MONTE CARLO STRATEGIES IN OPTION PRICING FOR SABR MODEL

Leicheng Yin

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Approved by:
Chuanshu Ji
Vidyadhar Kulkarni
Nilay Argon
Kai Zhang
Serhan Ziya
ABSTRACT

LEICHENG YIN: MONTE CARLO STRATEGIES IN OPTION PRICING FOR SABR MODEL
(Under the direction of Chuanshu Ji)

Option pricing problems have always been a hot topic in mathematical finance. The SABR model is a stochastic volatility model, which attempts to capture the volatility smile in derivatives markets. To price options under SABR model, there are analytical and probability approaches. The probability approach i.e. the Monte Carlo method suffers from computation inefficiency due to high dimensional state spaces. In this work, we adopt the probability approach for pricing options under the SABR model. The novelty of our contribution lies in reducing the dimensionality of Monte Carlo simulation from the high dimensional state space (time series of the underlying asset) to the 2-D or 3-D random vectors (certain summary statistics of the volatility path).
To Mom and Dad
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# TABLE OF CONTENTS

LIST OF TABLES ........................................................................................................ viii

LIST OF FIGURES.................................................................................................. ix

1 INTRODUCTION ................................................................................................. 1
  1.1 SABR model and its importance ................................................................. 1
  1.2 Summary of the analytical approach ......................................................... 2
  1.3 Monte Carlo dimension deduction using probability approximation ....... 3
  1.4 Outline .......................................................................................................... 3

2 LITERATURE REVIEW .................................................................................... 5
  2.1 Options and the Black-Scholes model ....................................................... 5
  2.2 Local volatility model ............................................................................... 7
  2.3 SABR model .............................................................................................. 8

3 MATHEMATICAL FORMULATION ................................................................ 11
  3.1 Conditional closed-form European call options prices under SABR model .. 12
    3.1.1 $\beta = 1$ stochastic log normal model ............................................. 13
    3.1.2 $\beta = 0$ stochastic normal model .................................................... 17
    3.1.3 Case $0 < \beta < 1$ ........................................................................... 20
  3.2 Moment matching ....................................................................................... 25
    3.2.1 Moments of $\int_0^{t_{ex}} \sigma^2(u)du$ .................................................... 27
    3.2.2 Moments of $\int_0^{t_{ex}} \sigma(u)dB_2(u)$ ............................................. 30
    3.2.3 Covariance of $\int_0^{t_{ex}} \sigma^2(u)du$ and $\int_0^{t_{ex}} \sigma(u)dB_2(u)$ ........... 32
  3.3 Joint distribution of $(\sigma^2, X_2)$ ............................................................... 35
    3.3.1 Gamma mixture of normals .............................................................. 36
3.3.2 Inverse Gamma mixture of normal ........................................ 39
3.3.3 Lognormal mixture of normal ........................................... 40

4 NUMERICAL COMPUTATION ................................................. 42
  4.1 Analytical approach ....................................................... 43
  4.2 Dimension reduction approach ......................................... 44
    4.2.1 $\beta = 1$ stochastic log normal model ......................... 45
    4.2.2 $\beta = 0$ stochastic normal model ............................. 48
    4.2.3 $0 < \beta < 1$ case .............................................. 49
  4.3 Original Monte Carlo method .......................................... 51

5 EMPIRICAL STUDY .......................................................... 53
  5.1 Model parameters ......................................................... 53
  5.2 Options on equity ....................................................... 57
    5.2.1 $\beta = 1$ Stochastic log normal model ....................... 58
    5.2.2 $\beta$ close to 1 ................................................. 63
  5.3 Fixed income options .................................................. 66
    5.3.1 $\beta = 0$ stochastic normal model ............................ 67
    5.3.2 $\beta$ close to 0 ................................................ 73
  5.4 Conclusion .............................................................. 77

BIBLIOGRAPHY .................................................................... 80
LIST OF TABLES

5.1 Model Parameters Microsoft Stock ........................................ 56
5.2 Model Parameters iShare 20+ Years Treasury Bond ETF ............... 57
5.3 $\beta = 1$ SABR model prices comparison................................. 59
5.4 $\beta$ close to 1 SABR model prices comparison............................ 64
5.5 $\beta = 0$ SABR model prices comparison................................. 68
5.6 $\beta$ close to 0 SABR model prices comparison............................ 74
5.7 Comparison between all pricing methods .................................... 77
LIST OF FIGURES

5.1 Microsoft Stock Option Chain ........................................ 54
5.2 iShare 20+ Year Treasury Bond Option Chain ..................... 55
5.3 Historical Volatilities .................................................. 56
5.4 Daily Treasury Yield Curve Rates ..................................... 58
5.5 $\beta = 1$ brute-force Monte Carlo and dimension reduction price difference:
$(\Sigma^2, X_2) \sim$ Gamma mixture of normal .......................... 63
5.6 $\beta = 1$ brute-force Monte Carlo and dimension reduction price difference:
$(\Sigma^2, X_2) \sim$ inverse Gamma mixture of normal ................. 64
5.7 $\beta = 1$ brute-force Monte Carlo and dimension reduction price difference:
$(\Sigma^2, X_2) \sim$ log normal mixture of normal ................... 66
5.8 $\beta = 1$ brute-force Monte Carlo and analytical price difference....... 67
5.9 $\beta$ close to 1: brute-force, original dimension reduction and approximated Monte Carlo prices ........................................... 68
5.10 $\beta$ close to 1 zoom-in: brute-force, original dimension reduction and approximated Monte Carlo prices ........................................... 73
5.11 $\beta = 0$ brute-force Monte Carlo and dimension reduction price difference:
$(\Sigma^2, X_2) \sim$ Gamma mixture of normal .......................... 74
5.12 $\beta = 0$ brute-force Monte Carlo and dimension reduction price difference:
$(\Sigma^2, X_2) \sim$ inverse Gamma mixture of normal ................. 76
5.13 $\beta = 0$ brute-force Monte Carlo and dimension reduction price difference:
$(\Sigma^2, X_2) \sim$ log normal mixture of normal ................... 77
5.14 $\beta = 0$ brute-force Monte Carlo and analytical price difference........ 78
5.15 $\beta$ close to 0: brute-force, original dimension reduction and approximated Monte Carlo prices ........................................... 78
5.16 $\beta$ close to 0: brute-force and original dimension reduction Monte Carlo prices ... 79
CHAPTER 1
INTRODUCTION

1.1 SABR model and its importance

In finance, an option is a contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date, depending on the form of the option. Because the values of option contracts depend on a number of different variables in addition to the value of the underlying asset, they are complex to price, and hence become one of the central topics in mathematical finance. There are many pricing models in use, although all essentially incorporate the concepts of rational pricing, moneyness and option time value etc.

Cases with closed-form pricing formulas are rare. The exceptions are Black-Scholes-Merton model, Hestons model, just name a few. Other problems almost always require efficient numerical computation and approximation techniques. There are basically two approaches. The analytical approach sets the price function as the solution to a PDE with boundary conditions. Then the numerical PDE are often solved by finite difference etc. The probabilistic approach tackles the option price problem as a conditional expectation under a risk neutral measure which needs to be computed using numerical integration. Such integration is often performed over a high dimensional state space in which state variables are time series of the underlying asset. In this situation, Monte Carlo simulation appears to be indispensible.

SABR model enjoys the popularity in the study of Managing Smile Risk (Hagan et al., 2002) [See, among others, Advanced Analytics For the SABR Model (Antonov and Spector, 2012), Probability Distribution In the SABR Model Of Stochastic Volatility (Hagan et al., 2005), Asymptotic Implied Volatility At the Second Order With Application To the SABR Model (Paulot, 2009), the SABR/LIBOR Market Model: Pricing, Calibration and Hedging For Complex Interest-rate Derivatives (Rebonato et al., 2011) etc.].
Prior to SABR model, stochastic volatility models were still struggling in predicting the correct dynamics of the market smile. For example, when the price of the underlying asset increases, CEV model predicts that the volatility smile shifts to lower prices which is opposite to the market behavior that smiles and skews always move in the same direction as the underlying asset. The main feature of the SABR model is to be able to reproduce the dynamics behavior of the volatility smiles and skews, and thus to yield stable pricing and hedges. It assumes that the volatility of the forward is a Geometric Brownian Motion, and is correlated to the underlying forward price. The approaches taken in these works are analytical, relying on singular perturbation of the pricing function under certain conditions.

In this work, we adopt the probability approach for pricing options under the SABR model. The novelty of our contribution lies in reducing the dimension for Monte Carlo simulation from the high dimensional state space (time series of the underlying asset) to the 2-D or 3-D random vectors (certain summary statistics of the sample path).

1.2 Summary of the analytical approach

In general, given the dynamics of the underlying asset and/or its volatility, there are always an analytical approach and a probabilistic approach to price the option on it.

Take the SABR model for instance and assume the underlying dynamics is under risk neutral measure. To price a European option under the probabilistic approach, one discretizes the price and volatility paths, and simulates the forward price on maturity $F(T)$ path by path. The European option price is then given by arithmetic mean of all $\max\{F(T) - K, 0\}$s.

In the analytical approach, singular perturbation technique is used to price European option under SABR model, which takes both the volatility $\sigma(t)$ and the volatility of volatility $\alpha$ to be small, i.e. $\epsilon \sigma(t)$ and $\epsilon \alpha$. The perturbation factor $\epsilon$ will be simply set to 1 to recast the original model variables.

The main idea of the analytical approach is to calculate the option price integration

$$V(t, f, \sigma) = \mathbb{E}\{\max\{F(t_{ex}) - K, 0\}|F(t) = f, \sigma(t) = \sigma\},$$
where $f$ and $\alpha$ are forward price and volatility at as-of-time $t$. This integration is then simplified to the following integration

$$P(t, f, \sigma; T, K) = \int_t^{t_{ex}} \int_{-\infty}^{\infty} A^2 p(t, f, \sigma; T, K, A) dA dT,$$

where $p(t, f, \sigma; T, F, A)$ is the probability density of $(F(T), \sigma(T))|_{F(t)=f, \sigma(t)=\sigma}$, and $T$ is any time satisfying $t < T < t_{ex}$. Notice that $P(t, f, \sigma; T, K)$ satisfies a backward Kolmogorov equation whose coefficients are fed from SABR model parameters, therefore, the initial option price integration is so solved. The resulting formula is further equated to the Black formula to give a closed-form formula for the implied volatility.

1.3 Monte Carlo dimension deduction using probability approximation

An alternative to the analytical method to price options under SABR model is Monte Carlo simulation. Pricing accuracy using Monte Carlo simulation is directly related to its dimensionality. To make it more specific, when using Euler approximation for the model, small steps in discretized forward and volatility paths are required to reduce the bias, which subsequently requires a far greater number of independent sample paths in order to reduce the variance. Such bias and variance issues present a great challenge for computational efficiency when dealing with stochastic volatility models.

To address this issue, we propose a Monte Carlo dimension reduction technique in this work. This is based on the observation that conditioning on the stochastic volatility path, the option pricing formula depends only on integrated volatilities and some other summary statistics instead of the entire volatility path. Therefore, it is enough to simulate only the low dimensional summary statistics instead of from the high dimensional state space (time series of the underlying asset).

1.4 Outline

In this work, we will study option pricing under SABR model using probability approach. Our contribution is to propose a probability approximation scheme for computing option prices in which the dimensionality of Monte Carlo is significantly reduced. Comparisons of numerical
computational results among different approaches are also presented. The main outline of the
dissertation is as follows:

- In Chapter 2, we briefly introduce related concepts in finance and summarize several afore-
  mentioned models.

- In Chapter 3, we introduce the SABR model and propose the dimension reduction strategy
  in derivative pricing under the model. We tackle the problem in multiple cases based on
  different choices of SABR model factor $\beta$. In the second half of this chapter, we propose
  distribution families of volatility summary statistics and introduce techniques for distribution
  specification.

- In Chapter 4, we introduce and implement our Monte Carlo dimension reduction technique
  as well as two alternative approaches, the brute force Monte Carlo method and the analytical
  approach, to price options under SABR model frame.

- In Chapter 5, we apply three pricing methods on different asset types and discuss the empirical
  studies, including pricing accuracy, computation efficiency and economic interpretation.
CHAPTER 2
LITERATURE REVIEW

There are so much work on option pricing that we couldn’t present them all. In this Chapter, we present a partial literature review related to our work. As mentioned in Introduction, the Black-Scholes-Merton model (Black and Scholes, 1973) derives a theoretical formula of the price of European options, which led to a boom in financial engineering and is the basis of almost all more complicated pricing and hedging strategies. Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work, and Black was ineligible for the prize because of his death in 1995. Black’s model presumes constant volatility for the underlying asset.

However, implied volatility always varies with strike price and time-to-maturity. Options whose strike price differs substantially from the underlying asset’s price command higher prices (and thus implied volatilities) than what is suggested by Black-Scholes-Merton model (Hull, 1997). An solution to this problem is provided by the local volatility model (Dupire, 1994). However, the local volatility model fails to predict implied volatility curve’s movement with underlying asset price changes. Both the Black-Scholes model and Hull’s local volatility model are presented here.

SABR model (Hagan et al., 2002) is the main framework we follow in this paper, it is discussed following the local volatility model. Hagan proposed the model to capture the dynamics of implied volatility and gave a theoretical estimate of the price of European-style options under this model, which inspired us to start our work.

In this chapter, we will present the three models mentioned above.

2.1 Options and the Black-Scholes model

An option is a contract which gives the buyer (the owner or holder of the option) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date, depending on the form of the option (Hull, 1997). A European option
is one that can be exercised only on the specified date. The simplest kind of option is one that grants the right to buy the underlying asset, which is referred to as a call option. The option we will discuss throughout the paper is the European call option.

Black-Scholes model assumes ideal conditions in the market, which are

1. The short-term interest rate is known and constant through time.
2. The instantaneous log return of stock price is a geometric Brownian motion, and we will assume that its drift and volatility are constant.
3. The underlying stock pays no dividends.
4. There is no arbitrage opportunity.
5. It is possible to borrow and lend any amount, even fractional, of cash at the risk-free rate.
6. There are no penalties to short selling.
7. There are no transaction costs or fees.

That being said, the Black-Scholes model assumes that under risk-neutral measure, the price of the underlying asset has following dynamics,

\[
\frac{dF(t)}{F(t)} = r dt + \sigma B(t) dW(t),
\]

where \( W(t) \) is a standard Brownian motion.

Under these assumptions, the value of a European call option for a non-dividend-paying underlying stock in terms of the Black-Scholes parameters is

\[
C(F_0, t_{ex}) = F_0 \Phi(d_1) - Ke^{-r_{ex}} \Phi(d_2),
\]

\[
d_1 = \frac{ln\left(\frac{F_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t_{ex}}{\sigma \sqrt{t_{ex}}},
\]

\[
d_2 = \frac{ln\left(\frac{F_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)t_{ex}}{\sigma \sqrt{t_{ex}}},
\]

whose notations have been specified in the list of abbreviations and symbols.
2.2 Local volatility model

In the paper pricing with a smile (Dupire, 1994) [see, among others, Asymptotics and Calibration Of Local Volatility Models (Berestycki et al., 2002), Reconstructing the Unknown Local Volatility Function (Coleman et al., 2001)], Dupire pointed that instead of setting constant implied volatility, one should replace $\sigma_B F(t)$ in (2.1) by local volatility $\sigma_{loc}(t, F(t)) F(t)$. In this local volatility model, the price of underlying asset is

$$dF(t) = rF(t)dt + \sigma_{loc}(t, F(t))F(t)dW(t),$$

(2.5)

and the local volatility function $\sigma_{loc}(t, F(t))$ is calibrated to market of liquid European options.

