Using Gram Matrices and Residues to Generate Symmetric Functions

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Abstract

Samuel Behrend: Using Gram Matrices and Residues
to Generate Symmetric Functions
(Under the direction of Dr. Richard Rimanyi)

In this paper we argue for the use of a symmetric bilinear map S on $\mathcal{Q}^{n+1}$ as a means of producing and manipulating symmetric functions; using certain vectors of rational functions we can produce Schur functions. We define S as the determinant of a specific Gram matrix, whose elements are the result of an antisymmetric bilinear map $\langle , \rangle$ on $\mathcal{Q}$ as well as a reversion map R on the same space. Ultimately, S allows us to derive an alternative construction of the Jacobi-Trudi identity (extending the identity to Schur functions) as well as a variant of the Cauchy identities.

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1. Introduction

The purpose of this paper is to summarize the concepts and results presented in [10] and [11]. Certain proofs required the consultation of additional texts ([1]-[9]), though the intention is for this paper to be as self-contained as possible.

In the first section we introduce an antisymmetric bilinear map on the space $\mathcal{Q}$ of rational functions of one complex variable We discuss two interpretations of the map, as an implementation of the Taylor functional and as the iterated residue of the product of the component functions at the roots of the first component.

We then prove properties of the inner product, including the removal of common factors and translation invariance. We also discuss the use of the differentiation operator as well as an involution of $\mathcal{Q}$ defined as $Rf(z) = (1/z)f(1/z)$.

Section 3 examines the use of our inner product and a general polynomial $w(z)$ as a divided difference functional sending $f(z)$ to the scalar $\left\langle \frac{1}{w}, f \right\rangle$. We then define a set of polynomials, known as Horner polynomials, which simplify the definition of the difference quotient of w, which we can use to interpolate a rational function at the roots of w.

In the next section we work through properties of Gram matrices and its relation to the difference quotient operator. We also work through a number of examples using certain bases and dual bases in spaces of polynomials that facilitate the construction of symmetric polynomials and Schur functions.

The paper concludes by explicitly defining and exploring the properties of the symetric bilinear map S. We supply an alternative construction of the Jacobi-Trudi identity (extending the identity to Schur functions), derive Schur functions and produce a variant of the Cauchy identities.

2. An inner product on $\mathcal{Q}$

Denote $\mathcal{P}$ as the complex vector space of all polynomials in one complex variable, z, and $\mathcal{P}_n$ as the subspace of the polynomials of degree at most $n \in \mathbb{Z}^{>0}$. Consider the complex vector space $\mathcal{R}$ generated by functions of the form

$$r_{a,k}(z) = \frac{1}{(z-a)^{k+1}} \qquad a \in \mathbb{C}, k \in \mathbb{N}. \tag{2.1}$$

Such functions are called *proper rational functions*. When applying a partial fraction decomposition, $\mathcal{R}$ can be seen as the set of rational functions of the form $p(z)/u(z)$ where p and u are polynomials sharing no roots and u is monic with degree strictly greater than that of p.

By the division algorithm of polynomials we know that every rational function can be written uniquely as the sum of a polynomial and a proper rational function, that is $\mathcal{Q} = \mathcal{P} \oplus \mathcal{R}$ where $\mathcal{Q}$ is the space of rational functions. This means that $\{z^n : n \in \mathbb{N}\} \cup \{r_{a,k} : a \in \mathbb{C}, k \in \mathbb{N}\}$, the union of the standard bases of $\mathcal{P}$ and $\mathcal{R}$, is a basis for $\mathcal{Q}$. With this in mind we define a bilinear map $\mathcal{Q} \times \mathcal{Q} \to \mathbb{C}$ on our basis elements as follows:

$$\begin{aligned}
\langle r_{a,k}(z), z^n \rangle &= \binom{n}{k} a^{n-k} \qquad a \in \mathbb{C}, \quad k,n \in \mathbb{N} && (2.2)\\
\langle z^n, r_{a,k}(z) \rangle &= -\binom{n}{k} a^{n-k} \qquad a \in \mathbb{C}, \quad k,n \in \mathbb{N} && (2.3)\\
\langle r_{a,k}(z), r_{b,m}(z) \rangle &= (-1)^k \binom{k+m}{k} \frac{1}{(a-b)^{k+m+1}} \qquad a \neq b, \quad k,m \in \mathbb{N} && (2.4)\\
\langle r_{a,k}(z), r_{a,m}(z) \rangle &= 0 \qquad a \in \mathbb{C}, \quad k,m \in \mathbb{N} && (2.5)\\
\left\langle z^n, z^k \right\rangle &= 0 \qquad k,n \in \mathbb{N}. && (2.6)
\end{aligned}$$

It is clear that $\langle z^n, z^k \rangle = 0 = -\langle z^k, z^n \rangle$ and moreover,

$$\begin{aligned}
\langle r_{a,k}(z), r_{b,m}(z) \rangle &= (-1)^k \binom{k+m}{k} \frac{1}{(a-b)^{k+m+1}} \\
&= (-1)^k \binom{m+k}{m} \frac{1}{(a-b)^{k+m+1}} \\
&= (-1)^k \binom{m+k}{m} \frac{(-1)^{-k-m-1}}{(b-a)^{k+m+1}} \\
&= (-1)^{-m-1} \binom{m+k}{m} \frac{1}{(b-a)^{k+m+1}} \\
&= (-1)^{m+1} \binom{m+k}{m} \frac{1}{(b-a)^{k+m+1}} \\
&= -\langle r_{b,m}(z), r_{a,k}(z) \rangle .
\end{aligned}$$

So by the above definitions and the extension by linearity, we see $\langle , \rangle$ is an anti-symmetric bilinear form on $\mathcal{Q}$.

Alternative Notation

Let $(a,k)^*$ denote the Taylor functional defined on the set of rational functions defined at a (so f does not have a singularity at a) as follows

$$(a,k)^* f = \frac{D^k f(a)}{k!} \qquad a \in \mathbb{C}, \quad k \in \mathbb{N} \tag{2.7}$$

where D denotes the usual differentiation operator. Note then that this means $(a,k)^* = \langle r_{a,k}(z), - \rangle$.

The Leibniz rule for differentiation affords us the following property

$$(a,n)^*(fg) = \sum_{k=0}^{n} [(a,k)^* f][(a,n-k)^* g] \tag{2.8}$$

for any rational functions f, g defined at a. In our earlier notation we would have written

$$\langle r_{a,n}(z), fg \rangle = \sum_{k=0}^{n} \langle r_{a,k}(z), f \rangle \langle r_{a,n-k}(z), g \rangle .$$

So if g is a rational function defined at a and $f(z) = (z-a)^m$ where $m \geq 0$ then by (2.8) we have $(a,n)^*(fg) = 0$ for $0 \leq n \leq m-1$. Further if we consider $f(z) = (z-a)^m$ and $g(z) = 1$ then $(a,k)^*(z-a)^m = \delta_{k,m}$ for $k, m \in \mathbb{N}$ because the m^{th} derivative of f will be just $m!$ (which cancels out the $m!$ in the denominator) and any subsequent derivative will just be 0.

In the next section we will begin discussing how our inner product is actually a residue calculation of the product of the two arguments. However, in order to do so, we will need to develop the algebra of taking products with our new notation. Specifically, our goal will be to develop a *partial fraction decomposition* formula for the product of two strictly rational functions. Let $r \geq 0$ and $a_0, a_1, \ldots, a_r$ be distinct complex numbers (roots) with corresponding positive integers $m_0, \ldots, m_r$ (their orders). Define

$$w(z) = \prod_{i=0}^{r} (z - a_i)^{m_i}$$

noting $n + 1 = \sum m_i$. For sake of ease define the index set

$$\mathcal{I} = \{(i, j) : 0 \leq i \leq r, 0 \leq j \leq m_i - 1\}$$

and for each $(i, j) \in \mathcal{I}$ let

$$q_{i,j}(z) = \frac{w(z)}{(z - a_i)^{m_i - j}}. \tag{2.9}$$

So $q_{i,j}(z)$ is a polynomial whose degree is at most n. Essentially $q_{i,j}(z)$ is our polynomial w but the root a_i now has multiplicity j. This will be most useful when we use the term $q_{i,0}(z)$, which effectively removes a_i as a root.

Next we define the linear functionals $L_{i,j}$ by

$$L_{i,j} p = (a_i, j)^* \frac{p}{q_{i,0}(z)} \qquad p \in \mathcal{P}, \quad (i, j) \in \mathcal{I}. \tag{2.10}$$

Now since

$$\frac{q_{k,s}(z)}{q_{i,0}(z)} = \frac{(z - a_i)^{m_i}}{(z - a_k)^{m_k - s}}$$

we see that $L_{i,j} q_{k,s}(z) = 0$ for $i \neq k$ and $L_{i,j} q_{i,s}(z) = (a_i, j)^* (z - a_i)^s = \delta_{j,s}$. Hence

$$L_{i,j} q_{k,s}(z) = \delta_{(i,j),(k,s)} \qquad (i, j), (k, s) \in \mathcal{I}. \tag{2.11}$$

This means that the set $\{q_{i,j}(z) : (i, j) \in \mathcal{I}\}$ is linearly independent. Note also that $|\mathcal{I}| = \sum_{i=0}^{r} m_i = n + 1$, which makes $\{q_{i,j}(z) : (i, j) \in \mathcal{I}\}$ a basis for the subspace $\mathcal{P}_n$. Hence we can describe our polynomials

$$p(z) = \sum_{(i,j) \in \mathcal{I}} \left[L_{i,j} p \right] q_{i,j}(z), \qquad p \in \mathcal{P}_n \tag{2.12}$$

so that when we divide the above equation by $w(z)$ we obtain the *partial fractions decomposition* formula (see 2.9)

$$\frac{p(z)}{w(z)} = \sum_{(i,j) \in \mathcal{I}} \frac{L_{i,j} p}{(z - a_i)^{m_i - j}}, \qquad p \in \mathcal{P}_n. \tag{2.13}$$

This says that every element of $\mathcal{R}$ can be written uniquely as a finite linear combination of functions of the form $(z-a)^{-k-1}$ where a is a complex number (root of w) and k is a nonnegative integer, i.e. our strictly rational functions $r_{a,k}(z)$. Consider a more compact notation for these strictly rational functions

$$(a,k) = (a,k)(z) = \frac{1}{(z-a)^{k+1}} \qquad a \in \mathbb{C} \quad k \in \mathbb{N}$$

Then if we let $p = 1$ and $w(z) = (a,k)(z)(b,m)(z)$ the partial fractions formula (2.13) gives us the multiplication formula

$$\begin{aligned}(a,k)(z)(b,m)(z) = &\left[\sum_{j=0}^{k}[(a,j)^*(b,m)][(a,k-j)]\right] \\ &+ \left[\sum_{i=0}^{m}[(b,i)^*(a,k)][(b,m-i)]\right] \qquad a \neq b.\end{aligned} \tag{2.14}$$

Also,

$$(a,k)(z)(a,m)(z) = \frac{1}{(z-a)^{k+1}(z-a)^{m+1}} = \frac{1}{(z-a)^{k+m+2}} = (a,k+m+1)(z).$$

Residue interpretation

Another powerful interpretation of our inner product is as the residue of the product of the component functions. Considering first two purely rational functions (where $a \neq b$), we want to show

$$\langle (a,k),(b,m) \rangle = \text{Residue at } a \text{ of } (a,k)(b,m). \tag{2.15}$$

Using the multiplication formula derived earlier, we see that

$$\begin{aligned}\operatorname*{Res}_a (a,k)(b,m) &= \operatorname*{Res}_a \left[\sum_{j=0}^{k}[(a,j)^*(b,m)][(a,k-j)] + \sum_{i=0}^{m}[(b,i)^*(a,k)][(b,m-i)]\right] \\ &= \operatorname*{Res}_a \left[\sum_{j=0}^{k}[(a,j)^*(b,m)][(a,k-j)]\right] + \operatorname*{Res}_a \left[\sum_{i=0}^{m}[(b,i)^*(a,k)][(b,m-i)]\right] \\ &= \sum_{j=0}^{k}[(a,j)^*(b,m)]\operatorname*{Res}_a (a,k-j) + \sum_{i=0}^{m}[(b,i)^*(a,k)]\operatorname*{Res}_a (b,m-i)\end{aligned}$$

Notice that the terms $(b,m-i)$ have no singularities at a, so their residues are zero.

Likewise, the residues of $(a, k - j)$ are all zero except for when $k = j$ (which leaves us with just a simple pole at a) because when we remove the singularity we're left to differentiate a constant.

$$\begin{aligned}
\operatorname*{Res}_{a}\,(a,k)(b,m) &= \sum_{j=0}^{k} [(a,j)^*(b,m)] \operatorname*{Res}_{a}\,(a,k-j) \\
&= [(a,k)^*(b,m)] \operatorname*{Res}_{a}\,(a,0) \\
&= (a,k)^*(b,m) \\
&= \langle (a,k),(b,m) \rangle .
\end{aligned}$$

Now assuming the second component function is a polynomial, we can see

$$\begin{aligned}
\langle (a,k), z^n \rangle &= \binom{n}{k} a^{n-k} \\
&= \frac{1}{k!} \cdot \left[\frac{n!}{(n-k)!} a^{n-k} \right] \\
&= \frac{1}{k!} \lim_{z \to a} \frac{d^k}{dz^k} (z^n) \\
&= \operatorname*{Res}_{a} \frac{z^n}{(z-a)^{k+1}}.
\end{aligned}$$

And obviously, if both component functions are polynomials there are no singularities, which means the residue of their product is just zero, which is consistent with our inner product definition. The following proposition takes the residue interpretation a step further.

Proposition 2.1. Let p, w be polynomials such that p/w is in $\mathcal{R}$ and let $f \in \mathcal{Q}$ such that f is defined at the roots of w. Then

$$\left\langle \frac{p}{w}, f \right\rangle = \left\langle \frac{1}{w}, pf \right\rangle = \sum_i \operatorname*{Res}_{a_i} \frac{pf}{w} \tag{2.16}$$

where the sum runs over the distinct roots a_i of w.

Proof. Let $w(z) = \prod_{i=0}^{r} (z - a_i)^{m_i}$ and notice that (2.13) - the partial fractions decomposition formula - can be rewritten, using the definition in (2.10), as

$$
\begin{aligned}
\frac{p(z)}{w(z)} &= \sum_{(i,j)\in\mathcal{I}} \frac{L_{i,j}p}{(z-a_i)^{m_i-j}} \\
&= \sum_{(i,j)\in\mathcal{I}} L_{i,j}p \cdot (a_i, m_i - j - 1)(z) \\
&= \sum_{(i,j)\in\mathcal{I}} (a_i, j)^* \frac{p}{q_{i,0}(z)} \cdot (a_i, m_i - j - 1)(z) \\
&= \sum_{(i,j)\in\mathcal{I}} \left\langle (a_i, j), \frac{p}{q_{i,0}(z)} \right\rangle \cdot (a_i, m_i - j - 1)(z).
\end{aligned}
$$

Then using the Leibniz rule (in the third and fourth upcoming steps) to shuffle the second component functions in our inner product we get

$$
\begin{aligned}
\left\langle \frac{p(z)}{w(z)}, f \right\rangle &= \sum_{(i,j)\in\mathcal{I}} \left\langle (a_i, j), \frac{p}{q_{i,0}(z)} \right\rangle \cdot (a_i, m_i - j - 1)(z)^* f \\
&= \sum_{(i,j)\in\mathcal{I}} (a_i, j)^* \frac{p}{q_{i,0}(z)} \cdot (a_i, m_i - j - 1)^* f \\
&= \sum_{i} \left\langle (a_i, m_i - 1), \frac{pf}{q_{i,0}(z)} \right\rangle \\
&= \sum_{(i,j)\in\mathcal{I}} (a_i, j)^* \frac{1}{q_{i,0}(z)} \cdot (a_i, m_i - j - 1)^* pf \\
&= \sum_{(i,j)\in\mathcal{I}} \left\langle (a_i, j), \frac{1}{q_{i,0}(z)} \right\rangle \cdot (a_i, m_i - j - 1)(z)^* pf \\
&= \left\langle \frac{1}{w(z)}, pf \right\rangle.
\end{aligned}
$$

Now we've previously shown that given a strictly rational function as our first component our inner product is equivalent to the residue of the product of the component functions (for any rational second component). So using the third equality above and the fact that $\frac{1}{q_{i,0}(z)} = \frac{(z-a_i)^{m_i}}{w(z)}$ we can complete our statment:

$$
\sum_{i} \left\langle (a_i, m_i - 1), \frac{pf}{q_{i,0}(z)} \right\rangle = \sum_{i} \operatorname*{Res}_{a_i} \frac{pf}{w}.
$$

□

3. Early Properties

Immediately, we would like to develop a means of transferring functions from one argument of the inner product to the other, akin to an adjoint operator to a linear transformation.

