Structure of Quiver Polynomials and Schur Positivity

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Abstract

RYAN KALISZEWSKI: Structure of Quiver Polynomials and Schur Positivity
(Under the direction of Richárd Rimányi)

Given a directed graph (quiver) and an association of a natural number to each vertex, one can construct a representation of a Lie group on a vector space. If the underlying, undirected graph of the quiver is a Dynkin graph of A-, D-, or E-type then the action has finitely many orbits. The equivariant fundamental classes of the orbit closures are the key objects of study in this paper. These fundamental classes are polynomials in universal Chern classes of a classifying space so they are referred to as "quiver polynomials."

It has been shown by A. Buch [B08] that these polynomials can be expressed in terms of Schur-type functions. Buch further conjectures that in this expression the coefficients are non-negative.

Our goal is to study the coefficients and structure of these quiver polynomials using an iterated residue description due to R. Rimányi [RR]. We introduce the Jacobi-Trudi transform, which creates an equivalence realtion on rational functions, to show that Buch's conjecture holds for a quiver polynomial if and only if there is a representative in the equivalence class that is Schur positive. Also we define a notion of strong Schur positivity and demonstrate the connection between this and Schur positivity, proving Schur positivity for some special cases of quiver polynomials.

Table of Contents

Introduction

Consider two complex vector bundles, F_1, F_2 over a complex projective variety X, with bundle map $\varphi : F_1 \to F_2$. Suppose that the bundle map satisfies some transversality conditions (that will be discussed later). Call the set $\Omega_r = \Omega_r(F_\bullet) = \{x \in X | \text{rank}(\varphi_x) \leq r\}$ to be a degeneracy locus of the bundles $F_\bullet$. The set Ω_r represents a cohomology class $[\Omega_r] \in H^*(X)$ and these classes will be the principal objects of study in this paper.

For us the concept of a degeneracy locus must be abstracted to other vector bundles. In the above example, the process for calculating $[\Omega_r]$ in terms of the characteristic class of F_1 and F_2 is the well-known Giambelli-Thom-Porteous formula. However, once these degeneracy loci are more abstract a method to compute these characteristic classes will have to be introduced. One of the methods of describing the characteristic class associated to a degeneracy locus is the iterated residue method of Rimányi, [RR], and is presented in Chapter 2.

0.1. History

The study of degeneracy loci is a natural extention of Bézout's Theorem. Bézout's Theorem states that the number of common zeroes of m polynomials of m variables is the product of the degrees. Of course one must consider the polynomials over an algebraically closed field, take the solutions to lie in projective space, and count multiplicities to achieve a correct statement.

More generally one considers homogeneous polynomials, $f_1, \ldots, f_m$ of degree $d_1, \ldots, d_m$ in $N+1$ variables, $N \geq m$. If one tries to describe the locus of zeroes $V = V(f_1, \ldots, f_m) \subset \mathbb{P}^N$ and all of the polynomials are general enough then V is an irreducible subvariety of codimension m. Here, Bézout's theorem says that the degree of V, the number of intersection points with a general linear subspace of complimentary dimension is $\prod d_i$.

In the ninteenth century there was further abstraction. Let A be an $l \times m$ matrix of homogeneous polynomials in $N+1$ variables. Describe the set V_r to be the locus of points in $\mathbb{P}^N$ where the rank of A is at most $r \leq min(l, m)$. This will be cut out by the minors of size $r+1$. If $r = 0$ this is just the zeroes of all of the entries, which is answered by Bézout's theorem.

Now consider two complex vector bundles over a complex projective variety, with a bundle map between. If one chooses a point on the variety, a neighborhood of the point, and fixes a basis for the bundles then the map can be represented as a matrix whose entries are regular functions–homogeneous polynomials. And if one studies where the rank of this map is at most r, this is equivalent to the ninteenth century problem. Presently, it is known how to express the cohomology class associated to the locus of points by using the Giambelli-Thom-Porteous formula.

However, what if there are more than two vector bundles with many maps between? First off, degeneracy loci begin to encode more information than just rank of the maps so one must be careful in defining the loci. In the case that the underlying shape of the bundles (replace a bundle with a vertex and a map with an arrow) is a quiver of ADE-type several recent papers have found a way to express the cohomology class associated to the loci. Studying this cohomology class, or quiver polynomial, may lead to further insight about the structure of the degeneracy locus itself and may hint at geometric phenomena.

Quiver polynomials of Dynkin type generalize several important polynomials in algebraic combinatorics. For example, the Giambelli-Thom-Porteous formulae [P], the double Schur and Schubert polynomials of Schubert calculus [F99], and the quantum [FGP] and universal Schubert polynomials [F99].

There has been a lot of focus in the past twelve years to find various formulae and algorithms to calculate quiver polynomials, such as [B02, B08, BFR, BF, BKTY, BR, BSY, FR02, FGP, KS06, KMS]. By now there are effective methods to find any particular quiver polynomial. The three papers that approach the problem for quivers of Dynkin type A,D,E are [FR02], [KS06], and [B08].

The structure of quiver polynomials. The goal is to understand the structure of these quiver polynomials. The following two phenomena have been discovered/conjectured:

- *Stability.* In [FR07] the authors study an analogous problem: Thom polynomials of singularities. They found that the equivariant fundamental classes display an unexpected stability property, which enables one to organize infinitely many such classes into a generating sequence. This phenomenon is developed under the name of Iterated Residue generating sequences in [BSz], [FR12], [K2]. This stability seems to hold for quiver polynomials.
- *Positivity.* In [B08] Buch proved that quiver polynomials are linear combinations of certain products of Schur polynomials. He conjectured that all of the coefficients in such an expression are all non-negative. These coefficients provide a wide generalization of several combinatorial constants in an similar way to Littlewood-Richardson coefficients. The usual techniques of proving positivity in equivariant cohomology (geometric intersection numbers, Gröbner degenerations, interpolation theory, counting arguments) so far failed to prove Buch's conjecture.

0.2. Quiver Polynomials

Let $Q = (Q_0, Q_1)$ be an oriented graph, or quiver, with vertex set $Q_0 = \{1, \ldots, n\}$ and finite edge set Q_1. An arrow $a \in Q_1$ has head $h(a) \in Q_0$ and tail $t(a) \in Q_0$. Fix a dimension vector $(e_1, \ldots, e_n) \in \mathbb{N}^n$ and associate to each vertex a vector space $E_i = \mathbb{C}^{e_i}$. The group $G = \times_{i=1}^n GL(E_i)$ acts on the vector space $V = \oplus_{a\in Q_1} Hom(E_{t(a)}, E_{h(a)})$ by

$$(g_i)_{i\in Q_0} \cdot (\phi_a)_{a\in Q_1} = \left(g_{h(a)} \circ \phi_a \circ g_{t(a)}^{-1}\right)_{a\in Q_1},$$

i.e. change of basis.

Consider an orbit of the G-action of V. If the underlying quiver is of ADE-type, that is its unoriented, underlying graph is a Dynkin diagram of type A_n, D_n, E_6, E_7, or E_8 then by Gabriel's theorem for any dimension vector there are only finitely many G-orbits in V, [BGP]. From now on we will restrict our attention to these quivers of ADE-type.

An *orbit closure*, S, is the closure of a G-orbit in V. Following the lectures of W. Fulton, [F], since the orbit closure is G-invariant it represents a class $[S] \in H_G^*(V)$, the G-equivariant cohomology of V.

The group G doesn't act freely on V, but one can find a contractable space EG on which G does act freely with quotient $BG = EG/G$. Then replace V with $V \times EG$ and let the action of G on $V \times EG$ be the diagonal action. The G-equivariant cohomology of V is defined to be the cohomology of $B_G V = V \times_G EG = (V \times EG)/G$. Since V is equivariantly contractable to a point, $H_G^*(V) = H_G^*(pt) = H^*(BG)$, the cohomology of the classifying space.

It is known that

$$H^*(BG) = \mathbb{Z}[c_j(\gamma_i) : i = 1, \dots, n; j = 1, \dots, e_i]$$

where $c_j(\gamma_i)$ denotes the universal Chern classes of the canonical bundle $EG \to BG$. Since $H^*(BG)$ is a polynomial ring the class $[S]$ is called the quiver polynomial corresponding to S.

0.3. Degeneracy Loci

Let X be a complex projective variety and

$$\begin{tikzcd} F = \oplus_{i=1}^n F_i \arrow[loop right, "\{\varphi_{j,k}\}_{(j,k)\in I}"] \arrow[d, "\pi"] \\ X \end{tikzcd}$$

be a collection of vector bundles for $1 \leq i \leq n$, with $rank(F_i) = e_i$. Suppose that $\{\varphi_{j,k}\}_{(j,k)\in I}$ are a finite collection of bundle maps with I some indexing set and

$\varphi_{j,k} : F_j \to F_k$. Further assume that the underlying quiver, $Q = (Q_0, Q_1)$ (by replacing bundle F_i with vertex i and maps with arrows), is a Dynkin quiver of ADE-type.

Construct the bundle $\xi = \oplus_{(j,k)\in I} Hom(F_j, F_k)$ over X, where each fiber is isomorphic to $V = \oplus_{(j,k)\in I} Hom(E_j, E_k)$, using the notation of section 2. Then the bundle maps $\{\varphi_{j,k}\}_{(j,k)\in I}$ define a section of ξ, $\varphi : X \to \xi$.

Suppose $S \subset V$ is an orbit closure. Define a subset of $\xi_S \subset \xi$ as follows. Let $x \in X$ and consider a trivialization of ξ at x. The point $(v, x) \in \xi_S$ if $v \in S$. This definition makes sense because S is invariant under change of basis, so the definition is independent of the choice of trivialization.

Definition 0.3.1. Continuing with the notation above, the **degeneracy locus of X corresponding to S** is $\Omega_S = \varphi^{-1}(\xi_S)$.

One can describe a degeneracy locus as "the points of X where the bundle maps are in S."

Consider the classifying map

$$\begin{tikzcd} F \arrow[r, "\hat{f}"] \arrow[d, "\rho"] & EG \arrow[d, "\rho'"] \\ X \arrow[r, "f"] & BG \end{tikzcd}.$$

Note that $BG = \oplus_{i=1}^{n} Gr(E_i)$ and $EG = \oplus_{i=1}^{n} \gamma_i$, the canonical bundle. Construct the bundle $B_G V = \oplus_{(j,k)\in I} Hom(\gamma_j, \gamma_k)$ over BG. The fibers in $B_G V$ are isomorphic to V.

Consider the following diagram

$$\begin{tikzcd} \xi \arrow[r, "\hat{g}"] \arrow[d, "\pi"] & B_G V \arrow[d, "\pi'"] \\ X \arrow[r, "g"] \arrow[u, bend left, "\varphi"] & BG \end{tikzcd}.$$

Since the fibers of $B_G V \to BG$ are isomorphic to V, one can construct $B_G S \subset B_G V$ in a similar way as the construction of ξ_S.

Proposition 0.3.2. *If $h : A \to B$, with $h \pitchfork C \subset B$ then $f^*[C] = [f^{-1}(C)]$.*

REMARK. If C is not smooth then we have to be a bit more clear about what $h \pitchfork C$ means. In this instance we mean that C is a stratified space

$$\emptyset = C_{-1} \subset C_0 \subset \ldots \subset C_n = C$$

where C_i is the singular part of C_{i+1}, and $C_i \diagdown C_{i+1}$ is smooth. Then $h \pitchfork C$ means $h \pitchfork C_i \diagdown C_{i+1}$ for each $i < n$. In this instance proposition 0.3.2 still holds.

THEOREM 0.3.3. *If $\varphi \pitchfork \xi_S$ then the class $[\Omega_S] \in H^*(X)$ is the quiver polynomial corresponding to S evaluated at the characteristic classes of the vector bundle, F. That is, replacing the $c_j(\gamma_i)$ with the j^{th} Chern class of the i^{th} vector bundle over X.*

PROOF. Let $\sigma : BG \to B_G V$ be the zero-section of the bundle.

From the previous section we defined the quiver polynomial as the class $[S] \in H^*_G(BG)$ by considering S as a subset of V. View S as a G-stable subvariety of the bundle $V \to pt$ and consider the following construction:

$$\begin{array}{ccccc} V & & V \times EG & & B_G V = V \times_G EG \\ \downarrow & & \downarrow & & \downarrow \\ pt & & EG = pt \times EG & & BG = pt \times_G EG \end{array} .$$

By definition $H^*_G(V) = H^*(BG)$, but since bundles are retractable $H^*(BG) \simeq H^*(B_G V)$. Also by definition $[S] \in H^*(BG) = H^*_G(V)$ is exactly $[B_G S] \in H^*(B_G V)$. Then σ^* is an isomorphism of $H^*(B_G V)$ to $H^*(BG)$, with $[S] = \sigma^*[B_G S]$.

Now $\hat{g}^{-1}(B_G S) = \xi_S$ and $\hat{g} \pitchfork B_G S$. Therefore

$$\hat{g}^*[B_G S] = [g^{-1}(B_G S)] = [\xi_S].$$

By assumption, $\varphi \pitchfork \xi_S$ so

$$\varphi^*[\xi_S] = [\varphi^{-1}(\xi_S)].$$

Recall that $\varphi^{-1}(\xi_S) = \Omega_S$, the degeneracy locus. This means that

$$\varphi^* \circ \hat{g}^*[B_G S] = [\Omega_S].$$

But by the commutativity of the diagram,

$$\varphi^* \circ \hat{g}^*[B_G S] = g^* \circ \sigma^*[B_G S] = g^*[S].$$

By rewriting, we have

$$[\Omega_S] = g^*[S],$$

which is that statement of the theorem. □

Summary The cohomology class associated to the degeneracy locus is a universal polynomial evaluated at the Chern classes of the bundles. This univeral polynomial is only determined by the underlying quiver, rank of the bundles of F, and orbit corresponding to the degeneracy locus.

CHAPTER 1

Basic Definitions and Notation

In this chapter we introduce definitions and notation that will be used throughout the rest of the document. In section 1.2 we define partitions and demonstrate a useful action of the symmetric group on the partitions. Then we introduce Schur functions and demonstrate several of their properties. Sections 1.3 and 1.4 detail two transformations of the Laurent polynomial ring that will be used to transform generating functions introduced in chapter 2.

1.1. Alphabets and Basic Functions

All of the functions and identities that we introduce will be described on abstract collections of indeterminates. However, many of these functions act independently on various subcollections of indeterminates, thus care has to be made in describing them.

Definition 1.1.1. An **alphabet** is an ordered set of indeterminates. A finite alphabet is indexed by a subset of the natural numbers while an infinite alphabet is indexed by the integers. A **positive** alphabet is an infinite alphabet $\mathbb{K}$ such that $k_0 = 1$ and $k_i = 0$ for $i < 0$.

Alphabets will be denoted with a doublestruck capital letter and its indeterminates will be denoted with the corresponding lower case letters. For example, if $\mathbb{A}$ is an alphabet with $\mathrm{card}(\mathbb{A}) = |\mathbb{A}| = n \in \mathbb{N}$ then $\mathbb{A} = (a_1, \ldots, a_n)$. For collections of alphabets the collection will have a subscripted index and the collection itself will be denoted with a Fraktur letter. For example, if $\mathfrak{A}$ is a collection of alphabets with $\mathrm{card}(\mathfrak{A}) = |\mathfrak{A}| = N$ then $\mathfrak{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_N)$.

Elements of alphabets belonging to a collection will be double subscripted where the first subscript denotes which alphabet the variable is from while the second denotes which variable. So $\mathbb{A}_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,n_1})$. The commas will be dropped when there is no confusion between subscripts.

Suppose $\mathbb{A}$ is an alphabet of cardinality at least n. If $\alpha \in \mathbb{Z}^n$ then the monomial associated to α is

$$\mathbb{A}^\alpha = \prod_{i=1}^{n} a_i^{\alpha_i}.$$

If $\mathrm{card}(\mathbb{A}) > n$, then append zeroes at the end of α so that $\alpha \in \mathbb{Z}^{|\mathbb{A}|}$. This yields the same result, but allows us to assume that α has as many entries as $\mathbb{A}$ has variables.

This same double subscript notation style mentioned above will be used for associated vectors.

The disjoint union of two alphabets will be a useful operation and it will be denoted $\mathbb{A}+\mathbb{B}$. It is important to note that the order of the indeterminates is preserved in $\mathbb{A}+\mathbb{B}$, meaning that if $\mathrm{card}(\mathbb{A}) = n$ and $\mathrm{card}(\mathbb{B}) = m$ then $\mathbb{A}+\mathbb{B} = (a_1, \ldots, a_n, b_1, \ldots, b_m)$.

At times it will be said that two alphabets are *equal*, written $\mathbb{A} = \mathbb{B}$. To be more precise about what this means, if $\mathbb{A} = \mathbb{B}$ then $\mathrm{card}(\mathbb{A}) = \mathrm{card}(\mathbb{B})$ and $a_i = b_i, \forall i$.

Let $\Pi_r(\mathbb{A})$ be the ring of Laurent polynomials in the indeterminates of $\mathbb{A}$ with coefficients in r and let $Symm_r(\mathbb{A})$ be the subring of symmetric Laurent polynomials. Then $\Pi_r(\mathbb{A})$ can be viewed as a $Symm_r(\mathbb{A})$-module. The notation Π_r^+ and $Symm_r^+$ will be used to distinguish the subrings of polynomials and symmetric polynomials with non-negative exponents.

If $\mathfrak{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_p)$ is a collection of alphabets, then define the sets $Symm_r(\mathfrak{A}) \subset \Pi_r(\mathfrak{A})$ as

$$\Pi_r(\mathfrak{A}) = \Pi_r\left(\bigcup_{i=1}^{p} \mathbb{A}_i\right),$$

$$Symm_r(\mathfrak{A}) = \bigcap_{i=1}^{p} Symm_{\Pi_r(\mathbb{A}_1, \ldots, \mathbb{A}_{i-1}, \mathbb{A}_{i+1}, \ldots, \mathbb{A}_p)}(\mathbb{A}_i).$$

Therefore, $Symm_r(\mathfrak{A})$ is symmetric in each alphabet, $\mathbb{A}_i$, but it is not symmetric between the alphabets. Define $\Pi_r^+(\mathfrak{A})$ and $Symm_r^+(\mathfrak{A})$ to be the subrings of polynomials and symmetric polynomials with non-negative exponents.

