:

\[
\dot{z}_t - z_t = \frac{\gamma(\gamma - 1)}{2} \int_0^t \pi_s^2 [\exp(\hat{Y}_s^1 + \hat{Y}_s^2) - \exp(Y_s^1, Y_s^2)] ds
\]

\[
\leq |\gamma(\gamma - 1)|L^2 u|\sigma_{p,y_1}(Y_t^1, Y_t^2) + \sigma_{p,y_2}(Y_t^1, Y_t^2)| \int_0^t |\hat{Y}_s^1 - Y_s^1 + \hat{Y}_s^2 - Y_s^2| ds
\]

For \( \sigma_i(Y_i) = (1 + Y_i) \) and for \( \hat{Y}_t - Y_t \) with continuous paths we have:

\[
d|\hat{Y}_t - Y_t| = 2(\hat{Y}_t - Y_t)[g_i(\hat{Y}_t) - g(Y_t) + (1-\gamma)\sqrt{1 - \rho^2(\hat{Y}_t - Y_t)} \pi_t[\exp(\hat{Y}_t + \hat{Y}_t) - \exp(Y_t + Y_t)]]
\]

As in the previous case, this implies that (again, we consider squared processes):

\[
|\hat{Y}_t - Y_t|^2 \leq \left| \hat{y}_t - y_t \right|^2 - 2K_i \int_0^t |\hat{Y}_s - Y_s|^2 ds + 2L|\hat{Y}_t - Y_t|(1-\gamma)\sqrt{1 - \rho^2}||\sigma_{p,y_1}(Y_t^1, Y_t^2) + \sigma_{p,y_2}(Y_t^1, Y_t^2)|| \int_0^t |\hat{Y}_s - Y_s|^2 ds
\]

Again, from Gronwall inequality we get:

\[
|\hat{Y}_t - Y_t| \leq \left| \hat{y}_t - y_t \right| \exp\left[-2(K_i - L)(\hat{Y}_t - Y_t)(1-\gamma)\sqrt{1 - \rho^2}\right]||\sigma_{p,y_1}(Y_t^1, Y_t^2) + \sigma_{p,y_2}(Y_t^1, Y_t^2)||t
\]

And then:

\[
\dot{z}_t - z_t \leq \frac{|\gamma(\gamma - 1)|L^2 u|\sigma_{p,y_1}(Y_t^1, Y_t^2) + \sigma_{p,y_2}(Y_t^1, Y_t^2)|}{2 \sum_{i=1}^2 [K_i - \frac{L}{2}||\hat{Y}_t^i - Y_t^i|| \sqrt{1 - \rho^2(1 - \gamma)||\sigma_{p,y_1}(Y_t^1, Y_t^2) + \sigma_{p,y_2}(Y_t^1, Y_t^2)||}]}
\]

\[
\left[1 - \exp\left[-2 \sum_{i=1}^2 (K_i - \frac{L}{2}||\hat{Y}_t^i - Y_t^i|| \sqrt{1 - \rho^2(1 - \gamma)||\sigma_{p,y_1}(Y_t^1, Y_t^2) + \sigma_{p,y_2}(Y_t^1, Y_t^2)||} + \sigma_{p,y_2}(Y_t^1, Y_t^2)||t \left| \hat{y}_t - y_t \right|\right]\right]
\]

The rest of the proof closely follows that of the first case. We find out that \( K \) is
independent of $L$ when:

$$K_i - L|\hat{Y}_i^i - Y_i^i|\sqrt{1 - \rho^2_i(1 - \gamma)||\sigma_{p,y_1}(y_1, y_2) + \sigma_{p,y_2}(Y_1^i, Y_2^i)| > 0}$$

