NONLINEAR GEOMETRIC OPTICS FOR REFLECTING AND EVANESCENT PULSES

Colton Willig

A dissertation submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the department of Mathematics.

Chapel Hill
2015

Approved by:

Mark Williams

Roberto Camassa

Christopher Jones

Jeremy Marzuola

Michael Taylor

ABSTRACT

Colton Willig: Nonlinear Geometric Optics for Reflecting and Evanescent Pulses.
(Under the direction of Mark Williams)

Weakly nonlinear geometric optics expansions of highly oscillatory reflecting and evanescent pulses are considered for a general class of differential operators. Through rigorous error analysis it is shown that the leading term in these expansions is suitably close to the uniquely determined exact solution. The pulses considered can have multiple components, some of which reflect off fixed non characteristic boundaries in a spectrally stable way (introduced in [10]), and some of which decay exponentially away from the boundary. The results in this paper provide a generalization to the work of Coulombel and Williams in [5], as the boundary frequency is considered not only in the hyperbolic region, but also in the mixed and elliptic regions. Furthermore the boundary data considered in this paper is more general than the boundary data considered in [5]. In fact, it is shown in this paper that θ-decay inheritance of the boundary data to the solution is in some cases not even possible.

TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION

Background

Highly oscillatory pulse solutions for a general class of hyperbolic equations are studied. These include quasilinear systems like the compressible Euler equations; in fact, an ongoing example through the thesis is considered for the Euler equations. Because they exhibit important qualitative features about the system, leading order weakly nonlinear geometric optics expansions of such highly oscillatory pulse solutions are sought. In a rigorous manner, these expansions must be justified (to show that they are close to the exact solution in a precise way). The approximate solution (the leading term of the expansion) yields a decomposition into hyperbolic and elliptic profile components, which exhibits important qualitative information about the system (the desired goal of geometric optics). The approximate solution will have the following form: $\sum \sigma_i(x, \theta_i) r_i$, where σ_i are the profiles and $\theta_i = \frac{\phi_i(x)}{\epsilon}$ (for interior phases $\phi_i(x)$ defined in section 1.3.) The hyperbolic profiles reflect off of the boundary and retain their "energy", while the elliptic profiles decay exponentially away from the boundary.

A single pulse colliding with a fixed noncharacteristic boundary in an $N \times N$ hyperbolic system will generally give rise to a family of reflected pulses traveling with several distinct group velocities. In this paper, the situation is studied when the underlying boundary problem is assumed to be uniformly spectrally stable in the sense of Kreiss. (Refer to [3] for a thorough treatment of Kreiss symmetrizers, along with the uniform Lopatinski condition: a very important assumption which is discussed in section 1.3.) A formal treatment of the boundary problem was given in [11], building on an earlier treatment of nonlinear geometric optics for pulses in free space in [8]. (It should be mentioned here that a rigorous treatment of the short-time propagation of a single pulse in free space were given in [1] and [6].) In the papers [11] and [8], systems of nonlinear equations for leading order profiles were derived, but their solvability was not discussed. Moreover, the questions of the existence of exact solutions on a fixed time interval independent of the wavelength of oscillations ϵ, and of the relation between exact and approximate solutions, were not studied there.

[5] gave a rigorous construction of leading pulse profiles in problems where pulses traveling with many distinct group velocities were unavoidably present. Additionally, [5] constructed exact solutions on a fixed time interval independent of ϵ, and provided a rigorous error analysis which yielded a rate of convergence of approximate to exact solutions as $\epsilon \to 0$.

The results in this paper appropriately generalize the results of Coulombel and Williams in [5], which is by far the primary motivation for this paper. The boundary frequency β appearing in the "θ-argument" of the boundary data is considered not only in the hyperbolic region, but also in the elliptic and the mixed regions. (Refer to the "Spectrum and Boundary Frequency Regions" discussion in section 1.3 for definitions of these regions.) As before the glancing set is a singular case which is not treated. The main difficulties arising in this generalization are the presence of both hyperbolic and elliptic modes. These mixed modes create difficulties when constructing the corrector to the approximate solution; the corrector possesses carefully constructed components, some of which only exist in a distributional sense. (Sections 2.2 and 3.3, along with Appendices B,D,E, discuss the construction of the corrector.) The corrector required moment-zero approximations, which were inspired by the "low-frequency cutoff" argument of [2].

The technique utilized to construct the exact solution and to justify leading term expansions of the system involves replacing the original system (0.1) with an associated singular system (0.3) involving coefficients of order ϵ^{-1} and a new unknown U_ϵ. The error analysis is accomplished via "simultaneous Picard iteration," a method first used in the study of geometric optics for wave trains in free space in [9].

Unlike wavetrains, interacting pulses do not produce resonances that affect leading order profiles. Therefore pulses interactions were assumed to be marginal at leading order. However, even though pulses are considered here, an interesting discovery is made that the presence of elliptic modes can disrupt θ-decay for the hyperbolic profiles. This behavior was not observed in the results of [5] as elliptic modes were not considered. This new behavior supplants previous intuitions that pulse interactions don't produce noticeable effects at leading order. Therefore, even though they decay exponentially away from the boundary, the elliptic profiles have a decided influence on the hyperbolic profiles as they pass into the interior. This new "spreading" phenomenon is discussed in Proposition 0.15. The adjective "spreading" indicates that the hyperbolic profile (after reflecting and losing its θ-decay) has oscillations which are more spread out from the zero set of the θ-argument of the profile.

The Relevant System and Its Perturbed Solutions

On $\overline{\mathbb{R}}^{d+1}_{+} = \{x = (x', x_d) = (t, y, x_d) = (t, x'') : x_d \geq 0\}$, consider the following $N \times N$ quasilinear hyperbolic boundary problem (where $x_0 = t$ indicates time):

$$\begin{aligned} &\sum_{j=0}^{d} A_j(v_\epsilon)\partial_{x_j} v_\epsilon = f(v_\epsilon) \\ &b(v_\epsilon)|_{x_d=0} = g_0 + \epsilon\, G(x', \frac{x' \cdot \beta}{\epsilon}) \\ &v_\epsilon|_{t<0} = u_0 \end{aligned} \tag{0.1}$$

The boundary data $G \in H^s(x', \theta_0)$ is called a *pulse*; it satisfies supp $G \subset \{t \geq 0\}$. $\beta \equiv (\underline{\tau}, \underline{\eta}) \in \mathbb{R}^d \backslash \{0\}$ indicates the *boundary frequency*; $\phi_0(t, y) \equiv \beta \cdot x'$ is called the *boundary phase*. The coefficients satisfy: $A_i \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N^2})$, $f \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, $b \in C^\infty(\mathbb{R}^N, \mathbb{R}^p)$.

Seeking a perturbed solution $v_\epsilon = u_0 + \epsilon u_\epsilon$ of a constant state u_0, such that $f(u_0) = 0$ and $b(u_0) = g_0$, gives (for modified A_j):

$$\begin{aligned}
&\sum_{j=0}^{d} A_j(\epsilon\, u_\epsilon)\partial_{x_j} u_\epsilon = \mathcal{F}(\epsilon\, u_\epsilon)u_\epsilon \\
&B(\epsilon u_\epsilon)u_\epsilon|_{x_d=0} = G(x', \frac{x'\cdot\beta}{\epsilon}) \\
&u_\epsilon = 0 \quad (t<0)
\end{aligned} \tag{0.2}$$

where the C^∞ $p \times N$ real matrix $B(v)$ is defined by: $b(u_0 + \epsilon u_\epsilon) = b(u_0) + B(\epsilon u_\epsilon)\epsilon u_\epsilon$ ($\mathcal{F}$ is defined analogously).

For any fixed $\epsilon_0 > 0$, the standard theory of hyperbolic boundary problems produces solutions of (0.2) on a fixed time interval $[0, T_{\epsilon_0}]$. However these time intervals $[0, T_\epsilon]$ shrink to zero as $\epsilon \to 0$, as the Sobolev norms of the boundary data "blow up" as $\epsilon \to 0$. Therefore, exact solutions of (0.2) are found by seeking solutions of the form $u_\epsilon(x) = U_\epsilon(x, \frac{x'\cdot\beta}{\epsilon})$ by solving the associated singular system for U_ϵ:

$$\begin{aligned}
&\sum_{j=0}^{d} A_j(\epsilon U_\epsilon)\partial_{x_j} U_\epsilon + \frac{1}{\epsilon}\sum_{j=0}^{d-1} A_j(\epsilon U_\epsilon)\beta_j\partial_{\theta_0} U_\epsilon = \mathcal{F}(\epsilon U_\epsilon)U_\epsilon \\
&B(\epsilon U_\epsilon)U_\epsilon|_{x_d=0} = G(x', \theta_0) \\
&U_\epsilon|_{t<0} = 0
\end{aligned} \tag{0.3}$$

[4] outlined a singular pseudo differential calculus for solving systems like (0.2) with pulse boundary data. Appendix F of this paper summarizes that calculus in order to properly prove Theorem 0.4. When applying this singular pseudo differential calculus, it's useful to rewrite

the singular system (0.3) as:

$$\begin{aligned}
&\partial_{x_d} U_\epsilon + \sum_{j=0}^{d-1} \widetilde{A}_j(\epsilon U_\epsilon)(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\epsilon})U_\epsilon = \partial_{x_d} U_\epsilon + \mathbb{A}(\epsilon U_\epsilon, \partial_{x'} + \frac{\beta \partial_{\theta_0}}{\epsilon})U_\epsilon = F(\epsilon U_\epsilon)U_\epsilon \\
&B(\epsilon U_\epsilon)U_\epsilon|_{x_d=0} = G(x', \theta_0) \\
&U_\epsilon|_{t<0} = 0
\end{aligned} \tag{0.4}$$

where $\widetilde{A}_j \equiv A_d^{-1}A_j$, $F \equiv A_d^{-1}\mathcal{F}$, and $\mathbb{A}$ is defined as above. $\mathbb{A}$ is an operator used in the singular calculus.

Examples 0.1. *Consider the isentropic, compressible Euler equations in three spacial dimensions on the half space $x_3 \geq 0$, for the unknowns,* density ρ *and* velocity $u = (u_1, u_2, u_3)$ *($p = p(\rho)$ is the pressure) :*

$$\partial_t \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \end{pmatrix} + \partial_{x_1} \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p(\rho) \\ \rho u_1 u_2 \\ \rho u_1 u_3 \end{pmatrix} + \partial_{x_2} \begin{pmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p(\rho) \\ \rho u_2 u_3 \end{pmatrix} + \partial_{x_3} \begin{pmatrix} \rho u_3 \\ \rho u_1 u_3 \\ \rho u_2 u_3 \\ \rho u_3^2 + p(\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{0.5}$$

It $v \equiv \begin{pmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$*, then, for appropriate functions f_i, (0.5) has the form:*

$$\partial_t f_0(v) + \partial_{x_1} f_1(v) + \partial_{x_2} f_2(v) + \partial_{x_3} f_3(v) = 0 \tag{0.6}$$

This in turn can be rewritten as:

$$A_0(v)\partial_t v + A_1(v)\partial_{x_1} v + A_2(v)\partial_{x_2} v + A_3(v)\partial_{x_3} v = 0 \tag{0.7}$$

where $A_i(v) \equiv f_i'(v)$. To ensure the proper hyperbolicity condition (as discussed in the following section), set $\rho > 0$. Also denote $c = \sqrt{\rho'}$. Seek the solution v as a perturbation about a constant state solution: $\underline{v} = \begin{pmatrix} \underline{\rho} \\ \underline{u_1} \\ \underline{u_2} \\ \underline{u_3} \end{pmatrix}$.

The Jacobians, $A_i(v)$, are computed as follows:

$$A_0(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 \\ u_2 & 0 & \rho & 0 \\ u_3 & 0 & 0 & \rho \end{pmatrix} \tag{0.8}$$

$$A_1(v) = \begin{pmatrix} u_1 & \rho & 0 & 0 \\ u_1^2 + p' & 2\rho u_1 & 0 & 0 \\ u_1 u_2 & \rho u_2 & \rho u_1 & 0 \\ u_1 u_3 & \rho u_3 & 0 & \rho u_1 \end{pmatrix} \tag{0.9}$$

$$A_2(v) = \begin{pmatrix} u_2 & 0 & \rho & 0 \\ u_1 u_2 & \rho u_2 & \rho u_1 & 0 \\ u_2^2 + p' & 0 & 2\rho u_2 & 0 \\ u_2 u_3 & 0 & \rho u_3 & \rho u_2 \end{pmatrix} \tag{0.10}$$

$$A_3(v) = \begin{pmatrix} u_3 & 0 & 0 & \rho \\ u_1 u_3 & \rho u_3 & 0 & \rho u_1 \\ u_2 u_3 & 0 & \rho u_3 & \rho u_2 \\ u_3^2 + p' & 0 & 0 & 2\rho u_3 \end{pmatrix} \tag{0.11}$$

Assumptions

The following 4 assumptions are made on the system (0.2):

(*Hyperbolicity*) Set $A_0 = I$. For an open neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^N$, $\exists\, q \geq 1$ and eigenvalues: $\lambda_1, \cdots, \lambda_q \in C^\infty(\mathcal{U} \times \mathbb{R}^d \backslash \{0\}, \mathbb{R})$, which are homogenous of degree 1 and analytic in ξ, so that for some positive integers $\nu_1, \cdots, \nu_q$:

$$\det L(u,\tau,\xi) = \det\,[\tau I + \sum_{j=1}^{d} \xi_j A_j(u)] = \prod_{k=1}^{q} (\tau + \lambda_k(u,\xi))^{\nu_k}$$

where $u \in \mathcal{U}$ and $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$. The eigenvalues $\lambda_1(u,\xi), \cdots, \lambda_q(u,\xi)$ are semi-simple and are ordered: $\lambda_1(u,\xi) < \cdots < \lambda_q(u,\xi)\, \forall u \in \mathcal{U}, \xi \in \mathbb{R}^d \backslash \{0\}$.

(*Noncharacteristic Boundary*) $\forall u \in \mathcal{U}$: $A_d(u)$ is invertible and the matrix $B(u)$ has maximal rank; its rank p equaling the number of positive eigenvalues of $A_d(u)$ (counted with multiplicity).

(*Diagonalizability / Nonsingularity*) For v in an open neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^N$ and $\zeta \equiv (\tau - i\lambda, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}$, consider the symbol:

$$\mathcal{A}(v,\zeta) \equiv -iA_d(v)^{-1}[(\tau - i\lambda)I + \sum_{j=1}^{d-1} \eta_j A_j(v)]$$

Denote its distinct eigenvalues as $i\omega_1(v,\zeta), \cdots, i\omega_M(v,\zeta)$.

Write $\mathcal{A}(\zeta) = \mathcal{A}(0,\zeta)$ and $\omega_i(\zeta) = \omega(0,\zeta)$.

The boundary frequency β is chosen so that there is a conic neighborhood O of β in $\mathbb{C} \times \mathbb{R}^{d-1} \backslash \{0\}$ on which eigenvalues of $-i\mathcal{A}(\beta)$ are semi-simple and given by smooth functions $\omega_m(\beta)\,(m = 1, \cdots, M)$, which are analytic in the $(\tau - i\lambda)$ argument, where $\omega_m(\beta)$ is of constant multiplicity ν_{k_m} on O. (For ease of notation, henceforth set $\omega_m = \omega_m(\beta)$.)

The *glancing set* $\mathcal{G}$ is a singular set that's defined as follows: Let G indicate the set of all $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$: $\xi \neq 0$ and $\exists\, 1 \leq k \leq q$:

$$\tau + \lambda_k(\xi) = 0 = \frac{\partial \lambda_k}{\partial \xi_d}(\xi)$$

Then the glancing set $\mathcal{G}$ is defined to be the projection $\pi(G)$ onto the first d coordinates of the elements of G. The diagonalizability assumption of β discussed above prevents it from lying in the glancing set: an important condition for the proof of (0.4).

(*Uniform Stability*) Define $\Xi \equiv \{\zeta = (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \backslash \{0\} : \gamma \geq 0\}$,

$\Xi_0 \equiv \Xi \cap \{\gamma = 0\}$, $\quad \Sigma \equiv \{\zeta = (\tau, \eta) \in \Xi : |\tau|^2 + |\eta|^2 = 1\}$

Due to a result by [12], when $\zeta \in \Xi \setminus \Xi_0$, $\mathcal{A}(\zeta)$ has no purely imaginary eigenvalues and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p. Furthermore, $\mathbb{E}^s(\zeta)$ defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ_0. The analysis in [12] also shows that, away from the glancing set $\mathcal{G} \subset \Xi_0$, $\mathbb{E}^s(\beta)$ depends analytically on $\zeta \in \Xi$.

With $\mathbb{E}^s$ properly defined, the assumption is made that (0.2) is *uniformly stable* at $u = 0$, meaning that:

$$B(0) : \mathbb{E}^s(\tau - i\gamma, \eta) \longrightarrow \mathbb{C}^p \quad \text{is an isomorphism}\, \forall\, (\tau - i\gamma, \eta) \in \Sigma$$

This is called the uniform Lopatinski condition. It's satisfied for the isentropic, compressible Euler equations in 3 spatial dimensions for physically relevant boundary operators.

Examples 0.2. *For the ongoing Euler equation example, the symbol $\mathcal{A}$ is:*

$$\mathcal{A}(\xi') = \mathcal{A}(\tau, \xi'') \equiv -iA_3^{-1}(\underline{v})[\tau A_0(\underline{v}) + \xi_1 A_1(\underline{v}) + \xi_2 A_2(\underline{v})]$$

$$= i \begin{pmatrix} -\frac{u_3^2\omega}{c^2-u_3^2} & \frac{\rho u_3 \xi_1}{c^2-u_3^2} & \frac{\rho u_3 \xi_2}{c^2-u_3^2} & -\frac{\rho u_3 \omega}{c^2-u_3^2} \\ -\frac{(\tau u_1+\xi_1(c^2+u_1^2)+u_1u_2\xi_2)}{\rho u_3} + \frac{u_1\omega}{\rho} & -\omega & 0 & 0 \\ -\frac{(\tau u_2+\xi_2(c^2+u_2^2)+u_1u_2\xi_1)}{\rho u_3} + \frac{u_2\omega}{\rho} & 0 & -\omega & 0 \\ -\frac{c^2u_3\omega}{\rho(c^2-u_3^2)} & \frac{-c^2\xi_1}{c^2-u_3^2} & 0 & \frac{u_3^2\omega}{c^2-u_3^2} \end{pmatrix} \tag{0.12}$$

where (for ease of notation and later relevance) $\omega \equiv \dfrac{(\tau + u_1\xi_1 + u_2\xi_2)}{u_3}$.

The symbol $L \equiv \tau A_0 + \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3$, *is related to* $\mathcal{S} \equiv A_0^{-1}[\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3]$ *and* $\mathcal{A} \equiv -iA_3^{-1}[\tau A_0 + \xi_1 A_1 + \xi_2 A_2]$ *in the following way:*

$$detL = det[\tau A_0 + \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3] = detA_0 det[\tau I + \mathcal{S}] = detA_3 det[i\mathcal{A} + \xi_d I] \tag{0.13}$$

The eigenvalues of $\mathcal{S}$ *are* $\lambda_m = \lambda_m(u,\xi)$: *(occurring with multiplicities 1,2,1, respectively)*

$$\lambda_1 = u \cdot \xi - c|\xi| \tag{0.14}$$

$$\lambda_2 = u \cdot \xi \tag{0.15}$$

$$\lambda_3 = u \cdot \xi + c|\xi| \tag{0.16}$$

The eigenvalues of $-i\mathcal{A}$ *are* $\omega_m = \omega_m(u,\xi')$: *(occurring with multiplicities 1,2,1, respectively)*

$$\omega_1 = \frac{u_3^2\omega - c\sqrt{(u_3^2 - c^2)(\xi_1^2 + \xi_2^2) + u_3^2\omega^2}}{c^2 - u_3^2} \tag{0.17}$$

$$\omega_2 = -\omega = \frac{-(\tau + u_1\xi_1 + u_2\xi_2)}{u_3} \tag{0.18}$$

$$\omega_3 = \frac{u_3^2\omega + c\sqrt{(u_3^2 - c^2)(\xi_1^2 + \xi_2^2) + u_3^2\omega^2}}{c^2 - u_3^2} \tag{0.19}$$

The eigenvalues are related as follows:

$$-\rho^3 \prod(\tau + \lambda_m) = detL = -detA_3 \prod(\xi_d - \omega_m) \tag{0.20}$$

Hence ω_m can be computed by solving $\tau + \lambda_m = 0$ in terms of ξ_d. Conversely, λ_m can be computed by solving $\xi_d - \omega_m = 0$ in terms of $-\tau$.

The eigenvectors s_m of $\mathcal{S}$ are easily computed as follows:

$$s_1 = \begin{pmatrix} -\frac{\rho|\xi|}{c\xi_3} \\ \frac{\xi_1}{\xi_3} \\ \frac{\xi_2}{\xi_3} \\ 1 \end{pmatrix} \qquad s_{2,1} = \begin{pmatrix} 0 \\ -\frac{\xi_2}{\xi_1} \\ 1 \\ 0 \end{pmatrix} \tag{0.21}$$

$$s_{2,2} = \begin{pmatrix} 0 \\ -\frac{\xi_3}{\xi_1} \\ 0 \\ 1 \end{pmatrix} \qquad s_3 = \begin{pmatrix} \frac{\rho|\xi|}{c\xi_3} \\ \frac{\xi_1}{\xi_3} \\ \frac{\xi_2}{\xi_3} \\ 1 \end{pmatrix} \tag{0.22}$$

Note that $s_{2,1}$ and $s_{2,2}$ are the 2 different eigenvectors corresponding to λ_2, which occurs which multiplicity 2.

Through the "shared" symbol L, the eigenvectors of $\mathcal{A}$ can be easily related to the eigenvectors of $\mathcal{S}$, which greatly simplifies the computation of the eigenvectors for $\mathcal{A}$.

To be clear, s_m is an eigenvector of $\mathcal{S}$ iff:

$$\mathcal{S}s_m = \lambda_m s_m \leftrightarrow [S - \lambda_m I]s_m = 0 \leftrightarrow [-\lambda_m A_0 + \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3]s_m = 0 \tag{0.23}$$

Conversely, r_m is an eigenvector of $\mathcal{A}$ iff:

$$i\mathcal{A}r_m = -\omega_m r_m \leftrightarrow [i\mathcal{A} + \omega_m I]r_m = 0 \leftrightarrow [\tau A_0 + \xi_1 A_1 + \xi_2 A_2 + \omega_m A_3]r_m = 0 \tag{0.24}$$

Hence an eigenvector s_m of $\mathcal{S}$ is an eigenvector of $\mathcal{A}$ when $\xi_3 = \omega_m$. (This also occurs exactly when $\tau = -\lambda_m$.) Thereby eigenvectors r_m of $\mathcal{A}$ are found by: $r_m = s_m|_{\xi_3=\omega_m}$.

$$r_1 = \begin{pmatrix} -\frac{\rho|(\xi'',\omega_1)|}{c\omega_1} \\ \frac{\xi_1}{\omega_1} \\ \frac{\xi_2}{\omega_1} \\ 1 \end{pmatrix} \qquad r_{2,1} = \begin{pmatrix} 0 \\ -\frac{\xi_2}{\xi_1} \\ 1 \\ 0 \end{pmatrix} \tag{0.25}$$

$$r_{2,2} = \begin{pmatrix} 0 \\ -\frac{\omega_2}{\xi_1} \\ 0 \\ 1 \end{pmatrix} \qquad r_3 = \begin{pmatrix} -\frac{\rho|(\xi'',\omega_3)|}{c\omega_3} \\ \frac{\xi_1}{\omega_3} \\ \frac{\xi_2}{\omega_3} \\ 1 \end{pmatrix} \tag{0.26}$$

(Subsonic Inflow) In this case, consider $0 < u_3 < c$. The boundary condition is taken to be: $b(\rho, u) = (\rho u_3, u_1, u_2)$, which has the linearized operator:

$$B(0)(\dot\rho, \dot u) = (\dot\rho u_3 + \rho \dot u_3, \dot u_1, \dot u_2) \tag{0.27}$$

so that:

$$B(0) = \begin{pmatrix} u_3 & 0 & 0 & \rho \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{0.28}$$

Notice that $p = 3$, where p is the rank of $B(0)$ (and also the number of positive eigenvalues of $A_3(\underline{v})$.)

The index set for the eigenvalues of $\mathcal{A}(\beta)$ can be partitioned as $\mathcal{O} \cup \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}$. $\mathcal{O}, \mathcal{I}$ indicate the outgoing/incoming *hyperbolic modes* (β, ω_m) for which the ω_m are real and $\mathcal{P}, \mathcal{N}$ indicate the *elliptic modes* (β, ω_m) for which the ω_m are non real, with positive/negative imaginary parts, respectively. For each $m \in \mathcal{O}, \mathcal{I}$, the hyperbolicity condition guarantees the existence of a unique k_m: $\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \omega_m) = 0$. The outgoing/incoming nature of these modes refers to whether $\partial_{\xi_d} \lambda_{k_m}(\beta, \omega_m) < 0$ or $\partial_{\xi_d} \lambda_{k_m}(\beta, \omega_m) > 0$, respectively. For each eigenvalue ω_m, define interior phases $\phi_m(x) \equiv \phi_0(t, y) + \omega_m x_d$. Finally, β is said to lie in either the hyperbolic, elliptic, or mixed region depending on whether $\mathcal{A}(\beta)$ has eigenvalues corresponding to modes that are entirely hyperbolic, entirely elliptic, or a mixture of both, respectively.

There are two useful decompositions from [7] and [5], respectively:

$$\mathbb{E}^s(\underline{\tau}, \underline{\eta}) = \bigoplus_{m \in \mathcal{I} \cup \mathcal{P}} \operatorname{Ker} L(d\phi_m) \tag{0.29}$$

$$\mathbb{C}^N = \bigoplus_{m=1}^{M} \operatorname{Ker} L(d\phi_m) \tag{0.30}$$

If $P_1, \cdots, P_M$ are the projectors associated with (0.30), then for $m = 1, \cdots, M$:

$\operatorname{Im} A_d^{-1}(0) L(d\phi_m) = \operatorname{Ker} P_m$. Denote a basis of Ker $L(d\phi_m)$ as $\{r_{m,k}\}_{k=1}^{\nu_{k_m}}$. For $m \in \mathcal{O}, \mathcal{I}$, these consist of real vectors. The approximate solution constructed for (0.2) will be written as an expansion of such basis vectors. For now, restrict attention to the strictly hyperbolic case (when $\nu_{k_m} = 1$ for $m = 1, \cdots, M$). The generalization (when at least one $\nu_{k_m} > 1$) is treated in Chapter 4.

