

COMBINATORIAL INTERPRETATION OF THE KUMAR-PETERSON LIMIT FOR
 $sl_n(\mathbb{C})$ DEMAZURE CHARACTERS AND GELFAND PATTERN DESCRIPTION OF
 $sl_n(\mathbb{C})$ DEMAZURE CHARACTERS.

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ABSTRACT

JOSEPH W. SEABORN III: Combinatorial Interpretation of the Kumar-Peterson Limit for $sl_n(\mathbb{C})$ Demazure Characters and Gelfand Pattern Description of $sl_n(\mathbb{C})$ Demazure Characters.

(Under the direction of Robert Proctor)

Given a semisimple Lie algebra, a dominant integral weight λ , and a Weyl group element w , the Kumar-Peterson identity expresses the limit of a certain sequence of Demazure characters as a certain product over a subset of the positive roots. In Type A_n , the Demazure characters for a given n -partition λ and a given permutation are also known as key polynomials. In this thesis we obtain a combinatorial interpretation of the Kumar-Peterson identity in Type A_n . Our combinatorial interpretation presents two product identities for the generating function for a set of certain reverse semistandard tableaux. The first right hand side is a product over the inversions of an inverse shuffle. The second right hand side is a product over the hooks of certain Hillman-Grassl boards. These identities can be viewed as combinatorial identities in and of themselves. We give bijective proofs of these identities which extend the Hillman-Grassl algorithm. We also give a Lie theoretic proof of the first identity by translating the Lie theoretic entities to combinatorial entities. The foremost special case of the second identity is equivalent to Gansner's identity for colored reverse plane partitions. That identity generalized identities found by Stanley, MacMahon, and Euler. Therefore this work places those identities in a Lie theoretic setting, and obtains the most general result of this kind within Type A . The foremost step in the Lie theoretic proof is to show that the limit of the Demazure polynomials can be found by calculating the direct limit of certain sets of Demazure tableaux. The principal tool we use in the Lie theoretic proof is Willis' scanning method for describing the Demazure tableaux. In the final chapter, we present a translation of this scanning method to Gelfand patterns.

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1 Introduction

The definitions of the combinatorial terms used in this introduction are given in Chapter 2. For the definitions of the Lie theoretic terms used in this thesis, consult Chapter 5. Fix $n \geq 1$ throughout this thesis.

1.1 Overview of thesis

Some famous identities in mathematics take the form of a sum equals a product. One such identity is due to Euler: For a nonnegative integer d , denote by $p_{\leq n}(d)$ the number of partitions of d with no more than n parts. Euler described and proved the following product identity for the generating function of these partitions [Sta1]:

$$\sum_{d=0}^{\infty} p_{\leq n}(d)t^d = \frac{1}{\prod_{i=1}^n (1-t^i)}.$$

The Kumar-Peterson (K-P) identity is an identity from Lie theory which is (not obviously) related to this partition generating function identity of Euler. To state the K-P identity, we need to specify 3 inputs: a semisimple Lie (or Kac-Moody) algebra X_n of rank n ; a dominant integral weight λ which determines a highest weight representation V_λ of X_n ; and a minimal length coset representative $w \in W^\lambda$. Let H be a Cartan subalgebra of X_n . Let Φ be the set of (real) roots of X_n . Let B denote the Borel subalgebra of X_n . Let $\mathcal{D}_\lambda(w)$ be the Demazure B -submodule of V_λ with lowest weight $w.\lambda$ and let $d_\lambda(w; y)$ be the formal character of this submodule with respect to the Cartan subalgebra H . Here the variable y indicates a generic n -variate coordinatization of the ring of formal exponentials of the integral weights for the algebra X_n . So for a positive root α , we are writing $\exp(\alpha) = y^\alpha$. And for the weight $-wm\lambda$, we are writing $\exp(-wm\lambda) = y^{-wm\lambda}$. Throughout this thesis, the variable z indicates the

coordinatization of the ring of formal exponentials in the simple root basis: For $1 \leq i \leq n$, we define $z_i := \exp(\alpha_i)$, the formal exponential of the simple root α_i . Define the “adjusted” Demazure character of the Demazure submodule $D_\lambda(w)$ to be $y^{-w\lambda}[\lambda(w; y)]$. The K-P identity is:

$$\lim_{m \rightarrow \infty} y^{-wm\lambda} \mathcal{D}_{m\lambda}(w; y) = \frac{1}{\prod_{\alpha \in \Phi(w)} (1 - y^\alpha)},$$

where $\Phi(w) := \Phi^+ \cap w(\Phi^-)$. Once the simple root basis has been fixed for the formal exponentials, the adjusted Demazure characters become polynomials in these variables and the right hand sides become formal power series in them. The K-P identity is explicitly stated in the literature only in [Pro1]; there it is indicated how it can be derived from an equation that appeared in the 1996 paper [Kum1] and then later in the book [Kum2].

In this thesis we consider only the Type A_n cases of the K-P identity. So the Lie algebra is $sl_{n+1}(\mathbb{C})$.

We introduce combinatorial objects to describe both sides of the K-P identity. Doing so enables us to reformulate the limit on the left side as a sum. The right side remains a product. The easiest cases of the K-P identity occur when the highest weight λ is a fundamental weight ω_b . Here in Type A_n , the combinatorial sum equals product identities can be interpreted as product identities for generating functions of reverse plane partitions on general shapes μ . In particular, when the highest weight λ is equal to the fundamental weight ω_n , the combinatorial identity can be interpreted as Euler’s partition generating function identity above. The combinatorial translations of the K-P identity for $\lambda = \omega_b$ give a Lie theoretic “explanation” of further product identities for reverse plane partition generating functions due to MacMahon [Mac], Stanley [Sta2] and Gansner [Ga]. By describing the K-P identity combinatorially for the general Type A_n case, we obtain a generalization of this sequence of generating function identities. Our viewpoints also provide explanations for the peculiar colored weighting introduced by Gansner.

The notion of “Young diagram” or “shape” arises in this thesis in two fundamentally different ways. On one hand, the dominant integral weight λ corresponds to an n -partition

λ whose shape we view in the xy -plane. It is well known that the weights of the irreducible representation V_λ can be described with reverse semistandard Young tableaux on the shape λ . We view these reverse semistandard tableaux in the xy -plane. In our combinatorial interpretation of the K-P identity, the limit becomes a sum over a certain set of “labelling tableaux”, which arises when taking the limit. It is interesting to note that this set of labelling tableaux decouples into a direct product of sets of “labelling subtableaux”; this was unexpected. The sets of labelling subtableaux consist of regions of columns of labelling tableaux of the same length. We view these labelling subtableaux in the xy -plane. On the other hand, in some of our combinatorial interpretations and bijective proofs, the tableaux viewed in the xy -plane are transformed via a 3-D picture to reverse plane partitions on a shape μ . As results of this 3-D transformation, both the shape μ and the reverse plane partitions lie in the xz -plane. This shape μ in the xz -plane is the shape on which the reverse plane partitions of MacMahon, Stanley and Gansner are defined.

In 1997, Dale Peterson developed and proved the K-P identity. This identity had already been found independently by Shrawan Kumar. For a poset P , reverse plane partitions generalize to P -partitions. Using the K-P identity, Peterson and Proctor proved [Pro1] that there are hook length product identities for the P -partition generating functions of colored d -complete posets. Shuji Okamura [Oka] and Kento Nakada later gave “Hillman-Grassl-style” bijective proofs of this result for most slant irreducible families of d -complete posets.

By 1990, Lascoux and Schützenberger [LS] had developed a combinatorial description for Type A_n Demazure characters. In 2010, Willis developed a simpler method to describe Demazure characters in Type A_n ; this is called the tableau scanning method. His tableau scanning method forms the foundation for the work in this thesis. We first use his notion of “Demazure tableaux” to derive our combinatorial interpretation of the K-P identity from the Lie theoretic K-P identity. Later, in Chapter 6, we also translate Willis’ tableau scanning method to a scanning method for Gelfand patterns. This enables us to describe Demazure characters as sums over Gelfand patterns.

Allen Knutson proposed describing Type A_n Demazure Characters as sums of weight

monomials of certain Gelfand patterns. In his proposal, one would define a pipe dream for each Gelfand pattern. One would then convert the pipe dream to a permutation. He proposed that one could describe the Demazure character $\lceil_{\lambda}(w; y)$ as the sum over the Gelfand patterns whose resulting permutations are weakly less than the permutation w in the Bruhat order. The author of this thesis was not able to describe a method to write Demazure polynomials in this way, but Knutson’s proposal led us to our work in Chapter 6. Jeffrey Ferreira gave [Fe] a description of constituent parts of Type A_n Demazure characters, called “atoms”, as sums over certain Gelfand patterns. We do not, however, address atoms of Type A_n Demazure characters in this thesis.

In the next section, we define a set of tableaux which we call “labelling tableaux”. We then state our combinatorial interpretations of the K-P identity. These are two product identities for a multivariate generating function for the set of labelling tableaux. In Section 1.3, we present Willis’ description of Demazure characters. In Sections 1.4 and 1.5, we describe how the partition and reverse plane partition generating function identities referred to above are special cases of our combinatorial version of the K-P identity. In Section 1.6, we present the various combinatorial and Lie theoretic structures associated to the K-P identity in Type A_n for the case when the highest weight λ is a fundamental weight ω_{n+1-b} for some $1 \leq b \leq n$. Combinatorially, the shape λ obtained from the highest weight $\lambda = \omega_{n+1-b}$ is one column of length b . In Section 1.7, we give a preview of our bijective proof of our first combinatorial interpretation of the K-P identity. In Section 1.8, we present an alternate description of the set of labelling tableaux. We call these tableaux “affine tableaux”. These tableaux provide a better description for the geometric interpretation of the K-P identity.

In Chapter 2, we first present the necessary background combinatorics to precisely state our combinatorial version of the K-P identity. Then we define the “weighted limit” of a “weighted direct system”. This allows us to state our first main result, Theorem 3.3, in Section 3.3: We can label the equivalence classes of the weighted limit of the direct system of sets of Demazure tableaux with the labelling tableaux. We then state our second main result, Theorem 3.4, in Section 3.4. This theorem gives two combinatorial interpretations

of the K-P identity for general Type A_n : the first interpretation gives a generating function identity for the set of labelling tableau in terms of “inversions” of an “ordered Q -partition”; the second gives an identity for the same generating function in terms of “hooks” of “Hillman-Grassl boards”. We then state our third main result, Theorem 3.7 in Section 3.4. This third result states that our first identity in Theorem 3.4 is a combinatorial translation of the Lie theoretic K-P identity for Type A_n . In Chapter 4, we present our bijective proof of Theorem 3.4. In this bijective proof, we use, among other things, Gansner’s colored version of the Hillman-Grassl algorithm. In Chapter 5, we present an alternate proof of the first identity of Theorem 3.4 in which we derive it from the K-P identity by translating Lie theoretic objects into combinatorial objects. In Chapter 6, we present our Gelfand pattern scanning method.

1.2 Our combinatorial Kumar-Peterson identity

To state our combinatorial version of the K-P identity in Type A_n , we need to specify three inputs: a positive integer n ; a subset $Q = \{q_1 < \dots < q_k\} \subseteq \{1, 2, \dots, n\}$; and an “ordered Q -partition” of the set $\{0, 1, \dots, n\}$:

$$\rho := \{\rho_n, \rho_{n-1}, \dots, \rho_{n+1-q_1}\}, \{\rho_{n-q_1}, \dots, \rho_{n+1-q_2}\}, \dots, \{\rho_{n-q_k}, \dots, \rho_0\}.$$

Define $q_0 := 0$ and $q_{k+1} := n + 1$. Then for $1 \leq r \leq k + 1$, the r^{th} block from the left has size $q_r - q_{r-1}$. We refer to the set of values in each individual block as a “cohort”. Our standard form for an ordered Q -partition ρ lists the values ρ_i within each block in decreasing order from left to right:

$$\rho := (\rho_n, \rho_{n-1}, \dots, \rho_{n+1-q_1}; \rho_{n-q_1}, \dots, \rho_{n+1-q_2}; \rho_{n-q_2}, \dots; \dots; \rho_{n-q_k}, \dots, \rho_0).$$

Here the ordered blocks of ρ are indicated with semicolons. Note that we index the positions of the values in ρ decreasing from n to 0 starting from the left. We refer to the set of positions for each individual block as a “carrel”. The set Q will arise as the set of column lengths appearing in the shape λ for the given highest weight λ used to form the K-P identity. Here

Q also lists the column lengths that appear in the “labelling tableaux” described below. In Chapter 5, we indicate the equivalence of the ordered Q -partitions ρ in standard form with the minimal length coset representatives $w \in W^\lambda$.

For $1 \leq r \leq k = |Q|$, we define the “ ρ -minimal column” $Y^{(r)}(\rho)$ of length q_r to be the vertical column of q_r boxes which contains the leftmost q_r values in ρ , decreasing from top to bottom. In particular, if $Q = \{b\}$, for each ordered Q -partition ρ there is only one ρ -minimal column. It has length b and contains the leftmost b values of ρ .

The three combinatorial inputs above determine a set $\mathcal{L}_Q(\rho)$ of reverse semistandard Young tableaux, which we call “labelling tableaux”. These tableaux T meet the following criteria: the set of distinct column lengths of T is equal to Q ; for each $1 \leq r \leq |Q| = k$, the tableau T has exactly one ρ -minimal column of length q_r ; and the values in a column appear in the rightmost ρ -minimal column to its left. The values in the columns of a labelling tableau T of the leftmost length q_k are merely bounded below by the values in the ρ -minimal column of this length; here the possible values are not restricted to come from a ρ -minimal column to the left.

Note that the values in the region of T of columns of length q_r are restricted only by the ρ -minimal columns of length q_r and q_{r+1} . Thus the values in the regions of various column lengths are independent of each other.

Our first main result, Theorem 3.3, proves that the “weighted limit” of Demazure tableaux used to describe the Demazure characters in the K-P identity may be regarded as being the set $\mathcal{L}_Q(\rho)$ of our labelling tableaux. Our second main result, Theorem 3.4 below, gives product identities for a multivariate generating function for this particular set $\mathcal{L}_Q(\rho)$ of reverse semistandard tableaux. This theorem was originally obtained as a result of calculating this weighted limit. It can, however, be considered as a combinatorial identity in and of itself. In Chapter 4, we temporarily forget the Lie theory background and give a bijective proof of Theorem 3.4. But the first identity of Theorem 3.4 also has a second proof, wherein it is viewed as a consequence of the K-P identity and Theorem 3.3. We present this proof in Chapter 5. Our two main results can be combined into the statement of Theorem 3.7: “a

combinatorial interpretation of the K-P identity in Type A_n in terms of reverse semistandard tableaux is:

Theorem 3.4. *Fix $n \geq 1$, a subset $Q \subseteq \{1, 2, \dots, n\}$, and an ordered Q -partition ρ . Let $\Phi(\rho)$ be the set of inversions of ρ . Then*

$$\begin{aligned} \sum_{T \in \mathcal{L}_Q(\rho)} z^T &= \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} z_{\rho_i+2} \dots z_{\rho_j})} \\ &= \prod_{r=1}^k \prod_{(i,j) \in \mu^{(r)}} \frac{1}{1 - z^{\text{hook}(i,j)}}. \end{aligned}$$

The ‘‘Hillman-Grassl boards’’ $\mu^{(r)}$ here are defined in terms of Q and ρ in Section 4.6. The hook weights $z^{\text{hook}(i,j)}$ are assigned to their boxes in Section 4.3. The weight z^T measures T with respect to the tableau of the same shape as T whose columns are ρ -minimal columns. Hence we call z^T the ‘‘ ρ -weight monomial’’ of T . The variables $\{z_i\}_{i=1}^n$ that form the ρ -weight monomials correspond to the simple root basis $\{\alpha_i\}_{i=1}^n$.