□

**Proof of Theorem 3.2:**

**Proof:** **Step 1:** In the first step get an estimate for $W_T(y_1, y_2, T)$:

For fixed and arbitrary $T > 0$, note that $z_t \leq -\bar{\alpha}t$. For each $(\pi, c) \in \mathcal{A}$ and $T' > T$ we get:

$$\tilde{J}(y_1, y_2; c, \pi, T') - \tilde{J}(y_1, y_2; c, \pi, T) = \mathbb{E} \int_T^{T'} e^{1 - \gamma t} \exp(z_t) dt$$

$$\leq c^{1 - \gamma} \int_T^{T'} \exp(-\bar{\alpha}t) dt$$

$$\leq c^{1 - \gamma} \exp(-\bar{\alpha}t)(T' - T)$$

Take $T' = T + h$ such that $h > 0$. Letting $h \to 0$, and dividing both sides by $h$ we have:

$$(***) 0 \leq W_T(y_1, y_2, T) \leq c_u^{1 - \gamma} \exp(-\bar{\alpha}t)$$

This last result provides us with the upper bound for $W_T(\cdot, \cdot, T)$. We are interested in showing that $|W_{y_i}(y_1, y_2, T)|$ is bounded independent of $T$ and $L$. To do so, we propose a new argument, as follows: fix $1 < R < \infty$, let $Z(y_1, y_2, T) = \ln W(y_1, y_2, T)$ for
\[ |y_i| \leq R - 1. \text{ Then:} \]

\[
Z_T + \alpha = g_1(y_1)Z_{y_1} + g_2(y_2)Z_{y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2}Z_{y_2,y_2}
+ \delta \sigma_1(y_1)\sigma_2(y_2)Z_{y_1,y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1}^2 + \frac{\sigma_2^2(y_2)}{2}Z_{y_2}^2
+ (1 - \gamma)\gamma + (1 - \gamma)\psi^L(y_1, y_2, Z_{y_1}, Z_{y_2}) - \gamma \exp(-\frac{Z}{\gamma})
\]

where:

\[
\psi^L(y_1, y_2, p, q) = \max_{\pi \in \Pi_\mu} [(\mu - r)\pi + \frac{-\gamma}{2}\pi \sigma_p^2(y_1, y_2)
+ [\sqrt{1 - \rho_1^2\sigma_1(y_1)\pi \sigma_p(y_1, y_2)}]p + [\sqrt{1 - \rho_2^2\sigma_2(y_2)\pi \sigma_p(y_1, y_2)}]q]
\]

Since \( \gamma \exp(-\frac{Z}{\gamma}) \), \( (1 - \gamma)\psi^L(y_1, y_2, Z_{y_1}, Z_{y_2}) \) and \( (1 - \gamma)r \) are all non-negative, then:

\[
Z_T + \alpha \geq g_1(y_1)Z_{y_1} + g_2(y_2)Z_{y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2}Z_{y_2,y_2}
+ \delta \sigma_1(y_1)\sigma_2(y_2)Z_{y_1,y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1}^2 + \frac{\sigma_2^2(y_2)}{2}Z_{y_2}^2
\]

If \( Z_{y_i}(y_1, y_2, T) \) has a local optimum at \( \tilde{y}_i \) such that \( |\tilde{y}_i| \leq R \), then \( Z_{y_i,\tilde{y}_i}(\tilde{y}_1, \tilde{y}_2, T) = 0 \) and \( Z_{y_i,\tilde{y}_i}(\tilde{y}_1, \tilde{y}_2, T) = 0 \). Letting \( \lambda_1 = \max \{ |g_i(y_i)| : |y_i| \leq R, i = 1, 2 \} \), from

\[
K_1 \leq W(y_1, y_2, T), W(y_1, y_2) \leq K_2
\]

and

\[
Z_T + \alpha \geq g_1(y_1)Z_{y_1} + g_2(y_2)Z_{y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2}Z_{y_2,y_2}
+ \delta \sigma_1(y_1)\sigma_2(y_2)Z_{y_1,y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1}^2 + \frac{\sigma_2^2(y_2)}{2}Z_{y_2}^2
\]
at \( \bar{y}_1 \) and \( \bar{y}_2 \) we have:

\[
\frac{\sigma_1^2(y_1)}{2} Z_{y_1}^2 + \frac{\sigma_2^2(y_2)}{2} Z_{y_2}^2 - \lambda_1 |Z_{y_1} + Z_{y_2}| - \lambda_2 \leq 0 \]

for some \( \lambda_2 \), which implies:

\[
|Z_{y_i}(y_1, y_2, T)| \leq C(R)
\]