Examples 0.3. *For the ongoing Euler equation example, it will be shown that there are two possibilities for $\beta = (\eta, \xi') = (\eta, \xi_1, \xi_2)$: the* hyperbolic *region and the* mixed *region. Because (for physically relevant solutions) $u \in \mathbb{R}^3$, ω_2 must always be real. Hence β can't lie in the elliptic region. Notice that the group velocity corresponding to ω_2 in incoming, as $\partial_{\xi_3} \lambda_2 = u_3 > 0$ in the subsonic inflow case. Whether or not ω_1 and ω_3 are non real depends* *on the discriminant:*

$$\Delta \equiv (u_3^2 - c^2)(\xi_1^2 + \xi_2^2) + u_3^2\omega^2 \tag{0.31}$$

If $\Delta > 0$, then β lies in the hyperbolic region. As mentioned in [H], this corresponds to the region:

$$\{\, (\tau, \xi') : |\tau + u_1\xi_1 + u_2\xi_2| > \sqrt{c^2 - u_3^2}|\xi''| \,\} \tag{0.32}$$

$\Delta = 0$ corresponds to the glancing set. Finally, if $\Delta < 0$, then β lies in the mixed region, where ω_1 and ω_3 are conjugate eigenvalues.

Main Results

First, note the following notations:

1. $\Omega \equiv \overline{\mathbb{R}^{d+1}_+} \times \mathbb{R}^1 \quad \Omega_T \equiv \Omega \cap \{-\infty < t < T\}$

2. $b\Omega \equiv \mathbb{R}^d \times \mathbb{R}^1 \quad b\Omega_T \equiv b\Omega \cap \{-\infty < t < T\}$

3. $\omega_T \equiv \overline{\mathbb{R}^{d+1}_+} \cap \{-\infty < t < T\} \qquad M_0 \equiv 3d + 5$

4. $H^s_T \equiv H^s(\Omega_T) \qquad bH^s_T \equiv H^s(b\Omega_T)$

5. $E^s_T = E^s_T(x, \theta_0) \equiv C(x_d, H^s(b\Omega_T)) \cap L^2(x_d, H^{s+1}(b\Omega_T))$

   (where $C(x_d, \cdot)$ denotes the space of bounded continuous functions in $x_d \geq 0$)

Theorem 0.4. *Suppose that the quasilinear boundary problem (0.2) satisfies the 4 assumptions of Section 1.3. Furthermore, suppose that* $G \in bH^{s+1}$ *satisfies supp* $G \subseteq \{t \geq 0\}$*, where* $s \geq [M_0 + \frac{d+1}{2}]$ *(the smallest integer* $\geq M_0 + \frac{d+1}{2}$*). Then* $\exists\, \epsilon_0 > 0, T_0 > 0$ *independent of* $\epsilon \in (0, \epsilon_0]$ *and* $\exists!\, U_\epsilon \in E^s_{T_0}$ *satisfying the associated singular problem (0.4), such that:*

$$u_\epsilon(x) \equiv U_\epsilon(x, \frac{x' \cdot \beta}{\epsilon})$$

is the unique C^1 *solution of (0.2) on* ω_{T_0}*.*

Remark 0.5. *The regularity requirement* $s \geq [M_0 + \frac{d+1}{2}]$ *is necessary to apply the singular pseudo differential calculus.*

Theorem 0.6. *Suppose that the same 4 assumptions hold and that* $G \in bH^{s+1}$*. Then* $\exists\, 0 < T_1 \leq T_0$ *and an (explicitly constructed) approximate solution* $u^a_\epsilon \equiv \mathcal{U}^0_\epsilon(x, \frac{x' \cdot \beta}{\epsilon}) \in H^s_{T_0}$*, defined by (0.107), such that:*

$$\lim_{\epsilon \to 0}(u_\epsilon - u^a_\epsilon) = 0 \quad \textit{in } L^\infty(\omega_{T_1})$$

where $u_\epsilon \in C^1(\omega_{T_0})$ *is the unique exact solution to (0.2), specified in Theorem 0.4*

The previous two theorems are proven in Chapter 3.

Remark 0.7. *Note that a rate of convergence couldn't be determined in (0.6), as was possible in [5]. However, Theorem 0.6 is more general than Theorem 1.14 of [5], as* β *is considered not only in the hyperbolic region, but also in the elliptic and mixed regions.*

Additionally more generalized boundary data G can be treated by this result, as frequency decay conditions (in the θ argument) on G are not necessary. (To be precise, it's not necessary that $G \in b\Gamma^{s+1}$, where $b\Gamma^s$ (and Γ^s) are defined below.) In fact, as discussed in section 2.3, there are times when, due to disruptive hyperbolic/elliptic boundary interactions, the profiles σ can't inherit the θ-decay properties of the boundary data G.

Definition 0.8. *For $s \in \mathbb{N}$, define:*

$$\Gamma^s \equiv \{ \, a(x,\theta) \in L^2(\mathbb{R}^{d+1}_+ \times \mathbb{R}) : (\theta, \partial_x, \partial_\theta)^\beta a \in L^2 \, \forall \, |\beta| \leq s \text{ and } a|_{t<0} = 0 \, \} \tag{0.33}$$

Similarly, define:

$$b\Gamma^s \equiv \{ \, a(x',\theta) \in L^2(\mathbb{R}^{d} \times \mathbb{R}) : (\theta, \partial_{x'}, \partial_\theta)^\beta a \in L^2 \, \forall \, |\beta| \leq s \text{ and } a|_{t<0} = 0 \, \} \tag{0.34}$$

Remark 0.9. *A related paper considering quasilinear, highly oscillatory IBVPs of the form (0.2) is [7]. [7] also solves (0.2) under the same 4 assumptions considered in this paper, along with the generalization that the boundary data G lies in a Sobolev space and the extension that the boundary frequency β can lie in any of the 3 regions (hyperbolic, elliptic, mixed). However [7] differs because it considers wave train boundary data G (data that's periodic in θ). Despite the similarities, many differences exist between [7] and this current paper. Namely, much effort of this paper goes into properly constructing and estimating the corrector $\mathcal{U}^1_p$ of the approximate solution $\mathcal{U}^0$, which plays a crucial role in the proof of (0.6). This corrector differs greatly from the corrector constructed in [7]. One major difference is that here the Fourier spectrum is continuous (and not discrete as in [7]). This poses new difficulties for the construction of the corrector $\mathcal{U}^1_p$ due to a different type of small-divisor problem, as κ (the dual variable to θ) can be close to 0. Thereby moment-zero approximations are necessary in constructing the required primitives, along with their own error analysis.*

CHAPTER 2: CONSTRUCTING THE APPROXIMATE SOLUTION $\mathcal{U}^0$

In this chapter the approximate solution u^a_ϵ to the system (0.2) is determined by solving (or in some cases approximately solving) suitable profile equations for the components ("profiles") of the eigen-decomposition of the approximate solution u^a_ϵ via (0.30). Important conditions on the profiles will be discussed.

To begin, the ansatz is made that the approximate solution $u^a_\epsilon(x)$ (to order ϵ^0) has the following form:

$$u^a_\epsilon(x) \equiv [\mathcal{U}^0(x,\theta_0,\xi_d)]_{\theta_0=\frac{\phi_0}{\epsilon},\xi_d=\frac{x_d}{\epsilon}} \tag{0.35}$$

Similarly, the corrected approximate solution $u^c_\epsilon(x)$ (to order ϵ^1) is written:

$$u^c_\epsilon(x) \equiv [\mathcal{U}^0(x,\theta_0,\xi_d) + \epsilon\,\mathcal{U}^1(x,\theta_0,\xi_d)]_{\theta_0=\frac{\phi_0}{\epsilon},\xi_d=\frac{x_d}{\epsilon}} \tag{0.36}$$

Taylor expanding the $\widetilde{A}_j$ and F about 0 for (0.4) and collecting ϵ terms yields the following ϵ^{-1} and ϵ^0 conditions, respectively, along with an ϵ^0 boundary condition:

$$\widetilde{\mathcal{L}}(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}^0 = 0 \tag{0.37}$$

$$\widetilde{\mathcal{L}}(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}^1 = F(0)\mathcal{U}^0 - \widetilde{L}(\partial_x)\mathcal{U}^0 - \mathcal{M}(\mathcal{U}^0,\partial_{\theta_0}\mathcal{U}^0) \equiv \mathcal{F}(x,\theta_0,\xi_d) \tag{0.38}$$

$$B(0)\mathcal{U}^0|_{x_d=0,\xi_d=0} = G(x',\theta_0) \tag{0.39}$$

where: $\widetilde{\mathcal{L}}(\partial_{\theta_0},\partial_{\xi_d}) \equiv \partial_{\xi_d} + i\mathcal{A}(\beta)\partial_{\theta_0}$, $\quad \widetilde{L}(\partial_x) \equiv \partial_{x_d} + \sum_{j=0}^{d-1}\widetilde{A}_j(0)\partial_{x_j}$,

$$\mathcal{M}(\mathcal{U},\partial_{\theta_0}\mathcal{V}) \equiv \sum_{j=0}^{d-1}\beta_j(d\widetilde{A}_j(0)\cdot\mathcal{U})\partial_{\theta_0}\mathcal{V}$$

ϵ^{-1} Interior Condition

Utilizing (0.30), the approximate solution decomposes as:

$$\mathcal{U}^0(x,\theta_0,\xi_d) = \sum_{m=1}^{M} \widetilde{\sigma}_m(x,\theta_0,\xi_d) r_m \tag{0.40}$$

Because the eigenvectors $\{r_m\}$ diagonalize $\mathcal{A}(\beta)$, (0.37) gives the equations:

$$(\partial_{\xi_d} - \omega_m \partial_{\theta_0})\widetilde{\sigma}_m = 0 \tag{0.41}$$

For ω_m corresponding to a hyperbolic mode, it's easy to see that the profiles have the form $\widetilde{\sigma}_m(x,\theta_0,\xi_d) = \sigma_m(x,\theta_m)$, where $\theta_m \equiv \theta_0 + \omega_m \xi_d$. However, for ω_m corresponding to an elliptic mode, in order to write same formula, the Fourier transform of (0.41) w.r.t. θ_0 is employed as follows:

$$(\partial_{\xi_d} - i\omega_m \kappa)\hat{\widetilde{\sigma}}_m = 0 \tag{0.42}$$

This equation is solved with the integrating factor $e^{-i\omega_m \kappa \xi_d}$, so that:

$$\hat{\widetilde{\sigma}}_m(x,\kappa,\xi_d) = e^{i\omega_m \kappa \xi_d} \hat{\widetilde{\sigma}}_m(x,\kappa,0)$$

Applying the inverse Fourier transform yields:

$$\widetilde{\sigma}_m(x,\theta_0,\xi_d) = (2\pi)^{-1} \int_{\mathbb{R}} e^{i\theta_m \kappa} \hat{\widetilde{\sigma}}_m(x,\kappa,0) d\kappa \equiv \sigma_m(x,\theta_m) \tag{0.43}$$

In order for the exponential term $e^{i\theta_m\kappa}$ in (0.43) to be decaying, it's necessary to require that:

$$\text{supp } \widetilde{\sigma}\hat{}_m(x,\kappa,0) \subseteq \{(\operatorname{Im}\omega_m)\kappa \geq 0\} \tag{0.44}$$

In light of (0.43), define:

$$\sigma\hat{}_m(x,\kappa) \equiv \widetilde{\sigma}\hat{}_m(x,\kappa,0) = e^{-i\omega_m\kappa\xi_d}\widetilde{\sigma}\hat{}_m(x,\kappa,\xi_d) \tag{0.45}$$

By (0.44), if $\sigma_m \in H^s_T$ for $s > \frac{d+1}{2}+1$ (the choice of $s > \frac{d+1}{2}$ guarantees continuity of σ_m in (x,θ)), then $\sigma_m(x,\theta)$ has an analytic extension in θ to the upper half-plane when $m \in \mathcal{P}$ (lower half-plane when $m \in \mathcal{N}$). The half-line support condition (0.44) on the elliptic profiles is an important condition which will be utilized extensively.

Remark 0.10. *Along with the half-line support conditions, (0.43) demonstrates the exponential decay of the elliptic pulses. In the case $m \in \mathcal{P}$, when κ is away from 0 (say $\kappa \geq \delta$ for a fixed $\delta > 0$), the modulus of the integrand of (0.43) is controlled by $e^{-\delta Im\omega_m\xi_d}|\sigma\hat{}_m(x,\kappa)|$, which rapidly decays to 0, as $\epsilon \to 0$, upon the substitution $\xi_d = \frac{x_d}{\epsilon}$.*

Remark 0.11. *It should also be noted here that the possible discontinuity of $\sigma\hat{}_m(x,\kappa)$ at $\kappa = 0$ can prevent θ_0-decay for $\sigma_m(x,\theta_0)$. In fact, as it will be demonstrated later, when β does not lie in the hyperbolic region, the approximate solution $\mathcal{U}^0$ may not lie in a Γ^s-space (see Definition 0.8), even if G does. Instead it will only be assumed that the boundary data G lies in a H^s-space. This generalization will be taken for β in all 3 possible regions.*

ϵ^0 Interior Condition

Similarly, (0.38) gives the following equations:

$$(\partial_{\xi_d} - \omega_m \partial_{\theta_0})\tau_m = \mathcal{F}_m \equiv l_m \cdot \mathcal{F} \tag{0.46}$$

where $\mathcal{U}^1(x, \theta_0, \xi_d) = \sum_{m=1}^{M} \tau_m(x, \theta_0, \xi_d) r_m$ and l_m denotes a left eigenvector of $i\mathcal{A}(\beta)$ associated to $-\omega_m$, which satisfies $l_m \cdot r_{m'} = \delta_{m,m'}$.

The elliptic profiles τ_m are once again solved by Fourier transforming to reduce to an ODE:

$$(m \in \mathcal{P}) \quad \hat{\tau}_m(x, \kappa, \xi_d) = e^{i\omega_m \kappa \xi_d} \hat{\tau}_m(x, \kappa, 0) + \int_0^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \hat{\mathcal{F}}_m(x, \kappa, s) ds \tag{0.47}$$

Suppose that Im $\omega_m > 0$. The exponential term $e^{i\omega_m \kappa(\xi_d - s)}$ is well-behaved when $\kappa > 0$. However, when $\kappa < 0$, to ensure this same behavior require that:

$$(m \in \mathcal{P}) \quad \hat{\tau}_m(x, \kappa, 0) = \int_\infty^0 e^{-i\omega_m \kappa s} \hat{\mathcal{F}}_m(x, \kappa, s) ds \tag{0.48}$$

So that the solution for $\kappa < 0$ becomes:

$$(m \in \mathcal{P}) \quad \hat{\tau}_m(x, \kappa, \xi_d) = \int_\infty^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \hat{\mathcal{F}}_m(x, \kappa, s) ds \tag{0.49}$$

So, when Im $\omega_m > 0$, the following full solution is found by setting $\hat{\tau}_m(x, \kappa, 0) \equiv 0$:

$$(m \in \mathcal{P}) \quad \hat{\tau}_m(x, \kappa, \xi_d) = \mathbb{1}_{\{\kappa < 0\}} \int_\infty^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \hat{\mathcal{F}}_m(x, \kappa, s) ds$$
$$+ \mathbb{1}_{\{\kappa > 0\}} \int_0^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \hat{\mathcal{F}}_m(x, \kappa, s) ds \tag{0.50}$$

For Im $\omega_m < 0$, the opposite formulae hold for $\hat{\tau_m}$ on the κ half-lines:

$$(m \in \mathcal{N}) \quad \hat{\tau_m}(x, \kappa, \xi_d) = \mathbb{1}_{\{\kappa<0\}} \int_0^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \hat{\mathcal{F}_m}(x, \kappa, s) ds$$

$$+\mathbb{1}_{\{\kappa>0\}} \int_\infty^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \hat{\mathcal{F}_m}(x, \kappa, s) ds \tag{0.51}$$

The following decomposition holds for $\mathcal{F}_m$ in terms of the σ_m:

(where $\sigma_i, \partial_{\theta_i}\sigma_i$ are evaluated at $(x, \theta_0 + \omega_i \xi_d)$)

$$\mathcal{F}_m = -X_{\phi_m}\sigma_m - \sum_i c_i^m \sigma_i \partial_{\theta_i}\sigma_i - \sum_{i \neq j} d_{i,j}^m \sigma_i \partial_{\theta_i}\sigma_j + \sum_i e_i^m \sigma_i - \sum_{i \neq m} V_i^m \sigma_i \tag{0.52}$$

where X_{ϕ_m} is the characteristic vector field:

$$X_{\phi_m} \equiv \partial_{x_d} + \sum_{i=0}^{d-1} -\partial_{\xi_i}\omega_m(\beta)\partial_{x_i} \tag{0.53}$$

and for $i \neq m$ the tangential vector field V_i^m is:

$$V_i^m \equiv \sum_{j=0}^{d-1} (l_m \widetilde{A}_j(0) r_i)\partial_{x_j} \tag{0.54}$$

$$c_i^m \equiv l_m \sum_{j=0}^{d-1} \beta_j (d\widetilde{A}_j(0) \cdot r_i) r_i, \; d_{i,j}^m \equiv l_m \sum_{j=0}^{d-1} \beta_j (d\widetilde{A}_j(0) \cdot r_i) r_j, \; e_i^m \equiv l_m F(0) r_i$$

In light of (0.47), the Fourier transform of $\mathcal{F}_m$ w.r.t. θ_0 is:

$$\mathcal{F}\hat{}_m(x,\kappa,s) = -X_{\phi_m} e^{i\omega_m \kappa s} \sigma\hat{}_m - \sum_i g_{m,i}(x,\kappa,s) - \sum_{i\neq j} h_{m,i,j}(x,\kappa,s)$$

$$+\sum_i e_i^m e^{i\omega_i \kappa s} \sigma\hat{}_i - \sum_{i\neq m} V_i^m e^{i\omega_i \kappa s} \sigma\hat{}_i \tag{0.55}$$

where $\sigma\hat{}_i$ is defined as in (0.45) and where (by the convolution theorem):

$$g_{m,i}(x,\kappa,s) \equiv c_i^m \int_{\mathbb{R}} e^{i\omega_i(\kappa-t)s} \sigma\hat{}_i(x,\kappa-t) e^{i\omega_i ts} (\partial_{\theta_i}\sigma_i)\hat{}(x,t)dt = c_i^m e^{i\omega_i \kappa s} \sigma\hat{}_i * (\partial_{\theta_i}\sigma_i)\hat{}(x,\kappa) \tag{0.56}$$

$$h_{m,i,j}(x,\kappa,s) \equiv d_{i,j}^m \int_{\mathbb{R}} e^{i\omega_i(\kappa-t)s} \sigma\hat{}_i(x,\kappa-t) e^{i\omega_j ts} (\partial_{\theta_i}\sigma_j)\hat{}(x,t)dt \tag{0.57}$$

From there it follows that (because $e^{i\omega_m \kappa s}$ commutes with differential operators in x and constants):

$$e^{i\omega_m \kappa(\xi_d - s)} \mathcal{F}\hat{}(x,\kappa,s) = e^{i\omega_m \kappa \xi_d}[-X_{\phi_m} \sigma\hat{}_m(x,\kappa) - \sum_i e^{-i\omega_m \kappa s} g_i(x,\kappa,s) - \sum_{i\neq j} e^{-i\omega_m \kappa s} h_{i,j}(x,\kappa,s)$$

$$+\sum_i e^{i(\omega_i - \omega_m)\kappa s} e_i^m \sigma\hat{}_i(x,\kappa) - \sum_{i\neq m} e^{i(\omega_i-\omega_m)\kappa s} V_i^m \sigma\hat{}_i(x,\kappa)] \tag{0.58}$$

Notice that when $\kappa < 0$ (for $m \in \mathcal{P}$):

$$\tau\hat{}_m(x,\kappa,\xi_d) = \int_{\infty}^{\xi_d} e^{i\omega_m \kappa(\xi_d - s)} \mathcal{F}\hat{}_m(x,\kappa,s)ds$$

$$= \int_{\infty}^{\xi_d} e^{i\omega_m \kappa \xi_d}[-X_{\phi_m} \sigma\hat{}_m(x,\kappa) - c_m^m \sigma\hat{}_i * (\partial_{\theta_i}\sigma_i)\hat{}(x,\kappa) + e_m^m \sigma\hat{}_i(x,\kappa)]\, ds + \cdots$$

$$= -e^{i\omega_m \kappa \xi_d}[X_{\phi_m} \sigma\hat{}_m(x,\kappa) + c_m^m \sigma\hat{}_i * (\partial_{\theta_i}\sigma_i)\hat{}(x,\kappa) - e_m^m \sigma\hat{}_i(x,\kappa)][\int_{\infty}^{\xi_d} ds] + \cdots \tag{0.59}$$

In Appendix C, it will be shown the terms included in the ellipses of (0.59) can be controlled. However, the rest of the terms highlighted in (0.59) are not even well-defined due to the presence of the (improper) integral $\int_{\infty}^{\xi_d} ds$. Hence the profile equations (0.79) must be considered.

The ϵ^0 interior condition will now be considered for the hyperbolic corrector profiles:

Motivated by [5], the attempt could be made to solve the hyperbolic corrector profiles τ_m ($m \in \mathcal{I} \cup \mathcal{O}$) by:

$$\tau_m(x, \theta_0, \xi_d) = \int_{\infty}^{\xi_d} \mathcal{F}_m(x, \theta_0 + \omega_m(\xi_d - s), s)ds \tag{0.60}$$

These indeed are solutions of (0.46).

However, the integral contribution to the RHS of (0.60) from:

$$\int_{\infty}^{\xi_d} [\widetilde{L}(\partial_x)\mathcal{U}^0 - F(0)\mathcal{U}^0]_m(x, \theta_0 + \omega_m(\xi_d - s), s)ds$$

contains (suppressing constants) integrals of the form:

$$\int_{\infty}^{\xi_d} \sigma_i(x, \theta_0 + \omega_i\xi_d + s(\omega_m - \omega_i))ds$$

In [5], the integrands above possessed nice decay properties in s. However, these decay properties relied upon the Γ^s regularity of the profiles σ_m. As will be seen in the next section 2.3, under the hypotheses of this paper, there are times when the profiles (even the hyperbolic profiles) fail to be in Γ^1.

Because the profiles σ may no longer belong to Γ^s, it can't be assumed that these (improper) integrals are well-defined. Instead, these integrals are replaced with primitives of moment-zero approximations, which have the form $\sigma^*_{i,p}(x, \theta_0 + \omega_m \xi_d)$ (these are defined in Definition 0.44). A small divisor problem requires these moment-zero approximations of the profiles (defined in Definition 0.43), in order for the primitives to be controlled for E^s_T-norms. Refer to Appendix B for a discussion of these constructions.

In addition to the modification of the primitives discussed above, the integral contribution to the RHS of (0.60) from:

$$\int_\infty^{\xi_d} M(\mathcal{U}^0_p, \partial_{\theta_0}\mathcal{U}^0_p)_m(x, \theta_0 + \omega_m(\xi_d - s), s)ds$$

contains (suppressing constants) integrals of the form:

$$\int_\infty^{\xi_d} \sigma_i(x, \theta_0 + \omega_i \xi_d + s(\omega_m - \omega_i))\partial_{\theta_0}\sigma_j(x, \theta_0 + \omega_j \xi_d + s(\omega_m - \omega_j))ds$$

Because the arguments of σ_i and $\partial_{\theta_0}\sigma_j$ differ, a difficulty arises when attempting to estimate these integrals. Definition 0.51 resolves this issue by introducing a hyperbolic transversal product modification. The results of Appendix E demonstrate that these modified products can be properly estimated in E^s_T spaces. Additionally, by the results of Appendix D, the error produced by substituting these modifications in the corrector is shown to be controlled.

Remark 0.12. *In [5], the above integrals were estimated via Proposition 4.10. However, the proof of that proposition relied upon the hypothesis that the profiles belonged to Γ^s spaces. The omission of the Γ^s hypothesis necessitates the careful constructions and resulting estimates present in this paper, outlined in Appendices D and E.*

The remaining obstacle to the construction of the hyperbolic corrector profiles are the

$X_{\phi_m}\sigma_m + c_m^m\sigma_m\partial_\theta\sigma_m - e_m^m\sigma_m$ terms which appear in (0.60). They have no s-dependency and so appear alongside a factor of $\int_\infty^{\xi_d} ds$. Hence the profile equations (0.79) are also necessary for the hyperbolic profiles $\sigma_m(m \in \mathcal{I} \cup \mathcal{O})$. Notice that the profiles equation hold for all profiles, both elliptic and hyperbolic. They are also decoupled, as there are no interactions between the elliptic and hyperbolic profiles (at order ϵ^0) when solving the approximate solution $\mathcal{U}^0$.

Motivated by the discussion in this section, the fully modified hyperbolic corrector profiles are defined in (0.121). Additionally, the modified elliptic corrector profiles are defined in (0.122) and (0.123). These formulae define the corrector $\mathcal{U}_p^1$, which, as seen in the proof of Theorem 0.29, is the appropriate corrector for the error analysis involving the approximate solution $\mathcal{U}^0$ and the exact solution U_ϵ.

ϵ^0 Boundary Condition

The following proposition will first be proven, in order to aid in the discussion of the ϵ^0 boundary condition:

Proposition 0.13. *All outgoing profiles σ_m ($m \in \mathcal{O}$) vanish.*

Proof: *Consider the iteration scheme (0.84), which is used in section 2.4 for solving the hyperbolic profiles:*

Initializing with $\sigma_m^0 = 0$, the $n = 1$ condition for (0.84) gives: $X_{\phi_m} \sigma_i^m = 0$.

Because X_{ϕ_m} is a real vector field when $i \in \mathcal{O}$, whose flow is "outward" (towards the boundary), integration along characteristics from the zero initial information $\sigma_m^1|_{t<0} = 0$ ensures that the outgoing iterate σ_m^1 vanishes throughout the interior $\mathbb{R}^{d+1}$.

By induction, its now clear that all outgoing iterates σ_m^n will also vanish, as does the limit $\sigma_m \equiv 0$ (say, in H_T^s for $s > \frac{d+1}{2}$). $\square$

Proposition 0.13 and the boundary condition at order ϵ^0 (0.39) require:

$$B(0) \left(\sum_{m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}} \sigma_m(x', 0, \theta_0) r_m \right) = G(x', \theta_0)$$

Fourier transforming (w.r.t. θ_0) yields:

$$\sum_{m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}} \hat{\sigma}_m(x', 0, \kappa) \left(B(0) r_m \right) = \hat{G}(x', \kappa) \tag{0.61}$$

Because the elliptic profiles are only supported on κ half-lines, write:

$$\sum_{m \in \mathcal{I} \cup \mathcal{P}} \hat{\sigma_m}(x', 0, \kappa)\,(B(0) r_m) = \hat{G}(x', \kappa) \quad \text{(when } \kappa > 0\text{)}$$

$$\sum_{m \in \mathcal{I} \cup \mathcal{N}} \hat{\sigma_m}(x', 0, \kappa)\,(B(0) r_m) = \hat{G}(x', \kappa) \quad \text{(when } \kappa < 0\text{)} \tag{0.62}$$

Thereby the elliptic profiles along with the "incoming" profiles are uniquely determined by the boundary data G.

For use in the discussion and formulae below, $\mathcal{O}$ refers to the index set $\{1, \cdots, |\mathcal{O}|\}$ corresponding to the outgoing modes. Similarly, $\mathcal{I}$ refers to $\{|\mathcal{O}|+1, \cdots, |\mathcal{O}|+|\mathcal{I}|\}$, $\mathcal{P}$ refers to $\{|\mathcal{O}|+|\mathcal{I}|+1, \cdots, |\mathcal{I}|+|\mathcal{O}|+|\mathcal{P}|\}$, and $\mathcal{N}$ refers to $\{|\mathcal{O}|+|\mathcal{I}|+|\mathcal{P}|+1, \cdots, |\mathcal{O}|+|\mathcal{I}|+|\mathcal{P}|+|\mathcal{N}|\}$, with each index set corresponding to the respectives modes.

