THEORY AND PHENOMENOLOGY OF KINETIC MIXING AT STRONG COUPLING

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ABSTRACT

Arada Malekian: THEORY AND PHENOMENOLOGY OF KINETIC MIXING AT STRONG COUPLING
(Under the direction of Jonathan J. Heckman)

In this dissertation, we present a class of string-based extensions of the Standard Model, which have a strongly coupled $U(1)$ as an extra sector that mixes with the $U(1)$ of the Standard Model. Calculating the mixing between these $U(1)$s is the main motivation of this dissertation. Generally, integrating out “messenger states”, which are charged under both the Standard Model and the extra $U(1)$, leads to electric and magnetic kinetic mixing between the two sectors. However, in the strongly coupled extra sector, the gauge couplings are of order one and electric and magnetic states have comparable mass. In this limit, we cannot use the well-known perturbative methods of quantum field theory and we have to resort to non-perturbative methods for mixing calculations. We introduce variation of a method originally postulated by N. Seiberg and E. Witten, which assimilates the stringy mixing between the two sectors to a $\mathcal{N} = 2$ superconformal field theory. We can then use the formal tools of $\mathcal{N} = 2$ supersymmetry to advance our calculation. We extend this method to compute the coupling constants between visible and extra sectors, by taking various flavor symmetry gauge groups to appear as our visible sector. The ultimate goal is to be able to find information about coupling constants for the case when the visible sector has a large enough flavor symmetry group that can contain all of Standard Model’s gauge group. Furthermore, we also investigate the leading order effects of supersymmetry breaking in these theories. We introduce a superpotential in the effective Lagrangian and witness that the masses of superpartners are indeed identical in the supersymmetric limit, and they diverge when we turn on the supersymmetry breaking factors. We also investigate gaugino mixing, which only appears if the supersymmetry breaking is turned on. In the end, we comment on the potential role of these extra sectors as a toy model scenario for Dark Matter, which is of interest for phenomenology.
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I dedicate this dissertation to

My Mother, for the unfailing impression that I’m making her proud.

My Father, for being a symbol of hard work and perseverance.

My Sister, for her courage to realize dreams.

My Grandmother, for incomprehensible amounts of love.

My Aunt, for always being there when nobody else is.

And to the memory of all innocent victims of police brutality all around the world.
# TABLE OF CONTENTS

**LIST OF FIGURES** ......................................................... ix

**LIST OF TABLES** ......................................................... xii

**LIST OF CONVENTIONS AND ABBREVIATIONS** ...................... xiii

1  Introduction ........................................................................ 1

2  A Look Into Supersymmetry ............................................... 6
   2.1  SUSY algebra ............................................................. 6
   2.2  SUSY Multiplets and Representations .......................... 8
       2.2.1  Massless multiplets ............................................ 8
       2.2.2  Massive BPS multiplets ...................................... 12
   2.3  Supersymmetric Field Formulation ............................. 15

3  Stringy Extensions To Standard Model ............................... 17
   3.1  Weak Coupling Limit ................................................ 18
   3.2  Strong Coupling Limit .............................................. 21
   3.3  \( N = 2 \) Superfield Formulation ............................... 24

4  The Seiberg-Witten Method .............................................. 28
   4.1  Flavor Symmetry Group: \( A_1 \) ..................................... 33
   4.2  Flavor Symmetry Group: \( A_2 \) ..................................... 39
   4.3  Flavor Symmetry Group: \( D_4 \) ..................................... 44
   4.4  Groups with Higher Rank Flavor Symmetry .................. 53

5  Breaking Supersymmetry ................................................. 54
3.1 Depiction of interaction between the Standard Model and the extra sector. a. Shows the general scheme and the “messenger” states. Under “Visible sector” can be any flavor symmetry group that partially includes SM gauge group $G_{SM}$. The “Extra sector” is the probe D3-brane. b. Pictures the model described in this chapter. The $\tau_{\text{vis}}$ seven-brane stack gives the SM gauge group (or replacements), and the $\gamma_{\text{hid}}$ stack corresponds to a flavor seven-brane. Their intersection creates the Yukawa points, and close to Yukawa point is the probe D3-brane as extra sector. The two sectors interact with each other via 3–7 messenger string states.

3.2 Feynman diagram showing the transition between $\gamma_1$ and $\gamma_2$ of two different $U(1)s$.

3.3 Vacuum polarization diagram of two different $U(1)s$ showing transition between $\gamma_1$ and $\gamma_2$.

4.1 (a). $\mathbb{C}/(1, \tau)$ plane. The lines of same color are considered equivalent, and thus construct the torus. The dashed lines are the “opened” version of cycles of the torus. (b). The torus created after connecting the equivalent lines of the $\mathbb{C}/(1, \tau)$ plane to each other. Loops $A$ and $B$ are the independent cycles of the torus.

4.2 The fundamental domain for the modular parameter $\tau$ is shaded in gray. Every point outside of this can be mapped into a point inside using an $SL(2, \mathbb{Z})$ transformation, and the $\tau$ values inside of the domain give all the values in the complex plane $\mathbb{C}$ for the $j$-function exactly once. This domain is restricted by conditions $\text{Im} \tau > 0$, $-\frac{1}{2} < \text{Re} \tau \leq \frac{1}{2}$ and $|\tau| \geq 1$, which come from $\mathbb{H}/SL(2, \mathbb{Z})$. The image is taken from Wikipedia.

4.3 Graphs of coupling constants $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ for the $A_1$ case when $f = u = 0$. (a) graph of $\tau_{\text{mix}}$. Points are concentrated at $\text{Re}\{\tau_{\text{mix}}\} = \pm 0.471405$. (b) graph of $\tau_{\text{extra}}$. All points are concentrated on $\tau_{\text{extra}} = e^{2\pi\sqrt{3}}$, which is expected as the argument of the $j$-function for $j = 0$.

4.4 Graph of $\tau_{\text{mix}}$ for the $A_1$ case when $g = 0$. At this limit, all the calculated points resulted in $\tau_{\text{mix}} = 0$. This means no mixing between the extra and visible sectors, and therefore it involves no useful information.

4.5 Graphs of components of coupling constant $\tau_{\text{mix}}$ vs. mass parameter $m$, when $u = 0.1$ (orange) and $u = 0.01$ (blue), and mass parameter $m$ takes over values from $[0, 10]$ range. Graphs are for the $A_1$ case. We notice that the curvature of component graphs (a and b) increase for smaller $u$. Graph c just gets closer to expected $\text{Im}\tau_{\text{mix}} = 0$, as $u$ gets closer to zero.

4.6 Graphs of coupling constants $\tau_{\text{mix},1}$ and $\tau_{\text{extra}}$ for the $A_2$ case when $f = u = 0$. (a) graph of $\tau_{\text{mix}}$. Points are localized in bands at $\text{Re}\{\tau_{\text{mix}}\} \approx 0, \pm 0.35, \pm 0.70$. (b) graph of $\tau_{\text{extra}}$. All points are concentrated on $\tau_{\text{extra}} = e^{2\pi\sqrt{3}/3} = -0.5 + 0.866i$, which is expected under the condition $j = 0$.

4.7 Graphs of $\tau_{\text{mix},1}$ and $\tau_{\text{mix},2}$ for $g = 0$ case in $A_2$. a. graph of $\tau_{\text{mix},1}$. b. graph of $\tau_{\text{mix},2}$. We notice that, although the distribution might not be identical, in both graphs the data points are concentrated around the same $\text{Re}\{\tau_{\text{mix}}\}$ values.

4.8 Graph of $\tau_{\text{extra}}$ for the $g = 0$ case in $A_2$. As expected, this results in only $\tau_{\text{extra}} = i$ in the fundamental domain.

4.9 Graphs of $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ for the $f = 0$ case of $D_4$. a. In the graph of $\tau_{\text{mix}}$ the data points are concentrated on six points, whereas in b. graph of $\tau_{\text{extra}}$ all points are concentrated on $-0.5 + 0.866i$, as expected for $j = 0$. 

List of Figures
4.10 Graphs of $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ for the $g = 0$ case of $D_4$. a. graph of $\tau_{\text{mix}}$ shows the values it takes when $j = 1$. b. graph of $\tau_{\text{extra}}$, showing that $\tau_{\text{extra}} = i$ when $j = 1$ as expected.

4.11 Graphs of the coupling constants for the weak coupling limit of $D_4$. a. The graph of $\tau_{\text{mix}}$ is divided into four rectangular regions in the four quadrants. b. The graph of $\tau_{\text{extra}}$ shows all the values of $\tau_{\text{extra}}$ concentrating in a band between $\text{Im}[\tau_{\text{extra}}] \approx 1.9$ and $\text{Im}[\tau_{\text{extra}}] \approx 2.8$.

4.12 $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ graphs for $D_4$ symmetry group when $\varphi$ is free to take on any value in a range equivalent to root range of $u$, no constraints. a. graph of $\tau_{\text{mix}}$. b. graph of $\tau_{\text{extra}}$.

5.1 Graph of gaugino mixing parameters $\varepsilon_{\text{mix}}$(blue) and $M_{\text{extra}}$(purple) for the $A_1$ case. On x-axis we have $|u| = [0, 10]$. In this range, $\varepsilon_{\text{mix}}$ values are rather small, but comparable to $M_{\text{extra}}$ values.

5.2 Graph of $|\partial V_{\text{bosonic}}/\partial u|$ with respect to $|u|$ for $|u| = [0, 10]$ for the case when there is no superpotential. For flavor symmetry group $A_1$.

5.3 Contour plot of magnitude of potential $V$ evaluated for 15000 values of $u$ between $|u| = [0, 1]$. This graph depicts the plot of values of $|V|$ with respect to real and imaginary parts of $u$. As it is evident, the magnitude of $|V|$ barely changes, and it is definitely not at minimum at origin $u = 0$.

5.4 Contour plots of potential $V_{\text{bosonic}}$ and its derivative $|\partial V_{\text{bosonic}}/\partial u|$ evaluated for 10000 points around $u = 1$. In both graphs, real part of $u$ lays on the x axis, and the imaginary part is on y axis. And in both graphs, we see a strong minimum at $u = 1$. 
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Maximum number of supercharges for each dimension for $d \leq 10$</td>
<td>8</td>
</tr>
<tr>
<td>6.1</td>
<td>Selection rules for electron scattering off a dyon, with assumptions that $l_z = 1$ and $l_e$ is minimal.</td>
<td>95</td>
</tr>
</tbody>
</table>
LIST OF CONVENTIONS AND ABBREVIATIONS

Our conventions follow [1]. Unless otherwise stated:

Metric signature: \((-, +, +, +)\)

Greek indices \(\mu, \nu, \ldots\): Lorentz indices. Run over four space-time coordinates 0, 1, 2, 3

Greek indices \(\alpha, \beta, \ldots\): Spinor indices. Usually run over 1, 2, for \(SO(3,1)\) Weyl spinors

Latin indices \(i, j, k, l, \ldots\): Moduli space indices. Usually run over 1, \ldots, \(r + 1\) where \(r\) is the rank of flavor symmetry group

Capital latin indices \(A, B, \ldots\): SUSY counters. Run over 1, \ldots, \(\mathcal{N}\) where \(\mathcal{N}\) is the number of supercharges

\(\theta_\alpha, \tilde{\theta}_\alpha\): Superspace coordinates

\(\epsilon^{0123}\): +1

SM: Standard Model

SUSY: Supersymmetry, Supersymmetric

SUGRA: Supergravity

DM: Dark Matter
CHAPTER 1: Introduction

As physicists, we like to group things together neatly in a way that they can be explained by as few universal laws as possible. This is the reason why Standard Model was created throughout the latter half of the 20th century in the first place. It was formulated as a gauge quantum field theory encompassing the electromagnetic, weak, and strong nuclear interactions, as well as classifying all the subatomic particles known at the time (mid-1970’s). But although a very triumphant achievement in the history of physics, it was not so satisfying because it left out gravity and the theory of general relativity. At this point, every law known in nature belonged in only one of the two boxes of physics: that of Standard Model (or alternatively, quantum physics), and that of general relativity. And laws from one box could not even be defined in the other. In addition, the core methodologies used in these two viewpoints were intrinsically different. This inconsistency was an obstacle in the attempts to explain nature as unity. Thus began a search for a theory that can combine quantum effects with general relativity and vice versa, i.e. a theory of quantum gravity.

But before we start discussing solutions to quantum gravity, let us talk a bit more about Standard Model (SM) and more reasons to why it is viewed as incomplete. SM was the result of impressive collaborations between both theoretical and experimental particle physicists from around the world. It was a renormalizable, mathematically self-consistent theory and it had the gauge symmetry group of $SU(3) \times SU(2) \times U(1)$. SM introduced and incorporated the concept of gluons and quantum chromodynamics (QCD) to modern physics. The combining of electromagnetic and weak forces, and constructing electroweak physics, which was definitely a milestone in modern physics, was also amongst the first steps of creation of SM. The theoretical side of Standard Model also predicted the existence of several particles, which were experimentally discovered decades later. But among all these successes, SM also had quite a few shortcomings. One of the main unsolved problems in SM is the hierarchy problem; The Standard Model leaves the mass of the Higgs boson as a parameter to be measured, rather than a value to be calculated – more of an empirical discovery, rather than a theoretical result. The only constraint on it is that it has to be of the order of 100 to 1000 GeV to ensure unitarity of SM. The problem occurs because quantum corrections require the Higgs boson to have a mass much higher than this, but for an inexplicable reason we have an almost-perfect cancellation of these corrections at $\sim 125$ GeV. The only possible explanations for this phenomenon are that there is either some underlying connections between these observations, which SM gives no explanation about, or that some

1 The top quark (1995), the tau neutrino (2000), and the Higgs boson (2012) are a few to name within those lines.
extremely precise fine-tuning must be applied on the mass parameter, which from theorists’ point of view is considered to be unnatural. Another problem with the SM are the neutrino masses; According to SM neutrinos are supposed to be massless, however neutrino oscillation experiments have shown that neutrinos actually do have mass! A third problem with SM is the inability to explain Dark Matter and Dark Energy. The SM does not have any candidates for weakly to not at all interacting Dark Matter, and the attempts to explain dark energy in terms of SM vacuum energy leads to a discrepancy of up to 120 orders of magnitude! And of course, last but not least, is the failure to couple to gravity; SM is just incompatible with gravity, or more accurately general relativity, and there is no way to incorporate gravity in the model without tearing it down with various assumptions first.

So enters the beautiful solution of String Theory. String theory, being a theory of quantum gravity, successfully answers the last shortcoming of SM listed in the previous paragraph, and also motivates various proposals to address the rest of the points. The idea is, that one can include gravity in a consistent quantum theory, if one gives up the notion that the fundamental particles in the theory must be point-like, and allow them to have one dimension, i.e. be strings. These fundamental strings can have a range of energies - or equivalently, masses - and thus can create different types of particles, and theories, depending on their state of oscillation. But one thing is for certain; all string theories contain a particle with zero mass and spin two. In SM formulation, there exists one massless spin two particle - the graviton! Hence, all string theories will indeed contain gravity.

Consistency puts a high level of constraints on possible string theories, however. For instance, there is a family of string theories, called bosonic string theories, that are only consistent in 26 dimensions (the name comes appropriately from the fact that these type of string theories consist only of bosons!). These theories, however, also contain tachyons – which means that their ground state has a negative mass-squared value – and that makes these theories unstable. Thus, to make the theories stable and tachyon-free, we must incorporate supersymmetry - which demands that a fermion exist for each boson and vice versa - and this results in the theory only existing in 10 dimensions. This is therefore called supersymmetric string theory, or Superstring theory. There are several types of superstrings, and therefore superstring theories, based on initial assumptions on whether or not allow our theory to be chiral (e.g. Type IIA superstring theory is nonchiral versus Type IIB superstring theory being chiral), or have specific gauge symmetries (e.g. Heterotic superstring theory has an $E_8 \times E_8$ gauge symmetry). Nevertheless, all of these different variations of superstrings have 10 dimensions.

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2 This type of supersymmetry is of course broken in low-energy limits, since we do not have equal number of fermions and bosons in nature. We will discuss an example of supersymmetry breaking in Chapter.

3 Heterotic superstring theories are consistent theories that are built by combining a bosonic string theory moving in one direction along the string, with a supersymmetric string theory moving in the opposite direction.
But as we know, the world around us in macroscopic level exists in only 4 dimensions, and in much lower energy levels than that of microscopic elements. A common approach to make contact with low energy physics, is to assume these extra dimensions are minuscule and do not appear at long distances. Mathematically, we *compactify* on the extra dimensions, and we get a low energy effective field theory as a result. A natural question then arises: How does string theory behave in \( d = 4 \) dimensions, i.e. the framework where Standard Model is formulated in? We know that String theory successfully couples gravity to SM, but can it provide suggestions for rest of SM shortcomings? It is undoubtedly important to examine possible low energy manifestations of String theory. In this thesis we are not going to give direct predictions of the strings, but rather we will present a well-motivated class of scenarios which canonically embed in string theory and we will use string theory to study these scenarios.

In a broad class of string-based models in F-theory, the Standard Model is given by a configuration of intersecting seven-branes \([5–10]\). In these configurations, the SM, also known as the *visible sector*, is realized on a stack of intersecting seven-branes, \( 7_{\text{vis}} \). Additionally, there could also exist flavor seven-branes, \( 7_{\text{hid}} \). This allows for physics beyond SM to potentially enter in specific ways; many SM extensions involve the existence of so-called “extra” sectors. Probe D3-branes are very well-motivated candidates for the extra sectors, because they are locally attracted to the points of triple intersections between these seven-branes \([11, 12]\). The physics *beyond* SM then is studied by the “interactions” between SM (visible sector) and the extra sector (the D3-brane). In this class of models, this corresponds to the 3–7 “messenger states”; the states which are charged under both SM and the natural \( U(1) \) of the D3-brane, \( U(1)_{D3} \). This encourages an investigation on mixing between these \( U(1) \)'s. Although, generally, the electric kinetic mixing has been heavily studied \([12–24]\), there is a substantial lack of analysis on magnetic kinetic mixing as far as we are aware \([25, 26]\).

It is shown in \([13]\) that the D3-brane is stable close to, but not directly on top of the \( 7_{\text{vis}} \) brane. This realizes an \( \mathcal{N} = 2 \) superconformal field theory (SCFT) with \( E_8 \) flavor symmetry to describe the model \([27, 28]\). The Standard Model gauge group then comes from breaking the \( E_8 \) flavor symmetry down to a \( G_{\text{SM}} \equiv SU(3) \times SU(2) \times U(1) \) subgroup of the \( E_8 \), which in turn has a \( U(1) \subset E_8 \) which is commonly referred to as the \( U(1) \)-hypercharge, or \( U(1)_Y \), of the SM.

Of course, because the extra sector is strongly coupled, we cannot use the perturbative methods prominent in quantum field theory for our calculations, and precisely for that reason, it is important to find calculable quantities in this limit. Fortunately, it turns out that four dimensional string theories with \( \mathcal{N} = 2 \) supersymmetry generators, have very interesting holomorphic properties that can be used to find exact results for these field theories \([29, 30]\). However, a remarkable feature of the probe theories described above then is that the \( \mathcal{N} = 2 \) SCFT which appears from D3-brane probing the seven-branes, is also the elliptic
curve appearing in a special class of F-theory compactifications, commonly known as the Seiberg-Witten curve. The **Seiberg-Witten (SW) method** then uses formal tools of elliptic curves to find the modular parameters $\tau$'s of the SW curve, which in turn act as the coupling constants between the probe D3-brane and the seven-branes, i.e. between the extra and visible sectors. Indeed, Seiberg and Witten used this method to write an exact low-energy effective Lagrangian for $\mathcal{N} = 2$ supersymmetric theory with $SU(2)$ gauge theory \[^{[37]}\]. This was later generalized to $SU(N)$ gauge groups in \[^{[38],[39]}\], and then even more generalized for the cases with exceptional Lie groups as gauge groups, or even cases where the gauge group is not specified and we cannot propose a Lagrangian \[^{[27],[40]}\].

The fine point of the SW method is that the $\mathcal{N} = 2$ supersymmetric gauge theory contains a complex adjoint matrix scalar, whose non-zero vacuum expectation value parameterizes physically distinct vacua. Thus, we can always write its effective field theory as a bundle of $U(1)$'s \[^{[41]}\]. This is because in four-dimensional $\mathcal{N} = 2$ theories, the definitive Kähler potential of the theory is also constrained by holomorphy \[^{[42]}\], hence we can make exact statements for four-dimensional strongly coupled field theories without knowing much about the details of the theory. This ability is what makes the four dimensional string theories, and specifically the $d = 4, \mathcal{N} = 2$ supersymmetric theory, much interesting subjects in field theory.

In our work, instead of $E_8$ flavor symmetry which has all of the SM gauge group as a subgroup, we assume different, smaller gauge groups with various flavor symmetries, with the presumption that the flavor symmetry of these groups would at least partially encompass the SM gauge group. Using the SW method then, with proper choice of parameters we can extract a $U(1)$ from these flavor symmetry groups; then the interaction between the visible and extra sectors is summarized in the interaction between $U(1)_{D3}$ and the extracted $U(1)$'s from the flavor symmetry group. For these flavor symmetry groups, we find the modular parameters of the SW curve, which in turn are the **coupling constants** $\tau_{ij}$'s between the probe D3-brane and the seven-branes, i.e. between the extra and visible sectors. These calculations form the heart of this thesis.

An additional scenario we investigate using this model is that of Dark Matter. It is estimated that about 27% of the mass and energy in the observable universe and 85% of the total matter in the Universe is Dark Matter (DM), but little to no information is known about the nature of DM. One of the reasons for this is because constructing a model for DM solely from astrophysical observations allows a wide range of possibilities concerning its nature, its mass, and its interactions. Furthermore, there can be a plethora of possibilities for mechanisms responsible for generating the observed Dark Matter abundance in the Universe. But the probe D3-brane described above is a strongly coupled U(1) gauge theory, which contains electrically and magnetically charged states of comparable mass. The strings stretched between the D3-brane and the visible sector stack of $7_{\text{vis}}$ then contribute massive states. Considering this and the fact that the D3-brane
is hardly interacting with the visible sector means this extra sector provides natural candidates for Dark Matter. The extra sector with these configurations is often referred to as the “Dark sector”. The electric kinetic mixing, from concepts similar to that of Dark sector, has been studied widely and resulted in several DM scenarios [22, 43–47]. However, in our scenario, we treat the physics of the Dark sector as one with $\mathcal{N} = 2$ supersymmetry, as well as both electric and magnetic kinetic mixing. This gives rise to very interesting possibilities for DM phenomenology that could be probed in DM experiments. For example, a recent review on composite dark matter scenarios with strong coupling dynamics can be found in [48].

The structure of this dissertation is as follows: in the next chapter I introduce supersymmetry and lay down the mathematical groundwork for four dimensional supersymmetric theories. In the following chapter I present details for our model and talk about the mixing terms between visible and extra sectors at both weak and strong coupling limits; the need to work in strong coupling then shapes the main motivation of this project. This brings the culmination to Chapter 3 where I introduce the Seiberg-Witten method and carry coupling constant calculations for various flavor symmetry groups, namely $A_1$, $A_2$ and $D_4$. Chapter 5 is dedicated to the effects of supersymmetry on the field theory; specifically, I calculate the bosonic and fermionic masses of the theory once in the supersymmetric limit, then I introduce supersymmetry breaking and see how the mass values diverge from each other when there is no supersymmetry. Eventually, in Chapter 6 I explore the concept of the dark sector; more specifically, I investigate Dark Rutherford scattering by treating a magnetically charged extra sector as a heavy classical source, and scatter charges (in case of direct detection experiments, most probably protons) off this source. The charges represent the visible sector and the model itself assimilates Dark Matter moving around the earth in galactic wind. We finish the dissertation with providing summary of results and an outlook of future prospective work in Chapter 7.

The Appendices include Mathematica code used for various calculations throughout this research; the main body is dedicated to magnetic mixing calculations presented in Chapter 4 but there is also code for the potential graphs and mass term calculations in Chapter 5.
CHAPTER 2: A Look Into Supersymmetry

Supersymmetry is one of the most compelling fields of study in modern physics, because it suggests solutions to several problems that are otherwise prominent in Standard Model. For instance, it has a natural solution to the hierarchy problem between the electroweak scale and the Planck scale. Supersymmetry (or SUSY, for short) also leads to gauge coupling unification at high energy (GUT scale) where Standard Model fails to do so. In addition, extensions of supersymmetry also provide Dark Matter candidates which are consistent with relic abundance calculations; we will explore a toy model of such candidates in Chapter 6. Because of these reasons and several more, supersymmetry has become the dominant framework in string theory.

In this chapter we will lay down the mathematical groundwork for formulating supersymmetry. This will begin by presenting the SUSY algebra. For the most part, we will focus on SUSY in 4 dimensions because this will be the most prominent case throughout this dissertation. However, generalizations to other dimensions will be discussed briefly.

Section 2.1: SUSY algebra

In its simplest form, SUSY in $d$ dimensions can be described by generator $Q_\alpha$ and its conjugate $\bar{Q}_{\dot{\alpha}}$. By Lorentz invariance these generators are spinor representations of $SO(d-1,1)$. In $d = 4$ therefore, they are $SO(3,1)$ Weyl spinors and they abide the following anti-commutation relations,

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu$$

(2.1)

where $P_\mu$ is the energy-momentum operator (the four-momentum in $d = 4$) and the index $\mu, (\nu,...)$ runs from one to four identifying Lorentz four-vectors. The indices $\alpha, \beta, ..., \dot{\alpha}, \dot{\beta}, ...$ run from one to two to denote the two-component Weyl spinors. Also $\sigma^\mu = (1, \sigma^i)$, where $\sigma^i, i = 1, 2, 3$ are the three Pauli matrices. It is also worthwhile to introduce the $\bar{\sigma}^\mu$ notation,

$$\sigma^\mu_{\alpha\dot{\beta}} = (1, \sigma^i)$$

$$\bar{\sigma}^{\mu\dot{\alpha}\dot{\beta}} = (1, -\sigma^i).$$
There is one more additional relation for SUSY algebra, and that comes from the fact that this symmetry does not depend on spacetime position; therefore

\[ [Q_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}}, P_\mu] = 0. \tag{2.2} \]

We also have the trivial \([P_\mu, P_\nu] = 0\).

The generator \(Q_\alpha\) is called the *supercharge* of the theory, since it exchanges superpartner states with each other. In other words, because \(Q_\alpha\) is a spinor, it turns bosons into fermions and vice versa according to the spin-statistics theorem. This is commonly written in a formal notation as

\[ \{(−1)^F, Q_\alpha\} = 0 \]

where

\[ (−1)^F \mid \text{boson} \rangle = +1 \mid \text{boson} \rangle \quad (−1)^F \mid \text{fermion} \rangle = −1 \mid \text{fermion} \rangle. \tag{2.3} \]

The relations (2.1) and (2.2) describe the formulation of the simplest form of supersymmetry, with only one supercharge \(Q_\alpha\). Generally we can have multiple supercharges governing the supersymmetry of the theory. In these cases we adopt an additional index for the supercharges and write them as \(Q^A_\alpha\), where \(A = 1, \ldots, \mathcal{N}\) for a theory with \(\mathcal{N}\) supercharges. Theories with \(\mathcal{N} > 1\) SUSY are then referred to as having *extended SUSY*; in return, the \(\mathcal{N} = 1\) SUSY described above adopts the name *unextended* supersymmetry.

The extended SUSY algebra in 4 dimensions then follows the following relations,

\[ \{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^{\mu\dot{\alpha}}_{\alpha\beta} P_\mu \delta^A_B \]
\[ \{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta} Z^{AB} \]
\[ \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = −2\epsilon_{\dot{\alpha}\dot{\beta}} Z^{AB\dagger} \]
\[ [Q^A_\alpha, P_\mu] = [\bar{Q}_{\dot{\alpha}A}, P_\mu] = 0 \tag{2.4} \]

with \([P_\mu, P_\nu] = 0\) and \(\epsilon_{\dot{\alpha}\dot{\beta}} = −\epsilon_{\dot{\beta}\dot{\alpha}}\). As already mentioned, \(A, B, \ldots = 1, \ldots, \mathcal{N}\); the rest of the indices run similar to the unextended case as described under (2.1). The \(Z^{AB}\) here are antisymmetric linear combinations of internal symmetry generators \(T^i\),

\[ Z_{AB} = \sum_i c^i_{AB} T^i \quad Z_{AB} = −Z_{BA} \]
Table 2.1: Maximum number of supercharges for each dimension for $d \leq 10$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\mathcal{N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
</tr>
</tbody>
</table>

and they commute with all other operators in the algebra. Because of this, the $Z^{AB}$ are called the central charges of the algebra.

There is a maximum number of supercharges a theory can have in each dimension, however. This number is decided by the constraint that no particle with spin larger than 2 should exist in the theory [49]. For $d = 4$ this number is $\mathcal{N} = 8$; this is equivalent to 32 degrees of freedom. Indeed, each Weyl spinor has four degrees of freedom ($Q_1, Q_2, \bar{Q}_1,$ and $\bar{Q}_2$) and $8 \times 4 = 32$. This is referred to as maximal supersymmetry. The maximal supersymmetry is closely associated with gravity; in some dimensions being the only case that includes the spin 2 particle graviton in its representations. Supersymmetric theories which include graviton (and its superpartner gravitino) are called supergravity theories, or SUGRA.

The maximum number of supercharges in dimensions other than four should also result in 32 maximum degrees of freedom for the theory. A list of maximum allowed number of supercharges in each dimension ($d \leq 10$) is shown in Table 2.1 borrowed partially from [50].

Section 2.2: SUSY Multiplets and Representations

2.2.1: Massless multiplets

We will now explain how to achieve the states in each supersymmetric theory using its SUSY generators $Q^A_\alpha$. For this we will start with the more straightforward case of massless multiplets first.

Since massless states do not have a rest frame, they are best described in the light frame; $P_\mu = (E, 0, 0, E)$ where $E$ is the energy of the state. A massless state is determined by its energy and its helicity $\lambda$. Combined it can be represented as $|E, \lambda\rangle$.

We now introduce an auxiliary operator called the Pauli-Lubanski pseudo-vector,

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}. \quad (2.5)$$
Here $M_{\mu\nu}$ is Lorentz rotation generator and it commutes with the SUSY generators like,

$$[Q^A_\alpha, M_{\mu\nu}] = \frac{1}{2} (\sigma_{\mu\nu})_\beta^\alpha Q^A_\beta$$

$$[\bar{Q}^A_\dot{\alpha}, M_{\mu\nu}] = -\frac{1}{2} (\bar{\sigma}_{\mu\nu})_\dot{\beta}^\dot{\alpha} \bar{Q}^A_{\dot{\beta}}.$$

The zeroth component of the pseudo-vector (2.5) has a convenient eigenvalue when acting on a massless state,

$$W_0 |E, \lambda\rangle = E\lambda |E, \lambda\rangle$$

the product of the energy and helicity of the state! Therefore we will now try acting with $W_0$ on the state $Q^A_\alpha |E, \lambda\rangle$. We will have

$$W_0 Q^A_\alpha |E, \lambda\rangle = \left( Q^A_\alpha W_0 + [W_0, Q^A_\alpha] \right) |E, \lambda\rangle = E \left( \lambda \delta^{\beta}_\alpha - \frac{1}{2} (\sigma^3)_\alpha^\beta \right) Q^A_\beta |E, \lambda\rangle \quad (2.6)$$

But we know that $(\sigma^3)_1 = (\sigma^3)_2 = -1$ and $(\sigma^3)_1 = (\sigma^3)_2 = 0$. Therefore this means that $Q^A_1$ lowers the helicity by $1/2$ and $Q^A_2$ raises it by $1/2$.

Alternatively, if we write (2.6) for the conjugate generators we will see that $\bar{Q}^A_{\dot{1}}$ would raise the helicity by $1/2$ and $\bar{Q}^A_{\dot{2}}$ would lower it by $1/2$. Therefore we can set $Q^A_2 = \bar{Q}^A_{\dot{2}} = 0$ and define “creation/annihilation” operators only using $Q^A_1, \bar{Q}^A_{\dot{1}}$. Assume

$$a^A = \frac{1}{2\sqrt{E}} Q^A_1$$

$$a^{A\dagger} = \frac{1}{2\sqrt{E}} \bar{Q}^A_{\dot{1}}.$$

These operators will then abide a simple Clifford algebra,

$$\{a^A, a^{B\dagger}\} = \delta^{AB}$$

$$\{a^A, a^B\} = \{a^{A\dagger}, a^{B\dagger}\} = 0.$$

Now that we have creation and annihilation operators we can start constructing our states starting by the ground state. By definition, a ground state $|E, \lambda_0\rangle$ would vanish by all annihilation operators. In other words,

$$a^A |E, \lambda_0\rangle = 0$$

$$\forall A = 1, ..., N$$

The rest of the states would then just be created by acting on the ground state by one or multiple creation operators. Each creation operator would raise the helicity by $1/2$.

$$a^{I_1\dagger}...a^{I_k\dagger} |E, \lambda_0\rangle = |E, \lambda_0 + k/2\rangle \quad (2.7)$$
The maximum number of creation operators acting on the ground state is of course capped by $\mathcal{N}$. Also, for each state $|E, \lambda_0 + k'/2\rangle$, there would be $\binom{\mathcal{N}}{k'}$ ways of achieving it. These two statements are what we use to construct our multiplets!

I will bring examples of constructing several multiplets that are pivotal to the work in this dissertation in the following.

$\mathcal{N} = 1$ Multiplets

When $\mathcal{N} = 1$, that means that $k$ in (2.7) has only one value, which in turn means only one creation operator.

- **Chiral Multiplet**
  This multiplet has the ground state helicity of $\lambda_0 = 0$. Therefore the states will be

$$|E, 0\rangle \quad a^\dag |E, 0\rangle = |E, \frac{1}{2}\rangle$$

The physical content of this multiplet is one complex scalar field and one Weyl spinor.

- **Vector Multiplet**
  This multiplet starts with ground state helicity of $\lambda_0 = \frac{1}{2}$. The states are

$$|E, \frac{1}{2}\rangle \quad a^\dag |E, \frac{1}{2}\rangle = |E, 1\rangle$$

The field content of this multiplet is a two-component spinor again (gaugino) and a vector boson (gauge boson).

These are the main $\mathcal{N} = 1$ multiplets we will encounter going forward.

$\mathcal{N} = 2$ Multiplets

According to (2.7), for $\mathcal{N} = 2$ we will have two creation operators: $a^\dag_1, a^\dag_2$.

- **The Vector Multiplet**
This multiplet starts with helicity $\lambda_0 = 0$ again. The states will be

$$|E, 0\rangle$$
$$a_1^\dagger |E, 0\rangle = |E, \frac{1}{2}\rangle$$
$$a_2^\dagger |E, 0\rangle = |E, \frac{1}{2}\rangle$$
$$a_1^\dagger a_2^\dagger |E, 0\rangle = |E, 1\rangle$$

The field content of this multiplet is a vector boson, two spin 1/2 fermions, and a complex scalar field.

- **The Hypermultiplet**

  Ground state of this multiplet has helicity $\lambda_0 = -\frac{1}{2}$. Therefore,

  $$|E, -\frac{1}{2}\rangle$$
  $$a_1^\dagger |E, -\frac{1}{2}\rangle = |E, 0\rangle$$
  $$a_2^\dagger |E, -\frac{1}{2}\rangle = |E, 0\rangle$$
  $$a_1^\dagger a_2^\dagger |E, -\frac{1}{2}\rangle = |E, \frac{1}{2}\rangle$$

  The field content would be a Weyl fermion $\psi$, two complex scalars $q$ and $\tilde{q}$, and another Weyl fermion $\tilde{\psi}$ in the conjugate representation of $\psi$.

  The $N=2$ multiplets can be decomposed in terms of $N=1$ multiplets. In the language of $N=1$ multiplet, the $N=2$ vector multiplet can be considered as the $N=1$ vector multiplet plus the chiral multiplet. The $N=2$ hypermultiplet can be thought of as the $N=1$ chiral multiplet plus its conjugate.

  **$N > 2$ Multiplets for Additional Examples**

  - $N=4$ Vector Multiplet

    This multiplet starts with helicity $\lambda_0 = -1$ for ground state and has four creation operators. The
states and multiplicities then are:

\[
\begin{align*}
\binom{4}{0} &= 1 \quad |E, -1) \\
\binom{4}{1} &= 4 \quad a_1^\dagger |E, -1) = |E, -\frac{1}{2}) \\
\binom{4}{2} &= 6 \quad a_1^\dagger a_2^\dagger |E, -1) = |E, 0) \\
\binom{4}{3} &= 4 \quad a_1^\dagger a_2^\dagger a_3^\dagger |E, -1) = |E, \frac{1}{2}) \\
\binom{4}{4} &= 1 \quad a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger |E, -1) = |E, +1) 
\end{align*}
\]

- $\mathcal{N} = 8$ SUGRA Multiplet

This is the maximal SUSY in $d = 4$ and it will have a spin 2 graviton. The states start with ground state helicity of $\lambda_0 = -2$. Then

\[
\begin{align*}
\binom{8}{0} &= 1 \quad |E, -2) \\
\binom{8}{1} &= 8 \quad (a_1^\dagger)^1 |E, -2) = |E, -\frac{3}{2}) \\
\binom{8}{2} &= 28 \quad (a_1^\dagger)^2 |E, -2) = |E, -1) \\
\binom{8}{3} &= 56 \quad (a_1^\dagger)^3 |E, -2) = |E, -\frac{1}{2}) \\
\binom{8}{4} &= 70 \quad (a_1^\dagger)^4 |E, -2) = |E, 0) \\
\binom{8}{8} &= 1 \quad (a_1^\dagger)^8 |E, -2) = |E, +2) \\
\binom{8}{7} &= 8 \quad (a_1^\dagger)^7 |E, -2) = |E, +\frac{3}{2}) \\
\binom{8}{6} &= 28 \quad (a_1^\dagger)^6 |E, -2) = |E, +1) \\
\binom{8}{5} &= 56 \quad (a_1^\dagger)^5 |E, -2) = |E, +\frac{1}{2})
\end{align*}
\]

All of the discussion brought in this section was about massless states and multiplets. Next we will study the massive multiplets.

2.2.2: Massive BPS multiplets

Let us refer back to the fact that in the massless multiplets, the number of states with each helicity was defined by $\binom{\mathcal{N}}{k}$. This means that the total number of states in a massless multiplet is

\[
\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = (1 + 1)^\mathcal{N} = 2^\mathcal{N}.
\]

In a massive multiplet however, since we will not be in the light-cone frame anymore, we do not have the freedom to set $Q_2^A = Q_2^{A^\dagger} = 0$. Therefore our creation and annihilation operators would be two-indexed, just
like the SUSY generators $Q^A_\alpha$. Thus the total number of states in a massive multiplet would be

$$\sum_{k=0}^{2N} \binom{2N}{k} = 2^{2N}.$$  

This is problematic for various reasons. Firstly, assuming fields become massive under Higgs mechanism, we cannot go from a multiplet with $2^N$ states to one with $2^{2N}$ states; quantum corrections cannot change the length of the multiplet. Furthermore, and we will get to talk about this concept more in the next chapter, while studying the strong coupling limit of supersymmetric theories, $2^{2N}$ includes too many degrees of freedom and result in inconsistent mass values. By contrast, multiplets with $2^N$ degrees of freedom have mass/charge ratios that are fixed by SUSY algebra and are preserved under a continuous variation in the gauge coupling.