Definition 1.1.2. Let $\mathbb{A}, \mathbb{B}$ be alphabets of cardinality n and m. Define the following functions:

$$\text{step}(\mathbb{A}) = \prod_{i=1}^{n} a_i^{n-i}, \tag{1.1}$$

$$V(\mathbb{A}) = \prod_{1 \leq i < j \leq n} (a_i - a_j), \tag{1.2}$$

$$Res(\mathbb{A}|\mathbb{B}) = \prod_{a \in \mathbb{A}} \prod_{b \in \mathbb{B}} (a - b). \tag{1.3}$$

Note that for $k = \frac{|\mathbb{A}|\cdot(|\mathbb{A}|-1)}{2}$, $Disc(\mathbb{A}) = (-1)^k \cdot V(\mathbb{A})$ is the (Vandermonde) discriminant and (1.3) is the resultant of $\mathbb{A}$ and $\mathbb{B}$.

Definition 1.1.3. Let $\mathbb{A}$ be an alphabet of cardinality n and $\alpha \in \mathbb{Z}^n$. Define the **anti-symmetrization of** $\mathbb{A}^\alpha$ to be

$$Asym_{\mathbb{A}}\left(\mathbb{A}^\alpha\right) = det(a_j^{\alpha_i})_{1 \leq i,j \leq n} \tag{1.4}$$

and extend $Asym_{\mathbb{A}}$ $\mathbb{Z}$-linearly to all of $\Pi_{\mathbb{Z}}(\mathbb{A})$.

Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$. If $\mathfrak{A} = \{\mathbb{A}_i\}_{i=1}^N$ is a collection of alphabets with $\text{card}(\mathbb{A}_i) = n_i$ and $\alpha_i \in \mathbb{Z}^{n_i}$ then

$$Asym_{\mathfrak{A}}\left(\prod_{i=1}^{N} \mathbb{A}_i^{\alpha_i}\right) := \prod_{i=1}^{N} Asym_{\mathbb{A}_i}\left(\mathbb{A}_i^{\alpha_i}\right)$$

and extend $Asym_{\mathfrak{A}}$ $\mathbb{Z}$-linearly to all of $\Pi_{\mathbb{Z}}(\mathfrak{A})$.

By using the Leibniz formula for the determinant it can be seen that

$$Asym_{\mathbb{A}}\left(\mathbb{A}^{\alpha}\right) = \sum_{\sigma \in \mathfrak{S}_n} \left(sgn(\sigma) \cdot \prod_{i=1}^{n} a_i^{\alpha_{\sigma(i)}} \right). \tag{1.5}$$

PROPOSITION 1.1.4. *$Asym_{\mathbb{A}}$ is an $Symm_{\mathbb{Z}}(\mathbb{A})$-module homomorphism of $\Pi_{\mathbb{Z}}(\mathbb{A})$.*

PROOF. The additivity of $Asym_{\mathbb{A}}$ comes from the definition.

Let $\mathbb{A}^{\alpha} \in \Pi_{\mathbb{Z}}(\mathbb{A})$ and $s \in Symm_{\mathbb{Z}}(\mathbb{A})$.

$$\begin{aligned} Asym_{\mathbb{A}}(s \cdot p) &= \sum_{\sigma \in \mathfrak{S}_n} \left(sgn(\sigma) \cdot s \cdot \prod_{i=1}^{n} a_i^{\alpha_{\sigma(i)}} \right) \\ &= s \cdot \sum_{\sigma \in \mathfrak{S}_n} \left(sgn(\sigma) \cdot \prod_{i=1}^{n} a_i^{\alpha_{\sigma(i)}} \right). \end{aligned}$$

Since s is invariant under any permutation of its variables. □

1.2. Partitions and Schur Functions

The ultimate goal of the dissertation is to prove results about quiver polynomials in terms of Schur determinants and Schur functions. Schur functions are defined in terms of partitions so we need to introduce some basic concepts from the study of partitions. Furthermore, we will introduce an action of the symmetric group on the set of partitions and n-vectors that will be very useful when we discuss the Jacobi-Trudi transform in section 1.3.

1.2.1. Partitions and the Bott Action.

DEFINITION 1.2.1. A **partition of k of length n** is a non-negative sequence of weakly decreasing integers, $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $k = |\lambda| = \sum_{i=1}^{k} \lambda_i$. The length, denoted $\ell(\lambda) = n$, is the location of the last non-zero entry of the partition.

Partitions will frequently be treated as n-vectors throughout this document. Furthermore, we can append zeroes on the end of a partition without inherently changing the partition. For example $(3, 2, 0, 0)$ is equal to $(3, 2)$ as partitions. This allows us to apply some operations to partitions whose length is not sufficient. For example, if $card(\mathbb{A}) = 3$ then $\mathbb{A}^{(2,1)}$ really means $\mathbb{A}^{(2,1,0)}$.

DEFINITION 1.2.2. There is an action of $\mathfrak{S}_n$ on $\mathbb{Z}^n$ defined by:

$$(\sigma \cdot \alpha)_i = \alpha_{\sigma^{-1}(i)} - \sigma^{-1}(i) + i.$$

This action will be referred to as the **Bott action**. If two n-vectors are in the same orbit under this action then we will say that the vectors are **Bott similar**.

PROPOSITION 1.2.3. *The Bott action is an action of* $\mathfrak{S}_n$ *.*

PROOF. The only question is composition. Let $\sigma, \tau \in \mathfrak{S}_n$.

$$\begin{aligned}
((\sigma \circ \tau) \cdot \alpha)_i &= (\sigma \cdot (\tau \cdot \alpha))_i \\
&= (\tau \cdot \alpha)_{\sigma^{-1}(i)} - \sigma^{-1}(i) + i \\
&= \alpha_{\tau^{-1}\sigma^{-1}(i)} - \tau^{-1}\sigma^{-1}(i) + \sigma^{-1}(i) - \sigma^{-1}(i) + i \\
&= \alpha_{(\sigma\tau)^{-1}(i)} - (\sigma\tau)^{-1}(i) + i \\
&= ((\sigma \circ \tau) \cdot \alpha)_i.
\end{aligned}$$

□

PROPOSITION 1.2.4. *Under the Bott action, if an orbit contains a weakly decreasing n-vector then it contains a* unique *weakly decreasing n-vector.*

PROOF. Suppose that λ is a weakly decreasing n-vector and $\sigma \in \mathfrak{S}_n$. If $\sigma \neq id$ then let i be the largest natural number (less than n) such that $\sigma(i) \neq i$. Therefore $\sigma(i) < i$ and $\sigma^{-1}(i) < i$. Hence

$$\begin{aligned}
(\sigma \cdot \lambda)_{\sigma(i)} - (\sigma \cdot \lambda)_i &= (\lambda_i - i + \sigma(i)) - \left(\lambda_{\sigma^{-1}(i)} - \sigma^{-1}(i) + i\right) \\
&= \left(\lambda_i - \lambda_{\sigma^{-1}(i)}\right) + (\sigma(i) - i) + \left(\sigma^{-1}(i) - i\right) \\
&< 0
\end{aligned}$$

because $\lambda_i - \lambda_{\sigma^{-1}(i)} < 0$ (λ is weakly decreasing) and the other two terms are also negative from above. Therefore $\sigma \cdot \lambda$ is not weakly decreasing. This shows that the only weakly decreasing n-vector that λ is Bott similar to is itself.

Note that existence of such a weakly decreasing n-vector will be discussed in proposition 1.3.3. $\square$

1.2.2. Schur Functions.

DEFINITION 1.2.5. Suppose that $\mathbb{A}$ is an alphabet and λ is a partition with $\ell(\lambda) = n \le |\mathbb{A}|$. Define the **Schur polynomial with parameter** λ, s_λ as

$$s_\lambda(\mathbb{A}) = \frac{Asym_{\mathbb{A}}\left(\mathbb{A}^\lambda \cdot \text{step}(\mathbb{A})\right)}{V(\mathbb{A})}. \tag{1.6}$$

PROPOSITION 1.2.6. *If λ is a partition then $s_\lambda(\mathbb{A}) \in Symm^+_{\mathbb{Z}}(\mathbb{A})$.*

PROOF. For any partition $\lambda, Asym_{\mathbb{A}}(\mathbb{A}^\lambda \cdot \text{step}(\mathbb{A})) \in \Pi^+_{\mathbb{Z}}(\mathbb{A})$ and it is anti-symmetric in $\mathbb{A}$ since applying $\sigma \in \mathfrak{S}_n$ to α is equivalent to column swaps in (1.4). Therefore $\frac{Asym_{\mathbb{A}}(\mathbb{A}^\lambda \cdot \text{step}(\mathbb{A}))}{V(\mathbb{A})} \in \Pi^+_{\mathbb{Z}}(\mathbb{A})$.

Let $\tau \in \mathfrak{S}_n$ be an adjacent transposition. $Asym_{\mathbb{A}}(\mathbb{A}^\lambda \cdot \text{step}(\mathbb{A})) = -Asym_{\mathbb{A}}(\mathbb{A}^{\tau\cdot\lambda} \cdot \text{step}(\mathbb{A}))$, and $V(\mathbb{A}) = -V(\tau \cdot \mathbb{A})$, so

$$\frac{Asym_{\mathbb{A}}(\mathbb{A}^\lambda \cdot \text{step}(\mathbb{A}))}{V(\mathbb{A})} = \frac{Asym_{\mathbb{A}}(\mathbb{A}^{\tau\cdot\lambda} \cdot \text{step}(\mathbb{A}))}{V(\tau \cdot \mathbb{A})}.$$

Therefore $s_\lambda(\mathbb{A}) = \frac{Asym_{\mathbb{A}}(\mathbb{A}^\lambda \cdot \text{step}(A))}{V(\mathbb{A})} \in Symm^+_{\mathbb{Z}}(\mathbb{A})$. $\square$

REMARK. The origin of the Bott action is the connection between Schur function and the anti-symmetrization function. When studying a Schur function with partition λ this corresponds to the anti-symmetrization of $\mathbb{A}^\lambda \cdot \text{step}(\mathbb{A}) = \prod_{i=1}^n a_i^{\lambda_i + n - i}$. Applying the canonical action of $\mathfrak{S}_n$ to $\mathbb{A}$ gives the same results as applying the Bott action to λ.

When $\lambda = (r), S_\lambda(\mathbb{A})$ is the complete symmetric funciton $h_r(\mathbb{A})$, and when $\lambda = (1^r), s_\lambda(\mathbb{A})$ is the elementary symmetric function $e_r(\mathbb{A})$. There are two alternate expressions for $S_\lambda(\mathbb{A})$ in terms of the complete and elementary symmetric functions:

$$\text{Jacobi-Trudi:} \quad s_\lambda(\mathbb{A}) = det\left(h_{\lambda_i - i + j}(\mathbb{A})\right)_{1 \leq i,j \leq n},$$

$$\text{Nägelsbach-Kostka}: \quad s_\lambda(\mathbb{A}) = det\left(e_{\lambda'_i - i + j}(\mathbb{A})\right)_{1 \leq i,j \leq m},$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ is the conjugate of the partition λ.

Theorem 1.2.7. *(Littlewood-Richardson Rule)*

If $\mathbb{A}$ is an alphabet and λ, μ are partitions with $\ell(\lambda), \ell(\mu) \leq card(\mathbb{A})$ then

$$s_\lambda(\mathbb{A}) \cdot s_\mu(\mathbb{A}) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(\mathbb{A}) \tag{1.7}$$

where $c^\nu_{\lambda,\mu} \geq 0$.

Theorem 1.2.8. *The Schur polynomials form a linear basis for the symmetric polynomials.*

Remark. The proof of these theorems will not be presented here. For the curious reader, see [M].

It is fairly clear that the product of two symmetric polynomials is symmetric. Thus the so-called Littlewood-Richardson coefficients, $c^\nu_{\lambda,\mu}$, are a way of expressing a product of Schur polynomials in the basis of Schur polynomials. Furthermore, they can be realized as counting the number of tableau of a prescribed type. This makes it clear that the coefficients must be positive.

Corollary 1.2.9. *If $\mathbb{A}$ is an alphabet and $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a list of partitions with $\ell(\lambda_i) = n_i$, and $n_i \leq card(\mathbb{A}), \forall i$ then*

$$\prod_{i=1}^N s_{\lambda_i}(\mathbb{A}) = \sum_{\nu_1, \ldots, \nu_{N-1}} \left(\prod_{j=1}^{N-1} c^{\nu_j}_{\nu_{j-1}, \lambda_{j+1}} \right) s_{\nu_{N-1}}(\mathbb{A}) \tag{1.8}$$

where the sum on the right is over all partitions ν_i for $1 \leq i < N$ and $\nu_0 = \lambda_1$.

Proof. (by induction on N)

The initial case when $N = 2$ is the Littlewood-Richardson Rule, (1.7).

Assume the theorem is true for all $N \leq k$,

$$\begin{aligned}
\prod_{i=1}^{k+1} s_{\lambda_i}(\mathbb{A}) &= \left(\prod_{i=1}^{k} s_{\lambda_i}(\mathbb{A})\right) \cdot s_{\lambda_{k+1}}(\mathbb{A}) \\
&= \left(\sum_{\nu_1,\ldots,\nu_{k-1}} \left(\prod_{j=1}^{k-1} c^{\nu_j}_{\nu_{j-1},\lambda_{j+1}}\right) s_{\nu_{k-1}}(\mathbb{A})\right) \cdot s_{\lambda_{k+1}}(\mathbb{A}) \\
&= \sum_{\nu_1,\ldots,\nu_{k-1}} \left(\prod_{j=1}^{k-1} c^{\nu_j}_{\nu_{j-1},\lambda_{j+1}}\right) \left(s_{\nu_{k-1}}(\mathbb{A}) \cdot s_{\lambda_{k+1}}(\mathbb{A})\right) \\
&= \sum_{\nu_1,\ldots,\nu_{k-1}} \left(\prod_{j=1}^{k-1} c^{\nu_j}_{\nu_{j-1},\lambda_{j+1}}\right) \left(\sum_{\nu_k} c^{\nu_k}_{\nu_{k-1},\lambda_{k+1}} s_{\nu_k}(\mathbb{A})\right) \\
&= \sum_{\nu_1,\ldots,\nu_k} \left(\prod_{j=1}^{k} c^{\nu_j}_{\nu_{j-1},\lambda_{j+1}}\right) s_{\nu_k}(\mathbb{A}).
\end{aligned}$$

□

Remark. If we let $\nu = (\nu_1, \ldots, \nu_{N-1})$ and define $c_\nu = \prod_{j=1}^{N-1} c^{\nu_j}_{\nu_{j-1},\lambda_{j+1}}$, we have a much more readable statement that $\prod_{i=1}^{N} s_{\lambda_i}(\mathbb{A}) = \sum_\nu c_\nu s_{\nu_{N-1}}(\mathbb{A})$. Furthermore, for each such ν note that $c^{\nu_j}_{\nu_{j-1},\lambda_{j+1}} \geq 0$ therefore $c_\nu \geq 0$.

1.3. The Jacobi-Trudi Transform

Recall the Jacobi-Trudi formula, above: $s_\lambda(\mathbb{A}) = det(h_{\lambda_i - i + j}(\mathbb{A}))_{1\leq i,j\leq n}$. If one writes the Schur functions in the algebraic basis of the complete symmetric polynomials then one can envision that the formula is actually a map that takes a vector and a finite alphabet and gives a polynomial in a positive alphabet–$\{h_i\}$ in this case. If instead of evaluating the right hand side of the equality in terms of the complete symmetric functions we chose to evaluate it in an arbitrary positive alphabet the resulting object is called a **Schur determinant**. The Schur determinant will be formally defined and discussed in further detail in Chapter 3.

The goal of this section is to consider the function that transforms the monomial $\mathbb{A}^\lambda$ into the corresponding Schur determinant. We then wish to extend this function to the entire Laurent polynomial ring of $\mathbb{A}$.

1.3.1. Definition and Basic Identities.

Definition 1.3.1. If $\mathbb{A}$ is an alphabet of cardinality n, $\alpha \in \mathbb{Z}^n$, and $\mathbb{K}$ is an infinite alphabet, the **Jacobi-Trudi transform** is a function, $\Delta_{\mathbb{A}}^{\mathbb{K}} : \Pi_{\mathbb{Z}}(\mathbb{A}) \to \Pi_{\mathbb{Z}}(\mathbb{K})$, given by

$$\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^\alpha) = det(k_{\alpha_i - i + j})_{1 \le i,j \le n} \tag{1.9}$$

on the monomials and extended $\mathbb{Z}$-linearly to all of $\Pi_{\mathbb{Z}}(\mathbb{A})$.

One can see that the Jacobi-Trudi transform matches the Jacobi-Trudi identity when α is the partition λ and $\mathbb{K}$ is the set of complete symmetric polynomials.

If $\mathfrak{A} = \{\mathbb{A}_i\}_{i=1}^N$ is a collection of finite alphabets, α_i appropriate length vectors, and $\mathfrak{K} = \{\mathbb{K}_i\}_{i=1}^N$ a collection of infinite alphabets then extend the Jacobi-Trudi transform as follows:

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}\left(\prod_{i=1}^N \mathbb{A}_i^{\alpha_i}\right) = \prod_{i=1}^N \Delta_{\mathbb{A}_i}^{\mathbb{K}_i}\left(\mathbb{A}_i^{\alpha_i}\right) \tag{1.10}$$

and extend the transform $\mathbb{Z}$-linearly.

Lemma 1.3.2. *If $\alpha \in \mathbb{Z}^n$ and $\sigma \in \mathfrak{S}_n$ then for any alphabet $\mathbb{A}$ of cardinality n and infinite alphabet $\mathbb{K}$*

$$\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^\alpha) = sgn(\sigma) \cdot \Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\sigma \cdot \alpha}),$$

where $\sigma \cdot \alpha$ signifies the Bott action.