Note that the first term in the expansion of the right hand side is 1. Given an ordered Q -partition ρ , there is a unique labelling tableau that is formed using one copy of each of the ρ -minimal columns for each of the column lengths in the set Q . This ‘‘minimal’’ labelling tableau has ρ -weight monomial equal to 1, and is accounted for by the first term of the right hand side. Note that this ρ -weight monomial is not the usual weight monomial for reverse semistandard tableaux; the usual weight monomial only considers the values in a tableau T itself.

As noted in Section 1.1, the variables z_i correspond to formal exponentials of the simple roots α_i . The ‘‘adjusted’’ Demazure characters inside the K-P limit are polynomials in the variables z_i once the characters have been coordinatized in the simple root basis. Hence the z_i variables are the natural choice for the coordinatization of the K-P identity. We could coordinatize the K-P identity in the variables x_i corresponding to the axis basis. However, the adjusted Demazure characters are not polynomials in these variables, and the right hand sides are not formal power series in them.

Given a subset $Q \subseteq \{1, 2, \dots, n\}$, there is a “minimal” ordered Q -partition which we denote ρ_0^Q . It is constructed as follows: Form an $(n + 1)$ -tuple of empty positions. Separate these positions into carrels using semicolons according to the set Q as above. Starting from the left, place the smallest values of $\{0, 1, 2, \dots, n\}$ in each cohort. Then, for instance, the values in the leftmost cohort are the numbers $\{q_1 - 1, q_1 - 2, \dots, 1, 0\}$ written in decreasing order. The simplest cases of our combinatorial identity occur when $\rho = \rho_0^Q$. In particular, when $Q = \{b\}$ and $Q = \{1, 2, \dots, n\}$, the right hand sides for ρ_0^Q are respectively

$$\prod_{1 \leq i \leq b \leq j \leq n} \frac{1}{1 - z_i z_{i+1} \dots z_j} \quad \text{and} \quad \prod_{1 \leq i \leq j \leq n} \frac{1}{1 - z_i z_{i+1} \dots z_j}.$$

For general Q , the indexing of the product for ρ_0^Q is over all (i, j) such that there does not exist an r such that $q_r < i \leq j < q_{r+1}$. The Weyl group element that corresponds to the ordered Q -partition ρ_0^Q is the longest element in the set W^J of minimal length representatives, where $J := \{1, 2, \dots, n\} - Q$.

1.3 Demazure tableaux and Willis’ scanning method

In this section we give a preview of Willis’ recent progress in combinatorially describing Demazure polynomials. It is described in detail for reverse semistandard tableaux in Section 2.5.

We need to first make a technical remark: In the papers [Wi] and [LS] referenced below, permutations $\pi \in S_n$ and semistandard tableaux were used. However, in the first five chapters of this thesis, we use ordered Q -partitions ρ of $\{0, 1, 2, \dots, n\}$ instead of permutations and reverse semistandard tableaux instead of semistandard tableaux. The translations from permutations to ordered Q -partitions and from semistandard tableaux to reverse semistandard tableaux are given in Chapter 5.

Once a basis for the space of weights has been chosen, Demazure characters become “Demazure polynomials”. The polynomial $\lceil_\lambda(\rho; x)$ is defined in [PW] to be the output of the Demazure character formula, wherein a sequence of divided difference operators is applied

to the weight monomial of λ , coordinatized in the axis basis. Lascoux and Schützenberger’s [LS] description of Demazure polynomials used the plactic algebra. Their description of Demazure polynomials relied on the construction of the “right key” of a tableau. For a tableau T , the right key $R(T)$ can be defined via a jeu de taquin process, as in Appendix A.5 of [Ful]. The “ λ -key of ρ ” is the particular key tableau on the shape λ whose columns are ρ -minimal columns. We denote this tableau by $Y_\lambda(\rho)$. Lascoux and Schützenberger proved that $\lceil_\lambda(\rho; x)$ is equal to the sum of the weight monomials of the reverse semistandard tableaux T on the shape λ whose right keys satisfy $R(T) \geq Y_\lambda(\rho)$. Following [PW], we say T is a “Demazure tableau” for ρ on the shape λ if $R(T) \geq Y_\lambda(\rho)$. If $\mathcal{D}_\lambda(\rho)$ denotes the set of Demazure tableaux for ρ on the shape λ , then $\lceil_\lambda(\rho; x)$ is equal to the sum of the weight monomials of $T \in \mathcal{D}_\lambda(\rho)$. The scanning method developed by Willis simplified the process of finding the right key $R(T)$. Thus his scanning method simplified the description of a Demazure tableau.

We first use the tableau scanning method to describe the limit of the adjusted characters in the left hand side of the K-P identity as a sum over the set of “labelling tableaux”: Fix an ordered Q -partition ρ and let $w \in W^\lambda$ be the equivalent minimal length representative. The adjusted Demazure polynomials $z^{-wm\lambda} \mathcal{D}_{m\lambda}(w; z)$ can be expressed as sums of our ρ -weight monomials z^T of the Demazure tableaux for ρ . We describe the sets $\mathcal{D}_{m\lambda}(\rho)$ of Demazure tableaux for $m \geq 1$ using the scanning method. We then calculate the “set direct limit” of the sequence of sets $\{\mathcal{D}_{m\lambda}(\rho)\}_{m \geq 1}$. In this limit, the Demazure tableau criterion $R(T) \geq Y_\lambda(\rho)$ simplifies to allow us to label the equivalence classes forming the direct limit with our labelling tableaux. These labels are not necessarily representatives of the equivalence classes, but they do have the same ρ -weight monomial as every member of the equivalence class which they label. We present this transition from limit to sum in our second proof of our combinatorial K-P identity, which is given in Chapter 5.

We also use the tableau scanning method in Chapter 6. There we translate it to a scanning method for Gelfand patterns, Algorithm 6.4. This algorithm takes a Gelfand pattern and from it produces a scanning pattern. Let π be a permutation. We define a Gelfand pattern

with top row λ whose scanning pattern entrywise dominates the “ λ -key pattern of π ” to be a “Demazure pattern” for π with top row λ . We prove that the Demazure polynomial for λ and π is equal to the sum of the weight monomials of the Demazure patterns by relating the tableau scanning method to the Gelfand pattern scanning method. We relate the two methods by using the standard bijection from tableaux on shape λ to Gelfand patterns with top row λ .

1.4 Principal specialization and two plane viewpoints

Theorem 3.4 gives product identities in the n variables z_i for a generating function for the set of labelling tableaux. When we coordinatize the K-P identity, the formal exponential of the simple root α_i becomes this variable z_i . Then the “principal specialization” of our combinatorial K-P identity is produced by setting $z_i = t$ for $1 \leq i \leq n$. Combinatorially, this amounts to ignoring the colors.

Fix an n -partition λ and a reverse semistandard tableau T on the shape λ . We visualize the shape λ in the xy -plane. Consider the following 3-D picture of T : Above the position (i, j) in the xy -plane, stack $T(i, j)$ cubes of side length 1. Move the blocks in the row $x = i$ up by $i - 1$ positions in the positive z direction. “Project the blocks” to the xz -plane: that is, record the number of blocks in the positive y direction above each integral position (x, z) in the xz -plane. This produces an upper triangular array of integers in the xz -plane when viewed from the positive y direction, standing on the xy -plane. If we rotate this array 135° clockwise we obtain a Gelfand pattern corresponding to T . The top row of this Gelfand pattern is equal to the n -partition λ .

Now let λ be the n -partition whose shape consists of just one column of length b , and let T be a reverse semistandard tableau on the shape λ . Consider the upper trapezoidal array of integers in the xz -plane before the 135° rotation. If we remove the entries below the line $z = b$, then we obtain an array of nonnegative integers contained in a (b^{n+1-b}) rectangle. From the vantage point of someone standing on the xy -plane in the positive y direction, the integers in this rectangle weakly increase when reading left to right in a row or top to bottom

in a column. Note that the indexing $(n + 1 - z, b + 1 - x)$ of the rows and columns here is the reverse of the indexing (z, x) provided by the x and z axes. Such an array is called a reverse plane partition on this rectangular shape (b^{n+1-b}) . This shape (b^{n+1-b}) should be thought of as lying in the xz -plane.

Let λ be as above. Then $Q = \{b\}$. Fix an ordered Q -partition ρ . Let T be the ρ -minimal column of length b . Convert the column T to a reverse plane partition P on the shape (b^{n+1-b}) as above. Let μ be the shape consisting of the boxes in the rectangle (b^{n+1-b}) with values in P that are equal to 0. This shape μ should also be thought of as lying in the xz -plane. We will see that labelling tableaux for $Q = \{b\}$ and ρ correspond to the reverse plane partitions on this shape μ . The reverse plane partition generating functions discussed in the next section should be thought of as generating functions for reverse plane partitions on this shape μ .

1.5 History and hierarchy of generating function identities

The product identities for the partition and reverse plane partition generating functions mentioned in Section 1.1 can now be viewed as special cases of our combinatorial K-P identity: In 1915, MacMahon [Mac] obtained a product identity for the generating function for the set of plane partitions contained in a rectangle. We can rotate his plane partitions 180° to obtain reverse plane partitions on the same rectangle. MacMahon's identity can be obtained from the principal specialization of our K-P identity as follows: Let λ be the n -partition whose shape consists of just one column of length b . Then we have $Q = \{b\}$. Again, we think of this λ in the xy -plane. Let $\rho = \rho_0^Q$ be the "minimal" ordered Q -partition defined in Section 1.2. This ordered Q -partition produces the rectangle $\mu = (b^{n+1-b})$ in the xz -plane by the process described in the previous section. Then we bijectively map our labelling tableaux to reverse plane partitions on the shape μ . This identification enables us to translate our labelling tableaux generating function to a generating function for the set of reverse plane partitions on μ . Applying the principal specialization to this identity produces MacMahon's identity. Here Euler's generating function identity in Section 1.1 is

a special case of MacMahon’s identity when $b = 1$. By 1971, Stanley [Sta2] had extended MacMahon’s result by obtaining a product identity for the generating function for reverse plane partitions on a general shape μ . Let $Q = \{b\}$ and let ρ be a general ordered Q -partition; these determine the shape μ in the xz -plane. Stanley’s result now arises as the principal specialization of our K-P identity for this Q and ρ . In 1981, Gansner obtained a multivariate version of Stanley’s identity. He first “colored” the diagonals of the shape μ and then introduced a multivariate weight for a reverse plane partition on μ in terms of these colors. In [Ga], the coloring of the diagonals of μ was unmotivated. However, when we convert the K-P identity to combinatorics, this coloring process arises naturally. This gives a Lie theory justification for coloring the diagonals as Gansner did. Gansner’s result now arises as our (now unspecialized) combinatorial K-P identity for general ρ with $Q = \{b\}$. As special cases, both Euler and MacMahon’s identities also have such colored versions. Our identity becomes these colored identities when $Q = \{1\}$ and $Q = \{b\}$ respectively and $\rho = \rho_0^Q$. In 2009, to begin to understand the K-P identity in Type A_n , Proctor extended MacMahon’s identity by formulating a precursor version of our labelling tableaux for general Q , but only for ρ_0 . Proctor’s tableaux essentially decomposed into k -tuples of reverse plane partitions of the kind that are handled by the “colored MacMahon” identity. Finally, to view our identity as a generating function for k -tuples of reverse plane partitions, we first construct k shapes $\{\mu^{(r)}\}_{r=1}^k$ for the given ordered Q -partition ρ . We then introduce a new set of colors for each of these k shapes. By using k of the xz -plane views as above, our combinatorial K-P identity can be viewed as an identity for the colored generating function of k -tuples of reverse plane partitions on the k general shapes $\{\mu^{(r)}\}_{r=1}^k$.

1.6 Structures for the one column λ case

Here we restrict our attention to the case when the shape λ in the xy -plane consists of just one column. We indicate how the known plane partition generating function identities (or proofs) due to MacMahon, Stanley, Hillman, Grassl and Gansner fit into a Lie theoretic framework as the foremost special case of the Kumar-Peterson identity. Along the way, we

mention most of the combinatorial and Lie theoretic structures that can be associated to this context. We also discuss the shapes that arise in the xz -plane. These are the shapes for the reverse plane partitions of Stanley and Gansner's generating function identities. Gansner assigned colors to the diagonals of the shape on which he defined his reverse plane partitions. We explain in this section how this peculiar coloring arises. We omit definitions and detailed explanations of the notation used; please refer to Chapters 2 and 5 of this thesis as well as to [Sta1], and [BB], and the appendix of [PW].

Fix $1 \leq b \leq n$. Let λ be the n -partition (1^b) whose shape consists of one column with b boxes. Here $Q = \{b\}$. As in Section 1.1, we view this shape λ in the xy -plane. We consider strictly decreasing tableaux on λ with values from $\{n, n-1, \dots, 0\}$. Let T be one tableau on the shape λ . Following the procedure in Section 1.4, we obtain a reverse plane partition on the rectangle (b^{n+1-b}) in the xz -plane that corresponds to T . This reverse plane partition consists of only 0's and 1's. Converting the 1's to dots, we can view this reverse plane partition as the Ferrer's diagram for an $(n+1-b)$ -partition ν with parts bounded by b . If we replace the 1's with boxes and rotate 180° , we obtain a Young diagram ν that fits into the rectangle (b^{n+1-b}) . Before the 180° rotation, these Young diagrams can also be viewed as order ideals in the product of chains $P_0 := (\mathbf{n+1-b}) \times \mathbf{b}$. Ordering these by inclusion produces the distributive lattice $J(P_0)$ that Stanley refers to as $L(n+1-b, b)$. The minimal element of this poset is the empty $(n+1-b)$ -partition $\nu = \phi$. This partition comes from the tableau T on the shape λ in the xy -plane which has decreasing values from $b-1$ to 0 down the column. The maximal element of this poset is the full rectangle $\nu = (b^{n+1-b})$. This partition comes from the tableau T on the shape λ in the xy -plane which has decreasing values from n to $n+1-b$ down the column.

In [Ga], Gansner colors the boxes in the rectangle (b^{n+1-b}) one diagonal at a time with the colors $-(n-b), -(n-1-b), \dots, -1, 0, 1, \dots, b-2, b-1$. Our colors for these same diagonals are respectively $1, 2, \dots, n$. The transition from tableaux in the xy -plane to reverse plane partitions in the xz -plane explains why the diagonals in (b^{n+1-b}) should be "isochromatic" in these fashions: Recall that given a tableau T on the shape λ , in Section 1.4 we viewed it

using piles of blocks on the xy -plane. We shifted these blocks up row by row and projected the y -censuses to the xz -plane. Ignoring the 1's below the line $z = b$, we obtained a reverse plane partition P on the rectangle (b^{n+1-b}) . One can see that the 1's in P along the diagonal colored i arise from values in T which are greater than or equal to i . Thus if a value of T were to be boosted from an $i - 1$ to an i , an additional 1 would be produced in the diagonal colored i . We will see in Chapter 4 that boosting a value $i - 1$ to a value i anywhere in a general tableau T has the effect of multiplying the ρ -weight monomial of both T and P by z_i . This corresponds Lie theoretically to adding the simple root α_i to the weight of T , since $z_i = \exp(\alpha_i)$.