To fix this issue let us return to (2.4). As mentioned, the central charges $Z^{AB}$ are anti-symmetric, and since they commute with everything they can be diagonalized. For this purpose we choose to work in a basis where the matrix $Z^{AB}$ can be written as

$$Z^{AB} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where the $D$-block is an $N/2 \times N/2$ diagonal matrix with eigenvalues $z_m$ on the diagonal, $m = 1, ..., N/2$.

$$D = \text{diag}(z_1, ..., z_{N/2})$$

We will now divide the indices $A = 1, ..., N$ into sets of double indices $(A, m)$ where $A = 1, 2$ and $m = 1, ..., N/2$. Therefore the extended SUSY algebra would then read

$$\{Q^{(A,m)}_\alpha, \bar{Q}^{(B,n)}_{\dot{\beta}}\} = 2\delta^{AB}\delta^{mn}\sigma^\mu\delta_{\alpha\dot{\beta}}P_\mu$$

$$\{Q^{(A,m)}_\alpha, Q^{(B,n)}_{\beta}\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\delta^{mn}z_m$$

$$\{\bar{Q}^{(A,m)}_{\dot{\alpha}}, \bar{Q}^{(B,n)}_{\dot{\beta}}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB}\delta^{mn}z_m$$

It is also worthwhile to mention that for massive states we can always operate in their rest frame, where $P_\mu = (M, 0, 0, 0)$. Therefore the first line of (2.8) becomes

$$\{Q^{(A,m)}_\alpha, \bar{Q}^{(B,n)}_{\dot{\beta}}\} = 2M\delta_{\alpha\dot{\beta}}\delta^{AB}\delta^{mn},$$

13
since $\sigma^0 = 1$.

We can make the sets of relations in (2.8) more compact by defining new generators,

$$\mathcal{Q}^\pm_{\alpha m} = \frac{1}{2}(Q^{(1,m)}_{\alpha} \pm \bar{Q}^{\dot{\alpha}(2,m)}),$$

(2.9)

and their Hermitian conjugates $\mathcal{Q}^{\pm}_{\dot{\alpha}n}$, where plus and minus act respectively. The dotted and undotted indices in (2.9) are mixed while preserving covariance; another way of writing this would be to replace $\bar{Q}^{\dot{\alpha}}$ with $\epsilon_{\alpha\dot{\beta}}\bar{Q}_{\dot{\beta}}$.

The SUSY algebra (2.8) with these new operators will then be

$$\{\mathcal{Q}^+_m, \mathcal{Q}^+_n\} = \{\mathcal{Q}^+_m, \mathcal{Q}^+_n\} = \{\mathcal{Q}^+_m, \mathcal{Q}^+_n\} = 0$$

and

$$\{\mathcal{Q}^+_m, \mathcal{Q}^+_n\} = \delta_{\alpha\dot{\beta}} \delta_{mn}(M \pm z_m).$$

(2.10)

But the left hand side of (2.10) is non-negative, so that requires

$$M \geq z_m$$

(2.11)

In other words, the mass of the state is bounded from bottom by the eigenvalues of central charges. The condition in (2.11) is known as the BPS bound, and states abiding by this bound as BPS states.

Suppose condition (2.11) is satisfied for $N$ eigenvalues $z_m$. Then we can rescale the operators $\mathcal{Q}^\pm_{\alpha m}$ for those $N$ eigenvalues by defining

$$a^\pm_{\alpha m} \equiv (M \pm z_m)^{-\frac{1}{2}}\mathcal{Q}^\pm_{\alpha m}$$

The operators $a^\pm_{\alpha m}$ are creation/annihilation operators whose degrees of freedom is reduced by $N$ bound states. Therefore they follow a Clifford algebra with $2(N - N)$ degrees of freedom. This means that the total number of states becomes

$$\sum_{k=0}^{2N} \binom{2(N - N)}{k} = 2^{2(N - N)}.$$ 

It is easy to see here that the maximum number of $z_m$ eigenvalues which can satisfy the BPS bound is $\mathcal{N}/2$ (since $m = 1, ..., \mathcal{N}/2$). If this is the case, then we will end up with $2^\mathcal{N}$ total states and $\mathcal{N}$ dimensional Clifford algebra; just like the massless case.
Section 2.3: Supersymmetric Field Formulation

Let us remember once again the field content of $\mathcal{N}=1$ SUSY multiplets:

- **chiral multiplet** - consists of a complex scalar field $\phi$ and a two-component spinor $\psi_\alpha$

- **vector multiplet** - consists of a massless vector field $A_\mu$ (gauge boson) and its superpartner $\lambda_\alpha$ (gaugino)

Here $\alpha = 1, 2$ and $\mu = 0, 1, 2, 3$ as usual.

In SUSY theories, for further calculation purposes, and to be able to write SUSY invariant Lagrangians, it is convenient to combine the consisting fields of these multiplets into so-called *superfields*. So let us investigate each of the aforementioned multiplets from QFT point of view.

For the chiral multiplet, we will introduce anticommuting variables $\theta^\alpha$ and $\bar{\theta}^\dot{\alpha}$, and an auxiliary field $F$ to assemble $\phi$ and $\psi$ into the *chiral superfield* $\Phi$.

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \quad (2.12)$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. One should realize that by $\theta^2$ we mean $\theta^a\theta^a$, the same way by $\theta\psi$ and $\theta\sigma^\mu\bar{\theta}$ we mean the summations $\theta^a\psi^a$ and $\theta^a\sigma^\mu\bar{\theta}^\dot{\alpha}$, respectively. Expanding the $y$-dependance would give us

$$\Phi = \phi(x) + i\theta^\mu\bar{\theta}\partial_\mu \phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 \phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2 (\partial_\mu \psi(x)\sigma^\mu \bar{\theta} + \theta^2 F(x) \quad (2.13)$$

Here we utilized the fact that $\theta$'s are anticommuting and $\theta^a\theta^a = 0$.

For the vector multiplet we will again use the $\theta$'s and another auxiliary field $D$ to assemble the *vector superfield* $V$.

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2 (\bar{\theta}\lambda) - i\bar{\theta}^2 (\theta\lambda) + \frac{i}{2}\theta^2 \bar{\theta}^2 D \quad (2.14)$$

The fields in the vector multiplet can be put together in a different way to create a more complex superfield which would prove to be very useful in further calculations. This superfield is

$$W_\alpha = -i\lambda_\alpha(y) + D(y)\theta_\alpha - i\sigma^\mu\nu\hat{\theta}\partial_\mu F_{\nu\nu} + \theta^2\sigma^\mu\nabla_\mu\bar{\lambda}_\alpha(y) \quad (2.15)$$

Here $\sigma^\mu\nu = \frac{1}{2}(\sigma^\mu\sigma^\nu - \sigma^\nu\sigma^\mu)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, $\nabla_\mu \lambda = \partial_\mu \lambda - ig[A_\mu, \lambda]$ and $g$ is the gauge coupling constant. The argument $y$ is also defined as before. This superfield is called the *spinorial superfield* in some literature, because it behaves like a spinor (notice the consistent index $\alpha$).
So now, since $\alpha$ is chiral, the contraction $W^\alpha W_\alpha$ would be SUSY invariant. One then is curious to calculate $\int d^2\theta W^\alpha W_\alpha$, knowing that $\int d\theta \equiv \frac{\partial}{\partial \theta}$. This means that the result of the integral would simply be the coefficient of the $\theta^2$ term of the $W^\alpha W_\alpha$ expansion. So,

$$W^\alpha W_\alpha \mid_{\theta^2} = \left[ -i\lambda^\alpha(y) + D(y)\theta^\alpha - i\sigma^{\mu\nu,\alpha\beta}\theta_\beta F_{\mu\nu} + \theta^2\sigma\nabla\bar{\lambda}^\alpha(y) \right] \times \left[ -i\lambda_\alpha(y) + D(y)\theta_\alpha - i\sigma^{\kappa\lambda}_{\alpha\beta}\theta^\beta F_{\kappa\lambda} + \theta^2\sigma\nabla\lambda_\alpha(y) \right]_{\theta^2} = (-2i\lambda^\mu\nabla_\mu\bar{\lambda} + D^2 - i(\sigma^{\mu\nu})^\alpha_\alpha DF_{\mu\nu} - \frac{1}{2}\sigma^{\mu\nu}\sigma^{\kappa\lambda}F_{\mu\nu}F_{\kappa\lambda})\theta^2$$

But $(\sigma^{\mu\nu})^\alpha_\alpha = \text{Tr}\sigma^{\mu\nu} = 0$. Also,

$$\sigma^{\mu\nu}\sigma^{\kappa\lambda} = \frac{1}{2}(g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\kappa}) - \frac{i}{2}\epsilon^{\mu\nu\kappa\lambda}$$

(with $\epsilon^{0123} = +1$). So eventually, we will get to the SUSY invariant result,

$$\int d^2\theta W^\alpha W_\alpha = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda^\mu\nabla_\mu\bar{\lambda} + D^2 + \frac{i}{2}F_{\mu\nu}\tilde{F}^{\mu\nu},$$

where we have replaced the dual $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$ in the last term.

This expression appears to have potential to be a Lagrangian, but it needs some changes. Most importantly, it’s last term is purely imaginary, which cannot happen in an invariant Lagrangian. However, if we introduce the complex coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2},$$

and calculate $\int d^2\tau W^\alpha W_\alpha$ instead, then we could get a real Lagrangian with correct coefficients. Indeed,

$$\frac{1}{8\pi} \Im \int d^2\tau W^\alpha W_\alpha = \frac{1}{8\pi} \left[ \frac{4\pi}{g^2}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda^\mu\nabla_\mu\bar{\lambda} + D^2\right) + \frac{\theta}{2\pi} \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right] = \frac{1}{g^2} \cdot \left( -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda^\mu\nabla_\mu\bar{\lambda} + \frac{1}{2}D^2\right) + \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.17}$$

The left-hand side of (2.17) will continue to be the way we set up Lagrangians in future. In the next chapter we will push forward a general discussion starting with a Lagrangian of a theory, and will make it around to discuss different aspects of the Lagrangian (2.17) in great detail.
CHAPTER 3: Stringy Extensions To Standard Model

Before we begin any discussion about physics beyond Standard Model, we first need to describe the Standard Model in the platform of string theory. In a broad class of string-based models, the Standard Model is given by a configuration of intersecting seven-branes via F-theory (see for example [5, 6], or for more extensive literature on model building in F-theory [7–10]). From this point of view then, many SM extensions involve the existence of so-called “extra” sectors. Let us describe an example of such configurations.

In these configurations, the SM is localized at the intersection of a stack of seven-branes $7_{\text{vis}}$; this is also known as the visible sector. Additionally, there could also exist extra flavor seven-branes $7_{\text{hid}}$. At the points of triple intersections between these seven-branes, Yukawa interactions are localized. This motivates another good choice for extra sector. Probe D3-branes are locally attracted to Yukawa points [11, 12], so we can include their “interactions” with the seven-branes. It is important for visualization to note that these branes are spacetime filling branes, with seven-branes filling four extra dimensions, while D3-branes are pointlike in extra dimensions. For a visual depiction of this scenario, see Figure 3.1.

Our goal is of course to study physics beyond the Standard Model; therefore we will focus on the “interactions” between SM (visible sector) and the extra sector, which is the D3-brane. The two communicate with each other via “messenger states”, which in the case of this model correspond to the $3–7_{\text{vis}}$ and $3–7_{\text{hid}}$ strings (see Figure 3.1b). The D3-brane has a natural $U(1)$ gauge group which we will denote by $U(1)_{D3}$. The $3–7_{\text{vis}}$ states are charged under both SM and $U(1)_{D3}$, while the $3–7_{\text{hid}}$ ones are only charged under $U(1)_{D3}$ and are SM singlets, and therefore are much lighter. Because of these properties, the $3–7_{\text{hid}}$ states form a natural candidate for Dark Matter, i.e. a dark sector; we will revisit this concept in Chapter [14]. But until then, we will explore more generally the physics of all states charged under $U(1)_{D3}$; which means the $3–3$, $3–7_{\text{vis}}$, and $3–7_{\text{hid}}$ string states.

It is worth reiterating again, that the $7–7$ intersections realize the Standard Model. More specifically, the $7_{\text{vis}}–7_{\text{vis}}$ string states are associated with force carriers of SM, while $7_{\text{vis}}–7_{\text{hid}}$ states represent the quarks and leptons. The $7_{\text{hid}}–7_{\text{hid}}$ intersection of the flavor branes is a weakly coupled extra sector, that can also be viewed as a possible scenario for Dark Matter. This possibility was studied in [51].

It was shown in [13] that the D3-brane is stable close to, but not directly on top of the $7_{\text{vis}}$ brane. In this case, the $3–7$ states are roughly $\mathcal{N} = 2$ supersymmetric and we will have a strongly coupled $\mathcal{N} = 2$ superconformal field theory (SCFT) with $E_8$ flavor symmetry. This theory is commonly known as the
Figure 3.1: Depiction of interaction between the Standard Model and the extra sector. 

(a) Shows the general scheme and the “messenger” states. Under “Visible sector” can be any flavor symmetry group that partially includes SM gauge group $G_{SM}$. The “Extra sector” is the probe D3-brane. 

(b) Pictures the model described in this chapter. The $7_{\text{vis}}$ seven-brane stack gives the SM gauge group (or replacements), and the $7_{\text{hid}}$ stack corresponds to a flavor seven-brane. Their intersection creates the Yukawa points, and close to Yukawa point is the probe D3-brane as extra sector. The two sectors interact with each other via 3–7 messenger string states.

“Minahan-Nemeschansky theory” [27, 28]. The Standard Model gauge group then comes from weakly gauging a $G_{SM} \equiv SU(3) \times SU(2) \times U(1)$ subgroup of $E_8$.

Instead of $E_8$, which has all of $SU(3) \times SU(2) \times U(1)$ as a subgroup, in this project we will start small! We will assume different gauge groups for our visible sector with various flavor symmetries, with the presumption that the flavor symmetry of the group would at least partially encompass the SM gauge group. With proper choice of parameters then, we can extract a $U(1)$ from the flavor symmetry groups too. The interaction between the visible and extra sector then, would be summarized in the interaction between the $U(1)_{D3}$ and the extracted $U(1)$ from the flavor symmetry group.

In the next chapter, we will investigate several different flavor symmetry groups and explore the model in each case. In this chapter, however, we will set up the framework that will be needed for these calculations.

**Section 3.1: Weak Coupling Limit**

The scenario described above suggests that we have two $U(1)$s interacting in the low-energy limit. One of them is associated with the Standard Model, indexed $U(1)_1$, and the other is the $U(1)_{D3}$, from hereon referred to as $U(1)_2$. Both Lorentz symmetry and gauge invariance allow for terms with mixed field-strengths of different $U(1)$s. Thus the most general renormalizable Lagrangian we can write for the $U(1)_1 \times U(1)_2$
\[
\mathcal{L} = -\frac{1}{4} \left\{ \frac{1}{g_{11}^2} F_{\mu\nu}^1 F^{1,\mu\nu} + \frac{1}{g_{22}^2} F_{\mu\nu}^2 F^{2,\mu\nu} + \frac{2}{g_{12}^2} F_{\mu\nu}^1 F^{2,\mu\nu} \right\} \\
+ \frac{1}{32\pi^2} \left\{ \theta_{11} F_{\mu\nu}^1 \tilde{F}_{1,\mu\nu} + \theta_{22} F_{\mu\nu}^2 \tilde{F}_{2,\mu\nu} + 2\theta_{12} F_{\mu\nu}^1 \tilde{F}_{2,\mu\nu} \right\},
\]

where \( F_{\mu\nu}^i \) are field strengths of \( U(1)_i \), and their duals are defined as \( \tilde{F}_{i,\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{i,\rho\sigma} \). The Lagrangian (3.1) can be rearranged to be written as

\[
\mathcal{L} = -\frac{1}{4g_{11}^2} F_{\mu\nu}^1 F^{1,\mu\nu} - \frac{1}{4g_{22}^2} F_{\mu\nu}^2 F^{2,\mu\nu} + \frac{\theta_{11}}{32\pi^2} F_{\mu\nu}^1 \tilde{F}_{1,\mu\nu} + \frac{\theta_{22}}{32\pi^2} F_{\mu\nu}^2 \tilde{F}_{2,\mu\nu} + \mathcal{L}_{\text{mix}},
\]

where

\[
\mathcal{L}_{\text{mix}} = -\frac{\chi_{12}^{\text{elec}}}{2} F_{\mu\nu}^1 F^{2,\mu\nu} - \frac{\chi_{12}^{\text{mag}}}{2} F_{\mu\nu}^2 \tilde{F}_{2,\mu\nu}.
\]

The terms with duplicate indices are simply the well-known formulation for single \( U(1) \) scenarios. The mixing effect between the two \( U(1) \)'s is represented in the terms proportional to \( \chi_{12}^{\text{elec}} \) and \( \chi_{12}^{\text{mag}} \). The \( \chi_{12}^{\text{elec}} \) term is the CP preserving electric mixing term between the two \( U(1) \)’s

\[
F_{\mu\nu}^1 F_{\rho\sigma}^2 = \vec{E}^1 \cdot \vec{E}^2 - \vec{B}^1 \cdot \vec{B}^2,
\]

while the term proportional to \( \chi_{12}^{\text{mag}} \), is CP violating and can be written as

\[
F_{\mu\nu}^1 \tilde{F}_{\rho\sigma}^2 = \vec{E}^1 \cdot \vec{B}^2 + \vec{B}^1 \cdot \vec{E}^2,
\]

and represents the mixing of the magnetic and electric fields of different \( U(1) \)’s with one another; this will be the magnetic mixing term.

Just like any other situation in quantum field theory, we first assume that these coupling constants are small and proceed to solve our problem using weak coupling methods. The most common tool at weak coupling is perturbation theory with Feynman diagrams. So let us figure out the Feynman diagrams associated with the Lagrangian (3.1).

If we had only one \( U(1) \), the lowest order Feynman diagram associated with an \( F_{\mu\nu} F^{\mu\nu} \) term would have just been a photon propagator. Here, however, the situation is slightly different. Due to the existence of two different \( U(1) \)'s, we will have terms showing interactions between two different classes of photons, i.e. the mixing terms \( \chi_{12}^{\text{elec}} \) and \( \chi_{12}^{\text{mag}} \). So aside from single \( U(1) \) photons, we will have terms that would transition the photons charged under \( U(1)_1 \), shown as \( \gamma_1 \), to photons charged under \( U(1)_2 \), as \( \gamma_2 \). The first order
Feynman diagram associated with these transitions would be

Figure 3.2: Feynman diagram showing the transition between $\gamma_1$ and $\gamma_2$ of two different $U(1)$s.

At second order in the case of two $U(1)$s, we would have the vacuum polarization diagram transitioning $\gamma_1$ to $\gamma_2$ through a loop of creation and annihilation.

Figure 3.3: Vacuum polarization diagram of two different $U(1)$s showing transition between $\gamma_1$ and $\gamma_2$.

The modified Green’s function for this diagram would be

$$
- \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ -ig_1 \gamma^\mu \cdot \frac{i}{p - m} - ig_2 \gamma^\nu \cdot \frac{i}{p' - m} \right],
$$

where $g_1$ and $g_2$ are charges of the fermions, i.e. $g_1 = e_1 q_1, g_2 = e_2 q_2$, with the $e_i$ being the unit charges under each $U(1)_i$. $p$ and $p'$ are the momenta of the fermion and anti-fermion created, and $m$ is their mass.

A review of a similar Green’s function calculation has been presented in [52]. The final result for a 4-dimensional spacetime with slight modification would match ours,

$$
- \frac{2\chi}{\pi} \int_0^1 dx x (1 - x) \left( \frac{2}{\epsilon} - \gamma - \log \Delta + \log 4\pi \right).
$$

Here, instead of the familiar $\alpha = \frac{e^2}{4\pi}$, we have introduced the modified $\chi = \frac{q_1 e_1 q_2 e_2}{4\pi}$.

In addition, $\Delta \equiv m^2 - x (1 - x) k^2$, with $k$ being the momentum of the photons. Since $x$ only ranges from 0 to 1, for large mass values, $\Delta \approx m^2$ and thus $\log \Delta \approx 2 \log m$. This will get rid of all the constant terms in the parenthesis\(^{1}\) and as a result we will be left with

$$
\frac{2\chi}{\pi} \cdot 2\log m \cdot \int_0^1 dx x (1 - x)
= \frac{q_1 e_1 q_2 e_2}{6\pi^2} \log \frac{m}{\mu},
$$

\(^{1}\) The terms $\frac{2}{\epsilon} - \gamma$ come from a Gamma function approximation and always cancel in observable quantities. The scheme used here is called modified minimal subtraction, or known by its acronym MS scheme.
where $\mu$ is a reference scale.

Now consider a toy model with four fermions $f_1, f_2, f_{12}, f'_{12}$ with respective charges $(q_1, 0), (0, q_2), (q_1, q_2), (q_1, -q_2)$ under $U(1)_1 \times U(1)_2$ gauge symmetry. And also respective masses of $m_1, m_2, m_{12}, m'_{12}$. According to $(3.3)$, the fermions $f_1$ and $f_2$ will not have a contribution to the Green’s function since the product of the charges would vanish. For $f_{12}$ and $f'_{12}$ on the other hand, we will have the combined contribution of

$$\frac{q_1 e_1 q_2 e_2}{6\pi^2} \log\frac{m_{12}}{m'_{12}}. \quad (3.4)$$

This result presents two very important ideas. One, that for there to be a mixing effect, matter needs to be charged under both $U(1)$s. Two, it gives rise to “non-decoupling effect”; meaning that even if the masses are too large to have an effect in the weak coupling limit, if they’re comparable (and charged oppositely under one of the $U(1)$s), the effect would still be present.

The expression written in $(3.4)$ is the coupling constant for the weak coupling limit. This result was first presented by Holdom in [21].

At this point one should note that unless there are magnetic monopoles in the theory, the $\theta$ terms and the entirety of second line in $(3.1)$, can be ignored because being a total derivative, they do not add any new information to the equations of motion. Therefore, in order to see effects of magnetic mixing, and to proceed with our calculations, it is necessary to assume that magnetic monopoles exist.

More so, it is important to remember that weak coupling, and therefore the method of Feynman diagrams, is a perturbative one. Indeed, in the weak coupling limit we assume that the monopoles are heavy. More specifically,

$$\frac{m_{\text{monopole}}}{m_{\text{electron}}} \propto \frac{1}{g^2}$$

where $g << 1$. It is then of natural curiosity (and a well-motivated one, as it will become clear later in this thesis), to explore the limits where the mass of electron and mass of monopole are comparable; i.e. when $g \approx 1$. In these limits, we cannot claim perturbation anymore, and thus we exit the weak coupling territory.

**Section 3.2: Strong Coupling Limit**

Let us have a quick review of Maxwell equations in presence of monopoles. The original Maxwell equations (without the monopoles) are summarized as

$$\partial_\mu F^{\mu\nu} = J_\nu$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0,$$ \quad (3.5)
where $F^{\mu\nu}$ is the electromagnetic field tensor, defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and $J^\nu_e$ is the electric current. The presence of monopoles, as we know, encourages a duality in the equations (3.5). They become

$$
\partial^\mu F^{\mu\nu} = J^\nu_e, \\
\partial^\mu \tilde{F}^{\mu\nu} = J^\nu_m,
$$

(3.6)

where $J^\nu_m$ is the magnetic current, which is defined analogous to the electric current. However, with the monopoles involved, the field tensor is no longer the antisymmetric difference of derivatives of the vector potential.

Unlike electric mixing, magnetic mixing does cause CP violation in the theory. Indeed, according to Dirac [53], if we have a particle with electric charge $q$ and another particle with magnetic charge $p$, the quantum mechanics of their interactions would only be consistent if $pq = 2\pi n$. Additionally, it was shown in [54, 55] that a generalized version of this condition also holds for dyons. Namely, if we have a dyon $d_1$, with electric charge $q_1$ and magnetic charge $p_1$ (we will incorporate the shorthand notation $d_1 = (q_1, p_1)$ for dyons), and another dyon $d_2 = (q_2, p_2)$, the quantum mechanics between them would only be consistent if $q_1 p_2 - p_1 q_2 = 2\pi n$. This quantity is often called dyon coupling between the two dyons,

$$
\langle d_1 | d_2 \rangle \equiv q_1 p_2 - p_1 q_2.
$$

It is of interest to note, that one can now treat a separated dyon as two distant special dyons, where one only has electric charge and the other only magnetic. In other words, one would have $d_1 = (q, 0)$ and $d_2 = (0, p)$. In this special case then, the dyon coupling is also going to be the system’s angular momentum, as we will have $\langle d_1 | d_2 \rangle = pq \equiv |\vec{l}|$. The derivation of this equation is shown in length in Chapter 6 but it is worthy to notice here that the quantization of the angular momentum of the system is equivalent to the quantization of electric and magnetic charges.

Now let us imagine a dyon with electric charge $q$ and minimum magnetic charge $p = \frac{2\pi}{e}$, i.e. $(q, \frac{2\pi}{e})$. When we act on this dyon with the CP operator, since electric and magnetic fields transform oppositely under parity, then the resulting dyon would have $(-q, \frac{2\pi}{e})$. Comparing these two particles then, we would have $q_1 p_2 - p_1 q_2 = \frac{4\pi q}{e}$. This is only a multiple of $2\pi$ if $q = ne$ or $q = (n + \frac{1}{2})e$. Therefore, dyons must all have either integer or half-integer electric charges.

Considering that CP is violated weakly in nature, one can reason that the electric charge of dyons, could be small variations away from the predicted options. However, including $\theta$ terms in the Lagrangian, discloses that this is not the case and reveals a major CP violation. To calculate this violation, it is convenient to
work in a basis in which all magnetic charges are integral and in which the physically measured electric
charges may contain shifts by various theta angles \[56\].

Let us first note that we can write the general $\theta$-terms as

$$-\frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{\theta}{8\pi^2} \mathbf{E} \cdot \mathbf{B}. \quad (3.7)$$

In a background with magnetic monopoles the $\mathbf{E}$ and $\mathbf{B}$ fields would be defined using the electromagnetic
potential $A^\mu = (A^0, \mathbf{A})$ as followed,

$$\mathbf{E} = \nabla A^0 \quad \quad \mathbf{B} = \nabla \times \mathbf{A} + \frac{\mathbf{r}}{r^3},$$

where $p$ is the magnetic charge of a monopole located at the origin. Now if we calculate the $\theta$-term contribution to the Lagrangian (using (3.7)), we will get

$$L_\theta = \frac{\theta}{8\pi^2} \int d^3r \mathbf{E} \cdot \mathbf{B} = \frac{\theta}{8\pi^2} \int d^3r (\nabla A^0) \cdot (\nabla \times \mathbf{A} + \frac{\mathbf{r}}{r^3})$$

$$= -\frac{\theta p}{8\pi^2} \int d^3r (\nabla A^0) \cdot \frac{\mathbf{r}}{r^3} = -\frac{\theta p}{2\pi} \int d^3r A^0 \delta^3(\mathbf{r}).$$

This is similar to the interaction of an electric point charge with the magnitude of $\frac{\theta p}{2\pi}$ located at the origin, i.e. where the monopole is, with the electrostatic potential $A^0$. Considering the agreed upon basis where $p$ is integer, then we can say that the dyon has acquired a shift in its electric charge by integer $\times \frac{\theta}{2\pi}$.

This calculation can be used for $\theta_{11}$ and $\theta_{22}$; in each case it would give us the electric charge of that monopole under its correspondent $U(1)$. The $\theta_{12}$ term however, would give us the “mixing” charges. To calculate this for example we will write the electric and magnetic fields of both $U(1)$s in presence of a monopole that has magnetic charge $p_2$ under $U(1)_2$. In other words,

$$\mathbf{E}_1 = \nabla A^0_1, \quad \mathbf{B}_1 = \nabla \times \mathbf{A}_1, \quad \mathbf{E}_2 = \nabla A^0_2, \quad \mathbf{B}_2 = \nabla \times \mathbf{A}_2 + p_2 \frac{\mathbf{r}}{r^3}.$$

We will then have

$$L_\theta = \frac{\theta_{12}}{8\pi^2} \int d^3r (\mathbf{E}_1 \cdot \mathbf{B}_2 + \mathbf{E}_2 \cdot \mathbf{B}_1) = -\frac{\theta_{12}}{8\pi^2} \int d^3r A^0_1 \nabla \cdot \frac{\mathbf{r}}{r^3}$$

$$= -\frac{\theta_{12}p_2}{2\pi} \int d^3r A^0_1 \delta^3(\mathbf{r}).$$

This indicates that this dyon has now acquired an electric charge shift by $\frac{\theta_{12}p_2}{2\pi}$. Remembering what our indices meant, this would mean that the electric charge of our visible sector, charged under $U(1)_1$, has now
acquired a shift in its electric charge affected by the monopole of the extra sector, which is charged under $U(1)_2$.

As we mentioned earlier in our discussion, with including monopoles we are enhancing a duality between electric and magnetic fields. This suggests that electric mixing and magnetic mixing, could be two faces of the same coin! In fact, we could nicely package the two together with the help of a complex matrix $\tau_{ij}$ defined as

$$\tau_{ij} = \frac{\theta_{ij}}{2\pi} + \frac{4\pi i }{g_{ij}}. \quad (3.8)$$

This is the most central expression of this thesis. From here on, whenever we mention “coupling constants” or “coupling constant matrix”, we will be referring to the matrix $\tau_{ij}$ and its elements.

Now, utilizing the coupling constant matrix $\tau_{ij}$, we can write the Lagrangian (3.1) as

$$L = \frac{1}{16\pi} \left( \text{Im} \tau_{ij} F_{\mu\nu}^{i} F_{j,\mu\nu} + \text{Re} \tau_{ij} F_{\mu\nu}^{i} \tilde{F}_{j,\mu\nu} \right) \quad (3.9)$$

It is easy to check that this will give us the same Lagrangian as (3.1). Indeed,

$$L = - \frac{1}{16\pi} \text{Im} \tau_{ij} F_{\mu\nu}^{i} F_{j,\mu\nu} + \frac{1}{16\pi} \text{Re} \tau_{ij} F_{\mu\nu}^{i} \tilde{F}_{j,\mu\nu}$$

$$= - \frac{1}{16\pi g_{ij}} F_{\mu\nu}^{i} F_{j,\mu\nu} + \frac{1}{16\pi \theta_{ij}} F_{\mu\nu}^{i} \tilde{F}_{j,\mu\nu}$$

$$= - \frac{1}{4g_{ij}} F_{\mu\nu}^{i} F_{j,\mu\nu} + \frac{\theta_{ij}}{32\pi^2} F_{\mu\nu}^{i} \tilde{F}_{j,\mu\nu}.$$

As one can notice, the real part of (3.8) gives the magnetic mixing terms ($\text{Re} \tau_{ij}$ is proportional to the magnetic coupling constants $\theta$), while the imaginary part of the statement gives the electric mixing terms ($\text{Im} \tau_{ij}$ is proportional to the electric coupling constants $\frac{1}{g}$). So it becomes evident that knowing the $\tau_{ij}$ will give us all we need to know about the mixing terms with the extra sector.

In the next chapter, we will introduce a method that will help us find the $\tau_{ij}$’s. But before that, we will need to introduce the necessary fields and framework in which this method operates and that is the $\mathcal{N} = 2$ supersymmetric quantum field theory. The next section is dedicated to superfield formulation of $\mathcal{N} = 2$ SUSY.

Section 3.3: $\mathcal{N} = 2$ Superfield Formulation

In Section 2.3 we talked about the $\mathcal{N} = 1$ SUSY field content and introduced the related superfields. Here we will take up where we left off and start our discussion of $\mathcal{N} = 2$ SUSY fields. The most common multiplet for $\mathcal{N} = 2$ SUSY actually comes from merging the two $\mathcal{N} = 1$ multiplets, the chiral multiplet and
the vector multiplet, into a single $\mathcal{N} = 2$ supersymmetric multiplet. Conceivably, this multiplet would have
two sets of fermions (spinors $\psi$ and gaugino $\lambda$) and two sets of bosons (the complex scalar $\phi$ and the gauge
boson $A_\mu$). A SUSY invariant Lagrangian for this multiplet then, would also come from combining SUSY
invariant expressions consisting of the $\mathcal{N} = 1$ superfields.

We have already found one such combination using the spinorial superfields, (2.17). Another SUSY
invariant expression can be constructed by combining the chiral and vector superfields as follows

$$\int d^2 \theta d^2 \bar{\theta} \Phi^+ e^{-2gV} \Phi.$$ 

These two expressions put together, with some coefficient adjustment, would create the $\mathcal{N} = 2$ super
Yang-Mills Lagrangian

$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{8\pi} \text{Im} \int d^2 \theta d^2 \bar{\theta} \frac{1}{2} \text{Tr} \Psi^2,$$ (3.10)

where $\Psi$ is the $\mathcal{N} = 2$ analogue of a chiral superfield, and $\tilde{\theta}_\alpha, \tilde{\bar{\theta}}^\dot{\alpha}$ are a new set of anticommuting variables
introduced for $\mathcal{N} = 2$ SUSY. The explicit form for $\Psi$ can therefore be obtained as

$$\Psi = \Phi(\tilde{y}, \theta) + \sqrt{2} \tilde{\theta}^\alpha W_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\dot{\alpha} \tilde{\bar{\theta}}^\dot{\alpha} G(\tilde{y}, \theta),$$ (3.11)

with

$$G(\tilde{y}, \theta) = -\frac{1}{2} \int d^2 \bar{\theta}[\Phi(\tilde{y} - i \theta \sigma \tilde{\theta}, \theta, \bar{\theta})]^+ \exp\{-2gV(\tilde{y} - i \theta \sigma \tilde{\theta}, \theta, \bar{\theta})\}$$

and $\tilde{y}^\mu = x^\mu + i \theta \sigma^\mu \tilde{\theta} + i \tilde{\theta} \sigma^\mu \bar{\theta} = y^\mu + i \theta \sigma^\mu \bar{\theta}$. The superfields $\Phi(y, \theta), \Phi(x, \theta, \bar{\theta}), V(x, \theta, \bar{\theta})$ and $W(y, \theta)$ are
given by their $\mathcal{N} = 1$ description, respectively (2.12), (2.13), (2.14) and (2.15). All the component fields are
in the adjoint representation of $SU(N)$ gauge symmetry.

One could notice from (3.10) that the integrand depends only on $\Psi$, and not $\Psi^+$. Indeed, the most
all-inclusive form of writing the $\mathcal{N} = 2$ SUSY Lagrangian is

$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{8\pi} \text{Im} \int d^2 \theta d^2 \bar{\theta} \mathcal{F}(\Psi),$$ (3.12)
where the function $F$ is a holomorphic function of $\Psi$, meaning that it is only a function of $\Psi$ and not $\Psi^+$. This function $F$ is called the $\mathcal{N} = 2$ prepotential and it is a very important object in $\mathcal{N} = 2$ supersymmetry. In particular, if we define it as $F(\Psi) \equiv \frac{1}{2} \text{Tr} \, \tau \Psi^2$, we will return to the Lagrangian \( (3.10) \).

Furthermore, if we expand $\Psi$ according to \( (3.11) \), and consequently \( (2.12) \), we will notice that its bosonic component involves the adjoint valued complex scalar field $\phi$. Let us now assume a ground state in which the adjoint field $\phi$ has a non-zero vacuum expectation value. At weak coupling limit, this can be parameterized in terms of a diagonal $N \times N$ matrix, where $N = r + 1$ and $r$ is rank of the gauge group. We have,

$$
\phi = \begin{pmatrix}
a_1 \\
\vdots \\
a_N
\end{pmatrix},
$$

(3.13)

with $a_1 + \ldots + a_N = 0$. The situation is slightly different in strong coupling limit, because there we should parameterize the vacuum only in terms of vacuum expectation values of gauge invariant operators. By multiplying this matrix by itself and taking the trace of the product diagonal matrices, we can create $N - 1 = r$ independent symmetric polynomials of $a_k$’s (Tr $\phi^k$ would not be independent anymore for $k > N$ and Tr$\phi = 0$). We will introduce these polynomials as

$$
u_k = \langle \text{Tr} \, \phi^k \rangle,
$$

(3.14)

These gauge invariant parameters act as coordinates on the Coulomb branch of the theory, which in case of $\mathcal{N} = 2$ SUSY for example, is the moduli space for the vector multiplets. Different expectation values for $a_k$’s, and more specifically for $u_k$’s, describe different physical theories. In case of an $SU(N)$ we have $N - 1$ independent coordinates, $u_2, \ldots, u_N$; this makes sense because $SU(N)$ is a rank $N - 1$ gauge group and with breaking the gauge symmetry it will become $U(1)^{N-1}$. In addition to their gauge invariance, a very useful aspect of introducing coordinates $u_k$ is that when we are in strong coupling limit and do not have a specific Lagrangian description, we can still talk about the Coulomb branch parameters and thus investigate the theory.

Let us refer back to the adjoint valued matrix $\phi$ in \( (3.13) \). From this definition we can say that the prepotential introduced above is a holomorphic function of the elements $a_k$ and thus define their dual fields $a_k^D$ as

$$
a_k^D \equiv \frac{\partial F}{\partial a_k}.
$$

This is in accordance to the Legendre transformation $F^*(a^D) = F(a) - aa^D$.

---

\[ \text{This is in accordance to the Legendre transformation } F^*(a^D) = F(a) - aa^D. \]
Then, by definition,

$$\tau_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = \frac{\partial a_i}{\partial a_j} = \frac{\partial a_j}{\partial a_i}, \quad (3.15)$$

and we can therefore state that having information about the prepotential is equivalent to having information about the $\tau$’s.

At this point, we have enough tools and information to introduce the method that Seiberg and Witten discovered for calculating the effective Lagrangian at strong coupling. This method, and several sample calculations, is what the next chapter is dedicated to.
CHAPTER 4: The Seiberg-Witten Method

Our main objective here is to seek out a way to find the coupling constant $\tau_{ij}$’s without knowing much details about the microscopic structure of our theory. This was achieved by Seiberg and Witten [37], by realizing two main symmetries about the behavior of $\tau$.

The first symmetry is based on the electric-magnetic duality, which states that Maxwell equations are symmetric under $E \rightarrow B$ and $B \rightarrow -E$ (or in short, $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu}$). The transition $\tau \rightarrow \frac{1}{\tau}$, switches $\theta \leftrightarrow g$ (up to factors), and this mimics switching between electric and magnetic fields, since $g$ and $\theta$ are their corresponding coupling constants. This transition is also used when switching from strong coupling limit to weak coupling limit, and vice versa. The second symmetry basically provides the periodicity of the $\theta$ angle, by being invariant under $\tau \rightarrow \tau + 1$.

These two symmetries together can combine and impose invariance under $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$. This transformation is the general $SL(2,\mathbb{Z})$ transformation under an $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrix. The genius in Seiberg-Witten work is realizing that $SL(2,\mathbb{Z})$ is also the group of modular transformations for theta functions defined on an elliptic curve.

The Seiberg-Witten method is based on the effective action on the Coulomb branch of the theory. From the properties of $\tau$ discussed at the beginning of this chapter, Seiberg and Witten realized that this moduli space is also where the modular transformations of $SL(2,\mathbb{Z})$ would be occurring; and that moduli space can be thought of as geometrically parameterized by the complex structure of a $T^2$. This $T^2$ can be written as the quotient space of the complex plane $\mathbb{C}$ by a group denoted by $\langle 1, \tau \rangle$ which ensures the periodicity of $\tau$ (see Fig. 4.1a for visual depiction). This $T^2 = \mathbb{C}/\langle 1, \tau \rangle$ space then parameterizes elliptic curves described by

$$y^2 = x^3 + f(u)x + g(u),$$

where $f$ and $g$ are polynomials in $u$, and their exact form is decided by the flavor symmetry group $G_{\text{flavor}}$ that we consider.

