WEIGHT STRETCHING IN MODULI OF PARABOLIC BUNDLES AND QUIVER REPRESENTATIONS

Cass Sherman

A dissertation submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2016

Approved by:

Prakash Belkale

Ivan Cherednik

Richard Rimányi

Justin Sawon

Jonathan Wahl

ABSTRACT

Cass Sherman: Weight Stretching in Moduli of Parabolic Bundles and Quiver Representations
(Under the direction of Prakash Belkale)

In their 2004 paper [KTT04], King et al. consider the effect of stretching each parameter in a Littlewood-Richardson coefficient by a positive integer N. When the L-R coefficient corresponding to $N = 1$ is small (0, 1, 2, or 3), they conjecture simple polynomial formulas determining the L-R coefficient for all $N \geq 1$. In this thesis, we consider generalizations of their conjectures to parabolic vector bundles and representations of quivers. In each instance, there is a polarized moduli space $(\mathcal{M}, \mathcal{L})$ with the property that $\dim H^0(\mathcal{M}, \mathcal{L}^{\otimes N})$ scales in the same way as the corresponding (generalized) L-R coefficient. The "simple polynomial formulas" then translate to simple geometric descriptions of $(\mathcal{M}, \mathcal{L})$. We prove that these descriptions hold in many cases.

TABLE OF CONTENTS

INTRODUCTION

One of the oldest problems in the representation theory of linearly reductive groups G, going back to at least Littlewood-Richardson in the 1930s, is the question of tensor product multiplicities. This question asks if V, W, and U are irreducible representations, then how many direct summands of U appear in the decomposition of $V \otimes W$ into irreducibles? For certain G, combinatorial formulas are known. For instance, when $G = \mathrm{GL}_r\mathbb{C}$, the Littlewood-Richardson rule computes tensor product multiplicities by counting the number of "reverse lattice words" on a certain skew Young diagram determined by V, W, U. Such formulas prove useful for computer calculations, but they do not tell the full story on the theory of tensor product multiplicities. Many interesting theoretical questions have only been answered recently by non-combinatorial (e.g. geometric) methods, and still more remain open. The proceedings [Kum10] provides a nice survey.

Our point of view on this problem begins with the trivial observation that by Schur's lemma, the multiplicity of U is the same as the dimension of the space of G-invariants of $V \otimes W \otimes U^*$. Symmetrizing, one then recasts the multiplicity problem as a special case of the invariant theory computation of

$$c_{\lambda^1,\ldots,\lambda^s} = \dim(V_{\lambda^1} \otimes \ldots \otimes V_{\lambda^s})^G.$$

Here $\lambda^1, \ldots, \lambda^s$ indicate the dominant weights corresponding to the irreducible representations $V_{\lambda^1}, \ldots, V_{\lambda^s}$. Scaling a dominant weight by a positive integer N produces another dominant weight. This gives rise to the *saturated* multiplicity question, in which one considers $P(N) := c_{N\lambda^1,\ldots,N\lambda^s}$ for arbitrary $N > 0$ and its relationship to $c_{\lambda^1,\ldots,\lambda^s}$ $(P(1))$. For $G = \mathrm{SL}_r\mathbb{C}$, the behavior of P has been studied extensively and many celebrated results produced. Several of these appear among the conjectures of King, Tollu, and Toumazet on P (stated for $s = 3$, but this is not expected to matter).

Conjecture. *[KTT04] For $G = \mathrm{SL}_r\mathbb{C}$, if $\lambda^1 + \ldots + \lambda^s$ is in the root lattice of G (a necessary condition for $P(1) \neq 0$), we have:*

- *(Polynomiality) P is a polynomial in N with rational coefficients.*
- *(Positivity) The coefficients of P are nonnegative.*
- *(Saturation) If $P(1) = 0$, then $P(N) = 0$ for all $N \geq 1$.*
- *(Fulton) If $P(1) = 1$, then $P(N) = 1$ for all $N \geq 1$.*
- *(KTT) If $P(1) = 2$, then $P(N) = N + 1$ for all $N \geq 1$.*
- *(KTT 2) If $P(1) = 3$, then $P(N) = 2N + 1$ or $P(N) = \frac{1}{2}N^2 + \frac{3}{2}N + 1$.*

The polynomiality conjecture was proven by Derksen and Weyman [DW02]. After a further GIT translation of the question, one can also deduce polynomiality from Teleman's higher cohomology vanishing [Tel00], in view of which P is the Hilbert polynomial of an ample line bundle on a projective variety. We shall pursue this point of view later. Several different proofs of saturation are known: combinatorial by the honeycomb model due to Knutson-Tao when $s = 3$ [KT99], algebraic by way of quiver representations due to Derksen-Weyman [DW00], and geometric through Schubert calculus due to Belkale [Bel06]. Additionally, Kapovich-Millson [KM04] deduce the saturation conjecture for $\mathrm{SL}_r\mathbb{C}$ from a saturation-type result which holds for any simple G.

The Fulton conjecture also has several proofs, the first combinatorial due to Knutson-Tao-Woodward [KTW04] when $s = 3$. Later geometric proofs appeared from Belkale [Bel07] (for arbitrary s) and Ressayre [Res11]. A non-obvious generalization (the obvious generalization is false) of the Fulton conjecture to connected reductive groups G was given by Belkale-Kumar-Ressayre [BKR12]. Fulton's conjecture holds additional interest beyond the behavior of P. It can be used to show that a certain set of linear inequalities in the components of the weights are irredundant (as well as necessary and sufficient) in describing the cone $\{(\lambda^1, ..., \lambda^s) : c_{\lambda^1,...,\lambda^s} \neq 0\} \subseteq (\mathbb{R}^r)^{\times s}$ [DW11].

The KTT conjecture was proven combinatorially by Ikenmeyer [Ike12] for $s = 3$ and geometrically by the author [She15] for arbitrary s. To the best of the author's knowledge, no attempts at the positivity conjecture or the second KTT conjecture exist in the literature. Some progress will be made toward these later in the thesis.

From the theory of Borel-Weil-Bott, each irreducible representation V_λ of $G = \mathrm{SL}_r\mathbb{C}$ can be realized as the space of sections of an ample G line bundle $\mathcal{L}_\lambda$ on a flag variety X_λ (actually it is conventional for the sections of $\mathcal{L}_\lambda$ to give the dual of V_λ). Thus, taking the GIT quotient of the product $\prod_{p=1}^s X_{\lambda^p}$ by the diagonal G action, with linearization $\widetilde{\mathcal{L}}_\lambda := \mathcal{L}_{\lambda^1} \boxtimes ... \boxtimes \mathcal{L}_{\lambda^s}$, one obtains a (projective, irreducible, normal) moduli space $\mathcal{M}_\lambda$ parametrizing classes of semistable s-tuples of flags. Since $\tilde{\mathcal{L}}_\lambda$ descends to an ample $\mathcal{L}_\lambda$ on $\mathcal{M}_\lambda$, one has $\dim H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = c_{\lambda^1,...,\lambda^s}$, and more generally $\dim H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda^{\otimes N}) = P(N)$. The above conjectures therefore have translations in terms of the polarized variety $(\mathcal{M}_\lambda, \mathcal{L}_\lambda)$.

Conjecture. *For $G = \mathrm{SL}_r\mathbb{C}$, if $\lambda^1 + ... + \lambda^s$ is in the root lattice of G, we have:*

- *(Saturation) If $\mathcal{M}_\lambda$ is nonempty, then $\dim H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda)$ is nonzero.*
- *(Fulton) If $\dim H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = 1$, then $\mathcal{M}_\lambda$ is a point.*
- *(KTT) If $\dim H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = 2$, then $(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = (\mathbb{P}^1, \mathcal{O}(1))$.*
- *(KTT 2) If $\dim H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = 3$, then $(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = (\mathbb{P}^1, \mathcal{O}(2))$ or $(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = (\mathbb{P}^2, \mathcal{O}(1))$.*

Taking this geometric approach to the conjectures as our jumping off point, we pursue generalizations in two directions. The first is to moduli of quiver representations. A quiver is simply a directed graph, and a representation is a collection of finite-dimensional vector spaces, one over each vertex, with linear maps over each arrow. As with s-tuples of flags, discrete weight data defines a notion of semistability for quiver representations. Again, one has a moduli space $\mathcal{M}$ parametrizing semistable quiver representations, and there is a line bundle $\mathcal{L}$ on $\mathcal{M}$ whose sections give semi-invariant functions on a certain space of quiver-representations, these being analogous to invariant vectors in a tensor product. Indeed, for well-chosen quivers and discrete weight data, the polarized space $(\mathcal{M}, \mathcal{L})$ for quiver representations is naturally isomorphic to $(\mathcal{M}_\lambda, \mathcal{L}_\lambda)$ as above. Thus, even if one is only interested in Littlewood-Richardson numbers, it may still be worthwhile to consider the corresponding quiver generalization, for, in the author's view, quiver representations are "more linear" and easier to work with. This point of view is supported by the early 2000s work of Derksen and Weyman cited above, in which e.g. polynomiality was proven first for quiver semi-invariants and then deduced for Littlewood-Richardson numbers.

The second direction in which the conjectures can be generalized is to moduli of parabolic bundles on $\mathbb{P}^1$. For our purposes, given a collection of marked points $S = \{p_1, ..., p_s\}$ on $\mathbb{P}^1$, a parabolic bundle is the data $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ of a vector bundle $\mathcal{V}$ on $\mathbb{P}^1$, a full flag in $\mathcal{V}|_p$ for each marked point p, and weights of an appropriate level attached to each vector space in the flags determined by $\mathcal{I}$. For fixed degree, rank, and weight+level data $\mathcal{I}$, there is again a moduli space $\mathcal{M} = \mathcal{M}(\mathcal{I})$ of classes of $\mathcal{I}$-semistable parabolic vector bundles $(\mathcal{V}, \mathcal{F}, \mathcal{I})$, and a natural ample line bundle $\mathcal{L}$ on $\mathcal{M}$ whose gloabal sections have dimension $\langle \mathcal{I} \rangle$ (we prove this in Section 2.8). Here $\langle \mathcal{I} \rangle$ denotes the quantum Schubert calculus number associated to $\mathcal{I}$, a count of certain maps from $\mathbb{P}^1$ into a Grassmannian. Since their discovery by Mehta-Seshadri [MS80] (in studying unitary representations of Fuchsian groups), parabolic bundles have been studied by many and for diverse reasons, including Narasimhan-Ramadas [NR93] (to establish fundamental properties of $(\mathcal{M}, \mathcal{L})$), Pauly [Pau96] (to compare with the physicists' spaces of conformal blocks), Boden-Hu [BH95] (in the context of variation of stability in GIT), Laszlo-Sorger [LS97] (to describe line bundles on the moduli *stack* of quasi-parabolic bundles), Belkale [Bel04a] (to solve a "recognition problem" for products of matrices in $\mathrm{SU}(n)$), Crawley-Boevey [CB04] (to partially solve the same problem for $\mathrm{GL}_n\mathbb{C}$), and many others. The connection to conformal blocks will be discussed in more detail later.

Note that for vector bundles of degree 0 and certain choices of $\mathcal{I}$, one recovers the moduli space of semistable s-tuples of flags from the previous set of conjectures. So once again, one can deduce facts about Littlewood-Richardson numbers from more general facts about moduli of parabolic bundles. However, unlike with quivers, it is more difficult to work in full generality with parabolic bundles.

The goal of this thesis is to describe the pairs $(\mathcal{M}, \mathcal{L})$ above, especially under the hypotheses of the Fulton and KTT conjectures. Our main result is the following.

Theorem 1. *Let $(\mathcal{M}, \mathcal{L})$ be a moduli space of either semistable quiver representations or semistable parabolic bundles as above.*

1. *If $\mathcal{M}$ is a moduli space of semistable parabolic bundles, and $h^0(\mathcal{M}, \mathcal{L}) = 1$, then $\mathcal{M}$ is a point.*

2. *If $\mathcal{M}$ is a moduli space of semistable quiver representations, and $h^0(\mathcal{M}, \mathcal{L}) = 2$, then $(\mathcal{M}, \mathcal{L}) = (\mathbb{P}^1, \mathcal{O}(1))$.*

3. *If $\mathcal{M}$ is a moduli space of semistable parabolic bundles of rank 2 or 3, then $\mathcal{L}$ is basepoint-free.*

We remark that item 1 is already known, as well as its quiver analogue. Indeed, in the quiver case, we do not even include a proof of item 1, as a proof using our methods would not differ significantly from the existing proof of [DW11]. In the parabolic bundle case, we include a proof of item 1. The existing proof by Belkale-Kumar [BK16] involves the multiplicative eigencone for $\mathrm{SU}(n)$. The inequalities defining facets of this cone are known to correspond to certain non-null quantum Schubert states $\mathcal{I}$, and Belkale-Kumar prove that one need only retain the inequalities with $\langle \mathcal{I} \rangle = 1$ and that every such inequality is needed (i.e. the set is *irredundant*). In the earlier paper [Bel04a], Belkale observes that quantum irredundancy implies our item 1, which in his terminology is the Quantum Fulton Conjecture. Thus, item 1 follows. However, the main argument of [Bel04a] actually goes the other way: Belkale shows that assuming QFC one can deduce quantum irredundancy. So our proof of item 1, which is independent of eigencone problems and considerations of redundancy, validates Belkale's original approach.

A second result relates to the positivity conjecture and has implications for KTT 2. We consider again the classical moduli space $(\mathcal{M}_\lambda, \mathcal{L}_\lambda)$.

Proposition 2. *If $(\mathcal{M}_\lambda, \mathcal{L}_\lambda)$ is a moduli space of λ-semistable complete flags, and if in addition $\mathcal{M}_\lambda$ contains a point corresponding to a stable flag, then the Weil divisor class $-K_{\mathcal{M}}$ is effective (note: $\mathcal{M}_\lambda$ is normal, so $K_{\mathcal{M}}$ makes sense). In particular, if $\mathcal{M}_\lambda$ is also smooth, then it is Fano.*

Note that the complete flag assumption was imposed for convenience. We have not seen this result to fail in the case of partial flags (and we do not expect it to).

Let us now draw some easy consequences. For degree 0 parabolic bundles, Pauly shows that $\dim H^0(\mathcal{M}, \mathcal{L})$ is the rank of the conformal block associated to the data $\mathcal{I}$ defining semistability [Pau96]. Item 1 in Theorem 1 shows in particular that if $\dim H^0(\mathcal{M}, \mathcal{L}) = 1$, then $\dim H^0(\mathcal{M}, \mathcal{L}^{\otimes N}) = 1$. Since $(\mathcal{M}, \mathcal{L}^{\otimes N})$ is the polarized moduli space associated to the data $N\mathcal{I}$, we have:

Corollary. *Let $\mathrm{V}_{\lambda^1,\ldots,\lambda^s,\ell}$ be an $\mathfrak{sl}_r$ conformal block vector space of level ℓ and weights λ^p associated to the stable curve $(\mathbb{P}^1, p_1, \ldots, p_s)$. Let $P'(N) := \dim \mathrm{V}_{N\lambda^1,\ldots,N\lambda^s,N\ell}$ for $N \geq 1$. If $P'(1) = 1$, then $P'(N) = 1$ for all $N \geq 1$.*

A similar result is implied by item 3 of Theorem 1. If $\dim H^0(\mathcal{M}, \mathcal{L}) = 2$ and $\mathcal{L}$ is basepoint-free, then $\dim \mathcal{M} \leq 1$ because $\mathcal{L}$ is ample. It follows from normality and rationality of $\mathcal{M}$ [BH95] that $(\mathcal{M}, \mathcal{L}) = (\mathbb{P}^1, \mathcal{O}(1))$. Therefore, P' behaves like the global sections of $\mathcal{O}(1)$.

Corollary. *Notation as in the previous corollary, and assume that $r = 2$ or $r = 3$. If $P'(1) = 2$, then $P'(N) = N + 1$ for all $N \geq 1$.*

Thanks to Belkale-Gibney-Kazanova [BGK15], the scaling results above have further implications for the conformal blocks *divisors* on the moduli space $\overline{M}_{0,s}$ of stable curves. We give an easy application of their work in Section 2.12.

As for item 2 in Theorem 1, it can be used to deduce the "classical" KTT conjecture.

Corollary. *If $\dim(V_{\lambda^1} \otimes ... \otimes V_{\lambda^s})^{\mathrm{SL}_r} = 2$, then $\dim(V_{N\lambda^1} \otimes ... \otimes V_{N\lambda^s})^{\mathrm{SL}_r} = N + 1$ for all positive integers N.*

The analogous item 2 for moduli of parabolic bundles also appears to be true. The author has not been able to prove it as of this writing, because of technical difficulties that arise when dealing with parameter spaces of vector bundles (whereas parameter spaces of vector spaces as in [She15] had been much more manageable). Thus, we state item 2 for parabolic bundles as a conjecture.

Conjecture. (**Quantum KTT**) *If $(\mathcal{M}, \mathcal{L})$ is a moduli space of semistable parabolic bundles as above, and $\dim H^0(\mathcal{M}, \mathcal{L}) = 2$, then $(\mathcal{M}, \mathcal{L}) = (\mathbb{P}^1, \mathcal{O}(1))$.*

Proposition 2 provides the first evidence towards the positivity conjecture of King et al. Indeed, it follows from the proposition and Nakai's criterion for ampleness that:

Corollary. *Assumptions as in Proposition 2. The coefficient of $N^{\deg P - 1}$ in $P(N) = h^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda^{\otimes N})$ is positive.*

Observe also that positivity of the second coefficient gives a weak variant of KTT 2. For suppose the second coefficient is positive and $h^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = 3$. Suppose further that $\mathcal{L}_\lambda$ is basepoint-free. This will be the case when $r = 2$ or $r = 3$ by Item 3 in Theorem 1 (it also seems likely to be the case for any r, provided an argument similar to the one in [She15] generalizes). It follows that $\dim \mathcal{M}_\lambda = 1$ or $\dim \mathcal{M}_\lambda = 2$. In the former case, it is easy to see that $(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = (\mathbb{P}^1, \mathcal{O}(2))$. In the latter case, the Hilbert polynomial P of $\mathcal{L}_\lambda$ has the form $P(N) = (a/2)N^2 + (b/2)N + 1$ for some *positive* integers a, b such that $a/2 + b/2 + 1 = 3$. This leaves only three possiblities for (a, b), namely $(1, 3)$, $(2, 2)$, and $(3, 1)$. The corresponding moduli spaces are $(\mathcal{M}_\lambda, \mathcal{L}_\lambda) = (\mathbb{P}^2, \mathcal{O}(1))$ or $|\mathcal{L}_\lambda|$ defines a generically 2:1 (resp. 3:1) morphism $\mathcal{M}_\lambda \to \mathbb{P}^2$.

We conclude this introduction with some remarks on the techniques. The starting point for our analysis of both the parabolic bundle and quiver representation moduli $(\mathcal{M}, \mathcal{L})$ is a simple description for the sections of $\mathcal{L}$. If $\mathcal{M}$ parametrizes parabolic bundles (resp. quiver representations) associated to some data ∂, then some sections of $\mathcal{L}$ are defined by parabolic bundles (resp. quiver representations) associated to complementary data ∂^c, and these sections together span $H^0(\mathcal{M}, \mathcal{L})$. The vanishing locus of such a section admits a particularly nice description. If W is a parabolic bundle (resp. quiver representation) for the data ∂^c, then the section s_W of $\mathcal{L}$ defined by W vanishes at precisely the $V \in \mathcal{M}$ for which there exists a nonzero morphism $V \to W$.

We use the above description to analyze the basepoints of $\mathcal{L}$ - particularly with the intention of showing that there are none in the cases we consider. If $V \in \mathcal{M}$ is a basepoint of $\mathcal{L}$, then s_W vanishes at V for all W associated to ∂^c, whence $\mathrm{Mor}(V, W) \neq 0$ for all such W. To derive a contradiction to the existence of the basepoint V, we show that $\mathrm{Mor}(V, W)$ is in fact 0 for a general W. The argument is inductive in flavor. The idea is to relate $\mathrm{Mor}(V, W)$ to $\mathrm{Mor}(S, W)$ where S is the kernel of a general morphism $V \to W$. Since by assumption a general morphism is nonzero, S will be strictly smaller than V. If the pair (S, W) is comparable to (V, W), we can say whatever we wanted to say about (V, W) about (S, W) by induction. The desired contradiction then follows. Note that in this overview, we have suppressed the role of semistability of V, which plays an important part in the actual proofs.

The relationship between $\mathrm{Mor}(V, W)$ and $\mathrm{Mor}(S, W)$ may be of independent interest. The statement of the relationship for quiver representations resembles similar statements of Schofield [Sch92]. Our results, obtained by dimension counting, are presented below.

Proposition. *Fix a representation V of a quiver Q. Let W be generic of its dimension vector with respect to V. If S denotes the kernel of a general morphism $V \to W$, then the canonical surjection $\mathrm{Ext}_Q(V, W) \twoheadrightarrow \mathrm{Ext}_Q(S, W)$ is an isomorphism.*

Proposition. *Fix a parabolic bundle $\widetilde{\mathcal{V}} = (\mathcal{V}, \mathcal{F}, \mathcal{I})$. Let $\widetilde{\mathcal{Q}} = (\mathcal{Q}, \mathcal{G}, \mathcal{I}^c)$ be a parabolic bundle which is generic with respect to $\widetilde{\mathcal{V}}$. If $\widetilde{\mathcal{S}}$ denotes the kernel of a general morphism $\widetilde{\mathcal{V}} \to \widetilde{\mathcal{Q}}$, then the canonical surjection $H^1(\underline{ParHom}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{Q}})) \twoheadrightarrow H^1(\underline{ParHom}(\widetilde{\mathcal{S}}, \widetilde{\mathcal{Q}}))$ is an isomorphism.*

See Sections 1.7 and 2.9 for more precise statements (in a slightly different notation in the parabolic bundle case).

Acknowledgements. The author would like to thank his thesis advisor, Prakash Belkale, for his many years of help and guidance. Thanks are also due Shrawan Kumar for a useful discussion on KTT 2.

Convention. *We work over an algebraically closed field* $\mathbb{C}$ *of characteristic* 0*. The word "scheme" will implicitly carry the meaning "of finite type over* $\mathbb{C}$*." The word "representation" is to be understood as a* $\mathbb{C}$ *vector space or, in the case of quiver representations, a collection of* $\mathbb{C}$ *vector spaces, one over each vertex of the quiver. Also, by "vector space," we only mean finite-dimensional* $\mathbb{C}$ *vector spaces. Finally, the word "subbundle" should be understood as a locally free subsheaf whose corresponding quotient is locally free.*

CHAPTER 1

Representations of Quivers

For α, β dimension vectors of a cycle-free quiver Q with Ringel product 0, the dimensions of the spaces of σ_β-semi-invariant functions $\mathrm{SI}(Q, \alpha)_{\sigma_\beta}$ on $\mathrm{Rep}(Q, \alpha)$ appear to exhibit the same behavior under stretching as the Littlewood-Richardson numbers (see Section 1.1 for notation and generalities on quiver representations). Thus, one has the familiar assertions for the function $\widetilde{P}(n) := \dim \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}}$.

- (Polynomiality) $\widetilde{P}$ is a polynomial with rational coefficients.
- (Saturation) If $\widetilde{P}(1) = 0$, then $\widetilde{P}(n) = 0$ for all $n \geq 1$.
- (Fulton) If $\widetilde{P}(1) = 1$, then $\widetilde{P}(n) = 1$ for all $n \geq 1$.

All of the above were proven by Derksen and Weyman in the papers [DW02], [DW00], and [DW11], respectively, the last of these being a reconstruction of a proof given by Belkale. The main object of this chapter is to establish the corresponding quiver generalization of the KTT Conjecture. That is, we prove:

Theorem 1.0.1. *Let α, β be dimension vectors of Q, a quiver without oriented cycles, such that $\langle \alpha, \beta \rangle_Q = 0$. If $\dim \mathrm{SI}(Q, \alpha)_{\sigma_\beta} = 2$, then $\dim \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}} = n + 1$ for all positive integers n.*

Our approach proceeds through geometric invariant theory, following similar proofs in [Bel07], [She15]. Along the way, we prove by dimension counting a result of general interest, Proposition 1.3.1. It has the flavor of results from Schofield's paper [Sch92], in that it equates $\mathrm{Ext}_Q(V, W)$ with $\mathrm{Ext}_Q(S, W)$, where S is a certain subrepresentation of V.

In the last section, we show how to deduce the main result of the author's paper [She15] (restated as Corollary 1.8.4 here) from Theorem 1.0.1.

1.1 Preliminaries and Notation on Quiver Representations

A *quiver* Q consists of the data of a pair finite sets Q_0 and Q_1 of vertices and arrows between vertices, respectively, along with maps $h, t : Q_1 \to Q_0$, where the head map h associates to each arrow the vertex of its pointer, and the tail map t associates to each arrow the vertex of its base. We will assume moreover that a quiver has no oriented cycles when regarded as a digraph.

A *dimension vector* α is a function $\alpha : Q_0 \to \mathbb{N} \cup \{0\}$. A *representation* of Q of dimension vector α is defined to be an element V of the set:

$$\mathrm{Rep}(Q, \alpha) := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha(ta)}, \mathbb{C}^{\alpha(ha)})$$

We will frequently regard $\mathrm{Rep}(Q, \alpha)$ as an affine variety, by the obvious identification with $\mathbb{A}^N$ for $N = \sum_a \alpha(ta)\alpha(ha)$.

If V and W are representations of Q of dimension vectors α and β, respectively, then a morphism $\phi : V \to W$ of quiver representations is, for each $x \in Q_0$, a homomorphism of vector spaces $\phi(x) : \mathbb{C}^{\alpha(x)} \to \mathbb{C}^{\beta(x)}$, where these must satisfy the commutativity property $\phi(ha) \circ V(a) = W(a) \circ \phi(ta)$ for every a in Q_1. The vector space $\mathrm{Hom}_Q(V, W)$ of all morphisms of quiver representation is then the kernel of the map

$$d_W^V = \oplus_{x \in Q_0} \mathrm{Hom}(\mathbb{C}^{\alpha(x)}, \mathbb{C}^{\beta(x)}) \to \oplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha(ta)}, \mathbb{C}^{\beta(ha)})$$

which sends $\{\phi(x)\}_{x \in Q_0}$ to the element $\{W(a) \circ \phi(ta) - \phi(ha) \circ V(a)\}_{a \in Q_1}$.

Let $\mathrm{Rep}(Q)$ denote the category with representations of Q (of any dimension vector) as objects and the above notion of morphism. It is an abelian category. For representations V and W, one has $\mathrm{Ext}^1(V, W) = \mathrm{coker}(d_W^V)$, and there is no higher Ext in this category, so we simply denote this cokernel by $\mathrm{Ext}_Q(V, W)$.

The (in general, nonsymmetric) *Ringel form* on the abelian group of functions $Q_0 \to \mathbb{Z}$ is the bilinear form:

$$\langle \alpha, \beta \rangle_Q = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha) \tag{1.1}$$

It is clear that if moreover α and β are dimension vectors, then $\langle \alpha, \beta \rangle_Q$ is the difference of the dimensions of the domain and codomain of d_W^V, whence

$$\langle \alpha, \beta \rangle_Q = \dim \operatorname{Hom}_Q(V, W) - \dim \operatorname{Ext}_Q(V, W) \tag{1.2}$$

for any representations V, W of dimensions α, β. In particular, the right hand side of (1.2) does not depend on V and W beyond their dimension vectors.

The affine variety $\operatorname{Rep}(Q, \alpha)$ has a natural action of

$$\operatorname{GL}(Q, \alpha) := \prod_{x \in Q_0} \operatorname{Aut}(\mathbb{C}^{\alpha(x)})$$

given by conjugation: $g = (A(x))_x$ sends $V = (V(a))_a$ to $gV = (A(ha)V(a)A(ta)^{-1})_a$. Let the subgroup $\operatorname{SL}(Q, \alpha)$ of $\operatorname{GL}(Q, \alpha)$ be the product of the determinant 1 subgroups of each factor in the product defining $\operatorname{GL}(Q, \alpha)$. We are interested in the rings of semi-invariants

$$\operatorname{SI}(Q, \alpha) = (H^0(\operatorname{Rep}(Q, \alpha), \mathcal{O}))^{\operatorname{SL}(Q, \alpha)},$$

where $\mathcal{O}$ is the structure sheaf. These decompose into direct sums of weight spaces, called spaces of σ *semi-invariants*:

$$\operatorname{SI}(Q, \alpha)_\sigma = \{f \in H^0(\operatorname{Rep}(Q, \alpha), \mathcal{O}) : g \cdot f = \sigma(g) f \text{ for all } g \in \operatorname{GL}(Q, \alpha)\},$$

for σ a multiplicative character of $\operatorname{GL}(Q, \alpha)$. Such a character must be a product over Q_0 of integral powers of the determinant characters on each factor of $\operatorname{GL}(Q, \alpha)$. A character σ may therefore be identified with a function or weight (also called σ) $Q_0 \to \mathbb{Z}$. Each such σ defines a notion of semistability on $\operatorname{Rep}(Q, \alpha)$.

Definition 1.1.1. Given two weights $\sigma, \gamma : Q_0 \to \mathbb{Z}$, one defines the evaluation of σ at γ to be

$$\sigma(\gamma) = \sum_{x \in Q_0} \sigma(x)\gamma(x).$$

A representation V of Q which satisfies $\sigma(\dim V) = 0$ is said to be σ-*semistable* if for every nonzero subrepresentation S of V, one has $\sigma(\dim S) \leq 0$. The representation V is σ-*stable* if the inequality is always strict.

To complete the notation for Theorem 1.0.1, we introduce the following definition.

Definition 1.1.2. For a dimension vector β of Q, we define the function $\sigma_\beta : Q_0 \to \mathbb{Z}$ by $\sigma_\beta(x) = -\beta(x) + \sum_{a:ta=x} \beta(ha)$.