Consider a shuffle σ of the $(n + 1)$ -tuple $(1^b, 0^{n+1-b})$. Here we index the positions of the $(n + 1)$ -tuple decreasing from n to 0 from left to right. Recording the positions of the 1's in a column decreasing from top to bottom produces a column tableau T on the shape λ considered above. The shuffle $(1^b, 0^{n+1-b})$ produces the column tableau with maximal values. The shuffle $(0^{n+1-b}, 1^b)$ produces the column tableau with minimal values. Counting the number of positions that each 1 has moved creates a b -partition μ' with parts bounded by $n + 1 - b$. Let μ denote the conjugate of the b -partition μ' ; it is an $(n + 1 - b)$ -partition with parts bounded by b . It can be seen that for a given column, its corresponding shapes ν and μ are “complementary” within the rectangle (b^{n+1-b}) . These “Hillman-Grassl boards” μ are the shapes on which the generating functions of Gansner and Stanley are defined; these shapes inherit the colors $1, 2, \dots, n$ from the full rectangle (b^{n+1-b}) . A shuffle can be viewed as a rearranging operation σ of $(n, n - 1, \dots, 1, 0)$ such that the values $n, n - 1, \dots, n + 1 - b$ and the values $n - b, n - b - 1, \dots, 1, 0$ respectively remain in decreasing order. We also index the positions in these $(n + 1)$ -tuples decreasing from n to 0 from left to right. The “inverse shuffle” σ^{-1} is a rearranging operation of $(n, n - 1, \dots, 1, 0)$ such that the values in the positions $n, n - 1, \dots, n + 1 - b$ and $n - b, n - b - 1, \dots, 1, 0$ decrease from left to right. For a fixed rearranging result, call these two sets of positions “carrels” and we call the sets of values in these positions “cohorts”. Such rearranging results produced from inverse shuffles are exactly the standard forms of our ordered Q -partitions ρ when $Q = \{b\}$. We form the

λ -key of ρ by filling the Young diagram of λ with the values in the leftmost cohort of ρ . This is the column tableau T on the shape $\lambda = (1^b)$ that was produced from σ near the beginning of this paragraph.

It can be seen that anytime that a new $(n+1-b)$ -partition μ' bounded by b is produced from μ by appending one box of color i , then in the shuffle picture a 1 in position i was swapped to the right with a 0 from position $i-1$. For $n \geq i \geq 1$, let s_i denote the operation of interchanging the values at positions i and $i-1$ in an $(n+1)$ -tuple. Here the symmetric group S_{n+1} of rearranging operations on $(n+1)$ -tuples can be viewed as the Weyl group W of Type A_n acting on its weight space E^{n+1} . Re-use the symbol λ to denote the fundamental weight $(1^b, 0^{n+1-b}) = \omega_{n+1-b}$ for $sl_{n+1}(\mathbb{C})$. Then the set of shuffles is the orbit $W\lambda$. The stabilizer of λ is the parabolic Weyl subgroup $W_J := W_\lambda = S_b \times S_{n+1-b}$, where $J = \{1, 2, \dots, n\} - \{n+1-b\}$. Here S_b and S_{n+1-b} denote the subgroups of S_{n+1} of rearranging operations which rearrange the sets of values within positions $n, n-1, \dots, n+1-b$ and within positions $n-b, \dots, 1, 0$ respectively. Given an ordered Q -partition ρ , view it as an inverse shuffle σ^{-1} . The corresponding $\sigma \in W$ is a minimal length representative of a coset in W/W_J . Let W^J denote the set of such minimal length representatives. Elements of W^J can be labelled with any of the previous structures associated to the column tableaux on the shape λ . To define the Bruhat order on W^J , view the simple reflections s_i as generators for S_{n+1} viewed as a Coxeter group. Now label the elements of W^J with their corresponding shapes ν . Then the order dual of the Bruhat order on W^J becomes $L(n+1-b, b)$. The identity element $\sigma = e$ of W^J is depicted as an $(n+1)$ -tuple by $(n, n-1, \dots, 1, 0)$. Its inverse shuffle σ^{-1} corresponds to $\nu = (b^{n+1-b})$. The longest element $\sigma = \sigma_0^J$ of W^J is depicted as an $(n+1)$ -tuple by $(b-1, b-2, \dots, 1, 0, n, n-1, \dots, n+2-b, n+1-b)$. Its inverse shuffle σ^{-1} corresponds to the empty $(n+1-b)$ -partition $\nu = \phi$.

Return to considering one fixed ordered Q -partition ρ for $Q = \{b\}$ and its corresponding objects $\sigma \in W$, ν , and μ . Consider the principal filter generated by σ in the order dual of W^J . This produces a lattice which we denote L_σ . To this corresponds the principal filter of $L(n+1-b, b)$ generated by ν . This lattice can be described as the poset of ideals of

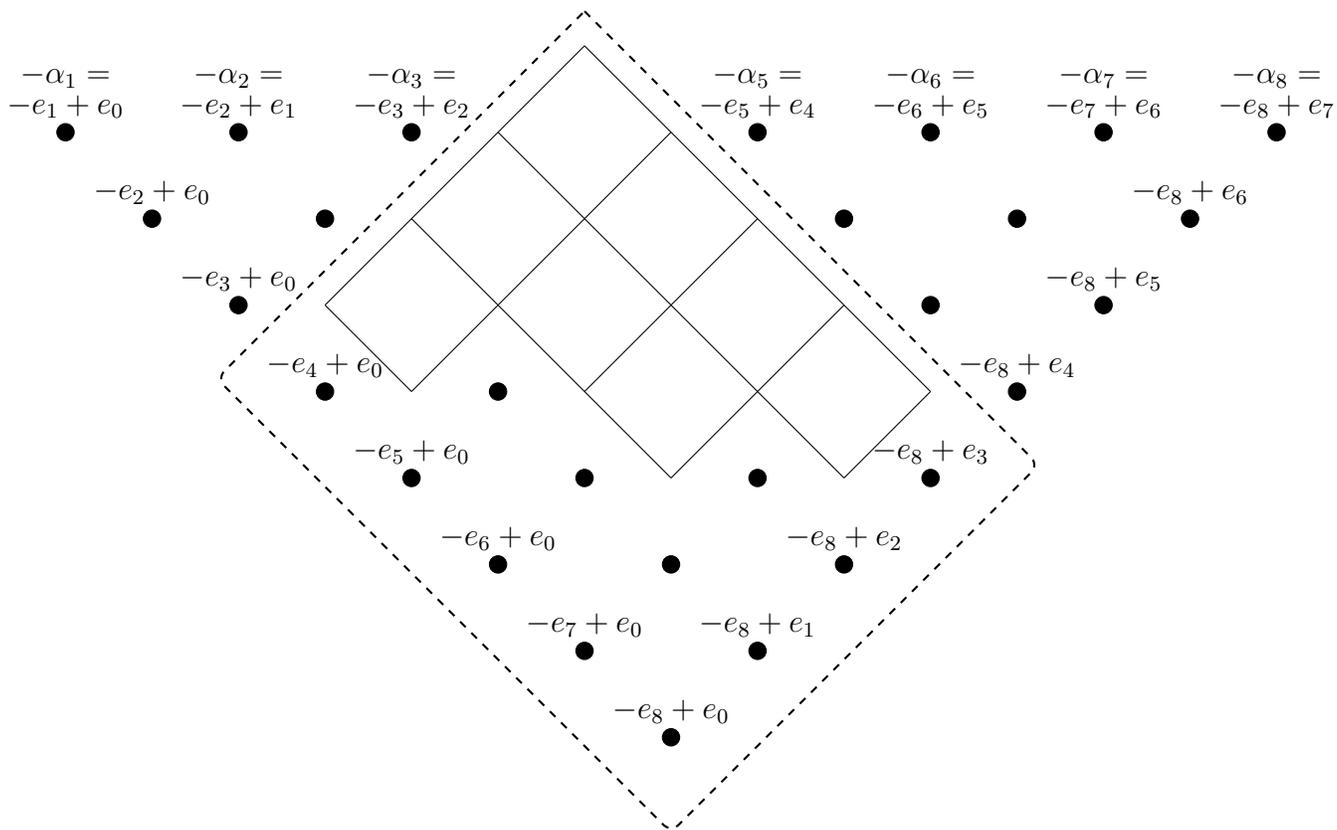
P_0 that contain ν , ordered by inclusion. Now view the shape μ as a poset whose box at location $(1, 1)$ is its unique maximal element. This is the poset of meet irreducibles of L_σ . We denote this poset P_σ . The ideals of P_σ correspond to reverse plane partitions contained in μ that are bounded by 1. These reverse plane partitions should be viewed as describing “augmentations” of ν in the xz -plane. In the one column λ case, the (inverse) Hillman-Grassl building procedure produces these reverse plane partitions by adding 1’s to existing reverse plane partitions. The 1’s are added in a manner that corresponds to multiplying by the variables $z_i = \exp(\alpha_i)$.

We choose the positive simple roots of Type A_n to be $e_i - e_{i-1}$ for $n \geq i \geq 1$. For $\sigma \in W^J$, set $\Phi(\sigma) := \Phi^+ \cap \sigma(\Phi^-)$. Then $\Phi^- \cap \sigma^{-1}(\Phi^+) = \sigma^{-1}\Phi(\sigma)$. Let $\sigma_0^J \in W^J$ be the longest element of W^J . The set $(\sigma_0^J)^{-1}\Phi(\sigma_0^J)$ of negative roots β such that $\sigma_0^J\beta \in \Phi^+$ consists of the negative roots $-e_j + e_{i-1}$ such that $n \geq j \geq n+1-b \geq i \geq 1$. There are $b(n+1-b)$ such negative roots. For an example, take $n = 8$ and $b = 5$. These negative roots form the 4×5 rectangle indicated by the dotted line in Figure 1.

Now fix an ordered Q -partition ρ for $Q = \{b\}$. For example, let $\rho = (8, 6, 4, 3, 1; 7, 5, 2, 0)$. This determines the corresponding $\sigma \in W^J$ and filter $\mu \subseteq P_0$. The corresponding μ is the shape $(4, 2, 1, 0)$ shown in Figure 2. Since A_n is simply laced, [Theorem 11, Pro2] (or [Theorem 2.4, BS]) implies that the poset P_σ of join irreducibles is isomorphic to the set of $\sigma^{-1}\Phi(\sigma)$ of negative roots β such that $\sigma.\beta \in \Phi^+$. So the order structure for our “encompassing Hillman-Grassl board” on which our reverse plane partitions are P -partitions comes from the order structure of the negative roots.

The right hand side of the K-P identity is a product over the positive roots $\Phi(\sigma) = \sigma[\sigma^{-1}\Phi(\sigma)]$. Each positive root $\alpha \in \Phi(\sigma)$ can be written as $e_{\rho_j} - e_{\rho_i}$ for some “inversion” (ρ_i, ρ_j) of ρ where $i > j$ and $\rho_i < \rho_j$. The multivariate weight monomial of any such positive root $e_{\rho_j} - e_{\rho_i}$ is $z_{\rho_{i+1}}z_{\rho_{i+2}} \dots z_{\rho_j}$. The colors which we assign to the boxes are indicated in Figure 2 by the boldface simple roots within the boxes. In Section 5.3, we will see that one can view the positive roots in $\Phi(w)$ as “being” the hooks of the shape μ . For example, in Figure 2 we view the positive root $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = w.(-e_5 + e_3)$ as the hook at

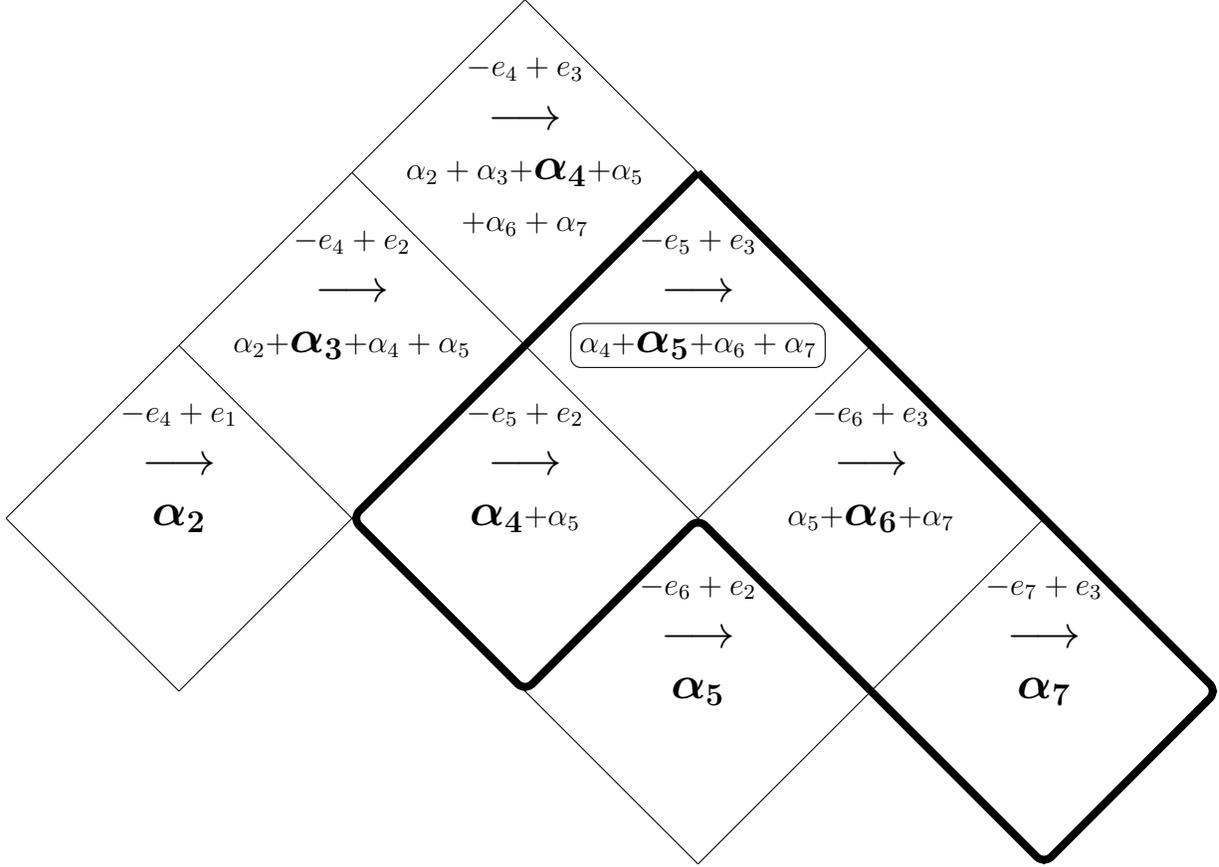
Figure 1: Negative roots and the shape μ



location $(1, 2)$ of the shape μ : One can see that the circled root $\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$ is equal to the sum of the colors along the hook at location $(1, 2)$ of μ . Thanks to the Hillman-Grassl algorithm, each $\alpha \in \Phi(\sigma)$ can be viewed as indicating a potential increment to a colored reverse plane partition on μ : the colors of the augmenting “blocks” must match the colors of the Hillman-Grassl board boxes as the strip of consecutive colors is “wiggled”.

Now let $m \geq 1$. The P_0 -partitions with values bounded by m correspond to the 3-dimensional Ferrers diagrams that fit inside an $(n+1-b) \times b \times m$ box. These 3-D Ferrers diagrams arise from the reverse semistandard Young tableaux with possible values $\{0, 1, \dots, n\}$ on the $b \times m$ rectangle in the xy -plane via the stacking, shifting, and truncating process of Section 1.4. MacMahon [Mac] found a quotient-of-products expression for the usual generating function for these P_0 -partitions. When $m = 1$, this quotient-of-products is the Gaussian coefficient $\binom{n+1}{b}_t$. Proctor and Stanley [Pro2] indicated how to rederive MacMahon’s result

Figure 2: Hooks on the shape μ



in a Lie theoretic context similar to the context of this thesis. They did this using a description of the highest weight representation $V_{m\omega_{n+1-b}}$ for $sl_{n+1}(\mathbb{C})$ resulting from Seshadri's standard monomial theorem for minuscule flag manifolds [Se]. There m -multichains in W^J were seen to correspond to the “wedding cake layers” of the P_0 -partitions at hand. In [Pro2], the poset P_0 and the lattice $L(n+1-b, b)$ are respectively viewed as the minuscule poset $a_n[r]$ and the minuscule lattice $A_n[r]$. Since m was finite, to obtain a product expression identity, it was necessary with the [Pro2] approach to restrict attention to the univariate principal specialization of the Weyl character for $V_{m\omega_{n+1-b}}$. Letting $m \rightarrow \infty$ in that result produces the principal specialization of Theorem 3.4 for $\lambda = (1^b)$.