Let Q_+ and Q_- denote the $p \times p$ matrices with columns: $\{B(0) r_m : m \in \mathcal{I} \cup \mathcal{P}\}$ and $\{B(0) r_m : m \in \mathcal{I} \cup \mathcal{N}\}$, respectively. Here the columns of Q_+ and Q_- are ordered by the index set $\mathcal{I}$ for the first $|\mathcal{I}|$ columns and then by either the index set $\mathcal{P}$ or $\mathcal{N}$, for Q_+ or Q_-, respectively. By the uniform Lopatinski condition and (0.29), $\{B(0) r_m : m \in \mathcal{I} \cup \mathcal{P}\}$ and $\{B(0) r_m : m \in \mathcal{I} \cup \mathcal{N}\}$ form bases of $\mathbb{C}^p$. Therefore Q_+^{-1} and Q_-^{-1} exist, yielding:

$$(m \in \mathcal{I}) \qquad \hat{\sigma_m}(x', 0, \kappa) = \mathbb{1}_{\{\kappa>0\}} (Q_+^{-1} \hat{G}(x', \kappa))_{m-|\mathcal{O}|} + \mathbb{1}_{\{\kappa<0\}} (Q_-^{-1} \hat{G}(x', \kappa))_{m-|\mathcal{O}|} \tag{0.63}$$

$$(m \in \mathcal{P}) \qquad \hat{\sigma_m}(x', 0, \kappa) = \mathbb{1}_{\{\kappa>0\}} (Q_+^{-1} \hat{G}(x', \kappa))_{m-|\mathcal{O}|} \tag{0.64}$$

$$(m \in \mathcal{N}) \qquad \hat{\sigma_m}(x', 0, \kappa) = \mathbb{1}_{\{\kappa<0\}} (Q_-^{-1} \hat{G}(x', \kappa))_{m-|\mathcal{O}|-|\mathcal{P}|} \tag{0.65}$$

where the indices m of the profiles σ_m above are indexed by $\mathcal{O} \cup \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}$, in that specific ordering.

Write this prescribed boundary data as:

$$(m \in \mathcal{I}) \qquad a_m(x', \theta_0) \equiv (2\pi)^{-1} \int_{\mathbb{R}} e^{i\theta_0 \kappa} [\mathbb{1}_{\{\kappa>0\}} (Q_+^{-1} G\hat{}(x', \kappa))_{m-|\mathcal{O}|}$$

$$+\mathbb{1}_{\{\kappa<0\}} (Q_-^{-1} G\hat{}(x', \kappa))_{m-|\mathcal{O}|} d\kappa \tag{0.66}$$

$$(m \in \mathcal{P}) \qquad a_m(x', \theta_0) \equiv (2\pi)^{-1} \int_{\mathbb{R}} e^{i\theta_0 \kappa} \mathbb{1}_{\{\kappa>0\}} (Q_+^{-1} G\hat{}(x', \kappa))_{m-|\mathcal{O}|} d\kappa \tag{0.67}$$

$$(m \in \mathcal{N}) \qquad a_m(x', \theta_0) \equiv (2\pi)^{-1} \int_{\mathbb{R}} e^{i\theta_0 \kappa} \mathbb{1}_{\{\kappa<0\}} (Q_-^{-1} G\hat{}(x', \kappa))_{m-|\mathcal{O}|-|\mathcal{P}|} d\kappa \tag{0.68}$$

So that the leading profiles must satisfy:

$$\sigma_m(x, \theta_0)|_{x_d=0} = a_m(x', \theta_0) \tag{0.69}$$

Remark 0.14. *The situation simplifies when β is not in the mixed region. When β is in the elliptic region, equations (0.67) and (0.68) still hold, except that the matrices $Q_\pm$ are simplified by the omission of the $B(0)r_{\mathcal{I}}$ columns. When β is in the hyperbolic region, the previous discussion is superfluous, as the boundary data for the incoming profiles σ can simply be expressed in terms of G without having to apply a Fourier transform w.r.t θ_0. Write $\sigma(x', 0, \theta_0) = Q^{-1} G(x', \theta_0)$ where Q is the matrix with columns $\{B(0)r_m : m \in \mathcal{I}\}$.*

An interesting "spreading" result is discussed in this section: the profiles σ_m are shown to not necessarily inherit the θ-decay of the boundary data G.

Proposition 0.15. *Let β be in the mixed region and $G \in \Gamma^s$, for $s > \frac{d+1}{2} + 1$.*

Then $\sigma_m \in H^s_T$ ($\forall\, m \in \mathcal{O} \cup \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}$) and the following two results hold:

$$\text{(1)} \quad \textit{If } \{B(0)r_m : m \in \mathcal{P} \cup \mathcal{N}\} \textit{ is a linearly independent set}$$

$$\textit{and } G\hat{\ }(x', 0) \notin span\{B(0)r_m : m \in \mathcal{I}\}, \ \textit{then } \sigma_m \notin \Gamma^1 \ (m \in \mathcal{I}) \tag{0.70}$$

$$\text{(2)} \quad G(x', \theta_0) \in span\{B(0)r_m : m \in \mathcal{I}\} \quad \Rightarrow \quad \sigma_m \in \Gamma^s \ (m \in \mathcal{I}) \tag{0.71}$$

Proof: Consider $s > \frac{d+1}{2} + 1$. First it will be shown that: $\sigma_m \in H^s_T \ \forall\, m$

The result holds trivially when $m \in \mathcal{O}$.

Proposition (0.19) gives the result when $m \in \mathcal{P} \cup \mathcal{N}$.

As will be discussed in section 2.4, the hyperbolic (incoming) profiles are solved for via an iteration argument. This argument utilizes the estimates of Theorem 0.20, which in particular show that the (Sobolev) regularity of the hyperbolic profiles derive from the regularity of the boundary data G. Thereby, in order to prove that $\sigma_m \in H^s$ (when $m \in \mathcal{I}$), it suffices to prove that $\sigma_m|_{x_d=0} \in bH^s_T$. Because κ commutes with $\mathbb{1}_{\{\kappa>0\}}$ and Q_+^{-1}, inspection of (0.66) demonstrates that (for $m \in \mathcal{I}$):

$$|\sigma_m|_{x_d=0}|_{bH^s_T} = |(0.66)|_{bH^s_T} \lesssim |G|_{bH^s_T}$$

Thereby $\sigma_m \in H^s$, when $m \in \mathcal{I}$, which concludes the first desired result.

Part (1) of Proposition 0.15 is proven as follows:

Let β be in the mixed region and $G \in \Gamma^s$, where $s > \frac{d+1}{2} + 1$.

Notice that, because $s > \frac{d+1}{2}$, G is continuous in (x', θ_0). Applying the dominated convergence theorem to the Fourier transform formula for $G\hat{}(x, \kappa)$ yields: $G\hat{}(x', 0^-) = G\hat{}(x', 0^+)$, so that (0.62) yields:

$$\sum_{m \in \mathcal{I} \cup \mathcal{N}} \sigma\hat{}_m(x', 0, 0^-)\,(B(0)r_m) = \sum_{m \in \mathcal{I} \cup \mathcal{P}} \sigma\hat{}_m(x', 0, 0^+)\,(B(0)r_m) \tag{0.72}$$

Suppose now, for the sake of contradiction, that $\sigma_m \in \Gamma^1$ (for some $m \in \mathcal{I}$), then also: $\sigma\hat{}_m(x', 0, 0^-) = \sigma\hat{}_m(x', 0, 0^+)$ a.e. in x'. Notice that, because $s > \frac{d+2}{2}$ and $\sigma_m \in H^s$, σ_m is continuous in (x, θ_0). This gives $\sigma\hat{}_m(x', 0, \kappa) \in L^2(x', C(\kappa))$.

The previous equality then simplifies to:

$$\sum_{m \in \mathcal{N}} \sigma\hat{}_m(x', 0, 0^-)\,(B(0)r_m) = \sum_{m \in \mathcal{P}} \sigma\hat{}_m(x', 0, 0^+)\,(B(0)r_m) \qquad \text{(a.e. in } x'\text{)} \tag{0.73}$$

But $\{B(0)r_m : m \in \mathcal{P} \cup \mathcal{N}\}$ was assumed to be a independent set, which implies that: $\sigma\hat{}_m(x', 0, 0^+) = 0$, $\sigma\hat{}_m(x', 0, 0^-) = 0$ a.e. in x' (for $m \in \mathcal{P}, \mathcal{N}$, respectively). This forces the polarization condition: $G\hat{}(x', 0) \in \mathrm{span}\{B(0)r_m : m \in \mathcal{I}\}$. (The a.e. in x' qualification can now be removed because $G\hat{}(x', \kappa)$ is continuous in x'.) However, this condition contradicts the hypothesis regarding $G\hat{}(x', 0)$. $\Rightarrow\Leftarrow$

Therefore $\sigma_m \notin \Gamma^1$.

Part (2) of Proposition 0.15 is proven as follows:

Suppose that G satisfies the polarization condition: $G(x', \theta_0) \in \text{span}\{B(0)r_m : m \in \mathcal{I}\}$.

As mentioned earlier in the proof, the (Sobolev) regularity of the hyperbolic profiles derive from the regularity of the boundary data G.

Therefore, in order to prove that $\sigma_m \in \Gamma^s$, it suffices to show that: $\sigma_m|_{x_d=0} = (0.66) \in b\Gamma^s$.

Let $\widetilde{Q}$ denote the $p \times |\mathcal{I}|$ matrix with columns $\{B(0)r_m : m \in \mathcal{I}\}$.

Let $\widetilde{Q}^{-1}_{left}$ denote the left matrix inverse of $\widetilde{Q}$.

Then the polarization hypothesis for G shows that:

$$[\mathbb{1}_{\{\kappa>0\}} Q_+^{-1} G\hat{}(x', \kappa) + \mathbb{1}_{\{\kappa<0\}} Q_-^{-1} G\hat{}(x', \kappa)]_m = [\widetilde{Q}^{-1}_{left} G\hat{}(x', \kappa)]_m \quad (\text{when } m = 1, \cdots, |\mathcal{I}|) \tag{0.74}$$

$$[\mathbb{1}_{\{\kappa>0\}} Q_+^{-1} G\hat{}(x', \kappa) + \mathbb{1}_{\{\kappa<0\}} Q_-^{-1} G\hat{}(x', \kappa)]_m = 0 \quad (\text{when } m = |\mathcal{I}| + 1, \cdots, p) \tag{0.75}$$

Thereby, for $|\beta| \leq s$, because $\widetilde{Q}^{-1}_{left}$ is a constant matrix:

$$|(\theta_0, \partial_{x'}, \partial_{\theta_0})^\beta \sigma_m|_{x_d=0}|_{L^2(x',\theta_0)} = |(\theta_0, \partial_{x'}, \partial_{\theta_0})^\beta (0.66)|_{L^2(x',\theta_0)}$$

$$\lesssim |(\partial_\kappa, \partial_{x'}, \kappa)^\beta \widetilde{Q}^{-1}_{left} G\hat{}(x', \kappa)|_{L^2(x',\theta_0)} \lesssim |(\theta_0, \partial_{x'}, \partial_{\theta_0})^\beta G(x', \theta_0)|_{L^2(x',\theta_0)}$$

which proves that $(0.66) \in b\Gamma^s$, as $G \in b\Gamma^s$. $\square$

Remark 0.16. *Therefore, when β does not lie in the hyperbolic region, the hyperbolic profiles may not belong to Γ^1, even if the boundary data belongs to $b\Gamma^s$ (for $s > \frac{d+1}{2} + 1$). Similarly, the elliptic profiles may not belong to Γ^1, even when the boundary data belongs to $b\Gamma^s$ (for $s > \frac{d+1}{2} + 1$), because the half-line support conditions on the elliptic profiles would require that the profiles vanish at $\kappa = 0$.*

Remark 0.17. *Notice that the discussion of Section 2.3 did not depend on the nonlinearity of the system (0.2). Therefore this "spreading" behavior is a linear phenomenon.*

Examples 0.18. *For the ongoing Euler equation example:*

$$B(0)s_1 = \begin{pmatrix} \rho - \frac{\rho u_3|\xi|}{c\xi_3} \\ \frac{\xi_1}{\xi_3} \\ \frac{\xi_2}{\xi_3} \end{pmatrix} \qquad B(0)s_3 = \begin{pmatrix} \rho + \frac{\rho u_3|\xi|}{c\xi_3} \\ \frac{\xi_1}{\xi_3} \\ \frac{\xi_2}{\xi_3} \end{pmatrix} \tag{0.76}$$

$$B(0)r_1 = \begin{pmatrix} \rho - \frac{\rho u_3|(\xi'',\omega_1)|}{c\omega_1} \\ \frac{\xi_1}{\omega_1} \\ \frac{\xi_2}{\omega_1} \end{pmatrix} \qquad B(0)r_3 = \begin{pmatrix} \rho - \frac{\rho u_3|(\xi'',\omega_3)|}{c\omega_3} \\ \frac{\xi_1}{\omega_3} \\ \frac{\xi_2}{\omega_3} \end{pmatrix} \tag{0.77}$$

It's clear that $B(0)r_1$ and $B(0)r_3$ can only be collinear if they are related by $\frac{\omega_1}{\omega_3}B(0)r_1 = B(0)r_3$. This occurs when:

$$(\frac{\omega_1}{\omega_3} - 1)\rho + \frac{\rho u_3(|(\xi'',\omega_3)| - |(\xi'',\omega_1)|)}{c\omega_3} = 0$$

$$\leftrightarrow (\frac{\omega_1}{\omega_3} - 1) + \frac{u_3(|(\xi'',\omega_3)| - |(\xi'',\omega_1)|)}{c\omega_3} = 0$$

$$\leftrightarrow c\omega_1 - c\omega_3 + u_3(|(\xi'',\omega_3)| - |(\xi'',\omega_1)|) = 0 \tag{0.78}$$

Because $\omega_1 - \omega_3$ is purely imaginary, while the other term is real, it follows that $\omega_1 = \omega_3$. By considering the definitions of ω_1 and ω_3, it's clear that this can only occur when Im $\omega_1 = 0 =$ Im ω_3. However (by definition), for a frequency β chosen in the mixed region, Im $\omega_1 \neq 0$. Hence $B(0)r_1$ and $B(0)r_3$ can never be collinear when β is chosen in the mixed region. (Notice that this argument did not have a dependency on u_3; the transversal relationship of $B(0)r_1$ and $B(0)r_3$ holds regardless of what values u_3 assumes.) Therefore $\{B(0)r_m : m \in \mathcal{P} \cup \mathcal{N}\}$ is a linearly independent set.

Thereby, via this extended Euler equation example, it's shown that there are physically relevant boundary operators B for important equations of mathematical physics for which the "spreading" phenomenon can occur.

At this point, it's instructive to remember that the boundary data G and the exact solution U_ϵ to the desired PDE should be real-valued in any physically relevant context. Thus, it should be verified that the approximate solution $\mathcal{U}^0$ is also real-valued (for it to be a valid, physically relevant approximation).

Using the Fourier inversion formula, it's easy to see that $\mathcal{U}^0$ is real-valued iff $\mathcal{U}^{0,\hat{}}(x,\kappa,\xi_d) = \overline{\mathcal{U}^{0,\hat{}}(x,-\kappa,\xi_d)}$. This is the condition that will be verified for $\mathcal{U}^0$. For $m \in \mathcal{I}$, one sees that the σ_m are real-valued via (0.66) and its use in Theorems 0.20, 0.22. It's already known that the r_m are real for $m \in \mathcal{I}$. Hence the hyperbolic contribution to $\mathcal{U}^0$ is real-valued.

Recall that there is a bijection between indices $m \in \mathcal{P}$ and indices $m' \in \mathcal{N}$. Therefore, to finish the desired verification, it suffices to show that $\hat{\sigma_m}(x,\kappa)r_m = \overline{\hat{\sigma_{m'}}(x,-\kappa)r_{m'}}$. It's already known that $r_m = \overline{r_{m'}}$, because the complex eigenvalues $\omega_m = \overline{\omega_{m'}}$ must exist in conjugate pairs. Additionally (0.67) and (0.68) show that $\hat{a_m}(x,\kappa) = \overline{\hat{a_{m'}}(x,-\kappa)}$. By how the elliptic profiles are constructed in (0.81), it follows that $\hat{\sigma_m}(x,\kappa) = \overline{\hat{\sigma_{m'}}(x,-\kappa)}$, as desired.

Solving for the Profiles σ

As was previously shown in Section 2.2, it's necessary to set the terms on the RHS of (0.58) which do not depend on the variable of integration equal to zero. Applying the Fourier inversion formula to this condition yields (along with the prescribed boundary data) the following profile equations:

$$\begin{aligned} &X_{\phi_m}\sigma_m + c^m_m\sigma_m\partial_\theta\sigma_m - e^m_m\sigma_m = 0 \\ &\sigma_m(x,\theta_0)|_{x_d=0} = a_m(x',\theta_0), \quad \sigma_m|_{t<0} = 0 \end{aligned} \tag{0.79}$$

As mentioned previously, these decoupled interior profile equations for the approximate profiles σ_m hold regardless of where β lies. Notice that the vector field X_{ϕ_m} governs the propagation of the profiles σ_m into the interior.

When β lies in the elliptic or mixed region, (0.79) must be solved for elliptic profiles σ_m. When X_ϕ is no longer a real vector field, (0.79) can no longer be solved in the interior. Instead, as in [7], (0.79) is solved to first order at the boundary by prescribing $[\partial x_d\sigma_{x_d}(x,\theta)]_m|_{x_d=0}$ to be:

$$b_m(x',\theta) \equiv \overline{X}_\phi a_m - c^m_m a_m\partial_\theta a_m + e^m_m a_m \tag{0.80}$$

where $X_\phi \equiv \partial_{x_d} - \overline{X}_\phi$. Notice that the RHS of (0.80) is well-known by (0.67) and (0.68). From there (the components of) a candidate solution for (0.79) can be constructed as:

$$\sigma_m(x,\theta) \equiv \chi(x_d)[a_m(x',\theta) + x_d b_m(x',\theta)] \tag{0.81}$$

where χ is a compactly supported cut-off function, which is identically 1 near $x_d = 0$. (This cut-off ensures the necessary boundedness of σ_m in x_d.) Notice that σ_m can be extended into the appropriate half-plane, due to the support property of a_m: supp $a_m\hat{}\subseteq \{\mathrm{Im}\omega_m\,\kappa \geq 0\}$

For use in the error analysis of Chapter 3, it's helpful to define an (elliptic) error function $\mathcal{R}(x,\theta_0,\xi_d) \equiv \sum_{m\in\mathcal{P}\cup\mathcal{N}} R_m(x,\theta_0,\xi_d)r_m$, which measures how well (0.81) solves (0.79), where:

$$R_m(x,\theta_0,\xi_d) \equiv [X_\phi\sigma_m + c_m^m\sigma_m\partial_\theta\sigma_m - e_m^m\sigma_m](x,\theta_0+\omega_m\xi_d) \tag{0.82}$$

As with σ_m, $R_m(x,\theta)$ inherits the desired support property from a_m, and can thereby be extended into the appropriate half-plane.

Proposition 0.19. *(Regularity of Elliptic Profiles)*

Suppose that $G \in bH_T^s$ *and that* $s > \frac{d+3}{2}$*. Then* $\sigma_m \in H_T^{s-1}$ *when* $m \in \mathcal{P}\cup\mathcal{N}$*, by the estimate:*

$$|\sigma_m(x,\theta_0)|_{H_T^{s-1}} \lesssim |G(x',\theta_0)|_{bH_T^s} \tag{0.83}$$

Proof: Let $G \in bH_T^s$.

By construction of a_m in (0.67) and (0.68), it follows that $a_m \in bH_T^s$, as:

$$|a_m(x',\theta_0)|_{bH_T^s} \sim \sum_{|\alpha|+\beta\leq s} |\partial_{x'}^\alpha \kappa^\beta (Q_\pm^{-1}G\hat{}(x',\kappa))_m \cdot \mathbb{1}_{\{\mathrm{Im}\omega_m\kappa\geq 0\}}|_{L_T^2(x',\kappa)}$$

$$\lesssim \sum_{|\alpha|+\beta\leq s} |\partial_{x'}^\alpha \kappa^\beta G\hat{}(x',\kappa))_m|_{L_T^2(x',\kappa)} \sim |G_m(x',\theta_0)|_{bH_T^s}$$

Because $s > \frac{d+3}{2}$, bH_T^{s-1} is an algebra. Thereby it follows that $b_m \in bH_T^{s-1}$.

By considering the estimate:

$$|\sigma_m|_{H_T^{s-1}} \leq |\chi(x_d)a_m|_{H_T^{s-1}} + |\chi(x_d)x_d b_m|_{H_T^{s-1}}$$

it suffices to show that $\psi(x_d)h(x',\theta) \in H_T^{s-1}$, when ψ is a compactly-supported smooth function in x_d and $h \in bH^{s-1}_T$. This is verified as follows:

$$|\psi(x_d)h(x',\theta_0)|_{H^{s-1}_T} \sim \sum_{\alpha+|\beta|+\gamma\leq s-1} |\,\partial^\alpha_{x_d}\partial^\beta_{x'}\kappa^\gamma[\psi(x_d)h\hat{}(x',\kappa)]\,|_{L^2_T(x,\theta_0)}$$

$$= \sum_{\alpha+|\beta|+\gamma\leq s-1} \sup_{x_d\geq 0} |\partial^\alpha_{x_d}\psi(x_d)|\,|\,\partial^\beta_{x'}\kappa^\gamma h\hat{}(x',\kappa)\,|_{L^2_T(x',\theta_0)} \lesssim |h(x',\theta_0)|_{bH^{s-1}_T} \quad \square$$

When β lies in the hyperbolic or mixed region, (0.79) must be solved for hyperbolic profiles σ using Picard iteration. The following iteration scheme is used to determine the hyperbolic profiles:

$$X_{\phi_m}\sigma^{n+1}_i + c^m_m\sigma^m_i\partial_\theta\sigma^{n+1}_m = e^m_m\sigma^n_m$$

$$\sigma^n_m|_{x_d=0} = a_m = (0.66), \quad \sigma^n_m|_{t<0} = 0 \quad (m \in \mathcal{I}) \tag{0.84}$$

An iteration argument is employed which utilizes standard L^2 "energy-style" estimates. The necessary Sobolev estimate and subsequent existence/uniqueness result will be stated here without proof. (Refer to Propositions 3.5 and 3.6 of [5] for proofs of these results.)

Theorem 0.20. *Let $T > 0$ and $s > \frac{d+2}{2} + 1$. Suppose that $G \in bH^s_T$ and $\sigma^n_m \in H^s_T$ vanish in $t \leq 0$. Then the system (0.84) has a unique solution $\sigma^{n+1}_m \in H^s_T$ vanishing in $t \leq 0$. Additionally, $\exists$ increasing functions $\gamma_0(K)$ and $C(K)$ of $K \equiv |\sigma^n_m|_{s,T} : \forall \gamma \geq \gamma_0(K)$:*

$$|\sigma^{n+1}_m|_{s,\gamma,T} \leq C(K)[\frac{\langle G \rangle_{s,\gamma,T}}{\sqrt{\gamma}} + \frac{|\sigma^n_m|_{s,\gamma,T}}{\gamma}] \tag{0.85}$$

where $|u|_{s,\gamma,T} \equiv |e^{-\gamma t}u|_{s,T} \equiv |e^{-\gamma t}u|_{H^s(\Omega_T)}$ and similarly $\langle u \rangle_{s,\gamma,T} \equiv \langle e^{-\gamma t}u \rangle_{s,T} \equiv \langle e^{-\gamma t}u \rangle_{H^s(b\Omega_T)}$.

Remark 0.21. *Note that, despite the same notation, this is not the Γ^s norm that was used in the construction of the hyperbolic profiles in [5].*

Theorem 0.22. *Under the hypotheses of Theorem 0.20, the iterates σ^n_m converge to a unique solution $\sigma_m \in H^s_{T_0}$ of the profile equations (0.84), for some $0 < T_0 \leq T$.*

This convergence is weak in $H^s_{T_0}$; it's only strong in $H^{s'}_{T_0}$ for $s' < s$. This fact will be used later in the error analysis of Chapter 3.

An iteration scheme for the elliptic profiles σ_m is necessary for the error analysis in the simultaneous Picard iteration. The elliptic iterates are constructed as follows:

$$X_\phi \sigma_m^{n+1} + c_m^m \sigma_m^n \partial_\theta \sigma_m^{n+1} - e_m^m \sigma_m^n = 0 \tag{0.86}$$

$$\sigma_m^n|_{x_d=0} = a_m = (0.67), (0.68) \tag{0.87}$$

This iteration scheme is adapted from [7], but simplifies greatly, as there is no "feedback" from the hyperbolic iterates when one solves for σ_m^n at the boundary. (Although it should be noted that the n-th incoming and elliptic iterates are solved simultaneously.) Instead all iterates must agree on the boundary with a_m (prescribed by G as in (0.67) and (0.68)).

This scheme is initialized with $\sigma_m^0 \equiv 0$, so that: $X_\phi \sigma_m^1 = 0$. σ_m^1 is solved at the boundary by prescribing $\partial_{x_d} \sigma_m^1|_{x_d=0} = [\overline{X}_\phi \sigma_m^1]_{x_d=0} = \overline{X}_\phi a_m$. Thereby, define: $\sigma_m^1 = \chi(x_d)(a_m + x_d \overline{X}_\phi a_m)$.

Next, solve (0.86) for $n = 1$ by requiring that:
$\partial_{x_d} \sigma_m^2|_{x_d=0} = [\overline{X}_\phi \sigma_m^2 - c_m^m \sigma_m^1 \partial_\theta \sigma_m^2 + e_m^m \sigma_m^1]_{x_d=0}$
$= \overline{X}_\phi a_m - c_m^m a_m \partial_\theta a_m + e_m^m a_m = b_m$
The prescriptions for $\partial_{x_d} \sigma_m^n|_{x_d=0}$, when $n > 2$, must also be b_m.

Thus, the equations for the profile iterates (solved to first order) are:

$$\sigma_m^0 = 0 \tag{0.88}$$

$$\sigma_m^1 = \chi(x_d)(a_m + x_d \overline{X}_\phi a_m) \tag{0.89}$$

$$\sigma_m^n = \chi(x_d)(a_m + x_d b_m) \quad (n \geq 2) \tag{0.90}$$

The elliptic iteration scheme also determine the error iterates

$\mathcal{R}^n(x,\theta_0,\xi_d) \equiv \sum_{m\in\mathcal{P}\cup\mathcal{N}} R^n_m(x,\theta_0,\xi_d) r_m$, defined component-wise as:

$$R^n_m(x,\theta_0,\xi_d) \equiv [X_\phi \sigma^{n+1}_m + c^m_m \sigma^n_m \partial_\theta \sigma^{n+1}_m - e^m_m \sigma^n_m](x,\theta_0+\omega_m\xi_d) \tag{0.91}$$

Note that the iterate components inherit the desired half-line support conditions:

supp $R^{\hat{n}}_m \subset \{\mathrm{Im}\omega_m\kappa \geq 0\}$ from the a_m.