Remark 1.1.1. Clearly $\sigma_{n\beta} = n\sigma_\beta$ for any positive integer n. Notice also that if $\alpha : Q_0 \to \mathbb{Z}$ is a function, one has $\sigma_\beta(\alpha) = -\langle \alpha, \beta \rangle$.

1.2 Translation via GIT

The goal of this section is to prove Proposition 1.2.1, which in turn gives Proposition 1.2.2, the latter translating the main theorem 1.0.1 into a form more adaptable to our geometric approach. Parts of 1.2.1 are known from the literature (and are credited suitably below), but the author could not find a reference for the descent of the line bundle L_{σ_β}, hence its full proof here.

Proposition 1.2.1. *Let α, β be dimension vectors for a quiver Q without oriented cycles, such that $\langle \alpha, \beta \rangle_Q = 0$. Let $\sigma_\beta : Q_0 \to \mathbb{Z}$ be the associated weight (Definition 1.1.2). If R^{SS} denotes the open set of σ_β-semistable points of $\mathrm{Rep}(Q, \alpha)$, then there is a good quotient $\pi : R^{SS} \to Y_{\alpha,\beta}$, where $Y_{\alpha,\beta}$ is an integral, projective $\mathbb{C}$-variety with rational singularities (in particular, is normal). If L_{σ_β} denotes the trivial line bundle $\mathrm{Rep}(Q, \alpha) \times \mathbb{C}$ with $\mathrm{GL}(Q, \alpha)$-equivariant structure provided by $g \cdot (V, z) = (g \cdot V, \sigma_\beta(g^{-1})v)$ (now viewing σ_β as a character, as in section 1.1), then there exists an ample line bundle L_Y on $Y_{\alpha,\beta}$ such that $\pi^* L_Y = L_{\sigma_\beta}|_{R^{SS}}$. Moreover, one has a canonical isomorphism $H^0(Y_{\alpha,\beta}, L_Y^{\otimes n}) = \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}}$. It follows from the saturation theorem of Derksen and Weyman [DW00] that $Y_{\alpha,\beta} = \emptyset$ if and only if $\mathrm{SI}(Q, \alpha)_{\sigma_\beta} = 0$.*

The next proposition paves the way for our geometric proof of Theorem 1.0.1. It follows from 1.2.1 by a simple argument which appears in [She15, Theorem 2.5].

Proposition 1.2.2. *Theorem 1.0.1 is equivalent to the following statement. If $\mathrm{SI}(Q, \alpha)_{\sigma_\beta}$ has dimension 2, then $Y_{\alpha,\beta}$ has dimension 1.*

To prove 1.2.1, we begin with some generalities for a reductive group G acting on the left on an affine $\mathbb{C}$-variety $V = \mathrm{Spec} A$. Let $\sigma : G \to \mathbb{C}^*$ be a character. Define a linearization L of the action of G on V by letting the underlying bundle of L be $V \times \mathbb{C}$ and defining the action on L such that $g \cdot (v, z) = (gv, \sigma(g^{-1})z)$. Writing L^{-1} (also a trivial bundle) as $\mathrm{Spec}A[x]$, one obtains from the induced action a rational representation of G on $A[x]$ by $g \cdot (fx^n) = (\sigma(g^{-n}))(f \circ g^{-1})x^n$; here we regard $f \in A$ as an algebraic function on V. Also, one has an action on global sections $s : V \to L$ by $g \cdot s = g \circ s \circ g^{-1}$. This gives rise to a grade-preserving action on $R = \oplus_{n=0}^{\infty} H^0(V, L^{\otimes n})$.

Lemma 1.2.3. *With the above actions, one has a G-equivariant, graded A-algebra isomorphism $A[x] \to R$ given by sending x to the constant section 1 of L in R_1.*

Proof. The polynomial fx^n goes to the section $s : v \mapsto (v, f(v))$ in R_n. The polynomial $g \cdot (fx^n)$ goes to to the section $v \mapsto (v, \sigma(g^{-n})(f \circ g^{-1})(v))$, which is $g \cdot s$. □

Remark 1.2.1. Since G acts rationally on $A[x]$, by the theorem of Hilbert/Nagata, R^G is a finitely generated $\mathbb{C}$ algebra.

Now, let Q be a quiver without oriented cycles, and fix dimension vectors α, β with $\langle \alpha, \beta \rangle_Q = 0$, and suppose σ_β is as in Definition 1.1.2. Define a $\mathrm{GL}(Q, \alpha)$-equivariant line bundle L_{σ_β} on $\mathrm{Rep}(Q, \alpha)$ as above.

Lemma 1.2.4. *For any $n \in \mathbb{N}$, we have $H^0(\mathrm{Rep}(Q, \alpha), L_{\sigma_\beta}^{\otimes n})^{\mathrm{GL}(Q,\alpha)} = \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}}$.*

Proof. A section f of $L_{\sigma_\beta}^{\otimes n}$ is simply a regular algebraic function on $\mathrm{Rep}(Q, \alpha)$. It is $\mathrm{GL}(Q, \alpha)$ invariant if and only if $f(gV) = (\sigma_\beta(g^{-1}))^n f(V)$ for all V, g. This happens if and only if $g^{-1} \cdot f = \sigma_{n\beta}(g^{-1})f$ for all g, i.e. if and only if f is a $\sigma_{n\beta}$ semi-invariant. □

Let $R_{\alpha,\beta} := \oplus_{n=0}^{\infty} H^0(\mathrm{Rep}(Q, \alpha), L_{\sigma_\beta}^n)^{\mathrm{GL}(Q,\alpha)} = \oplus_{n=0}^{\infty} \mathrm{SI}(Q, \alpha)_{\sigma_{n\beta}}$ be the homogeneous coordinate ring of $Y_{\alpha,\beta} := \mathrm{Proj}(R_{\alpha,\beta})$. Note that

- $(R_{\alpha,\beta})_0 = \mathbb{C}$ since Q has no oriented cycles [Sch08, Exercise 1.5.1.28].
- $Y_{\alpha,\beta}$ is a finite dimensional projective scheme over $\mathrm{Spec}((R_{\alpha,\beta})_0) = \mathrm{Spec}\mathbb{C}$ by Remark 1.2.1.
- $Y_{\alpha,\beta}$ is a good quotient of $\mathrm{Rep}(Q, \alpha)^{SS}_{L_{\sigma_\beta}}$ by $\mathrm{GL}(Q, \alpha)$ [Kin94]. Thus, $Y_{\alpha,\beta}$ is integral with rational singularities (in particular, is normal).
- The notion of L_{σ_β} GIT semistability agrees with the σ_β-semistability defined by inequalities; that is, $\mathrm{Rep}(Q, \alpha)^{SS}_{L_{\sigma_\beta}} = \mathrm{Rep}(Q, \alpha)^{\sigma_\beta - SS}$ [Kin94, Proposition 3.1].

Following closely the proof of Pauly [Pau96, Theorem 3.3] of the analogous fact for moduli of parabolic bundles, we will now show that the line bundle $L_{\sigma_\beta}|_{\mathrm{Rep}(Q,\alpha)^{\sigma_\beta - ss}}$ descends to an ample line bundle on $Y_{\alpha,\beta}$. To do this, we recall the descent lemma below due to Kempf [DN89, Theorem 2.3].

Lemma 1.2.5. *Let G be a reductive linear algebraic group acting on a $\mathbb{C}$-variety X. Let $f : X \to Y$ be a good quotient of X by G and E a G-equivariant vector bundle on X. Then E descends to Y if and only if for each closed point $x \in X$ whose orbit is closed, one has that the stabilizer of x in G acts trivially on the fiber $E|_x$.*

Let V be a σ_β-semistable representation whose orbit is closed in $\mathrm{Rep}(Q,\alpha)^{\sigma_\beta - SS}$. The stabilizer S_V of V in $\mathrm{GL}(Q,\alpha)$ is the group of invertible elements of $\mathrm{Hom}_Q(V,V)$. If V is stable, then we claim $S_V = \mathbb{C}^* {\cdot} \mathrm{Id}$, by the following simple argument. If $g : V \to V$ is an automorphism of Q representations, then choosing some x for which $V(x)$ is nonzero, the isomorphism $g(x) : V(x) \to V(x)$ has a nonzero eigenvalue λ. Thus, $g - \lambda \cdot \mathrm{Id}$ has a nontrivial kernel. By stability of V and general nonsense for abelian categories with stability structure [Rud97], it follows that $g - \lambda \cdot \mathrm{Id} = 0$, whence the claim. The automorphism $\lambda \cdot \mathrm{Id}$ acts on the fiber in L_{σ_β} over V by λ to the power of $-\sum_{x\in Q_0} \alpha(x)\sigma_\beta(x) = \langle \alpha, \beta \rangle_Q = 0$, as desired.

Now consider the general case where V may not be strictly stable. By [Kin94, Propostion 3.2], we can assume V is a direct sum of σ_β-stable representations $V = m_1V_1 \oplus ... \oplus m_tV_t$ which satisfy $\sigma_\beta(\dim V_i) = 0$. Here V_i is not isomorphic to V_j if $i \neq j$. The stabilizer S_V of V is the group of invertible elements of $\mathrm{Hom}_Q(V,V)$, which, arguing as above via [Rud97], is isomorphic to $\mathrm{GL}(m_1) \times ... \times \mathrm{GL}(m_t)$. Here we identify $\mathrm{GL}(m_1) \times ... \times \mathrm{GL}(m_t)$ with the subgroup of $\mathrm{GL}(Q,\alpha)$ consisting of g such that $g(x)$, taking an appropriate basis for the direct sum, is represented by a block diagonal matrix $\mathrm{diag}(A_1(x), ..., A_t(x))$, where $A_i(x)$ is a $m_i \cdot \dim(V_i(x)) \times m_i \cdot \dim(V_i(x))$ block matrix, with m_i^2-many scalar matrix blocks of size $\dim(V_i(x)) \times \dim(V_i(x))$. The scalars that appear in these blocks do not depend on $x \in Q_0$.

Since a 1-dimensional representation of the general linear group must be given by a power of the determinant, the action of $(g_1, ..., g_t) \in S_V$ on the fiber of L_{σ_β} over V is multiplication by $\prod_{i=1}^t \det(g_i)^{a_i}$ for some integers a_i. For $i = 1, .., t$, define 1-parameter subgroups h_i of S_V by $h_i(\lambda) = (\mathrm{Id}, ..., \mathrm{Id}, \lambda \cdot \mathrm{Id}, \mathrm{Id}, ..., \mathrm{Id})$, where the λ appears in the ith factor. On the one hand, h_i acts on $L_{\sigma_\beta}|_V$ by $\lambda^{m_i a_i}$. On the other hand, regarding S_V as a subgroup of $\mathrm{GL}(Q,\alpha)$ as above, we see that h_i acts on $L_{\sigma_\beta}|_V$ by λ^{y_i} where

$$y_i = -\sum_{x \in Q_0} m_i \cdot \dim(V_i(x))\sigma_\beta(x) = -m_i \cdot \sigma_\beta(\dim V_i) = 0.$$

Therefore, $a_i = 0$ for all $i = 1, ..., t$ and S_V acts trivially on $L_{\sigma_\beta}|_V$, as desired.

1.3 Useful Inductive Structure

The following proposition allows us to complete the proof of Theorem 1.0.1 by an induction argument. It may be of independent interest outside of this proof. For example, it can be used to simplify the existing proof of the quiver-generalized Fulton conjecture [DW11], although we do not do this here.

Proposition 1.3.1. *Fix $V \in \mathrm{Rep}(Q,\alpha)$. Let U_V be a dense open subset of $\mathrm{Rep}(Q,\beta)$ with the following properties:*

i. $\dim \mathrm{Hom}_Q(V,W)$ *does not depend on $W \in U_V$.*

ii. *There is a dimension vector γ such that for every $W \in U_V$, a dense open subset of $\mathrm{Hom}_Q(V,W)$ consists of morphisms ϕ of rank γ.*

Now, fix some W in U_V. If $\phi \in \mathrm{Hom}_Q(V,W)$ has rank γ and $\ker\phi = S \in \mathrm{Rep}(Q,\alpha-\gamma)$, then the canonical surjection $\mathrm{Ext}_Q(V,W) \twoheadrightarrow \mathrm{Ext}_Q(S,W)$ is an isomorphism.

Proof. Let

$$\mathbf{H} = \{(W',\phi') \in U_V \times \mathrm{Hom}_Q(V,W') : \phi' \text{ has rank } \gamma\}$$

Note that $\mathbf{H}$ is an open subset of the total space of a vector bundle over U_V, hence irreducible of dimension

$$\dim \mathbf{H} = \dim(\mathrm{Rep}(Q,\beta)) + \dim \mathrm{Hom}_Q(V,W). \tag{1.3}$$

Denote by $\mathrm{Gr}(\alpha-\gamma,V)$ the space of $(\alpha-\gamma)$-dimensional subrepresentations of V. Then we have a map $\mathbf{H} \to \mathrm{Gr}(\alpha-\gamma,V)$ which sends (W',ϕ') to $\ker\phi'$. The fiber over a point S' is an open subset of the space of points (W',ϕ'), where $\phi' \in \mathrm{Hom}_Q(V/S',W')$.

Define an intermediate space $\mathbf{H}'$ with $\mathbf{H} \to \mathbf{H}' \to \mathrm{Gr}(\alpha - \gamma, V)$ such that the fiber in $\mathbf{H}'$ over $S' \in \mathrm{Gr}(\alpha - \gamma, V)$ is given by the open subset of $\prod_{x \in Q_0} \mathrm{Hom}((V/S')(x), \mathbb{C}^{\beta(x)})$ consisting of ϕ' such that $\phi'(x)$ is injective for all x. Clearly, $\mathbf{H}' \to \mathrm{Gr}(\alpha - \gamma, V)$ is smooth ($\mathbf{H}'$ is an open subset of a vector bundle over $\mathrm{Gr}(\alpha - \gamma, V)$) and

$$\mathrm{reldim}(\mathbf{H}' \to \mathrm{Gr}(\alpha - \gamma, V)) = \sum_{x \in Q_0} \gamma(x)\beta(x). \tag{1.4}$$

Next observe that the fiber in $\mathbf{H}$ over $(S', \phi') \in \mathbf{H}'$ is given by the space of all $W' \in U_V$ such that ϕ' is an injective morphism of representations $V/S' \to W'$. The condition imposed on each arrow a in W' is that $\phi'(ha) \circ (V/S')(a) = W'(a) \circ \phi'(ta)$. Regarding $W'(a)$ as a $\beta(ta) \times \beta(ha)$ matrix with respect to appropriately chosen bases, this equation determines $\gamma(ta)\beta(ha)$ coordinates of $W'(a)$. Thus, we obtain:

$$\mathrm{reldim}(\mathbf{H} \to \mathbf{H}') = \dim(\mathrm{Rep}(Q, \beta)) - \sum_{a \in Q_1} \gamma(ta)\beta(ha). \tag{1.5}$$

Therefore combining (1.3), (1.4), and (1.5) we obtain:

$$\dim \mathrm{Hom}_Q(V, W) \leq \dim(\mathrm{Gr}(\alpha - \gamma, V) \text{ at } S) + \langle \gamma, \beta \rangle_Q, \tag{1.6}$$

where the first summand on the right hand side of (1.6) is the dimension of the largest irreducible component of $\mathrm{Gr}(\alpha - \gamma, V)$ passing through the point S. This is at most the dimension of the scheme-theoretic tangent space to $\mathrm{Gr}(\alpha - \gamma, V)$ at S, which is $\mathrm{Hom}_Q(S, V/S)$ [Sch92, Lemma 3.2]. From (1.6), it now follows that

$$\dim \mathrm{Hom}_Q(V, W) \leq \dim \mathrm{Hom}_Q(S, V/S) + \langle \gamma, \beta \rangle_Q. \tag{1.7}$$

The given map $\phi : V \to W$ with kernel S induces an injection $\mathrm{Hom}_Q(S, V/S) \hookrightarrow \mathrm{Hom}_Q(S, W)$. It follows that

$$\dim \mathrm{Hom}_Q(V, W) \leq \dim \mathrm{Hom}_Q(S, W) + \langle \gamma, \beta \rangle_Q. \tag{1.8}$$

Since $\langle \gamma, \beta \rangle = \langle \alpha, \beta \rangle - \langle \alpha - \gamma, \beta \rangle$, the inequality (1.8) can be rewritten as $\dim \mathrm{Ext}_Q(V, W) \leq \dim \mathrm{Ext}_Q(S, W)$. The proof is complete. $\square$

1.4 Outline of the Proof of Theorem 1.0.1 by Way of Proposition 1.2.2

Let Q be a quiver without oriented cycles, α, β dimension vectors with $\langle \alpha, \beta \rangle_Q = 0$. Assume $\dim \mathrm{SI}(Q, \alpha)_{\sigma_\beta} = 2$. By Proposition 1.2.2, it suffices for the proof of Theorem 1.0.1 to show that $\dim Y_{\alpha,\beta} = 1$. This will be done by contradiction in Section 1.6. If $\dim Y_{\alpha,\beta} \geq 2$, it forces L_Y to have a base locus. Take an irreducible component of the inverse image of the base locus in $\mathrm{Rep}(Q, \alpha)^{\sigma_\beta - SS}$ and let Z be its closure in $\mathrm{Rep}(Q, \alpha)$. Now for a general point of (V, W) of $Z \times \mathrm{Rep}(Q, \beta)$, we have $\mathrm{Hom}_Q(V, W) \neq 0$ (that is, the semi-invariant $\det d_W^{\square}$ vanishes at V). This statement is to be contradicted.

Indeed, the assumption $\langle \alpha, \beta \rangle_Q = 0$ ensures that $\dim \mathrm{Hom}_Q(V, W) = \dim \mathrm{Ext}_Q(V, W)$, so it suffices for the contradiction to show that $\mathrm{Ext}_Q(V, W) = 0$. By Proposition 1.3.1, this is equivalent to $\mathrm{Ext}_Q(S, W) = 0$, where S is the kernel of a general morphism $V \to W$. The tricky part is to show that (S, W) is generic enough in a closed subset of $\mathrm{Rep}(Q, \dim S) \times \mathrm{Rep}(Q, \beta)$ to apply 1.3.1 again. For this, we need a better understanding of Z. We show that Z is actually the image in $\mathrm{Rep}(Q, \alpha)$ of a natural map from a certain irreducible scheme $\mathbf{H}_*$, constructed in Section 1.5. The simple description (1.10) of $\mathbf{H}_*$ allows us to show that indeed (S, W) is generic enough for continued application of 1.3.1. After applying 1.3.1 enough times, using the semistability of V, one finds a subrepresentation S' of S (hence of V) such that

$$0 = \mathrm{Ext}_Q(S', W) \cong \mathrm{Ext}_Q(S, W) \cong \mathrm{Ext}_Q(V, W).$$

This gives our contradiction.

Before proceeding to the detailed proof, we isolate a basic principle from linear algebra which proves very useful in the work to follow. In fact, we've already used it once to get equation (1.5).

Basic Principle. *Let V_1 and V_2 be finite dimensional vector spaces. Given two subspaces $i_1 : S_1 \hookrightarrow V_1$ and $i_2 : S_2 \hookrightarrow V_2$ and a morphism $\phi : S_1 \to S_2$, the space of linear maps $\psi : V_1 \to V_2$ such that $i_2 \circ \phi = \psi \circ i_1$ is a closed nonempty subvariety of* $\mathrm{Hom}(V_1, V_2)$ *isomorphic to $\mathbb{A}^M$, where $M = \dim V_1 \dim V_2 - \dim S_1 \dim V_2$.*

1.5 Construction of $\mathbf{H}_*$

For dimension vectors α and δ, we will say $\delta \leq \alpha$ if for all $x \in Q_0$, $\delta(x) \leq \alpha(x)$. Choose dimension vectors α, δ, ϵ with $\epsilon \leq \delta \leq \alpha$. We will first construct a smooth, irreducible scheme

$$\mathbf{U}_{\alpha,\delta,\epsilon} = \{(V, S, S', T) : V \in \operatorname{Rep}(Q, \alpha), S, S' \in \operatorname{Gr}(\delta, V), T = S \cap S' \in \operatorname{Gr}(\epsilon, V)\}. \tag{1.9}$$

To begin the construction, recall from [She15, Appendix A] the space $A^r_{f,f,g}$ of triples of subspaces S, S', $T = S \cap S'$ of $\mathbb{C}^r$ with dimensions f, f, and g, respectively. It is shown there that this space is smooth and irreducible. Define

$$A_1 := \prod_{x \in Q_0} A^{\alpha(x)}_{\delta(x),\delta(x),\epsilon(x)}.$$

We will denote points in A_1 by (S, S', T), where $S = (S(x))_{x\in Q_0}$, a collection of $\delta(x)$ dimensional subspaces of $\mathbb{C}^{\alpha(x)}$ and similarly for S' and T.

For each x in Q_0, let $\mathcal{T}(x)$ be the appropriate rank $\epsilon(x)$ universal subbundle of $\mathcal{O}_{A_1} \otimes \mathbb{C}^{\alpha(x)}$. Letting a_1,...,$a_{|Q_1|}$ denote the arrows in Q, form the total space A_1^1 of the bundle $\underline{Hom}(\mathcal{T}(ta_1), \mathcal{T}(ha_1))$ over A_1. Over A_1^1, form the total space A_1^2 of the bundle $\underline{Hom}(\mathcal{T}(ta_2)|_{A_1^1}, \mathcal{T}(ha_2)|_{A_1^1})$. Continue in this fashion until all the arrows are expended. Call the resulting space A_2, which is evidently irreducible and smooth over A_1. It can be described as follows:

$$A_2 = \{(S, S', T, \{\varphi(a)\}) : (S, S', T) \in A_1 \text{ and } \{\varphi(a)\} \in \prod_{a\in Q_1} \operatorname{Hom}(T(ta), T(ha))\}.$$

Now we will attach morphisms to the arrows of S so that T with the arrows $\{\varphi(a)\}$ gives a subrepresentation of S with the attached morphisms.

To do this, the idea is to apply the Basic Principle of Section 1.4 at each point of A_2, once for each arrow in Q_1. More formally, let $\mathcal{S}(x)$ be the appropriate rank $\delta(x)$ universal subbundle of $\mathcal{O}_{A_2} \otimes \mathbb{C}^{\alpha(x)}$. For each $a \in Q_1$, let $\Phi(a) \in \operatorname{Hom}_{\mathcal{O}_{A_2}}(\mathcal{T}(ta), \mathcal{T}(ha))$ be the universal morphism. The inclusion of bundles $\mathcal{T}(x) \to \mathcal{S}(x)$ allows us to view $\Phi(a)$ as a section of the total space of $\underline{Hom}(\mathcal{T}(ta), \mathcal{S}(ha))$. Let A_2^1 be the inverse image of $\operatorname{Im}\Phi(a_1)$ under the smooth, surjective restriction map of total spaces $\underline{Hom}(\mathcal{S}(ta_1), \mathcal{S}(ha_1)) \to \underline{Hom}(\mathcal{T}(ta_1), \mathcal{S}(ha_1))$ over A_2. Thus, A_2^1 is a smooth and surjective over A_2 and closed in $\mathrm{Hom}(\mathcal{S}(ta_1), \mathcal{S}(ha_1))$. Moreover, since the restriction map is smooth with irreducible fibers (each isomorphic to an $\mathbb{A}^M$ as in the Basic Principle), we have that A_2^1 is irreducible. Similarly, build A_2^2 over A_2^1, etc. until all arrows are expended. Repeat the procedure for S' to finally obtain

$$\begin{aligned} A_3 = \{(S, S', T, \{\varphi(a)\}, \{\psi(a)\}, \{\psi'(a)\}) : (S, S', T, \{\varphi(a)\}) \in A_2, \\ \{\psi(a)\} \in \prod_{a \in Q_1} \mathrm{Hom}(S(ta), S(ha)), \{\psi'(a)\} \in \prod_{a \in Q_1} \mathrm{Hom}(S'(ta), S'(ha)), \\ \text{and } \psi(a)|_{T(ta)} = \psi'(a)|_{T(ta)} = \varphi(a) \text{ for all } a \in Q_1\}. \end{aligned}$$

It is irreducible, surjective, and smooth over A_2.

Finally, we construct $\mathbf{U}_{\alpha,\delta,\epsilon}$ as an irreducible, surjective, and smooth scheme over A_3 by a procedure similar to the construction of A_3 itself. The idea is to create a scheme A_3^1 over A_3 whose fiber over a point $(S, S', T, \{\varphi(a)\}, \{\psi(a)\}, \{\psi'(a)\})$ is the inverse image of $(\psi(a_1), \psi'(a_1))$ under the restriction $\mathrm{Hom}(\mathbb{C}^{\alpha(ta_1)}, \mathbb{C}^{\alpha(ha_1)})$ to $\mathrm{Hom}(S(ta_1), \mathbb{C}^{\alpha(ha_1)}) \oplus \mathrm{Hom}(S'(ta_1), \mathbb{C}^{\alpha(ha_1)})$. Because $\psi(a_1), \psi'(a_1)$ restrict to the same morphism on $T(ta_1)$, this fiber is irreducible of dimension independent of the point of A_3. Hence A_3^1 is irreducible, surjective, and smooth over A_3. As above, build an appropriate scheme A_3^2 over A_3^1, and so on, until the desired $\mathbf{U}_{\alpha,\delta,\epsilon}$ is reached.

Now define

$$\begin{aligned} \mathbf{H}_{\alpha,\beta,\delta,\epsilon} = \{(V, W, W', \phi, \phi') : V \in \mathrm{Rep}(Q, \alpha), W, W' \in \mathrm{Rep}(Q, \beta), \\ \phi \in \mathrm{Hom}_Q(V, W), \phi' \in \mathrm{Hom}_Q(V, W'), \\ \ker\phi, \ker\phi' \in \mathrm{Gr}(\delta, V), (\ker\phi) \cap (\ker\phi') \in \mathrm{Gr}(\epsilon, V)\} \end{aligned} \quad (1.10)$$

This can be constructed as an irreducible, smooth scheme over $\mathbf{U}_{\alpha,\delta,\epsilon}$ as follows. Letting $\mathcal{V}(x)$, $\mathcal{S}(x)$, and $\mathcal{S}'(x)$ denote the appropriate universal bundles on $\mathbf{U}_*$ for $x \in Q_0$, form the total space of the bundle

$$\prod_{x \in Q_0} (\underline{Hom}((\mathcal{V}/\mathcal{S})(x), \mathbb{C}^{\beta(x)} \otimes \mathcal{O}) \times \underline{Hom}((\mathcal{V}/\mathcal{S}')(x), \mathbb{C}^{\beta(x)} \otimes \mathcal{O})).$$

A point of this total space over $(V, S, S', T) \in \mathbf{U}_*$ is given by finite collections of linear maps

$\{\phi(x) : V/S(x) \to \mathbb{C}^{\beta(x)}\}$ and $\{\phi'(x) : V/S'(x) \to \mathbb{C}^{\beta(x)}\}$. Let $\mathbf{H}'_{\alpha,\beta,\delta\epsilon}$ denote the open locus of the total space where each of these linear maps is injective. It is clearly irreducible, surjective , and smooth over $\mathbf{U}_*$. We build an irreducible $\mathbf{H}_*$ smoothly over $\mathbf{H}'_*$ by attaching spaces of maps $\mathbb{C}^{\beta(ta)} \to \mathbb{C}^{\beta(ha)}$, so that $\{\phi(x)\}$ and $\{\phi'(x)\}$ become morphisms of representations.

To do this, the idea is again repeat applications of the Basic Principle with, for each arrow a, the vectors spaces "V_1," "V_2," "S_1," and "S_2" given by $\mathbb{C}^{\beta(ta)}$, $\mathbb{C}^{\beta(ha)}$, $(V/S)(ta)$, and $(V/S)(ha)$ respectively, and "ϕ" given by $(V/S)(a)$ (and similarly with S' in place of S). The formal argument mirrors the construction of A_3 over A_2.

1.6 Proof of Theorem 1.0.1

We proceed by contradiction via Proposition 1.2.2. That is, we suppose

$$2 = \dim \mathrm{SI}(Q, \alpha)_{\sigma_\beta},$$

and we assume to the contrary that $\dim Y_{\alpha,\beta} \geq 2$. Recall from Proposition 1.2.1 the ample line bundle L_Y on Y. Let $Z \subseteq \mathrm{Rep}(Q, \alpha)$ be the closure of an irreducible component of the preimage of the base locus of L_Y. This base locus is nonempty by the dimension assumption on Y. For a general element $(W, W') \in \mathrm{Rep}(Q, \beta)^{\times 2}$, the semi-invariants $\det d^{\square}_W, \det d^{\square}_{W'}$ form a basis for $\mathrm{SI}(Q, \alpha)_{\sigma_\beta} = H^0(Y, L_Y)$ (see Section 1.1 and [DW11, Section 2]). In particular, it follows that for a general element $(V, W) \in Z \times \mathrm{Rep}(Q, \beta)$, one has:

$$\mathrm{Hom}_Q(V, W) \neq 0, \tag{1.11}$$

i.e. d^V_W is noninjective. Let $\delta < \alpha$ be the dimension vector of the kernel of a general morphism of quiver representations $V \to W$, equivalently such a morphism has rank $\gamma := \alpha - \delta$. Also let ϵ be a dimension vector such that given a general element $(V, W, W') \in Z \times \mathrm{Rep}(Q, \beta)^{\times 2}$ and general pair of quiver morphisms $(\phi, \phi') \in \mathrm{Hom}_Q(V, W) \times \mathrm{Hom}_Q(V, W')$, the intersection of $\ker \phi$ and $\ker \phi'$ has dimension ϵ.

Constructed in Section 1.5, we have the irreducible smooth scheme $\mathbf{H}_{\alpha,\beta,\delta,\epsilon}$, whose closed points are given by (V, W, W', ϕ, ϕ') which satisfy:

$$V \in \mathrm{Rep}(Q,\alpha),\ W, W' \in \mathrm{Rep}(Q,\beta),\ \phi \in \mathrm{Hom}_Q(V,W),\ \phi' \in \mathrm{Hom}_Q(V,W'),$$
$$\ker\phi, \ker\phi' \in \mathrm{Gr}(\delta, V),\ (\ker\phi)\cap(\ker\phi') \in \mathrm{Gr}(\epsilon, V).$$

This scheme controls the base locus Z in the sense of the following proposition, whose proof is virtually identical to [She15, Proposition 5.2].