\( C(R) \) is independent of \( T \) and \( L \). On the other hand, from \( K_1 \leq W(y_1, y_2, T), W(y_1, y_2) \leq K_2 \) and Mean Value Theorem it follows that there exist \( y_{i-} \in (-R, -R + 1) \) and \( y_{i+} \in (R - 1, R) \) such that:

\[
|W_{y_i}(y_{i\pm}, T)| \leq K_2, |Z_{y_i}(y_{i\pm}, T)| \leq \frac{K_2}{K_1}
\]

Thus, if: \( y_{i-} \leq y_i \leq y_{i+} \)

then, \( |Z_{y_i}(y_1, y_2, T)| \leq \max \left\{ \frac{K_2}{K_1}, C(R) \right\} \)

hence, for: \( |y_i| \leq R - 1 \)

\[
|W_{y_i}(y_1, y_2, T)| \leq \max \left\{ \frac{K_2^2}{K_1}, K_2C(R) \right\}
\]

Since \( W(y_1, y_2, T) \) is a classical solution to (\( ?? \)), and given (\( ?? \)),

\[
|W_{y_i}(y_1, y_2, T)| \leq \max \left\{ \frac{K_2^2}{K_1}, K_2C(R) \right\}, \quad \text{and} \quad 0 \leq W_T(y_1, y_2, T) \leq c_{1-\gamma} \exp(-\alpha t),
\]

one can deduce that for \( |y_i| \leq R - 1, i = 1, 2; |W_{y_i,y_i}(y_1, y_2, T)| \) is uniformly bounded, independent from \( T \) and \( L \). In addition to the previous result, since \( \mathcal{H}(y_1, y_2, \omega, p, q) \) is locally Lipschitz, by Arzelá-Ascoli theorem we know that through some \( T_n \to \infty, W(\cdot, \cdot, T) \to \bar{W}(\cdot, \cdot) \) on compact sets uniformly, and so do its first and second derivatives.

Thus, \( \bar{W} \in C^2(\mathbb{R}) \) and it is a solution to

\[
\alpha \omega = \left( g_1(y_1)\omega y_1 + g_2(y_2)\omega y_2 + \frac{\sigma_1^2(y_1)}{2} \omega y_1 + \frac{\sigma_2^2(y_2)}{2} \omega y_2 + \delta \sigma_1(y_1)\sigma_2(y_2)\omega y_1 y_2 - \gamma \right)^{\frac{\gamma - 1}{\gamma}} + \mathcal{H}(Y_1, Y_2, \omega, \omega y_1, \omega y_2)
\]
**Step 2:** Let \( y \in \mathbb{R} \), it is intuitively obvious that \( \bar{W}(y_1, y_2) \leq W(y_1, y_2) \). The reverse will be:

For \((\pi, c) \in \mathcal{A}\) by Feynman-Kac we have:

\[
W(y_1, y_2) \geq \mathbb{E} \int_{t=0}^{T} c_t^{1-\gamma} \exp(z_t) dt + \mathbb{E}[\bar{W}(y_1, T, y_2, T) \exp(z_T)]
\]

Letting \( T \to \infty \) we notice that:

\[
W(y_1, y_2) \geq \mathbb{E} \int_{t=0}^{T} c_t^{1-\gamma} \exp(z_t) dt
\]

\[
\Rightarrow \bar{W}(y_1, y_2) \geq W(y_1, y_2)
\]

Given the above arguments, and considering the following feedback policies,

\[
\tilde{\pi}(y_1, y_2) = \arg \max_{\pi \in \Pi_L} \left\{ (1 - \gamma)[r + (\mu - r)\pi - \frac{\gamma^2}{2}\pi^2\sigma^2_p(y_1, y_2)]\bar{W}(y_1, y_2) 
+ \sigma_1(y_1)\sqrt{1 - \rho_1^2(1 - \gamma)}\pi\sigma_p(y_1, y_2)\bar{W}_{y_1}(y_1, y_2) 
+ \sigma_2(y_2)\sqrt{1 - \rho_2^2(1 - \gamma)}\pi\sigma_p(y_1, y_2)\bar{W}_{y_2}(y_1, y_2) \right\}
\]

\[
\tilde{c}(y_1, y_2) = \bar{W}(y_1, y_2)^{-\frac{1}{\gamma}}
\]

the reader will notice that both \( \tilde{\pi} \) and \( \tilde{c} \) are bounded and locally Lipschitz. Furthermore, these arguments show the optimality of these policies.