A certain class of functions will be considered, called type $\mathcal{F}$ functions, upon which an operator $\mathbb{E}$ can be defined.

Definition 0.23. *A function F is said to be of type $\mathcal{F}$ if it has the following form:*

$$F(x,\theta_0,\xi_d) = \sum_{i=1}^{M} F_i(x,\theta_0,\xi_d) r_i$$

where each F_i has the form:

$$F_i(x,\theta_0,\xi_d) = \sum_{j=1}^{M} f^i_j(x,\theta_0+\omega_j\xi_d) + \sum_{j\leq k=1}^{M} g^i_{j,k}(x,\theta_0+\omega_j\xi_d) h^i_{j,k}(x,\theta_0+\omega_k\xi_d) \tag{0.92}$$

where $f^i_j(x,\theta), g^i_{j,k}(x,\theta), h^i_{j,k}(x,\theta)$ are C^1 functions.

Definition 0.24. *For a type $\mathcal{F}$ function F, define the operator $\mathbb{E}$ by:*

$$\mathbb{E}F(x,\theta_0,\xi_d) \equiv \sum_{i=1}^{M} \widetilde{F}_i(x,\theta_0+\omega_i\xi_d) r_i \quad \textit{where} \quad \widetilde{F}_i(x,\theta) \equiv f^i_i(x,\theta) + g^i_{i,i}(x,\theta) h^i_{i,i}(x,\theta) \tag{0.93}$$

Remark 0.25. *Notice that this definition of the operator* $\mathbb{E}$ *agrees with the definition given in [5]. However, the definition here does not make use of "averaging" integrals, as the primitives which arise may not be exist, if* Γ^s *spaces are not used.*

With these definitions, the solvability conditions on $\mathcal{U}^0$ can now be formulated in the following concise form (which will be suitable for later error analysis):

$$\begin{aligned}
&\mathbb{E}\mathcal{U}^0 = \mathcal{U}^0 \\
&\mathbb{E}[\widetilde{L}(\partial)\mathcal{U}^0 + M(\mathcal{U}^0, \partial_{\theta_0}\mathcal{U}^0) - F(0)\mathcal{U}^0] = \mathcal{R} \\
&B(0)\mathcal{U}^0|_{x_d=0,\xi_d=0} = G \\
&\mathcal{U}^0|_{t<0} = 0
\end{aligned} \tag{0.94}$$

Remark 0.26. *Notice that unlike in [5], an error term is required for the system (0.94). As a reminder, this term needed to be introduced because the elliptic profile equations could only be approximately solved to first order at the boundary. Fortunately, the error is shown to be controlled in Proposition 0.149.*

CHAPTER 3: PROOF OF MAJOR THEOREMS 0.4 AND 0.6

Proof of Exact Solution Theorem 0.4:

As a reminder, a suitable singular pseudo-differential calculus is essential when solving systems of the form (0.3). [4] introduced a singular pseudo-differential calculus for solving such hyperbolic quasilinear problems with pulse boundary data. That calculus was used in [5] to prove Theorem 1.12 of that paper. A version of the [4] calculus is utilized in solving (0.2) for this paper. This calculus is summarized in Appendix F and is used in the following discussion of this section.

The hypotheses of the Exact Solution Theorem 0.4 are nearly identical to those made in Theorem 1.12 of [5]; the only difference being which boundary frequencies β are considered. (In [5] the boundary data G was also only assumed to lie in a H^s space for Theorem 1.12.) Therefore, the proof of Theorem 0.4 in this paper is nearly identical to the proof of Theorem 1.12 in [5], which was outlined in Chapter 2 of that paper. In fact, only one modification needs to be made to that proof, in order to apply its arguments in this paper. The modification involves the $L^\infty(L^2)$ estimate (part 2) in the proof of Theorem 2.3 of that paper. This was the only part of the proof of Theorem 1.12 that made use of the hypothesis that β must lie in the hyperbolic region.

Remark 0.27. *As a reminder, the normal (x_d) derivatives of $\chi^e_D U_\epsilon$ (where U_ϵ is the desired exact singular solution to (0.3)) were more difficult to estimate than the normal derivatives of* $(1-\chi^e_D)U_\epsilon$. *On the support of* $1-\chi^e$, $|\frac{\kappa\beta}{\epsilon}| \lesssim |\xi'|$, *so that* $|X| \lesssim |\xi'|$. *Thereby the equation* (0.4) *can be used to estimate normal derivatives of* $(1-\chi^e_D)U_\epsilon$. *On the support of* χ^e, *the direction of X is approximately the boundary frequency β. By the diagonalizability assumption on β, it can't lie in the glancing set. Thus the block structure (0.98) and ensuing energy estimate argument via Gårding inequalities is utilized in this case, as discussed in the following section.*

The linearized problem for the exact solution is first considered (refer to (2.2) in [5]):

$$\partial_{x_d} U_\epsilon + \sum_{j=0}^{d-1} \widetilde{A}_j(\epsilon V_\epsilon)(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\epsilon}) = f_\epsilon \tag{0.95}$$

Applying the "weight" function $e^{-\gamma t}$ to this equation yields:

$$[\partial_{x_d} - \mathcal{A}_D](e^{-\gamma t} U_\epsilon) = e^{-\gamma t} f_\epsilon \tag{0.96}$$

where $\mathcal{A}_D$ is the pseudo differential operator with singular symbol:

$$\mathcal{A}(\epsilon V_\epsilon, \tau - i\gamma + \frac{\beta_0 \kappa}{\epsilon}, \eta_1 + \frac{\beta_1 \kappa}{\epsilon}, \cdots, \eta_{d-1} + \frac{\beta_{d-1}\kappa}{\epsilon}) \tag{0.97}$$

as in Definition 0.60 and equation (0.169).

As discussed in Remark 0.27, $\mathcal{A}$ can be diagonalized when considering the $|\chi^e_s(D)(e^{-\gamma t} U_\epsilon)|_{\infty,0}$ estimate. This will allow (0.96) to be simplified by Proposition 0.68.

For β in any of the three regions, $|\chi_s^e(D)(e^{-\gamma t}U_\epsilon)|_{\infty,0}$ needs to be estimated. (Where $|\cdot|_{\infty,s}$ represents the norm for $C(x_d, H^s(b\Omega))$.) Let:

$$\Sigma \equiv \{z = (v, X, \gamma) \in B_R \times \mathbb{R}^N \times [0,\infty) : (X,\gamma) \neq 0\}$$

and $\mathcal{A}(z) = \mathcal{A}(v, \tau - i\gamma, \eta)$. (Recall that $X \equiv \xi' + \frac{\kappa\beta}{\epsilon}$.)

$\chi_s^e(D)$ is the Fourier multiplier associated to the cut-off function χ^e discussed in the final section of Appendix G. Following the construction of χ^e in that section, fix sufficiently small parameters $\delta_1 > 0$ and $\delta_2 > 0 : \forall v$ in a ball of radius δ_1 and $\forall(z,\eta)$ that are δ_2-close to $\beta :$

$$S(z)^{-1}\mathcal{A}(z)S(z) = M(z) \tag{0.98}$$

for a suitable invertible matrix S that's homogeneous of degree zero in (X,γ) and C^∞ for z in a conic neighborhood Γ of $\{(0,\beta,0),(0,-\beta,0)\}$ in Σ.

$M(z)$ has the following block diagonal form with blocks of dimension $|\mathcal{O}|, |\mathcal{P}|, |\mathcal{I}|, |\mathcal{N}|$:

$$M(z) = \begin{pmatrix} M_{\mathcal{O}} & 0 & 0 & 0 \\ 0 & M_{\mathcal{N}} & 0 & 0 \\ 0 & 0 & M_{\mathcal{I}} & 0 \\ 0 & 0 & 0 & M_{\mathcal{P}} \end{pmatrix}$$

where $M_{\mathcal{O}}$ and $M_{\mathcal{I}}$ are diagonal matrices, whose entries are the eigenvalues $\mu(z) \equiv i\omega_i(z)$ of $\mathcal{A}(z)$, which satisfy (for some $C > 0$ and $z \in \Gamma$):

$$\operatorname{Re}\mu_i(z) = \gamma H_i(z) \geq C\gamma \quad i \in \mathcal{O}$$

$$\operatorname{Re}\mu_i(z) = -\gamma H_i(z) \leq -C\gamma \quad i \in \mathcal{I} \tag{0.99}$$

where $H_i(z)$ is a homogeneous symbol of degree zero. Additionally:

$$\operatorname{Re} M_{\mathcal{N}}(z) \geq C\langle X, \gamma\rangle$$

$$\operatorname{Re} M_{\mathcal{P}}(z) \leq -C\langle X, \gamma\rangle \tag{0.100}$$

[4] showed that $\chi_s^e(D)(e^{-\gamma t}U_\epsilon)$ can be written as:

$$\chi_s^e(D)(e^{-\gamma t}U_\epsilon) = r_0\mathcal{W} \tag{0.101}$$

where r_0 denotes a bounded operator on $L^2(\Omega)$ whose operator norm is independent of ϵ and γ. Let $\mathcal{W} = (\mathcal{W}^{\mathcal{O}}, \mathcal{W}^{\mathcal{N}}, \mathcal{W}^{\mathcal{I}}, \mathcal{W}^{\mathcal{P}})$ correspond to the decomposition:

$S(z) = [S^{\mathcal{O}}(z) \;\; S^{\mathcal{N}}(z) \;\; S^{\mathcal{I}}(z) \;\; S^{\mathcal{P}}(z)]$, where $S^{\mathcal{O}}(z)$ is the matrix whose columns are $r_i(z)$, for $i \in \mathcal{O}$. (The other components of S have analogous columns for the respective index sets.)

For ease of exposition, the estimate for $\mathcal{W}^{\mathcal{N}}$ will be considered. (The estimate for $\mathcal{W}^{\mathcal{P}}$ is analogous; the estimates for the hyperbolic components have already been shown in [5].)

By the diagonalization of $\mathcal{A}$ and Proposition 0.68, $\mathcal{W}^{\mathcal{N}}$ satisfies the following:

$$\partial_{x_d}\mathcal{W}^{\mathcal{N}} - M_{\mathcal{N}}\,\mathcal{W}^{\mathcal{N}} = r_0(e^{-\gamma t}f_\epsilon) + r_0(e^{-\gamma t}U_\epsilon) \tag{0.102}$$

This equation can be used to derive useful energy estimates: First, take the real part of the tangential $L^2_T(x', \theta_0)$-inner product of (0.103) with $-\overline{\mathcal{W}}^{\mathcal{N}}$. Then, integrate this normed equation from x_d to ∞, in x_d:

$$|\mathcal{W}^{\mathcal{N}}(x_d)|^2_{L^2_T(x',\theta_0)} + \int_{x_d}^{\infty} \operatorname{Re}\langle M_{\mathcal{N}}\mathcal{W}^{\mathcal{N}}(y), \overline{\mathcal{W}}^{\mathcal{N}}\rangle_{L^2_T(x',\theta_0)}dy$$

$$= -\int_{x_d}^{\infty}\langle r_0(e^{-\gamma t}f_\epsilon(y)), \overline{\mathcal{W}}^{\mathcal{N}}\rangle_{L^2_T(x',\theta_0)}dy - \int_{x_d}^{\infty}\langle r_0(e^{-\gamma t}U_\epsilon(y)), \overline{\mathcal{W}}^{\mathcal{N}}\rangle_{L^2_T(x',\theta_0)}dy \tag{0.103}$$

(0.100) gives a positivity condition on $M_{\mathcal{N}}$, whereby Theorem 0.1 along with (0.170) can be applied to the previous equality to give:

$$|\mathcal{W}^{\mathcal{N}}(x_d)|^2_{L^2_T(x',\theta_0)} + \gamma\int_{x_d}^{\infty}|\mathcal{W}^{\mathcal{N}}(y)|^2_{L^2_T(x',\theta_0)}dy$$

$$\lesssim \int_{x_d}^{\infty}|\mathcal{W}^{\mathcal{N}}(y)|_{L^2_T(x',\theta_0)}|e^{-\gamma t}f_\epsilon(y)|_{L^2_T(x',\theta_0)}dy + \int_{x_d}^{\infty}|\mathcal{W}^{\mathcal{N}}(y)|_{L^2_T(x',\theta_0)}|e^{-\gamma t}U_\epsilon(y)|_{L^2_T(x',\theta_0)}dy \tag{0.104}$$

Because of the factor of γ, the contribution of $\mathcal{W}^{\mathcal{N}}$ on the RHS of 0.104 can be absorbed on the left by Young's inequality. Then $|\mathcal{W}^{\mathcal{N}}(,\cdot,x_d,\cdot)|_{L^2_T(x',\theta_0)}$ will be the only term appearing on the LHS of (0.104), which allows the $C(x_d, L^2_T(x',\theta_0))$ norm of $\mathcal{W}^{\mathcal{N}}$ to be estimated, as the RHS only involves tangential $L^2_T(x',\theta_0)$-norms:

$$|\mathcal{W}|^2_{\infty,0} \leq C(K)[\gamma^{-1}|e^{-\gamma t}f_\epsilon(y)|^2_{L^2_T(x,\theta_0)} + \gamma^{-2}|e^{-\gamma t}g_\epsilon(y)|^2_{L^2_T(x',\theta_0)}] \tag{0.105}$$

where $C(K)$ is the constant from Theorem 2.1 of [5]. The boundary g_ϵ appears here, because Theorem 2.1 of [5] controlled the $L^2_T(x,\theta_0)$ norm of U_ϵ by the $L^2_T(x,\theta_0)$ and $L^2_T(x',\theta_0)$ norms of f_ϵ and g_ϵ, respectively The desired result then follows:

$$|\chi^e_s(D)(e^{-\gamma t}U_\epsilon)|_{\infty,0} \leq C(K)(\gamma^{\frac{-1}{2}}|e^{-\gamma t}f_\epsilon|_{0,0} + \langle e^{-\gamma t}g_\epsilon\rangle_0) \tag{0.106}$$

thereby confirming the $L^\infty(L^2)$ estimate in the proof of Theorem 2.3 in [5] for the extension to β in the elliptic and mixed regions. $\square$

Remark 0.28. *Refer to p. 1961 of [5] for an explanation of why the factors of γ in (0.105) and (0.106) are different. The energy estimate argument changes slightly for $\mathcal{W}^{\mathcal{I}}$ and $\mathcal{W}^{\mathcal{P}}$. The positivity conditions which (0.99) and (0.100) give for $\mathcal{W}^{\mathcal{I}}$ and $\mathcal{W}^{\mathcal{P}}$, respectively, require that (0.103) instead have limits of integrations of 0 to x_d, in x_d. This introduces a $|\mathcal{W}^{\mathcal{I},\mathcal{P}}|_{x_d=0}|_{L^2_T(x',\theta_0)}$ term which must be controlled, introducing $L^2_T(x,\theta_0)$ and $L^2_T(x',\theta_0)$ norms of f_ϵ and g_ϵ, respectively, with different factors of γ.*

Proof of Approximation Theorem 0.6:

This section considers Theorem 0.29, which yields the desired Approximation Theorem (0.6) as a corollary. The proof of Theorem 0.29 is directly adapted from section 4.3 of [5], which proves Theorem 4.16 in that paper. The Picard Iteration argument is completely analogous; in fact, the conditions and estimates for the exact iterates are unaltered. Proofs of the necessary estimates for Theorem 0.29 are relegated to the appendices. One of the primary differences in this paper involves the error term $\mathcal{R}$, which arose from the failure of the elliptic profiles $\sigma_m \, (m \in \mathcal{P} \cup \mathcal{N})$ to exactly solve (0.79). The estimate of this error (considered in (0.149)) doesn't yield a specified rate of convergence. This estimate, along with the moment zero approximation estimates, prevents the determination of an exact rate of convergence between the exact and approximate solutions, as $\epsilon \to 0$ (in contrast to [5]). Another difference of this paper is the corrector $\mathcal{U}_p^1$, which differs from the corrector in [5]. As mentioned in Section 2.2, some of the components of the corrector are constructed from special primitives σ_m^*, which are only primitives in the distributional sense. Additionally, the corrector must be constructed with not only modified non transversal interaction terms, but also modified transversal interaction terms (defined in (0.120)). A third difference in this paper is the interaction terms in the corrector estimate involving elliptic profiles, which now need to be estimated on the Fourier transformed "κ-side," as shown in Appendix E.

Theorem 0.29. *For $M_0 = 3d+5$ and $s \geq 1 + [M_0 + \frac{d+1}{2}]$, suppose that $G \in bH^{s+1}_T$ vanishes for $t \leq 0$. Let $U_\epsilon \in E^s_{T_0}$ be the exact solution to the singular system (0.4) for $0 < \epsilon \leq \epsilon_0$ specified by Theorem 0.4. Let σ_j $(j \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N})$ be the profiles which were constructed in Chapter 2. Then define $\mathcal{U}^0_\epsilon \in E^{s-1}_{T_0}$ to be:*

$$\mathcal{U}^0_\epsilon(x,\theta_0) \equiv \sum_{j=1}^{M} \sigma_j(x, \theta_0 + \omega_j \frac{x_d}{\epsilon}) r_j \tag{0.107}$$

Here $0 < T_0 \leq T$ is the minimum of the existence times specified by Theorem 0.4 and Theorem 0.22. Under these hypotheses and definitions, the following result holds: The family $\mathcal{U}^0_\epsilon$ is uniformly bounded in $E^{s-1}_{T_0}$ for $0 < \epsilon \leq \epsilon_0$. Furthermore, $\exists\, 0 < T_1 \leq T_0$:

$$\lim_{\epsilon \to 0} |U_\epsilon - \mathcal{U}^0_\epsilon|_{E^{s-3}_{T_1}} = 0 \tag{0.108}$$

Proof of Precise Approximation Theorem 0.29:

In order to implement simultaneous Picard iteration, consider the iteration schemes for both the exact solution of (0.4) and the approximate solution of (0.94) (adapted for $\mathcal{U}^0$.)

$$\begin{gathered}
\partial_{x_d} U^{n+1}_\epsilon + \sum_{j=0}^{d-1} \widetilde{A}_j(\epsilon U^n_\epsilon)(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\epsilon}) U^{n+1}_\epsilon = F(\epsilon U^n_\epsilon) U^n_\epsilon \\
B(\epsilon U^n_\epsilon) U^{n+1}_\epsilon|_{x_d=0} = G(x',\theta_0) \\
U^{n+1}_\epsilon|_{t<0} = 0
\end{gathered} \tag{0.109}$$

$$\begin{aligned}
&\mathbb{E}\mathcal{U}^{0,n+1} = \mathcal{U}^{0,n+1} \\
&\mathbb{E}[\widetilde{L}(\partial)\mathcal{U}^{0,n+1} + M(\mathcal{U}^{0,n}, \partial_\theta \mathcal{U}^{0,n+1}) - F(0)\mathcal{U}^{0,n}] = \mathcal{R}^n \\
&B(0)\mathcal{U}^{0,n+1}|_{x_d=0,\xi_d=0} = G \\
&\mathcal{U}^{0,n+1}|_{t<0} = 0
\end{aligned} \tag{0.110}$$

Here $\mathcal{U}^{0,n}(x,\theta_0,\xi_d) \equiv \sum_{j=1}^{M} \sigma_j^n(x,\theta_0+\omega_j\xi_d)r_j$, $\mathcal{U}_\epsilon^{0,n}(x,\theta_0) \equiv \mathcal{U}^{0,n}(x,\theta_0,\frac{x_d}{\epsilon})$, and $\mathcal{R}^n$ is defined as in (0.91).

With the given hypotheses, in order to prove Theorem 0.29, it suffices to show the following 5 conditions:

1. Uniform boundedness of the family U_ϵ in $E_{T_0}^{s-2}$
2. Uniform boundedness of the family $\mathcal{U}_\epsilon^0$ in $E_{T_0}^{s-2}$
3. $\lim\limits_{n\to\infty} U_\epsilon^n = U_\epsilon$ in $E_{T_0}^{s-2}$ uniformly w.r.t $\epsilon \in (0,\epsilon_0]$
4. $\lim\limits_{n\to\infty} \mathcal{U}_\epsilon^{0,n} = \mathcal{U}_\epsilon^0$ in $E_{T_0}^{s-2}$ uniformly w.r.t $\epsilon \in (0,\epsilon_0]$
5. $\exists\, 0 < T_1 < T_0 : \forall\, n : \quad \lim\limits_{\epsilon\to 0} |U_\epsilon^n - \mathcal{U}_\epsilon^{0,n}|_{E_{T_1}^{s-3}} = 0$

Conditions 1 and 3 are confirmed in the construction of the exact solution. Conditions 2 and 4 follow from Proposition 0.41 and the convergence of the profiles $\sigma^n \to \sigma$ in $H_{T_0}^{s-1}$, because $\sigma \in H_{T_0}^s$ by Proposition 0.19. Condition 5 will be proven via induction using the following linear estimate:

Proposition 0.30. *Let $s \geq [M_0 + \frac{d+1}{2}]$ and consider the problem (0.109), where $G \in bH^{s+1}$ satisfies supp $G \subseteq \{t \geq 0\}$ and the RHS of (0.109a) is replaced by some $\mathcal{F} \in E_T^s$. Suppose that $U_\epsilon^n \in E_T^s$ and that $\exists\, K > 0, \epsilon_1 > 0$:*

$$|U_\epsilon^n|_{E_T^s} + |\epsilon\, \partial_{x_d} U_\epsilon^n|_{L^\infty} \leq K \quad \textit{for } 0 < \epsilon \leq \epsilon_1$$

Then $\exists\, T_0(K) > 0, \epsilon_0(K) \leq \epsilon_1 : \forall\, 0 < \epsilon \leq \epsilon_0, T \leq T_0$:

$$|U_\epsilon^{n+1}|_{E_T^s} + \sqrt{T}\langle U_\epsilon^{n+1}|_{x_d=0}\rangle_{s+1,T} \leq C(K)[T|\mathcal{F}|_{E_T^s} + \sqrt{T}\langle G\rangle_{s+1,T}]$$

The inductive argument will involve constructing an appropriate corrector $\epsilon\mathcal{U}^1_{p,\epsilon}$. To properly construct this corrector, $\mathcal{U}^{0,n}$ will be approximated by its moment-zero approximation $\mathcal{U}^{0,n}_p$, defined as:

$$\mathcal{U}^{0,n}_p(x,\theta_0,\xi_d) \equiv \sum_{j=1}^{M} \sigma^n_{j,p}(x,\theta_0+\omega_i\xi_d)r_j$$

The discussion of the moment-zero approximations and the resulting primitives is relegated to Appendix B.

Remark 0.31. *The "little o" notation o_ϵ, o_p indicates that the limits $\epsilon \to 0$, $p \to 0$, respectively, are being considered.*

The inductive hypothesis is: $\exists\, 0 < T_1 \leq T_0$:

$$\lim_{\epsilon\to 0} |U^n_\epsilon - \mathcal{U}^{0,n}_\epsilon|_{E^{s-3}_{T_1}} = 0 \tag{0.111}$$

1. This hypothesis along with the boundedness of the family U^n_ϵ in $E^{s-2}_{T_0}$ yields:

$$|F(\epsilon U^n_\epsilon)U^n_\epsilon - F(0)\mathcal{U}^{0,n}_\epsilon|_{E^{s-3}_{T_0}} = o_\epsilon(1)$$

Propositions 0.41 and 0.47 then yield:

$$|F(\epsilon U^n_\epsilon)U^n_\epsilon - F(0)\mathcal{U}^{0,n}_{p,\epsilon}|_{E^{s-3}_{T_0}} = o_\epsilon(1) + o_p(1) \tag{0.112}$$

2. Define:

$$\mathcal{G}_p \equiv \widetilde{L}(\partial_x)\mathcal{U}_p^{0,n+1} + M(\mathcal{U}_p^{0,n}, \partial_{\theta_0}\mathcal{U}_p^{0,n+1})$$

Remark 0.32. *Because the inductive step is shown for some fixed n, the dependency of $\mathcal{G}_p$ (and later of the corrector $\mathcal{U}_p^1$) on n is suppressed for ease of notation.*

Lemma 0.33.

$$|[\mathbb{E}(\mathcal{G}_p - F(0)\mathcal{U}_p^{0,n})]_\epsilon|_{E_{T_0}^{s-2}} = o_\epsilon(1) + o_p(1) \tag{0.113}$$

(The notation here indicates the norm is separately both $o_\epsilon(1)$ and $o_p(1)$; there's no dependency of one to the other.)

Proof: To begin, estimate as follows:

$$|[\mathbb{E}(\mathcal{G}_p - F(0)\mathcal{U}_p^{0,n})]_\epsilon|_{E_{T_0}^{s-2}}$$

$$\lesssim |[\mathbb{E}(\mathcal{G}_p - F(0)\mathcal{U}_p^{0,n})]_\epsilon - [\mathbb{E}(\mathcal{G} - F(0)\mathcal{U}^{0,n})]_\epsilon|_{E_{T_0}^{s-2}} + |[\mathbb{E}(\mathcal{G} - F(0)\mathcal{U}^{0,n})]_\epsilon|_{E_{T_0}^{s-2}} \tag{0.114}$$

The first term in (0.114) is estimated as follows:

$$|[\mathbb{E}(\mathcal{G}_p - F(0)\mathcal{U}_p^{0,n})]_\epsilon - [\mathbb{E}(\mathcal{G} - F(0)\mathcal{U}^{0,n})]_\epsilon|_{E_{T_0}^{s-2}}$$

$$\leq |\mathbb{E}(\mathcal{G} - \mathcal{G}_p) - [\mathbb{E}(F(0)\mathcal{U}_p^{0,n} - F(0)\mathcal{U}_p^{0,n})|_{H_{T_0}^{s-1}} \quad \text{by Proposition 0.41}$$

$$\leq |\mathbb{E}(\widetilde{L}(\partial_x)\mathcal{U}^{0,n+1} - \widetilde{L}(\partial_x)\mathcal{U}_p^{0,n+1})|_{H_{T_0}^{s-1}}$$

$$+|\mathbb{E}(M(\mathcal{U}^{0,n}, \partial_{\theta_0}\mathcal{U}^{0,n+1}) - M(\mathcal{U}_p^{0,n}, \partial_{\theta_0}\mathcal{U}_p^{0,n+1}))|_{H_{T_0}^{s-1}}$$

$$+|\mathbb{E}((F(0)\mathcal{U}^{0,n}) - F(0)\mathcal{U}_p^{0,n})|_{H_{T_0}^{s-1}}$$

$$\lesssim \sum_{m\in\mathcal{I}\cup\mathcal{P}\cup\mathcal{N}} \Big[\quad |\partial_{x_d}\sigma_m^{n+1} - \partial_{x_d}\sigma_{m,p}^{n+1}|_{H^{s-1}_{T_0}} + |\sigma_m^{n+1} - \sigma_{m,p}^{n+1}|_{H^{s}_{T_0}}$$

$$+ |\sigma_m^n \partial_{\theta_0}\sigma_m^{n+1} - \sigma_{m,p}^n \partial_{\theta_0}\sigma_{m,p}^{n+1}|_{H^{s-1}_{T_0}} + |\sigma_m^n - \sigma_{m,p}^n|_{H^{s-1}_{T_0}} \quad \Big]$$

Propositions 0.47, and 0.48 ensure that these terms are $o_p(1)$.