Lemma 1.6.1. *The morphism*

$$\mathbf{H}_{\alpha,\beta,\delta,\epsilon} \to \mathrm{Rep}(Q,\alpha)\times\mathrm{Rep}(Q,\beta)\times\mathrm{Rep}(Q,\beta) : (V, W, W', \phi, \phi') \mapsto (V, W, W') \tag{1.12}$$

factors through a dominant map pr *to* $Z \times \mathrm{Rep}(Q,\beta)\times\mathrm{Rep}(Q,\beta)$.

Now, there is an irreducible space $Z_{\delta,\epsilon}$ describing all pairs (S, T) consisting of a δ-dimensional representation S of Q with an ϵ-dimensional subrepresentation $T \hookrightarrow S$ (to see this, note that $Z_{\delta,\epsilon}$ is a fiber bundle over $\mathrm{Rep}(Q,\epsilon)$). Fix (S, T, W) a general element of $Z_{\delta,\epsilon}\times\mathrm{Rep}(Q,\beta)$. Since W is general, it follows from Lemma 1.6.1 that W has a γ-dimensional subrepsentation W' (which is the image of $V \xrightarrow{\phi} W$ for some V in Z), and W' has a $(\delta-\epsilon)$-dimensional subrepresentation W'' (which is the image of $S' \xrightarrow{\ker\phi'} V \xrightarrow{\phi} W$). Let $V_0 := S\oplus W'$ (an α-dimensional representation) and let $\phi_0 : V_0 \to W$ be the obvious map which has rank γ and kernel S. Observe that $S' := T \oplus W''$ is a second δ-dimensional subrepresentation of V_0 which intersects S in the ϵ-dimensional representation T. Therefore, $(V_0, S, S', T) \in \mathbf{U}_{\alpha,\delta,\epsilon}$. By the Basic Principle of Section 1.4, one can construct a β-dimensional representation W' and a morphism $\phi_0' : V_0 \to W'$ with kernel S'. Thus, $(V_0, W, W', \phi_0, \phi_0')$ is a point of $\mathbf{H}_{\alpha,\beta,\delta,\epsilon}$, where $(S = \ker\phi_0, T = (\ker\phi_0 \cap \ker\phi_0'), W)$ is a general element of $Z_{\delta,\epsilon}\times\mathrm{Rep}(Q,\beta)$. Thus, $\mathbf{H}_{\alpha,\beta,\delta,\epsilon}$ dominates $Z_{\delta,\epsilon}\times\mathrm{Rep}(Q,\beta)$, and we have the Lemma below.

Lemma 1.6.2. *Let (V, W, W', ϕ, ϕ') be a general element of $\mathbf{H}_{\alpha,\beta,\delta,\epsilon}$, with $S := \ker\phi$, $S' := \ker\phi'$, $T := S\cap S'$. Then (S, T, W) is a general element of $Z_{\delta,\epsilon}\times\mathrm{Rep}(Q,\beta)$ (e.g. the pair (S, W) is suitable for application of Proposition 1.3.1).*

Now take a general element of $(V, W, W', \phi, \phi') \in \mathbf{H}_{\alpha,\beta,\delta,\epsilon}$ with $S := \ker\phi$, $S' := \ker\phi'$, $T := S\cap S'$. The above discussion shows that we may assume:

i. V is σ_β-semistable.

ii. (V, W) is a general element of $Z \times \operatorname{Rep}(Q, \beta)$.

iii. (S, T, W) is a general element of $Z_{\delta,\epsilon} \times \operatorname{Rep}(Q, \beta)$.

By ii and Proposition 1.3.1, we have $\operatorname{Ext}_Q(V, W) \cong \operatorname{Ext}_Q(S, W)$. By i, every subrepresentation R of S satisfies $\langle \dim R, \beta \rangle \geq 0$. By iii, we may apply Proposition 1.7.1 to (S, T, W) and conclude that $\operatorname{Ext}_Q(V, W) = 0$. Hence,

$$\dim \operatorname{Hom}_Q(V, W) = \langle \alpha, \beta \rangle_Q + \dim \operatorname{Ext}_Q(V, W) = 0 + 0 \tag{1.13}$$

This contradicts (1.11).

1.7 Vanishing of Ext for S

We now ignore the previous context and prove Proposition 1.7.1 independent of the foregoing discussion. Let $\epsilon \leq \delta$ be dimension vectors. Let $Z_{\delta,\epsilon}$ be the irreducible space consisting of a δ-dimensional representation S with an ϵ-dimensional subrepresentation T. The following is a variant of [DW00, Theorem 3] (see also [She15, Proposition 6.2]).

Proposition 1.7.1. *Suppose* $(S_0, T_0, W_0) \in Z_{\delta,\epsilon} \times \operatorname{Rep}(Q, \beta)$ *is a general element. Suppose moreover that every subrepresentation* R *of* S_0 *satisfies the inequality* $\langle \dim R, \beta \rangle \geq 0$*. Then* $\operatorname{Ext}_Q(S_0, W_0) = 0$.

Proof. We proceed by induction on the number $M_\delta := \sum_{x \in Q_0} \delta(x)$. If $M_\delta = 0$, then the conclusion holds trivially. Assume $M_\delta \geq 1$. Let $\tilde{\delta} \leq \delta$ be a dimension vector such that if (S, T, W) is a general point of $Z_{\delta,\epsilon} \times \operatorname{Rep}(Q, \beta)$, then a general morphism of representations $\psi : S \to W$ has kernel of dimension $\tilde{\delta}$. If $\tilde{\delta} = \delta$, then for a general (S, T, W), one has $\operatorname{Hom}_Q(S, W) = 0$. On the other hand, by assumption

$$0 \leq \langle \delta, \beta \rangle = \dim \operatorname{Hom}_Q(S_0, W_0) - \dim \operatorname{Ext}_Q(S_0, W_0),$$

so the conclusion follows in this case. We may as well assume then that $M_{\tilde{\delta}} < M_\delta$.

Suppose also that for a general (S, T, W), the $\tilde{\delta}$-dimensional kernel $\tilde{S}$ of a general morphism $\psi : S \to W$ meets T in an $\tilde{\epsilon}$-dimensional subreprsentation $\tilde{T}$. Let $\mathbf{U}_{\delta,\epsilon,\tilde{\delta},\tilde{\epsilon}}$ be the irreducible smooth scheme whose points are $(S, T, \tilde{S}, \tilde{T})$ of the corresponding dimensions such that $S \supseteq T$,

$S \supseteq \tilde{S}$, and $\tilde{T} = T \cap \tilde{S}$ (the construction is identical to that of Section 1.5). Build over $\mathbf{U}_{\delta,\epsilon,\tilde{\delta},\tilde{\epsilon}}$ the smooth, irreducible scheme $\mathbf{H}_{\delta,\beta,\epsilon,\tilde{\delta},\tilde{\epsilon}}$ whose fiber over $(S, T, \tilde{S}, \tilde{T})$ is the space of (W, ψ) where W is a β-dimensional representation and $\psi : S \to W$ has kernel $\tilde{S}$. By choice of $\tilde{\delta}$, $\tilde{\epsilon}$, the map $\pi : \mathbf{H}_{\delta,\beta,\epsilon,\tilde{\delta},\tilde{\epsilon}} \to Z_{\delta,\epsilon} \times \mathrm{Rep}(Q, \beta)$ is dominant.

Let $(\tilde{S}, \tilde{T}, W)$ be a general element of $Z_{\tilde{\delta},\tilde{\epsilon}} \times \mathrm{Rep}(Q, \beta)$. Since W is general, it possesses a $(\delta - \tilde{\delta})$-dimensional subrepresentation W', which in turn has a $(\epsilon - \tilde{\epsilon})$-dimensional subrepresentation W'' (see the argument preceding Lemma 1.6.2). Now consider $S := \tilde{S} \oplus W'$ and the obvious morphism $\psi : S \to W$ with kernel S and rank $\delta - \tilde{\delta}$. If $T := \tilde{T} \oplus W''$, then $(S, T, \tilde{S}, \tilde{T}, W, \psi)$ is an element of $\mathbf{H}_{\delta,\beta,\epsilon,\tilde{\delta},\tilde{\epsilon}}$. Since $(\tilde{S}, \tilde{T}, W)$ is generic, this proves the map $\tilde{\pi} : \mathbf{H}_{\delta,\beta,\epsilon,\tilde{\delta},\tilde{\epsilon}} \to Z_{\tilde{\delta},\tilde{\epsilon}} \times \mathrm{Rep}(Q, \beta)$ is also dominant. In particular, if ψ_0 is a general element of the fiber in $\mathbf{H}_{\delta,\beta,\epsilon,\tilde{\delta},\tilde{\epsilon}}$ over the general element (S_0, T_0, W_0), we can assume that the induced element $(\tilde{S}_0 := \ker \psi_0, \tilde{T}_0 := \tilde{S}_0 \cap T_0, W_0)$ of $Z_{\tilde{\delta},\tilde{\epsilon}} \times \mathrm{Rep}(Q, \beta)$ is generic.

Now by Proposition 1.3.1, $\mathrm{Ext}_Q(S_0, W_0) \cong \mathrm{Ext}_Q(\tilde{S}_0, W)$. Clearly every subrepresentation of $\tilde{S}_0$, being also a subrepresentation of S_0, satisfies the appropriate inequality. By genericity of $(\tilde{S}_0, \tilde{T}_0, W_0)$, the inductive hypothesis now completes the proof. $\square$

1.8 Connection to Invariants of Tensor Products

We now show how Theorem 1.0.1 gives the main theorem of [She15] as a corollary. Indeed, the relationship between semi-invariants of so-called triple flag quivers and SL_r invariants of three-fold tensor products is well-known [DW00, Section 3 Proposition 1]. We prove a geometric generalization, namely that the polarized moduli space of semistable representations is isomorphic to the polarized moduli space of semistable parabolic vector spaces, where in both cases semistability is determined by given Young diagrams $\lambda^1, ..., \lambda^s$.

For $p = 1, ..., s$ with $s \geq 3$, let λ^p be a partition with at most $r - 1$ nonzero parts and no part greater than ℓ. Assume also the partitions satisfy the "codimension condition:"

$$r\ell - \sum_{p=1}^{s} \sum_{a=1}^{r-1} \lambda_a^p = 0 \tag{1.14}$$

Note that there must be some such ℓ if the tensor product corresponding to $\lambda^1, ..., \lambda^s$ is to have invariants.

Let $0 < \delta_1^p < ... < \delta_{C(\lambda^p)}^p < r$ be the distinct column lengths in λ^p and suppose that there are $b(\lambda^p)_i$-many columns of length δ_i^p. Let X^p be the partial flag variety of flags

$$0 \subset F^p_{\delta_1^p} \subset F^p_{\delta_2^p} \subset ... \subset F^p_{\delta^p_{C(\lambda^p)}} \subset \mathbb{C}^r,$$

where subscripts denote dimensions. We have $\mathrm{Pic}(X^p) = \oplus_{i=1}^{C(\lambda^p)} \mathbb{Z}[L_i]$, where L_i is the pullback of the ample generator of $\mathrm{Pic}(\mathrm{Gr}(\delta_i^p, \mathbb{C}^r)) \cong \mathbb{Z}$ along the canonical projection from X^p [Bri05]. Each L_i has a canonical SL_r-equivariant structure compatible with the usual SL_r action on X^p, so that every line bundle on X^p obtains such a structure. In the sequel, we will take this equivariant structure as implicit.

With the above description of $\mathrm{Pic}(X^p)$, one has an SL_r-equivariant line bundle

$$\tilde{\mathcal{L}}_\lambda = \prod_{p=1}^{s} (b(\lambda^p)_1, ..., b(\lambda^p)_{C(\lambda^p)})$$

on $X := \prod_{p=1}^s X^p$ (for SL_r acting diagonally). The semistable points with respect to this linearization are those $\mathcal{F} = \{F^p_\bullet\}_{p=1}^s \in X$ such that if S is an r' dimensional subspace of $\mathbb{C}^r$, then

$$\sum_{p=1}^{s} \sum_{i=1}^{C(\lambda^p)} b(\lambda^p)_i \dim(F^p_{\delta_i^p} \cap S) \leq r'\ell. \tag{1.15}$$

There is an integral, projective good quotient $\rho : X^{SS} \to \mathcal{M}_\lambda$ for the action of SL_r, where $\mathcal{M}_\lambda$ has rational singularities. The line bundle $\tilde{\mathcal{L}}_\lambda$ descends to an ample line bundle $\mathcal{L}_\lambda$ on $\mathcal{M}_\lambda$. Moreover, one has a natural isomorphism for each positive integer n:

$$H^0(\mathcal{M}_\lambda, \mathcal{L}_\lambda^{\otimes n}) = (V^*_{n\lambda^1} \otimes ... \otimes V^*_{n\lambda^s})^{\mathrm{SL}_r}.$$

See [She15, Section 2] for a summary with appropriate references.

We saw similarly in Section 1.2 that for a cycle-free quiver Q and dimension vectors α, β of Q, one has a moduli space with an ample line bundle $(Y_{\alpha,\beta}, L_Y)$, where sections of tensor powers of L_Y give σ_β semi-invariants of $\mathrm{Rep}(Q, \alpha)$. The goal now is to show that for the right of choice of Q, α, β, the polarized moduli spaces $(Y_{\alpha,\beta}, L_Y)$ and $(\mathcal{M}_\lambda, \mathcal{L}_\lambda)$ are actually the same. To this end, let

Q be the s-partial flag quiver of vertices labeled $1^p, 2^p, ..., C(\lambda^p)^p$ for $p = 1, ..., s$ and one additional vertex $r = (C(\lambda^1) + 1)^1 = ... = (C(\lambda^s) + 1)^s$, with arrows $i^p \to (i + 1)^p$ for all $p = 1, ..., s$ and $i = 1, ..., C(\lambda^p)$. Let α be the dimension vector given by $\alpha(i^p) = \delta_i^p$ and $\alpha(r) = r$. For example, if $r = 4$, $\ell = 5$, $s = 3$, $\lambda^1 = 5 \geq 2 \geq 1$, $\lambda^2 = \lambda^3 = 4 \geq 2$, an element of $\mathrm{Rep}(Q, \alpha)$ is depicted below.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\phi_1^1"] & \mathbb{C}^2 \arrow[r, "\phi_2^1"] & \mathbb{C}^3 \arrow[dr, "\phi_3^1"] & \\
& \mathbb{C} \arrow[r, "\phi_1^2"] & \mathbb{C}^2 \arrow[r, "\phi_2^2"] & \mathbb{C}^4 \\
& \mathbb{C} \arrow[r, "\phi_1^3"] & \mathbb{C}^2 \arrow[ur, "\phi_2^3"] &
\end{tikzcd}$$

Let β be the dimension vector $\beta(i^p) = \ell - \sum_{i=1}^{C(\lambda^p)} b(\lambda^p)_i$. Then $\langle \alpha, \beta \rangle_Q = 0$ because of (1.14). As in Section 1.1, we have an associated weight σ_β, given in this case by $\sigma_\beta(i^p) = b(\lambda^p)_i$ and $\sigma_\beta(r) = -\ell$.

Let $G = \prod_{p=1}^s \prod_{i=1}^{C(\lambda^p)} \mathrm{Aut}(\mathbb{C}^{\delta_i^p})$, so that $\mathrm{GL}(Q, \alpha) = G \times \mathrm{Aut}(\mathbb{C}^r)$. Denote by $\mathrm{Rep}(Q, \alpha)_{\mathrm{inj}}$ the open locus of $\phi \in \mathrm{Rep}(Q, \alpha)$ such that ϕ_i^p is injective for all $p = 1, ..., s$ and $i = 1, ..., C(\lambda^p)$. One has a map $f : \mathrm{Rep}(Q, \alpha)_{\mathrm{inj}} \to \prod_{p=1}^s X^p$ which sends ϕ to the s-tuple of flags whose pth flag $F_\bullet^p$ is given by $F_{\delta_i^p}^p = \mathrm{Im}(\phi_{C(\lambda^p)}^p \circ ... \circ \phi_{i+1}^p \circ \phi_i^p)$.

Lemma 1.8.1. *The map f is a geometric quotient of $\mathrm{Rep}(Q, \alpha)_{\mathrm{inj}}$ by $G \times \{\mathrm{Id}\}$.*

Proof. The case $s = 1$, $X = \mathrm{Gr}(a, \mathbb{C}^r)$ is standard (see e.g. [Muk03, Section 8.1]). The general case proceeds much the same way. Linearize the action of $G \times \{\mathrm{Id}\}$ on $\mathrm{Rep}(Q, \alpha)$ by the character which takes $(g_x)_{x \in Q_0 - \{r\}}$ to $\prod_{x \in Q_0 - \{r\}} \det(g_x)$. It is easy to see that with respect to this linearization, we have $\mathrm{Rep}(Q, \alpha)^{SS} = \mathrm{Rep}(Q, \alpha)^S = \mathrm{Rep}(Q, \alpha)_{\mathrm{inj}}$. Therefore, one has a geometric quotient Y of $\mathrm{Rep}(Q, \alpha)_{\mathrm{inj}}$. Clearly f is constant on $G \times \{\mathrm{Id}\}$ orbits, so f descends to $\tilde{f}$ on the geometric quotient of $\mathrm{Rep}(Q, \alpha)_{\mathrm{inj}}$. Now $\tilde{f}$ is surjective (because f is), and $\tilde{f}$ is injective by the following simple argument. If $\phi = \prod_{p=1}^s (\phi_1^p, ..., \phi_{r-1}^p)$ and $\phi' = \prod_{p=1}^s (\phi_1'^p, ..., \phi_{r-1}'^p)$ have the same image in X, then in particular, ϕ_{r-1}^p and $\phi_{r-1}'^p$ have the same image in $\mathbb{C}^r$. Thus, there is a $g_{r-1}^p \in \mathrm{GL}_{r-1}$ such that $\phi_{r-1}'^p = \phi_{r-1}^p \circ (g_{r-1}^p)^{-1}$. Similarly, we have $\psi_{r-2}^p := \phi_{r-1}^p \circ \phi_{r-2}^p$ equals $\psi_{r-2}'^p$ defined likewise, whence there is $g_{r-2}^p \in \mathrm{GL}_{r-2}$ such that $\psi_{r-2}'^p = \psi_{r-2}^p \circ (g_{r-2}^p)^{-1}$. Expanding this out and canceling ϕ_{r-1}^p on opposite sides, we obtain $\phi_{r-2}'^p = g_{r-1}^p \circ \phi_{r-2}^p \circ (g_{r-2}^p)^{-1}$. Continuing in this fashion, we see that ϕ and ϕ' are in the same orbit. By Zariski's main theorem, $\tilde{f}$ is an isomorphism. $\square$

Now consider a σ_β-semistable point $\phi \in \mathrm{Rep}(Q,\alpha)^{\sigma_\beta-SS}$. If for some $1 \le i_0 \le r-1$, $1 \le p_0 \le s$, the component $\phi_{i_0}^{p_0}$ has a kernel containing a nonzero vector $v \in \mathbb{C}^{\delta_{i_0}^{p_0}}$, then ϕ has a subrepresentation of dimension vector $(\dim\psi)(i^p) = 0$ for $i^p \neq i_0^{p_0}$ and $(\dim\psi)(i_0^{p_0}) = 1$. It is given by the map $\mathbb{C} \to \mathbb{C}^{\delta_{i_0}^{p_0}}$ which sends 1 to v. In this case we have $\sigma_\beta(\dim\psi) = b(\lambda^{p_0})_{i_0}$, which is strictly positive, violating semistability of ϕ. Thus, $\mathrm{Rep}(Q,\alpha)^{\sigma_\beta-SS} \subseteq \mathrm{Rep}(Q,\alpha)_{\mathrm{inj}}$. Now an easy calculation comparing the inequalities (1.15) to those of σ_β-semistability shows that $f^{-1}(X^{SS}) = \mathrm{Rep}(Q,\alpha)^{\sigma_\beta-SS}$. Therefore, by Lemma 1.8.1, the variety X^{SS} is the good quotient $\mathrm{Rep}(Q,\alpha)^{\sigma_\beta-SS}//(G\times\{\mathrm{Id}\})$. Since a morphism from $\mathrm{Rep}(Q,\alpha)$ is $\mathrm{GL}(Q,\alpha)$ invariant if and only if it is $G\times\mathrm{SL}_r$ invariant, one has:

$$Y_{\alpha,\beta} = \mathrm{Rep}(Q,\alpha)^{\sigma_\beta-SS}//(G\times\mathrm{SL}_r) = (X^{SS})//\mathrm{SL}_r = \mathcal{M}_\lambda.$$

It now remains to prove that the line $\mathcal{L}_\lambda$ and L_Y agree under this identification. We will need some lemmas.

Lemma 1.8.2. *A section s in $H^0(\mathrm{Rep}(Q,\alpha), L_{\sigma_\beta})$ is $\mathrm{GL}(Q,\alpha)$ invariant if and only if it is $G\times\mathrm{SL}_r$ invariant.*

Proof. Necessity is immediate. For sufficiency, suppose $\bar g \in \mathrm{GL}(Q,\alpha)$. We may write this element as $\bar g = \left(\times_{p=1}^{s}\times_{i=1}^{C(\lambda^p)} \bar g_i^p\right)\times \bar g_r$. Let t be an rth root of $\det \bar g_r$ and write

$$\bar g^1 = \left(\times_{p=1}^{s}\times_{i=1}^{C(\lambda^p)} t\cdot \mathrm{Id}_{\mathbb{C}^{\delta_i^p}}\right)\times(t\cdot\mathrm{Id}_{\mathbb{C}^r}) \text{ and } \bar g^2 = \left(\times_{p=1}^{s}\times_{i=1}^{C(\lambda^p)} \frac{\bar g_i^p}{t}\right)\times\frac{\bar g_r}{t},$$

so that $\bar g = \bar g^1\cdot\bar g^2$. Since $\bar g^2 \in G\times\mathrm{SL}_r$ and $\bar g^1$ acts trivially on sections of L_{σ_β} (by (1.14)), if s is $G\times\mathrm{SL}_r$ invariant, we have $\bar g\cdot s = s$, as desired. ☐

Lemma 1.8.3. *Let $0<\delta_1<\ldots<\delta_C<r$ be integers, let*

$$H = \mathrm{Hom}(\mathbb{C}^{\delta_1},\mathbb{C}^{\delta_2})\times\ldots\times\mathrm{Hom}(\mathbb{C}^{\delta_{C-1}},\mathbb{C}^{\delta_C})\times\mathrm{Hom}(\mathbb{C}^{\delta_C},\mathbb{C}^r),$$

and H_{inj} the locus where $\phi_1,\ldots,\phi_C$ are all injective. Observe that H_{inj} has a natural conjugation action of $\mathcal{G}\times\mathrm{SL}_r$ where $\mathcal{G} := \prod_{i=1}^C \mathrm{Aut}(\mathbb{C}^{\delta_i})$. For $i=1,\ldots,C$, let $\mathcal{G}$ act trivially on $\mathrm{Gr}(\delta_i,\mathbb{C}^r)$ and its ample generator $\mathcal{O}(1)$, and let SL_r act in the usual way on both of these. If $f_i : H_{\mathrm{inj}}\to\mathrm{Gr}(\delta_i,\mathbb{C}^r)$ is the $\mathcal{G}\times\mathrm{SL}_r$-equivariant map sending $(\phi_1,\ldots,\phi_C)$ to $\mathrm{Im}(\phi_C\circ\ldots\circ\phi_i)$, then the pullback along f_i of

$\mathcal{O}(1)$ is $\mathcal{G}\times\mathrm{SL}_r$ equivariantly isomorphic to L_i on H_{inj}. Here we denote by L_i the $\mathcal{G}\times\mathrm{SL}_r$ equivariant bundle whose underlying bundle is trivial and whose action is given by $g\cdot(\phi,z)=(g\cdot\phi,(\det g_i)^{-1}z)$.

Proof. Let $\mathcal{S}_i$ be the universal subbundle on $\mathrm{Gr}(\delta_i,\mathbb{C}^r)$, endowed with an equivariant structure by allowing $\mathcal{G}$ to act trivially and SL_r to act in the obvious way. Endow also the trivial rank i bundle $H_{\mathrm{inj}}\times\mathbb{C}^i$ with the action $g\cdot(\phi,v)=(g\cdot\phi,g_iv)$. One has a $\mathcal{G}\times\mathrm{SL}_r$ equivariant isomorphism $\rho: H_{\mathrm{inj}}\times\mathbb{C}^i\to f^*\mathcal{S}_i$ which sends (ϕ,v) to $(\phi_C\circ...\circ\phi_i)(v)$ in $\mathcal{S}|_{f_i(\phi)}$. Thus, $\det\rho$ is a $\mathcal{G}\times\mathrm{SL}_r$ equivariant isomorphism of $H_{\mathrm{inj}}\times\mathbb{C}$ (action given by $g\cdot(\phi,z)=(g\cdot\phi,(\det g_i)z)$) with $f_i^*\mathcal{O}(-1)$. The assertion follows. ☐

From Lemma 1.8.3, one deduces $f^*\tilde{\mathcal{L}}_\lambda^{\otimes n}$ is $G\times\mathrm{SL}_r$ equivariantly isomorphic to $L_{\sigma_\beta}^{\otimes n}$. Thus, using 1.8.2 in the first step below, we have:

$$H^0(Y_{\alpha,\beta},L_Y^{\otimes n})=H^0(\mathrm{Rep}(Q,\alpha)^{\sigma_\beta-SS},L_{\sigma_\beta}^{\otimes n})^{G\times\mathrm{SL}_r}=H^0(X^{SS},\tilde{\mathcal{L}}_\lambda^{\otimes n})^{\mathrm{SL}_r}=H^0(\mathcal{M}_\lambda,\mathcal{L}_\lambda^{\otimes n}). \quad (1.16)$$

It follows that $L_Y=\mathcal{L}_\lambda$.

Corollary 1.8.4. *[She15] If $\dim(V_{\lambda^1}^*\otimes...\otimes V_{\lambda^s}^*)^{\mathrm{SL}_r}=2$, then $\dim(V_{n\lambda^1}^*\otimes...\otimes V_{n\lambda^s}^*)^{\mathrm{SL}_r}=n+1$ for all positive integers n.*

Proof. The left hand side of (1.16) is $\mathrm{SI}(Q,\alpha)_{\sigma_{n\beta}}$ by Proposition 1.2.1 while the right hand side is $(V_{n\lambda^1}^*\otimes...\otimes V_{n\lambda^s}^*)^{\mathrm{SL}_r}$. The corollary now follows from Theorem 1.0.1. ☐

CHAPTER 2

Parabolic Vector Bundles

For a Schubert state $\mathcal{I}$, the quantum Schubert calculus number $\langle \mathcal{I} \rangle$ appears to exhibit the same behavior under stretching as the Littlewood-Richardson numbers (see Section 2.1 for notation and generalities on quantum Schubert calculus). Thus, one has the familiar assertions for the function $P'(N) := \langle N\mathcal{I} \rangle$.

- (Polynomiality) P' is a polynomial with rational coefficients.
- (Saturation) If $P'(1) = 0$, then $P'(N) = 0$ for all $N \geq 1$.
- (Fulton) If $P'(1) = 1$, then $P'(N) = 1$ for all $N \geq 1$.

Polynomiality follows from work of Teleman [Tel00] and the relationship, apparently due to Witten, but formulated in the math literature by Belkale [Bel08], [BGM15, Theorem 3.3], equating $\langle \mathcal{I} \rangle$ with the dimension of the space of global sections of the canonical theta line bundle on the moduli space of $\mathcal{I}$-semistable parabolic bundles. See Section 2.8 for a partial proof. Saturation is proven by Belkale in [Bel08], and the quantum Fulton conjecture is proven by Belkale and Kumar in [BK16] (see also [Bel04a]).

In this chapter, we give a different proof of the quantum Fulton conjecture, validating the initial approach of Belkale in [Bel04a] to "quantum irredundancy."

Theorem 2.0.1. (**Quantum Fulton**) *Let $\mathcal{I}$ be a Schubert state. If $\langle \mathcal{I} \rangle = 1$, then $\langle N\mathcal{I} \rangle = 1$ for all positive N.*

The proof proceeds by showing that the theta sections on $\mathcal{M}(\mathcal{I})$ are basepoint free if $\langle \mathcal{I} \rangle = 1$. Since the theta bundle is ample and has an $\langle \mathcal{I} \rangle$-dimensional space of sections, it follows that $\mathcal{M}(\mathcal{I})$ is a point, whence stretching (i.e. taking tensor powers of the theta bundle) does not change anything. We also show, with no hypothesis on $\langle \mathcal{I} \rangle$, the following theorem.

Theorem 2.0.2. *If $(\mathcal{M}(\mathcal{I}), \Theta)$ is a nonempty moduli space of $\mathcal{I}$-semistable, rank r parabolic bundles on $\mathbb{P}^1$, and $r = 2$ or $r = 3$, then the sections of the canonical line bundle Θ on $\mathcal{M}(\mathcal{I})$ have no base points.*

This chapter can be viewed as having roughly three parts. The first and longest part establishes the theory of parabolic vector bundles and their moduli, using an approach which emphasizes a connection to quantum Schubert calculus in the genus 0 case. We construct the moduli space of parabolic bundles as a GIT quotient of a modified quot scheme, and then prove properties of its canonical ample line bundle Θ. The results in this first part are well-known to the experts. They are included in the interest of clarity and consistency, particularly because there is no one source that assembles all of them, and notation for these objects tends to differ by author (ours will follow Belkale).

