Ergodic Theory of p-adic Transformations

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Abstract

JOANNA FURNO: Ergodic Theory of p-adic Transformations
(Under the direction of Jane Hawkins)

For a fixed prime p, we examine the ergodic properties and orbit equivalence classes of transformations on the p-adic numbers. Approximations and constructions are given that aid in the understanding of the ergodic properties of the transformations. Transformation types are calculated to give examples of transformations on measure spaces in various orbit equivalence classes. Moreover, we study the behavior of orbit equivalence classes under iteration. Finally, we give some preliminary investigations into the Haar measure and Hausdorff dimension of p-adic Julia sets and possible representations of the Chacon map as a 3-adic transformation.

Acknowledgements

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Introduction

The p-adic numbers were introduced over a century ago by Kurt Hensel. Since the p-adic valuation depends on divisibility by a prime p, the p-adic numbers have long been a tool in number theory. However, interest has recently spread to other branches of mathematics. For example, the ergodic properties of various transformations on the p-adic integers are studied with respect to Haar measure in [**4**–**6**, **9**, **15**, **26**, **27**, **30**, **41**, **42**, **44**–**46**]. This dissertation explores the ergodic properties and orbit equivalence of translation and multiplication maps with respect to measures other than Haar measure.

The rational numbers $\mathbb{Q}$ are contained in the p-adic numbers $\mathbb{Q}_p$, which has a construction similar to the construction of real numbers $\mathbb{R}$. Although the constructions are similar, the topological structure of $\mathbb{Q}_p$ has some important differences from the topological structure of $\mathbb{R}$. Chapter 1 introduces the p-adic numbers, defines independent and identically distributed (i.i.d.) product measures on the p-adic integers $\mathbb{Z}_p$, and discusses approximations and constructions of p-adic translations. Chapter 2 explores the ergodic properties of various transformations.

One theme that appears in the first two chapters is that translation by an element of $\mathbb{Z}$ can behave differently than translation by an element of $\mathbb{Z}_p \setminus \mathbb{Z}$. The first difference appears in Section 1.3, which gives a sequence of approximations of translations by periodic transformations. Proposition 1 states that the approximations converge differently in the strong topology on the set of endomorphisms, depending on whether the approximated transformation is translation by an integer or by another element of $\mathbb{Z}_p$. The second difference appears in Section 2.2, where Theorem 4 gives measures that are nonsingular

for translation by an integer but singular for translation by other rational numbers in $\mathbb{Z}_p$. This singularity result has consequences for other transformations considered in Section 2.3. Since nonsingularity is an important part of the definition of orbit equivalence, much of Chapter 3 focuses on translation by an integer.

Orbit equivalence is a weaker notion of equivalence than isomorphism for measurable systems. Chapter 3 gives examples of p-adic transformations in different orbit equivalence classes, using an invariant called transformation type. A transformation on $\mathbb{Q}_p$ is constructed to give an example for one of the orbit equivalence classes. This transformation preserves Haar measure on $\mathbb{Q}_p$, which is a σ-finite measure that is not finite. Besides being a representative for the orbit equivalence class, it is an example of a transformation satisfying properties from infinite-measure ergodic theory of current research interest.

Since all translations have the same transformation type with respect to Haar measure, it is necessary to consider translations with respect to other i.i.d. product measures in order to observe other transformation types. Moreover, translation by a positive integer is an iterate of translation by 1. After showing the existence of certain transformation types, Chapter 3 examines how the transformation type of an iterate is related to the transformation type of the original tranformation. The chapter concludes with a discussion of possible generalizations to the g-adic numbers, where g may be a composite number.

The final two chapters give some preliminary results that will lead to future work. The completion of the algebraic closure of the p-adic numbers $\mathbb{C}_p$ plays a role similar to the complex numbers $\mathbb{C}$. In particular, the Julia set of a polynomial with p-adic coefficients can be defined on $\mathbb{C}_p$. There are known results for the topological properties

of the certain Julia sets contained in $\mathbb{Z}_p$. Chapter 4 examines the Haar measure and Hausdorff dimension of these Julia sets.

Chapter 5 contains two more directions for future work. Although Theorem 4 states that translation by a rational number can be singular with respect to an i.i.d. product measure, Section 2.3 defines an averaged measure for which the translation is nonsingular. In Section 5.1, we discuss the possibility of calculating the transformation type of translation by a rational number with respect to this averaged measure. Section 5.2 is joint work with César Silva. It contains two possible descriptions of the Chacon map as a 3-adic transformation.

CHAPTER 1

Introduction to the p-adic Numbers

1.1. Definitions of the p-adic Numbers

In this section, we define the p-adic numbers and discuss their standard field operations, topology, and Haar measure. Further information and proofs of the facts stated in this section are found in [**5**,**47**,**57**,**59**]. For the sake of comparison, we recall the standard construction of the real numbers $\mathbb{R}$. The absolute value $|\cdot|$ on the rational numbers $\mathbb{Q}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0, \end{cases}$$

This definition uses the standard linear order on $\mathbb{Q}$. Then the set of real numbers $\mathbb{R}$ is defined to be the topological completion of $\mathbb{Q}$ with respect to $|\cdot|$.

In general, a *valuation* is a function v from a ring R to $\mathbb{R}^+$ such that

(a) $v(0) = 0$ and $v(a) > 0$ if $a \in R \setminus \{0\}$,

(b) $v(a+b) \leq v(a) + v(b)$ for all $a, b \in R$, and

(c) $v(ab) = v(a)v(b)$ for all $a, b \in R$.

Moreover, a valuation v that satisfies the strong triangle inequality

$$v(a+b) \leq \max\{v(a), v(b)\}. \tag{1}$$

is called *non-Archimedean* because the strong triangle inequality implies that

$$v(na) \leq v(a), \quad \text{for all } n \in \mathbb{N} \text{ and } a \in R.$$

For a fixed prime $p \geq 2$, the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ is a non-Archimedean valuation that is defined in terms of divisibility by p. If x is a nonzero rational number, then it can be written uniquely as $p^n(a/b)$, for some integer n and relatively prime integers a and b that are not divisible by p. Then the p-adic *order* is $\operatorname{ord}_p(x) = n$, and the *$p$-adic absolute value* is defined by

$$|x|_p = \begin{cases} p^{-\operatorname{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For example, the 2-adic absolute value gives

$$\begin{aligned} |6|_2 &= |2 \cdot 3|_2 = 1/2, \\ |3|_2 &= |2^0 \cdot 3|_2 = 1, \text{ and} \\ |1/4|_2 &= |2^{-2}|_2 = 4. \end{aligned}$$

The set of p-adic numbers $\mathbb{Q}_p$ is defined to be the topological completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. Addition and multiplication on $\mathbb{Q}_p$ are defined by extending the definition on $\mathbb{Q}$ by continuity. Finally, the set of p-adic integers is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Two valuations are *equivalent* if they induce the same topology. The *trivial valuation* assigns the value 0 to 0 and the value 1 to all other elements of the ring. Ostrowski's theorem gives a classification of all nontrivial valuations on $\mathbb{Q}$.

Ostrowski's Theorem. *Every nontrivial valuation on $\mathbb{Q}$ is equivalent to $|\cdot|$ or to $|\cdot|_p$ for some prime p.*

If v_1 and v_2 are equivalent valuations on a ring R, then the completion of R with respect to v_1 is homeomorphic to the completion of R with respect to v_2. Up to homeomorphism, Ostrowski's Theorem implies that $\mathbb{R}$ and $\mathbb{Q}_p$ for primes $p \geq 2$ are the only topological completions of $\mathbb{Q}$ with respect to nontrivial valuations.

Alternatively, $\mathbb{Q}_p$ may be defined as the set of formal Laurent series in p, with coefficients between 0 and $p-1$,

$$\mathbb{Q}_p = \left\{ \sum_{i=n}^{\infty} x_i p^i : n \in \mathbb{Z},\ 0 \leq x_i \leq p-1 \text{ for all } i \geq n \right\}.$$

As before, $|0|_p = |\sum_{i=0}^{\infty} 0 \cdot p^i|_p = 0$. If x is nonzero, then the order of x is $\operatorname{ord}(x) = \min\{i \in \mathbb{Z} : x_i \neq 0\}$. As above, the p-adic absolute value is defined by $|x|_p = p^{-\operatorname{ord}(x)} = p^{-n}$. Addition and multiplication are defined coordinatewise with carries. In this definition, the p-adic integers are the formal power series

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} x_i p^i :\ 0 \leq x_i \leq p-1 \text{ for all } i \geq 0 \right\}.$$

Although we use the formal series notation to express p-adic numbers, we also use the facts from the first definition that $\mathbb{Q} \subset \mathbb{Q}_p$ and $\mathbb{Z} \subset \mathbb{Z}_p$.

The set $\mathbb{Q}_p$ is a field under addition and multiplication. Since addition and multiplication are continuous on $\mathbb{Q}_p \times \mathbb{Q}_p$, the transformations

$$\begin{aligned} T_a : \mathbb{Z}_p &\to \mathbb{Z}_p \\ x &\mapsto x + a \end{aligned}$$

and

$$\begin{aligned} M_a : \mathbb{Z}_p &\to \mathbb{Z}_p \\ x &\mapsto ax \end{aligned}$$

are continuous for each $a \in \mathbb{Z}_p$. Although every element of $\mathbb{Z}_p$ has an inverse in $\mathbb{Q}_p$, the inverse may not be an element of the ring $\mathbb{Z}_p$. The set of *units* in $\mathbb{Z}_p$ is

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : \text{ there exists } y \in \mathbb{Z}_p \text{ such that } xy = 1\}.$$

Lemma 1. *The element $x = \sum_{i=0}^{\infty} x_i p^i \in \mathbb{Z}_p$ is a unit if and only if $x_0 \neq 0$.*

Remark 1. The map $\sum_{i=0}^{\infty} x_i p^i \mapsto \sum_{i=0}^{n-1} x_i p^i$ defines group homomorphism from $\mathbb{Z}_p$ onto $\mathbb{F}_{p^n}$, the finite field containing p^n elements. This group homomorphism motivates the definition of equality modulo p^n for two elements in $\mathbb{Z}_p$. For $x = \sum_{i=0}^{\infty} x_i p^i$ and $y = \sum_{i=0}^{\infty} y_i p^i$ in $\mathbb{Z}_p$, the equality $x = y \mod p^n$ holds if $\sum_{i=0}^{n-1} x_i p^i = \sum_{i=0}^{n-1} y_i p^i$. Thus, $x = y \mod p^n$ if and only if $|x - y|_p \leq p^{-n}$.

The p-adic absolute value $|\cdot|_p$ is used to define a metric by $d_p(x,y) = |x-y|_p$ for $x, y \in \mathbb{Q}_p$. For $r > 0$, the ball of radius r centered at $a \in \mathbb{Q}_p$ is

$$B_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}.$$

Since the p-adic absolute value of an element of $\mathbb{Q}_p$ is always a power of p, we can assume without loss of generality that $r = p^n$ for some $n \in \mathbb{Z}$. The strong triangle inequality (1) implies that

$$|a-b|_p = \max\{|a|_p, |b|_p\}, \text{ when } |a|_p \neq |b|_p.$$

Moreover, the strong triangle inequality implies that every element in a ball is a center of the ball. Thus, two balls of the same radius are either equal or disjoint. Then $\mathbb{Z}_p$ is a disjoint union of p^n balls; for example, $\mathbb{Z}_p = \bigcup_{i=0}^{p^n-1} B_{p^{-n}}(i)$. Since the "closed" balls $B_r(a)$ are both open and closed, the set of all "closed" balls forms a basis for the metric topology, which is totally disconnected.

Haar measure on a locally compact abelian group is a translation-invariant measure that is unique up to multiplication by a constant. Since $\mathbb{Q}_p$ is a locally compact abelian group under addition, there exists a unique Haar measure m that is normalized so that $m(\mathbb{Z}_p) = 1$. Under this normalization, the Haar measure of a "closed" ball in $\mathbb{Q}_p$ is equal to its radius, that is, $m(B_r(a)) = r$. Bryk and Silva give a nice construction of Haar measure on $\mathbb{Z}_p$ from this viewpoint in [**9**]. Since $\mathbb{Z}_p$ is a compact subset of $\mathbb{Q}_p$ that has Haar measure 1, its role in $\mathbb{Q}_p$ is similar to the role of $[0,1]$ in $\mathbb{R}$.

1.2. The p-adic Integers as a Product Space.

Besides the two definitions already given, the p-adic integers are isomorphic to a product space. This isomorphism motivates the definition of product measures on $\mathbb{Z}_p$. Haar measure is a special case of these product measures. Ergodic properties are known for certain p-adic transformations with respect to Haar measure, but other behaviors are possible with respect to other product measures. Some behaviors follow from known results for product spaces. However, the group structure of the p-adic numbers prompts more general questions that do not follow from known results. In this section, we give the definition of isomorphism for measurable systems, review product spaces, and give an isomorphism from $\mathbb{Z}_p$ to a product space.

If a transformation $T : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$ is measurable, then it is called an *endomorphism* of $(X, \mathcal{A}, \mu)$. If T is also invertible and measure-preserving, then T is called an *automorphism* or $(X, \mathcal{A}, \mu)$. We use the notation $(X, \mathcal{A}, \mu; T)$ for the system consisting of an endoorphism T of a measure space X, and a σ-algebra $\mathcal{A}$ of measurable sets in X. One notion of equivalence between such systems is isomorphism. For two systems $(X_1, \mathcal{A}_1, \mu_1; S_1)$ and $(X_2, \mathcal{A}_2, \mu_2; S_2)$, a measurable map $\phi : X_1 \to X_2$ is a *factor map* if there exist $Y_1 \in \mathcal{B}_1$ and $Y_2 \in \mathcal{B}_2$ such that $\mu(X_1 \backslash Y_1) = \nu(X_2 \backslash Y_2) = 0$, $\phi : Y_1 \to Y_2$ is surjective, $\phi \circ S_1 = S_2 \circ \phi$ on Y_1, and $\mu_2(A) = \mu_1(\phi^{-1}(A))$ for all $A \in \mathcal{A}_2$. If ϕ is a factor map that is also injective on Y_1, then ϕ is an *isomorphism* of measure spaces.

If $X_n = \{0, 1, \ldots, n-1\}$, then the product space $\prod_{i=0}^{\infty} X_n$ is the set of one-sided sequences $(x_0, x_1, x_2 \ldots)$, where $x_i \in X_n$ for all integers $i \geq 0$. If X_n has the discrete topology, then the standard product topology on $\prod_{i=0}^{\infty} X_n$ has a basis made of up cylinder

sets of the form

$$\prod_{i=0}^{k-1} a_i \prod_{i=k}^{\infty} X_n = \{(x_0, x_1, x_2, \ldots : x_i = a_i \text{ for } 0 \leq i < k\}.$$

The *odometer* $\mathcal{O} : \prod_{i=0}^{\infty} X_n \to \prod_{i=0}^{\infty} X_n$ is defined in the following manner. If we have $x \in \prod_{i=0}^{\infty} X_n$ such that $x_i = n-1$ for all $i \geq 0$, then $(\mathcal{O}\, x)_i = 0$ for all $i \geq 0$. Otherwise, there exists an index $j = \min\{i \in \mathbb{Z} : i \geq 0 \text{ and } x_i < n-1\}$. In this case,

$$(\mathcal{O}\, x)_i = \begin{cases} 0 & \text{if } i < j, \\ x_i + 1 & \text{if } i = j, \\ x_i & \text{if } i > j. \end{cases}$$

A probability vector $(q_0, q_1, \ldots, q_{n-1})$ defines a probability measure q on X_n by $q(i) = q_i$. This probability measure on X_n induces a premeasure ν_0 that is defined on cylinder sets by

$$\nu_0\left(\prod_{i=0}^{k-1} a_i \prod_{i=k}^{\infty} X_n\right) = \prod_{i=0}^{k-1} q(a_i).$$

Then the Caratheodory construction gives an independent and identically distributed (i.i.d.) product measure ν on the Borel σ-algebra $\mathcal{N}$ of measurable sets. Similarly, a probability vector $(q_0, q_1, \ldots, q_{p-1})$ defines a premeasure μ_0 on balls of $\mathbb{Z}_p$ by

$$\mu_0\left(B_{p^{-k}}\left(\sum_{i=0}^{\infty} a_i p^i\right)\right) = \prod_{i=0}^{k-1} q(a_i).$$

Again, the Caratheodory construction gives a measure μ on the Borel σ-algebra $\mathcal{M}$ of measurable sets. We also call $\mu = \prod_{i=0}^{\infty} \{q_0, q_1, \ldots, q_{p-1}\}$ an *i.i.d. product measure.* If all of the weights are equal to $1/p$, then the i.i.d. product measure equal to Haar measure.

If $(q_0, q_1, \ldots, q_{p-1})$ is a probability vector used to define both μ on $\mathbb{Z}_p$ and ν on $\prod_{i=0}^{\infty} X_p$, then

$$\begin{aligned} \Phi : (\mathbb{Z}_p, \mathcal{M}, \mu) &\to \left(\prod_{i=0}^{\infty} X_p, \mathcal{N}, \nu\right) \\ \sum_{i=0}^{\infty} a_i p^i &\mapsto (a_0, a_1, a_2, \ldots) \end{aligned}$$

acts on basis elements by $\Phi(B_{p^{-k}}(\sum_{i=0}^{\infty} a_i p^i)) = \prod_{i=0}^{k-1} a_i \prod_{i=k}^{\infty} X_n$. It follows from this equality that Φ is measurable and $\nu = \mu \circ \Phi^{-1}$. Moreover, Φ is an isomorphism from $(\mathbb{Z}_p, \mathcal{M}, \mu; T_1)$ to $(\prod_{i=0}^{\infty} X_p, \mathcal{N}, \nu; \mathcal{O})$. This isomorphism is used in Chapter 2 to transfer known results for the odometer on a product space to translation by 1 on $\mathbb{Z}_p$.

Finally, a goal of ergodic theory is to understand endomorphisms under iteration. For an endomorphism $T : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$, iterates are defined inductively by setting $T^0 = \mathrm{Id}$ and $T^n = T \circ T^{n-1}$ for all integers $n \geq 1$. If T is invertible, then negative iterates are defined by $T^{-n} = T^{-1} \circ T^{-n+1}$ for all integers $n \geq 1$.

1.3. Approximations and Constructions of Translations.

An important tool in ergodic theory is the construction of transformations by cutting and stacking the unit interval. In the first step of the construction, the unit interval is cut into subintervals, the subintervals are stacked into columns, and the transformation is defined on all but the top level by mapping linearly up the columns. The rest of the definition is done recursively, giving a method for cutting the columns from the previous step and stacking the subcolumns. Examples and more detailed descriptions of cutting and stacking can be found in [**28**, **29**]. Cutting and stacking are used to

construct transformations with specific properties, which is especially useful for giving counterexamples.

If we map the top level of each stack linearly to the bottom level at each step, rather than leaving it undefined, then we have a sequence of periodic transformations that approximate the final construction. This notion of approximation can be extended to measure spaces other than the unit interval with Lebesgue measure. In [**39**], Katok and Stepin prove that various ergodic properties hold for an automorphism of a Lebesgue space that has an approximation by periodic automorphisms that converges with a given speed. In this section, we construct perodic transformations that approximate translations. These approximations are also reminiscent of cutting and stacking. In cutting and stacking constructions, each interval is mapped linearly onto the one above. In a similar way, the approximations in this section will map each ball to the ball above by fixing all coordinates that are not determined by the center of the ball.

We consider two notions of convergence. A sequence of transformations S_n on a metric space (X, d) converges to S *uniformly in* x if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(S_n(x), S(x)) < \epsilon$$

for all $n \geq N$ and all $x \in X$. On the set of endomorphisms of a measure space $(X, \mathcal{A}, \mu)$, the metric defined by

$$d_\mu(S, T) = \mu \{x \in X : S(x) \neq T(x)\}$$

induces the *strong topology*. Hence, a sequence of transformations S_n on a measure space $(X, \mathcal{A}, \mu)$ converges to S in the strong topology if

$$\mu\left\{x \in X : S_n(x) \neq S(x)\right\} \to 0 \text{ as } n \to \infty.$$

We fix $a \in \mathbb{Z}_p^\times$ and approximate T_a by periodic transformations $\{t_n\}_{n\geq 1}$. For the nth approximation, we consider balls of radius p^{-n}. We stack the balls in the order that T_a visits them and define t_n so that it maps from one level of the stack to the next and fixes all coordinates with index greater than $n-1$. In coordinates, we define

$$(t_n(x))_i = \begin{cases} (T_a(x))_i & \text{if } 0 \leq i < n \\ \\ x_i & \text{if } i \geq n. \end{cases}$$

Since $a \in \mathbb{Z}_p^\times$, Lemma 1 states that $a_0 \neq 0$, so T_a cycles through all the balls of radius p^{-n}. Thus $t_n^{p^n}(x) = x$ for all $x \in \mathbb{Z}_p$, which means that t_n is periodic. By construction, $|T_a(x) - t_n(x)|_p \leq p^{-n}$ for all $x \in \mathbb{Z}_p$, so t_n converges uniformly to T_a in x.

Since t_n is completely determined by the order in which T_a visits balls of radius p^{-n}, a useful way to visualize the approximation is similar to a cutting and stacking diagram. A stack of p^n balls are labeled with the centers of the balls, beginning with 0. Figures 1.1 and 1.2 give the first and second stacks of $T_1 : \mathbb{Z}_3 \to \mathbb{Z}_3$ to illustrate this visualization.

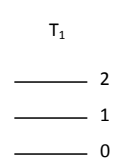

FIGURE 1.1. The first stack of T_1 on $\mathbb{Z}_3$.

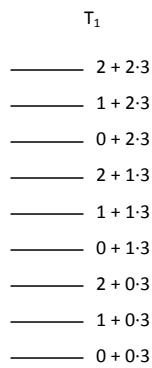

FIGURE 1.2. The second stack of T_1 on $\mathbb{Z}_3$.

For a rational number $a \in \mathbb{Z}_p^\times$ and a fixed radius p^{-n}, we construct the stack for T_a from the stack for T_1. Constructing the stack for T_a in this manner illustrates relationships between the two definitions of the p-adic integers. First, we consider the case where $a = k$ is a positive integer that is relatively prime to p. Writing $T_k = T_1^k$, we see that T_k is an iterate of T_1. We begin the stack for T_k with the ball centered at 0. Beginning at the bottom of the stack for T_1, we go up k levels to find the next level of T_k. We repeat this process until the stack for T_k is complete. When we reach the top of the stack for T_1, we simply return to the bottom and continue. The first two iterations are shown in Figure 1.3 and Figure 1.4 for the second stack for T_5 on $\mathbb{Z}_3$. The completed second stack for T_5 on $\mathbb{Z}_3$ is shown in Figure 1.5.

Next, we consider the case $a = 1/k$, where k is a positive integer in $\mathbb{Z}_p^\times$. Writing $T_1 = T_{1/k}^k$, we see that T_1 is the kth iterate of $T_{1/k}$. Again, we begin the stack for $T_{1/k}$ with the ball centered at 0. We label every kth level in the stack for $T_{1/k}$ with consecutive levels of T_1. When we reach the top of the stack for $T_{1/k}$, we return to the bottom and continue until all levels of the stack have been labels. Figure 1.6 illustrates the third

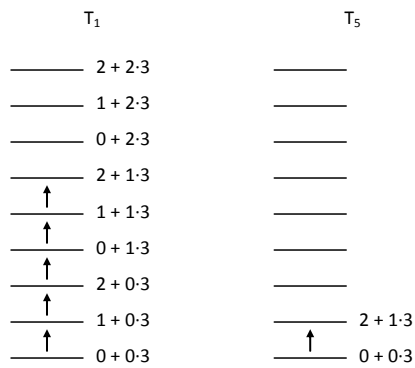

FIGURE 1.3. The first step in constructing the second stack for T_5 on $\mathbb{Z}_3$.

iteration of this process for the second stack of $T_{1/4}$ on $\mathbb{Z}_3$. The completed second stack is given in Figure 1.7

Similar considerations give a construction for any rational number $a \in \mathbb{Z}_p^\times$. If $a = j/k$ is a reduced fraction, then $a \in \mathbb{Z}_p^\times$ implies that both j and k are not divisible by p. If a is positive, then we begin the stack for $T_{j/k}$ with the ball centered at 0, as usual. In order to construct the rest of the stack, we go up j levels in the stack of T_1 for every k levels that we go up in the stack of $T_{j/k}$. If a is negative, then we can replace the stack for T_1 with the stack for T_{-1} and proceed as before.