Proof. Suppose τ is an adjacent transposition, that is $\tau(i) = i + 1$ for $1 \leq i < n$. Note that $\tau = \tau^{-1}$ because it is a transposition. Therefore

$$\begin{aligned}\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\tau\cdot\alpha}) &= det(k_{(\tau\cdot\alpha)_i - i + j})_{1\leq i,j\leq n}\\ &= det(k_{\alpha_{\tau(i)} - \tau(i) + i - i + j})_{1\leq i,j\leq n}\\ &= det(k_{\alpha_{\tau(i)} - \tau(i) + j})_{1\leq i,j\leq n}\\ &= -det(k_{\alpha_i - i + j})_{1\leq i,j\leq n}\end{aligned}$$

by swapping the rows i and $\tau(i)$. Every permutation can be written as a product of adjacent transpositions and the action is a group action so the lemma follows by setting $\sigma = \prod_{k=1}^{m} \tau_k$ and noting that $sgn(\sigma) = (-1)^m$. □

Proposition 1.3.3. *If $\alpha \in \mathbb{Z}^n$ then either α is Bott similar to a weakly decreasing n-vector or $\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\alpha}) = 0$ for any alphabet $\mathbb{A}$ of cardinality n and positive alphabet $\mathbb{K}$.*

Proof. Consider the matrix created in the calculation of the Jacobi-Trudi transform: $M = [k_{\alpha_i - i + j}]_{1\leq i,j\leq n}$. Let R_i be the i^{th} row of M. $det(M) = 0$ if and only if $\sum_{i=1}^{n} c_i R_i = 0$ for some $c_i \in \mathbb{C}$. However, since the entries of each R_i are completely defined by any one entry, we could simply look at the first entry and the linear relation becomes:

$$det(M) = 0 \iff \sum_{i=1}^{n} c_i \cdot k_{\alpha_i - i + 1} = 0.$$

But since the k's are indeterminates there are no relations among them. Therefore for the determinant to be 0 there must be s, t such that
$1 \leq s < t \leq n$ and $\alpha_s - s + 1 = \alpha_t - t + 1$, or $\alpha_t - s = \alpha_t - t$. Thus

$$\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\alpha}) = 0 \iff \alpha_t - t = \alpha_s - s \text{ for some } s < t.$$

Suppose that $\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\alpha}) \neq 0$. If we set $\bar{\alpha}_i = \alpha_i + n - i$ then for all $s < t, \bar{\alpha}_s \neq \bar{\alpha}_t$. Since the entries of $\bar{\alpha}$ are unique they can be ordered so that they are (strictly) decreasing. Call this permutation of the entries σ so $(\bar{\alpha})_{\sigma(i)} > (\bar{\alpha})_{\sigma(i+1)}$.

$\sigma \cdot \alpha$ is a semi-partition because

$$\begin{aligned}(\sigma \cdot \alpha)_i &= (\bar{\alpha})_{\sigma(i)} - n + i \\ &> (\bar{\alpha})_{\sigma(i+1)} - n + i \\ &= (\sigma \cdot \alpha)_{i+1} - 1.\end{aligned}$$

This means α is Bott similar to $\sigma \cdot \alpha$, which completes the proof. □

1.3.2. Δ-equivalence. One of the main strengths of the Jacobi-Trudi transform is that it creates an equivalence relation on the collection of power series. Thus when making certain claims about the Jacobi-Trudi transform, one can replace difficult power series or polynomials with simpler polynomials as long as they are in the same equivalence class.

PROPOSITION 1.3.4. *The Jacobi-Trudi transform creates an equivalence relation on power series. That is, if f, g are power series in the variables of $\mathfrak{A}$ then*

$$f \sim_\Delta g \iff \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f) = \Delta_{\mathfrak{A}}^{\mathfrak{K}}(g)$$

for a collection of positive alphabets $\mathfrak{K}$.

Note that on the left side the decorations for Δ have been dropped. This is done for readability and the decorations will be included if it is ever unclear as to which alphabets are being used.

PROOF. Note that $f \sim_\Delta g \iff \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f - g) = 0$ because Δ is $\mathbb{Z}$-linear. This immediately shows that the relation is both symmetric and reflexive.
If $f \sim_\Delta g$ and $g \sim_\Delta h$ then

$$\begin{aligned}\Delta_{\mathfrak{A}}^{\mathfrak{K}}(f - h) &= \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f - g + g - h) \\ &= \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f - g) + \Delta_{\mathfrak{A}}^{\mathfrak{K}}(g - h) \\ &= 0.\end{aligned}$$

□

In light of this equivalence relation and lemma 1.3.2, together they say that for any given monomial $\mathbb{A}^\alpha$, either

$$\mathbb{A}^\alpha \sim_\Delta 0 \text{ or } \mathbb{A}^\alpha \sim_\Delta \pm\mathbb{A}^\lambda$$

where λ is a weakly decreasing n-vector. However, if $\mathbb{K}$ is a positive alphabet this statement can be made much stronger. Suppose λ is weakly decreasing but not a partition, so $\lambda_n < 0$. Now

$$\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^\lambda) = det(k_{\lambda_i - i + j}).$$

But if one considers the last row of the matrix generated in the above definition, $\lambda_n - n + j < 0$ (since $j \leq n$), so $k_{\lambda_n - n + j} = 0$. Therefore the last row of the matrix is a row of zeroes and $\Delta_{\mathbb{A}}^{\mathbb{K}}(\lambda) = 0$.

So either

$$\mathbb{A}^\alpha \sim_\Delta 0 \text{ or } \mathbb{A}^\alpha \sim_\Delta \pm\mathbb{A}^\lambda \tag{1.11}$$

where λ is a **partition**.

Consider that a power series in the variables of $\mathfrak{A}$ can be written as

$$f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} \left(c_\alpha \cdot \prod_{i=1}^{n_i} \mathbb{A}_i^{\alpha_i} \right)$$

where each $\alpha_i \in \mathbb{Z}^{n_i}$. If we set

$$g = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} \left(d_\lambda \cdot \prod_{i=1}^{n_i} \mathbb{A}_i^{\lambda_i} \right)$$

where

$$d_\lambda = \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \left(c_{\sigma\cdot\lambda} \prod_{i=1}^{n} sgn(\sigma_i) \right) \text{ and } (\sigma \cdot \lambda)_i = \sigma_i \cdot \lambda_i$$

then $f \sim_\Delta g$.

This becomes clear if one simply replaces each $\mathbb{A}_i^{\alpha_i}$ with $sgn(\tau) \cdot \mathbb{A}_i^{\lambda_i}$ where λ is the unique partition with $\tau\cdot\alpha = \lambda$. Thus each equivalence class has a canonical representative where each n-vector is weakly decreasing (or a partition if $\mathfrak{K}$ is a collection of positive alphabets).

1.3.3. Connections of Δ to Schur Functions

and Anti-Symmetrization. Suppose that $\mathfrak{A} = \{\mathbb{A}_i\}_{i=1}^N$ is a collection of finite alphabets with $\mathrm{card}(\mathbb{A}_i) = n_i$ and $f \in \Pi_{\mathbb{Z}}^{+}(\mathfrak{A})$. Further suppose that $\mathfrak{K} = \{\mathbb{K}_i\}_{i=1}^N$ is a collection of disjoint positive alphabets and $\Delta_{\mathfrak{A}}^{\mathfrak{K}}(f) = 0$. Then it is immediate that for any other collection of positive alphabets, $\mathfrak{K}' = \{\mathbb{K}_i'\}_{i=1}^N$, $\Delta_{\mathfrak{A}}^{\mathfrak{K}'}(f) = 0$.

Set $H_i = H(\mathbb{A}_i)$ to be the set of complete symmetric polynomials in the indeterminates of $\mathbb{A}_i$. It is true that H_i forms an algebraic basis of $Symm_{\mathbb{Z}}^{+}(\mathbb{A}_i)$ so there are no algebraic relations among the polynomials. This means that $\mathfrak{H} = \{H_i\}_{i=1}^N$ is a collection of disjoint positive alphabets.

The following theorem and its corollary will be used extensively in later chapters.

Theorem 1.3.5. *Suppose $\mathfrak{A} = \{\mathbb{A}_i\}_{i=1}^N$ is a collection of finite alphabets with $card(\mathbb{A}_i) = n_i$. If $f \in \Pi_{\mathbb{Z}}(\mathfrak{A})$ and $\mathfrak{H} = \{H_i\}_{i=1}^N$ is the collection of complete symmetric polynomials in the indeterminates of $\mathbb{A}_i$ then*

$$Asym_{\mathfrak{A}}\left(f \cdot \prod_{i=1}^{N} step(\mathbb{A}_i)\right) = 0 \iff \Delta_{\mathfrak{A}}^{\mathfrak{H}}(f) = 0.$$

Proof. Consider

$$\begin{aligned} Asym_{\mathfrak{A}}\left(f \cdot \prod_{i=1}^{N} \mathrm{step}(\mathbb{A}_i)\right) = 0 &\iff \frac{Asym_{\mathfrak{A}}\left(f \cdot \prod_{i=1}^{N} \mathrm{step}(\mathbb{A}_i)\right)}{\prod_{i=1}^{N} V\left(\mathbb{A}_i\right)} = 0 \\ &\iff \Delta_{\mathfrak{A}}^{\mathfrak{H}}(f) = 0. \end{aligned}$$

□

For the following corollaries let $\mathfrak{K} = \{\mathbb{K}_i\}_{i=1}^N$ be a collection of disjoint positive alphabets.

Corollary 1.3.6. *If $f \in \Pi_{\mathbb{Z}}(\mathfrak{A})$, then*

$$Asym_{\mathfrak{A}}\left(f \cdot \prod_{i=1}^{N} step(\mathbb{A}_i)\right) = 0 \iff \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f) = 0.$$

Remark. In corollary 1.3.6 if one relaxes the disjoint requirement on the alphabets of $\mathfrak{K}$ then one still has the following:

$$Asym_{\mathfrak{A}}\left(f \cdot \prod_{i=1}^{N} step(\mathbb{A}_i)\right) = 0 \Rightarrow \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f) = 0.$$

Corollary 1.3.7. *If $f, g \in \Pi_{\mathbb{Z}}(\mathfrak{A})$ then*

$$Asym_{\mathfrak{A}}\left(f \cdot \prod_{i=1}^{N} step(\mathbb{A}_i)\right) = Asym_{\mathfrak{A}}\left(g \cdot \prod_{i=1}^{N} step(\mathbb{A}_i)\right)$$
$$\iff \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f) = \Delta_{\mathfrak{A}}^{\mathfrak{K}}(g).$$

Corollary 1.3.8. *If $f \in Symm_{\mathbb{Z}}(\mathfrak{A})$ and $g, h \in \Pi_{\mathbb{Z}}(\mathfrak{A})$ with $g \sim_{\Delta} h$ then*

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}(f \cdot g) = \Delta_{\mathfrak{A}}^{\mathfrak{K}}(f \cdot h).$$

Proof. Consider

$$Asym_{\mathfrak{A}}\left(f \cdot g \cdot \prod_{i=1}^{N} \mathrm{step}(\mathbb{A}_i)\right) = f \cdot Asym_{\mathfrak{A}}\left(g \cdot \prod_{i=1}^{N} \mathrm{step}(\mathbb{A}_i)\right).$$

□

1.3.4. The Schur Representative. If $\mathfrak{K}$ is a collection of positive alphabets it was previously stated that there is a canonical representative in each Δ-equivalence class comprised of monomials indexed by partitions. However, we are about to show that there is another representative that is equally as useful and it is convenient to change representatives at times.

Proposition 1.3.9. *Each Δ-equivalence class has a unique representative consisting of a $\mathbb{Z}$-linear combination of Schur polynomials.*

PROOF. Let $\mathbb{A}$ be a finite alphabet of cardinality n and let λ be a partition with $\ell(\lambda) \leq n$. Consider

$$Asym_{\mathbb{A}}(A^{\lambda} \cdot step(\mathbb{A})) = s_{\lambda}(\mathbb{A}) \cdot V(A) \tag{1.12}$$

$$= s_{\lambda} \cdot Asym_{\mathbb{A}}(step(\mathbb{A})) \tag{1.13}$$

$$= Asym_{\mathbb{A}}(s_{\lambda}(\mathbb{A}) \cdot step(\mathbb{A})). \tag{1.14}$$

This immediately implies that for any infinite alphabet, $\mathbb{K}$,

$$\Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\lambda}) = \Delta_{\mathbb{A}}^{\mathbb{K}}(s_{\lambda}(\mathbb{A})). \tag{1.15}$$

Since each canonical representative is unique, so too must these Schur representatives. □

1.4. The Chern Transform

The Chern transform is closely related to the Jacobi-Trudi transform. It is so called because it was initially a map from the Laurent polynomial ring directly to the Chern classes of a vector bundle. However, in this dissertation it is presented on arbitrary alphabets for abstraction purposes. Its relationship to the Jacobi-Trudi transform will be presented in detail in chapter 3.

DEFINITION 1.4.1. Suppose that $\mathbb{A}$ is an alphabet of cardinality n, $\alpha \in \mathbb{Z}^n$, and $\mathbb{K}$ is a positive alphabet. The **Chern transform** is a function, $C_{\mathbb{A}}^{\mathbb{K}} : \Pi_{\mathbb{Z}}(A) \to \Pi_{\mathbb{Z}}(\mathbb{K})$ given by

$$C_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^{\alpha}) = \prod_{i=1}^{n} k_{\alpha_i} \tag{1.16}$$

on the monomials and extended $\mathbb{Z}$-linearly to all of $\Pi_{\mathbb{Z}}(\mathbb{A})$.

If $\mathfrak{A} = \{\mathbb{A}_i\}_{i=1}^{N}$ is a collection of finite alphabets, α_i appropriate length vectors, and $\mathfrak{K} = \{\mathbb{K}_i\}_{i=1}^{N}$ a collection of positive alphabets then extend the Chern transform as follows:

$$C_{\mathfrak{A}}^{\mathfrak{K}}\left(\prod_{i=1}^{N} \mathbb{A}_i^{\alpha_i}\right) = \prod_{i=1}^{N} C_{\mathbb{A}_i}^{\mathbb{K}_i}\left(\mathbb{A}_i^{\alpha_i}\right) \tag{1.17}$$

and extend the transform $\mathbb{Z}$-linearly.

PROPOSITION 1.4.2. *Suppose that $\mathbb{A}, \mathbb{B}$ are alphabets with card$(\mathbb{A}) = n$ and card$(\mathbb{B}) = m$. If $\mathbb{K}$ is a positive alphabet then*

$$C^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}} = C^{\mathbb{K},\mathbb{K}}_{\mathbb{A},\mathbb{B}}.$$

PROOF. Let $\alpha \in \mathbb{Z}^{n+m}$. Set $\alpha_A = (\alpha_1, \ldots, \alpha_n)$ and $\alpha_B = (\alpha_{n+1}, \ldots, \alpha_m)$. Then

$$\begin{aligned} C^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}\left((\mathbb{A}+\mathbb{B})^{\alpha}\right) &= \left(\prod_{i=1}^{n} k_{\alpha_i}\right) \cdot \left(\prod_{i=n+1}^{n+m} k_{\alpha_i}\right) \\ &= C^{\mathbb{K}}_{\mathbb{A}}(\mathbb{A}^{\alpha_A}) \cdot C^{\mathbb{K}}_{\mathbb{B}}(\mathbb{B}^{\alpha_B}) \\ &= C^{\mathbb{K},\mathbb{K}}_{\mathbb{A},\mathbb{B}}\left((\mathbb{A}+\mathbb{B})^{\alpha}\right). \end{aligned}$$

$\square$

1.4.1. Connections Between the Jacobi-Trudi Transform and Chern Transform.

Here we show how the Jacobi-Trudi Transform and the Chern Transform are related.

LEMMA 1.4.3. *Suppose that $\mathbb{A}$ is a finite alphabet with card$(\mathbb{A}) = n$ and $\mathbb{K}$ is an infinite alphabet. If $f \in \Pi_{\mathbb{Z}}(\mathbb{A})$ then*

$$C^{\mathbb{K}}_{\mathbb{A}}\left(f \cdot Disc(\mathbb{A}) \cdot \prod_{i=1}^{n} a_i^{1-i}\right) = \Delta^{\mathbb{K}}_{\mathbb{A}}(f),$$

where $Disc(\mathbb{A}) = \prod_{1 \le i < j \le n}(a_j - a_i)$ is the (Vandermonde) discriminant.

PROOF. Since Δ and C are both linear, it suffices to prove this for a monomial $\mathbb{A}^{\alpha}$. Consider the following:

$$\Delta^{\mathbb{K}}_{\mathbb{A}}(\mathbb{A}^{\alpha}) = det(k_{\alpha_i+j-i}) = \sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\prod_{i=1}^{n} k_{\alpha_{\sigma(i)}+i-\sigma(i)} \tag{1.18}$$

$$= C^{\mathbb{K}}_{\mathbb{A}}\left(\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\prod_{i=1}^{n} a^{\alpha_{\sigma(i)}+i-\sigma(i)}\right). \tag{1.19}$$

But then we know that

$$Disc(\mathbb{A}) = \sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\prod_{i=1}^{n} a^{\sigma(i)-1}. \tag{1.20}$$

Therefore

$$\mathbb{A}^{\alpha}\cdot Disc(\mathbb{A})\cdot\prod_{i=1}^{n} a^{1-i} = \sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\cdot\prod_{i=1}^{n} a_i^{\alpha_i}\cdot a_i^{1-i}\cdot a_i^{\sigma(i)-1} \tag{1.21}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\cdot\prod_{i=1}^{n} a_i^{\alpha_i+\sigma(i)-i} \tag{1.22}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)\cdot\prod_{j=1}^{n} a_j^{\alpha_{\sigma(j)}+j-\sigma(j)}, \tag{1.23}$$

by letting $i=\sigma^{-1}(j)$, letting the sum run over all σ^{-1}, and relabeling σ^{-1} with σ. This completes the proof.

$\square$

CHAPTER 2

Generating Sequences

In this chapter we introduce the quiver, quiver representations, the idea of an orbit closure, quiver polynomials, and a way of generating the quiver polynomial related to an orbit closure.

2.1. Quiver Polynomials

Let $Q = (Q_0, Q_1)$ be an oriented graph (quiver), with vertex set $Q_0 = \{1, 2, \ldots, n\}$ and a finite set Q_1 of arrows. There are two canonical functions, $t, h : Q_1 \to Q_0$ that return the tail and head of an arrow, respectively. For a fixed dimension vector $e = (e_1, \ldots, e_n) \in \mathbb{N}^n$ a quiver representation associates a vector space $E_i = \mathbb{C}^{e_i}$ with each vertex and a linear map $\varphi_a : E_{t(a)} \to E_{h(a)}$ to each arrow.