The second part is devoted to proving an inductive structure on the space of parabolic morphisms, analogous to the structure 1.3.1 for quiver representations. This says very roughly that to compute the dimension of the space of parabolic morphisms $\widetilde{\mathcal{V}} = (\mathcal{V}, \mathcal{F}, \mathcal{I}) \to \widetilde{\mathcal{Q}} = (\mathcal{Q}, \mathcal{G}, \mathcal{I}^c)$ (here c denotes a "complementary" Schubert state: we will not actually use this notation in formal statements), it suffices to compute the dimension of the space of parabolic morphisms $\widetilde{\mathcal{S}} \to \widetilde{\mathcal{Q}}$, where $\widetilde{\mathcal{S}}$ is the kernel of a general morphism $\widetilde{\mathcal{V}} \to \widetilde{\mathcal{Q}}$. The statement is of some interest independent of this thesis, because for instance it can be used to give a simple proof of the quantum Horn conjecture (see [She15, Appendix B] for the classical counterpart). We state it here as a proposition.

Proposition 2.0.3. *Let $\mathbf{K}_{\mathcal{V}}$ be the sheaf on $\mathbb{P}^1$ of parabolic morphisms $\widetilde{\mathcal{V}} \to \widetilde{\mathcal{Q}}$ as above. If $(\mathcal{Q}, \mathcal{G})$ is generic with respect to $(\mathcal{V}, \mathcal{F})$, and a general morphism $\widetilde{\mathcal{V}} \to \widetilde{\mathcal{Q}}$ has kernel $\widetilde{\mathcal{S}}$, then the canonical surjection $H^1(\mathbf{K}_{\mathcal{V}}) \twoheadrightarrow H^1(\mathbf{K}_{\mathcal{S}})$ is an isomorphism.*

See Section 2.9 for a more precise statement of 2.0.3. From the above proposition, we deduce Theorem 2.0.2 as a fairly straightforward application.

The last part of the chapter concerns the proof of the quantum Fulton conjecture discussed above. As an epilogue, we combine our results with those of Belkale-Gibney-Kazanova [BGK15] to give a proposition on conformal blocks divisors on $\overline{M}_{0,s}$.

Convention. *For the rest of this chapter, we fix a smooth, projective, irreducible curve X with a subset $S = \{p_1, ..., p_s\}$ of s-many distinct closed points on X. The italicized capital S denoting* *a subset of X is* *not to be confused* *with the calligraphic $\mathcal{S}$, which will always be a subbundle of a bundle $\mathcal{V}$. In Section 2.1 and from Section 2.6 on, X will be the projective line $\mathbb{P}^1$. See Remark 2.6.1 for why not much is lost in assuming the genus to be* 0*.*

2.1 Quantum Schubert Calculus

Given an n-dimensional vector space W, classical Schubert calculus is concerned with the number of subspaces of W of given dimension satisfying some property. In particular, given an increasing complete flag $E_\bullet$ in W and a subset I of $[n] := \{1, ..., n\}$, say of cardinality r, one defines the associated *Schubert cell* $\Omega_I^\circ(E_\bullet)$ to be the smooth, locally closed subvariety of the Grassmannian whose pointwise description is

$$\Omega_I^\circ(E_\bullet) = \{S \in \mathrm{Gr}(r, W) : \dim(S \cap E_{I_a}) = a\}.$$

Here I_a is the ath largest integer in I. One is often interested in the *Schubert intersection* $\bigcap_{p=1}^s \Omega_{I^p}^\circ(E_\bullet^p)$, where for each $p = 1, ..., s$, the set I^p is again a subset of $[n]$ of cardinality r. We will use the notation $I^p \in \binom{[n]}{r}$ to indicate such an *index set* in the future.

Similarly, given a vector bundle $\mathcal{W}$ on $\mathbb{P}^1$, quantum Schubert calculus is concerned with the number of subbundles of given rank and degree satisfying some property (this is actually a generalization of the usual quantum Schubert calculus, where $\mathcal{W}$ is taken to be $W \otimes \mathcal{O}$). The *quantum Schubert intersection* is defined in much the same way, but we must first make sense of the Grassmannian. If $\mathcal{W}$ has rank n and degree $-D$, given integers d, r with $0 \leq r \leq n$, there is a quot scheme parametrizing coherent quotients of $\mathcal{W}$ with Hilbert polynomial $f(t) = (n-r)t + (d - D + n - r)$ with respect to $\mathcal{O}(1)$ on $\mathbb{P}^1$. This quot scheme has an open subscheme corresponding to locally free quotients, for which the kernel is a subbundle of $\mathcal{W}$ of degree $-d$ and rank r. This open subscheme shall be denoted by $\mathrm{Gr}(d, r, \mathcal{W})$ (it may be empty). Note that $\mathrm{Gr}(d, r, \mathcal{W})$ parametrizes degree *minus* d, rank r subbundles of $\mathcal{W}$. The minus sign appears so that degree d Gromov-Witten invariants may be computed by intersection theory on $\mathrm{Gr}(d, r, \mathcal{O}^{\oplus n})$.

To define the quantum Schubert intersection, let $I^p \in \binom{[n]}{r}$ for each $p \in S$, so that $I := \{I^p\}_{p \in S}$ is an *s-tuple of index sets*. We call the data $\mathcal{I} = (d, r, D, n, I)$ of integers d, r, D, n with $0 \leq r \leq n$ and an s-tuple of index sets I (with each index set in $\binom{[n]}{r}$) a *quantum Schubert state* or often just a

Schubert state. We also define

$$\mathrm{Fl}_S\mathcal{W} := \{\{E^p_\bullet\}_{p\in S} : E^p_\bullet \text{ is an increasing complete flag in the fiber } \mathcal{W}_p\}. \tag{2.1}$$

Elements of $\mathrm{Fl}_S\mathcal{W}$ will be indicated by a calligraphic letter $\mathcal{E}$. If $\pi_p : \mathrm{Gr}(d, r, \mathcal{W}) \to \mathrm{Gr}(r, \mathcal{W}_p)$ denotes the restriction to the fiber at p, the quantum Schubert intersection is then

$$\Omega^\circ_{\mathcal{I}}(\mathcal{E}) := \bigcap_{p\in S} \pi_p^{-1}(\Omega^\circ_{I^p}(E^p_\bullet)). \tag{2.2}$$

Observe the shorthand notation: no intersection sign appears, but rather the intersection should be understood from the Schubert state following the Ω°.

The scheme theoretic tangent space to $\mathrm{Gr}(d, r, \mathcal{W})$ at a point $\mathcal{V} \hookrightarrow \mathcal{W}$ is the same as the tangent space to the quot scheme at that point, which is $\mathrm{Hom}(\mathcal{V}, \mathcal{W}/\mathcal{V})$. The subspace which is tangent to the intersection $\Omega^\circ_{\mathcal{I}}(\mathcal{E})$ is given by:

$$T_{(\mathcal{V}\hookrightarrow\mathcal{W})}(\Omega^\circ_{\mathcal{I}}(\mathcal{E})) = \{\phi \in \mathrm{Hom}(\mathcal{V}, \mathcal{W}/\mathcal{V}) : \phi(E^p(\mathcal{V})_a) \subseteq E^p(\mathcal{W}/\mathcal{V})_{I^p_a - a} \text{ for } p \in S, a = 1, ..., r\}. \tag{2.3}$$

Note that $E^p(\mathcal{V})_a$ denotes the a-dimensional space in the flag on $\mathcal{V}_p$ induced by the flag $E^p_\bullet$ on $\mathcal{W}_p$. Similarly, $E^p(\mathcal{W}/\mathcal{V})_{I^p_a - a}$.

For arbitrary choices of parameters, the schemes $\mathrm{Gr}(d, r, \mathcal{W})$ and $\Omega^\circ_{\mathcal{I}}(\mathcal{E})$ may be very bad. For instance, they may be nonreduced, or different components may have different dimensions. But when the parameters are nice, so too are the schemes. In this case, a "nice" vector bundle $\mathcal{W}$ on $\mathbb{P}^1$ is one that is *evenly split*. This means that there exist integers a and $0 < b \leq n$ such that $\mathcal{W} = \mathcal{O}(a)^{\oplus b} \oplus \mathcal{O}(a+1)^{\oplus(n-b)}$. For every pair (D, n) consisting of an integer D and natural number n, there is a unique evenly split bundle of degree $-D$ and rank n up to isomorphism. It is denoted $\mathcal{Z}_{D,n}$. Evenly split bundles are the generic bundles of their rank and degree. That is, in a family of vector bundles on $\mathbb{P}^1$ parametrized by T, there is an open set (possibly empty) of $t \in T$ such that the bundle corresponding to t is evenly split.

What makes evenly split bundles so nice for Schubert calculus is first that $\mathrm{Gr}(d, r, \mathcal{Z}_{D,n})$ is smooth and integral, with a dense open subset of $\mathcal{V} \hookrightarrow \mathcal{Z}_{D,n}$ such that $\mathcal{V}$ and $\mathcal{Z}_{D,n}/\mathcal{V}$ are both evenly split. Moreover, we can apply Kleiman's theorem to infer that for generic $\mathcal{E} \in \mathrm{Fl}_S(\mathcal{Z}_{D,n})$, the intersection $\Omega^\circ_{\mathcal{I}}(\mathcal{E})$ is smooth and equidimensional of its expected dimension, which is:

$$\text{Expected Dimension of } \Omega^\circ_{\mathcal{I}} = r(n-r) + d(n-r) + r(d-D) - \sum_{p\in S}\sum_{a=1}^{r}(n-r+a-I^p_a). \quad (2.4)$$

In particular, if (2.4) is zero, then for some open dense subset U of $\mathrm{Fl}_S(\mathcal{Z}_{D,n})$, the intersection $\Omega^\circ_{\mathcal{I}}(\mathcal{E})$ is a reduced finite set of points and the number of points does not depend on $\mathcal{E} \in U$. We call this number $\langle \mathcal{I} \rangle$. For proofs of the above assertions, see [Bel08].

Finally, we remark that for $\mathcal{I} = (d, r, 0, n, I)$ (in other words, $\mathcal{I}$ is a Schubert state for $\mathcal{W} = \mathcal{O}^{\oplus n}$), $\langle \mathcal{I} \rangle$ gives the familiar Gromov-Witten number $\langle \omega_{I^{p_1}}, ..., \omega_{I^{p_s}} \rangle_d$.

2.2 An Important Vector Space

The tangent space in (2.3) can be abstracted from its context. This abstractified tangent space turns out to be central to our computations. For instance, the theta sections on the polarized moduli space of parabolic bundles will be described in terms of these spaces.

Definition 2.2.1. Let $\mathcal{I} = (d, r, D, n, I)$ be a Schubert state, $\mathcal{V}$ a vector bundle on X of rank r, $\mathcal{Q}$ a vector bundle on X of rank $n-r$, $\mathcal{F} \in \mathrm{Fl}_S\mathcal{V}$, and $\mathcal{G} \in \mathrm{Fl}_S\mathcal{Q}$. We define

$$\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}) = \{\phi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{Q}) : \phi_p(F^p_a) \subseteq G^p_{I^p_a - a} \text{ for } p \in S, a = 1, ..., r\},$$

a finite dimensional vector space over $\mathbb{C}$.

Later (Remark 2.7.1), we will see that $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ can be viewed as the space of parabolic morphisms from $(\mathcal{V}, \mathcal{F})$ to $(\mathcal{Q}, \mathcal{G})$ with appropriate weights determined by $\mathcal{I}$.

Now, let the assumptions be as in Definition 2.2.1. For each p in S, denote by $A(p)$ the sheaf on the point p whose sole fiber is the vector space

$$\frac{\mathrm{Hom}(\mathcal{V}_p, \mathcal{Q}_p)}{\{\phi : \phi(F^p_a) \subseteq G^p_{I^p_a - a},\ a = 1, ..., r\}}.$$

If i_p denotes the inclusion $\{p\} \hookrightarrow X$, then we have an exact sequence of coherent sheaves on X:

$$0 \to \mathbf{K}_{\mathcal{V}} \to \underline{Hom}(\mathcal{V}, \mathcal{Q}) \to \bigoplus_{p\in S} i_{p_*}A(p) \to 0$$

The kernel $\mathbf{K}_{\mathcal{V}}$ is locally free. Its global sections are $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V},\mathcal{Q},\mathcal{F},\mathcal{G})$. Thus, it is commonly referred to as the sheaf of parabolic morphisms $(\mathcal{V},\mathcal{F}) \to (\mathcal{Q},\mathcal{G})$ (with appropriate weights determined by $\mathcal{I}$).

Let $\mathcal{K} = (d', r', d, r, K)$ be another Schubert state. Let $\mathcal{S}$ be a subbundle of $\mathcal{V}$ such that $\mathcal{S} \in \Omega^{\circ}_{\mathcal{K}}(\mathcal{F}) \subset \mathrm{Gr}(d', r', \mathcal{V})$. For each $p \in S$, denote by $B(p)$ the sheaf on p whose sole fiber is the vector space:

$$\frac{\mathrm{Hom}(\mathcal{S}_p, \mathcal{Q}_p)}{\{\phi : \phi(F^p_a(\mathcal{S})) \subseteq G^p_{(I^p_{K^p_a} - K^p_a)},\ a = 1, ..., r'\}}.$$

Call $C(p)$ the kernel of the natural map $A(p) \to B(p)$, so that $C(p)$ is the sheaf whose sole fiber is the vector space

$$\frac{\mathrm{Hom}(\mathcal{V}_p/\mathcal{S}_p, \mathcal{Q}_p)}{\{\phi : \phi(F^p_a(\mathcal{V}/\mathcal{S})) \subseteq G^p_{(I^p_{H^p_a} - H^p_a)},\ a = 1, ..., r - r'\}}$$

Here H^p is the complement of K^p in $[r]$. We now have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{K}_{\mathcal{V}/\mathcal{S}} & \longrightarrow & \mathbf{K}_{\mathcal{V}} & \longrightarrow & \mathbf{K}_{\mathcal{S}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{Hom}(\mathcal{V}/\mathcal{S},\mathcal{Q}) & \longrightarrow & \underline{Hom}(\mathcal{V},\mathcal{Q}) & \longrightarrow & \underline{Hom}(\mathcal{S},\mathcal{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{p\in S} i_{p*}C(p) & \longrightarrow & \bigoplus_{p\in S} i_{p*}A(p) & \longrightarrow & \bigoplus_{p\in S} i_{p*}B(p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.5}$$

The sheaf $\mathbf{K}_{\mathcal{V}}$ of course depends on the flags $\mathcal{F}$ and $\mathcal{G}$. However, its Euler characteristic depends only on the Schubert state $\mathcal{I} = (d, r, D, n, I)$. We have:

$$\chi(\mathbf{K}_{\mathcal{V}}) = \chi(\mathcal{I}) = r(n-r)(1-g) + d(n-r) + r(d-D) - \sum_{p\in S}\sum_{a=1}^{r}(n - r + a - I^p_a). \tag{2.6}$$

Compare with (2.4) when $g = 0$. Similarly, let $\widetilde{\mathcal{I}}$ be the Schubert state $(d', r', D-d+d', n-r+r', \widetilde{I})$, where $\widetilde{I}^p_a = I^p_{K^p_a} - K^p_a + a$ for $p \in S$ and $a = 1, ..., r'$. Just as with $\mathbf{K}_{\mathcal{V}}$, we have $H^0(\mathbf{K}_{\mathcal{S}}) =$

$\mathrm{Hom}_{\widetilde{\mathcal{I}}}(\mathcal{S}, \mathcal{Q}, \mathcal{F}(\mathcal{S}), \mathcal{G})$. Once again, the Euler characteristic is determined by the Schubert state $\widetilde{\mathcal{I}}$.

$$\chi(\mathbf{K}_{\mathcal{S}}) = \chi(\widetilde{\mathcal{I}}) = r'(n-r)(1-g) + r'(d-D) + d'(n-r) - \sum_{p \in S} \sum_{a=1}^{r'} (n - r + K_a^p - I_{K_a^p}^p) \qquad (2.7)$$

Finally, let $\mathcal{I}' = (d-d', r-r', D-d', n-r', I')$, where $(I')_a^p = I_{H_a^p}^p - H_a^p + a$, so that $H^0(\mathbf{K}_{\mathcal{V}/\mathcal{S}}) = \mathrm{Hom}_{\mathcal{I}'}(\mathcal{V}/\mathcal{S}, \mathcal{Q}, \mathcal{F}(\mathcal{V}/\mathcal{S}), \mathcal{G})$ and $\chi(\mathbf{K}_{\mathcal{V}/\mathcal{S}}) = \chi(\mathcal{I}') = \chi(\mathcal{I}) - \chi(\widetilde{\mathcal{I}})$.

2.3 Quantum Hom Data and Genericity

Fix $\mathcal{V}$ and $\mathcal{Q}$ vector bundles on X of degrees $-d$ and $d-D$, ranks r and $n-r$, respectively. Given elements $\mathcal{F}$, $\mathcal{G}$ in $\mathrm{Fl}_S\mathcal{V}$, $\mathrm{Fl}_S\mathcal{Q}$, and a morphism $\phi : \mathcal{V} \to \mathcal{Q}$ in $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$, we can discuss the morphism's configuration in terms of the discrete quantities $\mathcal{K} = (d', r', d, r, K)$ the Schubert state of $\mathcal{S} = \ker\phi$ with respect to $\mathcal{F}$, the set theoretic map $\epsilon : S \to \mathbb{Z}_{\geq 0}$ which gives at each point p in S the dimension of the kernel of $\bar{\phi}_p : \mathcal{V}_p/\mathcal{S}_p \to \mathcal{Q}_p$, and $J = \{J^p\}_{p \in S}$ the Schubert positions of $\mathcal{B}(p) = \ker\bar{\phi}_p$ in $\mathcal{V}_p/\mathcal{S}_p$ with respect to the induced flags $F^p(\mathcal{V}/\mathcal{S})$. Let

$$\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}, \mathcal{K}, \epsilon, J)$$

denote the constructible locus in $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ of morphisms in the configuration specified by the last three parameters. Clearly, these loci disjointly stratify the vector space $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$. Moreover, only finitely many strata are nonempty. This follows from the fact that for any map $\mathcal{V} \to \mathcal{Q}$, the degree of the kernel is bounded above by a number depending on $\mathcal{V}$ and below a number depending on $\mathcal{Q}$. It follows that there are only finitely many possible d'. Finiteness of the remaining choices of parameters is immediate. In particular, we conclude that exactly one choice of $\mathcal{K}, \epsilon, J$ is such that $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}, \mathcal{K}, \epsilon, J)$ contains a dense open subset of $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$.

The above shows that for fixed $\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}$ there is a generic configuration in which most elements of $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ lie. Let us define the set theoretic map $\mathrm{hd}_{\mathcal{I}}^q$ on $\mathrm{Fl}_S\mathcal{V} \times \mathrm{Fl}_S\mathcal{Q}$ which assigns to $(\mathcal{F}, \mathcal{G})$ the quadruple $(R, \mathcal{K}, \epsilon, J)$, where R is the rank of $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ and the other entries comprise the generic configuration of elements in $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$. The tuple $\mathrm{hd}_{\mathcal{I}}^q(\mathcal{F}, \mathcal{G})$ will be called the *quantum data* of $(\mathcal{F}, \mathcal{G})$.

Definition 2.3.1. Let $(\mathcal{F}, \mathcal{G}) \in \mathrm{Fl}_S\mathcal{V} \times \mathrm{Fl}_S\mathcal{Q}$, and suppose $\mathrm{hd}^q_{\mathcal{I}}(\mathcal{F}, \mathcal{G}) = (R, \mathcal{K}, \epsilon, J)$. Then, we will say $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ is a *general element* if its configuration is described by $(\mathcal{K}, \epsilon, J)$.

2.4 Parabolic Vector Bundles

We will use a (cosmetically) different notion of parabolic bundles than appears in the standard references [MS80], [Pau96]. In particular, we prefer complete flags to partial flags. It will however be useful to go back and forth between them. Thus, the relationships to the standard notions of parabolic vector bundles are explained in the remarks following each definition.

Definition 2.4.1. A *parabolic bundle* $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ consists of a vector bundle $\mathcal{V}$ on X of degree $-d$ and rank r, an element $\mathcal{F} \in \mathrm{Fl}_S(\mathcal{V})$, and a Schubert state $\mathcal{I} = (d, r, D, n, I)$ with $I^p_1 \neq 1$ for all $p \in S$.

Remark 2.4.1. We note that to each parabolic bundle $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ as above, we may associate a parabolic bundle in the sense of [MS80]. The procedure is as follows. Define temporary weights $w^p_a = \frac{n-a+1-I^p_{r-a+1}}{n-r}$ for $a = 1, ..., r$, $p \in S$ and define weights W^p_a inductively by the prescription $W^p_1 = w^p_1$ and $W^p_i = w^p_{b^p_i}$ where $b^p_i = 1 + \max\{a : w^p_a = W^p_{i-1}\}$. In other words, the weights are determined from the temporary weights by discarding repeats. Let l^p be -1 plus the number of distinct weights W^p_i (the -1 is introduced to agree with the notation of [Pau96]). Define multiplicities by $m^p_i = |\{a : w^p_a = W^p_i\}|$. Finally, define (descending) partial flags $\tilde{F}^p$ from the full flags F^p by setting

$$\tilde{F}^p_i = F^p_{r-\sum_{j=1}^i m^p_j}$$

for $i = 0, 1, ..., l^p + 1$. It is easy to check that the data $(\mathcal{V}, \tilde{\mathcal{F}}, \{W^p_i, m^p_i\})$ defines a parabolic vector bundle in the sense of Mehta and Seshadri. We call this parabolic bundle the *induced MS parabolic bundle of type* $\mathcal{I}$.

We have a corresponding notion of parabolic subbundles.

Definition 2.4.2. A *parabolic subbundle* $(\mathcal{S}, \mathcal{F}(\mathcal{S}), \widetilde{\mathcal{I}})$ of $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ with

$$\widetilde{\mathcal{I}} = (d', r', D - d + d', n - r + r', \widetilde{I})$$

consists of a degree $-d'$, rank r' subbundle $\mathcal{S} \hookrightarrow \mathcal{V}$ such that, for all $p \in S$, the full flags $F^p_\bullet$ of $\mathcal{V}_p$ meet $\mathcal{S}_p$ in the full flags $F^p_\bullet(\mathcal{S})$ and such that $\mathcal{S}_p \in \Omega^\circ_{K^p}(F^p) \subseteq \mathrm{Gr}(r', \mathcal{V}_p)$ and where $I(K)^p_a = I^p_{K^p_a} - K^p_a + a$. Note in particular that a parabolic subbundle of $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is uniquely determined by the inclusion $\mathcal{S} \hookrightarrow \mathcal{V}$.

Remark 2.4.2. The induced MS parabolic bundle of type $\widetilde{\mathcal{I}}$ from $(\mathcal{S}, \mathcal{F}(\mathcal{S}), \widetilde{\mathcal{I}})$ (see Remark 2.4.1) is precisely the canonical MS parabolic subbundle of $(\mathcal{V}, \tilde{\mathcal{F}}, \{W^p_i, m^p_i\})$ induced by $\mathcal{S} \hookrightarrow \mathcal{V}$.

We have also the corresponding notions of parabolic degree, slope, and semistability.

Definition 2.4.3. The *parabolic degree* of the parabolic vector bundle $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is given by

$$\mathrm{pardeg}(\mathcal{V}) = -d + \sum_{p \in S} \sum_{a=1}^{r} \frac{n - r + a - I^p_a}{n - r},$$

and the *parabolic slope* is given by

$$\mu(\mathcal{V}) = \mathrm{pardeg}(\mathcal{V}) / \mathrm{rk}(\mathcal{V}).$$

The *weight* of a parabolic subbundle $\mathcal{S} \hookrightarrow \mathcal{V}$ in Schubert state $\mathcal{K}$ of a parabolic bundle $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is given by

$$\mathrm{wt}(\mathcal{S}, \mathcal{V}) = \frac{1}{n - r} \sum_{p \in S} \sum_{a \in K^p} (n - r + a - I^p_a).$$

It is easy to see that if $(\mathcal{S}, \mathcal{F}(\mathcal{S}), \widetilde{\mathcal{I}})$ is the induced structure, then

$$\mathrm{pardeg}(\mathcal{S}) = \deg \mathcal{S} + \mathrm{wt}(\mathcal{S}, \mathcal{V}).$$

A parabolic bundle $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is called *semistable* if $\mu(\mathcal{S}) \leq \mu(\mathcal{V})$ for all parabolic subbundles $\mathcal{S} \hookrightarrow \mathcal{V}$, and *stable* if this inequality is always strict.

Remark 2.4.3. Mehta-Seshadri define the parabolic degree of $(\mathcal{V}, \tilde{\mathcal{F}}, \{W^p_i, m^p_i\})$ to be

$$-d + \sum_{p \in S} \sum_{i=1}^{l^p+1} m^p_i W^p_i,$$

with slope and semistability defined analogously to Definition 2.4.3. The above quantity equals pardeg($\mathcal{V}$) as defined in Definition 2.4.3. Moreover, since taking induced MS parabolic bundles is compatible with subbundles (Remark 2.4.2), we have $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is semistable in the sense of Definition 2.4.3 if and only if $(\mathcal{V}, \tilde{\mathcal{F}}, \{W_i^p, m_i^p\})$ is semistable in the sense of Mehta and Seshadri.

To close this section, we define for a Schubert state $\mathcal{I} = (d, r, D, n, I)$ the *stretched* Schubert state

$$\langle N\mathcal{I} \rangle = (d, r, N(D-d) + d, N(n-r) + r, C) \tag{2.8}$$

where $C_a^p = N(I_a^p - a) + a$. This may seem like an unusual definition, but it is certainly the right one. For, the MS weights and multiplicites (Remark 2.4.1) defined by $N\mathcal{I}$ are the same as those defined by $\mathcal{I}$, and a parabolic bundle $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is (semi)stable if and only if $(\mathcal{V}, \mathcal{F}, N\mathcal{I})$ is semistable. We will see in Section 2.8 another reason to be happy with this definiton: namely that taking tensor powers of theta functions on $\mathcal{M}(\mathcal{I})$ is compatible with stretching $\mathcal{I}$ by N.

2.5 Moduli Space of Parabolic Bundles of Fixed Degree

Fix a point y not among the marked points of X and set $\mathcal{O}(1) := \mathcal{O}(y)$. The next lemma is well known (see, e.g. Lemma 2.3.2 of [Agn95]).

Lemma 2.5.1. *Let $\mathcal{I} = (d, r, D, n, I)$ be a Schubert state. There is an integer $k = k(\mathcal{I})$ depending only on $\mathcal{I}$ such that if $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is parabolic semistable, then $\mathcal{V}(k+j)$ is globally generated and $H^1(\mathcal{V}(k+j)) = 0$ for all $j \geq 0$.*

We will now construct the coarse moduli space of parabolic semistable bundles (up to grade equivalence) corresponding to the Schubert state $\mathcal{I} = (d, r, D, n, I)$. Choose $k = k(\mathcal{I})$ large enough as in Lemma 2.5.1. Let $\mathbf{Q}_{\mathcal{I}}$ be the quot scheme of coherent quotients of $\mathcal{O}^N$ with Hilbert polynomial $f(t) = rt - d + rk + r(1-g)$, where $N = -d + rk + r(1-g)$. Then $\mathbf{Q}_{\mathcal{I}}$ is projective and irreducible.

Choose $m = m(\mathcal{I})$ large enough so that for every $q \in \mathbf{Q}_{\mathcal{I}}$ represented by a quotient $q : \mathcal{O}^N \to \mathcal{V}(k)$ where $\mathcal{V}$ has degree $-d$ and rank r, one has $H^0(\mathcal{O}(m)^N) \twoheadrightarrow H^0(\mathcal{V}(k+m))$ surjective and $H^1(\mathcal{V}(k+m)) = 0$. Note that such an m exists and we have a closed immersion of $\mathbf{Q}_{\mathcal{I}}$ into the Grassmannian of $-d + r(k+m) + r(1-g)$ dimensional quotients of $H^0(\mathcal{O}(m)^N)$ by [Gro61].

Let $\mathscr{Q}$ denote the universal quotient sheaf on $\mathbf{Q}_{\mathcal{I}} \times X$. The group SL_N acts on $\mathbf{Q}_{\mathcal{I}} \times X$ by $g \cdot (q, x) = (q \circ g^{-1}, x)$ where we view g^{-1} as an automorphism of $\mathcal{O}^N = \mathcal{O} \otimes \mathbb{C}^N$. The fiber of $\mathscr{Q}$ over

$(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), x)$ is the quotient vector space $q|_x$ of $\mathcal{O}^N|_x$. The group SL_N acts on $\mathscr{Q}$ by sending the point $\overline{v}$ of $q|_x$ to the point $\overline{gv}$ of $(q \circ g^{-1})|_x$. Thus $\mathscr{Q}$ has the structure of an SL_N-equivariant bundle over $\mathbf{Q}_{\mathcal{I}} \times X$.