For each pair of maturity and strike price, calibration starts with a given local volatility function by evaluating the put-call parity

$$C = P + e^{-rt_{ex}}(F_0 - K)$$

until the theoretical prices match the actual market prices of the option for the maturity date and strike price. In practice, local volatility function $\sigma_{loc}(t, F(t))$ is always piecewise constant in time because market only exists for options with specific maturities, i.e.

$$\sigma_{loc}(t, F_0) = \sigma_{loc}(t_{ex}^1, F_0) \text{ for } t_{ex}^1 < t < t_{ex}^2,$$

where $t_{ex}^1$ and $t_{ex}^2$ are two consecutive maturity dates.

Once the local volatility function $\sigma_{loc}(t, F(t))$ has been obtained, the model can be used in option pricing without ambiguity. However, this model predicts the wrong dynamics of the implied volatility curve (Hagan et al., 2002). Now we illustrate this in a special case that the local volatility is a function of $F(t)$ only, i.e. $\sigma_{loc}(F(t))$.

In equivalent Black volatilities (Hagan and Woodward, 1999), the implied Black volatility could be related to local volatility function as

$$\sigma_{B}(K, F_0) = \sigma_{loc}(F_0 + K/2)\{1 + \cdots\}.$$  

(2.6)
Omitting higher order terms and calibrating (2.6) to today's implied volatility curve $\sigma^0_B(K, F_0)$, we have

$$\sigma_{loc}(F) = \sigma^0_B(2F - F_0, F_0)\{1 + ...\}. \quad (2.7)$$

According to (2.6) and (2.7), if forward price changes from $F_0$ to $F^*$, the new volatility curve would be

$$\sigma^*_B(K, F^*) = \sigma^0_B(K + F^* - F_0, F_0)\{1 + ...\}.$$

And this indicates that the smile curve moves in the opposite direction to the underlying asset, which is different from market observations.

2.3 SABR model

The SABR model we present here is a slight generalization from the original SABR model (Hagan et al., 2002). The failure of local volatility model reveals that smile risk is hardly managed by single factor models. In order to capture the volatility smile in derivatives markets, the SABR model selects volatility as a second factor, which suggests that volatility is a random function of time.

The name of SABR model stands for "stochastic alpha, beta, rho", which refers to the parameters of the model. It describes the dynamics of a single forward $F(t)$, such as a LIBOR forward rate, a forward swap rate, or a forward stock price. The volatility of the forward $F(t)$ over time is denoted by $\sigma(t)$. SABR model is a dynamic model in which both $F(t)$ and $\sigma(t)$ are represented by stochastic state variables whose time evolutions are given by the following system of stochastic differential equations

$$dF(t) = rF(t)\beta dt + \sigma(t)F(t)\beta dW_1(t), \quad (2.8)$$
$$d\sigma(t) = \alpha\sigma(t)dW_2(t), \quad (2.9)$$
with the prescribed initial values $F_0$ and $\sigma_0$. Under risk neutral measure, $r$ in (2.8) is the risk-free interest rate. And $W_1(t)$ and $W_2(t)$ are two correlated standard Brownian motions with correlation coefficient $-1 \leq \rho \leq 1$, i.e.

\[ dW_1(t)dW_2(t) = \rho dt. \]  

(2.10)

Constant parameter $\alpha$ is the volatility of volatility, which is usually calibrated to market, and $\beta$ sets the type of forward dynamics and mainly depends on the underlying asset. They satisfy conditions $\alpha \geq 0$, $0 \leq \beta \leq 1$.

To price an option under SABR model, Hagan adopted the analytical approach (Hagan et al., 2002). Recalling the price formula of a European call option,

\[ C(t, f, \sigma) = E\{\max\{F(t_{ex}) - K, 0\}|F(t) = f, \sigma(t) = \sigma}\}, \]  

(2.11)

where $f(t)$ and $\sigma(t)$ are forward price and its volatility as of time $t$. (2.11) is then simplified to the following one

\[ P(t, f, \sigma; T, K) = \int_t^{t_{ex}} \int_{-\infty}^{\infty} A^2 p(t, f, \sigma; T, K, A)dAdT, \]

where $p(t, f, \sigma; T, F, A)$ is the probability density of $(F(T), \sigma(T))|F(t) = f, \sigma(t) = \sigma$, where $T$ is any time satisfies $t < T < t_{ex}$. Notice that $P(t, f, \sigma; T, K)$ satisfies a backward Kolmogorov equation whose coefficients are fed from SABR model parameters, therefore the initial option price integration (2.11) is solved.

Considering this resulting formula is not very useful, it is equated to the Black-Scholes formula to give a closed-form formula for the implied volatility, which can be directly used in the Black-Scholes formula to price an option. The closed-form algebraic formula for the implied volatility
$\sigma_B(K, f)$ is

\[
\sigma_B(K, f) = \frac{\sigma}{(fK)^{(1-\beta)/2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 f/K + \frac{(1-\beta)^4}{1920} \log^4 f/K + \ldots \right\} \times \frac{z}{x(z)}
\times \left\{ 1 + \left[ \frac{(1-\beta)^2}{24} \frac{\sigma^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\rho \beta \nu \sigma}{(fK)^{(1-\beta)/2}} + \frac{2 - 3 \rho^2}{24} \nu^2 \right] t_{ex} + O(t_{ex}^2) \right\}
\]

\hspace{1cm} \tag{2.12}

where

\[
z = \frac{\nu}{\sigma} (fK)^{(1-\beta)/2} \log f/K \tag{2.13}
\]

and $x(z)$ is defined by

\[
x(z) = \log \left\{ \frac{\sqrt{1 - 2 \rho z + z^2} + z - \rho}{1 - \rho} \right\}. \tag{2.14}
\]

Hagan pointed out that higher order terms $O(t_{ex}^2)$ are usually omitted from implied volatility calculation because they are so small that they do not have material impact on the accuracy of implied volatility. This is the main conclusion of this model.

Pricing European call options using this method would be a straightforward implement of the implied volatility formula (2.12) and the Black-Scholes formula (2.2) to (2.4). It is worth noticing that the complexity of formula (2.12) is needed for accurate pricing. The omitted higher order terms are considered not material to pricing accuracy according to the authors.
CHAPTER 3
MATHEMATICAL FORMULATION

Our main contribution is to propose an revised Monte Carlo method to price options under SABR model. In the following three chapters, we will present our methodology, numerical implementation and empirical study results.

From this chapter and through out this paper, we will focus on the following SABR model, which describes the dynamics of a single forward $F(t)$, such as a LIBOR forward rate, a forward swap rate, or a forward stock price. The volatility of the forward $F(t)$ over time is denoted by $\sigma(t)$. The dynamics of $F(t)$ and $\sigma(t)$ are given by the following stochastic differential equations

\[
\begin{align*}
    dF(t) &= rF(t)^\beta dt + \sigma(t)F(t)^\beta dW_1(t), \quad (3.1) \\
    d\sigma(t) &= \alpha \sigma(t)dW_2(t), \quad (3.2)
\end{align*}
\]

with the prescribed initial values $F_0$ and $\sigma_0$. Under risk neutral measure, $r$ in (2.8) is the risk-free interest rate. And $W_1(t)$ and $W_2(t)$ are two correlated standard Brownian motions with correlation coefficient $-1 \leq \rho \leq 1$, i.e.

\[
dW_1(t)dW_2(t) = \rho dt. \quad (3.3)
\]

Constant parameter $\alpha$ is the volatility of volatility, which is usually calibrated to market. And $\beta$ sets the type of forward dynamics and itself closely depends on the type of underlying asset. $\alpha$ and $\beta$ satisfy conditions $\alpha \geq 0$, $0 \leq \beta \leq 1$.

The value of a European call option is defined by the expected value of discounted option payoff at maturity $t_{ex}$, i.e.

\[
C(F_0, K) = e^{-r t_{ex}} \mathbb{E}_\sigma \left\{ \mathbb{E}_{\sigma} \left\{ \max(F(t_{ex}) - K, 0) \right\} \right\}, \quad (3.4)
\]
where \( E_\sigma \) denotes the conditional expected value conditioning on \( \sigma(t) \), i.e.

\[
E_\sigma \{ \max(F(t_{ex}) - K, 0) \} = E\{ \max(F(t_{ex}) - K, 0) | \sigma(t), 0 \leq t \leq t_{ex} \}.
\]

The main focus of our work is to propose a new approach to solve (3.4) under SABR model. In what follows in this chapter, we will discuss the mathematical formulation of our dimension reduction Monte Carlo method.

### 3.1 Conditional closed-form European call options prices under SABR model

As we detailed in literature review Section 2.3, under SABR model, the price of a European option is priced by a closed-form algebraic formula for the implied volatility as a function of current forward price \( F_0 \) and the strike price \( K \), see managing smile risk (Hagan et al., 2002). However, although pricing accuracy depends on the complexity of the implied volatility formula, the omission of higher order terms in (2.12) is inevitable in practice.

An alternative method to price options under SABR model is Monte Carlo simulation. Pricing accuracy using Monte Carlo simulation is directly related to its dimensionality. To make it more specific, when using Euler approximation for (3.1) and (3.2), small steps in discretized forward and volatility paths are required to reduce the bias, which subsequently requires far greater number of independent sample paths in order to reduce the variance. Such bias and variance issues present a great challenge in computational efficiency when dealing with stochastic volatility model.

To address this issue, we propose a Monte Carlo dimension reduction technique in this work. This is based on the observation that, under SABR model, conditioning on the stochastic volatility path (3.2), the inner conditional expected value of price formula (3.4)

\[
E_\sigma \{ \max(F(t_{ex}) - K, 0) \}
\]

depends only on integrated volatilities and some other summary statistics instead of the entire volatility path. And we will show that this can largely reduce the number of independent sample paths needed in the simulation.
As we stated at the beginning of this chapter, in SABR model \( \beta \) can take any value between 0 and 1, i.e. \( 0 \leq \beta \leq 1 \). In our work, different \( \beta \) values require different treatments. In what follows, we are going to present our work in three cases: \( \beta = 1 \) (representing the stochastic log normal model), \( \beta = 0 \) (representing a stochastic normal model), and a more general case \( 0 < \beta < 1 \).

3.1.1 \( \beta = 1 \) stochastic log normal model

In this section, we will focus on a special case of SABR model when \( \beta = 1 \), i.e.

\[
\begin{align*}
    dF(t) &= rF(t)dt + \sigma(t)F(t)dW_1(t), \\
    d\sigma(t) &= \alpha\sigma(t)dW_2(t), \\
    dW_1(t)dW_2(t) &= \rho dt,
\end{align*}
\]

and solve (3.5) under it.

In model (3.6) - (3.8), \( W_1(t) \) and \( W_2(t) \) are correlated standard Brownian motions with correlation coefficient \( \rho \). To revise the model in terms of independent standard Brownian motions, let random vector \( W(t) = (W_1(t), W_2(t))^T \), and \( A(t) \) be the covariance matrix of \( W(t) \). According to Cholesky decomposition, \( A(t) = R(t)^TR(t) \), where \( R(t) \) is a unique upper-triangular matrix with positive diagonal entries, i.e.

\[
A(t) = \begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix}
= \begin{bmatrix}
\sqrt{1-\rho^2} & \rho \\
0 & 1
\end{bmatrix} \times \begin{bmatrix}
\sqrt{1-\rho^2} & 0 \\
\rho & 1
\end{bmatrix}
= R(t)^TR(t)
\]

Let random vector \( B(t) = (B_1(t), B_2(t))^T \), where \( B_1(t) \) and \( B_2(t) \) are independent standard Brownian motions. Then \( R(t) \times B(t) \) defines a random vector of two correlated standard Brownian
motions with correlation coefficient $\rho$, i.e.

$$R(t) \times B(t) = \begin{bmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$$

Therefore, we can rewrite model (3.6) - (3.8) to the following equivalent form

$$dF(t) = rF(t)dt + \sigma(t)F(t)[\sqrt{1-\rho^2}dB_1(t) + \rho dB_2(t)] \quad (3.9)$$

$$d\sigma(t) = \alpha \sigma(t)dB_2(t) \quad (3.10)$$

where $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions and all other notations remain the same as defined in (3.6) - (3.8). From now on, we will use this form of SABR model in our discussion.

Before calculating (3.5), we will first introduce some notations for computational convenience. We will be using the following notations through out this paper:

$$F(t_{ex})^+ = \begin{cases} F(t_{ex}) & F(t_{ex}) > K \\ 0 & F(t_{ex}) \leq K \end{cases} \quad (3.11)$$

$$K^+ = \begin{cases} K & F(t_{ex}) > K \\ 0 & F(t_{ex}) \leq K \end{cases} \quad (3.12)$$

$$\Sigma^2 = \int_0^{t_{ex}} \sigma(u)^2 du, \quad (3.13)$$

$$X_1 = \int_0^{t_{ex}} \sigma(u)dB_1(u), \quad (3.14)$$

$$X_2 = \int_0^{t_{ex}} \sigma(u)dB_2(u), \quad (3.15)$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions. Sebsequently, $X_1$ is a normal random variable $N(0, \Sigma^2)$, and $\Sigma^2$ and $X_2$ are constants conditioning on volatility path (3.10).
Theorem 1. Conditioning on volatility path (3.10), the price of a European call option under SABR model, i.e. formula (3.5) is

\[
E_\sigma \{ \max (F(t_{ex}) - K, 0) \}
= F_0 e^{r_{tx} + \rho X_2 + \frac{1}{2} (1 - \rho) \Sigma^2} \Phi(d_1) - K \Phi(d_2),
\]

(3.16)

where \( \Phi \) is the cumulative distribution function of standard normal distribution, and \( d_1 \) and \( d_2 \) are defined by

\[
d_1 = \ln \left( \frac{F_0}{K} \right) - \frac{1}{2} \left( \rho + \sqrt{1 - \rho^2} \right) \Sigma^2 + \left( 1 - \rho^2 \right) \Sigma^2 + \rho X_2 + \sqrt{1 - \rho^2} \Sigma,
\]

(3.17)

\[
d_2 = \ln \left( \frac{F_0}{K} \right) - \frac{1}{2} \left( \rho + \sqrt{1 - \rho^2} \right) \Sigma^2 + \rho X_2 + \sqrt{1 - \rho^2} \Sigma.
\]

(3.18)
Proof. Notice that by notations (3.11) and (3.12), we have

$$
\mathbb{E}_\sigma \left\{ \max(F(t_{ex}) - K, 0) \right\} = \mathbb{E}_\sigma \left\{ F(t_{ex})^+ \right\} - \mathbb{E}_\sigma \left\{ K^+ \right\}
$$

(3.19)

Solving (3.9) conditioning on (3.9), we get

$$
F(t) = F_0 e^{rt + \sqrt{1-\rho^2} \int_0^t \sigma(u) dB_1(u) + \rho \int_0^t \sigma(u) dB_2(u) - \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \int_0^t \alpha(u)^2 du},
$$