We now consider the case of the K-P identity that becomes Gansner's identity. Let ρ be an ordered Q -partition and continue to fix $m \geq 1$. Consider the highest weight $\lambda := m\omega_{n+1-b}$

for an irreducible representation of $sl_{n+1}(\mathbb{C})$. The corresponding Young diagram λ is the rectangle (m^b) with sole column length b . When λ has only one column length b , it has long been known that the character of the Demazure module of $sl_{n+1}(\mathbb{C})$ with lowest weight $\rho \cdot \lambda$ is described with the reverse semistandard tableaux on the shape λ which entrywise dominate the λ -key of ρ . Here it is straightforward to form the “direct limit” of the set of such tableaux. This direct limit can be described with the labelling tableaux presented in Section 1.2: In this case these are the reverse semistandard tableaux on the rectangular shapes $b \times c$ for $c \geq 1$ whose entries are bounded below by the column depiction of ρ and which contain exactly one copy of the column depiction of ρ . Since forming the direct limit commutes with projecting the tableaux to the xz -plane, we can alternatively calculate the direct limit of the set of reverse plane partitions as $m \rightarrow \infty$. The direct limit can be described as the set of unbounded reverse plane partitions on the shape μ . Hence the left hand side of the K-P identity may be combinatorially interpreted as the sum of a multivariate weight over these reverse plane partitions.

The two families of Lie theoretic cases which are the most amenable to combinatorial description consist of the Type A cases and the λ -minuscule cases. This thesis describes the generalization of the [Pro2] situation described above from $\lambda = \omega_{n+1-b}, 2\omega_{n+1-b}, 3\omega_{n+1-b}, \dots$ to the general Type A situation for $\lambda, 2\lambda, 3\lambda, \dots$. Dale Peterson independently developed the K-P identity to help Proctor obtain [Pro1] the generalization from the [Pro2] cases to P -partitions on all d -complete posets; this generalization used the λ -minuscule representations of the simply-laced Kac-Moody algebras.

1.7 Preview of the bijective proof

In this section we present a preview of the bijective proof of our combinatorial interpretation of the K-P identity, Theorem 3.4.

Fix $n \geq 1$, a subset $Q = \{q_1 < \dots < q_k\} \subseteq \{1, 2, \dots, n\}$, and an ordered Q -partition ρ .

The left hand side, as originally stated, is a sum over the labelling tableaux:

$$\sum_{T \in \mathcal{L}_Q(\rho)} z^T.$$

As we noted in Section 1.2, the values in a labelling tableau in each region of a distinct column length are independent from the values in the regions of a different column length. This enables us to write the set of labelling tableaux as a Cartesian product of k sets, denoted $\mathcal{L}_Q^{(r)}(\rho)$ for $1 \leq r \leq k$. The set $\mathcal{L}_Q^{(r)}(\rho)$ consists of the tableaux $T^{(r)}$ such that the only column length for $T^{(r)}$ is q_r , the values of $T^{(r)}$ come from the ρ -minimal column of length q_{r+1} , and $T^{(r)}$ contains exactly 1 ρ -minimal column of length q_r . These tableaux appear as “subtableaux” of the labelling tableaux. We can then write $\mathcal{L}_Q(\rho) = \prod_{r=1}^k \mathcal{L}_Q^{(r)}(\rho)$. This enables us to rewrite the sum on the left hand side as

$$\prod_{r=1}^k \left(\sum_{T \in \mathcal{L}_Q^{(r)}(\rho)} z^T \right).$$

Starting with $r = k$ and proceeding to $r = 1$, we produce k identities that equate these factors of the left hand side to k products over certain subsets of inversions of ρ . When $r = k$, we construct a shape $\mu^{(k)}$ which we call a “Hillman-Grassl board”. This Hillman-Grassl board is defined by the values in the ρ -minimal column of length q_k . Here the shape $\mu^{(k)}$ is constructed from the ordered Q -partition ρ as in Section 1.4. We now “color” the diagonals of $\mu^{(k)}$. For now, each color is a one element subset of $\{1, 2, \dots, n-1, n\}$. For $r < k$, each new “composite” color will be a subset of consecutive integers of $\{1, 2, \dots, n-1, n\}$. Now we construct three weight-preserving bijections. For our first bijection when $r = k$, we map tableaux in the set $\mathcal{L}_Q^{(k)}(\rho)$ to reverse plane partitions on the shape $\mu^{(k)}$. This map is given by the transition from reverse semistandard tableaux in the xy -plane to reverse plane partitions in the xz -plane as described in Section 1.4. At this stage we employ the “colored” Hillman-Grassl algorithm as developed by Gansner [Ga]. This algorithm takes a multiset of “hooks” on the shape $\mu^{(k)}$ and constructs a reverse plane partition on $\mu^{(k)}$. For our proof,

we use the opposite direction. That is, for our second weight-preserving bijection, we use the colored Hillman-Grassl algorithm to map reverse plane partitions on $\mu^{(k)}$ to the set of multisets of “hooks” on $\mu^{(k)}$. To set up our third weight-preserving bijection, we construct a map from the set of hooks of $\mu^{(k)}$ to a certain subset of the set $\Phi(\rho)$ of inversions of ρ . We denote this subset by $\Phi^{(k)}(\rho)$. This induces our third weight-preserving bijection: a map from the set of multisets of hooks of $\mu^{(k)}$ to the set of multisets of elements of the set $\Phi^{(k)}(\rho)$. Denote by $M(\Phi^{(k)}(\rho))$ the set of multisets of elements of $\Phi^{(k)}(\rho)$. These three bijections give us the following identity:

$$\sum_{T \in \mathcal{L}_Q^{(k)}(\rho)} z^T = \sum_{S \in M(\Phi^{(k)}(\rho))} z^S.$$

The current right hand side of this identity is the naive expansion of

$$\frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(k)}(\rho)} (1 - z_{\rho_i+1} z_{\rho_i+2} \cdots z_{\rho_j})}.$$

Thus we have

$$\sum_{T \in \mathcal{L}_Q^{(k)}(\rho)} z^T = \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(k)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})}.$$

Now for $k-1 \geq r \geq 1$, we similarly construct Hillman-Grassl boards $\mu^{(r)}$. For fixed $k-1 \geq r \geq 1$, these Hillman-Grassl boards are constructed from the values in the ρ -minimal columns of length q_r and q_{r+1} . However, we need to carefully form the colors for these boards. The colors assigned to the diagonals of the Hillman-Grassl board $\mu^{(r)}$ are formed from unions of colors that were assigned to the board $\mu^{(r+1)}$. For $k-1 \geq r \geq 1$, we construct subsets $\Phi^{(r)}(\rho) \subset \Phi(\rho)$. Then for $k-1 \geq r \geq 1$, we define three weight-preserving bijections as above. For $k-1 \geq r \geq 1$, these three bijections give us the following identities:

$$\sum_{T \in \mathcal{L}_Q^{(r)}(\rho)} z^T = \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(r)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})}.$$

Finally, the proof can be completed with the observation that $\Phi(\rho) = \sqcup_{r=1}^k \Phi^{(r)}(\rho)$. Thus we have

$$\begin{aligned} \sum_{T \in \mathcal{L}_Q(\rho)} z^T &= \prod_{r=1}^k \left(\sum_{T \in \mathcal{L}_Q^{(r)}(\rho)} z^T \right) = \prod_{r=1}^k \left(\frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(k)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})} \right) \\ &= \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})}. \end{aligned}$$

1.8 Geometry and affine labelling tableaux

Here we present another choice of tableaux for labelling the equivalence classes that arise when we form the weighted limit of the sequence of sets of Demazure tableaux. The specification of these tableaux is only slightly different from the definition of the labelling tableaux presented in Section 1.2; this formulation is more closely related to the geometric structures that provided the original context for the Kumar-Peterson identity.

Fix a subset $Q \subseteq \{1, 2, \dots, n\}$ and an ordered Q -partition ρ . For $1 \leq r \leq k$, recall that the ρ -minimal column of length q_r is the vertical column of q_r boxes which contains the leftmost q_r values in ρ , decreasing from top to bottom. This column is denoted $Y^{(r)}(\rho)$. We define the set of affine labelling tableaux to be the tableaux T which meet the following criteria: each column length of T is in Q ; the allowable columns of length q_r are those which dominate but do not equal $Y^{(r)}(\rho)$; and the values in the columns of length q_r come from the ρ -minimal column $Y^{(r+1)}(\rho)$. To convert a labelling tableau S to the corresponding affine labelling tableau T , delete the one copy of each ρ -minimal column from S . Note that the null tableau \emptyset is an affine labelling tableau for every choice of n , Q , and ρ .

Example 1.1. Fix $n = 2$, $Q = \{1\}$, and $\rho = \rho_0^Q = (0; 2, 1)$. The ρ -minimal column of length 1 is $\boxed{0}$. Here the labelling tableaux are

$$\boxed{0}, \boxed{10}, \boxed{20}, \boxed{110}, \boxed{210}, \boxed{220}, \dots$$

The affine labelling tableaux are

$$\emptyset, \boxed{1}, \boxed{2}, \boxed{1\ 1}, \boxed{2\ 1}, \boxed{2\ 2}, \dots$$

These affine labelling tableaux can be obtained from the labelling tableaux by deleting the ρ -minimal column $\boxed{0}$ from each tableau respectively. ■

The $Q = \{1\}$ and $\rho = \rho_0^Q$ case of our work provides a combinatorial model for the usual realization of affine n -space within projective n -space: Set $V := \mathbb{C}^{n+1}$. Let f_0, \dots, f_n denote the axis basis for V^* . These are global sections on $\mathbb{P}^n := \mathbb{P}(V)$ for its standard line bundle. The homogeneous coordinate ring for \mathbb{P}^n is $\bigoplus_{m \geq 0} S^m V^*$. Here $S^m V^*$ consists of the homogeneous polynomials of degree m in the f_i . The group $GL(n+1)$ acts on V^* via the contragredient of the natural representation. Its torus subgroup T consists of the diagonal matrices $t := (t_0, \dots, t_n)$. Define characters $x_i : T \rightarrow \mathbb{C}^*$ by $x_i(t) := t_i^{-1}$. The character of T under the induced action of $GL(n+1)$ on $S^m V^*$ is the homogeneous symmetric function $h_m(x_0, \dots, x_n)$. This may be depicted with the $(n+1)$ -reverse semistandard Young tableaux (defined in Section 2.4) on the one row shape (m^1) . The affine space A^n may be realized within \mathbb{P}^n by requiring that $f_0 = 1$. Its affine coordinate ring is $R := \mathbb{C}[f_1, \dots, f_n]$. Here the character for the induced action of T on R is the sum of all monomials in f_1, \dots, f_n . This may be depicted with the one row affine labelling tableaux, as in the example above for $n = 2$: Here the value i in a tableau is to be interpreted as the variable x_i , and so these tableaux depict the monomials $1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, \dots$

The $Q = \{b\}$ and general ρ case of our work provides a combinatorial model for the affine coordinate ring of a Bruhat cell within the Grassman manifold $G_{b,n+1}$: Here the global sections on all of $G_{b,n+1}$ of its standard line bundle are certain $b \times b$ “minor” polynomials that are formed from certain variables that are drawn from an $(n+1) \times b$ array of variables; for projective space the f_0, \dots, f_n were 1×1 minor polynomials drawn from the column of variables $(f_0, f_1, \dots, f_n)^T$.

These minors are depicted by the $(n + 1)$ -reverse semistandard Young tableaux on the one column shape (1^b) . Fix an ordered Q -partition ρ and consider the corresponding ρ -minimal column $Y_{(1^b)}(\rho)$. The Schubert subvariety of $G_{b,n+1}$ indexed by ρ is determined by setting to zero the minors that are indexed by the one column tableaux that do not dominate $Y_{(1^b)}(\rho)$. Here a basis for the space of global sections of degree m for the standard line bundle restricted to the subvariety can be obtained by choosing the m -fold products of minors that are indexed by the $(n + 1)$ -reverse semistandard Young tableaux on the shape (m^b) which are comprised of the “surviving” columns. The corresponding Bruhat cell arises when the minor indexed by $Y_{(1^b)}(\rho)$ is set equal to 1. The character for the action of T on the affine coordinate ring of this cell is the sum of our weight monomials x^T over the affine labelling tableaux for $Q = \{b\}$ and ρ . Here the shapes of the affine labelling tableaux are all the rectangular shapes with b rows along with the empty shape (which can be viewed as the shape with b rows and no columns.) Here the exclusion of the ρ -minimal column from the affine labelling tableaux corresponds to setting this minor to 1.

Let Q be arbitrary. This specifies a certain flag manifold of Type A_n . Choosing an ordered Q -partition ρ specifies a Schubert subvariety. David Lax has used Willis’ scanning method to simplify Reiner’s and Shimozono’s Demazure tableaux derivation of bases of “standard monomials” of minors for the spaces of global sections of the standard line bundles [Lax]. Here the Bruhat cell indexed by ρ arises when the product of the minors that correspond to the ρ -minimal columns is set to 1.

Part of the proof [Pro1] of the Kumar-Peterson identity produces the right hand side of that identity by directly describing the character for the action of T on the coordinate ring of this cell. Then the bijective proof presented in this thesis can be used to show that the character for the action of T on the coordinate ring of this Bruhat cell is the sum of the x^T over the affine labelling tableaux for Q and ρ . If describing this character with reverse semistandard Young tableaux is the only goal, then the notions of Demazure modules and characters and limits thereof may be bypassed in this manner.

2 Definitions and combinatorial background

In this chapter, we introduce the combinatorial structures needed to state our main results in Chapter 3. In Sections 2.1 to 2.3, we define n -partitions, ordered Q -partitions, and inversions of ordered Q -partitions. The first right hand side of our second main result, Theorem 3.4, is a product over the inversions of an ordered Q -partition; the second right hand side is a product over the “hooks” of certain “Hillman-Grassl boards”. We define the boards in Section 4.6. In Section 2.4 we introduce reverse semistandard tableaux. In Section 2.5, we present Willis’ tableau scanning method in terms of reverse semistandard tableaux. This allows us to give our definition of Demazure tableaux in Section 2.6. In Section 2.7, we define the usual weight monomial of a tableau and the ρ -weight monomial of a Demazure tableau. These monomials constitute the terms of the Demazure polynomials and “adjusted” Demazure polynomials respectively. We give a definition of the “weighted limit” of a direct system of sets equipped with maps in Section 2.8. Finally, in Section 2.9, we define the set of labelling tableaux.

2.1 n -partitions

Denote the set of integers from 1 to n by $[n]$ and the set of integers from 0 to n by $[0, n]$. For integers $i < j$, define $(i, j] := \{i + 1, \dots, j\}$, and $[i, j) := \{i, \dots, j - 1\}$. An n -partition λ is an n -tuple $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Fix an n -partition λ . For $m \geq 1$, define $m\lambda := (m\lambda_1, m\lambda_2, \dots, m\lambda_n)$. The *Young diagram* (or *shape*) λ , also denoted λ , consists of λ_i left justified boxes in the i^{th} row for $1 \leq i \leq n$. Let (i, j) denote the box in the i^{th} row and the j^{th} column in λ . For $1 \leq j \leq \lambda_1$, let ζ_j be the number of boxes in the j^{th} column in λ . Let k denote the number of distinct column lengths in λ . Denote the set of distinct column lengths of λ by $Q(\lambda) := \{q_1, q_2, \dots, q_k\}$, where $q_i < q_{i+1}$. Note that $q_1 = \zeta_{\lambda_1}$ and $q_k = \zeta_1$. Further, we define $q_0 := 0$ and $q_{k+1} := n + 1$. Note that λ and

$m\lambda$ have the same set of distinct column lengths.