Proposition 3.1. Let f be an element of $\mathcal{Q}$ and let $p, q \in \mathcal{P}$. Then

$$\langle pf, q \rangle = \langle f, pq \rangle . \tag{3.1}$$

Proof. We will conduct this proof on the basis elements of $\mathcal{Q} = \mathcal{R} \oplus \mathcal{P}$ and then use linearity to achieve our goal. Let k, n and m be nonnegative integers and let $a \in \mathbb{C}$. Then

$$\langle (a,k)(z), z^n z^m \rangle = (a,k)^* z^{n+m} = \binom{n+m}{k} a^{n+m-k}.$$

On the other hand, writing $z^n = (a + z - a)^n$ we can use the binomial formula to see

$$z^n (a,k)(z) = \left[\sum_{j=0}^{n} \binom{n}{j} a^{n-j} (z-a)^j \right] (a,k)(z) = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} (z-a)^{j-k-1},$$

and since the inner prduct of two polynomials is zero we only need to consider the terms in which j does not exceed k, that is, where $r = \min\{n, k\}$:

$$\begin{aligned}
\langle z^n (a,k)(z), z^m \rangle &= \sum_{j=0}^{r} \binom{n}{j} a^{n-j} \langle (a, k-j), z^m \rangle \\
&= \sum_{j=0}^{r} \binom{n}{j} a^{n-j} \binom{m}{k-j} a^{m-k+j} \\
&= \sum_{j=0}^{r} \binom{n}{j} \binom{m}{k-j} a^{m+n-k} \\
&= \binom{n+m}{k} a^{n+m-k}.
\end{aligned}$$

The last equality requires some combinatorics, known formally as the Chu-Vandermonde convolution formula. Thus

$$\langle (a,k)(z), z^n z^m \rangle = \langle z^n (a,k)(z), z^m \rangle .$$

Finally, notice that $\langle z^s, z^n z^m \rangle = 0 = \langle z^n z^s, z^m \rangle$ – where s is a nonnegative integer – so (3.1) follows by linearity. □

We can use (2.16) now to prove two similar properties: the first is known as Popoviciu's reduction and the second is a decomposition formula for the product of two strictly rational functions. Recall that a function f is said to be defined on the roots of a polynomial $u(z)$ if for each root a of u with multiplicity m, the derivatives $D^k f$ for $0 \le k < m$ are defined (i.e. have no singularities) at a.

Proposition 3.2. (Popoviciu's Reduction) Let $u, v \in \mathcal{P}$ and f be a function defined on the roots of uv. Then

$$\left\langle \frac{1}{uv}, vf \right\rangle = \left\langle \frac{1}{u}, f \right\rangle . \tag{3.2}$$

Proof. Assume the roots of u and v are indexed as $a_i \in A$ and $b_j \in B$ respectively (it is possible that $a_i = b_j$ for some i, j). Using our residue interpretation,

$$\begin{aligned} \left\langle \frac{1}{uv}, vf \right\rangle &= \sum_i \operatorname*{Res}_{\{a_i\}\cup\{b_j\}} \frac{vf}{uv} \\ &= \sum_i \operatorname*{Res}_{\{a_i\}\cup\{b_j\}} \frac{f}{u} \\ &= \sum_i \operatorname*{Res}_{\{a_i\}} \frac{f}{u} \\ &= \left\langle \frac{1}{u}, f \right\rangle . \end{aligned}$$

This penultimate equality holds because for $b_j \in B \setminus A$, the residue of $\dfrac{f}{u}$ is 0 (f does not have a singularity at any b_j and u has no roots in $B \setminus A$). □

Proposition 3.3. (Popoviciu's Decomposition) Let $u, v \in \mathcal{P}$ and f be a function defined on the roots of uv. If u, v have no common roots, then

$$\left\langle \frac{1}{uv}, f \right\rangle = \left\langle \frac{1}{u}, \frac{f}{v} \right\rangle + \left\langle \frac{1}{v}, \frac{f}{u} \right\rangle . \tag{3.3}$$

Proof. Assume the unique roots of u and v are indexed as a_i ad b_j respectively ($a_i \neq b_j$ for all i, j). Then with our residue interpretation we can see

$$\begin{aligned}\left\langle \frac{1}{uv}, f \right\rangle &= \sum_i \operatorname*{Res}_{a_i} \frac{f}{uv} + \sum_j \operatorname*{Res}_{b_j} \frac{f}{uv} \\ &= \left\langle \frac{1}{u}, \frac{f}{v} \right\rangle + \left\langle \frac{1}{v}, \frac{f}{u} \right\rangle .\end{aligned}$$

□

The next property we consider is the behavior of the operator D, differentiation with respect to z.

Proposition 3.4. Let $f, g \in \mathcal{Q}$. Then

$$\langle f, Dg \rangle = \langle -Df, g \rangle . \tag{3.4}$$

Proof. We will again be considering the basis elements of $\mathcal{Q} = \mathcal{R} \oplus \mathcal{P}$ and then use linearity to finish the proof.

Case 1: $\langle z^n, z^m \rangle$. We know the inner product of two polynomials is just zero, so it's easy to see

$$\left\langle z^n, mz^{m-1} \right\rangle = 0 = \left\langle -nz^{n-1}, z^m \right\rangle$$

.

Case 2: $\langle (a,k), z^n \rangle$. Using linearity and the definition of our inner product,

$$\begin{aligned}\langle (a,k), Dz^n \rangle &= \left\langle (a,k), nz^{n-1} \right\rangle \\ &= n \left\langle (a,k), z^{n-1} \right\rangle \\ &= n \binom{n-1}{k} a^{n-k-1} \\ &= \left(\frac{n!}{k!(n-k-1)!} \right) a^{n-k-1} \\ &= (k+1) \binom{n}{k+1} a^{n-k-1} \\ &= (k+1) \langle (a, k+1), z^n \rangle \\ &= \langle (k+1)(a, k+1), z^n \rangle \\ &= \langle -D(a,k), z^n \rangle .\end{aligned}$$

Because of antisymmetry we also have proof of the $\langle z^n, (a,k) \rangle$ case.

Case 3: $\langle (a,k),(b,m)\rangle$. Again we use linearity and the definition of our inner product to see

$$\begin{aligned}
\langle (a,k), D(b,m)\rangle &= \langle (a,k), -(m+1)(b,m+1)\rangle \\
&= -(m+1)\,\langle (a,k),(b,m+1)\rangle \\
&= -(m+1)(-1)^k \binom{k+m+1}{k} \frac{1}{(a-b)^{k+m+2}} \\
&= (-1)^{k+1}\left(\frac{(k+m+1)!}{k!m!}\right)\frac{1}{(a-b)^{k+m+2}} \\
&= (k+1)(-1)^{k+1}\binom{k+m+1}{k+1}\frac{1}{(a-b)^{k+m+2}} \\
&= (k+1)\,\langle (a,k+1),(b,m)\rangle \\
&= \langle -D(a,k),(b,m)\rangle .
\end{aligned}$$

So by linearity we have proved our hypothesis. □

Another operator with similar behavior regarding our inner product is the reversion map $R : \mathcal{Q} \to \mathcal{Q}$ which is defined by

$$Rf(z) = \frac{1}{z} f\left(\frac{1}{z}\right), \qquad f \in \mathcal{Q}. \tag{3.5}$$

Note that

$$Rz^n = \frac{1}{z^{n+1}} = (0,n)(z), \qquad n \in \mathbb{N} \tag{3.6}$$

and

$$R(a,k)(z) = \frac{1}{z}\cdot\frac{1}{\left(\frac{1}{z}-a\right)^{k+1}} = \frac{z^k}{(1-az)^{k+1}}, \qquad a\in\mathbb{C}, \quad k \in \mathbb{N}. \tag{3.7}$$

Further we can see that $R^2 = I$. This allows us to prove the following

Proposition 3.5. Let f, g be any rational functions. Then

$$\langle f, Rg\rangle = \langle -Rf, g\rangle . \tag{3.8}$$

Proof. We need to preface the work in this proof with a simple lemma.

Lemma 3.1. Let f be a rational function. Then

$$\operatorname*{Res}_{1/a} f(z) = \operatorname*{Res}_{a} \frac{-1}{z^2} f(1/z). \tag{3.9}$$

Proof. Assume $f : \mathbb{C} \to \mathbb{C}$ is some smooth function mapping a to $f(a)$ and that we have some smooth one-form $\omega = f(z)dz$. We know that under the pullback, the following holds true

$$\operatorname*{Res}_{f(a)} \omega = \operatorname*{Res}_{a} f^* \omega.$$

Then if $f(z) = \dfrac{1}{z}$ we can see

$$\begin{aligned} \operatorname*{Res}_{1/a} f(z)dz &= \operatorname*{Res}_{a} f(1/z)d(1/z) \\ &= \operatorname*{Res}_{a} f(1/z) \cdot \frac{-1}{z^2} dz, \end{aligned}$$

and we can just supress the dz terms in the residues on either side of the equality to finish our proof. $\square$

With this Lemma in mind we proceed case-by-case with the basis elements of $\mathcal{Q}$ using properties (3.6) and (3.7).

Case 1: $f = z^n; g = z^m$. Using our definition of the reversion map and inner product we see

$$\begin{aligned} \langle f, Rg \rangle &= \langle z^n, Rz^m \rangle \\ &= \langle z^n, (0, m)(z) \rangle \\ &= -\langle (0, m)(z), z^n \rangle \\ &= -\binom{n}{m} 0^{n-m} \\ &= -\binom{m}{n} 0^{m-n} \\ &= -\langle (0, n)(z), z^m \rangle \\ &= \langle -Rz^n, z^m \rangle \\ &= \langle -Rf, g \rangle \end{aligned}$$

Case 2: $f = (a, k)(z); g = z^n$. First we use our definition of the reversion map and the residue interpretation of the inner product. The fourth proceeding step is simply a convenient rephrasing of each term so we can use the lemma.

$$
\begin{aligned}
\langle f, Rg \rangle &= \langle (a,k)(z), Rz^n \rangle \\
&= \langle (a,k)(z), (0,n)(z) \rangle \\
&= \operatorname*{Res}_{a} \, [(a,k)(z)][(0,n)(z)] \\
&= \operatorname*{Res}_{a} \, \left[\frac{1}{z} R(a,k)(1/z)\right] \left[\frac{1}{z}\left(\frac{1}{z}\right)^n\right] \\
&= \operatorname*{Res}_{a} \, \frac{1}{z^2} R(a,k)(1/z) \left(\frac{1}{z}\right)^n \\
&= \operatorname*{Res}_{1/a} \, -R(a,k)(z) z^n \\
&= \langle -R(a,k)(z), z^n \rangle \\
&= \langle -Rf, g \rangle .
\end{aligned}
$$

Case 3: $f = (a,k)(z); g = (b,m)(z)$. We proceed much the same way as the previous case, using our lemma and the residue interpretation.

$$
\begin{aligned}
\langle f, Rg \rangle &= \langle (a,k)(z), R(b,m)(z) \rangle \\
&= \operatorname*{Res}_{a} \, [(a,k)(z)][R(b,m)(z)] \\
&= \operatorname*{Res}_{a} \, \left[\frac{1}{z} R(a,k)(1/z)\right] \left[\frac{1}{z}(b,m)(1/z)\right] \\
&= \operatorname*{Res}_{a} \, \frac{1}{z^2} \left[R(a,k)(1/z)\right] \left[(b,m)(1/z)\right] \\
&= \operatorname*{Res}_{1/a} \, -\left[R(a,k)(z)\right] \left[(b,m)(z)\right] \\
&= \langle -R(a,k)(z), (b,m)(z) \rangle \\
&= \langle -Rf, g \rangle .
\end{aligned}
$$

□

Then with (3.8) we can prove

$$
\begin{aligned}
\langle f, g \rangle &= -\langle g, f \rangle \\
&= -\left\langle R^2 g, f \right\rangle \\
&= \langle Rg, Rf \rangle .
\end{aligned}
$$

We will finish up by proving the translation invariance property of $\langle , \rangle$.

Proposition 3.6. Let $f, g \in \mathcal{Q}$ and $a \in \mathbb{C}$. Then

$$
\langle f(z), g(z) \rangle = \langle f(z+a), g(z+a) \rangle . \tag{3.10}
$$

Proof. As with the previous proposition, we will take this one case at a time, considering the basis elements of $\mathcal{Q} = \mathcal{R} \oplus \mathcal{P}$ and then use linearity to finish the proof. So let $f, g \in \mathcal{Q}$, $a, b, c \in \mathbb{C}$ and $k, m, n \in \mathbb{Z}^{>0}$.

Case 1: $\langle z^n, z^m \rangle$. We know the inner product of two polynomials is just zero, so it's easy to see

$$\langle z^n, z^m \rangle = 0 = \langle (z+a)^n, (z+a)^m \rangle .$$

Case 2: $\langle (a,k)(z), z^n \rangle$. Note that if we translate a strictly rational function $(a,k) = \dfrac{1}{(z-a)^{k+1}}$ by b then we're left with $\dfrac{1}{(z+b-a)^{k+1}} = (a-b,k)(z)$. Then using our rules we see

$$\begin{aligned} \langle (a-b,k)(z), (z+b)^n \rangle &= \binom{n}{k} ((a-b)+b)^{n-k} \\ &= \binom{n}{k} a^{n-k} \\ &= \langle (a,k), z^n \rangle . \end{aligned}$$

Because of antisymmetry we also have proof of the $\langle (z+b)^n, (a-b,k)(z) \rangle$ case.

Case 3: $\langle (a,k)(z), (b,m)(z) \rangle$. The evaluation of this inner product is more dependant on the exponents k and m than the roots themselves, so we see

$$\begin{aligned} \langle (a+c,k)(z), (b+c,m)(z) \rangle &= (-1)^k \binom{k+m}{k} \frac{1}{((a+c)-(b+c))^{k+m+1}} \\ &= (-1)^k \binom{k+m}{k} \frac{1}{(a-b)^{k+m+1}} \\ &= \langle (a,k)(z), (b,m)(z) \rangle . \end{aligned}$$

So by linearity we have proved our hypothesis. $\square$

4. Divided Difference Functional and the Difference Quotient

With all this information we now have regarding our inner product, we can discuss the specific linear functional that sends a rational function f to $\left\langle \frac{1}{w}, f \right\rangle$ where $w(z) = \prod_{i=0}^{r}(z - a_i)^{m_i}$ and f is defined on the roots of w. We call this the *divided differences functional*; in [8] Verde-Star introduces a whole seperate notation for this functional, stating $\Delta_w f(z) = \left\langle \frac{1}{w(z)}, f(z) \right\rangle$. Interestingly enough we can develop an explicit expression for this functional as a linear combination of Taylor functionals at the roots of w.