Let V be the vector space $V = \oplus_{a \in Q_1} Hom(E_{t(a)}, E_{h(a)})$. The group

$$G = \times_{i=1}^{n} GL(E_i) \circlearrowright V$$

by

$$(g_i)_{i=1}^{n} \cdot (\varphi_a)_{a \in Q_1} = \left(g_{h(a)} \circ \varphi_a \circ g_{t(a)}^{-1} \right)_{a \in Q_1}.$$

A Dynkin quiver is a quiver whose underlying unoriented graph is one of the simply-laced Dynkin graphs $A_n, D_n, E_6, E_7, or E_8$. It is known that such quivers have finitely many orbits when acted on by G–for any orientation and any dimension vector. This result is known as Gabriel's Theorem, [BGP]. Furthermore the orbits have an explicit description, as follows. Consider the set Φ^+ of positive roots, and the set $\{\alpha_i : i = 1, \ldots, n\}$ of simple roots for the corresponding root system. For a positive root α define $d(\alpha) \in \mathbb{N}^n$ by $\alpha = \sum_{i=1}^{n} d_i(\alpha)\alpha_i$. The orbits of V with Dynkin graph Q and dimension vector e are in one-to-one correspondence with the vectors

$$(2.1) \qquad (m_\alpha) \in \mathbb{N}^{\Phi^+}, \quad \text{for which} \quad \sum_{\alpha \in \Phi^+} m_\alpha d_i(\alpha) = e_i, \ \text{for } i = 1, \dots, n.$$

Note that the list of orbits does not depend on the orientation of Q. The closure of the orbit corresponding to the vector $m = (m_\alpha) \in \mathbb{N}^{\Phi^+}$ will be denoted S_m.

Consider the G-equivariant cohomology class represented by the orbit closure S_m in V as described in the introduction,

$$[S_m] \in H^*_G(V; \mathbb{Z}).$$

It is known that

$$H^*_{GL(E_i)}(V) = \mathbb{Z}[c_j(\gamma_i) : j = 1, \dots, e_i],$$

the polynomial ring in the universal Chern classes of the canonical bundle over the classifying space, BGL. Therefore

$$H^*_G(V) = \otimes_{i=1}^n \mathbb{Z}[c_j(\gamma_i) : j = 1, \dots, e_i] = \mathbb{Z}[c_j(\gamma_i) : i = 1, \dots, n; j = 1, \dots, e_i].$$

Since $[S_m] \in H^*_G(V)$, a polynomial ring, the class $[S_m]$ is called the quiver polynomial corresponding to the orbit closure S_m.

Following [B08] and [RR], the presentation of quiver polynomials will be in terms of some special polynomials in $H^*_G(V)$. For $i \in Q_0$ define

$$T(i) = \{j \in Q_0 | \exists a \in Q_1 \text{ with } t(a) = j, h(a) = i\},$$

$$H(i) = \{j \in Q_0 | \exists a \in Q_1 \text{ with } t(a) = i, h(a) = j\},$$

and let $\tau_i = \oplus_{j \in T(i)} \gamma_j$.

We express quiver polynomials in terms of the Chern classes of the dual bundles $c_j(\tau_i^* - \gamma_i^*)$, which are defined by the formal expansion

$$\sum_{k=0}^{\infty} c_k(\tau_i^* - \gamma_i^*)t^k = \frac{\sum_{k=0}^{dim\tau_i} c_k(\tau_i)(-t)^k}{\sum_{k=0}^{e_i} c_k(\gamma_i)(-t)^k} = \frac{\prod_{j\in T(i)} \left(\sum_{k=0}^{e_j} c_k(\gamma_j)(-t)^k\right)}{\sum_{k=0}^{e_i} c_k(\gamma_i)(-t)^k}.$$

2.2. Construction of Generating Sequences

In this section we present methods for constructing generating sequences corresponding to any simply-laced Dynkin quiver as well as present examples using these methods.

2.2.1. A Generating Sequence Corresponding to an Orbit (Closure). Fix a Dynkin quiver $Q = (Q_0, Q_1)$ with vertex set $Q_0 = \{1, 2, \ldots, n\}$.

2.2.1.1. *Resolution Pairs.* If $e, f \in \mathbb{N}^n$ are dimension vectors let

$$\langle e, f\rangle = \sum_{i=1}^{n} e_i f_i - \sum_{a\in Q_1} e_{t(a)} f_{h(a)} \tag{2.2}$$

denote the Euler form for Q. By identifying a positive root for the underlying Dynkin diagram α with its dimension vector $d(\alpha) \in \mathbb{N}^n$ where $\alpha = \sum_{i=1}^n d_i(\alpha)\alpha_i$ for simple roots α_i we can extend the Euler form to the positive roots.

Definition 2.2.1. Let $\Phi' \subset \Phi^+$ be a subset of the positive roots of Q. A partition

$$\Phi' = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_s$$

is called **directed** if $\langle \alpha, \beta\rangle \geq 0$ for all $\alpha, \beta \in \mathcal{I}_j$ for $(1 \leq j \leq n)$, and $\langle\alpha, \beta\rangle \geq 0 \geq \langle\beta, \alpha\rangle$ for all $\alpha \in \mathcal{I}_i, \beta \in \mathcal{I}_j$ with $1 \leq i < j \leq s$.

Proposition 2.2.2. *(Reineke, [Re]) A directed partition of any subset of the positive roots exists for any Dynkin quiver.*

Here we will observe that if $\mathcal{I}_1 \cup \ldots \cup \mathcal{I}_s$ is a directed partition of Φ^+ then removing roots from the directed partition does not change the directed property. Therefore it suffices to show that a directed partition of the positive roots exists for any Dynkin quiver.

For a complete proof see [Re].

Fix a dimension vector, e. Next, we will define a *resolution pair* for an orbit closure, $S_m \subset V$.

Choose the vector $m = (m_\alpha) \in \mathbb{N}^{\Phi^+}$ corresponding to an orbit closure S_m by (2.1). Let $\Phi' = \{\alpha | m_\alpha \neq 0\} \subset \Phi^+$ and let $\Phi' = \mathcal{I}(S_m) = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_s$ be a directed partition. For each $j \in \{1, \ldots, s\}$ write

$$\sum_{\alpha \in I_j} m_\alpha \alpha = (p_1^{(j)}, \ldots, p_n^{(j)}) \in \mathbb{N}^n$$

and let $\mathbf{i}^{(j)} = (i_1, \ldots, i_l)$ be any sequence of the vertices $i \in Q_0$ for which $p_i^{(j)} \neq 0$, with no vertices repeated and ordered so that the tail of any arrow of Q comes before its head. Then for each j let $\mathbf{r}^{(j)} = (p_{i_1}^{(j)}, \ldots, p_{i_l}^{(j)})$.

Then let $\mathbf{i}$ and $\mathbf{r}$ be the concatenated sequences $\mathbf{i} = \mathbf{i}^{(1)}\mathbf{i}^{(2)} \ldots \mathbf{i}^{(s)}$ and $\mathbf{r} = \mathbf{r}^{(1)}\mathbf{r}^{(2)} \ldots \mathbf{r}^{(s)}$.

Definition 2.2.3. The pair $\mathbf{i}, \mathbf{r}$ is called a **resolution pair** for S_m.

2.2.1.2. *Resolution Functions.* For a given resolution pair

$$\mathbf{i} = (i_1, \ldots, i_p), \qquad \mathbf{r} = (r_1, \ldots, r_p)$$

define the base set of variables $\mathbb{A}_k = \{a_{k1}, \ldots, a_{kr_k}\}$ for $k = 1, \ldots, p$. Set

$$\mathbb{B}_k = \bigcup_{\substack{l>k \\ i_l = i_k}} \mathbb{A}_l \qquad \mathbb{C}_k = \bigcup_{\substack{l>k \\ i_l \in H(i_k)}} \mathbb{A}_l \qquad n_k = \sum_{\substack{l>k \\ i_l \in T(i_k)}} r_l - \sum_{\substack{l>k \\ i_l = i_k}} r_l.$$

Next we define various functions in the $\cup_k \mathbb{A}_k$ variables. For $1 \leq k \leq p$ define • the monomial factors

$$M_k = \prod_{s=1}^{r_k} (a_{ks})^{n_k},$$

• and the interference factors

$$I_k = \prod_{a \in \mathbb{A}_k} \frac{\prod_{b \in \mathbb{B}_k} \left(1 - \frac{a}{b}\right)}{\prod_{c \in \mathbb{C}_k} \left(1 - \frac{a}{c}\right)}.$$

2.2.1.3. *Generating Sequences.*

DEFINITION 2.2.4. The **(iterated residue) generating sequence** associated to an orbit closure S_m with directed partition $\mathcal{I}(S_m)$ is $G_{\mathcal{I}} = \prod_{k=1}^{p} M_k I_k$.

This next theorem describes how to use the generating functions associated to $\mathcal{I}(S_m)$ to compute the quiver polynomial associated to S_m. Consider the Taylor expansion of $G_{\mathcal{I}}$, that is replace all $\frac{1}{1-x/u}$ factors with $\sum_{k=0}^{\infty}\left(\frac{x}{u}\right)^k$. If we set $\mathbb{K}_j = \{c_n(\tau_{i_j}^* - \gamma_{i_j}^*)\}_n$ then we have the following theorem.

THEOREM 2.2.5. *(Rimányi)*

The quiver polynomial can be computed as

$$[S_m] = \Delta_{\mathbb{A}_1,\dots,\mathbb{A}_p}^{\mathbb{K}_1,\dots,\mathbb{K}_p}\left(G_{\mathcal{I}}\right). \tag{2.3}$$

PROOF. The proof of this theorem is in [RR]. □

2.3. Generating Sequence Examples

2.3.1. Example 1. Consider the inward-pointing A_3 quiver: $1 \to 2 \leftarrow 3$. The (non-simple) positive roots of the root system A_3 are $\alpha_{ij} = \sum_{i\le u\le j}\alpha_u$, for $i,j \in \{1,2,3\}$ with $i<j$. Consider the orbit corresponding to the linear combination $\sum_{1\le i\le j\le 3} m_{ij}\alpha_{ij}$. One directed partition is

$$\Phi^+ = \mathcal{I}(S_m) = \{\alpha_2\}\cup\{\alpha_{12},\alpha_{23},\alpha_{13}\}\cup\{\alpha_1,\alpha_3\}.$$

This generates the resolution pair $\mathbf{i} = (2,1,3,2,1,3), \mathbf{r} = (m_2, m_{12}+m_{13}, m_{23}+m_{13}, m_{12}+m_{13}+m_{23}, m_1, m_3)$. For the special case $m_{13} = m_{12} = m_2 = m_3 = 1$ and all other $m_\bullet$ are zero we have that

$$\mathbb{A}_1 = \{a_{1,1}\},\ \mathbb{A}_2 = \{a_{2,1}, a_{2,2}\},\ \mathbb{A}_3 = \{a_{3,1}\},$$

$$\mathbb{A}_4 = \{a_{4,1}, a_{4,2}\},\ \mathbb{A}_5 = \emptyset,\ \mathbb{A}_6 = \{a_{6,1}\}.$$

For readability suppose we rename the variables so that

$$\mathbb{A}_1 = \{u\},\ \mathbb{A}_2 = \{v_1, v_2\},\ \mathbb{A}_3 = \{w\},\ \mathbb{A}_4 = \{x_1, x_2\},\ \mathbb{A}_5 = \emptyset,\ \mathbb{A}_6 = \{z\}.$$

This gives the generating sequence

$$G_{\mathcal{I}} = \frac{u^2 x_1 x_2 \left(1 - \frac{u}{x_1}\right)\left(1 - \frac{u}{x_2}\right)\left(1 - \frac{w}{z}\right)}{w\left(1 - \frac{v_1}{x_1}\right)\left(1 - \frac{v_2}{x_1}\right)\left(1 - \frac{v_1}{x_2}\right)\left(1 - \frac{v_2}{x_2}\right)\left(1 - \frac{w}{x_1}\right)\left(1 - \frac{w}{x_2}\right)}.$$

The quiver polynomial is

$$[S_m] = \Delta^{\mathbb{K}_2, \mathbb{K}_1, \mathbb{K}_3, \mathbb{K}_2, \mathbb{K}_3}_{\{u\},\{v_1,v_2\},\{w\},\{x_1,x_2\},\{z\}}(G_{\mathcal{I}})$$

Since each $\mathbb{K}_i$ is a positive alphabet, the part of G_m that isn't immediately mapped to zero by Δ is $-u^3 + u^2 x_2 + u^2 v_2$. Hence the corresponding quiver polynomial is

$$-c_3(\gamma_1^* + \gamma_3^* - \gamma_2^*) + c_2(\gamma_1^* + \gamma_3^* - \gamma_2^*)c_1(\gamma_1^* + \gamma_3^* - \gamma_2^*) + c_2(\gamma_1^* + \gamma_3^* - \gamma_2^*)c_1(-\gamma_1^*) \tag{2.4}$$

2.3.2. Example 2. Now consider the same orbit with same choice of m, but with choice of directed partition

$$\Phi^+ = \mathcal{J}(S_m) = \{\alpha_2, \alpha_{12}, \alpha_{23}\} \cup \{\alpha_{13}, \alpha_1, \alpha_3\}.$$

This generates the resolution pair $\mathbf{i} = (1, 3, 2, 1, 3, 2)$, $\mathbf{r} = (m_{12}, m_{23}, m_{12} + m_2 + m_{23}, m_1 + m_{13}, m_{13} + m_3, m_{13})$. So we have

$$\mathbb{A}_1 = \{a_{1,1}\},\ \mathbb{A}_2 = \emptyset,\ \mathbb{A}_3 = \{a_{3,1}, a_{3,2}\},$$

$$\mathbb{A}_4 = \{a_{4,1}\},\ \mathbb{A}_5 = \{a_{5,1}, a_{5,2}\},\ \mathbb{A}_6 = \{a_{6,1}\}.$$

For readability suppose we rename the variables so that

$$\mathbb{A}_1 = \{u\},\ \mathbb{A}_2 = \emptyset,\ \mathbb{A}_3 = \{w_1, w_2\},\ \mathbb{A}_4 = \{x\},\ \mathbb{A}_5 = \{y_1, y_2\},\ \mathbb{A}_6 = \{z\}.$$

Thus we have a second generating sequence

$$G_{\mathcal{J}} = \frac{w_1^2 w_2^2 \left(1-\frac{u}{x}\right)\left(1-\frac{w_1}{z}\right)\left(1-\frac{w_2}{z}\right)}{u\left(1-\frac{u}{w_1}\right)\left(1-\frac{u}{w_2}\right)\left(1-\frac{x}{z}\right)\left(1-\frac{u}{z}\right)\left(1-\frac{y_1}{z}\right)\left(1-\frac{y_2}{z}\right)}.$$

By theorem 2.2.5

$$[S_m] = \Delta^{\mathbb{K}_1,\mathbb{K}_2,\mathbb{K}_1,\mathbb{K}_3,\mathbb{K}_2}_{\{u\},\{w_1,w_2\},\{x\},\{y_1,y_2\},\{z\}}(G_{\mathcal{J}}).$$

Note that the variable z only appears in the denominators. Therefore multiplying by any factor that contains a z will create a term that has a negative exponent and is mapped to zero by Δ. This technique is then repeated with the variable x. This gives

$$\begin{aligned} G_{\mathcal{J}} \sim_\Delta G_{\mathcal{J}}^{(2)} &= \frac{w_1^2 w_2^2 \left(1-\frac{u}{x}\right)}{u\left(1-\frac{u}{w_1}\right)\left(1-\frac{u}{w_2}\right)} \\ \sim_\Delta G_{\mathcal{J}}^{(3)} &= \frac{w_1^2 w_2^2}{u\left(1-\frac{u}{w_1}\right)\left(1-\frac{u}{w_2}\right)} \\ \sim_\Delta G_{\mathcal{J}}^{(4)} &= w_1 w_2^2 + w_2^2 u - w_1^3 \\ \sim_\Delta G_{\mathcal{I}}. \end{aligned}$$

This is the same result as the previous example.