The equation (4.1) corresponds to a genus-one Riemann surface; more specifically a torus. Evidently, a torus has two independent close cycles; one that loops through the hole - we will name it loop $A$, and another one that goes around the torus - loop $B$. These loops are direct result of manipulating the quotient
Figure 4.1: (a). $\mathbb{C}/(1,\tau)$ plane. The lines of same color are considered equivalent, and thus construct the torus. The dashed lines are the “opened” version of cycles of the torus. (b). The torus created after connecting the equivalent lines of the $\mathbb{C}/(1,\tau)$ plane to each other. Loops $A$ and $B$ are the independent cycles of the torus.

We can also define a modular invariant function for elliptic curves, called the Klein invariant $j$-function, as

$$j(\tau) = \frac{4f^3}{4f^3 + 27g^2}. \quad (4.2)$$

The $j$-function is only a function of the modular parameter $\tau$ and is therefore $SL(2,\mathbb{Z})$ invariant. To avoid $SL(2,\mathbb{Z})$ redundancy, it is common to define a “fundamental domain” for the $\tau$ parameter and work only in that domain. Mathematically, this domain is defined by $\mathbb{H}/SL(2,\mathbb{Z})$, where $\mathbb{H} = \{\text{Im} \tau > 0\}$ is the upper half plane. Geometrically, the domain is marked by the shaded area in Figure 4.2. All the values of $\tau$ lying outside the fundamental domain can be mapped to a point inside the domain using an $SL(2,\mathbb{Z})$ transformation. Additionally, when $\tau$ is restricted to this domain, the $j$-function takes on every value in the complex plane $\mathbb{C}$ exactly once.

For a representation $\mathcal{R}$ of $G_{\text{flavor}}$, we can then define the Seiberg-Witten (SW) differential, $\lambda_\mathcal{R}$. According to [10] (which is in turn a review of [27]), the SW differential for a representation $\mathcal{R}$ can be written as

$$\lambda_\mathcal{R} = (c_1 u + c_3 B(u)) \frac{dx}{y} + c_2 \sum_a \frac{m_a y_a(u)}{x - x_a(u)} \frac{dx}{y}, \quad (4.3)$$

where $c_1, c_2$ and $c_3$ are normalization constants, and $B(u)$ depends on the symmetry group $G_{\text{flavor}}$. For most smaller rank symmetry groups, $B = 0$. The sections $(x_a(u), y_a(u))$ are the poles of the SW differential, where $a = 1, \ldots, \dim \mathcal{R}$, and $m_a$ are the mass parameters at these poles.

The dual elements $a$ and $a^D$ now can be written as exact integrals of the SW differential along the $A$
Figure 4.2: The fundamental domain for the modular parameter $\tau$ is shaded in gray. Every point outside of this can be mapped into a point inside using an $SL(2, \mathbb{Z})$ transformation, and the $\tau$ values inside of the domain give all the values in the complex plane $\mathbb{C}$ for the $j$-function exactly once. This domain is restricted by conditions $\text{Im}\tau > 0$, $-\frac{1}{2} < \text{Re}\tau \leq \frac{1}{2}$ and $|\tau| \geq 1$, which come from $\mathbb{H}/SL(2, \mathbb{Z})$. The image is taken from Wikipedia.

and B loops of the torus, see Figure 4.1b

$$a = \oint_{A} \lambda_{\mathcal{R}}$$

$$a^D = \oint_{B} \lambda_{\mathcal{R}}.$$  \hspace{1cm} (4.4)

In this setting, $a$ and $a^D$ are called the SW periods. An additional expression gives the mass parameters $m_a$ around the poles

$$\frac{1}{k_{\mathcal{R}} 2\sqrt{2}} m_a = \oint_{\lambda_{\mathcal{R}}}. $$

Alternatively, the mass parameters for an irreducible representation $\mathcal{R}$ of $G_{\text{flavor}}$, can be found using the roots and weights of $G_{\text{flavor}}$. Define $\vec{\lambda}_{a}$ as the weight space vector of $\mathcal{R}$ and $\vec{\alpha}_f$ as the root space vector of $G_{\text{flavor}}$. Then,

$$m_a = \vec{\lambda}_{a} \cdot \vec{\varphi}$$  \hspace{1cm} (4.5)

where $\vec{\varphi} = \sum_{f} \vec{\alpha}_f \varphi^f$, with $a = 1, \ldots, \text{dim } \mathcal{R}$ and $f = 1, \ldots, \text{rank } G_{\text{flavor}}$.

At this point it is worth mentioning that knowledge of SW periods can also aid us in extracting the mass $M$ of BPS states introduced in Section 2.2.2. Indeed, assume a state has both electric and magnetic charges and is also charged under the flavor symmetry group. Let us denote these charges by $q_e$, $q_m$, and $q_a$ respectively. The central charge $Z$ of such a state would then be

$$Z = q_m a^D + q_e a + \frac{1}{\sqrt{2}} \sum_{a} q_a m_a $$  \hspace{1cm} (4.6)
where \(a\) and \(a^D\) are the SW periods, and the \(m_a\) are the flavor symmetry characteristic mass parameters (4.5). The BPS mass is then given by

\[
M = \sqrt{2}|Z|.
\] (4.7)

Let us take a second look at eq. (4.6). The last term in (4.6) can be rewritten using the expression for mass parameters (4.5). We will have,

\[
\frac{1}{\sqrt{2}} \sum_a q_a m_a = \frac{1}{\sqrt{2}} \sum_a q_a \tilde{\lambda}_a \cdot \bar{\varphi} = \frac{1}{\sqrt{2}} \sum_a q_a \tilde{\lambda}_a \cdot \sum_f \sigma_f \varphi^f
\]

Now if we combine all the coefficients as \(w_f\), we can rewrite this as

\[
\frac{1}{\sqrt{2}} \sum_f w_f \varphi^f
\]

At this point we can define new “mass parameters” \(a_f\) as

\[
a_f \equiv \frac{1}{\sqrt{2}} \varphi^f,
\]

So that the overall BPS mass formula will become

\[
M = \sqrt{2}|q_m a^D + q_e a + \sum_f w_f a_f|,
\] (4.8)

The main benefit of writing the mass formula in this form is that it provides a very clear distinction and a straightforward understanding of the two types of coupling constants we investigate in this project. The \(\tau_{\text{extra}}\), or \(\tau_{r+1,r+1}\) where \(r\) is the rank of the flavor symmetry group, which describes the coupling of the extra sector with itself, and the so-called \(\tau_{\text{mix}}\)’s, which define the mixing between the extra sector and the flavor symmetry. In other words,

\[
\tau_{\text{mix}} = \tau_{f,r+1} \equiv \frac{\partial a^D}{\partial a_f}, \quad \text{for } f = 1, \ldots, r
\]

\[
\tau_{\text{extra}} = \tau_{r+1,r+1} \equiv \frac{\partial a^D}{\partial a}
\]

where \(r\) is the rank of \(G_{\text{flavor}}\).

To explore the values of coupling constants we need to be able to write expressions for SW periods.
For that, we will introduce two classes of contour integrals which we will face while integrating the SW differential \((4.3)\). These are

\[
I_{1}^{(A), (B)}(A, B) = \oint_{(A), (B)} \frac{dx}{y}, \quad I_{3}^{(A), (B)}(c) = \oint_{(A), (B)} \frac{dx}{y(x - c)},
\]

where the \(A\) and \(B\) indices refer to different cycles encompassing two out of three roots of the cubic \((4.1)\), if we write the cubic equation as

\[
y^2 = x^3 + fx + g = (x - e_1)(x - e_2)(x - e_3).
\]

By convention, cycle \(A\) encircles \(e_2\) and \(e_3\), while cycle \(B\) encircles \(e_1\) and \(e_2\). We can then use the exact expressions given in \([57]\), to write explicit forms for contour integrals \((4.9)\) in terms of elliptic integrals.

\[
I_{1}^{(A)} = \frac{4}{(e_1 - e_3)^{1/2}} K(k) \quad \quad \quad \quad I_{3}^{(A)} = \frac{4}{(e_1 - e_3)^{3/2}} \left[ \frac{1}{1 - \tilde{c} + k^2} K(k) + \frac{4k'}{1 + k^2 (1 - \tilde{c})^2 - k^2} \Pi_1(\nu(c), \frac{1 - k'}{1 + k'}) \right]
\]

with

\[
k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = 1 - k^2 = \frac{e_2 - e_1}{e_3 - e_1}, \\
\tilde{c} = \frac{c - e_3}{e_1 - e_3}, \quad \nu(c) = \frac{1}{1 - \tilde{c} + k'^2} \left( 1 - k'^2 \right)^2.
\]

And the \(I^{(B)}\) integrals are similar to \(I^{(A)}\)’s with \(e_1\) and \(e_3\) interchanged. In these equations, \(K\) and \(\Pi_1\) are standard elliptic integrals, known as the incomplete elliptic integrals of the first and third kind, respectively. Their full integral representation is given as followed

\[
K(k) = \int_{0}^{1} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)^{1/2}}} \\
\Pi_1(\nu, k) = \int_{0}^{1} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)^{1/2}(1 + \nu x^2)}}
\]

In \textbf{Mathematica} however, these integrals are defined a bit differently, with the correspondence being \(K(k) = \text{EllipticK}[k^2]\) and \(\Pi_1(\nu, k) = \text{EllipticPi}[\nu, k^2]\). The reader will notice this subtlety in the \textbf{Mathematica} code presented in the Appendix.

Once we have expressions for the periods we can then talk about the coupling constants using the
Or as we differentiated between two classes of the coupling constants

\[ \tau_{f,r+1} = \frac{\partial a^D}{\partial u} = \sqrt{2} \frac{\partial a^D}{\partial \varphi} \]  

which we will often refer to as \( \tau_{\text{mix}} \), for \( f = 1, \ldots, r \) for a flavor symmetry group of rank \( r \). And in turn

\[ \tau_{r+1,r+1} = \frac{\partial a^D}{\partial u} = \frac{\partial a^D}{\partial a/\partial u} \]

which we will often refer to as \( \tau_{\text{extra}} \). The reason for the last equality is because finding the dependence of \( a^D \) and \( a \) on the Coulomb branch parameter \( u \) is much easier than their dependence on each other. In case we work in a regime that has more than one Coulomb branch coordinate, i.e. a higher rank than \( U(1) \), this relation simply becomes

\[ \tau_{\text{extra}} = \frac{\partial a^D}{\partial a} = \sum_k \frac{\partial a^D}{\partial a/\partial u_k}. \]

The rest of this chapter will be dedicated to doing this calculation for various flavor symmetry groups \( G_{\text{flavor}} \). In all of these cases it is implied that at least a part of \( G_{\text{SM}} \) is a subset in \( G_{\text{flavor}} \), so that the flavor symmetry group poses as our visible sector. Each of these \( G_{\text{flavor}}'s \) are also strongly coupled to exactly one \( U(1) \), which poses as our extra sector. In other words, our models will have an overall group construct of \( G_{\text{flavor}} \times U(1)_{\text{extra}} \).

We will look into \( G_{\text{flavor}} \) as Lie algebra groups \( A_1, A_2 \) and \( D_4 \), and notice how complexities arise as we get higher in the rank. The Mathematica code for each section will also be provided in the Appendix section.

Section 4.1: Flavor Symmetry Group: \( A_1 \)

We will start with rank one symmetry group \( A_1 \) as the simplest case in this series of calculations. The gauge group of \( A_1 \), otherwise known as \( SU(2) \), can be broken to a single \( U(1) \subset SU(2) \) with a proper choice of parameters. Posing as our visible sector, this could be thought of as the \( U(1)_{\varphi} \) or \( U(1)\)-hypercharge, of the SM gauge group. Thus the situation in \( A_1 \) is quite simple: \( U(1)_{\varphi} \times U(1)_{\text{extra}} \).

For the rank one group \( A_1 \) we have \( r = 1 \) and so, as previously defined,

\[ \tau_{\text{mix}} \equiv \tau_{12} = \sqrt{2} \frac{\partial a^D}{\partial m} \quad \tau_{\text{extra}} \equiv \tau_{22} = \frac{\partial a^D}{\partial a} = \frac{\partial a^D}{\partial u}. \]
The parameters $f$ and $g$ from the cubic equation (4.1) for different flavor symmetry groups have been listed in [40]. For the case of $A_1$ symmetry group they are $f = u$ and $g = 4m^2$. Therefore we have,

$$y^2 = x^3 + ux + 4m^2. \quad (4.12)$$

The SW differential for $A_1$ case then, according to [40], can be written as

$$\lambda_2 = c_2 \left( \frac{u}{3} + \frac{m_1 y_1}{x - x_1} \right) \frac{dx}{y},$$

with a single pole at $(x_1, y_1) = (0, 2m_1)$, which will simplify the differential to

$$\lambda_2 = \frac{\sqrt{2}}{4\pi i} \left( \frac{u}{3} + \frac{2m_1^2}{x} \right) \frac{dx}{y} \quad (4.13)$$

where $c_2 = \frac{\sqrt{2}}{4\pi i}$ is the renormalization coefficient.

To get analytical results for $A_1$ coupling constants we will proceed by using perturbation theory. Let us set $w \equiv g = 4m^2$, and introduce a dimension-less parameter $\epsilon = \frac{1}{3\sqrt{2\pi^2}} \frac{w^{4/3}}{m_1^{2/3}}$ as our expansion parameter. Then the cubic (4.12) becomes

$$y^2 = x^3 + \epsilon w^{2/3}x + w. \quad (4.14)$$

The $\frac{1}{3\sqrt{2\pi^2}}$ coefficient in $\epsilon$ is a posteriori for convenience.

For the leading order of perturbation, we will solve the cubic (4.14) with $\epsilon = 0$. We have,

$$x^3 + w = 0,$$

and therefore,

$$x_i = (-w)^{1/3} = \xi_i w^{1/3} \quad (4.15)$$

where $\xi_i$ are the roots of $\xi^3 = -1$. Namely,

$$\xi_i = \begin{cases} 
  e^{2\pi i/6} \\
  -1 \\
  e^{-2\pi i/6}
\end{cases}. \quad (4.16)$$
Now we will move to next order of perturbation in $x_i$'s (4.15). We will have

$$x_i = \xi_i w^{1/3} + \epsilon \cdot \alpha_i,$$  \hspace{1cm} (4.17)

where $\alpha_i$ are to be found. Substituting (4.17) back in the cubic (4.14) and requiring it to be set to zero, we will get

$$(\xi_i w^{1/3} + \epsilon \cdot \alpha_i)^3 + 3\epsilon w^{2/3}(\xi_i w^{1/3} + \epsilon \cdot \alpha_i) + w = 0$$

$$(\xi_i w^{1/3})^3 + 3\epsilon \alpha_i (\xi_i w^{1/3})^2 + 3\epsilon w^{2/3} \xi_i w^{1/3} + w = 0$$

But by definition, $(\xi_i w^{1/3})^3 + w = 0$. Therefore we are left with

$$3\epsilon \alpha_i \xi_i + 3w^{1/3} = 0,$$

and thus,

$$\alpha_i = -\frac{w^{1/3}}{\xi_i}.$$

Having $\alpha_i$, we can plug them back in (4.17) and have our “perturbed” roots of the cubic,

$$e_i = \xi_i w^{1/3} + \epsilon \cdot \left( -\frac{w^{1/3}}{\xi_i} \right) = w^{1/3} \left[ \xi_i - \frac{\epsilon}{\xi_i} \right].$$

Remembering that $\xi_i^3 = -1$, one can check that $-\frac{1}{\xi_i} = \xi_i^2$. Thus, we can rewrite the above roots as

$$e_i = w^{1/3} [\xi_i + \epsilon \cdot \xi_i^2].$$  \hspace{1cm} (4.18)

Let us rewrite the SW differential for the $A_1$ case

$$\lambda_2 = \frac{\sqrt{2}}{4\pi i} \left( \frac{u}{3} + \frac{w}{2x} \right) \frac{dx}{y}. \hspace{1cm} \text{(4.13 revisited)}$$

Having the differential, we can write the periods using (4.4)

$$a = \frac{\sqrt{2}}{4\pi i} \left( \frac{u}{3} \int_A \frac{dx}{y} + \frac{w}{2} \int_A \frac{dx}{xy} \right)$$

$$a^D = \frac{\sqrt{2}}{4\pi i} \left( \frac{u}{3} \int_B \frac{dx}{y} + \frac{w}{2} \int_B \frac{dx}{xy} \right)$$
Using our expansion parameter
\[ \epsilon = \frac{1}{3} \frac{u}{m^4/3} = \frac{1}{3} \frac{u}{w^{2/3}} \]
we can rewrite the above expressions in terms of only the Coulomb branch parameter \( u \). We will therefore have

\[
a = \sqrt{2} \left( \frac{u}{3} I_1^{(A)} + \frac{1}{2} \left( \frac{3\epsilon}{u} \right)^{3/2} I_3^{(A)}(0) \right)
\]
\[
a^D = \sqrt{2} \left( \frac{u}{3} I_1^{(B)} + \frac{1}{2} \left( \frac{3\epsilon}{u} \right)^{3/2} I_3^{(B)}(0) \right)
\]

(4.19)

where \( I_j^{(A)}, I_j^{(B)} \) are the contour integrals defined in (4.9).

Using the perturbative expression we found for the cubic roots in (4.18), we can also write perturbative expressions for the various parameters needed for evaluating these integrals.

\[
k^2 = \frac{e_2 - e_3}{e_1 - e_3} \left( \frac{\xi_2 - \xi_3}{\xi_1 - \xi_3} \right) \left[ 1 + \epsilon(\xi_2 - \xi_1) \right]
\]
\[
k'^2 = \frac{e_2 - e_1}{e_3 - e_1} \left( \frac{\xi_2 - \xi_1}{\xi_3 - \xi_1} \right) \left[ 1 + \epsilon(\xi_2 - \xi_3) \right]
\]
\[
\bar{c}|_{c=0} = \frac{e_3}{e_3 - e_1} \left( \frac{\xi_3}{\xi_3 - \xi_1} \right) \left[ 1 - \epsilon \xi_1 \right]
\]

Considering the form of the roots in (4.16), we can introduce the constant \( \xi \equiv e^{2\pi i/6} \) and rewrite the roots as \( \xi_i = \{ \xi, -1, \bar{\xi} \} \). Using this notation we can simplify the above parameters even more

\[
k^2 = \frac{-1 - \xi}{\xi - \xi} \left[ 1 + \epsilon(-1 - \xi) \right] = -\frac{1 + \xi}{2i \text{ Im} \xi} \left[ 1 - \epsilon(1 + \xi) \right]
\]
\[
k'^2 = \frac{-1 - \bar{\xi}}{\xi - \bar{\xi}} \left[ 1 + \epsilon(-1 - \bar{\xi}) \right] = \frac{1 + \bar{\xi}}{2i \text{ Im} \bar{\xi}} \left[ 1 - \epsilon(1 + \bar{\xi}) \right]
\]
\[
\bar{c}|_{c=0} = \frac{\bar{\xi}}{\xi - \bar{\xi}} \left[ 1 - \epsilon \bar{\xi} \right] = -\frac{\bar{\xi}}{2i \text{ Im} \bar{\xi}} \left[ 1 - \epsilon \bar{\xi} \right]
\]

In addition to the analytic approach that we incorporated here, one could also have a graphical representation of how the coupling constants are distributed. As we move to more complicated flavor symmetry groups the analytic computations of coupling constants will become more and more tedious to accomplish. Hence, the graphical approach will prove to be a very useful tool to complete our objective.

The general approach for graphical representation of coupling constants, is that we would examine the behavior of \( \tau_{\text{mix}} \) when \( \tau_{\text{extra}} \) takes on critical values. These critical conditions come from the Klein invariant \( j \)-function we introduced in (4.2).

\[
j(\tau) = \frac{4f^3}{4f^3 + 27g^2}.
\]
One notices that the conditions \( f = 0 \) and \( g = 0 \) would respectively correspond to \( j = 0 \) and \( j = 1 \). These values are the extremes of the \( j \)-function, and referring back to the geometric properties of the \( j \)-function they associate with

\[
j(e^{2\pi i/3}) = 0 \quad \text{and} \quad j(i) = 1.
\]

So with putting \( f = 0 \) and \( g = 0 \), we should get \( \tau_{\text{extra}} = e^{2\pi i/3} \) and \( \tau_{\text{extra}} = i \), respectively. There is also another condition that results in an additional extreme value for the \( j \)-function. That is,

\[
j(i\infty) \simeq \infty
\]

which corresponds to the condition that the denominator of the Klein invariant \( j \)-function vanishes, i.e. \( 4f^3 + 27g^2 = 0 \). The coupling constant limit \( \tau = i\infty \) is actually associated with weak coupling limit. So we should see interesting results in this limit, in cases where we can test it.

So, to summarize, we have three special values we are interested in

1. \( \tau_{\text{extra}} = e^{2\pi i/3} \) results in \( j = 0 \) associated with \( f = 0 \) (4.20a)
2. \( \tau_{\text{extra}} = i \) results in \( j = 1 \) associated with \( g = 0 \) (4.20b)
3. \( \tau_{\text{extra}} \simeq i\infty \) results in \( j \simeq \infty \) associated with \( 4f^3 + 27g^2 = 0 \) (4.20c)

For graphical representations of coupling constants we use Mathematica for plotting. In this approach, we generate data points abiding the conditions in the last column of equations (4.20), and find the values \( \tau_{\text{mix}} \)'s take on under these conditions. Additionally, we could also check if we get the values in the first column for \( \tau_{\text{extra}} \) using these points; from this point of view, we are utilizing the opposite reasoning of conditions (4.20) in a sense. One should also note, that we will not need all three of these conditions for every flavor symmetry group. For example, we will see that in the case of \( A_1 \), it only suffices to look into the first condition (4.20a).

As mentioned before, in the case of \( A_1 \), the function \( f \) is simply \( f = u \). Therefore, putting \( f = u = 0 \) simplifies the criteria of point generation because we would only need to randomize \( m \). We generate 2000 points using this condition and then plot graphs for \( \tau_{\text{mix}} \) and \( \tau_{\text{extra}} \). The resulting graphs are pictured in Figure 4.3. The Mathematica code we used to produce these graphs is brought in Appendix A.1.

It is evident from fig. 4.3b that all the randomly generated points give \( \tau_{\text{extra}} = e^{2\pi i/3} = -0.5 + 0.866i \) when \( j = 0 \), as expected. Figure 4.3a also determines the values of \( \tau_{\text{mix}} \) at this limit, showing concentration on \( \tau_{\text{mix}} = \pm 0.471405 \).
Figure 4.3: Graphs of coupling constants $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ for the $A_1$ case when $f = u = 0$. (a) graph of $\tau_{\text{mix}}$. Points are concentrated at $\text{Re}[\tau_{\text{mix}}] = \pm0.471405$. (b) graph of $\tau_{\text{extra}}$. All points are concentrated on $\tau_{\text{extra}} = e^{2\pi i/3}$, which is expected as the argument of the $j$-function for $j = 0$. 
As previously stated, the $j = 1; g = 0$ condition does not have a significant importance in the $A_1$ case and it will not add much information to the model. Indeed, putting $g = 4m^2$ equal to zero, is equivalent with putting $m = 0$. This is not a desirable limit for us, because regardless of what range we choose to randomize the Coulomb branch parameter $u$ in, it will have a larger magnitude than the mass parameter $m$, and that is not something we are interested in. Plus, as it is evident from figure 4.4 having $m = 0$ results in all $\tau_{\text{mix}}$ values to vanish; and $\tau_{\text{mix}} = 0$ simply means no mixing!

![Graph of $\tau_{\text{mix}}$ for the $A_1$ case when $g = 0$. At this limit, all the calculated points resulted in $\tau_{\text{mix}} = 0$. This means no mixing between the extra and visible sectors, and therefore it involves no useful information.](image)

Although it is clear that $\tau_{\text{mix}}$ vanishes when $m = 0$, and that is indeed what was expected, there is still lots of information that can be extracted from the $A_1$ case. One useful way to learn more information about $A_1$ coupling constants is to hold $u$ constant at a small value and find the (graphical) dependance of $\tau_{\text{mix}}$ with respect to the mass parameter. In Figure 4.5 we found the dependance of components of $\tau_{\text{mix}}$ to the mass parameter while the mass parameter took real values between $[0, 10]$. The Coulomb branch parameter $u$ was set equal to 0.1 (orange) and 0.01 (blue), for comparison. Again as expected, the graph becomes more extreme as we get closer to $u = 0$. The Mathematica code for these graphs is provided at the end of Appendix [A.1]

**Section 4.2: Flavor Symmetry Group: $A_2$**

The next symmetry group to examine is $A_2$, also known as $SU(3)$. As we go higher in the rank of flavor symmetry groups, the analytical computation of coupling constants becomes more complicated. This is why, for the case of $A_2$ and higher, we only focus on the graphical representation.

With a proper choice of parameters, the $SU(3)$ can be broken into $SU(3) \supset SU(2) \times U(1)$; or eventually,
Figure 4.5: Graphs of components of coupling constant $\tau_{\text{mix}}$ vs. mass parameter $m$, when $u = 0.1$ (orange) and $u = 0.01$ (blue), and mass parameter $m$ takes over values from $[0, 10]$ range. Graphs are for the $A_1$ case. We notice that the curvature of component graphs (a and b) increase for smaller $u$. Graph c just gets closer to expected $\text{Im}\tau_{\text{mix}} = 0$, as $u$ gets closer to zero.
as we saw in last section, \( SU(3) \supset U(1)^2 \). To find a motivated choice of parameters, we will have a reminder of the roots and weights of \( SU(3) \), and focus on its adjoint representation. With the following choice of roots and weights for \( SU(3) \)

- roots: \( \vec{a}_1 = \sqrt{2}(\frac{1}{2}, \frac{\sqrt{3}}{2}) \), \( \vec{a}_2 = \sqrt{2}(\frac{1}{2}, -\frac{\sqrt{3}}{2}) \)

- weights: \( \vec{\lambda}_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})\sqrt{2} \), \( \vec{\lambda}_2 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}})\sqrt{2} \), \( \vec{\lambda}_3 = (0, -\frac{1}{\sqrt{3}})\sqrt{2} \)

and the help of relation (4.5) written in the following form

\[
m_a = \sum_f \varphi_f \vec{\lambda}_a \cdot \vec{\alpha}_f, \tag{4.21}
\]

where \( a = 1, 2, 3 \) since the adjoint of \( SU(3) \) has dimension three, and \( f = 1, 2 \) since \( SU(3) \) is of rank \( r = 2 \), we can find the following parameters as our proper choice

\[
m_1 = \varphi_1 \quad m_2 = -\varphi_2 \quad m_3 = -\varphi_1 + \varphi_2.
\]

With this choice of parameters, the adjoint scalar of \( A_2 \) is then given by

\[
\phi = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}
\]

with \( \sum_a m_a = 0 \).

Now that we have broken \( SU(3) \) to \( U(1)^2 \), we can proceed with the technical computation. The \( f \) and \( g \) functions in the cubic (4.1) for \( A_2 \), are given in [40] as

\[
f = 4(\varphi_1^2 + \varphi_2^2 - \varphi_1\varphi_2)
\]
\[
g = 8\varphi_1\varphi_2(\varphi_1 - \varphi_2) - u^2. \tag{4.22}
\]

Thus the cubic for \( A_2 \) is

\[
y^2 = x^3 + 4x(\varphi_1^2 + \varphi_2^2 - \varphi_1\varphi_2) + 8\varphi_1\varphi_2(\varphi_1 - \varphi_2) - u^2,
\]
and the SW differential, for the adjoint representation of $SU(3)$ is given by

$$\lambda_3 = \frac{\sqrt{2}}{8\pi i} \left[ \frac{3}{2} \frac{dx}{y} + \left( \frac{m_1 u}{x - \frac{m_2}{2}} + \frac{m_2 u}{x - \frac{m_3}{2}} + \frac{m_3 u}{x - \frac{m_1}{2}} \right) \right] \frac{dx}{y},$$

with the pole sections $(x_a, y_a) = (\frac{m_a}{2}, u)$.

Following the same steps as before, we can then write expressions for SW periods $a$ and $a^D$. The general gauge group of the model in case of $A_2$ is $\sim U(1)^2 \times U(1)_{\text{extra}}$. Let us have a consistency check of indices here.

We are working with a rank 2 flavor symmetry, thus $r = 2$. The period $a$ that we calculate using the SW differential is associated with the extra $U(1)_{D3}$. If we index the $U(1)$’s, the indices 1 and 2 associate with the visible $U(1)$’s, and the 3rd index is for $U(1)_{D3}$. Respectively, $a_1$ and $a_2$ are related to $\propto \varphi_1$ and $\propto \varphi_2$, whereas $a_3$. For the coupling constants, we have as before,

$$\tau_{\text{mix}1} = \tau_{13} = \sqrt{2} \frac{\partial a^D}{\partial \varphi_1}, \quad \tau_{\text{mix}2} = \tau_{23} = \sqrt{2} \frac{\partial a^D}{\partial \varphi_2},$$

$$\tau_{\text{extra}} = \tau_{33} = \frac{\partial a^D}{\partial a} = \frac{\partial a^D/\partial u}{\partial a/\partial u}.$$

Now we can start the calculations in Mathematica (the code for this section is given in Appendix A.2). Here in the case of $A_2$, we would examine the behavior of $\tau_{\text{mix}}$, when $\tau_{\text{extra}}$ takes on the critical values (4.20). We would find out that in this case, it is sufficient to only examine the conditions when $j = 0$ and $j = 1$, i.e. (4.20a) and (4.20b) respectively.

We generate 4000 points under each condition using Mathematica, and then plot complex graphs for $\tau_{\text{mix}}$, and $\tau_{\text{extra}}$. Looking at the definition of $f$ and $g$ for $A_2$ in (4.22), one could notice that $f$ is symmetric under $\varphi_1 \leftrightarrow \varphi_2$ exchange, so without loss of generality, we can show the graph of only of the $\tau_{\text{mix}}$’s in the case of $f = 0$. This is not true for $g$, however - although we will see that the graphs of $\tau_{\text{mix}1}$ and $\tau_{\text{mix}2}$ are still pretty similar in the case of $g = 0$.

Let us investigate each of these cases individually.

- **The case where $j = 0; f = 0$:** This corresponds to condition (4.20a) associated with $\tau_{\text{extra}} = e^{2\pi i/3}$. To impose the condition on mass parameters we solve the equation $f = 0$ based on (4.22). Solving for one of the $\varphi$’s, we will have

$$f = 4(\varphi_1^2 + \varphi_2^2 - \varphi_1 \varphi_2) = 0 \quad \Rightarrow \quad \varphi_2 = \varphi_1 \pm \frac{i \sqrt{3}}{2}.$$

Using Mathematica then we generate 4000 data points with this condition on the mass parameters. We
evaluate the coupling constants associated with these data points and graph these values. The results are given in Figure 4.6. Specifically, Figure 4.6a shows the graph of $\tau_{mix_1}$; as it was stated above, the graph of $\tau_{mix_2}$ looks almost identical (up to numerical fluctuations). Figure 4.6b confirms $\tau_{extra} = e^{2\pi i/3}$ under this condition.

Figure 4.6: Graphs of coupling constants $\tau_{mix_1}$ and $\tau_{extra}$ for the $A_2$ case when $f = u = 0$. (a) graph of $\tau_{mix}$. Points are localized in bands at $Re[\tau_{mix}] \approx 0, \pm 0.35, \pm 0.70$. (b) graph of $\tau_{extra}$. All points are concentrated on $\tau_{extra} = e^{2\pi i/3} = -0.5 + 0.866i$, which is expected under the condition $j = 0$.

- The case where $j = 1; g = 0$: This case corresponds to condition (4.20b) with $\tau_{extra} = i$. To
find the condition to impose we will solve the $g = 0$ equation. Referring to (4.22) again, this will result in

$$g = 8\varphi_1\varphi_2(\varphi_1 - \varphi_2) - u^2 = 0 \quad \Rightarrow \quad \varphi_2 = \begin{cases} \frac{u^2}{8\varphi_1} \\ \varphi_1\left(1 - \frac{u^2}{8\varphi_1}\right) \end{cases}.$$  

Again, we generate a total of 4000 points with these conditions on the mass parameters and graph the coupling constant values corresponding to these points. We get Figures 4.7a, 4.7b for $\tau_{\text{mix}}$’s, and Figure 4.8 for the $\tau_{\text{extra}}$.

These graphs conclude our computation of coupling constants in the case of $A_2$ flavor symmetry group. Next, we will explore the flavor symmetry group $D_4$. As we have mentioned before, since we are getting higher in the rank of gauge group, we will face even more complexity in our computations.

**Section 4.3: Flavor Symmetry Group: $D_4$**

The proper parameters needed to break the symmetry group $D_4$, also known as $SO(8)$, into a bundle of $U(1)$’s can be found using the roots and weights of $SO(8)$ and the help of relation (4.21). We will get

$$m_1 = \varphi_1 \quad m_2 = -\varphi_1 + \varphi_2 \quad m_3 = -\varphi_2 + \varphi_3 + \varphi_4 \quad m_4 = -\varphi_3 + \varphi_4.$$  

The $f$ and $g$ functions for $D_4$ are again imported from [40]. We have

$$f = u^2 + \tilde{w}_4$$
$$g = w_2u^2 + w_4u + w_6,$$

which result in the cubic

$$y^2 = x^3 + (u^2 + \tilde{w}_4)x + (w_2u^2 + w_4u + w_6).$$

(4.23)
Figure 4.7: Graphs of $\tau_{\text{mix}_1}$ and $\tau_{\text{mix}_2}$ for $g = 0$ case in $A_2$.  

(a) graph of $\tau_{\text{mix}_1}$ when $g = 0$  

(b) graph of $\tau_{\text{mix}_2}$ when $g = 0$  

We notice that, although the distribution might not be identical, but in both graphs the data points are concentrated around the same Re[$\tau_{\text{mix}}$] values.
The various parameters mentioned in these expressions are defined as

\[ w_2 = \frac{1}{3} z_2 \]
\[ w_4 = \tilde{z}_4 \]
\[ \tilde{w}_4 = z_4 - 3w_2^2 \]
\[ w_6 = z_6 + w_2^3 + w_2 \tilde{w}_4 \]

where

\[ z_2 = -\sum_{i=1}^{4} m_i^2 \]
\[ z_6 = -\sum_{i<j<k} m_i^2 m_j^2 m_k^2 \]
\[ z_4 = \sum_{i<j} m_i^2 m_j^2 \]
\[ \tilde{z}_4 = -2im_1m_2m_3m_4. \]

We write the SW differential for the \( D_4 \) case using the pole sections \( (x_f, y_f) = (m_f^2 - w_2, m_f u + \frac{w_2}{2m_f}) \). It will be

\[ \lambda_8 = \frac{\sqrt{2}}{8\pi i} \left[ u \frac{dx}{y} + \frac{1}{2} \sum m_f y_f \frac{dx}{x - x_f \frac{dx}{y}} \right], \]

where \( f = 1, \ldots, 4 \) since \( D_4 \) is a rank 4 group [40]. The SW periods would then be determined by \( a = \oint_A \lambda_8 \) and \( a^O = \oint_B \lambda_8 \). Given how things are set up at the moment, we will end up with five independent coupling
constants originally. Trivially, they are defined as below

\[ \tau_{\text{mix}} \equiv \tau_f = \frac{\partial a}{\partial \phi_f} \quad \tau_{\text{extra}} \equiv \tau_5 = \frac{\partial a}{\partial a/\partial u} . \]

The above choice of parameters will break \( D_4 \) into \( SU(2) \times U(1) \times U(1) \). This means, that the interaction of \( D_4 \) with the \( U(1)_{\text{extra}} \), could be simplified to interaction of only one of the \( U(1) \)'s with \( U(1)_{D3} \). Therefore, for the sake of numerical simplicity we can focus only on that part of \( D_4 \) by setting

\[ \varphi_3 = \varphi_4 = 0 \]

Furthermore one can notice from (4.23), that both functions \( f \) and \( g \) of \( D_4 \) are symmetric under exchange of \( \phi_i \)'s. Therefore, we will furthermore simplify the theory by setting

\[ \varphi_1 = \varphi_2 \equiv \varphi . \]

These choices simplify \( D_4 \) all the way to \( U(1)^4 \). As a result, we will only have two independent coupling constants to investigate graphically, \( \tau_{\text{mix}} \equiv \tau_{\text{mix}} \) and \( \tau_{\text{extra}} \).

We will use Mathematica to generate points again. The codes for this section are brought in Appendix A.3. In the case of \( D_4 \), we would need to investigate all three of the critical conditions in (4.20). Let us get through each of them individually.

**Condition (4.20a):** \( j = 0; f = 0 \):

To find the exact conditions to impose on the mass parameter, we will use (4.23) with conditions (4.25) and (4.26) to solve the \( f = 0 \) equation. We will get

\[ -\frac{\varphi^4}{3} + u^2 = 0 , \]

which has four roots for the mass parameter in relation to \( u \). These are

\[ \varphi = \left\{ -3^{1/4} \sqrt{\bar{u}}, -i3^{1/4} \sqrt{\bar{u}}, i3^{1/4} \sqrt{\bar{u}}, 3^{1/4} \sqrt{\bar{u}} \right\} . \]

We generate 4000 combined points with these conditions, and we get Figures 4.9a and 4.9b as graphs for \( \tau_{\text{mix}} \) and \( \tau_{\text{extra}} \) respectively.
Figure 4.9: Graphs of $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ for the $f = 0$ case of $D_4$. a. In the graph of $\tau_{\text{mix}}$ the data points are concentrated on six points, whereas in b. graph of $\tau_{\text{extra}}$ all points are concentrated on $-0.5 + 0.866i$, as expected for $j = 0$. 
Condition (4.20b): \( j = 1; g = 0 \): 

We will now have to solve the \( g = 0 \) equation with the same conditions as above. Using (4.23), (4.25) and (4.26) we get 

\[
\frac{-2\varphi^6}{27} - \frac{2\varphi^2u^2}{3} = 0
\]

\[
\text{\Rightarrow }
\begin{align*}
\varphi^2 &= 0 \\
\varphi &= (-1)^{1/4}\sqrt{3u} \\
\varphi &= (-1)^{3/4}\sqrt{3u}
\end{align*}
\]

We generate 4000 combined data points using the nontrivial solutions above, which in a more compact way can be written as \( \varphi = \{\pm(-1)^{1/4}\sqrt{3u}, \pm(-1)^{3/4}\sqrt{3u}\} \). The resulting graphs for \( \tau_{\text{mix}} \) and \( \tau_{\text{extra}} \) are depicted in Figure 4.10 for \( g = 0 \) case of \( D_4 \).

Condition (4.20c): \( 4f^3 + 27g^2 = 0 \) (Weak coupling limit) 

This condition is associated with \( \tau_{\text{extra}} \approx i\infty \), which is the limit for weak coupling. To achieve this condition we have to set the denominator of the \( j \)-function equal to zero. Using (4.23), (4.25) and (4.26) again we have 

\[
4f^3 + 27g^2 = 0 \quad \Rightarrow \quad 4u^2(\varphi^4 + u^2)^2 = 0
\]

which gives the following solutions for \( \varphi \),

\[
\varphi = (-1)^{1/4}\sqrt{u} \quad \varphi = (-1)^{1/4}\sqrt{u} \quad \varphi = (-1)^{3/4}\sqrt{u} \quad \varphi = (-1)^{3/4}\sqrt{u},
\]

or, more compactly, \( \varphi = \{\pm(-1)^{1/4}\sqrt{u}, \pm(-1)^{3/4}\sqrt{u}\} \).