First, we repeatedly use (3.3) to obtain

$$\left\langle \frac{1}{w}, f \right\rangle = \sum_{i=0}^{r} \left\langle (a_i, m_i - 1)(z), \frac{f}{q_{i,0}(z)} \right\rangle . \tag{4.1}$$

Then we apply Leibniz's rule (2.8) to get the following formalization.

Proposition 4.1. The divided differences functional sending rational function f to $\left\langle \frac{1}{w}, f \right\rangle$ can be expressed as the following linear combination of Taylor functionals at the roots of w:

$$\left\langle \frac{1}{w}, f \right\rangle = \sum_{i=0}^{r} \sum_{k=0}^{m_i - 1} \left\langle (a_i, k)(z), \frac{1}{q_{i,0}(z)} \right\rangle \langle (a_i, m_i - 1 - k)(z), f \rangle . \tag{4.2}$$

If all the multiplicities m_i are equal to one then,

$$\begin{aligned}
\left\langle \frac{1}{w}, f \right\rangle &= \sum_{i=0}^{r} \left\langle (a_i,0)(z), \frac{1}{q_{i,0}(z)} \right\rangle \langle (a_i,0)(z), f \rangle \\
&= \sum_{i=0}^{r} \left\langle (a_i,0)(z), \frac{1}{q_{i,0}(z)} \right\rangle f(a_i) \\
&= \sum_{i=0}^{r} \left[\operatorname*{Res}_{a_i} \frac{1}{w(z)} \right] f(a_i) \\
&= \sum_{i=0}^{r} \left[\frac{1}{\prod_{i \neq j}(a_i - a_j)} \right] f(a_i) \\
&= \sum_{i=0}^{r} \left[\frac{1}{w'(a_i)} \right] f(a_i) \\
&= \sum_{i=0}^{r} \frac{f(a_i)}{w'(a_i)}.
\end{aligned}$$

Notice this implies that

$$\left\langle \frac{1}{w}, w' f \right\rangle = \sum_{i=0}^{r} f(a_i) \tag{4.3}$$

and in particular

$$\left\langle \frac{1}{w(z)}, w'(z) z^m \right\rangle = \sum_{i=0}^{r} a_i^m = \sigma_m(a_0, a_1, \ldots, a_n) \tag{4.4}$$

where $\sigma_m(a_0, a_1, \ldots, a_n)$ is a the power sum symmetric polynomial on elements $(a_0, a_1, \ldots, a_n)$. We will see this property of the inner product later in our discussion of Gram matrices. We will also return to these symmetric polynomials in our discussion of partitions and Schur functions.

However, we first use our divided difference functional with specific monomials to develop coefficients we will find useful in our discussion of difference quotients.

Toeplitz Matrices

Let $n + 1 = \sum_i m_i$. Then define

$$h_k = \left\langle \frac{1}{w(z)}, z^{n+k} \right\rangle. \tag{4.5}$$

Proposition 4.2. Let w be defined as usual, then

$$h_k = \sum \prod_{i=0}^{r} \binom{k_i + m_i - 1}{k_i} a_i^{k_i} \tag{4.6}$$

where the sum runs over the multi-indices $(k_0, k_1, \ldots, k_r)$ that satisfy $\sum_i k_i = k$.

Proof. Let x be any number such that $w(x) \neq 0$. Let $f(z) = \dfrac{1}{x-z}$ and use (2.16) to first bring the polynomial p to the other component. Then we use our Liebniz rule (4.2) and evaluate the second inner product. Finally we can use the partial fractions decomposition formual (2.13) to simplify:

$$\begin{aligned}
\left\langle \frac{p(z)}{w(z)}, \frac{1}{x-z} \right\rangle &= \left\langle \frac{1}{w(z)}, \frac{p(z)}{x-z} \right\rangle \\
&= \sum_{i=0}^{r} \sum_{k=0}^{m_i-1} \left\langle (a_i, k)(z), \frac{p(z)}{q_{i,0}(z)} \right\rangle \left\langle (a_i, m_i - 1 - k)(z), \frac{1}{x-z} \right\rangle \\
&= \sum_{i=0}^{r} \sum_{k=0}^{m_i-1} \left\langle (a_i, k)(z), \frac{p(z)}{q_{i,0}(z)} \right\rangle \frac{1}{(x-a_i)^{m_i-k}} \\
&= \sum_{(i,k)} \frac{L_{i,k} p(z)}{(x-a_i)^{m_i-k}} \\
&= \frac{p(x)}{w(x)}.
\end{aligned}$$

Then let $p(z) = z^n$ where $n+1$ is the degree of w. Then we have

$$\frac{x^n}{w(x)} = \left\langle \frac{1}{w(z)}, \frac{z^n}{x-z} \right\rangle.$$

Multiply both sides of the equation by x (the inner product is dependent on z, so x is just a scalar) and replace x by $1/y$ to get the following generating function:

$$
\begin{aligned}
\frac{x^{n+1}}{w(x)} &= x\left\langle \frac{1}{w(z)}, \frac{z^n}{x-z} \right\rangle \\
x^{n+1}\prod_{i=0}^{r}(x-a_i)^{-m_i} &= \left\langle \frac{1}{w(z)}, \frac{xz^n}{x-z} \right\rangle \\
\frac{1}{y^{n+1}}\prod_{i=0}^{r}(\frac{1}{y}-a_i)^{-m_i} &= \left\langle \frac{1}{w(z)}, \frac{xz^n}{x-z} \right\rangle \\
\prod_{i=0}^{r}(1-ya_i)^{-m_i} &= \left\langle \frac{1}{w(z)}, \frac{xz^n}{x-z} \right\rangle \\
&= \left\langle \frac{1}{w(z)}, \frac{\frac{1}{y}z^n}{\frac{1}{y}-z} \right\rangle \\
&= \left\langle \frac{1}{w(z)}, \frac{z^n}{1-yz} \right\rangle .
\end{aligned}
$$

Then we apply the Taylor functional $(0,k)^*$ with respect to y to both sides. On the left hand side we use the Leibniz rule (2.8) over and over to seperate factors in the product, applying them to decreasing orders of $(0,k_i)$ giving us

$$
\begin{aligned}
(0,k)^*\prod_{i=0}^{r}(1-ya_i)^{-m_i} &= \sum_{(k_0,\ldots,k_r)}\prod_{i=0}^{r}\langle (0,k_i),(1-ya_i)^{-m_i}\rangle \\
&= \sum_{(k_0,\ldots,k_r)}\prod_{i=0}^{r}\left\langle (0,k_i), \left(\frac{-1}{a_i}\right)^{m_i}(y-\frac{1}{a_i})^{-m_i}\right\rangle \\
&= \sum_{(k_0,\ldots,k_r)}\prod_{i=0}^{r}\left(\frac{-1}{a_i}\right)^{m_i}\langle (0,k_i),(1/a_i,m_i-1)\rangle \\
&= \sum_{(k_0,\ldots,k_r)}\prod_{i=0}^{r}\left(\frac{-1}{a_i}\right)^{m_i}(-1)^{k_i}\binom{k_i+m_i-1}{k_i}\frac{1}{\left(\frac{-1}{a_i}\right)^{k_i+m_i}} \\
s &= \sum_{(k_0,\ldots,k_r)}\prod_{i=0}^{r}\binom{k_i+m_i-1}{k_i}a_i^{k_i}
\end{aligned}
$$

where the multiindices in the sum satisfy $\sum_i k_i = k$. On the right hand side, because we're taking the Taylor functional with respect to y, the inner product (which is with respect to z) treats it like a constant, and so by bilinearity we apply $(0,k)$ to the second component and see

$$
\begin{aligned}
\left\langle \frac{1}{w(z)}, (0,k)^* \frac{z^n}{1-yz} \right\rangle &= \left\langle \frac{1}{w(z)}, \left. \frac{D^k\left(\frac{z^n}{1-yz}\right)}{k!} \right|_{y=0} \right\rangle \\
&= \left\langle \frac{1}{w(z)}, \left. \frac{(-1)^{2k}k!\left(\frac{z^{n+k}}{(1-yz)^{k+1}}\right)}{k!} \right|_{y=0} \right\rangle \\
&= \left\langle \frac{1}{w(z)}, \left. \frac{z^{n+k}}{(1-yz)^{k+1}} \right|_{y=0} \right\rangle \\
&= \left\langle \frac{1}{w(z)}, z^{n+k} \right\rangle .
\end{aligned}
$$

Thus by the definition (4.5)

$$
h_k = \left\langle \frac{1}{w(z)}, z^{n+k} \right\rangle = \sum_{(k_0,\ldots,k_r)} \prod_{i=0}^{r} \binom{k_i + m_i - 1}{k_i} a_i^{k_i}.
$$

□

Notice then if again all the multiplicities m_i are one then h_k becomes the complete homogeneous symmetric polynomial of order k in the variables a_i. Also, the proposition shows that $h_0 = 1$ (this will be useful later). With this information we can begin to develop the concept of the Toeplitz matrix. First note that from (4.5) and properties of the reversion map (3.10) we know

$$
\begin{aligned}
h_k &= \left\langle \frac{1}{w(z)}, z^{n+k} \right\rangle \\
&= \left\langle Rz^{n+k}, R\frac{1}{w(z)} \right\rangle \\
&= \left\langle \frac{1}{z^{n+k+1}}, \frac{1}{zw(1/z)} \right\rangle \\
&= \left\langle \frac{1}{z^{k+1}}, \frac{1}{z^{n+1}w(1/z)} \right\rangle .
\end{aligned}
$$

The last step uses Popoviciu's reduction (3.2).

Proposition 4.3. The function $\dfrac{1}{z^{n+1}w\left(\dfrac{1}{z}\right)}$ is the generating function of h_k, that is $\dfrac{1}{z^{n+1}w\left(\dfrac{1}{z}\right)} = h_0 + h_1 z + \ldots$.

Proof. Let $w(z) = z^{n+1} + b_1 z^n + \ldots + b_{n+1}$. Then $z^{n+1}w\left(\dfrac{1}{z}\right) = 1 + b_1 z + \ldots + b_{n+1}z^{n+1}$. It is sufficient then to show

$$\frac{1}{1 + b_1 z + \ldots + b_{n+1}z^{n+1}} = h_0 + h_1 z + \ldots$$

In other words, we need to prove that h_k is the k^{th} coefficient of the Taylor expansion of $\dfrac{1}{z^{n+1}w\left(\dfrac{1}{z}\right)}$ at $z = 0$. Let $f(z) = \frac{1}{z^{n+1}w(\frac{1}{z})}$, so that $h_k = \left\langle \frac{1}{z^{k+1}}, f(z) \right\rangle$. Let $\hat{h}(z) = \frac{1}{kz^k}$ which means $D\hat{h}(z) = \frac{-1}{z^{k+1}}$ where D is differentiaion with respect to z. Then we use Proposition 3.4 to show

$$\begin{aligned}
h_k &= \left\langle -D\hat{h}(z), f(z) \right\rangle \\
&= \left\langle \hat{h}(z), Df(z) \right\rangle \\
&= \left\langle \frac{1}{kz^k}, Df(z) \right\rangle \\
&= \operatorname*{Res}_{z=0} \frac{Df(z)}{kz^k} \\
&= \frac{1}{(k-1)!} \lim_{z\to 0} \frac{d^{(k-1)}}{dz^{(k-1)}} \left[z^k \frac{Df(z)}{kz^k} \right] \\
&= \frac{1}{k!} \lim_{z\to 0} \frac{d^{(k-1)}}{dz^{(k-1)}} Df(z) \\
&= \frac{1}{k!} \lim_{z\to 0} \frac{d^k(f(z))}{dz^k} \\
&= \frac{f^{(k)}(0)}{k!}
\end{aligned}$$

which is by definition the k^{th} coefficient of the Taylor expansion of $f(z)$ at $z = 0$. □

Furthermore, if we let $w(z) = z^{n+1} + b_1 z^n + \ldots + b_{n+1}$ so that $z^{n+1}w(1/z) = 1 + b_1 z + b_2 z^2 + \ldots + b_{n+1}z^{n+1}$, then

$$1 = \left[\sum_{k=0}^{\infty} h_k z^k \right] \left[z^{n+1} w(1/z) \right] = \left[\sum_{k=0}^{\infty} h_k z^k \right] \left[\sum_{l=0}^{n+1} b_l z^l \right].$$

Thus,

$$\sum_{j=0}^{m} b_j h_{m-j} = \delta_{0,m}, \qquad m \geq 0. \tag{4.7}$$

Then for $k \geq 0$ we can solve for h_k using Cramer's rule on the finite system of equations that corresponds to $m = 0, 1, \ldots, k$ to get

$$h_k = \frac{\begin{vmatrix} 1 & 0 & \ldots & & 1 \\ b_1 & 1 & 0 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ b_{k-1} & b_{k-2} & \ldots & 1 & 0 \\ b_k & b_{k-1} & \ldots & b_1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \ldots & & 0 \\ b_1 & 1 & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \\ b_{k-1} & b_{k-2} & \ldots & 1 & 0 \\ b_k & b_{k-1} & \ldots & b_1 & 1 \end{vmatrix}} = (-1)^k \begin{vmatrix} b_1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ b_{k-1} & b_{k-2} & \ldots & 1 \\ b_k & b_{k-1} & \ldots & b_1 \end{vmatrix}, \qquad k \geq 1. \tag{4.8}$$

By the symmetry of the system in (4.7) we also have

$$b_k = (-1)^k \begin{vmatrix} h_1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \\ h_{k-1} & h_{k-2} & \ldots & 1 \\ h_k & h_{k-1} & \ldots & h_1 \end{vmatrix}, \qquad k \geq 1. \tag{4.9}$$

We call these matrices (lower triangular) *Toeplitz matrices* because each left-right diagonal is a constant value. Also if $k > n+1$ then $b_k = 0$.

Difference Quotient

Before we introduce the difference quotient terminology and ultimately a means of interpolating a polynomial using our inner product, we need to expound on our divided difference operator. Let $w(z) = z^{n+1} + b_1 z^n + \ldots + b_{n+1}$ ($b_j = 0$ for $j > n+1$) be a fixed monic polynomial of degree $n+1$. Then define the sequence $\{w_k\}$ of *Horner polynomials* of w as follows

$$w_k(z) = z^k + b_1 z^{k-1} + \ldots + b_k. \tag{4.10}$$

It is clear that $\{w_k\}$ is a basis of the vector space $\mathcal{P}$ and that $w_{n+1} = w$ and $w_{n+1+k}(z) = z^k w(z)$ for $k \geq 0$. Furthermore, the Horner polynomials satisfy the following recurrence relation

$$w_{k+1}(z) = z w_k(z) + b_{k+1}, \qquad k \geq 0$$

Now we can prove a few properties of the divided difference linear functional $\left(f \mapsto \left\langle \frac{1}{w(z)}, f \right\rangle\right)$ on these Horner polynomials using our definitions of h_k and (4.7):

$$\begin{aligned}
\left\langle \frac{1}{w(z)}, w_k(z) \right\rangle &= \left\langle \frac{1}{w(z)}, z^k + b_1 z^{k-1} + \ldots + b_k \right\rangle \\
&= b_0 \left\langle \frac{1}{w(z)}, z^k \right\rangle + b_1 \left\langle \frac{1}{w(z)}, z^{k-1} \right\rangle + \ldots + b_k \left\langle \frac{1}{w(z)}, 1 \right\rangle \\
&= b_0 h_{k-n} + b_1 h_{k-1-n} + \ldots + b_k h_{-n} \\
&= \sum_{j=0}^{k} b_j h_{k-n-j} \\
&= \delta_{0,k-n} \\
&= \delta_{k,n}.
\end{aligned}$$

Next, if we consider a polynomial f with degree less than n, we can express it as a linear combination of $\{w_k\}$ as follows

$$\begin{aligned}
\left\langle \frac{1}{w(z)}, f(z) \right\rangle &= \left\langle \frac{1}{w(z)}, \sum_{k=0}^{n-1} a_k w_k(z) \right\rangle \\
&= \sum_{k=0}^{n-1} a_k \left\langle \frac{1}{w(z)}, w_k(z) \right\rangle \\
&= 0
\end{aligned}$$

since $k < n$. Finally, we can see that any polynomial multiple of w also produces a value of 0 under the divided difference with respect to w:

$$\begin{aligned}
\left\langle \frac{1}{w(z)}, g(z) w(z) \right\rangle &= \left\langle \frac{1}{w(z)}, \sum_{k=0}^{m} a_k z^k w(z) \right\rangle \\
&= \sum_{k=0}^{m} a_k \left\langle \frac{1}{w(z)}, z^k w(z) \right\rangle \\
&= \sum_{k=0}^{m} a_k \left\langle \frac{1}{w(z)}, w_{n+1+k}(z) \right\rangle \\
&= 0.
\end{aligned}$$

Proposition 4.4. The divided difference of polynomial multiples of Horner polynomials can be described as:

$$\left\langle \frac{1}{w(z)}, z^{n-j} w_k(z) \right\rangle = \delta_{j,k} \qquad 0 \leq j, k \leq n.$$

Proof. We will proceed in three cases:

Case 1: $k < j$. In this case $z^{n-j} w_k(z)$ is a polynomial of degree less than n, which we have seen produces 0 under the divided difference operator.