\[
\square
\]

**Proof of Theorem 3.3:**

**Proof:** Since \( W^L \) satisfies both ?? and \(|W_{y_i}(y_1, y_2, T)| \leq \max \left\{ \frac{K_2^2}{K_1^3}, K_2C(R) \right\} \) indepen-
dently from \( L \), and is also a smooth solution to:

\[
\alpha \omega = g_1(y_1)\omega_{y_1} + g_2(y_2)\omega_{y_2} + \frac{\sigma_1^2(y_1)}{2}\omega_{y_1} + \frac{\sigma_2^2(y_2)}{2}\omega_{y_2} + \delta_1(y_1)\sigma_2(y_2)\omega_{y_1,y_2} - \gamma \omega^{\frac{\gamma - 1}{\gamma}} + \mathcal{H}(y_1, y_2, \omega, \omega_{y_1}, \omega_{y_2})
\]

s.t.

\[
\mathcal{H}(y_1, y_2, l, p, q) = \sup_{\pi \in \Pi_L} \left[ (1 - \gamma)[r + (\mu - r)\pi - \frac{\gamma}{2}\pi^2\sigma_p^2(y_1, y_2)]l \right. \\
\left. + [(1 - \gamma)\sigma_1(y_1)\sqrt{1 - \rho_1^2\pi_1\sigma_p(y_1, y_2)p + (1 - \gamma)\sigma_2(y_2)\sqrt{1 - \rho_2^2\pi_2\sigma_p(y_1, y_2)q}} \right]
\]

then, \(|W_{y_{i,y}}^L|\) is bounded independent of \( L \) for \(|y_i| \leq R - 1\) for each \( 1 < R < \infty\).

Therefore, as \( L \to \infty \), on compact sets, by Arzelá-Ascoli we have: \( W^L \uparrow W \) and \( W_{y_i}^L \to W_{y_i} \). Moreover, for \(|y_i| \leq R - 1\), and sufficiently large \( M \), the supremum in definition of \( \mathcal{H}(y_1, y_2, W^L(y_1, y_2), W_{y_i}^L(y_1, y_2)) \) is achieved at:

\[
\pi(y_1, y_2) = \frac{\mu - r}{\gamma\sigma_p^2(y_1, y_2)} + \frac{\sigma_1(y_1)\sqrt{1 - \rho_1^2W_{y_1}^L(y_1, y_2)} + \sigma_2(y_2)\sqrt{1 - \rho_2^2W_{y_2}^L(y_1, y_2)}}{\gamma\sigma_p(y_1, y_2)W^L(y_1, y_2)}
\]

This implies that

\[
\mathcal{H}(y_1, y_2, W^L(y_1, y_2), W_{y_i}^L(y_1, y_2)) \to (1 - \gamma)rW + (1 - \gamma)W\left[ \frac{\mu - r + \sigma_1(y_1)(1 - \gamma)\sqrt{1 - \rho_1^2\sigma_p(y_1, y_2)W_{y_1}^L}}{2\gamma\sigma_p^2(y_1, y_2)} \right. \\
\left. + \frac{\sigma_2(y_2)(1 - \gamma)\sqrt{1 - \rho_2^2\sigma_p(y_1, y_2)W_{y_2}^L}}{2\gamma\sigma_p^2(y_1, y_2)} \right]
\]

uniformly in compact sets. These results show that \( W \) is indeed a positive, classical solution to the above PDE.

\[\square\]
Proof of Lemma 3.4 and Theorem 3.5:

**Proof:** {for Lemma 3.4} Let $Z(y_1, y_2) = \ln W(y_1, y_2)$ as before. Then, $Z$ is a bounded classical solution to:

\[
(i) : \alpha = \tilde{g}_1(y_1)Z_{y_1} + \tilde{g}_2(y_2)Z_{y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2}Z_{y_2,y_2} + \delta \sigma_1(y_1)\sigma_2(y_2)Z_{y_1,y_2} + \frac{\tilde{\sigma}_1^2(y_1)}{2}Z_{y_1}y_1 + \frac{\tilde{\sigma}_2^2(y_2)}{2}Z_{y_2}y_2 - \gamma \exp\left(\frac{Z}{\gamma}\right) + \phi(y_1, y_2)
\]

Thus, \[
\tilde{\sigma}_i(y_i) = \sigma_i(y_i)\sqrt{1 - \left(1 - \gamma\right)^2\rho^2} \quad \gamma \quad \phi(y_1, y_2) = (1 - \gamma)r + \frac{(1 - \gamma)(\mu - r)^2}{2\gamma\sigma_\rho^2(y_1, y_2)}
\]

Since $Z \in \mathbb{C}^2(\mathbb{R})$ and $\tilde{g}_i(y_i) \leq -\tilde{k}_i$, it is easy to see that $Z \in \mathbb{C}^2(\mathbb{R})$. This point is important since if $Z_{y_i}(y_i)$ has a positive maximum at $y_i^*$, then $Z_{y_i,y_i}(y_i^*) = 0$ and $Z_{y_i,y_i,y_i}(y_i^*) \leq 0$. If we differentiate equation (i) with respect to $y_i$ at $y_i = y_i^*$, and given $\tilde{g}_i(y_i) \leq -\tilde{k}_i$, we have:

\[
0 \leq \sum_{i=1}^{2} \tilde{g}_{i,y_i}(y_i^*) - \exp\left(\frac{-Z}{\gamma}\right)Z_{y_1} + \exp\left(\frac{-Z}{\gamma}\right)Z_{y_2} + \phi_{y_1} + \phi_{y_2}
\]

Let $\varsigma$ be a lower bound for $\exp\left(\frac{-Z}{\gamma}\right)$, then:

\[
(\tilde{k}_1 + \varsigma)Z_{y_1}(y_1^*) + (\tilde{k}_1 + \varsigma)Z_{y_2}(y_2^*) \leq \|\phi_{y_1} + \phi_{y_2}\|_\infty
\]

Similarly, if $Z_{y_i}(y_i)$ has a negative minimum at $y_i^*$, the using similar arguments, we have:

\[
0 \geq \sum_{i=1}^{2} \tilde{g}_{i,y_i}(y_i^{**}) - \exp\left(\frac{-Z}{\gamma}\right)Z_{y_1} + \exp\left(\frac{-Z}{\gamma}\right)Z_{y_2} + \phi_{y_1} + \phi_{y_2}
\]
As a result of the above arguments,

\[(\tilde{k}_1 + \varsigma)Z_{y_1}(y_1^*) + (\tilde{k}_1 + \varsigma)Z_{y_2}(y_2^*) \leq \|\phi_{y_1} + \phi_{y_2}\|_{\infty}\]

Thus, we can rewrite the results as:

\[
[\tilde{K} + \varsigma][Z_{y_i}(\vec{y})] \leq \|\phi_{y_1} + \phi_{y_2}\|_{\infty}
\]

where:

\[
\tilde{K} + \varsigma = \begin{pmatrix} \tilde{k}_1 + \varsigma & 0 \\ 0 & \tilde{k}_2 + \varsigma \end{pmatrix}
\]

\[
Z_{y_i}(\vec{y}) = \begin{pmatrix} Z_{y_1}(y_1^*, y_2^*) \\ Z_{y_2}(y_1^*, y_2^*) \end{pmatrix}
\]

and \(\mathcal{R}\) denotes either a maximum or a minimum. Since \(Z\) is bounded and \(\lim \inf_{|y|\to\infty} |Z_{y_i}(y_1, y_2)| = 0\), it is easy to notice that \(Z_{y_i}(y_1, y_2)\) is bounded in \(\mathbb{R}\), such that in fact:

\[
[Z_{y}(\vec{y})] \leq (\tilde{K} + \varsigma)^{-1}\|\phi_{y_1} + \phi_{y_2}\|_{\infty}
\]

□

**Proof**: for Theorem 3.5 Suppose that \(\hat{W}\) is a classical solution, and \(\hat{W}, \hat{W}^{-1}\) and \(\hat{W}_{y_i}\) are bounded. We need to show that \(\hat{W} = W\) when \(W\) is defined as before.