2.3.3. Example 3. Consider the inward-oriented D_4 quiver with vertices labelled as follows:

$$\begin{tikzcd} & 2 \arrow[d] & \\ & 1 & \\ 4 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}.$$

The simple roots are $\alpha_1, \alpha_2, \alpha_3$, and α_4. The remaining positive roots of the root system are

$$\begin{aligned}
\alpha_{1k} &= \alpha_1 + \alpha_k, && \text{for } k = 2, 3, 4, \\
\alpha_{ij} &= \alpha_1 + \alpha_i + \alpha_j, && \text{for } i, j \in \{2, 3, 4\} \text{ and } i < j, \\
\beta &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\
\delta &= 2 \cdot \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.
\end{aligned}$$

A directed partition for this quiver is

$$\mathcal{I} = \{\alpha_1\} \cup \{\alpha_{12}, \alpha_{13}, \alpha_{14}, \delta\} \cup \{\alpha_{23}, \alpha_{24}, \alpha_{34}\} \cup \{\alpha_2, \alpha_3, \alpha_4, \beta\}.$$

Let us select the special case that $m_{12} = m_{13} = m_{34} = m_4 = 1$ and all other $m_\bullet$ are 0. This gives the reduced directed partition

$$\mathcal{J} = \{\alpha_{12}, \alpha_{13}\} \cup \{\alpha_{34}\} \cup \{\alpha_3\},$$

with resolution pair

$$\mathbf{i} = (2, 3, 1, 3, 4, 1, 4),$$

$$\mathbf{r} = (m_{12}, m_{13}, m_{12} + m_{13}, m_{34}, m_{34}, m_{34}, m_4) = (1, 1, 2, 1, 1, 1, 1).$$

Calculating the alphabets (and relabeling the variables for readability), we have

$$\mathbb{A}_1 = \{a\}, \ \ \mathbb{A}_2 = \{b\}, \ \ \mathbb{A}_3 = \{c_1, c_2\}, \ \ \mathbb{A}_4 = \{d\},$$

$$\mathbb{A}_5 = \{x\}, \ \ \mathbb{A}_6 = \{y\}, \ \ \mathbb{A}_7 = \{z\}.$$

This gives us the generating sequence

$$G_{\mathcal{J}} = \frac{c_1^2 c_2^2 y}{bx} \cdot \frac{(1 - b/d)(1 - c_1/y)(1 - c_2/y)(1 - x/z)}{(1 - a/c_1)(1 - a/c_2)(1 - b/c_1)(1 - b/c_2)(1 - d/y)(1 - x/y)}.$$

Therefore, the quiver polynomial is

$$[S_m] = \Delta^{\mathbb{K}_2,\mathbb{K}_3,\mathbb{K}_1,\mathbb{K}_3,\mathbb{K}_4,\mathbb{K}_1,\mathbb{K}_4}_{\{a\},\{b\},\{c_1,c_2\},\{d\},\{x\},\{y\},\{z\}}(G_{\mathcal{J}}).$$

Notice that z and d only appear in denominators. Thus, by the argument in the previous examples,

$$G_{\mathcal{J}} \sim_\Delta G^{(2)}_{\mathcal{J}} = \frac{c_1^2c_2^2y}{bx} \cdot \frac{(1-c_1/y)(1-c_2/y)}{(1-a/c_1)(1-a/c_2)(1-b/c_1)(1-b/c_2)(1-d/y)(1-x/y)}.$$

The only way to get x out of the denominator is to select a $\left(\frac{x}{y}\right)^n$ term with $n > 0$ from $(1-x/y) = \sum_{k=0}^{\infty}\left(\frac{x}{y}\right)^k$. Therefore,

$$G^{(2)}_{\mathcal{J}} \sim_\Delta G^{(3)}_{\mathcal{J}} = \frac{c_1^2c_2^2}{b}\left(\sum_{k=0}^{\infty}\left(\frac{x}{y}\right)^k\right)\frac{(1-c_1/y)(1-c_2/y)}{(1-a/c_1)(1-a/c_2)(1-b/c_1)(1-b/c_2)(1-d/y)}.$$

But now, y only appears in denominators. So once again we can state

$$\begin{aligned} G^{(3)}_{\mathcal{J}} \sim_\Delta G^{(4)}_{\mathcal{J}} &= \frac{c_1^2c_2^2}{b} \cdot \frac{1}{(1-a/c_1)(1-a/c_2)(1-b/c_1)(1-b/c_2)} \\ \sim_\Delta G^{(5)}_{\mathcal{J}} &= 2a+b+c_1+c_2 \end{aligned}$$

So the quiver polynomial is

$$\Delta^{\mathbb{K}_2,\mathbb{K}_3,\mathbb{K}_1}_{\{a\},\{b\},\{c_1,c_2\}}(2a+b+c_1+c_2) = 2\cdot k_{2,1} + k_{3,1} + k_{1,1} \tag{2.5}$$

$$= 2\cdot c_1(\gamma_2^*) + c_1(\gamma_3^*) + c_1(\gamma_2^*+\gamma_3^*+\gamma_4^*-\gamma_1^*). \tag{2.6}$$

2.4. Truncating the Generating Series

We introduce a shorthand notation for some of the previous products appearing in our functions so that we can discuss them directly. Throughout this section fix a collection of alphabets $\mathfrak{A} = \{\mathbb{A}_1,\dots,\mathbb{A}_N\}$ with $\mathrm{card}(\mathbb{A}_i) = n_i$ and a collection of positive alphabets $\mathfrak{K} = \{\mathbb{K}_1,\dots,\mathbb{K}_N\}$.

Recall the following:

$$\mathbb{B}_k = \bigcup_{\substack{l>k \\ i_l = i_k}} \mathbb{A}_l, \qquad \mathbb{C}_k = \bigcup_{\substack{l>k \\ i_l \in H(i_k)}} \mathbb{A}_l.$$

DEFINITION 2.4.1. Define the **Cauchy product**

$$\mathbf{K}(\mathbb{A}_k, \mathbb{A}_l^\vee) = \prod_{\substack{a \in \mathbb{A}_k \\ b \in \mathbb{A}_l}} \left(1 - \frac{a}{b}\right)^{-1}.$$

In our above discussion we would consider the Taylor expansion of such a function,

$$\mathbf{K}(\mathbb{A}_k, \mathbb{A}_l^\vee) = \prod_{\substack{a \in \mathbb{A}_k \\ b \in \mathbb{A}_l}} \sum_{i=0}^{\infty} \left(\frac{a}{b}\right)^i.$$

However, we will show that one can replace such a power series with a polynomial without affecting the Δ-image of the quiver polynomial.

Consider that one can rewrite any power series in the basis of monomials, so for $\mathbf{K}$:

$$\mathbf{K}(\mathbb{A}_k, \mathbb{A}_l^\vee) = \sum_{\alpha, \beta \in \mathbb{Z}^n} c_{\alpha,\beta} \mathbb{A}_k^\alpha \mathbb{A}_l^\beta.$$

Define

$$\mathbf{K}_p(\mathbb{A}_k, \mathbb{A}_l^\vee) = \sum_{\alpha, \beta \in \mathbb{Z}^n} d_{\alpha,\beta} \mathbb{A}_k^\alpha \mathbb{A}_l^\beta$$

where

$$d_{\alpha,\beta} = \begin{cases} c_{\alpha,\beta} & \text{if } \sum \alpha_i \leq p \\ 0 & \text{otherwise} \end{cases}.$$

We need not worry about the size of β because if $\sum \beta_i \neq \sum \alpha_i$ then $c_{\alpha,\beta} = 0$.

2.4.1. Truncating the Interference Factors. In light of our new notation, the interference factors can be written:

$$I_k = \mathbf{K}(\mathbb{A}_k, \mathbb{C}_k^\vee) \prod_{\substack{a \in \mathbb{A}_k \\ b \in \mathbb{B}_k}} \left(1 - \frac{a}{b}\right).$$

Define

$$I_{k,p} = \mathbf{K}_p(\mathbb{A}_k, \mathbb{C}_k^\vee) \prod_{\substack{a\in\mathbb{A}_k \\ b\in\mathbb{B}_k}} \left(1 - \frac{a}{b}\right)$$

to be a truncated interference factor.

THEOREM 2.4.2. *For each $k = 1, \dots, N$ there is a p_k such that*

$$\prod_{k=1}^{N} M_k I_k \sim_{\Delta_{\mathfrak{A}}^{\mathfrak{K}}} \prod_{k=1}^{N} M_k I_{k,p_k}.$$

We will suspend the proof of this until we have shown a few more facts.

2.4.2. Laurent Series are Δ-Similar to Power Series. Let f be a Laurent series in the variables of $\mathfrak{A}$. Then

$$f = \sum_{\alpha=(\alpha_1,\dots,\alpha_p)} c_\alpha \prod_{k=1}^{N} \mathbb{A}_k^{\alpha_k}.$$

Choose a term of f: $T = c_\alpha \prod_{k=1}^N \mathbb{A}_k^{\alpha_k}$. Suppose for some k that there is an i with $1 \le i \le n_k$ and $(\alpha_k)_i \le -n_k$. Now α_k is Bott similar to a unique non-increasing vector, call it β. Because of the assumptions on α_k, β will have a negative entry and thus not be a partition. Therefore

$$\Delta_{\mathbb{A}_k}^{\mathbb{K}_k}(\mathbb{A}_k^{\alpha_k}) = \pm\Delta_{\mathbb{A}_k}^{\mathbb{K}_k}(\mathbb{A}_k^{\beta}) = \Delta_{\mathbb{A}_k}^{\mathbb{K}_k}(0) \Rightarrow \Delta_{\mathfrak{A}}^{\mathfrak{K}}(T) = \Delta_{\mathfrak{A}}^{\mathfrak{K}}(0).$$

See statement (1.11).

PROPOSITION 2.4.3. *Suppose that g is a polynomial in the variables of $\mathfrak{A}$ and $1 \le s \le N$. There is a $p \in \mathbb{N}$ such that*

$$g \cdot \prod_{k=1}^{s} \boldsymbol{K}(\mathbb{A}_k, \mathbb{C}_k^\vee) \sim_{\Delta_{\mathfrak{A}}^{\mathfrak{K}}} g \cdot \boldsymbol{K}_p(\mathbb{A}_s, \mathbb{C}_s^\vee) \cdot \prod_{k=1}^{s-1} \boldsymbol{K}(\mathbb{A}_k, \mathbb{C}_k^\vee).$$

PROOF. Let f be the Laurent series

$$f = g \cdot \prod_{k=1}^{s} \mathbf{K}(\mathbb{A}_k, \mathbb{C}_k^\vee) = g \cdot \prod_{k=1}^{s} \left(\prod_{\substack{a\in\mathbb{A}_k \\ c\in\mathbb{C}_k}} \sum_{i=0}^{\infty} \left(\frac{a}{c}\right)^i \right).$$

So any given term of h appears as

$$d_{\alpha,\delta} \prod_{k=1}^{p} \frac{\mathbb{A}_k^{\alpha_k}}{\mathbb{C}_k^{\delta_k}} = \bar{d}_{\bar{\alpha}} \prod_{k=1}^{p} \mathbb{A}_k^{\bar{\alpha}_k}.$$

Where $\alpha = (\alpha_1, \ldots, \alpha_N), \delta = (\delta_1, \ldots, \delta_N)$, and $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_N)$.

Consider f as a Laurent series in the variables of $\mathbb{C}_s$.

Note that each term of the series of $\mathbf{K}(\mathbb{A}_s, \mathbb{C}_s^\vee)$ reduces the exponents of the indeterminants of $\mathbb{C}_s$, but no terms of f increase these exponents. This is because the numerators are comprised of only those $\mathbb{A}_l$ with $l \leq s$, while $\mathbb{C}_s$ is comprised of certain $\mathbb{A}_l$ with $l > s$.

Since g is a polynomial, it has only finitely many terms. Thus there are only finitely many terms where $\forall a_{ki} \in \mathbb{C}_s$, $\bar{\alpha}_{ki} > -r_k$ (to paraphrase: "the exponents stay sufficiently large"). The terms where the exponents are not "suffienctly large" will be Δ-equivalent to zero, therefore all but finitely many terms of f (as a Laurent series in the variables of $\mathbb{C}_s$) are Δ-equivalent to 0. This is exactly the claim of the proposition. □

2.4.3. Proof of Theorem 2.4.2. Now we are prepared to prove theorem 2.4.2:

Proof. Let $g = \prod_{k=1}^{N} \left(M_k \cdot \prod_{\substack{a \in \mathbb{A}_k \\ b \in \mathbb{B}_k}} \left(1 - \frac{a}{b}\right) \right)$. Note that g is a polynomial. By proposition 2.4.3 there is a p_N such that:

$$g \cdot \prod_{k=1}^{N} I_k \sim_{\Delta_{\mathfrak{A}}^{\mathfrak{K}}} g \cdot I_{N,p_N} \cdot \prod_{k=1}^{N-1} I_k.$$

Then note that $g \cdot I_{N,p_N}$ is a polynomial and use proposition 2.4.3, again. And thus through "N" applications of proposition 2.4.3 one has completed the proof. □

Therefore, if

$$[S_m] = \Delta_{\mathfrak{A}}^{\mathfrak{K}} \left(\prod_{k=1}^{N} M_k I_k \right) \quad \text{then} \quad [S_m] = \Delta_{\mathfrak{A}}^{\mathfrak{K}} \left(\prod_{k=1}^{N} M_k I_{k,p_k} \right), \tag{2.7}$$

for some $p_k \in \mathbb{N}$. Further, $\prod_{k=1}^{N} M_k I_{k,p_k} \in \Pi_{\mathbb{Z}}^{+}(\mathfrak{A})$.

2.5. The Numerator Lemma and Gluing Property

This section introduces a rather powerful tool which is called "the Numerator Lemma." This lemma describes how to collapse variables from the same vertex into one alphabet.

2.5.1. The Lemma.

DEFINITION 2.5.1. Define the **rational resultant** of two alphabets, $\mathbb{A}$ and $\mathbb{B}$, to be

$$RRes(\mathbb{A}|\mathbb{B}) = \prod_{a\in\mathbb{A}}\prod_{b\in\mathbb{B}}\left(1-\frac{b}{a}\right).$$

REMARK. Note that $RRes(\mathbb{A}|\mathbb{B})\cdot\prod_{a\in\mathbb{A}} a^{|\mathbb{B}|} = Res(\mathbb{A}|\mathbb{B})$.

LEMMA 2.5.2. *(Numerator Lemma)*

Suppose that $\mathfrak{A} = (\mathbb{A}_1,\ldots,\mathbb{A}_N)$ *is a collection of finite alphabets with* $card(\mathbb{A}_i) = n_i$ *and* $\mathfrak{K} = (\mathbb{K}_1,\ldots,\mathbb{K}_N)$ *is a collection of infinite alphabets* that are all equal. *If* $f\in\Pi_{\mathbb{Z}}(\mathfrak{A})$ *then*

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}\left(f\cdot\prod_{j>i}RRes(A_j|A_i)\right) = \Delta_{\mathbb{A}_1+\ldots+\mathbb{A}_N}^{\mathbb{K}_1}(f).$$

PROOF. For readability in the following proof, let $Disc(\mathfrak{A}) = \prod_{\mathbb{A}\in\mathfrak{A}} Disc(\mathbb{A})$ and $Disc\left(\sum\mathbb{A}\right) = Disc(\mathbb{A}_1+\ldots+\mathbb{A}_N)$.

We know from chapter one, lemma 1.4.3, that

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}\left(f\cdot\prod_{j>i}RRes(A_j|A_i)\right) = C_{\mathfrak{A}}^{\mathfrak{K}}\left(f\cdot Disc(\mathfrak{A})\cdot\prod_{j>i}RRes(A_j|A_i)\cdot\prod_{i=1}^{k}\prod_{j=1}^{n_j}a_{i,j}^{1-j}\right) \tag{2.8}$$

$$= C_{\mathfrak{A}}^{\mathfrak{K}}\left(f\cdot Disc(\mathfrak{A})\cdot\prod_{j>i}\frac{Res(A_j|A_i)}{\prod_{a\in\mathbb{A}_j}a^{n_i}}\cdot\prod_{i=1}^{k}\prod_{j=1}^{n_j}a_{i,j}^{1-j}\right) \tag{2.9}$$

$$= C_{\mathfrak{A}}^{\mathfrak{K}}\left(f\cdot Disc\left(\sum\mathbb{A}\right)\cdot\prod_{i=1}^{k}\prod_{j=1}^{n_j}a_{i,j}^{1-j+\sum_{q<i}n_q}\right) \tag{2.10}$$

$$= C_{\mathbb{A}_1+\ldots+\mathbb{A}_k}^{\mathbb{K}_1}\left(f\cdot Disc\left(\sum\mathbb{A}\right)\cdot\prod_{i=1}^{k}\prod_{j=1}^{n_j}a_{i,j}^{1-j+\sum_{q<i}n_q}\right) \tag{2.11}$$

$$= \Delta_{\mathbb{A}_1+\ldots+\mathbb{A}_k}^{\mathbb{K}_1}(f). \tag{2.12}$$

$\square$

Corollary 2.5.3. *(The Gluing Property)*

Suppose that $\mathbb{A}$ *and* $\mathbb{B}$ *are finite alphabets of cardinality* n *and* m*, respectively. If* $\mathbb{K}$ *is a positive alphabet and* λ, μ *are partitions with* $\ell(\lambda) \leq n, \ell(\mu) \leq m$*, then*

$$\Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_\lambda(\mathbb{A}) \cdot s_\mu(\mathbb{B})) = \Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_\nu(\mathbb{A}+\mathbb{B}))$$

where

$$\nu_i = \begin{cases} \lambda_i & \text{if } i \leq \ell(\lambda) \\ \mu_{i-n} & \text{if } n < i \leq n + \ell(\mu) \\ 0 & \text{otherwise} \end{cases} .$$

Example. *Suppose* $\mathbb{A} = \{a_1, a_2\}, \mathbb{B} = \{b_1, b_2, b_3\}, \lambda = (2), \mu = (2,1)$*. Then* $\nu = (2,0,2,1,0)$ *and*

$$\Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_{(2)}(\mathbb{A}) \cdot s_{(2,1)}(\mathbb{B})) = \Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_{(2,0)}(\mathbb{A}) \cdot s_{(2,1,0)}(\mathbb{B})) \tag{2.13}$$

$$= \Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_{(2,0,2,1,0)}(\mathbb{A}+\mathbb{B})) \tag{2.14}$$

$$= \Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(-s_{(2,1,1,1,0)}(\mathbb{A}+\mathbb{B})). \tag{2.15}$$

Proof. (of Corollary 2.5.3)

By the numerator lemma,

$$\Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_\lambda(\mathbb{A}) \cdot s_\mu(\mathbb{B})) = \Delta^{\mathbb{K},\mathbb{K}}_{\mathbb{A},\mathbb{B}}(s_\lambda(\mathbb{A}) \cdot s_\mu(\mathbb{B}) \cdot RRes(\mathbb{B}|\mathbb{A})).$$

Now $RRes(\mathbb{B}|\mathbb{A}) \in Symm_{\mathbb{Z}[\mathbb{B}]}(\mathbb{A}) \cap Symm_{\mathbb{Z}[\mathbb{A}]}(\mathbb{B})$ so

$$\begin{aligned} &Asym_{\mathbb{A},\mathbb{B}}(s_\lambda(\mathbb{A}) step(\mathbb{A}) \cdot s_\mu(\mathbb{B}) step(\mathbb{B}) \cdot RRes(\mathbb{B}|\mathbb{A})) \\ &\qquad = Asym_{\mathbb{A},\mathbb{B}}(\mathbb{A}^\lambda step(\mathbb{A}) \cdot \mathbb{B}^\mu step(\mathbb{B}) \cdot RRes(\mathbb{B}|\mathbb{A})). \end{aligned} \tag{2.16}$$

This gives that

$$\Delta^{\mathbb{K},\mathbb{K}}_{\mathbb{A},\mathbb{B}}(s_\lambda(\mathbb{A})\cdot s_\mu(\mathbb{B})\cdot RRes(\mathbb{B}|\mathbb{A})) = \Delta^{\mathbb{K},\mathbb{K}}_{\mathbb{A},\mathbb{B}}(\mathbb{A}^\lambda\cdot\mathbb{B}^\mu\cdot RRes(\mathbb{B}|\mathbb{A})) \tag{2.17}$$

$$= \Delta^{\mathbb{K},\mathbb{K}}_{\mathbb{A},\mathbb{B}}((\mathbb{A}+\mathbb{B})^\tau\cdot RRes(\mathbb{B}|\mathbb{A})) \tag{2.18}$$

$$= \Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}((\mathbb{A}+\mathbb{B})^\nu) \tag{2.19}$$

$$= \Delta^{\mathbb{K}}_{\mathbb{A}+\mathbb{B}}(s_\nu(\mathbb{A}+\mathbb{B})) \tag{2.20}$$

by reversing the previous arguments. □

2.5.2. The Resultant-Free Generating Sequence. Consider an orbit closure, S_m, with corresponding directed partition, $\mathcal{I}(S_m)$, and generating sequence, $G_\mathcal{I}$. Let $\mathbf{i}=(i_1,\ldots,i_q)$, $\mathbf{r}$ be the resolution pair derived from $\mathcal{I}$. One can construct the **resultant-free generating sequence** through the following method. Set

$$\mathbb{D}_k = \sum_{i_l=k}\mathbb{A}_l, \qquad \text{and} \qquad \mathbb{K}_k = \{c_n(\tau^*_{i_k}-\gamma^*_{i_k})\}_n. \tag{2.21}$$

Now define the resultant-free interference factors for $1\le k\le p$:

$$\bar{I}_k = \frac{1}{\prod_{\substack{a\in\mathbb{A}_k\\ c\in\mathbb{C}_k}}\left(1-\frac{a}{c}\right)} = I_k\cdot RRes(\mathbb{B}_k|\mathbb{A}_k).$$

The resultant-free generating sequence associated to $\mathcal{I}(S_m)$ is defined as

$$H_\mathcal{I} = \prod_{k=1}^{p} M_k\bar{I}_k.$$

THEOREM 2.5.4. *The quiver polynomial can be obtained from the resultant-free generating sequence. That is, for $N = card(Q_0)$,*

$$[S_m] = \Delta^{\mathbb{K}_1,\ldots,\mathbb{K}_N}_{\mathbb{D}_1,\ldots,\mathbb{D}_N}(H_\mathcal{I}).$$

PROOF. The proof is just repeated application of the numerator lemma. □

One can further truncate the intereference factors as discussed in section 2.4 and to have

$$[S_m] = \Delta^{\mathbb{K}_1,\ldots,\mathbb{K}_N}_{\mathbb{D}_1,\ldots,\mathbb{D}_N}\left(\prod_{k=1}^{N} M_k \bar{I}_{k,p_k}\right),$$

for some $p_k \in \mathbb{N}$.