Inspection of the system (0.110b) yields:

$$|[\mathbb{E}(\mathcal{G} - F(0)\mathcal{U}^{0,n})]_\epsilon|_{E^{s-2}_{T_0}} = |\mathcal{R}^n_\epsilon|_{E^{s-2}_{T_0}}$$

Proposition 0.148 shows that this norm is $o_\epsilon(1)$, which completes the lemma. □

3. The following estimate holds for the singular exact solution:

$$|\mathbb{L}_0 U_\epsilon^{n+1} - F(\epsilon U_\epsilon^n)U_\epsilon^n|_{E^{s-3}_{T_0}} = o_\epsilon(1) + o_p(1) \tag{0.115}$$

where $\mathbb{L}_0 \equiv \widetilde{L}(\partial_x) + \frac{1}{\epsilon}\widetilde{L}(d\phi_0)\partial_{\theta_0} + M(\mathcal{U}^{0,n}_{p,\epsilon}, \partial_{\theta_0})$ provides an approximation to the operator appearing on the LHS of (0.109a). This estimate follows from (0.109a) and the following estimates:

$$|\widetilde{A}_j(\epsilon U_\epsilon^n)\partial_{x_j}U_\epsilon^{n+1} - \widetilde{A}_j(0)\partial_{x_j}U_\epsilon^{n+1}|_{E^{s-1}_{T_0}} \lesssim \epsilon$$

$$|\epsilon^{-1}\widetilde{A}_j(\epsilon U_\epsilon^n)\beta_j\partial_{\theta_0}U_\epsilon^{n+1} - (\epsilon^{-1}\widetilde{A}_j(0)\beta_j\partial_{\theta_0}U_\epsilon^{n+1} + d\widetilde{A}_j(0)\cdot U_\epsilon^n\beta_j\partial_{\theta_0}U_\epsilon^{n+1})|_{E^{s-1}_{T_0}} \lesssim \epsilon$$

$$|d\widetilde{A}_j(0)\cdot(U_\epsilon^n - \mathcal{U}^{0,n}_{p,\epsilon})\beta_j\partial_{\theta_0}U_\epsilon^{n+1}|_{E^{s-3}_{T_0}} \lesssim |U_\epsilon^n - \mathcal{U}^{0,n}_{p,\epsilon}|_{E^{s-3}_{T_0}} = o_\epsilon(1) + o_p(1) \tag{0.116}$$

4. Constructing the Corrector: $\widetilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{\xi_d})\mathcal{U}_p^{0,n+1} = 0$ implies that $\mathbb{L}_0\mathcal{U}_{p,\epsilon}^{0,n+1} = \mathcal{G}_{p,\epsilon}$, which yields:

$$\mathbb{L}_0\mathcal{U}_{p,\epsilon}^{0,n+1} - F(0)\mathcal{U}_{p,\epsilon}^{0,n} = \mathcal{G}_{p,\epsilon} - F(0)\mathcal{U}_{p,\epsilon}^{0,n}$$

$$= [\mathbb{E}(\mathcal{G}_p - F(0)U_p^{0,n})]_\epsilon + [(I - \mathbb{E})(\mathcal{G}_p - F(0)U_p^{0,n})]_\epsilon \tag{0.117}$$

Lemma 0.33 shows that the first term on the RHS of (0.117) is $o_\epsilon(1) + o_p(1)$. As discussed in Section 2.2, the attempt is made to construct a corrector which solves away the second term on the RHS of (0.117). However, as also noted in Section 2.2, several difficulties arose when trying to properly define and estimate such a corrector, particularly for its hyperbolic components τ_m ($m \in \mathcal{O} \cup \mathcal{I}$).

5. Therefore modifications to the function $(I - \mathbb{E})\mathcal{G}_p$ are needed, where as a reminder:

$$(I - \mathbb{E})\mathcal{G}_p = \sum_{i=1}^{M}[-\sum_{k\neq i} V_k^i \sigma_{k,p}^{n+1} - \sum_{k\neq i} c_k^i \sigma_{k,p}^n \partial_{\theta_0}\sigma_{k,p}^{n+1} - \sum_{k\neq j} d_{k,j}^i \sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1}]r_i \tag{0.118}$$

Consider the following modification of $(I - \mathbb{E})\mathcal{G}_p$:

$$[(I - \mathbb{E})\mathcal{G}_p]_{mod} \equiv (1 - \mathbb{E})\widetilde{L}(\partial_x)\mathcal{U}_p^{0,n+1} + [(1 - \mathbb{E})M(\mathcal{U}_p^{0,n}, \partial_{\theta_0}\mathcal{U}_p^{0,n+1})]_{mod} \tag{0.119}$$

$$\text{where} \qquad [(1 - \mathbb{E})M(\mathcal{U}_p^{0,n}, \partial_{\theta_0}\mathcal{U}_p^{0,n+1})]_{mod} \equiv \sum_{i=1}^{M}[-\sum_{k\neq i} c_k^i (\sigma_{k,p}^n \partial_{\theta_0}\sigma_{k,p}^{n+1})_p$$

$$- \sum_{k\neq j: k \text{ or } j\in\mathcal{P}\cup\mathcal{N}} d_{k,j}^i (\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1})_{p_e} - \sum_{k\neq j: k \text{ and } j\in\mathcal{I}\cup\mathcal{O}} d_{k,j}^i (\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1})_{p_h}]r_i \tag{0.120}$$

Note that $\sigma_{i,p}^n = \sigma_{i,p}^n(x, \theta_0 + \omega_i\xi_d)$.

$(\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1})_{p_e}(x, \theta_0, \xi_d)$ is defined so that its Fourier transform w.r.t θ_0 is $h_{i,j,k,p}^n(x, \kappa, \xi_d)$, as defined in (0.160). $(\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1})_{p_h}(x, \theta_0, \xi_d)$ is well-defined by Definition 0.51.

The error produced by this modification can be controlled; it's estimated in (0.126).

Remark 0.34. *Notice that in this paper, the transversal interaction terms must also be modified, in addition to the non transversal interaction terms, in order to obtain a corrector that can be suitably estimated in an* E_T^s *space.*

Remark 0.35. *Due to the differing arguments of* $\sigma_{k,p}^n$ *and* $\partial_{\theta_0}\sigma_{j,p}^{n+1}$ *in (0.120), the moment-zero approximations defined in Definition 0.43 could not be used. Instead, the* p_e *and* p_h *approximations were used similar, which resemble the approximations of Definition 0.43.*

The (appropriately modified) corrector $\mathcal{U}_p^1 = \sum_{m=1}^M \tau_{m,p} r_m$ is (explicitly) constructed as:

$$(m \in \mathcal{O} \cup \mathcal{I}) \quad \tau_{m,p}(x,\theta_0,\xi_d) \equiv \int_\infty^{\xi_d} ([(1-\mathbb{E})M(\mathcal{U}_p^{0,n}, \partial_{\theta_0}\mathcal{U}_p^{0,n+1})]_{mod})_m(x, \theta_0 + \omega_m(\xi_d - s), s)ds$$

$$+(1-\mathbb{E})[\widetilde{L}(\partial_x)\mathcal{U}_p^{0,n+1} - F(0)\mathcal{U}_p^{0,n}]_m^*(x,\theta_0,\xi_d) \tag{0.121}$$

$$(m \in \mathcal{P}) \quad \tau_{m,p}\hat{}\,(x,\kappa,\xi_d) \equiv \mathbb{1}_{\{\kappa<0\}}[\int_\infty^{\xi_d} e^{i\omega_m\kappa(\xi_d - s)}([(I-\mathbb{E})\mathcal{G}_p]_{mod} - F(0)\mathcal{U}_p^{0,n})\hat{}_m(x,\kappa,s)ds]$$

$$+\mathbb{1}_{\{\kappa>0\}}[\int_0^{\xi_d} e^{i\omega_m\kappa(\xi_d - s)}([(I-\mathbb{E})\mathcal{G}_p]_{mod} - F(0)\mathcal{U}_p^{0,n})\hat{}_m(x,\kappa,s)ds] \tag{0.122}$$

$$(m \in \mathcal{N}) \quad \tau_{m,p}\hat{}\,(x,\kappa,\xi_d) \equiv \mathbb{1}_{\{\kappa<0\}}[\int_0^{\xi_d} e^{i\omega_m\kappa(\xi_d - s)}([(I-\mathbb{E})\mathcal{G}_p]_{mod} - F(0)\mathcal{U}_p^{0,n})\hat{}_m(x,\kappa,s)ds]$$

$$+\mathbb{1}_{\{\kappa>0\}}[\int_\infty^{\xi_d} e^{i\omega_m\kappa(\xi_d - s)}([(I-\mathbb{E})\mathcal{G}_p]_{mod} - F(0)\mathcal{U}_p^{0,n})\hat{}_m(x,\kappa,s)ds] \tag{0.123}$$

Remark 0.36. $\mathcal{U}_p^{1,\hat{}}$ *needs to only be defined almost everywhere. Hence the lack of a definition for* $\mathcal{U}_p^{1,\hat{}}(x,0,\xi_d)$ *is not problematic.*

By construction (outlined in Section 2.2) this corrector $\mathcal{U}_p^1$ satisfies:

$$\widetilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{x_d})\mathcal{U}_p^1 = (I - \mathbb{E})F(0)\mathcal{U}_p^{0,n} - [(I - \mathbb{E})\mathcal{G}_p]_{mod} \tag{0.124}$$

The error produced by this modified corrector is:

$$D(x, \theta_0, \xi_d) \equiv \widetilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{x_d})\widetilde{\mathcal{U}}_p^1 - \widetilde{\mathcal{L}}(\partial_{\theta_0}, \partial_{x_d})\mathcal{U}_p^1 = [(I - \mathbb{E})\mathcal{G}_p]_{mod} - [(I - \mathbb{E})\mathcal{G}_p]$$

$$\begin{aligned}
&= \sum_{i=1}^{M}[-\sum_{k\neq i} c_k^i[\sigma_{k,p}^n \partial_{\theta_0}\sigma_{k,p}^{n+1} - (\sigma_{k,p}^n \partial_{\theta_0}\sigma_{k,p}^{n+1})_p] \\
&- \sum_{k\neq j: k \text{ or } j \in \mathcal{P}\cup\mathcal{N}} d_{k,j}^i[\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1} - (\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1})_{p_e}] \\
&- \sum_{k\neq j: k \text{ and } j \in \mathcal{I}\cup\mathcal{O}} d_{k,j}^i[\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1} - (\sigma_{k,p}^n \partial_{\theta_0}\sigma_{j,p}^{n+1})_{p_h}]\,]r_i
\end{aligned} \tag{0.125}$$

In Appendix D, the following estimate is proven:

$$|D_\epsilon|_{E_{T_0}^{s-3}} \lesssim \sqrt{p} \tag{0.126}$$

6.

Proposition 0.37.

$$|\epsilon\,\mathcal{U}_{p,\epsilon}^1|_{E_T^{s-2}} \lesssim \frac{\epsilon}{p} \tag{0.127}$$

$$|\epsilon\,[\partial_{x_d}\mathcal{U}_p^1]_\epsilon|_{E_T^{s-3}} \lesssim \frac{\epsilon}{p} \tag{0.128}$$

Proof: The proof of (0.127) is relegated to Appendix E. The estimate of (0.128) is found by differentiating all the components of $\mathcal{U}_p^1$ by x_d and then proceeding analogously as in the arguments of Appendix E.

7.

$$|\mathbb{L}_0(\mathcal{U}^{0,n+1}_{p,\epsilon}+\epsilon\mathcal{U}^1_{p,\epsilon})-F(0)\mathcal{U}^{0,n}_{p,\epsilon}|_{E^{s-3}_{T_0}}$$

$$\leq |[\mathbb{E}(\mathcal{G}_p-F(0)\mathcal{U}^{0,n}_p)]_\epsilon|_{E^{s-3}_{T_0}}+|D_\epsilon|_{E^{s-3}_{T_0}}+|(\widetilde{L}(\partial_x)\epsilon\mathcal{U}^1_p)_\epsilon|_{E^{s-3}_{T_0}}+|M(\mathcal{U}^{0,n}_{p,\epsilon},\partial_{\theta_0})(\epsilon\mathcal{U}^1_{p,\epsilon})|_{E^{s-3}_{T_0}}$$

$$=o_\epsilon(1)+o_p(1)+O_{\epsilon,p}(\frac{\epsilon}{p})+O_p(\sqrt{p}) \tag{0.129}$$

Hence by (0.112), (0.115), (0.129) and setting (for example) $p=\sqrt{\epsilon}$, it follows that:

$$\lim_{\epsilon\to 0}|\mathbb{L}_0[U^{n+1}_\epsilon-(\mathcal{U}^{0,n+1}_{p,\epsilon}+\epsilon\mathcal{U}^1_{p,\epsilon})]|_{E^{s-3}_{T_0}}=0 \tag{0.130}$$

8. The following estimates hold:

$$\lim_{\epsilon\to 0}|(\partial_{x_d}+\mathbb{A}(\epsilon\mathcal{U}^{0,n}_{p,\epsilon},\partial_{x'}+\frac{\beta\partial_{\theta_0}}{\epsilon}))(U^{n+1}_\epsilon-(\mathcal{U}^{0,n+1}_{p,\epsilon}+\epsilon\mathcal{U}^1_{p,\epsilon}))|_{E^{s-3}_{T_0}}=0$$

$$\lim_{\epsilon\to 0}|B(\epsilon\mathcal{U}^{0,n}_{p,\epsilon})(U^{n+1}_\epsilon-(\mathcal{U}^{0,n+1}_{p,\epsilon}+\epsilon\mathcal{U}^1_{p,\epsilon}))|_{H^{s-2}_{T_0}}=0 \tag{0.131}$$

Applying Proposition 0.30 yields:

$$\lim_{\epsilon\to 0}|U^{n+1}_\epsilon-(\mathcal{U}^{0,n+1}_{p,\epsilon}+\epsilon\mathcal{U}^1_{p,\epsilon})|_{E^{s-3}_{T_0}}=0$$

So that by (0.127):

$$\lim_{\epsilon\to 0}|U^{n+1}_\epsilon-\mathcal{U}^{0,n+1}_{p,\epsilon}|_{E^{s-3}_{T_0}}=0$$

Finally, by applying Propositions 0.41 and 0.47 (and once more setting, e.g., $p=\sqrt{\epsilon}$)

$$\lim_{\epsilon\to 0}|U^{n+1}_\epsilon-\mathcal{U}^{0,n+1}_\epsilon|_{E^{s-3}_{T_0}}\leq\lim_{\epsilon\to 0}|U^{n+1}_\epsilon-\mathcal{U}^{0,n+1}_\epsilon|_{E^{s-3}_{T_0}}+\lim_{\epsilon\to 0}|\mathcal{U}^{0,n+1}_\epsilon-\mathcal{U}^{0,n+1}_{p,\epsilon}|_{E^{s-3}_{T_0}}$$

$$\leq\lim_{\epsilon\to 0}|U^{n+1}_\epsilon-\mathcal{U}^{0,n+1}_\epsilon|_{E^{s-3}_{T_0}}+\lim_{\epsilon\to 0}|\mathcal{U}^{0,n+1}-\mathcal{U}^{0,n+1}_p|_{H^{s-2}_{T_0}}=0 \tag{0.132}$$

This completes the inductive argument and hence the proof of Theorem 0.29. $\square$

CHAPTER 4: EXTENSION TO HIGHER MULTIPLICITY CASE

Here is treated the extension of (0.2) with the four assumptions of section 1.3 to the case of higher eigenvalue multiplicity. To be precise, an extension is needed when at least one eigenvalue ω_m of $\mathcal{A}(\beta)$ has a corresponding (algebraic) multiplicity $\nu_{k_m} > 1$.

For $m \in \{1, \cdots, M\}$, let $l_{m,k} \quad (k = 1, \cdots, \nu_{k_m})$ denoted a basis of vectors for the left eigenspace of the matrix $i\mathcal{A}(\beta)$ associated to the eigenvalue $-\omega_m$, chosen to satisfy:

$$l_{m,k} \cdot r_{m',k'} = \begin{cases} 1 & \text{if}\, m = m' \,\text{and}\, k = k' \\ 0 & \text{otherwise} \end{cases}$$

For $v \in \mathbb{C}^N$, define:

$$P_{m,k} v \equiv (l_{m,k} \cdot v) r_{m,k}$$

Functions of type $\mathcal{F}$ have the following form:

$$F(x, \theta_0, \xi_d) = \sum_{m=1}^{M} \sum_{k=1}^{\nu_{k_m}} F_{m,k}(x, \theta_0, \xi_d) r_{m,k}$$

where each $F_{m,k}$ has the form:

$$F_{m,k}(x,\theta_0,\xi_d) = \sum_{m'=1} f_{m'}^{m,k}(x,\theta_0+\omega_{m'}\xi_d)$$

$$+ \sum_{m',k',m'',k''} g^{m,k}_{m',k',m'',k''}(x,\theta_0+\omega_{m'}\xi_d)h^{m,k}_{m',k',m'',k''}(x,\theta_0+\omega_{m''}\xi_d)$$

where $m' \in \{1,\cdots,M\}$, $k' \in \{1,\cdots,\nu_{k'_m}\}$ (similarly for m'',k'') and $f_m^{m,k}(x,\theta), g^{m,k}_{m',k',m'',k''}(x,\theta), h^{m,k}_{m',k',m'',k''}(x,\theta)$ are C^1 functions.

The operator $\mathbb{E}$ acts on functions of type $\mathcal{F}$ by:

$$\mathbb{E}F(x,\theta_0,\xi_d) \equiv \sum_{i=1}^{M} \widetilde{F}_{m,k}(x,\theta_0+\omega_i\xi_d)r_{m,k}$$

where:

$$\widetilde{F}_{m,k}(x,\theta) \equiv f_m^{m,k}(x,\theta) + \sum_{k',k''} g^{m,k}_{m,k',m,k''}(x,\theta)h^{m,k}_{m,k',m,k''}(x,\theta)$$

It will be seen that the general form of (0.94) still holds.

If $W(x,\theta_0,\xi_d) = \sum_{m,k} w_{m,k}(x,\theta_0,\xi_d)r_{m,k}$, then:

$$\widetilde{L}(\partial_x)W = \sum_{m,k}(X_{\phi_m}w_{m,k})r_{m,k} + \sum_{m,k}\left(V^{m,k}_{m',k'}w_{m',k'}\right)r_{m,k}$$

where $V^{m,k}_{m',k'}$ is the tangential vector field:

$$V^{m,k}_{m',k'} \equiv \sum_{j=1}^{d-1}(l_{m,k}\widetilde{A}_j(0)r_{m',k'})\partial_{x_j}$$

The approximate solution generalizes as follows:

$$\mathcal{U}^{0,n}(x,\theta_0,\xi_d) = \sum_{m=1}^{M}\sum_{k=1}^{\nu_{k_m}} \sigma^n_{m,k}(x,\theta_0+\omega_m\xi_d)r_{m,k}$$

By (0.29), even in the higher multiplicity case, $\{B(0)r_{m,k} : k \in \{1, \cdots, \nu_{k_m}\}, m \in \mathcal{I} \cup \mathcal{P}\}$ and $\{B(0)r_{m,k} : k \in \{1, \cdots, \nu_{k_m}\}, m \in \mathcal{I} \cup \mathcal{N}\}$ are both still bases of $\mathbb{C}^p$, so that the arguments of section 2.3 still hold and prescribe boundary data $a_{m,k}$ for $m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}$ and $k \in \{1, \cdots, \nu_{k_m}\}$: $\sigma^n_{m,k}|_{x_d=0} = a_{m,k}$.

So now the profiles equations (0.84) take the generalized form:

$$X_{\phi_m}\sigma^{n+1}_{m,l} + \sum_{j=0}^{d-1}\sum_{k,k'=1}^{\nu_{k_m}} b^{k,k'}_{m,l,j}\sigma^n_{m,k}\partial_\theta\sigma^{n+1}_{m,k'} = \sum_{k=1}^{\nu_{k_m}} e^k_{m,l}\sigma^n_{m,k}$$

$$\sigma^n_{m,k}|_{x_d=0} = a_{m,k} = (0.66), \quad \sigma^n_i|_{t<0} = 0 \quad (\forall\, m, k) \tag{0.133}$$

where the coefficients $b^{k,k'}_{m,l,j}$ are defined by

$$b^{k,k'}_{m,l,j} \equiv l_{m,l} \cdot \beta_j(d\widetilde{A}_j(0)r_{m,k})r_{m,k'} \tag{0.134}$$

Remark 0.38. *There is a potentially serious obstacle to proving estimates for (0.133). When taking the L^2 pairing of (0.133a) (the first equation) with $\sigma^{n+1}_{m,l}(x,\theta)$, it's not clear how to use integration by parts in θ to move the θ-derivative in the sum on the left onto the n-th iterate. Notice that this problem did not arise in the estimate for (0.84). However, the next proposition will remove this difficulty by demonstrating that there is a symmetry in the coefficients $b^{k,k'}_{m,l,j}$ that appears after regrouping.*

Definition 0.39. *For u near 0 let $-\omega_m$ $(m = 1, \cdots, M)$ be the eigenvalues of:*

$$i\mathcal{A}(u,\beta) \equiv A_d^{-1}(u)(\underline{\tau}I + \sum_{j=1}^{d-1}\underline{\eta}_j A_j(u))$$

with corresponding projectors $P_m(u)$. It's assumed that the functions $\omega_m(u)$ and $P_m(u)$ are C^∞ for u near 0.

Proposition 0.40. *Let $w \in \mathbb{R}^N$ be written as $w = \sum_{m,k} w_{m,k} r_{m,k} = \sum w_m$ and define:*

$$B^m_{l,k'}(w) \equiv \sum_{j=0}^{d-1} \sum_{k=1}^{\nu_{k_m}} b^{k,k'}_{m,l,j} w_{m,k} \tag{0.135}$$

where the $b^{k,k'}_{m,l,j}$ are defined as in (0.134). Then the following holds:

$$B^m_{l,k'}(w) = \begin{cases} -d\omega_m(0) \cdot w_m & if k' = l \\ 0 \quad otherwise \end{cases} \tag{0.136}$$

(For the proof of this proposition, refer to Proposition 5.3 of [5].)

This proposition allows:

$$\sum_{j=0}^{d-1} \sum_{k,k'=1}^{\nu_{k_m}} b^{k,k'}_{m,l,j} \sigma^n_{m,k} \partial_\theta \sigma^{n+1}_{m,k'} = B^m_{l,l}(\mathcal{W}^{0,n}) \partial_\theta \sigma^{n+1}_{m,l} \tag{0.137}$$

where $\mathcal{W}^{0,n} \equiv \sum_{m,k} \sigma^n_{m,k} r_{m,k}$. Thereby the θ-derivative can be shifted, which facilitates the integration by parts as discussed in Remark 0.38. Hence the results of section 2.4 for the approximate solution still hold, where now $\sigma_{m,k} = 0 \; \forall \, k$ when $m \in \mathcal{O}$. Otherwise, minor changes are needed for the error analysis in the proof of Theorem 0.29. For example, the self-interactions terms $d^i_{k,j} \sigma^n_{k,p} \partial_{\theta_0} \sigma^{n+1}_{j,p}$ are replaced by terms of the form $d^i_{m,m',k,k'} \sigma^n_{m,k,p} \partial_{\theta_0} \sigma^{n+1}_{m',k',p}$. These terms are once more handled by replacing $(I - \mathbb{E})\mathcal{G}$ with $[(I - \mathbb{E})\mathcal{G}]_{mod}$, which is composed of terms of the form $d^i_{m,m',k,k'} (\sigma^n_{m,k,p} \partial_{\theta_0} \sigma^{n+1}_{m',k',p})_p$. The ensuing corrector and modification error can be controlled as in the estimates in the proof of Theorem 0.29.

APPENDIX A: ESTIMATES AND RELATIONS FOR THEOREM 0.29

Proposition 0.41. *(Relating Norms) Fix* $s > \frac{d+1}{2}$ *and* $\omega \in \mathbb{C}$*. When* $\omega \notin \mathbb{R}$*, require the support condition: supp* $\hat{\sigma}(x,\kappa) \subset \{Im\omega\,\kappa \geq 0\}$*. Then the following estimate holds for functions of the form* $\sigma_\epsilon(x,\theta_0) = \sigma(x, \theta_0 + \omega\frac{x_d}{\epsilon})$*:*

$$|\sigma_\epsilon(x,\theta_0)|_{E^s_T} \leq |\sigma(x,\theta_0)|_{H^{s+1}_T} \tag{0.138}$$

Proof: This result has already been verified for the hyperbolic profiles (by Proposition 4.3 of [5]) in the case $\omega \in \mathbb{R}$. To prove the estimates for the elliptic profiles, the support condition on $\hat{\sigma}$ will be utilized as follows:

$$\sup_{x_d \geq 0} |\sigma(x, \theta_0 + \omega\frac{x_d}{\epsilon})|_{bH^s_T} \sim \sup_{x_d\geq 0} \sum_{|\alpha|+\beta\leq s} |\partial^\alpha_{x'}\kappa^\beta e^{i\omega\kappa\frac{x_d}{\epsilon}}\hat{\sigma}(x,\kappa)|_{L^2_T(x',\kappa)}$$

$$\leq \sup_{x_d\geq 0} \sum_{|\alpha|+\beta\leq s} |\partial^\alpha_{x'}\kappa^\beta \hat{\sigma}(x,\kappa)|_{L^2_T(x',\kappa)} \sim \sup_{x_d\geq 0} |\sigma(x,\theta_0)|_{bH^s_T} \leq |\sigma(x,\theta_0)|_{H^{s+1}_T}$$

$$\sqrt{\int_0^\infty |\widetilde{\sigma}(x,\theta_0 + \omega\frac{x_d}{\epsilon})|^2_{bH^{s+1}_T} dx_d} \sim \sqrt{\int_0^\infty \sum_{|\alpha|+\beta\leq s+1} |\partial^\alpha_{x'}\kappa^\beta e^{i\omega\kappa\frac{x_d}{\epsilon}}\hat{\sigma}(x,\kappa)|^2_{L^2_T(x',\kappa)} dx_d}$$

$$\leq \sqrt{\int_0^\infty \sum_{|\alpha|+\beta\leq s+1} |\partial^\alpha_{x'}\kappa^\beta \hat{\sigma}(x,\kappa)|^2_{L^2_T(x',\kappa)} dx_d} \sim \sqrt{\int_0^\infty |\sigma(x,\theta_0)|^2_{bH^{s+1}_T} dx_d} \leq |\sigma(x,\theta_0)|_{H^{s+1}_T}$$

These two inequalities yield Proposition 0.41. $\square$

Thereby, for an elliptic profile $\sigma_\epsilon(x,\theta_0) = \sigma(x,\theta_0+\omega\frac{x_d}{\epsilon})$, Propositions 0.19 and 0.41 yield:

$$|\sigma_\epsilon(x,\theta_0)|_{E_T^{s-2}} \lesssim |G(x',\theta_0)|_{bH_T^s} \tag{0.139}$$

Proposition 0.42. *Let m be the Lebesgue measure on $\mathbb{R}$. The following property holds:*

$$\textit{Given a nonnegative integrable function}\, f: \quad \forall\,\epsilon>0:\,\exists\,\delta>0:$$

$$\forall\,\textit{measurable sets}\, E:\, m(E)<\delta \quad \Rightarrow \quad \int_E f dm < \epsilon \tag{0.140}$$

Proof: Consider an arbitrary nonnegative integrable function f and a measurable set E. Fix some $M>0$, so that:

$$\begin{aligned} \int_E f dm \quad &= \quad \int_{E\cap\{f\le M\}} f dm + \int_{E\cap\{f>M\}} f dm \\ \le Mm(E) + \int_{E\cap\{f>M\}} f dm \quad &\le \quad Mm(E) + \int_{\{f>M\}} f dm \end{aligned}$$

$\forall\, M>0$: $f\cdot\mathbb{1}_{\{f>M\}} \le f \in L^1(\mathbb{R})$. Additionally, because f can only have values of $+\infty$ on a set of measure 0, it follows that *a.e.* $f\cdot\mathbb{1}_{\{f>M\}} \to 0$ pointwise. Hence, by the Dominated Convergence Theorem, $\int_{\{f>M\}} f dm \to 0$, as $M\to\infty$.