It is sufficient to consider control policies \((\pi, c) \in \mathcal{A}\) such that \(\tilde{\mathbb{E}} \int_{t=0}^{\infty} c_t^{1-\gamma} \exp(z_t)dt < \infty\)

Notice that since \(c_t^{1-\gamma} \geq c_t^{1-\gamma} > 0\), then \(\tilde{\mathbb{E}} \int_{t=0}^{\infty} \exp(z_t)dt < \infty\). This directly implies that \(\lim_{T \to \infty} \tilde{\mathbb{E}} \exp(z_T) = 0\) through some sequence. From Feynman-Kac, we have:

\[
\hat{W}(y_1, y_2) \geq \tilde{\mathbb{E}} \int_{t=0}^{T} c_t^{1-\gamma} \exp(z_t)dt + \tilde{\mathbb{E}} \exp(z_T)\hat{W}(y_{1,T}, y_{2,T})
\]
Since $\tilde{E}\exp(z_T)\tilde{W}(y_1,T,y_2,T) \geq 0$, we directly get $\hat{W}(y_1,y_2) \geq \tilde{E}\int_{t=0}^{T} c_{i}^{1-\gamma}\exp(z_t)dt$, and thus, $\hat{W}(y_1,y_2) \geq W(y_1,y_2)$.

Define $\pi^*(y_1,y_2)$ and $c^*(y_1,y_2)$ as above, with $W(y_1,y_2)$ replaced by $\hat{W}(y_1,y_2)$.

Since $\hat{W}, \hat{W}^{-1}$ and $\hat{W}_{yi}$ are bounded, then $(\pi^*(y_1,y_2)|_{\hat{W}}, c^*(y_1,y_2)|_{\hat{W}}) \in A$. Thus,

\[
\alpha = \tilde{g}_1(y_1)Z_{y_1} + \tilde{g}_2(y_2)Z_{y_2} + \frac{\sigma_1^2(y_1)}{2}Z_{y_1,y_1} + \frac{\sigma_2^2(y_2)}{2}Z_{y_2,y_2} + \delta \sigma_1(y_1)\sigma_2(y_2)Z_{y_1,y_2} \]

\[
+ \frac{\tilde{\sigma}_1^2(y_1)}{2}Z_{y_1}^2 + \frac{\tilde{\sigma}_2^2(y_2)}{2}Z_{y_2}^2 - \gamma \exp\left(\frac{Z}{\gamma}\right) + \phi(y_1,y_2)
\]

\[
\tilde{\sigma}_i(y_i) \doteq \sigma_i(y_i)\sqrt{1 - (1 - \gamma)\rho_i^2/\gamma}
\]

\[
\phi(y_1,y_2) = (1 - \gamma)r + \frac{(1 - \gamma)(\mu - r)^2}{2\gamma\sigma_p^2(y_1,y_2)}
\]

holds for $(\pi^*, c^*)$, since as $T \to \infty$, $\phi(y_1,y_2) \to 0$. This means that by letting $T \to \infty$, we have:

\[
\hat{W}(y_1,y_2) = \tilde{E}\int_{t=0}^{\infty} c_{i}^{1-\gamma}\exp(z_t^*)dt \geq W(y_1,y_2)
\]

Thus, we have $\hat{W}(y_1,y_2) = W(y_1,y_2)$, and $(\pi^*, c^*)$ are optimal controls.

\[\square\]

**Proof of Theorem 3.7:**

**Proof:** The main task in this proof is finding bounds for $W_{yi}$, which are independent of bounds on optimal portfolio weights, $L$.

Fix $1 < R < \infty$, and prove that $|\tilde{W}_{yi}(y_1,y_2)|$ is bounded independent of $L$. That is, for $|y_i| \leq R - 1$, let $Z = \ln W$. 