2.5.3. Example of the Resultant-Free Generating Sequence. Consider the source-sink A_4 quiver: $1 \to 2 \leftarrow 3 \to 4$. The simple roots of the A_4 root system are $\alpha_1, \alpha_2, \alpha_3$, and α_4. The remaining positive roots are $\alpha_{ij} = \sum_{i\le u\le j} \alpha_u$, for $i, j \in \{1, 2, 3, 4\}$ with $i < j$.

A directed partition for this quiver is

$$\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_{24}\} \cup \{\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}\} \cup \{\alpha_1, \alpha_3, \alpha_{34}\}.$$

If we look at a generic case (rather than selecting values for the $m_\bullet$), we get that the resolution pair is

$$\mathbf{i} = (3, 2, 4, 1, 3, 2, 4, 1, 3, 4),$$

$$\begin{aligned}\mathbf{r} = (&m_{24}, m_2, m_4, m_{12} + m_{13} + m_{14}, m_{13} + m_{14} + m_{23},\\ &\qquad m_{12} + m_{13} + m_{14} + m_{23}, m_{14}, m_1, m_3 + m_{34}, m_{34}).\end{aligned}$$

The monomial term is

$$M = \frac{\mathbb{A}_2^{(m_{13}+m_{14}+m_1+m_3+m_{34})}\mathbb{A}_3^{(m_{13}+m_{23}+m_3)}\mathbb{A}_6^{(m_{14}+m_1+m_{34})}\mathbb{A}_7^{(m_3)}}{\mathbb{A}_1^{(m_{13}+m_{14}+m_{23}+m_3+m_{34})}\mathbb{A}_4^{(m_1)}\mathbb{A}_5^{(m_3+m_{34})}}$$

and the non-trivial resultant-free interference factors are

$$\bar{I}_1 = \prod_{\substack{a\in\mathbb{A}_1\\ b\in\mathbb{A}_2+\mathbb{A}_3+\mathbb{A}_6+\mathbb{A}_7+\mathbb{A}_{10}}} \frac{1}{(1-a/b)}, \qquad \bar{I}_4 = \prod_{\substack{a\in\mathbb{A}_4\\ b\in\mathbb{A}_6}} \frac{1}{(1-a/b)},$$

$$\bar{I}_5 = \prod_{\substack{a\in\mathbb{A}_5\\ b\in\mathbb{A}_6+\mathbb{A}_7+\mathbb{A}_9}} \frac{1}{(1-a/b)}, \qquad \bar{I}_9 = \prod_{\substack{a\in\mathbb{A}_9\\ b\in\mathbb{A}_{10}}} \frac{1}{(1-a/b)}.$$

If we define $(1-\mathbb{A}/\mathbb{B}) = \prod_{\substack{a\in\mathbb{A}\\ b\in\mathbb{B}}}(1-a/b)$, then

$$H_{\mathcal{I}} = M \cdot \frac{1}{\left(1-\frac{\mathbb{A}_1}{\mathbb{A}_2+\mathbb{A}_3+\mathbb{A}_6+\mathbb{A}_7+\mathbb{A}_{10}}\right)\left(1-\frac{\mathbb{A}_4}{\mathbb{A}_6}\right)\left(1-\frac{\mathbb{A}_5}{\mathbb{A}_6+\mathbb{A}_7+\mathbb{A}_9}\right)\left(1-\frac{\mathbb{A}_9}{\mathbb{A}_{10}}\right)},$$

and

$$\Delta^{\mathbb{K}_3,\mathbb{K}_2,\mathbb{K}_4,\mathbb{K}_1}_{\mathbb{A}_1+\mathbb{A}_5+\mathbb{A}_9,\mathbb{A}_2+\mathbb{A}_6,\mathbb{A}_3+\mathbb{A}_7+\mathbb{A}_{10},\mathbb{A}_4+\mathbb{A}_8}(H_{\mathcal{I}})$$

is the quiver polynomial.

CHAPTER 3

Schur Positivity and Strong Schur Positivity Results

The purpose of this chapter is to present Schur positivity and strong Schur positivity in terms of quiver polynomials. After introducing basic definitions in section 1, sections 2 through 4 will discuss basic theorems involving Schur positivity and strong Schur positivity. Finally section 5 will address which quivers and orbits are known to be strongly Schur positive and any new Schur positivity results.

3.1. Schur Determinants

Recall the Jacobi-Trudi transform $\Delta_{\mathfrak{A}}^{\mathfrak{K}} : \Pi_{\mathbb{Z}}(\mathfrak{A}) \to \Pi_{\mathbb{Z}}(\mathfrak{K})$. We want to describe the image of the generating sequences discussed in Chapter 2 so we need to introduce some new terminology.

Definition 3.1.1. Suppose $\mathbb{K}$ is an infinite alphabet. If $\lambda \in \mathbb{Z}^n$ then the *Schur determinant with parameter* λ is

$$\mathcal{S}_\lambda(\mathbb{K}) = \Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^\lambda)$$

for some finite alphabet, $\mathbb{A}$ with $\mathrm{card}(\mathbb{A}) = n$.

If $\mathfrak{K} = (\mathbb{K}_1, \dots, \mathbb{K}_N)$ is a collection of infinite alphabets and $\lambda = (\lambda_1, \dots, \lambda_N)$ with $\lambda_i \in \mathbb{Z}^{n_i}$, then one can simply extend the above definition to get

$$\mathcal{S}_\lambda(\mathfrak{K}) = \Delta_{\mathfrak{A}}^{\mathfrak{K}}(\mathbb{A}_1^{\lambda_1} \cdots \mathbb{A}_N^{\lambda_N})$$

with $\mathbb{A}_i$ disjoint finite alphabets with $\mathrm{card}(\mathbb{A}_i) = n_i$.

Essentially the monomial functions form a linear basis of the polynomials and the Schur determinants are the images of the monomials under the Jacobi-Trudi transform.

Since the Jacobi-Trudi transform is linear then the Schur determinants spand the image of the Jacobi-Trudi transform.

3.2. Schur Positivity

Let $f \in Symm_{\mathbb{Z}}^{+}(\mathbb{A})$ for some finite alphabet $\mathbb{A}$. Suppose that $f = \sum_{\lambda} c_{\lambda} s_{\lambda}(\mathbb{A})$ is written in terms of the Schur polynomials. We know that such an expression exists and is unique since the Schur polynomials are a linear basis for the space of symmetric polynomials. If each $c_{\lambda} \geq 0$ then f is said to be **Schur positive**.

One can extend this notion to Schur positivity for polynomials that are symmetric in each variable set of $\mathfrak{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_N)$ with $\text{card}(A_i) = n_i$. If $f \in Symm_{\mathbb{Z}}^{+}(\mathfrak{A})$, then $f = \sum_{\lambda=(\lambda_1,\ldots,\lambda_N)} \left(c_{\lambda} \cdot \prod_{i=1}^{N} s_{\lambda_i}(\mathbb{A}_i) \right)$. Thus f can be said to be Schur positive if and only if each $c_{\lambda} \geq 0$.

Definition 3.2.1. Let $f \in \mathbb{Z}[[\mathfrak{A}]]$ be a power series symmetric in each $\mathbb{A}_i$, then $f = \sum_{k=0}^{\infty} f_k(\mathfrak{A})$ where $f_k \in \Pi_{\mathbb{Z}}(\mathfrak{A})$ is homogeneous of degree k. Then f is **Schur positive** if and only if $f_k(\mathfrak{A})$ is Schur positive for all k.

Definition 3.2.2. Let f be a Laurent series (polynomial) in the variables of $\mathfrak{A}$. Then $f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_p)} c_{\alpha} \cdot \prod_{i=1}^{p} \mathbb{A}_p^{\alpha}$, where $\alpha_i \in \mathbb{Z}^{n_i}$. The **integer part of** f is $f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_p)} c_{\alpha} \cdot \prod_{i=1}^{p} \mathbb{A}_p^{\alpha}$, where $\alpha_i \in (\mathbb{Z}^{\geq 0})^{n_i}$. The integer part of f is denoted $ip(f)$.

Definition 3.2.3. Let f be a Laurent series (polynomial) in the variables of $\mathfrak{A}$. Then f is **Schur positive** if and only if the integer part of f is Schur positive.

3.3. Expressing a Quiver Polynomial in Terms of Schur Determinants

It was proven by A. Buch [B08] that quiver polynomials can be expressed uniquely in terms of Schur determinants. That is, if we set $\mathbb{K}_k = \{c_n(\tau_{i_k}^{*} - \gamma_{i_k}^{*})\}_n$, where $\tau_{i_k} = \oplus_{j \in T(i_k)} \gamma_j$, then

$$[S_m] = \sum_{\lambda=(\lambda_1,\ldots,\lambda_N)} \left(c_{\lambda} \cdot \prod_{i=1}^{N} \mathcal{S}_{\lambda_i}(\mathbb{K}_i) \right).$$

The cohomological part of Buch's conjecture is that when a quiver polynomial is expressed in this way that each c_λ is non-negative.

PROPOSITION 3.3.1. *If* $\mathbb{A}$ *is a finite alphabet with card*$(\mathbb{A}) = n$*,* $\mathbb{K}$ *is a positive alphabet, and* $\lambda \in \mathbb{Z}^n$ *then*

$$\Delta_{\mathbb{A}}^{\mathbb{K}}(s_\lambda(\mathbb{A})) = \mathcal{S}_\lambda(\mathbb{K}).$$

PROOF. We know that $\Delta_{\mathbb{A}}^{\mathbb{K}}(s_\lambda(\mathbb{A})) = \Delta_{\mathbb{A}}^{\mathbb{K}}(\mathbb{A}^\lambda) = \mathcal{S}_\lambda(\mathbb{K})$. □

This makes it clear that for a given quiver and orbit closure, S_m, Buch's conjecture holds if and only if there is a Schur positive function, $f \in \Pi_{\mathbb{Z}}(\mathfrak{A})$, such that

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}(f) = [S_m].$$

Thus one can say that the quiver polynomial is Schur positive without confusion. Using this language, Buch's claim is that quiver polynomials associated to all quivers of ADE-type are Schur positive.

3.4. Strong Schur Positivity

The goal of this section is to extend the idea of Schur positivity to Laurent polynomials and Laurent series. This is done by decomposing the series (or polynomial) into a sum of polynomials that are homogeneous and then reimagining Schur positivity in terms of these homogeneous polynomials.

DEFINITION 3.4.1. Suppose $\mathfrak{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_N)$ is a collection of finite alphabets with card$(A_i) = n_i$. Let f be a Laurent series (polynomial) in the variables of $\mathfrak{A}$. Then f is **strongly Schur positive** if

$$f \cdot \prod_{i=1}^{N} \mathbb{A}_i^{(k_i)}$$

is Schur positive for all $k_i \geq 0$.

Recall that $\mathbb{A}^{(k)} = \prod_{a \in \mathbb{A}} a^k$.

Remark. If $\mathbb{A}$ is an alphabet of cardinality n and $\lambda \in \mathbb{Z}^n$ is non-increasing with $\lambda_n < 0$, then let $\nu \in \mathbb{Z}^n$ be $\nu = (\lambda_n, \ldots, \lambda_n)$. We can define

$$s_\lambda(\mathbb{A}) = \frac{s_{\lambda+\nu}(\mathbb{A})}{s_\nu(\mathbb{A})}.$$

Keeping this in mind, the definition of f being strongly Schur positive is equivalent to f being able to be written in the following form:

$$f = \sum_{\lambda=(\lambda_1,\ldots,\lambda_N)} c_\lambda \prod_{i=1}^{N} s_{\lambda_i}(\mathbb{A}_i),$$

where $c_\lambda \geq 0, \forall \lambda$ with $\lambda_i \in \mathbb{Z}^{|\mathbb{A}_i|}$ non-increasing.

Proposition 3.4.2. *If f, g are two Laurent series that are strongly Schur positive then their product is also strongly Schur positive.*

Proof. Suppose that $\mathfrak{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_N)$ is a collection of finite alphabets. Write

$$f = \sum_{\lambda=(\lambda_1,\ldots,\lambda_N)} c_\lambda \cdot \prod_{j=1}^{N} s_{\lambda_j}(\mathbb{A}_j),$$

$$g = \sum_{\mu=(\mu,\ldots,\mu_N)} c_\mu \cdot \prod_{j=1}^{N} s_{\mu_j}(\mathbb{A}_j).$$

Therefore,

$$\begin{aligned} f \cdot g &= \sum_{\lambda,\mu} c_\lambda c_\mu \cdot \prod_{j=1}^{N} s_{\lambda_j}(\mathbb{A}_j) s_{\mu_j}(\mathbb{A}_j) \\ &= \sum_{\nu} \left(\sum_{\lambda,\mu} c_\lambda c_\mu c^{\nu_j}_{\lambda_j,\mu_j} \right) \cdot \prod_{j=1}^{N} s_{\nu_j}(\mathbb{A}_j). \end{aligned}$$

Since each $c_\lambda, c_\mu, c^{\nu_j}_{\lambda_j,\mu_j} \geq 0$ then $\sum_{\lambda,\mu} c_\lambda c_\mu c^{\nu_j}_{\lambda_j,\mu_j} \geq 0$. $\square$

Again, we wish to extend this idea to orbit closures. Let $\mathfrak{D} = \{\mathbb{D}_k\}$ with $\mathbb{D}_k = \sum_{i_l=k} \mathbb{A}_l$, and let $\mathfrak{K} = (\mathbb{K}_1, \ldots, \mathbb{K}_N)$ be a collection of disjoint infinite alphabets.

Definition 3.4.3. Let Q be a quiver, S_m be an orbit closure associated to Q and $\mathcal{I}(S_m)$ be a directed partition obtained from the orbit closure. We say that the **directed partition is strongly Schur positive** if the resultant-free generating sequence $H_{\mathcal{I}}$ is $\Delta_{\mathfrak{D}}^{\mathfrak{K}}$-equivalent to a strongly Schur positive Laurent series $h_{\mathcal{I}}$.

The function $h_{\mathcal{I}}$ is the result of applying the gluing property, corollary 2.5.3, to the resultant-free generating sequence, $H_{\mathcal{I}}$, and then selecting the Schur representative.

Definition 3.4.4. Consider an orbit arising from a quiver. We say the **orbit is strongly Schur positive** if there is a strongly Schur positive directed partition associated to the orbit.

Remark. Notice that there is a significant difference between stating that an orbit is Schur positive and stating that it is strongly Schur positive. When one states that an orbit is Schur positive it means that *all* generating sequences derived from *all* directed partitions are Schur positive. Stating that an orbit is strongly Schur positive is simply an existence statement–that there is an associated directed partition with a generating sequence that is strongly Schur positive.

Proposition 3.4.5. *Any orbit closure that is strongly Schur positive is also Schur positive.*

Proof. Let S_m be an orbit closure that is strongly Schur positive and let $\mathcal{I}(S_m)$ be the associated strongly Schur positive directed partition. Then the derived resultant-free generating sequence $H_{\mathcal{I}}$ satisfies

$$\Delta_{\mathfrak{D}}^{\mathfrak{K}}(H_{\mathcal{I}}) = \Delta_{\mathfrak{D}}^{\mathfrak{K}}(h_{\mathcal{I}})$$

where $h_{\mathcal{I}}$ is a strongly Schur positive Laurent series.

Since $h_{\mathcal{I}}$ is a Laurent series it can be written as a sum of monomials. As per (1.11) each of these monomials is Δ-similar to a monomial with non-negative exponents or it is Δ-similar to 0. Thus $h_{\mathcal{I}}$ is Δ-equivalent to the integer part of $h_{\mathcal{I}}$. Since $h_{\mathcal{I}}$ is strongly Schur positive, it's integer part must be Schur positive.