Section 1.1 gives two definitions of the p-adic numbers. In one definition, the p-adic numbers are defined as the completion of the rational numbers with respect to the p-adic absolute value. In the other definition, the p-adic numbers are defined as formal Laurent series in p. One step in the proof of the equivalence of these definitions is expressing rational numbers as formal power series. Any rational number can be written in the form $p^n(j/k)$, where $n \in \mathbb{Z}$ and j, k are relatively integers that are not divisible by p. If j is a positive integer, then it has a finite expansion in p, so it ends in repeating 0's. Similarly, a negative integer ends in repeating $p-1$'s. In particular, we note that

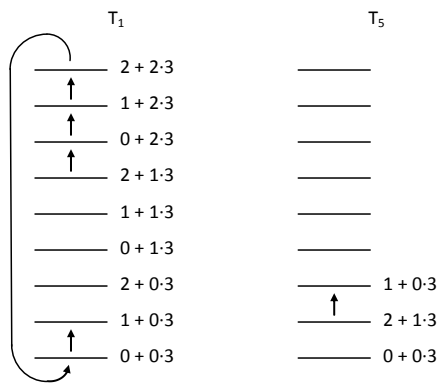

FIGURE 1.4. The second step in constructing the second stack for T_5 on $\mathbb{Z}_3$.

$-1 = \sum_{i=0}^{\infty}(p-1)p^i$. A power of p shifts the coordinates. The main difficulty is expressing $1/k$ as formal Laurent series. We find the coefficients of this power series by using the Euclidean algorithm.

The Euclidean algorithm is the process of finding coefficients that satisfy the linear combination in the following theorem from basic number theory:

THEOREM 1. *Let a and b be two natural numbers such that* $\gcd(a,b) = d$. *Then there exist integers x and y such that $xa + yb = d$.*

Since the k chosen above is not divisible by p, we have $\gcd(k, p^n) = 1$ for all $n \in \mathbb{N}$. Using the Euclidean algorithm and rearranging the equation, we find integers x_n and y_n such that $x_n k = 1 + y_n p^n$. This linear combination implies that $1/k \equiv x_n \mod p^n$, so the first n coordinates of $1/k$ agree with the first n coordinates of x_n.

For the balls of a fixed radius p^{-n}, the construction of the stack for $T_{1/k}$ from the stack for T_1 is a process that serves the same purpose as the Euclidean algorithm. The ball that contains $1/k$ is the level directly above the base of the stack for $T_{1/k}$. In order to label the p^n levels of the stack, we cycle through the levels k at a time. Hence, we label the first level above the base when we reach a multiple of k that is congruent to 1

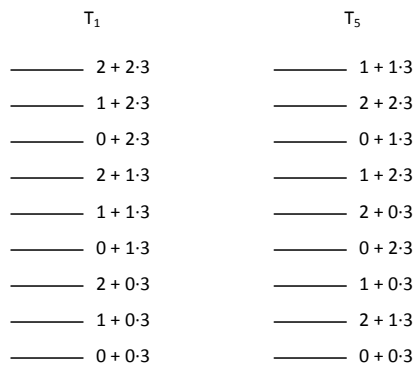

FIGURE 1.5. The second stacks of T_1 and T_5 on $\mathbb{Z}_3$.

modulo p^n. In terms of the equation $x_n k = 1 + y_n p^n$, going up k levels x_n times cycles through the entire stack y_n times and ends at the first level above the base.

For example, in the construction of the second stack for $T_{1/4}$, we assign a ball to every fourth level. On the seventh iteration of this process, we have cycled through the stack three times and assign a ball to the first level above the base. Since $7 \cdot 4 = 1 + 3 \cdot 3^2$, we conclude that $7 = 1 + 2 \cdot 3 \equiv 1/4 \mod 3^2$.

For a fixed $a \in \mathbb{Z}_p^\times$, the periodic endomorphisms $\{t_n\}_{n\geq 1}$ converge to T_a uniformly in $x \in \mathbb{Z}_p$. By construction, if $n \geq N$, then $|T_a(x) - t_n(x)|_p \leq p^N$ for all $x \in \mathbb{Z}_p$. However, the convergence of the approximations in the strong topology differentiates the natural integers $\mathbb{Z}$ from the other elements of $\mathbb{Z}_p$, as we see in Proposition 1.

PROPOSITION 1. *For $a \in \mathbb{Z}_p^\times$, we define $T_a(x) = x + a$ and define $t_n(x)$ by*

$$(t_n(x))_i = \begin{cases} (T_a(x))_i & \text{if } 0 \leq i < n \\ x_i & \text{if } i \geq n. \end{cases}$$

With respect to an i.i.d. product measure $\mu = \prod_{i=0}^{\infty} \{q_0, \ldots, q_{p-1}\}$, the sequence of endomorphisms $\{t_n\}_{n\geq 1}$ converges to T_a in the strong topology if and only if $a \in \mathbb{Z}$.

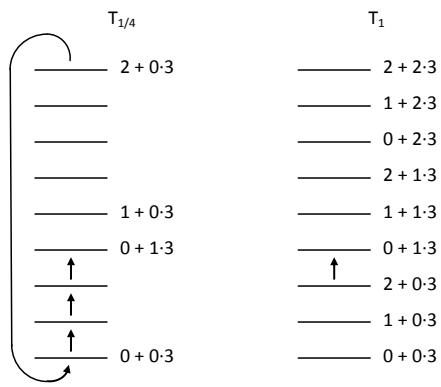

FIGURE 1.6. The third step in constructing the second stack for $T_{1/4}$ on $\mathbb{Z}_3$.

PROOF. By definition, $T_a(x) = t_n(x)$ implies that $(x+a)_i = x_i$ for all $i \geq n$. This equality can occur in one of two ways. First, we consider the case that $\sum_{i=0}^{n-1} x_i p^i + \sum_{i=0}^{n-1} a_i p^i < p^n$, so that the addition does not result in a carry to the nth coordinate. Then $(x+a)_n = x_n + a_n$ is equal to x_n if and only if $a_n = 0$. If $a_n = 0$, then $x_n + a_n < p$, which again does not result in a carry. This serves as the base case for an induction argument. As an induction hypothesis, we suppose that $x_{i-1} + a_{i-1} < p$ for some $i \geq n$. Then $(x+a)_i = x_i + a_i$ is equal to x_i if and only if $a_i = 0$. Moreover, if $a_i = 0$, then $x_i + a_i < p$. It follows by induction that if $x_{n-1} + a_{n-1} < p$, then $T_a(x) = t_n(x)$ if and only if $a_i = 0$ for all $i \geq n$. If there exists an $n \in \mathbb{N}$ such that $a_i = 0$ for all $i \geq n$, then a is a positive integer.

On the other hand, we consider the case that $\sum_{i=0}^{n-1} x_i p^i + \sum_{i=0}^{n-1} a_i p^i \geq p^n$. This inequality implies that there is a carry to the nth coordinate. Hence, we have $(x+a)_n = x_n + a_n + 1$, which is equal to x_n if and only if $a_n = p-1$. Then $(x+a)_n = x_n + a_n + 1 = x_n + p \geq p$ results in a carry to the next coordinate. An induction argument similar to the previous case implies that $T_a(x) = t_n(x)$ if and only if $a_i = p-1$ for all $i \geq n$.

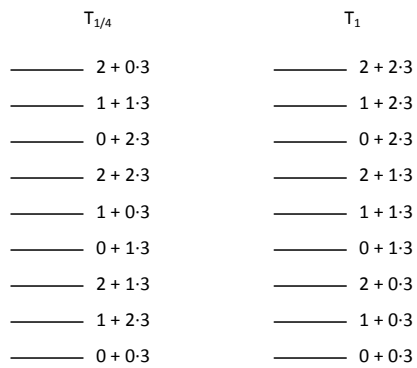

FIGURE 1.7. The second stacks of $T_{1/4}$ and T_1 on $\mathbb{Z}_3$.

Similarly, if there exists an $n \in \mathbb{N}$ such that $a_i = p - 1$ for all $i \geq n$, then a is a negative integer.

Thus, if a is not an integer, then $T_a(x) \neq t_n(x)$ for all $x \in \mathbb{Z}_p$. In terms of the metric on the space of endomorphisms, we have

$$d_\mu(T_a, t_n) = \mu(\mathbb{Z}_p) = 1$$

for all $n \in \mathbb{N}$. Therefore, if a is not an integer, then t_n fails to converge to T_a in the strong topology.

It remains to show that convergence holds when a is an integer. If $a \in \mathbb{Z}_p^\times$ is a positive integer, then Lemma 1 implies that a is not divisible by p. For $n > \log_p(a)$, we consider t_n approximating T_a. Since $n > \log_p(a)$ implies that $a < p^n$, we have $a_i = 0$ for all $i \geq n$. If $\sum_{i=0}^{n-1} x_i p^i + a < p^n$, then there is no carry to the the nth coordinate. An induction argument, similar to the one in the first paragraph, shows that $(x + a)_i = x_i + a_i = x_i$ for all $i \geq n$. In this case, $T_a(x) = t_n(x)$. On the other hand, if $\sum_{i=0}^{n-1} x_i p^i + a \geq p^n$, then there is a carry. Thus, we have $(x + a)_n = 1 + x_n + a_n = 1 + x_n \neq x_n$, which implies that

$T_a(x) \neq t_n(x)$. If $W = \max_{0 \leq i < p} q_i$ is the maximal weight, then

$$d_\mu(T_a, t_n) = \mu\left(\bigcup_{k=p^n-a}^{p^n-1} B_{p^{-n}}(k)\right) \leq aW^n.$$

Since $W < 1$, we conclude that $aW^n \to 0$ as $n \to \infty$. Therefore t_n converges to T_a in the strong topology.

Keeping a and n as in the previous paragraph, we now consider t_n approximating T_{-a}. Since $p^n > a$, we have $(-a)_i = p - 1$ for all $i \geq n$. We note that $-a \equiv p^n - a \mod p^n$ and that $p^n - a > 0$. If $\sum_{i=0}^{n-1} x_i p^i + \sum_{i=0}^{n-1} (-a)_i p^i = \sum_{i=0}^{n-1} x_i p^i + p^n - a \geq p^n$, then there is a carry to the nth coordinate. A standard induction argument, similar to the one in the second paragraph, shows that $(x+a)_i = 1 + x_i + a_i \mod p = 1 + x_i + p - 1 \mod p = x_i$ for all $i \geq n$. This equality implies that $T_{-a}(x) = t_n(x)$. On the other hand, if $\sum_{i=0}^{n-1} x_i p^i + p^n - a < p^n$, then there is not a carry, so $T_{-a}(x) \neq t_n(x)$. Thus, we have

$$d_\mu(T_{-a}, t_n) = \mu\left(\bigcup_{k=0}^{a-1} B_{p^{-n}}(k)\right) \leq aW^n.$$

Since $aW^n \to 0$ as $n \to \infty$, it follows that t_n converges to T_{-a} in the strong topology. $\square$

When a is an integer, an approximation t_n is the same as T_a on some levels but different on other levels. In order to have a finer structure, we give a construction with multiple stacks, so that t_n is the same as T_a on every level except the top levels. For positive integers $a, k \in \mathbb{N}$ such that $0 \leq k < p^n$, we consider $x \in B_{p^{-n}}(k)$. In the proof of Proposition 1, we see that $x + a$ has a carry after the nth coordinate if and only if $p^n - a \leq k < p^n$. With this observation as motivation, we use multiple stacks to construct a translation by a positive integer, so that $\{B_{p^{-n}}(k)\}_{k=p^n-a}^{p^n-1}$ are the top levels of the stacks. In Chapter 2, we define the Radon-Nikodým derivative and show in Theorem

2 that it is constant on each level of the stacks except for the top levels. A collection of stacks at a particular step is called a tower.

We set $N = \min\{n \in \mathbb{N} : a < p^n\}$. The first tower in the construction has a stacks containing balls of radius p^{-N}. The bases of the stacks are $B_{p^{-N}}(k)$ for $0 \leq k < a$. The rest of the balls are added to the stacks so that $B_{p^{-N}}(k+a)$ is directly above $B_{p^{-N}}(k)$ for all $0 \leq k < p^n - a$. We note that $t_N(x) = T_a(x)$ for x in all levels except the top levels. The top levels of the stacks are the balls where $x + a$ has a carry after the Nth coordinate. Since $2^2 > 3$, Figure 1.8 illustrates the first tower for T_3 on $\mathbb{Z}_2$.

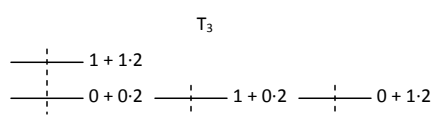

FIGURE 1.8. The first tower of T_3 on $\mathbb{Z}_2$, with cuts.

If we have a tower with balls of radius p^{-n} for some $n \geq N$, then we constuct the next tower by a type of cutting and stacking. Each stack with base $B_{p^{-n}}(i)$ for $0 \leq i < p$ is cut into p substacks with bases $B_{p^{-n-1}}(i + j \cdot p^n)$ for $0 \leq j < p$. For $a \leq k < p^n$, if $B_{p^{-n}}(k)$ is in the stack with base $B_{p^{-n}}(i)$, then $B_{p^{-n-1}}(k + j \cdot p^n)$ is in the substack with base $B_{p^{-n-1}}(i + j \cdot p^n)$. Again, the bases of the stacks for the new tower are the balls $B_{p^{-n-1}}(i)$ for $0 \leq i < a$. The substacks from the previous tower are stacked on each other so that $B_{p^{-n-1}}(i + a)$ is directly above $B_{p^{-n-1}}(i)$ for all $0 \leq i < p^{n+1} - a$. Again, the top levels of each tower are the balls where $x + a$ has a carry after the nth coordinate. With cuts as in Figure 1.8, the substacks are then stacked to give the second tower in Figure 1.9. These towers help us understand the Radon-Nikodým derivative and motivate density arguments in Chapters 2 and 3.

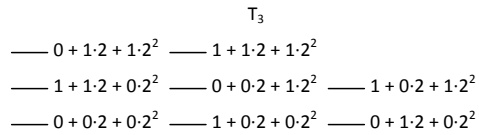

FIGURE 1.9. The second tower of T_3 on $\mathbb{Z}_2$.

CHAPTER 2

Ergodic Properties of p-adic Translation and Multiplication Maps

2.1. Ergodic Properties of Translation by an Integer

In this section, we define the ergodic properties that are the focus of this chapter and are necessary for the definition of orbit equivalence in Chapter 3. We discuss known results and then prove statements for translations by $a \in \mathbb{Z}$. In particular, Theorem 2 and the density argument in the proof of Theorem 3 play key roles in the proof of Theorem 8 in Chapter 3.

A measure μ on a σ-algebra $\mathcal{A}$ is *absolutely continuous* with respect to another measure ν if $\nu(A) = 0$ implies that $\mu(A) = 0$ for all $A \in \mathcal{A}$. The two measures are mutually absolutely continuous or *equivalent* when $\mu(A) = 0$ if and only if $\nu(A) = 0$ for all $A \in \mathcal{A}$. For an endomorphism T on $(X, \mathcal{A}, \mu)$, we define a new measure μT^{-1} by $\mu T^{-1}(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$. If μ and μT^{-1} are equivalent measures, then T is *nonsingular* with respect to μ. If μ and μT^{-1} are not equivalent measures, then T is *singular* with respect to μ. In other words, T is singular with respect to μ if there exists $A \in \mathcal{A}$ such that one of $\mu(A)$ or $\mu T^{-1}(A)$ is zero, but the other is nonzero. We also say that μ is nonsingular (resp. singular) for T when T is nonsingular (resp. singular) with respect to μ.

Radon-Nikodým Theorem *If μ_1 on $(X, \mathcal{A})$ is absolutely continuous with respect to μ_2 on $(X, \mathcal{A})$, then there exists a measurable function $d\mu_1/d\mu_2$ such that*

$$\mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2} d\mu_2, \text{ for all } A \in \mathcal{A}.$$

The function $d\mu_1/d\mu_2$ is called the *Radon-Nikodým derivative* of μ_1 with respect to μ_2. For a nonsingular endomorphism T, the function $d\mu T^{-1}/d\mu$ is called the Radon-Nikodým derivative of T with respect to μ. To aid the study of iterates, Proposition 2 gives a basic fact about compositions of endomorphisms.

PROPOSITION 2. *For endomorphisms T and S on a measure space $(X, \mathcal{B}, \mu)$, if both T and S are nonsingular with respect to μ, then $T \circ S$ is nonsingular with respect to μ.*

PROOF. For a measurable set A, we have $\mu(A) = 0$ if and only if $\mu T^{-1}(A) = 0$ because T is nonsingular with respect to μ. Since S is nonsingular with respect to μ, we have $\mu T^{-1}(A) = \mu(T^{-1}(A)) = 0$ if and only if $\mu S^{-1}(T^{-1}(A)) = 0$. Thus, $\mu(A) = 0$ if and only if $\mu(T \circ S)^{-1}(A) = \mu S^{-1}(T^{-1}(A)) = 0$, so $T \circ S$ is nonsingular with respect to μ. $\square$

Since $\mu \operatorname{Id}^{-1}(A) = \mu(A)$ for all $A \in \mathcal{A}$, the identity map is nonsingular with respect to every measure. If an endomorphism T is invertible, then T is nonsingular with respect to μ if and only if T^{-1} is nonsingular with respect to μ. Finally, an induction argument using Proposition 2 implies that if an invertible endomorphism T is nonsingular with respect to μ, then T^n is nonsingular with respect to μ for all integers n.

It is well-known that the odometer is nonsingular with respect to i.i.d. product measures. For example, Aaronson shows that the odometer on $\prod_{i=0}^{\infty} X_2$ is nonsingular and

ergodic with respect to i.i.d. product measures in [**1**]. Hence, translation by 1 is nonsingular for all i.i.d. product measures on $\mathbb{Z}_p$, by the isomorphism in Section 1.2. For $a \in \mathbb{Z}$, the equality $T_a = T_1^a$ implies that T_a is nonsingular with respect to i.i.d. product measures μ. Therefore, the Radon-Nikodým derivative of T_a with respect to μ exists. Theorem 2 and Corollary 1 give an explicit description of the Radon-Nikodým derivative of T_a^{-1} with respect to μ.

THEOREM 2. *For $a \in \mathbb{N} \subsetneq \mathbb{Z}_p$, we take the translation $T_a : (\mathbb{Z}_p, \mu) \to (\mathbb{Z}_p, \mu)$ and an i.i.d. product measure μ. For n and k in $\mathbb{N}$ such that $0 \leq k < p^n - a$, the Radon-Nikodým derivative is*

$$\frac{d\mu \circ T_a}{d\mu} \equiv \frac{\mu(B_{p^{-n}}(k+a))}{\mu(B_{p^{-n}}(k))} \tag{1}$$

on the ball $B_{p^{-n}}(k)$.

PROOF. For a positive integer a, the proof of Proposition 1 shows that $x+a$ does not have a carry after the nth coordinate for $x \in B_{p^{-n}}(k)$ when $0 \leq k < p^n - a$. In Section 1.3, this fact is used in the construction of T_a by multiple columns. Moreover, this fact is the reason that the Radon-Nikodým derivative is constant on these balls.

We fix $n \geq N = \min\{n \in \mathbb{N} : a \leq p^n\}$ and consider a ball of radius p^{-n} and center $x = \sum_{i=0}^{n-1} x_i p^i < p^n - a$. For ease of notation, we define $y = \sum_{i=0}^{n-1} y_i p^i$ by

$$T_a\left(\sum_{i=0}^{n-1} x_i p^i\right) = \sum_{i=0}^{n-1} y_i p^i < p^n,$$

so $B_{p^{-n}}(x+a) = B_{p^{-n}}(y)$. For an i.i.d. product measure μ defined by a probability vector $(q_0, q_1, \ldots, q_{p-1})$, we use a monotone class argument to show that

$$\frac{d\mu \circ T_a}{d\mu} \equiv \frac{\mu(B_{p^{-n}}(k+a))}{\mu(B_{p^{-n}}(k))} \text{ on } B_{p^{-n}}(x).$$

We let $\mathcal{A}$ be the collection of measurable $A \subset B_{p^{-n}}(x)$ such that (1) holds. For a subball $B = B_{p^{-n-m}}\left(\sum_{i=0}^{n-1} x_i p^i + \sum_{i=n}^{n+m-1} z_i p^i\right)$, we have

$$\begin{aligned} \mu \circ T_a(B) &= \mu(B_{p^{-n-m}}(\sum_{i=0}^{n-1} y_i p^i + \sum_{i=n}^{n+m-1} z_i p^i)) \\ &= \prod_{i=0}^{n-1} q(y_i) \prod_{i=n}^{n+m-1} q(z_i). \end{aligned}$$

Multiplying and dividing by $\prod_{i=0}^{n-1} q(x_i)$ yields

$$\begin{aligned} \mu \circ T_a(B) &= \int_B \frac{\prod_{i=0}^{n-1} q(y_i)}{\prod_{i=0}^{n-1} q(x_i)} d\mu \\ &= \int_B \frac{\mu(B_{p^{-n}}(k+a))}{\mu(B_{p^{-n}}(k))} d\mu. \end{aligned}$$

Thus, $\mathcal{A}$ contains all subballs of $B_{p^{-n}}(x)$. We note that the subballs form a generating algebra for the measurable subsets of $B_{p^{-n}}(x)$. Moreover, $\mathcal{A}$ is a monotone class. Hence, $\mathcal{A}$ contains all measurable subsets of $B_{p^{-n}}(x)$, so equation (1) holds on $B_{p^{-n}}(x)$. □

The equation

$$\frac{d\mu T_{-a}}{d\mu}(x) = \frac{d\mu T_a^{-1}}{d\mu}(x) = \frac{1}{\frac{d\mu T_a}{d\mu}(T_{-a}x)}$$

and Theorem 2 imply the following corollary.

Corollary 1. *For $a \in \mathbb{N}$, we take the translation $T_{-a} : (\mathbb{Z}_p, \mu) \to (\mathbb{Z}_p, \mu)$ and an i.i.d. product measure μ. For n and k in $\mathbb{N}$ such that $a \leq k < p^n$, the Radon-Nikodým*

derivative

$$\frac{d\mu \circ T_{-a}}{d\mu} \equiv \frac{\mu(B_{p^{-n}}(k-a))}{\mu(B_{p^{-n}}(k))}$$

on the ball $B_{p^{-n}}(k)$.

An endomorphism T on $(X, \mathcal{A}, \mu)$ is *ergodic* with respect to μ if $A \in \mathcal{A}$ and $T^{-1}A = A$ imply that $\mu(A) = 0$ or $\mu(X \backslash A) = 0$. We also say that μ is ergodic for T when T is ergodic with respect to μ. An endomorphism T is *totally ergodic* with respect to μ if T^n is ergodic with respect to μ for all $n \in \mathbb{N}$.