Thus, for a given $\epsilon>0$, $\exists\, M_\epsilon>0$: $\int_{\{f>M_\epsilon\}} f dm < \frac{\epsilon}{2}$.

Thereby, if δ is chosen so that: $\delta < \frac{\epsilon}{2M_\epsilon}$, then $\int_E f dm < \epsilon$.

(Notice that M_ϵ depends only on ϵ and not on any particular choice of measurable set E.) $\square$

APPENDIX B: MOMENT-ZERO APPROXIMATIONS

When constructing the corrector to the approximate solution $\mathcal{U}^0$, primitives of the profiles $\sigma(x,\theta)$ must be considered, which may not $\to 0$ as $|\theta| \to 0$, even though the σ do. The failure of a primitive to properly decay is due to a small divisor problem on the Fourier side κ. Hence a primitive of a *moment-zero approximation* to σ can be utilized, which possesses the desired decay.

Definition 0.43. *Consider $\phi(\kappa) \in C^\infty(\mathbb{R})$ which has support in $[-2,2]$ and is identically 1 on $[-1,1]$. For $p \in (0,1)$, define $\phi_p(\kappa) \equiv \phi(\frac{\kappa}{p})$ and set $\chi_p \equiv 1 - \phi_p$. For $\sigma(x,\theta) \in L^2(\Omega_T)$, define the* moment-zero approximation *of σ, $\sigma_p(x,\theta)$ (more precisely it's Fourier transform w.r.t. θ_0) by:*

$$\sigma\hat{}_p(x,\kappa) \equiv \chi_p(\kappa)\sigma\hat{}(x,\kappa) \tag{0.141}$$

Definition 0.44. *The primitive of σ_p, σ_p^*, is then defined (its Fourier transform w.r.t. θ_0 is defined) as:*

$$\hat{\sigma^*}_p(x,\kappa) \equiv \frac{\sigma\hat{}_p(x,\kappa)}{i\kappa} = \frac{\chi_p(\kappa)}{i\kappa}\sigma\hat{}(x,\kappa) \tag{0.142}$$

whereby $\widehat{(\partial_\theta \sigma_p^*)} = i\kappa\widehat{\sigma_p^*} = \widehat{\sigma_p}$, indicating that σ_p^* is indeed a primitive of σ_p, in the distributional sense.

Remark 0.45. *For hyperbolic profiles in Γ^s spaces, this definition directly corresponds to the unique primitive constructed in Proposition 4.7 of [5], which decays to zero as $|\theta| \to \infty$. The primitive definition in this paper is necessary because the profiles may not possess the required θ-decay to define antiderivatives of the form $\int_\infty^\theta \sigma_p(x,s)ds$ (as the profiles may not belong to Γ^s spaces).*

Proposition 0.46. *(Moment-Zero Estimates)*

The following two estimates relate the H_T^s-norms of σ_p, σ_p^* back to the H_T^s-norms of σ, *and are easily proven. For $p \in (0,1)$:*

$$|\sigma_p||_{H^s_T} \leq |\sigma||_{H^s_T} \tag{0.143}$$

$$|\sigma_p^*|_{H^s_T} \lesssim \frac{|\sigma|_{H^s_T}}{p} \tag{0.144}$$

H^s_T is a Banach Algebra for $s > \frac{d+1}{2}$.

Therefore, if $\sigma, \tau \in H^s_T$ for $s > \frac{d+1}{2}$, then (0.144) yields:

$$|(\sigma\tau)_p^*|_{H^s_T} \leq \frac{|\sigma|_{H^s_T}|\tau|_{H^s_T}}{p} \tag{0.145}$$

Proposition 0.47. *(Moment-Zero Approximation Error Estimates)*

For $\sigma \in H^s_T(x, \theta_0)$:

$$\lim_{p\to 0} |\sigma - \sigma_p|_{H^s_T} = 0$$

$$\lim_{p\to 0} |\partial_{x_d}\sigma - \partial_{x_d}\sigma_p|_{H^{s-1}_T} = 0 \tag{0.146}$$

Proof:

$$|\sigma - \sigma_p|_{H^s_T} \sim \sum_{|\alpha|+\beta\leq s} |\partial_x^\alpha \kappa^\beta [1 - \chi_p(\kappa)]\sigma\hat{}(x,\kappa)|_{L^2_T(x,\kappa)}$$

$$= \sum_{|\alpha|+\beta\leq s} [\int_{-2p}^{2p} \kappa^\beta \phi_p(\kappa) |\partial_x^\alpha \sigma\hat{}(x,\kappa)|^2_{L^2_T(x)} d\kappa]^{1/2} \lesssim \sum_{|\alpha|+\beta\leq s} [\int_{-2p}^{2p} |\partial_x^\alpha \sigma\hat{}(x,\kappa)|^2_{L^2_T(x)} d\kappa]^{1/2}$$

The result now holds by Proposition 0.42. $\square$

The following proposition is helpful in estimating the error for moment-zero approximations involving products of functions.

Proposition 0.48. *For $\sigma, \tau \in H^s_T$ and $s > \frac{d+1}{2}$:*

$$|\sigma\tau - (\sigma\tau)_p|_{H^s_T} \lesssim \sqrt{p}\, |\sigma|_{H^s_T} |\tau|_{H^s_T} \tag{0.147}$$

Proof:

$$
\begin{aligned}
|\sigma\tau - (\sigma\tau)_p|_{H^s_T} &\sim \sum_{|\alpha|+\beta \le s} |\partial_x^\alpha \kappa^\beta [1 - \chi_p(\kappa)](\sigma\hat{} * \tau\hat{})(x,\kappa)|_{L^2_T(x,\kappa)} \\
&\lesssim \sum_{|\alpha|+\beta \le s} [\int_{-2p}^{2p} |\partial_x^\alpha (\sigma\hat{} * \tau\hat{})(x,\kappa)|^2_{L^2_T(x)} d\kappa]^{1/2} \\
&\lesssim [\int_{-2p}^{2p} (\int_{\mathbb{R}} |\sigma\hat{}(x,\kappa - s)\tau\hat{}(x,s)|_{H^s_T(x)} ds)^2 d\kappa]^{1/2} \\
&\lesssim [\int_{-2p}^{2p} (\int_{\mathbb{R}} |\sigma\hat{}(x,\kappa - s)|_{H^s_T(x)} |\tau\hat{}(x,s)|_{H^s_T(x)} ds)^2 d\kappa]^{1/2} \\
&\lesssim [\int_{-2p}^{2p} |\sigma\hat{}(x,\kappa - s)|^2_{L^2(s,H^s_T(x))} |\tau\hat{}(x,s)|^2_{L^2(s,H^s_T(x))} d\kappa]^{1/2} \\
&\le |\sigma\hat{}|_{H^s_T(x,\kappa)} |\tau\hat{}|_{H^s_T(x,\kappa)} [\int_{-2p}^{2p} d\kappa]^{1/2} \\
&\lesssim \sqrt{p}\, |\sigma|_{H^s_T} |\tau|_{H^s_T} \quad \square
\end{aligned}
$$

APPENDIX C: ERROR ESTIMATES FOR $\mathcal{R}$

This appendix proves the following error estimate:

$$\lim_{\epsilon \to 0} |\mathcal{R}^n_\epsilon(x, \theta_0, \frac{x_d}{\epsilon})|_{E^s_T} = 0 \tag{0.148}$$

(0.148) is a corollary of the following proposition, as the components of $\mathcal{R}^n_\epsilon$ satisfy the hypotheses of Proposition 0.49.

Proposition 0.49. *Fix some $\omega \in \mathbb{C}$. Suppose that $f(x, \theta) \in H^{s+1}_T$ vanishes at the boundary $x_d = 0$ and satisfies supp $\hat{f}(x, \kappa) \subset \{Im\,\omega\, \kappa \geq 0\}$ (so that f can be analytically extended in its θ argument into the half plane $Im\,\omega\, z > 0$.) Then, independently of $p \in (0, 1)$:*

$$\lim_{\epsilon \to 0} |f(x, \theta_0 + \omega \frac{x_d}{\epsilon})|_{E^s_T} = 0 \tag{0.149}$$

where (as a reminder) $E^s_T(x, \theta_0) \equiv C(x_d, bH^s_T) \cap L^2(x_d, bH^{s+1})$.

<u>Proof:</u> WLOG, consider the case where Im $\omega > 0$. Setting $\theta \equiv \theta_0 + \omega \frac{x_d}{\epsilon}$, write $\mathcal{F}_{\theta_0} f(x, \theta) = e^{i\omega\kappa x_d \epsilon^{-1}} \hat{f}(x, \kappa)$. Thereby the useful norm relation follows:

$$|f(x, \theta)|_{bH^s_T} \lesssim \sum_{|\alpha| + \beta \leq s} |\partial^\alpha_{x'} \kappa^\beta e^{i\omega\kappa x_d \epsilon^{-1}} \hat{f}(x, \kappa)|_{L^2_T(x', \kappa)} \tag{0.150}$$

Consider first the sup of (0.150) in x_d over $[0, \sqrt{\epsilon}]$. Because the exponential is bounded by 1 on the support of $\hat{f}$, it follows that:

$$\sup_{0 \leq x_d \leq \sqrt{\epsilon}} |f(x, \theta)|_{bH^s_T} \leq \sup_{0 \leq x_d \leq \sqrt{\epsilon}} \sum_{|\alpha| + \beta \leq s} |\partial^\alpha_{x'} \kappa^\beta \hat{f}(x, \kappa)|_{L^2_T(x', \kappa)} \leq \sup_{0 \leq x_d \leq \sqrt{\epsilon}} |f(x, \theta_0)|_{bH^s_T} \tag{0.151}$$

Thereby, from the Sobolev embedding: $f \in H^{s+1}_T \subset H^1(x_d, bH^s_T) \subset C(x_d, bH^s_T)$, it follows that: $f|_{x_d = 0} = 0 \Rightarrow |f(x', 0, \theta_0)|_{bH^s_T} = 0 \Rightarrow \lim_{x_d \to 0} |f(x', x_d, \theta_0)|_{bH^s_T} = 0$.

Hence $\lim_{\epsilon\to 0}$ [RHS of (0.151)] $=0$. Notice that the Extreme Value Theorem can be utilized here because the sup in x_d is taken over the compact interval $[0,\sqrt{\epsilon}]$.

Next consider the sup of (0.150) in x_d over $(\sqrt{\epsilon},\infty)$:

$$\sup_{x_d\geq\sqrt{\epsilon}}|f(x,\theta)|_{bH^s_T}\leq\sup_{x_d\geq\sqrt{\epsilon}}\sum_{|\alpha|+\beta\leq s}|e^{-(\mathrm{Im}\omega)\kappa\epsilon^{-1/2}}\partial^\alpha_{x'}\kappa^\beta \hat{f}(x,\kappa)|_{L^2_T(x',\kappa)} \tag{0.152}$$

A beneficial exponential can be extracted from the terms on the RHS of (0.152) (along with a bound by $|f(x,\theta_0)|_{bH^s_T}$) away from $\kappa=0$). To be precise, the integral contribution of the RHS of (0.152) over I^c in the κ-variable (where $I\equiv[0,\epsilon^{1/3}]$, in this case where Im $\omega>0$) is controlled by:

$$e^{(-Im\omega)\epsilon^{-1/6}}\sup_{x_d\geq\sqrt{\epsilon}}\sum_{|\alpha|+\beta\leq s}|\partial^\alpha_{x'}\kappa^\beta \hat{f}(x,\kappa)|_{L^2_{T,I^c}(x',\kappa)}\leq e^{(-Im\omega)\epsilon^{-1/6}}\sup_{x_d\geq 0}\sum_{|\alpha|+\beta\leq s}|\partial^\alpha_{x'}\kappa^\beta \hat{f}(x,\kappa)|_{L^2_T(x',\kappa)}$$

$$\lesssim e^{(-Im\omega)\epsilon^{-1/6}}\sup_{x_d\geq 0}|f(x,\theta_0)|_{bH^s_T}\leq e^{(-Im\omega)\epsilon^{-1/6}}|f(x,\theta_0)|_{H^{s+1}_T}$$

The regularity hypothesis on f and the decaying exponential ensures that this term $\to 0$ as $\epsilon\to 0$. Notice that the half-line support condition on $\hat{f}$ is being used here to extract the decaying exponential.

Close to $\kappa=0$, using the above Sobolev embedding once more yields:

$$\sup_{x_d\geq\sqrt{\epsilon}}\sum_{|\alpha|+\beta\leq s}[\int_0^{\epsilon^{1/3}} e^{(-2Im\omega)\kappa\epsilon^{-1/2}}\kappa^{2\beta}|\partial^\alpha_{x'}\hat{f}(x',x_d,\kappa)|^2_{L^2_T(x')}d\kappa]^{1/2}$$

$$\lesssim\sum_{|\alpha|\leq s}[\int_0^{\epsilon^{1/3}}\sup_{x_d\geq 0}|\partial^\alpha_{x'}\hat{f}(x',x_d,\kappa)|^2_{L^2_T(x')}d\kappa]^{1/2}\lesssim\sum_{|\alpha|+\gamma\leq s+1}[\int_0^{\epsilon^{1/3}}|\partial^\alpha_{x'}\partial^\gamma_{x_d}\hat{f}(x',x_d,\kappa)|^2_{L^2_T(x)}d\kappa]^{1/2}$$

Because the integrands are nonnegative, integrable functions over $\kappa\in\mathbb{R}$, Proposition 0.42 may be used to conclude that this term $\to 0$, as $\epsilon\to 0$.

Now the $L^2(x_d, bH_T^{s+1})$ estimate must be considered. The decomposition is analogous:

$$||f(x,\theta)|_{bH_T^{s+1}}|_{L^2(x_d)} \leq |\sum_{|\alpha|+\beta\leq s+1} |\partial_{x'}^\alpha \kappa^\beta e^{i\omega\kappa x_d \epsilon^{-1}} f\hat{}(x,\kappa)|_{L^2_T(x',\kappa)}|_{L^2(x_d)}$$

For the integral in x_d over $[0, \sqrt{\epsilon}]$:

$$[\int_0^{\sqrt{\epsilon}} \sum_{|\alpha|+\beta\leq s+1} |\partial_{x'}^\alpha \kappa^\beta e^{i\omega\kappa x_d \epsilon^{-1}} f\hat{}(x,\kappa)|^2_{L^2_T(x',\kappa)} dx_d]^{1/2} \leq [\int_0^{\sqrt{\epsilon}} \sum_{|\alpha|+\beta\leq s+1} |\partial_{x'}^\alpha \kappa^\beta f\hat{}(x,\kappa)|^2_{L^2_T(x',\kappa)} dx_d]^{1/2}$$

This term $\to 0$ as $\epsilon \to 0$, by Proposition 0.42.

For the integral in x_d over $(\sqrt{\epsilon}, \infty)$:

$$[\int_{\sqrt{\epsilon}}^{\infty} \sum_{|\alpha|+\beta\leq s+1} |\partial_{x'}^\alpha \kappa^\beta e^{i\omega\kappa x_d \epsilon^{-1}} f\hat{}(x,\kappa)|^2_{L^2_T(x',\kappa)} dx_d]^{1/2} \leq \sum_{|\alpha|+\beta\leq s+1} |\partial_{x'}^\alpha \kappa^\beta e^{(-\mathrm{Im}\omega)\kappa\epsilon^{-1/2}} f\hat{}(x,\kappa)|_{L^2_T(x,\kappa)}$$

As before, the integral contribution of the RHS terms over I^c in the κ-variable is controlled by: $e^{-\mathrm{Im}\omega\epsilon^{-1/6}}|f(x,\theta_0)|_{H_T^{s+1}}$, which goes to 0 as ϵ goes to 0. The rest is controlled by:

$$\sum_{|\alpha|+\beta\leq s+1} [\int_0^{\epsilon^{1/3}} |\partial_{x'}^\alpha f\hat{}(x,\kappa)|^2_{L^2_T(x)} d\kappa]^{1/2}$$

This term is handled once more by Proposition 0.42. $\square$

APPENDIX D: ESTIMATING THE MODIFICATION ERROR D

In order to estimate (0.126), the three parts of (0.125) will be estimated separately. Similarly to many estimates in this paper, the $bH^{s'}_T$ estimate is first determined. Then the $E^{s'}_T$ estimate is concluded by applying $\sup_{x_d \geq 0}$ and $|\cdot|_{L^2_{x_d}}$ to the $bH^{s'}_T$, $bH^{s'+1}_T$ estimates, respectively.

1. Nontransversal Modification Error Estimate

In the following discussion of the non transversal modification error estimate, for ease of notation, let $[\sigma^n_{k,p}]_\epsilon = \sigma^n_{k,p}(x, \theta_0 + \omega_k \dfrac{x_d}{\epsilon})$, $\sigma^n_{k,p} = \sigma^n_{k,p}(x, \theta_0)$.

By Propositions 0.41 and 0.48, along with the uniform boundedness of the profile iterates:

$$|c^i_k[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p})_p]_\epsilon|_{E^{s-3}_{T_0}} \lesssim |\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{k,p})_p|_{H^{s-2}_{T_0}}$$

$$\leq \sqrt{p}\,|\sigma^n_{k,p}|_{H^{s-2}_{T_0}}|\sigma^{n+1}_{k,p}|_{H^{s-1}_{T_0}} \lesssim \sqrt{p}$$

2. Elliptic Transversal Modification Error Estimate

By the definition of $(\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_e}$, and thereby by the definition of $h^n_{i,j,k,p}$ in (0.160):

$$|d^i_{k,j}[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_e}]_\epsilon|_{bH^{s'}_{T_0}}$$

$$= |d^i_{k,j}[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_e}]_\epsilon\hat{}|_{bH^{s'}_{T_0}}$$

$$\lesssim |\phi_p(\kappa)\int_{\mathbb{R}} e^{i\omega_k(\kappa-t)\frac{x_d}{\epsilon}}\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)e^{i\omega_j t\frac{x_d}{\epsilon}}(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)dt|_{H^{s'}_{T_0}(x',\kappa)}$$

Because the exponential terms are uniformly bounded by 1 on their support:

$$|d^i_{k,j}[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_e}]_\epsilon|^2_{bH^{s'}_{T_0}}$$

$$\lesssim |\phi_p(\kappa)\int_{\mathbb{R}}\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)dt|^2_{H^{s'}_{T_0}(x',\kappa)}$$

$$\leq \sum_{|\alpha|+\beta\leq s'}\int_{-2p}^{2p}\int_{\mathbb{R}^d}[\partial^\alpha_{x'}\kappa^\beta\phi_p(\kappa)\int_{\mathbb{R}}\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)dt]^2\,dx'\,d\kappa$$

$$\lesssim \sum_{|\alpha|\leq s'}\int_{-2p}^{2p}\int_{\mathbb{R}^d}[\partial^\alpha_{x'}\int_{\mathbb{R}}\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)dt]^2\,dx'\,d\kappa$$

$$\lesssim \int_{-2p}^{2p}|\int_{\mathbb{R}}\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)dt|^2_{H^{s'}_{T_0}(x')}d\kappa$$

$$\leq \int_{-2p}^{2p}[\int_{\mathbb{R}}|\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{H^{s'}_{T_0}(x')}dt]^2\,d\kappa$$

$$\lesssim \int_{-2p}^{2p}[\int_{\mathbb{R}}|\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)|_{H^{s'}_{T_0}(x')}|(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{H^{s'}_{T_0}(x')}dt]^2\,d\kappa$$

$$\leq \int_{-2p}^{2p}|\sigma^{n,\hat{}}_{k,p}(x,\kappa-t)|^2_{L^2(t,H^{s'}_{T_0}(x'))}|(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}(x,t)|^2_{L^2(t,H^{s'}_{T_0}(x'))}\,d\kappa$$

$$\leq |\sigma^{n,\hat{}}_{k,p}|^2_{bH^{s'}_{T_0}}|(\partial_{\theta_k}\sigma^{n+1}_{j,p})\hat{}|^2_{bH^{s'}_{T_0}}\int_{-2p}^{2p}d\kappa$$

$$\lesssim \sqrt{p}\,|\sigma^n_{k,p}|^2_{bH^{s'}_{T_0}}|\partial_{\theta_k}\sigma^{n+1}_{j,p}|^2_{bH^{s'}_{T_0}}$$

The desired result follows once more by uniform boundedness of the profile iterates and by applying $\sup_{x_d\geq 0}$ and $|\cdot|_{L^2_{x_d}}$ to the $bH^{s'}_T$, $bH^{s'+1}_T$ estimates, as in part 1 of this appendix:

$$|d^i_{k,j}[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p}-(\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_e}]_\epsilon|_{E^{s-3}_{T_0}}\lesssim\sqrt{p} \tag{0.153}$$

3. Hyperbolic Transversal Modification Error Estimate

By Definition 0.51, which properly defines $(\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_h}$, the following estimate holds:

$$|d^i_{k,j}[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_h}]_\epsilon|_{bH^{s'}_{T_0}}$$

$$= |(2\pi)^{\frac{-1}{2}}\beta^{-1}\int_{\mathbb{R}} \phi_p(\kappa)e^{i\theta\kappa'}e^{it\frac{1-\beta}{\beta}(\theta_0+\omega_i\xi_d)}\mathcal{F}_\theta\sigma^n_{k,p}(x,\frac{t}{\beta})\mathcal{F}_\theta(\partial_{\theta_0}\sigma^{n+1}_{j,p})(x,\kappa-t)dt|_{H^{s'}_{T_0}(x',\kappa)}$$

$$\lesssim |\phi(\kappa)\int_{\mathbb{R}} \sigma^{n,\hat{}}_{k,p}(x,\frac{t}{\beta})(\partial_{\theta_0}\sigma^{n+1}_{j,p})\hat{}(x,\kappa-t)dt|_{H^{s'}_{T_0}(x',\kappa)}$$

It's now clear that a computation analogous to part 2 yields the desired result:

$$|d^i_{k,j}[\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p} - (\sigma^n_{k,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_h}]_\epsilon|_{E^{s-3}_{T_0}} \lesssim \sqrt{p}$$

APPENDIX E: ESTIMATING THE CORRECTOR $\mathcal{U}_p^1$

In order to prove (0.127), the individual components occurring in the τ profile formulae (0.121), (0.122), and (0.123) will be separately estimated. More precisely, in order to estimate $|\mathcal{U}^1_{p,\epsilon}|_{E_T^{s-2}}$, first estimate the $bH_T^{s'}$ norm of the components appearing in (0.121), (0.122), and (0.123). Then, upon the substitution $\xi_d = \frac{x_d}{\epsilon}$, apply the $L^2_{x_d}$ and C_{x_d} norms to the bH_T^{s-1} estimate and bH_T^{s-2} estimate (resp.) to obtain the E_T^{s-2} estimate.

To begin, for $m \in \mathcal{O} \cup \mathcal{I}$, estimate $|\tau_{m,p}|_{bH_T^{s'}}$ as follows using (0.144):

$$|(1-\mathbb{E})[\widetilde{L}(\partial_x)\mathcal{U}_p^{0,n+1} - F(0)\mathcal{U}_p^{0,n}]^*_m|_{bH_T^{s'}} \leq \sum_{i\neq m} |(\omega_i - \omega_m)^{-1}(V_i^m \sigma_i^{n+1,*} - e_i^m \sigma_i^{n,*})|_{bH_T^{s'}}$$

$$\lesssim \sum_{i\neq m} [|\sigma_i^{n+1,*}|_{bH_T^{s-1}} + |\sigma_i^{n,*}|_{bH_T^{s'}}] \leq \sum_{i\neq m} p^{-1}[|\sigma_i^{n+1}|_{bH_T^{s-1}} + |\sigma_i^{n}|_{bH_T^{s'}}]$$

The linear portion of the corrector is thereby easily controlled.

The nonlinear portion of the corrector is now considered:

$$|\int_\infty^{\frac{x_d}{\epsilon}} ([(1-\mathbb{E})M(\mathcal{U}_p^{0,n}, \partial_{\theta_0}\mathcal{U}_p^{0,n+1})]_{mod})_m(x, \theta_0 + \omega_m(\frac{x_d}{\epsilon} - s), s)ds|_{bH_T^{s'}}$$

$$\lesssim \sum_{i\neq m} |\int_\infty^{\frac{x_d}{\epsilon}} (\sigma^n_{i,p}\partial_{\theta_0}\sigma^{n+1}_{i,p})_p ds|_{bH_T^{s'}} + \sum_{i\neq j: i \text{ or } j\in\mathcal{P}\cup\mathcal{N}} |\int_\infty^{\frac{x_d}{\epsilon}} (\sigma^n_{i,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_e} ds|_{bH_T^{s'}}$$

$$+ \sum_{i\neq j: i \text{ and } j\in\mathcal{I}\cup\mathcal{O}} |\int_\infty^{\frac{x_d}{\epsilon}} (\sigma^n_{i,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_h} ds|_{bH_T^{s'}} \equiv (A) + (B) + (C)$$

Note that here $\sigma^n_{i,p}$ has argument $(x, \theta_0 + (\omega_i - \omega_m)s + \omega_m\frac{x_d}{\epsilon})$. Therefore the integrands in (A) have the common argument $(x, \theta_0 + \omega_i\frac{x_d}{\epsilon})$ and so a primitive can be considered, which is controlled as follows:

$$(A) \leq \sum_{i\neq m} p^{-1}|\sigma^n_{i,p}\partial_{\theta_0}\sigma^{n+1}_{i,p}|_{bH_T^{s'}} \lesssim \sum_{i\neq m} p^{-1}|\sigma^n_i|_{H_T^{s'}(x',\theta_0)}|\partial_{\theta_0}\sigma_i^{n+1}|_{bH_T^{s'}}$$

(Once more (0.144) was used, along with the algebra property of H_T^s, when $s > \frac{d+3}{2}$.)

Due to the presence of an elliptic profile factor in each term in (B), an analogous argument to the estimate of (3) for the elliptic corrector profiles may be used. This estimate is outlined in (0.161), Proposition 0.56, and the ensuing $L^2(H^s)$ estimates: Propositions 0.57 and 0.58.

From consideration of (0.120) and (0.121), it's clear that moment-zero approximations of products of transversal hyperbolic profiles, along with their primitives, must be properly defined and estimated.