Thus

$$[S_m] = \Delta_{\mathfrak{D}}^{\mathfrak{K}}(H_{\mathcal{I}}) = \Delta_{\mathfrak{D}}^{\mathfrak{K}}(h_{\mathcal{I}}) = \Delta_{\mathfrak{D}}^{\mathfrak{K}}(ip(h_{\mathcal{I}}))$$

for $\mathfrak{K} = \{\mathbb{K}_k\}$ and $\mathbb{K}_k = \{c_n(\tau_{i_k}^* - \gamma_{i_k}^*)\}_n$. □

3.5. n-Step Desingularizations

In Rimányi's paper [RR] he constructs a multi-step desingularization. In the final step of the desingularization, the quiver polynomial is trivial. Then Rimanyi undoes his desingularization step-by-step and keeps track of how each step affects the quiver polynomial. If the process required n steps then it is referred to as an n-step desingularization. One immediate consequence is that if the desingularization is n-step, then the associated directed partition will be comprised of $(n+1)$ sets (and vice-versa).

Definition 3.5.1. Suppose that Q is a quiver of ADE type with a n-step desingularization and associated directed partition $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_{n+1}$. A **next-step** desingularization is any quiver, orbit, and associated directed partion, $\mathcal{J}$ that admits an $(n+1)$-step desingularization with $\mathcal{J} = \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_{n+1} \cup \mathcal{J}_{n+2}$ where $\mathcal{J}_i$ is equal to $\mathcal{I}_i$ (except possibly up to relabeling) for all $i \leq n+1$.

Example. *Consider the equi-oriented A_3 quiver: $E_1 \to E_2 \to E_3$. This admits a 2-step desingularization with directed partition*

$$\mathcal{I} = \{\alpha_{22}\} \cup \{\alpha_{33}, \alpha_{23}, \alpha_{13}\} \cup \{\alpha_{11}, \alpha_{33}\}.$$

Even though the A_4 quiver: $E_1 \to E_2 \leftarrow E_3 \leftarrow E_4$, with directed partition

$$\mathcal{J}_{\Phi^+} = \{\alpha_{22}\} \cup \{\alpha_{12}, \alpha_{23}, \alpha_{13}, \alpha_{24}\} \cup \{\alpha_{11}, \alpha_{33}, \alpha_{14}\} \cup \{\alpha_{34}, \alpha_{44}\}$$

*is **not** a next-step desingularization, if we choose an orbit with $m_{24} = m_{14} = 0$ then the directed partition becomes*

$$\mathcal{J}(S_m) = \{\alpha_{22}\} \cup \{\alpha_{12}, \alpha_{23}, \alpha_{13}\} \cup \{\alpha_{11}, \alpha_{33}\} \cup \{\alpha_{34}, \alpha_{44}\}.$$

This directed partition is a next-step desingularization of $\mathcal{I}$.

LEMMA 3.5.2. *(Next-Step)*

Suppose that $S1_m$ is an orbit with a strongly Schur positive directed partition $\mathcal{I}$ and $S2_m$ is an orbit with directed partition $\mathcal{J}$ that is a next-step desingularization of $\mathcal{I}(S1_m)$. Then $S2_m$ is Schur positive.

Observe that if we let $\mathbf{i}$,$\mathbf{r}$ be the resolution pair for $\mathcal{I}$ and $\mathbf{j}$,$\mathbf{s}$ be the resolution pair for $\mathcal{J}$. Set p equal to the length of $\mathbf{i}$ and q equal to the length of $\mathbf{j}$. Note that $i_k = j_k$ and $r_k = s_k$ for all k up to the length of $\mathbf{i}$. Finally, let $\mathfrak{A} = (\mathbb{A}_1, \ldots, \mathbb{A}_q)$ be finite alphabets with $\mathrm{card}(\mathbb{A}_t) = s_t$ and $\mathbb{K}_k = \{c_n(\tau^*_{j_k} - \gamma^*_{j_k})\}_n, \mathfrak{K} = (\mathbb{K}_1, \ldots, \mathbb{K}_q)$.

Define the following for $1 \le k \le p$,

$$\hat{n}_k = \sum_{\substack{l>p \\ j_l \in T(i_k)}} s_l - \sum_{\substack{l>p \\ j_l = i_k}} s_l, \qquad L_k = \prod_{t=1}^{r_k} a_{k,t}^{\hat{n}_k}.$$

and for $p+1 \le t \le q$,

$$\hat{\mathbb{B}}_t = \prod_{\substack{l<t \\ j_l = j_t}} \mathbb{A}_l \qquad \hat{\mathbb{C}}_t = \prod_{\substack{l<t \\ j_l \in T(j_t)}} \mathbb{A}_l.$$

Notice that if $i_k = i_l$ then $\hat{n}_k = \hat{n}_l$. This means that all variables associated to the vertex i_k have their exponent increased (or decreased) by the same amount.

Suppose that $G_{\mathcal{I}}$ is the generating sequence associated to $\mathcal{I}$ and $G_{\mathcal{J}}$ is the generating sequence associated to $\mathcal{J}$. Then

$$G_{\mathcal{J}} = G_{\mathcal{I}} \cdot \left(\prod_{k=1}^{p} L_k \right) \cdot \prod_{k=p+1}^{q} \frac{\prod_{\substack{a \in \mathbb{A}_k \\ b \in \hat{\mathbb{B}}_k}} (1 - b/a)}{\prod_{\substack{a \in \mathbb{A}_k \\ c \in \hat{\mathbb{C}}_k}} (1 - c/a)}.$$

Since we are trying to show that $G_{\mathcal{J}}$ is Schur positive we can disregard any terms with negative exponents as the Jacobi-Trudi transform will map these to zero, by (1.11). Let

$$g_t = G_{\mathcal{I}} \cdot \left(\prod_{k=1}^{p} L_k \right) \cdot \prod_{k=p+1}^{t} \frac{\prod_{\substack{a \in \mathbb{A}_k \\ b \in \hat{\mathbb{B}}_k}} (1 - b/a)}{\prod_{\substack{a \in \mathbb{A}_k \\ c \in \hat{\mathbb{C}}_k}} (1 - c/a)}$$

so $g_q = G_{\mathcal{J}}$. Now let's put the proof on hold to show the following.

Proposition 3.5.3. *For $p < t < q$, $\Delta_{\mathfrak{A}}^{\mathfrak{K}}(g_t) = \Delta_{\mathfrak{A}}^{\mathfrak{K}}(g_{t+1})$.*

Proof. As stated above, we can disregard any terms with negative exponents. The factor

$$\frac{\prod_{\substack{a \in \mathbb{A}_{t+1} \\ b \in \hat{\mathbb{B}}_{t+1}}} (1 - b/a)}{\prod_{\substack{a \in \mathbb{A}_{t+1} \\ c \in \hat{\mathbb{C}}_{t+1}}} (1 - c/a)}$$

is the only instance of the variables from $\mathbb{A}_{t+1}$ in all of g_{t+1}–and they can only occur with non-positive exponents. Thus when g_{t+1} is expanded out as a Laurent series, the only terms that aren't mapped to zero by the Jacobi-Trudi transform must have received a factor of 1 from this piece. This proves the proposition. □

Proof. (of Lemma 3.5.2)

Proposition 3.5.3 shows that

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}(G_{\mathcal{J}}) = \Delta_{\mathfrak{A}}^{\mathfrak{K}} \left(G_{\mathcal{I}} \cdot \prod_{k=1}^{p} L_k \right).$$

Let J be the maximal entry of j, that is the number of verticies in the quiver associated to $\mathcal{J}$. Define $\mathfrak{D} = (\mathbb{D}_1, \ldots, \mathbb{D}_J)$ with $\mathbb{D}_k = \sum_{i_l = k} \mathbb{A}_l$ (where order is preserved) and $\mathbb{L}_k = \{c_n(\tau_k^* - \gamma_k^*)\}_n, \mathfrak{L} = (\mathbb{L}_1, \ldots, \mathbb{L}_J)$. After applying the numerator lemma we get the resultant-free generating sequence:

$$\begin{aligned} \Delta_{\mathfrak{A}}^{\mathfrak{K}}(G_{\mathcal{J}}) &= \Delta_{\mathfrak{D}}^{\mathfrak{L}} \left(H_{\mathcal{I}} \cdot \prod_{k=1}^{J} \prod_{d \in \mathbb{D}_k} d^{\hat{n}_k} \right) \\ &= \Delta_{\mathfrak{D}}^{\mathfrak{L}} \left(h_{\mathcal{I}} \cdot \prod_{k=1}^{J} \prod_{d \in \mathbb{D}_k} d^{\hat{n}_k} \right), \end{aligned}$$

where $h_{\mathcal{I}}$ is a strongly Schur positive Laurent series in the variables of $\mathfrak{D}$. This last step is justified because $\prod_{k=1}^{J}\prod_{d\in\mathbb{D}_k} d^{\hat{n}_k}$ is symmetric in each $\mathbb{D}_k$ and the alphabets of $\mathfrak{L}$ are disjoint.

Now $h_{\mathcal{I}}$ is strongly Schur positive so, by definition, the integer part of $h_{\mathcal{I}}\cdot\prod_{k=1}^{J}\prod_{d\in\mathbb{D}_k} d^{\hat{n}_k}$ is Schur positive. But $\Delta_{\mathfrak{D}}^{\mathfrak{L}}(f) = \Delta_{\mathfrak{D}}^{\mathfrak{L}}(ip(f))$ for any Laurent series f. This completes the proof. □

3.5.1. Schur Postitivity of 1-step Desingularizations.

PROPOSITION 3.5.4. *All 0-step desingularizations are strongly Schur positive.*

PROOF. Consider a quiver, Q, and orbit closure, S_m, that admit a 0-step desingularization with directed partition $\mathcal{I}(S_m)$. Note that the partition $\mathcal{I}$ is comprised of exactly one set so we can refer to that set as $\mathcal{I}$ without confusion. Using the notation of chapter 2 and by theorem 2.2.5 we have

$$[S_m] = \Delta_{\mathfrak{A}}^{\mathfrak{K}}\left(\prod_{k=1}^{N}\left(\mathbb{A}_k^{(n_k)}\cdot\frac{\prod_{\substack{a\in\mathbb{A}_k\\ b\in\mathbb{B}_k}}(1-a/b)}{\prod_{\substack{a\in\mathbb{A}_k\\ b\in\mathbb{C}_k}}(1-a/c)}\right)\right).$$

However, since there is only one set in the directed partition, $\mathbb{B}_k=\emptyset$ for each k and each alphabet in $\mathfrak{K}$ is disjoint.

Now each factor,

$$\begin{aligned}\mathbb{A}_k^{(n_k)}\cdot\mathbf{K}(\mathbb{A}_k,\mathbb{C}_k^{\vee}) &= s_{(n_k)}(\mathbb{A}_k)\cdot\sum_{\lambda}s_{\lambda}(\mathbb{A}_k)s_{\lambda}(\mathbb{C}_k)\\ &= \sum_{\lambda}s_{\lambda+(n_k)}(\mathbb{A}_k)s_{\lambda}(\mathbb{C}_k)\end{aligned}$$

is strongly Schur positive. Therefore, by proposition 3.4.2, the product is strongly Schur positive. □

COROLLARY 3.5.5. *All 1-step desingularizations are Schur positive.*

PROOF. Every 1-step desingularization is a next-step desingularization of a 0-step desingularization. By Lemma 3.5.2 this completes the proof. □

Note that if we could extend the conclusion of Lemma 3.5.2 to show strong Schur positivity, this would completely prove Buch's conjecture. This is because every n-step desingularization is a next-step desingularization of a $(n-1)$-step desingularization.

3.5.2. Results for A_n. The "equi-oriented A_n quiver" is the quiver: $\bullet \to \bullet \to \ldots \to \bullet$, where there are n verticies (by convention the vertices of any A_n quiver are labelled from left to right). The equi-oriented A_n quiver has a directed partition, $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_n$ where $\mathcal{I}_{n-j+1} = \{\alpha_{jk}\}_{k=j}^{n}$, for any n. For example, if $n = 3$ then $\mathcal{I} = \{\alpha_{33}\} \cup \{\alpha_{22}, \alpha_{23}\} \cup \{\alpha_{11}, \alpha_{12}, \alpha_{13}\}$.

Therefore the equi-oriented A_{n+1} quiver with the directed partition $\mathcal{J} = \mathcal{J}_0 \cup \ldots \cup \mathcal{J}_n$ where $\mathcal{J}_{n-j} = \{\alpha_{jk}\}_{k=j}^{n}$ is a next-step desingularization of the $\mathcal{I}$ desingularization. Continuing with the previous example $\mathcal{J} = \{\alpha_{33}\} \cup \{\alpha_{22}, \alpha_{23}\} \cup \{\alpha_{11}, \alpha_{12}, \alpha_{13}\} \cup \{\alpha_{00}, \alpha_{01}, \alpha_{02}, \alpha_{03}\}$.

Theorem 3.5.6. *The directed partition $\mathcal{I}$ is strong Schur positive.*

Proof. Consider the equi-oriented A_n quiver with directed partition $\mathcal{I}$ and the equi-oriented A_{n+1} quiver with directed partition $\mathcal{J}$ as described above, where $\mathcal{J}$ is a next-step desingularization of $\mathcal{I}$. If $\mathbf{j}, \mathbf{s}$ is the resolution pair associated to $\mathcal{J}$ consider $\mathbf{j}^{(n+1)}, \mathbf{s}^{(n+1)}$ the contribution that $\mathcal{J}_{n+1}$ makes to the resolution pair. By construction of the resolution pair,

$$\mathbf{j}^{(n+1)} = (0, 1, 2, \ldots, n), \qquad \mathbf{s}^{(n+1)} = \left(\sum_{k=0}^{n} m_{0k}, \sum_{k=1}^{n} m_{0k}, \ldots, m_{0n} \right).$$

Set $N = card(Q_0)$ and $\mathfrak{D} = (\mathbb{D}_1, \ldots, \mathbb{D}_N)$ with $\mathbb{D}_k = \sum_{i_l = k} \mathbb{A}_l$ (where order is preserved) and compute $\hat{n}_k = s_{j_{k-1}}^{(n+1)} - s_{j_k}^{(n+1)} = m_{0(k-1)}$. Therefore

$$\Delta_{\mathfrak{A}}^{\mathfrak{K}}(G_{\mathcal{J}}) = \Delta_{\mathfrak{D}}^{\mathfrak{L}} \left(H_{\mathcal{I}} \cdot \prod_{k=1}^{N} \prod_{d \in \mathbb{D}_k} d^{\hat{n}_k} \right) \tag{3.1}$$

$$= \Delta_{\mathfrak{D}}^{\mathfrak{L}} \left(H_{\mathcal{I}} \cdot \prod_{k=1}^{N} \prod_{d \in \mathbb{D}_k} d^{m_{0(k-1)}} \right). \tag{3.2}$$

It is known that all orbits arising from quivers of type equi-oriented A_n are Schur positive, [KMS]. This means that $\Delta^{\mathfrak{K}}_{\mathfrak{A}}(G_{\mathcal{J}}) = \Delta^{\mathfrak{L}}_{\mathfrak{D}}(f)$ where f is Schur positive. Again, this holds for any orbit and therefore any choices of $m_{0(k-1)}$ for $k = 1, \ldots, N$. Thus $H_{\mathcal{I}} \cdot \prod_{k=1}^{r} \prod_{d \in \mathbb{D}_k} d^{m_{0(k-1)}}$ is Schur positive for any values of $m_{0(k-1)} \geq 0$. This is exactly the definition of $H_{\mathcal{I}}$ being strongly Schur positive, which implies that $\mathcal{I}$ is a strongly Schur positive directed partition. Though this orbit was assumed to have no roots of multiplicity zero, if any of the roots do have multiplicy zero simply substitute zero into those variables associated to those roots in $H_{\mathcal{I}}$. □

COROLLARY 3.5.7. *Every orbit of the equi-oriented A_n quiver is strongly Schur positive.*

3.6. Simple Sinks and Simple Sources

In this section we discuss how some simple roots affect the Schur positivity or strong Schur positivity of an orbit.

DEFINITION 3.6.1. Suppose $Q = (Q_0, Q_1)$ is a quiver of ADE-type. A **simple sink** is a simple root $\alpha_i \in \Phi^+$ such that the vertex $i \in Q_0$ is a sink. A **simple source** is a simple root $\alpha_i \in \Phi^+$ such that the vertex $i \in Q_0$ is a source.

3.6.1. Simple Sinks. Let $Q = (Q_0, Q_1)$ be a quiver of ADE-type, α be a simple sink at vertex $i \in Q_0$, and $\beta \in \Phi^+$ any positive root. Consider the Euler form applied to α and β:

$$< \alpha, \beta > = \sum_{j=1}^{|Q_0|} \alpha_j \beta_j - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)} \tag{3.3}$$

$$= \alpha_i \beta_i, \tag{3.4}$$

because $(\alpha)_{t(a)} = 0$, since vertex i is a sink. Therefore $< \alpha, \beta > \geq 0, \forall \beta$.

This means that when constructing a directed partition, one can always put the simple sinks as the first set. That is, suppose $\Phi' \subset \Phi^+$ is a subset of the positive roots, and

$\Phi^{sk} \subset \Phi^+$ is the set of simple sinks. If $\Phi' = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_s$ is a directed partition, then $\Phi' = (\Phi^{sk} \cap \Phi') \cup (\mathcal{I}_1 - \Phi^{sk}) \cup \ldots \cup (\mathcal{I}_s - \Phi^{sk})$ is another directed partition.

Proposition 3.6.2. *Let $Q = (Q_0, Q_1)$ be a quiver with orbit, S_m, and directed partition $\Phi' = \mathcal{I}$. If $\mathcal{I}(S_m)$ is strongly Schur positive then the orbit with associated directed partition $\mathcal{J}(S_{m'}) = (\Phi^{sk} - \Phi') \cup \mathcal{I}$ is strongly Schur positive.*

Essentially this proposition asserts that if (after applying the numerator lemma) the generating sequence for a directed partition is strongly Schur positive then recalculating the generating sequence with added simple sinks in front of the directed partition will not change strong Schur positivity.

Proof. First, observe that if $\alpha, \beta \notin \Phi'$ are simple sinks at vertices k, l, respectively, and $\mathcal{J}_1 = \{\alpha, \beta\} \cup \mathcal{I}$, $\mathcal{J}_2 = \{\alpha\} \cup \{\beta\} \cup \mathcal{I}$ then the generating funcitons are equal. That is, $G_{\mathcal{J}_1} = G_{\mathcal{J}_2}$. This is clear by constructing the resolution pairs for each directed partition. If $\mathbf{i}$=$(i_1, \ldots, i_p)$,$\mathbf{r}$=$(r_1, \ldots, r_p)$ is the resolution pair for $\mathcal{I}$ then **i'**=$(k, l, i_1, \ldots, i_p)$, **r'** $= (m_k, m_l, r_1, \ldots, r_p)$ is the resolution pair for both $\mathcal{J}_1$ and $\mathcal{J}_2$.