Numerically, however, we cannot generate data using these conditions directly, since they produce a zero denominator in calculations. Instead, we will solve the equation for when \( 4f^3 + 27g^2 \) is equal to a variable \( \text{eps}=0.001 \), and then keep only the points for which the magnitude of \( j \)-function is greater than another variable \( \text{cap}=100 \) (this is to approximate \( j \approx \infty \)). The Mathematica code for this subsection brought in Appendix A.3 outlines this procedure. We plot the coupling constant graphs for these selected points. The resulting graphs are depicted in Figure 4.11.
Figure 4.10: Graphs of $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ for the $g = 0$ case of $D_4$. **a.** graph of $\tau_{\text{mix}}$ shows the values it takes when $j = 1$. **b.** graph of $\tau_{\text{extra}}$, showing that $\tau_{\text{extra}} = i$ when $j = 1$ as expected.
Figure 4.11: Graphs of the coupling constants for the weak coupling limit of $D_4$. 

\textbf{a.} The graph of $\tau_{\text{mix}}$ is divided into four rectangular regions in the four quadrants. 

\textbf{b.} The graph of $\tau_{\text{extra}}$ shows all the values of $\tau_{\text{extra}}$ concentrating in a band between $\text{Im}[\tau_{\text{extra}}] \approx 1.9$ and $\text{Im}[\tau_{\text{extra}}] \approx 2.8$. 
However, the weak coupling limit means that the $\tau_{\text{mix}}$ values should be small. We do not get a sound confirmation on this from Figure 4.11. Therefore, a different approach to take here, separate from the three conditions (4.20), is to not constrain the mass parameter $\varphi$ to take any specific values in relation to $u$; we only require it to be comparable to $u^{1/2}$ (the power of $1/2$ comes from checking the dimensionality analysis in (4.24)). So in other words, we are letting $\varphi$, as well as $u$, to take on values from all over the range, without constraints. For example, in the following sample graphs $|u| = [0, 100]$, and thus $|\varphi| = [0, 10]$. Turns out in this case we achieve interesting results which include small $\tau_{\text{mix}}$ values, but they are no longer in $\varphi^2 << u$ regime. See Figure 4.12.

Figure 4.12: $\tau_{\text{mix}}$ and $\tau_{\text{extra}}$ graphs for $D_4$ symmetry group when $\varphi$ is free to take on any value in a range equivalent to root range of $u$, no constraints. a. graph of $\tau_{\text{mix}}$. b. graph of $\tau_{\text{extra}}$. 

52
Section 4.4: Groups with Higher Rank Flavor Symmetry

A very interesting extension of this procedure, would be to follow the calculation in case of groups with higher ranked flavor symmetry, namely $E_6$ and $E_8$. The case of $E_8$ is particularly interesting because the largest subgroup of $E_8$ is $SU(5) \times SU(5) \subset E_8$, and the $SU(5)$ can itself be broken into

$$SU(5)_{GUT} \supset SU(3)_C \times SU(2)_L \times U(1)_Y.$$  

The right-hand side of the above expression is exactly the flavor symmetry group of Standard Model, $G_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$. So weakly gauging the $G_{\text{SM}}$ subgroup of $E_8$ will provide us with our desired $U(1)_Y \times U(1)_{\text{extra mixing}}$.

We leave computations for both $E_6$ and $E_8$ cases for future work. In the next chapter, we will shift our focus to supersymmetry breaking, and the dependence of effects of supersymmetry breaking on coupling constants.
CHAPTER 5: Breaking Supersymmetry

In our calculation of kinetic mixing in the previous chapter we were consistently working in the limit of exact $\mathcal{N} = 2$ supersymmetry. Supersymmetry (SUSY) in general suggests that each elementary particle has a superpartner particle: there is a boson for each fermion, and vice versa. Of course such symmetries are “broken” in nature. In this Chapter we discuss a toy model for SUSY breaking in the extra sector. We will do this by introducing a class of auxiliary parameters, which we will activate when we want to shift from the supersymmetric limit. Breaking supersymmetry generally refers to a phenomenon as a result of which the masses of superpartners would end up differing. To be able to keep the appealing features of supersymmetry while also seeing the effects of SUSY breaking, this breaking must be spontaneous - this means that the Lagrangian would still remain supersymmetric.

We will first lay out the groundwork for SUSY breaking calculations with a small introduction section. Then we will dedicate a section each to the Gauge Lagrangian and Coulomb branch Lagrangian without the superpotential. In Section 5.4 we will introduce the superpotential to the Lagrangian and calculate the bosonic and fermionic mass values in the supersymmetric limit. In the last section we break the supersymmetry by activating a perturbation in the fields and see the change in mass values.

Section 5.1: Introduction

As we already discussed in Chapter 4, an $\mathcal{N} = 2$ SUSY multiplet includes both chiral and vector multiplets of $\mathcal{N} = 1$ SUSY. However, the Lagrangian provided in (3.12) has a very general form and from the analytical point of view, not very practical to work with. A better and more specific but still general, supersymmetric Lagrangian for $\mathcal{N} = 2$ SUSY is given by

$$\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} K(\Phi^i, \Phi^i) + \left[ \int d^2 \theta P(\Phi^i) + \text{h.c.} \right] + \left[ \frac{1}{16 \pi^2} \int d^2 \tau_{ij} W^i_{\alpha} W^\alpha_{\beta} + \text{h.c.} \right],$$  \hspace{1cm} (5.1)

where $\Phi^i$ are the chiral superfields introduced in (2.12), $W^i_{\alpha}$ are the spinorial superfields as defined in (2.15), and h.c. stands for hermitian conjugate. Also $K(\Phi^i, \Phi^i)$ here is the Kähler potential which relates to the metric and for $\mathcal{N} = 2$ SUSY theories with $A_1$ gauge group is given by

$$K = \frac{\bar{a} a^D - a a^D}{2i},$$  \hspace{1cm} (5.2)
Eventually, $P(Φ^i)$ is the superpotential, a differentiable function of the Coulomb branch parameter $u$ and the fields, which we will discuss in the upcoming sections. Throughout all of this chapter, the indices $i, j, k,...$ go over values 1, 2 only.

To distinctly track the effects of SUSY breaking, we will introduce auxiliary parameters $F^i$ such that

$$a^i \mapsto a^i + θ^2 F^i.$$ (5.3)

With this notation when $F^i = 0$ we are in the supersymmetric limit, and for non-zero $F^i$ we can see the SUSY breaking effects in each step by tracking the terms associated with $F^i$'s.

In this chapter we will mainly explore the $A_1$ symmetry group. Since $A_1$ is a rank one group, we have $r = 1$, and thus only one mass field and parameter $a^1 \propto m$, and we also have $a \equiv a^{r+1} \equiv a^2$ associated with the extra sector. Therefore we will have

$$m \mapsto m + θ^2 F^1 \quad \quad a \mapsto a + θ^2 F^2.$$ (5.4)

The higher symmetry groups would simply have more than one mass parameter.

Also, as a reminder, the last term in (5.1) can be rewritten as

$$\frac{1}{16πi} \int d^2θτ_{ij} W^i W^j + \text{h.c.} = \text{Re} \int d^2θ \frac{1}{8π} τ_{ij} W^i W^j = \frac{1}{8π} \text{Im} \int d^2θ τ_{ij} W^i W^j,$$

which is the same Lagrangian given in (2.17). This portion of the Lagrangian [5.1], is commonly known as the *Gauge Lagrangian*, and the first two terms are the *Coulomb branch Lagrangian*.

**Section 5.2: Gauge Lagrangian**

Let us rewrite the Gauge Lagrangian as we just introduced,

$$L_{\text{gauge}} = \frac{1}{8π} \text{Im} \int d^2θ τ_{ij} W^i W^j.$$ (5.5)

where $τ_{ij} = \frac{θ_{ij}}{2π} + \frac{2πi}{g_{ij}}$ are the generalized coupling constants, and

$$W^i_α = -iλ^i_α(y) + D^i(y)θ_α - iσ^{μν}_αβ F^i_{μν} + θ^2σ^μ \nabla_μ \bar{λ}^i_α(y).$$ (5.6)
are the generalized superfields. This will result in

\[ L_{\text{gauge}} = \frac{1}{g_{ij}} \left( -\frac{1}{4} F_{\mu\nu}^i F^{j,\mu\nu} - i \lambda^i \sigma^\mu \partial_\mu \bar{\lambda}^j + \frac{1}{2} D^2 \right) + \frac{\theta_{kl}}{32\pi^2} F_{\mu\nu}^k \tilde{F}^{l,\mu\nu}. \]

We will now take the coupling constants \( \tau_{ij} \) to supersymmetry breaking limit by expanding it with respect to its components \( m \) and \( a \), using the transformations (5.4). This expansion will look like

\[ \tau_{ij} \mapsto \tau_{ij} + \sqrt{2} \frac{\partial \tau_{ij}}{\partial m} \theta^2 F^1 + \frac{\partial \tau_{ij}}{\partial a} \theta^2 F^2, \tag{5.7} \]

where in the second term we have used the fact that \( a^1 = \sqrt{2} m \). The terms related to \( a^2 = a \), and in turn \( F^2 \), are charged under the \( U(1)_{\text{extra}} \) and are therefore heavy. As we will get to it more in the next section of this chapter, we “integrate out” the heavy states in order to have a meaningful computation. In this section, however, we will suffice to saying that we will only keep one of the additional terms in (5.7), namely \( \frac{\partial \tau_{ij}}{\partial m} \theta^2 F^1 \). Also, to avoid redundancy and with misuse of notation, we will from here on replace \( \sqrt{2} F^1 \) with \( F^1 \).

Now we would have to incorporate the expansion (5.7) in Lagrangian (5.5). It is easy to note, however, that in the subsequent expansion of \( \tau_{ij} W^i W^j \), we would only need the first term of \( W_\alpha \)'s in (5.6), since the expanded terms of \( \tau_{ij} \) already provide the factor of \( \theta^2 \). Thus, the contributing factor of \( W^\alpha W_\alpha \) would be \( W^i W^j \big|_{\theta_0} = -\lambda^i \lambda^j \), and the new Lagrangian can be written as

\[ L_{\text{gauge}}^\text{new} = \frac{1}{8\pi} \text{Im} \int d^2 \theta(\tau_{ij} + \frac{\partial \tau_{ij}}{\partial m} \theta^2 F^1) W^i W^j \]

\[ = \frac{1}{8\pi} \text{Im} \int d^2 \theta \tau_{ij} W^i W^j - \frac{1}{8\pi} \text{Im} F^1 \frac{\partial \tau_{ij}}{\partial m} \lambda^i \lambda^j \tag{5.8} \]

From (3.15) we know that

\[ \tau_{ij} = \frac{\partial a_i^D}{\partial a_j} = \frac{\partial^2 F}{\partial a_i \partial a_j}, \]

where \( F \) is a holomorphic function of the elements \( a_k \) called the prepotential. We can now introduce elements \( F_{ijk} \) such that

\[ F_{ijk} = \frac{\partial^3 F}{\partial a^i \partial a^j \partial a^k} = \frac{\partial \tau_{ij}}{\partial a_k} \tag{5.9} \]

\(^1\) Compare with equations (2.15) and (2.17).
Incorporating this in (5.8) we will have,

\[ L_{\text{new gauge}} = L_{\text{old gauge}} - \frac{1}{8\pi} \text{Im}(F^1_iF^{-1}_{ij}\lambda^i\lambda^j) \]

\[ = L_{\text{old gauge}} - \frac{1}{8\pi} F^1(F_{111}\lambda^1\lambda^1 + 2F_{112}\lambda^1\lambda^2 + F_{122}\lambda^2\lambda^2) \]  

(5.10)

Here \( L_{\text{old gauge}} \) is the original Lagrangian in (5.5) before entering the SUSY breaking limit. Let us rewrite this Lagrangian again, this time separating the real and imaginary parts of the \( W^iW^j \) product. We will have

\[ L_{\text{old gauge}} = \frac{1}{8\pi} \text{Im} \int d^2\theta \tau_{ij} W^iW^j \]

\[ = \frac{1}{8\pi} \left[ \text{Im}(\tau_{ij})\left( -\frac{1}{2}F^i_{\mu\nu}\tilde{F}^{j,\mu\nu} - 2i\lambda^i\sigma^\mu\partial_\mu\bar{\lambda}^j + D^2 \right) + \text{Re}(\tau_{ij})\left( \frac{1}{2}F^i_{\mu\nu}\tilde{F}^{j,\mu\nu} \right) \right] \]

Introducing a \( C_{ij} \) matrix such that \( C_{ij} \equiv \text{Im}(\tau_{ij}) \) we can rewrite this as,

\[ L_{\text{old gauge}} = \frac{1}{16\pi} \text{Re}(\tau_{ij})F^i_{\mu\nu}\tilde{F}^{j,\mu\nu} - \frac{1}{16\pi} C_{ij}F^i_{\mu\nu}F^{j,\mu\nu} + \frac{1}{8\pi} C_{ij}D^2 - \frac{i}{4\pi} C_{ij}\lambda^i\sigma^\mu\partial_\mu\bar{\lambda}^j \]

Substituting this back in (5.10), we have

\[ L_{\text{new gauge}} = \frac{1}{16\pi} \text{Re}(\tau_{ij})F^i_{\mu\nu}\tilde{F}^{j,\mu\nu} - \frac{1}{16\pi} C_{ij}F^i_{\mu\nu}F^{j,\mu\nu} + \frac{1}{8\pi} C_{ij}D^2 \]

\[ - \frac{i}{4\pi} C_{ij}\lambda^i\sigma^\mu\partial_\mu\bar{\lambda}^j - \frac{1}{8\pi} \text{Im} F^1F_{ij}\lambda^i\lambda^j \]  

(5.11)

Both terms written on the second line of Lagrangian (5.11) are related to the fields \( \lambda^i \). As mentioned in Chapter 2, the \( \lambda^i \) are the gauginos in \( \mathcal{N} = 2 \) theories. Therefore these two terms represent the fermionic terms in the Lagrangian. Let us narrow our focus on them.

\[ \Lambda = -\frac{1}{8\pi} \left( 2iC_{ij}\lambda^i\sigma^\mu\partial_\mu\bar{\lambda}^j + F^1F_{ij}\lambda^i\lambda^j \right) \]

(5.12)

At this point, we need to find a more helpful way to represent the matrix \( C_{ij} \). We know that the imaginary part of \( \tau_{\text{visible}} \) is \( \alpha_{EM}^{-1} \), the inverse fine-structure constant; i.e. \( C_{11} = 137 \). Assuming the rest of the elements in the matrix are of the order of 1, we can replace the \( C_{ij} \) matrix with the following approximation

\[ \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \approx \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} \equiv \hat{C}_{ij} \]
Indeed, since $C_{11}$ is much larger than $C_{12} = C_{21}$, we have

$$\det C_{ij} = C_{11}C_{22} - C_{12}^2 \approx C_{11}C_{22} = \det \hat{C}_{ij}$$

$$\text{Tr} C_{ij} = C_{11} + C_{22} = \text{Tr} \hat{C}_{ij}$$

Therefore, replacement of $C_{ij}$ with $\hat{C}_{ij}$ is a valid, well-motivated approximation. The fermionic Lagrangian terms $\Lambda$ [5.12] now becomes

$$\Lambda = 2i\hat{C}_{ij} \lambda^i \sigma^\mu \partial_\mu \bar{\lambda}^j + F^1 F_{1ij} \lambda^i \lambda^j.$$

Writing out each term explicitly we will have

$$\Lambda = 2iC_{11} \lambda^1 \sigma^\mu \partial_\mu \bar{\lambda}^1 + 2iC_{22} \lambda^2 \sigma^\mu \partial_\mu \bar{\lambda}^2 + F^1 F_{111} \lambda^1 \lambda^1 + 2F^1 F_{112} \lambda^1 \lambda^2 + F^1 F_{122} \lambda^2 \lambda^2.$$

We will now introduce the "effective photini" $\eta^i$, such that $\eta^i = \lambda^i \sqrt{C_{ii}}$; we can rewrite the above expression as

$$\Lambda = 2i\eta^1 \sigma^\mu \partial_\mu \eta^1 + 2i\eta^2 \sigma^\mu \partial_\mu \eta^2 + F^1 \left( \frac{F_{111}}{C_{11}} \eta^1 \eta^1 + \frac{2F_{112}}{\sqrt{C_{11}C_{22}}} \eta^1 \eta^2 + \frac{F_{122}}{C_{22}} \eta^2 \eta^2 \right).$$

To see a mixing effect between the gauginos, or equivalently photini, of the visible and extra sectors we need the second term in the parenthesis, the term proportional to $\eta^1 \eta^2$, to have a significant effect in this expression. The matrix element associated fully with the visible sector $C_{11}$, is by definition a large quantity, so the first term in the parenthesis is already small. Thus, we will need to deal with two quantities,

$$M_{\text{extra}} \equiv \frac{F_{122}}{C_{22}} \quad \quad \varepsilon_{\text{mix}} \equiv \frac{F_{112}}{\sqrt{C_{11}C_{22}}}.$$

We will need $\varepsilon_{\text{mix}}$ to be non-zero and at least comparable to $M_{\text{extra}}$.

To check this condition we use Mathematica to graph the values of $|M_{\text{extra}}|$ and $|\varepsilon_{\text{mix}}|$. We generated 2000 data points with Coulomb branch parameter values limited to a range of $|u| = [0, 10]$. The result is brought in Figure 5.1. The blue trail represents the $\varepsilon_{\text{mix}}$ values calculated with these inputs of $u$, and the purple trail is the one corresponding to $M_{\text{extra}}$ values. The effect of $\varepsilon_{\text{mix}}$, although not significant, but is in visible comparison with $M_{\text{extra}}$ within these ranges of values.

The Mathematica code used for producing this graph is brought in Appendix B.1.

\footnote{There is no summation on $i$ of course. The elements of the matrix are just acting as coefficients.}
Figure 5.1: Graph of gaugino mixing parameters $\epsilon_{\text{mix}}$ (blue) and $M_{\text{extra}}$ (purple) for the $A_1$ case. On x-axis we have $|u| = [0, 10]$. In this range, $\epsilon_{\text{mix}}$ values are rather small, but comparable to $M_{\text{extra}}$ values.

Section 5.3: Coulomb Branch Lagrangian

Now, as mentioned earlier in the chapter, we will examine the rest of the Lagrangian \( \mathcal{L} \) called the Coulomb Lagrangian.

\[
\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\Phi_i, \Phi_i^\dagger) + \int d^2\theta P(\Phi^i) + \text{h.c.}
\]

According to [1], for $\mathcal{N} = 2$ SUSY theories, this Lagrangian can be written as,

\[
\mathcal{L}_{\text{Coulomb}} = g_{ij} F^i F^{*j} + \frac{1}{4} g_{ij,k\bar{l}} \chi^i \bar{\chi}^k \bar{\chi}^j \chi^l - F^i \left( \frac{1}{2} g_{ij,k} \bar{\chi}^k \bar{\chi}^{j*} \chi^i \right) - F^{*i} \left( \frac{1}{2} g_{ij,k} \chi^i \bar{\chi}^k \chi^j \right) - g_{ij} \partial_{\mu} a^i \partial_{\bar{\alpha}} a^j - i g_{ij} \bar{\partial}_{\bar{\mu}} a^i \partial_\alpha a^j - \frac{1}{2} \partial^2 P \partial_\alpha a^i \chi^i \bar{\chi}^j \chi^j - \frac{1}{2} \partial^2 \bar{P} \partial_{\bar{\alpha}} a^j \bar{\chi}^j \chi^i \chi^i. \tag{5.13}
\]

In these equations the $g_{ij}$ is the metric and is related to the Kähler potential as follows

\[
g_{ij} = \frac{\partial^2 K}{\partial a^i \partial a^j},
\]

and its derivatives are given by

\[
g_{ij,k} = \frac{\partial}{\partial a^k} g_{ij}, \quad g_{ij,\bar{k}} = \frac{\partial}{\partial \bar{a}^k} g_{ij}, \quad g_{ij,ki} = \frac{\partial g_{ij}}{\partial a^k \partial a^i}.
\]
Let us calculate these derivatives for our case of Kähler potential, given in (5.2).

\[
g_{ij} = \frac{\partial}{\partial a^i} \frac{\partial}{\partial \bar{a}^j} \left( \frac{\partial_D^k - a^k D^j}{2i} \right) = \frac{1}{2i} \left( \frac{\partial_D^k}{\partial a^i} - \frac{\partial_D^k}{\partial \bar{a}^j} \right) = \frac{1}{2i} (\tau_{ij} - \tau_{ij}) = \text{Im}(\tau_{ij}) \equiv C_{ij}
\]

(5.14)

and its derivatives would be

\[
g_{ij,k} = \frac{\partial}{\partial a^k} g_{ij} = \frac{1}{2i} \left( \frac{\partial}{\partial a^k} (\tau_{ij} - \tau_{ij}) \right) = \frac{1}{2i} \left( \frac{\partial \tau_{ij}}{\partial a^k} - 0 \right) = \frac{1}{2i} F_{ijk}
\]

\[
g_{ij,k} = \frac{\partial}{\partial \bar{a}^k} g_{ij} = \frac{1}{2i} \left( \frac{\partial}{\partial \bar{a}^k} (\tau_{ij} - \tau_{ij}) \right) = \frac{1}{2i} \left( 0 - \frac{\partial \tau_{ij}}{\partial \bar{a}^k} \right) = \frac{1}{2i} \bar{F}_{ijk}
\]

\[g_{ij,k\bar{k}} = 0\]  

As it was mentioned earlier in the chapter, \(F_{ijk}\) and \(\bar{F}_{ijk}\) are respectively holomorphic and anti-holomorphic derivatives of the holomorphic \(F\) introduced in (5.9), and they are defined as

\[
F_{ijk} = \frac{\partial^3 F}{\partial a^i \partial a^j \partial a^k}, \quad \bar{F}_{ijk} = \frac{\partial^3 F}{\partial \bar{a}^i \partial \bar{a}^j \partial \bar{a}^k}.
\]

Because the first derivatives of \(g_{ij}\) are either fully holomorphic or anti-holomorphic, the mixed second derivative of the metric in the last line is zero.

Substituting these back into the Lagrangian (5.13), we will get

\[
\mathcal{L} = \text{Im}(\tau_{ij}) F^i F^{*j} + \frac{1}{4i} \bar{F}_{ijk} F^{i} \chi^j \bar{\chi}^k - \frac{1}{4i} F_{ijk} F^{*i} \bar{\chi}^j \chi^k
\]

\[ - g_{ij} \partial_\mu a^i \partial_\mu \bar{a}^j - ig_{ij} \bar{\chi}^j \sigma^\mu \partial_\mu \chi^i
\]

\[ + F^{i} \partial P + \bar{F}^{i} \frac{\partial \bar{P}}{\partial a^i} - \frac{1}{2} \partial^2 P \lambda^i \lambda^j - \frac{1}{2} \partial^2 \bar{P} \bar{\lambda}^i \bar{\lambda}^j.\]

(5.15)

In this section, we will examine the case when there is no superpotential, in other words \(P = 0\). We will bring back the superpotential in the next section. Here, with \(P = 0\), the last line in the above Lagrangian vanishes

\[
\mathcal{L} = \text{Im}(\tau_{ij}) F^i F^{*j} + \frac{1}{4i} \bar{F}_{ijk} F^{i} \chi^j \bar{\chi}^k - \frac{1}{4i} F_{ijk} F^{*i} \bar{\chi}^j \chi^k
\]

\[ - g_{ij} \partial_\mu a^i \partial_\mu \bar{a}^j - ig_{ij} \bar{\chi}^j \sigma^\mu \partial_\mu \chi^i.\]

(5.16)
We will now proceed to “integrate out” the heavy terms associated with the *extra* sector. We will do this by making these terms into a full square form. To do this we separate the first three terms of the Lagrangian \( \mathbb{V} \) where there is \( F^i \) dependance; Let us name them \( \mathbb{V} \),

\[
\mathbb{V} \equiv \text{Im}(\tau_{ij}) F^i F^{*j} + \frac{1}{4i} \mathcal{F}_{ijk} F^i \bar{\chi}^j \bar{\chi}^k - \frac{1}{4i} \mathcal{F}_{ijk} F^{*i} \chi^j \chi^k.
\]

We will then expand \( \mathbb{V} \) such that we separate the visible and extra terms. As we have already discussed, the index \( i = 1 \) associates with the visible sector and \( i = 2 \) with the extra sector. We will have

\[
\mathbb{V} = \text{Im}(\tau_{11}) F^1 F^{*1} + \text{Im}(\tau_{12}) [F^1 F^{*2} + F^2 F^{*1}] + \text{Im}(\tau_{22}) F^2 F^{*2} + \frac{1}{4i} F^i F^j \bar{\chi}^j \bar{\chi}^k + \frac{1}{4i} F^j F^k \bar{\chi}^i \bar{\chi}^k - \frac{1}{4i} F^{*i} F_{ijk} \chi^j \chi^k - \frac{1}{4i} F^{*j} F_{ijk} \chi^i \chi^k + + F^2 F^{*1} \text{Im}(\tau_{12}) + \frac{1}{4i} \mathcal{F}_{2jk} \bar{\chi}^j \bar{\chi}^k + F^2 \{ \text{Im}(\tau_{12}) - \frac{1}{4i} \mathcal{F}_{2jk} \chi^j \chi^k \} \]

To avoid repetition, we will denote the first line by \( \mathbb{V} = \text{Im}(\tau_{11}) F^1 F^{*1} + \frac{1}{4i} F^i \mathcal{F}_{1jk} \bar{\chi}^j \bar{\chi}^k - \frac{1}{4i} F^{*i} \mathcal{F}_{1jk} \chi^j \chi^k \); this represents the terms solely associated with the visible sector. Hence,

\[
\mathbb{V} = \mathbb{V} + \text{Im}(\tau_{22}) \left[ F^2 F^{*2} + F^2 (\text{Im}(\tau_{12}) \text{Im}(\tau_{22}) + \frac{1}{4i} \mathcal{F}_{2jk} \bar{\chi}^j \bar{\chi}^k) + F^{*2} \left( \text{Im}(\tau_{12}) - \frac{1}{4i} \mathcal{F}_{2jk} \chi^j \chi^k \right) \right].
\]

Now we will proceed to make a full square from the remaining terms.

\[
\mathbb{V} = \mathbb{V} + \text{Im}(\tau_{22}) \left\{ \left[ F^2 + (F^1 \text{Im}(\tau_{12}) \text{Im}(\tau_{22}) - \frac{1}{4i} \mathcal{F}_{2jk} \bar{\chi}^j \bar{\chi}^k) \right] \left[ F^{*2} + (F^{*1} \text{Im}(\tau_{12}) \text{Im}(\tau_{22}) + \frac{1}{4i} \mathcal{F}_{2jk} \bar{\chi}^j \bar{\chi}^k) \right] \right\}
\]

\[
- \left[ F^1 F^{*1} \left( \frac{\text{Im}(\tau_{12})}{\text{Im}(\tau_{22})} \right)^2 + \frac{1}{16} \mathcal{F}_{2jk} \mathcal{F}_{2mn} \bar{\chi}^j \bar{\chi}^k \bar{\chi}^m \bar{\chi}^n + \frac{1}{16} \text{Im}(\tau_{12}) \text{Im}(\tau_{22})^2 (F^1 \mathcal{F}_{2mn} \bar{\chi}^m \bar{\chi}^n - F^{*1} \mathcal{F}_{2jk} \chi^j \chi^k) \right] \}
\]

The expressions in the first two square brackets are each other’s conjugate, thus

\[
\mathbb{V} = \mathbb{V} + \text{Im}(\tau_{22}) \left( F^2 + (F^1 \text{Im}(\tau_{12}) \text{Im}(\tau_{22}) - \frac{1}{4i} \mathcal{F}_{2jk} \bar{\chi}^j \bar{\chi}^k) \right)^2
\]

\[
- \frac{1}{\text{Im}(\tau_{22})} \left\{ |F|^2 \text{Im}(\tau_{12})^2 + \frac{1}{16} \mathcal{F}_{2jk} \mathcal{F}_{2mn} \bar{\chi}^j \bar{\chi}^k \bar{\chi}^m \bar{\chi}^n + \frac{\text{Im}(\tau_{12})}{4i} (F^1 \mathcal{F}_{2mn} \bar{\chi}^m \bar{\chi}^n - F^{*1} \mathcal{F}_{2jk} \chi^j \chi^k) \right\}
\]

(5.18)

We will also set

\[
\mathcal{X} \equiv \text{Im}(\tau_{22}) \left( F^2 + F^1 \text{Im}(\tau_{12}) \text{Im}(\tau_{22}) - \frac{1}{4i} \mathcal{F}_{2jk} \bar{\chi}^j \bar{\chi}^k \right)^2
\]
which will be a full square term encompassing all the terms associated with the extra sector. Thus, (5.18) could be written as

\[
\mathcal{V} = \mathcal{V} + \mathcal{X} - \frac{1}{\text{Im}(\tau_{22})} \left\{ |F^1|^2 \text{Im}(\tau_{12})^2 + \frac{1}{16} \mathcal{F}_{2jk} \mathcal{F}_{2nm} \chi^j \chi^k \bar{\chi}^m \bar{\chi}^n + \frac{\text{Im}(\tau_{12})}{2} \text{Im}[F^1 \mathcal{F}_{2mn} \bar{\chi}^m \chi^n] \right\}.
\]

The full Lagrangian (5.16) can now be written as

\[
\mathcal{L} = \mathcal{V} - \frac{1}{\text{Im}(\tau_{22})} \left\{ |F^1|^2 \text{Im}(\tau_{12})^2 + \frac{1}{16} \mathcal{F}_{2jk} \mathcal{F}_{2nm} \chi^j \chi^k \bar{\chi}^m \bar{\chi}^n + \frac{\text{Im}(\tau_{12})}{2} \text{Im}[F^1 \mathcal{F}_{2mn} \bar{\chi}^m \chi^n] \right\}
- g_{ij} \partial_\mu a^i \partial_\mu \bar{a}^j - ig_{ij} \bar{\chi}^j \bar{\sigma}^\mu \partial_\mu \chi^i + \mathcal{X}.
\]

We would now like to shift our focus to the bosonic fields. To write a bosonic effective potential we would pick all the potential terms that do not have a fermion dependence. This would be equivalent to terms that are proportional to $|F^1|^2$. Thus, the bosonic effective potential would be

\[
V_{\text{boson}} = -\left[ \text{Im}(\tau_{11}) - \frac{\text{Im}(\tau_{12})^2}{\text{Im}(\tau_{22})} \right] |F^1|^2.
\]

Or, using the $C_{ij}$ matrix notation,

\[
V_{\text{boson}} = -\frac{\det C}{C_{22}} |F^1|^2. \quad (5.19)
\]

The negative sign in front comes from the fact that a general Lagrangian has the form of $\mathcal{L} = K - U$, with $K$ and $U$ being the kinetic and potential terms respectively.

Naturally, it is the next logical step to try to find a minimum point for the effective potential $V_{\text{boson}} (= V$, to avoid writing the subscript for the rest of the chapter). Before moving forward, let us write down expressions for the derivatives of $C_{ij}$. Reverting to its definition, we can rewrite $C_{ij}$ as,

\[
C_{ij} = \text{Im}(\tau_{ij}) = \frac{\tau_{ij} - \bar{\tau}_{ij}}{2i}.
\]

And therefore its derivatives will be given by,

\[
\frac{\partial C_{ij}}{\partial a} = \frac{1}{2i} \frac{\partial \tau_{ij}}{\partial a} = \frac{\mathcal{F}_{ij2}}{2i}, \quad \frac{\partial C_{ij}}{\partial \bar{a}} = -\frac{1}{2i} \frac{\partial \bar{\tau}_{ij}}{\partial \bar{a}} = -\frac{\mathcal{F}_{ij2}}{2i}. \quad (5.21)
\]

To find a minimum point for the effective potential we would set its derivative equal to zero. Using the
above expressions we have,

\[
\frac{\partial V}{\partial a} = -|F^1|^2 \left\{ -\det C \frac{\partial C_{22}}{\partial a} + \frac{1}{C_{22}} \left( \frac{\partial C_{11}}{\partial a} C_{22} + C_{11} \frac{\partial C_{22}}{\partial a} - 2C_{12} \frac{\partial C_{12}}{\partial a} \right) \right\}
\]

\[
= - \frac{1}{2i} \frac{|F^1|^2}{C_{22}^2} \left\{ -\det C \cdot F_{222} + C_{22} \left( F_{112} C_{22} + C_{11} F_{222} - 2C_{12} F_{122} \right) \right\}
\]

\[
= - \frac{1}{2i} \frac{|F^1|^2}{C_{22}^2} \left\{ -(C_{11} C_{22} - C_{12}^2) F_{222} + C_{22}^2 F_{112} + C_{11} C_{22} F_{222} - 2C_{12} C_{22} F_{122} \right\}
\]

Rearranging this we would eventually have

\[
\frac{\partial V}{\partial a} = - \frac{1}{2i} \frac{|F^1|^2}{C_{22}^2} \left\{ C_{12}^2 F_{222} + C_{22}^2 F_{112} - 2C_{12} C_{22} F_{122} \right\} = 0.
\]

It is time to use Mathematica again to graph this expression. Figure 5.2 depicts a graph of magnitude of the potential derivative with respect to magnitude of the Coulomb branch parameter \(u\), with \(|u|\) ranging from 0 to 10. It appears that \(|\partial V/\partial a| = 0\) at origin. To check whether or not \(u^* = 0\) is truly a minimum point we would need to write out the second-order derivative Hessian matrix for the potential \(V\) and evaluate it at \(u^*\).

Figure 5.2: Graph of \(|\partial V/\partial a|\) with respect to \(|u|\) for \(|u| = [0, 10]\) for the case when there is no superpotential. For flavor symmetry group \(A_1\)

We have to be delicate however, because most of the functions needed for this calculation are indeterminate at the limit of \(u^* = 0\). To overcome this issue, we find the data point that produced the smallest \(|\partial V/\partial a|\) value, and find the Coulomb branch parameter associated with it. In the case of data presented here, this point was \(u^* = 0.001609585188387274 + 0.00216759594830534i\). The Mathematica code for this graph and the calculation that follows is included in Appendix B.2.
To get the Hessian matrix, we have to calculate the second-order derivatives of potential $V$. One has to notice, however, that although $a$ is itself eventually a function of $m$ and $u$, $V$ is a function of real and imaginary parts of $a$, rather than $a$ and $\bar{a}$. Therefore, if we write $a = x + iy$ and $\bar{a} = x - iy$, the Hessian matrix would rather be,

$$H_{ij}(V) = \begin{pmatrix}
\frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\
\frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial y^2}
\end{pmatrix}.$$  

The sign of eigenvalues of this matrix would let us know what type of critical point is $u^*$; if both of these eigenvalues are positive that would refer to a minimum, both negative to a maximum, and a negative and a positive to a saddle point. To find this, let us rewrite these derivatives in terms of derivatives with respect to $a$ and $\bar{a}$.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial}{\partial \bar{a}} \frac{\partial \bar{a}}{\partial x} = \frac{\partial}{\partial a} + \frac{\partial}{\partial \bar{a}},$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial}{\partial \bar{a}} \frac{\partial \bar{a}}{\partial y} = i \left( \frac{\partial}{\partial a} - \frac{\partial}{\partial \bar{a}} \right).$$

Thus,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial a} + \frac{\partial V}{\partial \bar{a}}, \quad \frac{\partial V}{\partial y} = i \left( \frac{\partial V}{\partial a} - \frac{\partial V}{\partial \bar{a}} \right).$$

The second-order derivatives would be,

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial V}{\partial a} + \frac{\partial V}{\partial \bar{a}} \right) = \frac{\partial^2 V}{\partial a^2} + \frac{\partial^2 V}{\partial \bar{a}^2} + 2 \frac{\partial^2 V}{\partial a \partial \bar{a}} = 2 \text{Re} \left[ \frac{\partial^2 V}{\partial a^2} \right] + 2 \frac{\partial^2 V}{\partial a \partial \bar{a}},$$

$$\frac{\partial^2 V}{\partial x \partial y} = i \left( \frac{\partial V}{\partial a} - \frac{\partial V}{\partial \bar{a}} \right) = i \left[ \frac{\partial^2 V}{\partial a^2} - \frac{\partial^2 V}{\partial \bar{a}^2} \right] = -2 \text{Im} \left[ \frac{\partial^2 V}{\partial a^2} \right],$$

$$\frac{\partial^2 V}{\partial y^2} = i \left( \frac{\partial V}{\partial a} - \frac{\partial V}{\partial \bar{a}} \right) = - \left( \frac{\partial^2 V}{\partial a^2} + \frac{\partial^2 V}{\partial \bar{a}^2} - 2 \frac{\partial^2 V}{\partial a \partial \bar{a}} \right) = -2 \text{Re} \left[ \frac{\partial^2 V}{\partial a^2} \right] + 2 \frac{\partial^2 V}{\partial a \partial \bar{a}}.$$ 

Hence, the Hessian matrix can be rewritten in terms of these expressions as,

$$H_{ij}(V) = 2 \begin{pmatrix}
\text{Re} \left[ \frac{\partial^2 V}{\partial a^2} \right] + \frac{\partial^2 V}{\partial a \partial \bar{a}} & -\text{Im} \left[ \frac{\partial^2 V}{\partial a^2} \right] \\
-\text{Im} \left[ \frac{\partial^2 V}{\partial a^2} \right] & -\text{Re} \left[ \frac{\partial^2 V}{\partial a^2} \right] + \frac{\partial^2 V}{\partial a \partial \bar{a}}
\end{pmatrix} \quad (5.22)$$

So now, let us calculate the terms that show up in the Hessian matrix. These second-order derivatives
can be found from the first-order terms we calculated before. We have,

$$\frac{\partial^2 V}{\partial a^2} = \frac{\partial}{\partial a} \frac{\partial V}{\partial a} = -\frac{|F|^2}{2i} \frac{\partial}{\partial a} \left( \frac{C_{12}^2 F_{222} + C_{22}^2 F_{112} - 2C_{12}C_{22} F_{122}}{C_{22}^2} \right)$$

This results in

$$\frac{\partial^2 V}{\partial a^2} = -\frac{|F|^2}{2i} \left\{ -\frac{2}{C_{22}^3} (C_{12}^2 F_{222} + C_{22}^2 F_{112} - 2C_{12}C_{22} F_{122}) \frac{\partial C_{22}}{\partial a} + \right.$$

$$\left. + \frac{1}{C_{22}^2} \left[ 2C_{12} \frac{\partial C_{12}}{\partial a} F_{222} + C_{12}^2 \frac{\partial F_{222}}{\partial a} + 2C_{22} \frac{\partial C_{22}}{\partial a} F_{112} + C_{22} \frac{\partial F_{112}}{\partial a} - 2 \frac{\partial C_{12}}{\partial a} C_{22} F_{122} - 2C_{12} \frac{\partial C_{22}}{\partial a} F_{122} - 2C_{12} C_{22} \frac{\partial F_{122}}{\partial a} \right] \right\}$$

Now we can substitute the derivatives of $C_{ij}$ using (5.21). We will get

$$\frac{\partial^2 V}{\partial a^2} = \frac{|F|^2}{4C_{22}^3} \left\{ -2(C_{12}^2 F_{222} + C_{22}^2 F_{112} - 2C_{12}C_{22} F_{122}) F_{222} + 

+ C_{22} \left[ 2C_{12} C_{22} F_{222} + C_{12}^2 F_{2222} + 2i \left( 2C_{22} F_{222} F_{112} + C_{22}^2 F_{112} \right) \right]

- 2C_{12} F_{222} F_{122} - 2C_{12} C_{22} F_{122} - 2C_{12} C_{22} F_{122} + 2i \right\}$$

Repackaging these terms in a more convenient way we will have

$$\frac{\partial^2 V}{\partial a^2} = \frac{|F|^2}{4C_{22}^3} \left\{ -2C_{12}^2 F_{222}^2 - 2C_{22}^2 F_{112} F_{222} + 4C_{12} C_{22} F_{122} F_{222} + 

+ 2C_{12} C_{22} F_{122} F_{222} + 2C_{22}^2 F_{222} F_{112} + 2C_{12}^2 F_{222}^2 - 2C_{12} C_{22} F_{222} F_{122} + 

+ 2i \left( C_{12}^2 C_{22} F_{2222} + C_{22}^3 F_{112} - 2C_{12} C_{22}^2 F_{122} \right) \right\}$$

$$= \frac{|F|^2}{2C_{22}^3} \left\{ -C_{12}^2 F_{222}^2 + 2C_{12} C_{22} F_{122} F_{222} - C_{22}^2 F_{112}^2 + i \left( C_{12}^2 C_{22} F_{2222} + C_{22}^3 F_{112} - 2C_{12} C_{22}^2 F_{122} \right) \right\}$$

which can eventually be written as

$$\frac{\partial^2 V}{\partial a^2} = \frac{|F|^2}{2C_{22}^3} \left\{ - (C_{12} F_{222} - C_{22} F_{122})^2 + i \left( C_{12}^2 C_{22} F_{2222} + C_{22}^3 F_{112} - 2C_{12} C_{22}^2 F_{122} \right) \right\} \quad (5.23)$$

One should definitely remember that $(F_{ijk})^2 \neq |F_{ijk}|^2$.