Definition 0.50. *For $\sigma_p = \sigma_{i,p}$ and $\tau_p = \sigma_{j,p}$, define an ancillary function to σ_p, $\overline{\sigma}_p$, as:*

$$\overline{\sigma}_p(x, \theta) \equiv \sigma_p(x, \beta\theta + (1-\beta)(\theta_0 + \omega_i \xi_d))$$

where $\beta = (\omega_j - \omega_i)^{-1}$.

Notice that: $\overline{\sigma}_p(x, \theta_0 + \omega_j \xi_d) = \sigma_p(x, \theta_0 + \omega_i \xi_d)$.

Because $\overline{\sigma}_p$ and τ_p share the same argument $\theta_0 + \omega_j \xi_d$, the moment-zero approximation of the product $\overline{\sigma}_p \tau_p$, $(\overline{\sigma}_p \tau_p)_p(x, \theta)$ is well-defined by (0.43).

Definition 0.51. *The hyperbolic moment-zero approximation of a product of hyperbolic profiles $\sigma_p = \sigma_{i,p}$ and $\tau_p = \sigma_{j,p}$ is defined to be:*

$$(\sigma_p \tau_p)_{p_h}(x, \theta) \equiv (2\pi)^{\frac{-1}{2}} \beta^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_p(\kappa') e^{i\theta\kappa'} e^{it\frac{1-\beta}{\beta}(\theta_0 + \omega_i \xi_d)} \mathcal{F}_\theta \sigma_p(x, \frac{t}{\beta}) \mathcal{F}_\theta \tau_p(x, \kappa' - t) dt d\kappa'$$

Lemma 0.52. *The hyperbolic moment-zero approximation of a product of hyperbolic profiles agrees with the moment-zero approximation of the product involving the over-line definition:*

$$(\sigma_p\tau_p)_{p_h}(x,\theta) = (\overline{\sigma}_p\tau_p)_p(x,\theta) \tag{0.154}$$

<u>Proof:</u>

$$\mathcal{F}_\theta\overline{\sigma_p}(x,\kappa') = (2\pi)^{\frac{-1}{2}}\int_{\mathbb{R}} e^{-i\kappa'\theta}\sigma_p(x,\beta\theta+(1-\beta)(\theta_0+\omega_i\xi_d))d\theta$$

$$= (2\pi)^{\frac{-1}{2}}\beta^{-1}\int_{\mathbb{R}} e^{-i\frac{\kappa'}{\beta}(\theta-[(1-\beta)(\theta_0+\omega_i\xi_d)]}\sigma_p(x,\theta)d\theta = \beta^{-1}e^{i\kappa'(\frac{1-\beta}{\beta})(\theta_0+\omega_i\xi_d)}\mathcal{F}_\theta\sigma_p(x,\frac{\kappa'}{\beta})$$

$$\Rightarrow \mathcal{F}_\theta(\sigma_p\tau_p)_p(x,\kappa') = \chi_p(\kappa')\mathcal{F}_\theta\overline{\sigma_p} * \mathcal{F}_\theta\tau_p(x,\kappa')$$

$$= \chi_p(\kappa')(2\pi)^{\frac{-1}{2}}\int_{\mathbb{R}} \mathcal{F}_\theta\overline{\sigma_p}(x,t)\mathcal{F}_\theta\tau_p(x,\kappa'-t)dt$$

$$= (2\pi)^{\frac{-1}{2}}\beta^{-1}\int_{\mathbb{R}} \chi_p(\kappa')e^{it\frac{1-\beta}{\beta}(\theta_0+\omega_i\xi_d)}\mathcal{F}_\theta\sigma_p(x,\frac{t}{\beta})\mathcal{F}_\theta\tau_p(x,\kappa'-t)dt \quad \square$$

By (0.44), the primitive of $(\overline{\sigma}_p\tau_p)_p$ is well-defined. Furthermore, (0.52) allows the primitive to be written in the following integral form:

$$(\sigma_p\tau_p)^*_{p_h}(x,\theta) \equiv (\overline{\sigma}_p\tau_p)^*_p(x,\theta)$$

$$= (2\pi)^{\frac{-1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\chi_p(\kappa')}{i\beta\kappa}e^{i\theta\kappa'}e^{it(\frac{1-\beta}{\beta})(\theta_0+\omega_i\xi_d)}\mathcal{F}_\theta\sigma_p(x,\frac{t}{\beta})\mathcal{F}_\theta\tau_p(x,\kappa'-t)dtd\kappa' \tag{0.155}$$

Proposition 0.53.

$$|(\sigma_p\tau_p)^*_{p_h}|_{L^2_{\theta_0}} \lesssim p^{-1/2}|\sigma_p|_{L^2_{\theta_0}}|\tau_p|_{L^2_{\theta_0}} \tag{0.156}$$

Proof: Compute the Fourier transform w.r.t θ_0 of (0.155) as:

$$\mathcal{F}_{\theta_0}(\sigma_p\tau_p)^*_{p_h}(x,\kappa) = (2\pi i\beta)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\chi_p(\kappa')}{\kappa'}e^{-i\kappa\theta_0}e^{i\theta_0\kappa'}e^{it(\frac{1-\beta}{\beta})(\theta_0+\omega_i\frac{x_d}{\epsilon})}$$

$$\mathcal{F}_\theta(\sigma_p)(x,\frac{t}{\beta})\mathcal{F}_\theta(\tau_p)(x,\kappa'-t)\,dt\,d\kappa'\,d\theta_0$$

$$= (2\pi i\beta)^{-1}\frac{\chi_p(\kappa)}{\kappa}\int_{\mathbb{R}} e^{it(\frac{1-\beta}{\beta})(\theta_0+\omega_i\frac{x_d}{\epsilon})}\mathcal{F}_\theta(\sigma_p)(x,\frac{t}{\beta})\mathcal{F}_\theta(\tau_p)(x,\kappa-t)\,dt$$

Thereby:

$$|\mathcal{F}_{\theta_0}(0.155)| \lesssim |\mathcal{F}_\theta(\sigma_p)(x,\frac{t}{\beta})|_{L^2_t}|\frac{\chi_p(\kappa)}{\kappa}\mathcal{F}_\theta(\tau_p)(x,\kappa-t)|_{L^2_t}$$

$$= |\mathcal{F}_\theta(\sigma_p)|_{L^2_\kappa}|\frac{\chi_p(t'+t)}{(t'+t)}\mathcal{F}_\theta(\tau_p)(x,t')|_{L^2_{t'}} \leq (\sup_t \frac{\chi_p(t'+t)}{(t'+t)})|\mathcal{F}_\theta(\sigma_p)|_{L^2_\kappa}|\mathcal{F}_\theta(\tau_p)|_{L^2_\kappa}$$

$$\Rightarrow\ |\mathcal{F}_{\theta_0}(0.155)|_{L^2_\kappa} \lesssim \sqrt{\int_{\mathbb{R}}\sup_t \frac{\chi_p(t'+t)^2}{(t'+t)^2})dt'}\,|\sigma_p|_{L^2_{\theta_0}}|\tau_p|_{L^2_{\theta_0}}$$

Claim: $\sup_t \int_{\mathbb{R}} \frac{\chi_p^2(t'+t)}{(t'+t)^2}dt' \lesssim p^{-1}$

Proof of Claim: Fix t. Then:

$$\int_{\mathbb{R}} \frac{\chi_p^2(t'+t)}{(t'+t)^2}dt' \leq \int_{-\infty}^{-p-t}\frac{dt'}{(t'+t)^2} + \int_{p-t}^{\infty}\frac{dt'}{(t'+t)^2}$$

$$= \frac{-1}{t'+t}|_{-\infty}^{-p-t} + \frac{-1}{t'+t}|_{p-t}^{\infty} = \frac{-1}{(-p-t)+t} + \frac{1}{(p-t)+t} = 2p^{-1} \quad \square$$

Thus: $|(\sigma_p\tau_p)^*_{p_h}|^2_{L_{\theta_0}} \lesssim p^{-1/2}|\sigma_p|_{L^2_{\theta_0}}|\tau_p|_{L^2_{\theta_0}}$. $\square$

The following norm relation is useful in estimating the $bH^{s'}_T$ norm of the transversal hyperbolic interactions:

$$|\cdot|_{bH^s_T} \lesssim |\cdot|_{L^2_T(x',H^s(\theta_0))} + |\cdot|_{L^2(\theta_0,H^s_T(x'))} \tag{0.157}$$

The following two estimates complete the estimate of (C), which in turn completes the estimation of the hyperbolic components of the corrector $\mathcal{U}^1_p$.

Compute the $L^2(\theta_0, H^{s'}_T(x'))$ estimate of $(\sigma^n_{i,p}\partial_\theta\sigma^{n+1}_{j,p})^*_{p,\epsilon}(x, \theta_0 + \omega_j\frac{x_d}{\epsilon})$ for some $i \neq j$: i and $j \in \mathcal{I} \cup \mathcal{O}$:

Proposition 0.54.

$$|(\sigma^n_{i,p}\partial_\theta\sigma^{n+1}_{j,p})^*_{p_h}(x, \theta_0 + \omega_j\frac{x_d}{\epsilon})|_{L^2(\theta_0,H^{s'}_T(x'))} \lesssim p^{-1/2}|\sigma^n_{i,p}|_{bH^{s'}_T}|\partial_\theta\sigma^{n+1}_{j,p}|_{bH^{s'}_T}$$

Proof:

$$|(\sigma^n_{i,p}\partial_\theta\sigma^{n+1}_{j,p})^*_{p_h}(x, \theta_0 + \omega_j\frac{x_d}{\epsilon})|_{L^2(\theta_0,H^{s'}_T(x'))} = |(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\chi_p(\kappa')}{i\beta\kappa'}e^{i(\theta_0+\omega_j\frac{x_d}{\epsilon})\kappa'}e^{it(\frac{1-\beta}{\beta})(\theta_0+\omega_i\frac{x_d}{\epsilon})}$$

$$\mathcal{F}_\theta(\sigma^n_{i,p})(x,\frac{t}{\alpha})\mathcal{F}_\theta(\partial_{\theta_0}\sigma^{n+1}_{j,p})(x,\kappa'-t)\,dt\,d\kappa'|_{L^2(\theta_0,H^{s'}_T(x'))}$$

$$\lesssim |p^{-1/2}|\sigma^n_{i,p}|_{L^2(\theta_0)}|\partial_\theta\sigma^{n+1}_{j,p}|_{L^2(\theta_0)}\,|_{H^{s'}_T(x')} \quad \textit{(by Proposition 0.53)}$$

$$\lesssim p^{-1/2}|\sigma^n_{i,p}|_{H^{s'}_T(x',L^2(\theta_0))}|\partial_\theta\sigma^{n+1}_{j,p}|_{H^{s'}_T(x',L^2(\theta_0))}$$

$$\lesssim p^{-1/2}|\sigma^n_{i,p}|_{bH^{s'}_T}|\partial_\theta\sigma^{n+1}_{j,p}|_{bH^{s'}_T} \quad \square$$

Compute the $L^2_T(x', H^{s'}(\theta_0))$ estimate of $(\sigma^n_{i,p}\partial_\theta\sigma^{n+1}_{j,p})^*_{p,\epsilon}(x, \theta_0 + \omega_j \frac{x_d}{\epsilon})$ for some $i \neq j$: i and $j \in \mathcal{I} \cup \mathcal{O}$:

Proposition 0.55. *For $s_0 > \dfrac{d}{2}$:*

$$|\kappa^\beta \mathcal{F}_{\theta_0}[(\sigma^n_{i,p}\partial_\theta\sigma^{n+1}_{j,p})^*_{p_h}(x, \theta_0 + \omega_j \frac{x_d}{\epsilon})]|_{L^2(x',\kappa)}$$

$$\lesssim p^{-1/2}(|\sigma^n_{i,p}|_{bH^{s'}_T}|\partial_\theta\sigma^{n+1}_{j,p}|_{bH^{s_0}_T} + |\sigma^n_{i,p}|_{bH^{s_0}_T}|\partial_\theta\sigma^{n+1}_{j,p}|_{bH^{s'}_T})$$

Proof:

$$|\kappa^\beta \mathcal{F}_{\theta_0}[(\sigma^n_{i,p}\partial_\theta\sigma^{n+1}_{j,p})^*_{p_h}(x, \theta_0 + \omega_j \frac{x_d}{\epsilon})]|_{L^2(x',\kappa)} = |\kappa^\beta \mathcal{F}_{\theta_0}[(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\chi_p(\kappa')}{i\beta\kappa'}$$

$$e^{i(\theta_0+\omega_j\frac{x_d}{\epsilon})\kappa'} e^{it(\frac{1-\beta}{\beta})(\theta_0+\omega_i\frac{x_d}{\epsilon})} \mathcal{F}_\theta(\sigma^n_{i,p})(x, \frac{t}{\alpha}) \mathcal{F}_\theta(\partial_\theta\sigma^{n+1}_{j,p})(x, \kappa' - t)\, dt\, d\kappa']|_{L^2(\kappa,x')}$$

Use the inequality $|\kappa|^\beta \lesssim |t|^\beta + |\kappa - t|^\beta$ to modify Proposition 0.53, in order to estimate $|\kappa^\beta(\sigma_p\tau_p)^*_p|^2_{L_{\theta_0}}$, so that:

$$|\kappa^\beta(\sigma_p\tau_p)^*_{p_h}|^2_{L_{\theta_0}} \leq 2p^{-1/2}(|\sigma_p|_{H^{s'}(\theta_0)}|\tau_p|_{L^2_{\theta_0}} + |\sigma_p|_{L^2_{\theta_0}}|\tau_p|_{H^{s'}(\theta_0)})$$

(Note: this is analogous to the use of this inequality in Proposition 0.58.) Thereby:

$$|\kappa^\beta \mathcal{F}_{\theta_0}(\sigma_p\tau_p)^*_{p_h}|_{L^2(x',\kappa)}$$

$$\lesssim |p^{-1/2}(|\sigma^n_{i,p}|_{H^{s'}(\theta_0)}|\partial_\theta\sigma^{n+1}_{j,p}|_{L^2_{\theta_0}} + |\sigma^n_{i,p}|_{L^2_{\theta_0}}|\partial_\theta\sigma^{n+1}_{j,p}|_{H^{s'}(\theta_0)})|_{L^2_T(x')}$$

$$\lesssim p^{-1/2}(|\sigma^n_{i,p}|_{L^2(x',H^{s'}(\theta_0))}|\partial_\theta\sigma^{n+1}_{j,p}|_{L^\infty(x',L^2_{\theta_0})} + |\sigma^n_{i,p}|_{L^\infty(x',L^2_{\theta_0})}|\partial_\theta\sigma^{n+1}_{j,p}|_{L^2(x',H^{s'}(\theta_0))})$$

$$\lesssim p^{-1/2}(|\sigma^n_{i,p}|_{bH^{s'}_T}|\partial_\theta\sigma^{n+1}_{j,p}|_{bH^{s_0}_T} + |\sigma^n_{i,p}|_{bH^{s_0}_T}|\partial_\theta\sigma^{n+1}_{j,p}|_{bH^{s'}_T}) \quad \square$$

Now $|\tau_{m,p}|_{bH^{s'}_T}$ will be estimated when $m \in \mathcal{P} \cup \mathcal{N}$:

Consider $m \in \mathcal{P}$; the other case $m \in \mathcal{N}$ is completely analogous.

$$
\begin{aligned}
|\tau_{m,p}|_{bH^{s'}_T} = |\mathbb{1}_{\{\kappa<0\}} \int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)}([(I-\mathbb{E})\mathcal{G}_p]_{mod} - F(0)\mathcal{U}^{0,n}_p)\hat{\ }_m(x,\kappa,s)\,ds|_{H^{s'}_T(x',\kappa)} \\
\lesssim \sum_{i\neq m} |\mathbb{1}_{\{\kappa<0\}} \int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)} e^{i\omega_i\kappa s}\sigma^{n+1,\hat{\ }}_{i,p}\,ds|_{H^{s'}_T(x',\kappa)} \\
+ \sum_{i\neq m} |\mathbb{1}_{\{\kappa<0\}} \int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)} g^n_{m,i,p}(x,\kappa,s)\,ds|_{H^{s'}_T(x',\kappa)} \\
+ \sum_{i\neq j:i \text{ or } j\in\mathcal{P}\cup\mathcal{N}} |\mathbb{1}_{\{\kappa<0\}} \int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)} h^n_{m,i,j,p}(x,\kappa,s)ds|_{H^{s'}_T(x',\kappa)} \\
+ \sum_{i\neq j:i \text{ and } j\in\mathcal{I}\cup\mathcal{O}} |\mathbb{1}_{\{\kappa<0\}} \int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)} \mathcal{F}_{\theta_0}\,(\sigma^n_{i,p}\partial_{\theta_0}\sigma^{n+1}_{j,p})_{p_h}\,ds|_{H^{s'}_T(x',\kappa)} \\
+ \sum_{i\neq m} |\mathbb{1}_{\{\kappa<0\}} \int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)} e^{i\omega_i\kappa s}\sigma^{n,\hat{\ }}_{i,p}\,ds|_{H^{s'}_T(x',\kappa)} \\
\equiv (1)+(2)+(3)+(4)+(5)
\end{aligned}
\tag{0.158}
$$

Here, analogously to the definitions in Chapter 2:

$$
\begin{aligned}
g^n_{m,i,p}(x,\kappa,s) &\equiv c^m_i\chi_p(\kappa)\int_{\mathbb{R}} e^{i\omega_i(\kappa-t)s}\sigma^{n,\hat{\ }}_{i,p}(x,\kappa-t)e^{i\omega_i ts}(\partial_{\theta_i}\sigma^{n+1}_{i,p})\hat{\ }(x,t)dt \\
&= c^m_i\chi_p(\kappa)e^{i\omega_i\kappa s}\sigma^{n,\hat{\ }}_{i,p} * (\partial_{\theta_i}\sigma^{n+1}_{i,p})\hat{\ }(x,\kappa)
\end{aligned}
\tag{0.159}
$$

$$
h^n_{m,i,j,p}(x,\kappa,s) \equiv d^m_{i,j}\chi_p(\kappa)\int_{\mathbb{R}} e^{i\omega_i(\kappa-t)s}\sigma^{n,\hat{\ }}_{i,p}(x,\kappa-t)e^{i\omega_j ts}(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{\ }(x,t)dt \tag{0.160}
$$

The 5 parts of (0.158) are estimated separately (for ease of notation the sum over $i \neq m$ or $i \neq j$ is suppressed):

Because:

$$|\mathbb{1}_{\{\kappa<0\}}\int_{\infty}^{\frac{x_d}{\epsilon}} e^{i\omega_m\kappa(\frac{x_d}{\epsilon}-s)}\mathcal{F}_{\theta_0}\left(\sigma_{i,p}^n\partial_{\theta_0}\sigma_{j,p}^{n+1}\right)_{p_h}ds|_{H_T^{s'}(x',\kappa)} \lesssim |\mathcal{F}_{\theta_0}\left(\sigma_{i,p}^n\partial_{\theta_0}\sigma_{j,p}^{n+1}\right)_{p_h}^*|_{H_T^{s'}(x',\kappa)}$$

(4) is estimated in the same way that the transversal hyperbolic interactions present in the hyperbolic corrector profiles were estimated.

$$\begin{aligned}
(1) &= |\mathbb{1}_{\{\kappa<0\}}e^{i\omega_m\kappa\frac{x_d}{\epsilon}}\sigma_{i,p}^{n+1,\hat{}}(x,\kappa)\int_{\infty}^{\frac{x_d}{\epsilon}} e^{i(\omega_i-\omega_m)\kappa s}ds|_{H_T^{s'}(x',\kappa)} \\
&= |\mathbb{1}_{\{\kappa<0\}}\frac{e^{i\omega_i\kappa\frac{x_d}{\epsilon}}}{i(\omega_i-\omega_m)\kappa}\sigma_{i,p}^{n+1,\hat{}}(x,\kappa)|_{H_T^{s'}(x',\kappa)} \lesssim |\frac{\sigma_{i,p}^{n+1,\hat{}}(x,\kappa)}{i\kappa}|_{H_T^{s'}(x',\kappa)} \\
&= |\sigma_{i,p}^{n+1,*}|_{bH_T^{s'}} \leq p^{-1}|\sigma_i^{n+1}|_{bH_T^{s'}}
\end{aligned}$$

Note that this same estimate works for (5).

$$\begin{aligned}
(2) &\lesssim |\mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)e^{i\omega_m\kappa\frac{x_d}{\epsilon}}\sigma_{i,p}^{n,\hat{}}*(\partial_{\theta_i}\sigma_{i,p}^{n+1})\hat{}(x,\kappa)\int_{\infty}^{\frac{x_d}{\epsilon}} e^{i(\omega_i-\omega_m)\kappa s}ds|_{H_T^{s'}(x',\kappa)} \\
&= |\mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)\frac{e^{i\omega_i\kappa\frac{x_d}{\epsilon}}}{i(\omega_i-\omega_m)\kappa}\sigma_{i,p}^{n,\hat{}}*(\partial_{\theta_i}\sigma_{i,p}^{n+1})\hat{}(x,\kappa)|_{H_T^{s'}(x',\kappa)} \\
&\lesssim |\chi_p(\kappa)\frac{[\sigma_{i,p}^n(\partial_{\theta_i}\sigma_{i,p}^{n+1})]\hat{}(x,\kappa)}{i\kappa}|_{H_T^{s'}(x',\kappa)} \\
&= |[\sigma_{i,p}^n(\partial_{\theta_i}\sigma_{i,p}^{n+1})]^*|_{bH_T^{s'}} \leq p^{-1}|\sigma_{i,p}^n(\partial_{\theta_i}\sigma_{i,p}^{n+1})|_{bH_T^{s'}} \lesssim p^{-1}|\sigma_{i,p}^n|_{bH_T^{s'}}|(\partial_{\theta_i}\sigma_{i,p}^{n+1})|_{bH_T^{s'}}
\end{aligned}$$

$$\begin{aligned}
(3) &\lesssim |\mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)e^{i\omega_m\kappa\frac{x_d}{\epsilon}}\int_{\mathbb{R}}\sigma_{i,p}^{n,\hat{}}(x,\kappa-t)(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t) \\
&\quad \left(\int_{\infty}^{\frac{x_d}{\epsilon}} e^{is[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]}ds\right)dt|_{H_T^{s'}(x',\kappa)} \\
&= |\mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)\int_{\mathbb{R}}\sigma_{i,p}^{n,\hat{}}(x,\kappa-t)(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)\left(\frac{e^{i\frac{x_d}{\epsilon}[\omega_i(\kappa-t)+\omega_j t]}}{i[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]}\right)dt|_{H_T^{s'}(x',\kappa)}
\end{aligned} \tag{0.161}$$

To estimate (0.161), the norm relation (0.157) will again be utilized. But first the modulus of the integrand of (0.161) will be estimated by the following proposition:

Proposition 0.56.

$$|\mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)\int_{\mathbb{R}} \sigma_{i,p}^{n,\hat{}}(x,\kappa-t)(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)\left(\frac{e^{i\frac{x_d}{\epsilon}[\omega_i(\kappa-t)+\omega_j t]}}{i[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]}\right)dt|$$

$$\lesssim |\kappa|^{-1}|\sigma_{i,p}^{n,\hat{}}| * |(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,\kappa)| \tag{0.162}$$

Proof: Write the integrand as:

$$\mathcal{H}_{i,j}(x,\kappa) \equiv \mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)\int_{\mathbb{R}} \Omega_{i,j}(x,\kappa,t)H_{i,j}(x_d,\kappa,t)\,dt$$

$$\text{where:}\quad \Omega_{i,j}(x,\kappa,t) \equiv \sigma_{i,p}^{n,\hat{}}(x,\kappa-t)(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t),$$

$$H_{i,j}(x_d,\kappa,t) \equiv \frac{\mathbb{1}_{\{\kappa<0\}}\chi_p(\kappa)e^{i\frac{x_d}{\epsilon}[\omega_i(\kappa-t)+\omega_j t]}}{i[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]}$$

So that:

$$|\mathcal{H}_{i,j}(x,\kappa)| \lesssim \int_{\mathbb{R}} |\Omega_{i,j}(x,\kappa,t)H_{i,j}(x_d,\kappa,t)|\,dt \le |H_{i,j}(x_d,\kappa)|_{L_t^\infty}|\Omega_{i,j}|_{L_t^1}$$

To finish the estimate, the various cases for the indices m,i,j must be considered. One case is $i,m\in\mathcal{P}$, $j\in\mathcal{N}$, considered in the following:

$$|i[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]| \ge |\mathrm{Im}[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]| = \mathrm{Im}\,\omega_i(\kappa-t)-\mathrm{Im}\,\omega_m\kappa+\mathrm{Im}\,\omega_j t$$

$$\ge \mathrm{Im}\,\omega_j t-\mathrm{Im}\,\omega_m\kappa \ge (\mathrm{Im}\,\omega_j-\mathrm{Im}\,\omega_m)\kappa = (\mathrm{Im}\,\omega_m-\mathrm{Im}\,\omega_j)|\kappa|$$

$$\Rightarrow\quad |i[(\omega_i-\omega_m)\kappa+(\omega_j-\omega_i)t]|^{-1} \lesssim |\kappa|^{-1}$$

Notice that $(\mathrm{Im}\,\omega_m-\mathrm{Im}\,\omega_j)>0$ can't vanish if $i\neq m$.