112
From the previous theorem we know that $Z$ solves:

$$
\alpha = \sum_{i=1}^{2} \left[ g_i(y_1, y_2) Z_{y_i} + \frac{\sigma_i^2(y_i)}{2} Z_{y_i, y_i} + \frac{\sigma_i^2(y_i)}{2} Z_{y_i}^2 \right] + \delta \sigma_1(y_1) \sigma_2(y_2) Z_{y_1, y_2} + (1 - \gamma) r \\
+ (1 - \gamma) \psi^L(y_1, y_2, Z_{y_1}, Z_{y_2}) - \gamma \exp\left( -\frac{Z}{\gamma} \right)
$$

where:

$$
\psi^L(y_1, y_2, p, q) = \max_{\pi \in \Pi^L} \left\{ (\mu - r) \pi - \frac{\gamma}{2} \pi^2 \sigma_p^2(y_1, y_2) + 1 - \rho^2 \sigma_p(y_1, y_2) \right\}
$$

since:

$$
\psi^L(y_1, y_2, p, q) \leq \psi(y_1, y_2, p, q) = \frac{[(\mu - r) + \sigma_1(y_1) \sqrt{1 - \rho^2} \sigma_p(y_1, y_2) + \sigma_2(y_2) \sqrt{1 - \rho^2} \sigma_p(y_1, y_2)]^2}{2 \gamma \sigma_p^2(y_1, y_2)}
$$

we get:

$$
\alpha \geq \sum_{i=1}^{2} \left[ \tilde{g}_i(y_1, y_2) Z_{y_i} + \frac{\tilde{\sigma}_i^2(y_i)}{2} Z_{y_i, y_i} + \frac{\tilde{\sigma}_i^2(y_i)}{2} Z_{y_i}^2 \right] + \sigma_1(y_1) \sigma_2(y_2) Z_{y_1, y_2} + (1 - \gamma) r + \gamma \exp\left( -\frac{Z}{\gamma} \right) + \frac{(1 - \gamma)(\mu - r)^2}{2 \gamma \sigma_p^2(y_1, y_2)}
$$

where:

$$
\tilde{g}_i(y_i) = g_i(y_i) + \frac{(1 - \gamma) \sigma_i(y_i) \sqrt{1 - \rho_i^2}(\mu - r)}{\gamma \sigma_p(y_1, y_2)}
$$

$$
\tilde{\sigma}_i = \sigma_i(y_i) \sqrt{\frac{1 - (1 - \gamma) \rho_i^2}{\gamma}}
$$

Following almost identical arguments used in proof of Theorem 3.2, and given $|y_i| \leq R - 1$, with $C(R)$ independent of $L$, we may get an estimate for:

$$
|\bar{W}_{y_i}(y_i)| \leq \max \left\{ \frac{K^2_1}{K_2}, K_1 C(R) \right\}
$$

Next, prove that $|\bar{W}_{y_1}(y_1, y_2)| \to 0$, as $|y_i| \to \infty$ at a rate independent of $L$. A proof is
given for \( y_i > 0 \), and the same arguments apply to the case where \( y_i < 0 \).

If \( \bar{W}(y_1, y_2) \) has a positive local maximum at \( y_i = \hat{y}_i \), then \( \bar{W}_{y_i,y_i}(\hat{y}_i) = 0 \).

Since \( \mathcal{H}(y_1, y_2, W^L(y_1, y_2), W^L_{y_1,y_2}(y_1, y_2)) \leq 0 \) in Theorem 3.6, and because \( K_1 \leq \bar{W}(y_1, y_2, T) \), \( \bar{W}(y_1, y_2) \leq K_2 \) holds, then:

\[
g_i(\hat{y}_i)\bar{W}_{y_i}(\hat{y}_i) \geq -C_i^*; \quad C_i^* > 0
\]

Since:

\[
g_{y_i}(y_i) \leq -k_i, \quad \text{and} \quad g_i(\hat{y}_i) < 0
\]

then for large \( \hat{y}_i \),

\[
|g_i(y_i)|\bar{W}_{y_i}(\hat{y}_i) \leq C_i^*
\]