For each marked point $p \in S$, let $\mathcal{R}_p \to \mathbf{Q}_{\mathcal{I}} \times \{p\}$ denote Grothendieck's full, locally free quotient flag bundle of $\mathscr{Q}|_{\mathbf{Q}_{\mathcal{I}} \times \{p\}}$ and similarly $\mathcal{R}_{p,MS}$ the flag bundle of type I^p (see [GD71, Section 9.9]). Using the projections from $\mathbf{Q}_{\mathcal{I}} \times \{p\}$ to $\mathbf{Q}_{\mathcal{I}}$, form the fiber products

$$\mathcal{R}_{\mathcal{I}} := \prod_{p \in S} \mathcal{R}_p \qquad \text{and} \qquad \mathcal{R}_{\mathcal{I},MS} := \prod_{p \in S} \mathcal{R}_{p,MS}.$$

The action of SL_N on $\mathscr{Q}$ induces actions on $\mathcal{R}_{\mathcal{I}}$ and $\mathcal{R}_{\mathcal{I},MS}$. A point of $\mathcal{R}_{\mathcal{I},MS}$ is given by a quotient bundle $q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k)$ of degree $-d + rk$, rank r, and for each $p \in S$, a flag of the appropriate type determined by $\mathcal{I}$ in $\mathcal{V}|_p$ (clearly equivalent to giving such a flag in $\mathcal{V}(k)|_p$). By the above, we obtain an SL_N equivariant embedding of $\mathcal{R}_{\mathcal{I},MS}$ into $\mathscr{G}$, where

$$\begin{aligned}\mathscr{G} := \mathrm{Gr}^{-d+r(k+m)+r(1-g)}(H^0(\mathcal{O}(m)^N)) \times \\ \prod_{p \in S} \{\mathrm{Gr}^{m_1^p}(\mathcal{O}(-k)^N|_p) \times \mathrm{Gr}^{m_1^p+m_2^p}(\mathcal{O}(-k)^N|_p) \times ... \\ \times \mathrm{Gr}^{m_1^p+m_2^p+...+m_{l^p}^p}(\mathcal{O}(-k)^N|_p) \times \mathrm{Gr}^r(\mathcal{O}(-k)^N|_p)\}.\end{aligned} \tag{2.9}$$

This is given of course by sending $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \tilde{\mathcal{F}})$ to $H^0(\mathcal{O}(m)^N) \twoheadrightarrow H^0(\mathcal{V}(k+m))$ in the first factor of $\mathscr{G}$ and in the subsequent factors take the quotients $\mathcal{V}|_p/\tilde{F}_i^p$ defined by the tuple of flags $\tilde{\mathcal{F}}$. The above embedding allows us to linearize the action of SL_N on $\mathcal{R}_{\mathcal{I},MS}$ by selecting an SL_N-equivariant line bundle on $\mathscr{G}$.

The Picard group of $\mathscr{G}$ is given by a product of $\mathbb{Z}$'s, each generated by the determinant bundle of the universal quotient on the corresponding Grassmannian factor. Since each of these determinants has a canonical structure of an SL_N-equivariant line bundle, every line bundle on $\mathscr{G}$ can be endowed with an SL_N-equivariant structure. In particular, we linearize the action of SL_N on $\mathcal{R}_{\mathcal{I},MS}$ by the pullback of the element

$$\mathcal{L}_{\mathscr{G}} := M \cdot ((\ell + k(n-r))/m, \prod_{p \in S} \{d_1^p, ..., d_{l^p+1}^p\}) \in \mathrm{Pic}(\mathscr{G}) \tag{2.10}$$

where (taking $W^p_{l^p+2} = 1$) we set $d^p_i = (n-r)(W^p_{i+1} - W^p_i)$, ℓ is the rational number defined by the equation

$$r\ell = (-d + r(1-g))(n-r) - \sum_{p \in S} \sum_{i=1}^{l^p+1} d^p_i \{\sum_{j=1}^{i} m^p_j\}, \tag{2.11}$$

and M is a positive integer chosen so that $M(\ell + k(n-r))/m$ is an integer. We may suppose that k was chosen large enough that the first coordinate of $\mathcal{L}_{\mathscr{G}}$ is positive, and hence $\mathcal{L}_{\mathscr{G}}$ is ample. Then, it can be shown (Proposition 2.3.3 in [Agn95], Appendix A in [NR93] in the rank 2 case) that the GIT (semi)stable points

$$(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \tilde{\mathcal{F}})$$

of $(\mathcal{R}_{\mathcal{I},MS}, \mathcal{L}_{\mathscr{G}})$ are precisely those such that $\mathcal{V}$ is locally free, $H^0(\mathcal{O}^N) \to H^0(\mathcal{V}(k))$ is an isomorphism, and $(\mathcal{V}, \tilde{\mathcal{F}}, \{W^p_i, m^p_i\})$ is MS parabolic (semi)stable. The semistable locus $\mathcal{R}^{SS}_{\mathcal{I},MS}$ is connected and can be seen to be smooth by Lemma 2.5.1. The GIT quotient

$$\mathcal{M}(\mathcal{I}) := \operatorname{Proj}((\oplus_{j=0}^{\infty} H^0(\mathcal{R}_{\mathcal{I},MS}, \mathcal{L}_{\mathscr{G}}^{\otimes j}))^{\mathrm{SL}_N}) \tag{2.12}$$

of $(\mathcal{R}_{\mathcal{I},MS}, \mathcal{L}_{\mathscr{G}})$ is thus a projective, integral, normal scheme parametrizing MS parabolic semistable bundles up to an equivalence relation [MFK94, Remarks following Amplification 1.11]. In fact, $\mathcal{M}(\mathcal{I})$ has rational singularities [Bou87].

Note that the choice of M does not affect the GIT quotient (2.12) since a nonnegative integer graded ring S has the same Proj as any of its Veronese subrings $\oplus_{k=0}^{\infty} S_{dk}$. Further, the SL_N equivariant, smooth, projective surjective map

$$\mathcal{R}_{\mathcal{I}} \to \mathcal{R}_{\mathcal{I},MS} : (q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \mathcal{F}) \mapsto (q, \tilde{\mathcal{F}})$$

has geometrically connected and reduced fibers, all isomorphic to $\times_{p \in S} P^p/B$ where P^p is the parabolic in SL_r corresponding to the partial flag at p and B is a standard Borel of SL_r. So the pullback of $\mathcal{L}_{\mathscr{G}}$ to $\mathcal{R}_{\mathcal{I}}$ has the same space of global sections as $\mathcal{L}_{\mathscr{G}}$ (see Chapter 28 of the online notes [Vak13]). Hence $\mathcal{M}(\mathcal{I})$ can be viewed as a quotient of $\mathcal{R}_{\mathcal{I}}$. By Remark 2.4.3, the semistable points are precisely the $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \mathcal{F})$ such that $\mathcal{V}$ is locally free, $H^0(\mathcal{O}^N) \to H^0(\mathcal{V}(k))$ is an isomorphism, and $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is parabolic semistable.

2.6 Descending Line Bundle

If a numerical condition is satisfied, a canonical ample line bundle $\tilde{\Theta}$ on the semistable locus $\mathcal{R}^{SS}_{\mathcal{I},MS}$ descends to an ample line bundle on $\mathcal{M}(\mathcal{I})$. Note that expanding out the summation on the right hand side of (2.11), we obtain:

$$\sum_{p\in S}\sum_{i=1}^{l^p+1} d_i^p\{\sum_{j=1}^{i} m_i^p\} = r(n-r)|S| - \sum_{p\in S}\sum_{a=1}^{r} n-r+a-I_a^p \tag{2.13}$$

So ℓ will be guaranteed to be an integer provided r divides

$$-d(n-r)+\sum_{p\in S}\sum_{a=1}^{r}(n-r+a-I_a^p). \tag{2.14}$$

We make the following convention.

Convention. *From this point onward, assume $g=0$ ($X=\mathbb{P}^1$) and that r divides (2.14).*

Note that for any g there is a descending $\tilde{\Theta}$ for moduli of parabolic bundles of fixed determinant rather than fixed degree. For $\mathbb{P}^1$ of course these notions are the same. We will try to point out where the genus 0 assumption is used.

Remark 2.6.1. Also note that the restriction to genus 0 is very mild in light of the problems considered in this thesis. We consider the problem of computing $h^0(\mathcal{M},\Theta^{\otimes N})$ when $h^0(\mathcal{M},\Theta)$ is very small (1, 2, or 3). Boysal [Boy08] gives a lower bound for $h^0(\mathcal{M},\Theta)$ in positive genus, from which it follows that (a) $h^0(\mathcal{M},\Theta)$ is never 0 in positive genus, and (b) $h^0(\mathcal{M},\Theta)$ is at least 4 when $g\geq 2$, except in a small finite number of cases. Thus, $g=0$ is indeed the main case of interest for the problems considered here.

Let $\pi_{\mathcal{R}_{\mathcal{I},MS}}:\mathcal{R}_{\mathcal{I},MS}\times X\to\mathcal{R}_{\mathcal{I},MS}$ be the projection. We have a family of parabolic bundles on $\mathcal{R}_{\mathcal{I},MS}\times X$ given as follows. The forgetful map $\rho:\mathcal{R}_{\mathcal{I},MS}\times X\to\mathbf{Q}_{\mathcal{I}}\times X$ which discards the flags gives a bundle $\mathscr{V}$ on $\mathcal{R}_{\mathcal{I},MS}\times X$ by pulling back $\mathscr{Q}(-k)$ along ρ; here $\mathscr{Q}(-k)$ is the universal quotient of $\mathcal{O}(-k)^N$ of Hilbert polynomial $rt-d+r(1-g)$. Also we get bundles $\{\mathscr{Q}_i^p\}_{1\leq i\leq l^p+1}^{p\in S}$ on $\mathcal{R}_{\mathcal{I},MS}$ by pulling back the universal quotients on the Grassmannian factors after the first in $\mathscr{G}$ (see (2.9)). Note $\mathscr{Q}^p_{l^p+1}=\mathscr{V}|_{\mathcal{R}_{\mathcal{I},MS}\times\{p\}}$.

For a point $x \in X$ and a bundle $\mathscr{M}$ on $\mathcal{R}_{\mathcal{I},MS} \times X$, abbreviate $\mathscr{M}|_{\mathcal{R}_{\mathcal{I},MS}\times\{x\}}$ by $\mathscr{M}|_x$. Define the line bundle $\tilde{\Theta}$ on $\mathcal{R}_{\mathcal{I},MS}$ by the formula:

$$\tilde{\Theta} := (\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V})^{n-r} \otimes \{\otimes_{p\in S} \otimes_{i=1}^{l^p+1} (\det \mathscr{Q}_i^p)^{d_i^p}\} \otimes (\det \mathscr{V}|_y)^{\ell}. \tag{2.15}$$

The first factor of $\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V}$ is the so-called determinant of cohomology of $\mathscr{V}$. It is a bundle whose fiber over a point $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \tilde{\mathcal{F}})$ is the line $\det H^0(X,\mathcal{V})^{-1} \otimes \det H^1(X,\mathcal{V})$.

We first remark that $\{\otimes_{p\in S} \otimes_{i=1}^{l^p+1} (\det \mathscr{Q}_i^p)^{d_i^p}\}$ is the pullback of the line bundle

$$\delta := (0, \prod_{p\in S}\{d_1^p, ..., d_{l^p+1}^p\}) \in \mathrm{Pic}(\mathscr{G}) \tag{2.16}$$

Next, observe that the immersion

$$\mathbf{Q}_{\mathcal{I}} \hookrightarrow \mathrm{Gr}^{-d+r(k+m)+r(1-g)}(H^0(\mathcal{O}(m)^N))$$

of Section 2.5 is given by the quotient bundle $\pi_*\mathscr{V}(k+m)$ (guaranteed to be a bundle by the choice of m and the theorems of cohomology and base change) of the trivial bundle $\mathcal{O}^N \otimes H^0(\mathcal{O}(m))$. The determinant of this bundle is $\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}(\mathscr{V}(k+m))^{-1}$. An easy long exact sequence argument shows that

$$\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V}(k+m) = (\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V}(k)) \otimes (\det \mathscr{V}|_y)^{-m}.$$

Now we may write the polarization

$$\mathcal{L}_{\mathscr{G}} = (\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V}(k))^{-M(\ell+k(n-r))/m} \otimes (\det \mathscr{V}|_y)^{M(\ell+k(n-r))} \otimes \delta^M$$

Similarly, rewrite

$$\tilde{\Theta} = (\det R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V}(k))^{n-r} \otimes \delta \otimes (\det \mathscr{V}|_y)^{\ell+k(n-r)}$$

Since $X = \mathbb{P}^1$, and $\mathcal{O}^N \to \mathcal{V}(k)$ induces isomorphisms on H^0 and H^1 for semistable $\mathcal{V}$ (Lemma 2.5.1), the line bundle $\det R\pi_{\mathcal{R}^{SS}_{\mathcal{I},MS}*}\mathscr{V}(k)$ is trivial on the semistable locus. We conclude: *On the semistable locus, $\tilde{\Theta}$ is an Mth root of polarization defining the GIT quotient.*

If $\lambda \in \mathbb{C}^* \cap \mathrm{SL}_N$, then λ acts on the fibers of δ by multiplication by λ^α where

$$\alpha = \sum_{p \in S} \sum_{i=1}^{l^p+1} d_i^p \{\sum_{j=1}^{i} m_j^p\}.$$

Also, λ acts on the fibers of $R\pi_{\mathcal{R}_{\mathcal{I},MS}*}\mathscr{V}(k)$ by multiplication by λ to the power $-\chi(\mathcal{V}(k)) = -N$ (hence acts trivially). Combining with the action of λ on the fibers of $\det(\mathscr{V}|_y)$ (by λ^r), one sees that scalars in SL_N act trivially on the fibers of $\tilde{\Theta}$, so that $\tilde{\Theta}$ is a PSL_N equivariant bundle.

We will say that a G-equivariant line bundle $\tilde{\Theta}$ on a G-eqivariantly polarized space $(\mathcal{R}, \mathcal{L})$ *descends* to the GIT quotient $\mathcal{R}//G$ if there exists a line bundle Θ on $\mathcal{R}//G$ such that the pullback of Θ to $\mathcal{R}^{SS}$ agrees with the restriction of $\tilde{\Theta}$ to $\mathcal{R}^{SS}$.

Proposition 2.6.1. *The SL_N-equivariant bundle $\tilde{\Theta}$ on $\mathcal{R}_{\mathcal{I},MS}$ descends to an ample line bundle Θ on $\mathcal{M}(\mathcal{I})$.*

Proof. By Lemma 1.2.5, for descent it suffices to show the stabilizer of each point of $\mathcal{R}_{\mathcal{I},MS}^{SS}$ with closed orbit acts trivially on the fibers of $\tilde{\Theta}$. The stabilizer of a *stable* point of $\mathcal{R}_{\mathcal{I},MS}$ consists of the scalars in SL_N [Ses82, Troisiéme Partie], which we have already seen act trivially on the fibers. In the more general case of a point of $\mathcal{R}_{\mathcal{I},MS}^{SS} - \mathcal{R}_{\mathcal{I},MS}^{S}$ with closed orbit, calculate as in [Pau96] or the descent proof in Section 1.2, which was modeled on Pauly's argument. Thus, $\tilde{\Theta}$ descends to a line bundle Θ on $\mathcal{M}(\mathcal{I})$.

By [MFK94, Theorem 1.10], some power of the polarization descends to an ample line bundle on $\mathcal{M}(\mathcal{I})$. Since $\tilde{\Theta}$ is an Mth root of the polarization on the semistable locus, Θ must too be ample. $\square$

2.7 Theta Sections

Fix $\mathcal{Q}$ a vector bundle on X of rank $n-r$ and let $\mathcal{G} \in \mathrm{Fl}_S\mathcal{Q}$. It will be useful to connect Definition 2.2.1 to the partial flag notation we have been using up to this point.

Lemma 2.7.1. *With the notation as in Remark 2.4.1, if $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{Q})$, we have $\phi(F_a^p) \subset G^p_{I_a^p - a}$ for $a = 1, ..., r$ if and only if $\phi(\tilde{F}_i^p) \subseteq G^p_{(n-r)(1-W_{i+1}^p)}$ for $i = 0, ..., l^p$.*

Proof. It is clear from Remark 2.4.1 that

$$W^p_{i+1} = w^p_{1+\sum_{j=1}^i m^p_j} = \frac{1}{n-r}(n - \sum_{j=1}^{i}(m^p_j - I^p_{r-\sum_{j=1}^i m^p_j})) \tag{2.17}$$

So, using the above and the fact that $\tilde{F}^p_i = F^p_{r-\sum_{j=1}^i m^p_j}$, the Lemma amounts to the claim that $\phi(F^p_a) \subseteq G^p_{I^p_a - a}$ for all $a = 0, 1..., r$ if and only if

$$\phi(F^p_{(r-\sum_{j=1}^i m^p_j)}) \subseteq G^p_{I^p_{(r-\sum_{j=1}^i m^p_j)} - (r-\sum_{j=1}^i m^p_j)}$$

for all $i = 0, 1, ..., l^p$. One implication is immediate. For the other, suppose a is strictly between $r - \sum_{j=1}^{i+1} m^p_j$ and $r - \sum_{j=1}^{i} m^p_j$ for some $i = 0, ..., l^p - 1$. Then, by definition of multiplicity, $w^p_{r-a+1} = W^p_{i+1}$. Examining the formula for w^p in Remark 2.4.1 and equation (2.17), we obtain the equation

$$I^p_a - a = I^p_{(r-\sum_{j=1}^i m^p_j)} - (r - \sum_{j=1}^{i} m^p_j).$$

It follows that

$$\phi(F^p_a) \subseteq \phi(F^p_{(r-\sum_{j=1}^i m^p_j)}) \subseteq G^p_{I^p_{(r-\sum_{j=1}^i m^p_j)} - (r-\sum_{j=1}^i m^p_j)} = G^p_{I^p_a - a},$$

as needed. □

Remark 2.7.1. If $(\mathcal{Q}, \mathcal{G})$ is given an MS parabolic structure with multiplicities $\tilde{m}^p_j = (n-r)(W^p_j - W^p_{j-1})$ for $j = 1, ..., l^p + 1$, then by Remark 2.4.1, $\tilde{G}^p_i = G^p_{(n-r)(1-W^p_i)}$. If the MS parabolic weights of $(\mathcal{Q}, \mathcal{G})$ are $\tilde{W}^p_i = W^p_{i-1}$ for $i = 1, ..., l^p + 1$, then Lemma 2.7.1 shows that $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ is precisely the set of MS parabolic morphisms (see [MS80, 1.5]) from $(\mathcal{V}, \tilde{\mathcal{F}}, \{W^p_i, m^p_i\})$ to $(\mathcal{Q}, \tilde{\mathcal{G}}, \{\tilde{W}^p_i, \tilde{m}^p_i\})$.

We now define a sheaf $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})$ on $\mathcal{R}^{SS}_{\mathcal{I},MS} \times X$ with the following property. For each $\mathbf{q} := (q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \mathcal{F})$ in $\mathcal{R}^{SS}_{\mathcal{I}}$ with image (also denoted $\mathbf{q}$) $(q, \tilde{\mathcal{F}})$ in $\mathcal{R}^{SS}_{\mathcal{I},MS}$, the sheaf $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})|_{\{\mathbf{q}\} \times X}$ has global sections isomorphic to $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$. This is a "family" version of the construction given in 2.2.

To this end, note that $\mathscr{V}$ is locally free on $\mathcal{R}^{SS}_{\mathcal{I},MS} \times X$, as it has constant fiber dimension on a smooth scheme. For each $p \in S$, let $i_p : \mathcal{R}^{SS}_{\mathcal{I},MS} \times \{p\} \to \mathcal{R}^{SS}_{\mathcal{I},MS} \times X$ denote the inclusion. One has a surjective map of sheaves

$$\underline{Hom}(\mathscr{V}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}\times X}, \pi_X^*\mathcal{Q}) \twoheadrightarrow \oplus_{p\in S} i_{p*}\underline{Hom}(\mathscr{V}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}\times\{p\}}, \pi_X^*(\mathcal{Q}|_p)).$$

For convenience, denote $\mathscr{V}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}\times\{p\}}$ by $\mathscr{V}|_p$. For $i = 1, ..., l^p$, let $\tilde{\mathscr{F}}_i^p$ denote the (locally free) kernel of $\mathscr{V}|_p \twoheadrightarrow \mathscr{Q}_i^p|_{\mathcal{R}^{SS}_{\mathcal{I},MS}} \to 0$, and set $\tilde{\mathscr{F}}_0^p = \mathscr{V}|_p$. Then we have a map

$$\underline{Hom}(\mathscr{V}|_p, \pi_X^*(\mathcal{Q}|_p)) \to \oplus_{i=0}^{l^p}\underline{Hom}(\tilde{\mathscr{F}}_i^p, \pi_X^*(\mathcal{Q}|_p/G^p_{(n-r)(1-W^p_{i+1})})).$$

Let $\mathscr{K}^p$ denote the kernel. We then have a composite surjection:

$$\underline{Hom}(\mathscr{V}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}\times X}, \pi_X^*\mathcal{Q}) \twoheadrightarrow \oplus_{p\in S} i_{p*}\frac{\underline{Hom}(\mathscr{V}|_p, \pi_X^*(\mathcal{Q}|_p))}{\mathscr{K}^p} \tag{2.18}$$

Denote the kernel of (2.18) by $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$. Then $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$ is locally free (Proposition 2.7.2) and is easily seen to have the desired property (Lemma 2.7.1).

We now record the important properties of $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$. Similar results appear in [Pau98] and [Gav04].

Proposition 2.7.2. *$\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$ is locally free on $\mathcal{R}^{SS}_{\mathcal{I},MS}\times X$ of rank equal to the rank of $\underline{Hom}(\mathscr{V}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}\times X}, \pi_X^*\mathcal{Q})$.*

Proof. Denote $\mathcal{R} := \mathcal{R}^{SS}_{\mathcal{I},MS}$. We shorthand $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$ and $\underline{Hom}(\mathscr{V}|_{\mathcal{R}\times X}, \pi_X^*\mathcal{Q})$ by $\mathscr{H}$ and $\underline{H}$, respectively. Let $(\mathbf{q}, x)$ be a point of $\mathcal{R}\times X$. Clearly we have an isomorphism of stalks $\mathscr{H}_{(\mathbf{q},x)} \to \underline{H}_{(\mathbf{q},x)}$ when x is not among the points of S. It suffices then to show that for each marked point $p \in S$ that the stalk $\mathscr{H}_{(\mathbf{q},p)}$ is free.

By the local criterion for flatness, we will be done if we can show that

$$\mathrm{Tor}_1^{\mathcal{O}_{\mathcal{R}\times X,(\mathbf{q},p)}}(\mathscr{H}_{(\mathbf{q},p)}, k(\mathbf{q},p)) = 0.$$

Passing to $(\mathbf{q}, p)$ stalks in (2.18) and taking the Tor sequence, we see that the condition above is equivalent to

$$0 = \mathrm{Tor}_2^{\mathcal{O}_{\mathcal{R}\times X,(\mathbf{q},p)}}((i_{p*}\underline{H}|_p/\mathscr{K}^p)_{(\mathbf{q},p)}, k(\mathbf{q},p)) = \mathrm{Tor}_2^{\mathcal{O}_{\mathcal{R},\mathbf{q}}}((\underline{H}|_p/\mathscr{K}^p)_{\mathbf{q}}, k(\mathbf{q})).$$

This holds, since by Lemma 2.7.4, the sheaf $\underline{H}|_p/\mathscr{K}^p$ is locally free on $\mathcal{R}$. $\square$

Proposition 2.7.3. *Assume $g = 0$ ($X = \mathbb{P}^1$), $\mathcal{I}$ satisfies the codimension condition (2.22), and $\deg \mathcal{Q} = d - D$. Then, $\det R\pi_{\mathcal{R}^{SS}_{\mathcal{I},MS}*}\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$ and $\tilde{\Theta}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}}$ are canonically isomorphic.*

Proof. We will omit reference to the semistable locus for convenience. Also we will use the shorthand $\mathscr{H}$ and $\underline{H}$ for $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q},\mathcal{G})$ and $\underline{Hom}(\mathscr{V},\pi_X^*\mathcal{Q})$, respectively. By the defining expression (2.18), we have

$$\det R\pi_*\mathscr{H} = \det R\pi_*\underline{H} \otimes \otimes_{p\in S}(\det R\pi_*(i_{p*}\underline{H}|_p/\mathscr{K}^p))^{-1} \tag{2.19}$$

Since $i_{p*}\underline{H}|_p/\mathscr{K}^p$ is only supported on $\mathcal{R}^{SS}_{\mathcal{I},MS} \times \{p\}$, we have:

$$\det R\pi_*(i_{p*}\underline{H}|_p/\mathscr{K}^p) = (\det(\underline{H}|_p/\mathscr{K}^p))^{-1}.$$

Equation (2.19) can now be rewritten:

$$\det R\pi_*\mathscr{H} = \det R\pi_*\underline{H} \otimes \otimes_{p\in S}(\det(\underline{H}|_p/\mathscr{K}^p)). \tag{2.20}$$

Use Lemma 2.7.4 to write:

$$\det(\underline{H}|_p/\mathscr{K}^p) = (\det \mathscr{V}|_p)^{-(n-r)(W_1^p)-(\sum_{i=1}^{l^p} d_i^p)} \otimes \{\otimes_{i=1}^{l^p}(\det \mathscr{Q}_i^p)^{d_i^p}\}.$$

It is easy to see from the definitions that the first tensor factor above can be rewritten as $(\det \mathscr{Q}^p_{l^p+1})^{d^p_{l^p+1}} \otimes (\det \mathscr{V}|_p)^{-(n-r)}$. Substituting into (2.20), we obtain:

$$\det R\pi_*\mathscr{H} = \det R\pi_*\underline{H} \otimes \delta \otimes \{\otimes_{p\in S}(\det \mathscr{V}|_p)^{-(n-r)}\}. \tag{2.21}$$

By Mumford's Seesaw Theorem, we have $\det \mathscr{V}|_p$ canonically isomorphic to $\det \mathscr{V}|_y$ for any $p \in X$. By Serre duality, $\det R\pi_*\underline{H} = \det R\pi_*(\mathscr{V} \otimes \pi_X^*(\mathcal{Q}^\vee \otimes K_X))$. From [Pau98, Lemme 3.5] it follows that the right hand side of this equation is canonically isomorphic to $(\det R\pi_*\mathscr{V})^{n-r} \otimes (\det \mathscr{V}|_y)^{d-D+2(n-r)}$. Combine this with (2.21) and the codimension condition (2.22), and compare with definitions (2.15), (2.16), and (2.11) to complete the proof. $\square$

Remark 2.7.2. The condition $g = 0$ was necessary to identify the various $\det \mathscr{V}|_x$ line bundles via the Seesaw Theorem. In higher genus, since we consider bundles $\mathcal{V}$ of fixed degree but not fixed determinant, the line bundles $\det \mathscr{V}|_x$ will depend a priori on x.

Proposition 2.7.3 gives a way of producing a SL_N invariant sections of $\tilde{\Theta}|_{\mathcal{R}^{SS}_{\mathcal{I},MS}}$ and hence of Θ. With assumptions as in 2.7.3, let $\mathcal{Q}$ be a vector bundle of rank $n-r$, degree $d-D$, with flags $\mathcal{G} \in \mathrm{Fl}_S\mathcal{Q}$ such that there exists a semistable parabolic vector bundle $(\mathcal{V}_0, \mathcal{F}_0)$ with $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}_0, \mathcal{Q}, \mathcal{F}_0, \mathcal{G}) = 0$ (we will produce such $(\mathcal{Q}, \mathcal{G})$ in Section 2.8). Using the assumptions, one computes $\chi(\mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})|_{\mathbf{q}}) = 0$ for any $\mathbf{q} \in \mathcal{R}^{SS}_{\mathcal{I},MS}$.

We proceed as in the det/div theory of [KM76], see [Pau98, Section 3] for the relevant aspects. Suppose $f : L_0 \to L_1$ is a map of locally frees on $\mathcal{R}^{SS}_{\mathcal{I},MS}$ such that

$$\ker f = \pi_* \mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G}),\ \mathrm{coker} f = R^1\pi_* \mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G}).$$

By cohomology and base change, at a point $\mathbf{q}_0$ such that $H^i(\mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})|_{\mathbf{q}_0}) = 0$ for $i = 0, 1$, the map f is an isomorphism. Any preimage of $(\mathcal{V}_0, \mathcal{F}_0) \in \mathcal{M}(\mathcal{I})$ will do for such a $\mathbf{q}_0$. In particular, L^0 and L^1 have the same rank, so we obtain a map $\det(f)$ of their determinant bundles. Tensoring both sides of $\det(f)$ by $\det L_0^{-1}$, we obtain a section of $R\pi_* \mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})$. Up to a nonzero complex scalar, it does not depend on the choice of L_0, L_1, or f, so we denote the (scalar class of the) section $\theta(\mathcal{Q}, \mathcal{G})$. The vanishing locus of $\theta(\mathcal{Q}, \mathcal{G})$ is the set of $\mathbf{q}$ such that f fails to be surjective on the $\mathbf{q}$ fiber, equivalently $R^1\pi_* \mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})|_{\mathbf{q}} \neq 0$. Using the trivial Euler characteristic of $\mathscr{H}om_{\mathcal{I}}(\mathcal{Q}, \mathcal{G})$ and the theory of cohomology and base change, one sees that this is precisely the set of $\mathbf{q}$ such that if the image of $\mathbf{q}$ in $\mathcal{M}(\mathcal{I})$ is $(\mathcal{V}, \mathcal{F})$ then $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}) \neq 0$.

All of the above works equally well in families (see e.g. [Pau98, Section 5]) so $\theta(\mathcal{Q}, \mathcal{G})$ descends to a section of Θ. It is clear in any case from our construction that the divisor of $\theta(\mathcal{Q}, \mathcal{G})$ is SL_N invariant.

We now prove a technical lemma that was needed above.

Lemma 2.7.4. *Let $\mathcal{V}$ be a vector bundle on a scheme $\mathcal{R}$ with a filtration by subbundles*

$$\mathcal{V} = \tilde{\mathcal{F}}_0 \supseteq \tilde{\mathcal{F}}_1 \supseteq \ldots \supseteq \tilde{\mathcal{F}}_l \supseteq \tilde{\mathcal{F}}_{l+1} = 0$$

Let Q be a vector space with a filtration

$$Q \supseteq G_{\alpha_0} \supseteq G_{\alpha_1} \supseteq ... \supseteq G_{\alpha_l}.$$

Consider the map:

$$\underline{Hom}(\mathcal{V}, Q \otimes \mathcal{O}) \to \oplus_{i=0}^{l} \underline{Hom}(\tilde{\mathcal{F}}_i, Q/G_{\alpha_i} \otimes \mathcal{O}).$$

Denote its image by $\underline{H}/\mathscr{K}$. Then, after choosing a splitting of the vector space

$$Q = Q/G_{\alpha_0} \oplus G_{\alpha_0}/G_{\alpha_1} \oplus ... \oplus G_{\alpha_l},$$

one has a canonical isomorphism

$$\underline{H}/\mathscr{K} = \oplus_{i=0}^{l} (\tilde{\mathcal{F}}_i^{\vee} \otimes G_{\alpha_{i-1}}/G_{\alpha_i}),$$

where we take $G_{\alpha_{-1}} = Q$.