There are many known results for ergodicity of the odometer with respect to i.i.d. product measures. For example, it is known that the odometer on $\prod_{i=0}^{\infty} X_2$ is ergodic with respect to i.i.d. product measures, and it is known that the odometer on $\prod_{i=0}^{\infty} X_n$ is not totally ergodic with respect to i.i.d. product measures. These results can be stated for translation by 1, through the isomorphism in Section 1.2. Moreover, conditions for ergodicity are stated for translations on $\mathbb{Z}_p$ with respect to Haar measure in [**4**, **9**, **26**, **27**]. These results are special cases of Theorem 3, which states the conditions for ergodicity in the full generality required for Chapter 3. Moreover, the proof given for Theorem 3 is a density argument that reappears in the proof of Theorem 3.8. We begin with Lemma 2, which states that a set of positive measure is arbitrarily dense in some ball. Lemma 2 is a consequence of the Lebesgue Density Theorem. Since the full strength of the Lebesgue Density Theorem is not needed in what follows, we give a short proof of the lemma.

Lemma 2. *If μ is an i.i.d. measure on $\mathbb{Z}_p$, then we take A to be a measurable subset of $\mathbb{Z}_p$ such that $\mu(A) > 0$. For all $0 < \alpha < 1$, there exists a ball B_α such that*

$$\frac{\mu(A \cap B_\alpha)}{\mu(B_\alpha)} > \alpha.$$

Proof. Since $\alpha < 1$ and $\mu(A) > 0$, we can set $\epsilon = \mu(A)(1-\alpha) > 0$. Since μ is a regular measure, there exists an open set U such that $A \subset U$ and $\mu(U \backslash A) < \epsilon$. Then

$$\begin{aligned} \mu(U) &= \mu(U \backslash A) + \mu(A \cap U) \\ &< \epsilon + \mu(A \cap U). \end{aligned}$$

Since $A \subset U$, we have $\mu(A) \leq \mu(U)$, so

$$\frac{\mu(A \cap U)}{\mu(U)} > \frac{\mu(U) - \epsilon}{\mu(U)} = 1 - \frac{\epsilon}{\mu(U)}.$$

By the definition of ϵ, we have

$$\frac{\mu(A \cap U)}{\mu(U)} \geq 1 - \frac{\mu(A)(1-\alpha)}{\mu(A)} = \alpha.$$

The open set U is a countable union of disjoint balls, $U = \bigcup_{i=0}^{\infty} B_i$. Since $\mu(A \cap U)/\mu(U) > \alpha$, there exists a ball B_i of positive measure such that $\mu(A \cap B_i)/\mu(B_i) > \alpha$. □

In Theorem 3, we prove a condition equivalent to ergodicity. An invertible, nonsingular endomorphism T on $(X, \mathcal{A}, \mu)$ is ergodic with respect to μ if for all sets A_1 and A_2 of positive measure, there exists $n \in \mathbb{N}$ such that $\mu(T^{-n} A_1 \cap A_2) > 0$ [**28**]. In the proof, we consider sets A_1 and A_2 of positive measure. Each of these sets is arbitrarily dense in some ball. In terms of the tower construction, if we cut and stack enough times, then parts of each ball end up in the same stack. Since the Radon-Nikodým derivative is constant on each level of the stack except the top, the density of A_1 in the one ball does not change as it is mapped up the stack by the translation to the ball in which A_2 is dense.

THEOREM 3. *For an i.i.d. product measure μ on $\mathbb{Z}_p$ and an integer $a \in \mathbb{Z} \subsetneq \mathbb{Z}_p$, the translation T_a is ergodic with respect to μ if and only if $a \in \mathbb{Z}_p^\times$.*

PROOF. Since T_a is invertible, the transformation T_a is ergodic with respect to μ if and only if T_{-a} is ergodic with respect to μ. Hence, we can assume that $a \geq 0$, without loss of generality.

If a is not a p-adic unit, then $a_0 = 0$. For all $x \in B_{p^{-1}}(0)$, we have $T_a^{-1}(x) = x - a = 0 \mod p$, so $T_a^{-1}(x) \in B_{p^{-1}}(0)$. Thus, $B_{p^{-1}}(0)$ is an invariant set for T_a. Since $0 < \mu(B_{p^{-1}}(0)) < 1$, the translation T_a is not ergodic with respect to μ.

If a is a p-adic unit, then we begin by fixing $N \in \mathbb{N}$ such that $0 < a < p^N$. Next, we set $\beta = \min_{0 \leq i < p^N} \mu(B_{p^{-N}}(i))$ and $\alpha = 1 - \beta/4$. Given two measurable sets A_1 and A_2 of positive measure, we find an $n \in \mathbb{N}$ such that $\mu(T^{-n}A_1 \cap A_2) > 0$. By Lemma 2, there exist balls B_i such that

$$\frac{\mu(A_i \cap B_i)}{\mu(B_i)} > \alpha, \text{ for } i = 1,\ 2. \tag{2}$$

Since each ball is a disjoint union of balls of a smaller radius, we can assume that B_1 and B_2 are balls of the same radius. There exist $k \in \mathbb{N}$ and $0 \leq b_i < p^k$, such that $B_i = B_{p^{-k}}(b_i)$. The set B_i has a partition $\left\{B_{p^{-k-N}}(b_i + j \cdot p^k)\right\}_{j=0}^{p^N-1}$. Since $a \in \mathbb{Z}_p^\times$ and a is an integer, we have $\gcd(p, a) = 1$, so $\gcd(p^k, a) = 1$. Thus, p^k is invertible modulo a. Therefore, there exist integers c_i such that $0 \leq c_i < a < p^N$ such that $b_i + c_i \cdot p^k \equiv 0 \mod a$ for $i = 1, 2$. In terms of the tower construction, this equivalence means that $B_{p^{-k-N}}(b_i + c_i \cdot p^k)$ is in the first stack of a tower—the stack with base $B_{p^{-k-N}}(0)$.

Without loss of generality, we suppose that $b_2 + c_2 \cdot p^k \geq b_1 + c_1 \cdot p^k$. Hence, there exists an integer $0 \leq m < a$ such that $m \cdot a = (b_2 + c_2 \cdot p^k) - (b_1 + c_1 \cdot p^k)$. Thus

$b_1 + c_1 \cdot p^k + i \cdot a \leq b_2 + c_2 \cdot p^k - a < p^{k+N} - a$ for all $0 \leq i \leq m-1$. By Theorem 2, this implies that $d\mu T_a/d\mu$ is constant on $B_{p^{-k-N}}(b_1 + c_1 \cdot p^k + i \cdot a)$ for all $0 \leq i < m$. Since the Radon-Nikodým derivative $d\mu T_a^m/d\mu$ is constant on $B_{p^{-k}}(b_1 + c_1 \cdot p^k)$, we have

$$\begin{aligned} (3) \qquad \frac{\mu(T_a^m(A_1) \cap B_{p^{-k-N}}(b_2 + c_2 p^k))}{\mu(B_{p^{-k-N}}(b_2 + c_2 p^k))} &= \frac{\mu(T_a^m(A_1 \cap B_{p^{-k-N}}(b_1 + c_1 p^k)))}{\mu(T_a^m(B_{p^{-k-N}}(b_1 + c_1 p^k)))} \\ &= \frac{\mu(A_1 \cap B_{p^{-k-N}}(b_1 + c_1 p^k))}{\mu(B_{p^{-k-N}}(b_1 + c_1 p^k))}. \end{aligned}$$

By (2), the choice of α, and

$$\begin{aligned} \mu(B_{p^{-k-N}}(b_i + c_i \cdot p^k)) &= \mu(B_{p^{-N}}(c_i))\mu(B_{p^{-k}}(b_i)) \\ &\geq \beta\mu(B_i), \end{aligned}$$

it follows that

$$(4) \qquad \mu(A_i \cap B_{p^{-k-N}}(b_i + c_i p^k)) > 0.75\mu(B_{p^{-k-N}}(b_i + c_i p^k)).$$

Then (3) and (4) imply that $\mu(T_a^m A_1 \cap A_2) > 0.5 > 0$. Hence, T_a is ergodic with respect to μ. □

2.2. Ergodic Properties of Translation by a Rational Number

Since Haar measure is translation-invariant, translations are certainly nonsingular with respect to Haar measure. In Section 2.1, we extend known results for the odometer to show that translation by an integer is nonsingular with respect to i.i.d. product measures. Moreover, any translation can be approximated by translations by an integer. For $a = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p$, we define the partial sums $s_n = \sum_{i=0}^{n-1} a_i p^i \in \mathbb{Z}$. Then $s_n \to a$ as

$n \to \infty$, from which it follows that $T_{s_n}(x)$ converges uniformly in x to $T_a(x)$. Based on this convergence, we might guess that μ and μT_a^{-1} are mutually absolutely continuous. However, this guess would be incorrect. In fact, translation by any rational number, other than an integer, is singular with respect to any product measure, other than Haar measure. We call $a \in \mathbb{Q} \setminus \mathbb{Z}$ a *nonintegral rational number*. This singularity result is the main theorem of this section.

THEOREM 4. *If $a \in \mathbb{Z}_p$ is a nonintegral rational number, and if μ is an i.i.d. product measure other than Haar measure, then $T_a : \mathbb{Z}_p \to \mathbb{Z}_p$ is singular with respect to μ.*

Before giving the proof, we discuss the distinguishing characteristics of integers and rational numbers in $\mathbb{Z}_p$. These characteristics are then used in the proof of a technical result (Proposition 3). After giving examples that illustrate the proof of Proposition 3, we review the Birkhoff Ergodic Theorem and the shift map. Finally, we use these elements to give a proof of Theorem 4.

We recall from Section 1.3 that the Euclidean algorithm is the tool used to express a rational number as a formal power series. A positive integer has a finite expansion, so its coordinates eventually end in repeating zeros. Similarly, the coordinates of a negative integer eventually end in repeating $p-1$'s. In general, an element in $\mathbb{Z}_p$ is rational if and only if there is eventually a block of repeating coordinates [**51**,**57**]. Thus, if $a = \sum_{i=0}^{\infty} a_i p^i$ is also an element of $\mathbb{Q}$, then there exist integers l and r such that $a_{i+r} = a_i$ for all $i \geq l$. Although l and r are not unique, there are unique minimal choices for each. For a fixed rational number $a \in \mathbb{Z}_p$ and a fixed choice of l and r, we call $\sum_{i=0}^{l-1} a_i p^i$ the leading part of a, $c = \sum_{i=0}^{r-1} a_{l+i} p^i$ the repeating segment of a, and $A = \sum_{i=0}^{\infty} c p^{ir}$ the repeating part of a.

The following technical proposition gives the measurable sets that define characteristic functions in the proof of Theorem 4.

PROPOSITION 3. *Assuming the hypotheses from Theorem 4 and notation from the previous paragraph, there exists a ball $B \subset \mathbb{Z}_p$ such that*

$$\mu(B) > \mu(T_A B) + \mu(T_{1+A} B). \tag{5}$$

PROOF. Since the repeating segment has length r, we begin by considering balls of radius p^{-r}. Since translations are invertible isometries, we have $T_A(B_{p^{-r}}(x)) = B_{p^{-r}}(T_A(x))$. Since a ball of radius p^{-r} is determined by the first r coordinates of its center, we also have $B_{p^{-r}}(T_A(x)) = B_{p^{-r}}(c + x)$. Similarly, we have $T_{1+A}(B_{p^{-r}}(x)) = B_{p^{-r}}(1 + c + x)$. For a ball $B_{p^{-r}}(x)$ that has maximal measure among the balls of radius p^{-r}, we set

$$\begin{aligned} M &= \mu(B_{p^{-r}}(x)), \\ m_0 &= \mu(B_{p^{-r}}(c + x)), \text{ and} \\ m_1 &= \mu(B_{p^{-r}}(1 + c + x)). \end{aligned}$$

Using this notation, we define the following three conditions on the ball $B_{p^{-r}}(x)$:

(i) $M > m_0$, $M > m_1$, and $x = p^r - c - 1$,

(ii) $M > m_0$ and $0 \leq x < p^r - c - 1$, or

(iii) $M > m_1$ and $p^r - c - 1 < x < p^r$.

First, we show that if there is a ball of maximal measure satisfying one of these conditions, then we can find a ball satisfying (5). Next, we consider various cases for the measure

μ, showing that in each case we can find at least one ball of maximal measure satisfying one of the three conditions.

If there is a ball of maximal measure $B_{p^{-r}}(x)$ that satisfies Condition (i), then we define $m = \max\{m_0, m_1\}$ and fix an integer

$$N > \log_{M/m} 2.$$

The ball $B = B_{p^{-rN}}(\sum_{i=0}^{N-1} xp^{ir})$ has measure

$$\mu(B) = \prod_{i=0}^{N-1} \mu(B_{p^{-r}}(x)) = M^N.$$

If $x = p^r - c - 1$, then $c + x = p^r - 1 < p^r$, so adding the first r coefficients of A to the first r coefficients of x does not result in a carry. Thus, each of the following groups of coefficients taken r at a time from $A + x$ are the same as the first group of r coefficients of $c + x$, so $\mu(T_A B) = m_0^N$. Similarly, we have $1 + c + x = p^r$, so adding the first r coefficients of $1 + A$ to the first r coefficients of x does result in a carry. Thus, each of the next groups of coefficients taken r at a time from $1 + A + x$ are the same as the first r coefficients of $1 + c + x$, so $\mu(T_{1+A} B) = m_1^N$. Finally, the choice of N implies that

$$\begin{aligned}
\mu(B) &= M^N \\
&= \left(\frac{M}{m}\right)^N m^N \\
&> 2m^N \\
&\geq m_0^N + m_1^N \\
&= \mu(T_A B) + \mu(T_{1+A}(B)),
\end{aligned}$$

so (5) is satisfied.

If there is a ball of maximal measure $B_{p^{-r}}(x)$ that satisfies Condition (ii), then we fix an integer

$$N > \log_{M/m_0} \frac{m_0 + m_1}{m_0}.$$

Again, the ball $B = B_{p^{-rN}}(\sum_{i=0}^{N-1} xp^{ir})$ has measure M^N. If $x < p^r - c - 1$, then $c + x < p^r$, so adding the first r coefficients of A to the first r coefficients of x does not result in a carry. Thus, it again follows that $\mu(T_A B) = m_0^N$. Similarly, we have $1 + c + x < p^r$, so adding the first r coefficients of $1 + A$ to the first r coefficients of x does not result in a carry. Thus, each of the following groups of coefficients taken r at a time from $1 + A + x$ are the same as the first r coefficients of $c + x$, so $\mu(T_{1+A} B) = m_1 m_0^{N-1}$. Finally, the choice of N implies that

$$\begin{aligned}
\mu(B) &= M^N \\
&= \left(\frac{M}{m_0}\right)^N m_0^N \\
&> \frac{m_0 + m_1}{m_0} m_0^N \\
&= m_0^N + m_1 m_0^{N-1} \\
&= \mu(T_A B) + \mu(T_{1+A}(B)),
\end{aligned}$$

so (5) is satisfied.

A similar argument proves that Condition (iii) implies that (5) is satisfied. The only changes are switching m_0 and m_1, switching the defining inequalities for x, and observing that the additions do result in carries after each group of coefficients taken r at a time.

So far, we know that each of the three conditions on a ball of radius p^{-r} implies that we can find a ball, possibly of smaller radius, that satisfies (5). Now, we show that it is always possible find a ball of radius p^{-r} that satisfies one of the three conditions. We split the remainder of the proof into cases that depend on the measure μ. Since μ is not Haar measure, it is determined by a probability vector $(q_0, q_1, \ldots, q_{p-1})$ such that the weights q_i are not all equal. We let $Q = \max_i q_i$ be the largest weight. Either the probability vector that defines μ has a unique largest weight or it does not. If there is a unique largest weight, then a ball satisfying one of the three conditions has an explicit description. We now prove this case.

If there exists a unique largest weight, then there exists a weight q_j such that $q_j = Q$ and $q_i < q_j$ for all $i \neq j$. Then $B_{p^{-r}}(\sum_{i=0}^{r-1} jp^i)$ is the unique ball of radius p^{-r} that has maximal measure. If a is a positive integer or zero, then a ends in repeating zeros, which gives $A = 0$. If a is a negative integer, then a ends in repeating $p-1$'s, which gives $A = -1$. By the assumption that a is not an integer, A is not equal to 0 or -1. Since A is not zero, $B_{p^{-r}}(A + \sum_{i=0}^{r-1} jp^i)$ is not equal to $B_{p^{-r}}(\sum_{i=0}^{r-1} jp^i)$. Uniqueness then implies that $M > m_0$. Similarly, since A is also not -1, $B_{p^{-r}}(1 + A + \sum_{i=0}^{r-1} jp^i)$ is not equal to $B_{p^{-r}}(\sum_{i=0}^{r-1} jp^i)$. Again, uniqueness implies that $M > m_1$. Thus, $B_{p^{-r}}(\sum_{i=0}^{r-1} jp^i)$ satisfies Condition (i) if $\sum_{i=0}^{r-1} jp^i = p^r - c - 1$, Condition (ii) if $\sum_{i=0}^{r-1} jp^i < p^r - c - 1$, or Condition (iii) if $\sum_{i=0}^{r-1} jp^i > p^r - c - 1$.

If $p = 2$ and μ is not Haar measure, then the two weights are not equal. Thus, there is a unique largest weight and the proof of this case is complete. Thus, we can assume that $p \geq 3$ for the remainder of the proof of the proposition.

If there is not a unique largest weight, then we let I be the set of $i \in X_p$ such that $q(i) = Q$ and let k be the cardinality of I. If there is not a unique largest weight,

then $k > 1$. Since μ is not Haar measure, we must also have $k < p$. Since we have k possibilities for maximal coefficients and since a ball of radius p^{-r} is determined by r coefficients, there are k^r distinct balls of radius p^{-r} of maximal measure. We observe that p does not divide k^r, because p is prime and $1 < k < p$. By not requiring that r is the minimal period, we can assume that $r \geq 2$. If $k \geq 2$ and $r \geq 2$, then $k^r \geq 2k$. Thus, either it is the case that

$$\mathcal{A}_0 = \{B_{p^{-r}}(x) : 0 \leq x < p^r - c - 1\}$$

contains at least k balls of maximal measure, or it is the case that

$$\mathcal{A}_1 = \{B_{p^{-r}}(x) : p^r - c - 1 < x < p^r - 1\}$$

contains at least k balls of maximal measure.

Before we consider these two cases, we prove one more fact. For the collection $\mathcal{A}_i$, we suppose that for each $j \in I$ there exists a ball $B_{p^{-r}}(x_j)$ in $\mathcal{A}_i$ such that $x_j = j \mod p$ and $T_{i+c}(B_{p^{-r}}(x_j)) = B_{p^{-r}}(T_{i+c}(x_j))$ has maximal measure. We define a group homomorphism $T_{i+c} \mod p$ on $\mathbb{F}_p$ by $k \mapsto k+i+c \mod p$. If a ball has maximal measure, then the first coordinate must also have maximal weight. Thus, the orbit of each $j \in I$ under $T_{i+c} \mod p$ is contained in I. Since $T_{i+c} \mod p$ is a group homomorphism of $\mathbb{F}_p$, the minimal period of each $j \in I$ divides p. Since μ is not Haar measure, I does not contain all indices. Hence, the minimal period is not p, so every $j \in I$ is fixed point. Since $j + (i + c) = j \mod p$, it follows that p divides $i + c$.

The previous paragraph shows that if there are k maximal balls in $\mathcal{A}_i$ that map to maximal balls under T_{i+c}, such that every maximal index is equal modulo p to the

center of one of these balls, then $i + c$ is divisible by p. For future reference, we give the contrapositive of this statement. For a collection of k maximal balls in $\mathcal{A}_i$ such that every maximal index is equal modulo p to the center of one of the balls, if $i + c$ is not divisible by p, then one of the balls in $\mathcal{A}_i$ does not map to a maximal ball under T_{i+c}. With these observations, we proceed to prove the last two cases.

First, we suppose that $\mathcal{A}_0$ contains all balls of maximal measure. If $T_c^n(B_{p^{-r}}(x)) = B_{p^{-r}}(T_c^n(x))$ is a ball of maximal measure for all integers n, then $x \mod p^r$ is periodic under $T_c \mod p^r$. Since $T_c \mod p^r$ is a group homomorphism of the finite group $\mathbb{F}_{p^r}$, the period must be divisible by p. There are k^r balls of maximal measure, with k^r distinct centers x. Since p does not divide k^r, there must be an x that is not periodic under $T_c \mod p^r$. Thus, there exists an $x < p^r - c - 1$ such that $\mu(B_{p^{-r}}(T_c^{n-1}x))$ is maximal but $\mu(B_{p^{-r}}(T_c^n(x)))$ is not, that is, $M > m_0$. Since $B_{p^{-r}}(T_c^{n-1}x)$ has maximal measure, our initial assumption implies that it must be in $\mathcal{A}_0$. Thus, $B_{p^{-r}}(T_c^{n-1}x)$ is a ball satifying Condition (ii).

Next, we suppose that $\mathcal{A}_0$ contains at least k balls of maximal measure, but none of the balls satisfy Condition (ii). It follows from the previous paragraph that there are balls of maximal measure $B_{p^{-r}}(x)$ such that $p^r - c - 1 \leq x < p^r$. We show that one of these balls must satisfy Condition (iii). If $\mathcal{A}_0$ contains at least k balls of maximal measure but none of them satisfy Condition (ii), then we have k maximal balls that map to maximal balls under T_c, such that every maximal index is equal modulo p to the center of one of the balls. Thus, c is divisible by p. We now argue that $B_{p^{-r}}(p^r - c - 1)$ cannot be the only ball of maximal measure with center x such that $p^r - c - 1 \leq x < p^r$. Since c is divisible by p, it follows that $x = p^r - c - 1 = p - 1 \mod p$. Thus, if $B_{p^{-r}}(p^r - c - 1)$ has maximal measure, then $p-1$ must have maximal weight. Thus, $B_{p^{-r}}(p^r - 1) = B_{p^{-r}}(\sum_{i=0}^{r-1}(p-1)p^i)$

also has maximal measure. We have shown that whether or not $p-1$ has maximal weight, there exists a ball of maximal measure in $\mathcal{A}_1$. Suppose that this ball has center $\sum_{i=0}^{r-1} x_i p^i$. For all $j \in I$, the ball with center $j + \sum_{i=1}^{r-1} x_i p^i$ will also have maximal measure. Since c is a multiple of p, if $p^r - c - 1 < \sum_{i=0}^{r-1} x_i p^i < p^r$, then it is also true that $p^r - c - 1 < j + \sum_{i=1}^{r-1} x_i p^i < p^r$. Thus, every maximal index is equal mod p to the center of a ball in $\mathcal{A}_1$. Since p divides c, it cannot divide $1+c$. This implies that there must be a maximal ball $B_{p^{-r}}(x)$ such that $M > m_1$ and $p^r - c - 1 < x < p^r$, so we have satisfied Condition (iii).

If it is the case that $\mathcal{A}_1$ contains at least k balls of maximal measure, then the argument is similar to the case for $\mathcal{A}_0$. The only changes are switching $\mathcal{A}_0$ and $\mathcal{A}_1$, c and $c+1$, Conditions (ii) and (iii), and the defining inequalities for x. $\square$

Before giving the proof of Theorem 4, we illustrate Proposition 3 with three examples.