(3.20)

which gives the conditional underlying forward price at any given time $t$. Applying formula (3.20) on $t_{ex}$ and plugging $F(t_{ex})$ into (3.19), the first term on the right hand side is

$$
\mathbb{E}_\sigma \left\{ F(t_{ex})^+ \right\}
$$

$$
= \mathbb{E}_\sigma \left\{ \left( F_0 e^{rt_{ex} + \sqrt{1-\rho^2} \int_0^{t_{ex}} \sigma(u) dB_1(u) + \rho \int_0^{t_{ex}} \sigma(u) dB_2(u) - \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \int_0^{t_{ex}} \alpha(u)^2 du} \right)^+ \right\}
$$

$$
= \mathbb{E}_\sigma \left\{ \left( F_0 e^{rt_{ex} + \sqrt{1-\rho^2} X_{1} + \rho X_2 - \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2} \right)^+ \right\}
$$

$$
= \mathbb{E}_\sigma \left\{ F_0 e^{rt_{ex} + \rho X_2 - \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2} \right\}
$$

$$
\cdot \mathbb{E}_\sigma \left\{ e^{\sqrt{1-\rho^2} X_1} 1_{\{\sqrt{1-\rho^2} X_1 > \ln \left( \frac{K}{F_0} \right) + \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2 - \rho X_2 - rt_{ex} \}} \right\}
$$

We can show by direct integration that

$$
\mathbb{E}_\sigma \left\{ e^{\sqrt{1-\rho^2} X_1} 1_{\{\sqrt{1-\rho^2} X_1 > \ln \left( \frac{K}{F_0} \right) + \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2 - \rho X_2 - rt_{ex} \}} \right\}
$$

$$
= \int_{\ln \left( \frac{K}{F_0} \right) + \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2 - \rho X_2 - rt_{ex}}^{\infty} e^{x} d \mathbb{F}_{\sqrt{1-\rho^2} X}(x)
$$

$$
= e^{\frac{1}{2}(1-\rho^2) \Sigma^2} \Phi \left( \frac{\ln \left( \frac{K}{F_0} \right) + \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2 - \rho X_2 - rt_{ex}}{\sqrt{1-\rho^2} \Sigma} \right)
$$

$$
= e^{\frac{1}{2}(1-\rho^2) \Sigma^2} \Phi \left( \frac{\ln \left( \frac{F_0}{K} \right) - \frac{1}{2}(\rho+\sqrt{1-\rho^2}) \Sigma^2 + (1-\rho^2) \Sigma^2 + \rho X_2 + rt_{ex}}{\sqrt{1-\rho^2} \Sigma} \right)
$$

$$
= e^{\frac{1}{2}(1-\rho^2) \Sigma^2} \Phi (d_1)
$$
Putting previous calculations together, we get

\[
\mathbb{E}_\sigma \left\{ F(T)^+ \right\} = F_0 e^{rt_{ex} + \rho X_2 + \frac{1}{2} (1 - \rho - \sqrt{1 - \rho^2}) \Sigma^2 d_1} \tag{3.21}
\]

where \( d_1 \) is defined by equation (4.12).

The second term on the right hand side of (3.19) is

\[
\mathbb{E}_\sigma \{ K^+ \} = \mathbb{E}_\sigma \{ K 1_{\{F(t_{ex}) > K\}} \}
\]

\[
= K \mathbb{P} \{ \sqrt{1 - \rho^2} X_1 > ln(\frac{K}{F_0}) + \frac{1}{2} (\rho + \sqrt{1 - \rho^2}) \Sigma^2 - \rho X_2 - rt_{ex} \}
\]

\[
= K \Phi \left( \frac{ln(\frac{F_0}{K}) - \frac{1}{2} (\rho + \sqrt{1 - \rho^2}) \Sigma^2 + \rho X_2 + rt_{ex}}{\sqrt{1 - \rho^2} \Sigma} \right)
\]

\[
= K \Phi(d_2) \tag{3.22}
\]

where \( d_2 \) is as given in equation (3.18). Substituting (3.21) and (3.22) into equation (3.19), we get equation (3.16).

Theorem 1 suggests that under model (3.9) - (3.10), conditional expected value (3.5) depends on volatility in the form of its integrals \( X_2 \) and \( \Sigma^2 \) instead of the entire volatility path \( \sigma(t) \).

3.1.2 \( \beta = 0 \) stochastic normal model

In this section, we will focus on another special case of SABR model when \( \beta = 0 \), the stochastic normal model, i.e.

\[
dF(t) = r dt + \sigma(t) dW_1(t), \tag{3.23}
\]

\[
d\sigma(t) = \alpha \sigma(t) dW_2(t), \tag{3.24}
\]

\[
dW_1(t) dW_2(t) = \rho dt \tag{3.25}
\]

and solve (3.5) under it.

Notice that (3.23) is also a special case of Vasicek model (Vasicek, 1977). The Vasicek model describes the evolution of the instantaneous interest rate in the following stochastic differential
equation (we use $F(t)$ to denote instantaneous interest rate for convenience although it is not usual):

$$dF(t) = a(b - F(t)) + \sigma dW_1(t).$$

Parameters $a, b$ and $\sigma$ together with the initial condition $F_0$ characterize the dynamics of interest rate:

$b$: long term mean level, all future trajectories of $F(t)$ will evolve around $b$ in the long run;

$a$: speed of reversion, which characterizes the velocity at which such trajectories will regroup around $b$ in time;

$\sigma$: volatility.

Applying the same Cholesky decomposition, we can rewrite (3.23)–(3.25) as what follows,

$$dF(t) = r dt + \sigma(t)\left[\sqrt{1 - \rho^2} dB_1(t) + \rho dB_2(t)\right]$$

(3.26)

$$d\sigma(t) = \alpha \sigma(t) dB_2(t)$$

(3.27)

where $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions and all other notations remain the same as defined in (3.23) - (3.25). And we will calculate (3.5) in the following theorem.

**Theorem 2.** Conditioning on volatility path (3.27), the price of a European call option under SABR model, i.e. formula (3.5) is

$$\mathbb{E}_{\sigma}\{\max(F(t_{ex}) - K, 0)\}$$

$$= [rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2})\Sigma^2 - K]\Phi(d) + \frac{\sqrt{1 - \rho^2}\Sigma^2 e^{-\frac{d^2}{2}}}{\sqrt{2\pi}},$$

(3.28)

where $\Phi$ is the cumulative distribution function of standard normal distribution, and $d$ is define as

$$d = \frac{rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2})\Sigma^2 - K}{\sqrt{1 - \rho^2}\Sigma}.$$  

(3.29)

**Proof.** Using the same decomposition (3.19) in the proof of Theorem 1,

$$\mathbb{E}_{\sigma}\{\max(F(t_{ex}) - K, 0)\} = \mathbb{E}_{\sigma}\{F(t_{ex})\} - \mathbb{E}_{\sigma}\{K^+\}$$

(3.30)

18
Before being able to calculate (3.30), we need the price of underlying asset \( F(t) \) conditioning on volatility path (3.27). Applying Ito’s formula and conditioning in (3.27), the underlying price at any given time \( t \), \( F(t) \), in model (3.26) - (3.27) is

\[
F(t) = rt_{ex} + F_0 + \sqrt{1 - \rho^2} \int_0^t \sigma(s) dB_1(u) + \rho \int_0^t \sigma(s) dB_2(u) - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \int_0^t \sigma^2(s) ds.
\]

Subsequently, at maturity,

\[
F(t) = rt_{ex} + F_0 + \sqrt{1 - \rho^2} X_1 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2.
\] (3.31)

Now, applying (3.31) to (3.30), the first term on its right hand side is

\[
\mathbb{E}_\sigma \{ F(t_{ex})^+ \}
= \mathbb{E}_\sigma \{ (rt_{ex} + F_0 + \sqrt{1 - \rho^2} X_1 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2) 1_{\{ F(t_{ex}) > K \}} \}
= [rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2] \times \mathbb{E}_\sigma \{ 1_{\{ \sqrt{1 - \rho^2} X_1 > K - F_0 - \rho X_2 + \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2 - rt_{ex} \}} \}
+ \mathbb{E}_\sigma \{ \sqrt{1 - \rho^2} X_1 1_{\{ \sqrt{1 - \rho^2} X_1 > K - F_0 - \rho X_2 + \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2 - rt_{ex} \}} \}
= [rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2] \Phi \left( \frac{F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2 - K + rt_{ex}}{\sqrt{1 - \rho^2} \Sigma} \right)
+ \mathbb{E}_\sigma \{ \sqrt{1 - \rho^2} X_1 1_{\{ \sqrt{1 - \rho^2} X_1 > K - F_0 - \rho X_2 + \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2 - rt_{ex} \}} \}
= [rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2] \Phi \left( 
\right)
+ \mathbb{E}_\sigma \{ \sqrt{1 - \rho^2} X_1 1_{\{ \sqrt{1 - \rho^2} X_1 > K - F_0 - \rho X_2 + \frac{1}{2}(\rho + \sqrt{1 - \rho^2}) \Sigma^2 - rt_{ex} \}} \}.
\] (3.32)
Now we show the expected value in (3.32) in by direct integration:

\[
E_\sigma \left\{ \sqrt{1 - \rho^2} X_1 \mathbf{1}_{\{\sqrt{1 - \rho^2} X_1 > K - F_0 - \rho X_2 + \frac{1}{2} (\rho + \sqrt{1 - \rho^2}) \Sigma^2 - r t_{ex} \}} \right\}
= \int_{-\infty}^{\infty} x dF \sqrt{1 - \rho^2} X_1 (x)
= \sqrt{1 - \rho^2} \sum \frac{[K - r t_{ex} - F_0 - \rho X_2 + \frac{1}{2} (\rho + \sqrt{1 - \rho^2}) \Sigma^2]^2}{2 (1 - \rho^2) x^2}
= \sqrt{1 - \rho^2} \sum \frac{d^2}{2 \pi}
\]

Plotting (3.33) back into (3.32), we get

\[
E_\sigma \left\{ F(t_{ex})^+ \right\} = \left[ r t_{ex} + F_0 + \rho X_2 - \frac{1}{2} (\rho + \sqrt{1 - \rho^2}) \Sigma^2 \right] \Phi (d) + \frac{\sqrt{1 - \rho^2} \sum}{\sqrt{2 \pi}} e^{-\frac{d^2}{2}}
\] (3.34)

where \(d\) is defined in (3.29).

Next, we calculate the second term on the right hand side of (3.30):

\[
E_\sigma \{ K^+ \} = E_\sigma \{ K \mathbf{1}_{\{F(T) > K\}} \}
= K \mathbb{P} \left\{ \sqrt{1 - \rho^2} X_1 > K - r t_{ex} - F_0 - \rho X_2 + \frac{1}{2} \Sigma^2 \right\}
= K \Phi \left( \frac{r t_{ex} + F_0 + \rho X_2 - \frac{1}{2} \Sigma^2 - K}{\sqrt{1 - \rho^2} \Sigma} \right)
= K \Phi (d)
\] (3.35)

where \(d\) is as defined in equation (3.29). Substituting (3.34) and (3.35) in equation (3.30), we get (3.28).

As we have expected, Theorem 2 shows that under model (3.26) - (3.27), conditional expected value (3.5) depends on volatility in the form of its integrals \(X_2\) and \(\Sigma^2\) instead of the entire volatility path \(\sigma(t)\).

\subsection{Case 0 < \beta < 1}

In this subsection, we will discuss a more general case of SABR model, when \(\beta\) takes any value between 0 and 1. Based on the discussions in previous to subsections, let \(B_1\) and \(B_2\) be
two independent standard Brownian motions and all other notations as defined in (3.1) – (3.3), we rewrite these equations into following equivalent form

\[ dF(t) = rF(t) \beta + \sigma F(t) \beta (\sqrt{1 - \rho^2} dB_1(t) + \rho dB_2(t)) \]

(3.36)

\[ d\sigma(t) = \alpha \sigma(t) dB_2(t) \]

(3.37)

Conditioning on volatility path \(\sigma(t)\) and applying Ito’s Lemma, we solve the underlying forward price \(F(t)\) at any given time \(t\),

\[ F(t) = \left[ rt_{ex}(1 - \beta) + F^1_0 - \beta + (1 - \beta)\sqrt{1 - \rho^2} \int_0^t \sigma(s) dB_1(s) \right. \]

\[ \left. + (1 - \beta)^2 \rho \int_0^t \sigma(s) dB_2(s) - \frac{1 - \beta}{2}(\rho + \sqrt{1 - \rho^2}) \int_0^t \sigma^2(u) du \right]^{1/\beta}. \]

And more specifically, at maturity \(t_{ex}\), the forward price \(F(t_{ex})\) would be

\[ F(t_{ex}) = \left[ rt_{ex}(1 - \beta) + F^1_0 - \beta + (1 - \beta)\sqrt{1 - \rho^2} X_1 \right. \]

\[ \left. + (1 - \beta)^2 \rho X_2 - \frac{1}{2}(1 - \beta)(\rho + \sqrt{1 - \rho^2})\Sigma^2 \right]^{1/\beta}, \]

(3.38)

where notations are same as defined from (3.13) to (3.15).

For calculation convenience, we introduce a new notation

\[ L = rt_{ex}(1 - \beta) + F^1_0 - \beta + (1 - \beta)\rho X_2 - \frac{1}{2}(1 - \beta)(\rho + \sqrt{1 - \rho^2})\Sigma^2, \]

And (3.38) can be further rewritten as

\[ F(t_{ex}) = \left[ L + (1 - \beta)\sqrt{1 - \rho^2} X_1 \right]^{1/\beta}. \]

(3.39)

Notice that, conditioning on volatility path (3.37), \(X_1\) is a normal random variable \(N(0, \sigma^2)\), \(\sigma^2\) and \(X_2\) are constants. The following theorem gives (3.5) under model (3.36) – (3.37).
Theorem 3. Conditioning on volatility path (3.37), the price of a European call option under SABR model, i.e. the conditional expected value (3.5) is

\[ \int_{K^{1-\beta}-L}^{\infty} (x + L)^{1-\beta} dF_{(1-\beta)\sqrt{1-\rho^2}}(x) - K \Phi(d) \] (3.40)

where \( \Phi \) is the cumulative distribution function of standard normal distribution, where \( L \) and \( d \) are defined as

\[ L = rt_{ex}(1-\beta) + F^{1-\beta}_0 + (1-\beta)\rho X_2 - \frac{1}{2}(1-\beta)(\rho + \sqrt{1-\rho^2})\Sigma^2, \] (3.41)

\[ d = \frac{L - K^{1-\beta}}{(1-\beta)\sqrt{1-\rho^2}} \Sigma. \] (3.42)

Proof. First using the definitions (3.11) and (3.12), we decompose (3.5) as

\[ \mathbb{E}_\sigma\{ \max(F(t_{ex}) - K, 0) \} = \mathbb{E}_\sigma\{ F(t_{ex})^+ \} - \mathbb{E}_\sigma\{ K^+ \} \] (3.43)

Calculate the two terms on the right hand side of (3.43), we get

\[
\mathbb{E}_\sigma\{ F(t_{ex})^+ \} = \mathbb{E}_\sigma\{ [[(1-\beta)\sqrt{1-\rho^2}X_1 + L]^{\frac{1}{1-\beta}}]^+ \} \\
= \mathbb{E}_\sigma\{ [[(1-\beta)\sqrt{1-\rho^2}X_1 + L]^{\frac{1}{1-\beta}} 1_{\{F(t_{ex})>K\}} \} \\
= \mathbb{E}_\sigma\{ [[(1-\beta)\sqrt{1-\rho^2}X_1 + L]^{\frac{1}{1-\beta}} 1_{\{(1-\beta)\sqrt{1-\rho^2}X_1>K^{1-\beta}-L\}} \} \\
= \int_{K^{1-\beta}-L}^{\infty} (x + L)^{\frac{1}{1-\beta}} dF_{(1-\beta)\sqrt{1-\rho^2}}(x) \] (3.44)

and

\[
\mathbb{E}_\sigma\{ K^+ \} = \mathbb{E}_\sigma\{ K1_{\{F(t_{ex})>K\}} \} \\
= KP\{(1-\beta)\sqrt{1-\rho^2}X_1 > K^{1-\beta} - L \} \\
= K\Phi\left( \frac{L - K^{1-\beta}}{(1-\beta)\sqrt{1-\rho^2}} \right) \\
= K\Phi(d) \] (3.45)

Substituting (3.44) and (3.45) into equation (3.43), we have proved (3.40). \[\square\]
Unless Theorem 1 and Theorem 2, Theorem 3 does not get rid of $X_1$ and give us a direct function of $X_2$ and $\Sigma^2$. Instead, in (3.40), $X_1$ still appears in the integral

$$\int_{K^{1-\beta}-L}^{\infty} (x + L)^{\frac{1}{1-\beta}} dF(1-\beta)\sqrt{1-\rho^2}X_1(x).$$

(3.46)

However, given $X_2$ and $\Sigma^2$ sample and the distribution of normal random variable $X_1$, this one-dimensional integral (3.46) is easy and quick to carry out numerically.