Proof. We induct on l. The case $l = 0$ is clear. Assume $l \geq 1$ and assume that the image of

$$\underline{Hom}(\tilde{\mathcal{F}}_1, \mathcal{G}_{\alpha_0} \otimes \mathcal{O}) \xrightarrow{\phi} \oplus_{i=1}^{l} \underline{Hom}(\tilde{\mathcal{F}}_i, G_{\alpha_0}/G_{\alpha_i} \otimes \mathcal{O})$$

is isomorphic to $\oplus_{i=1}^{l} (\tilde{\mathcal{F}}_i^{\vee} \otimes G_{\alpha_{i-1}}/G_{\alpha_i})$. Now consider the commutative diagram

$$\begin{array}{ccc}
\underline{Hom}(\mathcal{V}, G_{\alpha_0} \otimes \mathcal{O}) & \xrightarrow{a} & \oplus_{i=1}^{l} \underline{Hom}(\tilde{\mathcal{F}}_i, G_{\alpha_0}/G_{\alpha_i} \otimes \mathcal{O}) \\
b \downarrow & & \downarrow c \\
\underline{Hom}(\mathcal{V}, Q \otimes \mathcal{O}) & \xrightarrow{d} & \oplus_{i=0}^{l} \underline{Hom}(\tilde{\mathcal{F}}_i, Q/G_{\alpha_i} \otimes \mathcal{O}).
\end{array}$$

We would like to compute $\mathrm{Im}(d)$. An easy diagram chase shows that $\ker(d) = b \circ \ker(a)$. Thus, we get an inclusion $\mathrm{Im}(a) \to \mathrm{Im}(d)$ whose cokernel, by the snake lemma, is isomorphic to the cokernel of b. We then have an exact sequence:

$$0 \to \mathrm{Im}(a) \to \mathrm{Im}(d) \to \mathrm{coker}(b) \to 0.$$

It is easy to see that $\mathrm{Im}(a) = \mathrm{Im}(\phi)$ and that $\mathrm{coker}(b) = \mathcal{V}^{\vee} \otimes Q/G_{\alpha_0}$ (or equivalently $\mathrm{coker}(b) =$

$\tilde{\mathcal{F}}_0^\vee \otimes G_{\alpha_{-1}}/G_{\alpha_0})$. So by the inductive hypothesis, we have written $\mathrm{Im}(d)$ as an extension of the desired form. Using the splitting of Q, we have a section $\mathrm{coker}(b) \to \underline{Hom}(\mathcal{V}, Q \otimes \mathcal{O})$, which then splits the extension $\mathrm{Im}(d)$. $\square$

2.8 Basis of Theta Sections in Genus 0

Convention. *From this point onward, assume $\mathcal{I} = (d, r, D, n, I)$ satisfies the codimension condition:*

$$0 = r(n-r) + d(n-r) + r(d-D) - \sum_{p \in S} \sum_{a=1}^{r} (n - r + a - I_a^p) \tag{2.22}$$

Now, recall from Section 2.7 that for each pair $(\mathcal{Q}, \mathcal{G})$ consisting of a rank $n-r$, degree $d-D$ vector bundle $\mathcal{Q}$ on $\mathbb{P}^1$ and an element $\mathcal{G} \in \mathrm{Fl}_S\mathcal{Q}$, one has a (possibly zero) section $\theta(\mathcal{Q}, \mathcal{G})$ in $H^0(\mathcal{M}(\mathcal{I}), \Theta)$ whose vanishing locus is precisely the set $(\mathcal{V}, \mathcal{F})$ in $\mathcal{M}(\mathcal{I})$ such that $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}) \neq 0$. The proposition below is a straightforward modification of Theorem 4.3 in [Bel04b].

Proposition 2.8.1. *The theta sections $\theta(\mathcal{Q}, \mathcal{G})$ span $H^0(\mathcal{M}(\mathcal{I}), \Theta)$, as $(\mathcal{Q}, \mathcal{G})$ range over all degree $d - D$ rank $n - r$ bundles $\mathcal{Q}$ equipped with flags $\mathcal{G} \in \mathrm{Fl}_S\mathcal{Q}$. Moreover, one can take a theta basis of $H^0(\mathcal{M}(\mathcal{I}), \Theta)$ of the form $\theta(\mathcal{Q}, \mathcal{G}_1), ..., \theta(\mathcal{Q}, \mathcal{G}_{\langle \mathcal{I} \rangle})$ (same $\mathcal{Q}$ for all of them) where $\mathcal{Q} = \mathcal{Z}_{D-d,n-r}$ is evenly split and the $\mathcal{G}_j$ are general elements of $\mathrm{Fl}_S\mathcal{Q}$.*

Proof. Let $\mathcal{W} = \mathcal{Z}_{D,n}$ be the evenly split bundle of degree $-D$ and rank n and choose a general element $\mathcal{E} \in \mathrm{Fl}_S\mathcal{W}$. The Schubert intersection $\Omega_{\mathcal{I}}^\circ(\mathcal{W}, \mathcal{E}) \subseteq \mathrm{Gr}(d, r, \mathcal{W})$ is a reduced, finite set of $\langle \mathcal{I} \rangle$-many points. Label the points $\mathcal{V}_1, ..., \mathcal{V}_{\langle \mathcal{I} \rangle}$. We may assume all of the $(\mathcal{V}_j, \mathcal{E}(\mathcal{V}_j))$ define semi stable bundles with respect to $\mathcal{I}$ since $\mathcal{E}$ is generic. For the same reason, we may assume that each quotient pair $(\mathcal{W}/\mathcal{V}_j, \mathcal{E}(\mathcal{W}/\mathcal{V}_j))$ consists of an evenly split vector bundle and a general s-tuple of flags on it. [Bel08, Lemma 4.5].

Now the tangent space to $\Omega^\circ(\mathcal{W}, \mathcal{E})$ at the point $\mathcal{V}_j$ is $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}_j, \mathcal{W}/\mathcal{V}_j, \mathcal{E}(\mathcal{V}_j), \mathcal{E}(\mathcal{W}/\mathcal{V}_j))$, which is 0 by the transversality of the intersection. Thus, we get sections $\theta_j := \theta(\mathcal{W}/\mathcal{V}_j, \mathcal{E}(\mathcal{W}/\mathcal{V}_j))$ such that θ_j does not vanish at $(\mathcal{V}_j, \mathcal{E}(\mathcal{V}_j))$ in $\mathcal{M}(\mathcal{I})$. On the other hand, for $k \neq j$, the morphism $\mathcal{V}_k \to \mathcal{W} \to \mathcal{W}/\mathcal{V}_j$ defines a nonzero element of $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}_k, \mathcal{W}/\mathcal{V}_j, \mathcal{E}(\mathcal{V}_k), \mathcal{E}(\mathcal{W}/\mathcal{V}_j))$, hence θ_j does vanish at $(\mathcal{V}_k, \mathcal{E}(\mathcal{V}_k))$. It follows that the θ_j are $\langle \mathcal{I} \rangle$-many linearly independent elements of $H^0(\mathcal{M}(\mathcal{I}), \Theta)$.

It is known from the work of Witten [Wit95] and Agnihotri [Agn95] (more accessible references are [Bel08, Section 3] and [BGM15, Theorem 3.3]) that $\langle \mathcal{I} \rangle = \dim H^0(\mathcal{M}(\mathcal{I}), \Theta)$. This is the quantum generalization of the better known fact (when $d = D = 0$) that the ordinary Schubert product $\omega_{I^1} \cdot \ldots \cdot \omega_{I^s}$ in $H^*(\mathrm{Gr}(r,n), \mathbb{Z})$ is $\dim(V_{I^1}^* \otimes \ldots \otimes V_{I^s}^*)^{\mathrm{SL}_r}$ times the class of a point [Ful00]. The proposition now follows. □

We note that if $\mathcal{I}$ is replaced by $N\mathcal{I}$ (defined at the end of Section 2.4), the codimension condition (2.22) remains satisfied, the MS parabolic weights and multiplicities stay the same, and the same quot scheme can be used to parametrize $N\mathcal{I}$ semistable bundles (which are precisely $\mathcal{I}$-semistable bundles). However, the numbers $n - r$, d_i^p (2.10), and ℓ (2.11) are multiplied by N, so that the corresponding tilde theta bundle (2.15) for $N\mathcal{I}$ is $\tilde{\Theta}^{\otimes N}$. The lemma below now follows from 2.8.1.

Lemma 2.8.2. *For all positive integers N, we have $h^0(\mathcal{M}(\mathcal{I}), \Theta^{\otimes N}) = \langle N\mathcal{I} \rangle$.*

2.9 A Useful Inductive Structure

Convention. *For this section, we fix $\mathcal{V}$ a vector bundle on $\mathbb{P}^1$ of rank r, degree $-d$, $\mathcal{F} \in \mathrm{Fl}_S\mathcal{V}$, and $\mathcal{Q} = \mathcal{Z}_{D-d,n-r}$. Let $\mathcal{K} = (d', r', d, r, K)$ be a Schubert state, and for $p \in S$, let H^p denote the complement of K^p in $[r]$.*

The purpose of this section is to prove Proposition 2.9.2. This gives our main computational trick. The proposition says roughly that in order to compute the dimension of the space of parabolic morphisms $(\mathcal{V}, \mathcal{F}) \to (\mathcal{Q}, \mathcal{G})$ (what we have been calling $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$, see Remark 2.7.1), where $\mathcal{G}$ is generic, it suffices to compute the dimension of the space of parabolic morphisms $(\mathcal{S}, \mathcal{F}(\mathcal{S})) \to (\mathcal{Q}, \mathcal{G})$ where $\mathcal{S}$ is a certain proper subbundle of $\mathcal{V}$. To prove this, we need the following calculation.

Lemma 2.9.1. *Let $\phi : \mathcal{V} \to \mathcal{Q}$ be a morphism such that $\ker\phi$ is in Schubert state $\mathcal{K}$ with respect to $\mathcal{F}$ and such that the fiberwise maps at the marked points $(\mathcal{V}/\ker\phi)_p \to \mathcal{Q}_p$ have kernels of dimension $\epsilon(p)$ and lie in the Schubert cells $\Omega^\circ_{J^p}(F^p(\mathcal{V}/\ker\phi))$. Then, the number*

$$\dim \mathrm{Fl}_S\mathcal{Q} + \sum_{p\in S}\sum_{a=1}^{\epsilon(p)}(n - r + H^p_{J^p_a} - I_{H^p_{J^p_a}}) + \sum_{p\in S}\sum_{a=1}^{r'}(n - r + K^p_a - I^p_{K^p_a})$$
$$- \sum_{p\in S}\sum_{a=1}^{r}(n - r + a - I^p_a)$$

is the dimension of the space of $\mathcal{G} \in \mathrm{Fl}_S\mathcal{Q}$ such that $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$, if this space is nonempty.

Proof. Assume the space (call it Y) is nonempty. It is clear that Y can be constructed by building first the space of G^{p_1} with the property that $\phi(F_a^{p_1}) \subseteq G^{p_1}_{I_a^{p_1}-a}$ for $a = 1, ..., r$, building over this the space of G^{p_2} with the appropriate property, etc. So it suffices to verify the Lemma in the case where S has only one point (and hence we will omit reference to p).

Set $\ell_a = I_{H_a} - H_a$ for $a = 1, ..., r - r'$. The condition that $\phi(F_a) \subset G_{I_a - a}$ for $a = 1, ..., r$ is equivalent to the condition that $\phi(F_a(\mathcal{V}/\mathcal{S})) \subseteq G_{\ell_a}$ for $a = 1, .., r - r'$. To build the appropriate flag space, start by building the space of $(n-r-1)$-dimensional subspaces G_{n-r-1} of Q such that G_{n-r-1} contains $\phi(F_a(\mathcal{V}/\mathcal{S}))$ whenever $\ell_a \leq n - r - 1$. If $c_t = \max\{a : \ell_a \leq t\}$, this will happen if and only G_{n-r-1} contains $\phi(F_{c_{n-r-1}}(\mathcal{V}/\mathcal{S}))$. The space of all such G_{n-r-1} is then the Grassmannian of $n - r - 1 - \dim \phi(F_{c_{n-r-1}}(\mathcal{V}/\mathcal{S}))$ dimensional subspaces of $Q/\phi(F_{c_{n-r-1}}(\mathcal{V}/\mathcal{S}))$, a projective space of dimension $n - r - 1 - \dim \phi(F_{c_{n-r-1}}(\mathcal{V}/\mathcal{S}))$. Now over each point G_{n-r-1} in this projective space, we build the space of all $(n-r-2)$-dimensional subspaces of G_{n-r-1} which contain $\phi(F_a(\mathcal{V}/\mathcal{S}))$ whenever $\ell_a \leq n-r-2$, equivalently, contain $\phi(F_{c_{n-r-2}}(\mathcal{V}/\mathcal{S}))$. We obtain a projective space bundle over the previously constructed space, which has relative dimension $n-r-2-\dim\phi(F_{c_{n-r-2}}(\mathcal{V}/\mathcal{S}))$. Continue in this way until obtaining the desired flag space Y. In total, we see that Y has dimension:

$$\sum_{t=1}^{n-r-1} (n - r - t - \dim(\phi(F_{c_{n-r-t}}(\mathcal{V}/\mathcal{S})))) = \dim \mathrm{Fl}(Q) - \sum_{t=1}^{n-r-1} \dim \phi(F_{c_t}(\mathcal{V}/\mathcal{S})). \tag{2.23}$$

We now compute the sum on the right hand side of (2.23).

Clearly $c_t = a$ if $t \in [\ell_a, \ell_{a+1} - 1]$ for $a = 1, ..., r - r'$ (here we define $\ell_{r-r'+1} = n - r$). Thus, viewing the sum in (2.23) as a sum over t in $[1, \ell_1 - 1]$ plus a sum over t in $[\ell_1, \ell_2 - 1]$ and so on, we obtain:

$$\sum_{t=1}^{n-r-1} \dim \phi(F_{c_t}(\mathcal{V}/\mathcal{S})) = \sum_{a=1}^{r-r'} (\ell_{a+1} - \ell_a) \dim \phi(F_a(\mathcal{V}/\mathcal{S})).$$

Let $0 \leq b(a) \leq \epsilon$ be the unique integer such that $J_{b(a)} \leq a < J_{b(a)+1}$. Then,

$$\sum_{t=1}^{n-r-1} \dim \phi(F_{c_t}(\mathcal{V}/\mathcal{S})) = \sum_{a=1}^{r-r'} (\ell_{a+1} - \ell_a)(a - b(a)).$$

The sum $\sum_{a=1}^{r-r'} a(\ell_{a+1}-\ell_a)$ is easily seen to equal $(r-r')(n-r)-\sum_{a=1}^{r-r'} \ell_a$. The sum $\sum_{a=1}^{r-r'} b(a)(\ell_{a+1}-$

ℓ_a), breaking up into a sum over a such that $b(a) = 0$ plus a sum over a such that $b(a) = 1$ and so forth, is seen to equal

$$\begin{aligned}(\ell_{J_2} - \ell_{J_1}) + 2(\ell_{J_3} - \ell_{J_2}) + ... + \epsilon((n-r) - \ell_{J_\epsilon}) &= \epsilon(n-r) - \sum_{a=1}^{\epsilon} \ell_{J_a} \\ &= \sum_{a=1}^{\epsilon}(n - r + H_{J_a} - I_{H_{J_a}}).\end{aligned}$$

Combining these results with (2.23), we get

$$\dim Y = \mathrm{Fl}(Q) + \sum_{a=1}^{\epsilon}(n - r + H_{J_a} - I_{H_{J_a}}) - (r - r')(n - r) + \sum_{a=1}^{r-r'} \ell_a.$$

It is easy to see that

$$-(r - r')(n - r) + \sum_{a=1}^{r-r'} \ell_a = \sum_{a=1}^{r'}(n - r + K_a^p - I_{K_a^p}^p) - \sum_{a=1}^{r}(n - r + a - I_a^p).$$

So the proof is complete. □

Proposition 2.9.2. *Let $(\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V},\mathcal{F}}$ be a nonempty Zariski open subset of $\mathrm{Fl}_S \mathcal{Q}$ such that the quantum hom data $\mathrm{hd}_{\mathcal{I}}^q(\mathcal{V}, \mathcal{F}, \mathcal{G})$ is constant $(R, \mathcal{K}, \epsilon, J)$ over $\mathcal{G} \in (\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V},\mathcal{F}}$. Now fix some $\mathcal{G} \in (\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V},\mathcal{F}}$ and let $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ be a general element (2.3.1) with kernel $\mathcal{S}$. Then the canonical surjection $H^1(\mathbf{K}_{\mathcal{V}}) \to H^1(\mathbf{K}_{\mathcal{S}})$ coming from the diagram 2.5 is an isomorphism.*

Proof. Let $\mathbf{H}$ be the irreducible scheme over $(\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V},\mathcal{F}}$ whose fiber over a point $\mathcal{G}'$ is the subset $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}', \mathcal{K}, \epsilon, J)$ of $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}')$, which by assumption is a dense subset whose dimension does not vary with $\mathcal{G}' \in (\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V},\mathcal{F}}$. We thus have

$$\dim \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}) = \dim \mathbf{H} - \dim \mathrm{Fl}_S \mathcal{Q} \tag{2.24}$$

We now compute an upper bound for $\dim \mathbf{H}$ by fibering $\mathbf{H}$ over $\Omega_{\mathcal{K}}^{\circ}(\mathcal{F})$. Let $\mathbf{H}'$ be the scheme whose closed points are given by morphisms $\phi' : \mathcal{V} \to \mathcal{Q}$ with kernel $\mathcal{S}'$ in Schubert position $\mathcal{K}$ such that $\bar{\phi}'_p : (\mathcal{V}/\mathcal{S}')_p \to \mathcal{Q}_p$ has kernel in Schubert position J^p in $\mathrm{Gr}(\epsilon(p), (\mathcal{V}/\mathcal{S}')_p)$ for all $p \in \mathcal{S}$. We have an obvious morphism $\mathbf{H} \to \mathbf{H}'$ given by discarding the $\mathrm{Fl}_S \mathcal{Q}$ component. By Lemma 2.9.1, we have:

$$\dim(\mathrm{Fiber}(\mathbf{H} \to \mathbf{H}')) = \dim \mathrm{Fl}_S \mathcal{Q} + \sum_{p \in S} \sum_{a=1}^{\epsilon(p)} (n - r + H^p_{J^p_a} - I^p_{H^p_{J^p_a}})$$
$$+ \sum_{p \in S} \sum_{a=1}^{r'} (n - r + K^p_a - I^p_{K^p_a}) - \sum_{p \in S} \sum_{a=1}^{r} (n - r + a - I^p_a). \quad (2.25)$$

Let $\mathbf{B}$ be the scheme over $\mathrm{Gr}(d', r', \mathcal{V})$ whose fiber over a closed point $\mathcal{S}'$ is the product (over $\mathrm{Spec}(k(\mathcal{S}')))$ of the ordinary Schubert cells $\Omega^\circ_{\mathcal{J}^p}(F^p(\mathcal{V}/\mathcal{S}'))$ as p runs through the marked points. Its points can be described by tuples $(\mathcal{S}', (\mathcal{B}(p))_{p \in S})$. We have an obvious map $\mathbf{H}' \to \mathbf{B}$ which takes a point ϕ' of $\mathbf{H}'$ to $(\ker \phi', (\ker \bar{\phi}'_p)_{p \in S})$ in $\mathbf{B}$. I claim that if the fiber over a point $(\mathcal{S}', (\mathcal{B}(p))_{p \in S})$ is nonempty, then:

$$\dim(\mathrm{Fiber}(\mathbf{H}' \to \mathbf{B})) = (r - r')(n - r) + (d - d')(n - r)$$
$$+ (r - r')(d - D) - \sum_{p \in S} \epsilon(p)(n - r). \quad (2.26)$$

Indeed, the fiber over $(\mathcal{S}', (\mathcal{B}(p))_{p \in S})$ can be identified with the space of injective morphisms $\phi' : \mathcal{V}/\mathcal{S}' \to \mathcal{Q}$ with kernel at a marked point p given by $\mathcal{B}(p)$. By [Bel08, Appendix A], this is an open subset of the space of morphisms $\widetilde{\mathcal{V}/\mathcal{S}'} \to \mathcal{Q}$ where $\widetilde{\mathcal{V}/\mathcal{S}'}$ is the shift of $\mathcal{V}/\mathcal{S}'$ along $\mathcal{B}(p)$. This has the same rank $r - r'$ as $\mathcal{V}/\mathcal{S}'$, but has degree $\deg(\mathcal{V}/\mathcal{S}') + \sum_{p \in S} \epsilon(p)$. Thus, the fiber has dimension:

$$\dim H^0(\mathbb{P}^1, \underline{Hom}(\widetilde{\mathcal{V}/\mathcal{S}'}, \mathcal{Q})) = \chi(\underline{Hom}(\widetilde{\mathcal{V}/\mathcal{S}}, \mathcal{Q})) + H^1(\mathbb{P}^1, \underline{Hom}(\widetilde{\mathcal{V}/\mathcal{S}'}, \mathcal{Q}))$$
$$= (n - r + d - D)(r - r') + (d - d' - \sum_{p \in S} \epsilon(p))(n - r) + H^1(\mathbb{P}^1, \underline{Hom}(\widetilde{\mathcal{V}/\mathcal{S}'}, \mathcal{Q})).$$

The claim will be proven once we show that the H^1 term is 0. To see this, we note that nonemptiness of the fiber in $\mathbf{H}'$ over $(\mathcal{S}', (\mathcal{B}(p))_{p \in S})$ implies the existence of an injective morphism $\widetilde{\mathcal{V}/\mathcal{S}'} \to \mathcal{Q}$ by [Bel08, Lemma A.2(3)]. Since $\mathcal{Q}$ is evenly split, say $\mathcal{Q} = \mathcal{O}(a-1)^t \oplus \mathcal{O}(a)^{n-r-t}$, the bundle $\widetilde{\mathcal{V}/\mathcal{S}'}$ has only summands $\mathcal{O}(b)$ where $b \leq a$ in its Grothendieck decomposition. It follows that the Grothendieck decomposition of $\underline{Hom}(\widetilde{\mathcal{V}/\mathcal{S}'}, \mathcal{Q})$ has only summands $\mathcal{O}(c)$ where $c \geq -1$. Such summands have no H^1.

Now it easy to see that:

$$\dim(\mathrm{Fiber}(\mathbf{B} \to \mathrm{Gr}(d', r', \mathcal{V}))) = \sum_{p \in S} \sum_{a=1}^{\epsilon(p)} J_a^p - a. \tag{2.27}$$

If $\mathrm{Im}(\mathbf{H})$ denotes the (irreducible) image of $\mathbf{H}$ in $\Omega^\circ_{\mathcal{K}}(\mathcal{F}) \subseteq \mathrm{Gr}(d', r', \mathcal{V})$, then:

$$\begin{aligned}
\dim \mathbf{H} \leq \dim \mathrm{Im}(\mathbf{H}) &+ \dim(\mathrm{Fiber}(\mathbf{B} \to \mathrm{Gr}(d', r', \mathcal{V}))) \\
&+ \dim(\mathrm{Fiber}(\mathbf{H}' \to \mathbf{B})) + \dim(\mathrm{Fiber}(\mathbf{H} \to \mathbf{H}')) \\
= \dim \mathrm{Im}(\mathbf{H}) &+ \dim \mathrm{Fl}_S \mathcal{Q} + (r - r')(n - r) + (d - d')(n - r) + (r - r')(d - D) \\
&+ \sum_{p \in S} \sum_{a=1}^{r'} (n - r + K_a^p - I^p_{K_a^p}) - \sum_{p \in S} \sum_{a=1}^{r} (n - r + a - I_a^p) + \sum_{p \in S} \mathrm{Disc}(p),
\end{aligned} \tag{2.28}$$

where the "local discrepancy" $\mathrm{Disc}(p)$ is defined as

$$\mathrm{Disc}(p) = \sum_{a=1}^{\epsilon(p)} (H^p_{J_a^p} - I^p_{H^p_{J_a^p}} + J_a^p - a).$$

The same quantity appears in [Bel08, Section 9.1]. As there, $\mathrm{Disc}(p) \leq 0$ by the following simple argument. Since the given ϕ is such that for all $m = 1, ..., r - r'$, one has

$$\bar{\phi}_p(F_m^p(\mathcal{V}/\mathcal{S})) \subseteq G^p_{I^p_{H^p_m} - H^p_m},$$

the dimension of the left hand side when $m = J_a^p$ must be less than or equal to the dimension of the right hand side for the same m. The former is $J_a^p - a$, while the latter is $I^p_{H^p_{J_a^p}} - H^p_{J_a^p}$, which completes the simple argument. Recalling the formulas (2.6) and (2.7) for $\chi(\mathbf{K}_\mathcal{V})$ and $\chi(\mathbf{K}_\mathcal{S})$ respectively, one obtains from (2.28):

$$\dim \mathbf{H} \leq \dim \mathrm{Im}(\mathbf{H}) + \dim \mathrm{Fl}_S \mathcal{Q} + \chi(\mathbf{K}_\mathcal{V}) - \chi(\mathbf{K}_\mathcal{S}). \tag{2.29}$$

Combining with (2.24) and recalling that $H^0(\mathbf{K}_\mathcal{V}) = \mathrm{Hom}_\mathcal{I}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$, we find the inequality

$$h^1(\mathbf{K}_\mathcal{V}) \leq \dim \mathrm{Im}(\mathbf{H}) - \chi(\mathbf{K}_\mathcal{S}), \tag{2.30}$$

Finally, we note that $\mathrm{Im}(\mathbf{H})$ is an irreducible subscheme of $\Omega^\circ_{\mathcal{K}}(\mathcal{F})$ which contains the point $\mathcal{S}$, so certainly its dimension does not exceed the dimension of the tangent space to $\mathcal{S}$ in $\Omega^\circ_{\mathcal{K}}(\mathcal{F})$, which is $\mathrm{Hom}_{\mathcal{K}}(\mathcal{S}, \mathcal{V}/\mathcal{S}, \mathcal{F}(\mathcal{S}), \mathcal{F}(\mathcal{V}/\mathcal{S}))$. Under the injection $\mathrm{Hom}(\mathcal{S}, \mathcal{V}/\mathcal{S}) \to \mathrm{Hom}(\mathcal{S}, \mathcal{Q})$ induced by ϕ, the subspace $\mathrm{Hom}_{\mathcal{K}}(\mathcal{S}, \mathcal{V}/\mathcal{S}, \mathcal{F}(\mathcal{S}), \mathcal{F}(\mathcal{V}/\mathcal{S}))$ maps into $H^0(\mathbf{K}_{\mathcal{S}})$. Inequality (2.30) now gives $h^1(\mathbf{K}_{\mathcal{V}}) \leq h^1(\mathbf{K}_{\mathcal{S}})$, and the proposition follows. □

2.10 Analysis of the Cases $r = 2$ and $r = 3$

Here we explore simple applications of Proposition 2.9.2 to the cases where $r = 2$ or $r = 3$. Suppose first that $\mathcal{I} = (d, 2, D.n, I)$ is a Schubert state with $\langle \mathcal{I} \rangle \neq 0$.

Proposition 2.10.1. *For $r = 2$, the line bundle Θ on $\mathcal{M}(\mathcal{I})$ is base point free.*

Proof. Suppose to the contrary that $Z \neq \emptyset$ denotes an irreducible component the base locus. By the description of the theta sections in Section 2.8, for each $(\mathcal{V}, \mathcal{F}, \mathcal{G}) \in Z \times \mathrm{Fl}_S(\mathcal{Q})$, the general element $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ is nonzero. The rank r_0', degree $-d_0'$, and Schubert position $\mathcal{K}_0$ of the kernel of such a ϕ will be constant over an open subset U_0 of $Z \times \mathrm{Fl}_S \mathcal{Q}$. Since r_0' equals 0 or 1, there is a unique bundle $\mathcal{S}_0$ of rank r_0' and degree $-d_0'$, and $\mathrm{Fl}_S \mathcal{S}_0$ consists of a single point $\mathcal{F}(\mathcal{S}_0)$. Therefore, shrinking U_0 if necessary, we can assume that for every $(\mathcal{V}, \mathcal{F}, \mathcal{G}) \in U_0$, we have $\mathcal{G} \in (\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V}, \mathcal{F}} \cap (\mathrm{Fl}_S \mathcal{Q})_{\mathcal{S}_0, \mathcal{F}(\mathcal{S}_0)}$, an intersection of nonempty open subsets as described in 2.9.2.

Now choose $(\mathcal{V}, \mathcal{F}, \mathcal{G}) \in U_0$, and let $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ be a general element. Then $\ker \phi \cong \mathcal{S}_0$, and we have $h^1(\mathbf{K}_{\mathcal{V}}) = h^1(\mathbf{K}_{\mathcal{S}_0})$ by 2.9.2. If $r_0' = 0$, then $h^1(\mathbf{K}_{\mathcal{V}}) = 0$ follows. In the case where $r_0' = 1$, we observe that a general element of $\mathrm{Hom}_{\tilde{\mathcal{I}}}(\mathcal{S}_0, \mathcal{Q}, \mathcal{F}(\mathcal{S}_0), \mathcal{G}) = H^0(\mathbf{K}_{\mathcal{S}_0})$ is injective or zero. If the general element is injective, then use the construction of U_0 to apply 2.9.2 again, obtaining $0 = h^1(\mathbf{K}_{\mathcal{S}_0}) = h^1(\mathbf{K}_{\mathcal{V}})$. If the general element is zero, then we have $h^0(\mathbf{K}_{\mathcal{S}_0}) = 0$. But by semistability of $(\mathcal{V}, \mathcal{F})$, we also have $\chi(\mathbf{K}_{\mathcal{S}_0}) \geq 0$. Again, $0 = h^1(\mathbf{K}_{\mathcal{S}_0}) = h^1(\mathbf{K}_{\mathcal{V}})$ follows.