Case 2: $k = j$. This means $z^{n-k} w_k(z)$ is a monic polynomial of degree n. Because $\{w_k\}$ forms a basis of $\mathcal{P}_n$ we see

$$z^{n-k} w_k(z) = w_n(z) + \sum_{i=0}^{n-1} a_i w_i(z).$$

Hence, because $\left\langle \frac{1}{w(z)}, w_k(z) \right\rangle = \delta_{k,n}$,

$$\left\langle \frac{1}{w(z)}, z^{n-k} w_k(z) \right\rangle = \left\langle \frac{1}{w(z)}, w_n(z) \right\rangle + \sum_{i=0}^{n-1} a_i \left\langle \frac{1}{w(z)}, w_i(z) \right\rangle = 1.$$

Case 3: $k > j$. Let $k = j + i$ where $i \geq 1$. Then

$$\begin{aligned} z^{n-j} w_k(z) &= z^{n-k+i}(z^k + b_1 z^{k-1} + \ldots + b_k) \\ &= z^{i-1}(z^{n+1} + b_1 z^n + \ldots + b_k z^{n-k+1}) \\ &= z^{i-1} w(z) - z^{i-1}(b_{k+1} z^{n-k} + \ldots + b_{n+1}). \end{aligned}$$

The first summand is a multiple of w so when we introduce our divided difference operator, we get a 0. The remaining polynomial has order $n - j - 1 < n$ so as in case 1, we get 0 for our divided difference. $\square$

Now for any polynomial $u(z)$ we define the *difference quotient*

$$u[z, t] = \frac{u(z) - u(t)}{z - t}. \tag{4.11}$$

It is easy to see that

$$z^{k+1} - t^{k+1} = (z - t) \sum_{j=0}^{k} z^j t^{k-j} \Rightarrow \frac{z^{k+1} - t^{k+1}}{z - t} = \sum_{j=0}^{k} z^j t^{k-j}, \tag{4.12}$$

which means that for any polynomial u of degree $m + 1$, the difference quotient $u[z, t]$ is a homogeneous symmetric polynomial in z and t of degree m. If we rearrange some terms, we can introduce our Horner polynomials as follows

$$w[z,t] = \sum_{k=0}^{n} w_k(z)t^{n-k} = \sum_{k=0}^{n} w_k(t)z^{n-k}. \tag{4.13}$$

Isaac Newton used the concept of divided differences to calculate the coefficients of the polynomial (of least possible degree) that interpolates a given set of data points. That is, he described a means of developing a polynomial that reaches every point in a given set. Charles Hermite took this a step further by describing a method of producing a polynomial that not only matches an unknown function at certain points, but whose first k derivatives match the unknown function's first k derivatives at those points.

Specifically, if $f(z) = g(z)w(z) + r(z)$, note that for any linear functional L on $\mathcal{P}$ such that $L\{g(z)w(z)\} = 0$ for any $g(z)$ then $Lf(z) = L(g(z)w(z) + r(z)) = Lr(z)$. If a is a root of w with multiplicity m, then applying Leibniz's rule, we see that $\langle (a,k)(z), g(z)w(z)\rangle = 0$ for any polynomial $g(z)$ and $0 \leq k \leq m-1$. This last qualification is critical, because if k cant exceed $m-1$ then, thinking of the inner product from a Taylor functional standpoint, when we differentiate $g(z)w(z)$ k times each summand (assuming we split things naturally using the product rule) will have a $w(z)$ term and will zero out when evaluated at a. Thus $D^k f(a) = D^k r(a)$ for $0 \leq k \leq m-1$. This means that $r(z)$ is the element of $\mathcal{P}_n$ that interpolates $f(z)$, in the sense of Hermite, at the roots of $w(z)$.

Proposition 4.5. [Hermite Interpolation] Let f be a rational function with polynomials g, r such that $f = gw + r$ with $r \in \mathcal{P}_n$ and w defined as usual (f is defined on the roots of w). Then

$$\left\langle \frac{1}{w(z)}, f(z)w[z,t] \right\rangle = \left\langle \frac{w[z,t]}{w(z)}, f(z) \right\rangle = r(t) \tag{4.14}$$

is the unique polynomial of degree less than or equal to n that interpolates f at the roots of w in the sense of Hermite.

Proof. Since applying our divided difference operator to any multiple of w yields zero, it is clear that

$$\left\langle \frac{1}{w(z)}, f(z)w[z,t] \right\rangle = \left\langle \frac{1}{w(z)}, r(z)w[z,t] \right\rangle.$$

Then since $r \in \mathcal{P}_n$ it is suffient to show that

$$\left\langle \frac{1}{w(z)}, z^j w[z,t] \right\rangle = t^j, \qquad 0 \leq j \leq n.$$

By (4.13) and Proposition 4.4 we have

$$\begin{aligned}
\left\langle \frac{1}{w(z)}, z^j w[z,t] \right\rangle &= \left\langle \frac{1}{w(z)}, z^j \left[\sum_{k=0}^{n} w_{n-k}(z) t^k \right] \right\rangle \\
&= \sum_{k=0}^{n} \left\langle \frac{1}{w(z)}, z^j w_{n-k}(z) \right\rangle t^k \\
&= \sum_{k=0}^{n} \delta_{n-j,n-k} t^k \\
&= \sum_{k=0}^{n} \delta_{j,k} t^k \\
&= t^j.
\end{aligned}$$

Note that the second step pulls out t^k from the inner product because the operator functions on variable z, making t just a constant. Then for the last equality, because $w[z,t]$ is just a polynomial, by (3.1) we have

$$\left\langle \frac{1}{w(z)}, f(z) w[z,t] \right\rangle = \left\langle \frac{w[z,t]}{w(z)}, f(z) \right\rangle.$$

We call (4.14) the *general interpolation formula*. □

There is another interesting property concerning the difference quotient we can now prove.

Proposition 4.6. Let f, g and h be rational functions such that f and gh have no common poles. Then

$$\langle f, gh \rangle = \langle \langle -f[x,y], g(x) \rangle, h(y) \rangle. \tag{4.15}$$

Proof. First, given $n \geq 0$, we use (4.12) to show

$$\begin{aligned}
(a,n)[x,y] &= \frac{\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(y-a)^{n+1}}}{x-y} \\
&= \frac{(y-a)^{n+1} - (x-a)^{n+1}}{(x-y)(x-a)^{n+1}(y-a)^{n+1}} \\
&= -\left[\frac{(x-a)^{n+1} - (y-a)^{n+1}}{(x-y)}\right] \frac{1}{(x-a)^{n+1}(y-a)^{n+1}} \\
&= -\sum_{k=0}^{n} (x-a)^k (y-a)^{n-k} \cdot \frac{1}{(x-a)^{n+1}(y-a)^{n+1}} \\
&= -\sum_{k=0}^{n} (x-a)^{k-n-1} (y-a)^{-k-1} \\
&= -\sum_{k=0}^{n} (a,n-k)(x)(a,k)(y) \\
&= -\sum_{k=0}^{n} (a,k)(x)(a,n-k)(y).
\end{aligned}$$

So if $f(z)$ is of the form $(a,n)(z)$ then by Leibniz's rule (2.8) we see

$$\begin{aligned}
\langle \langle -(a,n)[x,y], g(x) \rangle, h(y) \rangle &= \left\langle \left\langle \sum_{k=0}^{n} (a,k)(x)(a,n-k)(y), g(x) \right\rangle, h(y) \right\rangle \\
&= \left\langle \sum_{k=0}^{n} \langle (a,k)(x), g(x) \rangle \, (a,n-k)(y), h(y) \right\rangle \\
&= \sum_{k=0}^{n} \langle (a,k)(x), g(x) \rangle \, \langle (a,n-k)(y), h(y) \rangle \\
&= \sum_{k=0}^{n} \langle (a,k), g \rangle \, \langle (a,n-k), h \rangle \\
&= \langle (a,n), gh \rangle .
\end{aligned}$$

Now if f is a polynomial, say $f(t) = t^i$, then by (4.12) and antisymmetry we see

$$\begin{aligned}
\langle\langle -f[x,y], g(x)\rangle, h(y)\rangle &= \left\langle \left\langle -\sum_{k=0}^{i-1} x^k y^{i-1-k}, g(x) \right\rangle, h(y) \right\rangle \\
&= \left\langle -\sum_{k=0}^{i-1} \left\langle x^k, g(x) \right\rangle y^{i-1-k}, h(y) \right\rangle \\
&= -\sum_{k=0}^{i-1} \left\langle x^k, g(x) \right\rangle \left\langle y^{i-1-k}, h(y) \right\rangle \\
&= -\sum_{k=0}^{i-1} \left\langle g(x), x^k \right\rangle \left\langle h(x), x^{i-1-k} \right\rangle \\
&= -\left\langle g(x)h(x)), x^i \right\rangle \\
&= \left\langle x^i, g(x)h(x) \right\rangle \\
&= \langle f, gh \rangle .
\end{aligned}$$

Note that the variable change midway through the proof is for convenience. As long as the change is consistent in both arguments, the inner product is unaffected. Also, the next step is a generalization of Leibniz's rule. Even though g and h can be any rational function, the rule holds because any rational function can be split into the sum of a strictly rational function and a polynomial and the inner product of two polynomials is just zero (essentially we can assume g and h are just strictly rational, and use (2.8) as we always do). So now that we've proven the statement for basis elements of the polynomials and strictly rational functions, we use linearity to finish the proposition. □

For an explicit example of the operation of this proposition, see Appendix B.

5. Gram Matrices and Polynomial Bases

We now introduce a specific matrix construction known as a *Gram matrix* which will set us up for the next section on symmetric polynomials using our inner product and difference quotient. First, let $\mathcal{E}$ be a complex vector space of dimension m and let $\mathcal{E}^*$ be its dual vector space. The elements of the Cartesian products $\mathcal{E}^m$ and $(\mathcal{E}^*)^m$ are written in bold as column vectors. Let $\mathcal{M}$ be the algebra of $m \times m$ matrices with complex entries. Then for each pair $(\mathbf{L}, \mathbf{f}) \in (\mathcal{E}^*)^m \times \mathcal{E}^m$ we define the *Gram matrix* $[\mathbf{L} : \mathbf{f}]$ as the element of $\mathcal{M}$ whose (j,k) entry is the number $L_j f_k$.

Proposition 5.1. The following are basic properties of Gram matrices:

(i) If A and B are in $\mathcal{M}$ then $[A\mathbf{L} : B\mathbf{f}] = A[\mathbf{L} : \mathbf{f}]B^T$.

(ii) The matrix $[\mathbf{L} : \mathbf{f}]$ is nonsingular if and only if $\mathbf{L}$ and $\mathbf{f}$ are ordered bases of $\mathcal{E}^*$ and $\mathcal{E}$ respectively.

(iii) For each basis $\mathbf{f}$ of $\mathcal{E}$ there exists a unique basis $\mathbf{f}^*$, called the dual basis of $\mathbf{f}$, that satisfies $[\mathbf{f}^* : \mathbf{f}] = I$.

(iv) If $\mathbf{f}$ is a basis of $\mathcal{E}$ and $\mathbf{g} = B\mathbf{f}$ for some $B \in \mathcal{M}$, then $B^T = [\mathbf{f}^* : \mathbf{g}]$.

(v) If $\mathbf{g}^*$ is a basis of $\mathcal{E}^*$ and $\mathbf{L} = A\mathbf{g}^*$ for some $A \in \mathcal{M}$, then $A = [\mathbf{L} : \mathbf{g}]$.

(vi) Let $\mathbf{g}$ be a basis of $\mathcal{E}$, let $\mathbf{L}$ be in $(\mathcal{E}^*)^m$ and $\mathbf{f}$ in $\mathcal{E}^m$. Then $[\mathbf{L} : \mathbf{f}] = [\mathbf{L} : \mathbf{g}][\mathbf{g}^* : \mathbf{f}]$.

(vii) If $\mathbf{f}$ and $\mathbf{g}$ are bases of $\mathcal{E}$ then $[\mathbf{f}^* : \mathbf{g}][\mathbf{g}^* : \mathbf{f}] = I$.

(viii) If $\mathbf{f}$ is a basis of $\mathcal{E}$ and A, B are nonsingular elements of $\mathcal{M}$, then the bases of $A\mathbf{f}^*$ and $B\mathbf{f}$ are dual of each other if and only if $AB^T = I$.

Proof. First let $\mathcal{E}$ be a complex m-dimensional vector space with standard basis $\mathcal{B} = \{e_i\}$ while $\mathcal{E}^*$ is the associated dual space with dual basis $\mathcal{B}^* = \{\epsilon_i\}$ such that $\epsilon_i(e_j) = \delta_{i,j}$.

(i) Let $A, B \in \mathcal{M}$ with $A = (a_{ij})$ and $B = (b_{ij})$. Further, let $\mathbf{L}^T = (\ell_1, \ldots, \ell_m)$ and $\mathbf{f}^T = (f_1, \ldots, f_m)$. From this we see

$$A\mathbf{L} = \begin{pmatrix} \sum_{i=1}^m a_{1i}\ell_i \\ \vdots \\ \sum_{i=1}^m a_{mi}\ell_i \end{pmatrix} \qquad B\mathbf{f} = \begin{pmatrix} \sum_{i=1}^m b_{1i}f_i \\ \vdots \\ \sum_{i=1}^m b_{mi}f_i \end{pmatrix},$$

$$[A\mathbf{L} : B\mathbf{f}] = \begin{pmatrix} \sum_{i=1}^m \sum_{j=1}^m a_{1i} b_{1j} \ell_i f_j & \dots & \sum_{i=1}^m \sum_{j=1}^m a_{1i} b_{mj} \ell_i f_j \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^m \sum_{j=1}^m a_{mi} b_{1j} \ell_i f_j & \dots & \sum_{i=1}^m \sum_{j=1}^m a_{mi} b_{mj} \ell_i f_j \end{pmatrix} = A[\mathbf{L} : \mathbf{f}]B^T.$$

The second equality results from the definition of the Gram matrix and simple inspection.