Unlike the above expression, the mixed holomorphic second-order derivative would be a real number.
Indeed, we have
\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = \frac{\partial}{\partial a} \frac{\partial V}{\partial \bar{a}} = -\frac{|F_1|^2}{2i} \frac{\partial}{\partial \bar{a}} \left( \frac{C_{12}^2 F_{222} + C_{22}^2 F_{112} - 2C_{12} C_{22} F_{122}}{C_{22}^2} \right)
\]

Carrying out the derivative we will get
\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = -\frac{|F_1|^2}{2i} \left\{ -2 C_{12} \frac{\partial C_{12}}{\partial \bar{a}} F_{222} + C_{22} \frac{\partial F_{222}}{\partial \bar{a}} + 2C_{22} \frac{\partial C_{22}}{\partial \bar{a}} F_{112} + C_{22}^2 \frac{\partial F_{112}}{\partial \bar{a}} \\
-2 \frac{\partial C_{12}}{\partial \bar{a}} C_{22} F_{122} - 2C_{12} \frac{\partial C_{22}}{\partial \bar{a}} F_{122} - 2C_{12} C_{22} \frac{\partial F_{122}}{\partial \bar{a}} \right\}
\]

But a holomorphic function does not depend on anti-holomorphic elements, thus its derivative with respect to anti-holomorphic variables is zero, i.e.
\[
\frac{\partial F_{ij}}{\partial \bar{a}} = 0, \quad \text{for all } i, j = 1, 2.
\]

Hence,
\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = -\frac{|F_1|^2}{4 C_{22}^3} \left\{ -2 C_{12}^2 F_{222} + C_{22}^2 F_{112} - 2C_{12} C_{22} F_{122} \right\} + C_{22} \left( 2C_{12}^2 F_{222} + 2C_{22}^2 F_{112} - 2F_{112} C_{22} F_{122} - 2C_{12} C_{22} F_{122} \right) \}
\]

Basic expanding and rearranging will get us
\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = -\frac{|F_1|^2}{4 C_{22}^3} \left\{ -2 C_{12}^2 F_{222} + 2C_{12} C_{22} F_{112} - 2C_{12} C_{22} F_{122} - 2C_{22}^2 F_{122} \right\}
\]

which can be factorized nicely as
\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = -\frac{|F_1|^2}{2 C_{22}^3} \left\{ C_{12} F_{222} (-C_{12} F_{222} + C_{22} F_{112}) + C_{22} F_{112} (C_{12} F_{222} - C_{22} F_{122}) \right\}
\]

\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = \frac{|F_1|^2}{2 C_{22}^3} \left\{ (C_{12} F_{222} - C_{22} F_{112}) (C_{12} F_{222} - C_{22} F_{122}) \right\}
\]

Remembering that $C_{ij}$’s are real numbers, one would notice that the two expressions in parenthesis are complex conjugates of each other. Thus we can eventually write
\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = \frac{|F_1|^2}{2 C_{22}^3} |C_{12} F_{222} - C_{22} F_{122}|^2,
\]

(5.24)
Now, using (5.23) and (5.24) to compute the terms in the Hessian matrix (5.22) and evaluating each element at \( u = u^* \simeq 0.00161 + 0.00217i \) we will have

\[
H_{ij}(V)|_{u=u^*} = \begin{pmatrix}
-0.210312 & -0.111189 \\
-0.111189 & 0.460089
\end{pmatrix}
\]

The eigenvalues for this matrix are \( \lambda^H_1 = 0.478049 \) and \( \lambda^H_2 = -0.228272 \), which means that \( u^* \) is an extremum saddle point and not a minimum.

Another graphical evidence for this observation is presented in Figure 5.3. Here, we generated a contour plot of the magnitude of potential \( |V| \) evaluated for 15000 points in the range of \( |u| = [0, 1] \). As it is evident from the graph, the magnitude of the potential does not have a minimum at origin, or anywhere else in this region. The Mathematica code for these graphs is provided in Appendix B.3.

![Figure 5.3: Contour plot of magnitude of potential V evaluated for 15000 values of u between |u| = [0, 1]. This graph depicts the plot of values of |V| with respect to real and imaginary parts of u. As it is evident, the magnitude of |V| barely changes, and it is definitely not at minimum at origin u = 0.](image)

**Section 5.4: Superpotential Deformation**

Since we were not able to achieve a minimum point for potential while the superpotential \( P \) was set equal to zero, we will now try the same calculation for a non-zero superpotential. Let us refer back to our general
Lagrangian \((5.15)\),

\[
\mathcal{L}_{\text{Coulomb}} = \text{Im}(\tau_{ij}) F^i F^{*j} - F^i \left( \frac{-1}{4i} \frac{\partial \chi^j}{\partial \bar{a}^i} \chi^k - \frac{\partial P}{\partial a^i} \right) - F^{*i} \left( \frac{1}{4i} \frac{\partial \chi^j}{\partial \bar{a}^i} \chi^k - \frac{\partial \bar{P}}{\partial \bar{a}^i} \right) - g_{ij} \partial_{\mu} a^i \partial^\mu \bar{a}^j - ig_{ij} \chi^j \bar{\sigma}^a \partial_{\mu} \chi^i - \frac{1}{2} \frac{\partial^2 P}{\partial a^i \partial \bar{a}^j} \chi^i \chi^j. 
\]

Similar to before, we can proceed to make a full square from the terms associated with the extra sector in the first line so we can integrate them out. So, following the same steps as before,

\[
\mathcal{V} = \text{Im}(\tau_{ij}) F^i F^{*j} - F^i \left( \frac{-1}{4i} \frac{\partial \chi^j}{\partial \bar{a}^i} \chi^k - \frac{\partial P}{\partial a^i} \right) - F^{*i} \left( \frac{1}{4i} \frac{\partial \chi^j}{\partial \bar{a}^i} \chi^k - \frac{\partial \bar{P}}{\partial \bar{a}^i} \right) = \\
= \text{Im}(\tau_{11}) F^1 F^{*1} + \text{Im}(\tau_{12}) \left[ F^1 F^{*2} + F^2 F^{*1} \right] + \text{Im}(\tau_{22}) F^2 F^{*2} + \\
+ \frac{1}{4i} F^1 \overline{\tau}_{1jk} \chi^j \chi^k + \frac{1}{4i} F^2 \overline{\tau}_{2jk} \chi^j \chi^k + F^1 \frac{\partial P}{\partial m} + F^2 \frac{\partial \bar{P}}{\partial \bar{a}} + \\
- \frac{1}{4i} F^{*1} \overline{\tau}_{1jk} \chi^j \chi^k - \frac{1}{4i} F^{*2} \overline{\tau}_{2jk} \chi^j \chi^k + F^{*1} \frac{\partial \bar{P}}{\partial \bar{m}} + F^{*2} \frac{\partial P}{\partial m} 
\]

Packaging the terms proportional to \(F^2\) and \(F^{*2}\) separately, we will have

\[
\mathcal{V} = \text{Im}(\tau_{11}) F^1 F^{*1} + \frac{1}{4i} F^1 \overline{\tau}_{1jk} \chi^j \chi^k - \frac{1}{4i} F^{*1} \overline{\tau}_{1jk} \chi^j \chi^k + F^1 \frac{\partial P}{\partial m} + F^{*1} \frac{\partial \bar{P}}{\partial \bar{m}} + \\
+ F^2 \left\{ \text{Im}(\tau_{12}) + \frac{1}{4i} \overline{\tau}_{2jk} \chi^j \chi^k + \frac{\partial P}{\partial a} \right\} + \\
+ F^{*2} \left\{ \text{Im}(\tau_{12}) - \frac{1}{4i} \overline{\tau}_{2jk} \chi^j \chi^k + \frac{\partial \bar{P}}{\partial \bar{a}} \right\} + F^2 F^{*2} \text{Im}(\tau_{22}) 
\]

Now if we denote all the terms associated with the visible sector with \(\mathcal{V}\) again,

\[
\mathcal{V} \equiv \text{Im}(\tau_{11}) |F^1|^2 + \frac{1}{4i} F^1 \overline{\tau}_{1jk} \chi^j \chi^k - \frac{1}{4i} F^{*1} \overline{\tau}_{1jk} \chi^j \chi^k + F^1 \frac{\partial P}{\partial m} + F^{*1} \frac{\partial \bar{P}}{\partial \bar{m}},
\]

then we can write

\[
\mathcal{V} = \mathcal{V} + \text{Im}(\tau_{22}) \left[ F^2 F^{*2} + F^2 \left\{ \text{Im}(\tau_{12}) + \frac{1}{4i} \overline{\tau}_{2jk} \chi^j \chi^k + \frac{\partial P}{\partial a} \right\} + \\
+ F^{*2} \left\{ \text{Im}(\tau_{12}) - \frac{1}{4i} \overline{\tau}_{2jk} \chi^j \chi^k + \frac{\partial \bar{P}}{\partial \bar{a}} \right\} \right], 
\]

(5.25)

Here we assumed that \(F^1\) is real. We will carry out this assumption for the rest of the chapter. In this case, then \(\mathcal{V}\) becomes \(\mathcal{V} = \text{Im}(\tau_{11}) |(F^1)|^2 + \frac{1}{4i} \overline{\tau}_{1jk} \chi^j \chi^k - \overline{\tau}_{1jk} \chi^j \chi^k + 2F^1 \text{Re} \left( \frac{\partial P}{\partial m} \right)\).
Now if we set \( \mathcal{B} \equiv \frac{1}{\Im(\tau_{22})} \left[ \Im(\tau_{12})F^1 - \frac{1}{4i}\mathcal{F}_{2jk}\chi^j\chi^k + \frac{\partial P}{\partial a} \right] \), then (5.25) can be written as

\[
\mathcal{V} = \mathcal{V} + \Im(\tau_{22}) \left[ |F^2|^2 + F^2\mathcal{B} + F^*\mathcal{B} \right]
\]

\[
= \mathcal{V} + \Im(\tau_{22}) |F^2 + \mathcal{B}|^2 - \Im(\tau_{22}) |\mathcal{B}|^2.
\]

Hence the Lagrangian can be written as

\[
\mathcal{L}_{\text{Coulomb}} = \mathcal{V} + \Im(\tau_{22}) |F^2 + \mathcal{B}|^2 - \Im(\tau_{22}) |\mathcal{B}|^2
\]

\[
- g_{ij}\partial_\mu a^i \partial^\mu a^j - ig_{ij}\chi^j \partial_\mu a^i \chi^i - \frac{1}{\Im(\tau_{22})} \frac{\partial^2 P}{\partial a^i} \chi^i \chi^j
\]

\[
- \frac{1}{\Im(\tau_{22})} \left[ \Im(\tau_{12})F^1 - \frac{1}{4i}\mathcal{F}_{2jk}\chi^j\chi^k + \frac{\partial P}{\partial a} \right]^2
\]

\[
- g_{11}\partial_\mu a^1 \partial^\mu a^1 - g_{12}\partial_\mu a^1 \partial^\mu a^2 - g_{21}\partial_\mu a^2 \partial^\mu a^1 - g_{22}\partial_\mu a^2 \partial^\mu a^2
\]

\[
- ig_{11}\chi^1 \partial_\mu a^1 \chi^1 - ig_{12}\chi^2 \partial_\mu a^1 \chi^1 - ig_{21}\chi^1 \partial_\mu a^2 \chi^2 - ig_{22}\chi^2 \partial_\mu a^2 \chi^2
\]

\[
- \frac{1}{2} \frac{\partial^2 P}{\partial a^i \partial a^j} \chi^i \chi^j - \frac{1}{2} \frac{\partial^2 P}{\partial a^i \partial a^j} \chi^i \chi^j - \frac{1}{2} \frac{\partial^2 P}{\partial a^i \partial a^j} \chi^i \chi^j
\]

\[
V_{\text{boson}} = - \Im(\tau_{11})(F^1)^2 - 2F^1 \text{Re} \left[ \frac{\partial P}{\partial m} \right] - \Im(\tau_{12})^2 \left( F^1 \right)^2 + 2F^1 \text{Re} \left[ \frac{\partial P}{\partial a} \right] + \frac{1}{\Im(\tau_{22})} \left| \frac{\partial P}{\partial a} \right|^2
\]

69
This can be repackaged and rewritten into

\[ V_{\text{boson}} = \left( \frac{\text{Im}(\tau_{12})^2}{\text{Im}(\tau_{22})} - \frac{\text{Im}(\tau_{11})}{\text{Im}(\tau_{22})} \right) (F_1)^2 - 2F_1 \text{Re} \left[ \frac{\partial P}{\partial m} \right] + 2F_1 \frac{\text{Im}(\tau_{12})}{\text{Im}(\tau_{22})} \text{Re} \left[ \frac{\partial P}{\partial a} \right] + \frac{1}{\text{Im}(\tau_{22})} \left| \frac{\partial P}{\partial a} \right|^2, \]

or equivalently,

\[ V_{\text{boson}} = \frac{1}{C_{22}} \left( \frac{\partial P}{\partial a} \right)^2 - \text{det} \frac{C}{C_{22}} (F_1)^2 - 2F_1 \text{Re} \left[ \frac{\partial P}{\partial m} \right] + 2F_1 \frac{C_{12}}{C_{22}} \text{Re} \left[ \frac{\partial P}{\partial a} \right]. \]

We would need to take an example function for the superpotential \( P \) at this point. We will demand \( \frac{\partial P}{\partial u} \) to have a critical point at \( u = m^{\frac{4}{3}} \); this corresponds to the situation where the D3-brane is attracted to the flavor brane. The powers are based on dimensional analysis for the \( A_1 \) case. A simple example of a function satisfying this condition would be the quadratic potential, namely

\[ P = \kappa \cdot (u - m^{\frac{4}{3}})^2. \]

The choice of quadratic is to avoid trivial derivatives. The proportionality constant \( \kappa \) here has dimensions of \( \text{mass}^{1/3} \), since the superpotential itself is of the dimension \( \text{mass}^3 \).

We will start by investigating the limit where there is no supersymmetry breaking, i.e. when \( F_1 = 0 \). This will give us the supersymmetric basis we need for the next subsections. We will explore the bosonic and fermionic effective potentials individually.

5.4.1: The Bosonic Effective Potential

At the supersymmetric limit, i.e. when \( F_1 = 0 \), all but the first term in the effective potential expression (5.28) vanish. We end up with

\[ V_{\text{boson}}^* = \frac{1}{\text{Im}(\tau_{22})} \left| \frac{\partial P}{\partial a} \right|^2. \]

At this limit, we incorporate a calculation technique called the background field method. In this method the quantum fields under investigation are expanded around a classical background value and the small fluctuations around this value are the dynamical fields we will focus on. This procedure is thus useful in calculating the effective action caused by these dynamical fields. In our case, we use the background field method slightly differently for each of our fields. For the fields with the indices \( i = 1 \), i.e. those associated with the visible sector, we consider the whole field being frozen at its classical background value \( \propto m \). For the fields associated with the extra sector, i.e. those with \( i = 2 \) indices, we impose the general background field method.
Since \( m(\times a^1) \) is considered to be a fixed value, then its differentials will indeed vanish. Thus, the kinetic term of the bosonic fields in (5.26)
\[-g_{ij} \partial_\mu a^i \partial^\mu \bar{a}^j,\]
or the fourth line of (5.27), will simply become
\[-g_{22} \partial a \partial \bar{a}.\]
Additionally, using the background field method, we can write \( a \) as expanded over a background value \( \langle a \rangle \) with small dynamic fluctuations \( \delta a \). In other words,
\[a \rightarrow \langle a \rangle + \delta a.\]
The kinetic term now becomes
\[-g_{22} \partial (\delta a) \partial (\delta \bar{a}).\]
Now, in a similar manner to the photini case, we replace the dynamical field with rescaled normalized field \( \delta \hat{a} \), which will incorporate the metric;
\[\delta \hat{a} = \sqrt{g_{22}} \delta a.\] (5.31)
Therefore our kinetic term can finally be written as
\[-\partial (\delta \hat{a}) \partial (\delta \hat{a}).\]
Thus the fourth line in (5.27), associated with the term \(-g_{ij} \partial_\mu a^i \partial^\mu \bar{a}^j\) from (5.26), is simplified to the only one term above.

Now, let us go back to the effective potential \( V(a, \bar{a}) \) and calculate its variation around a critical point \( a = a^* \). Namely,
\[\delta V|_{a^*} = V(a^* + \delta a) - V(a^*) = \frac{\partial^2 V}{\partial a \partial \bar{a}}|_{a^*} \delta a \delta \bar{a}\]
Changing to the new normalized fields we will have
\[\delta V = \frac{1}{g_{22}|_{a^*}} \partial^2 V|_{a^*} \delta \hat{a} \delta \hat{a} \]
We will name the coefficient \( \frac{1}{3} \frac{\partial^2 V}{\partial a \partial \bar{a}} \bigg|_{a^*} \) as the mass-squared value of the normalized fields \( \delta \hat{a} \), so that

\[
\delta V_{\text{eff}} = m_a^2 \delta \hat{a} \delta \bar{a}.
\]

To calculate the above mass-squared value we will need to evaluate the mentioned coefficient at the critical point \( a^* \). This is the point where the derivative of \( V_{\text{eff}}(a) \) vanishes. From the expressions for the potential at (5.30), and superpotential \( P \) at (5.29), it becomes clear that this critical point happens when \( u = \frac{m^4}{3} \). Indeed,

\[
\frac{\partial P}{\partial a} \propto \frac{\partial P}{\partial u} = 2\kappa \cdot (u - \frac{m^4}{3})
\]

Thus, \( a^* \) is the value of the field \( a \) when \( u = \frac{m^4}{3} \). To find numerical values for these derivatives we will refer to Mathematica again. Since Mathematica has difficulty calculating derivatives of commands like \( \text{Im} \) and \( \text{Abs} \), we will generate a table of derivatives using gradual increments for components of \( u \). Also one needs to note, that \( u \) is our independent variable, so instead of directly calculating \( \frac{\partial^2 V}{\partial a \partial \bar{a}} \), we will break it into pieces using the chain rule

\[
\frac{\partial^2 V}{\partial a \partial \bar{a}} = \frac{\partial^2 V}{\partial u \partial \bar{u}} \frac{\partial u}{\partial a} \frac{\partial \bar{u}}{\partial \bar{a}}
\]

Where \( \frac{\partial u}{\partial a} \) and \( \frac{\partial \bar{u}}{\partial \bar{a}} \) could be interpreted as \( \frac{1}{\partial a / \partial u} \) and \( \frac{1}{\partial a / \partial \bar{u}} \), respectively. Additionally, the second-order derivative \( \frac{\partial^2 V}{\partial a \partial \bar{a}} \) can be written in terms of the real and imaginary components of \( u \) to make the computation process easier. Namely,

\[
\frac{\partial^2 V}{\partial u \partial \bar{u}} = \frac{1}{4} \left( \frac{\partial^2 V}{\partial (\text{Re}\{u\})^2} + \frac{\partial^2 V}{\partial (\text{Im}\{u\})^2} \right).
\]

The Mathematica code for evaluation of the mass-squared values is provided in Appendix C.1. We can collect the following information from the code,

1. \( \frac{\partial u}{\partial a} \bigg|_{u=1} = -4.99462 + 7.97591i \)
2. \( \frac{\partial \bar{u}}{\partial a} \bigg|_{u=1} = -4.99462 - 7.97591i \)
3. \( \frac{\partial^2 V}{\partial (\text{Re}\{u\})^2} \bigg|_{u=1} = 787.703\kappa^2 \)
4. \( \frac{\partial^2 V}{\partial (\text{Im}\{u\})^2} \bigg|_{u=1} = 787.527\kappa^2 \)
5. \( \frac{1}{\text{Im}\{\tau_{22}\}} \bigg|_{u=1} = 1.11156 \)

Putting these pieces all together we will find that the bosonic mass-squared value, at the supersymmetric
limit, would be

\[ m^2_a = \left[ \frac{1}{4} \left( \frac{\partial^2 V}{\partial (\text{Re}[u])^2} + \frac{\partial^2 V}{\partial (\text{Im}[u])^2} \right) \right]_{u=1} \bigg|_{u=1} = 38766.8 \kappa^2 \]  

(5.32)

We were also able to produce the contour plots 5.4 for the magnitude of the potential \( V_{\text{bosonic}} \) and its derivative \( \partial V_{\text{bosonic}} / \partial u \), and they both show a strong minimum at \( u = 1 \) (i.e. \( \text{Re}[u]=1, \text{Im}[u]=0 \)), as expected. See Figure 5.4 for the graphs, and Appendix B.4 for the Mathematica code.

5.4.2: The Fermionic Effective Potential

We could use a similar procedure to the bosonic case to find the mass-squared value for the effective fermionic potential. Since we are working at the supersymmetric limit \( F^1 = 0 \), the numerical value for these two masses should be equal to each other.

To find the fermionic mass-squared value, we will refer back to equation (5.26) or (5.27). Incorporating the same background field consideration, i.e. having fields with \( i = 1 \) indices frozen and those with \( i = 2 \)'s dynamic, the fermionic potential terms in (5.27) become,

\[ V_{\text{fermion}} = -\frac{1}{2} g^2 \partial^2 P \partial a^2 (\chi^2)^2 - \frac{1}{2} g^2 \tilde{P} \partial \tilde{a}^2 (\tilde{\chi}^2)^2. \]

Following the same steps as for the bosonic case, we will introduce the normalized fermionic fields \( \chi^2 \equiv \sqrt{g_{22}} \chi^2 \) which will transform the fermionic kinetic term from \( -ig_{22} \chi^2 \partial \chi^2 \) to \( -i \chi^2 \partial \chi^2 \). In turn, the potential terms become

\[ V_{\text{fermion}} = -\frac{1}{2} g^2 \partial^2 P \partial a^2 (\chi^2)^2 - \frac{1}{2} g^2 \tilde{P} \partial \tilde{a}^2 (\tilde{\chi}^2)^2. \]

The coefficients \( \frac{1}{g_{22}} \partial^2 P / \partial a^2 \) and its conjugate, are then defined as the mass values for these fermionic fields, i.e.

\[ m_\chi \equiv \left[ \frac{1}{4 \text{Im}[\tau_{22}]} \partial^2 P \right]_{u=1}. \]

Because there are two conjugate terms for fermions in the Lagrangian, the real, comparable to bosonic mass-squared value would actually be \( |m_\chi|^2 \). To find this value, we will need to evaluate the second derivative of the superpotential \( P \) at the point \( u = 1 \). Remembering its definition from (5.29), we can write for the derivatives,

\[ \frac{\partial P}{\partial a} = \frac{\partial P}{\partial u} \frac{\partial u}{\partial a} = 2\kappa(u - m^{4/3}) \frac{\partial u}{\partial a}. \]
Figure 5.4: Contour plots of potential $V_{\text{bosonic}}$ and its derivative $|\partial V_{\text{bosonic}}/\partial u|$ evaluated for 10000 points around $u = 1$. In both graphs, real part of $u$ lays on the x axis, and the imaginary part is on y axis. And in both graphs, we see a strong minimum at $u = 1$. 

74
and subsequently,
\[ \frac{\partial^2 P}{\partial a^2} = 2\kappa \left( \frac{\partial u}{\partial a} \right)^2 + 2\kappa (u - m^{4/3}) \frac{\partial^2 u}{\partial a^2} . \]

Where \( \frac{\partial u}{\partial a} \) and \( \frac{\partial^2 u}{\partial a^2} \) are meant to be interpreted as \( \frac{1}{\partial a/\partial u} \) and \( \frac{1}{\partial a/\partial u}^2 \), respectively. But at the critical \( u = m^{4/3} = 1 \) the second term vanishes, therefore

\[ \left. \frac{\partial^2 P}{\partial a^2} \right|_{u=1} = \frac{2\kappa}{\left( \partial a/\partial u \right)^2}_{u=1} \]

and so,

\[ m_\chi \equiv \left. \left[ \frac{1}{\text{Im} [\tau_{22}]} \frac{\partial^2 P}{\partial a^2} \right] \right|_{u=1} = 2\kappa \left[ \frac{1}{\text{Im} [\tau_{22}]} \frac{1}{\left( \partial a/\partial u \right)^2} \right]_{u=1} . \]

(5.33)

This can now easily be evaluated using the values we calculated with the help of Mathematica. We will get

\[ m_\chi = 2\kappa \left[ \frac{1}{\text{Im} [\tau_{22}]} \frac{1}{\left( \partial a/\partial u \right)^2} \right]_{u=1} = \kappa(-85.9654 - 177.123i) . \]

As mentioned earlier, in the supersymmetric limit, we should have \( m_\alpha^2 = |m_\chi|^2 \). And indeed, for the \( m_\chi \) value calculated above, we will get

\[ |m_\chi|^2 = 38762.5\kappa^2 \]

which is within .01% of what we got for \( m_\alpha^2 \) in (5.32) when \( F^1 = 0 \).

One can find the Mathematica code for both fermionic and bosonic mass-squared value calculation in Appendix C.1.

Section 5.5: Breaking Supersymmetry

Now that we have shown that the theory behaves as expected in the supersymmetric limit, let us explore what happens when the SUSY breaking is turned on, i.e. when \( F^1 \neq 0 \). Both mass values would now have corrections depending on \( F^1 \), and these corrections should be different from each other which will result in divergence of bosonic and fermionic mass-squared values and in turn, break the supersymmetry!

Going back to (5.28), the bosonic potential to the first order in \( F^1 \), is given by

\[ V_{boson}^F = \frac{1}{\text{Im} [\tau_{22}]} \left[ \frac{\partial P}{\partial a} \right]^2 + 2F^1 \frac{\text{Im} [\tau_{12}]}{\text{Im} [\tau_{22}]} \text{Re} \left[ \frac{\partial P}{\partial a} \right] . \]
However, the bosonic mass-squared value is related to the potential just as before,

\[ m_a^2 = \left[ \frac{1}{g_{22}} \frac{\partial^2 V}{\partial a \partial \bar{a}} \right]_{a = u^*} = \frac{1}{4} \left[ \frac{1}{\text{Im}[\tau_{22}]} \left( \frac{\partial^2 V}{\partial (\text{Re}[u])^2} + \frac{\partial^2 V}{\partial (\text{Im}[u])^2} \right) \right] \left| \frac{\partial u}{\partial a} \right|^2 \bigg|_{u = u^*} \]  

(5.34)

One should notice, however, that \( u^* \neq u_0 = 1 \) anymore, since the minimum point of the potential would also change when we add the superpotential \( P \). We can assume that the new minimum is at a point

\[ u^* = u_0 + \delta u; \]

or equivalently at \( x^* = x_0 + \delta x \) and \( y^* = y_0 + \delta y \), where \( x \) and \( y \) are the real and imaginary parts of \( u \) respectively. Evidently, \( x_0 = 1, y_0 = 0 \). Now, if we assume that the potential is eventually a function of real and imaginary parts of \( u \), we can write the perturbation theory for the new potential with respect to each component. This new potential can be written as

\[ V(u) = V^0(u) + \epsilon V^1(u), \]

where \( V^0(u) \) is the original potential at (5.30) when \( F^1 = 0 \), and \( V^1(u) \) is the first perturbative term of the potential, and \( \epsilon \equiv \frac{F^1}{m_0^2} \) is the dimensionless expansion parameter with \( m_0^2 \approx 38765 \kappa^2 \). To avoid repetition, we will proceed the calculation with only the real part of \( u \). The calculation for the imaginary part would be identical. Thus we have

\[ V(x) = V^0(x) + \epsilon V^1(x) \]

\[ x^* = x_0 + \delta x \]

We require the new potential to have a minimum at \( x^* \), therefore

\[ \frac{\partial V}{\partial x}(x^*) = \frac{\partial V^0}{\partial x}(x^*) + \epsilon \frac{\partial V^1}{\partial x}(x^*) = 0 \]

\[ = \frac{\partial V^0}{\partial x}(x_0) + \frac{\partial^2 V^0(x_0)}{\partial x^2} \delta x + \epsilon \frac{\partial V^1}{\partial x}(x_0) + O(\epsilon^3) = 0 \]

But the first term on the second line vanishes since it is the minimum condition for the unperturbed potential

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3 This is a very legitimate assumption. Since \( V \) is a function of \( a \) and \( \bar{a} \), which are holomorphic with respect to \( u \), therefore we can treat them separately and make this claim.
at the unperturbed critical point. Thus, we will have that
\[ \delta x = -\epsilon \frac{\partial V^1(x_0)}{\partial x^2}(x_0) \partial x^2. \]  
\[ (5.35) \]

Now, to find the second-order derivatives, as they appear in (5.34), we will continue with perturbation
method. We will have (to the second order in \( \epsilon \)),
\[ \frac{\partial^2 V}{\partial x^2}(x^*) = \frac{\partial^2 V}{\partial x^2}(x_0) + \frac{\partial^3 V(x_0)}{\partial x^3} \delta x \frac{\partial^4 V(x_0)(\delta x)^2}{2}. \]  
\[ (5.36) \]

Where each of the derivatives can in term be written as,
\[ \frac{\partial^2 V}{\partial x^2}(x_0) = \frac{\partial^2 V^0}{\partial x^2}(x_0) + \epsilon \frac{\partial^2 V^1}{\partial x^2}(x_0) \]
\[ \frac{\partial^3 V}{\partial x^3}(x_0) = \frac{\partial^3 V^0}{\partial x^3}(x_0) + \epsilon \frac{\partial^3 V^1}{\partial x^3}(x_0) \]
\[ \frac{\partial^4 V}{\partial x^4}(x_0) = \frac{\partial^4 V^0}{\partial x^4}(x_0) + \epsilon \frac{\partial^4 V^1}{\partial x^4}(x_0) \]

So, substituting these and (5.35) back in (5.36), we will eventually get the following expression for the
second-order derivative of potential, up to the second order in \( \epsilon \),
\[ \frac{\partial^2 V}{\partial x^2}(x^*) = \frac{\partial^2 V^0}{\partial x^2}(x_0) + \epsilon \frac{\partial^2 V^1}{\partial x^2}(x_0) + \delta x \frac{\partial^3 V^0}{\partial x^3}(x_0) \frac{\partial^4 V^0(x_0)(\delta x)^2}{2} + O(\epsilon^3). \]

We can now use Mathematica to evaluate each of these derivatives numerically. The code attached in
Appendix C.2 calculates these derivatives and we will get
\[ \frac{\partial^2 V}{\partial x^2}(x^*) = 787.703\kappa^2 - 1.13575\kappa\epsilon + 0.000262\epsilon^2. \]

Following similar steps with the imaginary part \( y \), we will end up with \( \frac{\partial^2 V}{\partial y^2}(y^*) = 787.527\kappa^2 + 1.47694\kappa\epsilon - 0.000804\epsilon^2. \)

We would also need Mathematica to evaluate the rest of the terms in (5.34) at the new critical point \( a^* \).
These expressions each come up to
\[ \left| \frac{\partial u}{\partial a} \right|^2 = 88.561 - 0.0050687\kappa \epsilon + O(\epsilon^3) \]
\[ \frac{1}{\text{Im}(\tau_{22})} = 1.11156 - 0.000700976\kappa \epsilon + O(\epsilon^3) \]
Putting it all together using (5.34), we will eventually have for the bosonic mass squared value,

\[ m_a^2 = 38766.8\kappa^2 + 3.733\kappa \epsilon - 0.01412\epsilon^2 + O(\epsilon^3), \]  

(5.37)

where, as a reminder, \( \epsilon = \frac{F_1}{m_0^2} \).

For the fermionic case, we will refer back to (5.27) again, and we will have for the fermionic potential to the first order in \( F_1 \)

\[ V_{\text{fermion}} = \frac{1}{\text{Im}(\tau_{22})} \left[ -\frac{1}{2} \frac{\partial^2 P}{\partial a^2} - \frac{1}{4\text{Im}(\tau_{22})} \frac{\partial P}{\partial a} F_{222} + \frac{F_1}{4i} \left( \text{Im}(\tau_{12}) F_{222} - \frac{\text{Im}(\tau_{22})}{\text{Im}(\tau_{22})} F_{222} \right) \right] (\tilde{\chi}^2)^2 + \frac{1}{\text{Im}(\tau_{22})} \left[ -\frac{1}{2} \frac{\partial^2 \bar{P}}{\partial \bar{a}^2} + \frac{1}{4\text{Im}(\tau_{22})} \frac{\partial \bar{P}}{\partial \bar{a}} F_{\bar{2}\bar{2}\bar{2}} + \frac{F_1}{4i} \left( \text{Im}(\tau_{12}) F_{\bar{2}\bar{2}\bar{2}} - \frac{\text{Im}(\tau_{22})}{\text{Im}(\tau_{22})} F_{\bar{2}\bar{2}\bar{2}} \right) \right] (\tilde{\chi}^2)^2. \]

Extracting the fermionic mass from the above expression we will have,

\[ m_\chi = \frac{1}{\text{Im}(\tau_{22})} \left[ 2\kappa \left( \frac{\partial u}{\partial a} \right)^2 + \frac{1}{2\text{Im}(\tau_{22})} \cdot 2\kappa(u - 1) \frac{\partial u}{\partial a} \cdot F_{222} + \frac{F_1}{2i} \left( \text{Im}(\tau_{12}) F_{222} - \frac{\text{Im}(\tau_{22})}{\text{Im}(\tau_{22})} F_{222} \right) \right] \bigg|_{u^* = u_0 + \delta u} \]

Again, referring to the Mathematica code in Appendix C.2 to evaluate these quantities, we will get for fermionic mass up to second order in \( \epsilon \),

\[ m_\chi = (-85.9654 - 177.123i)\kappa + [(0.0418761 + 0.0129025i) - (5908.66 - 1977.96i)\kappa^2] \epsilon + (2.33271 - 1.43946i)\kappa \epsilon^2 + O(\epsilon^3). \]

We need to multiply this mass value by its conjugate in order for it to be real and comparable to the bosonic mass-squared value. Doing that we will get

\[ |m_\chi|^2 = 38762.5\kappa^2 + (-11.7704\kappa + 315197\kappa^3) \epsilon + (0.00177555 - 334.965\kappa^2 + 3.88246 \times 10^7 \kappa^4) \epsilon^2 + O(\epsilon^3). \]  

(5.38)

As it is evident from (5.38), the fermionic mass squared value has a very unique and strong dependence on the parameter \( \kappa \). However, for small values of \( \kappa \), e.g. for \( \epsilon \ll \kappa \ll 1 \), the expression (5.38) turns into

\[ |m_\chi|^2 = 38762.5\kappa^2 - 11.7704\kappa \epsilon + 0.00177555 \epsilon^2 + O(\epsilon^3). \]  

(5.39)

Comparing (5.37) and (5.39), it can be clearly seen that these two mass-squared values diverge in different directions from the supersymmetric mass value \( m_0^2 \sim 38765\kappa^2 \) when SUSY is broken. And this difference would only amplify in the case of larger \( \kappa \) parameters.
In the next chapter, we take a rather different route from the rest of the thesis. We comment on the potential role of the extra sectors as a toy model scenario for Dark Matter and investigate scattering associated with these sectors.
CHAPTER 6: Dark Rutherford Scattering

In this chapter we investigate the model of Dark Rutherford scattering. We treat a magnetically charged extra sector as a heavy classical source, and we scatter charges off this source. The source being heavy assimilates Dark Matter (DM), and the model itself assimilates Dark Matter moving around the earth in galactic wind; this is why we investigate the case of a moving source in Section 7.3. The charges being scattered off the source - or in case of direct detection experiments, most probably protons - would represent visible matter, and act as the visible sector.

We will start this chapter by reviewing the original case of Rutherford scattering using quantum electrodynamics. For that we assume a classical, stationary, potential field, in this case the Coulomb potential. We will then generalize this assumption in following sections, by adding a degree of complexity in each step. In section 7.2, we will switch the nucleus source of Coulomb potential to a dyon source, which is a basis of our DM model. In 7.3, we will add movement of the source to the problem. In section 7.4 we will investigate some dyon properties and conclude the physical meaning of the results achieved in previous sections.

Section 6.1: Coulomb Scattering

The general QED Lagrangian can be written as

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}}$$

$$= \bar{\psi}(i\slashed{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \bar{\psi} A^{\text{vis}}_{\mu} \gamma^\mu \psi$$

Here $A^{\text{vis}}_{\mu}$ is the electromagnetic vector potential of the visible sector, $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, and $e = -|e|$ is the electron charge. In interacting field theories such as QED, many new interactions become possible when we add vector fields to the theory. The vector-spinor interaction of $e \bar{\psi} \gamma^\mu \psi A_{\mu}$ is one of the most recurring types in QED. The interaction Hamiltonian for such an interaction is simply

$$H_I = \int d^3 x e \bar{\psi} \gamma^\mu \psi A_{\mu},$$

(6.1)

where $\psi(x)$ is a general quantized Dirac field.

To start with the most classical case of Coulomb scattering, we will assume $A_{\mu}(x)$ is a given, localized,
classical potential and calculate the \( T\)-matrix element of an electron scattering off this potential.

As a reminder, the \( T\)-matrix is the part of \( S\)-matrix only associated with nontrivial interactions. The \( S\)-matrix is a sequence of unitary operators which relates the in and out states with the following structure

\[
\langle \mathbf{p}_1 \mathbf{p}_2 \ldots | k_A k_B \ldots \rangle_{\text{in}} \equiv \langle \mathbf{p}_1 \mathbf{p}_2 \ldots | S | k_A k_B \ldots \rangle.
\]

It can be deduced from here that if particles in the in and out states do not interact at all, \( S \) becomes the identity operator. The probability of particles missing each other even if the states do interact is also probable, so we define \( T \) as \( S = 1 + iT \) so that \( T \) includes all the information from \( S \) that is strictly due to real interactions.