Since in all cases $h^1(\mathbf{K}_{\mathcal{V}}) = 0$, and $\chi(\mathbf{K}_{\mathcal{V}}) = 0$ by (2.22), we have $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G}) = 0$. This contradicts the fact that ϕ is nonzero. It must be that the base locus is empty. □

Now suppose $\mathcal{I} = (d, 3, D, n, I)$ is a Schubert state for SL_3 with $\langle \mathcal{I} \rangle \neq 0$.

Proposition 2.10.2. *For $r = 3$, the line bundle Θ on $\mathcal{M}(\mathcal{I})$ is base point free.*

Proof. As in 2.10.1, suppose contrariwise that Z is an irreducible component of the base locus. The isomorphism class $\mathcal{S}_0$ and Schubert position $\mathcal{K}_0 = (d'_0, r'_0, d, r, K)$ of the kernel of a general $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ remain constant over an open subset U_0 of $Z \times \mathrm{Fl}_S \mathcal{Q}$. Choose a general point $(\mathcal{V}, \mathcal{F}, \mathcal{G}) \in U_0$ and a general element $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ with kernel $\mathcal{S}_0$ in Schubert position $\mathcal{K}_0$. We have $r'_0 = 0$, 1, or 2. If $r'_0 = 0, 1$, derive a contradiction exactly as in 2.10.1.

If $r'_0 = 2$, a different argument is required. Consider the exact sequence coming from (2.5):

$$0 \to H^0(\mathbf{K}_{\mathcal{V}/\mathcal{S}_0}) \to H^0(\mathbf{K}_{\mathcal{V}}) \to H^0(\mathbf{K}_{\mathcal{S}_0}) \to H^1(\mathbf{K}_{\mathcal{V}/\mathcal{S}_0})$$

Note that $\mathcal{V}/\mathcal{S}_0$ has rank 1 and that the space $\mathrm{Fl}_S(\mathcal{V}/\mathcal{S}_0)$ is a point. We can therefore assume that U_0 was chosen sufficiently small such that the $\mathcal{G}$ component of each element of U_0 lies in the open set $(\mathrm{Fl}_S \mathcal{Q})_{\mathcal{V}/\mathcal{S}_0, \mathcal{F}(\mathcal{V}/\mathcal{S}_0)}$ where $\mathrm{hd}^q_{\mathcal{I}'}$ is constant. Since by assumption, one has an injective morphism $\phi \in \mathrm{Hom}_{\mathcal{I}'}(\mathcal{V}/\mathcal{S}_0, \mathcal{Q}, \mathcal{F}(\mathcal{V}/\mathcal{S}_0), \mathcal{G})$, applying Proposition 2.9.2, we obtain $H^1(\mathbf{K}_{\mathcal{V}/\mathcal{S}_0}) = 0$. On the other hand, by semistability of $(\mathcal{V}, \mathcal{F})$, we have $\chi(\mathbf{K}_{\mathcal{V}/\mathcal{S}_0}) \leq 0$. The only possibility then is that $0 = H^0(\mathbf{K}_{\mathcal{V}/\mathcal{S}_0}) = \mathrm{Hom}_{\mathcal{I}'}(\mathcal{V}/\mathcal{S}_0, \mathcal{Q}, \mathcal{F}(\mathcal{V}/\mathcal{S}_0), \mathcal{G})$, a contradiction. □

Corollary 2.10.3. *If $r = 2, 3$, the moduli space $\mathcal{M}(\mathcal{I})$ has dimension at most $\langle \mathcal{I} \rangle - 1$. In particular, if $\langle \mathcal{I} \rangle = 1$, then $(\mathcal{M}(\mathcal{I}), \Theta) = (\mathrm{Pt}, \mathcal{O})$. If $\langle \mathcal{I} \rangle = 2$, then $(\mathcal{M}(\mathcal{I}), \Theta) = (\mathbb{P}^1, \mathcal{O}(1))$. Therefore, the quantum Fulton and KTT conjectures hold when $r = 2, 3$.*

Proof. If $\dim \mathcal{M}(\mathcal{I})$ were greater than or equal to $\langle \mathcal{I} \rangle$, then the $\langle \mathcal{I} \rangle$ independent sections of 2.8.1 would have a common vanishing locus, contradicting 2.10.1 or 2.10.2. Thus, we have the inequality. The first "in particular" statement follows. Assume $\langle \mathcal{I} \rangle = 2$. By the construction in Section 2.5, $\mathcal{M}(\mathcal{I})$ is an integral, rational, normal, projective scheme. The inequality imposes the condition $\dim \mathcal{M}(\mathcal{I}) \leq 1$, and Θ has a 2 dimensional space of sections. The only possibility is then $(\mathcal{M}(\mathcal{I}), \Theta) = (\mathbb{P}^1, \mathcal{O}(1))$. □

Remark 2.10.1. In Propositions 2.10.1 and 2.10.2, we use critically that either $\mathcal{S}_0$ or $\mathcal{V}/\mathcal{S}_0$ has rank 1. This allows us to say that certain induced structures will be generic, since the resulting flag space is a single point. The author encountered serious technical difficulties trying to prove the KTT conjecture for arbitrary r precisely because it did not appear obvious that the induced structures would be generic enough for continued application of 2.9.2 (neither the approach of [She15] in the classical case, nor that of Section 1.6 in the quiver case seemed to generalize easily to the quantum setting).

2.11 Proof of Quantum Fulton Conjecture

We now prove the quantum analogue of Fulton's conjecture for any r using Proposition 2.9.2. The general structure resembles Belkale's proof [Bel07] of the "classical" Fulton conjecture, where $d = D = 0$. Fix again $\mathcal{Q} = \mathcal{Z}_{D-d,n-r}$ and let $\mathcal{K}$ be the Schubert state (d', r', d, r, K). To begin, we will need the following technical lemma. The construction in its proof is a relative version of the construction given in the proof of 2.9.2.

Lemma 2.11.1. *Consider quadruples of the form $(q, \mathcal{F}, \mathcal{G}, \phi)$, where $q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k)$ is a quotient bundle on $\mathbb{P}^1$ of rank r and degree $-d + rk$ (here k and N are as in Section 2.5), the flags $\mathcal{F}$ and $\mathcal{G}$ are elements of $\mathrm{Fl}_S\mathcal{V}$ and $\mathrm{Fl}_S\mathcal{Q}$, respectively, and ϕ is an element of $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ whose kernel is in Schubert state $\mathcal{K}$ with respect to $(\mathcal{V}, \mathcal{F})$ and such that the maps $\bar{\phi}_p : (\mathcal{V}/\ker\phi)_p \to \mathcal{Q}_p$ have kernels of dimension $\epsilon(p)$ and lie in the Schubert cells $\Omega^\circ_{J^p}(F^p(\mathcal{V}/\ker\phi))$. If there is even one such quadruple, then an irreducible, smooth scheme $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J}$ parametrizing such quadruples exists. It admits a morphism $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J} \to \mathcal{R}_{\mathcal{I}} \times \mathrm{Fl}_S\mathcal{Q}$ by forgetting ϕ. See Section 2.5 for the relevant notation.*

Proof. Form the partial flag quot scheme $\mathbf{Q}_{\mathcal{I},\mathcal{K}}$ parametrizing quotient coherent sheaves $q_1 : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k)$, $q_2 : \mathcal{V}(k) \twoheadrightarrow (\mathcal{V}/\mathcal{S})(k)$ on $\mathbb{P}^1$, where $\mathcal{V}$ has Hilbert polynomial $rt + r - d$ and $\mathcal{S}$ has Hilbert polynomial $r't + r' - d'$. Such a space exists as a smooth, projective, irreducible scheme (see [Kim95] and the related [KP01]). Let $\mathscr{V}$ be the twist of the first universal quotient on $\mathbf{Q}_{\mathcal{I},\mathcal{K}} \times \mathbb{P}^1$ by $\mathrm{pr}^*_{\mathbb{P}^1}\mathcal{O}(-k)$. Let $\mathscr{S} \hookrightarrow \mathscr{V}$ be the kernel of the second universal quotient twisted by $\mathrm{pr}^*_{\mathbb{P}^1}\mathcal{O}(-k)$. Let $\mathbf{Q}'_{\mathcal{I},\mathcal{K}}$ be the largest open subscheme of $\mathbf{Q}_{\mathcal{I},\mathcal{K}}$ with points corresponding to quotients where both $\mathcal{V}$ and $\mathcal{V}/\mathcal{S}$ are locally free. Over $\mathbf{Q}'_{\mathcal{I},\mathcal{K}} \times \mathbb{P}^1$, the map $\mathscr{S} \hookrightarrow \mathscr{V}$ is an inclusion of a vector subbundle.

For each $p \in S$, let $\mathbf{F}(\mathscr{S})_p$ denote the complete quotient flag bundle of $\mathscr{S}|_{\mathbf{Q}'_{\mathcal{I},\mathcal{K}} \times \{p\}}$, and let $\mathbf{F}(\mathscr{S})$ denote the fiber product of these over $\mathbf{Q}'_{\mathcal{I},\mathcal{K}}$ as p runs through the points of S. Then closed points of $\mathbf{F}(\mathscr{S})$ are given by $(q_1, q_2, \mathcal{F}(\mathcal{S}))$ where $q_1 : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k)$, $q_2 : \mathcal{V}(k) \twoheadrightarrow (\mathcal{V}/\mathcal{S})(k)$ are locally free quotients on $\mathbb{P}^1$ and $\mathcal{F}(\mathcal{S})$ is an element of $\mathrm{Fl}_S\mathcal{S}$ (here we identify a complete quotient flag with a complete subspace flag by taking kernels). The scheme $\mathbf{F}(\mathscr{S})$ is smooth and irreducible since it is relatively smooth with irreducible fibers over the smooth and irreducible scheme $\mathbf{Q}'_{\mathcal{I},\mathcal{K}}$.

Over $\mathbf{F}(\mathscr{S})$, one can build a relatively smooth, irreducible scheme $\mathbf{U}_{\mathcal{K}}$, whose fiber over $(q_1, q_2, \mathcal{F}(\mathcal{S}))$ consists of all of $\mathcal{F} \in \mathrm{Fl}_S\mathcal{V}$ such that $\mathcal{S} \in \Omega^\circ_{\mathcal{K}}(\mathcal{V}, \mathcal{F})$ and such that the flag induced on $\mathcal{S}$ by $\mathcal{F}$ is $\mathcal{F}(\mathcal{S})$. A classical analogue of this scheme (of the same name) appears in [Bel06]. Indeed, the same proof as there, by successive applications of [Bel06, Lemma A.3] can be used to show its existence, smoothness, and irreducibility (see also [She15, Lemma A.1]). The quotient bundle on $\mathbf{U}_{\mathcal{K}} \times \mathbb{P}^1$ obtained by pulling back the first universal quotient on $\mathbf{Q}'_{\mathcal{I},\mathcal{K}} \times \mathbb{P}^1$ along with the universal flag bundles of $\mathscr{V}|_{\mathbf{Q}'_{\mathcal{I},\mathcal{K}} \times \{p\}}$ define a morphism $\mathbf{U}_{\mathcal{K}} \to \mathcal{R}_{\mathcal{I}}$ which on closed points sends $(q_1, q_2, \mathcal{F})$ to $(q_1, \mathcal{F})$ (note we may ignore the $\mathcal{F}(\mathcal{S})$ coordinate of points of $\mathbf{U}_{\mathcal{K}}$, since it is determined by the $\mathcal{F}$ coordinate).

To complete the proof, it suffices to build a relatively smooth scheme over $\mathbf{U}_{\mathcal{K}}$ whose fiber over $(q_1, q_2, \mathcal{F})$ as above is the set of all $\mathcal{G}$ in $\mathrm{Fl}_S\mathcal{Q}$ and $\phi \in \mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ such that the $\ker \phi = \ker(q_2(-k))$, and it has the required properties at the marked points. By assumption, there is a quadruple $(q_0, \mathcal{F}_0, \mathcal{G}_0, \phi_0)$ of the asserted form. Suppose $\bar{\phi}_p : (\mathcal{V}/\ker \phi_0)_p \to \mathcal{Q}_p$ has kernel $B_0(p)$ for $p \in S$.

Now, let $\mathbf{B}$ be the relatively smooth scheme over $\mathbf{U}_{\mathcal{K}}$ whose fiber over a point $(q_1, q_2, \mathcal{F})$ as above is the (irreducible) product of the ordinary Schubert cells $\Omega^\circ_{J^p}(F^p(\mathcal{V}/\mathcal{S}))$ as p ranges through S. As an intermediate step on the way to $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J}$, we build a scheme $\mathbf{H}''$ over $\mathbf{B}$ whose fiber over $(q_1, q_2, \mathcal{F}, (B(p))_{p \in S})$ is the set of homomorphisms $\phi : \mathcal{V} \to \mathcal{Q}$ with kernel equal to $\ker(q_2(-k))$ and $\ker(\bar{\phi}_p) = B(p)$. To this end, for $p \in S$, consider the tautological subbundle $\mathcal{B}(p)$ of $(\mathscr{V}/\mathscr{S})|_{\mathbf{B} \times \{p\}}$ whose fiber over $(q_1, q_2, \mathcal{F}, (B(p))_{p \in S}) \times \{p\}$ is the vector space $B(p)$. Using the theory developed in [Bel08, Appendix A], we can shift $(\mathscr{V}/\mathscr{S})|_{\mathbf{B} \times \mathbb{P}^1}$ along $\mathcal{B}$ to obtain a vector bundle $(\mathscr{V}/\mathscr{S})^{\mathrm{shift}}$ on $\mathbf{B} \times \mathbb{P}^1$ which has the property that for any vector bundle A on $\mathbf{B} \times \mathbb{P}^1$ and point

$$\mathbf{b} = (q_1 : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), q_2 : \mathcal{V}(k) \twoheadrightarrow (\mathcal{V}/\mathcal{S})(k), \mathcal{F}, (B(p))_{p \in S}) \in \mathbf{B},$$

the space $\mathrm{Hom}((\mathscr{V}/\mathscr{S})^{\mathrm{shift}}|_{\mathbf{b}}, A|_{\mathbf{b}})$ is canonically isomorphic to the space of $\phi \in \mathrm{Hom}(\mathcal{V}/\mathcal{S}, A|_{\mathbf{b}})$ with $B(p) \subseteq \ker(\bar{\phi}_p)$, and injective elements map to injective elements under the isomorphism. By semicontinuity, there is an open subset $\mathbf{B}'$ of $\mathbf{B}$ whose points $\mathbf{b}'$ satisfy

$$H^1(\{\mathbf{b}'\} \times \mathbb{P}^1, \underline{Hom}((\mathscr{V}/\mathscr{S})^{\mathrm{shift}}|_{\mathbf{b}'}, \mathcal{Q})) = 0;$$

here we shorthand $\mathrm{pr}_{\mathbb{P}^1}^* \mathcal{Q}|_{\mathbf{b}'}$ by $\mathcal{Q}$. This open set is *nonempty*, since the point

$$\mathbf{b}'_0 = (q_0, \mathrm{Im}(\phi_0(k)), \mathcal{F}_0, (B_0(p))_{p\in S})$$

has this property, which one sees by exactly the argument used to show H^1 vanished in the proof of Proposition 2.9.2. Over $\mathbf{B}'$, by cohomology and base change, the sheaf $\mathrm{pr}_{\mathbf{B}*}(\underline{Hom}((\mathscr{V}/\mathscr{S})^{\mathrm{shift}}, \mathrm{pr}_{\mathbf{P}^1}^* \mathcal{Q}))$ is a vector bundle. Form its total space $\mathbf{H}'$ over $\mathbf{B}'$.

The space $\mathbf{H}'$ parametrizes pairs $(\mathbf{b}', \phi)$ where $\mathbf{b}' \in \mathbf{B}'$ and $\phi \in \mathrm{Hom}(\mathcal{V}_{\mathbf{b}'}, \mathcal{Q})$ is such that $\ker\phi$ contains $\mathcal{S}_{\mathbf{b}'}$, and $\ker(\bar{\phi}_p)$ contains $B(p)_{\mathbf{b}'}$. Let $\mathbf{H}''$ be the open subset where all of these containments are equalities. Again, $\mathbf{H}''$ is nonempty, since $(\mathbf{b}'_0, \phi_0) \in \mathbf{H}''$. It is now easy to obtain the desired space $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J}$ as a relatively smooth scheme over $\mathbf{H}''$ by doing the procedure of the proof of Lemma 2.9.1 over each point of $\mathbf{H}''$ (indeed the number computed in the lemma is the relative dimension of $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J} \to \mathbf{H}''$). □

We proceed to the proof of the Quantum Fulton Conjecture.

Proof. (Theroem 2.0.1) Suppose $h^0(\mathcal{M}(\mathcal{I}), \Theta) = 1$ and suppose to the contrary that $\mathcal{M}(\mathcal{I})$ is not a point. Then, if $\mathcal{G}$ is a general element of $\mathrm{Fl}_S\mathcal{Q}$, the section $\theta(\mathcal{Q}, \mathcal{G}) \neq 0$ gives a basis of $H^0(\mathcal{M}(\mathcal{I}), \Theta)$. Since $\dim \mathcal{M}(\mathcal{I}) \geq 1$ and Θ is ample, this section must have a nonempty vanishing locus Z', which is then the base locus of Θ. Fix once and for all the closure Z of an irreducible component of the preimage of Z' in $\mathcal{R}_{\mathcal{I}}$ (see Section 2.5). Let $\bar{U}$ be a dense open subset of $Z \times \mathrm{Fl}_S\mathcal{Q}$ over which the quantum hom data $\mathrm{hd}^q_{\mathcal{I}}$ is constant, say $(R, \mathcal{K}, \epsilon, J)$ where $\mathcal{K} = (d', r', d, r, K)$. Note that $r' < r$, since $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ is nonzero for general $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \mathcal{F}, \mathcal{G}) \in Z \times \mathrm{Fl}_S\mathcal{Q}$.

Claim 2.11.2. *The map $\pi : \mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J} \to \mathcal{R}_{\mathcal{I}} \times \mathrm{Fl}_S\mathcal{Q}$ of Lemma 2.11.1 factors through a dominant morphism to $Z \times \mathrm{Fl}_S\mathcal{Q}$.*

Proof. By construction, $\bar{U}$ is contained in the image of π. On the other hand, if $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}, \mathcal{F}, \mathcal{G}) \in \mathcal{R}_{\mathcal{I}} \times \mathrm{Fl}_S\mathcal{Q}$ has a nonempty fiber in $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J}$, then $(q, \mathcal{F})$ is in the vanishing locus of $\theta(\mathcal{Q}, \mathcal{G})$ (since $r' < r$), and hence $(q, \mathcal{F})$ is in the inverse image of Z' (denote this inverse image by $Z'|_{\mathcal{R}_{\mathcal{I}}}$). So $\mathrm{Im}(\pi)$ is irreducible, contained in $Z'|_{\mathcal{R}_{\mathcal{I}}} \times \mathrm{Fl}_S\mathcal{Q}$, and contains a dense open subset of the irreducible component $Z \times \mathrm{Fl}_S\mathcal{Q}$ of $Z'|_{\mathcal{R}_{\mathcal{I}}} \times \mathrm{Fl}_S\mathcal{Q}$. The claim follows. □

Choose a general point $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \mathcal{F}, \mathcal{G}, \phi)$ in $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J}$ and let $\ker \phi = \mathcal{S}$. By 2.11.2, this lies over a general point of $Z \times \mathrm{Fl}_S \mathcal{Q}$, so in particular $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ is semistable.

Claim 2.11.3. *We may assume that $\mathcal{S} \cong \mathcal{Z}_{d',r'}$ is evenly split and that $(\mathcal{F}(\mathcal{S}), \mathcal{G})$ is a general element of $\mathrm{Fl}_S \mathcal{S} \times \mathrm{Fl}_S \mathcal{Q}$.*

Proof. The degree of $\mathcal{S}$ is bounded above (by semistability of $\mathcal{V}$) and bounded below (by virtue of being the kernel of a map to $\mathcal{Q}$) where both bounds depend only on the fixed data of the Schubert state $\mathcal{I}$. Thus, we can assume that $k = k(\mathcal{I})$ from Section 2.5 was chosen sufficiently large that the numbers $\delta_1 = -d' + r'k + r'$ and $\delta_2 = (d' - d) + (r - r')k + (r - r')$ are both positive. Let $(\mathcal{S}(k))_{\mathrm{ES}}$ be the unique evenly split bundle of rank r', degree $-d' + r'k$, and let $((\mathcal{V}/\mathcal{S})(k))_{\mathrm{ES}}$ be the unique evenly split bundle of rank $r - r'$ and degree $(d' - d) + (r - r')k$. It is easy to see that $(\mathcal{S}(k))_{\mathrm{ES}}$ admits a surjection from $\mathcal{O}^{\delta_1}$, and $((\mathcal{V}/\mathcal{S})(k))_{\mathrm{ES}}$ admits a surjection from $\mathcal{O}^{\delta_2}$. Since $N = \delta_1 + \delta_2$, we have a surjection $\mathcal{O}^N \twoheadrightarrow (\mathcal{S}(k))_{\mathrm{ES}} \oplus ((\mathcal{V}/\mathcal{S})(k))_{\mathrm{ES}}$. It follows that there exists an element of the partial flag quot scheme $\mathbf{Q}_{\mathcal{I},\mathcal{K}}$ (see the proof of Lemma 2.11.1) whose $\mathcal{S}$ coordinate corresponds to an evenly split bundle. The set $U_{\mathcal{S},\mathrm{ES}}$ of all such elements is open by [Bel08, Lemma 12.1].

Now let $\mathbf{H}_{\mathcal{S},\mathrm{ES}}$ be the (open, dense) inverse image of $U_{\mathcal{S},\mathrm{ES}}$ in $\mathbf{H}_{\mathcal{I},\mathcal{K},\epsilon,J}$. On $\mathbb{P}^1 \times \mathbf{H}_{\mathcal{S},\mathrm{ES}}$, there is a universal kernel $\mathscr{S}$. Choosing arbitrarily an isomorphism ψ_x of $\mathscr{S}$ with $\mathrm{pr}_{\mathbb{P}^1}^* \mathcal{Z}_{d',r'}$ over one point x of $\mathbf{H}_{\mathcal{S},\mathrm{ES}}$, we can extend this to an isomorphism ψ over an open neighborhood $\mathbf{H}'_{\mathcal{S},\mathrm{ES}}$ [loc. cit.]. Then, we have a map $\sigma : \mathbf{H}'_{\mathcal{S},\mathrm{ES}} \to \mathrm{Fl}_S(\mathcal{Z}_{d',r'}) \times \mathrm{Fl}_S(\mathcal{Q})$ which sends $(q, \mathcal{F}, \mathcal{G}, \phi)$ to $(\mathcal{F}(\mathcal{S}), \mathcal{G})$, where $\mathcal{F}(\mathcal{S})$ is identified with a tuple of flags on $\mathcal{Z}_{d',r'}$ by ψ.

It follows from the Claim 2.11.2 that σ composed with the projection to $\mathrm{Fl}_S \mathcal{Q}$ is dominant. Let $\mathcal{G}$ be a general element of $\mathrm{Fl}_S \mathcal{Q}$ and let $(q : \mathcal{O}^N \twoheadrightarrow \mathcal{V}(k), \mathcal{F}, \mathcal{G}, \phi)$ be any fixed element of $\mathbf{H}'_{\mathcal{S},\mathrm{ES}}|_{\mathcal{G}}$. To complete the proof of Claim 2.11.3, we will show that the map $\sigma|_{q,\mathcal{G},\phi} : \mathbf{H}'_{\mathcal{S},\mathrm{ES}}|_{q,\mathcal{G},\phi} \to \mathrm{Fl}_S(\mathcal{Z}_{d',r'})$ is dominant.

To this end, for $p \in S$, consider the subgroup G_p of $\mathrm{Aut}(\mathcal{V}_p)$ which sends $\mathcal{S}_p$ to itself and acts trivially on $(\mathcal{V}/\mathcal{S})_p$. Then, $G := \times_{p \in S} G_p$ acts on $\mathbf{H}'_{\mathcal{S},\mathrm{ES}}|_{q,\mathcal{G},\phi}$ such that $(g_p)_{p \in S}$ sends $(q, \mathcal{F}, \mathcal{G}, \phi)$ to $(q, \{g_p F^p\}_{p \in S}, \mathcal{G}, \phi)$. The map $\sigma|_{q,\mathcal{G},\phi}$ is equivariant with respect to the given action of G on $\mathbf{H}'_{\mathcal{S},\mathrm{ES}}|_{q,\mathcal{G},\phi}$ and its obvious action on $\mathrm{Fl}_S(\mathcal{S}) \cong_\psi \mathrm{Fl}_S(\mathcal{Z}_{d',r'})$. The latter action is transitive, so we have the desired dominance. $\square$

Now by Proposition 2.9.2, for the chosen general point $(q, \mathcal{F}, \mathcal{G}, \phi)$, we have:

$$H^1(\mathbf{K}_{\mathcal{V}}) \cong H^1(\mathbf{K}_{\mathcal{S}}) \tag{2.31}$$

By Claim 2.11.3, $\mathcal{S}$ is evenly split and the flags $\mathcal{F}(\mathcal{S}), \mathcal{G}$ are mutually generic, so we can use the quantum Horn conjecture [Bel08, Theorem 2.8] to compute the right hand side to be zero.

Indeed, suppose $\mathcal{N} = (d'', r'', d', r', N)$ is a non-null Schubert state with $0 < r'' < r'$. Let $\mathcal{Y} \in \Omega^{\circ}_{\mathcal{N}}(\mathcal{F}(\mathcal{S}))$. Then, using the inclusion $\mathcal{Y} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{V}$, we see that $\mathcal{Y} \in \Omega^{\circ}_{\mathcal{K}_{\mathcal{N}}}(\mathcal{F})$, where $\mathcal{K}_{\mathcal{N}} = (d'', r'', d, r, K_N)$, $(K_N)^p_a = K^p_{N^p_a}$. Semistability of $(\mathcal{V}, \mathcal{F}, \mathcal{I})$ now gives the inequality

$$\frac{1}{r''}(-d'' + \frac{1}{n-r}\sum_{p\in S}\sum_{a=1}^{r''}(n - r + K^p_{N^p_a} - I^p_{K^p_{N^p_a}})) \leq \frac{1}{r}(-d + \frac{1}{n-r}\sum_{p\in S}\sum_{a=1}^{r}(n - r + a - I^p_a)).$$

Use the codimension condition (2.22) to rewrite the right hand side as $\frac{1}{n-r}(n-r+d-D)$. Multiplying both sides of the resulting inequality by $r''(n-r)$, we obtain

$$-d''(n-r) + \sum_{p\in S}\sum_{a=1}^{r''}(n - r + K^p_{N^p_a} - I^p_{K^p_{N^p_a}}) \leq r''(n-r) + r''(d-D).$$

This is easily seen to be the Horn inequality [loc. cit., Theorem 2.8b] for $\mathcal{Y}$ with respect to the Schubert state $\tilde{\mathcal{I}} = (d', r', D - d + d', n - r + r', \tilde{I})$, where $\tilde{I}^p_a = I^p_{K^p_a} - K^p_a + a$. Therefore, the Schubert state $\tilde{\mathcal{I}}$ is non-null. It now follows from [loc. cit., Proposition 5.5] that $H^0(\mathbf{K}_{\mathcal{S}})$ has its expected dimension, which is the same as saying $H^1(\mathbf{K}_{\mathcal{S}}) = 0$. Thus, by (2.31) the same is true of $\mathbf{K}_{\mathcal{V}}$. We conclude that $\mathrm{Hom}_{\mathcal{I}}(\mathcal{V}, \mathcal{Q}, \mathcal{F}, \mathcal{G})$ has its expected dimension, *which is zero* because of the codimension condition. This contradicts the fact that ϕ was nonzero (its kernel was the proper subbundle $\mathcal{S}$). So the proof is complete. □

2.12 Application to Vector Bundles of Conformal Blocks

Given a stable s-pointed curve of arithmetic genus 0 along with a non-null Schubert state $\mathcal{I} = (0, r, D, n, I)$ (here I is an s-tuple of index sets), one can associate a nonzero finite dimensional vector space $\mathrm{V}_{\mathcal{I}}$, called a conformal block, coming from the representation theory of affine Lie algebras. These fit together into a vector bundle $\mathbb{V}_{\mathcal{I}}$ over the space of such stable curves, $\overline{M}_{0,s}$.

From [Pau96], one knows that for each *smooth* curve $x \in M_{0,s}$, there is a natural isomorphism $\mathbb{V}_{\mathcal{I}}^*|_x \cong H^0(\mathcal{M}(\mathcal{I}), \Theta)$. In the paper [BGK15], the question is taken up as to whether points on the boundary $\partial := \overline{M}_{0,s} - M_{0,s}$ also admit such *geometric interpretations*, i.e. for all $x \in \partial$, is there a projective variety $\mathcal{M}_x$ and ample line bundle Θ_x such that $\mathbb{V}_{\mathcal{I}}^*|_x \cong H^0(\mathcal{M}_x, \Theta_x)$, and if so, what are the implications of this? It turns out that nice things happen when $(\mathcal{M}(\mathcal{I}), \Theta) = (\mathbb{P}^n, \mathcal{O}(1))$, and one has a simple criterion for whether geometric interpretations exist.

Proposition 2.12.1. *Let $\mathcal{I} = (0, r, D, n, I)$ be a Schubert state with $\langle \mathcal{I} \rangle = 2$ and suppose the quantum KTT conjecture holds for r (known to be so when $r = 2, 3$ by Section 2.10), then geometric interpretations for $\mathbb{V}_{\mathcal{I}}$ exist at almost all boundary points. Consequently, one has the Chern class identity:*

$$c_1(\mathbb{V}_{\mathcal{I}}[N]) = \frac{N(N+1)}{2} c_1(\mathbb{V}_{\mathcal{I}})$$

for all positive integers N.