In a similar fashion, let \( Z = \ln(\bar{W}) \), and suppose that \( Z_{y_i} \) has a negative local minimum at \( \hat{y}_i \). Then, \( Z_{y_i}(\hat{y}_i) = 0 \) and from

\[
\alpha \geq \sum_{i=1}^{2} \left[ \hat{g}_i(y_i)Z_{y_i} + \frac{\hat{\sigma}_i^2(y_i)}{2}Z_{y_i,y_i} + \frac{\hat{\sigma}_i^2(y_i)}{2}Z_{y_i}^2 \right] + \delta \sigma_1(y_1)\sigma_2(y_2)Z_{y_1,y_2} + (1 - \gamma)r
\]

\[
-\gamma \exp \left( -\frac{Z}{\gamma} \right) + \frac{(1 - \gamma)(\mu - r)^2}{2\gamma \sigma_p^2(y_1, y_2)}
\]

it follows that for some \( C^{**} > 0 \), \( \bar{g}_i(\hat{y}_i)Z_{y_i}(\hat{y}_i) \leq \alpha - \psi(\hat{y}_i) \leq C^{**} \). Since \( g_i(\hat{y}_i) < 0 \), for \( \hat{y}_i \) large, then:

\[
|\bar{g}_i(\hat{y}_i)||\bar{W}_{y_i}(\hat{y}_i)| \leq C^{**}\bar{W}(\hat{y}_i) \leq C^{**}K_1
\]

Given \( \varepsilon > 0 \), \( \exists R_\varepsilon \) independent of \( L \) such that \( |\bar{W}_{y_i}(y_1, y_2)| < \varepsilon, \forall y_i > R_\varepsilon \). Since \( 0 < \varepsilon \),
\[ K_1 \leq \bar{W}(y_1, y_2) \leq K_2 \text{ for each } n \in \mathbb{N}, \exists y_{i,n} \in (n, 2n) \text{ such that} \]

\[
|\bar{W}_{y_i}(y_{1n}, y_{2n})| \leq \frac{K_2 - K_1}{2}
\]

and

\[
|Z_{y_i}(y_{1n}, y_{2n})| = \frac{|\bar{W}_{y_i}(y_{1n}, y_{2n})|}{\bar{W}(y_{1n}, y_{2n})} \leq \frac{K_2 - K_1}{nK_1}
\]

If \( \bar{W}_{y_i} \) has a positive local maximum at \( y_{i,1n} > y_{i,n} \), then by the fact that \( g_i(y_i) \) is bounded and there exists \( k_i > 0 \) such that \( g_i(y_i) \leq -k_i \) and by \( g_i(y_i)\bar{W}_{y_i}(y_1, y_2) \geq C_i^* \), there exists \( b_{1,i} \) such that for large enough \( n \),

\[
\bar{W}_{y_i}(y_1, y_2) < \frac{b_{1,i}}{n}
\]

In a similar fashion, from

\[
|\tilde{g}_i(\bar{y}_i)||\bar{W}_{y_i}(\bar{y}_i)| \leq C^{**}\bar{W}(\bar{y}_i) \leq C^{**}K_1
\]

There exists \( b_{2,i} > 0 \) such that if \( Z_{y_i} \) has a negative local minimum at \( y_{i,2n} > y_{i,n} \), then for large enough \( n \),

\[
Z_{y_i}(y_{1,2n}, y_{2,2n}) \geq -\frac{b_{2,i}}{n}
\]

All these leads us to the conclusion that \( |\bar{W}_{y_i}(y_1, y_2)| < \varepsilon \), for each \( y_i \in R_\varepsilon \). This means that \( \pi^*(y_1, y_2) \) is bounded and independent of \( L \).

Let \( L_o > \|\pi^*\|_\infty \). Then for each \( L_o \leq L \), the infimum in \( \mathcal{H} \) is attained at \( \pi^* \). Substituting the result for the previous theorem, we obtain \( \tilde{\pi}(y_1, y_2) \) and \( \tilde{c}(y_1, y_2) \). For small enough \( |\rho_1|, k_i - L|\sigma_i(y_i)\sqrt{1 - \rho_1^2}(1 - \gamma)||\sigma_p(y_1, y_2)| > 0 \), holds for \( L = L_o \).

□