(ii) Consider $\mathbf{L}$ and $\mathbf{f}$ to be ordered bases of $\mathcal{E}^*$ and $\mathcal{E}$ respectively. This means there exist linear isomorphisms A and B from the standard bases to $\mathbf{L}$ and $\mathbf{f}$, i.e. $\mathbf{L} = A(\epsilon_1, \dots, \epsilon_m)^T$ and $\mathbf{f} = B(e_1, \dots, e_m)^T$. By (i) we know $[\mathbf{L} : \mathbf{f}] = [A\epsilon : Be] = A[\epsilon : e]B^T$. Now the determinant of $[\epsilon : e]$ is just 1, and A and B are invertible matrices, so $det[\mathbf{L} : \mathbf{f}] \neq 0$ which means it is nonsingular. Conversely, assuming $\mathbf{L} = A\epsilon$ and $\mathbf{f} = Be$ (where A and B are not necessarily linear isomorphisms) we can use the fact that $[\mathbf{L} : \mathbf{f}]$ is non singular to prove A and B^T have non-zero determinants, and are thus invertible. Thus $\epsilon = A^{-1}L$ and $e = B^{-1}f$ which means L and $\mathbf{f}$ are just ordered bases of our spaces.

(iii) We know for any basis $\mathbf{f}$ of $\mathcal{E}$ there will be a unique dual basis $\mathbf{f}^*$ of $\mathcal{E}^*$. As with the previous part, we have invertible matrices $A, B \in \mathcal{M}$ such that $\mathbf{f}^* = A$**ffl** and $\mathbf{f} = B\mathbf{e}$, however, from linear algebra we know that $AB^T = I$. Hence

$$[\mathbf{f}^* : \mathbf{f}] = [A\epsilon : Be] = A[\epsilon : e]B^T = AIB^T = I.$$

(iv) Let $\mathbf{f}$ be a basis of $\mathcal{E}$ with unique dual basis $\mathbf{f}^*$ and $\mathbf{g} = B\mathbf{f}$ with $B \in \mathcal{M}$. Then using our previous proofs we see

$$[\mathbf{f}^* : \mathbf{g}] = [\mathbf{f}^* : B\mathbf{f}] = [\mathbf{f}^* : \mathbf{f}]B^T = IB^T = B^T.$$

(v) Let $\mathbf{g}^*$ be a basis of $\mathcal{E}^*$ with unique dual basis $\mathbf{g}$ and $\mathbf{L} = A\mathbf{g}^*$ with $A \in \mathcal{M}$. Then using our previous proofs we see

$$[\mathbf{L} : \mathbf{g}] = [A\mathbf{g}^* : \mathbf{g}] = A[\mathbf{g}^* : \mathbf{g}] = AI = A.$$

(vi) Let $\mathbf{g}$ be a basis of $\mathcal{E}$ with $\mathbf{L} \in (\mathcal{E}^*)^m$ and $\mathbf{f} \in \mathcal{E}^m$. Then we know there exists a unique dual basis $\mathbf{g}^*$ of $\mathcal{E}^*$ such that $[\mathbf{g}^* : \mathbf{g}] = I$ and there exists $A, B \in \mathcal{M}$ such that $\mathbf{L} = A\mathbf{g}^*$ and $\mathbf{f} = B\mathbf{g}$. So

$$[A\mathbf{g}^* : B\mathbf{g}] = A[\mathbf{g}^* : \mathbf{g}]B^T = A[\mathbf{g}^* : \mathbf{g}][\mathbf{g}^* : \mathbf{g}]B^T = [A\mathbf{g}^* : \mathbf{g}][\mathbf{g}^* : B\mathbf{g}] = [\mathbf{L} : \mathbf{g}][\mathbf{g}^* : \mathbf{f}].$$

(vii) Let $\mathbf{f}$ and $\mathbf{g}$ be bases of $\mathcal{E}$ which means there are unique dual bases such that

$[\mathbf{f}^* : \mathbf{f}] = I = [\mathbf{g}^* : \mathbf{g}]$. Then by the previous proof we see

$$I = [\mathbf{f}^* : \mathbf{f}] = [\mathbf{f}^* : \mathbf{g}][\mathbf{g}^* : \mathbf{f}].$$

(viii) If $AB^T = I$ then because we have a unique dual basis $\mathbf{f}^*$ of $\mathbf{f}$ such that $[\mathbf{f}^* : \mathbf{f}] = I$ we see $I = A[\mathbf{f}^* : \mathbf{f}]B^T = [A\mathbf{f}^* : B\mathbf{f}]$. Now because Bf is a basis of $\mathcal{E}$ (B is invertible) this means $A\mathbf{f}^*$ is a dual basis of $B\mathbf{f}$. The converse is a similar proof.

□

From now on the space $\mathcal{E}$ will be the set $\mathcal{P}_n$, the polynomials of degree less than or equal to n. Let J be the permutation matrix of order $n+1$ that reverses order, that is $J(a_0, a_1, \ldots, a_n)^T = (a_n, a_{n-1}, \ldots, a_0)^T$. Note that J is symmetric and that $J^2 = I$. Letting $\mathbf{w} = (w_0, w_1, \ldots, w_n)^T$ and $\mathbf{p}(z) = (1, z, \ldots, z^n)^T$ we can rewrite (4.13) as

$$w[z,t] = \mathbf{w}^T(z)J\mathbf{p}(t) = \mathbf{p}^T(t)J\mathbf{w}(z). \tag{5.1}$$

Further, based on our definition of the Horner polynomials we can rewrite the difference quotient further as $w[z,t] = \mathbf{p}^T(z)JB\mathbf{p}(t)$ where

$$B = \begin{vmatrix} 1 & 0 & \ldots & & 0 \\ b_1 & 1 & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \\ b_{k-1} & b_{k-2} & \ldots & 1 & 0 \\ b_k & b_{k-1} & \ldots & b_1 & 1 \end{vmatrix}.$$

Matrix JB is what we call a *Hankel matrix* (a matrix with constant skew-diagonals – closely related to Toeplitz matrices) and is symmetric. Now define *the reversal* of $A \in \mathcal{M}$ by

$$A^{\#} = JAJ. \tag{5.2}$$

This leads us to our next proposition.

Proposition 5.2. For any ordered basis $\mathbf{f}$ of $\mathcal{P}_n$ there exists a unique ordered basis $\mathbf{F}$ of $\mathcal{P}_n$ such that

$$w[z,t] = \mathbf{f}^T(z)J\mathbf{F}(t). \tag{5.3}$$

Proof. Let $\mathbf{f} = A\mathbf{p}$ with $A \in \mathcal{M}$. Hence

$$\begin{aligned} \mathbf{f}^T &= \mathbf{p}^T A^T \\ &= \mathbf{p}^T[\mathbf{p}^* : A\mathbf{p}] \\ &= \mathbf{p}^T[\mathbf{p}^* : \mathbf{f}] \\ \mathbf{f}^T[\mathbf{f}^* : \mathbf{p}] &= \mathbf{p}^T. \end{aligned}$$

Substitute this into (5.1) to get

$$w[z,t] = \mathbf{f}^T(z)[\mathbf{f}^* : \mathbf{p}]J\mathbf{w}(t).$$

Define $\mathbf{F} = [\mathbf{f}^* : \mathbf{p}]^\# \mathbf{w}$. Because $\mathbf{f}^*$ and $\mathbf{p}$ are bases of $\mathcal{E}^*$ and $\mathcal{E}$ respectively, $[\mathbf{f}^* : \mathbf{p}]^\#$ is nonsingular by Proposition 5.1 (ii). Then because $\mathbf{w}$ is a basis of $\mathcal{P}_n$, we see $\mathbf{F}$ is another basis of $\mathcal{P}_n$ and that

$$w[z,t] = \mathbf{f}^T(z)J\mathbf{F}(t).$$

As for uniqueness, assume there were another basis $\mathbf{G}(t)$ such that $w[z,t] = \mathbf{f}^T(z)J\mathbf{F}(t) = \mathbf{f}^T(z)J\mathbf{G}(t)$. Applying $\mathbf{f}^*(z)$ to both sides leaves us with $J\mathbf{F}(t) = J\mathbf{G}(t)$, and since J is nonsingular $\mathbf{F}(t) = \mathbf{G}(t)$. $\square$

Since $\mathbf{w} = B\mathbf{p}$, the bases $\mathbf{f}$ and $\mathbf{F}$ of the above proposition are related by

$$\mathbf{F} = A^\# B A^T \mathbf{f}$$

where $A = [\mathbf{f}^* : \mathbf{p}]$. Note also that $[\mathbf{f}^* : \mathbf{F}]^T = [\mathbf{f}^* : \mathbf{F}]^\#$.

Proposition 5.3. Let $\mathbf{f}$ and $\mathbf{F}$ be bases of $\mathcal{P}_n$. Then the following statments are equivalent:

(i) There exists a symmetric permutation matrix P such that

$$w[z,t] = \mathbf{f}^T(z)P\mathbf{F}(t).$$

(ii) There exists a permutation σ of $\{0,1,\ldots,n\}$ such that $\sigma^2 = 1$ and

$$\left\langle \frac{1}{w}, f_j F_{\sigma(k)} \right\rangle = \left\langle \frac{F_{\sigma(k)}}{w}, f_j \right\rangle = \delta_{j,k}, \qquad 0 \le j,k \le n.$$

Proof. First assume (i) holds and let $A = [\mathbf{f}^* : \mathbf{p}]$. Then $\mathbf{p} = A^T\mathbf{f}$. From the previous proposition we found that

$$w[z,t] = \mathbf{f}^T(z)[\mathbf{f}^* : \mathbf{p}]J\mathbf{w}(t) = \mathbf{f}^T(z)AJ\mathbf{w}(t),$$

so by the cancellation technique used to prove uniqueness in the previous proposition we get $P\mathbf{F} = AJ\mathbf{w}$. Let $A = (a_{ij}), A^{-1} = (c_{ij})$ and let σ be the permutation represented by the matrix P, i.e.

$$P(y_0, y_1, \ldots, y_n)^T = (y_{\sigma(0)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})^T.$$

Then we can see

$$\mathbf{f}^T P\mathbf{F} = \left(\mathbf{p}^T A^{-1}\right)(AJ\mathbf{w}) \Rightarrow f_j F_{\sigma(k)} = \sum_{i=0}^{n} c_{ij} p_i \sum_{r=0}^{n} a_{kr} w_{n-r}.$$

By Proposition 4.4 we obtain

$$\left\langle \frac{1}{w}, f_j F_{\sigma(k)} \right\rangle = \sum_{i=0}^{n}\sum_{r=0}^{n} a_{kr} c_{ij} \left\langle \frac{1}{w}, z^i w_{n-r} \right\rangle = \sum_{i=0}^{n}\sum_{r=0}^{n} a_{kr} c_{ij} \delta_{i,r} = \sum_{i=0}^{n} a_{ki} c_{ij} = \delta_{k,j}.$$

Now assume (ii) holds. Let P be the matrix that corresponds to the permutation σ as before. Changing the definition in the proof of the previous proposition, given the basis $\mathbf{f}$ and the symmetric permutation matrix P, then $\mathbf{G} = P[\mathbf{f}^* : \mathbf{p}]J\mathbf{w}$ is the unique basis that satisfies

$$w[z,t] = \mathbf{f}^T(z) P\mathbf{G}(t) = \sum_{j=0}^{n} f_j(z) G_{\sigma(j)}(t)$$

Then because f_j and $F_{\sigma(k)}$ are just elements of $\mathcal{P}_n$, then using (4.14) and hypothesis (ii) we get

$$\begin{aligned}
F_{\sigma(k)}(t) &= \left\langle \frac{1}{u(z)}, F_{\sigma(k)}(z) w[z,t] \right\rangle \\
&= \left\langle \frac{1}{w(z)}, F_{\sigma(k)}(z) \sum_{j=0}^{n} f_j(z) G_{\sigma(j)}(t) \right\rangle \\
&= \left\langle \frac{1}{w(z)}, F_{\sigma(k)}(z) \sum_{j=0}^{n} f_j(z) \right\rangle G_{\sigma(j)}(t) \\
&= \sum_{j=0}^{n} \left\langle \frac{1}{w(z)}, F_{\sigma(k)}(z) f_j(z) \right\rangle G_{\sigma(j)}(t) \\
&= \sum_{j=0}^{n} \delta_{j,k} G_{\sigma(j)}(t) \\
&= G_{\sigma(k)}(t).
\end{aligned}$$

Therefore $\mathbf{F} = \mathbf{G}$ and (i) follows. □

We say that the bases **f** and **F** are *biorthonormal* with respect to w, or that they are w-reciprocal since from the last proposition $f_k^* = F_{n-k}/w$ and $F_k^* = f_{n-k}/w$ for $0 \le k \le n$. A more general statement can be written as follows:

Corollary 5.1. If **f** and **F** are bases of $\mathcal{P}$ which are biorthonormal with respect to w then

$$F_k^* g = \left\langle \frac{1}{w}, g f_{\sigma(k)} \right\rangle, \qquad 0 \le k \le n, g \in \mathcal{P}_n,$$

and

$$f_k^* g = \left\langle \frac{1}{w}, g F_{\sigma(k)} \right\rangle, \qquad 0 \le k \le n, g \in \mathcal{P}_n,$$

where σ is as in the above proposition.

Next we generalize the Leibniz rule for divided differences (2.8).

Corollary 5.2. Let **f** and **F** be bases of $\mathcal{P}_n$ which are biorthonormal with respect to w, and let σ be as in proposition 5.3. Then for any g and h in $\mathcal{P}_n$ we have

$$\left\langle \frac{1}{w}, gh \right\rangle = \sum_{j=0}^{n} f_j^* g F_{\sigma(j)}^* h \tag{5.4}$$

Proof. Let $g, h \in \mathcal{P}_n$. Then since

$$g = \sum_{j=0}^{n} f_j^* g f_j \qquad \text{and} \qquad h = \sum_{k=0}^{n} F_{\sigma(k)}^* h F_{\sigma(k)}$$

we get

$$\begin{aligned}
\left\langle \frac{1}{w}, gh \right\rangle &= \left\langle \frac{1}{w}, \sum_{j=0}^{n} \sum_{k=0}^{n} f_j^* g f_j F_{\sigma(k)}^* h F_{\sigma(k)} \right\rangle \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} \left\langle \frac{1}{w}, f_j^* g f_j F_{\sigma(k)}^* h F_{\sigma(k)} \right\rangle \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} f_j^* g F_{\sigma(k)}^* h \left\langle \frac{1}{w}, f_j F_{\sigma(k)} \right\rangle \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} f_j^* g F_{\sigma(k)}^* h \delta_{j,k} \\
&= \sum_{j=0}^{n} f_j^* g F_{\sigma(j)}^* h.
\end{aligned}$$

□

Now before we move on to some examples using Propositions 5.2 and 5.3 we prove a statement about the relationship between the bases developed therein.

Proposition 5.4. Let $\mathbf{f}, \mathbf{F}$ and $\mathbf{g}, \mathbf{G}$ be two pairs of bases of $\mathcal{P}_n$ that are biorthonormal with respect to u. Let P and Q be symmetric permutation matrices for which

$$w[z,t] = \mathbf{F}^T(z)P\mathbf{f}(t) = \mathbf{G}^T(z)Q\mathbf{g}(t).$$

Then we have

$$P[\mathbf{F}^* : \mathbf{G}]Q[\mathbf{f}^* : \mathbf{g}]^T = I.$$

Proof. As we saw in the proof of Proposition 5.2, $\mathbf{G}^T = \mathbf{F}^T[\mathbf{F}^* : \mathbf{G}]$ and $\mathbf{g} = [\mathbf{f}^* : \mathbf{g}]^T\mathbf{f}$. Hence from our hypothesis we have

$$\mathbf{F}^T(z)P\mathbf{f}(t) = \mathbf{F}^T(z)[\mathbf{F}^* : \mathbf{G}]Q[\mathbf{f}^* : \mathbf{g}]^T\mathbf{f}(t).$$

Using the same cancellation property as in previous propositions we see

$$P\mathbf{f}(t) = [\mathbf{F}^* : \mathbf{G}]Q[\mathbf{f}^* : \mathbf{g}]^T\mathbf{f}(t),$$

and since $P^2 = I$,

$$\mathbf{f}(t) = P[\mathbf{F}^* : \mathbf{G}]Q[\mathbf{f}^* : \mathbf{g}]^T\mathbf{f}(t),$$

which proves our statement. □

Notice that if $\mathbf{F} = \mathbf{g}$ and $\mathbf{f} = \mathbf{G}$ we obtain

$$P[\mathbf{F}^* : \mathbf{f}]P[\mathbf{f}^* : \mathbf{F}]^T = I,$$

which is equivalent to

$$P[\mathbf{F}^* : \mathbf{f}]P = [\mathbf{F}^* : \mathbf{f}]^T.$$

Examples

Our first example uses a simple monomial for w and the standard basis of $\mathcal{P}_n$.