However, numerically integrating (3.46) has its limitations under certain circumstances. We will show in later chapters that, as $\beta$ approaching 1, integral (3.46) would explode or even impossible to carry out a result. This is probably because that the exponent of integrand $\frac{1}{1-\beta}$ diverges so fast as $\beta$ approaching 1.

To resolve this problem in numerical computation when $\beta$ is very close to 1, we approximate the integrand in (3.46)

$$(x + L)^{\frac{1}{1-\beta}}$$

and calculate the integral of the proxy instead in this case.

For calculation convenience, we introduce the following notation:

$$A = \sqrt{1-\rho^2}X_1 + \rho X_2 - \frac{1}{2}\Sigma^2(\rho + \sqrt{1-\rho^2}) + rt_{ex},$$

(3.47)

$$B = \rho X_2 - \frac{1}{2}(\rho + \sqrt{1-\rho^2})\Sigma^2 + rt_{ex}$$

(3.48)

And the following theorem indicates how to approximate (3.40) in numerical calculation when $\beta$ is extremely close 1.

**Theorem 4.** **Conditioning on volatility path (3.37), the price of a European call option under SABR model, i.e. the conditional expected value (3.5) can be approximated by**

$$\int_{K^{1-\beta}-L}^{\infty} e^{B + x + \log F_0 - \frac{1}{2}(B+x)^2 + 2(B+x)\log F_0}(1-\beta) dF \frac{dF}{\sqrt{1-\rho^2}X_1(x)} - K \Phi(d),$$

(3.49)
where \( \Phi \) is the cumulative distribution function of standard normal distribution, where \( L \) and \( d \) are defined as

\[
B = \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2})\Sigma^2 + rt_{ex}, 
\]

\[
L = rt_{ex}(1 - \beta) + F_{0}^{1-\beta} + (1 - \beta)\rho X_2 - \frac{1}{2}(1 - \beta)(\rho + \sqrt{1 - \rho^2})\Sigma^2, 
\]

\[
d = \frac{L - K^{1-\beta}}{(1 - \beta)\sqrt{1 - \rho^2}}. 
\]

**Proof.** It suffices to show that

\[
\int_{K^{1-\beta} - L}^{\infty} (x + L)^{\frac{1}{1-\beta}} dF_{\frac{1}{1-\beta}}(x) 
\]

\[
\approx \int_{K^{1-\beta} - L}^{\infty} e^{B+x+\log F_{0} - \frac{1}{2}(B+x)^2 + 2(B+x)\log F_{0}(1-\beta)^2} dF_{\frac{1}{1-\rho^2}}(x) 
\]

Let \( \epsilon = 1 - \beta \), according to (3.39),

\[
F(t_{ex}) = (\epsilon A + F_{0}^{\epsilon})^{\frac{1}{\epsilon}} = e^{\frac{1}{\epsilon} \log(\epsilon A + F_{0}^{\epsilon})}. 
\]

Applying Taylor’s expansion on the exponent of (3.54) and omit higher order terms \( O(\epsilon^2) \), we have

\[
\frac{1}{\epsilon} \log(\epsilon A + F_{0}^{\epsilon}) 
\]

\[
= \frac{1}{\epsilon} \log [1 + (A + \log F_{0})\epsilon + \frac{1}{2}(\log F_{0})^2\epsilon^2 + O(\epsilon^3)] 
\]

\[
= \frac{1}{\epsilon} [(A + \log F_{0})\epsilon - \frac{1}{2}(A^2 + 2A\log F_{0})\epsilon^2 + O(\epsilon^3)] 
\]

\[
= A + \log F_{0} - \frac{1}{2}(A^2 + 2A\log F_{0})\epsilon + O(\epsilon^2) 
\]

Plugging (3.55) back into (3.54), we get

\[
F(t_{ex}) = e^{A + \log F_{0} - \frac{1}{2}(A^2 + 2A\log F_{0})\epsilon + O(\epsilon^2)} 
\]
Substituting (3.56) into (3.43), the first term on its right hand side becomes

\[
\mathbb{E}_\sigma \{ F(t_{ex})^+ \} = \mathbb{E}_\sigma \{ e^{A + \log F_0 - \frac{1}{2}(A^2 + 2A \log F_0) \epsilon + O(\epsilon^2)} 1_{\{F(t_{ex}) > K\}} \} 
\]

\[
= \mathbb{E}_\sigma \{ e^{B + \sqrt{1-\rho^2} X_1 + \log F_0 - \frac{1}{2}[(B+\sqrt{1-\rho^2} X_1)^2 + 2(B+\sqrt{1-\rho^2} X_1) \log F_0] \epsilon + O(\epsilon^2)} \times P\{\sqrt{1-\rho^2} X_1 > \frac{K^{1-\beta} - L}{1-\beta}\} \}
\]

\[
\approx \int_{\frac{K^{1-\beta} - L}{1-\beta}}^\infty e^{B+x+\log F_0 - \frac{1}{2}[(B+x)^2 + 2(B+x) \log F_0] \epsilon} dF_{\sqrt{1-\rho^2} X_1 (x)}, \quad (3.57)
\]

and this proves (3.53).

We will see in later chapters that Theorem 4 works perfectly to calculate approximated option prices when \(\beta\) is very close to 1, where (3.40) would fail to work.

### 3.2 Moment matching

Recalling the general SABR model (3.1) – (3.3)

\[
dF(t) = rF(t)^\beta dt + \sigma(t)F(t)^\beta dW_1(t),
\]

\[
d\sigma(t) = \alpha \sigma(t) dW_2(t),
\]

\[
dW_1(t)dW_2(t) = \rho dt,
\]

and its equivalent form

\[
dF(t) = rF(t)^\beta dt + \sigma(t)F(t)^\beta [\sqrt{1-\rho^2} dB_1(t) + \rho dB_2(t)], \quad (3.58)
\]

\[
d\sigma(t) = \alpha \sigma(t) dB_2(t). \tag{3.59}
\]

We want to calculate (3.4), the value of a European call option under this model

\[
C(F_0, K) = e^{-r t_{ex}} \mathbb{E}\left\{ \mathbb{E}_\sigma \{ \max(F(t_{ex}) - K, 0) \} \right\}.
\]
In Section 3.1, conditioning on volatility path (3.59), we gave the inner conditional expected value of (3.4), as in (3.5)

$$E_\sigma \{ \max(F(t_{ex}) - K, 0) \}$$

for all $0 \leq \beta \leq 1$. And we notice that for all $\beta$, (3.5) is a function of the following two volatility summary statistics as defined in (3.13) and (3.15), i.e.

$$\Sigma^2 = \int_0^{t_{ex}} \sigma(u)^2 du,$$
$$X_2 = \int_0^{t_{ex}} \sigma(u) dB_2(u).$$

In principle, the final result of option pricing formula under SABR model, i.e. formula (3.4), is an expected value with respect to the joint distribution of $(\Sigma^2, X_2)$. Since the exact distribution of $(\Sigma^2, X_2)$ is never known, our strategy is to approximate this joint distribution. It is worth noticing that this would be a Monte Carlo method for 2-D random vector and understandably would be much faster than the brute-force Monte Carlo method for the high dimensional sample path of $\sigma(t)$ ($0 < t < T$). This is the major contribution of our work. We are going to present such probability approximation schemes in what follows by proposing certain distribution families with moment matching.

To price European options under SABR model, we need to calculate (3.4). This computation is done by Monte Carlo method, which will be introduced in the next chapter.

Before proposing joint distribution of $\Sigma^2$ and $X_2$, we need to know more information on these two random variables and study how they are correlated with each other. In this section, we compute the moments of $(\Sigma^2, X_2)$, which are mainly applications of stochastic calculus [see Stochastic Calculus For Finance II: Continuous-time Models (Shreve, 2004), Brownian Motion and Stochastic Calculus (Karatzas and Shreve, 2012) etc.].
To start with, we notice that given the dynamics of volatility in (3.59) and initialization, the system

\begin{align*}
    d\sigma(t) &= \alpha \sigma(t) dB_2(t), \\
    \sigma(0) &= \sigma_0,
\end{align*}

indicates that volatility \( \sigma(t) \) is a geometric Brownian motion

\[ \sigma(t) = \sigma_0 e^{-\frac{\alpha^2}{2} t + \alpha B_2(t)}. \]

### 3.2.1 Moments of \( \int_0^t \sigma^2(u) du \)

The following theorem gives first three moments of \( \Sigma^2 \).

**Theorem 5.** Let \( B_2(t) \) be a standard Brownian motion, \( \sigma(t) \) be the volatility process defined in (3.59)

\begin{align*}
    d\sigma(t) &= \alpha \sigma(t) dB_2(t) \\
    \sigma(0) &= \sigma_0,
\end{align*}

and \( \Sigma^2 \) as defined in (3.13). Then the first three moments of \( \Sigma^2 \) are:

\begin{align*}
    \mathbb{E}\{\Sigma^2\} &= \frac{\sigma_0^2}{2} (e^{\alpha^2 t_{ex}} - 1), \\
    \mathbb{E}\{(\Sigma^2)^2\} &= 2\frac{\sigma_0^4}{5} \left( \frac{1}{6} e^{6\alpha^2 t_{ex}} - e^{\alpha^2 t_{ex}} + \frac{5}{6} \right), \\
    \mathbb{E}\{(\Sigma^2)^3\} &= \frac{\sigma_0^6}{315} \left( e^{15\alpha^2 t_{ex}} - 7 e^{6\alpha^2 t_{ex}} + 27 e^{\alpha^2 t_{ex}} - 21 \right).
\end{align*}

**Proof.** By direct integration.
First moment

\[
E\{\Sigma^2\} = E\left\{ \int_0^{t_{ex}} \sigma^2(u)du \right\} \\
= \int_0^{t_{ex}} E\{\sigma_0^2 e^{-\alpha^2u + 2\alpha B_2(u)}\} du \\
= \sigma_0^2 \int_0^{t_{ex}} E\{e^{2\alpha B_2(u)}\} e^{-\alpha^2u} du \\
= \sigma_0^2 \int_0^{t_{ex}} e^{2\alpha^2u - \alpha^2u} du \\
= \sigma_0^2 \int_0^{t_{ex}} e^{\alpha^2u} du \\
= \frac{\sigma_0^2}{\alpha^2} (e^{\alpha^2 t_{ex}} - 1) \quad (3.63)
\]

Second moment

\[
E\{\Sigma^4\} = E\left\{ \left[ \int_0^{t_{ex}} \sigma^2(u)du \right]^2 \right\} \\
= E\left\{ \int_0^{t_{ex}} \int_0^{t_{ex}} \sigma^2(u)\sigma^2(v)dvdu \right\} \\
= 2E\left\{ \int_0^{t_{ex}} \int_0^{t_{ex}} \sigma^2(u)\sigma^2(v)dvdu \right\} \\
= 2 \int_0^{t_{ex}} \int_0^{t_{ex}} E\{\sigma^2(u)\sigma^2(v)\} dudv \quad (3.64)
\]

In order to solve (3.64), we first need to rewrite its integrand from expected value to a function of \(u\) and \(v\). Without loss of generality, we assume that \(0 \leq u \leq v \leq t_{ex}\). Then,

\[
E\{\sigma^2(u)\sigma^2(v)\} = E\{\sigma(u)^4 e^{-\alpha^2(v-u) + 2\alpha B_2(v-u)}\} \\
= \sigma_0^4 e^{-2\alpha^2 u - \alpha^2(v-u)} E\{e^{4\alpha B_2(u)}\} E\{e^{2\alpha B_2(v-u)}\} \\
= \sigma_0^4 e^{-2\alpha^2 u - \alpha^2(v-u)} e^{8\alpha^2 u} e^{2\alpha^2(v-u)} \\
= \sigma_0^4 e^{5\alpha^2 u + \alpha^2 v} \quad (3.65)
\]
Substituting (3.65) back into (3.64) gives

\[
\begin{align*}
\mathbb{E}\{\Sigma^4\} &= 2 \int_0^{t_{ex}} \int_0^v \sigma_0^4 e^{5\alpha^2 u + \alpha^2 v} du dv \\
&= 2\sigma_0^4 \int_0^{t_{ex}} e^{\alpha^2 v} \int_0^v e^{5\alpha^2 u} du dv \\
&= 2\sigma_0^4 \int_0^{t_{ex}} e^{\alpha^2 v} \frac{1}{5\alpha^2}(e^{5\alpha^2 v} - 1) dv \\
&= \frac{2\sigma_0^4}{5\alpha^2} \int_0^{t_{ex}} (e^{5\alpha^2 v} - e^{\alpha^2 v}) dv \\
&= \frac{2\sigma_0^4}{5\alpha^2} \left[ \frac{1}{6\alpha^2}(e^{6\alpha^2 t_{ex}} - 1) - \frac{1}{\alpha^2}(e^{\alpha^2 t_{ex}} - 1) \right] \\
&= \frac{2\sigma_0^4}{5\alpha^4} \left( \frac{1}{6} e^{6\alpha^2 t_{ex}} - e^{\alpha^2 t_{ex}} + \frac{5}{6} \right) \\
&= \frac{2\sigma_0^4}{5\alpha^4} \left( \frac{1}{6} e^{6\alpha^2 t_{ex}} - e^{\alpha^2 t_{ex}} + \frac{5}{6} \right) (3.66)
\end{align*}
\]