Example 1. For the i.i.d. product measure $\mu = \prod_{i=0}^{\infty} \left\{\frac{2}{11}, \frac{3}{11}, \frac{2}{11}, \frac{2}{11}, \frac{2}{11}\right\}$ on $\mathbb{Z}_5$, we consider $a = A = \sum_{n=0}^{\infty} 3 \cdot 5^n = -3/4$ and take $c = 3$. Since $q_1 = 3/11$ is the unique greatest weight, we consider the ball $B_{p^{-1}}(1)$. Since

$$\mu(B_{p^{-1}}(1)) = \frac{3}{11} > \frac{2}{11} = \mu(B_{p^{-1}}(T_3(1))) = \mu(B_{p^{-1}}(T_4(1)))$$

and $1 = 5 - 3 - 1$, the ball $B_{p^{-1}}(1)$ satisfies Condition (i). If we take $N = 2 > \log_{3/2} 2$, then we have

$$\begin{aligned}
\mu(B_{p^{-2}}(1 + 1 \cdot 5)) &= \frac{3}{11} \cdot \frac{3}{11} = \frac{9}{121}, \\
\mu(B_{p^{-2}}(T_A(1 + 1 \cdot 5))) &= \mu(B_{p^{-2}}(3 + 3 \cdot 5 + \cdots)) \\
&= \frac{2}{11} \cdot \frac{2}{11} = \frac{4}{121}, \text{ and} \\
\mu(B_{p^{-2}}(T_{1+A}(1 + 1 \cdot 5))) &= \mu(B_{p^{-2}}(4 + 4 \cdot 5 + \cdots)) \\
&= \frac{2}{11} \cdot \frac{2}{11} = \frac{4}{121},
\end{aligned}$$

so $B_{p^{-2}}(1 + 1 \cdot 5)$ satisfies (5).

FIGURE 2.1. Tables for Examples 2 and 3.

Example 2

x	$T_3(x)$ $0 \le x < 21$	$T_4(x)$ $21 < x < 24$
$0+0\cdot 5$	$3+0\cdot 5$	
$1+0\cdot 5$	$4+0\cdot 5$	
$2+0\cdot 5$	$0+1\cdot 5$	
$3+0\cdot 5$	$\mathbf{1+1\cdot 5}$	
$4+0\cdot 5$	$\mathbf{2+1\cdot 5}$	
$0+1\cdot 5$	$\mathbf{3+1\cdot 5}$	
$\mathbf{1+1\cdot 5}$	$4+1\cdot 5$	
$\mathbf{2+1\cdot 5}$	$0+2\cdot 5$	
$\mathbf{3+1\cdot 5}$	$\mathbf{1+2\cdot 5}$	
$4+1\cdot 5$	$\mathbf{2+2\cdot 5}$	
$0+2\cdot 5$	$\mathbf{3+2\cdot 5}$	
$\mathbf{1+2\cdot 5}$	$4+2\cdot 5$	
$\mathbf{2+2\cdot 5}$	$0+3\cdot 5$	
$\mathbf{3+2\cdot 5}$	$\mathbf{1+3\cdot 5}$	
$4+2\cdot 5$	$\mathbf{2+3\cdot 5}$	
$0+3\cdot 5$	$\mathbf{3+3\cdot 5}$	
$\mathbf{1+3\cdot 5}$	$4+3\cdot 5$	
$\mathbf{2+3\cdot 5}$	$0+4\cdot 5$	
$\mathbf{3+3\cdot 5}$	$1+4\cdot 5$	
$4+3\cdot 5$	$2+4\cdot 5$	
$0+4\cdot 5$	$3+4\cdot 5$	
$1+4\cdot 5$		
$2+4\cdot 5$		$1+0\cdot 5$
$3+4\cdot 5$		$2+0\cdot 5$
$4+4\cdot 5$		$3+0\cdot 5$

Example 3

x	$T_{15}(x)$ $0 \le x < 9$	$T_{16}(x)$ $9 < x < 24$
$\mathbf{0+0\cdot 5}$	$\mathbf{0+3\cdot 5}$	
$\mathbf{1+0\cdot 5}$	$\mathbf{1+3\cdot 5}$	
$2+0\cdot 5$	$2+3\cdot 5$	
$\mathbf{3+0\cdot 5}$	$\mathbf{3+3\cdot 5}$	
$\mathbf{4+0\cdot 5}$	$\mathbf{4+3\cdot 5}$	
$\mathbf{0+1\cdot 5}$	$\mathbf{0+4\cdot 5}$	
$\mathbf{1+1\cdot 5}$	$\mathbf{1+4\cdot 5}$	
$2+1\cdot 5$	$2+4\cdot 5$	
$\mathbf{3+1\cdot 5}$	$\mathbf{3+4\cdot 5}$	
$\mathbf{4+1\cdot 5}$		
$0+2\cdot 5$		$\mathbf{1+0\cdot 5}$
$1+2\cdot 5$		$2+0\cdot 5$
$2+2\cdot 5$		$\mathbf{3+0\cdot 5}$
$3+2\cdot 5$		$\mathbf{4+0\cdot 5}$
$4+2\cdot 5$		$\mathbf{0+1\cdot 5}$
$\mathbf{0+3\cdot 5}$		$\mathbf{1+1\cdot 5}$
$\mathbf{1+3\cdot 5}$		$2+1\cdot 5$
$2+3\cdot 5$		$\mathbf{3+1\cdot 5}$
$\mathbf{3+3\cdot 5}$		$\mathbf{4+1\cdot 5}$
$\mathbf{4+3\cdot 5}$		$0+2\cdot 5$
$\mathbf{0+4\cdot 5}$		$1+2\cdot 5$
$\mathbf{1+4\cdot 5}$		$2+2\cdot 5$
$2+4\cdot 5$		$3+2\cdot 5$
$\mathbf{3+4\cdot 5}$		$4+2\cdot 5$
$\mathbf{4+4\cdot 5}$		$\mathbf{0+3\cdot 5}$

EXAMPLE 2. For the i.i.d. product measure $\mu = \prod_{i=0}^{\infty} \left\{\frac{1}{6}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{6}\right\}$ on $\mathbb{Z}_5$, we consider $a = A = \sum_{n=0}^{\infty}(3+0\cdot 5)5^{2n} = -1/8$ and take $c = 3+0\cdot 5$. The centers of the balls of radius 5^{-2} are listed in the first column of the left table in Figure 2.1. The second column contains the images of the balls in $\mathcal{A}_0$ under the translation T_c. Finally, the third column contains the images of the balls in $\mathcal{A}_1$ under the translation T_{1+c}. The centers of maximal balls are in bold.

Since all balls of maximal measure have center less than $p^r - c - 1 = 25 - 3 - 1 = 21$, all balls of maximal measure are in $\mathcal{A}_0$. There are two balls of maximal measure that map to other balls of maximal measure under T_c. These correspond to rows that have bold entries in both the first and second column. However, there are seven balls of maximal measure that map to balls of smaller measure under T_c. These correspond to rows that have a bold entry in the first column but not in the second column. Thus, we have seven balls that satisfy Condition (ii).

For example, we could pick $B_{5^{-2}}(1 + 1 \cdot 5)$. Taking $N = 3 > \log_{4/3} 2$, we make the following additions with carries:

$$\begin{array}{rl} & 1 + 1 \cdot 5 + 1 \cdot 5^2 + 1 \cdot 5^3 + 1 \cdot 5^4 + 1 \cdot 5^5 \\ + & \underline{3 + 0 \cdot 5 + 3 \cdot 5^2 + 0 \cdot 5^3 + 3 \cdot 5^4 + 0 \cdot 5^5 + \cdots} \\ & 4 + 1 \cdot 5 + 4 \cdot 5^2 + 1 \cdot 5^3 + 4 \cdot 5^4 + 1 \cdot 5^5 + \cdots \end{array}$$

$$\begin{array}{rl} & 1 \quad 1 \\ & 1 + 1 \cdot 5 + 1 \cdot 5^2 + 1 \cdot 5^3 + 1 \cdot 5^4 + 1 \cdot 5^5 \\ + & \underline{3 + 0 \cdot 5 + 3 \cdot 5^2 + 0 \cdot 5^3 + 3 \cdot 5^4 + 0 \cdot 5^5 + \cdots} \\ & 0 + 2 \cdot 5 + 4 \cdot 5^2 + 1 \cdot 5^3 + 4 \cdot 5^4 + 1 \cdot 5^5 + \cdots . \end{array}$$

By examining the first six coefficients of the center of each ball, we find

$$\begin{aligned}
\mu\left(B_{5^{-6}}\left(\sum_{i=0}^{5} 1\cdot 5^i\right)\right) &= q_1^6 = \left(\frac{2}{9}\right)^6, \\
\mu\left(B_{5^{-6}}\left(A+\sum_{i=0}^{5} 1\cdot 5^i\right)\right) &= (q_4q_1)^3 = \frac{1}{27^3}, \text{ and} \\
\mu\left(B_{5^{-6}}\left(1+A+\sum_{i=0}^{5} 1\cdot 5^i\right)\right) &= q_0q_2(q_4q_1)^2 = \frac{1}{27^3}.
\end{aligned}$$

Since $\left(\frac{2}{9}\right)^6 > \frac{1}{27^3} + \frac{1}{27^3}$, the ball $B_{5^{-6}}(\sum_{i=0}^{5} 1\cdot 5^i)$ satisfies (5).

Example 3. For the i.i.d. product measure $\mu = \prod_{i=0}^{\infty}\left\{\frac{3}{14}, \frac{3}{14}, \frac{1}{7}, \frac{3}{14}, \frac{3}{14}\right\}$ on $\mathbb{Z}_5$, we consider $a = A = \sum_{n=0}^{\infty}(0+3\cdot 5)5^{2n} = -5/8$ and take $c = 0+3\cdot 5$.

The centers of balls of radius 5^{-2} are listed in the first column of the left table in Figure 2.1. The second column contains the images of the balls in $\mathcal{A}_0$ under the translation T_c. Finally, the third column contains the images of the balls in $\mathcal{A}_1$ under the translation T_{1+c}. The centers of maximal balls are in bold.

In this example, all balls of maximal measure with center less than $p^r - c - 1 = 25 - 15 - 1 = 9$ map to another ball of maximal measure under T_c. As we expect from the proof of Proposition 3, $p = 5$ divides $c = 15$. On the other hand, there are five balls of maximal measure that map to balls of smaller measure under T_{1+c}. These correspond to rows that have a bold entry in the first column but not in the third column. Thus, we have five balls that satisfy Condition (iii).

For example, we could pick $B_{5^{-2}}(1+3\cdot 5)$. Taking $N=3>\log_{3/2}(5/2)$, we make the following additions with carries:

$$
\begin{array}{rl}
 & \phantom{1+3\cdot 5+{}}1 \qquad\qquad\qquad 1 \\
 & 1+3\cdot 5+1\cdot 5^2+3\cdot 5^3+1\cdot 5^4+3\cdot 5^5 \\
+ & \underline{0+3\cdot 5+0\cdot 5^2+3\cdot 5^3+0\cdot 5^4+3\cdot 5^5+\cdots} \\
 & 1+1\cdot 5+2\cdot 5^2+1\cdot 5^3+2\cdot 5^4+1\cdot 5^5+\cdots
\end{array}
$$

$$
\begin{array}{rl}
 & 1 \qquad\qquad 1 \qquad\qquad\qquad 1 \\
 & 1+3\cdot 5+1\cdot 5^2+3\cdot 5^3+1\cdot 5^4+3\cdot 5^5 \\
+ & \underline{0+3\cdot 5+0\cdot 5^2+3\cdot 5^3+0\cdot 5^4+3\cdot 5^5+\cdots} \\
 & 2+1\cdot 5+2\cdot 5^2+1\cdot 5^3+2\cdot 5^4+1\cdot 5^5+\cdots .
\end{array}
$$

By examining the first six coefficients of the center of each ball, we find

$$
\begin{aligned}
\mu\left(B_{5^{-6}}\left(\sum_{i=0}^{2}(1+3\cdot 5)5^{2i}\right)\right) &= q_1^3q_3^3=\left(\frac{3}{14}\right)^6, \\
\mu\left(B_{5^{-6}}\left(A+\sum_{i=0}^{2}(1+3\cdot 5)5^{2i}\right)\right) &= q_1^2(q_2q_1)^2=\left(\frac{3}{14}\right)^2\left(\frac{3}{98}\right)^2, \text{ and} \\
\mu\left(B_{5^{-6}}\left(1+A+\sum_{i=0}^{2}(1+3\cdot 5)5^{2i}\right)\right) &= (q_2q_1)^3=\left(\frac{3}{98}\right)^3.
\end{aligned}
$$

Since $\left(\frac{3}{14}\right)^6>\left(\frac{3}{14}\right)^2\left(\frac{3}{98}\right)^2+\left(\frac{3}{98}\right)^3$, the ball $B_{5^{-6}}(\sum_{i=0}^{5}1\cdot 5^i)$ satisfies (5).

Next, we review the Birkhoff Ergodic Theorem and the definition of the shift. The proof of Theorem 4 is done when we apply the Birkhoff Ergodic Theorem to an iterate of the shift and characteristic functions.

Birkhoff Ergodic Theorem. *If $(X, \mathcal{B}, \mu)$ is a probability space, $S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and measure-preserving, and $f \in L^1(\mu)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(S^i x) = \int_X f d\mu$$

almost everywhere.

The one-sided *shift* is an endomorphism $\sigma : \prod_{i=0}^{\infty} X_n \to \prod_{i=0}^{\infty} X_n$, which is defined by $(\sigma x)_i = x_{i+1}$. We note that the shift is not invertible. The shift on a product space $(\prod_{i=0}^{\infty} X_n, \mathcal{N}, \nu)$ is a standard example of a system where the transformation is measure-preserving and ergodic with respect to the measure. The shift is also totally ergodic with respect to i.i.d. product measures. For a fixed prime p, the isomorphism in Section 1.2 defines a shift σ on $\mathbb{Z}_p$, which acts by $\sigma(\sum_{i=0}^{\infty} x_i p^i) = \sum_{i=0}^{\infty} x_{i+1} p^i$. By the isomorphism, the shift σ is measure-preserving and totally ergodic with respect to i.i.d. product measures. Thus, for all $n \in \mathbb{N}$, the iterate σ^n satisfies the conditions on S in the Birkhoff Ergodic Theorem. We finally give the proof of Theorem 4.

Proof of Theorem 4. Assuming that $a \in \mathbb{Z}_p$ is a nonintegral rational number and that μ is an i.i.d. product measure other than Haar measure, our goal is to show that T_a is singular with respect to μ. Since T_a is invertible, we do this by finding a set X such that $\mu(X) > 0$ but $\mu(T_a X) = 0$.

If A is the repeating part of a, then Proposition 3 gives a ball B of radius p^{-rN} for some $N \in \mathbb{N}$ such that $\mu(B) > \mu(T_A B) + \mu(T_{1+A} B)$. Since B and $T_A B \cup T_{1+A} B$ are measurable sets, the characteristic functions $\mathbb{1}_B$ and $\mathbb{1}_{T_A B \cup T_{1+A} B}$ are in $L^1(\mu)$. Since the shift σ is totally ergodic and measure-preserving with respect to the i.i.d. product measure μ, the iterate σ^{rN} is ergodic and measure-preserving with respect to μ. By the Birkhoff Ergodic Theorem, the sets

$$\begin{aligned} X &= \left\{ z \in \mathbb{Z}_p : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_B(\sigma^{l+rNi} z) = \mu(B) \right\} \text{ and} \\ Y &= \left\{ z \in \mathbb{Z}_p : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{T_A B \cup T_{1+A} B}(\sigma^{l+rNi} z) = \mu(T_A B \cup T_{1+A} B) \right\} \end{aligned}$$

have full measure.

For $x \in X$, if $\sigma^{l+rNi} x \in B$, then there are two possibilities for $\sigma^{l+rNi} T_a x$. If adding a to x does not result in a carry after the $l + rNi - 1$st coordinate, then $\sigma^{l+rNi} T_a x \in T_A B$. If adding a to x does result in a carry after the $l + rNi - 1$st coordinate, then $\sigma^{l+rNi} T_a x \in T_{1+A} B$. In either case, $\sigma^{l+rNi} T_a x \in T_A B \cup T_{1+A} B$. This inclusion implies that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{T_A B \cup T_{1+A} B}(\sigma^{l+rNi} T_a x) &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_B(\sigma^{l+rNi} z) \\ &= \mu(B) > \mu(T_A B \cup T_{1+A} B). \end{aligned}$$

Thus, $T_a(x)$ is not in Y. Since $T_a(X) \subset \mathbb{Z}_p \setminus Y$ and $\mu(Y) = 1$, it follows that $\mu(T_a(X)) = 0$. Since $\mu(X) = 1 > 0$ but $\mu(T_a X) = 0$, the translation T_a is singular with respect to μ.

$\square$

2.3. Other Transformations and Measures

Since the proof of Theorem 4 depends heavily on the repetitive structure of rational numbers, the proof does not generalize to $\mathbb{Z}_p \setminus \mathbb{Q}$. It is still an open question as to whether or not translation by an irrational p-adic integer is nonsingular with respect to product measures other than Haar measure. The method of proof does not seem to extend to translation by elements of $\mathbb{Z}_p \setminus \mathbb{Q}$, but it can be used to show which i.i.d. product measures are singular for another transformation P. This transformation is then used to determine which i.i.d. product measures are nonsingular for multiplication by -1. We give this proof and then give Theorem 5, which completely describes when product measures are nonsingular for multiplication by a rational p-adic integer.

We define a transformation P that switches k with $p-1-k$ at every coordinate of a p-adic integer. A probability vector $(q_0, q_1, \ldots q_{p-1})$ is *palindromic* if $q(k) = q(p-1-k)$ for all $0 \leq k \leq p-1$.

Proposition 4. *For an i.i.d. product measure μ on $\mathbb{Z}_p$ defined by a probability vector $(q_0, q_1, \ldots q_{p-1})$, if the probability vector is palindromic, then the transformation*

$$\begin{aligned} P : \mathbb{Z}_p &\to \mathbb{Z}_p \\ \sum_{i=0}^{\infty} x_i p^i &\mapsto \sum_{i=0}^{\infty} (p-1-x_i) p^i \end{aligned}$$

preserves μ. If the probability vector is not palindromic, then P is singular with respect to μ.

Proof. Since

$$\begin{aligned} P^2(\sum_{i=0}^{\infty} x_i p^i) &= P(\sum_{i=0}^{\infty}(p-1-x_i)p^i) \\ &= \sum_{i=0}^{\infty}(p-1-(p-1-x_i))p^i \\ &= \sum_{i=0}^{\infty} x_i p^i, \end{aligned}$$

we have $P^{-1} = P$. If the probability vector is palindromic, then $q(k) = q(p-1-k)$ for all $0 \leq k \leq p-1$. On balls in $\mathbb{Z}_p$,

$$\begin{aligned} \mu\left(PB_{p^{-n}}\left(\sum_{i=0}^{\infty} a_i p^i\right)\right) &= \mu\left(B_{p^{-n}}\left(\sum_{i=0}^{\infty}(p-1-a_i)p^i\right)\right) \\ &= \prod_{i=0}^{n-1} q(p-1-a_i) \\ &= \prod_{i=0}^{n-1} q(a_i) \\ &= \mu\left(B_{p^{-n}}\left(\sum_{i=0}^{\infty} a_i p^i\right)\right). \end{aligned}$$

Since the set of balls form a semi-algebra that generates the Borel sets, the transformation P preserves μ.

If the probability vector is not palindromic, then there exists an index k such that $q(k) \neq q(p-1-k)$. Without loss of generality, we suppose that $q(k) > q(p-1-k)$.

Applying the Birkhoff Ergodic Theorem to the sets

$$\begin{aligned} X &= \left\{ z \in \mathbb{Z}_p : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{B_{p^{-1}}(k)}(\sigma^i z) = q(k) \right\} \text{ and} \\ Y &= \left\{ z \in \mathbb{Z}_p : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{B_{p^{-1}}(p-1-k)}(\sigma^i z) = q(p-1-k) \right\}, \end{aligned}$$

we conclude that $\mu(X) = 1$ and $\mu(Y) = 1$. If $x \in X$, then $\sigma^i x \in B_{p^{-1}}(k)$ implies that $\sigma^i Px \in B_{p^{-1}}(p-1-k)$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{B_{p^{-1}}(p-1-k)}(\sigma^i Rx) \geq q(k) > q(p-1-k).$$

It follows that $PX \subset \mathbb{Z}_p \backslash Y$, so $\mu(PX) = 0$. Since $\mu(X) = 1$ but $\mu(PX) = 0$, the transformation P is singular with respect to μ. □

THEOREM 5. *For an i.i.d. product measure μ on $\mathbb{Z}_p$ defined by a probability vector $(q_0, q_1, \ldots q_{p-1})$, the multiplication $M_{-1} : \mathbb{Z}_p \to \mathbb{Z}_p$ is nonsingular with respect to μ if and only if the probability vector is palindromic. Moreover, if $a \in \mathbb{Z}_p^\times \setminus \{1, -1\}$ is a rational number, then the multiplication $M_a : \mathbb{Z}_p \to \mathbb{Z}_p$ is nonsingular with respect to μ if and only if μ is Haar measure.*

PROOF. 2 If $x = \sum_{i=0}^{\infty} x_i p^i$, then

$$(Px + x)_i = p - 1 - x_i + x_i = p - 1$$

for all integers $i \geq 0$. Thus, we have $Px + x = -1$ for all $x \in \mathbb{Z}_p$, so $Px = -x - 1 = M_{-1} \circ T_1 x$ for all $x \in \mathbb{Z}_p$. Since T_1 is nonsingular with respect to μ, the multiplication M_{-1} is nonsingular with respect to μ if and only if P is nonsingular with respect to μ,

by Proposition 2. By Proposition 4, P is nonsingular with respect to μ if and only if the probability vector is palindromic.

If μ is Haar measure and $a \in \mathbb{Z}_p^\times$, then M_a preserves Haar measure, as shown in [**9**, **15**, **26**, **27**]. If M_a preserves Haar measure, then M_a is certainly nonsingular with respect to Haar measure.

To prove the converse, we suppose that μ is not Haar measure and $a \in \mathbb{Z}_p^\times \setminus \{1, -1\}$. Since M_a is invertible, M_a is nonsingular with respect to μ if and only if $M_a^{-1} = M_{a^{-1}}$ is nonsingular with respect to μ. If a is an integer other than 1 or -1, then a^{-1} is not an integer. Thus, without loss of generality, we can assume that a is not an integer. Note that $T_a = M_a \circ T_1 \circ M_a^{-1}$. By Theorem 4, T_a is nonsingular with respect to μ, because μ is not Haar measure and a is a rational number but not an integer. On the other hand, the translation T_1 is nonsingular with respect to μ. By Proposition 2, M_a is singular with respect to μ. $\square$

For a rational number $a \in \mathbb{Z}_p$, there exist integers r and s such that $a = r/s$. If r and s are relatively prime and $s > 0$, then we say that $a = r/s$ is in *reduced form*. Since a is not an integer, Theorem 4 states that T_a is singular for all i.i.d. product measures other than Haar measure. However, we can define another measure that is nonsingular for T_a. If $\mu_a = (1/s)\sum_{i=0}^{s-1} \mu T_a^{-i}$, then

$$\begin{aligned}
\mu_a T_a^{-1} &= \sum_{i=0}^{s-1} \mu T_a^{-i-1} \\
&= \mu T_a^{-s} + \sum_{i=1}^{s-1} \mu T_a^{-i} \\
&= \mu T_r^{-1} + \sum_{i=1}^{s-1} \mu T_a^{-i}.
\end{aligned}$$

Since r is an integer, T_r is nonsingular with respect to μ. Since μT_r^{-1} and μ are equivalent, it follows that $\mu_a T_a^{-1}$ and μ_a are equivalent. Thus, T_a is nonsingular with respect to μ_a.