Example 5.1. Let $\mathbf{p}$ be the standard basis of powers for $\mathcal{P}_n$ as before, and $w(z) = z^{n+1}$. Then from (4.13) we know

$$w[z,t] = \sum_{k=0}^{n} t^{n-k}z^k$$

which means, by proposition 5.2, our biorthonormal basis $\mathbf{F}(t) = (t^n, t^{n-1}, \ldots, 1)$ is the same basis $\mathbf{p}$ we started with. Then proposition 5.3 tells us $p_k^* = z^{n-k}/z^{n+1} = 1/z^{k+1}$, which corresponds to the Taylor functional at zero of order k.

Now we consider the monic polynomial w weve been using for most of this paper and whose Horner polynomials we rely on greatly. Again we use the standard basis of $\mathcal{P}_n$.

Example 5.2. Let $w(z) = z^{n+1} + b_1 z^n + \ldots + b_{n+1}$. Recall from (4.13)

$$w[z,t] = \sum_{k=0}^{n} w_{n-k}(t) z^k$$

where w_k are the Horner polynomials associated with w. In this case we find $p_k^* = w_{n-k}/w$ and $w_k^* = z^{n-k}/w(z)$.

We can also express the difference quotient of w in terms of the Lagrange interpolation polynomials. As before, w is a product of simple roots.

Example 5.3. Let $w(z) = \prod_{j=0}^{n}(z - a_j)$ where the a_j are distinct complex numbers. The Lagrange interpolation polynomials associated with nodes a_j are defined by

$$\ell_k(z) = \frac{w[z, a_k]}{w'(a_k)}, \qquad 0 \le k \le n. \tag{5.5}$$

By Lagrange's interpolation formula we have

$$w[z,t] = \sum_{k=0}^{n} w[t, a_k] \ell_k(z).$$

Further, proposition 5.3 tells us $\ell_k^*(z) = \dfrac{w[z, a_k]}{w(z)} = 1/(z - a_k)$ which is just evaluation at a_k.

Next we use the same w from the previous eample, but instead of Lagrange polynomials, we define a set of polynomials that are increasing factors of w (given a sequence of its roots).

Example 5.4. Let w be as in the previous example. Define $N_0(z) = 1$ and

$$N_k(z) = (z - a_0)(z - a_1) \ldots (z - a_{k-1}), \qquad 1 \le k \le n. \tag{5.6}$$

These are what we call the *Newton polynomials* associated with the sequence of roots $a_0, a_1, \ldots, a_n$. Let F_k be the Newton polynomials associated with the reversal of this sequence, i.e. $a_n, a_{n-1}, \ldots, a_0$. Then by a simple telescopic summation we have

$$w[z,t] = \sum_{k=0}^{n} F_{n-k}(t) N_k(z)$$

which means $N_k^* = F_{n-k}/w = 1/N_{k+1}$ and $F_k^* = N_{n-k}/w = 1/F_{k+1}$.

Our last example assumes that w's roots are not necessarily simple, and requires us to develop another set of interpolation polynomials.

Example 5.5. Let $w(z) = \prod_{j=0}^{s}(z - a_j)^{m_j}$ where a_j are distinct roots and m_j are postiive integers that satisfy $\sum_j m_j = n + 1$. In (2.9) we defined the polynomials

$$q_{i,k}(z) = \frac{w(z)}{(z - a_i)^{m_i - k}}, \qquad 0 \le i \le s, \quad 0 \le k \le m_i - 1$$

which is our original polynomial w with all but k of the a_i terms removed ($q_{i,0}(z)$ then becomes a convenient means of removing a root from w). Then

$$w[z, t] = \sum_{i=0}^{s} \sum_{k=0}^{m_i - 1} H_{i,k}(t) q_{i, m_i - 1 - k}(z)$$

where the $H_{i,k}$ are the basic *Hermite interpolation polynomials* associated with the nodes a_j with multiplicities m_j. Therefore $H_{i,k}^* = 1/(z - a_i)^{1+k}$.

Determinants of Specific Gram Matrices

Using the Lagrange polynomials we defined in the previous section we can express the product of the derivatives of w evaluated at its roots as the square of a specific Vandermonde matrix.

Recall at the beginning of this section we defined a symmetric reversal matrix J. Note that $\det J = (-1)^{\binom{n+1}{2}}$. Let $\mathbf{a} = (a_0, a_1, \ldots, a_n)^T$ be a vector of distinct complex coordinates. Now denote by $\ell_{\mathbf{a}}$ the basis of Lagrange polynomials associated with the nodes a_j, that is $\ell_{\mathbf{a}} = (\ell_0, \ell_1, \ldots, \ell_n)^T$. We define

$$V_{\mathbf{a}} = \det\,[\ell_{\mathbf{a}}^* : \mathbf{p}] = \det[a_j^k] \tag{5.7}$$

where $\mathbf{p}$ is the standard power basis of $\mathcal{P}_n$. The second equality comes from Example 5.3 of the previous section in which we found that the associated dual basis of the Lagrange polynomials amounted to evaluation of functions at the roots a_j. We call $[a_j^k]$ the Vandermonde matrix of $\mathbf{a}$, and in Appendix C we prove its determinant is

$$V_{\mathbf{a}} = \prod_{0 \le i < j \le n} (a_j - a_i). \tag{5.8}$$

Now let $\mathbf{u} = (u_0, u_1, \ldots, u_n)^T$ where u_k is a monic polynomial of degree k for $0 \le k \le n$. Then $\mathbf{u} = A\mathbf{p}$ where A is a lower triangular matrix with determinant one and for which $\mathbf{p}^* = A^T \mathbf{u}^*$. Hence

$$[\ell_{\mathbf{a}}^* : \mathbf{u}] = [\ell_{\mathbf{a}}^* : A\mathbf{p}] = [\ell_{\mathbf{a}}^* : \mathbf{p}]\, A^T$$

and, taking the determinant of both sides,

$$\det\left[\ell_{\mathbf{a}}^* : \mathbf{u}\right] = \det\left[\ell_{\mathbf{a}}^* : \mathbf{p}\right] = V_{\mathbf{a}}. \tag{5.9}$$

So if $\mathbf{u} = (u_0, u_1, \ldots, u_n)^T$ is the vector of Horner polynomials of u, then for any $\mathbf{a}$ defined as above, $\det\left[\ell_{\mathbf{a}}^* : \mathbf{u}\right] = V_{\mathbf{a}}$ and

$$\det\left[\ell_{\mathbf{a}}^* : J\mathbf{u}\right] = (-1)^{\binom{n+1}{2}} V_{\mathbf{a}}. \tag{5.10}$$

Take now $w(z) = \prod_{j=0}^n (z - a_j)$ with distinct roots a_j. Recall from Example 5.2 that $p_k^* = w_{n-k}/w$, which means the inverse of the Vandermonde matrix $\left[\ell_{\mathbf{a}}^* : \mathbf{p}\right]$ (see Proposition 5.1 part (vii)) is

$$\begin{aligned}
\left[\mathbf{p}^* : \ell_{\mathbf{a}}\right] &= \left[p_j^* \ell_k\right] \\
&= \left[\left\langle \frac{w_{n-j}}{w}, \ell_k \right\rangle\right] \\
&= \left[\left\langle \frac{1}{w}, w_{n-j}\ell_k \right\rangle\right] \\
&= \left[\frac{w_{n-j}(a_k)\ell_k(a_k)}{w'(a_k)}\right] \\
&= \left[\frac{w_{n-j}(a_k)}{w'(a_k)}\right]
\end{aligned}$$

where the second to last equality follows from a simple case of (4.2) where all $m_i = 1$, and because $\ell_j(a_k) = \delta_{j,k}$. Computing determinants we obtain

$$\begin{aligned}
V_{\mathbf{a}}^{-1} &= \det\left[\mathbf{p}^* : \ell_{\mathbf{a}}\right] \\
&= \det\left[\frac{w_{n-j}(a_k)}{w'(a_k)}\right] \\
&= \left(\prod_{k=0}^n w'(a_k)\right)^{-1} \det\left[w_{n-j}(a_k)\right] \\
&= \left(\prod_{k=0}^n w'(a_k)\right)^{-1} \det\left[\ell_{\mathbf{a}}^* : J\mathbf{w}\right] \\
&= \left(\prod_{k=0}^n w'(a_k)\right)^{-1} (-1)^{\binom{n+1}{2}} V_{\mathbf{a}}.
\end{aligned}$$

This means

$$\prod_{k=0}^n w'(a_k) = (-1)^{\binom{n+1}{2}} V_{\mathbf{a}}^2. \tag{5.11}$$

6. Symmetric Functions

We are now prepared to discuss how the determinants of certain Gram matrices (which implicitly use our inner product) can produce rather interesting symmetric functions such as the factorial Schur functions. This will require some discussion of partitions, Schur polynomials and a new bilinear map $S : \mathcal{Q}^{n+1} \times \mathcal{Q}^{n+1} \to \mathbb{C}$.

So let $\mathbf{f}$ and $\mathbf{g}$ be elements of $\mathcal{Q}^{n+1}$. Since $\langle f, g \rangle \in \mathbb{C}$ we can really consider the bilinear map $\langle , \rangle$ as a linear functional on $\mathcal{Q}$ allowing us to rewrite our Gram matrices as follows

$$[\mathbf{f} : \mathbf{g}] = [\langle f_j, g_k \rangle]. \tag{6.1}$$

Now define $S : \mathcal{Q}^{n+1} \times \mathcal{Q}^{n+1} \to \mathbb{C}$ as

$$S(\mathbf{f}, \mathbf{g}) = \det[\mathbf{f} : R\mathbf{g}], \qquad \mathbf{f}, \mathbf{g} \in \mathcal{Q}^{n+1} \tag{6.2}$$

where R is the reversion map defined previously, which acts coordinate-wise on $\mathbf{g}$. Moreover, recalling (3.8) and (3.10), we can see

$$\begin{aligned} S(\mathbf{f}, \mathbf{g}) &= \det[\mathbf{f} : R\mathbf{g}] \\ &= \det[\langle f_j, Rg_k \rangle] \\ &= \det[\langle -Rf_j, g_k \rangle] \\ &= \det[\langle g_k, Rf_j \rangle] \\ &= \det[\mathbf{g} : R\mathbf{f}] \\ &= S(\mathbf{g}, \mathbf{f}). \end{aligned}$$

Hence S is a symmetric map.

Partitions and Schur Functions

For the remainder of this section let $w(z) = (z - x_0)(z - x_1) \ldots (z - x_n)$ where x_j are all distinct. Then define $\mathbf{w}$ as the basis of Horner polynomials of w (see Example 5.2) and let $\mathbf{u} = J\mathbf{w}$. Then we know $u_j^* = z^j / w(z)$ for $0 \leq j \leq n$.

We define a partition as a vector with integer components $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ with $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_n \geq 0$. The standard partition we will denote as $\delta = (n, n-1, \ldots, 1, 0)$. A *Young tableau* is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths weakly decreasing (each row has the same or shorter length than its predecessor). As such we can see that a partition λ can be represented by a Young tableau with row i containing λ_i boxes – this we will call *shape* λ. A *semistandard Young tableau* T of shape λ is a filling of the Young diagram of λ with positive integers that are weakly increasing along rows and strictly increasing along columns. The *weight* of a tableau T is defined as $x^T = x_0^{T_0} x_1^{T_1} x_2^{T_2} \dots$ where T_i is the total number of times i appears in the tableau.

We can then define the Schur function of a partition λ as

$$s_\lambda(x_0, \dots, x_n) = \sum_T x^T \tag{6.3}$$

where the sum is taken over all semistandard Young tableaux of shape λ. Then we can see $s_{(k)}(x_0, \dots, x_n) = h_k(x_0, \dots, x_n)$ where (k) is the tableau of one row of k boxes and h_k is the complete homogenous symmetric polynomial of order k on elements $(x_0, \dots, x_n)$ – see (4.5).

Another approach to developing the Schur functions concerns our S functional. For any partition λ define $\mathbf{z}_\lambda$ as the element of $\mathcal{P}^{n+1}$ whose k-th component is z^{λ_k}. We can see that

$$\begin{aligned}
S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta}) &= \det[\mathbf{u}^* : \mathbf{z}_{\lambda+\delta}] \\
&= \det[\left\langle u_j^*, z^{\lambda_k+n-k} \right\rangle] \\
&= \det[\left\langle \frac{z^j}{w(z)}, z^{\lambda_k+n-k} \right\rangle] \\
&= \det[\left\langle \frac{1}{w(z)}, z^{\lambda_k+n-k+j} \right\rangle] \\
&= \det[h_{\lambda_k-k+j}(x_0, \dots, x_n)].
\end{aligned}$$

By Giambelli's Formula (a consequence of Pieri's Formula) we know that the last term is the Schur function associated with the partition λ, i.e. s_λ. Hence the same is true for $S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta})$. This leads us to an important proposition classically presented with respect to partitions, but can be generalized to rational functions and a general basis of $\mathcal{P}_n$.

Proposition 6.1. [Generalized Jacobi-Trudi Identity] Let $\mathbf{u}$ be a basis of $\mathcal{P}_n$ and let $\mathbf{f}$ be in $\mathcal{Q}^{n+1}$. Then for any basis $\mathbf{g}$ of $\mathcal{P}_n$ we have

$$S(\mathbf{u}^*, \mathbf{f}) = \frac{S(\mathbf{g}^*, \mathbf{f})}{S(\mathbf{g}^*, R\mathbf{u})}. \tag{6.4}$$

Proof. By Proposition 5.1 (vi) we know $[\mathbf{u}^* : R\mathbf{f}] = [\mathbf{u}^* : \mathbf{g}][\mathbf{g}^* : R\mathbf{f}]$ and by part (vii) $[\mathbf{u}^* : \mathbf{g}] = [\mathbf{g}^* : \mathbf{u}]^{-1}$. Therefore using the definition of S we see

$$\begin{aligned}
\frac{S(\mathbf{g}^*, \mathbf{f})}{S(\mathbf{g}^*, R\mathbf{u})} &= \frac{\det[\mathbf{g}^* : R\mathbf{f}]}{\det[\mathbf{g}^* : \mathbf{u}]} \\
&= \det\left([\mathbf{g}^* : \mathbf{u}]^{-1}[\mathbf{g}^* : R\mathbf{f}]\right) \\
&= \det\left([\mathbf{u}^* : \mathbf{g}][\mathbf{g}^* : R\mathbf{f}]\right) \\
&= \det[\mathbf{u}^* : R\mathbf{f}] \\
&= S(\mathbf{u}^*, \mathbf{f}).
\end{aligned}$$

□

If we let $\mathbf{f} = R\mathbf{z}_{\lambda+\delta}$, $\mathbf{g} = \ell_{\mathbf{x}}$ and $\mathbf{u} = J\mathbf{w}$ then using (5.7) and (5.10) we can obtain the classical Jacobi-Trudi identity

$$\begin{aligned}
S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta}) &= \frac{S(\ell_{\mathbf{x}}^*, R\mathbf{z}_{\lambda+\delta})}{S(\ell_{\mathbf{x}}^*, RJ\mathbf{w})} \\
&= \frac{\det[\ell_{\mathbf{x}}^* : \mathbf{z}_{\lambda+\delta}]}{\det[\ell_{\mathbf{x}}^* : J\mathbf{w}]} \\
&= \frac{\det[\ell_{\mathbf{x}}^* : \mathbf{z}_{\lambda+\delta}]}{(-1)^{\binom{n+1}{2}} V_{\mathbf{x}}} \\
&= \frac{\det[x_j^{\lambda_k+n-k}]}{(-1)^{\binom{n+1}{2}} V_{\mathbf{x}}} \\
&= (-1)^{\binom{n+1}{2}} V_{\mathbf{x}}^{-1} \det[x_j^{\lambda_k+n-k}] \\
&= V_{\mathbf{x}}^{-1} \det[x_j^{k+\lambda_{n-k}}].
\end{aligned}$$

Alternatively let $\mathbf{g} = \mathbf{N}$, the basis of Newton polynomials associated with $x_0, x_1, \ldots, x_n$. Then from Example 5.4 we know $N_k^* = F_{n-k}/w$, which means by Proposition 5.3 (ii) $\left\langle N_j^*, w_k \right\rangle = \delta_{j,k}$. Thus $[\mathbf{N}^* : \mathbf{w}] = I$ and $\det[\mathbf{N}^* : \mathbf{w}] = 1$. So by Proposition 6.1 we have

$$\begin{aligned}
S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta}) &= \frac{S(\mathbf{N}^*, R\mathbf{z}_{\lambda+\delta})}{S(\mathbf{N}^*, RJ\mathbf{w})} \\
&= \frac{\det[\mathbf{N}^* : \mathbf{z}_{\lambda+\delta}]}{\det[\mathbf{N}^* : J\mathbf{w}]} \\
&= (-1)^{\binom{n+1}{2}} \det[\mathbf{N}^* : \mathbf{z}_{\lambda+\delta}] \\
&= \det[h_{\lambda_{n-k}+k-j}(x_0, \ldots, x_j)].
\end{aligned}$$

This is another representation of the Schur functions in terms of the complete homogeneous symmetric functions. Even further, the Schur functions can be expressed in terms of the determinants of the power sum symmetric functions σ_k, which we show next.