Third moment

\[
\begin{align*}
\mathbb{E}\{\Sigma^6\} &= \mathbb{E}\{\int_0^{t_{ex}} \sigma^2(u) du \}^3 \\
&= \mathbb{E}\{ \int_0^{t_{ex}} \sigma^2(u) du \int_0^{t_{ex}} \sigma^2(v) dv \int_0^{t_{ex}} \sigma^2(w) dw \} \\
&= 6\mathbb{E}\{ \int_0^{t_{ex}} \int_0^w \int_0^v \sigma^2(u)\sigma^2(v)\sigma^2(w) du dv dw \} \\
&= 6 \int_0^{t_{ex}} \int_0^w \int_0^v \mathbb{E}\{\sigma^2(u)\sigma^2(v)\sigma^2(w)\} du dv dw \\
&= 6 \int_0^{t_{ex}} \int_0^w \int_0^v \mathbb{E}\{\sigma^2(u)\sigma^2(v)\sigma^2(w)\} du dv dw \\
&= \mathbb{E}\{\sigma^2(u)\sigma^2(v)\sigma^2(w)\} (3.67)
\end{align*}
\]

Without loss of generality, we assume \(0 \leq u \leq v \leq w \leq 1\). Then the integrand of (3.67) can be further solved as

\[
\begin{align*}
\mathbb{E}\{\sigma^2(u)\sigma^2(v)\sigma^2(w)\} &= \mathbb{E}\{\sigma^6(u)\} \frac{1}{\sigma_0^6} \mathbb{E}\{\sigma^4(v - u)\} \frac{1}{\sigma_0^4} \mathbb{E}\{\sigma^2(w - v)\} \\
&= \frac{1}{\sigma_0^6} \mathbb{E}\{\sigma^6(u)\} \mathbb{E}\{\sigma^4(v - u)\} \mathbb{E}\{\sigma^2(w - v)\} \\
&= \sigma_0^6 e^{15\alpha^2 u + 6\alpha^2 (v - u) + \alpha^2 (w - v)} \\
&= \sigma_0^6 e^{9\alpha^2 u + 5\alpha^2 v + \alpha^2 w} . (3.68)
\end{align*}
\]
Substituting (3.68) back into (3.67), we get

\[
\mathbb{E}\{\left[\int_0^{t_{ex}} \sigma^2(u)du\right]^3\} = 6 \int_0^{t_{ex}} \int_0^w \int_0^v \sigma_0^6 e^{9\alpha^2 u + 5\alpha^2 v + \alpha^2 w} du dv dw
\]

\[
= 6\sigma_0^6 \int_0^{t_{ex}} e^{\alpha^2 w} \int_0^w e^{5\alpha^2 v} \int_0^v e^{9\alpha^2 u} du dv dw
\]

\[
= \frac{6\sigma_0^6}{9\alpha^2} \int_0^T e^{\alpha^2 w} \int_0^w e^{5\alpha^2 v}(e^{9\alpha^2 v} - 1) dv du
\]

\[
= \frac{\sigma_0^6}{105\alpha^4} \int_0^T e^{\alpha^2 w} (5e^{14\alpha^2 w} - 14e^{5\alpha^2 w} + 9) du
\]

\[
= \frac{\sigma_0^6}{315\alpha^9} (e^{15\alpha^2 t_{ex}} - 7e^{6\alpha^2 t_{ex}} + 27e^{\alpha^2 t_{ex}} - 21).
\]

Thus, the first three moments of \(\Sigma^2\) are solved.

\[\square\]

### 3.2.2 Moments of \(\int_0^{t_{ex}} \sigma(u)dB_2(u)\)

In this subsection, we study the moments of Ito integral \(X_2\). The following theorems make a very general conclusion by giving all moments of \(X_2\).

**Theorem 6.** Let \(B_2(t)\) be a standard Brownian motion, \(\sigma(t)\) be the volatility process defined in (3.59)

\[
d\sigma(t) = \alpha \sigma(t) dB_2(t),
\]

\[
\sigma(0) = \sigma_0.
\]

And \(X_2\) as defined in (3.15). Then the first two moments of \(X_2\) are:

\[
\mathbb{E}\{X_2\} = 0,
\]

\[
\mathbb{E}\{X_2^2\} = \frac{\sigma_0^2}{\alpha^2} (e^{\alpha^2 t_{ex}} - 1).
\]

**Proof.** Use properties of Ito’s integral.

**First moment**

The proof is trivial, using the fact that \(X_2\) is a martingale.
Second moment

Using Ito’s isometry, we have

\[
\mathbb{E}\{X_2^2\} = \mathbb{E}\{\int_0^{t_{ex}} \sigma(u) dB_2(u)^2\} = \mathbb{E}\{\int_0^{t_{ex}} \sigma^2(u) du\} = \frac{\sigma_0^2}{\alpha^2} e^{\alpha^2 t_{ex}} - 1 \tag{3.69}
\]

Now, we obtain a more general conclusion by calculating all moments of \(X_2\). And the result is stated in the following theorem.

**Theorem 7.**

\[
\mathbb{E}\{X_2^n\} = \frac{\sigma_0^n}{\alpha^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{\frac{1}{2} k(k-1) \alpha^2 T}.
\]

**Proof.** Integrating both sides of (3.59) from 0 to \(t_{ex}\), we get

\[
\int_0^{t_{ex}} d\sigma(u) = \int_0^{t_{ex}} \alpha \sigma(u) dB_2(u). \tag{3.70}
\]

Rearranging (3.70), we have

\[
\int_0^{t_{ex}} \sigma(u) dB_2(u) = \frac{1}{\alpha} [\sigma(t_{ex}) - \sigma_0].
\]

Therefore, the \(n^{th}\) moment of \(X_2\) is

\[
\mathbb{E}\{X_2^n\} = \mathbb{E}[\int_0^{t_{ex}} \sigma(u) dB_2(u)]^n = \frac{1}{\alpha^n} \mathbb{E}[\sigma(t_{ex}) - \sigma_0]^n = \frac{1}{\alpha^n} \mathbb{E} \left\{ \sum_{k=0}^{n} \binom{n}{k} \sigma^k(t_{ex})(-\sigma_0)^{n-k} \right\} = \frac{1}{\alpha^n} \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\{\sigma^k(t_{ex})\} (-\sigma_0)^{n-k} \tag{3.71}
\]

\[
= \frac{1}{\alpha^n} \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\{\sigma^k(t_{ex})\} (-\sigma_0)^{n-k} \tag{3.71}
\]
Now, it suffices to calculate $E\{\sigma^k(t_{ex})\}$. Applying Ito’s formula to $d\sigma^k(t)$, we have

$$d\sigma^k(t) = k\sigma^{k-1}(t)d\sigma(t) + \frac{1}{2}k(k-1)\sigma^{k-2}(t)(d\sigma(t))^2$$

$$= \frac{1}{2}k(k-1)\sigma^{k-2}(t)\alpha^2\sigma^2(t)dt + k\sigma^{k-1}(t)\alpha\sigma(t)dB_2(t)$$

$$= \frac{1}{2}k(k-1)\alpha^2\sigma^k(t)dt + k\alpha\sigma^k(t)dB_2(t). \quad (3.72)$$

And notice that (3.72) indicates that $\sigma^k(t)$ is a geometric Brownian Motion with drift term $\frac{1}{2}k(k-1)\alpha^2$. Therefore,

$$E\{\sigma(t_{ex})^k\} = \sigma_0^k e^{\frac{1}{2}k(k-1)\alpha^2 t_{ex}}.$$ 

Substituting previous equation back into (3.71) and rearranging, we get the formula for $X_2$’s $n^{th}$ moment that

$$E\{X_2^n\} = \frac{\sigma_0^n}{\alpha^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{\frac{1}{2}k(k-1)\alpha^2 t_{ex}}$$

Using Theorem 7, the first four moments of $X_2$ are as what follow. And the first two moments match the conclusion of Theorem 6 as we have expected.

$$E\{X_2\} = 0$$

$$E\{X_2^2\} = \frac{\sigma_0^2}{\alpha^2} (e^{\alpha^2 t_{ex}} - 1)$$

$$E\{X_2^3\} = \frac{\sigma_0^3}{\alpha^3} (e^{3\alpha^2 t_{ex}} - 3e^{\alpha^2 t_{ex}} + 2)$$

$$E\{X_2^4\} = \frac{\sigma_0^4}{\alpha^4} (e^{6\alpha^2 t_{ex}} - 4e^{3\alpha^2 t_{ex}} + 6e^{\alpha^2 t_{ex}} - 3)$$

3.2.3 Covariance of $\int_0^{t_{ex}} \sigma^2(u)du$ and $\int_0^{t_{ex}} \sigma(u)dB_2(u)$

In this subsection, we will study the covariance between $X_2$ and $\Sigma^2$. The basic idea of calculation is to divide the domain of integral into two triangle areas and integrate them seperately. The result is presented in the following theorem.
Theorem 8. Let $B_2(t)$ be a standard Brownian motion, and $\sigma(t)$ be the volatility process defined in (3.59)

$$d\sigma(t) = \alpha \sigma(t) dB_2(t),$$

$$\sigma(0) = \sigma_0.$$ 

$\sigma^2$ and $X_2$ are as defined in (3.13) and (3.15). Then,

$$\text{Cov}(\Sigma^2, X_2) = \frac{\sigma_0^3}{3\alpha^3} (e^{3\alpha^2 t_{ex}} - 3e^{\alpha^2 t_{ex}} + 2)$$

Proof. Using the result of Theorem 5 – Theorem 6 and dividing the domain of integral $[0, T] \times [0, T]$ into two triangle areas, we have

$$\text{Cov}(\Sigma^2, X_2) = \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) du - \sigma_0^2 (e^{\alpha^2 t_{ex}} - 1) \int_0^T \sigma(u) dB_2(u) \right\}$$

$$= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) du \int_0^{t_{ex}} \sigma(u) dB_2(u) \right\}$$

$$= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) du \int_0^{t_{ex}} \sigma(v) dB_2(v) \right\}$$

$$= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) dB_2(u) \right\} + \mathbb{E}\left\{ \int_0^{v<u<t_{ex}} \sigma^2(v) \sigma(u) dB_2(v) du \right\}$$

$$= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) dB_2(u) \right\} + \mathbb{E}\left\{ \int_0^{v<u<t_{ex}} \sigma^2(v) \sigma(u) dB_2(v) du \right\}$$

$$= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) dB_2(u) \right\} + \mathbb{E}\left\{ \int_0^{v<u<t_{ex}} \sigma^2(v) \sigma(u) dB_2(v) du \right\}$$

$$= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) dB_2(u) \right\} + \mathbb{E}\left\{ \int_0^{v<u<t_{ex}} \sigma^2(v) \sigma(u) dB_2(v) du \right\}$$

$$\tag{3.73}$$

$$\mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) dB_2(u) \right\} + \mathbb{E}\left\{ \int_0^{v<u<t_{ex}} \sigma^2(v) \sigma(u) dB_2(v) du \right\}$$

$$\tag{3.74}$$
Conditioning on \(\sigma(t), 0 \leq t \leq v\), we can further expand integral I in (3.73) as

\[
I = \mathbb{E}\{\int_0^{t_{ex}} \int_0^{t_{ex}} \sigma^2(v)\sigma(u)dB_2(u)dv\}
\]

\[
= \mathbb{E}\{\mathbb{E}\{\int_0^{t_{ex}} \int_0^{t_{ex}} \sigma^2(v)\sigma(u)dB_2(u)dv|\sigma(t), 0 \leq t \leq v\}\}\}
\]

\[
= \mathbb{E}\{\mathbb{E}\{\int_0^{t_{ex}} \sigma^2(v)\int_0^{t_{ex}} \sigma(u)dB_2(u)dv|\sigma(t), 0 \leq t \leq v\}\}\}
\]

\[
= \mathbb{E}\{\mathbb{E}\{\int_0^{t_{ex}} \sigma^2(v)\frac{1}{2} [\sigma(t_{ex}) - \sigma(v)] dv|\sigma(t), 0 \leq t \leq v\}\}\}
\]

\[
= \frac{1}{\alpha} \mathbb{E}\{\mathbb{E}\{\int_0^{t_{ex}} \sigma^2(v)\sigma(t_{ex})dv|\sigma(t), 0 \leq t \leq v\}\} - \frac{1}{\alpha} \mathbb{E}\{\int_0^{t_{ex}} \sigma^3(v)dv\} \quad (3.75)
\]

Using Fubini’s theorem and switching the order of the two integrations inside the first expected value in (3.75), we have

\[
\frac{1}{\alpha} \mathbb{E}\{\mathbb{E}\{\int_0^{t_{ex}} \sigma^2(v)\sigma(t_{ex})dv|\sigma(t), 0 \leq t \leq v\}\} = \frac{1}{\alpha} \mathbb{E}\{\int_0^{t_{ex}} \mathbb{E}\{\sigma^2(v)\sigma(t_{ex})|\sigma(t), 0 \leq t \leq v\}dv\}
\]

\[
= \frac{1}{\alpha} \mathbb{E}\{\int_0^{t_{ex}} \sigma^3(v)dv\} \quad (3.76)
\]

And substituting (3.76) back in (3.75), we get

\[
I = 0. \quad (3.77)
\]
Therefore, only integral II is left in (3.74). We show by direct integration and result from Theorem 5 that

\[
\begin{align*}
\text{II} &= \mathbb{E}\left\{ \int_0^{t_{ex}} \int_0^v \sigma^2(u) \sigma(v) dB_2(u) dv \right\} \\
&= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) \int_0^v \sigma(v) dB_2(u) dv \right\} \\
&= \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(u) \frac{1}{\alpha} [\sigma(v) - \sigma_0] dv \right\} \\
&= \frac{1}{\alpha} \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^3(v) dv \right\} - \frac{\sigma_0}{\alpha^3} \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(v) dv \right\} \\
&= \frac{1}{\alpha} \int_0^{t_{ex}} \mathbb{E}\left\{ \sigma^3(v) \right\} dv - \frac{\sigma_0}{\alpha^3} \mathbb{E}\left\{ \int_0^{t_{ex}} \sigma^2(v) dv \right\} \\
&= \frac{\sigma_0^3}{\alpha} \int_0^{t_{ex}} \mathbb{E}\left\{ e^{3\alpha B_2(u)} \right\} e^{-\frac{3}{2} \alpha^2 u} du - \frac{\sigma_0^3}{\alpha^3} (e^{\alpha^2 t_{ex}} - 1) \\
&= \frac{\sigma_0^3}{\alpha} \int_0^{t_{ex}} e^{3\alpha B_2(u)} e^{-\frac{3}{2} \alpha^2 u} du - \frac{\sigma_0^3}{\alpha^3} (e^{\alpha^2 t_{ex}} - 1) \\
&= \frac{\sigma_0^3}{\alpha} \int_0^{t_{ex}} e^{3\alpha^2 u} du - \frac{\sigma_0^3}{\alpha^3} (e^{\alpha^2 t_{ex}} - 1) \\
&= \frac{\sigma_0^3}{3\alpha^3} (e^{3\alpha^2 t_{ex}} - 1) - \frac{\sigma_0^3}{\alpha^3} (e^{\alpha^2 t_{ex}} - 1) \\
&= \frac{\sigma_0^3}{3\alpha^3} (e^{3\alpha^2 t_{ex}} - 3e^{\alpha^2 t_{ex}} + 2).
\end{align*}
\]

Substituting (3.77) and (3.78) back into (3.73) and (3.74), we have proved that

\[
\text{Cov}(\Sigma^2, X_2) = \frac{\sigma_0^3}{3\alpha^3} (e^{3\alpha^2 t_{ex}} - 3e^{\alpha^2 t_{ex}} + 2)
\]

\(\square\)

### 3.3 Joint distribution of \((\sigma^2, X_2)\)

In the beginning of this chapter, we wanted to price European options under SABR model by calculating (3.4)

\[
C(F_0, K) = e^{-rt_{ex}} \mathbb{E}\left\{ \mathbb{E}_\sigma \left\{ \max(F(t_{ex}) - K, 0) \right\} \right\}.
\]
We start with solving (3.5)

\[ \mathbb{E}_{\sigma} \{ \max(F(t_{ex}) - K, 0) \} \]

for different \( \beta \) values in Section 3.1. We then noticed that the European option price formula (3.4) is actually an expected value with respect to the joint distribution of \((\Sigma^2, X_2)\).