If a is an integer, then $\mu_a = \mu$. Also, if μ is Haar measure, then the invariance of Haar measure under translations implies that $\mu_a = \mu$. Interpreting μ_a appropriately, Theorem 3 implies a more general statement.

THEOREM 6. *For an i.i.d. product measure μ and a rational number $a = r/s \in \mathbb{Z}_p$ in reduced form, the transformation T_a is ergodic with respect to μ_a if and only if $a \in \mathbb{Z}_p^\times$.*

PROOF. As observed in the proof of Theorem 3, if $a \notin \mathbb{Z}_p^\times$, then $B_{p^{-1}}(0)$ is an invariant set for T_a. Since $T_a^{-i}(B_{p^{-1}}(0))$ is another ball of radius p^{-1}, we have $0 < \mu T_a^{-i}(B_{p^{-1}}(0)) < 1$ for all $0 \leq i < s$. Thus $0 < \mu_a(B_{p^{-1}}(0)) < 1$, so T_a is not ergodic with respect to μ_a.

If a is in $\mathbb{Z}_p^\times$, then r is also in $\mathbb{Z}_p^\times$. Thus, Theorem 3 implies that μ is ergodic for T_r. If A is an invariant set for T_a, then $T_a^{-i}(A) = A$ for all $i \in \mathbb{Z}$. Since $T_a^{-s} = T_r^{-1}$, the set A is also invariant for T_r. By ergodicity, it follows that $\mu(A)$ is either 0 or 1. Moreover, $T_a^{-i}(A) = A$ implies that

$$\begin{aligned} \mu_a(A) &= \frac{1}{s}\sum_{i=0}^{s-1} \mu T_a^{-i}(A) \\ &= \frac{1}{s}\sum_{i=0}^{s-1} \mu(A) \\ &= \mu(A). \end{aligned}$$

Hence, $\mu_a(A)$ is either 0 or 1, so T_a is ergodic with respect to μ_a. $\square$

CHAPTER 3

Orbit Equivalence and Transformation Types of p-adic Translation Maps

3.1. Existence of Type II p-adic Transformations

In this section, we define orbit equivalence and an invariant called transformation type. The survey article [**40**] by Katznelson and Weiss contains more information on orbit equivalence and transformation types and uses Rokhlin towers to prove the invariance of transformation types. The type II transformations divide into two subtypes that are complete invariants. It is trivial to give an example for one of the subtypes. The focus of the section is an example for the other subtype, an example that preserves an infinite measure. We define some properties and tools that are used in the ergodic theory of infinite measure preserving transformations. Then Theorem 7 states the desired ergodic properties satisfied by the example that is constructed in the proof.

Orbit equivalence is a weak notion of equivalence that is defined between invertible, nonsingular, and ergodic transformations on measure spaces. Two such transformations, T on $(X, \mathcal{B}, \mu)$ and S on $(Y, \mathcal{C}, \nu)$, are *orbit equivalent* if there exists a bimeasurable, nonsingular map $\Phi : X \to Y$ such that for almost every $x \in X$

$$\{\Phi(T^n x) : n \in \mathbb{Z}\} = \{S^m(\Phi x) : m \in \mathbb{Z}\}.$$

A transformation on a nonatomic measure space is a *type II transformation* if it is orbit equivalent to a transformation that preserves a σ-finite measure. A transformation on a nonatomic measure space is a *type III transformation* if it is not orbit equivalent to a transformation that preserves a σ-finite measure. These transformation types are invariants for orbit equivalence.

The type II transformations can be further subdivided into two subtypes. A type II transformation is *type* II_1 if it is orbit equivalent to a transformation that preserves a finite measure. A type II transformation is *type* II_∞ if it is orbit equivalent to a transformation that preserves an infinite σ-finite measure. The transformation types II_1 and II_∞ are complete invariants for orbit equivalence. The fact that any two measure-preserving transformations are orbit equivalent is called Dye's Theorem [**21**, **22**]. This section gives a representative transformation for each these two transformation types. The first example of a type III transformation was given by Ornstein in [**52**]. Type III transformations and their subtypes are discussed further in Section 3.2.

For $a \in \mathbb{Z}_p^\times$, the translation T_a is ergodic and measure-preserving with respect to Haar measure, so T_a on $(\mathbb{Z}_p, \mathcal{B}, m)$ is a type II_1 transformation. Diao and Silva prove in [**18**] that no rational function—a quotient of two polynomials—is both measure-preserving and ergodic on $\mathbb{Q}_p$ with respect to Haar measure. Thus, we cannot look to polynomials or rational functions for an example of a type II_∞ transformation. Hence, the example of a type II_∞ transformation on $\mathbb{Q}_p$ that we construct in Theorem 7 is not a polynomial or rational function. Before constructing the transformation, we discuss some desirable properties for transformations on infinite-measure spaces.

For a nonsingular transformation T on a σ-finite measure space $(X, \mathcal{A}, \mu)$, a set $W \in \mathcal{A}$ is a *wandering set* if the sets $\{T^{-i}W\}_{i=0}^{\infty}$ are pairwise disjoint. A nonsingular transformation is *conservative* if every wandering set has measure zero. A set $A \in \mathcal{A}$ is a *sweep-out set* for T if $\mu(A) < \infty$ and $X \setminus \bigcup_{i=0}^{\infty} T^{-i}A$ is a set of measure zero. If T is measure-preserving and $(X, \mathcal{A}, \mu)$ is σ-finite, then Maharam's Recurrence Theorem states that the existence of a sweep-out set for T implies that T is conservative [**50**].

Since it is often easier to work with finite-measure systems than infinite-measure systems, induced transformations are an important tool in infinite ergodic theory. For a conservative transformation T on $(X, \mathcal{A}, \mu)$ and a set $A \in \mathcal{A}$ of positive measure, the *return-time function* $n_A(x) = \min\{n \in \mathbb{N} : T^n x \in A\}$ is defined μ-almost everywhere. Then the *induced transformation* T_A is defined on A by $T_A(x) = T^{n_A(x)}(x)$. The *return-time partition* of A consists of the sets

$$R_i = \{x \in A : n_A(x) = i\}\,, \text{ for } i \in \mathbb{N}\,.$$

More generally, a *partition* of a measure space $(X, \mathcal{A}, \mu)$ is a collection of pairwise-disjoint, measurable sets whose union is all of X, up to a set of measure zero. To define entropy, we assume that T is measure-preserving transformation on a finite measure space $(X, \mathcal{A}, \mu)$. The *entropy of a partition* $\alpha = \{A_i\}_{i=0}^{\infty}$ is

$$H(\alpha) = -\sum_{i=0}^{\infty} \mu(A_i) \log \mu(A_i).$$

For a finite partition $\alpha = \{A_0, A_2, \ldots, A_{k-1}\}$, the refinement $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ is the partition with sets of the form $\bigcap_{i=0}^{n-1} T^{-1}A_{j_i}$. Then the entropy of T with respect to α is

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

Finally, the *entropy of* T is a supremum over measurable partitions,

$$h(T) = \sup_{\alpha} h(T, \alpha).$$

This definition of entropy does not extend immediately to infinite measure spaces. In particular, any finite partition of an infinite measure space contains a set of infinite measure. If T is a conservative transformation on a σ-finite measure space $(X, \mathcal{A}, \mu)$, then the *Krengel entropy* is the entropy of the induced transformation on a set of positive and finite measure. Any set of postive measure is a sweep-out set for a conservative transformation T. In [**48**], Ulrich Krengel shows that any sweep-out set yields the same entropy for the induced transformation, so the Krengel entropy is well-defined.

Krengel entropy is not the only attempt to define entropy on an infinite measure space. Other notions of entropy are Parry entropy [**54**] and Poisson entropy, defined by Roy Emmanuel. These definitions do not always give the same number, as seen in [**36**]. However, Janvresse et al. show in [**37**] that the three definitions do give the same entropy when the transformation is quasi-finite. A conservative transformation is *quasi-finite* if the entropy of the return-time partition is finite.

We can define Haar measure by defining the measure of a ball to be equal to the radius of the ball. This definition extends naturally to define Haar measure on $\mathbb{Q}_p$. Since Haar measure is translation invariant,it is also possible to construct Haar measure on

$\mathbb{Q}_p$ from Haar measure on $\mathbb{Z}_p$ in the same way Lebesgue measure on $\mathbb{R}$ is constructed from Lebesgue measure on the unit interval. Then the Haar measure of a measurable set $A \subset \mathbb{Q}_p$ is found by considering the intersection of A with $\mathbb{Z}_p$ and with the translates of $\mathbb{Z}_p$ by $\{i/p^n : i, n \in \mathbb{N}, 0 < i < p^n\}$. Since $\mathbb{Z}_p$-Haar measure is the same as the restriction of the $\mathbb{Q}_p$-Haar measure to $\mathbb{Z}_p$, we use m for both Haar measures. For example, the radius of the ball tells us that $m(B_3(0)) = 3$. Moreover, we can write $B_3(0)$ the disjoint union of three balls of radius one

$$B_3(0) = \bigcup_{i=0}^{2} B_1\left(\frac{i}{3}\right) = \bigcup_{i=0}^{2} T_{i/3}(\mathbb{Z}_3),$$

which also implies that $m(B_3(0)) = 3$.

We define spheres in $\mathbb{Q}_p$ by

$$S_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p = r\}.$$

Since the p-adic absolute value of an element of $\mathbb{Q}_p$ is always a power of p, it follows that a sphere in $\mathbb{Q}_p$ is empty unless r is a power of p. Moreover, we have

$$\mathbb{Q}_p = \mathbb{Z}_p \bigcup \left(\bigcup_{n=1}^{\infty} S_{p^n}(0) \right).$$

Theorem 7. *There exists a transformation $f : \mathbb{Q}_p \to \mathbb{Q}_p$ that is invertible, (infinite) measure-preserving, conservative, ergodic, and quasi-finite with respect to Haar measure. Moreover, the transformation has Krengel entropy 0 with respect to Haar measure.*

Proof. We define a transformation $f : \mathbb{Q}_p \to \mathbb{Q}_p$ in steps, so that standard induction proofs easily give the desired properties. The definition has two main goals. The first

goal is to have $\mathbb{Z}_p$ as a sweep-out set. To this end, iterates of balls of $\mathbb{Z}_p$ sweep out part of $\mathbb{Q}_p$ and then return to $\mathbb{Z}_p$. In each sphere of $\mathbb{Q}_p$, we send some balls back to $\mathbb{Z}_p$ and use another to sweep out the rest of the sphere and continue to the next sphere. The second goal is to define an ergodic transformation. To avoid invariant sets, the transformation "rotates" the points when they are sent back to $\mathbb{Z}_p$. The remaining properties follow from understanding the induced transformation on $\mathbb{Z}_p$.

The first step defines f on balls of radius p^{-1} in $\mathbb{Z}_p$. For all $0 \leq j < p-1$, we define

$$f\left(j + \sum_{i=1}^{\infty} x_i p^i\right) = j + 1 + \sum_{i=1}^{\infty} x_i p^i.$$

This "rotates" the balls, so that $S_1(-1)$ is mapped to $S_1(0)$. For the final ball centered at $p-1$, we define

$$f\left(p - 1 + \sum_{i=1}^{\infty} x_i p^i\right) = \frac{1}{p} + \sum_{i=1}^{\infty} x_i p^i,$$

so the ball is sent to $S_p(0)$. For example, Figure 3.1 illustrates the action of f on each ball of radius $1/3$ in $\mathbb{Z}_3$.

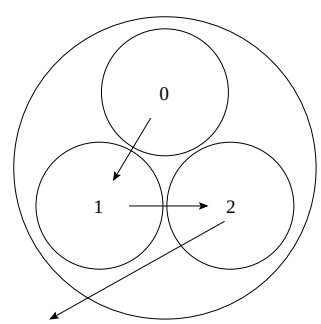

FIGURE 3.1. Construction of f on $\mathbb{Q}_3$, step 1.

For $n \geq 2$, the nth step defines f on balls of radius p^{-n} in $S_{p^{n-1}}(0)$. For all $0 \leq j < p-1$, we define

$$f\left(\frac{1}{p^{n-1}} + jp^{n-1} + \sum_{i=n}^{\infty} x_i p^i\right) = (j+1)p^{n-1} + \sum_{i=n}^{\infty} x_i p^i,$$

which sends $p-1$ balls to $\mathbb{Z}_p$, with a "rotation" in the coefficient of p^{n-1}. Next, we define f so that iterates of $B_{p^{-n}}((p-1)/p^{n-1} + jp^{n-1})$ sweep out the rest of the sphere. In order to have a conservative transformation, it does not really matter in which order we iterate through the balls. For simplicity, we use the dictionary order on the coordinates that determine the balls. If $a_{-n+1}a_{-n+2}\ldots a_{n-2}a_{n-1}$ is a word of length $2n-1$ and j is the largest index such that $a_j < p-1$, then the word with next largest dictionary order has $b_i = 0$ for all $j < i \leq n-1$, $b_j = a_j + 1$, and $b_i = a_i$ for all $-n+1 \leq i < j$. For balls $B_{p^{-n}}(\sum_{i=-n+1}^{n-1} a_i p^i)$ such that $a_{-n+1}a_{-n+2}\ldots a_{n-2}a_{n-1}$ has dictionary greater than or equal to $(p-1)0\ldots 1$ and strictly less than $(p-1)(p-1)\ldots(p-1)$, we define

$$f\left(\sum_{i=-n+1}^{n-1} a_i p^i + \sum_{i=n}^{\infty} x_i p^i\right) = \sum_{i=-n+1}^{n-1} b_i p^i + \sum_{i=n}^{\infty} x_i p^i.$$

Finally, we define

$$f\left(\sum_{i=-n+1}^{n-1} (p-1)p^i + \sum_{i=n}^{\infty} x_i p^i\right) = \frac{1}{p^n} + \sum_{i=n}^{\infty} x_i p^i,$$

which sends $B_{p^{-n}}(\sum_{i=-n+1}^{n-1}(p-1)p^i)$ into the next largest sphere, $S_{p^n}(0)$.

For $p = 3$, the second step defines f on balls of radius $1/9$ in $S_3(0) \subset \mathbb{Q}_3$. Two of these balls are mapped back into $\mathbb{Z}_3$. Then f acts on the balls according to the dictionary order on the first three coordinates of the centers. Thus, the transformation

f maps $B_{3^{-2}}(\frac{1}{3}+0+2\cdot 3)$ to $B_{3^{-2}}(\frac{1}{3}+1+0\cdot 3)$ to $B_{3^{-2}}(\frac{1}{3}+1+1\cdot 3)$ to $B_{3^{-2}}(\frac{1}{3}+1+2\cdot 3)$ and so on. Finally, the last ball is mapped into $S_9(0)$. Figure 3.2 extends Figure 3.1 to illustrate the action of f on the balls in $B_3(0)$.

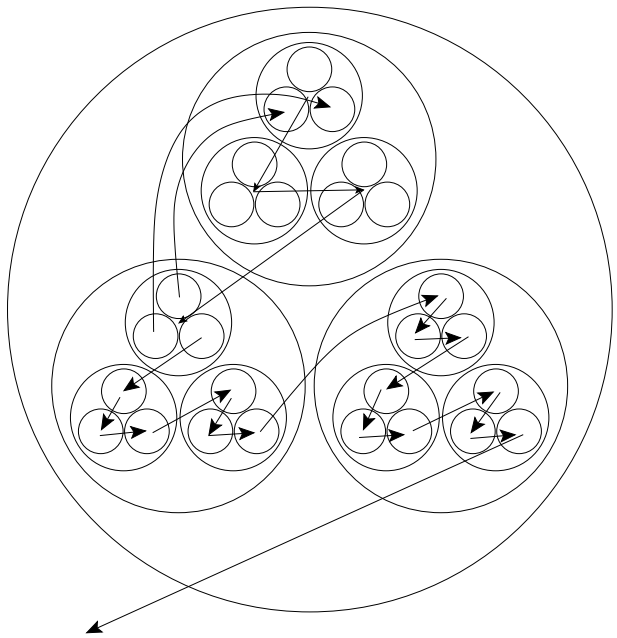

FIGURE 3.2. Construction of f on $\mathbb{Q}_3$, step 2.

Since each step of the definition of f maps balls in $\mathbb{Q}_p$ to balls of the same radius, the transformation f preserves the Haar measures of these balls. Since finite unions of balls in $\mathbb{Q}_p$ form a generating algebra of the Borel sets of $\mathbb{Q}_p$, it follows that f preserves Haar measure.

We next define a transformation $g : \mathbb{Q}_p \to \mathbb{Q}_p$ such that $g = f^{-1}$. The definition of g is also given in steps, so that it is easy to compute the composition with f. The first step defines g on balls of radius p^{-1} in $S_1(0)$ by

$$g\left(j + \sum_{i=1}^{\infty} x_i p^i\right) = j - 1 + \sum_{i=1}^{\infty} x_i p^i$$

for all $0 < j \leq p - 1$.

For $n \geq 2$, the nth step defines g on balls of radius p^{-n} in $S_{p^{-n+1}}(0) \cup S_{p^{n-1}}(0)$. First, we define g on the balls in $S_{p^{-n+1}}(0)$ by

$$g\left(jp^{n-1} + \sum_{i=n}^{\infty} x_i p^i\right) = \frac{1}{p^{n-1}} + (j-1)p^{n-1} + \sum_{i=n}^{\infty} x_i p^i$$

for all $0 < j \leq p-1$. If $b_{-n+1}b_{-n+2}\dots b_{n-2}b_{n-1}$ is a word of length $2n-1$ and j is the largest index such that $b_j > 0$, then the word with next smallest dictionary order has $a_i = p-1$ for all $j < i \leq n-1$, $a_j = b_j - 1$, and $a_i = b_i$ for all $-n+1 \leq i < j$. For balls $B_{p^{-n}}(\sum_{i=-n+1}^{n-1} b_i p^i)$ such that $b_{-n+1}b_{-n+2}\dots b_{n-2}b_{n-1}$ has dictionary strictly greater than $(p-1)0\dots 1$ and less than or equal to $(p-1)(p-1)\dots(p-1)$, we define

$$g(\sum_{i=-n+1}^{n-1} b_i p^i + \sum_{i=n}^{\infty} x_i p^i) = \sum_{i=-n+1}^{n-1} a_i p^i + \sum_{i=n}^{\infty} x_i p^i.$$

Finally, we define

$$g\left(\frac{1}{p^{n-1}} + \sum_{i=n-1}^{\infty} x_i p^i\right) \quad = \quad \sum_{i=-n+2}^{n-2} (p-1)p^i + \sum_{i=n-1}^{\infty} x_i p^i.$$

which sends $B_{p^{-n+1}}(p^{-n+1})$ into the next smallest sphere, $S_{p^{n-1}}(0)$.

Comparing the nth step of both constructions, we see that $g \circ f = \mathrm{Id}$ on all but the final ball, $B_{p^{-n}}(\sum_{i=-n+1}^{n-1}(p-1)p^i)$. For this ball, we compare the nth step in the construction of f with the $n+1$st step in the construction of g to see that $g \circ f = \mathrm{Id}$. Moreover, g also preserves Haar measure. Thus, f is invertible and g is the inverse.

In $\mathbb{Z}_p$, we have $-1 = \sum_{i=0}^{\infty}(p-1)p^i$. To prove that $\mathbb{Z}_p$ is a sweep-out set, we consider $S_{p^{-n}}(-1)$ for $n \geq 0$ and count how many iterations of f it takes to reach $S_{p^{-n}}(0)$. In this way, we calculate the return times and the induced transformation. We begin by

considering the first step in the definition of f, which defines f on $\mathbb{Z}_p$. In this step, we see that f maps the sphere $S_1(-1) = \bigcup_{j=0}^{p-2} B_{p^{-1}}(j)$ to the sphere $S_1(0) = \bigcup_{j=1}^{p-1} B_{p^{-1}}(j)$, so $n_{\mathbb{Z}_p}(x) = 1$ for all $x \in S_1(-1)$. Moreover, we observe that the induced transformation is $f_{\mathbb{Z}_p}(x) = x + 1 = T_1(x)$ for all $x \in S_1(-1)$. On the other hand, f maps $B_{p^{-1}}(-1)$ to $B_{p^{-1}}(p^{-1}) \subset S_p(0)$ by

$$f\left((p-1) + \sum_{i=1}^{\infty} x_i p^i\right) = \frac{1}{p} + \sum_{i=1}^{\infty} x_i p^i.$$

Proceeding to the second step, we have

$$\begin{aligned} f^2\left((p-1) + jp + \sum_{i=2}^{\infty} x_i p^i\right) &= f\left(\frac{1}{p} + jp + \sum_{i=2}^{\infty} x_i p^i\right) \\ &= (j+1)p + \sum_{i=2}^{\infty} x_i p^i, \end{aligned}$$

for all $0 \leq j < p-1$. Thus $n_{\mathbb{Z}_p}(x) = 2$ and $f_{\mathbb{Z}_p}(x) = T_1(x)$ for all $x \in S_{p^{-1}}(-1)$. Using dictionary order on the first three coordinates, the iterates of f map $f(B_{p^{-2}}(-1)) = B_{p^{-2}}(p^{-1} + (p-1)p)$ through the rest of the balls of radius p^{-2} in $S_p(0)$, fixing all coordinates with index greater than 1. In order to count iterates, we need to know how many balls are left in $S_p(0) \backslash B_{p^{-1}}(p^{-1})$. A ball $B_{p^{-2}}(a) \subset S_p(0)$ is determined by the three coordinates a_{-1}, a_0, and a_1. Since a_{-1} must be nonzero, there are $(p-1)p^2$ balls of radius p^{-2} in $S_p(0)$. However, there are p balls of radius p^{-2} in $B_{p^{-1}}(p^{-1})$, so there are only $(p-1)p^2 - p$ balls of radius p^{-2} in $S_p(0) \backslash B_{p^{-1}}(p^{-1})$. Hence

$$\begin{aligned} f^{(p-1)p^2-p+1}\left((p-1) + (p-1)p + \sum_{i=2}^{\infty} x_i p^i\right) &= f^{(p-1)p^2-p}\left(\frac{1}{p} + (p-1)p + \sum_{i=2}^{\infty} x_i p^i\right) \\ &= \sum_{i=-1}^{1} (p-1)p^i + \sum_{i=2}^{\infty} x_i p^i. \end{aligned}$$

Having considered these base cases, we proceed by induction to calculate the return times and induced transformation. For notational convenience, we define $s(n) = \sum_{k=2}^{n}[(p-1)p^{2k-2} - (p-1)]$ for $n \geq 2$. For some $n \geq 2$, we suppose that

$$f^{s(n)}\left(\sum_{i=0}^{n-1}(p-1)p^i + \sum_{i=n}^{\infty} x_i p^i\right) = \sum_{i=-n+1}^{n-1}(p-1)p^i + \sum_{i=n}^{\infty} x_i p^i.$$

From the nth step of the definition, we have

$$\begin{aligned} f^{s(n)+1}\left(\sum_{i=0}^{n-1}(p-1)p^i + \sum_{i=n}^{\infty} x_i p^i\right) &= f\left(\sum_{i=-n+1}^{n-1}(p-1)p^i + \sum_{i=n}^{\infty} x_i p^i\right) \\ &= \frac{1}{p^n} + \sum_{i=n}^{\infty} x_i p^i. \end{aligned}$$

From the $n+1$st step of the definition of f, we have

$$\begin{aligned} f^{s(n)+2}\left(\sum_{i=0}^{n-1}(p-1)p^i + jp^n \sum_{i=n+1}^{\infty} x_i p^i\right) &= f(\frac{1}{p^n} + jp^n + \sum_{i=n+1}^{\infty} x_i p^i) \\ &= (j+1)p^n + \sum_{i=n+1}^{\infty} x_i p^i \end{aligned}$$

for all $0 \leq j < p-1$. Thus $n_{\mathbb{Z}_p}(x) = s(n) + 2 = 2 + \sum_{k=2}^{n}[(p-1)p^{2k-2} - (p-1)]$ and $f_{\mathbb{Z}_p}(x) = T_1(x)$ for all $x \in S_{p^{-n}}(-1)$. In the $n+1$st step in the construction of f, iterates of f send $B_{p^{-n-1}}(p^{-n} + (p-1)p^n)$ through all of the balls of radius $p^{-(n+1)}$ in $S_{p^{n+1}}(0) \backslash B_{p^{-n}}(p^{-n})$, according to the dictionary order of their first $2n+1$ coordinates. Since the first coordinate is nonzero and $B_{p^{-n}}(p^{-n})$ contains p balls of radius $p^{-(n+1)}$,

there are $(p-1)p^{2n+1} - p$ balls in $S_{p^{n+1}}(0) \backslash B_{p^{-n}}(p^{-n})$. Thus

$$
\begin{aligned}
f^{s(n+1)} &\left(\sum_{i=0}^{n} (p-1)p^i + \sum_{i=n+1}^{\infty} x_i p^i \right) \\
&= f^{s(n)+(p+1)p^{2n+1}-p+1} \left(\sum_{i=0}^{n} (p-1)p^i + \sum_{i=n+1}^{\infty} x_i p^i \right) \\
&= f^{(p-1)p^{2n+1}-p+1} \left(\sum_{i=-n+1}^{n} (p-1)p^i + \sum_{i=n+1}^{\infty} x_i p^i \right) \\
&= f^{(p-1)p^{2n+1}-p} \left(\frac{1}{p^n} + (p-1)p^n + \sum_{i=n+1}^{\infty} x_i p^i \right) \\
&= \sum_{i=-n}^{n} (p-1)p^i + \sum_{i=n+1}^{\infty} x_i p^i.
\end{aligned}
$$

We observe that this is our induction hypothesis with $n+1$ in place of n.