If $i = m$, then the above estimate still holds, by noting that: $|i[(\omega_i - \omega_m)\kappa + (\omega_j - \omega_i)t]| \geq (\mathrm{Im}\,\omega_j - \mathrm{Im}\,\omega_m)t \geq (\mathrm{Im}\,\omega_m - \mathrm{Im}\,\omega_j)|\kappa|$

Thus: $|e^{\frac{x_d}{\epsilon}\alpha}| = e^{\frac{x_d}{\epsilon}(-\mathrm{Im}\,\omega_i(\kappa-t)-\mathrm{Im}\,\omega_j t)} \leq 1 \Rightarrow |\widetilde{H}_{i,j}(\xi_d,\kappa)|_{L^\infty_t} \lesssim |\kappa|^{-1} \quad \square$

Use Proposition 0.56 to compute the $L^2(\theta_0, H^{s'}_T(x'))$ estimate of $\mathcal{H}_{i,j}$:

Proposition 0.57.

$$|(0.161)|_{L^2(\kappa, H^{s'}_T(x'))} \lesssim p^{-1/2}|\sigma^{n,\hat{}}_{i,p}|_{bH^{s'}_T}|(\partial_{\theta_i}\sigma^{n+1}_{j,p})|_{bH^{s'}_T} \tag{0.163}$$

Proof:

$$|(0.161)|_{L^2(\kappa,H^{s'}_T(x'))} \leq [\int_p^\infty |\int_{\mathbb{R}} \sigma^{n,\hat{}}_{i,p}(x,\kappa-t)(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)H_{i,j}(\frac{x_d}{\epsilon},\kappa,t)\,dt\,|^2_{H^{s'}_T(x')}\,d\kappa]^{1/2} \tag{0.164}$$

$$\lesssim [\int_p^\infty |\,|\kappa|^{-1}\int_{\mathbb{R}} |\sigma^{n,\hat{}}_{i,p}(x,\kappa-t)(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|\,dt\,|^2_{H^{s'}_T(x')}\,d\kappa]^{1/2} \quad (by\,(0.162))$$

$$\lesssim [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |\sigma^{n,\hat{}}_{i,p}(x,\kappa-t)(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{H^{s'}_T(x')}\,dt]^2\,d\kappa]^{1/2}$$

$$\lesssim [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |\sigma^{n,\hat{}}_{i,p}(x,\kappa-t)|_{H^{s'}_T(x')}|(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{H^{s'}_T(x')}\,dt]^2\,d\kappa]^{1/2} \quad (H^{s'}_T \textit{ is an algebra})$$

$$\leq |\sigma^{n,\hat{}}_{i,p}|_{L^2(\kappa,H^{s'}_T(x'))}|(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}|_{L^2(\kappa,H^{s'}_T(x'))}[\int_p^\infty \kappa^{-2}d\kappa]^{1/2}$$

$$\leq p^{-1/2}|\sigma^{n,\hat{}}_{i,p}|_{bH^{s'}_T}|(\partial_{\theta_i}\sigma^{n+1}_{j,p})|_{bH^{s'}_T} \quad \square$$

Also use Proposition 0.56 to compute the $L^2_T(x', H^{s'}(\theta_0))$ estimate of $\mathcal{H}_{i,j}$:

Proposition 0.58.

$$|\kappa^\beta(0.161)|_{L^2_T(x',\kappa)} \lesssim p^{-1/2}|\sigma^n_{i,p}|_{bH^{s_0}}\,|\sigma^{n+1}_{j,p}|_{bH^{s'+1}_T} \tag{0.165}$$

<u>*Proof:*</u>

$$|\kappa^\beta(0.161)|_{L^2_T(x',\kappa)} = |\int_{\mathbb{R}} \kappa^\beta \sigma^{n,\hat{}}_{i,p}(x,\kappa-t)(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)H_{i,j}(\frac{x_d}{\epsilon},\kappa,t)dt\,|_{L^2_T(x',\kappa)}$$

$$\lesssim [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |\kappa^\beta \sigma^{n,\hat{}}_{i,p}(x,\kappa-t)(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{L^2_T(x')}\,dt]^2\,d\kappa]^{1/2} \lesssim (1)+(2)$$

$$(1) = [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |(\kappa-t)^\beta \sigma^{n,\hat{}}_{i,p}(x,\kappa-t)(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{L^2_T(x')}\,dt]^2\,d\kappa]^{1/2}$$

$$\leq [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |(\kappa-t)^\beta \sigma^{n,\hat{}}_{i,p}(x,\kappa-t)|_{L^2_T(x')}\,|(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{L^\infty_T(x')}\,dt]^2\,d\kappa]^{1/2}$$

$$\leq [\int_p^\infty \kappa^{-2}\,|(\kappa-t)^\beta \sigma^{n,\hat{}}_{i,p}(x,\kappa-t)|^2_{L^2(t,L^2_T(x'))}\,|(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|^2_{L^2(t,L^\infty_T(x'))}\,d\kappa]^{1/2}$$

$$= |\kappa^\beta \sigma^{n,\hat{}}_{i,p}|_{L^2(x',\kappa)}\,|(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{L^2(\kappa,L^\infty_T(x'))}\,[\int_p^\infty \kappa^{-2}d\kappa]^{1/2}$$

$$\lesssim p^{-1/2}|\kappa^\beta \sigma^{n,\hat{}}_{i,p}|_{L^2(x',\kappa)}\,|(\partial_{\theta_i}\sigma^{n+1}_{j,p})\hat{}(x,t)|_{L^2(\kappa,H^{s_0}_T(x'))} \qquad (\textit{for } s_0 > \frac{d}{2})$$

$$\leq p^{-1/2}|\sigma^n_{i,p}|_{bH^{s'}}\,|\sigma^{n+1}_{j,p}|_{bH^{s_0+1}_T}$$

$$(2) = [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |t^\beta \sigma_{i,p}^{n,\hat{}}(x,\kappa-t)(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)|_{L^2_T(x')}\,dt]^2\,d\kappa]^{1/2}$$

$$\leq [\int_p^\infty \kappa^{-2}[\int_{\mathbb{R}} |\sigma_{i,p}^{n,\hat{}}(x,\kappa-t)|_{L^\infty_T(x')}\,|t^\beta(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)|_{L^2_T(x')}\,dt]^2\,d\kappa]^{1/2}$$

$$\leq [\int_p^\infty \kappa^{-2}\,|\sigma_{i,p}^{n,\hat{}}(x,\kappa-t)|_{L^2(t,L^\infty_T(x'))}\,|t^\beta(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)|_{L^2(t,L^2_T(x'))}\,d\kappa]^{1/2}$$

$$= |\sigma_{i,p}^{n,\hat{}}|_{L^2(\kappa,L^\infty_T(x'))}\,|t^\beta(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)|_{L^2(x',\kappa)}\,[\int_p^\infty \kappa^{-2}d\kappa]^{1/2}$$

$$\lesssim p^{-1/2}|\sigma_{i,p}^{n,\hat{}}|_{L^2(\kappa,H^{s_0}_T(x'))}\,|t^\beta(\partial_{\theta_i}\sigma_{j,p}^{n+1})\hat{}(x,t)|_{L^2(x',\kappa)} \qquad (for\ s_0 > \frac{d}{2})$$

$$\leq p^{-1/2}|\sigma_{i,p}^{n}|_{bH^{s_0}_T}\,|\sigma_{j,p}^{n+1}|_{bH^{s'+1}_T} \quad \square$$

The estimate of: $|\mathbb{1}_{\{\kappa>0\}}[\int_0^{\xi_d} e^{i\omega_m\kappa(\xi_d-s)}([(I-\mathbb{E})\mathcal{G}_p]_{mod} - F(0)\mathcal{U}_p^{0,n})\hat{}_m(x,\kappa,s)ds]|_{H^{s'}_T(x',\kappa)}$ is directly analogous to the estimate already proven for (0.158); therefore that estimate is omitted.

The estimates of this appendix have shown that:

$$
\begin{aligned}
|\epsilon \mathcal{U}^1_{p,\epsilon}|_{bH^{s'}_T} \quad &\lesssim \quad \frac{\epsilon}{p}|\mathcal{U}^{0,n+1}_\epsilon|_{bH^{s'}_T} + \frac{\epsilon}{p}|\mathcal{U}^{0,n}_\epsilon|_{bH^{s'}_T}|\mathcal{U}^{0,n+1}|_{bH^{s'+1}_T} \\
&+\frac{\epsilon}{\sqrt{p}}\,|\mathcal{U}^{0,n}_\epsilon|_{bH^{s'}_T}|\mathcal{U}^{0,n+1}_\epsilon|_{bH^{s'+1}_T} + \frac{\epsilon}{p}|\mathcal{U}^{0,n}_\epsilon|_{bH^{s'}_T} \qquad (0.166)
\end{aligned}
$$

Applying the $L^2_{x_d}$ and C_{x_d} norms:

$$
\begin{aligned}
\sup_{x_d\geq 0}(|\mathcal{U}^{0,n}_\epsilon|_{bH^{s'}_T}\,|\mathcal{U}^{0,n+1}_\epsilon|_{bH^{s'+1}_T}) &\leq (\sup_{x_d\geq 0}|\mathcal{U}^{0,n}_\epsilon|_{bH^{s'}_T})\,(\sup_{x_d\geq 0}|\mathcal{U}^{0,n+1}_\epsilon|_{bH^{s'+1}_T}) \\
&\leq |\mathcal{U}^{0,n}_\epsilon|_{E^{s'}_T}\,|\mathcal{U}^{0,n+1}_\epsilon|_{E^{s'+1}_T} \\
|\,|\mathcal{U}^{0,n}_\epsilon|_{bH^{s'+1}_T}|\mathcal{U}^{0,n+1}_\epsilon|_{bH^{s'+2}_T}|_{L^2(x_d)} &\leq |\mathcal{U}^{0,n}_\epsilon|_{L^\infty(x_d,bH^{s'+1}_T)}\,|\mathcal{U}^{0,n+1}_\epsilon|_{L^2(x_d,bH^{s'+2}_T)} \\
&\leq |\mathcal{U}^{0,n}_\epsilon|_{E^{s'+1}_T}\,|\mathcal{U}^{0,n+1}_\epsilon|_{E^{s'+1}_T}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&|\epsilon \mathcal{U}^1_{p,\epsilon}|_{E^{s-2}_T} \lesssim \\
&\frac{\epsilon}{p}|\mathcal{U}^{0,n+1}_\epsilon|_{E^{s-1}_T} + \frac{\epsilon}{p}|\mathcal{U}^{0,n}_\epsilon|_{E^{s-1}_T}|\mathcal{U}^{0,n+1}_\epsilon|_{E^{s-1}_T} \\
&+\frac{\epsilon}{\sqrt{p}}\,|\mathcal{U}^{0,n}_\epsilon|_{E^{s-1}_T}|\mathcal{U}^{0,n+1}_\epsilon|_{E^{s-1}_T} + \frac{\epsilon}{p}|\mathcal{U}^{0,n}_\epsilon|_{E^{s-1}_T} \qquad (0.167)
\end{aligned}
$$

Uniform boundedness of the iterates $\mathcal{U}^{0,n}$ yields the desired estimate (0.127). $\square$

APPENDIX F: SINGULAR PSEUDO DIFFERENTIAL CALCULUS

This appendix summarizes and slightly modifies the singular pseudo-differential calculus of [4]; this appendix closely resembles Appendix A of [5].

First the singular Sobolev spaces used to describe mapping properties are defined. The variable in $\mathbb{R}^{d+1}$ is denoted (x, θ) ($x \in \mathbb{R}^d$, $\theta \in \mathbb{R}$), with associated frequency (ξ, κ). In this new context, the singular Sobolev spaces are defined as follows:

Consider a vector $\beta \in \mathbb{R}^d \setminus \{0\}$. Then for $s \in \mathbb{R}$ and $\epsilon \in [0,1]$, the anisotropic Sobolev space $H^{s,\epsilon}(\mathbb{R}^{d+1})$ is defined as:

$$H^{s,\epsilon}(\mathbb{R}^{d+1}) \equiv \Big\{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) \,/\, \hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^{d+1})$$

$$\text{and} \quad \int_{\mathbb{R}^{d+1}} \left(1 + |\xi + \frac{k\,\beta}{\epsilon}|^2\right)^s |\hat{u}(\xi,k)|^2 \, d\xi \, d\kappa < \infty \Big\}$$

Here $\hat{u}$ denotes the Fourier transform of u on $\mathbb{R}^{d+1}$. The space $H^{s,\epsilon}(\mathbb{R}^{d+1})$ is equipped with the following family of norms: $\forall\, \gamma \geq 1$, $\forall\, u \in H^{s,\epsilon}(\mathbb{R}^{d+1})$:

$$|u|^2_{H^{s,\epsilon},\gamma} \equiv \frac{1}{(2\,\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \left(\gamma^2 + |\xi + \frac{k\,\beta}{\epsilon}|^2\right)^s \left|\hat{u}(\xi,k)\right|^2 d\xi \, dk \,.$$

When $m \in \mathbb{Z}$, the space $H^{m,\epsilon}(\mathbb{R}^{d+1})$ coincides with the space of functions $u \in L^2(\mathbb{R}^{d+1})$ such that the derivatives:

$$\left(\partial_{x_1} + \frac{\beta_1}{\epsilon}\,\partial_\theta\right)^{\alpha_1} \cdots \left(\partial_{x_d} + \frac{\beta_d}{\epsilon}\,\partial_\theta\right)^{\alpha_d} u \,, \quad \alpha_1 + \cdots + \alpha_d \leq m \,,$$

belong (in the sense of distributions) to $L^2(\mathbb{R}^{d+1})$. In the definition of the norm $|\cdot|_{H^{m,\epsilon},\gamma}$, one power of γ counts as much as one derivative.

The singular symbols are built from the following sets of classical symbols:

Definition 0.59. *Let $O \subset \mathbb{R}^N$ be an open subset which contains the origin. For $m \in \mathbb{R}$ let $\boldsymbol{S}^m(O)$ denote the class of all functions $\sigma : O \times \mathbb{R}^d \times [1, \infty) \to \mathbb{C}^M$ ($M \geq 1$), such that σ is*

C^∞ on $O \times \mathbb{R}^d$ and for all compact sets $K \subset O$:

$$\sup_{v \in K} \sup_{\xi \in \mathbb{R}^d} \sup_{\gamma \geq 1} (\gamma^2 + |\xi|^2)^{-(m-|\nu|)/2} \, |\partial_v^\alpha \partial_\xi^\nu \sigma(v, \xi, \gamma)| \leq C_{\alpha,\nu,K}.$$

Let $\mathcal{C}_b^k(\mathbb{R}^{d+1})$ $(k \in \mathbb{N})$, denote the space of continuous and bounded functions on $\mathbb{R}^{d+1}$, whose derivatives up to order k are continuous and bounded. Next define the singular symbols:

Definition 0.60. *Fix $\beta \in \mathbb{R}^d \setminus 0$, and let $m \in \mathbb{R}$, $n \in \mathbb{N}$. Then denote S_n^m as the set of families of functions $(a_{\epsilon,\gamma})_{\epsilon \in [0,1], \gamma \geq 1}$, which are constructed as follows:*

$$\forall \, (x, \theta, \xi, \kappa) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} : \quad a_{\epsilon,\gamma}(x, \theta, \xi, \kappa) \equiv \sigma\left(\epsilon \, V(x, \theta), \xi + \frac{k \, \beta}{\epsilon}, \gamma\right) \tag{0.168}$$

where $\sigma \in \boldsymbol{S}^m(O)$ and V belongs to the space $\mathcal{C}_b^n(\mathbb{R}^{d+1})$. Furthermore V takes its values in a convex compact subset K of O that contains the origin. (For instance, K can be a closed ball centered round the origin.)

Remark 0.61. *The results that follow can be extended to the case where, in place of a function V that's independent of ϵ, the representation of $a_{\epsilon,\gamma}$ in Definition 0.60 is considered for a function V_ϵ that's indexed by ϵ, provided that all functions ϵV_ϵ are assumed to take values in a fixed convex compact subset K of O that contains the origin, and that $(V_\epsilon)_{\epsilon \in [0,1]}$ is a bounded family of $\mathcal{C}_b^n(\mathbb{R}^{d+1})$. Such singular symbols with a function V_ϵ are exactly those utilized in the construction of the exact solution for (0.3).*

To each symbol $a = (a_{\epsilon,\gamma})_{\epsilon \in [0,1], \gamma \geq 1} \in S^m_n$ specified in Definition 0.60 and with values in $\mathbb{C}^{N \times N}$, associate a singular pseudodifferential operator $\mathrm{Op}^{\epsilon,\gamma}(a)$, where $\epsilon \in [0,1], \gamma \geq 1$, whose action on a function $u \in \mathcal{S}(\mathbb{R}^{d+1}, \mathbb{C}^N)$ is defined by:

$$\mathrm{Op}^{\epsilon,\gamma}(a)\, u\,(x,\theta) \equiv \frac{1}{(2\,\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} e^{i\,(\xi\cdot x + \kappa\,\theta)}\, \sigma\left(\epsilon\, V(x,\theta), \xi + \frac{\kappa\,\beta}{\epsilon}, \gamma\right)\, \hat{u}(\xi,\kappa)\, d\xi\, d\kappa \qquad (0.169)$$

Note that for the Fourier multiplier $\sigma(v,\xi,\gamma) = i\,\xi_1$, the corresponding singular operator is $\partial_{x_1} + \frac{\beta_1}{\epsilon}\,\partial_\theta$. The action of singular pseudo-differential operators on Sobolev spaces is now described.

Proposition 0.62. *Let $n \geq d+1$, and let $a \in S^m_n$ with $m \leq 0$. Then $Op^{\epsilon,\gamma}(a)$ in (0.169) defines a bounded operator on $L^2(\mathbb{R}^{d+1})$: $\exists\, C > 0$ depending on σ and V (as in Definition 0.60): $\forall \epsilon \in [0,1]$, $\forall\, \gamma \geq 1$:*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}) : \quad |\,Op^{\epsilon,\gamma}(a)\, u|_0 \leq \frac{C}{\gamma^{|m|}}\, |u|_0$$

Note that the constant C in Proposition 0.62 depends uniformly on the compact set in which V takes its values and on the norm of V in $\mathcal{C}^{d+1}_b$. For operators defined by symbols of order $m > 0$:

Proposition 0.63. *Let $n \geq d+1$, and let $a \in S^m_n$ with $m > 0$. Then $Op^{\epsilon,\gamma}(a)$ in (0.169) defines a bounded operator from $H^{m,\epsilon}(\mathbb{R}^{d+1})$ to $L^2(\mathbb{R}^{d+1})$: $\exists\, C > 0$ depending on σ and V (as in Definition 0.60): $\forall \epsilon \in [0,1]$, $\forall\, \gamma \geq 1$:*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}) : \quad |\,Op^{\epsilon,\gamma}(a)\, u|_0 \leq C\, |u|_{H^{m,\epsilon},\gamma}$$

The next proposition describes the smoothing effect of operators of order -1.

Proposition 0.64. *Let $n \geq d+2$, and let $a \in S_n^{-1}$. Then $Op^{\epsilon,\gamma}(a)$ in (0.169) defines a bounded operator from $L^2(\mathbb{R}^{d+1})$ to $H^{1,\epsilon}(\mathbb{R}^{d+1})$: $\exists\, C > 0$ depending on σ and V (as in Definition 0.60): $\forall \epsilon \in [0,1]$, $\forall\, \gamma \geq 1$:*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}): \quad |Op^{\epsilon,\gamma}(a)\, u|_{H^{1,\epsilon},\gamma} \leq C\, |u|_0$$

Remark 0.65. *When applying the pulse calculus, it's verified that for V (as in Definition 0.60): $V \in \mathcal{C}_b^n(\mathbb{R}^{d+1})$, by showing that $V \in H^s(\mathbb{R}^{d+1})$ for some $s > \frac{d+1}{2} + n$, where $n \geq 1$.*

The two first results deal with adjoints of singular pseudo-differential operators, while the last two results deal with products.

Proposition 0.66. *Let $a = \sigma(\epsilon V, \xi + \frac{\kappa\beta}{\epsilon}, \gamma) \in S_n^0$ ($n \geq 2\,(d+1)$), where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Let a^* denote the conjugate transpose of the symbol a. Then $Op^{\epsilon,\gamma}(a)$ and $Op^{\epsilon,\gamma}(a^*)$ are bounded on L^2 and $\exists\, C > 0 : \forall \epsilon \in [0,1]$, $\forall\, \gamma \geq 1$:*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}): \quad |Op^{\epsilon,\gamma}(a)^*\, u - Op^{\epsilon,\gamma}(a^*)\, u|_0 \leq \frac{C}{\gamma}\, |u|_0$$

If $n \geq 3d+3$, then for another constant C:

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}): \quad |Op^{\epsilon,\gamma}(a)^*\, u - Op^{\epsilon,\gamma}(a^*)\, u|_{H^{1,\epsilon},\gamma} \leq C\, |u|_0$$

uniformly in ϵ and γ.

Proposition 0.67. *Let $a = \sigma(\epsilon\, V, \xi + \frac{\kappa\beta}{\epsilon}, \gamma) \in S_n^1$ ($n \geq 3d+4$), where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Let a^* denote the conjugate transpose of the symbol a. Then $Op^{\epsilon,\gamma}(a)$ and $Op^{\epsilon,\gamma}(a^*)$ map $H^{1,\epsilon}$ into L^2 and there exists a family of operators $R^{\epsilon,\gamma}$ that satisfies:*

- $\exists\, C > 0 : \forall \epsilon \in [0,1]$, $\forall\, \gamma \geq 1$:

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}): \quad |R^{\epsilon,\gamma}\, u|_0 \leq C\, |u|_0$$

- *The following duality property holds:*

$$\forall\, u, v \in \mathcal{S}(\mathbb{R}^{d+1}) : \quad \langle Op^{\epsilon,\gamma}(a)\, u, v\rangle_{L^2} - \langle u, Op^{\epsilon,\gamma}(a^*)\, v\rangle_{L^2} = \langle R^{\epsilon,\gamma}\, u, v\rangle_{L^2}$$

In particular, the adjoint $Op^{\epsilon,\gamma}(a)^*$ for the L^2 scalar product maps $H^{1,\epsilon}$ into L^2.

Proposition 0.68. *(a) Let $a, b \in S^0_n$ $(n \geq 2(d+1))$, and suppose $b = \sigma(\epsilon V, \xi + \frac{\kappa\beta}{\epsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then $\exists\, C > 0 : \forall \epsilon \in [0, 1], \forall\, \gamma \geq 1$:*

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1})\,, \quad |Op^{\epsilon,\gamma}(a)\; Op^{\epsilon,\gamma}(b)\, u - Op^{\epsilon,\gamma}(a\, b)\, u|_0 \leq \frac{C}{\gamma}\, |u|_0$$

If $n \geq 3d + 3$, then for another constant C:

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}) : \quad |Op^{\epsilon,\gamma}(a)\; Op^{\epsilon,\gamma}(b)\, u - Op^{\epsilon,\gamma}(a\, b)\, u|_{H^{1,\epsilon},\gamma} \leq C\, |u|_0$$

uniformly in ϵ and γ.

(b) Let $a \in S^1_n, b \in S^0_n$ or $a \in S^0_n, b \in S^1_n$ $(n \geq 3d+4)$. In each case suppose $b = \sigma(\epsilon\, V, \xi + \frac{\kappa\beta}{\epsilon}, \gamma)$ where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2} + 1$. Then $\exists\, C > 0 : \forall \epsilon \in [0, 1], \forall\, \gamma \geq 1$:

$$\forall\, u \in \mathcal{S}(\mathbb{R}^{d+1}) : \quad |Op^{\epsilon,\gamma}(a)\; Op^{\epsilon,\gamma}(b)\, u - Op^{\epsilon,\gamma}(a\, b)\, u|_0 \leq C\, |u|_0$$

The final result is Gårding's inequality:

Theorem 0.1. *(a) Let $\sigma \in \boldsymbol{S}^0$ satisfy $Re\,\sigma(v,\xi,\gamma) \geq C_K > 0$ for all v in a compact subset K of $\mathcal{O}$. Let now $a \in S^0_n$, $n \geq 2\,d+2$ be given by Definition 0.60, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2}+1$ and is valued in a convex compact subset K. Then for all $\delta > 0$, there exists γ_0 which depends uniformly on V, the constant C_K and δ, such that for all $\gamma \geq \gamma_0$ and all $u \in \mathcal{S}(\mathbb{R}^{d+1})$, there holds:*

$$Re\,\langle Op^{\epsilon,\gamma}(a)\,u;u\rangle_{L^2} \geq (C_K - \delta)\,|u|_0^2\,.$$

(b) Let $\sigma \in \boldsymbol{S}^1$ satisfy $Re\,\sigma(v,\xi,\gamma) \geq C_K\langle\xi,\gamma\rangle$ for all v in a compact subset K of $\mathcal{O}$. Let now $a \in S^1_n$, $n \geq 3d+4$ be given by Definition 0.60, where $V \in H^{s_0}(\mathbb{R}^{d+1})$ for some $s_0 > \frac{d+1}{2}+1$ and is valued in a convex compact subset K. Then for all $\delta > 0$, there exists γ_0 which depends uniformly on V, the constant C_K and δ, such that for all $\gamma \geq \gamma_0$ and all $u \in \mathcal{S}(\mathbb{R}^{d+1})$, there holds:

$$Re\,\langle Op^{\epsilon,\gamma}(a)\,u;u\rangle_{L^2} \geq (C_K - \delta)\,|\Lambda_D^{\frac{1}{2}}u|_0^2\,.$$

Remark 0.69. *Notice that, in case (b) of Theorem 0.1, because Λ is simply a Fourier multiplier:*

$$|\Lambda^{1/2}u|_0^2 = |\,|\xi + \frac{\beta\kappa}{\epsilon},\gamma|^{1/2}u\hat{}\,|_0^2 \geq |\gamma^{1/2}u\hat{}\,|_0^2 = \gamma|u|_0^2 \tag{0.170}$$

For given parameters $0 < \delta_1 < \delta_2 < 1$, choose a cutoff $\chi^e(\xi', \frac{\kappa\beta}{\epsilon}, \gamma)$:

$$
\begin{aligned}
&0 \le \chi^e \le 1 \\
&\chi^e\left(\xi', \frac{\kappa\beta}{\epsilon}, \gamma\right) = 1 \text{ on } \left\{(\gamma^2 + |\xi'|^2)^{1/2} \le \delta_1 |\frac{\kappa\beta}{\epsilon}|\right\} \\
&\operatorname{supp} \chi^e \subset \left\{(\gamma^2 + |\xi'|^2)^{1/2} \le \delta_2 |\frac{\kappa\beta}{\epsilon}|\right\}
\end{aligned}
\tag{0.171}
$$

and define a corresponding Fourier multiplier χ^e_D in the extended calculus by the formula (0.169) with $\chi^e(\xi', \frac{\kappa\beta}{\epsilon}, \gamma)$ in place of $\sigma(\epsilon V, X, \gamma)$. Part (a) of Proposition 0.68 still holds when either a or b is replaced by an extended cutoff χ^e.

REFERENCES

[1] D. Alterman and J. Rauch. Nonlinear geometric optics for short pulses. *J. Differential Equations*, 178(2):437–465, 2002.

[2] D. Alterman and J. Rauch. Diffractive nonlinear geometric optics for short pulses. *SIAM J. Math. Anal.*, 34(6):1477–1502, 2003.

[3] J. Chazarain and A. Piriou. *Introduction to the Theory of Linear Partial Differential Equations*. North-Holland, 1982.

[4] J.-F. Coulombel, O. Guès, and M. Williams. Singular pseudifferential calculus for wavetrains and pulses. *Bull. Soc. Math. France*, 142:719–776, 2014.

[5] J.-F. Coulombel and M. Williams. Nonlinear geometric optics for reflecting uniformly stable pulses. *J. Differential Equations*, 255:1939–1987, 2013.

[6] O. Guès and J. Rauch. Nonlinear asymptotics for hyperbolic internal waves of small width. *J. Hyperbolic Differ. Equ.*, 3(2):269–295, 2006.

[7] M. Hernandez. Resonant leading term geometric optics expansions with boundary layers for quasilinear hyperbolic boundary problems. *Comm. Partial Differential Equations*, 40(3):387–437, 2015.

[8] J.-K. Hunter, A. Majda, and R. Rosales. Resonantly interacting, weakly nonlinear hyperbolic waves. ii. several space variables. *Stud. Appl. Math.*, 75(3):187–226, 1986.

[9] J.-L. Joly, G. Métivier, and J. Rauch. Coherent and focusing multidimensional nonlinear geometric optics. *Ann. Sci. École Norm. Sup. (4)*, 28(1):51–113, 1995.

[10] H.-O. Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.

[11] A. Majda and M. Artola. Nonlinear geometric optics for hyperbolic mixed problems. In *Analyse mathématique et applications*, pages 319–356. Gauthier-Villars, Paris, 1998.

[12] G. Mètivier. The block structure condition for symmetric hyperbolic systems. *Bull. Lond. Math. Soc.*, 32(6):689–702, 2000.

[13] M. Williams. Singular pseudodifferential operators, symmetrizers, and oscillatory multidimensional shocks. *J. Funct. Anal.*, 191(1):132–209, 2002.