Proof. We check the criteria (1), (2), (3), (4) of [BGK15, Theorem 7.1]. Since we assume the quantum KTT conjecture, $\mathbb{V}_{\mathcal{I}}$ has projective space scaling, and (1) holds. Now let $J = \{j_1, ..., j_t\}$ be a subset of $[s]$, with complement $K = \{k_1, ..., k_{s-t}\}$. The factorization property of conformal blocks gives:

$$\sum_{\mu \in P_{n-r}(\mathfrak{sl}_r)} \mathrm{rk}\mathbb{V}(\mathfrak{sl}_r, \lambda^{j_1}, ..., \lambda^{j_t}, \mu, n-r)\mathrm{rk}\mathbb{V}(\mathfrak{sl}_r, \lambda^{k_1}, ..., \lambda^{k_{s-t}}, \mu^*, n-r) = \mathrm{rk}\mathbb{V}_{\mathcal{I}} = 2.$$

At most two summands on the left hand side can be nonzero. The criteria (2) and (3) follow immediately. For (4), any socle must have rank 2, and hence satisfies projective space rank scaling by the assumed KTT conjecture. $\square$

CHAPTER 3

Classical Moduli and Positivity

Consider a nonempty moduli space $\mathcal{M}(\mathcal{I})$ of parabolic vector bundles, with $\delta := \dim \mathcal{M}(\mathcal{I})$ and resolution of singularities $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}(\mathcal{I})$. By Grothendieck-Riemann-Roch and the cohomology vanishing of Teleman [Tel00] (calling also by Θ the associated Cartier divisor):

$$h^0(\mathcal{M}(\mathcal{I}), \Theta^{\otimes N}) = \frac{\int_{\widetilde{\mathcal{M}}}(\pi^*\Theta)^{\delta}}{\delta!} N^{\delta} + \frac{\int_{\widetilde{\mathcal{M}}}(\pi^*\Theta)^{\delta-1}.c_1(\mathcal{T}_{\widetilde{\mathcal{M}}})}{2\cdot(\delta-1)!} N^{\delta-1} + ... + \chi(\mathcal{O}_{\tilde{\mathcal{M}}}). \tag{3.1}$$

Also, we have used above that $\mathcal{M}(\mathcal{I})$ has rational singularities [Bou87]. Note that since Θ is ample, the lead coefficient must be positive. The rationality of the moduli spaces $\mathcal{M}(\mathcal{I})$ established by Boden and Hu [BH95] shows the constant term to be 1.

In the classical case where $d = D = 0$, the left hand side of (3.1) is $\dim(V_{N\lambda(I^1)} \otimes ... \otimes V_{N\lambda(I^s)})^{\mathrm{SL}_r}$, also known as the Littlewood-Richardson number $c_{N\lambda(I^1),...,N\lambda(I^s)}$. Also, in this case, one need not go through the quot scheme construction of Section 2.5. The moduli space can be realized as a quotient of a product of flag varieties, see the Introduction and [She15] for more details. *We will now denote the classical polarized moduli space by* $(\mathcal{M}, \mathcal{L})$.

When $s = 3$, King, Tollu, and Toumazet [KTT04, Conjecture 3.1] conjecture that all of the coefficients in (3.1) are nonnegative (although one does not expect the assumption on s to matter). We will show that under some conditions for $\lambda(I^1), ..., \lambda(I^s)$, the coefficient of $N^{\delta-1}$ is positive. In particular, if in addition $\mathcal{M}$ is a surface, their Conjecture 3.1 holds.

The key point is that the anticanonical divisor $-K_{\mathcal{M}} = c_1(\mathcal{T}_{\mathcal{M}})$ on the moduli space should be effective. This follows from the naive argument that since $\mathcal{M}$ is a GIT quotient of $\mathrm{Fl}(V)^{\times s}$, and the anticanonical bundle on $\mathrm{Fl}(V)^{\times s}$ possesses invariant sections (Lemma 3.1.1), one ought to obtain sections of the line bundle corresponding to $-K_{\mathcal{M}}$. This is essentially correct, although there are some delicate points to be ironed out. Indeed, these remarks implicitly assume $\mathcal{M}$ is smooth with many stable points, which is less than the generality we want.

Proposition 3.0.1. *If $d = D = 0$, $I^p_{a+1} \neq I^p_a + 1$ for any $p \in S$ and $a = 1, ..., r-1$ (equivalently, if for each $p \in S$ the Young diagram associated to I^p has no repeat rows), and if $\mathcal{M}$ contains a point corresponding to a stable orbit on $\mathrm{Fl}(V)^{\times s}$, then the coefficient of $N^{\delta-1}$ in the polynomial (3.1) is positive.*

Positivity of the second coefficient has implications for KTT 2, as discussed in the introduction.

3.1 Proof of Proposition 3.0.1

For the proof, we recall for a good quotient $q : X \to X//G$, one has an *invariant direct image* functor q^G_* which associates to a G-equivariant quasi-coherent sheaf $\mathcal{N}$ the sheaf whose sections over $U \subseteq X//G$ are $\mathcal{N}(q^{-1}U)^G$. See [Tel00, Section 3] for details.

Proof. We must show $\int_{\widetilde{\mathcal{M}}} (\pi^*\mathcal{L})^{\delta-1}.c_1(\mathcal{T}_{\widetilde{\mathcal{M}}}) > 0$. Note that $c_1(\mathcal{T}_{\widetilde{\mathcal{M}}}) = -K_{\widetilde{\mathcal{M}}}$, the anticanonical divisor on $\widetilde{\mathcal{M}}$, and that

$$\int_{\widetilde{\mathcal{M}}} (\pi^*\mathcal{L})^{\delta-1}.(-K_{\widetilde{\mathcal{M}}}) = \int_{\mathcal{M}} \pi_*((\pi^*\mathcal{L})^{\delta-1}.(-K_{\widetilde{\mathcal{M}}})) = \int_{\mathcal{M}} \mathcal{L}^{\delta-1}.(\pi_*(-K_{\widetilde{\mathcal{M}}})),$$

where we use the projection formula in the last step. Since $\mathcal{L}$ is ample, it suffices by Nakai's criterion to prove that $\pi_*(-K_{\widetilde{\mathcal{M}}})$ is effective.

To this end, let $j : U := \mathcal{M} - \mathcal{M}_{Sing} \to \mathcal{M}$ be the inclusion of the nonsingular locus. Since $\mathcal{M}$ is normal, the complement of U has codimension 2, and therefore it suffices to show that $j^*\pi_*(-K_{\widetilde{\mathcal{M}}})$ is effective. By standard push-pull formulas [Ful98, Proposition 1.7], this divisor is the same as $\pi'_*(-K_{\widetilde{\mathcal{M}}}|_{\pi^{-1}U})$, where π' is the restriction of π to $\pi^{-1}U$. The latter will be effective provided $\underline{Hom}(\omega_{\widetilde{\mathcal{M}}}, \mathcal{O})|_{\pi^{-1}U}$ has a nonzero section. Since π' is an isomorphism onto U, we are reduced to proving that $\mathrm{Hom}(\omega^\circ_{\mathcal{M}}|_U, \mathcal{O}_U)$ is nonzero.

Under the hypotheses of the theorem, by Lemma 3.1.2, for $q : (\mathrm{Fl}(V)^{\times s})^{\mathcal{I}-SS} \to \mathcal{M}$, the invariant direct image of the canonical bundle $q^{\mathrm{SL}_r}_* \omega_{\mathrm{Fl}}$ is the dualizing sheaf $\omega^\circ_{\mathcal{M}}$. A nonzero invariant section of $\omega^{-1}_{\mathrm{Fl}}$ (which exists by Lemma 3.1.1) defines a nonzero invariant morphism $\omega_{\mathrm{Fl}}|_{q^{-1}U} \to \mathcal{O}_{\mathrm{Fl}}|_{q^{-1}U}$. Applying $q^{\mathrm{SL}_r}_*$ to this, we obtain a nonzero element of $\mathrm{Hom}(\omega^\circ_{\mathcal{M}}|_U, \mathcal{O}_U)$, as desired. □

For the proof, we needed the lemmas below, which will be proven in the sections that follow.

Lemma 3.1.1. *The space of invariants* $H^0(\mathrm{Fl}(V)^{\times s}, \omega_{\mathrm{Fl}}^{-1})^{\mathrm{SL}_r}$ *is nonzero for the diagonal action of* SL_r *on* $\mathrm{Fl}(V)^{\times s}$.

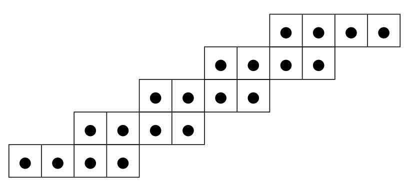
Lemma 3.1.2. *Under the hypotheses of Proposition 3.0.1, the invariant direct image preserves dualizing sheaves. That is,* $q_*^{\mathrm{SL}_r}\omega_{\mathrm{Fl}} = \omega_{\mathcal{M}}^{\circ}$.

3.2 Invariant Sections of the Anticanonical Bundle on $\mathrm{Fl}(V)^{\times s}$

The anticanonical bundle on the full flag variety $\mathrm{Fl}(V) = \mathrm{SL}_r/B$ is the ample equivariant line bundle $L_{2\rho}$ associated to the dominant weight 2ρ, where ρ is half the sum of the positive roots. It follows that the anticanonical bundle $\omega_{\mathrm{Fl}}^{-1}$ on $\mathrm{Fl}(V)^{\times s}$ is the tensor product of the pullbacks of $L_{2\rho}$ on each of the factors. With this, we prove Lemma 3.1.1.

Proof. By the Küneth formula and the theorem of Borel-Weil [Ful97, Section 9.3], the space of invariants in Lemma 3.1.1 is isomorphic to $(V_{2\rho}^{\otimes s})^{\mathrm{SL}_r}$. To show that the latter is nonzero, it suffices to show that in the sum decomposition of $V_{2\rho} \otimes V_{2\rho}$ into irreducible SL_r representations, the representations $V_{2\rho}$ and the trivial representation both appear with nonzero multiplicity. The appearance of the trivial representation follows from the fact that $V_{2\rho}$ is a self-dual SL_r representation. As for the appearance of $V_{2\rho}$ in the sum, we will prove this using the Littlewood-Richardson rule [Ful97, Section 5.2].

We must show that there exists a Littlewood-Richardson skew tableau of content ρ on the shape



That is, the skew shape with r-many rows, each of length $r-1$, such that row i has exactly two boxes which do not lie over any boxes in row $i+1$ for $i = 1, ..., r-1$ (the above shows the case $r = 5$). We exhibit this explicitly in the cases $r = 4$, $r = 5$, and $r = 6$, trusting the reader will discern the pattern.

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & 1 & 1 \\
 & & & & 1 & 2 & 2 & & \\
 & & 1 & 2 & 3 & & & & \\
1 & 2 & 3 & & & & & &
\end{array}
\qquad
\begin{array}{cccccccccccc}
 & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & 1 & 2 & 2 & 2 & & \\
 & & & & 1 & 2 & 3 & 3 & & & & \\
 & & 1 & 2 & 3 & 4 & & & & & & \\
1 & 2 & 3 & 4 & & & & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & & 1 & 2 & 2 & 2 & 2 & & \\
 & & & & & & 1 & 2 & 3 & 3 & 3 & & & & \\
 & & & & 1 & 2 & 3 & 4 & 4 & & & & & & \\
 & & 1 & 2 & 3 & 4 & 5 & & & & & & & & \\
1 & 2 & 3 & 4 & 5 & & & & & & & & & &
\end{array}
\tag{3.2}
$$

This completes the proof. $\square$

Remark 3.2.1. We remark that rearranging the bottom two rows in the skew diagrams of (3.2) gives in fact many summands of $V_{2\rho}$ in $V_{2\rho} \otimes V_{2\rho}$.

3.3 Invariant Direct Image of the Canonical Bundle on $\mathrm{Fl}(V)^{\times s}$

We prove a proposition of general interest from which Lemma 3.1.2 follows. Teleman uses this result implicitly in [Tel00], and we thank him for an email communication in which he affirmed its validity.

Proposition 3.3.1. *Let G be a connected, complex, reductive group. Let X be a smooth, irreducible, affine G-variety. Suppose*

- *There exists a point x_0 in X such that the orbit Gx_0 is closed, and the stabilizer G_{x_0} is finite, i.e. x_0 is G-stable.*
- *The stabilizer of any point with closed orbit in Y is abelian.*

Then the set $Y := \{y \in X : \dim G_y^0 > 0\}$ has codimension at least 2 in X.

Let us now deduce Lemma 3.1.2 from Proposition 3.3.1. We first claim that SL_r acting on $(\mathrm{Fl}(V)^{\times s})^{\mathcal{I}-SS}$ has abelian stabilizers for points with closed orbits. Let $\mathcal{F}$ be an s-tuple of flags defining a parabolic $\mathcal{I}$-semistable vector space $(V, \mathcal{F}, \mathcal{I})$. The latter has a Jordan-Hölder filtration by parabolic vector subspaces, such that successive quotients in the filtration $(V_j, \mathcal{F}_j, \mathcal{I}_j)$ are stable with respect to the weights induced by $\mathcal{I}$. It is known [Ses82, Troisiéme Partie] that if $\mathcal{F}$ has a closed SL_r orbit, then $(V, \mathcal{F}, \mathcal{I})$ is isomorphic to its so-called grade decomposition, that is, the direct sum $\oplus_j (V_j, \mathcal{F}_j, \mathcal{I}_j)$. Note that *no summand* is parabolically isomorphic to any other summand. This is because the weights of the direct sum at $p \in S$ are given by the union of the weights on the factors. Since the weights $\lambda(I^p)$ on $(V, \mathcal{F})$ at p are assumed to contain no repeats, the weights at p on the summands are distinct from one another, and hence no two summands can be isomorphic. Nonisomorphic stable parabolic vector spaces admit only the zero parabolic morphism between them. Therefore, the stabilizer $\mathrm{Aut}((V, \mathcal{F}, \mathcal{I}))_{\det=1} \subset \mathrm{SL}_r$ is isomorphic to the product of the automorphism groups of the $(V_j, \mathcal{F}_j, \mathcal{I}_j)$. But these are stable, so each such automorphism group is isomorphic to $\mathbb{C}^*$, giving us the desired abelian stabilizer condition. We may now apply Propostion 3.3.1 to deduce that locus of s-tuples of flags with positive dimensional stabilizer has codimension at least 2 in $(\mathrm{Fl}(V))^{\mathcal{I}-SS}$. This is the condition needed to apply [Kno89, Korollar 2], which gives the conclusion of Lemma 3.1.2.

To prove Proposition 3.3.1, we will need some lemmas. The first is a résumé of results which appear in Drézet's notes [Dre04] on Luna's slice theorem. The results there are more general; we only quote what we need.

Lemma 3.3.2. *Let G be an reductive linear algebraic group acting on a smooth (not necessarily irreducible) affine variety X. Let $x_0 \in X$ have a closed G orbit. Then there exists a smooth, affine, G_{x_0}-invariant closed subvariety (again not necessarily irreducible) V of X containing x_0, and of these objects, the following are true.*

1. *G_{x_0} is reductive. [Dre04, 2.19]*
2. *Let $X' := G \times^{G_{x_0}} V$ have the left G-action given by $g' \cdot \overline{(g, v)} = \overline{(g'g, v)}$. The image of the equivariant multiplication $\psi : X' \to X$ is a saturated open subset U of X. [Dre04, 5.3.iii]*
3. *The restriction of ψ to $\psi^{-1}U$ is strongly étale. [Dre04, 5.3.iv]*
4. *The stabilizer of a point $\overline{(g, v)} \in X'$ is $g(G_{x_0} \cap G_v)g^{-1}$. [Dre04, 4.9.3]*
5. *There is a linear G_{x_0} representation Z, and a G_{x_0} equivariant morphism $\phi : V \to T$ whose image is a saturated open subset W of Z. [Dre04, 5.4.vi]*
6. *The restriction of ϕ to $\phi^{-1}W$ is strongly étale. [Dre04, 5.4.vii]*
7. *A point $\overline{(g, v)} \in X'$ has closed G orbit if and only if v has closed G_{x_0} orbit, in which case $G \cdot \overline{(g, v)} = G \times^{G_{x_0}} G_{x_0} v$. [Dre04, 4.9.5] [Dre04, 4.9.3]*

8. *A strongly étale H-equivariant morphism φ (for H an algebraic group) has the property that z has closed H-orbit if and only if $\varphi(z)$ has closed H-orbit. Moreover, the stabilizers z and $\varphi(z)$ are the same. [Dre04, 4.15.3]*

Lemma 3.3.3. *Let G be a linear algebraic group acting on an affine variety X. If x is G-stable, then it is G^0-stable, where G^0 is the identity component.*

Proof. Since Gx is closed in X, it suffices to prove that G^0x is closed in Gx. In other words, we may assume that $X = Gx$. By [Spr09, 2.3.3], $\overline{G^0x} - G^0x$ is the union of G^0-orbits of strictly smaller dimension than that of G^0x. But the G^0 stabilizer of any point in Gx is finite, so every G^0 orbit has the same dimension, and we conclude that $\overline{G^0x} - G^0x = \emptyset$. $\square$

We now proceed to the proof of Proposition 3.3.1. The idea is to prove it in the most basic case of a torus acting linearly on a vector space. Then we use the results above to reduce the general case down to this basic one.

Lemma 3.3.4. *Proposition 3.3.1 is true when $G = T$ is a torus and X a linear representation of T.*

Proof. There exist characters $\lambda_1, ..., \lambda_n$ of T and linear coordinates for X such that $t \cdot (x_1, ..., x_n) = (\lambda_1(t)x_1, ..., \lambda_n(t)x_n)$. Since the generic point of X is stable, we may take $x_0 = (x_1^0, x_2^0, ..., x_n^0)$ to be a stable point with no coordinate equal to 0. It follows that $\bigcap_{i=1}^n \ker \lambda_i$ is a finite subgroup of T, and therefore every point with no coordinate equal to 0 has finite stabilizer. It suffices now to show that the hyperplanes $H_j = \{x_j = 0\}$ each contain points whose stabilizers are finite (in which case, a generic point of the hyperplane will have this property).

Suppose to the contrary that H_j is a hyperplane all of whose points have positive dimensional stabilizer; for convenience, suppose $j = 1$. Then, in particular, $x_{00} = (0, x_2^0, ..., x_n^0)$ has a positive dimensional stabilizer $H := \bigcap_{i=2}^n \ker \lambda_i$. Let μ be a nontrivial one parameter subgroup of H. Then, $\mu(z) \cdot x_0 = (z^\alpha x_1^0, x_2^0 ..., x_n^0)$ for some *nonzero* integer α (since x_0 has finite stabilizer). Thus, either $\lim_{z\to 0} \mu(z) \cdot x_0 = x_{00}$ or $\lim_{z\to\infty} \mu(z) \cdot x_0 = x_{00}$, depending on whether α is positive or negative. In either case, we have x_{00} in the orbit closure of x_0. But the orbit of x_0 is closed, so $x_{00} \in Tx_0$ and thus x_{00} has finite stabilizer, a contradiction. $\square$

Proof of Proposition 3.3.1. It suffices to find a finite open cover of $\{U_i\}$ of X with the property each set in the cover is met by Y in codimension at least 2 (hereafter "codimension 2+"). Let U_0 be the open set of stable points. Pick a point y_1 of Y with closed orbit in X. Lemma 3.3.2 (2) produces a G-invariant open subset U_1 of X containing y_1, which we will call a "slice neighborhood of y_1".Therefore, $Y_2 := Y - U_1$ is a G-invariant closed subset of X. So there exists a point y_2 of Y_2 with closed orbit in X. Let U_2 be a slice neighborhood in X of y_2, and set $Y_3 := Y - (U_1 \cup U_2)$. Continue in this fashion. After finitely many steps, the sequence $Y_1 \supset Y_2 \supset Y_3 \supset ...$ of proper containments of closed subsets terminates in the empty set, and we have a cover $U_1, U_2, ...$ of $X - U_0$ by slice neighborhoods in X of points in Y with closed orbit. Thus it suffices to prove that any slice neighborhood through a point of Y is met by Y in codimension 2+.

To this end, pick a point $y_0 \in Y$ with closed orbit. We obtain ψ, X', and U as in 3.3.2(2). Let

$$Y' := \{\overline{(g,v)} \in X' : \text{Stabilizer in } G \text{ of } \overline{(g,v)} \text{ has positive dimension}\}$$

By 3.3.2(3) and 3.3.2(8), we have $\psi^{-1}(Y \cap U) = Y'$. Since ψ is étale, $Y \cap U$ has codimension 2+ in U if and only if Y' has codimension 2+ in X'. Also, since U meets the stable locus in X, there exists a stable point in X' (3.3.2(8)), which - by 3.3.2(7) - we can take to be of the form $\overline{(e, v_0)}$ where v_0 is a G_{y_0} stable point of V. We have reduced to proving the proposition for G acting on X'.

Now we further reduce to proving the proposition for $G^0_{y_0}$ acting on V (note that $G^0_{y_0}$ is reductive by 3.3.2(1)), although observe that V may not be irreducible. In any case, we have already proven that V contains G_{y_0}-stable points and hence $G^0_{y_0}$-stable points (Lemma 3.3.3). Let $Y_V := \{v \in V : \dim(G_{y_0} \cap G_v) > 0\}$. From 3.3.2(4), we have $\overline{(g,v)} \in Y'$ if and only if $v \in Y_V$. It suffices to prove that Y_V has codimension 2+ in V.

Finally, 3.3.2(5) and 3.3.2(6) give a strongly etale morphism $\phi : V \to W \subseteq Z$, where Z is a $G^0_{y_0}$ representation. Note that $G^0_{y_0} = T$ is a torus acting linearly on Z, which has T-stable points by 3.3.2(8). From Lemma 3.3.4, the set of points Y_Z with positive dimensional stabilizers in Z has codimension 2+. Since $\phi^{-1}(Y_Z \cap W) = Y_V$, the proof is complete. $\square$

REFERENCES

[Agn95] S. Agnihotri, *Quantum cohomology and the verlinde algebra*, Ph.D. thesis, University of Oxford, 1995.

[Bel04a] Prakash Belkale, *Extremal unitary local systems on* $\mathbb{P}^1 - \{p_1, ..., p_s\}$, Proceedings of the International Colloquium on Algebraic Groups (2004).

[Bel04b] ———, *Invariant theory of GL(n) and intersection theory of Grassmannians*, International Math Research Notices **69** (2004), 3709–3721.

[Bel06] ———, *Geometric proof of Horn and Satutration conjectures*, Journal of Algebraic Geometry **15** (2006), 133–173.

[Bel07] ———, *Geometric proof of a conjecture of Fulton*, Advances in Mathematics **216** (2007), 346–357.

[Bel08] ———, *Quantum generalization of the horn conjecture*, Journal of the AMS (2008).

[BGK15] Prakash Belkale, Angela Gibney, and Anna Kazanova, *Scaling of conformal blocks and generalized theta functions*, arXiv eprint (2015), http://arxiv.org/abs/1412.7204.

[BGM15] Prakash Belkale, Angela Gibney, and Swarnava Mukhopadhyay, *Vanishing and identities of conformal blocks divisors*, Algebraic Geometry **2** (2015), 62–90.

[BH95] Hans Boden and Yi Hu, *Variations of moduli of parabolic bundles*, Mathematische Annalen **301** (1995), 539–559.

[BK16] Prakash Belkale and Shrawan Kumar, *The multiplicative eigenvalue problem and deformed quantum cohomology*, Advances in Mathematics **288** (2016), 1309–1359.

[BKR12] Prakash Belkale, Shrawan Kumar, and Nicolas Ressayre, *A generalization of Fulton's conjecture for arbitrary groups*, Mathematische Annalen **354** (2012), 401–425.

[Bou87] Jean-François Boutot, *Singularitiés rationelles et quotients par les groupes réductifs*, Inventiones Mathematicae **88** (1987), 65–68.

[Boy08] Arzu Boysal, *Nonabelian theta functions of positive genus*, Proceedings of the American Mathematical Society **136** (2008).

[Bri05] Michel Brion, *Lectures on the geometry of flag varieties*, Topics in the Cohomological Study of Algebraic Varieties (2005), 33–85.

[CB04] William Crawley-Boevey, *Indecomposable parabolic bundles*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **100** (2004), 171–207.

[DN89] Jean-Marie Drezet and Mudumbai Seshachalu Narasimhan, *Groupe de Picard des variétiés de modules de fibrés semistables sur les courbes algébriques*, Inventiones Mathematicae **97** (1989), 53–94.

[Dre04] Jean-Marie Drezet, *Luna's slice theorem and applications*, Algebraic Group Actions and Quotients (2004), 39–90.

[DW00] Harm Derksen and Jerzy Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, Journal of the AMS **13** (2000), 467–479.

[DW02] ______, *On Littlewood-Richardson polynomials*, Journal of Algebra **255** (2002), 247–257.

[DW11] ______, *The combinatorics of quiver representations*, Annales de l'Institut Fourier **61** (2011), 1061–1131.

[Fuj90] Takao Fujita, *Classification theories of polarized varieties*, London Mathematical Society Lecture Notes, no. 155, Cambridge University Press, 1990.

[Ful97] William Fulton, *Young tableaux: With applications to representation theory and geometry*, London Mathematical Society Student Texts, no. 35, Cambridge University Press, 1997.

[Ful98] ______, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

[Ful00] ______, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bulletin of the American Mahtematical Society **37** (2000), 209–249.

[Gav04] Francesca Gavioli, *Theta functions on the modulia space of parabolic bundles*, International Journal of Mathematics **15** (2004), 259–287.

[GD71] Alexander Grothendieck and Jean Dieudonné, *Eléments de géométrie algébrique, I*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 166, Springer-Verlag, 1971.

[Gro61] Alexander Grothendieck, *Techniques de construction et théorèmes d'existence en géomeétrie algébrique IV: les schémas de Hilbert*, Séminaire Bourbaki **221** (1961), 249–276.

[Ike12] Christian Ikenmeyer, *Small Littlewood-Richardson coefficients*, arXiv eprint (2012), http://arxiv.org/abs/1209.1521.

[Kim95] Bumsig Kim, *Quot schemes for flags and Gromov invariants for flag varieties*, arXiv eprint (1995), http://arxiv.org/abs/alg-geom/9512003.

[Kin94] Alastair King, *Moduli of representations of finite dimensional algebras*, The Quarterly Journal of Mathematics **45** (1994), 515–530.

[KM76] Finn Knudsen and David Mumford, *The projectivity of the moduli space of stable curves I*, Mathematica Scandinavia **39** (1976), 19–55.

[KM04] Michael Kapovich and John J. Milson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, arXiv eprint (2004), http://arxiv.org/abs/math/0411182.

[Kno89] Friedrich Knop, *Der kanonische Modul eines Invariantenrings*, Journal of Algebra **127** (1989), 40–54.

[KP01] Bumsig Kim and Rahul Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, Symplectic Geometry and Mirror Symmetry (Seoul, 2000) (2001), 187–201.

[KT99] Allen Knutson and Terence Tao, *The honeycomb model of GLn tensor products I: Proof of the saturation conjecture*, Journal of the AMS **12** (1999), 1055–1090.

[KTT04] Ronald King, Christophe Tollu, and Frédéric Toumazet, *Stretched Littlewood-Richardson and Kostka coefficients*, CRM Proceedings and Lecture Notes **34** (2004), 99–112.

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of GLn tensor products II: Puzzles determine facets of the Littlewood-Richardson cone*, Journal of the AMS **17** (2004), 19–48.

[Kum10] Shrawan Kumar, *Tensor product decomposition*, Proceedings of the International Congress of Mathematicians. Volume III, 2010, pp. 1226–1261.

[LS97] Yves Laszlo and Christoph Sorger, *The line bundles on the moduli of parabolic G-bundles over curves*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série **30** (1997), 499–525.

[MFK94] David Mumford, John Fogarty, and Frances Clare Kirwan, *Geometric invariant theory*, 3rd ed., vol. 34, Springer Science & Business Media, 1994.

[MS80] Vikram Bhagvandas Mehta and Conjeevaram Srirangachari Seshadri, *Moduli of vector bundles on curves and parabolic structures*, Mathematische Annalen **248** (1980), 205–239.

[Muk03] Shigeru Mukai, *An introduction to invariants and moduli*, Cambridge Studies in Advanced Mathematics, no. 81, Cambridge University Press, 2003.

[NR93] Mudumbai Seshachalu Narasimhan and Trivandrum Ramakrishnan Ramadas, *Factorisation of generalized theta functions, I*, Inventiones Mathematicae **114** (1993), 565–623.

[Pau96] Christian Pauly, *Espaces de modules de fibrés paraboliques et blocs conformes*, Duke Mathematical Journal **84** (1996), 217–236.

[Pau98] ———, *Fibrés paraboliques de rang 2 et fonctions thêta généralisées*, Mathematische Zeitschrift **228** (1998), 31–50.

[Res11] Nicolas Ressayre, *A short geometric proof of a conjecture of Fulton*, L'Enseignement Mathématique. Revue Internationale. 2e Série **57** (2011).

[Rud97] Alexei Rudakov, *Stability for an abelian category*, Journal of Algebra **197** (1997), 231–245.

[Sch92] Aidan Schofield, *General representations of quivers*, Proceedings of the London Mathematical Society **3** (1992), 46–64.

[Sch08] Alexander H. W. Schmitt, *Geometric invariant theory and decorated principal bundles*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2008.

[Ses82] Conjeevaram Srirangachari Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérique **96** (1982).

[She15] Cass Sherman, *Geometric proof of a conjecture of King, Tollu, and Toumazet*, arXiv eprint (2015), http://arxiv.org/abs/1505.06551.

[Spr09] Tonny Albert Springer, *Linear algebraic groups*, 2nd ed., Modern Birkhäuser Classics, Springer Science & Business Media, 2009.

[Tel00] Constantin Teleman, *The quantization conjecture revisited*, Annals of Mathematics **152** (2000), 1–43.

[Vak13] Ravi Vakil, *MATH 216: Foundations of Algebraic Geometry*, 2013.

[Wit95] Edward Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, Geometry, Topology, and Physics, Conference Proceedings and Lecture Notes in Geometric Topology **IV** (1995).