It follows by induction that $\mathbb{Z}_p$ is a sweep-out set. If $x \in \mathbb{Z}_p$ and x is not -1, then $x \in S_{p^{-n}}(-1)$ for some n. Thus $n_{\mathbb{Z}_p}(x)$ is defined, and $x \in f^{-n_{\mathbb{Z}_p}(x)} \mathbb{Z}_p$. If $x \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, then $x \in S_{p^n}(0)$ for some $n > 0$. If x_i is not $p-1$ for some $i > n$, then we let k be the minimal such index. It follows that x is in the orbit of an element $y \in S_{p^{-k}}(-1)$, so $x \in \bigcup_{i=1}^{n_{\mathbb{Z}_p}(y)} f^{-i} \mathbb{Z}_p$. Since there are only countably many elements of $\mathbb{Q}_p$ that end in repeating $p-1$'s, the set $\mathbb{Q}_p \setminus \cup_{i=1}^{\infty} f^{-i} \mathbb{Z}_p$ has Haar measure zero, so $\mathbb{Z}_p$ is a sweep-out set. Since a sweep-out set exists, f is conservative by Maharam's Recurrence Theorem.

Moreover, we have $f_{\mathbb{Z}_p}(x) = T_1(x)$ for all $x \in \mathbb{Z}_p \backslash \{-1\}$. We recall that T_1 is ergodic with respect to Haar measure on $\mathbb{Z}_p$. Since the induced measure on $\mathbb{Z}_p$ is Haar measure on $\mathbb{Z}_p$, we also have that $f_{\mathbb{Z}_p}$ is ergodic with respect to the induced measure on $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is a sweep-out set for the conservative transformation f, the ergodicity of the induced transformation with respect to the induced measure implies that f is also ergodic with respect to Haar measure on $\mathbb{Q}_p$.

From calculating the return-time function, we see that the return-time partition is $\alpha = \{S_{p^{-n}}(-1)\}_{n=0}^{\infty}$. The entropy of this partition,

$$H(\alpha) = -\sum_{n=0}^{\infty} m(S_{p^{-n}}(-1)) \log m(S_{p^{-n}}(-1)) = -\sum_{n=0}^{\infty} \frac{p-1}{p^{n+1}} \log \frac{p-1}{p^{n+1}},$$

is finite. Thus, the transformation f is quasi-finite with respect to Haar measure. Since f is quasi-finite, all definitions of entropy for infinite measure spaces give the same result. Since $f_{\mathbb{Z}_p}$ is a translation on a compact group, it has entropy 0 with respect to Haar measure on $\mathbb{Z}_p$. Since the induced transformation has entropy 0, the transformation f has Krengel entropy 0. $\square$

Since f is invertible, conservative, and measure-preserving, and since m is a σ-finite measure, f is isomorphic to the Kakutani skyscraper over $f_{\mathbb{Z}_p} = T_1$ with height function $n_{\mathbb{Z}_p}$. In general, if S is a conservative, nonsingular transformation on a σ-finite measure space $(X, \mathcal{A}, \mu)$, then the Kakutani skyscraper over S with height function $h : X \to \mathbb{N}$ is the transformation

$$T(x,n) = \begin{cases} (x, n+1) & \text{if } n < h(x), \\ (S(x), 1) & \text{if } n = h(x) \end{cases}$$

on the set

$$Y = \{(x,n) : x \in X, 1 \leq n \leq h(x)\}$$

with the measure ν such that for $A \in \mathcal{A}$

$$\nu(A \times \{n\}) = \mu(A).$$

The induced transformation T_1 on $\mathbb{Z}_2$ is isomorphic to the odometer on $\prod_{i=0}^{\infty} X_2$, which is isomorphic to the odometer on the unit interval. The Hajian-Kakutani transformation is also a Kakutani skyscraper over the odometer on the unit interval. However, it is not the same Kakutani skyscraper as the Kakutani skyscraper over $f_{\mathbb{Z}_p} = T_1$ with height function $n_{\mathbb{Z}_p}$, because the height functions are different. Hajian and Kakutani define their skyscraper in [**31**] to give an example of an ergodic, measure-preserving transformation on an infinite measure space that has an exhaustive weakly wandering set of measure 1. A set W is *weakly wandering* for an invertible transformation T on a measure space $(X, \mathcal{A}, \mu)$ if there exists a sequence of nonnegative integers $\{n_i\}_{i=0}^{\infty}$ such that $T^{-n_i}W \cap T^{-n_j}W = \emptyset$ for all $i \neq j$. A weakly wandering set W for T is *exhaustive* if $(\bigcup_{i=0}^{\infty} T^{-n_i}W = X$, up to a set of μ-measure 0. Eigen, Hajian, and Prasad study the weakly wandering sets of general Kakutani skyscrapers over the odometer in [**23**]. For future work, their paper could be applied to investigate the weakly wandering sets of f.

3.2. Existence of Type III p-adic Transformations

In this section, we discuss type III transformations. We begin by defining the ratio set, which then defines subtypes. Theorem 8 simplifies the calculation of the ratio set to the calculation of the measures for finitely many balls. The remainder of the section gives examples for the transformation types that are complete invariants for orbit equivalence.

Since translation by an integer is an iterate of translation by 1, these examples are used to examine how orbit equivalence classes behave under iteration. Measure-theoretic isomorphism classes are preserved by iteration. If $\phi : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ is an isomorphism from T on $(X, \mathcal{B}, \mu)$ to S on $(Y, \mathcal{C}, \nu)$, then the definition and the invertibility of ϕ

imply that $S = \phi \circ T \circ \phi^{-1}$. For $k > 0$, this equality implies that

$$S^k = (\phi \circ T \circ \phi^{-1})^k = \phi \circ T^k \circ \phi^{-1},$$

so T^k on $(X, \mathcal{B}, \mu)$ is isomorphic to S^k on $(Y, \mathcal{C}, \nu)$. Thus, if two transformations of measures spaces are isomorphic, then all iterations are also isomorphic. In Theorem 9, we see that this is not the case for orbit equivalence. We find transformations on measure spaces that are orbit equivalent, but an iteration of the transformations breaks the orbit equivalence.

Since all translations on $\mathbb{Z}_p$ are type II_1 transformations with respect to Haar measure, we examine translations with respect to other i.i.d. product measures to find examples of type III transformations. Although not all type III transformations are orbit equivalent, the ratio set defines subtypes, many of which are complete invariants for orbit equivalence.

Definition 1. For an invertible, nonsingular and ergodic transformation T on a measure space $(X, \mathcal{A}, \mu)$, a real number $r \in [0, \infty]$ is in the *ratio set* $\mathcal{R}(T, \mu)$ if for all $\epsilon > 0$ and for every measurable set A of postive measure, there exists $B \subset A$ of positive measure and an $n \in \mathbb{Z} \setminus \{0\}$ such that $T^n B \subset A$ and $|(d\mu T^n / d\mu)(x) - r| < \epsilon$ for all $x \in B$.

The ratio set $\mathcal{R}(T, \mu)$ depends only on the absolute continuity equivalence class of μ. The ratio set is closed, and $\mathcal{R}(T, \mu) \cap (0, \infty)$ is a multiplicative subgroup of $(0, \infty)$. The possibilities for multiplicative subgroups give the following possibilities for ratio sets, which are used to define the transformation types III_λ for $0 \leq \lambda \leq 1$.

Transformation Type	Ratio Set $\mathcal{R}(T,\mu)$
Type II	$\{1\}$
Type III_0	$\{0,\ 1,\ \infty\}$
Type III_λ	$\{\lambda^n : n \in \mathbb{Z}\} \cup \{0,\ \infty\}$, $\lambda \in (0,\ 1)$
Type III_1	$[0,\ \infty]$

The transformation types III_λ for $0 < \lambda \leq 1$ are complete invariants for orbit equivalence. Although the definition of the ratio set is complicated, Theorem 8 below reduces the problem to calculating a generating set. Before giving the theorem, we make some comments about the notation. If an ergodic transformation T on a measure space is invertible, then it follows from the definition and properties of the Radon-Nikodým derivative that T and T^{-1} have the same transformation type. For a negative integer a, the transformation type of translation by a is the same as the transformation type of translation by the positive integer $-a$. Therefore, Theorem 8 is stated for positive integers, which simplifies the proof. Since the i.i.d. product measures are required to be probability measures, knowing $p-1$ of the weights is enough information to calculate the pth weight. To reflect this fact and simplify calculations, we can write the i.i.d. product measure in the form $\mu = \prod_{i=0}^{\infty} \left\{ \frac{1}{Q}, \frac{q_1}{Q}, \ldots, \frac{q_{p-1}}{Q} \right\}$, where $Q = 1 + \sum_{i=1}^{p-1} q_i$.

Theorem 8. *For $a \in \mathbb{N} \cap \mathbb{Z}_p^\times$, $N \in \mathbb{N}$ such that $p^N > a$, and an i.i.d. product measure $\mu = \prod_{i=0}^{\infty} \left\{ \frac{1}{Q}, \frac{q_1}{Q}, \ldots, \frac{q_{p-1}}{Q} \right\}$ with $Q = 1 + \sum_{i=1}^{p-1} q_i$, the ratio set $\mathcal{R}(T_a,\mu)$ is the closure of the multiplicative subgroup generated by*

$$r(T_a,\mu) = \left\{ \frac{\mu T_a(B_{1/p^{N+1}}(k))}{\mu(B_{1/p^{N+1}}(k))} q_{p-1}^j : j \geq 0, 0 \leq k < p^{N+1} - a \right\}.$$

Proof. We begin by showing that the elements of $r(T_a, \mu)$ are the values of the Radon-Nikodým derivative of T_a^{-1} with respect to μ. More precisely, we show that $\frac{d\mu T_a}{d\mu}(x) \in r(T_a, \mu)$ for almost every $x \in \mathbb{Z}_p$. If $W = \max\{1/Q, q_1/Q, \ldots, q_{p-1}/Q\}$, then

$$\mu\left(\bigcup_{i=1}^{a} B_{p^{-n}}(p^n - i)\right) < aW^n$$

for all $n > N$. Since $\lim_{n\to\infty} aW^n = 0$, the set

$$Z = \bigcup_{n=N}^{\infty} \bigcup_{i=1}^{p^n-a-1} B_{p^{-n}}(i)$$

has μ-measure 1. Thus, for almost every $x = \sum_{i=0}^{\infty} x_i p^i$, there exists an $n > N$ such that $\sum_{i=0}^{n-1} x_i p^i < p^n - a$. If $\sum_{i=0}^{N-1} x_i p^i < p^N - a$, then adding a does not result in a carry to the Nth coordinate. If $T_a(x) = \sum_{i=0}^{N-1} y_i p^i \mod p^N$, then

$$\begin{array}{rl} & x_0 + x_1 p^1 + \cdots + x_{N-1} p^{N-1} + x_N p^N + \cdots + x_n p^n + \cdots \\ + & \underline{a_0 + a_1 p^1 + \cdots + a_{N-1} p^{N-1} + \ 0\ p^N \ + \cdots + \ 0\ p^n + \cdots} \\ & y_0 + y_1 p^1 + \cdots + y_{N-1} p^{N-1} + x_N p^N + \cdots + x_n p^n + \cdots . \end{array}$$

Thus, Proposition 2 implies that

$$\begin{aligned} \frac{d\mu T_a}{d\mu}(x) &= \frac{\mu T_a(B_{p^{-n}}(\sum_{i=0}^{n-1} x_i p^i))}{\mu(B_{p^{-n}}(\sum_{i=0}^{n-1} x_i p^i))} \\ &= \frac{\mu T_a(B_{1/p^{N+1}}(\sum_{i=0}^{N} x_i p^i))}{\mu(B_{1/p^{N+1}}(\sum_{i=0}^{N} x_i p^i))}, \end{aligned}$$

which is an element of $r(T_a, \mu)$. If $\sum_{i=0}^{N-1} x_i p^i \geq p^N - a$, then adding a does result in a carry to the Nth coordinate. We let k be the least index such that $k \geq N$ and $x_k \neq p-1$.

Then the carry to the Nth coordinate results in further carries until the $k-1$st digit, so

$$
\begin{array}{rl}
 & \qquad\qquad\qquad\qquad 1 \qquad \cdots \qquad 1 \qquad\qquad 1 \\
 & x_0 + \cdots + x_{N-1}p^{N-1} + (p-1)p^N + \cdots + (p-1)p^{k-1} + x_k p^k + \cdots \\
+ & \underline{a_0 + \cdots + a_{N-1}p^{N-1} + \qquad 0\ p^N \ + \cdots + \qquad 0\ p^{k-1} \ + \ 0\ p^k + \cdots} \\
 & y_0 + \cdots + y_{N-1}p^{N-1} + \qquad 0\ p^N \ + \cdots + \qquad 0\ p^{k-1} \ + (x_k+1)p^k + \cdots .
\end{array}
$$

Thus, Proposition 2 implies that

$$
\begin{aligned}
\frac{d\mu T_a}{d\mu}(x) &= \frac{\mu T_a(B_{p^{-n}}(\sum_{i=0}^{n-1} x_i p^i))}{\mu(B_{p^{-n}}(\sum_{i=0}^{n-1} x_i p^i))} \\
&= \frac{\mu T_a(B_{1/p^{N+1}}(\sum_{i=0}^{N-1} x_i p^i + x_k p^N))}{\mu(B_{1/p^{N+1}}(\sum_{i=0}^{N-1} x_i p^i + x_k p^N))} q_{p-1}^{k-N},
\end{aligned}
$$

which is an element of $r(T_a, \mu)$.

Next, we show that $r = d\mu T_a/d\mu(x) \in r(T_a, \mu)$ is an element of the ratio set $\mathcal{R}(T_a, \mu)$. With n chosen for x as in the previous paragraph, we set $\bar{x} = \sum_{i=0}^{n-1} x_i p^i$, $\beta = \min\{\mu(B_{p^{-n}}(x)), \mu T_a(B_{p^{-n}}(x))\}$ and $\alpha = 1 - \beta/4$. By Lemma 2, we can find a ball C such that $\mu(A \cap C) > \alpha\mu(C)$. Without loss of generality, there exists an integer $m \geq 0$ and an integer $0 \leq k < p^m$ such that $C = B_{p^{-m}}(k)$. Since $\bar{x} + a \leq p^n - 1$, we have

$$
\begin{aligned}
k + p^m\bar{x} + p^m a &= k + p^m(\bar{x} + a) \\
&\leq k + p^m(p^n - 1) \\
&= k - p^m + p^{n+m} \\
&< p^{n+m}.
\end{aligned}
$$

Since $k + p^m\bar{x} < p^{n+m} - p^m a$, Proposition 2 implies that for $y \in B_{1/p^{n+m}}(k + p^n\bar{x})$

$$\begin{aligned}\frac{d\mu T_a^{p^m}}{d\mu}(y) &= \frac{\mu(B_{1/p^{n+m}}(T_a^{p^m}(j + p^m\bar{x})))}{\mu(B_{1/p^{n+m}}(j + p^m\bar{x}))} \\ &= \frac{\mu(B_{1/p^n}(T_a(\bar{x})))}{\mu(B_{1/p^n}(\bar{x}))} = r.\end{aligned}$$

To satisfy the definition of the ratio set, we set $B = A \cap T_a^{-p^m}(A \cap T_a^{p^m}(B_{1/p^{n+m}}(k+p^n\bar{x})))$. By construction, we have $B \subset A$. Since T_a is invertible, the construction implies that $T_a^{p^m} B \subset A$. Since $\mu T_a^{p^m}(B_{1/p^{n+m}}(k + p^n\bar{x})) = \mu T_a(B_{1/p^n}(\bar{x}))\mu(C)$ and

$$\mu(A \cap C) > \left(1 - \frac{\beta}{4}\right)\mu(C) \geq \left(1 - \frac{\mu(T_a B_{1/p^n}(\bar{x})}{4}\right)\mu(C),$$

it follows that

$$\begin{aligned}\mu(A \cap T_a^{p^m} B_{1/p^{n+m}}(k + p^n\bar{x})) &\geq (3/4)\mu T_a(B_{1/p^n}(\bar{x}))\mu(C) \\ &= (3/4)\mu T_a^{p^m}(B_{1/p^{n+m}}(k + p^n\bar{x})).\end{aligned}$$

By Proposition 2, the Radon-Nikodým derivative of $T_a^{p^m}$ is constant on $B_{1/p^{n+m}}(k+p^n\bar{x})$, so

$$\begin{aligned}\mu(T_a^{-p^m}(A \cap T_a^{p^m} B_{1/p^{n+m}}(k + p^n\bar{x}))) &\geq (3/4)\mu(B_{1/p^{n+m}}(k + p^n\bar{x})) \\ &= (3/4)\mu(B_{1/p^n}(\bar{x}))\mu(C).\end{aligned}$$

Since

$$\mu(A \cap C) > \left(1 - \frac{\beta}{4}\right)\mu(C) \geq (3/4)\mu(B_{1/p^n}(\bar{x}))\mu(C),$$

it follows that

$$\begin{aligned}\mu(B) &= \mu(A \cap T_a^{-p^m}(A \cap T_a^{p^m}(B_{1/p^{n+m}}(k+p^n\bar{x})))) \\ &\geq (1/2)\mu(B_{1/p^n}(\bar{x}))\mu(C) > 0.\end{aligned}$$

Finally, since $B \subset B_{1/p^{n+m}}(k+p^n\bar{x})$, we have $d\mu T_a^{p^m}/d\mu(y) = r$ for all $y \in B$.

Since every element of $r(T_a, \mu)$ is an element of the ratio set, the closed multiplicative subgroup generated by $r(T_a, \mu)$ is contained in the ratio set. To show the reverse inclusion, we suppose that $r \in \mathcal{R}(T_a, \mu)$. For each $n \in \mathbb{N}$, the definition of the ratio set implies that there exists a set $B_n \subset \mathbb{Z}_p$ of positive measure and an integer j_n such that $|d\mu T_a^{j_n}/d\mu(y) - r| < 1/n$ for all $y \in B_n$. Since T_a and T_a^{-1} are nonsingular, $\mu(Z) = 1$ implies that $\mu(\bigcap_{i\in\mathbb{Z}} T_a^i Z) = 1$. Since B_n has positive measure, there exists $x_n \in B_n$ such that $x_n \in \bigcap_{i\in\mathbb{Z}} T_a^i Z$. Thus $T_a^i(x_n) \in Z$ for all $i \in \mathbb{Z}$. If $j^n \geq 0$, then the chain rule for Radon-Nikodým derivatives, proved in [**60**], implies that

$$\frac{d\mu T_a^{j_n}}{d\mu}(x_n) = \frac{d\mu T_a}{d\mu}(x_n)\frac{d\mu T_a}{d\mu}(T_a^1 x_n)\cdots\frac{d\mu T_a}{d\mu}(T_a^{j_n-1}x_n).$$

Hence $d\mu T_a^{j_n}/d\mu(x_n)$ is in the multiplicative subgroup generated $r(T_a, \mu)$. If $j^n < 0$, then

$$\frac{d\mu T_a^{j_n}}{d\mu}(x_n) = \frac{1}{\frac{d\mu T_a^{-j_n}}{d\mu}(T^{j_n}x_n)}$$

is in the multiplicative subgroup generated by $r(T_a, \mu)$. Since $d\mu T_a^{j_n}/d\mu(x_n) \to r$ as $n \to \infty$, it follows that r is in the closure of the multiplicative subgroup generated by the values of the Radon-Nikodým derivative. □

If μ in Theorem 8 is Haar measure, then $q_{p-1} = 1$ and all of the Radon-Nikodým derivatives equal 1. Hence, Theorem 8 implies that $\mathcal{R}(T_a, m) = \{1\}$, which implies that translation is type II with respect to Haar measure. This agrees with the discussion at the beginning of Section 3.1.

Next, we use Theorem 8 to give examples of type III_λ transformations, for $0 < \lambda < 1$. For $a = 1$, Propositions 5 and 6 are generalizations of the examples given by Hamachi and Osikawa for the odometer on the product spaces $\prod_{i=0}^{\infty} X_2$ and $\prod_{i=0}^{\infty} X_3$. Moreover, we see that it is possible for all ergodic iterates to have the same transformation type.

Proposition 5. *For $0 < \lambda < 1$ and an integer $a \in \mathbb{Z}_p^\times$, the transformation T_a on $\mathbb{Z}_p$ is type III_λ with respect to the measure $\mu_1 = \prod_{i=0}^{\infty} \left\{ \frac{1}{Q_1}, \ldots, \frac{1}{Q_1}, \frac{\lambda}{Q_1} \right\}$, where $Q_1 = p-1+\lambda$.*

Proof. We fix N such that $p^N > a$. Using Theorem 8, we want to show that

$$r(T_a, \mu) = \left\{ \frac{\mu_1 T_a(B_{1/p^{N+1}}(k))}{\mu_1(B_{1/p^{N+1}}(k))} \lambda^j : j \geq 0, 0 \leq k < p^{N+1} - a \right\}$$

generates $\{\lambda^i : i \in \mathbb{Z}\}$. For all $0 \leq k < p^{N+1}$, if $n_{p-1}(k)$ is the number of $p-1$s in the p-adic expansion of i, then $\mu_1(B_{1/p^{N+1}}(k)) = \lambda^{n_{p-1}(k)}/Q_1^{N+1}$. In other words, the measure of every ball of radius p^{-N-1} has a power of λ in the numerator and Q_1^{N+1} in the denominator. Then the fraction

$$\frac{\mu_1 T_1(B_{1/p^{N+1}}(k))}{\mu_1(B_{1/p^{N+1}}(k))} = \frac{\mu_1(B_{1/p^{N+1}}(k+1))}{\mu_1(B_{1/p^{N+1}}(k))}$$

is a power of λ for all $0 \leq i < p^{N+1} - a$. Thus, we have $r(T_a, m_1) \subset \left\{\lambda^k : k \in \mathbb{Z}\right\}$.