It is also useful at this point to find an expression that will help us calculate \( T\)-matrix elements. According to the definition of the \( S\)-matrix

\[
\langle \mathbf{p}_1 \mathbf{p}_2 \ldots | S | k_A k_B \ldots \rangle \equiv \langle \mathbf{p}_1 \mathbf{p}_2 \ldots | e^{-i \int dt H_I(t)} | k_A k_B \ldots \rangle
\]

To the lowest order we can expand this equation to,

\[
\langle \mathbf{p}_1 \mathbf{p}_2 \ldots | S | k_A k_B \ldots \rangle = \langle \mathbf{p}_1 \mathbf{p}_2 \ldots | 1 - i \int dt H_I(t) | k_A k_B \ldots \rangle
\]

Thus we extract an expression for computing the \( T\)-matrix element, to the lowest order,

\[
\langle \mathbf{p}_1 \mathbf{p}_2 \ldots | i T | k_A k_B \ldots \rangle = \langle \mathbf{p}_1 \mathbf{p}_2 \ldots | -i \int dt H_I(t) | k_A k_B \ldots \rangle
\]

So now we are finally equipped to compute the \( T\)-matrix element for the interaction described in (6.1). From here on and for the most of this chapter (unless stated otherwise) we will use \( p \) and \( p' \) to denote the four-momenta associated with in and out states respectively.
The result of the integral in the last line is $\tilde{A}_\mu(p' - p)$, the four-dimensional Fourier transform of $A_\mu(x)$. Since $A_\mu(x)$ is time independent, we can rewrite the integral as,

$$
\tilde{A}_\mu(p' - p) = \int d^4xe^{i(p'\cdot x - p\cdot x)}A_\mu(x) = \int dx e^{i(p'\cdot x - p\cdot x)} \int d^3xe^{i(p' - p)\cdot \vec{x}}A_\mu(\vec{x}) = 2\pi\delta(E_f - E_i) \cdot \tilde{A}_\mu(p' - p),
$$

where $\tilde{A}_\mu(p' - p)$ is the three-dimensional Fourier transform of $A_\mu(x)$. Combining these expressions we can write the T-matrix element to be

$$
\langle \vec{p}' | iT | \vec{p} \rangle \equiv iM \cdot 2\pi\delta(E_f - E_i),
$$

where we introduced the invariant matrix element $M$, as

$$
iM = -ie\bar{u}(p')\gamma^\mu u(p) \cdot \tilde{A}_\mu(p' - p).
$$

(6.2)

According to [52], the scattering cross section for an interaction with two in states $A$ and $B$, and multiple out states, can be written as

$$
d\sigma = \frac{1}{2E_A2E_B|v_A - v_B|} \left( \prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |M(p_A, p_B \rightarrow p_f)|^2 (2\pi)^4\delta^{(4)}(p_A + p_B - \sum p_f)
$$

We will write an analog of this expression for our model of one in, one out interaction. Since we have only one in state, everything with indices $A, B$ would now only have a single index of $i$ (initial). We also have only one out state, so there is no product of final states anymore. Hence we have

$$
d\sigma = \frac{1}{2E_i} \cdot \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \cdot |M|^2 \cdot 2\pi\delta(E_f - E_i)
$$

Switching from Cartesian coordinates for momenta to polar ones, we will have $d^3p_f = p_f^2 dp_f d\Omega$. And thus

$$
\frac{d\sigma}{d\Omega} = \frac{1}{4E_i} \frac{1}{(2\pi)^2} \int dp_f \frac{p_f^2}{E_f} \delta(E_f - E_i)|M|^2
$$

One can also note that, in units where $c = 1$, we have the relation $p_f^2 = E_f^2 - m^2$, therefore $dp_f = \frac{E_f dE_f}{\sqrt{E_f^2 - m^2}}$. 

82
Incorporating this relation we can then integrate above expression,

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4E_i v_i (2\pi)^2} \int \frac{E_f^2 - m^2}{E_f} \frac{E_f dE_f}{\sqrt{E_f^2 - m^2}} \delta(E_f - E_i) |M|^2
\]

\[
= \frac{1}{4E_i v_i (2\pi)^2} \cdot \sqrt{E_f^2 - m^2} \cdot |M|^2
\]

\[
= \frac{|M|^2}{16\pi^2} \cdot p_f \cdot \frac{E_i v_i}{E_i v_i},
\]

(6.3)

Now let us focus on the specific case of Coulomb potential. The scalar, time-independent, Coulomb potential is given by \( A^0 = \frac{Ze}{4\pi r} \) and \( A^i = 0 \). We will need to convert to the momentum-space counterpart so we can use it in our calculations. The Fourier transform gives us,

\[
\tilde{A}^0(\vec{k}) = \int d^3x e^{i\vec{k} \cdot \vec{r}} \cdot \frac{Ze}{4\pi r} = \frac{Ze}{4\pi} \int 2\pi r^2 dr d\theta \cdot \frac{e^{i\vec{k} \cdot \vec{r}}}{r}
\]

\[
= \frac{Ze 4\pi}{4\pi k^2}
\]

\[
\tilde{A}^0(\vec{k}) = \frac{Ze}{k^2}
\]

Similarly, we would have \( \tilde{A}^i(\vec{k}) = 0 \), for \( i = 1, 2, 3 \). We can now use these expressions to find the invariant matrix element. Using (6.2),

\[
|M| = -ie\tilde{u}(p')\gamma^0 u(p)\tilde{A}_0(\vec{k}).
\]

In the non-relativistic limit \( \tilde{u}(p')\gamma^0 u(p) = u^+(p')u(p) = 2m \). Therefore,

\[
|M| = -2m \cdot \frac{Ze^2}{k^2}
\]

(6.4)

Before writing down the final expression for the scattering amplitude \( \frac{d\sigma}{d\Omega} \), let’s get a few things straight. First of all, \( \vec{k} = \vec{p}_f - \vec{p}_i \). Assuming \( |\vec{p}_f| = |\vec{p}_i| = p \), this relationship can be written as

\[
\vec{k} = \vec{p}_f - \vec{p}_i = p[(1 - \cos \theta)\hat{i} + \sin \theta \hat{j}].
\]

Thus,

\[
k^2 = 2p^2(1 - \cos \theta) = 4p^2 \sin^2 \frac{\theta}{2}.
\]

Substituting this back into \( |M| \), we could write for \( |M|^2 \)

\[
|M|^2 = 4m^2 \cdot \frac{Z^2 e^4}{k^4}.
\]

83
Additionally, since we are working in non-relativistic limit, \( p_f = p_i = mv \) and \( E_i = E_f = m(c = 1) \). Therefore, the fraction \( \frac{p_f}{E_i} \) becomes equal to 1. Putting this all together and using (6.3), we can eventually write for the scattering amplitude,

\[
\frac{d\sigma}{d\Omega} = \frac{|M|^2}{16\pi^2} = \frac{Z^2e^4}{16\pi^2} \frac{4m^2}{16(mv)^2\sin^2\frac{\theta}{2}}.
\]

In natural units, where \( \hbar = c = \epsilon_o = 1 \), the fine structure constant is \( \alpha = \frac{e^2}{4\pi} \). Noting this, we can rewrite the scattering amplitude formula to get to the form known as the Rutherford formula,

\[
\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4m^2v^4\sin^2\frac{\theta}{2}}.
\]  \hfill (6.5)

The form of Coulomb potential written as \( A_0 \) comes from solving the Maxwell’s equation \( \Box A^\mu = J^\mu \). This equation, in its most general case, solves to

\[
A^\mu(\vec{r}) = \frac{1}{4\pi} \int d^3x J^\mu(\vec{r}) \left| \vec{r} - \vec{r}' \right|.
\]

In cases of more complex four-currents and potentials, this would be the more general expression to use. It is, however, easy to show that in the case of Coulomb potential and Rutherford scattering this comes down to the expression we used before. Indeed, assuming we have a nucleus with charge \( Ze \) located at a position \( \vec{r}' \); the four-current can be written as \( J^\mu = (Ze\delta(\vec{r}'), \vec{0}) \). It can be seen that after integrating with this four-current, the integral will result in the expression we had earlier, i.e. \( A^0 = \frac{Ze}{4\pi r} \).

**Section 6.2: Stationary Dark Dyons**

Let us make things more interesting, and consider the case of a dyon. A dyon is a four-particle with both electric and magnetic charges. A dyon with electric charge \( q \) and magnetic charge \( p \), is conventionally written as \( (\varepsilon_e q + \varepsilon_m p) \), where \( \varepsilon_e \) and \( \varepsilon_m \) are, respectively, the electric and magnetic base charges of dyons.

First we will examine the case where the electric and magnetic base charge are located at the same position. Hence the four-current is \( J^\mu = (\varepsilon_e q\delta(\vec{r}) + \varepsilon_m p\delta(\vec{r}), \vec{0}) \). Mathematically, this is very similar to our previous case, so following the same steps we will have

\[
A^0 = \frac{\varepsilon_e q + \varepsilon_m p}{4\pi r},
\]
and the scattering cross section will be

\[
\frac{d\sigma}{d\Omega} = \frac{e^2 (\varepsilon_e q + \varepsilon_mp)^2}{64\pi^2 m^2 v^4 sin^4 \frac{\theta}{2}}.
\]

Note that the power of electron charge \(e\) has changed when comparing to \((6.5)\). This is because only one of the \(e\)'s written in \((6.4)\) is due to charge of the target, the first one comes originally with the theory since we are investigating scattering of an electron and this originates in \((6.1)\). As a result, the \(e^4\) power in Rutherford’s formula \((6.5)\), is constructed by two separately originated \(e^2\)'s, only one of which will get replaced by the dyon’s charge. One should also notice that the charge of nucleus is actually \(Ze\), so when replacing that with the expression for dyon, the \(\frac{1}{4\pi}\) factors that were included in \(\alpha\) would also come out.

Now let us focus on the case where the electric and magnetic charge are a distant \(d = |\vec{r}_q - \vec{r}_p|\) apart from each other. In the case of only one pair, the corresponding four-current is \(J^\mu = (\varepsilon_e q \delta(\vec{r}_q) + \varepsilon_mp \delta(\vec{r}_p), \vec{0})\). The electromagnetic vector potential for this four-current is

\[
A^\mu(\vec{r}) = \frac{1}{4\pi} \int d^3x' J^\mu(\vec{r}') = \frac{1}{4\pi} \varepsilon_e q \int d^3x' \frac{\delta(\vec{r}_q)}{|\vec{r}' - \vec{r}|} + \frac{1}{4\pi} \varepsilon_mp \int d^3x' \frac{\delta(\vec{r}_p)}{|\vec{r}' - \vec{r}|}.
\]

And \(A^i = 0\) for \(i = 1, 2, 3\). The Fourier transform of \(A^0\) will then be

\[
\tilde{A}^0(\vec{k}) = \left( \int d^3x e^{i\vec{k} \cdot \vec{r}} \frac{1}{|\vec{r} - \vec{r}_q|} \right) \cdot \frac{\varepsilon_e q}{4\pi} + \left( \int d^3x e^{i\vec{k} \cdot \vec{r}} \frac{1}{|\vec{r} - \vec{r}_p|} \right) \cdot \frac{\varepsilon_mp}{4\pi}
\]

In the first integral, if we replace \(\vec{r}' \equiv \vec{r} - \vec{r}_q\), and \(r' = |\vec{r}'|\), we will get

\[
\int d^3x' e^{i\vec{k} \cdot \vec{r}'} \frac{1}{r'} \cdot e^{i\vec{k} \cdot \vec{r}_q}
\]

The integrand now looks like a Coulomb potential, so the result will be

\[
\int d^3x' e^{i\vec{k} \cdot \vec{r}'} \frac{1}{r'} \cdot e^{i\vec{k} \cdot \vec{r}_q} = \frac{4\pi}{k^2} e^{i\vec{k} \cdot \vec{r}_q}.
\]

Similarly for the second integral, we will have \(\int d^3x' e^{i\vec{k} \cdot \vec{r}'} \frac{1}{r'} \cdot e^{i\vec{k} \cdot \vec{r}_p} = \frac{4\pi}{k^2} e^{i\vec{k} \cdot \vec{r}_p}\). So eventually, we have a full
expression for the momentum-space vector potential,

\[ \tilde{A}^0(k) = \frac{\varepsilon e^{ik \cdot r_q} + \varepsilon_m p e^{ik \cdot r_p}}{k^2}, \]

with \( \tilde{A}^i = 0 \) for \( i = 1, 2, 3 \) as previously mentioned.

We can follow the previously shown steps again now, noticing that (6.2) will now become,

\[ |\mathcal{M}| = -e^2 m \frac{\varepsilon e^{ik \cdot r_q} + \varepsilon_m p e^{ik \cdot r_p}}{k^2}. \]

Consequently, the scattering cross section for this case will be

\[ \frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2} = \frac{e^2 (\varepsilon e^{ik \cdot r_q} + \varepsilon_m p e^{ik \cdot r_p})^2}{64\pi^2 m^2 v^4 sin^4 \frac{\theta}{2}}. \]

Section 6.3: Moving Target

It is valid to assume that the galactic wind moving around the earth would contain Dark Matter. Thus we can model earth in the wind as the protons (our visible sector) interacting with moving Dark Matter target (our extra sector). In this case however, the four-potential is not just a Coulomb potential anymore and it’s spatial components do not vanish.

Indeed, in the case of a particle moving with four-velocity \( \beta^\mu = (1, \vec{v}) \), the four-potential can be written to the first order as

\[ A^\mu(r) = \frac{1}{4\pi} \frac{Q\beta^\mu}{r}, \]

where \( Q \) is the charge of our moving particle. The momentum-space four-potential then, by a Fourier transform similar to the original case, will be

\[ \tilde{A}^\mu = \frac{Q\beta^\mu}{k^2}. \]

The corrections to the \( \mathcal{M} \) matrix element can be written as

\[ i\mathcal{M} = -ie\bar{u}(p')\Gamma^\mu u(p) \cdot \tilde{A}^\mu(p' - \vec{p}), \]

where \( \Gamma^\mu \) has replaced \( \gamma^\mu \) and is defined as

\[ \Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{\gamma^\nu q^\nu}{2m} F_2(q^2), \]

86
where, from this point on, \( q^\mu = p^\mu - p'^\mu \), \( \gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \), and \( F_1, F_2 \) are form factors, which to the lowest order are given as \( F_1 = 1, F_2 = 0 \). With this information, the square of \( iM \) matrix element will be

\[
|M|^2 = M M^\dagger \nonumber
\]

\[
= e^2 \left[ \bar{u}(p') \gamma^\mu u(p) - \frac{1}{2m} \bar{u}(p') \gamma^{\mu\nu} q_\nu \bar{u}(p) \right] \tilde{\Lambda}_\mu \nonumber
\]

\[
= e^2 \left[ \bar{u}(p') \gamma^\mu u(p) - \frac{1}{2m} \bar{u}(p') \gamma^{\mu\nu} q_\nu u(p) \right] \tilde{\Lambda}_\alpha \tilde{\Lambda}_\alpha \nonumber
\]

\[
= e^2 \left[ \bar{u}(p') \gamma^\mu u(p) - \frac{1}{2m} \bar{u}(p') \gamma^{\mu\nu} q_\nu u(p) \right] \tilde{\Lambda}_\alpha \nonumber
\]

\[
= e^2 \left[ \bar{u}(p') \gamma^\mu u(p) - \frac{1}{2m} \bar{u}(p') \gamma^{\mu\nu} q_\nu u(p) \right] \tilde{\Lambda}_\alpha \nonumber
\]

In these lines we have used the properties of gamma matrices \((\gamma^\alpha)^\dagger = \gamma^\alpha, (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0, (\gamma^{\mu\nu})^\dagger = \gamma^0 \gamma^{\mu\nu} \gamma^0,\) and \((\gamma^0)^2 = 1\). Factoring out the common terms we will eventually end up with,

\[
|M|^2 = e^2 \bar{u}(p') \left( \gamma^\mu - \frac{1}{2m} \gamma^{\mu\nu} q_\nu \right) u(p) \tilde{\Lambda}_\mu \nonumber
\]

\[
\times \bar{u}(p) \left( \gamma^\alpha - \frac{1}{2m} q_\beta \gamma^{\beta\alpha} \right) u(p') \nonumber
\]

\[
(6.6)
\]

The more detailed form of writing the spinors would be \( u^s(p) \) and \( u^{s'}(p') \) (and conjugates), where \( s \) and \( s' \) represent the spins of incoming and outgoing protons respectively. In this step, we are mainly interested in the momenta of the protons; much more than their spins. So we can take the spin average sum of our expression. This will be

\[
\sum_s u^s(p) \bar{u}^s(p) = \not{p} + m,
\]

where \( \not{p} = \gamma^\rho p_\rho \). This average sum considerably simplifies the rest of the computation. We can rewrite the above expression (we will drop the \( e^2 \) factor here for convenience and bring it back while writing the full \( |M|^2 \) expression),

\[
\frac{1}{2} \sum_{s,s'} \text{Tr} \left[ \bar{u}(p') \left( \gamma^\mu - \frac{1}{2m} \gamma^{\mu\nu} q_\nu \right) u(p) \tilde{\Lambda}_\mu \tilde{\Lambda}_\alpha \bar{u}(p) \left( \gamma^\alpha - \frac{1}{2m} q_\beta \gamma^{\beta\alpha} \right) u(p') \right]
\]

The trace of a product is invariant under the rotation of its components while keeping the same permutation. Therefore, we can bring \( u(p') \) from the end to the beginning of the product and the trace will not change.
Therefore we have,

\[
\frac{1}{2} \sum_{\sigma,\tau} \text{Tr} \left[ u(p') \bar{u}(p') \left( \gamma^\mu - \frac{1}{2m} \gamma^{\mu\nu} q_\nu \right) \bar{A}_\mu u(p) \bar{u}(p) \tilde{A}_\alpha \left( \gamma^\alpha - \frac{1}{2m} q_\beta \gamma^{\beta\alpha} \right) \right]
\]

which according to the sum introduced above is equal to

\[
= \frac{1}{2} \text{Tr} \left[ (p' + m) \left( \gamma^\mu - \frac{1}{2m} \gamma^{\mu\nu} q_\nu \right) \bar{A}_\mu (p + m) \tilde{A}_\alpha \left( \gamma^\alpha - \frac{1}{2m} q_\beta \gamma^{\beta\alpha} \right) \right] = \\
= \frac{1}{2} \text{Tr} \left[ \left| \left( (p' + m) \gamma^\mu (p + m) \gamma^\alpha \right) + \frac{1}{4m^2} \left| (p' + m) \gamma^{\mu\nu} (p + m) \gamma^{\beta\alpha} q_\nu q_\beta \right| \right| - \frac{1}{2m} \left| (p' + m) \gamma^{\mu\nu} (p + m) \gamma^{\beta\alpha} q_\nu \right| \right] \tilde{A}_\mu \tilde{A}_\alpha \right]
\]

Let us examine each term in \([6.7]\) individually. Starting from the first term

\[
\text{Tr}((p' + m)\gamma^\mu (p + m)\gamma^\alpha \bar{A}_\mu \tilde{A}_\alpha) \\
= \text{Tr}((p' \gamma^\mu \rho \gamma^\alpha + m(p' \gamma^\mu \gamma^\alpha + \gamma^\mu \rho \gamma^\alpha) + m^2 \gamma^\mu \gamma^\alpha) \bar{A}_\mu \tilde{A}_\alpha)
\]

Here, the term linear to \(m\) would vanish because \(\rho = \gamma^\rho p_\rho\) and the trace of any odd number of gamma matrices is equal to zero. Hence we will only focus on the first and last term.

\[
\text{Tr}((p' \gamma^\mu \rho \gamma^\alpha + m^2 \gamma^\mu \gamma^\alpha) \bar{A}_\mu \tilde{A}_\alpha) = \\
\text{Tr}((\gamma^\mu \gamma^\rho \gamma^\alpha p'_\rho p_\sigma + m^2 \gamma^\mu \gamma^\alpha) \bar{A}_\mu \tilde{A}_\alpha) = \\
= 4(g^{\mu\rho} g^{\sigma\alpha} - g^{\sigma\rho} g^{\mu\alpha} + g^{\rho\alpha} g^{\mu\sigma}) p'_\rho p_\sigma \bar{A}_\mu \tilde{A}_\alpha + 4m^2 g^{\mu\alpha} \bar{A}_\mu \tilde{A}_\alpha = \\
= 4((p' \cdot \bar{A})(p \cdot \tilde{A}) - (p' \cdot p) \bar{A}^2 + (p' \cdot \bar{A})(p \cdot \tilde{A}) + m^2 \tilde{A}^2) \\
= 4(2(p' \cdot \bar{A})(p \cdot \tilde{A}) + (m^2 - (p' \cdot p)) \tilde{A}^2).
\]

This is the final expression for the first term of \([6.7]\). The second term is similar but with slightly more complexity.

\[
\text{Tr}((p' + m)\gamma^{\mu\nu} (p + m)\gamma^{\beta\alpha} q_\nu q_\beta \bar{A}_\mu \tilde{A}_\alpha) \\
= \text{Tr}((p' \gamma^{\mu\nu} \rho \gamma^{\beta\alpha} + m(p' \gamma^{\mu\nu} \gamma^{\beta\alpha} + \gamma^{\mu\nu} \rho \gamma^{\beta\alpha}) + m^2 \gamma^{\mu\nu} \gamma^{\beta\alpha} q_\nu q_\beta) \bar{A}_\mu \tilde{A}_\alpha)
\]

Since \(\gamma^{\mu\nu} = \frac{1}{2} (\gamma^{\mu\nu} - \gamma^{\nu\mu})\), for the same reason as before the term in the middle will vanish. Thus we will
have
\[
Tr((p'\gamma^{\mu\nu}p\gamma^{\beta\alpha} + m^2\gamma^{\mu\nu}\gamma^{\beta\alpha})q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
= \frac{1}{4}Tr((\gamma^\rho(\gamma^{\mu\nu} - \gamma^{\nu\mu})\gamma^\sigma(\gamma^{\beta\gamma} - \gamma^{\gamma\beta})p'_\mu p_\sigma + m^2(\gamma^{\mu\nu} - \gamma^{\nu\mu})(\gamma^{\beta\gamma} - \gamma^{\gamma\beta})q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
\]
(6.9)

Let’s break this into parts again and calculate the last term first.
\[
\frac{m^2}{4}Tr((\gamma^\rho(\gamma^{\mu\nu} - \gamma^{\nu\mu})\gamma^\sigma(\gamma^{\beta\gamma} - \gamma^{\gamma\beta})q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
= \frac{m^2}{4}Tr((\gamma^\rho(\gamma^{\mu\nu} - \gamma^{\nu\mu})\gamma^\sigma(\gamma^{\beta\gamma} - \gamma^{\gamma\beta})q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
= m^2\left[(g^{\mu\nu}g^{\beta\alpha} - g^{\mu\beta}g^{\nu\alpha} + g^{\mu\alpha}g^{\nu\beta}) - (g^{\mu\nu}g^{\beta\alpha} - g^{\mu\beta}g^{\nu\alpha} + g^{\mu\alpha}g^{\nu\beta}) - \right.
\left. - (g^{\nu\beta}g^{\mu\alpha} + g^{\nu\alpha}g^{\mu\beta} + g^{\nu\beta}g^{\mu\alpha})\right]q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha
= m^2\left[4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})\right]q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha
= 4m^2(\tilde{A}_\mu \tilde{A}_\alpha - (q \cdot \tilde{A})^2)
\]

The first term of \(6.9\), however, which is proportional to \(p'p\), has six gamma matrices involved. So it will be
\[
\frac{1}{4}Tr((\gamma^\rho(\gamma^{\mu\nu} - \gamma^{\nu\mu})\gamma^\sigma(\gamma^{\beta\gamma} - \gamma^{\gamma\beta})p'_\mu p_\sigma q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
= \frac{1}{4}Tr((\gamma^\rho(\gamma^{\mu\nu} - \gamma^{\nu\mu})\gamma^\sigma(\gamma^{\beta\gamma} - \gamma^{\gamma\beta})p'_\mu p_\sigma q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
= \frac{1}{4}16(g^{\mu\nu}g^{\beta\alpha} - g^{\mu\beta}g^{\nu\alpha} - g^{\nu\beta}g^{\mu\alpha} + g^{\nu\alpha}g^{\mu\beta} + g^{\rho\sigma}g^{\mu\alpha}g^{\nu\beta} - g^{\rho\sigma}g^{\mu\beta}g^{\nu\alpha} - \right.
\left. - g^{\rho\beta}g^{\mu\alpha}g^{\nu\sigma} + g^{\rho\sigma}g^{\mu\alpha}g^{\nu\beta} + g^{\rho\alpha}g^{\mu\beta}g^{\nu\sigma} - g^{\rho\alpha}g^{\mu\sigma}g^{\nu\beta})p'_\mu p_\sigma q_\nu q_\beta \tilde{A}_\mu \tilde{A}_\alpha
= 4\left[2((p' \cdot \tilde{A})(q \cdot \tilde{A})(p,q) - (p' \cdot \tilde{A})q^2(\tilde{A} \cdot p) + (p' \cdot q)(q \cdot \tilde{A})(p \cdot \tilde{A}) + (p' \cdot p)(\tilde{A}^2 q^2 - (q \cdot \tilde{A})^2)\right]
\]

So the overall contribution of the second term of \(6.7\) would be
\[
\frac{1}{4m^2} \cdot 4 \cdot \left[2((p' \cdot \tilde{A})(q \cdot \tilde{A})(p,q) - (p' \cdot \tilde{A})q^2(\tilde{A} \cdot p) - (p' \cdot q)(\tilde{A}^2 q^2 - m^2 \tilde{A}^2 q^2 - m^2(q \cdot \tilde{A})^2)\right]
\]
(6.10)
Now it is time to examine the cross terms in (6.7). First,

\[-\frac{1}{2m} \text{Tr}( (p' + m) \gamma^\mu (\gamma^\alpha + m) q^\beta q_\alpha \tilde{A}_\mu \tilde{A}_\alpha) =
\]

\[-\frac{1}{2m} \text{Tr}( p' \gamma^\mu \gamma^\beta + m(p' \gamma^\mu \gamma^\alpha + \gamma^\mu \gamma^\beta) + m^2 \gamma^\mu \gamma^\beta) q_\beta \tilde{A}_\mu \tilde{A}_\alpha
\]

In this case the first and last terms will vanish because of trace of odd number of \(\gamma\)'s. The term in the middle can be written as

\[-\frac{1}{2m} \text{Tr}(m(p' \gamma^\mu \gamma^\beta + \gamma^\mu \gamma^\beta) q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
\]

\[-\frac{1}{4} \text{Tr}( (\gamma^\rho \gamma^\mu \gamma^\beta \gamma^\alpha - \gamma^\sigma \gamma^\beta \gamma^\alpha) p'_\rho + \gamma^\mu \gamma^\sigma (\gamma^\beta \gamma^\alpha - \gamma^\alpha \gamma^\beta) p_\sigma q_\beta \tilde{A}_\mu \tilde{A}_\alpha)
\]

\[-2(p' \cdot \tilde{A}) (q \cdot \tilde{A}) - (p' \cdot q) \tilde{A}^2 + (p \cdot q) \tilde{A}^2 - (p \cdot \tilde{A}) (q \cdot \tilde{A})
\]

Remembering \(q = p' - p\), this would simplify to

\[2(q^2 \tilde{A}^2 - (q \cdot \tilde{A})^2).
\]

(6.11)

The second cross term of (6.7) is similar in nature to this one, but slightly different in order.

\[-\frac{1}{2m} \text{Tr}( (p' + m) \gamma^\mu (\gamma^\alpha + m) q_\rho \tilde{A}_\rho \tilde{A}_\alpha) =
\]

\[-\frac{1}{2m} \text{Tr}( p' \gamma^\mu \gamma^\alpha + m(p' \gamma^\mu \gamma^\alpha + \gamma^\mu \gamma^\rho) + m^2 \gamma^\mu \gamma^\rho) q_\rho \tilde{A}_\rho \tilde{A}_\alpha
\]

\[-\frac{1}{2m} \text{Tr}(m(p' \gamma^\mu \gamma^\alpha + \gamma^\mu \gamma^\rho) q_\rho \tilde{A}_\rho \tilde{A}_\alpha)
\]

\[-\frac{1}{4} \text{Tr}( (\gamma^\rho \gamma^\mu \gamma^\rho \gamma^\alpha - \gamma^\rho \gamma^\rho \gamma^\alpha) p'_\rho + (\gamma^\mu \gamma^\rho \gamma^\rho \gamma^\alpha - \gamma^\rho \gamma^\rho \gamma^\alpha) p_\sigma q_\rho \tilde{A}_\rho \tilde{A}_\alpha)
\]

\[-2((p' \cdot \tilde{A}) (q \cdot \tilde{A}) - (p' \cdot q) \tilde{A}^2 + (p \cdot q) \tilde{A}^2 - (p \cdot \tilde{A}) (q \cdot \tilde{A})
\]

\[2(q^2 \tilde{A}^2 - (q \cdot \tilde{A})^2).
\]

(6.12)

Now that we have all the terms examined, we can write an expression for the corrections. Substituting
Now we need to enter the moving potential field as \((6.8), (6.10), (6.11)\) back in \((6.7)\) we will have

\[
\frac{1}{2} \left[4(2(p' \cdot \vec{A})(p \cdot \vec{A}) + (m^2 - (p' \cdot p))\vec{A}^2) + \\
+ \frac{1}{m^2} (2((p' \cdot \vec{A})(q \cdot \vec{A})(p \cdot q) - (p' \cdot \vec{A})q^2(p \cdot \vec{A}) - (p' \cdot q)A^2(p \cdot q) + (p' \cdot q)(q \cdot \vec{A})(p \cdot \vec{A})) \\
+ (p' \cdot p)\vec{A}^2q^2 - (p' \cdot p)(q \cdot \vec{A})^2 + m^2\vec{A}^2q^2 - m^2(q \cdot \vec{A})^2) + \\
+ 4(q^2\vec{A}^2 - (q \cdot \vec{A})^2) \right].
\]

Multiplying this by a factor of \(e^2\) would give us the matrix element \(|M|^2\), needed to determine the scattering cross section,

\[
\frac{d\sigma}{d\Omega} = \frac{|M|^2}{16\pi^2}.
\]

Now we need to enter the moving potential field as \(\vec{A}' = \frac{Q\vec{\beta}^\mu}{q^2}\), where

\[
\vec{\beta}^\mu = (1, \vec{v}_{DM}) = (1, v_{DM} \cos \zeta \cos \xi, v_{DM} \cos \zeta \sin \xi, v_{DM} \sin \zeta).
\]

Here we used polar coordinates with polar angles \(\zeta\) and \(\xi\). One must also note that we are working in natural units where \(c = 1\). Defining the angle between incoming and outgoing momenta to be \(\theta\), (i.e. \(\vec{p}' \cdot \vec{p} = p^2 \cos \theta\)), we can expand the final expression for non-relativistic limit, i.e. when \(v_{DM} \rightarrow 0\) and \(v \equiv \frac{p}{m} \rightarrow 0\). We will have

\[
\frac{d\sigma}{d\Omega} = \frac{e^2Q^2}{16\pi^2} \left[ \frac{m^2 + \rho^2 - 2\rho^2(1 - \cos \theta)}{4p^4 \sin^4 \left(\frac{\theta}{2}\right)} + \\
+ \frac{\sqrt{m^2 + \rho^2}v_{DM} \cos \zeta (\cos \theta - \xi) + \cos \xi)}{4p^4 \sin^4 \left(\frac{\theta}{2}\right)} + \\
+ \frac{v_{DM}^2(\cos^2 \zeta \cos^2 \left(\frac{\theta}{2}\right) - \xi)(1 + \sin^2 \frac{\theta}{2}) + 4\sin^2 \frac{\theta}{2} \sin^2 \zeta}{4p^4 \sin^4 \left(\frac{\theta}{2}\right)} \right]
\]

\[
= \frac{e^2Q^2}{16\pi^2} \left[ \frac{1 + v^2(2 \cos \theta - 1)}{4m^2v^4 \sin^4 \left(\frac{\theta}{2}\right)} + \\
+ \frac{v_{DM} \sqrt{1 + v^2} \cos \zeta (\cos \theta - \xi) + \cos \xi)}{4m^2v^3 \sin^4 \left(\frac{\theta}{2}\right)} + \\
+ \frac{v_{DM}^2(\cos^2 \zeta \cos^2 \left(\frac{\theta}{2}\right) - \xi)(1 + \sin^2 \frac{\theta}{2}) + 4\sin^2 \frac{\theta}{2} \sin^2 \zeta}{4m^2v^2 \sin^4 \left(\frac{\theta}{2}\right)} \right]
\]

\[
= \frac{e^2Q^2}{16\pi^2} \left[ \frac{1 + v^2(2 \cos \theta - 1)}{4m^2v^4 \sin^4 \left(\frac{\theta}{2}\right)} + \\
+ \frac{v_{DM} \sqrt{1 + v^2} \cos \zeta (\cos \theta - \xi) + \cos \xi)}{4m^2v^3 \sin^4 \left(\frac{\theta}{2}\right)} + \\
+ \frac{v_{DM}^2(\cos^2 \zeta \cos^2 \left(\frac{\theta}{2}\right) - \xi)(1 + \sin^2 \frac{\theta}{2}) + 4\sin^2 \frac{\theta}{2} \sin^2 \zeta}{4m^2v^2 \sin^4 \left(\frac{\theta}{2}\right)} \right].
\]
Using the original Rutherford formula,

\[ \frac{d\sigma}{d\Omega} \bigg|_{\text{Ruth}} = \frac{e^2 Q^2}{16\pi^2 4m^2 v^4 \sin^4 \left( \frac{\theta}{2} \right)}, \]

we can rewrite the above result as

\[ \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega} \bigg|_{\text{Ruth}} \left[ 1 + v^2 (2 \cos \theta - 1) + vv_{DM} \sqrt{1 + v^2} \cos \zeta \left( \cos(\theta - \xi) + \cos \xi \right) \right. \]

\[ \left. + v^2_{DM} \left( \cos^2 \zeta \cos^2 \left( \frac{\theta}{2} - \xi \right)(1 + \sin^2 \frac{\theta}{2}) + 4 \sin^2 \frac{\theta}{2} \sin^2 \zeta \right) \right]. \] (6.13)

And thus we have the penultimate result of this chapter. In the next section, however, we will return to the case of a stationary dyon target to investigate what empirical results we can expect from scattering protons off a Dark Matter dyon.

Section 6.4: Dyon Angular Momentum

In Section 4.2, we briefly mentioned that the angular momentum of a dyon \( d = (q, p) \) is the same as the dyon coupling between two distant dyons \( d_1 = (q, 0) \) and \( d_2 = (0, p) \), i.e.

\[ |\vec{l}_d| = \langle d_1 | d_2 \rangle = pq. \]

Let us first show that this is indeed true. According to the general angular momentum expression for electromagnetic particles we have

\[ \vec{l} = \int \vec{r} \times (\vec{E} \times \vec{B}) d^3 \vec{r}. \]

The electric and magnetic fields in our case are simply

\[ \vec{E} = \frac{q}{r_q^2} \vec{r}_q, \quad \vec{B} = \frac{p}{r_p^3} \vec{r}_p. \]

One should remember here that we are working in natural units. Without loss of generality, we will put our electric charge at the origin and set \( \hat{z} \) in the direction of the “rod” connecting the charges, so that
\[ \vec{d} = \vec{r}_q - \vec{r}_p = d \hat{z}. \] As a result, \( \vec{r}_q = \vec{r} \) and \( \vec{r}_p = \vec{r} - \vec{d} \), and the integral becomes

\[
\vec{l} = \int \vec{r} \times (\vec{E} \times \vec{B}) \, d^3 \vec{r}
\]

\[
= \int \vec{r} \times \left( \frac{q}{r^3} \vec{r}_q \times \frac{p}{r^3} \vec{r}_p \right) \, d^3 \vec{r}
\]

\[
= q \int \frac{1}{r^3} \vec{r} \times (\vec{r} \times \vec{B}) \, d^3 \vec{r}.
\]

Using vector identities we can rewrite the cross term as \( \vec{r} \times (\vec{r} \times \vec{B}) = \vec{r}(\vec{r} \cdot \vec{B}) - r^2 \vec{B} \). Thus, we will have

\[
\vec{l} = q \int \frac{1}{r} \left( \vec{r}(\vec{r} \cdot \vec{B}) - r^2 \vec{B} \right) \, d^3 \vec{r},
\]

(6.14)

where \( \hat{r} = \frac{\vec{r}}{r} \). According to the identity

\[
(\vec{a} \cdot \nabla) \hat{n} f(r) = \frac{f(r)}{r} [\hat{a} - \hat{n}(\hat{a} \cdot \hat{n})] + \hat{n}(\hat{a} \cdot \hat{n}) \frac{\partial f}{\partial r},
\]

if we set \( f(r) = 1 \) and \( \vec{a} = \vec{B} \), we will get

\[
(\vec{B} \cdot \nabla) \hat{n} = \frac{1}{r} (\vec{B} - \hat{n}(\vec{B} \cdot \hat{n})).
\]

Thus, (6.14) can be rewritten as

\[
\vec{l} = -q \int (\vec{B} \cdot \nabla) \hat{n} d^3 \vec{r}
\]

Eventually integrating this by parts, we will end up with

\[
\vec{l} = -q \left( \oint_S \hat{n}(\vec{n} \cdot \vec{B}) d^2 x - \int \hat{n}(\vec{\nabla} \cdot \vec{B}) d^3 \vec{r} \right)
\]

The first term vanishes because \( \hat{n} \) is pointed radially outwards and will average to zero while integrated over the surface of a sphere. The second term is simply Gauss’s law for magnetism, \( \vec{\nabla} \cdot \vec{B} = p \delta(\vec{r}_p) = p \delta(\vec{r} - \vec{d}) \). Thus,

\[
\vec{l} = q \int \hat{n}(\vec{\nabla} \cdot \vec{B}) d^3 \vec{r}
\]

\[
= pq \int \hat{n} \delta(\vec{r} - d \hat{z}) d^3 \vec{r}
\]

\[
\vec{l} = pq \hat{z}.
\]

(6.15)
An important outcome of this derivation is that it indicates that magnitude of the angular momentum of a dyon cannot change unless we manipulate its charges. The natural question that comes up here is then how would an electron scattering off of it, affect the angular momentum of the dyon. According to (6.15), the dyon is only left with the option of changing its orientation, i.e. the direction of its distance vector \( \mathbf{d} = d \hat{z} \). Therefore, conservation of angular momentum will lead to non-trivial selection rules on possible interaction terms. This can be viewed as a generalization of the Callan-Rubakov effect [58–60]. In the following section we will study these possibilities.