However, the joint distribution of \((\Sigma^2, X_2)\) is unknown and not easily carried out. In this section, we will approximate this distribution by using the moments of \(\Sigma^2\) and \(X_2\) we have calculated in Section 3.2.

Many proposals can be made to approximate the joint distribution of \((\Sigma^2, X_2)\). For example, bivariate Gaussian distribution can be used. (Johnson et al., 2002) In this section, we discuss Gamma mixture of normals and log-normal mixture of normals based on the consideration that \(\Sigma^2\) is a positive random variable with a noticeable skewness, whereas \(X_2\) is a Normal random variable conditioning on \(\sigma(t), 0 < t < t_{ex}\). For the similar reason, other joint distributions may also work, such as \(\Sigma^2\) following an inverse Gamma distribution \((1/\Sigma^2\) follow a Gamma distribution).

### 3.3.1 Gamma mixture of normals

In this subsection, we assume that the joint distribution of \((\Sigma^2, X_2)\) is a Gamma mixture of normal. That being said, \(\Sigma^2\) is a Gamma random variable and \(X_2|\Sigma^2\) is normally distributed conditioning on \(\Sigma^2\). Given this assumption and all parameters of SABR model (3.1) – (3.3), our purpose is to determine the parameters of \((\Sigma^2, X_2)\)’s joint distribution. This is accomplished by moment matching, which connects joint distribution parameters and SABR model parameters and initializations. Subsequently, we solve joint distribution parameters in terms of \(\alpha, \sigma_0\) and \(t_{ex}\).

For calculation convenience, we introduce following notations:

\[
\begin{align*}
\mathbb{E}(X_2^2) &= S, \\
\mathbb{E}(\Sigma^2) &= S, \\
\mathbb{E}[(\Sigma^2)^2] &= \Delta, \\
\text{Cov}(\Sigma^2, X_2) &= \Gamma.
\end{align*}
\] (3.79) (3.80) (3.81) (3.82)
And notice that $E(X_2^2) = E(\sigma^2)$ due to Ito’s isometry, and $S \Delta \geq \Gamma$ simply applying Cauchy-Schwarz inequality.

The following theorem determines the joint distribution of $(\Sigma^2, X_2)$ from specified SABR model.

**Theorem 9.** Assume that $\Sigma^2 \sim \text{Gamma}(k, \theta)$, $X_2$ is a normal random variable conditioning on $\Sigma^2$, i.e. $X_2|\Sigma^2 \sim N(a_0 + a_1 \Sigma^2, b \Sigma^2)$ with constants $a_0 \in R$, $a_1 \in R$ and $b > 0$. Then these joint distribution parameters can be expressed as functions of $\alpha, \sigma_0$ and $t_{ex}$, i.e.

$$
\begin{align*}
    k &= \frac{S^2}{\Delta - S^2}, \\
    \theta &= \frac{\Delta - S^2}{S}, \\
    a_0 &= \frac{-S \Gamma}{\Delta - S^2}, \\
    a_1 &= \frac{\Gamma}{\Delta - S^2}, \\
    b &= 1 - \frac{\Gamma^2}{S(\Delta - S^2)}.
\end{align*}
$$

where $S, \Delta$ and $\Gamma$ are defined in (3.79) – (3.82), and according to Section 3.2, they are functions of $\alpha, \sigma_0$ and $t_{ex}$.

**Proof.** Because $\Sigma^2$ is $\text{Gamma}(k, \theta)$, we have

$$
\begin{align*}
    \mathbb{E}\{\Sigma^2\} &= k\theta, \\
    \text{Var}\{\Sigma^2\} &= k\theta^2.
\end{align*}
$$

Solving these equations gives

$$
\begin{align*}
    k &= \frac{S^2}{\Delta - S^2}, \\
    \theta &= \frac{\Delta - S^2}{S}.
\end{align*}
$$

Now we establish a set of three equations to solve $a_0, a_1$ and $b$. Notice the fact about conditional expected value that

$$
\mathbb{E}X_2 = \mathbb{E}[\mathbb{E}(X_2|\sigma^2)],
$$

37
we then have the first equation

\[ a_0 + a_1 S = 0. \]  \hspace{1cm} (3.89)

Rewriting the covariance between \( \Sigma^2 \) and \( X_2 \)

\[
\text{Cov}(\Sigma^2, X_2) = \mathbb{E}\{(\Sigma^2 - \mathbb{E}\{\Sigma^2\})(X_2 - \mathbb{E}\{X_2\})\}
\]

\[
= \mathbb{E}\{(\Sigma^2 - \mathbb{E}\{\Sigma^2\})X_2\}
\]

\[
= \mathbb{E}(\Sigma^2X_2) - \mathbb{E}\{\Sigma^2\}\mathbb{E}\{X_2\}
\]

\[
= \mathbb{E}(\Sigma^2X_2)
\]

\[
= \mathbb{E}[\Sigma^2\mathbb{E}(X_2|\Sigma^2)],
\]

and we have the second equation

\[ a_0 S + a_1 \Delta = \Gamma. \]  \hspace{1cm} (3.90)

Using the formula of conditional variance that

\[
\mathbb{E}(X_2^2) = \mathbb{E}[\mathbb{E}(X_2^2|\Sigma^2)]
\]

\[
= \mathbb{E}\{\text{Var}(X_2|\Sigma^2) + [\mathbb{E}(X_2|\Sigma^2)]^2\},
\]

we have the third equation we need

\[ bS + a_0^2 + 2a_0a_1 S + a_1^2 \Delta = S. \]  \hspace{1cm} (3.91)

Solving equations (3.89), (3.90) and (3.91) directly leads to (4.32), (4.33) and (3.87).

Theorem 9 determines the Gamma mixture of normal joint distribution of \( \sigma^2 \) and \( X_2 \), and enable us to sample \((\Sigma^2, X_2)\) once the SABR model is set up.
Remark 3.3.1. We observe that \( b > 0 \) in (3.87) follows from Cauchy-Schwarz inequality

\[
\frac{\Gamma^2}{S(\Delta - S^2)} = \frac{[\text{Cov}\{\Sigma^2, X_2\}]^2}{\text{Var}\{X_2\} \text{Var}\{\Sigma^2\}} \leq 1
\]

The equality does not hold because there is no linear relationship between \( \Sigma^2 \) and \( X_2 \) with probability one.

3.3.2 Inverse Gamma mixture of normal

In this subsection, we assume that the joint distribution of \((\Sigma^2, X_2)\) is an inverse Gamma mixture of normal. That being said, \( \Sigma^2 \) is an inverse Gamma random variable and \( X_2|\Sigma^2 \) is normally distributed conditioning on \( \Sigma^2 \). Given this assumption and all parameters of SABR model (3.1) – (3.3), our purpose is to determine the parameters of \((\Sigma^2, X_2)\)’s joint distribution. This is accomplished by moment matching, which connects joint distribution parameters and SABR model parameters and initializations. Subsequently, we solve joint distribution parameters in terms of \( \alpha, \sigma_0 \) and \( t_{\text{ex}} \).

Theorem 10. Assume that \( \Sigma^2 \sim \text{inv-Gamma}(k, \theta) \), \( X_2 \) is a normal random variable conditioning on \( \Sigma^2 \), i.e. \( X_2|\Sigma^2 \sim N(a_0 + a_1 \Sigma^2, b \Sigma^2) \) with constants \( a_0 \in \mathbb{R}, a_1 \in \mathbb{R} \) and \( b > 0 \). Then these joint distribution parameters can be expressed as functions of \( \alpha, \sigma_0 \) and \( t_{\text{ex}} \), i.e.

\[
k = \frac{S^2}{\Delta - S^2} + 2, \quad (3.92)
\]

\[
\theta = \frac{S\Delta}{\Delta - S^2}, \quad (3.93)
\]

\[
a_0 = \frac{-S\Gamma}{\Delta - S^2},
\]

\[
a_1 = \frac{\Gamma}{\Delta - S^2},
\]

\[
b = 1 - \frac{\Gamma^2}{S(\Delta - S^2)}.
\]

where \( S, \Delta \) and \( \Gamma \) are defined in (3.79) – (3.82), and according to Section 3.2, they are functions of \( \alpha, \sigma_0 \) and \( t_{\text{ex}} \).
Proof. It suffices to prove (3.92) – (3.93). Given that $\Sigma^2$ is $\text{inv - Gamma}(k, \theta)$, we have

$$
\mathbb{E}\{\Sigma^2\} = \frac{1}{(k-1)\theta},
$$

$$
\text{Var}\{\Sigma^2\} = \frac{1}{(k-1)^2(k-2)\theta^2}.
$$

Solving these equations gives

$$
k = \frac{S^2}{\Delta - S^2} + 2,
$$

$$
\theta = \frac{S\Delta}{\Delta - S^2}.
$$

\[\Box\]

**Remark 3.3.2.** $X \sim \text{inv - Gamma}(k, \theta)$ is equivalent to $\frac{1}{X} \sim \text{Gamma}(k, \frac{1}{\theta})$. In practice, we simulate random variables from $\text{Gamma}(k, \frac{1}{\theta})$ and take their inverse numbers to get inverse Gamma random variables.

### 3.3.3 Lognormal mixture of normal

In this subsection, we assume that the joint distribution of $(\Sigma^2, X_2)$ is a lognormal mixture of normal. That being said, $\Sigma^2$ is a lognormal random variable and $X_2|\Sigma^2$ is normally distributed conditioning on $\Sigma^2$. Given this assumption and all parameters of SABR model (3.1) – (3.3), our purpose is to determine the parameters of $(\Sigma^2, X_2)$’s joint distribution. This is accomplished by moment matching, which connects joint distribution parameters and SABR model parameters and initializations. Subsequently, we solve joint distribution parameters in terms of $\alpha, \sigma_0$ and $t_{ex}$.

**Theorem 11.** Assume that $\Sigma^2 \sim \log N(\mu, \theta)$, $X_2$ is a normal random variable conditioning on $\Sigma^2$, i.e. $X_2|\Sigma^2 \sim N(a_0 + a_1 \Sigma^2, b\Sigma^2)$ with constants $a_0 \in R$, $a_1 \in R$ and $b > 0$. Then these joint
distribution parameters can be expressed as functions of \( \alpha, \sigma_0 \) and \( t_{ex} \), i.e.

\[
\begin{align*}
\mu &= 2 \log(S) - \frac{1}{2} \log(\Delta), \\
\theta &= \sqrt{\log\left(\frac{\Delta}{S^2}\right)}, \\
a_0 &= -\frac{S \Gamma}{\Delta - S^2}, \\
a_1 &= \frac{\Gamma}{\Delta - S^2}, \\
b &= 1 - \frac{\Gamma^2}{S(\Delta - S^2)}.
\end{align*}
\]

where \( S, \Delta \) and \( \Gamma \) are defined in (3.79) – (3.82), and according to Section 3.2, they are functions of \( \alpha, \sigma_0 \) and \( t_{ex} \).

Proof. It suffices to prove (3.94) – (3.95). Knowing that \( \Sigma^2 \) is \( \log N(\mu, \theta) \), we have

\[
\begin{align*}
\mathbb{E}\{\Sigma^2\} &= e^{\mu + \frac{\theta^2}{2}}, \\
\text{Var}\{\Sigma^2\} &= (e^{\theta^2} - 1)e^{2\mu + \theta^2}.
\end{align*}
\]

Solving these equations gives

\[
\begin{align*}
\mu &= 2 \log(S) - \frac{1}{2} \log(\Delta), \\
\theta &= \sqrt{\log\left(\frac{\Delta}{S^2}\right)}.
\end{align*}
\]
CHAPTER 4
NUMERICAL COMPUTATION

European options are often priced and hedged using Black-Scholes model. In this model, the underlying forward price follows

\[
    dF(t) = rF(t)dt + \sigma_B F(t)dW(t),
\]

and the price of a European call option is

\[
    C(S_0, t_{ex}) = S_0 \Phi(d_1) - Ke^{-r(t_{ex}-t)} \Phi(d_2),
\]

\[
    d_1 = \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma_B^2}{2})t_{ex}}{\sigma_B \sqrt{t_{ex}}},
\]

\[
    d_2 = \frac{\ln \frac{S_0}{K} + (r - \frac{\sigma_B^2}{2})t_{ex}}{\sigma_B \sqrt{t_{ex}}}.
\]

Notice that in Black-Scholes model, the price of a European option is a strict monotone function of the implied volatility \(\sigma_B\). Therefore, option prices are usually quoted by the implied volatility \(\sigma_B\), also referred as the Black vol, the unique value of the volatility which yields the option’s price when used in Black-Scholes model.

As we briefly mentioned in Section 2.2, however, instead of being a constant, the implied volatility \(\sigma_B\) actually highly depends on strikes \(K\) and terms \(t_{ex}\). And this volatility-strike relationship is observed to be a convex curve and referred as the volatility smile. And SABR model was first introduced to better capture the volatility smile. Recalling and restating the SABR model,

\[
    dF(t) = \sigma(t)F^\beta dW_1(t),
\]

\[
    d\sigma = \alpha \sigma(t)dW_2(t),
\]

\[
    dW_1(t)dW_2(t) = \rho dt..\]
In this chapter, we will discuss and compare three numerical computation of pricing procedures under SABR model:

1. The analytical approach, as proposed in managing smile risk (Hagan et al., 2002);
2. The dimension reduction approach we have introduced in Chapter 3;
3. The original Monte Carlo method.

### 4.1 Analytical approach

The main idea of the analytical approach is to calculate the option price integral

\[
V(t, f, \sigma) = \mathbb{E}\{\max\{F(t_{ex}) - K, 0\}|F(t) = f, \sigma(t) = \sigma},
\]

where \(f(t)\) and \(\sigma(t)\) are forward price and volatility as of time \(t\). This integral is then simplified to the following

\[
P(t, f, \sigma; T, K) = \int_t^{t_{ex}} \int_{-\infty}^{\infty} A^2 p(t, f, \sigma; T, K, A) dA dT,
\]

where \(p(t, f, \sigma; T, F, A)\) is the probability density of \((F(T), \sigma(T))|F(t)=f,\sigma(t)=\sigma\), where \(T\) is any time satisfies \(t < T < t_{ex}\). Notice that \(P(t, f, \sigma; T, K)\) satisfies a backward Kolmogorov equation whose coefficients are fed from SABR model parameters, therefore the initial option price integral is solved.