2.2 Ordered Q -partitions

Fix an integer $1 \leq k \leq n$. Let $Q = \{q_1, q_2, \dots, q_k\}$ be a subset of $[n]$ such that $q_i < q_{i+1}$. We define an *ordered Q -partition* ρ to be a partition of the set $[0, n]$ into $k + 1$ parts such that the r^{th} part has size $q_r - q_{r-1}$ for $1 \leq r \leq k + 1$. We notate ρ as an $(n + 1)$ -tuple

$$(\rho_n, \rho_{n-1}, \dots, \rho_{n+1-q_1}; \rho_{n-q_1}, \dots, \rho_{n+1-q_2}; \rho_{n-q_2}, \dots; \dots; \rho_{n-q_k}, \dots, \rho_0),$$

where the parts are separated by semicolons. The *standard form* of ρ is the unique such $(n + 1)$ -tuple whose entries decrease from left to right within each part. Throughout this thesis, we assume that ordered Q -partitions are in standard form. Denote the set of ordered Q -partitions by S_{n+1}^Q . We call the set of positions (values) within each part a *carrel* (*cohort*). So the number of carrels (cohorts) in $\rho \in S_{n+1}^Q$ is $k + 1$.

The ordered Q -partition ρ in standard form will play the role in our interpretation of the K-P identity that the minimal length representative $w \in W^J$ plays in the Lie theory K-P identity, where $J := [n] - Q$. We show in Chapter 5 how to obtain an ordered Q partition ρ from w .

Given an ordered Q -partition ρ , we define a *Q -chain of ordered subsets* $B_1 \subset B_2 \subset \dots \subset B_{k+1}$ by $B_i := B_{i,\rho} := \{\rho_n, \rho_{n-1}, \dots, \rho_{n+1-q_i}\}$. So the set B_i contains the leftmost q_i entries of ρ . Henceforth the elements of B_i are to be listed in decreasing order from left to right. Note that $B_{k+1} = [0, n]$. We define $B_0 = \emptyset$. For $1 \leq i \leq k + 1$, we denote the i^{th} cohort of ρ by H_i . Note that we have $H_i = B_i \setminus B_{i-1}$ and $B_i = H_1 \sqcup H_2 \sqcup \dots \sqcup H_i$ for $1 \leq i \leq k + 1$. Define $B_i(j)$ and $H_i(j)$ to be the j^{th} largest entries of B_i and of H_i respectively.

Throughout this thesis, whenever we specify an n -partition λ we are also implicitly specifying the set $Q := Q(\lambda)$ and the integer $k := k_\lambda := |Q|$. In the following two examples, we start with an n -partition λ and from λ we produce the set $Q := Q(\lambda)$. For our combinatorial K-P identity, each n -partition λ with the same set $Q(\lambda)$ of distinct column lengths produces

the same combinatorial identity. For this reason, in our identity, we take the set Q as our starting point instead of λ .

Example 2.1. Fix $n = 8$. Let λ be the 8-partition $(1, 1, 1, 1, 1, 0, 0, 0)$. Here λ has just one column of length 5. Hence $Q := Q(\lambda) = \{5\}$. Since Q contains $k = 1$ integer, there are two cohorts for any ordered Q -partition $\rho \in S_9^Q$. Consider $\rho = (7, 6, 4, 2, 1; 8, 5, 3, 0) \in S_9^Q$. Here $\rho_8 = 7$ and $\rho_0 = 0$. The Q -chain of subsets produced from ρ is $B_1 = \{7, 6, 4, 2, 1\} \subset B_2 = \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$. Here we have $H_1 = \{7, 6, 4, 2, 1\}$ and $H_2 = \{8, 5, 3, 0\}$.

Example 2.2. Fix $n = 9$. Let λ be the 9-partition $(4, 4, 2, 2, 1, 1, 1, 0, 0)$. We have $Q := Q(\lambda) = \{2, 4, 7\}$. Since Q contains $k = 3$ integers, each $\rho \in S_{10}^Q$ has four cohorts. Let $\rho = (5, 1; 8, 4; 9, 3, 0; 7, 6, 2)$; here $\rho \in S_{10}^Q$ and from ρ we produce the Q -chain of subsets

$$\{5, 1\} \subset \{8, 5, 4, 1\} \subset \{9, 8, 5, 4, 3, 1, 0\} \subset \{9, 8, 7, 6, 5, 4, 3, 2, 1, 0\}.$$

We then have, for example, $B_3(2) = 8$.

2.3 Inversions of an ordered Q -partition

An *inversion* of an ordered Q -partition ρ is defined to be an ordered pair (ρ_i, ρ_j) such that $i > j$ and $\rho_i < \rho_j$. That is, an inversion is a pair of values in the $(n + 1)$ -tuple ρ which increase from left to right. Note that no inversion can consist of a pair of values in the same cohort since the values in each cohort decrease from left to right. Denote the set of inversions of ρ by $\Phi(\rho)$. The right hand side of the generating function identity in Theorem 3.4 is a product over the set of inversions of an ordered Q -partition ρ .

Example 2.3. Fix $n = 8$. As in Example 2.1, let $Q = \{5\}$ and $\rho = (7, 6, 4, 2, 1; 8, 5, 3, 0)$. The set of inversions of ρ is

$$\Phi(\rho) = \{(7, 8), (6, 8), (4, 8), (2, 8), (1, 8), (4, 5), (2, 5), (1, 5), (2, 3), (1, 3)\}.$$

Example 2.4. Fix $n = 9$. As in Example 2.2, let $Q = \{2, 4, 7\}$ and $\rho = (5, 1; 8, 4; 9, 3, 0; 7, 6, 2)$.

Here we have

$$\begin{aligned} \Phi(\rho) = & \{(5, 7), (1, 7), (4, 7), (3, 7), (0, 7), (5, 6), (1, 6), (4, 6), (3, 6), (0, 6), \\ & (1, 2), (0, 2), (5, 9), (1, 9), (8, 9), (4, 9), (1, 3), (5, 8), (1, 8), (1, 4)\}. \end{aligned}$$

Let z_1, z_2, \dots, z_n be variables. We define the *weight monomial of an inversion* (ρ_i, ρ_j) to be $z_{\rho_i+1}z_{\rho_i+2}\dots z_{\rho_j} =: z^{(\rho_i, \rho_j)}$.

Given $\rho \in S_{n+1}^Q$, in Section 4.9 we decompose the set of inversions of ρ into $k = |Q|$ disjoint subsets. This decomposition will form a crucial step in the bijective proof of Theorem 3.4.

2.4 Reverse semistandard tableaux

Fix an n -partition λ . A *reverse $(n+1)$ -semistandard tableau* T on the shape λ is a filling of the shape λ with elements from $[0, n]$ such that the values $T(i, j)$ satisfy $T(i, j) \geq T(i, j+1)$ and $T(i, j) > T(i+1, j)$ whenever both values are defined. Denote the shape of T by $\text{shape}(T)$. Let \mathcal{T}_λ denote the set of all reverse $(n+1)$ -semistandard tableaux on the shape λ . We refer to a reverse $(n+1)$ -semistandard tableau simply as a tableau. For $1 \leq j \leq \lambda_1$, let T_j denote the j^{th} column of T . If the shape λ consists of only one column, then we refer to a tableau on the shape λ as a *column*. If $\lambda = (0, 0, \dots, 0)$ is the n -partition consisting of all 0's, then we refer to the shape λ as the *empty shape*, denoted ϕ . This shape ϕ does not contain any boxes. There is a unique tableau on the shape ϕ which we call the *null tableau*. Since the empty shape contains no boxes, the null tableau contains no values. For $T, U \in \mathcal{T}_\lambda$, write $T \leq U$ if $T(i, j) \leq U(i, j)$ for all $(i, j) \in \lambda$. A tableau is a *key* if all the values in a column also appear in every column to the left of that column.

As noted earlier, the n -partition λ determines a subset $Q \subseteq [n]$ and an integer $k := |Q|$. Now fix an ordered Q -partition $\rho \in S_{n+1}^Q$. This determines the Q -chain of subsets $B_1 \subset B_2 \subset \dots \subset B_{k+1}$ for ρ . The λ -*key* of ρ , denoted $Y_\lambda(\rho)$, is the reverse semistandard tableau on the shape λ defined as follows: for $1 \leq r \leq k$, each column of length q_r is obtained by

placing the values of B_r in descending order from top to bottom. We denote the value in $Y_\lambda(\rho)$ in position (i, j) by $Y_\lambda(\rho; i, j)$. So $Y_\lambda(\rho; i, j) = B_r(i)$ if $\zeta_j = q_r$. Note that any two columns of $Y_\lambda(\rho)$ of the same length are equal. For $1 \leq r \leq k$, we call a column of length q_r the ρ -minimal column of length q_r if it appears as one of these equal columns in $Y_\lambda(\rho)$ of length q_r . Denote this ρ -minimal column of length q_r by $Y^{(r)}(\rho)$. Note that the number of columns in λ of length q_r is equal to $\lambda_{q_{r+1}} - \lambda_{q_r}$ for $1 \leq r \leq k$. So $Y_\lambda(\rho)$ contains $\lambda_{q_r} - \lambda_{q_{r+1}}$ columns equal to $Y^{(r)}(\rho)$ for $1 \leq r \leq k$.

Example 2.5. Fix $n = 9$. Let $\lambda = (4, 4, 2, 2, 1, 1, 1, 0, 0)$ and $\rho = (5, 1; 8, 4; 9, 3, 0; 7, 6, 2) \in S_{10}^Q$ as in Example 2.2. Here $Q = \{2, 4, 7\}$ and

$$Y_\lambda(\rho) = \begin{array}{cccc} \boxed{9} & \boxed{8} & \boxed{5} & \boxed{5} \\ \boxed{8} & \boxed{5} & \boxed{1} & \boxed{1} \\ \boxed{5} & \boxed{4} & & \\ \boxed{4} & \boxed{1} & & \\ \boxed{3} & & & \\ \boxed{1} & & & \\ \boxed{0} & & & \end{array} .$$

The three ρ -minimal columns in this case are

$$Y^{(3)}(\rho) = \begin{array}{c} \boxed{9} \\ \boxed{8} \\ \boxed{5} \\ \boxed{4} \\ \boxed{3} \\ \boxed{1} \\ \boxed{0} \end{array}, Y^{(2)}(\rho) = \begin{array}{c} \boxed{8} \\ \boxed{5} \\ \boxed{4} \\ \boxed{1} \end{array}, Y^{(1)}(\rho) = \begin{array}{c} \boxed{5} \\ \boxed{1} \end{array}.$$

2.5 Tableau scanning method

Fix an n -partition λ . Let $T \in \mathcal{T}_\lambda$. In this section we construct the *scanning tableau* $S(T)$ of T using the scanning method developed by Willis [Wi]. The tableau scanning method is defined in [Wi] in terms of semistandard tableaux. However, here we state its analog for reverse semistandard tableaux.

We need the following preliminary definition: Given a sequence (s_1, s_2, s_3, \dots) , its *earliest weakly decreasing subsequence* (EWDS) is $(s_{i_1}, s_{i_2}, s_{i_3}, \dots)$, where $i_1 = 1$ and for $j > 1$ the

index i_j is the smallest index such that $s_{i_{j-1}} \geq s_{i_j}$.

Given $T \in \mathcal{T}_\lambda$, construct the scanning tableau $S(T)$ as follows: Draw an empty Young diagram of the shape λ . Viewing the bottom values of the columns of T from left to right as forming a sequence, find the EWDS of this sequence. Whenever a value is added to the EWDS, put a dot above it. When this EWDS ends, write its last member in the diagram for the scanning tableau of T in the lowest available box in the leftmost available column. Repeat the process as if the boxes with the dots and the values in them are no longer a part of T . Once every box in T has a dot, the leftmost column of the right key has been formed. To find the values of the next column in the right key: ignore the leftmost column of T , erase the dots in the remaining boxes, and repeat the above process. Continue in this manner until the Young diagram has been filled with values; this is the scanning tableau $S(T)$ of T .

The *right key* of T , denoted $R(T)$, is a special key tableau that is defined in e.g. Section 3 of [RS]. By the main result Theorem 2.5.5 of [Wi], we know that $R(T) = S(T)$. Thus the scanning method gives a direct description of $R(T)$, and in this thesis we calculate the right key of T using only the scanning method. We denote the value in $R(T)$ at position (i, j) by $R(T; i, j)$. It is a known result, stated in Corollary 3.4 of [PW], that $T \geq R(T) = S(T)$.

2.6 Demazure tableaux

Demazure tableaux are defined in [PW] for n -semistandard tableaux. Here a reverse $(n + 1)$ -semistandard tableau T on the shape λ is defined to be a *Demazure tableau* for ρ if $R(T) \geq Y_\lambda(\rho)$. The set of Demazure tableaux for ρ on the shape λ is denoted $\mathcal{D}_\lambda(\rho)$.

Example 2.6. Fix $n = 9$. As in Example 2.5, let $\lambda = (4, 4, 2, 2, 1, 1, 1, 0, 0)$. The following

are two tableaux on the shape λ :

$$T = \begin{array}{|c|c|c|c|} \hline 9 & 9 & 8 & 6 \\ \hline 8 & 6 & 2 & 1 \\ \hline 6 & 5 & & \\ \hline 5 & 2 & & \\ \hline 3 & & & \\ \hline 2 & & & \\ \hline 1 & & & \\ \hline \end{array}, \quad U = \begin{array}{|c|c|c|c|} \hline 9 & 8 & 4 & 3 \\ \hline 8 & 4 & 2 & 1 \\ \hline 6 & 3 & & \\ \hline 4 & 1 & & \\ \hline 3 & & & \\ \hline 2 & & & \\ \hline 0 & & & \\ \hline \end{array}.$$

We use the scanning method to construct the right keys of these tableaux:

$$R(T) = \begin{array}{|c|c|c|c|} \hline 9 & 8 & 6 & 6 \\ \hline 8 & 6 & 1 & 1 \\ \hline 6 & 5 & & \\ \hline 5 & 1 & & \\ \hline 3 & & & \\ \hline 2 & & & \\ \hline 1 & & & \\ \hline \end{array}, \quad R(U) = \begin{array}{|c|c|c|c|} \hline 9 & 8 & 3 & 3 \\ \hline 8 & 3 & 1 & 1 \\ \hline 6 & 2 & & \\ \hline 3 & 1 & & \\ \hline 2 & & & \\ \hline 1 & & & \\ \hline 0 & & & \\ \hline \end{array}.$$

We now consider an ordered Q -partition ρ and determine if $\mathcal{D}_\lambda(\rho)$ contains T and/or U from Example 2.6:

Example 2.7. Fix $n = 9$ and λ as in Example 2.6. Let $\rho = (5, 1; 8, 4; 9, 3, 0; 7, 6, 2) \in S_{10}^Q$. The λ -key of ρ was constructed in Example 2.5. For all $(i, j) \in \lambda$, we have $R(T; i, j) \geq Y_\lambda(\rho; i, j)$. Hence T is a Demazure tableau for ρ of shape λ . The tableau U is not a Demazure tableau for ρ since, for instance, we have $R(U; 1, 3) = 3 < 5 = Y_\lambda(\rho; 1, 3)$.

Calculating the set of Demazure tableaux is particularly simple when λ consists of just one column:

Example 2.8. Fix $n = 8$ and $\lambda = (1, 1, 1, 1, 1, 0, 0, 0, 0)$. Let $\rho = (7, 6, 4, 2, 1; 8, 5, 3, 0) \in S_9^Q$. Then the λ -key of ρ is the following:

$$Y_\lambda(\rho) = \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}.$$

We want to determine the set of Demazure tableaux for this λ and ρ . To do so, we begin

by calculating the right keys of the tableaux $T \in \mathcal{T}_\lambda$ via the scanning method. However, it can be seen that $R(T) = T$ for any column tableau. Hence with this particular λ , we have $R(T) \geq Y_\lambda(\rho)$ if and only if $T \geq Y_\lambda(\rho)$. So the set of Demazure tableaux $\mathcal{D}_\lambda(\rho)$ here is the set of tableau on the shape λ with values from $[0, 8]$ which are entrywise greater than $Y_\lambda(\rho)$.