In particular, if $k = 0$ and $j = n_{p-1}(0) - n_{p-1}(a) + 1$, then

$$\begin{aligned}\frac{\mu_1 T_a(B_{1/p^{N+1}}(0))}{\mu_1(B_{1/p^{N+1}}(0))}\lambda^j &= \frac{\mu_1(B_{1/p^{N+1}}(a))}{\mu_1(B_{1/p^{N+1}}(0))}\lambda^j \\ &= \frac{\lambda^{n_{p-1}(a)}}{\lambda^{n_{p-1}(0)}}\lambda^j = \lambda.\end{aligned}$$

Thus, the set $r(T_a, \mu_1)$ generates $\left\{\lambda^k : k \in \mathbb{Z}\right\}$. Therefore, $\mathcal{R}(T_a, \mu_1) = \{\lambda^n : n \in \mathbb{Z}\} \cup \{0,\ \infty\}$, so T_a has transformation type III_λ with respect to μ_1. □

Two real numbers $r_1, r_2 \in \mathbb{R}$ are *rationally independent* if $q_1, q_2 \in \mathbb{Q}$ such that $q_1 r_1 + q_2 r_2 = 0$ implies that $q_1 = q_2 = 0$.

Proposition 6. *For $p \geq 3$, if $\lambda_1, \lambda_2 \in (0,1)$ such that $\log \lambda_1$ and $\log \lambda_2$ are rationally independent, then the transformation T_1 on $\mathbb{Z}_p$ is type III_1 with respect to the measure $\mu_2 = \prod_{i=0}^{\infty}\left\{\frac{1}{Q_2}, \ldots, \frac{1}{Q_2}, \frac{\lambda_2}{Q_2}, \frac{\lambda_1}{Q_2}\right\}$, where $Q_2 = p - 2 + \lambda_1 + \lambda_2$.*

Proof. For all $0 \leq i < p^{N+1}$, if $n_j(i)$ is the number of j's in the p-adic expansion of i, then $\mu_1(B_{1/p^{N+1}}(i)) = \lambda_2^{n_{p-2}(i)}\lambda_1^{n_{p-1}(i)}/Q_1^{N+1}$. For $k = \sum_{i=0}^{N}(p-2)p^i < p^{N+1}$, we have

$$\begin{aligned}\frac{\mu T_1(B_{1/p^{N+1}}(k-1))}{\mu(B_{1/p^{N+1}}(k-1))} &= \frac{\mu(B_{1/p^{N+1}}(k))}{\mu(B_{1/p^{N+1}}(k-1))} \\ &= \lambda_2^{n_{p-2}(k)-n_{p-2}(k-1)}\lambda_1^{n_{p-1}(k)-n_{p-1}(k-1)} \\ &= \lambda_2^{N-n_{p-2}(k-1)}\lambda_1^{-n_{p-1}(k-1)}.\end{aligned}$$

Since $n_{p-2}(k-1) < N$, the multiplicative subgroup generated by $\lambda_2^{N-n_{p-2}(k-1)}$ and λ_1 is contained in $\mathcal{R}(T_1, \mu_2)$. If $\log \lambda_1$ and $\log \lambda_2$ are rationally independent, then $(N - n_{p-2}(k-1)) \log \lambda_2$ and $\log \lambda_1$ are also rationally independent. Hence, the values $\lambda_2^{N-n_{p-2}(k-1)}$ and λ_1 generate a dense multiplicative subgroup of $(0, \infty)$. Thus, Theorem 8 implies that

$\mathcal{R}(T_1, \mu_2) = [0, \infty]$. By the definition of the transformation types, T_1 is type III_1 with respect to μ_2. $\square$

For the measure μ_1, the ergodic iterates have the same transformation type as the original transformation. In general, it follows from the definition of the ratio set that $\mathcal{R}(T^n, \mu) \subset \mathcal{R}(T, \mu)$. In the next proposition, we see that it is possible to have a strict inclusion.

PROPOSITION 7. *For $0 < \lambda < 1$, the translation T_1 on $\mathbb{Z}_3$ has transformation type III_λ with respect to $\mu_3 = \prod_{i=0}^{\infty} \left\{ \frac{1}{2+\lambda}, \frac{\lambda}{2+\lambda}, \frac{1}{2+\lambda} \right\}$. However, the iterate $T_2 = T_1^2$ on $\mathbb{Z}_3$ has transformation type III_{λ^2} with respect to μ_3.*

PROOF. Since both 1 and 2 are less than 3, we take $N = 1$ in Theorem 8. Since $q_2 = 1$, we have

$$\begin{aligned} r(T_1, \mu_3) &= \left\{ \frac{1}{\lambda}, \lambda \right\} \\ r(T_2, \mu_3) &= \left\{ \frac{1}{\lambda^2}, 1, \lambda^2 \right\}. \end{aligned}$$

Therefore, $\mathcal{R}(T_1, \mu_3) = \{\lambda^n : n \in \mathbb{Z}\} \cup \{0, \ \infty\}$, so T_1 has transformation type III_λ with respect to μ_3. However, $\mathcal{R}(T_2, \mu_3) = \{\lambda^{2n} : n \in \mathbb{Z}\} \cup \{0, \ \infty\}$, so T_2 has transformation type III_{λ^2} with respect to μ_3. $\square$

For $0 < \lambda \leq 1$, the transformation type III_λ is a complete invariant for orbit equivalence. Hence, the previous propositions imply that orbit equivalence is not preserved by iteration.

THEOREM 9. *For $0 < \lambda < 1$, there exist transformations T on $(X, \mathcal{B}, \mu)$ and S on $(Y, \mathcal{C}, \nu)$ that both have transformation type III_λ, but the iterates T^2 on $(X, \mathcal{B}, \mu)$ and S^2 on $(Y, \mathcal{C}, \nu)$ have different transformation types.*

PROOF. For $0 < \lambda < 1$, the transformations T_1 on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ and T_1 on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$ both have transformation type III_λ, so they are orbit equivalent. However, the iterate $T_2 = T_1^2$ on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ has transformation type III_λ, but the iterate $T_2 = T_1^2$ on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$ has transformation type III_{λ^2}. Hence, the iterates are not orbit equivalent. $\square$

If two transformations on measure spaces are isomorphic, then the isomorphism between them also gives an isomorphism of the iterates. To emphasize that this may fail for orbit equivalent transformations, we give an orbit equivalence between T_1 on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ and T_1 on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$. Then we show that it is not an orbit equivalence between T_2 on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ and T_2 on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$. The measures μ_1 and μ_3 differ by switching the weights of 1 and 2. Thus, we examine the transformation $\eta : (\mathbb{Z}_3, \mathcal{M}, \mu_1) \to (\mathbb{Z}_3, \mathcal{M}, \mu_3)$ defined by switching 1's and 2's; that is,

$$(\eta(x))_i = \begin{cases} 0 & \text{if } x_i = 0, \\ 2 & \text{if } x_i = 1, \\ 1 & \text{if } x_i = 2. \end{cases}$$

Since $\eta^2(x) = x$ for all $x \in \mathbb{Z}_3$, it follows that η is invertible and $\eta^{-1} = \eta$. The measures μ_3 and $\mu_1 \circ \eta^{-1}$ are not only equivalent, but also equal. For a fixed $x \in \mathbb{Z}_3$, the composition $\eta \circ T_1 \circ \eta^{-1}$ changes only finitely many coordinates of x. Thus, there exists $k \in \mathbb{Z}$, depending on x, such that $\eta \circ T_1 \circ \eta(x) = T_k(x) = T_1^k(x)$. Since η is invertible, this implies

that η preserves orbits. Therefore, η is an orbit equivalence between T_1 on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ and T_1 on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$. However, η is not an orbit equivance between T_2 on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ and T_2 on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$. If $x \in B_{1/3}(0)$, then

$$\begin{aligned}
\eta^{-1} \circ T_2 \circ \eta(x) &= \eta \circ T_2 \circ \eta(0 + \sum_{i=1}^{\infty} x_i \cdot 3^i) \\
&= \eta \circ T_2(0 + \eta(\sum_{i=1}^{\infty} x_i \cdot 3^i)) \\
&= \eta \circ (2 + \eta(\sum_{i=1}^{\infty} x_i \cdot 3^i)) \\
&= 1 + \sum_{i=1}^{\infty} x_i \cdot 3^i = x + 1.
\end{aligned}$$

Since $T_1(x) = x + 1$ is not in the orbit of x under T_2, the transformation η does not preserve orbits for $x \in B_{1/3}(0)$. Since $\mu_1(B_{1/3}(0)) > 0$, we conclude that η is not an orbit equivalence between T_2 on $(\mathbb{Z}_3, \mathcal{M}, \mu_1)$ and T_2 on $(\mathbb{Z}_3, \mathcal{M}, \mu_3)$.

3.3. Generalizations to the g-adic Numbers

In the first part of [**51**], Mahler discusses the g-adic numbers in great detail, where $g \geq 2$ is a fixed integer that may be composite. Most of the preceding work does not depend on the fact that p is a prime number. For a composite number g, the definitions follow as in Section 1.1, with a couple of exceptions. As before, the pseudo-valuation $|\cdot|_g$ is defined on $\mathbb{Q}$ in terms of divisibility by g. If x is a nonzero rational number, then it can be written uniquely as $g^n(a/b)$, for some integer n and relatively prime integers a and b that are not divisible by g. The g-adic *order* of x is $\mathrm{ord}_g(x) = n$ and the g-adic

absolute value is

$$|x|_g = \begin{cases} g^{-\operatorname{ord}_g(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

If g is a composite number, then $|\cdot|_g$ may fail to be a valuation. In particular, the multiplicative condition

(3) $v(ab) = v(a)v(b)$ for all $a, b \in R$

may fail to hold. For example, we have $|2|_6 = 1$ and $|3|_6 = 1$, but $|2 \cdot 3|_6 = |6|_6 = 1/6$. Instead of being a valuation, $|\cdot|_g$ may only be a psuedo-valuation, in which the multiplicative condition of a valuation is weakened to a submultiplicative condition

(3') $v(ab) \leq v(a)v(b)$ for all $a, b \in R$.

Nevertheless, a pseudo-valuation still defines a metric, so $\mathbb{Q}_g$ is defined as the completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_g$. Moreover, $\mathbb{Q}_g$ may not be a field. It is a ring, but it may have zero divisors. As before, we define the set of g-adic integers and its set of units by

$$\begin{aligned} \mathbb{Z}_g &= \{x \in \mathbb{Q}_g : |x|_g \leq 1\} \text{ and} \\ \mathbb{Z}_g^\times &= \{x \in \mathbb{Z}_g : \text{ there exists } y \in \mathbb{Z}_g \text{ such that } xy = 1\}. \end{aligned}$$

Then Lemma 1 is generalized as follows.

Lemma 3. *The element* $a = \sum_{i=0}^{\infty} a_i g^i \in \mathbb{Z}_g$ *is a unit if and only if* $\gcd(a_0, g) = 1$.

The ring $\mathbb{Q}_g$ can also be defined as the set of formal Laurent series in g, with $\mathbb{Z}_g$ defined as the set of formal power series in g. The pseudo-valuation $|\cdot|_g$ is still non-Archimedean and defines a totally disconnected topology. The g-adic integers are isomorphic to the product space $\prod_{i=0}^{\infty} X_g$. Again, the Caratheodory construction defines i.i.d. product measures on $\mathbb{Z}_g$, beginning with a probability vector $(q_0, q_1, \ldots, q_{g-1})$ that defines a premeasure μ_0 on balls of $\mathbb{Z}_g$ by

$$\mu_0\left(B_{g^{-k}}\left(\sum_{i=0}^{\infty} a_i g^i\right)\right) = \prod_{i=0}^{k-1} q(a_i).$$

After replacing $a_0 \neq 0$ with $\gcd(a_0, g) = 1$, the approximations and constructions in Section 1.3 follow verbatim.

THEOREM 10. *For $a \in \mathbb{Z}_g$, we define $T_a : \mathbb{Z}_g \to \mathbb{Z}_p$ by $T_a(x) = x + a$ and consider an i.i.d. product measure μ.*

(A) For $a \in \mathbb{Z}_g^{\times}$ and $n \in \mathbb{N}$, we define $t_n : \mathbb{Z}_g \to \mathbb{Z}_p$ by

$$(t_n(x))_i = \begin{cases} (T_a(x))_i & \text{if } 0 \leq i < n \\ x_i & \text{if } i \geq n. \end{cases} \tag{1}$$

With respect to μ, the sequence of endomorphisms $\{t_n\}_{n\geq 1}$ converges to T_a in the uniform topology if and only if $a \in \mathbb{Z}$.

(B) For $a \in \mathbb{N} \subsetneq \mathbb{Z}_g$ and $0 \leq i < g^n - a$, the Radon-Nikodým derivative is

$$\frac{d\mu \circ T_a}{d\mu} \equiv \frac{\mu(B_{g^{-n}}(i+a))}{\mu(B_{g^{-n}}(i))}$$

on the ball $B_{g^{-n}}(i)$.

(C) For $a \in \mathbb{N}$ and $a \leq i < g^n$, the Radon-Nikodým derivative

$$\frac{d\mu \circ T_{-a}}{d\mu} \equiv \frac{\mu(B_{g^{-n}}(i-a))}{\mu(B_{g^{-n}}(i))}$$

on the ball $B_{g^{-n}}(i)$.

(D) For $a \in \mathbb{Z} \subsetneq \mathbb{Z}_p$, the translation T_a is ergodic with respect to μ if and only if $a \in \mathbb{Z}_g^\times$.

(E) For a rational number $a = r/s \in \mathbb{Z}_g$ in reduced form, the transformation T_a is ergodic with respect to $\mu_a = (1/s)\sum_{i=0}^{s-1} \mu T_a^{-i}$ if and only if $a \in \mathbb{Z}_g^\times$.

(F) There exists a transformation $f : \mathbb{Q}_g \to \mathbb{Q}_g$ that is invertible, (infinite) measure-preserving, conservative, ergodic, and quasi-finite with respect to Haar measure. Moreover, the transformation has Krengel entropy 0 with respect to Haar measure.

(G) For $a \in \mathbb{N} \cap \mathbb{Z}_g^\times$, $N \in \mathbb{N}$ such that $g^N > a$, and $\mu = \prod_{i=0}^{\infty} \left\{\frac{1}{Q}, \frac{q_1}{Q}, \ldots, \frac{q_{g-1}}{Q}\right\}$ with $Q = 1 + \sum_{i=1}^{g-1} q_i$, the ratio set $\mathcal{R}(T_a, \mu)$ is the closure of the multiplicative subgroup generated by

$$r(T_a, \mu) = \left\{ \frac{\mu T_a(B_{1/g^{N+1}}(k))}{\mu(B_{1/g^{N+1}}(k))} q_{g-1}^j : j \geq 0, 0 \leq k < g^{N+1} - a \right\}.$$

(H) For $0 < \lambda < 1$ and an integer $a \in \mathbb{Z}_g^\times$, the transformation T_a on $\mathbb{Z}_g$ is type III_λ with respect to the measure $\mu_1 = \prod_{i=0}^{\infty} \left\{\frac{1}{Q_1}, \ldots, \frac{1}{Q_1}, \frac{\lambda}{Q_1}\right\}$, where $Q_1 = g - 1 + \lambda$.

(I) For $g \geq 3$, if $\lambda_1, \lambda_2 \in (0,1)$ such that $\log \lambda_1$ and $\log \lambda_2$ are rationally independent, then the transformation T_1 on $\mathbb{Z}_g$ is type III_1 with respect to the measure $\mu_2 = \prod_{i=0}^{\infty} \left\{\frac{1}{Q_2}, \ldots, \frac{1}{Q_2}, \frac{\lambda_1}{Q_2}, \frac{\lambda_2}{Q_2}\right\}$, where $Q_2 = g - 2 + \lambda_1 + \lambda_2$.

Proof. The proofs of most of these results are exactly the same as the proofs already given for prime g. The only exceptions are the forward directions of (D) and (E). For completeness, we give the more general proof of the contrapostive of the forward direction

of (D) for composite g. In other words, we prove that if $a \in \mathbb{Z}_g^\times$, then T_a is not ergodic with respect to μ. The proof for the forward direction of (E) follows similarly.

If $a \notin \mathbb{Z}_g^\times$, then $\gcd(a, g) = \gcd(a_0, g) > 1$. For $k = g/\gcd(a, g)$, the set

$$A = \bigcup_{i=0}^{k-1} B_{1/g}(i \cdot \gcd(a, g))$$

is invariant under T_a. Since the balls are disjoint,

$$\mu(A) = \sum_{i=0}^{k-1} \mu(B_{1/q}(i \cdot \gcd(a, g))).$$

Since $0 < k < g$, this sum is strictly between 0 and 1. Therefore T_a is not ergodic. $\square$

It is not known whether or not rational translations are singular; i.e. whether Theorem 4 in Section 2.2 extends completely to the case where g is composite. In particular, the proof of Proposition 3 in Section 2.2 does not generalize, and the proposition was needed to prove Theorem 4. In Proposition 3, the integer k is the number of maximal weights in the probability vector that defines the measure μ. When μ is not Haar measure, we have $1 \leq k < p$, which implies that the powers k^r and p^r are relatively prime. The corresponding statement can fail for a composite number g.

With regard to the algebraic structure, if $g = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ is the prime decomposition, then $\mathbb{Q}_g$ and $\mathbb{Q}_{p_1^{n_1}} \times \mathbb{Q}_{p_1^{n_1}} \times \cdots \times \mathbb{Q}_{p_1^{n_1}}$ are isomorphic as rings. Moreover, we see that $\mathbb{Q}_p$ and $\mathbb{Q}_{p^n}$ are isomorphic as rings via the isomorphism $\sum_{i=0}^{\infty} x_i p^i \mapsto \sum_{i=0}^{\infty} (\sum_{j=0}^{n-1} x_{ni+j}) p^{ni}$. For the purposes of studying the ring structure, we can simply study $\mathbb{Q}_p$. Moreover, for the purposes of studying the topological structure, Ostrowski's theorem states that it is enough to study $\mathbb{Q}_p$. However, the measure-theoretic structure may be more complicated. For example, we could put an i.i.d. product measure μ_i on $\mathbb{Z}_{p_i}$

for each $1 \leq i \leq k$, then take the product of these measures $\prod_{i=1}^{k} \mu_i$ to put a measure on $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$. Although there is a ring isomorphism from $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ to $\mathbb{Z}_{p_1 p_2 \cdots p_k} = \mathbb{Z}_g$, the ring isomorphism may not be an isomorphism of measure spaces from $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ with the measure $\prod_{i=1}^{k} \mu_i$ to $\mathbb{Z}_g$ with any i.i.d. product measure. Since the measurable structures could be different, it is left for future work to determine ergodic properties and the transformation types of translations on $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ with respect to measures of the form $\prod_{i=1}^{k} \mu_i$.

CHAPTER 4

Haar Measure and Hausdorff Dimension of p-adic Julia Sets

This chapter examines two standard examples of Julia sets of polynomials on $\mathbb{C}_p$. Unlike the translation and multiplication maps in the previous chapters, polynomial maps of degree greater than or equal to 2 are not invertible, since $\mathbb{C}_p$ is algebraically closed. We begin by defining the Fatou and Julia sets for polynomial functions, then we review the known topological structure of the Julia sets for the examples. New work appears in Proposition 8 and Theorem 11. Since the Julia set of each example is contained in $\mathbb{Z}_p$, in Remark 2 and Proposition 8 we calculate the Haar measure m of the Julia set, as a subset of $\mathbb{Z}_p$. After reviewing the definition of Hausdorff dimension, we calculate the Hausdorff dimension of balls in Proposition 9 and the Hausdorff dimensions of the Julia sets in Corollary 2 and Theorem 11.

4.1. Definitions and the Haar Measure of Julia Sets

When considering a polynomial function with coefficients in the real numbers, we often consider it as a function on the complex numbers, which is the algebraic closure of the real numbers. Similarly, when considering polynomial functions with coefficients in the p-adic numbers, we would like to work over an algebraically closed field that contains $\mathbb{Q}_p$. Unlike the complex numbers, the algebraic closure of $\mathbb{Q}_p$ is neither a finite extension of $\mathbb{Q}_p$ nor complete. However, the completion of the algebraic closure of $\mathbb{Q}_p$ remains algebraically closed and is denoted by $\mathbb{C}_p$. In general, $\mathbb{Z}_p$ in $\mathbb{Q}_p$ is like an interval in the

real line, and $\mathbb{Q}_p$ in $\mathbb{C}_p$ is like the real line in the complex plane. Since $\mathbb{C}_p$ is not a locally compact group, it does not have a Haar measure. To avoid this difficulty, we restrict our attention to polynomial functions such that the Julia set is contained in $\mathbb{Z}_p$, which does have a Haar measure m. More information on p-adic analogues of complex dynamics can be found in Silverman's book [**62**].

The *p-adic chordal metric* on $\mathbb{C}_p$ is defined by

$$\rho_p(y,x) = \frac{|y-x|_p}{\max\{|x|_p, 1\}\max\{|y|_p, 1\}}.$$

This metric is the same as $d_p(x,y) = |y-x|_p$ for $x, y \in \mathbb{Z}_p$. Moreover, points near infinity—points with large p-adic absolute value—are near each other under ρ_p.

Definition 2. The *Fatou set* of a polynomial function is the largest open set in $\mathbb{C}_p$ on which the iterates of the polynomial function are equicontinuous under the p-adic chordal metric. The *Julia set* is the complement of the Fatou set.

For a polynomial function γ, we denote the Fatou set by $\mathcal{F}(\gamma)$ and the Julia set by $\mathcal{J}(\gamma)$. As in complex dynamics on $\mathbb{C}$, the Fatou and Julia sets of a polynomial are completely invariant under the polynomial. In other words, we have

$$\begin{aligned} \mathcal{F}(\gamma) &= \gamma(\mathcal{F}(\gamma)) = \gamma^{-1}(\mathcal{F}(\gamma)) \text{ and} \\ \mathcal{J}(\gamma) &= \gamma(\mathcal{J}(\gamma)) = \gamma^{-1}(\mathcal{J}(\gamma)). \end{aligned}$$

Hence, the polynomial function restricted to the Julia set is a dynamical system in its own right. The following examples have Julia sets contained in $\mathbb{Z}_p$. Thus, we consider the polynomial restricted to the Julia set, with respect to Haar measure on $\mathbb{Z}_p$.

The first example is

$$\begin{aligned} \phi_p(z) : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto \frac{z^p - z}{p}. \end{aligned}$$

In [**35**], Hsia shows that $\mathcal{J}(\phi_p) = \mathbb{Z}_p$ and that ϕ_p on $\mathbb{Z}_p$ is homeomorphic to the shift σ on $\prod_{i=0}^{\infty} X_p$. In [**65**], Woodcock and Smart construct such a homeomorphism independently of Hsia and also show that the system $(\mathbb{Z}_p, \mathcal{M}, m; \phi_p)$ is isomorphic to the system $(\prod_{i=0}^{\infty} X_p, \mathcal{N}, \prod_{i=0}^{\infty} \{1/p, \ldots, 1/p\}; \sigma)$.

Remark 2. It is a simple observation that $m(\mathcal{J}(\phi_p)) = m(\mathbb{Z}_p) = 1$.