Proposition 6.2. For any partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ we have

$$S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta}) = (-1)^{\binom{n+1}{2}} V_{\mathbf{x}}^{-2} \det\left[\sigma_{\lambda_k+n-k+j}(x_0, \ldots, x_n)\right].$$

Proof. Let us define $\mathbf{g}$ by $g_k(z) = w'(z)z^{\lambda_k+n-k}$, for $0 \le k \le n$. Then by (4.4),

$$\begin{aligned}
[\mathbf{u}^* : \mathbf{g}] &= \left[\left\langle u_j^*(z), g_k(z)\right\rangle\right] \\
&= \left[\left\langle \frac{z^j}{w(z)}, w'(z)z^{\lambda_k+n-k}\right\rangle\right] \\
&= \left[\left\langle \frac{1}{w(z)}, w'(z)z^{\lambda_k+n-k+j}\right\rangle\right] \\
&= \left[\sum_{i=0}^{n} x_i^{\lambda_k+n-k+j}\right] \\
&= \left[\sigma_{\lambda_k+n-k+j}(x_0, \ldots, x_n)\right].
\end{aligned}$$

On the other hand we can write $[\mathbf{u}^* : \mathbf{g}] = [\mathbf{u}^* : \ell_{\mathbf{x}}][\ell_{\mathbf{x}}^* : \mathbf{g}] = [\ell_{\mathbf{x}}^* : \mathbf{u}]^{-1}[\ell_{\mathbf{x}}^* : \mathbf{g}]$. Then we obtain

$$\begin{aligned}
\det[\mathbf{u}^* : \mathbf{g}] &= \frac{\det[\ell_{\mathbf{x}}^* : \mathbf{g}]}{\det[\ell_{\mathbf{x}}^* : \mathbf{u}]} \\
&= \frac{\det[w'(x_j)x_j^{\lambda_k+n-k}]}{\det[\ell_{\mathbf{x}}^* : \mathbf{u}]} \\
&= \prod_{j=0}^{n} w'(x_j)\frac{\det[x_j^{\lambda_k+n-k}]}{\det[\ell_{\mathbf{x}}^* : \mathbf{u}]} \\
&= \prod_{j=0}^{n} w'(x_j)\frac{\det[\ell_{\mathbf{x}}^* : \mathbf{z}_{\lambda+\delta}]}{\det[\ell_{\mathbf{x}}^* : \mathbf{u}]} \\
&= (-1)^{\binom{n+1}{2}} V_{\mathbf{x}}^2 \frac{\det[\ell_{\mathbf{x}}^* : \mathbf{z}_{\lambda+\delta}]}{\det[\ell_{\mathbf{x}}^* : \mathbf{u}]} \\
&= (-1)^{\binom{n+1}{2}} V_{\mathbf{x}}^2 S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta}).
\end{aligned}$$

The penultimate step used (5.11) and the last step used Proposition 6.1. $\square$

Now using the generalized Leibniz rule (Proposition 5.4) we can actually factor our Schur functions. Using the Newton polynomials N_k associated with the roots $(x_0, \ldots, x_n)$ – see Example 5.4 – we get

$$\begin{aligned}\left\langle \frac{1}{w(z)}, z^{\lambda_k - k} z^{n+j} \right\rangle &= \sum_{i=0}^{n} \left\langle N_i^*(z), z^{\lambda_k - k} \right\rangle \left\langle F_{n-k+1}^*, z^{n+j} \right\rangle \\ &= \sum_{i=0}^{n} \left\langle \frac{1}{N_{i+1}(z)}, z^{\lambda_k - k} \right\rangle \left\langle \frac{1}{F_{n-i+1}(z)}, z^{n+j} \right\rangle .\end{aligned}$$

Then using our previous derivations of Schur polynomials this becomes

$$h_{\lambda_k - k + j}(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} h_{\lambda_k - k - i}(x_0, x_1, \ldots, x_i) h_{j+i}(x_i, x_{i+1}, \ldots, x_n),$$

and thus

$$S(\mathbf{u}^*, R\mathbf{z}_{\lambda+\delta}) = \det[h_{\lambda_k - k - i}(x_0, x_1, \ldots, x_i)] \det[h_{j+i}(x_i, x_{i+1}, \ldots, x_n)].$$

Taylor Series Expansions and Cauchy Identities

Recall in (2.7) we defined the Taylor series expansion of a function f around the point a. Taking $a = 0$ and using our reversion map R, we find

$$\begin{aligned} f(z) &= \sum_{k \geq 0} \frac{D^k f(0)}{k!} z^k \\ &= \sum_{k \geq 0} \left\langle \frac{1}{t^{k+1}}, f(t) \right\rangle z^k \\ &= \sum_{k \geq 0} \left\langle Rf(t), R\frac{1}{t^{k+1}} \right\rangle z^k \\ &= \sum_{k \geq 0} \left\langle Rf(t), t^k \right\rangle z^k. \end{aligned}$$

Then because $\langle Rf(t), t^k \rangle$ is a scalar, for any rational function g, we have

$$\langle g, f \rangle = \sum_{k \geq 0} \left\langle Rf(t), t^k \right\rangle \left\langle g(z), z^k \right\rangle . \tag{6.5}$$

Bringing back in our discussion of partitions, let Λ be the set of all partitions $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ such that $\lambda_0 > \lambda_1 > \ldots > \lambda_n$. Also let Π be the group of permutations on $n+1$ elements. These sets help us develop an expansion formula for our S operator.

Theorem 6.1. Let $\mathbf{f}$ and $\mathbf{g}$ be elements of $\mathcal{Q}^{n+1}$ such that each Rf_i has a Taylor series expansion at zero. Then

$$S(\mathbf{g}, \mathbf{f}) = \sum_{\lambda \in \Lambda} S(\mathbf{g}, R\mathbf{z}_\lambda) S(\mathbf{f}, R\mathbf{z}_\lambda). \tag{6.6}$$

Proof. By definition (6.2) and the discussion immediately prior to Proposition 6.1 we know $S(\mathbf{g}, \mathbf{f}) = \det[\mathbf{g} : R\mathbf{f}] = \det[\langle g_j, Rf_i \rangle]$. Then using Leibniz's formula for determinants and (6.5) we see

$$\begin{aligned}
S(\mathbf{g}, \mathbf{f}) &= \sum_{\tau \in \Pi} \operatorname{sign}(\tau) \prod_{j=0}^{n} \left\langle g_j, Rf_{\tau(i)} \right\rangle \\
&= \sum_{\tau \in \Pi} \operatorname{sign}(\tau) \sum_{\mathbf{k} \in \mathbb{N}^{n+1}} \prod_{j=0}^{n} \left\langle f_{\tau(j)}(t), t^{k_j} \right\rangle \prod_{j=0}^{n} \left\langle g_j(z), z^{k_j} \right\rangle \\
&= \sum_{\mathbf{k} \in \mathbb{N}^{n+1}} \left(\sum_{\tau \in \Pi} \operatorname{sign}(\tau) \prod_{j=0}^{n} \left\langle f_{\tau(j)}(t), t^{k_j} \right\rangle \right) \prod_{j=0}^{n} \left\langle g_j(z), z^{k_j} \right\rangle \\
&= \sum_{\mathbf{k} \in \mathbb{N}^{n+1}} \det[\left\langle f_i(t), t^{k_j} \right\rangle] \prod_{j=0}^{n} \left\langle g_j(z), z^{k_j} \right\rangle \\
&= \sum_{\lambda \in \Lambda} \det[\left\langle f_i(t), t^{\lambda_j} \right\rangle] \sum_{\rho \in \Pi} \operatorname{sign}(\rho) \prod_{j=0}^{n} \left\langle g_j(z), z^{\lambda_{\rho(j)}} \right\rangle \\
&= \sum_{\lambda \in \Lambda} \det[\mathbf{f} : \mathbf{z}_\lambda] \det[\mathbf{g} : \mathbf{z}_\lambda].
\end{aligned}$$

The second to last step comes from the fact that reordering the rows of the matrix to turn the $(n+1)$-tuple $\mathbf{k}$ into a partition λ by a series of permutations affects the determinant by a product of the signs of those permutations. $\square$

A specific application of Proposition 6.1 gives us an equality we can use with Theorem 6.1 to develop the Cauchy identities (an application of the Robinson-Schensted-Knuth correspondence).

Proposition 6.3. Let f be a rational function and define $\mathbf{v} \in \mathcal{Q}^{n+1}$ by $v_j = z^{n-j} f(z)$ for $0 \leq j \leq n$. Then

$$S(\mathbf{u}^*, R\mathbf{v}) = \prod_{k=0}^{n} f(x_k). \tag{6.7}$$

Proof. By Proposition 6.1 if we take the basis of Lagrange polynomials ℓ,

$$S(\mathbf{u}^*, R\mathbf{v}) = \frac{S(\ell^*, R\mathbf{v})}{S(\ell^*, R\mathbf{u})}.$$

As for the numerator, we recall that the action of these Lagrange polynomials amounts to evaluation at a given root:

$$\begin{aligned} S(\ell^*, R\mathbf{v}) &= \det\left[\left\langle \frac{1}{z - x_k}, z^{n-j} f(z) \right\rangle\right] \\ &= \det\left[x_k^{n-j} f(x_k)\right] \\ &= \det[x_k^{n-j}] \prod_{k=0}^{n} f(x_k). \end{aligned}$$

Then the denominator, by (5.9) and (5.7) gives us

$$S(\ell^*, R\mathbf{u}) = \det[\ell^* : \mathbf{u}] = \det[x_k^{n-j}].$$

After cancelation we see our proposition holds. $\square$

Corollary 6.1. [Cauchy Identities] Let $r \leq n$ and let $y_0, y_1, \ldots, y_r$ be distinct numbers. Let $\mathbf{u}_\mathbf{y}^*$ be the elements of $\mathcal{Q}^{n+1}$ whose j-th component is $\dfrac{z^j}{z^{n-r} \prod_{i=0}^{r}(z - y_i)}$ and denote by $\mathbf{u}_\mathbf{x}^*$ the vector with k-th component equal to $\dfrac{z^k}{\prod_{i=0}^{n}(z - x_i)}$. Then

$$S(\mathbf{u}_\mathbf{x}^*, \mathbf{u}_\mathbf{y}^*) = \prod_{k=0}^{n} \prod_{i=0}^{r} (1 - x_k y_i)^{-1} = \sum_{\lambda \in \Lambda} S(\mathbf{u}_\mathbf{x}^*, R\mathbf{z}_\lambda) S(\mathbf{u}_\mathbf{y}^*, R\mathbf{z}_\lambda). \tag{6.8}$$

Proof. Let $f(z) = \dfrac{1}{\prod_{i=0}^{r}(1 - zy_i)} = \prod_{i=0}^{r}(1 - zy_i)^{-1}$ with $r \leq n$. By Proposition 6.3 we know $v_j = \dfrac{z^{n-j}}{\prod_{i=0}^{r}(1 - zy_i)}$ and

$$\begin{aligned} Rv_j &= \frac{1}{z}\left[\frac{1}{z^{n-j} \prod_{i=0}^{r}(1 - \frac{y_i}{z})}\right] \\ &= \frac{1}{z^{n-j+1} \prod_{i=0}^{r}(1 - \frac{y_i}{z})} \\ &= \frac{z^{r+1}}{z^{n-j+1} \prod_{i=0}^{r}(z - y_i)} \\ &= \frac{z^j}{z^{n-r} \prod_{i=0}^{r}(z - y_i)} \\ &= u_{y_j}^*. \end{aligned}$$

Thus $R\mathbf{v} = \mathbf{u}^*_{\mathbf{y}}$ and by Proposition 6.3

$$S(\mathbf{u}^*_{\mathbf{x}}, \mathbf{u}^*_{\mathbf{y}}) = \prod_{k=0}^{n} f(x_k) = \prod_{k=0}^{n} \prod_{i=0}^{r} (1 - x_k y_i)^{-1}$$

As for the second equality, note that $R\mathbf{u}^*_{y_j} = v_j = \dfrac{z^{n-j}}{\prod_{i=0}^{r}(1 - zy_i)}$ is a rational function defined at zero. This means we can develop its Taylor series expansion and use Theorem 6.1 (with $\mathbf{u}^*_{\mathbf{x}}, \mathbf{u}^*_{\mathbf{y}}$ as $\mathbf{g}$ and $\mathbf{f}$ respectively) to show

$$S(\mathbf{u}^*_{\mathbf{x}}, \mathbf{u}^*_{\mathbf{y}}) = \sum_{\lambda \in \Lambda} S(\mathbf{u}^*_{\mathbf{x}}, R\mathbf{z}_\lambda) S(\mathbf{u}^*_{\mathbf{y}}, R\mathbf{z}_\lambda).$$

$\square$

Appendix A: Methods of Deriving h_k terms.