6.4.1: Selection Rules

In the interaction between the scattering electron and the dark dyon, there are two sets of angular momenta to focus on. One is the electron’s spin plus its orbital angular momentum; the other one is the orbital angular momentum of the connecting rod between the charges of the dyon. We can denote these quantities with \( \vec{j}_e \) and \( \vec{j}_d \) respectively,

\[
\vec{j}_e = \vec{s}_e + \vec{l}_e \quad \quad \quad \quad \vec{j}_d = \vec{d}_d = pq \hat{z}.
\]

The total angular momentum of the system \( \vec{J}_{tot} \), which can be written as the sum of these two, has to be conserved throughout this interaction. Namely,

\[
\vec{J}_{tot} = \vec{j}_e \otimes 1 + 1 \otimes \vec{j}_d = \text{const}.
\]

Now let us assume that both \( p \) and \( q \), and thus their product, is equal to unity in (6.15). In this particular case, \( l_z = 1 \) and thus \( m_l = -1, 0, 1 \). So the maximum amount of “jumps” the rod can take is 2 units. Also let us assume, for the sake of simplicity and without a real loss of generality, that all the incoming electrons are in their ground orbital state, meaning that their orbital angular momentum can only increase or stay the same, but not decrease. With these assumptions, we can then create a table of Selection Rules for this example case, see Table 6.1.

In this table, we depicted spin of the electron with an arrow pointing up or down, and the angular number of the rod with an arrow pointing from \( p \) to \( q \) in case of \( m_l = 1 \), from \( q \) to \( p \) in case of \( m_l = -1 \), and upwards in case of \( m_l = 0 \). We also showed the increase in orbital angular momentum of the electron with \( l_+ \). From point of view of the rod, when the incoming electron is at spin \( -\frac{1}{2} \), it does not make a difference if it increases its spin by one unit or its orbital angular momentum; that is why we combined these cases as one option for each \( m_l \).
Table 6.1: Selection rules for electron scattering off a dyon, with assumptions that $l_z = 1$ and $l_e$ is minimal.

We marked banned outcomes with a big red circle. It is evident that only 2/3 of the possibilities are allowed by selection rules.
CHAPTER 7: Summary and Conclusions

In this dissertation, we have aimed to explore a branch of physics beyond the Standard Model, by studying its interactions with extensions of stringy SM models based on F-theory. In these models SM is generally localized on a stack of seven-branes and there is an “extra” sector in the form of probe D3-branes interacting with the seven-branes. The “messenger states” governing these interactions are approximately $\mathcal{N} = 2$ supersymmetric so we can use the $\mathcal{N} = 2$ SUSY formal tools to investigate these interactions, which in the language of gauge theory, are mixing terms between the gauge group of the visible sector (generally SM) and the $U(1)$ of the probe D3-brane. At this point then, we perform the Seiberg-Witten solution to find the coupling constants describing these mixings, in $\mathcal{N} = 2$ strongly coupled theories.

Section 7.1: Summary of Concepts

We started in Chapter 2 by giving a review of Supersymmetry, as a build-up to $\mathcal{N} = 2$ SUSY and its superfields. We discussed the general algebra for unextended and extended supersymmetric gauge theories, and presented the important BPS bound condition that must be met in order to keep the massive multiplets short, with stable masses. We also presented the superfield content of $\mathcal{N} = 1$ SUSY as pillars to build the $\mathcal{N} = 2$ SUSY on.

The goal of Chapter 3 was to set up the $\mathcal{N} = 2$ formalism. We started this by describing in good detail the string-based models for extensions to the SM. We discussed the strong coupling limit between the gauge groups of visible and extra sector, and most importantly, we defined the complex coupling constants $\tau_{ij}$ \[[3.8]\]. We then used these coupling constants and the $\mathcal{N} = 1$ superfields given in the previous chapter, to write the superfield formation for $\mathcal{N} = 2$ SUSY. Breaking the gauge symmetry, we also introduced new parameters $u_k$ \[[3.14]\], which serve as coordinates on the Coulomb branch of the theory. This naturally sets us up for discussing the Seiberg-Witten solution.

The Seiberg-Witten (SW) solution is the heart of this project. We investigated this in Chapter 4. The solution was first suggested in \[37\], and later generalized in \[38\]–\[40\]. The gist of the SW method lies in the realization that an $\mathcal{N} = 2$ SUSY strongly coupled field theory, can be written as an elliptic curve, with holomorphic Kähler geometry, where the coupling constants $\tau_{ij}$ become modular coordinates obeying $SL(2, \mathbb{Z})$ transformations (see Figure 4.1). In Chapter 4 we presented the analytical derivation of the method
in full detail; Then we proceeded the calculation for sample flavor symmetry groups $A_1, A_2$ and $D_4$ as the “mixing” visible sector in our model. We provided graphical representations of the coupling constants values in each case using the Mathematica code presented in Appendix A.

Chapter 5 then was set up to numerically prove that supersymmetry really does what it is supposed to do; i.e. the mass values for bosonic and fermionic states are equal. We accomplished this by introducing a parameter-based shift in the susy component fields and studied the effects of this shift on the theory, and specifically the mass values. This was done for $A_1$ flavor symmetry group. We began the chapter by “gauge Lagrangian” portion of the general Lagrangian (5.1) and showed that the mixing conditions hold to a reasonable extent for gauginos. Then we focused on the Coulomb branch Lagrangian (5.13). With use of Mathematica again, we found out that the Coulomb branch potential by itself does not have a minimum point in our case (Figure 5.3). We then added a proper “superpotential” which indeed gave us minimum points in the supersymmetric case, i.e. when the shift was set to zero (Figure 5.4). We then calculated the mass values for bosonic and fermionic states using Mathematica again, and proved their equity. At the end of the chapter, we turned the SUSY breaking parameter back on and witnessed the fermionic and bosonic mass values diverging. This phenomenon (and other possible procedures that lead to similar results) is known as supersymmetry breaking.

Chapter 6 explored a possible implication for the stringy SM extension model we have been building. In this chapter we postulated that the extra sector in our model can be a toy model candidate for Dark Matter; we named it the “dark” sector. We studied a parallel to Rutherford scattering problem, by assuming a dyon populated heavy target to be similar to Dark Matter (DM), and the process of scattering was assumed to be assimilated by DM moving around earth in galactic wind. Therefore we calculated the problem with a moving target with charge properties of a dyon (both electric and magnetic charges, $(q_e, q_m)$), and determined a result similar to the Rutherford formula with expansions, (6.13). We then investigated the possible outcomes of a similar hypothetical “collision” between the dyons and matter (electron, proton, etc.) from conservation of angular momentum point of view, and listed the results in Table 6.1.

Section 7.2: Outlook and Future Directions

Perhaps the most important step to expanding our work would be to consider a symmetry group large enough, that includes all of SM gauge group $G_{SM} = SU(3) \times SU(2) \times U(1)$ as a subgroup, for our visible sector. This of course hints to $E_8$ gauge group. Under a proper choice of parameters $E_8$ can be written as $E_7 \times U(1)$. Our work has set up the foundation for calculating the coupling constants between visible and extra sectors, and most certainly for cases where we can separate a $U(1)$ of the gauge group of the
visible sector; but it will be a useful and well motivated expansion to perform the SW calculation brought in Chapter 4 for larger symmetry groups, in particular $E_6, E_7$, and eventually $E_8$.

Another interesting scenario for expansion of this project is in turn related to the extra sector. It would be interesting to explore the possibility of the extra sector having a more complex gauge group than $U(1)$. The extra sector would still need to keep its nature as a probe D3-brane because that makes it attracted to the Yukawa points of intersecting seven-branes, so it would be interesting to see how we can break the overall $U(1)$. Alternatively, we might need to find other possibilities for the extra sector which would interact with the seven-branes in the same or similar manner as to that of the probe D3-brane.

Lastly, it would be beneficial to investigate more DM phenomena using our toy model, partially introduced in Chapter 6 to improve its legitimacy as a candidate for DM. One path towards these investigations is via the idea that the position of D3-brane in regards to seven-branes can act as *inflaton*; and then D3-brane “rolling” closer to seven-branes would be responsible for *slow roll inflation*. It would be quite interesting to develop such a phenomenological scenario.
APPENDIX A: Code for Coupling Constant Graphs

All of the code in this appendix chapter share several predefined variables. We will bring the few lines of Mathematica code defining these variables here, under Prelude, and we will refer to it in the upcoming sets of code. These lines refer to the $I$ contour integrals defined in (4.10), and the corresponding variables used for computing them.

Prelude

Defining necessary variables:

\[
\begin{align*}
\text{k} &= \text{Sqrt}\left[\frac{r_2-r_3}{r_1-r_3}\right]; \\
\text{ksqu} &= \frac{r_2-r_3}{r_1-r_3}; \\
\text{kp} &= \text{Sqrt}\left[\frac{r_2-r_3}{r_2-r_1}\right]; \\
\text{kpsqu} &= \frac{r_2-r_1}{r_2-r_1}; \\
\text{ctilde} &= \frac{r_1+r_2}{r_1-r_3}; \\
\text{nu} &= -\left(\frac{1-\text{ctilde}+\text{kp}}{1-\text{ctilde}-\text{kp}}\right)^2 * \left(\frac{1-\text{kp}}{1+\text{kp}}\right)^2;
\end{align*}
\]

The $I$-integrals (as defined in Chapter 5):

\[
\begin{align*}
\text{IONEONE} &= \frac{4}{(r_1-r_3)^{3/2}} \cdot \text{EllipticK}[\text{ksqu}]; \\
\text{IONETHR} &= \frac{4}{(r_1-r_3)^{3/2}} \cdot \left(\frac{\text{EllipticK}[\text{ksqu}]}{1-\text{ctilde}+\text{kp}} + \left(\frac{4\text{kp}}{1+\text{kp}}\right) \cdot \frac{-\text{EllipticPi}\left[-\text{nu},\left(\frac{1-\text{kp}}{1+\text{kp}}\right)^2\right]}{(1-\text{ctilde})^2-\text{kpsqu}}\right); \\
\text{ITWOONE} &= \text{IONEONE}/.\{r_1 \rightarrow r_3, r_3 \rightarrow r_1\}; \\
\text{ITWOTH} &= \text{IONETHR}/.\{r_1 \rightarrow r_3, r_3 \rightarrow r_1\};
\end{align*}
\]

The roots for general $f$ and $g$:

\[
\text{SOLVE} = \text{Solve}\left[x^3 + f \cdot x + g == 0, x\right];
\]

\[
\begin{align*}
\begin{pmatrix}
\text{r}_1 \\
\text{r}_2 \\
\text{r}_3
\end{pmatrix} &= \begin{pmatrix}
x / \text{SOLVE}[2] \\
x / \text{SOLVE}[1] \\
x / \text{SOLVE}[3]
\end{pmatrix};
\end{align*}
\]

sanity check on roots:

\[
r_3 + r_2 + r_1 // \text{FullSimplify}
\]

\[
0
\]

% End of Prelude

Now we will focus on code specific to each flavor symmetry group.
Section A.1: $A_1$ Flavor Symmetry Group

Prelude;

$f,g,a$ and $a^D$ for the case of $A_1$

$f = u$;

$g = 4m^2$;

$a = \frac{\sqrt{2}}{4\pi} \left( \frac{2\pi}{4m} + g \right)$;

$aD = \frac{\sqrt{2}}{4\pi} \left( \frac{2\pi}{4m} + g \right)$;

The $A_1$ coupling constants:

$\tau_{\text{mix}} = D[aD, m]$;

$\tau_{\text{extra}} = D[aD, u]/D[a, u]$;

- $A_1$ when $f = 0$

Table of $\tau_{\text{extra}}$ and $\tau_{\text{mix}}$ for $A_1$ case when $f = u = 0$

tA1 = Table[{\tau_{\text{extra}}, \tau_{\text{mix}}, m} /., u -> 0, m -> (RandomReal[{0.5, 10}] * Exp[I * RandomReal[{0, 2\pi}]]), \{i, 2000\}]

% condition to only keep points which produce $\tau_{\text{extra}}$ in the fundamental domain

TABA1 = Select[tA1, Abs[Re[#1]] <= 0.5 && Abs[#2] <= 1 && Im[#2] >= 0 && Abs[#1] <= 1 &][All, All]

Graph of $\tau_{\text{extra}}$ when $f = 0$. Should give us $\tau_{\text{extra}} = e^{2\pi i/3}$.

ListPlot[Table[{Re[TABA1[[i, 1]]], Im[TABA1[[i, 1]]}, \{i, 2000\}],

PlotRange -> {{-1, 0}, {0, 1.5}}, AxesOrigin -> {-1, 0}, PlotStyle -> PointSize[Large],

AxesLabel -> {"Re[\tau_{\text{extra}}]", "Im[\tau_{\text{extra}}]"}, PlotLabel -> "Im[\tau_{\text{extra}}] vs. Re[\tau_{\text{extra}}] for $u=0$"

Im[\tau_{\text{extra}}] vs. Re[\tau_{\text{extra}}] for $u=0$
Graph of $\tau_{\text{mix}}$ for A1 case, when $\tau_{\text{extra}} = e^{2\pi i/3}$.

```
ListPlot[Table[{Re[TABA1[[i, 2]]], Im[TABA1[[i, 2]]]}, {i, 2000}], AxesLabel -> {"Re[\tau_{\text{mix}}]", "Im[\tau_{\text{mix}}]"}, PlotRange -> {{-1, 1}, {-3, 3}}, PlotLabel -> "Im[\tau_{\text{mix}}] vs. Re[\tau_{\text{mix}}] zoomed in", LabelStyle -> Directive[Bold], PlotStyle -> PointSize[Large]]
```

- **$A_1$ when $g = 0$**

Table of $\tau_{\text{extra}}$ and $\tau_{\text{mix}}$ for A1 case when $g = 4m^2 = 0$

```
TA1g = Table[{\tau_{\text{extra}}, \tau_{\text{mix}}, m} /. 
{u -> (RandomReal[{0.001, 1}]) * Exp[I * RandomReal[{0, 2\pi}]], m -> 0}, {i, 2000}]
```

%condition to only keep points which produce $\tau_{\text{extra}}$ in the fundamental domain

```
TAB = Select[TA1g, Abs[Re[#1]] <= 0.5 && Abs[#1] >= 1 && Im[#1] >= 0 && Abs[#1][[All, All]]
```

Graph of $\tau_{\text{extra}}$ when $g = 0$. Should give us $\tau_{\text{extra}} = i$, and its modular transformations.

```
ListPlot[Table[{Re[TAB[[i, 1]]], Im[TAB[[i, 1]]]}, {i, 2000}], PlotStyle -> PointSize[Large], AxesLabel -> {"Re[\tau_{\text{extra}}]", "Im[\tau_{\text{extra}}]"}, PlotLabel -> "Im[\tau_{\text{extra}}] vs. Re[\tau_{\text{extra}}] for m=0"]
```
Graph of $\tau_{\text{mix}}$ for $A_1$ case, when $g = 0.$

\begin{verbatim}
ListPlot[Table[{Re[TAB[i, 2]], Im[TAB[i, 2]]}, {i, 2000}], AxesLabel -> {"Re[\tau_{\text{mix}}]", "Im[\tau_{\text{mix}}]"},
PlotLabel -> "Im[\tau_{\text{mix}}] vs. Re[\tau_{\text{mix}}]", LabelStyle -> Directive[Bold]]
\end{verbatim}

- $A_1$ when $u$ is small and $m$ runs from $[0, 10]$

\begin{verbatim}
t1A1 = Table[{{\tau_{\text{extra}}, \tau_{\text{mix}}, m} /. {u -> 0.01, m -> RandomReal[{0, 10}]}, {i, 2000}};
t2A1 = Table[{{\tau_{\text{extra}}, \tau_{\text{mix}}, m} /. {u -> 0.1, m -> RandomReal[{0, 10}]}, {i, 2000}};
TABA1 = Select[t1A1, Abs[Re[#[[1]]]] <= 0.5 && Abs[#[[1]]] >= 1 && Im[#[[1]]] >= 0 &][All, All];
TA2A1 = Select[t2A1, Abs[Re[#[[1]]]] <= 0.5 && Abs[#[[1]]] >= 1 && Im[#[[1]]] >= 0 &][All, All];
ListPlot[{Table[{{TABA1[[i, 3]], Im[TABA1[[i, 2]]]}}, {i, Length[TABA1]}], AxesLabel -> {m, "Im[\tau_{\text{mix}}]"},
PlotLabel -> "Im[\tau_{\text{mix}}] vs. m"}, LabelStyle -> Directive[Bold], PlotStyle -> PointSize[Large],
PlotLegends -> Placed[SwatchLegend[{{"u=0.01", "u=0.1"}}, {Right, Center}]}]
\end{verbatim}
ListPlot[
  ListPlot[
    Table[
      {Re[TABA1[i, 3]], Re[TABA1[i, 2]]},
      {i, Length[TABA1]}]
    , AxesLabel -> {m, "Re[τ mix]"}
    , PlotLabel -> "Re[τ mix] vs. m",
    LabelStyle -> Directive[Bold],
    PlotStyle -> PointSize[Large],
    PlotLegends -> Placed[SwatchLegend[{"u=0.01", "u=0.1"}], {Right, Center}]]
]

ListPlot[
  ListPlot[
    Table[
      {Im[TABA1[i, 3]], Im[TABA1[i, 2]]},
      {i, Length[TABA1]}]
    , AxesLabel -> {m, "Im[τ mix]"}
    , PlotLabel -> "Im[τ mix] vs. m",
    LabelStyle -> Directive[Bold],
    PlotStyle -> PointSize[Large],
    PlotLegends -> Placed[SwatchLegend[{"u=0.01", "u=0.1"}], {Right, Center}]]
]
$|\tau_{\text{mix}}| \text{ vs. } m$

- $u=0.01$
- $u=0.1$
Section A.2: \( A_2 \) Flavor Symmetry Group

Prelude:

\( f.g.a \) and \( a^D \) for the case of \( A_2 \)

\[
a = \frac{\sqrt{2}}{8\pi} \left( 3 + \text{SINONE} \right) + u \left( (\phi_1 + \text{IONETHR} \cdot c \rightarrow 2 \phi_1) + ((-\phi_2) + \text{IONETHR} \cdot c \rightarrow -2 \phi_2) \right) + ((\phi_2 - \phi_1) \cdot \text{IONETHR} \cdot c \rightarrow 2 \phi_2) \right) \right); \\
aD = \frac{\sqrt{2}}{8\pi} \left( \frac{3 + \text{ITWOONE}}{2} + u \left( (\phi_1 + \text{ITWOTH} \cdot c \rightarrow 2 \phi_1) + ((-\phi_2) + \text{ITWOTH} \cdot c \rightarrow -2 \phi_2) \right) + ((\phi_2 - \phi_1) \cdot \text{ITWOTH} \cdot c \rightarrow 2 \phi_2) \right); \\
f = -4 \left( \phi_1^2 + \phi_2^2 - \phi_1 \phi_2 \right); \\
g = u^2 - 8 \phi_1 \phi_2 \left( \phi_1 - \phi_2 \right);
\]

The \( A_2 \) coupling constants:

\[
\tau_{\text{extra}} = D[aD, u]/D[a, u]; \\
\tau_{\text{mix1}} = D[aD, \phi_1]; \\
\tau_{\text{mix2}} = D[aD, \phi_2];
\]

- \( A_2 \) when \( f = 0 \)

We will solve \( f=0 \) for the \( A_2 \) case, and use the roots of \( f=0 \) to generate tables with the roots acting as constraining conditions for \( \phi_2 \).

Solve \( f == 0, \phi_2 \)

\[
\{ \phi_2 \rightarrow \frac{1}{3} (\phi_1 - i\sqrt{3}\phi_1) \}, \{ \phi_2 \rightarrow \frac{1}{3} (\phi_1 + i\sqrt{3}\phi_1) \}
\]

\( \text{TABf1} = \text{Table} \left[ \{ \tau_{\text{mix1}}, \tau_{\text{mix2}}, \tau_{\text{extra}}, u, \phi_1, \phi_2 \} / \{ \phi_2 \rightarrow \frac{1}{3} (1 - i\sqrt{3}) \} \right] / . \\
\{ \phi_1 \rightarrow (\text{RandomReal}[\{0.5, 1\}] \cdot \text{Exp}[I \cdot \text{RandomReal}[\{0, 2\pi\}])], \right] \\
u \rightarrow (\text{RandomReal}[\{0.001, 1\}] \cdot \text{Exp}[I \cdot \text{RandomReal}[\{0, 2\pi\}]) \}, \{ i, 1, 2000 \} \)
\]

\( \text{TABf2} = \text{Table} \left[ \{ \tau_{\text{mix1}}, \tau_{\text{mix2}}, \tau_{\text{extra}}, u, \phi_1, \phi_2 \} / \{ \phi_2 \rightarrow \frac{1}{3} (1 + i\sqrt{3}) \} \right] / . \\
\{ \phi_1 \rightarrow (\text{RandomReal}[\{0.5, 1\}] \cdot \text{Exp}[I \cdot \text{RandomReal}[\{0, 2\pi\}])], \right] \\
u \rightarrow (\text{RandomReal}[\{0.001, 1\}] \cdot \text{Exp}[I \cdot \text{RandomReal}[\{0, 2\pi\}]) \}, \{ i, 1, 2000 \} \)
\]

\( \text{TABfa} = \text{Join}[\text{TABf1}, \text{TABf2}] \)

condition to only keep points which produce \( \tau_{\text{extra}} \) in the fundamental domain:

\( \text{TABf} = \text{Select}[\text{TABf}, \text{Abs}[\text{Re}[\#[3]]] <= 0.5 && \text{Abs}[\#[3]] >= 1 && \text{Im}[\#[3]] >= 0 \&\& [\text{All}, \text{All}] \)

Graph of values of \( \tau_{\text{mix1}} \) in the complex plane when \( f=0 \) in \( A_2 \)

\( \text{ListPlot} [\text{Table}[\{\text{Re}[\text{TABf}[\{i, 1\}]], \text{Im}[\text{TABf}[\{i, 1\}]]\}, \{i, \text{Length}[\text{TABf}]\}], \text{AxesLabel} \rightarrow \{ "\text{Re}[\tau_{\text{mix1}}]", "\text{Im}[\tau_{\text{mix1}}]" \}, \)

\( \text{PlotLabel} \rightarrow \{ \text{"Im}[\tau_{\text{mix1}}] \] vs. \text{Re}[\tau_{\text{mix1}}], \text{when } \phi_2 = \phi_1 \frac{1 + i\sqrt{3}}{2} \} \)
Graph of values of $\tau_{\text{mix2}}$ in the complex plane when $f=0$ in A2

ListPlot[Table[{Re[TABf[[i, 2]]], Im[TABf[[i, 2]]]}, {i, Length[TABf]}], AxesLabel -> {"Re[$\tau_{\text{mix2}}$]", "Im[$\tau_{\text{mix2}}$]"}, PlotLabel -> "Im[$\tau_{\text{mix2}}$] vs. Re[$\tau_{\text{mix2}}$], when $\phi_2 = \phi_1 \frac{1 \pm i \sqrt{3}}{2}\)"

Graph of values of $\tau_{\text{extra}}$ in the complex plane when $f=0$. Should give us $\tau_{\text{extra}} = e^{2\pi i/3}$.

ListPlot[Table[{Re[TABf[[i, 3]]], Im[TABf[[i, 3]]]}, {i, Length[TABf]}], AxesLabel -> {"Re[$\tau_{\text{extra}}$]", "Im[$\tau_{\text{extra}}$]"}, PlotLabel -> "Im[$\tau_{\text{extra}}$] vs. Re[$\tau_{\text{extra}}$], when $\phi_2 = \phi_1 \frac{1 \pm i \sqrt{3}}{2}\), PlotRange -> {{-.6, -.4}, {0.75, 0.95}}, PlotStyle -> PointSize[Large]]
\* \textit{A}₂ \textbf{when} \ \( g = 0 \)

We will solve \( f=0 \) for the \( A² \) case, and use the roots of \( f=0 \) to generate tables with the roots acting as constraining conditions for \( \phi_2 \).

\[
\text{Solve}[ g == 0, \phi_2 ]
\]

\[
\left\{ \phi_2 \rightarrow \frac{2 \phi_1^2 - 2 \sqrt{-u^2 \phi_1 + 2 u^2}}{4 \phi_1}, \phi_2 \rightarrow \frac{2 \phi_1^2 + 2 \sqrt{-u^2 \phi_1 + 2 u^2}}{4 \phi_1} \right\} \]

\( \text{TABg1} = \text{Table}[\{\tau_{\text{mix1}}, \tau_{\text{mix2}}, \tau_{\text{extra}}, u, \phi_1, \phi_2\} /. \{\phi_2 \rightarrow \phi_1 - \frac{u^2}{\phi_1} \}, \{\phi_1 \rightarrow (\text{RandomReal}[\{0.5, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]], u \rightarrow (\text{RandomReal}[\{0.001, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]]\} , \{i, 1, 2000\}] \)

\( \text{TABg2} = \text{Table}[\{\tau_{\text{mix1}}, \tau_{\text{mix2}}, \tau_{\text{extra}}, u, \phi_1, \phi_2\} /. \{\phi_2 \rightarrow \frac{u^2}{\phi_1 - 2}, \phi_1 \rightarrow (\text{RandomReal}[\{0.5, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]], u \rightarrow (\text{RandomReal}[\{0.001, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]]\} , \{i, 1, 2000\}] \)

\( \text{TABgg} = \text{Join}[\text{TABg1}, \text{TABg2}] \)

condition to only keep points which produce \( \tau_{\text{extra}} \) in the fundamental domain:

\( \text{TABg} = \text{Select}[\text{TABgg}, \text{Abs}[\text{Re}[[3]]] <= 0.5 && \text{Abs}[[3]] >= 1 && \text{Im}[[3]] >= 0 && \text{All} \land \text{All}] \)

Graph of values of \( \tau_{\text{mix1}} \) in the complex plane when \( g=0 \) in \( A² \)

\[
\text{ListPlot}[[\text{Re}[\text{TABg}[\{i, 1\}]], \text{Im}[\text{TABg}[\{i, 1\}]], \{i, \text{Length}[\text{TABg}]\}], \text{AxesLabel} \rightarrow \{"\text{Re}[\tau_{\text{mix1}}]", "\text{Im}[\tau_{\text{mix1}}]\}"}, \text{PlotLabel} \rightarrow "\text{Im}[\tau_{\text{mix1}}] \text{ vs.} \ \text{Re}[\tau_{\text{mix1}}] \text{, when} \ \phi_2 = \frac{\phi_1}{2} \left( 1 \pm \frac{u^2}{4 \phi_1} \right)\"], \text{PlotRange} \rightarrow \{\{-1, 1\}, \{-0.03, 0.03\}\}] \)
Graph of values of $\tau_{\text{mix}2}$ in the complex plane when $g=0$ in A2

$\tau_{\text{mix}2}$ vs. $\text{Re}[\tau_{\text{mix}2}]$, when $\phi_2 = \frac{\phi_1}{2}(1 \pm \left(1 - \frac{u^2}{4\phi_1^3}\right))$

Graph of values of $\tau_{\text{extra}}$ in the complex plane when $g=0$. Should give us $\tau_{\text{extra}} = i$.

$\tau_{\text{extra}}$ vs. $\text{Re}[\tau_{\text{extra}}]$, when $\phi_2 = \frac{\phi_1}{2}(1 \pm \left(1 - \frac{u^2}{4\phi_1^3}\right))$
\[ \text{Im}[r_{\text{extra}}] \text{ vs. Re}[r_{\text{extra}}], \text{ when } \phi_2 = \frac{\phi_1}{2} \left(1 \pm \left(1 - \frac{u^2}{4 \phi_1^3}\right)\right) \]
Section A.3:  $D_4$ Flavor Symmetry Group

Prelude:

Information and parameters for $D_4$:

$$m_1 = \phi_1; m_2 = -\phi_1 + \phi_2; m_3 = -\phi_2 + \phi_3 + \phi_4; m_4 = -\phi_3 + \phi_4;$$

$$u_4 = (m_1^2m_2^2 + m_1^2m_3^2 + m_1^2m_4^2 + m_2^2m_3^2 + m_2^2m_4^2 + m_3^2m_4^2);$$

$$u_6 = - (m_1^2m_2^2m_3^2 + m_1^2m_3^2m_4^2 + m_2^2m_3^2m_4^2);$$

$$u_2 = - (m_1^2 + m_2^2 + m_3^2 + m_4^2);$$

$$u_4 = -2lm_1m_2m_3m_4;$$

$$wt_4 = u_4 - 3w_2^2;$$

$$w_4 = u_4;$$

$$w_6 = u_6 + w_2^3 + w_2w_4;$$

$$w_2 = -\frac{1}{3}u_2;$$

$$g = w_2 * u_2^2 + w_4 * u + w_6;$$

$$f = u_2^2 + wt_4;$$

$$f/ \{ \phi_1 \rightarrow t, \phi_2 \rightarrow t, \phi_3 \rightarrow 0, \phi_4 \rightarrow 0 \}$$

$$-t^2/5 + u^2$$

Section: $x_a = m_a^2 - w_2$; $y_a = m_a u + \frac{w_4}{2m_a}$;

$$x_1 = (m_1)^2 - w_2; y_1 = m_1 u + \frac{m_4}{2m_1};$$

$$x_2 = (m_2)^2 - w_2; y_2 = m_2 u + \frac{m_4}{2m_2};$$

$$x_3 = (m_3)^2 - w_2; y_3 = m_3 u + \frac{m_4}{2m_3};$$

$$x_4 = (m_4)^2 - w_2; y_4 = m_4 u + \frac{m_4}{2m_4};$$

$TA1g = Table[\{\{\tau_{extra}, \tau_{mix}, m\}\}/.$

$$\{ u \rightarrow (\text{RandomReal}[[0,0.001,1]]) * \text{Exp}[I * \text{RandomReal}[[0,2\pi]], m \rightarrow 0], \{i, 2000\}]$$

$a$ and $a^D$ for $D_4$:

$$a = \frac{\sqrt{2}}{8\pi} * (u * \text{IONEONE} + \frac{1}{2} * ((m_1 * y_1 * \text{IONETHR} / . c \rightarrow x_1) + (m_2 * y_2 * \text{IONETHR} / . c \rightarrow x_2) +$$

$$(m_3 * y_3 * \text{IONETHR} / . c \rightarrow x_3) + (m_4 * y_4 * \text{IONETHR} / . c \rightarrow x_4)));$

$$aD = \frac{\sqrt{2}}{8\pi} * (u * \text{ITWOONE} + \frac{1}{2} * ((m_1 * y_1 * \text{ITWOHR} / . c \rightarrow x_1) + (m_2 * y_2 * \text{ITWOHR} / . c \rightarrow x_2) +$$

$$(m_3 * y_3 * \text{ITWOHR} / . c \rightarrow x_3) + (m_4 * y_4 * \text{ITWOHR} / . c \rightarrow x_4)));$$

Coupling constants for $D_4$:

$$\tau_{mix1} = D[aD, \phi_1] / \{ \phi_1 \rightarrow t, \phi_2 \rightarrow t, \phi_3 \rightarrow 0, \phi_4 \rightarrow 0 \};$$

$$\tau_{mix2} = D[aD, \phi_2] / \{ \phi_1 \rightarrow t, \phi_2 \rightarrow t, \phi_3 \rightarrow 0, \phi_4 \rightarrow 0 \};$$

$$\tau_{mix3} = D[aD, \phi_3] / \{ \phi_1 \rightarrow t, \phi_2 \rightarrow t, \phi_3 \rightarrow 0, \phi_4 \rightarrow 0 \};$$

$$\tau_{mix4} = D[aD, \phi_4] / \{ \phi_1 \rightarrow t, \phi_2 \rightarrow t, \phi_3 \rightarrow 0, \phi_4 \rightarrow 0 \};$$

Note that with the condition $\{ \phi_1 \rightarrow t, \phi_2 \rightarrow t, \phi_3 \rightarrow 0, \phi_4 \rightarrow 0 \}$, we would have only one independent $\tau_{mix},$

110
and the rest would be duplicates of it with values +/- \( \tau_{mix} \).

\[
\tau_{extra} = D[aD, u]/D[a, u]. \{ \phi_1 \to t, \phi_2 \to t, \phi_3 \to 0, \phi_4 \to 0 \};
\]

- \( D_4 \) case when \( f = 0 \)

\[
\text{Solve}[(f/. \{ \phi_1 \to t, \phi_2 \to t, \phi_3 \to 0, \phi_4 \to 0 \}) == 0, t]
\]

\[
\{ t \to -3^{1/4} \sqrt{u} \}, \{ t \to -i3^{1/4} \sqrt{u} \}, \{ t \to i3^{1/4} \sqrt{u} \}, \{ t \to 3^{1/4} \sqrt{u} \}
\]

\[ f1stbr = \text{Table} \left[ \{ \tau_{extra}, \tau_{mix1}, u, t \} /. \{ t \to -3^{1/4} \sqrt{u} \} \right]. \\
\{ u \to (\text{RandomReal}[\{0.001, 1\}]) \times \text{Exp}[I \times \text{RandomReal}[\{0, 2 \times \pi\}]], \{ i, 1000 \} \}
\]

\[ f2ndbr = \text{Table} \left[ \{ \tau_{extra}, \tau_{mix1}, u, t \} /. \{ t \to -i3^{1/4} \sqrt{u} \} \right]. \\
\{ u \to (\text{RandomReal}[\{0.001, 1\}]) \times \text{Exp}[I \times \text{RandomReal}[\{0, 2 \times \pi\}]], \{ i, 1000 \} \}
\]

\[ f3rdb = \text{Table} \left[ \{ \tau_{extra}, \tau_{mix1}, u, t \} /. \{ t \to i3^{1/4} \sqrt{u} \} \right]. \\
\{ u \to (\text{RandomReal}[\{0.001, 1\}]) \times \text{Exp}[I \times \text{RandomReal}[\{0, 2 \times \pi\}]], \{ i, 1000 \} \}
\]

\[ f4thbr = \text{Table} \left[ \{ \tau_{extra}, \tau_{mix1}, u, t \} /. \{ t \to 3^{1/4} \sqrt{u} \} \right]. \\
\{ u \to (\text{RandomReal}[\{0.001, 1\}]) \times \text{Exp}[I \times \text{RandomReal}[\{0, 2 \times \pi\}]], \{ i, 1000 \} \}
\]

\[ \text{ft1} = \text{Join}[f1stbr, f2ndbr, f3rdb, f4thbr]
\]

\[ \text{smt} = \text{Select}[\text{ft1}, \text{Abs}[	ext{Re}[\#]]<0.5 \&\& \text{Abs}[\#[1]]<1 \&\& \text{Im}[\#[[1]]]>0 \&\& [\text{All}, \text{All}]]
\]

%condition to only keep points which produce \( \tau_{extra} \) in the fundamental domain

\[ \text{Length}[\text{smt}]
\]

1933

\[ \text{fext} = \text{Table}[\{ \text{Re}[\text{smt}[[j, 1]]], \text{Im}[\text{smt}[[j, 1]]] \}, \{ j, \text{Length}[\text{smt}] \}]
\]

\[ \text{fmix} = \text{Table}[\{ \text{Re}[\text{smt}[[j, 2]]], \text{Im}[\text{smt}[[j, 2]]] \}, \{ j, \text{Length}[\text{smt}] \}]
\]

\[ \text{ListPlot}[\text{fext, AxesLabel} \to \{ "\text{Re}[\tau_{extra}]", "\text{Im}[\tau_{extra}]" \}, \text{PlotLabel} \to \{ "\text{Im}[\tau_{extra}]" \text{ vs. Re}[\tau_{extra}] \}, \text{when f=0"}, \text{PlotStyle} \to \text{PointSize}[\text{Large}]]
\]
ListPlot[fmix, AxesLabel -> {"Re[τ_{mix1}]", "Im[τ_{mix1}]"}, PlotLabel -> "Im[τ_{mix1}] vs. Re[τ_{mix1}]", when f=0", PlotStyle -> PointSize[Medium]]

- $D_4$ case when $g = 0$

$g/. \{\phi_1 \to t, \phi_2 \to t, \phi_3 \to 0, \phi_4 \to 0\}$

$\frac{2t^6}{3\pi} + 2t^2 u^2$

Solve[(g/. \{\phi_1 \to t, \phi_2 \to t, \phi_3 \to 0, \phi_4 \to 0\}) == 0, t]

$\{t \to 0\}$, $\{t \to -(-1)^{1/4}\sqrt{3} \sqrt{u}\}$, $\{t \to -(-1)^{1/4}\sqrt{3} \sqrt{u}\}$, $\{t \to -(-1)^{3/4}\sqrt{3} \sqrt{u}\}$

g1stbr = Table[\{τ_{extra}, τ_{mix1}, u, t\} /. \{t \to -(-1)^{1/4}\sqrt{3} \sqrt{u}\} /.

$\{u \to (\text{RandomReal}[\{0.001, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]], \{i, 1000\}]$

g2ndbr = Table[\{τ_{extra}, τ_{mix1}, u, t\} /. \{t \to -(-1)^{1/4}\sqrt{3} \sqrt{u}\} /.

$\{u \to (\text{RandomReal}[\{0.001, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]], \{i, 1000\}]$

g3rdbr = Table[\{τ_{extra}, τ_{mix1}, u, t\} /. \{t \to -(-1)^{3/4}\sqrt{3} \sqrt{u}\} /.