For $p > 2$, the second example is

$$\begin{aligned} \psi_p(z) : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto \frac{z^2 - z}{p}. \end{aligned}$$

In [**62**], Silverman throroughly discusses the Julia set of ψ_p. He proves that the set

$$\Lambda = \left\{ z \in \mathbb{C}_p : \psi_p^n(z) \text{ is bounded for all } n \geq 0 \right\} \subset B_{1/p}(0) \cup B_{1/p}(1)$$

is contained in $\mathbb{Q}_p$ and defines the itinerary map

$$\begin{aligned} \beta : \Lambda &\to \prod_{i=0}^{\infty} X_2 \\ z &\mapsto [\beta_0, \beta_1, \beta_2, \ldots], \end{aligned}$$

if $\psi_p^n(z) \in B_{1/p}(\beta_n)$ for all $n \geq 0$. For any prime $p > 2$, he proves that ψ_p on Λ is homeomorphic to the shift map on $\prod_{i=0}^{\infty} X_2$. Since the shift map is uniformly expanding, the Julia set of the shift map is all of $\prod_{i=0}^{\infty} X_2$. It follows from the homeomorphism that $\mathcal{J}(\psi_p) = \Lambda$. The same method of proof can be used to show that $\mathcal{J}(\phi_p) = \mathbb{Z}_p$.

The map ψ_p is an example of a general result of Benedetto, Briend, and Perdry in [**8**]. They prove that if φ is a quadratic polynomial on $\mathbb{C}_p$ such that the Julia set is nonempty, then φ on $\mathcal{J}(\varphi)$ is homeomorphic to σ on $\prod_{i=0}^{\infty} X_2$. Since $\mathbb{Z}_p$ is itself homeomorphic to the product space $\prod_{i=0}^{\infty} X_2$, the measure of the Julia set is not immediately clear. Examining preimages of these balls gives the next result.

Proposition 8. *For $p > 2$, the transformation $\psi_p : \mathbb{C}_p \to \mathbb{C}_p$ is defined by $\psi_p(z) = (z^2 - z)/p$. With respect to Haar measure on $\mathbb{Z}_p$, the Julia set $\mathcal{J}(\psi_p) \subsetneq \mathbb{Z}_p$ has measure zero.*

Proof. First, we examine how ψ_p acts on balls. For $a, b \in \mathbb{C}_p$, factoring gives

$$\psi_p(a) - \psi_p(b) = \frac{1}{p}(a - b)(a + b - 1).$$

If a and b are in $B_{1/p}(0)$ or $B_{1/p}(1)$, then $a + b$ is equal to 0 or 2 modulo p. In either case, it follows that $|a + b - 1|_p = 1$. Thus, if a, b are both in $B_{1/p}(0)$ or $B_{1/p}(1)$, then

$$|\psi_p(a) - \psi_p(b)|_p = p|a - b|_p. \tag{1}$$

The next step of the proof is an induction argument that hinges on equation (1).

Since $\mathcal{J}(\psi_p) \subset B_{1/p}(0) \cup B_{1/p}(1)$, the remaining $p-2$ balls of radius $1/p$ in $\mathbb{Z}_p$ are contained in the Fatou set. Hence, we begin the base case by setting

$$J_0 = B_{1/p}(0) \cup B_{1/p}(1).$$

For each $i = 0, 1$, equation (1) implies that ψ_p maps distinct balls of radius $1/p^2$ in $B_{1/p}(i)$ to distinct balls of radius $1/p$ in $\mathbb{Z}_p$. The function ψ_p maps exactly 2 of these balls to J_0. We define J_1 to be the 2^2 balls of radius $1/p^2$ in J_0 that map to ball of radius $1/p$ in J_0 under ψ. Since the Fatou and Julia sets are invariant under iteration and preimages, we conclude that $\mathcal{J}(\psi_p) \subset J_1$.

For an induction hypothesis, we assume that there exists an integer $k \geq 1$ such that the Julia set is contained in a set J_k that consists of 2^k balls of radius $1/p^k$ such that 2 of these balls are contained in each ball of radius $1/p^{k-1}$ in J_{k-1}. If $B_{1/p^k}(a) \subset J_k$, then equation (1) and the definition of ψ_p imply that the p balls of radius $1/p^{k+1}$ in $B_{1/p^k}(a)$ are mapped to the p balls of radius $1/p^k$ in $B_{1/p^{k-1}}(a)$. By the induction hypothesis, 2 of these balls are in J_{k-1}. Then we define J_{k+1} to be the union of the 2 disjoint balls of radius $1/p^{k+1}$ from each of the 2^k balls of radius $1/p^k$ in J_k. Again, the invariance of the Julia set under preimages implies that $\mathcal{J}(\psi_p) \subset J_{k+1}$.

The final step of the proof calculates the measure of $\mathcal{J}(\psi_p)$ in $\mathbb{Z}_p$. Since $\mathcal{J}(\psi_p) \subset J_k$ for all $k \geq 0$, the Haar measure of the Julia set is bounded above by

$$m(J_k) = \frac{2^k}{p^k}.$$

Since $p > 2$, the upper bound implies that $m(\mathcal{J}(\psi_p)) = 0$.

$\square$

4.2. Hausdorff Dimension of Julia Sets

Although $m(\mathcal{J}(\psi_p)) = 0$ for all $p > 2$, the Hausdorff dimension of $\mathcal{J}(\psi_p)$ varies with p. Since the distance induced by the p-adic absolute value makes $\mathbb{Q}_p$ a separable metric space, it is possible to calculate the Hausdorff dimension of subsets of $\mathbb{Q}_p$. In order to set notation, we begin by reviewing the Caratheodory construction of Hausdorff measures and the definition of Hausdorff dimension. First, an outer measure is defined from countable covers of a set by balls with diameter less than $\delta > 0$. Since every point in a ball is a center of the ball, the diameter of a ball is equal to its radius. For $A \subset \mathbb{Q}_p$ and $s \geq 0$,

$$h^*_{s,\delta}(A) = \inf \left\{ \sum_{i=0}^{\infty} (r_i)^s : A \subset \bigcup_{i=0}^{\infty} B_{r_i}(a_i), r_i \leq \delta, a_i \in \mathbb{Q}_p \right\}$$

is an outer measure. Then

$$h^*_s(A) = \lim_{\delta \to 0} h^*_{s,\delta}(A)$$

is also an outer measure. The restrictions of $h^*_{s,\delta}$ and h^*_s to measurable sets are the measure $h_{s,\delta}$ and the s-dimensional Hausdorff measure h_s, respectively. Finally, the *Hausdorff dimension* of a measurable set $A \subset \mathbb{Q}_p$ is

$$\mathrm{Hdim}(A) = \sup \{ s \geq 0 : h_s(A) > 0 \}.$$

As might be expected, balls in $\mathbb{Q}_p$ have Hausdorff dimension 1.

Proposition 9. *If $B_{p^N}(a)$ is a ball in $\mathbb{Q}_p$, then the Hausdorff dimension of $(B_{p^N}(a))$ is* 1.

Proof. For one inequality, we use specific covers to give a bound on the s-dimensional Hausdorff measure. The ball of radius p^N is covered by $p^N p^n$ pairwise disjoint balls of

radius $1/p^n$. If $1/p^n < \delta$, then

$$\begin{aligned} h_{s,\delta}(B_{p^N}(a)) &\leq p^N p^n \left(\frac{1}{p^n}\right)^s \\ &= p^N p^{n(1-s)}. \end{aligned}$$

Thus, the upper bound

$$\begin{aligned} h_s(B_{p^N}(a)) &= \lim_{\delta \to 0} h_{s,\delta}(B_{p^N}(a)) \\ &\leq \lim_{n \to \infty} p^N p^{n(1-s)} \\ &= \begin{cases} 0 & \text{if } s > 1 \\ p^N & \text{if } s = 1 \\ \infty & \text{if } 0 \leq s < 1 \end{cases} \end{aligned}$$

implies that $\mathrm{Hdim}(B_{p^N}(a)) \leq 1$.

Next, we show that $h_1(B_{p^N}(a)) > 0$. If $\{B_{r_i}(a_i)\}_{i \geq 1}$ is a countable cover of $B_{p^n}(a)$ by balls, then

$$\begin{aligned} \sum_{i=1}^{\infty} r_i &= \sum_{i=1}^{\infty} m(B_{r_i}(a_i)) \\ &\geq m(\bigcup_{i=1}^{\infty} B_{r_i}(a_i)) \\ &\geq m(B_{p^N}(a)) = p^N. \end{aligned}$$

For all $\delta > 0$, these inequalities imply that $h_{1,\delta}(B_{p^N}(a)) \geq p^N$, which implies that $h_1(B_{p^N}(a)) = p^N > 0$. Therefore $\mathrm{Hdim}(B_{p^N}(a)) = 1$. $\square$

Corollary 2. *The Hausdorff dimension of $\mathbb{Z}_p$ is 1.*

As $\mathcal{J}(\phi_p) = \mathbb{Z}_p$ is a ball in $\mathbb{Q}_p$, Proposition 9 implies that $\mathrm{Hdim}(\mathcal{J}(\phi_p)) = 1$. Next, we calculate $\mathrm{Hdim}(\mathcal{J}(\psi_p))$.

Theorem 11. *For $p > 2$, $Hdim(\mathcal{J}(\psi_p)) = \log 2/\log p$.*

Proof. In the proof of Proposition 8, the set $J_n = \mathbb{Z}_p \setminus \bigcup_{k=0}^{n} F_k$ is a cover of $\mathcal{J}(\psi_p)$ made up of 2^n disjoint balls of radius $1/p^n$. For $1/p^n < \delta$, this cover yields the inequality

$$h_{s,\delta}(\mathcal{J}(\psi_p)) \leq 2^n \left(\frac{1}{p^n}\right)^s = \left(\frac{2}{p^s}\right)^n,$$

which implies that

$$\begin{aligned} h_s(\mathcal{J}(\psi_p)) &\leq \lim_{n\to\infty} \left(\frac{2}{p^s}\right)^n \\ &= \begin{cases} 0 & \text{if } s > \log 2/\log p \\ 1 & \text{if } s = \log 2/\log p \\ \infty & \text{if } 0 \leq s < \log 2/\log p. \end{cases} \end{aligned}$$

Therefore $\mathrm{Hdim}(\mathcal{J}(\psi_p)) \leq \log 2/\log p$.

For $s = \log 2/\log p$, we show that $h_s(\mathcal{J}(\psi_p)) \geq 1$. By the proof of Proposition 8, $\mathcal{J}(\psi_p)$ is a totally bounded subset of the metric space $\mathbb{Q}_p$. Hence, the set $\mathcal{J}(\psi_p)$ is relatively compact. Thus, a countable cover of $\mathcal{J}(\psi_p)$ by balls contains a finite subcover, $B_{1/p^{n_i}}(a_i)$ where $n_i \in \mathbb{N}$ and $a_i \in \mathbb{Z}_p$ for $i = 0, \ldots, k$. If $m = \max_{0\leq i\leq k} n_i$, then a ball $B_{1/p^{n_i}}(a_i)$ contains $p^m/p^{n_i} = p^{m-n_i}$ ball of radius $1/p^m$. If $J_m \cap B_{1/p^{n_i}}(a_i)$ is nonempty,

then it is equal to 2^{m-n_i} disjoint balls of radius $1/p^m$. Since $2 = p^s$, we have

$$\begin{aligned} m(B_{1/p^{n_i}}(a_i))^s &= \left(\frac{1}{p^{n_i}}\right)^s \\ &= \left(\frac{p^{m-n_i}}{p^m}\right)^s \\ &= 2^{m-n_i}\left(\frac{1}{p^m}\right)^s. \end{aligned}$$

Thus,

$$\begin{aligned} h_s(\mathcal{J}(\psi_p)) &\geq \sum_{i=0}^{k} m(B_{1/p^{n_i}}(a_i))^s \\ &\geq 2^m\left(\frac{1}{p^m}\right)^s = 1. \end{aligned}$$

Therefore $\mathrm{Hdim}(\mathcal{J}(\psi_p)) = \log 2/\log p$. □

4.3. Further Examples of Haar Measure for Julia Sets

So far, we have seen two examples of polynomials where the Julia set is contained in $\mathbb{Z}_p$. One Julia set had Haar measure 0, and the other had Haar measure 1. Is there a polynomial such that the Julia set is contained in $\mathbb{Z}_p$ and has Haar measure strictly between 0 and 1? If ϕ_p is conjugated by $M_p(z) = pz$, then $\Phi_k = M_p^k \circ \phi_p \circ M_p^{-k}$ is the polynomial

$$\begin{aligned} \Phi_k : \mathbb{C}_p &\to \mathbb{C}_p \\ z &\mapsto \frac{z}{p}\left(\left(\frac{z}{p^k}\right)^{p-1} - 1\right). \end{aligned}$$

Since $\mathcal{J}(\phi_p) = \mathbb{Z}_p$ and M_p is a homeomorphism on $\mathbb{C}_p$, it follows that

$$\mathcal{J}(\Phi_k) = M_p^k(\mathcal{J}(\phi_p)) = B_{1/p^k}(0).$$

Thus, the Julia set of Φ_k is contained in $\mathbb{Z}_p$ and has Haar measure $1/p^k$, which is strictly between 0 and 1.

From this family of examples, we see that $\mathbb{Z}_p$ plays the role in $\mathbb{Q}_p$ that the interval $[-1, 1]$ plays in $\mathbb{R}$. In this regard, ϕ_p is similar to $\chi_2(z) = 2z^2 - 1$, which is a Chebyshev polynomial. The Julia set of χ_2 is the interval $[-1, 1]$. Moreover, if χ_2 is conjugated by multiplication by a real number $r > 0$, then the resulting map has Julia set $[-r, r]$. Thus, we can shrink or expand the measure of the Julia set of χ_2, with respect to the 1-dimensional Lebesgue measure on the real line.

A more difficult question is whether or not there exists a polynomial map on $\mathbb{C}$ such that the Julia set has positive 2-dimensional Lebesgue measure as a subset of the Riemann sphere but is not the entire Riemann sphere. An example of such a polynomial is given by Buff and Chéritat in [**10**–**12**]. While $\mathbb{C}_p$ is a complete, algebraically closed set containing $\mathbb{Q}_p$, it is not locally compact and lacks other desirable properties for working with measures. Berkovich space is a compact, simply connected metric space that contains $\mathbb{C}_p$. It has shown promise as a more appropriate setting to ask dynamical questions and to prove results similar to those in complex dynamics. A polynomial map can be extended to Berkovich space and used to construct a natural measure that is supported on the Julia set and used to prove equidistribution results [**7**, **62**]. To ask an analogous question for polynomial maps on Berkovich space, an analog of Lebesgue measure on the Riemann sphere needs to be defined and understood.

Additionally, Berkovich space has a tree structure on which there exist inequivalent metrics. If these metrics are separable, then each would define a Hausdorff dimension. One possible future direction is to determine whether the different metrics give different Hausdorff dimensions to the Julia set of a polynomial function.

CHAPTER 5

Future Work

5.1. Transformation Type of Translation by a Rational Number

Theorem 4 in Section 2.2 states that translation by a nonintegral rational number is singular with respect to i.i.d. product measures other than Haar measure. Nevertheless, we define an averaged measure in Section 2.3. If $a = r/s \in \mathbb{Z}_p^\times$ is in reduced form and μ is an i.i.d. product measure, then T_a is nonsingular and ergodic with respect to

$$\mu_a = \frac{1}{s}\sum_{i=0}^{s-1} \mu T_a^{-i}.$$

However, T_a does not preserve μ_a, unless μ is Haar measure. Thus, the transformation type of T_a with respect to μ_a is not obvious. Since T_a is nonsingular with respect to μ_a, we know that the Radon-Nikodým derivative exists. We can approximate the Radon-Nikodým derivative by taking ratios of balls of smaller and smaller radii. The linearization

$$\begin{aligned} \varphi : \mathbb{Z}_p &\to [0,1] \\ \sum_{i=0}^{\infty} x_i p^i &\mapsto \sum_{i=0}^{\infty} \frac{x_i}{p^{i+1}}, \end{aligned}$$

is given by Robert in [**57**]. To plot the approximation, we identify $x = \sum_{i=0}^{\infty} x_i p^i \in \mathbb{Z}_p$ with $\varphi(x) \in [0,1]$. For example, we find a very clear picture of the values in the ratio set by using ratios of balls of radius $1/2^8$ to approximate the Radon-Nikodým derivative of

$T_3 : \mathbb{Z}_2 \to \mathbb{Z}_2$ with respect to the measure $\mu = \prod_{i=0}^{\infty} \{1/3, 2/3\}$. Using the identification $x \sim \varphi(x)$, Figure 5.1 is a plot of the points $(x, \mu(B_{2^{-8}}(x))/\mu(B_{2^{-8}}(x+3)))$, for integers x such that $0 \leq x < 256$. Since the values of the ratio set appear to be powers of 2, we can correctly guess from the plot that T_3 has transformation type $\mathrm{II}_{1/2}$ with respect to μ.

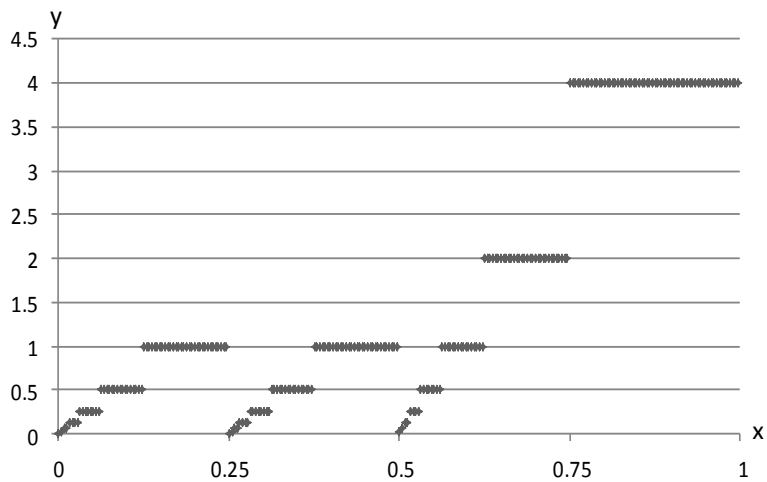

FIGURE 5.1. Approximation of $d\mu T_3^{-1}/d\mu$.

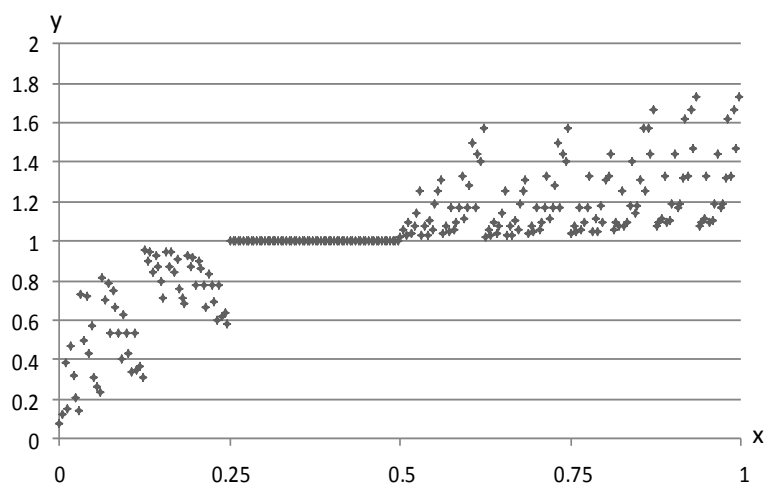

FIGURE 5.2. Approximation of $d\mu_{1/3} T_{1/3}^{-1}/d\mu_{1/3}$.

However, this approximation does not give as clear a picture for rational numbers. Figure 5.2 is a plot of the points $(x, \mu_{1/3}(B_{2^{-8}}(x))/\mu_{1/3}(B_{2^{-8}}(x+1/3)))$ for integers x such that $0 \leq x < 256$. It is clear from the plot that 1 should be in the ratio set, but 1 is always in the ratio set. Even for this example, the transformation type is unknown.

5.2. The Chacon Map as a 3-adic Transformation

This section contains preliminary work in progress with César Silva on representing the Chacon map as a 3-adic transformation. Chacon defines his map in [**14**] to give an example of a transformation that is weakly mixing but not strongly mixing. The Chacon map is defined on an interval by a cutting and stacking construction with spacers. For the base case, the first stack is the unit interval. Inductively, we cut the nth stack into three substacks of equal width, and we take a spacer that is the same width as the substacks. From left to right, the second substack is placed over the first, a spacer is placed above the second substack, and the third substack is placed above the spacer. Adding up the lengths of the spacers, we find that the Chacon map C is a transformation on the interval $[0, 3/2]$. We can also start the construction with the interval $[0, 2/3]$, so that the final transformation is a map on $[0, 1]$. We call this construction the normalized Chacon map $\bar{C}$. Since the Chacon map preserves a finite measure, it has transformation type II_1. For $0 \leq \lambda \leq 1$, type III_λ and type II_∞ versions of the Chacon map are constructed and studied in [**17**, **34**]. These transformations are contructed by cutting each stack into substacks of unequal widths.

The Chacon map can also be represented as a tower over $T_1 : \mathbb{Z}_3 \to \mathbb{Z}_3$. For example, as a tower over T_1 with the height function

$$h(x) = \begin{cases} 1 & \text{if } x = \sum_{i=0}^{k-1} 2 \cdot 3^i + 1 \cdot 3^k + \sum_{i=k+1}^{\infty} x_i 3^i \text{ for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The representation of the Chacon map in terms of T_1 on $\mathbb{Z}_3$ can be useful for studying its ergodic properties. For example, this description is used in [**38**, **56**] to study the spectral properties of the Chacon map. This section gives two possible representation of the Chacon map as a p-adic transformation. Unlike the tower representation, these representations are defined as transformations on a subset of $\mathbb{Q}_p$, without resorting to towers. Since these representations act on a subset of $\mathbb{Q}_p$, perhaps more of the algebraic structure of the p-adic numbers can be used with these representations to explore the ergodic properties of the Chacon map.

For the first representation C_1, we use elements of $\mathbb{Q}_3 \setminus \mathbb{Z}_3$ to play the role of the second level of the tower. We let

$$S = \mathbb{Z}_3 \cup \left\{ \frac{1}{3} + \sum_{i=0}^{k-1} 2 \cdot 3^i + 1 \cdot 3^k + \sum_{i=k+1}^{\infty} x_i 3^i : k \geq 1 \right\}.$$

In coordinates, we define $C_1 : S \to S$ by

$$C_1(x) = \begin{cases} T_1(x) & \text{if } x = 0 + \sum_{i=1}^{\infty} x_i 3^i \\ 1/3 + x & \text{if } x = \sum_{i=0}^{k-1} 2 \cdot 3^i + 1 \cdot 3^k + \sum_{i=k+1}^{\infty} x_i 3^i \text{ for some } k \geq 1 \\ T_1(x - 1/3) & \text{if } x = 1/3 + \sum_{i=0}^{k-1} 2 \cdot 3^i + 1 \cdot 3^k + \sum_{i=k+1}^{\infty} x_i 3^i \text{ for some } k \geq 1. \end{cases}$$

While C_1 may be a useful representation, we would also like a representation that is an automorphism on $\mathbb{Z}_3$. This can be done by using the isomorphism $\varphi : \mathbb{Z}_3 \to [0,1]$ given in Section 5.1. We define $C_2 : \mathbb{Z}_3 \to \mathbb{Z}_3$ by $C_2(x) = \varphi^{-1} \circ \bar{C} \circ \varphi(x)$. While this representation is an automorphism of $\mathbb{Z}_3$, it is unclear how to write C_2 in coordinates. In fact, preliminary calculations suggest that C_2 is not a continuous function on $\mathbb{Z}_3$.

For future work with these representations, we could try to generalize them to include representations of the type III_λ and type II_∞ versions of the Chacon map. Examining the coefficients of the Mahler or van der Put series for a continuous representation could be used as an aid in studying the ergodic properties of the Chacon map. Previous work on the relationship between ergodicity and the coefficients of the Mahler or van der Put series can be found in [**6**, **42**, **46**].

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