We recall the definition of the h_k terms in (4.5):

$$h_k = \left\langle \frac{1}{w(z)}, z^{k+n} \right\rangle,$$

where the order of w is $n+1 = \sum_j m_j$, the sum of the orders of the roots of w. The first method we use is just to refer to the axioms of the inner product and its residue interpretation. The second method is to take determinants of increasingly larger lower triangular matrices whose terms are the coefficients of the expanded polynomial $w(z)$:

$$h_k = (-1)^k \det \begin{bmatrix} b_1 & 1 & & & 0 \\ b_2 & b_1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ b_{k-1} & b_{k-2} & b_{k-3} & \dots & 1 \\ b_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{bmatrix}.$$

Example 6.1. $w(z) = (z-1)^2 = z^2 - 2z + 1$. This means $b_0 = 1, b_1 = -2, b_2 = 1$ and $n+1 = 2 \Rightarrow n = 1$. Using our first method we have

$$\begin{aligned} h_k &= \left\langle \frac{1}{(z-1)^2}, z^{k+1} \right\rangle \\ &= \left\langle (1,1), z^{k+1} \right\rangle \\ &= k+1. \end{aligned}$$

This means $h_0 = 1, h_1 = 2, h_2 = 3$. We can support this claim using the second method (though we assume $h_0 = 1$):

$$\begin{aligned} h_1 &= (-1)^1 \det[-2] = 2 \\ h_2 &= (-1)^2 \det \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = 3. \end{aligned}$$

Example 6.2. $w(z) = (z-1)(z-2)(z-3) = z^3 - 6z^2 + 11z - 6$. This means $b_0 = 1, b_1 = -6, b_2 = 11, b_3 = -6$ and $n+1 = 3 \Rightarrow n = 2$. Using our first method with (2.14) we have

$$
\begin{aligned}
h_k &= \left\langle \frac{1}{(z-1)(z-2)(z-3)}, z^{k+2} \right\rangle \\
&= \left\langle (\langle (1,0),(2,0)\rangle\,(1,0) + \langle (2,0),(1,0)\rangle\,(2,0))(3,0), z^{k+2} \right\rangle \\
&= \left\langle (-(1,0)+(2,0))(3,0), z^{k+2} \right\rangle \\
&= -\left\langle (1,0)(3,0), z^{k+2} \right\rangle + \left\langle (2,0)(3,0), z^{k+2} \right\rangle \\
&= -\left\langle \frac{-1}{2}(1,0) + \frac{1}{2}(3,0), z^{k+2} \right\rangle + \left\langle -(2,0)+(3,0), z^{k+2} \right\rangle \\
&= \frac{1}{2}\left\langle (1,0), z^{k+2} \right\rangle - \frac{1}{2}\left\langle (3,0), z^{k+2} \right\rangle - \left\langle (2,0), z^{k+2} \right\rangle + \left\langle (3,0), z^{k+2} \right\rangle \\
&= \frac{1}{2} - \frac{9}{2}(3^k) - 4(2^k) + 9(3^k) = \frac{1}{2} + \frac{9}{2}3^k - 4\cdot 2^k.
\end{aligned}
$$

This means $h_0 = 1, h_1 = 6, h_2 = 25, h_3 = 90$. We support this claim using the second method:

$$
\begin{aligned}
h_1 &= (-1)^1 \det[-6] = 6 \\
h_2 &= (-1)^2 \det \begin{bmatrix} -6 & 1 \\ 11 & -6 \end{bmatrix} = 25 \\
h_3 &= (-1)^3 \det \begin{bmatrix} -6 & 1 & 0 \\ 11 & -6 & 1 \\ -6 & 11 & -6 \end{bmatrix} = 90.
\end{aligned}
$$

Example 6.3. $w(z) = (z-1)^2(z-2) = z^3 - 4z^2 + 5z - 2$. This means $b_0 = 1, b_1 = -4, b_2 = 5, b_3 = -2$ and $n+1 = 3 \Rightarrow n = 2$. Using our first method we have

$$
\begin{aligned}
h_k &= \left\langle \frac{1}{(z-1)^2(z-2)}, z^{k+2} \right\rangle \\
&= \left\langle \langle (1,0),(2,0)\rangle\,(1,1) + \langle (1,1),(2,0)\rangle\,(1,0) + \langle (2,0),(1,1)\rangle\,(2,0), z^{k+2} \right\rangle \\
&= \left\langle -(1,1) - (1,0) + (2,0), z^{k+2} \right\rangle \\
&= -\left\langle (1,1), z^{k+2} \right\rangle - \left\langle (1,0), z^{k+2} \right\rangle + \left\langle (2,0), z^{k+2} \right\rangle \\
&= -(2+k) - 1 + 4(2^k) = -3 - k + 4(2^k).
\end{aligned}
$$

This means $h_0 = 1, h_1 = 4, h_2 = 11, h_3 = 26$. We can support this claim using the second method:

$$\begin{aligned}
h_1 &= (-1)^1 \det[-4] = 4 \\
h_2 &= (-1)^2 \det \begin{bmatrix} -4 & 1 \\ 5 & -4 \end{bmatrix} = 11 \\
h_3 &= (-1)^3 \det \begin{bmatrix} -4 & 1 & 0 \\ 5 & -4 & 1 \\ -2 & 5 & -4 \end{bmatrix} = 26.
\end{aligned}$$

Of special note is that h_k represents the complete homogeneous symmetric polynomial of order k on the root values a_i, i.e.

$$\begin{aligned}
h_0 &= 1 \\
h_1 &= a_0 + a_1 + \ldots + a_n \\
h_2 &= a_0^2 + \ldots + a_n^2 + a_0 a_1 + a_0 a_2 + \ldots + a_{n-1} a_n
\end{aligned}$$

and so forth. Also, while the second method (matrix determinants) is generally easier computationally, the axiomatic method of decomposition allows us to get a closed form for h_k. With a little work up front, the first method gives us a formula into which we can simply plug values of k to get an answer, which is far easier than calculating determinants of large matrices. Hence for small values of k it makes sense to calculate determinants of small matrices as opposed to grinding out the costly up-front calculations of the axiomatic method, but for large values it might make sense to choose the first method (especially if you can get a computer program to break things down).

Appendix B: An Example of Statement (4.15)

Proposition 3 states that for rational functions f, g, h with f and gh having no common poles, that:

$$\langle f, gh \rangle = \langle \langle -f[x,y], g(x) \rangle, h(y) \rangle.$$

Now the use of different variables is actually important as we know the inner product is bilinear in a single variable from both components. This means if both the first and second component are functions of x and one of the components has a factor of another variable y, we can pull that out of the inner product as if it were a constant.

Claim: $\left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-2)^2(z-3)^2} \right\rangle = \left\langle \left\langle -\frac{\frac{1}{(x-2)^2} - \frac{1}{(y-1)^2}}{x-y}, \frac{1}{(x-2)^2} \right\rangle, \frac{1}{(y-3)^2} \right\rangle.$

LHS: Partial Fractions The first approach we take is to break up the second component of the LHS so that we can use bilinearity of the inner product to break the product up into something manageable. Therefore we start with the following:

$$\frac{1}{(z-2)^2(z-3)^2} = \frac{A}{(z-2)} + \frac{B}{(z-2)^2} + \frac{C}{(z-3)} + \frac{D}{(z-3)^2}.$$

Once we find a common denominator, we can identify the numerators:

$$1 = A(z-2)(z-3)^2 + B(z-3)^2 + C(z-2)^2(z-3) + D(z-2)^2.$$

When we expand this out and combine like terms we're left with a system of equations we can solve using matrices:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -8 & 1 & -7 & 1 \\ 21 & -6 & 16 & -4 \\ -18 & 9 & -12 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which is solved by $A = 2, B = 1, C = -2, D = 1$. Then proceed axiomatically:

$$
\begin{aligned}
\left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-2)^2(z-3)^2} \right\rangle &= \left\langle \frac{1}{(z-1)^2}, \frac{2}{(z-2)} + \frac{1}{(z-2)^2} - \frac{2}{(z-3)} + \frac{1}{(z-3)^2} \right\rangle \\
&= 2\left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-2)} \right\rangle + \left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-2)^2} \right\rangle \\
&\quad -2\left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-3)} \right\rangle + \left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-3)^2} \right\rangle \\
&= 2\left\langle (1,1),(2,0) \right\rangle + \left\langle (1,1),(2,1) \right\rangle \\
&\quad -2\left\langle (1,1),(3,0) \right\rangle + \left\langle (1,1),(3,1) \right\rangle \\
&= 2(-1) + 1(-2) - 2(-1/4) + 1(-2/-8) \\
&= \frac{3}{4}.
\end{aligned}
$$

We will now check that against the other methods for consistency.

LHS: Expansion using (2.8) The next approach uses (2.8), Liebniz's rule with our vocabulary. This allows us to again split the second component of the inner product to apply our axioms more directly:

$$
\begin{aligned}
\left\langle \frac{1}{(z-1)^2}, \frac{1}{(z-2)^2(z-3)^2} \right\rangle &= \sum_{k=0}^{1} \left\langle (1,k), \frac{1}{(z-2)^2} \right\rangle \left\langle (1,1-k), \frac{1}{(z-3)^2} \right\rangle \\
&= \left\langle (1,0), \frac{1}{(z-2)^2} \right\rangle \left\langle (1,1), \frac{1}{(z-3)^2} \right\rangle \\
&\quad + \left\langle (1,1), \frac{1}{(z-2)^2} \right\rangle \left\langle (1,0), \frac{1}{(z-3)^2} \right\rangle \\
&= \left\langle (1,0),(2,1) \right\rangle \left\langle (1,1),(3,1) \right\rangle + \left\langle (1,1),(2,1) \right\rangle \left\langle (1,0),(3,1) \right\rangle \\
&= (1)(1/4) + (2)(1/4) = \frac{3}{4}.
\end{aligned}
$$

RHS: Axiomatically We now tackle the RHS of the equality strictly using the bilinearity of the inner product and other axioms:

$$
\begin{aligned}
&\left\langle\left\langle -\frac{\frac{1}{(x-2)^2}-\frac{1}{(y-1)^2}}{x-y}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
=\ &\left\langle\left\langle \frac{x+y-2}{(x-1)^2(y-1)^2}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
=\ &\left\langle \frac{1}{(y-1)^2}\left\langle \frac{x+y-2}{(x-1)^2}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
=\ &\left\langle \frac{1}{(y-1)^2}\left\langle \frac{x}{(x-1)^2}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
&+\left\langle \frac{y}{(y-1)^2}\left\langle \frac{1}{(x-1)^2}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
&-\left\langle \frac{2}{(y-1)^2}\left\langle \frac{1}{(x-1)^2}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
=\ &\left\langle \frac{1}{(y-1)^2}\operatorname*{Res}_{x=1} \frac{x}{(x-1)^2(x-2)^2}, \frac{1}{(y-3)^2}\right\rangle \\
&+\left\langle \frac{y}{(y-1)^2}\left\langle (1,1),(2,1)\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
&-\left\langle \frac{2}{(y-1)^2}\left\langle (1,1),(2,1)\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
=\ &\left\langle \frac{1}{(y-1)^2}(3), \frac{1}{(y-3)^2}\right\rangle + \left\langle \frac{y}{(y-1)^2}(2), \frac{1}{(y-3)^2}\right\rangle - \left\langle \frac{2}{(y-1)^2}(2), \frac{1}{(y-3)^2}\right\rangle \\
=\ &3\left\langle (1,1),(3,1)\right\rangle + 2\operatorname*{Res}_{y=1} \frac{y}{(y-1)^2(y-3)^2} - 4\left\langle (1,1),(3,1)\right\rangle \\
=\ &3(1/4)+2(1/2)-4(1/4) = \frac{3}{4}.
\end{aligned}
$$

RHS: Iterated Residue Finally we will tackle the RHS iterated inner product by simply treating it as a residue calculation:

$$
\begin{aligned}
\left\langle\left\langle -\frac{\frac{1}{(x-2)^2}-\frac{1}{(y-1)^2}}{x-y}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle &= \left\langle\left\langle \frac{x+y-2}{(x-1)^2(y-1)^2}, \frac{1}{(x-2)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
&= -\left\langle\left\langle \frac{1}{(x-2)^2}, \frac{x+y-2}{(x-1)^2(y-1)^2}\right\rangle, \frac{1}{(y-3)^2}\right\rangle \\
&= -\left\langle \operatorname*{Res}_{x=2} \frac{x+y-2}{(x-2)^2(x-1)^2(y-1)^2}, \frac{1}{(y-3)^2}\right\rangle \\
&= \left\langle \frac{1}{(y-3)^2}, \operatorname*{Res}_{x=2} \frac{x+y-2}{(x-2)^2(x-1)^2(y-1)^2}\right\rangle \\
&= \operatorname*{Res}_{y=3}\left[\operatorname*{Res}_{x=2} \frac{x+y-2}{(x-2)^2(x-1)^2(y-1)^2(y-3)^2}\right] \\
&= \operatorname*{Res}_{y=3} \frac{1-2y}{(y-3)^2(y-1)^2} \\
&= \frac{3}{4}.
\end{aligned}
$$

We can then conclude that the proposition indeed holds, at least with the given example. We can also see certain methods of resolution are faster computationally than others.

Appendix C: Determinant of a Vandermonde Matrix

Let $\mathbf{a} = (a_1, \ldots, a_n)^T$ be a vector of distinct complex coordinates.

$$V_{\mathbf{a}} = \det[a_j^k] = \prod_{1 \le i < j \le n} (a_j - a_i). \tag{6.9}$$

Proof. Let

$$V_n = \begin{vmatrix} 1 & a_1 & \ldots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \ldots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_{n-1} & \ldots & a_{n-1}^{n-2} & a_{n-1}^{n-1} \\ 1 & a_n & \ldots & a_n^{n-2} & a_n^{n-1} \end{vmatrix}.$$

We can subtract row 1 from each of the other rows and leave V_n unchanged:

$$V_n = \begin{vmatrix} 1 & a_1 & \ldots & a_1^{n-2} & a_1^{n-1} \\ 0 & a_2 - a_1 & \ldots & a_2^{n-2} - a_1^{n-2} & a_2^{n-1} - a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n-1} - a_1 & \ldots & a_{n-1}^{n-2} - a_1^{n-2} & a_{n-1}^{n-1} - a_1^{n-1} \\ 0 & a_n - a_1 & \ldots & a_n^{n-2} - a_1^{n-2} & a_n^{n-1} - a_1^{n-1} \end{vmatrix}.$$

Similarly without changing the value of V_n we can subtract, in order, a_1 times column $n-1$ from column n, a_1 times column $n-2$ from column $n-1$, and so on, until we subtract a_1 times column 1 from column 2. This eliminates the first row save for the first element which is just 1.

$$V_n = \begin{vmatrix} 1 & 0 & \ldots & 0 & \\ 0 & a_2 - a_1 & \ldots & (a_2 - a_1)a_2^{n-3} & (a_2 - a_1)a_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n-1} - a_1 & \ldots & (a_{n-1} - a_1)a_{n-1}^{n-3} & (a_{n-1} - a_1)a_{n-1}^{n-2} \\ 0 & a_n - a_1 & \ldots & (a_n - a_1)a_n^{n-3} & (a_n - a_1)a_n^{n-2} \end{vmatrix}.$$

Notice then that every row i save for the first is a multiple of $(a_i - a_1)$, so we can extract these factors, which means our determinant is just multiplied by these elements.

$$V_n = \prod_{i=2}^{n}(a_i - a_1)\begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & a_2^{n-3} & a_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & a_{n-1}^{n-3} & a_{n-1}^{n-2} \\ 0 & 1 & \dots & a_n^{n-3} & a_n^{n-2} \end{vmatrix}.$$

Expand the determinant along the first row or column to get

$$V_n = \prod_{i=2}^{n}(a_i - a_1)\begin{vmatrix} 1 & a_2 & \dots & a_2^{n-3} & a_2^{n-2} \\ 1 & a_3 & \dots & a_3^{n-3} & a_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_{n-1} & \dots & a_{n-1}^{n-3} & a_{n-1}^{n-2} \\ 1 & a_n & \dots & a_n^{n-3} & a_n^{n-2} \end{vmatrix}.$$

Repeating this process were next left with

$$V_n = \prod_{i=2}^{n}(a_i - a_1)\prod_{i=3}^{n}(a_i - a_2)\begin{vmatrix} 1 & a_3 & \dots & a_3^{n-4} & a_3^{n-3} \\ 1 & a_4 & \dots & a_4^{n-4} & a_4^{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_{n-1} & \dots & a_{n-1}^{n-4} & a_{n-1}^{n-3} \\ 1 & a_n & \dots & a_n^{n-4} & a_n^{n-3} \end{vmatrix}.$$

So we continue until we're left with a 2x2 matrix

$$\begin{vmatrix} 1 & a_{n-1} \\ 1 & a_n \end{vmatrix} = a_n - a_{n-1}.$$

The result follows. $\square$

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