The entrywise smallest such tableau is

$$\begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}.$$

The entrywise largest such tableau is

$$\begin{array}{|c|} \hline 9 \\ \hline 8 \\ \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline \end{array}.$$

2.7 Weight monomials and ρ -weight monomials

Fix an n -partition λ and an ordered Q -partition ρ . In this section we define weight monomials for reverse semistandard tableaux and ρ -weight monomials for Demazure tableaux. Let x_0, x_1, \dots, x_n be variables. Given $T \in \mathcal{T}_\lambda$, its *weight monomial* is defined to be $x^T := \prod_{i=0}^n x_i^{c_i}$, where c_i is the number of times the value i occurs in T . This is the traditional weight monomial of a tableau.

Example 2.9. Let T be the tableau in Example 2.6. Here we have $x^T = x_9^2 x_8^2 x_6^3 x_5^2 x_3 x_2^3 x_1^2$.

Let λ be an n -partition and $\rho \in S_{n+1}^Q$ be an ordered Q -partition. A certain *key polynomial* $k_{\rho, \lambda}(x)$ is defined in [PW] and [RS] by applying a sequence of divided difference operators to the weight monomial of the tableau on λ with the largest possible values. According to Theorem 1 of [RS] we have

Theorem 2.10.

$$k_{\lambda, \rho}(x) = \sum_{T \in \mathcal{D}_\lambda(\rho)} x^T.$$

We define the *Demazure polynomial* for λ and ρ to be $\lceil_\lambda(\rho; x) := \sum_{T \in \mathcal{D}_\lambda(\rho)} x^T$.

Introduce variables z_1, z_2, \dots, z_n by $z_i := x_{i-1}^{-1} x_i$. Given $T \in \mathcal{D}_\lambda(\rho)$, we define its ρ -*weight monomial* to be $z^T := (x^{Y_\lambda(\rho)})^{-1} x^T$. We write the ρ -weight monomials in terms on the variables z_i .

Note that the ρ -weight monomial of T depends on ρ and T , while the weight monomial depends only on T . Further, the adjusted weight monomial is only defined for Demazure tableaux.

We now calculate the ρ -weight monomial of the tableau T from Example 2.6:

Example 2.11. Fix $n = 9$. As in Example 2.5, let $\lambda = (4, 4, 2, 2, 1, 1, 1, 0, 0)$ and $\rho = (5, 1; 8, 4; 9, 3, 0; 7, 6, 2) \in S_{10}^Q$. Let T be the tableau in Example 2.6. The ρ -weight monomial of T is given by

$$z^T = \frac{x^T}{x^{Y_\lambda(\rho)}} = \frac{x_9^2 x_8^2 x_6^3 x_5^2 x_3 x_2^3 x_1^2}{x_9 x_8^2 x_5^4 x_4^2 x_3 x_1^4 x_0} = \frac{x_9 x_6^3 x_2^3}{x_5^2 x_4^2 x_1^2 x_0} = z_1 z_2^3 z_5^2 z_6^4 z_7 z_8 z_9.$$

Given λ and ρ , we define the *adjusted Demazure polynomial* to be $\mathcal{D}_\lambda(\rho; z) := \sum_{T \in \mathcal{D}_\lambda(\rho)} z^T$.

We then obviously have the relationship $\mathcal{D}_\lambda(\rho; z) = (x^{Y_\lambda(\rho)})^{-1} \lceil_\lambda(\rho; x)$. It will be seen in Lemma 2.12 that the adjusted Demazure polynomials are actually polynomials in the z_i variables.

Fix $T \in \mathcal{T}_\lambda$. Let C be a column of T of length q_r for some $1 \leq r \leq k$. The ρ -weight monomial of C is $z^C := (x^{Y^{(r)}(\rho)})^{-1} x^C$. Note that $z^T = \prod_C z^C$, where the product is over the columns C of the tableau T . We use this method to calculate the ρ -weight monomials of the ‘‘labelling tableaux’’ later in this chapter. Calculating the ρ -weight monomials of Demazure tableaux in this way, we see that ρ -minimal columns play a special role in the ρ -weight calculation:

Lemma 2.12. *Fix an n -partition λ and an ordered Q -partition $\rho \in S_{n+1}^Q$. Let $T \in \mathcal{D}_\lambda(\rho)$. Then a column C of T contributes at least one factor z_t for some $1 \leq t \leq n$ to z^T if and only if C is not a ρ -minimal column. Further, all of the z_t variables in the monomial z^T*

have nonnegative exponents.

Proof. Suppose C is a column of T of length q_r which is not equal to $Y^{(r)}(\rho)$. For $1 \leq i \leq q_r$, let $C(i)$ and $Y^{(r)}(\rho; i)$ denote the values in row i of columns C and $Y^{(r)}(\rho)$ respectively. We can calculate the ρ -weight of the column C as a product over the individual boxes of C as follows:

$$z^C = \left(x^{Y^{(r)}(\rho)}\right)^{-1} x^C = \prod_{i=1}^{q_r} \left(x_{Y^{(r)}(\rho; i)}\right)^{-1} x_{C(i)}.$$

Recall that we have $T \geq R(T) \geq Y_\lambda(\rho)$. Then the column C must satisfy $C(i) \geq Y^{(r)}(\rho; i)$ for all $1 \leq i \leq q_r$. Since $C \neq Y^{(r)}(\rho)$, for some $1 \leq i \leq q_r$, we must have $C(i) > Y^{(r)}(\rho; i)$. This value $C(i)$ contributes $\left(x_{Y^{(r)}(\rho; i)}\right)^{-1} x_{C(i)} = z_{Y^{(r)}(\rho; i)+1} z_{Y^{(r)}(\rho; i)+2} \cdots z_{C(i)}$ to the ρ -weight of C . Also, since $C(i) \geq Y^{(r)}(\rho; i)$ for all $1 \leq i \leq q_r$, no value of C contributes a negative power of any z_t variable for $1 \leq t \leq n$. Hence a column which is not a ρ -minimal column contributes at least one z_t factor to the ρ -weight monomial of C . On the other hand, clearly any ρ -minimal column contributes only a factor of 1 to the ρ -weight monomial of T . Since every column contributes a nonnegative factor of some z_i variable, clearly z^T is a product of the z_i variables with nonnegative exponents. ■

2.8 Weighted limit of a direct system

A set of *weighted* combinatorial objects is a finite set B of combinatorial objects equipped with a function $wt : B \rightarrow \mathbb{N}^n$. Given $b \in B$ we refer to $wt(b)$ as the *weight of b* . Suppose $wt(b) = (t_1, t_2, \dots, t_n)$. Define the *weight monomial of b* to be $z^b := z^{wt(b)} := z_1^{t_1} z_2^{t_2} \cdots z_n^{t_n}$. We define $|wt(b)| := t_1 + t_2 + \cdots + t_n$. The generating function for B is defined to be $F_B(z) := \sum_{b \in B} z^b$. This polynomial is an element of $\mathbb{C}[[z_1, z_2, \dots, z_n]]$, the ring of formal power series in the variables z_1, z_2, \dots, z_n .

Let $\{A_m\}_{m \geq 1}$ be a family of finite sets. For $1 \leq i \leq j$, let $\phi_{i,j} : A_i \rightarrow A_j$ be injective maps such that for $1 \leq i \leq j \leq p$, the maps satisfy $\phi_{j,p} \circ \phi_{i,j} = \phi_{i,p}$. Following [AM, Exercise 2.14] (but also assuming injectivity), we call $(\{A_m\}, \{\phi_{i,j}\})$ a *direct system* of sets. Note that injectivity and the composition rule imply that $\phi_{i,i}(a) = a$ for all $i \geq 1$ and $a \in A_i$.

Define a relation \sim on the elements of the set $A := \bigsqcup_{m \geq 1} A_m$ as follows: Let $a, b \in A$. Define $a \sim b$ if there exists $1 \leq i \leq j$ such that $a \in A_i$ and $b \in A_j$ and $\phi_{i,j}(a) = b$ or $b \in A_i$ and $a \in A_j$ and $\phi_{i,j}(b) = a$. This is an equivalence relation on A . We define the *limit of the direct system* $(\{A_m\}, \{\phi_{i,j}\})$ to be the set A/\sim of equivalence classes E of \sim .

Let $E \in A/\sim$. It can be seen that E is a set of elements from consecutive sets A_m : Injectivity implies that we can write $E = \{a_t, a_{t+1}, a_{t+2}, \dots\}$ for some $t \geq 1$ such that $a_i \in A_i$ for $i \geq t$.

Now suppose the sets A_m in the direct system $(\{A_m\}, \{\phi_{i,j}\})$ are equipped with weight functions $wt : A_m \rightarrow \mathbb{N}^n$. For each $\nu \in \mathbb{N}^n$, denote by $A_{m,\nu}$ the subset of A_m of elements of weight ν . We refer to this structure $(\{A_m\}, \{\phi_{i,j}\}, wt)$ as a *weighted direct system* of sets. Let $E \in A/\sim$. We say that E is a *tame class* if there exists a weight $\nu \in \mathbb{N}^n$ and $M_E \geq 1$ such that for all $i \geq M_E$, we have $wt(a_i) = \nu$. In this case we define $wt(E) := \nu$ and $z^E := z^\nu$. We define the *weighted limit* of the weighted direct system $(\{A_m\}, \{\phi_{i,j}\}, wt)$ to be the subset of tame classes in A/\sim . We denote this weighted limit by $\lim_{m \rightarrow \infty} (A_m, \phi, wt)$. We call a weighted limit of a weighted direct system *stable* if for every weight $\nu \in \mathbb{N}^n$, there exists $f_\nu \geq 0$ and $M_\nu \geq 1$ such that the number of tame classes of weight ν in the weighted limit is equal to f_ν and for $m \geq M_\nu$, we have $|A_{m,\nu}| = f_\nu$. We define the generating function of a stable weighted limit of a weighted direct system $(\{A_m\}, \{\phi_{i,j}\}, wt)$ to be

$$F \lim_{m \rightarrow \infty} (A_m, \phi, wt)(z) := \sum_{E \in \lim_{m \rightarrow \infty} (A_m, \phi, wt)} z^E.$$

Given a formal power series $F(z)$ and $\nu \in \mathbb{N}^n$, define $\langle F(z), \nu \rangle$ to be the coefficient of z^ν in $F(z)$. Note that for a set A_m of weighted combinatorial objects, we have $\langle F_{A_m}(z), \nu \rangle = |A_{m,\nu}|$. A sequence of multivariate polynomials $\{F_{A_m}(z)\}_{m \geq 1}$ is said to *converge* to a formal power series $F(z)$ as $m \rightarrow \infty$ if for each $\nu \in \mathbb{N}^n$ there exists L_ν such that for $m \geq L_\nu$, we have $\langle F_{A_m}(z), \nu \rangle = \langle F(z), \nu \rangle$. Then this limit is denoted $\lim_{m \rightarrow \infty} F_{A_m}(z)$.

We have the following:

Lemma 2.13. *If the weighted limit as $m \rightarrow \infty$ of the weighted direct system $(\{A_m\}, \{\phi_{i,j}\}, wt)$ is stable, then*

$$\lim_{m \rightarrow \infty} F_{A_m}(z) = F \lim_{m \rightarrow \infty} (A_m, \phi, wt)(z),$$

where the limit on the left hand side is found in the ring of formal power series.

Proof. Fix some $\nu \in \mathbb{N}^n$. Since the weighted limit is stable there exists $f_\nu \geq 0$ such that the number of tame classes of weight ν in the weighted limit is equal to f_ν . That is, we have $\langle F \lim_{m \rightarrow \infty} (A_m, \phi, wt)(z), \nu \rangle = f_\nu$. Also from the stability of the weighted limit, there exists an $M_\nu \geq 1$ such that for $m \geq M_\nu$, we have $\langle F_{A_m}(z), \nu \rangle = |A_{m,\nu}| = f_\nu$. Now let $L_\nu := M_\nu$. Then clearly for $m \geq L_\nu$, we have $\langle F \lim_{m \rightarrow \infty} (A_m, \phi, wt)(z), \nu \rangle = \langle F_{A_m}(z), \nu \rangle$. Thus the polynomials $F_{A_m}(z)$ converge to the formal power series $F \lim_{m \rightarrow \infty} (A_m, \phi, wt)(z)$. ■

From this lemma, we know that if the weighted limit of a weighted direct system is stable, then the limit of the generating functions of the sets exists in the ring of formal power series. Once we know the weighted limit is stable, it may be useful to label the equivalence classes of the weighted limit with objects from a (not necessarily finite) set \mathcal{B} of weighted combinatorial objects. We do so by defining a map $\Psi : \lim_{m \rightarrow \infty} (A_m, \phi, wt) \rightarrow \mathcal{B}$. We call this ordered pair (Ψ, \mathcal{B}) a labelling of $\lim_{m \rightarrow \infty} (A_m, \phi, wt)$ if Ψ is injective and $wt(\Psi(E)) = wt(E)$ for all equivalence classes $E \in \lim_{m \rightarrow \infty} (A_m, \phi, wt)$. Elements of $\Psi(\lim_{m \rightarrow \infty} (A_m, \phi, wt))$ are called the labels of the equivalence classes of the weighted limit.

2.9 Labelling tableaux

Re-fix an n -partition λ and an ordered Q -partition ρ . For $1 \leq r \leq k$, we define g_r to be the number of columns of length q_r in the shape λ . Note that we have $g_r = \lambda_{q_r} - \lambda_{q_r+1}$.

In Chapter 3 we consider the sets of weighted combinatorial objects $A_m = \mathcal{D}_{m\lambda}(\rho)$ for $m \geq 1$. We define $wt : \mathcal{D}_{m\lambda}(\rho) \rightarrow \mathbb{N}^n$ to be the ρ -weight of a Demazure tableau for ρ . We equip these sets with maps $\gamma_{i,j}$ defined as follows: Let $1 \leq i \leq j$ and let $T \in \mathcal{D}_{i\lambda}(\rho)$. To construct $\gamma_{i,j}(T)$, for each $1 \leq r \leq k$ insert $(j-i)g_r$ duplicates of the rightmost column of length q_r into T to the right of that column to fill the shape $j\lambda$. These maps are often

not weight preserving. However, we can see that these maps are injective and that they satisfy the direct system requirement. Moreover, successive applications of the first part of the following lemma imply that $\gamma_{i,j}(T) \in \mathcal{D}_{j\lambda}(\rho)$:

Lemma 2.14. *Fix a tableau $U \in \mathcal{T}_\lambda$ and an integer $1 \leq j \leq \lambda_1$. Define S to be the tableau obtained from U by inserting a duplicate of column U_j into U directly to the right of column U_j . Then $U \in \mathcal{D}_\lambda(\rho)$ if and only if $S \in \mathcal{D}_{\text{shape}(S)}(\rho)$. This implies that $(\{\mathcal{D}_{m\lambda}(\rho)\}, \{\gamma_{i,j}\}, wt)$ is a weighted direct system.*

Proof. First note that when we insert a duplicate of a column directly to its right, the resulting tableau is still a reverse semistandard tableau. Let $1 \leq l \leq j$ be an integer. Then S_l is a column in S which appears weakly to the left of column S_j . Consider the scanning paths starting with locations in column S_l . Since $S_j = S_{j+1}$ for all $1 \leq i \leq \zeta_j$, the scanning path starting at a location in column S_l contains the location (i, j) if and only if it contains the location $(i, j+1)$. Thus it is clear that the l^{th} column of the right key of S is equal to the l^{th} column of the right key of U . That is, $R(S)_l = R(U)_l$ for $1 \leq l \leq j$. Further, inserting the extra column U_j into U does not affect the columns to the right of the column U_j when calculating scanning paths. Thus we also have $R(S)_{l+1} = R(U)_l$ for $j \leq l \leq \lambda_1$. Now for all $1 \leq r \leq k$, every column of length q_r in $R(U)$ is equal to every column of length q_r in $R(S)$. Hence $U \in \mathcal{D}_\lambda(\rho)$ if and only if $S \in \mathcal{D}_{\text{shape}(S)}(\rho)$.