Considering the above result formula is not straightforward and useful, it is equated to the Black formula to give a closed-form formula for the implied volatility, which can be directly used in the Black formula (4.2) to price an option. The closed-form algebraic formula for the implied volatility \(\sigma_B(K, f)\) is

\[
\sigma_B(K, f) = \frac{\sigma}{(fK)^{(1-\beta)/2}} \left\{1 + \frac{(1-\beta)^2}{24} \log^2 f/K + \frac{(1-\beta)^4}{1920} \log^4 f/K + ...\right\} \times \frac{z}{x(z)} \\
\times \left\{1 + \left[\frac{(1-\beta)^2}{24} \frac{\sigma^2}{(fK)^{1-\beta}} + \frac{\rho \beta \nu \sigma}{4 (fK)^{(1-\beta)/2}} + \frac{2 - 3 \rho^2}{24} \nu^2\right] t_{ex}\right\} + O(t_{ex}^2),
\]

(4.6)
where
\[ z = \nu \sigma (fK)^{(1-\beta)/2} \log \frac{f}{K}, \tag{4.7} \]
and \( x(z) \) is defined by
\[ x(z) = \log \left\{ \frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right\}. \tag{4.8} \]

Notice that the higher order term \( O(t_{ex}^2) \) is usually omitted from implied volatility calculation because this term is so small that it does not have material impact on the accuracy of implied volatility, see (Hagan et al., 2002).

Pricing European call options using analytical procedure would be a straight forward implement of the implied volatility formula (4.6) and the Black formula (4.2). It is worth noticing that the complexity of formula (4.6) is needed for accurate pricing.

### 4.2 Dimension reduction approach

In this section, we will discuss the numerical computation of our dimension reduction Monte Carlo method proposed in Chapter 3. The idea of dimension reduction for SABR model is to simulate low dimensional summary statistics of the volatility to get the option price
\[ C(F_0, K) = e^{-rt_{ex}} \mathbb{E}\left\{ \mathbb{E}_\sigma \left\{ \max(F(t_{ex}) - K, 0) \right\} \right\}. \tag{4.9} \]

And for each \((\Sigma^2, X_2)\) sample drawn from the proposed joint distribution, calculate conditional option payoff
\[ \mathbb{E}_\sigma \left\{ \max(F(t_{ex}) - K, 0) \right\}. \tag{4.10} \]

As we have showed in Section 3.1.1 to Section 3.1.3, different \( \beta \) values lead to different conditional option payoff (4.10). The remainder of this section will be organized by \( \beta \) values.
4.2.1 $\beta = 1$ stochastic log normal model

Let $P(\sigma(t), t_{ex})$ denote the payoff of a European call option on maturity $t_{ex}$, conditioning on volatility path $\sigma(t), 0 < t < t_{ex}$. According to Theorem 1, when $\beta = 1$,

$$P(\sigma(t), t_{ex}) = \mathbb{E}_\sigma \{ \max(F(t_{ex}) - K, 0) \} = F_0 e^{rt_{ex} + \rho X_2 + \frac{1}{2}(1 - \rho - \sqrt{1 - \rho^2}) \Sigma^2 \Phi(d_1) - K \Phi(d_2)}, \quad (4.11)$$

where $\Phi$ is the cumulative distribution function of standard normal distribution, and $d_1$ and $d_2$ are defined by

$$d_1 = \ln \left( \frac{F_0}{K} \right) - \frac{1}{2} \left( \rho + \sqrt{1 - \rho^2} \right) \Sigma^2 + \left( 1 - \rho^2 \right) \Sigma^2 + \rho X_2 + rt_{ex}, \quad (4.12)$$

$$d_2 = \ln \left( \frac{F_0}{K} \right) - \frac{1}{2} \left( \rho + \sqrt{1 - \rho^2} \right) \Sigma^2 + \rho X_2 + rt_{ex}, \quad (4.13)$$

$$\Sigma^2 = \int_0^{t_{ex}} \sigma(u)^2 du, \quad (4.14)$$

$$X_2 = \int_0^{t_{ex}} \sigma(u) dB_2(u). \quad (4.15)$$

According to (4.9) and (4.11) to (4.15), the European call option price $C(F_0, K, t_{ex})$ is then given by

$$C(F_0, K, t_{ex}) = e^{-rt_{ex}} \mathbb{E} \{ P(\sigma(t), t_{ex}) \} = \mathbb{E} \left\{ F_0 e^{\rho X_2 + \frac{1}{2}(1 - \rho - \sqrt{1 - \rho^2}) \Sigma^2 \Phi(d_1) - e^{-rt_{ex}} K \Phi(d_2)} \right\}. \quad (4.16)$$

Recall that in Section 3.3, we made assumptions about the joint distribution of $\Sigma^2$ and $X_2$ and discussed three distribution families: Gamma mixture of normal distribution, inverse Gamma mixture of normal distribution and log normal mixture of normal distribution. In what follows, we present parameters of all three joint distributions given SABR model set up.

Case Gamma mixture of normal
According to Theorem 5 – Theorem 9,
\[
\Sigma^2 \sim \text{Gamma}(k, \theta),
\]
\[
X_2|\Sigma^2 \sim N(a_0 + a_1\Sigma^2, b\Sigma^2),
\]
where
\[
k = \frac{S^2}{\Delta - S^2}, \quad (4.17)
\]
\[
\theta = \frac{\Delta - S^2}{S}, \quad (4.18)
\]
\[
a_0 = \frac{\Delta - S^2}{\Delta - S^2}, \quad (4.19)
\]
\[
a_1 = \frac{\Gamma}{\Delta - S^2}, \quad (4.20)
\]
\[
b = 1 - \frac{\Gamma^2}{S(\Delta - S^2)}, \quad (4.21)
\]
\[
S = \frac{\sigma_0^2}{\alpha^2}(e^{\alpha^2t_{ex}} - 1), \quad (4.22)
\]
\[
\Delta = \frac{2\sigma_0^4}{5\alpha^4}(\frac{1}{6}e^{6\alpha^2t_{ex}} - e^{\alpha^2t_{ex}} + \frac{5}{6}), \quad (4.23)
\]
\[
\Gamma = \frac{\sigma_0^3}{3\alpha^3}(e^{3\alpha^2t_{ex}} - 3e^{\alpha^2t_{ex}} + 2). \quad (4.24)
\]

Case inverse Gamma mixture of normal

According to Theorem 5 – Theorem 8 and Theorem 10,
\[
\Sigma^2 \sim \text{Gamma}(k, \theta),
\]
\[
X_2|\Sigma^2 \sim N(a_0 + a_1\Sigma^2, b\Sigma^2),
\]
where

\[ k = \frac{S^2}{\Delta - S^2} + 2, \quad (4.25) \]
\[ \theta = \frac{S\Delta}{\Delta - S^2}, \quad (4.26) \]
\[ a_0 = \frac{-S\Gamma}{\Delta - S^2}, \quad (4.27) \]
\[ a_1 = \frac{\Gamma}{\Delta - S^2}, \quad (4.28) \]
\[ b = 1 - \frac{\Gamma^2}{S(\Delta - S^2)}, \quad (4.29) \]

and \( S, \Delta \) and \( \Gamma \) are as defined by (4.22) – (4.24).

**Case log normal mixture of normal**

According to Theorem 5 – Theorem 8 and Theorem 11,

\[ \Sigma^2 \sim \Gamma(\mu, \theta), \]
\[ X_2|\Sigma^2 \sim N(a_0 + a_1\Sigma^2, b\Sigma^2), \]

where

\[ \mu = 2\log(S) - \frac{1}{2}\log(\Delta), \quad (4.30) \]
\[ \theta = \sqrt{\log(\frac{\Delta}{S^2})}, \quad (4.31) \]
\[ a_0 = \frac{-S\Gamma}{\Delta - S^2}, \quad (4.32) \]
\[ a_1 = \frac{\Gamma}{\Delta - S^2}, \quad (4.33) \]
\[ b = 1 - \frac{\Gamma^2}{S(\Delta - S^2)}, \quad (4.34) \]

and \( S, \Delta \) and \( \Gamma \) are as defined by (4.22) – (4.24).

Now, we can implement Monte Carlo method on (4.16) to price European call options by only sampling the two-dimensional random variable \((\sigma^2, X_2)\). The dimension reduction Monte Carlo procedure is:
1. Specify SABR model parameters $\alpha, \beta$ and $\rho$, model initialization $F_0$ and $\sigma_0$, and risk-free interest rate $r$;

2. Sample $(\Sigma^2, X_2)$ from one of the three joint distribution families we discussed above, with parameters defined from (4.17) to (4.34);

3. Use formula (4.11) to calculate the present value of option payoff $e^{-rt_{ex}} P(\sigma(t), t_{ex})$ for each $(\Sigma^2, X_2)$ pair;

4. Price of the European call option $C(F_0, K, t_{ex})$ is the average of all $e^{-rt_{ex}} P(\sigma(t), t_{ex})$ calculated from last step.

4.2.2 $\beta = 0$ stochastic normal model

Use the same notations defined in last subsection. Let $P(\sigma(t), t_{ex})$ denote the payoff of a European call option on maturity $t_{ex}$, conditioning on volatility path $\sigma(t), 0 < t < t_{ex}$. According to Theorem 2, when $\beta = 1$,

$$
P(\sigma(t), t_{ex}) = \mathbb{E}_\sigma \{ \max \{ F(t_{ex}) - K, 0 \} \}
= [rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2})\Sigma^2 - K] \Phi(d)
+ \frac{\sqrt{1 - \rho^2}\Sigma}{\sqrt{2\pi}} e^{-\frac{d^2}{2}},
$$

(4.35)

where $\Phi$ is the cumulative distribution function of standard normal distribution, and $d$ is defined by

$$
d = \frac{rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2})\Sigma^2 - K}{\sqrt{1 - \rho^2}\Sigma},
$$

(4.36)

and $\Sigma^2$ and $X_2$ are defined by (4.14) and (4.15).
According to (4.9) and (4.35) – (4.36), the European call option price \( C(F_0, K, t_{ex}) \) is then given by

\[
C(F_0, K, t_{ex}) = e^{-rt_{ex}} \mathbb{E}\{ P(\sigma(t), t_{ex}) \} \\
= \mathbb{E}\{ |rt_{ex} + F_0 + \rho X_2 - \frac{1}{2}(\rho + \sqrt{1 - \rho^2})\Sigma^2 - K|e^{-rt_{ex}} \Phi(d) \\
+ \frac{\sqrt{1 - \rho^2} \Sigma}{\sqrt{2\pi}} e^{-rt_{ex} - \frac{d^2}{2}} \}.
\]

(4.37)

Now, we can implement Monte Carlo method on (4.37) to price European call options by only sampling the two-dimensional random variable \((\sigma^2, X_2)\). The dimension reduction Monte Carlo procedure is:

1. Specify SABR model parameters \(\alpha, \beta\) and \(\rho\), model initialization \(F_0\) and \(\sigma_0\), and risk-free interest rate \(r\);

2. Sample \((\Sigma^2, X_2)\) from one of the three joint distribution families we discussed above, with parameters defined from (4.17) to (4.34);

3. Use formula (4.35) to calculate the present value of option payoff \(e^{-rt_{ex}} P(\sigma(t), t_{ex})\) for each \((\Sigma^2, X_2)\) pair;

4. Price of the European call option \(C(F_0, K, t_{ex})\) is the average of all \(e^{-rt_{ex}} P(\sigma(t), t_{ex})\) calculated from last step.

### 4.2.3 \(0 < \beta < 1\) case

Again use the same notations as defined in previous subsections. Let \(P(\sigma(t), t_{ex})\) denote the payoff of a European call option on maturity \(t_{ex}\), conditioning on volatility path \(\sigma(t), 0 < t < t_{ex}\). According to Theorem 3, for any \(0 < \beta < 1\),

\[
P(\sigma(t), t_{ex}) = \mathbb{E}_\sigma\{ \max(F(t_{ex}) - K, 0) \} \\
= \int_{K_{1-\beta - L}}^{\infty} (x + L) \frac{1}{\Gamma(\beta)} e^{(1-\beta)\sqrt{1-\rho^2}X_1(x) - K} \Phi(d) \\
(4.38)
\]
where $\Phi$ is the cumulative distribution function of standard normal distribution, $L$ and $d$ are defined by

$$L = rt_{ex}(1 - \beta) + F_0^{1-\beta} + (1 - \beta)\rho X_2 - \frac{1}{2}(1 - \beta)(\rho + \sqrt{1 - \rho^2})\Sigma^2,$$  \hspace{1cm} (4.39)

$$d = \frac{L - K^{1-\beta}}{(1 - \beta)\sqrt{1 - \rho^2}\Sigma},$$  \hspace{1cm} (4.40)

and $\Sigma^2$ and $X_2$ are defined by (4.14) and (4.15).

According to (4.9) and (4.38) – (4.40), the European call option price $C(F_0, K, t_{ex})$ is then given by

$$C(F_0, K, t_{ex}) = e^{-rt_{ex}}\mathbb{E}\{P(\sigma(t), t_{ex})\}$$

$$= e^{-rt_{ex}}\mathbb{E}\left\{ \int_{K^{1-\beta} - L}^{\infty} (x + L)^{\frac{1-\alpha}{2}} dF_{(1-\beta)\sqrt{1-\rho^2}X_1}(x) - K\Phi(d) \right\}. \hspace{1cm} (4.41)$$

Now, we can implement Monte Carlo method on (4.37) to price European call options by only sampling the two-dimensional random variable $(\sigma^2, X_2)$. The dimension reduction Monte Carlo procedure is:

1. Specify SABR model parameters $\alpha, \beta$ and $\rho$, model initialization $F_0$ and $\sigma_0$, and risk-free interest rate $r$;

2. Sample $(\Sigma^2, X_2)$ from one of the three joint distribution families we discussed above, with parameters defined from (4.17) to (4.34);

3. Use formula (4.38) to calculate the present value of option payoff $e^{-rt_{ex}}P(\sigma(t), t_{ex})$ for each $(\Sigma^2, X_2)$ pair;

4. Price of the European call option $C(F_0, K, t_{ex})$ is the average of all $e^{-rt_{ex}}P(\sigma(t), t_{ex})$ calculated from last step.

As we discussed in Section 3.1.3 Theorem 4, the integral in (4.41) might be troublesome in numerical calculations when $\beta$ is very close to 1, in which case we use the following pricing formula to approximate it. Under such circumstances, the European call option price $C(F_0, K, t_{ex})$ is then
given by

\[
C(F_0, K, t_{ex}) = e^{-r_{ex}} \mathbb{E}\left\{ P(\sigma(t), t_{ex}) \right\} \\
= e^{-r_{ex}} \mathbb{E}\left\{ \int_0^\infty e^{B+x+\log F_0 - \frac{1}{2}(B+x)^2 + 2(B+x) \log F_0} (1-\beta) dF \right\} \\
= e^{-r_{ex}} \mathbb{E}\left\{ \int_0^\infty K \Phi(d) \right\},
\]

(4.42)

where

\[
B = \rho X - \frac{1}{2} \left( \rho + \sqrt{1 - \rho^2} \right) \Sigma^2 + rt_{ex}.
\]

(4.43)

Generally, when $\beta$ is close to 1, use (4.42) when applying step 3 of the dimension reduction Monte Carlo procedure above.