$\{u \to (\text{RandomReal}[\{0.001, 1\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}]], \{i, 1000\}]$
\[u \rightarrow (\text{RandomReal}([0.001, 1])) \times \text{Exp}[I \times \text{RandomReal}([0, 2 \times \text{Pi}])], \{i, 1000\}\]

\[g4thbr = \text{Table} \left[ \{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t \} /./ \{t \rightarrow (-1)^{3/4} \sqrt{3} \sqrt{u}\} /.
\right.

\[\{u \rightarrow (\text{RandomReal}([0.001, 1])) \times \text{Exp}[I \times \text{RandomReal}([0, 2 \times \text{Pi}])], \{i, 1000\}\]

\[g2 = \text{Join}[g1stbr, g2ndbr, g3rdbr, g4thbr] \]

\[smtg = \text{Select}[g2, \text{Abs}[	ext{Re}[\#][[1]]] <= 0.5 \& \& \text{Abs}[	ext{Im}[\#][[1]]] >= 1 \& \& \text{Im}[[\#][[1]]] >= 0\\] \[\{\text{All, All}\}\]

%condition to only keep points which produce \(\tau_{\text{extra}}\) in the fundamental domain

\[\text{Length}[smtg]\]

1384

\[gextra = \text{Table}[[\text{Re}[\text{smtg}[[j, 1]]], \text{Im}[\text{smtg}[[j, 1]]]], \{j, \text{Length}[\text{smtg}]\}]; \]

\[gmix = \text{Table}[[\text{Re}[\text{smtg}[[j, 2]]], \text{Im}[\text{smtg}[[j, 2]]]], \{j, \text{Length}[\text{smtg}]\}]; \]

\[
\text{ListPlot} [\text{gextra}, \text{AxesLabel} \rightarrow \{"Re[\tau_{\text{extra}}]", "Im[\tau_{\text{extra}}]"\}, \text{PlotLabel} \rightarrow "\text{Im}[\tau_{\text{extra}}] \text{ vs. Re}[\tau_{\text{extra}}], \text{when } g=0", \text{PlotStyle} \rightarrow \text{PointSize}[\text{Large}], \text{PlotRange} \rightarrow \{\{-0.5, 0.5\}, \{0, 2\}\}]
\]

\[
\text{ListPlot} [\text{gmix}, \text{AxesLabel} \rightarrow \{"Re[\tau_{\text{mix1}}]", "Im[\tau_{\text{mix1}}]"\}, \text{PlotLabel} \rightarrow "\text{Im}[\tau_{\text{mix1}}] \text{ vs. Re}[\tau_{\text{mix1}}], \text{when } g=0", \text{PlotStyle} \rightarrow \text{PointSize}[\text{Medium}], \text{PlotRange} \rightarrow \{\{-0.7, 0.7\}, \text{Automatic}\}]
\]
\[ D_4 \text{ case when } j \approx \infty \]

\[
(27g^2 + 4f^3)/.\{\phi_1 \to t, \phi_2 \to t, \phi_3 \to 0, \phi_4 \to 0\} //\text{FullSimplify}
\]

\[
4u^2 (t^4 + u^2)^2
\]

\[ \text{eps} = 0.001; \]

\[
\text{Solve}\left((27g^2 + 4f^3)/.\{\phi_1 \to t, \phi_2 \to t, \phi_3 \to 0, \phi_4 \to 0\}\right) == \text{eps, } t\]
\[
\left\{\begin{aligned}
& t \to -0.265915 \left(\frac{-\sqrt[4]{1.6228}}{u} - 200. u^2\right), \\
& t \to 0.265915 \left(\frac{-\sqrt[4]{1.6228}}{u} - 200. u^2\right), \\
& t \to (0. - 1.i) \left(\frac{\sqrt[4]{0.0158114}}{u} - 1. u^2\right), \\
& t \to (0. + 1.i) \left(\frac{\sqrt[4]{0.0158114}}{u} - 1. u^2\right), \\
& t \to \left(\frac{\sqrt[4]{0.0158114}}{u} - 1. u^2\right),
\end{aligned}\right.
\]

\[
d1stbr = \text{Table}\left[\left\{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t\right\}/.\{t \to -"0.265915" \left(\frac{\sqrt[4]{1.6228}}{u} - "200." u^2\right)\}\right] /.\}
\]

\[
\{u \to (\text{RandomReal}[\{0.001, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \Pi\}]], \{i, 500\}\}
\]

\[
d2ndbr = \text{Table}\left[\left\{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t\right\}/.\{t \to ("0." - "0.265915" i) \left(\frac{\sqrt[4]{1.6228}}{u} - "200." u^2\right)\}\right] /.\}
\]

\[
\{u \to (\text{RandomReal}[\{0.001, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \Pi\}]], \{i, 500\}\}
\]

\[
d3rdbr = \text{Table}\left[\left\{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t\right\}/.\{t \to ("0." + "0.265915" i) \left(\frac{\sqrt[4]{1.6228}}{u} - "200." u^2\right)\}\right] /.\}
\]

\[
\{u \to (\text{RandomReal}[\{0.001, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \Pi\}]], \{i, 500\}\}
\]

\[
d4thbr = \text{Table}\left[\left\{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t\right\}/.\{t \to "0.265915" \left(\frac{\sqrt[4]{1.6228}}{u} - "200." u^2\right)\}\right] /.\}
\]

\[
\{u \to (\text{RandomReal}[\{0.001, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \Pi\}]], \{i, 500\}\}
\]

\[
d5thbr = \text{Table}\left[\left\{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t\right\}/.\{t \to "-1." \left(\frac{\sqrt[4]{0.0158114}}{u} - "1." u^2\right)\}\right] /.\}
\]

\[
\{u \to (\text{RandomReal}[\{0.001, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \Pi\}]], \{i, 500\}\}
\]

\[
d6thbr = \text{Table}\left[\left\{\tau_{\text{extra}}, \tau_{\text{mix1}}, u, t\right\}/.\{t \to ("0." - "1." i) \left(\frac{\sqrt[4]{0.0158114}}{u} - "1." u^2\right)\}\right] /.\}
\]

\[
\{u \to (\text{RandomReal}[\{0.001, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \Pi\}]], \{i, 500\}\}
d7thbr = Table[{t → ("0.0158114" - "1."u^2)^(1/4) /. 
{u → (RandomReal[{0.001, 1}]) * Exp[I * RandomReal[{0, 2 * Pi}]]}, {i, 500}]

d8thbr = Table[{t → ("0.0158114" - "1."u^2)^(1/4) /. 
{u → (RandomReal[{0.001, 1}]) * Exp[I * RandomReal[{0, 2 * Pi}]]}, {i, 500}]

fulltab = Join[d1stbr, d2ndbr, d3rdb, d4thbr, d5thbr, d6thbr, d7thbr, d8thbr]

selt = Select[fulltab, Abs[Re[#1[[1]]]] <= 0.5 && Abs[#1[[1]]] >= 1 && Im[#1[[1]]] >= 0 &][[All, All]]

% condition to only keep points which produce \( \tau_{\text{extra}} \) in the fundamental domain

cap = 100;

sel = Select[selt, Abs[KleinInvariantJ[First[#]]] > cap &][[All, All]]

dext = Table[{Re[sel[[j, 1]]], Im[sel[[j, 1]]]}, {j, Length[sel]}];
dmix = Table[{Re[sel[[j, 2]]], Im[sel[[j, 2]]]}, {j, Length[sel]}];

ListPlot[{dext, AxesLabel \[Rule] {"Re[\( \tau_{\text{extra}} \)]", "Im[\( \tau_{\text{extra}} \)]"},
PlotLabel \[Rule] "Im[\( \tau_{\text{extra}} \)] vs. Re[\( \tau_{\text{extra}} \)], when 4 \( f^3 + 27 b^2 = 0 \), PlotRange \[Rule] \{-5, 5\}, {0, 3}\}]

ListPlot[dmix, AxesLabel \[Rule] {"Re[\( \tau_{\text{mix1}} \)]", "Im[\( \tau_{\text{mix1}} \)]"},
PlotLabel \[Rule] "Im[\( \tau_{\text{mix1}} \)] vs. Re[\( \tau_{\text{mix1}} \)], when 4 \( f^3 + 27 b^2 = 0 \), PlotRange \[Rule] Automatic]
• \( D_4 \) case with free parameters

This is the case where both Coulomb branch parameter \( u \) and the mass parameter \( \varphi \) are free to take random values in relative ranges.

\[
tab = \text{Table}[\{\tau_{\text{extra}}, \tau_{\text{mix1}}, \text{adt}/N, \text{at}/N, (\text{adt}/\text{at}), u, t\} / .
\]
\[
\{u \rightarrow (\text{RandomReal}[\{1, 100\}]) \cdot \text{Exp}[I \cdot \text{RandomReal}[\{0, 2 \cdot \pi\}]],
\]
\[
t \rightarrow (\text{RandomReal}[\{1, 10\}]) \cdot \text{Exp}[I \cdot \text{RandomReal}[\{0, 2 \cdot \pi\}]]\}, \{i, 4000\}\]
\]
\[
\text{sel = Select[tab, Abs[Re[#[[1]]]] < 0.5 && Abs[#[[1]]] > 1 && Abs[#[[1]]] > 0] & \{\text{All, All}\}}
\]
% the condition to only keep points which produce \( \tau_{\text{extra}} \) in the fundamental domain.

\[
\text{ListPlot[Table[\{Re[sel[[i, 1]]], Im[sel[[i, 1]]]\}, \{i, \text{Length[sel]}\}], \text{AxesLabel} \rightarrow \{"Re[\tau_{\text{extra}}]", "Im[\tau_{\text{extra}}]\},}
\]
\[
\text{PlotLabel} \rightarrow \text{"Im[\tau_{\text{extra}}] vs. Re[\tau_{\text{extra}}], when \( \phi \) is free"}]
\]

\[
\text{ListPlot[Table[\{Re[sel[[i, 2]]], Im[sel[[i, 2]]]\}, \{i, \text{Length[sel]}\}], \text{AxesLabel} \rightarrow \{"Re[\tau_{\text{mix1}}]", "Im[\tau_{\text{mix1}}]\},}
\]
\[
\text{PlotLabel} \rightarrow \text{"Im[\tau_{\text{mix1}}] vs. Re[\tau_{\text{mix1}}], when \( \phi \) is free"}]
\]
$\text{Im}[\tau_{\text{mix1}}]$ vs. $\text{Re}[\tau_{\text{mix1}}]$, when $\phi$ is free
APPENDIX B: Code for Effective Potential Graphs

All the codes brought in this Appendix chapter would also share the same starting lines, since they are all performed for $A_1$ symmetry group case. We will name them $A_1$ Essentials.

$A_1$ Essentials:

The I-integrals and necessary variables:

$$
k = \text{Sqrt} \left[ \frac{r_2-r_3}{r_1-r_2} \right];
$$

$$
ksqu = \frac{r_2-r_3}{r_1-r_2};
$$

$$
kp = \text{Sqrt} \left[ \frac{r_3-r_1}{r_3-r_1} \right];
$$

$$
kpsqu = \frac{r_2-r_3}{r_3-r_1};
$$

$$
ctilde = \frac{r_2-r_3}{r_1-r_3};
$$

$$
u = - \left( \frac{1-ctilde+kp}{1-ctilde-kp} \right)^2 \left( \frac{1-kp}{1+kp} \right)^2;
$$

$$
IONEONE = \frac{4}{(r_1-r_3)^7} \cdot \text{EllipticK}[ksqu];
$$

$$
IONETHR = \frac{4}{(r_1-r_3)^7} \cdot \left( \frac{\text{EllipticK}[ksqu]}{1-ctilde+kp} + \left( \frac{4kp}{1+kp} \right) \cdot \frac{\text{EllipticPi}\left[\text{-nu},(\frac{1+kp}{1-kp})^2\right]}{1-ctilde-kp} \right);
$$

$$
ITWOONE = IONEONE/. \{ r_1 \rightarrow r_3, r_3 \rightarrow r_1 \};
$$

$$
ITWOTHR = IONETHR/. \{ r_1 \rightarrow r_3, r_3 \rightarrow r_1 \};
$$

The roots of general cubic:

$$
\text{SOLVE} = \text{Solve} \left[ x^3 + f \cdot x + g \text{==} 0, x \right];
$$

$$
\begin{pmatrix}
  r_1 \\
  r_2 \\
  r_3 
\end{pmatrix} = \begin{pmatrix}
  x/.\text{SOLVE}[[2]] \\
  x/.\text{SOLVE}[[1]] \\
  x/.\text{SOLVE}[[3]]
\end{pmatrix};
$$

sanity check on roots:

$$
r_3 + r_2 + r_1/.\text{FullSimplify}
$$

0

$$
f = u;
$$

$$
g = 4m^2;
$$

'\alpha' and $a^D$ for the case of $A_1$

$$
a = \frac{\sqrt{2}}{4\pi} \cdot \left( \frac{\pi}{3} \cdot \text{IONEONE} + \left( \frac{\pi}{3} \cdot \text{IONETHR/.} \{ c \rightarrow 0 \} \right) \right);
$$

$$
aD = \frac{\sqrt{2}}{4\pi} \cdot \left( \frac{\pi}{3} \cdot \text{ITWOONE} + \left( \frac{\pi}{3} \cdot \text{ITWOOTH/.} \{ c \rightarrow 0 \} \right) \right);
$$

The $A_1$ coupling constants and other useful variables:

$$
\tau_{12} = D[aD, m];
$$

$$
\tau_{22} = D[aD, u]/D[a, u];
$$

$$
F222 = D[\tau_{22}, u]/D[a, u];
$$

$$
F122 = D[\tau_{12}, u]/D[a, u];
$$

118
F121 = \( D[\tau_{12}, m] \);
F2222 = \( D[F222, u]/D[a, u] \);
F1212 = \( D[F121, u]/D[a, u] \);
F1222 = \( D[F122, u]/D[a, u] \);

\[ M = \frac{F121}{\sqrt{137 + \text{Im}[\tau_{22}]}} \}
\]
\[ \{m \to 1\} \]
\[ \text{eps} = \frac{F121}{\sqrt{137 + \text{Im}[\tau_{22}]}} \}
\]
\[ \{m \to 1\} \]
\[ M = \frac{F122}{\text{Im}[\tau_{22}]} \}
\]
\[ \{m \to 1\} \]
\[ \text{tn1} = \text{Table} \{(u, \text{eps}, M, \tau_{12}, \tau_{22}) \}
\]
\[ \{m \to 1, u \to (\text{RandomReal}[\{0, 10\}) * \text{Exp}[I * \text{RandomReal}[\{0, 2 * \text{Pi}\}])]/N, \{i, 2000\} \]

condition to only keep points which produce \( \tau_{\text{extra}} \) in the fundamental domain

\[ \text{tph} = \text{Select}[\text{tn1}, \text{Abs}[\text{Re}[\#[[5]]]] < 0.5 && \text{Abs}[\#[[5]]] >= 1 && \text{Abs}[\#[[5]]] >= 0 && \text{All} \{\text{All} \}; \text{tn1eps} = \text{Table} \{\text{Abs}[\text{tph}[\{j, 1\}]], \text{Abs}[\text{tph}[\{j, 2\}]], \{j, \text{Length}[\text{tph}]\}] \}; \text{tn1M} = \text{Table} \{\text{Abs}[\text{tph}[\{j, 1\}], \text{Abs}[\text{tph}[\{j, 3\}]], \{j, \text{Length}[\text{tph}]\}] \}; \]
\[ \text{ListPlot} \{(\text{tn1eps}, \text{tn1M}), \text{PlotStyle} \to \{\text{Default}, \text{Purple}\}, \text{AxesLabel} \to \{"|u|", \}, \]
\[ \text{PlotLabel} \to \"graph of |\epsilon_{\text{mix}}| and |M_{\text{extra}}| vs. |u|", \]
\[ \text{PlotLegends} \to \text{Placed} \{\text{SwatchLegend} \{"|\epsilon_{\text{mix}}|", "|M_{\text{extra}}|"\}, \{\text{Right}, \text{Center}\}] \]

Section B.2: Effective Potential Derivative with no Superpotential

\[ |\epsilon_{\text{mix}}| \]
\[ |M_{\text{extra}}| \]
A1 Essentials:

DerV = \left( \frac{1}{2\pi} * (\text{Im}[\tau_{12}])^2 \right) \cdot ((\text{Im}[\tau_{12}])^2 \cdot F222 + (\text{Im}[\tau_{22}])^2 \cdot F121 - 2 \cdot \text{Im}[\tau_{12}] \cdot \text{Im}[\tau_{22}] \cdot F122) / \{m \to 1\};

Vtab = Table[{DerV, u, \tau_{22}} /. 

\{m \to 1, u \to (\text{RandomReal}[\{0, 10\}]) * \text{Exp}[I * \text{RandomReal}[\{0, 2\pi\}]]/N, \{i, 5000}\}

condition to only keep points which produce \(\tau_{\text{extra}}\) in the fundamental domain

TAVB = Select[Vtab, Abs[Re[#]][3]] <= 0.5 && Abs[#][3] >= 1 && Abs[#][3] > 0 \&\& [[All, All]] 

ListPlot[Table[{Abs[TAVB[[j, 2]]], Abs[TAVB[[j, 1]]]}, \{j, Length[TAVB]\}], AxesLabel \to \{"|u|", "|\partial V/\partial a|"\}],

PlotLabel \to "Graph of |\partial V/\partial a| vs. |u|"

\[|\partial V/\partial a| \text{ vs. } |u|\]

\[t = Table[{Abs[TAVB[[j, 2]]], Abs[TAVB[[j, 1]]] * Abs[D[u, u]]}/.u \to TAVB[[j, 2]], m \to 1, \{j, \text{Length}[TAVB]\}]

ListPlot[t, AxesLabel \to \{"|u|", "|\partial V/\partial u|"\}], PlotLabel \to "Graph of |\partial V/\partial u| vs. |u|"

\[|\partial V/\partial u| \text{ vs. } |u|\]

Select[TAVB, Abs[#][1]] < 0.0001 \&\& [[All, All]]

\{3.573855892875055^8-6 + 0.0000048978i, 0.00160959 + 0.0021676i, 0.496509 + 0.872561i\}

ustar = "0.00160959" + "0.0021676"i;
\[ \text{DDerVaa} = \frac{1}{2(\text{Im}[\tau_{22}])^2} \ast ((\text{Im}[\tau_{12}] \ast F222 - \text{Im}[\tau_{22}] \ast F122)^2 + I \ast ((\text{Im}[\tau_{12}])^2 \ast \text{Im}[\tau_{22}] \ast F2222 + (\text{Im}[\tau_{22}])^3 \ast F1212 - 2 \ast \text{Im}[\tau_{12}] \ast (\text{Im}[\tau_{22}])^2 \ast F1222)) / \{m \to 1, u \to u\ast\} \]

\[ \text{DDerVab} = \frac{1}{2(\text{Im}[\tau_{22}])^2} \ast (\text{Abs}[\text{Im}[\tau_{12}] \ast F222 - \text{Im}[\tau_{22}] \ast F122])^2 / \{m \to 1, u \to u\ast\} \]

\[ \{2 \ast (\text{Re}[\text{DDerVaa}] + \text{DDerVab}), -2 \ast \text{Im}[\text{DDerVaa}]\}, \{ -2 \ast \text{Im}[\text{DDerVaa}], 2 \ast (\text{Re}[\text{DDerVaa}] + \text{DDerVab})\} \]

\[
\begin{pmatrix}
-0.210312 & -0.111189 \\
-0.111189 & 0.460089
\end{pmatrix}
\]

Eigenvalues[\%]

\{0.478049, -0.228272\}

Section B.3: Effective Potential Contour plots with no Superpotential

A1 Essentials;

\[ V = \frac{\text{Im}[\tau_{22}]}{\text{Im}[\tau_{22}]} \ast \text{Abs}[\text{Re}[\text{DDerVaa}]] / \{m \to 1\} \]

\[ \text{amod} = a / \{m \to 1\} \]

\[ \text{TAB} = \text{Table} \{\{V, u, \text{amod}, \tau_{22}\} / \{m \to 1, u \to (\text{RandomReal}[\{0, 1\}]) \ast \text{Exp}[I \ast \text{RandomReal}[\{0, 2 \ast \pi\}]]\} / \text{N}, \{i, 15000\}\}
\]

condition to only keep points which produce \(\tau_{\text{extra}}\) in the fundamental domain

\[ \text{TABP} = \text{Select}[\text{TAB}, \text{Abs}[\text{Re}[\text{Tab}[\{4\}]]] < 0.5 \&\& \text{Abs}[\text{Tab}[\{4\}]] > 1 \&\& \text{Im}[\text{Tab}[\{4\}]] > 0 \&\& \{\text{All}, \text{All}\} \]

\[ \text{Creim} = \text{Table}[\{\text{Re}[\text{TABP}[\{j, 2\}], \text{Im}[\text{TABP}[\{j, 2\}], \text{Abs}[\text{TABP}[\{j, 1\}] - 137\}, \{j, \text{Length}[\text{TABP}]\}]\]

\[ \text{ListContourPlot}[\text{Creim}, \text{PlotLegends} \to \text{Automatic}, \text{Contours} \to 10, \text{FrameLabel} \to \{\text{"Re[u"]}, \text{"Im[u"]}\}, \text{LabelStyle} \to \text{Directive[Bold, Medium]}, \text{ColorFunction} \to \text{"BlueGreenYellow"}] \]
Section B.4: Effective Potential Contour plots with Superpotential

**A1 Essentials;**

\[ V = \frac{(\text{Im}[\tau_{22}])^2}{\text{Im}[\tau_{22}]} \cdot \{m \to 1\} \]

\[ \text{amod} = a/\{m \to 1\}; \]

A table of 10000 points:

\[ \text{alltab} = \text{Table}\{\{V, u, \text{Im}[\tau_{22}], \text{amod}, P\} / \{m \to 1, b \to -4\} / \{u \to x + I \cdot y\} / N, \]
\[ \{x, 0.5, 1.5, 0.01\}, \{y, -0.5, 0.5, 0.01\}\]

Graph of \(|V|\) vs. \(|u|\)

\[ \text{flattab} = \text{Flatten}[\text{alltab}, 1]; \]

condition to only keep points which produce \(\tau_{\text{extra}}\) in the fundamental domain

\[ \text{tab} = \text{Select}[\text{flattab}, \text{Abs}[\text{Re}[\#][3]]] < 0.5 && \text{Abs}[\#][3] > 1 && \text{Im}[\#][3] > 0 ] [[\text{All}, \text{All}]] \]

\[ \text{VvsU} = \text{Table}\{\{\text{Re}[\text{tab}[i]][2], \text{Im}[\text{tab}[i]][2], \text{tab}[i][1]\}, \{i, \text{Length}[\text{tab}]\}\]

\[ \text{ListContourPlot}[\text{VvsU}, \text{Contours} \to 50, \text{PlotLegends} \to \text{Automatic}, \text{FrameLabel} \to \{\text{“Re}[u]”, \text{“Im}[u]”\}] \]
Graph of Derivative of $V$

Derivatives with respect to real and imaginary parts of $u$ evaluated separately.

$$\text{delV} = \text{Table}\left[\left\{\text{alltab}[i][j][2], \text{alltab}[i][j][1] - \text{alltab}[i][j][2]\right\}, \{j, \text{Length[alltab]}[1] - 1\}, \{i, \text{Length[alltab]}\}\right]$$

$$\text{delV2} = \text{Table}\left[\left\{\text{alltab}[i][j][2], \frac{\text{delV}[i][j][2] + \text{delV2}[i][j][2]}{2}\right\}, \{j, \text{Length[alltab]}[1]\}, \{i, \text{Length[alltab]} - 1\}\right]$$

$$\text{dV} = \text{Flatten}\left[\text{Table}\left[\left\{\text{alltab}[i][j][2], \frac{\text{delV}[i][j][2] + \text{delV2}[i][j][2]}{2}\right\}, \{j, \text{Length[alltab]}[1] - 1\}, \{i, \text{Length[alltab]} - 1\}\right], 1\right]$$

$$\text{ListContourPlot[Table[\{\text{Re}[dV[i][1]], \text{Im}[dV[i][1]], \text{Abs}[dV[i][2]]\}, \{i, \text{Length[dV]}\}], \text{Contours} \to 20, \text{PlotLegends} \to \text{Automatic}, \text{FrameLabel} \to \{\text{"Re[u"]}, \text{"Im[u"]}\}, \text{ColorFunction} \to \text{"BlueGreenYellow"}]$$
APPENDIX C: Code for Calculating Mass-Squared Values

This appendix chapter contains two sections. Majority of the code needed for these two sections is identical and the results are based on the same data; only the presentation way differs. Therefore, in this chapter, we will import the majority of the code here, and only include the specialized parts in the separate sections.

A1 Essentials:

\[ \text{amod} = a/\{m \to 1\}/\{u \to x + I \times y\}; \]
\[ P0 = \kappa (u - m^{4/3})^2/\{m \to 1\}/\{u \to x + I \times y\}; \]
\[ P = \text{Refine}[P0, \kappa \in \text{Reals}]; \]
\[ \text{derP} = D[P,x] - I \times D[P,y]; \]
\[ \text{dera} = D[\text{amod},x] - I \times D[\text{amod},y]; \]
\[ \text{dPda} = \frac{\text{derP}}{\text{dera}}. \]

For bosonic potential \( V_{\text{boson}} = V0 + F1 \times V1. \)

\[ V0 = \left( \frac{1}{\text{Im}[\text{D}]} \right) \times (\text{Abs}[\text{dPda}]^2)/\{m \to 1\}/\{u \to x + I \times y\}; \]
\[ V1 = \left( 2 \times \frac{\text{Im}[\text{D}]}{\text{Im}[\text{D}]} \right) \times \text{Re}[\text{dPda}]/\{m \to 1\}/\{u \to x + I \times y\}; \]
\[ V = V0 + F1 \times V1; \]

check
\[ V/\{x \to 1, y \to 0\}///N \]

0.

The Original table:

\[ \text{VF} = \text{Table}[\{\{V0, V1, x, y\}/\{m \to 1\}/\{\text{Abs}[\kappa]^2 \to \kappa^2\} ///N, \{x, 0.8, 1.2, 0.01\}, \{y, -0.2, 0.2, 0.01\}] \]

\[ \text{Re}[u]: \]
\[ y0 = \text{Flatten}[\text{Table}[\text{Select}[\text{VF}[\text{i}], \#[[4]] == 0\&], \{\text{i}, \text{Length}[\text{VF}]\}]]; \]
\[ \text{DV0x} = \text{Table}[\{y0[[\text{\text{i}}][[4]]] - y0[[\text{\text{j}}][[4]]], y0[[\text{\text{j}}]][[3]]\}, \{\text{j}, \text{Length}[y0] - 1\}]; \]
\[ \text{DV0} = \text{Select}[\text{DV0x}, \#[[2]] == 1\&][[1, 1]]///\text{Simplify} \]

0. + 3.93793\( \kappa^2 \)
\[ \text{D2V0x} = \text{Table}[\{\text{D2V0x}[[\text{j}][[1]]] - \text{D2V0x}[[\text{j}][[1]]], \text{D2V0x}[[\text{j}]][[2]]\}, \{\text{j}, \text{Length}[\text{DV0x}] - 1\}]; \]
\[ \text{D2V0} = \text{Select}[\text{D2V0x}, \#[[2]] == 1\&][[1, 1]]///\text{Simplify} \]

0. + 787.703\( \kappa^2 \)
\[ \text{D3V0x} = \text{Table}[\{\text{D3V0x}[[\text{j}][[1]]] - \text{D3V0x}[[\text{j}][[1]]], \text{D3V0x}[[\text{j}]][[2]]\}, \{\text{j}, \text{Length}[\text{D2V0x}] - 1\}]; \]
\[ \text{D3V0} = \text{Select}[\text{D3V0x}, \#[[2]] == 1\&][[1, 1]]///\text{Simplify} \]

0. + 18.041\( \kappa^2 \)
D4V0x = Table \[ \{ \text{D4V0}[i+1][3][1]-D3V0x[i][3][1], D3V0x[i][j][2] \} \}/\text{Simplify}, \{ j, \text{Length}[D3V0x]-1 \} \]

D4V0 = Select[D4V0x, #[[2]] == 1\&\&[[1, 1]]/\text{Simplify}

0. + 53.9479 \kappa^2

D6V1x = Table \[ \{ \text{D6V1}[i+1][2]-y0[i][j][2], y0[i][j][3] \}, \{ j, \text{Length}[y0]-1 \} \]

DV1 = Select[DV1x, #[[2]] == 1\&\&[[1, 1]]/\text{Simplify}

0. - 0.629128 \kappa

D2V1x = Table \[ \{ \text{D2V1}[i+1][2]-D1V1x[i][2], D1V1x[i][j][2] \}, \{ j, \text{Length}[D1V1x]-1 \} \]

D2V1 = Select[D2V1x, #[[2]] == 1\&\&[[1, 1]]/\text{Simplify}

0. - 1.15016 \kappa

D3V1x = Table \[ \{ \text{D3V1}[i+1][2]-D2V1x[i][2], D2V1x[i][j][2] \} \}/\text{Simplify}, \{ j, \text{Length}[D2V1x]-1 \} \]

D3V1 = Select[D3V1x, #[[2]] == 1\&\&[[1, 1]]/\text{Simplify}

0. + 0.307118 \kappa

dx = \(- \epsilon * \text{D4V0}/\text{D2V0}\) /\text{N}/\text{FullSimplify}

0. + 0.00079808 \epsilon \kappa

D2V = D2V0 + \epsilon * D2V1 + dx * D3V0 + \epsilon * dx * D3V1 + \frac{1}{2} * (dx)^2 * D4V0;

D2V /\text{N}/\text{FullSimplify}

0. + 0.000262498 \epsilon^2 - 1.13575 \epsilon \kappa + 787.703 \kappa^2

\text{Im}[u]:

x1 = Flatten[Table[Select[VF[[i]], #[[3]] == 1\&\&, \{ i, \text{Length}[VF] \}], 1]

dV0y = Table \[ \{ x1[i+1][2]-x1[i][2], x1[i][4] \}, \{ j, \text{Length}[x1]-1 \} \]

DV0 = Select[dV0y, #[[2]] == 0\&\&[[1, 1]]/\text{Simplify}

0. + 3.93664 \kappa^2

d2V0y = Table \[ \{ dV0y[i+1][2]-dV0y[i][2], dV0y[i][j][2] \}, \{ j, \text{Length}[dV0y]-1 \} \]

d2V0 = Select[d2V0y, #[[2]] == 0\&\&[[1, 1]]/\text{Simplify}

0. + 787.527 \kappa

D3V0y = Table \[ \{ dV0y[i+1][2]-dV0y[i][2], dV0y[i][j][2] \} \}/\text{Simplify}, \{ j, \text{Length}[dV0y]-1 \} \]

d3V0 = Select[d3V0y, #[[2]] == 0\&\&[[1, 1]]/\text{Simplify}

0. - 0.324968 \kappa^2

d4V0y = Table \[ \{ dV0y[i+1][2]-dV0y[i][2], dV0y[i][3][2] \}, \{ j, \text{Length}[dV0y]-1 \} \]

d4V0 = Select[d4V0y, #[[2]] == 0\&\&[[1, 1]]/\text{Simplify}

0. - 21.6675 \kappa^2
\[
dV1y = \text{Table} \left\{ \frac{x1[[j+1][4]]-x1[[j][4]]}{x1[[j]][4]}, x1[[j]][4] \right\}, \{j, \text{Length}[x1] - 1\}
\]
\[
dV1 = \text{Select}[dV1y, \#[[2]] == 0\&\&[[1, 1]]/\text{Simplify}
\]
0. \(-0.988047\kappa\)
\[
d2V1y = \text{Table} \left\{ \{\frac{dV1y[[j+1][4]]-dV1y[[j][4]]}{0.01}, dV1y[[j]][4]\}, \{j, \text{Length}[dV1y] - 1\}\right\}, \{j, \text{Length}[dV1y] - 1\}\)
\[
d2V1 = \text{Select}[d2V1y, \#[[2]] == 0\&\&[[1, 1]]/\text{Simplify}
\]
0. \(+1.47735\kappa\)
\[
d3V1y = \text{Table} \left\{ \{\frac{d2V1y[[j+1][4]]-d2V1y[[j][4]]}{0.01}, d2V1y[[j]][4]\}, \{j, \text{Length}[d2V1y] - 1\}\right\}, \{j, \text{Length}[d2V1y] - 1\}\)
\[
d3V1 = \text{Select}[d3V1y, \#[[2]] == 0\&\&[[1, 1]]/\text{Simplify}
\]
0. \(-0.627316\kappa\)
\[
dy = \left( -\epsilon \cdot \frac{dV1}{d\epsilon}\right) /N//\text{FullSimplify}
\]
0. \(+0.00125462\epsilon\)
\[
d2V = d2V0 + \epsilon \cdot d2V1 + dy \cdot d3V0 + \epsilon \cdot dy \cdot d3V1 + \frac{1}{2} \cdot (dy)^2 \cdot d4V0;
\]
\[
d2V//N//\text{FullSimplify}
\]
0. \(-0.000804097\epsilon^2 + 1.47694\epsilon\kappa + 787.527\kappa^2\)

Section C.1: Bosonic and Fermionic Mass-Squared Values, at the Supersymmetric Limit

\[
\text{Series}[(1/\text{dera})/\{x \to 1, y \to 0\}]/N, \{\epsilon, 0, 2\}]
\]
\[-4.99462 + 7.97591i\]
\[
ua = \text{"-4.99462" + "7.97591"i};
\]
\[
uad = \text{"-4.99462" - "7.97591"i};
\]
\[
\text{Series}[\tau22/\{m \to 1\}/\{u \to x + I \cdot y\}/\{x \to 1, y \to 0\}]/N, \{\epsilon, 0, 2\}]
\]
\[0.436634 + 0.899639i\]
\[
\text{Int22} = 0.899639;
\]
\[
g22 = \text{Series} \left[ \frac{1}{\text{Int22}}, \{\epsilon, 0, 2\} \right]
\]
\[1.11156\]
\[
\text{Series}[\tau12/\{m \to 1\}/\{u \to x + I \cdot y\}/\{x \to 1, y \to 0\}]/N, \{\epsilon, 0, 2\}]
\]
\[-0.491275 + 0.0280705i\]
\[
\text{Int12} = \text{"0.0280705"};
\]
\[
f22 = \text{Series}[F222/\{m \to 1\}/\{u \to x + I \cdot y\}/\{x \to 1, y \to 0\}]/N, \{\epsilon, 0, 2\}]
\]
\[0.0735233 - 0.680554i\]
\[
f12 = \text{Series}[F122/\{m \to 1\}/\{u \to x + I \cdot y\}/\{x \to 1, y \to 0\}]/N, \{\epsilon, 0, 2\}]
\]
-0.0895189 - 0.295503i 
\[ pt = f122 - \text{Int12} \cdot g22 \cdot f22 \] 
-0.091813 - 0.274268i

**THE BOSONIC MASS**

\[ mA = \frac{1}{2} \cdot g22 \cdot (D2V + d2V) \cdot u a \cdot uad; \]
\[ \text{Series}[mA, \{\epsilon, 0, 2\}] \text{Simplify} \]
38766.8\(\kappa^2\)

**THE FERMIONIC MASS**

\[ mX0 = g22 \cdot 2\kappa \cdot (ua)^2 \]
(-85.9654 - 177.123i)\(\kappa\)
\[ \text{Series}[(-"85.9654" - "177.123i\kappa) \ast ((-"85.9654" + "177.123i\kappa), \{\epsilon, 0, 2\})] \text{Simplify} \]
(3876.5 + 0.\kappa)^2

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Section C.2: Bosonic and Fermionic Mass-Squared Values, with SUSY Breaking Turned On

\[ \text{Series}[(1/\text{dera}) / \{x \rightarrow 1 + dx, y \rightarrow dy\} / N, \{\epsilon, 0, 2\}] \]
\[-(4.99462 - 7.97591i) - \frac{(0.000301896 + 0.0000506802)i}{\kappa} + \frac{(8.77715545659317^\ast - 9 - 2.662464703657753^\ast - 8i)\kappa^2}{\kappa^2} + O[\epsilon]^3 \]
\[ \text{ua} = -(4.99462 - 7.97591i) - \left( \frac{0.000301896 + 0.0000506802}{\kappa} \right) + \] 
\[ \frac{(8.77715545659317^\ast - 2.662464703657753^\ast - 8i)\kappa^2}{\kappa^2} ; \]
\[ \text{ua} = -(4.99462 - 7.97591i) - \left( \frac{0.000301896 - 0.0000506802}{\kappa} \right) + \] 
\[ \frac{(8.77715545659317^\ast + 2.662464703657753^\ast - 8i)\kappa^2}{\kappa^2} ; \]
\[ \text{Series}[\tau22, \{m \rightarrow 1\}] / \{u \rightarrow x + I \ast y\} / \{x \rightarrow 1 + dx, y \rightarrow dy\} / N, \{\epsilon, 0, 2\}] \]
\[ (0.436634 + 0.899639i) - \frac{(0.000921108 + 0.0000567335)}{\kappa} + \frac{(5.8328924576086865^\ast - 9 - 1.299700769261508^\ast - 9i)\kappa^2}{\kappa^2} + O[\epsilon]^3 \]
\[ \text{Int22} = 0.899639 + "0.0000567335" \; \kappa^2 ; \]
\[ g22 = \text{Series} \left[ \frac{1}{\text{Int22}}, \{\epsilon, 0, 2\} \right] \]
1.11156 - 9.0009700976\(\kappa\) + 4.420527614073487^\ast - 9\(\kappa^2\) + O[\epsilon]^3
\[ \text{Series}[\tau12, \{m \rightarrow 1\}] / \{u \rightarrow x + I \ast y\} / \{x \rightarrow 1 + dx, y \rightarrow dy\} / N, \{\epsilon, 0, 2\}] \]
\[-(0.491275 - 0.0280705i) - \frac{(0.000482472 + 7.30572560501539^\ast - 6i)}{\kappa} + \frac{(8.40749190959112^\ast - 9 + 1.688956409283623^\ast - 10i)\kappa^2}{\kappa^2} + O[\epsilon]^3 \]
Int12 = "0.0280705" - 7.305725060501539*^6*κ; 

f22 = Series[F22/.{m → 1}/.{u → x + I * y}/.{x → 1 + dx, y → dy}//N,{ε, 0, 2}]
(0.0735233 - 0.680554i) + (0.0000628572 + 0.0000100444i)κ - (4.0793686239472671*^4*κ + 1.622773190008971*^8*κ)*κ^2 + O[ε]^3

f12 = Series[F12/.{m → 1}/.{u → x + I * y}/.{x → 1 + dx, y → dy}//N,{ε, 0, 2}]
-(0.0895189 + 0.295503i) + (0.0000628572 + 0.0000100444i)κ - (2.144816209673046*^4*κ + 5.722127999917844*^9*κ)*κ^2 + O[ε]^3

pt = f12 - Int12 * g22 * f22
-(0.091813 + 0.274268i) + (0.0000610268 + 0.0000998368i)κ - (1.938708595099071*^4*κ + 3.94398493604365*^9*κ)*κ^2 + O[ε]^3

**THE BOSONIC MASS**

mA = 1/4 * g22 * (D2V + d2V) * ua * uad
((38766.8 + 2.7981704033299277*^4*κ + (3.73332 + 9.47652170877663*^4*κ)*κ^2 + (0.0141169 + 8.672413972193371*^6*κ - 19i)*κ^2) + O[ε]^3

**THE FERMIONIC MASS**

mX0 = g22 * 2κ * (ua)^2
(-(85.9654 - 177.123i)κ + (0.030098 + 0.0117184i)κ - (1.517161630918473*^4*κ - 8.438068394771261*^4*κ - 7i)κ^2) + O[ε]^3

Series[ua * uad, {ε, 0, 2}]/Simplify
((85.5614 + 0.4κ - 9.095087ε - 1.643995870931913*^4*κ - 7κ^2 + O[ε]^3

Series[ua * uad, {ε, 0, 2}]/Simplify
((85.5614 + 0.4κ - 9.095087ε - 1.643995870931913*^4*κ - 7κ^2 + O[ε]^3

mX = g22 * (2κ * (ua)^2 - I * κ * (dx + Idy) * ua * g22 * f22 + "38762.5"*κ^2 * pt)
(-(85.9654 - 177.123i)κ + ((0.00542119 + 0.0111698i)κ + 1.11156 + 0.00155878i - (5315.66 - 1779.45i)κ^2) + O[ε]^3

Series[mX, {ε, 0, 2}]/Simplify

\((-85.9654 - 177.123i)\kappa + \left( (0.0418761 + 0.0129025i) - (5908.66 - 1977.96i)\kappa^2 \right) \epsilon +
\left( \frac{4.833060912423989 \times 10^{-6} - 2.7537076518432195 \times 10^{-6}i}{\kappa} + (2.33271 - 1.43946i)\kappa \right) \epsilon^2 + O[\epsilon]^3
\)

Series[
\((-85.9654 - 177.123i)\kappa +
\left( (0.0418761 + 0.0129025i) - (5908.66 - 1977.96i)\kappa^2 \right) \epsilon +
\left( \frac{4.833060912423989 \times 10^{-6} - 2.7537076518432195 \times 10^{-6}i}{\kappa} + (2.33271 - 1.43946i)\kappa \right) \epsilon^2 \right) *
\left( (-85.9654 + "177.123"i)\kappa +
\left( (0.0418761 - "0.0129025"i) - (5908.66 - "1977.96"i)\kappa^2 \right) \epsilon +
\left( \frac{4.833060912423989 \times 10^{-6} + 2.7537076518432195 \times 10^{-6}i}{\kappa} + ("2.33271" - "1.43946"i)\kappa \right) \epsilon^2 \right) ,
\{\epsilon, 0, 2\}]/Simplify
38762.5\kappa^2 + (-11.7704\kappa + 315197.\kappa^3) \epsilon + (0.00177555 - 334.965\kappa^2 + 3.88246 \times 10^7 \kappa^4) \epsilon^2 + O[\epsilon]^3
REFERENCES