Let $1 \leq i \leq j \leq p$ and let $T \in \mathcal{D}_{i\lambda}(\rho)$. Then the tableaux $\gamma_{j,p} \circ \gamma_{i,j}(T)$ and $\gamma_{i,p}(T)$ are both formed by duplicating the rightmost columns of each length in the tableau T to fill the shape $p\lambda$. Hence we have $\phi_{j,p} \circ \phi_{i,j} = \phi_{i,p}$. Thus $(\{\mathcal{D}_{m\lambda}(\rho)\}, \{\gamma_{i,j}\}, wt)$ is a weighted direct system. ■

Here and in Chapter 3, we consider the weighted direct system $(\{\mathcal{D}_{m\lambda}(\rho)\}, \{\gamma_{i,j}\}, wt)$. In the first lemma of Chapter 3, we show that $\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)$ is stable. After we show that the weighted limit is stable, we label the equivalence classes with certain tableaux in Lemma 3.2. Then in Theorem 3.3, we provide a simpler description for the set of tableaux of Lemma 3.2. We now present this simpler description:

Fix a subset $Q \subseteq [n]$ and $\rho \in S_{n+1}^Q$. We define a new set of tableaux, denoted $\mathcal{L}_Q(\rho)$. A

tableau T with values from $[0, n]$ is defined to be an element of $\mathcal{L}_Q(\rho)$ if the following are true:

1. the set of distinct column lengths of T is equal to Q .
2. T contains exactly one ρ -minimal column of length q_r for each $1 \leq r \leq k$.
3. For all $1 \leq r \leq k - 1$, every value in a column of length q_r is a value in the ρ -minimal column of length q_{r+1} .

In anticipation of Theorem 3.3, we call each tableau $T \in \mathcal{L}_Q(\rho)$ a *labelling tableau*.

Note that these tableaux depend only on the set Q of distinct column lengths of λ and not on λ itself.

We will consider the generating function for the labelling tableaux in which the weight monomial of a labelling tableau is defined to be its ρ -weight monomial. We define the ρ -weight monomial of a labelling tableau column by column. Let C be a column of length q_r of a labelling tableau T . Then

$$z^C := (x^{Y^{(r)}(\rho)})^{-1} x^C$$

We then define the ρ -weight monomial of T to be $z^T := \prod_C z^C$ where the product is over all the columns of T . Consider the following labelling tableau ρ -weight monomial:

Example 2.15. Fix $n = 9$. As in Example 2.5, let $Q = \{2, 4, 7\}$ and $\rho = (5, 1; 8, 4; 9, 3, 0; 7, 6, 2) \in S_{10}^Q$. One example of a labelling tableau for this Q and ρ is the following:

$$T = \begin{array}{cccccc} \boxed{9} & \boxed{9} & \boxed{9} & \boxed{8} & \boxed{8} & \boxed{5} & \boxed{5} \\ \boxed{8} & \boxed{8} & \boxed{8} & \boxed{5} & \boxed{5} & \boxed{4} & \boxed{1} \\ \boxed{7} & \boxed{5} & \boxed{5} & \boxed{4} & \boxed{4} & & \\ \boxed{5} & \boxed{4} & \boxed{3} & \boxed{3} & \boxed{1} & & \\ \boxed{3} & \boxed{3} & & & & & \\ \boxed{2} & \boxed{1} & & & & & \\ \boxed{0} & \boxed{0} & & & & & \end{array} .$$

Then

$$Y_{\text{shape}(T)}(\rho) = \begin{array}{|c|c|c|c|c|c|c|} \hline 9 & 9 & 8 & 8 & 8 & 5 & 5 \\ \hline 8 & 8 & 5 & 5 & 5 & 1 & 1 \\ \hline 5 & 5 & 4 & 4 & 4 & & \\ \hline 4 & 4 & 1 & 1 & 1 & & \\ \hline 3 & 3 & & & & & \\ \hline 1 & 1 & & & & & \\ \hline 0 & 0 & & & & & \\ \hline \end{array} .$$

We calculate the ρ -weight monomial of this tableau column by column. The first column contributes the the following to the ρ -weight monomial: $\frac{x_9 x_8 x_7 x_5 x_3 x_2 x_0}{x_9 x_8 x_5 x_4 x_3 x_1 x_0} = z_2 z_5 z_6 z_7$. Here the denominator is the weight monomial of the ρ -minimal column of length 7. After calculating the ρ -weights for the other columns and multiplying, the ρ -weight monomial of this labelling tableau is $z_2^4 z_3^3 z_4 z_5^2 z_6^2 z_7^2 z_8 z_9$.

3 Main results

In this chapter we present the main results of this thesis. In our first main result, Theorem 3.3, we show that we can label the equivalence classes of the weighted limit of the weighted direct system $(\{\mathcal{D}_{m\lambda}(\rho)\}, \{\gamma_{i,j}\}, wt)$ defined in Section 2.9 with the labelling tableaux. In our second main result, Theorem 3.4, we give two product identities for a multivariate generating function for the set of labelling tableaux. We give bijective proofs of both identities in the next chapter. We present an alternate proof of the first identity of Theorem 3.4 in Chapter 5, where we derive it from the Lie theory K-P identity and Theorem 3.3. Our third main result, Theorem 3.7, states that Theorem 3.4 is a combinatorial translation of the Lie theoretic Kumar-Peterson identity.

Throughout this chapter, fix an n -partition λ and an ordered Q -partition $\rho \in S_{n+1}^Q$.

3.1 Preliminary results

In our first lemma, we apply the definition of the weighted limit to the weighted direct system $(\{\mathcal{D}_{m\lambda}(\rho)\}, \{\gamma_{i,j}\}, wt)$ defined in Section 2.9. Recall that the maps $\gamma_{i,j}$ duplicate the rightmost column of each column length to fill the shape $j\lambda$. We have the following:

Lemma 3.1. *The weighted limit $\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)$ is stable.*

Proof. Fix $\nu = (t_1, t_2, \dots, t_n) \in \mathbb{N}^n$. For $m \geq 1$, let $\mathcal{D}_{m\lambda}(\rho)_\nu$ denote the subset of $\mathcal{D}_{m\lambda}(\rho)$ of tableaux T such that $wt(T) = \nu$. For $1 \leq i \leq j$, define $\gamma_{i,j,\nu}$ to be the map $\gamma_{i,j}$ restricted to the set $\mathcal{D}_{i\lambda}(\rho)_\nu$. Note that these maps are injective since the maps $\gamma_{i,j}$ are injective.

We will first show that there exists $M_\nu \geq 1$ and $f_\nu \geq 0$ such that for each $m \geq M_\nu$, we have $|\mathcal{D}_{m\lambda}(\rho)_\nu| = f_\nu$. We will then show that f_ν is the number of tame classes of weight ν of the weighted limit.

Define $M_\nu := |\nu| + 1 = t_1 + t_2 + \dots + t_n + 1$. For some $i \geq M_\nu$, let $T \in \mathcal{D}_{i\lambda}(\rho)_\nu$. By Lemma 2.12, every column of T which is not ρ -minimal contributes at least one z_h factor to the ρ -weight monomial of T . Since $z^T = z^\nu = z_1^{t_1} \dots z_n^{t_n}$, the tableau T must have at most $|\nu| = t_1 + t_2 + \dots + t_n$ columns which are not ρ -minimal. Since T is a tableau on the shape $i\lambda$ with $i \geq M_\nu$, the tableau T has at least $M_\nu = |\nu| + 1$ columns of length q_r for $1 \leq r \leq k$. Hence for $1 \leq r \leq k$, the tableau T contains at least one ρ -minimal column of length q_r . Since these ρ -minimal columns are the entrywise smallest columns for a given length, they must appear as the rightmost columns of that length.

Let $M_\nu \leq i \leq j$. Since for every $T \in \mathcal{D}_{i\lambda}(\rho)_\nu$ and all $1 \leq r \leq k$, the rightmost column of T of length q_r is a ρ -minimal column, the maps $\gamma_{i,j,\nu}$ only insert duplicates of ρ -minimal columns. Thus by Lemma 2.12, the maps $\gamma_{i,j,\nu}$ are weight preserving. That is, $\gamma_{i,j,\nu}(\mathcal{D}_{i\lambda}(\rho)_\nu) \subseteq \mathcal{D}_{j\lambda}(\rho)_\nu$. Now let $T \in \mathcal{D}_{j\lambda}(\rho)_\nu$. From the preceding paragraph we know that any Demazure tableau with ρ -weight equal to ν has at most $|\nu|$ non- ρ -minimal columns. We define the tableau $\gamma_{i,j,\nu}^{-1}(T)$ to be the tableau obtained from T by deleting the rightmost $(j-i)g_r$ columns of length q_r for $1 \leq r \leq k$. The shape of the resulting tableau is $i\lambda$. After deleting these columns there are ig_r columns of length q_r for $1 \leq r \leq k$. We have $|\nu| < ig_r$, since $|\nu| < M_\nu \leq i$ and $g_r \geq 1$. Since the tableau $\gamma_{i,j,\nu}^{-1}(T)$ must have at most $|\nu|$ non- ρ -minimal columns of any given length, it must contain at least one ρ -minimal column of each length. Hence the map $\gamma_{i,j,\nu}^{-1}$ deleted only ρ -minimal columns, and so it is weight preserving. Thus $\gamma_{i,j,\nu}^{-1}(T) \in \mathcal{D}_{i\lambda}(\rho)_\nu$. It is clear that $\gamma_{i,j,\nu}(\gamma_{i,j,\nu}^{-1}(T)) = T$. Therefore $\gamma_{i,j,\nu}(\mathcal{D}_{i\lambda}(\rho)_\nu) = \mathcal{D}_{j\lambda}(\rho)_\nu$. Hence $|\mathcal{D}_{i\lambda}(\rho)_\nu| = |\mathcal{D}_{j\lambda}(\rho)_\nu|$ since the maps $\gamma_{i,j,\nu}$ are injective. Define $f_\nu = |\mathcal{D}_{M_\nu\lambda}(\rho)_\nu|$. Then for all $m \geq M_\nu$, we have $|\mathcal{D}_{m\lambda}(\rho)_\nu| = f_\nu$, since the maps $\gamma_{M_\nu,m,\nu}$ are bijections from $\mathcal{D}_{M_\nu\lambda}(\rho)_\nu$ to $\mathcal{D}_{m\lambda}(\rho)_\nu$.

Now we need to show that the number of tame classes of weight ν in the weighted limit is equal to f_ν . Consider the set $\mathcal{D}_{M_\nu\lambda}(\rho)_\nu$. Each $T \in \mathcal{D}_{M_\nu\lambda}(\rho)_\nu$ is a member of a distinct equivalence class. Since the maps $\gamma_{M_\nu,m,\nu}$ for $m \geq M_\nu$ are weight preserving injections, each equivalence class that contains a $T \in \mathcal{D}_{M_\nu\lambda}(\rho)_\nu$ is a tame class of weight ν . Hence there are at least f_ν tame classes of weight ν in the weighted limit. Since for all $m \geq M_\nu$, we have

$|\mathcal{D}_{m\lambda}(\rho)_\nu| = f_\nu$, there cannot be more than f_ν tame classes of weight ν in the weighted limit. Hence the number of tame classes of weight ν in the weighted limit is equal to f_ν , and thus the weighted limit is stable. ■

It can be seen from the work in the preceding proof that each tame class of the weighted limit $\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)$ consists of tableaux which differ only by the number of ρ -minimal columns of each length they contain. Now that we know that the weighted limit of our weighted direct system is stable, we describe a labelling for the equivalence classes of the weighted limit. Let \mathcal{T} denote the set of all tableaux on all n -partitions. We have the following:

Lemma 3.2. *There exists a labelling (Ψ, \mathcal{T}) of $\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)$ such that the set of labels $\Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt))$ is equal to the set of tableaux $T \in \mathcal{T}$ such that*

1. *the set of distinct column lengths of T is equal to Q ,*
2. *T contains exactly one ρ -minimal column of length q_r for $1 \leq r \leq k$, and*
3. *$T \in \mathcal{D}_{\text{shape}(T)}(\rho)$.*

Proof. Define $\Psi : \lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt) \rightarrow \mathcal{T}$ as follows: Let E be a tame class of $\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)$ of weight ν , and let $T \in E$. Since E is tame, every rightmost column of a given length in T must be a ρ -minimal column. Define $\Psi(E)$ to be the tableau constructed from T by deleting from T all but one copy of $Y^{(r)}(\rho)$ for all $1 \leq r \leq k$. The map Ψ is well defined on tame classes since each tame class consists of tableaux which differ only by the number of ρ -minimal columns of each length that they contain. By Lemma 2.12, for all equivalence classes E we have $wt(E) = wt(\Psi(E))$, since the map Ψ deletes only ρ -minimal columns.

We next show that Ψ is injective: Let E' be another equivalence class of weight ν and suppose $\Psi(E) = \Psi(E')$. Let $T \in E$ and $S \in E'$. Then deleting all but one ρ -minimal column of each length in T and S yields the same tableau. Without loss of generality, let $i \leq j$ be such that $T \in D_{i\lambda}(\rho)_\nu$ and $S \in D_{j\lambda}(\rho)_\nu$. Clearly we have $\gamma_{i,j,\nu}(T) = S$. Hence $T \sim S$ and $E' = E$. Thus Ψ is injective. Therefore Ψ is a labelling.

It is clear that every $T \in \Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt))$ satisfies criteria 1 and 2 of the lemma. From Lemma 2.14, we see that every $T \in \Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt))$ is a Demazure tableau for ρ on $shape(T)$, and hence satisfies criterion 3.

Conversely, consider any tableau $T \in \mathcal{T}$ which meets the three criteria of Lemma 3.2. Via duplication it generates a tame class E , and it is the label $\Psi(E)$ of that class. Thus we know that the set of labels $\Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt))$ is exactly equal to the set of tableaux which meet the criteria of the lemma. ■

3.2 Main result 1: Labelling tableaux label equivalence classes

The description of the labels in Lemma 3.2 requires that T be a Demazure tableau for ρ on its shape. To determine if a tableau is a Demazure tableau for some ρ , one must calculate its right key. However, we show in the following theorem that this condition can be simplified to the third condition in the definition of the labelling tableaux:

Theorem 3.3. *Fix an n -partition λ and an ordered Q -partition ρ . Then*

$$\Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)) = \mathcal{L}_Q(\rho).$$

Proof. Note that the first two criteria for both sets of tableaux are the same. Let $T \in \mathcal{T}$ satisfy the first two criteria. We need to show that T satisfies the third membership criterion for tableaux in $\Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt))$ if and only if T satisfies the third membership criterion of the labelling tableaux $\mathcal{L}_Q(\rho)$.

First suppose T violates criterion 3 of the labelling tableau definition. Then for some $1 \leq r \leq k$ there exists a value $T(i, j)$ in a column of length q_r of T such that $T(i, j)$ is not a value in the ρ -minimal column $Y^{(r+1)}(\rho)$. We need to show that T cannot be a Demazure tableau for ρ on its shape. To do so, we show that at least one of the values in the ρ -minimal column of length q_{r+1} “is decreased” when the right key of T is calculated. This creates a violation of the Demazure condition, since it requires that every column of $R(T)$ of length q_{r+1} dominates the ρ -minimal column of length q_{r+1} .