### 4.3 Original Monte Carlo method

Recall the rewritten SABR model

\[
dF(t) = rF(t)^\beta + \sigma F(t)^\beta \left( \sqrt{1 - \rho^2} dB_1(t) + \rho dB_2(t) \right),
\]

(4.44)

\[
d\sigma(t) = \alpha \sigma(t) dB_2(t).
\]

(4.45)

We want to use Monte Carlo method to calculate the European call option price (4.9)

\[
C(F_0, K) = e^{-r_{ex}} \mathbb{E}\left\{ \mathbb{E}_\sigma \left\{ \max(F(t_{ex}) - K, 0) \right\} \right\}.
\]

(4.46)

The major work of applying Monte Carlo method is to sample underlying asset price on maturity $F(t_{ex})$, where the underlying asset’s price and volatility follow the SABR model (4.44) – (4.45). For general Monte Carlo method in finance, see Monte Carlo Methods in Financial Engineering (Glasserman, 2003). Discretizing the model gives us

\[
\sigma(t + \Delta t) = \sigma(t) + \alpha \sigma(t) \sqrt{\Delta t} X_1
\]

(4.47)

\[
F(t + \Delta t) = F(t) + rF(t)^\beta \Delta t + \sigma(t) F(t)^\beta \left[ \sqrt{(1 - \rho^2) \Delta t} X_2 + \rho \sqrt{\Delta t} X_1 \right]
\]

(4.48)
where $X_1$ and $X_2$ are independent standard normal random variables and $\Delta t$ is the time ticker. Given initialization $F(0), \sigma(0)$ and other model parameters, a sample of $F(t_{ex})$ can be generated by iterating (4.47) and (4.48) repeatedly. The present value of sample mean converges to the European call option price $C(F_0, K)$ we are looking for.
CHAPTER 5
EMPIRICAL STUDY

In this chapter, we will price European call options on both equity and fixed income products using all three approaches: the brute-force Monte Carlo method, the dimension reduction Monte Carlo methods we presented in Chapter 3 and Chapter 4, and the analytical approach first introduced in the paper managing smile risk (Hagan et al., 2002). We will compare the three joint distribution families of $(\Sigma^2, X_2)$ we introduced in Section 3.3. Also, as we have briefly mentioned in Section 3.1.3 Theorem 4, we are going to compare the original and approximated dimension reduction Monte Carlo methods when $\beta$ is between 0 and 1, and figure out the circumstances under which the approximated dimension reduction Monte Carlo method has to be used.

5.1 Model parameters

As we have discussed in previous chapters, all pricing methods require the SABR model parameters as inputs. In this section, we set model parameters for empirical studies in the rest of this chapter, and briefly introduce the data we use.

For different underlying asset types, there are different channels to obtain related option contracts information. For equity, all trading information of any specified equity product can be found on YAHOO! Finance channel. As to other underlying asset type such as energy, foreign exchange, interest rates and weather, option contracts in trading are often listed on CME Group website. In our study, we will use Microsoft stock, MSFT, as an example for equity, and iShare 20+ years treasury bond ETF, TLT, as an example of fixed income product. We will price the European options on these to assets as of March 28th 2016 that will expire on May 20th 2016.

The following two charts show option chains of MSFT and TLT on March 28th 2016 that will expire on May 20th 2016. An option chain is simply a listing of all the call and put option strike prices along with their premiums for a given maturity period. (Harris, 2003)
To fully describe the SABR model, we still need initial volatility risk-free interest rate. We use the historical volatility up to the as-of-date as a proxy of volatility initialization. This piece of information is provided by Option Strategist website to each traded underlying asset.

In the following chart, each underlying asset has three historical at-the-money volatilities based on the window length used to calculate that volatility. For example, there are three historical volatilities associated with MSFT, 24%, 31% and 26%, which are calculated from 20, 50 and 100 days historical volatility respectively as of March 24th 2016. And the volatility of MSFT on that day is 20.82%.
Risk-free interest rate is the minimum rate of return an investor should expect for any investment. In practice, three-month U.S. Treasury bill is often used as a proxy of risk-free interest rate. U.S. Department of the Treasury releases daily treasury yield curves on its website, where we quote our risk-free interest rate.

Fitting SABR model parameters is not always straightforward because some of them are not observable from market. Therefore, we determine reasonable ranges for each of $\alpha$ and $\rho$, and simulate European call option prices in scenarios with different price-volatility correlation $\rho$ and vol of vol $\alpha$ combinations.

The following two tables specify SABR model parameters for MSFT and TLT.
Table 5.1: Model Parameters Microsoft Stock

<table>
<thead>
<tr>
<th>Item</th>
<th>Symbol</th>
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### Table 5.2: Model Parameters iShare 20+ Years Treasury Bond ETF

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<th>Item</th>
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<td>Closing Price</td>
<td>$F_0$</td>
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<tr>
<td>Initial Volatility</td>
<td>$\sigma_0$</td>
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</tr>
<tr>
<td>Time to Maturity</td>
<td>$t_{ex}$</td>
<td>$39$</td>
</tr>
<tr>
<td>Risk-free Interest Rate</td>
<td>$r$</td>
<td>$0.29%$</td>
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<tr>
<td>Correlation</td>
<td>$\rho$</td>
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<tr>
<td>Vol of Vol</td>
<td>$\alpha$</td>
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### 5.2 Options on equity

In this section, we conduct empirical study of our dimension reduction Monte Carlo method on SABR model on two cases: $\beta = 1$ and $\beta$ close to 1. We use Microsoft stock as the underlying asset, because the forward price dynamics of equity is widely believed to be log normal. That being said, it suits the models $\beta = 1$ and $\beta$ close to 1 better than other underlying assets classes.
5.2.1 $\beta = 1$ Stochastic log normal model

The following table shows prices of the May 20th 2016 European call option on Microsoft stocks as of March 28th 2016 using log normal SABR model, i.e. $\beta = 1$. Model parameters are specified in Table 5.1. In the following table, each row is a scenario of a combination of $\alpha$ and $\rho$. Each scenario contains four prices: the first one is derived from the brute-force Monte Carlo Method, and the other three are from dimension reduction Monte Carlo method, whose joint distributions of $(\Sigma^2, X_2)$ are Gamma mixture of normal, inverse Gamma mixture of normal and log normal mixture of normal respectively. $\Sigma^2$ and $X_2$ are integrals of volatility path, and they were defined from (3.13) – (3.15).
Table 5.3: $\beta = 1$ SABR model prices comparison

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</table>
Table 5.3 shows that the dimension reduction approach and Monte Carlo method give relatively close results. However, among three joint distribution families of $(\Sigma^2, X_2)$, gamma mixture of normal appeared unstable and not so accurate and reliable at pricing. While log normal mixture of normal and inverse gamma mixture of normal gave closer and stabler prices benchmarking to brute-force Monte Carlo prices.

However, the analytical approach did not give similar prices to Monte Carlo methods. Setting the Monte Carlo method price as a benchmark, the following four plots give a close look of the pricing accuracy of the dimension reduction Monte Carlo method and the analytical approach.
In this subsection, we implement dimension reduction monte Carlo method on SABR model when $0 < \beta < 1$ and close to 1. Recall our discussion in Section 4.2.3. In order to price an European option under this model, we have to conduct a low-dimensional integral on

$$
\int_{K^{1-\beta-L}}^{\infty} (x + L) \frac{1}{(1-\beta)} dF \left( 1 - \frac{1}{2} \right) \sqrt{1 - \rho^2 X_1} (x),
$$

where $L$ and $d$ are defined by (4.39) and (4.40).

We mentioned in that same section that the above integral could be troublesome in numerical calculations when $\beta$ is very close to 1. Therefore, in that case, we use the following integral to approximate the original one,

$$
\int_{K^{1-\beta-L}}^{\infty} e^{B+x+\log F_0 - \frac{1}{2} ((B+x)^2 + 2(B+x) \log F_0) (1-\beta)} dF \left( 1 - \frac{1}{2} \right) \sqrt{1 - \rho^2 X_1} (x),
$$

Figure 5.5: $\beta = 1$ brute-force Monte Carlo and dimension reduction price difference: $(\Sigma^2, X_2) \sim$ Gamma mixture of normal
where $L$ and $d$ are defined the same as above, and $B$ is defined by (4.43).

In what follows, we will price the same underlying asset defined in Table 5.1, but only for one scenario of $\alpha = 0.3$ and $\rho = -0.3$ and assume the koint distribution of $(\Sigma^2, X_2)$ is log normal mixture of normal. We will try 20 different $\beta$ values that are close to 1 and reasonable for equity underlying asset. In each case, we use four different methods: Monte Carlo method, initial dimension reduction method, approximated dimension reduction method and the analytical method. Results are listed in the following table.

Table 5.4: $\beta$ close to 1 SABR model prices comparison

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Monte Carlo</th>
<th>Integral</th>
<th>Approximation</th>
<th>Analytical</th>
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<td>1.78</td>
<td>1.73</td>
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</table>
Notice that when $\beta = 0.98$ and 0.99, the original dimension reduction Monte Carlo method using pricing formula (4.41) fails. However, the approximated dimension reduction Monte Carlo method using pricing formula (4.42) works and gives almost the same numbers as the brute-force Monte Carlo method does. This is because when $\beta$ approaching 1, the higher order terms $O\left((1-\beta)^2\right)$ we omit in approximation (3.57) becomes neglectable, while at the same time the approximation succeeds in resolving the problem brought by $\frac{1}{1-\beta}$ going to infinity in the original integral.
However, this advantage of the approximated dimension reduction Monte Carlo method is weakened as $\beta$ moving away from 1, in which case higher order terms omitted are not insignificant anymore. This aligns with the above table. When $\beta < 0.95$, we are able to observe significant differences between Monte Carlo prices and approximated dimension reduction Monte Carlo prices whilst the original dimension reduction Monte Carlo worked perfectly.

In conclusion, when $\beta$ is very close to 1, the approximated dimension reduction Monte Carlo method serves as a perfect proxy to the original dimension reduction Monte Carlo method, and gives satisfactory pricing accuracy and efficiency compared with brute-force Monte Carlo method. The two methods complement each other in different ranges of $\beta$.

### 5.3 Fixed income options

In this section, we conduct empirical study of our dimension reduction Monte Carlo method on SABR model on two cases: $\beta = 0$ and $\beta$ close to 0. We use iShare 20+ years treasury bond ETF as the underlying asset, because the forward price dynamics of fixed income or interest rate product
is likely to be a normal model [See Interest Rates and FX Models, the Pricing Of Options On Debt Securities(Lesniewski, 2013), (Rendleman, 1980)]. That being said, it suits the models $\beta = 0$ and $\beta$ close to 0 better than other underlying assets types.

### 5.3.1 $\beta = 0$ stochastic normal model

The following table shows prices of the May 20th 2016 European call option on iShare 20+ years treasury bond ETF as of March 28th 2016 using log normal SABR model, i.e. $\beta = 1$. Model parameters are specified in Table 5.1. In the following table, each row is a scenario of a combination of $\alpha$ and $\rho$. Each scenario contains four prices: the first one is derived from the brute-force Monte Carlo Method, and the other three are from dimension reduction Monte Carlo method, whose joint distributions of $(\Sigma^2, X_2)$ are Gamma mixture of normal, inverse Gamma mixture of normal and log normal mixture of normal respectively. $\Sigma^2$ and $X_2$ are integrals of volatility path, and they were defined from (3.13) – (3.15).
Figure 5.9: $\beta$ close to 1: brute-force, original dimension reduction and approximated Monte Carlo prices

Table 5.5: $\beta = 0$ SABR model prices comparison

<table>
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<tr>
<th>$\alpha$</th>
<th>$\rho$</th>
<th>Monte Carlo</th>
<th>Gamma</th>
<th>Inverse Gamma</th>
<th>Log Normal</th>
<th>Analytical</th>
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</table>
Table 5.5 indicates that all three pricing methods gave very close results. As to the dimension reduction method, three joint distribution families of \((\Sigma^2, X_2)\) did not make too much difference in pricing, although we observe that log normal mixture of normal and inverse gamma mixture of normal gave closer and stabler prices benchmarking to brute-force Monte Carlo prices.

The analytical approach also appears stable with \(\alpha\) and \(\rho\), although price differences are larger compared with that between the other two approaches.

Setting the Monte Carlo method price as a benchmark, the following four plots give a close look of the pricing accuracy of the dimension reduction approach and the analytical approach.
In this subsection, we implement dimension reduction Monte Carlo method on SABR model when $0 < \beta < 1$ and close to 0. In what follows, we will price the same underlying asset defined in Table 5.2, but only for one scenario of $\alpha = 0.3$ and $\rho = -0.3$ and assume the joint distribution of $(\Sigma^2, X_2)$ is log normal mixture of normal. We will try 20 different $\beta$ values that are close to 0 and reasonable for equity underlying asset. In each case, we use four different methods: Monte Carlo method, initial dimension reduction method, approximated dimension reduction method and the analytical method. Results are listed in the following table.
Figure 5.11: $\beta = 0$ brute-force Monte Carlo and dimension reduction price difference: $(\Sigma^2, X_2) \sim$ Gamma mixture of normal

Table 5.6: $\beta$ close to 0 SABR model prices comparison

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Monte Carlo</th>
<th>Integral</th>
<th>Approximation</th>
<th>Analytical</th>
</tr>
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<tbody>
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<td>5.11</td>
<td>5.87</td>
<td>5.18</td>
</tr>
</tbody>
</table>
First we notice that the approximated dimension reduction Monte Carlo method does not really work when $\beta$ is close to 0. Actually, it does not work whenever $\beta$ is not close to 1 because higher order terms omitted are not insignificant anymore in such cases.
First we notice that the approximated dimension reduction Monte Carlo method does not really work when $\beta$ is close to 0. Actually, it does not work whenever $\beta$ is not close to 1 because higher order terms omitted are not insignificant anymore in such cases. However, as the above table shows, the original dimension reduction Monte Carlo method already works fine. It gave almost the same results as the brute-force Monte Carlo method did for all cases we tested, and consumed way less time in computation that the brute-force one did.
Figure 5.13: $\beta = 0$ brute-force Monte Carlo and dimension reduction price difference: $(\Sigma^2, X_2) \sim \log$ normal mixture of normal

In conclusion, when $0 < \beta < 1$ and $\beta$ not close to 1, our dimension reduction Monte Carlo method works and gives satisfactory results as we expected.

5.4 Conclusion

In the following table, we summarize and compare the results of empirical studies of the brute force Monte Carlo method, our dimension reduction method and the analytical method, focusing on computation efficiency and pricing accuracy.

<table>
<thead>
<tr>
<th>Method</th>
<th>Accuracy</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brute force MC</td>
<td>Good</td>
<td>Slow</td>
</tr>
<tr>
<td>Dimension Reduction</td>
<td>Good</td>
<td>Fast</td>
</tr>
<tr>
<td>Analytical Method</td>
<td>Questionable</td>
<td>Fast</td>
</tr>
</tbody>
</table>

We observe that compared with the brute force Monte Carlo method, our dimension reduction method is able to gives close option prices but consumes far less computation time. Furthermore, the prices are usually close to those observed from the market.
Figure 5.14: $\beta = 0$ brute-force Monte Carlo and analytical price difference

Figure 5.15: $\beta$ close to 0: brute-force, original dimension reduction and approximated Monte Carlo prices
Figure 5.16: $\beta$ close to 0: brute-force and original dimension reduction Monte Carlo prices
BIBLIOGRAPHY


Paulot, L. (2009). Asymptotic implied volatility at the second order with application to the sabr model. *Available at SSRN 1413649*.