Since the resolution pairs are the same, so are the generating sequences. Thus it suffices to prove this theorem adding only one simple sink at a time.

Let $G_{\mathcal{I}}$ be the generating sequence for $\mathcal{I}(S_m)$ and let $\alpha \notin \Phi'$ be a simple sink at vertex j. As mentioned previously, the resolution pair for the directed partition $\mathcal{J} = \{\alpha\} \cup \mathcal{I}$ will be **i'**=$(j = i_0, i_1, \ldots, i_p)$,**r'**=$(m_j = r_0, r_1, \ldots, r_p)$. Therefore,

$$G_{\mathcal{J}} = \mathbb{A}_0^{(n_0)} \cdot \prod_{\substack{a \in \mathbb{A}_0 \\ b \in \mathbb{B}_0}} \left(1 - \frac{a}{b}\right) \cdot G_{\mathcal{I}}.$$

This is because vertex j is a sink, so $\mathbb{C}_0 = \emptyset$. After applying the numerator lemma, we have

$$H_{\mathcal{J}} = \mathbb{A}_0^{(n_0)} \cdot H_{\mathcal{I}}.$$

Let $k = min_{l>0}\{i_l = j\}$. Then

$$n_0 = n_k - r_k + \sum_{\substack{k>l>0 \\ i_l \in T(j)}} r_l.$$

But α is the only simple root associated with vertex j. Therefore, for each vertex associated to j that contributes a positive value to r_k, it must contribute the same value to at least one vertex adjacent to j. Since vertex j is a sink, there must be an arrow from the adjacent vertex to vertex j, meaning that

$$\sum_{\substack{k>l>0 \\ i_l \in T(j)}} r_l \geq r_k \Rightarrow n_0 \geq n_k.$$

Let $\mathfrak{L} = (\mathbb{L}_1, \ldots, \mathbb{L}_{|Q_0|})$ be a collection of infinite alphabets and define

$$\mathfrak{D} = (\mathbb{D}_1, \ldots, \mathbb{D}_{|Q_0|}), \qquad \text{with} \qquad \mathbb{D}_t = \bigcup_{\substack{l>0 \\ i_l = t}} \mathbb{A}_l,$$

$$\mathfrak{D}' = (\mathbb{D}_1, \ldots, \mathbb{D}_{j-1}, \mathbb{A}_0 + \mathbb{D}_j, \mathbb{D}_{j+1} \ldots, \mathbb{D}_{|Q_0|}).$$

Now consider all of the variables in $\mathbb{D}_j$. Because vertex j is a sink, there are no arrows coming from it. Therefore, these variables never appear in the numerators of the Cauchy products appearing in $H_{\mathcal{I}}$ which means that the exponents of these variables are always less than n_k.

If $n_0 \geq n_k \geq \mu_1$, for some $\mu \in \mathbb{Z}^{|\mathbb{D}_j|}$, non-increaing, and

$$\Delta^{\mathbb{L}_j}_{\mathbb{D}_j}(\mathbb{D}_j^{\mu}) = \Delta^{\mathbb{L}_j}_{\mathbb{D}_j}(s_\mu(\mathbb{D}_j)), \text{ then}$$

$$\Delta^{\mathbb{L}_j}_{\mathbb{A}_0+\mathbb{D}_j}(\mathbb{A}_0^{(n_0)} \cdot \mathbb{D}_j^{\mu}) = \Delta^{\mathbb{L}_j}_{\mathbb{A}_0+\mathbb{D}_j}(s_{\{\mathbb{A}_0,\mu\}}(\mathbb{A}_0 + \mathbb{D}_j))$$

with $\{n_0, \mu\}$ non-increasing.

So if we write

$$\Delta^{\mathfrak{L}}_{\mathfrak{D}}(H_{\mathcal{I}}) = \Delta^{\mathfrak{L}}_{\mathfrak{D}}\left(\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} c_\lambda \prod_{l=1}^{|Q_0|} \mathbb{D}_l^{\lambda_l}\right),$$

for $\lambda_l \in \mathbb{Z}^{|\mathbb{D}_l|}$ non-increasing, then

$$\Delta^{\mathfrak{L}}_{\mathfrak{D}'}(\mathbb{A}_0^{(n_0)} \cdot H_{\mathcal{I}}) = \Delta^{\mathfrak{L}}_{\mathfrak{D}'}\left(\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} c_\lambda s_{\{(n_0),\lambda_j\}}(\mathbb{A}_0+\mathbb{D}_j)\prod_{l\neq j} s_{\lambda_l}(\mathbb{D}_l)\right).$$

This completes the proof. □

Corollary 3.6.3. *Let $Q=(Q_0,Q_1)$ be a quiver with orbit, S_m, and directed partition $\Phi' = \mathcal{I}$. If $\mathcal{I}(S_m)$ is Schur positive then the orbit with associated directed partition $\mathcal{J}(S_{m'}) = (\Phi^{sk}-\Phi')\cup\mathcal{I}$ is Schur positive.*

Proof. The proof is the same except one can assume that the λ_l are partitions and $\mathfrak{L}$ is a collection of positive alphabets. □

3.6.2. Simple Sources. Let $Q=(Q_0,Q_1)$ be a quiver of ADE-type, α a simple source at vertex $i\in Q_0$, and $\beta\in\Phi^+$ any positive root. Consider the Euler form applied to β and α:

$$< \beta,\alpha > = \sum_{j=1}^{|Q_0|}\beta_j\alpha_j - \sum_{a\in Q_1}\beta_{t(a)}\alpha_{h(a)} \tag{3.5}$$

$$= \alpha_i\beta_i, \tag{3.6}$$

because $\alpha_{h(a)}=0$, since vertex i is a source. Therefore $<\beta,\alpha>\geq 0,\forall\beta$.

This means that when construting a directed partition, one can always put the simple sources as the last set. That is, suppose $\Phi'\subset\Phi^+$ is a subset of the positive roots, and $\Phi^{sr}\subset\Phi^+$ is the set of simple sources. if $\Phi'=\mathcal{I}_1\cup\ldots\cup\mathcal{I}_s$ is a directed partition, then $\Phi'=(\mathcal{I}_1-\Phi^{sr})\cup\ldots\cup(\mathcal{I}_s-\Phi^{sr})\cup(\Phi^{sr}-\Phi')$ is another directed partition.

Proposition 3.6.4. *Let $Q=(Q_0,Q_1)$ be a quiver with orbit, S_m, and directed partition $\Phi'=\mathcal{I}$. If $\mathcal{I}(S_m)$ is strongly Schur positive then the orbit with associated directed partition $\mathcal{J}(S_{m'})=\mathcal{I}\cup(\Phi^{sr}-\Phi')$ is strongly Schur positive.*

We will hold off on the proof to make one observation.

Lemma 3.6.5. *Suppose that $\mathbb{A}$ is a finite alphabet and λ is a partition with card$(\mathbb{A}) = |\lambda| = n$ and define μ to be the $(n+k)$-vector with*

$$\mu_i = \begin{cases} \lambda_i - k & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}.$$

That is, $\mu = \{\lambda - (k)^n, (0)^k\}$. If $\lambda_n < k$ then $s_\mu(\mathbb{A}) = 0$.

Proof. Suppose that $\lambda_n < k$. But λ is a partition, so $\lambda_n \geq 0$. Therefore, $-k \leq \mu_n = \lambda_n - k < 0$. Consider the transposition $\sigma \in \mathfrak{S}_n$ such that $\sigma(n) = n - \mu_n$. Under the Bott action,

$$(\sigma \cdot \mu)_n = \mu_{n-\mu_n} + n - (n - \mu_n) = \mu_n, \tag{3.7}$$

$$(\sigma \cdot \mu)_{n-\mu_n} = \mu_n + (n - \mu_n) - n = 0 = \mu_{n-\mu_n}. \tag{3.8}$$

So $\sigma \cdot \mu = \mu$ and $s_\mu(\mathbb{A}) = -s_{\sigma\cdot\mu}(\mathbb{A}) = -s_\mu(\mathbb{A})$, which must mean that $s_\mu(\mathbb{A}) = 0$. $\square$

Proof. (of proposition 3.6.4)

First, observe that if $\alpha, \beta \notin \Phi'$ are simple sources at vertices k, l, respectively, and $\mathcal{J}_1 = \mathcal{I} \cup \{\alpha, \beta\}$, $\mathcal{J}_2 = \mathcal{I} \cup \{\alpha\} \cup \{\beta\}$ then the generating funcitons are equal. That is, $G_{\mathcal{J}_1} = G_{\mathcal{J}_2}$. This is clear by constructing the resolution pairs for each directed partition. If $\mathbf{i}$=$(i_1, \ldots, i_p)$,$\mathbf{r}$=$(r_1, \ldots, r_p)$ is the resolution pair for $\mathcal{I}$ then $\mathbf{i'}$=$(i_1, \ldots, i_p, k, l)$, $\mathbf{r'}$ = $(r_1, \ldots, r_p, m_k, m_l)$ is the resolution pair for both $\mathcal{J}_1$ and $\mathcal{J}_2$.

Since the resolution pairs are the same, so are the generating sequences. Thus it suffices to prove this theorem adding only one simple source at a time.

Let $G_{\mathcal{I}}$ be the generating sequence for $\mathcal{I}(S_m)$ and let $\alpha \notin \Phi'$ be a simple source at vertex j. As mentioned previously, the resolution pair for the directed partition $\mathcal{J} = \mathcal{I} \cup \{\alpha\}$ will be $\mathbf{i'}$=$(i_1, \ldots, i_p, i_{p+1} = j)$,$\mathbf{r'}$=$(r_1, \ldots, r_p, r_{p+1}$

$= m_j)$. Define $\bar{\mathbb{D}} = \sum_{l \in H(j)} \mathbb{D}_l$. Therefore,

$$G_{\mathcal{J}} = \left(\prod_{c \in \bar{\mathbb{D}}} c^{m_j}\right) \left(\prod_{b \in \mathbb{D}_j} b^{-m_j}\right) \left(\prod_{\substack{a \in \mathbb{A}_{p+1} \\ b \in \mathbb{D}_j}} \left(1 - \frac{b}{a}\right)\right) \cdot G_{\mathcal{I}}.$$

This is because vertex j is a source, so $\mathbb{C}_l \cap \mathbb{A}_{p+1} = \emptyset$ for all l. After applying the numerator lemma, we have

$$H_{\mathcal{J}} = \left(\prod_{c \in \bar{\mathbb{D}}} c^{m_j}\right) \left(\prod_{b \in \mathbb{D}_j} b^{-m_j}\right) \cdot H_{\mathcal{I}}.$$

Let $\mathfrak{L} = (\mathbb{L}_1, \ldots, \mathbb{L}_{|Q_0|})$ be a collection of infinite alphabets and define

$$\mathfrak{D} = (\mathbb{D}_1, \ldots, \mathbb{D}_{|Q_0|}), \qquad \text{with} \qquad \mathbb{D}_t = \bigcup_{\substack{l < p+1 \\ i_l = t}} \mathbb{A}_l,$$

$$\mathfrak{D}' = (\mathbb{D}_1, \ldots, \mathbb{D}_{j-1}, \mathbb{D}_j + \mathbb{A}_{p+1}, \mathbb{D}_{j+1} \ldots, \mathbb{D}_{|Q_0|}).$$

First consider all of the variables in $\mathbb{D}_j$. Because vertex j is a source, there are no arrows going into it. Therefore, these variables never appear in the denominators of the Cauchy products appearing in $H_{\mathcal{I}}$ which means that the exponents of all of these variables are at least as big as n_k.

By lemma 3.6.5, if μ is a partition then either $\nu = \{\mu - (m_j)^{|\mathbb{D}_j|}, (0)^{m_j}\}$ is a partition or $s_\nu(\mathbb{D}_j + \mathbb{A}_{p+1}) = 0$. Thus, if

$$\Delta_{\mathbb{D}_j}^{\mathbb{L}_j}(\mathbb{D}_j^\mu) = \Delta_{\mathbb{D}_j}^{\mathbb{L}_j}(s_\mu(\mathbb{D}_j)), \text{ then}$$

$$\Delta_{\mathbb{D}_j + \mathbb{A}_{p+1}}^{\mathbb{L}_j}\left(\mathbb{D}_j^\mu \cdot \prod_{b \in \mathbb{D}_j} b^{-m_j}\right) = \Delta_{\mathbb{D}_j + \mathbb{A}_{p+1}}^{\mathbb{L}_j}(s_\nu(\mathbb{A}_0 + \mathbb{D}_j))$$

with ν a partition.

So if we write

$$\Delta_{\mathfrak{D}}^{\mathfrak{L}}(H_{\mathcal{I}}) = \Delta_{\mathfrak{D}}^{\mathfrak{L}}\left(\sum_{\lambda = (\lambda_1, \ldots, \lambda_n)} c_\lambda \prod_{l=1}^{|Q_0|} \mathbb{D}_l^{\lambda_l}\right),$$

for $\lambda_l \in \mathbb{Z}^{|\mathbb{D}_l|}$ non-increasing, then

$$\Delta^{\mathfrak{L}}_{\mathfrak{D}'}\left(H_{\mathcal{I}} \cdot \prod_{b\in\mathbb{D}_j} b^{-m_j}\right) = \Delta^{\mathfrak{L}}_{\mathfrak{D}'}\left(\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} c_\lambda s_\nu(\mathbb{D}_j + \mathbb{A}_{p+1}) \prod_{l\neq j} s_{\lambda_l}(\mathbb{D}_l)\right).$$

Next consider that $\prod_{c\in\bar{\mathbb{D}}} c^{m_j} \in Symm_{\mathbb{Z}}(\mathfrak{D}')$ so,

$$\Delta^{\mathfrak{L}}_{\mathfrak{D}'}\left(H_{\mathcal{I}} \cdot \left(\prod_{b\in\mathbb{D}_j} b^{-m_j}\right)\left(\prod_{c\in\bar{\mathbb{D}}} c^{m_j}\right)\right) \tag{3.9}$$
$$= \Delta^{\mathfrak{L}}_{\mathfrak{D}'}\left(\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} c_\lambda s_\nu(\mathbb{D}_j + \mathbb{A}_{p+1}) \left(\prod_{l\in\bar{\mathbb{D}}} s_{\lambda_l+(m_j)}(\mathbb{D}_l)\right)\left(\prod_{\substack{l\neq j\\ l\notin\bar{\mathbb{D}}}} s_{\lambda_l}(\mathbb{D}_l)\right)\right).$$

Since $\nu, \lambda_l + (m_j)$, and λ_l are non-increasing, this completes the proof. □

3.7. Results for A_3

THEOREM 3.7.1. *Consider the inward-oriented A_3 quiver: $\bullet \to \bullet \leftarrow \bullet$, with directed partition $\mathcal{I} = \{\alpha_2\} \cup \{\alpha_{12}, \alpha_{23}, \alpha_{13}\} \cup \{\alpha_1, \alpha_3\}$. Any orbit associated to $\mathcal{I}$ is strongly Schur positive.*

PROOF. The inward-oriented A_3 quiver with directed partition

$$\mathcal{J} = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}$$

with any orbit associated to $\mathcal{J}$ is strongly Schur positive because it is a 0-step desingularization. By propositions 3.6.4 and 3.6.2, adding simple sinks to the beginning and simple sources to the end of $\mathcal{J}$ doesn't change strong Schur positivity. This completes the proof. □

THEOREM 3.7.2. *Consider the outward-oriented A_3 quiver: $\bullet \leftarrow \bullet \to \bullet$, with directed partition $\mathcal{I} = \{\alpha_1, \alpha_3\} \cup \{\alpha_{12}, \alpha_{23}, \alpha_{13}\} \cup \{\alpha_2\}$. Any orbit associated to $\mathcal{I}$ is strongly Schur positive.*

PROOF. The outward-oriented A_3 quiver with directed partition

$$\mathcal{J} = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}$$

with any orbit associated to $\mathcal{J}$ is strongly Schur positive because it is a 0-step desingularization. By propositions 3.6.4 and 3.6.2, adding simple sinks to the beginning and simple sources to the end of $\mathcal{J}$ doesn't change strong Schur positivity. This completes the proof. $\square$

COROLLARY 3.7.3. *Any orbit of any quiver of type A_3 with any orientation is strongly Schur positive.*

PROOF. Corollary 3.5.7 and theorems 3.7.1 and 3.7.2 show that for any orbit of any A_3 quiver there is a directed patition such that together they are strongly Schur positive. $\square$

3.8. Results for A_4

Let us name the A_4 quivers:

(1) Equi-oriented: $\bullet \to \bullet \to \bullet \to \bullet$.
(2) Single-sink: $\bullet \to \bullet \leftarrow \bullet \leftarrow \bullet$.
(3) Sink-source: $\bullet \to \bullet \leftarrow \bullet \to \bullet$.
(4) Single-source: $\bullet \leftarrow \bullet \to \bullet \to \bullet$.

One can see by exhaustion that these are the only quivers of type A_4.

THEOREM 3.8.1. *Any orbit that excludes simple sources that is associated to a quiver of A_4-type, with any orientation, is Schur positive.*

PROOF. This is already known for the equi-oriented A_4 quivers. For the others consider the following.

In light of corollary 3.5.5, all one-step desingularizations are Schur positive. Further, from corollary 3.6.3, beginning a directed partition with simple sinks doesn't change

Schur positivity for associated orbits. Thus it suffices to find a directed partition for each quiver $\Phi^+ = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ where $\mathcal{I}_1$ is comprised of only simple sinks .

For the single-sink A_4 quiver let

$$\Phi^+ = \{\alpha_2\} \cup \{\alpha_{12}, \alpha_{13}, \alpha_{23}\} \cup \{\alpha_3, \alpha_{24}, \alpha_{14}, \alpha_{34}\}.$$

For the sink-source A_4 quiver let

$$\Phi^+ = \{\alpha_2, \alpha_4\} \cup \{\alpha_{12}, \alpha_{24}\} \cup \{\alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{34}\}.$$

For the single-source A_4 quiver let

$$\Phi^+ = \{\alpha_1, \alpha_4\} \cup \{\alpha_{14}, \alpha_{24}, \alpha_{34}\} \cup \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}.$$

This suffices for the proof. □

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