We consider the scanning paths which start in the ρ -minimal column of T of length q_{r+1} . No value in this column is equal to $T(i, j)$. Note that every location to the right of the ρ -minimal column is contained in exactly one scanning path starting from that column. First consider scanning paths which originate from locations in this column with values which are less than $T(i, j)$. Since we are finding the EWDS, none of these scanning paths will contain the location (i, j) . So the scanning path which passes through (i, j) must originate in the ρ -minimal column of length q_{r+1} from a location with a value which is larger than $T(i, j)$. The value in the scanning tableau $R(T)$ in the location from which the scanning path originated is now less than or equal to $T(i, j)$. However, the value $T(i, j)$ was smaller than the minimal value for the location from where the scanning path originated. Therefore this value in $R(T)$ is too small, and so T cannot be a Demazure tableau for ρ . Hence if T is a Demazure tableau which satisfies criteria 1 and 2, then for all r with $1 \leq r \leq k - 1$, every value in a column of length q_r is a value in the ρ -minimal column of length q_{r+1} . Thus $\Psi(\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)) \subseteq \mathcal{L}_Q(\rho)$.

Now suppose that T satisfies the third criterion for a labelling tableau. We need to show that T is a Demazure tableau on its shape. Let $1 \leq r \leq k$. First note that since $R(T)$ is a key, there is only one distinct column of each length q_r in $R(T)$ (which is possibly repeated). Thus we only need to calculate one column of $R(T)$ of length q_r and the rest of the columns of this length will be equal to this column. For that reason, let us find the rightmost column of length q_r of the right key of T . Using the tableau scanning method, the values of this column will be a decreasing sequence of integers, each of which comes from a column of T weakly to the right of this column. By hypothesis, all the values in these columns appear in the column $Y^{(r)}(\rho)$. Hence this column cannot change when the right key is being calculated, and so every column of length q_r in $R(T)$ is equal to this column $Y^{(r)}(\rho)$. Thus each column of $R(T)$ is a ρ -minimal column and hence $R(T) = Y_{\text{shape}(T)}(\rho)$. Thus T is a Demazure tableau on its shape. ■

3.3 Main result 2: Our combinatorial K-P identity

Independently of Lie theory, we can state and prove the identity below for a generating function for the labelling tableaux. This is a combinatorial version of the K-P identity; this claim is verified in Chapter 5. Note that this generating function depends only on the set Q and not on the n -partition λ .

Theorem 3.4. *Fix $n \geq 1$, a subset $Q \subseteq [n]$, and an ordered Q -partition $\rho \in S_{n+1}^Q$. Then*

$$\begin{aligned} \sum_{T \in \mathcal{L}_Q(\rho)} z^T &= \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} z_{\rho_i+2} \cdots z_{\rho_j})} \\ &= \prod_{r=1}^k \prod_{(i,j) \in \mu^{(r)}} \frac{1}{1 - z^{\text{hook}(i,j)}}. \end{aligned}$$

The ‘‘Hillman-Grassl boards’’ $\mu^{(r)}$ are defined in terms of Q and ρ in Section 4.6. The hook weights $z^{\text{hook}(i,j)}$ are assigned to their boxes in Section 4.3.

In the following two examples, we use the first identity in Theorem 3.4 to describe the generating function for the set of labelling tableaux for each (n, Q, ρ) triple presented.

Example 3.5. Fix $n = 4$. Let $Q = \{3\}$ and $\rho = (3, 2, 0; 4, 1) \in S_5^Q$. Then

$$\Phi(\rho) = \{(3, 4), (2, 4), (0, 4), (0, 1)\}.$$

So the right hand side of the equation in Theorem 3.4 in this case is

$$\frac{1}{(1 - z_4)(1 - z_3 z_4)(1 - z_1 z_2 z_3 z_4)(1 - z_1)}.$$

The only ρ -minimal column in this case is

$$Y^{(1)}(\rho) = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 0 \\ \hline \end{array}.$$

Thus for the left hand side, we have that $\mathcal{L}_Q(3, 2, 0; 4, 1)$ is the set of tableau T such that

1. T contains only columns of length 3,
2. T has exactly 1 column equal to $\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 0 \\ \hline \end{array}$; necessarily the rightmost column, and
3. the values in T come from the set $[0, 4]$.

Let us consider the monomials on the left hand side that come from labelling tableaux with only 1 or 2 columns. The only labelling tableau with one column is $\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 0 \\ \hline \end{array}$. In the sum on the

left hand side, we have $z^{\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 0 \\ \hline \end{array}} = \frac{x_3 x_2 x_0}{x_3 x_2 x_0} = 1$.

The only labelling tableaux with two columns are

$$\begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 2 & 0 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & 2 \\ \hline 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & 2 \\ \hline 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & 0 \\ \hline \end{array}.$$

For the first tableau in this list, we have $z^{\begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & 2 \\ \hline 2 & 0 \\ \hline \end{array}} = \frac{x_4 x_3^2 x_2^2 x_0}{x_3^2 x_2^2 x_0} = \frac{x_4}{x_0} = z_1 z_2 z_3 z_4$.

Calculating the adjusted weights for all 6 of these tableau gives us respectively

$$z_1 z_2 z_3 z_4, \quad z_1 z_3 z_4, \quad z_3 z_4, \quad z_1 z_4, \quad z_4, \quad z_1.$$

■

We next consider another (n, Q, ρ) triple to illustrate Theorem 3.4.

Example 3.6. Fix $n = 4$. Let $Q = \{1, 3\}$ and $\rho = (3; 4, 0; 2, 1) \in S_5^Q$. Here we have $\Phi(\rho) = \{(0, 2), (0, 1), (3, 4)\}$. Thus the right hand side is equal to

$$\frac{1}{(1 - z_1 z_2)(1 - z_1)(1 - z_4)}.$$

The ρ -minimal column of length 1 is $\boxed{3}$ and the ρ -minimal column of length 3 is $\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 0 \\ \hline \end{array}$. There is only one labelling tableau with 2 columns:

$$\begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & \\ \hline 0 & \\ \hline \end{array}$$

Since both of these columns are ρ -minimal, we have $z^{\begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & \\ \hline 0 & \\ \hline \end{array}} = \frac{x_4 x_3^2 x_0}{x_4 x_3^2 x_0} = 1$.

The only labelling tableaux with exactly 3 columns are

$$\begin{array}{|c|c|c|} \hline 4 & 4 & 3 \\ \hline 3 & 3 & \\ \hline 1 & 0 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & 4 & 3 \\ \hline 3 & 3 & \\ \hline 2 & 0 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & 4 & 3 \\ \hline 3 & & \\ \hline & & \\ \hline \end{array}.$$

Calculating the ρ -weight monomials for these three tableaux gives us respectively

$$z_1, z_1 z_2, z_4.$$

■

3.4 Main Result 3: It is a combinatorial interpretation of the K-P identity

The following statement confirms that Theorem 3.4 is a combinatorial interpretation of the K-P identity:

Theorem 3.7. *Let λ be a dominant integral weight and let $w \in W^\lambda$. Let λ be the n -partition obtained from the weight λ . Let $Q = Q(\lambda) \subseteq [n]$ and let ρ be the ordered Q -partition produced from w . Then Theorem 3.4 is a combinatorial interpretation of the Kumar-Peterson identity; i.e. in the simple root basis coordinatization the Kumar-Peterson identity becomes*

$$\sum_{T \in \mathcal{L}_Q(\rho)} z^T = \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} z_{\rho_i+2} \cdots z_{\rho_j})}.$$

4 Bijective proof of our combinatorial K-P identity

Throughout this chapter, fix a subset $Q \subseteq [n]$ and an ordered Q -partition $\rho \in S_{n+1}^Q$. We define further combinatorial objects for the bijective proof of Theorem 3.4. In Section 4.1, we define reverse plane partitions on a shape μ . In Section 4.2 we describe how to write the set of labelling tableaux $\mathcal{L}_Q(\rho)$ as a Cartesian product of k sets of “labelling subtableaux”. This decomposition allows us to work with just one of the k sets at a time in Sections 4.5 to 4.11. In Section 4.3 we present a template which we will later use to construct “Hillman-Grassl boards”. We also define template “colors” for these boards. Reverse plane partitions on these Hillman-Grassl boards will provide intermediate combinatorial objects which allow us to complete the bijective proof of both identities in Theorem 3.4. In Section 4.4, we define a template map from certain tableaux to reverse plane partitions on a shape μ constructed in Section 4.3. Now that we have defined the template boards and maps, we begin our bijective proof. We give our bijective proof of the second identity in Theorem 3.4 first. To do so, we construct three weight preserving bijections. In Section 4.5, we present a map from labelling subtableaux to “shrunk tableaux” by “shrinking” the values in the subtableaux. In Section 4.6 we define the Hillman-Grassl boards $\mu^{(r)}$ using the procedure from Section 4.3. We then define a map from the shrunk tableaux to reverse plane partitions on the Hillman-Grassl board $\mu^{(r)}$ in Section 4.6 by using the template map constructed in Section 4.4. In Section 4.7, we cite the Hillman-Grassl algorithm. This algorithm provides a bijection from reverse plane partitions on $\mu^{(r)}$ to multisets of “hooks” of the shape $\mu^{(r)}$. In Section 4.8, we finish the proof of the identity in Theorem 3.4 which describes the generating function of the labelling tableaux as a product over the hooks of the Hillman-Grassl boards. In Section 4.9 we describe how to decompose the set of inversions of ρ in a way which is compatible with the decomposition of tableaux into subtableaux. To complete the proof of the first identity in Theorem 3.4, we construct two additional weight preserving bijections. We will compose

these bijections to form a bijection from the set of hooks on the Hillman-Grassl board $\mu^{(r)}$ to the subset of inversions $\Phi^{(r)}(\rho)$. In Section 4.9, we define a map from multisets of inversions of ρ to multisets of shrunken inversions of ρ . In Section 4.10, we define a bijection from multisets of hooks of $\mu^{(r)}$ to shrunken inversions of ρ . This map induces a bijection from multisets of hooks to multisets of such shrunken inversions. In Section 4.11, we complete the proof of the first identity in Theorem 3.4 by composing the maps defined in Sections 4.9 and 4.10.

4.1 Reverse plane partitions

Let $d \geq 1$ and fix a d -partition μ . A *reverse plane partition* R on the shape μ is a filling of the boxes $(i, j) \in \mu$ with entries $R(i, j) \in \mathbb{N}$ such that $R(i, j) \leq R(i, j + 1)$ and $R(i + 1, j) \leq R(i, j)$ wherever the inequalities are defined. Let $rpp(\mu)$ denote the set of reverse plane partitions on the shape μ . For $R_1, R_2 \in rpp(\mu)$ let $R_1 + R_2$ be the entrywise sum of the two reverse plane partitions. One can see that $rpp(\mu)$ is closed under addition: for the first inequality in the definition of reverse plane partition, we have $(R_1 + R_2)(i, j) = R_1(i, j) + R_2(i, j) \leq R_1(i, j + 1) + R_2(i, j + 1) = (R_1 + R_2)(i, j + 1)$. The second inequality can be shown similarly.

4.2 Decomposition of labelling tableaux

Let $T \in \mathcal{L}_Q(\rho)$. For $1 \leq r \leq k$, define $T^{(r)}$ to be the subtableau consisting of the columns of T of length q_r . We define $\mathcal{L}_Q^{(r)}(\rho) = \{T^{(r)} | T \in \mathcal{L}_Q(\rho)\}$. This is the set of subtableaux of column length q_r of labelling tableaux. We decompose the set of labelling tableaux as follows:

Lemma 4.1. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. Then $\mathcal{L}_Q(\rho) = \prod_{r=1}^k \mathcal{L}_Q^{(k+1-r)}(\rho)$.*

Proof. It is clear that we can write any labelling tableau T as the k -tuple of its k subtableaux $T^{(r)}$. Let $(T_{(k)}, T_{(k-1)}, \dots, T_{(1)}) \in \prod_{r=1}^k \mathcal{L}_Q^{(k+1-r)}(\rho)$. These tableaux $T_{(r)}$ are chosen independently from the sets $\mathcal{L}_Q^{(r)}(\rho)$. We need to show that these tableaux $T_{(r)}$ can be con-

catenated to form a labelling tableaux T : To form a tableau T , start with the tableau $T_{(k)}$. The rightmost column of this tableau is the ρ -minimal column $Y^{(k)}(\rho)$. Since $T_{(k-1)} \in \mathcal{L}_Q^{(k-1)}$, the tableau $T_{(k-1)}$ consist of values which come from the column $Y^{(k)}(\rho)$. In particular, the leftmost column of $T_{(k-1)}$ is entrywise weakly less than the column tableau consisting of the q_{k-1} largest values of $Y^{(k)}(\rho)$. Hence placing the tableau $T_{(k-1)}$ to the right of the tableau $T_{(k)}$ produces a reverse semistandard tableau. Then for $k-2 \geq r \geq 1$, we can similarly concatenate the tableau $T_{(r)}$ to the right of the tableau $T_{(r+1)}$. Hence we obtain a reverse semistandard tableau T obtained from concatenating the k tableaux $T_{(r)}$. It is clear that this tableau T is a labelling tableau. ■

4.3 Hillman-Grassl board and color template

To give a bijective proof of Theorem 3.4, for each $1 \leq r \leq k$ we ultimately need to produce a weight preserving bijection from the set $\mathcal{L}_Q^{(r)}$ of labelling subtableaux to a certain set of multisets of inversions $(\rho_i, \rho_j) \in \Phi^{(r)}(\rho)$. We define the subset $\Phi^{(r)}(\rho)$ in Section 4.9. We will construct this bijection as a composition of five weight preserving bijections. We construct the first bijection by “shrinking” the values in a labelling subtableaux so that the possible values in a subtableau come from a set of consecutive integers $\{0, 1, \dots, p\}$. We then define a bijection from the tableaux with consecutive values to sets of reverse plane partitions on certain shapes; these shapes are called *Hillman-Grassl boards*. In this section we present a template to define these boards and the reverse plane partitions on them. In the following, if $p = n$ and $b = q_1$, then this template describes the sole Hillman-Grassl board for the case $Q = \{q_1\}$ and $\rho \in S_{n+1}^Q$.

Fix integers $p \geq 1$ and $1 \leq b \leq p$. We define the *encompassing board* $\xi_{p,b}$ to be the rectangular shape $(p+1-b) \times b$. The shape $\xi_{p,b}$ contains p diagonals. We color the boxes along these p diagonals respectively with the colors $\langle 1 \rangle, \langle 2 \rangle, \dots, \langle p \rangle$. So the box at location $(p+1-b, 1) \in \xi_{p,b}$ is assigned the color $\langle 1 \rangle$ and the box at location $(1, b)$ is colored $\langle p \rangle$. In general, to the box $(i, j) \in \xi_{p,b}$ we assign the color $c(i, j) := \langle p+1-b+j-i \rangle$. So the box at position $(1, 1)$ is colored $\langle p+1-b \rangle$. Now let B be a subset of $[0, p]$ such that

$|B| = b$. Define $H := [0, p] - B$. Note that the number of rows in $\xi_{p,b}$ is $|H| = p + 1 - b$ and the number of columns in $\xi_{p,b}$ is $|B| = b$. Corresponding to B there is an ordered Q -partition ρ of $[0, p]$, where $Q = \{b\}$: The first cohort of ρ is $H_1 := B$, the second cohort is $H_2 := H$, the first subset is $B_1 := B$, and $B_2 := [0, p]$. Let $B(i)$ and $H(i)$ be the i^{th} largest values of B and H respectively. So for $1 \leq j \leq b$, we see that $B(b + 1 - j)$ is the j^{th} smallest value in B . Define a shape μ from the sets B and H as follows: For $1 \leq j \leq b$, the length of column j of μ is defined to be $\tau_j := |\{i \in H | i > B(b + 1 - j)\}|$. Here τ_j is the number of inversions of ρ attributed to $B(b + 1 - j)$. Hence the length of the j^{th} column of μ is equal to the number of values in H larger than the j^{th} smallest number in B . So as j increases from 1 to b , this count will weakly decrease. So μ is a shape. Since τ_j is the number of inversions in ρ attributed to $B(b + 1 - j)$, it can be seen that the shape μ sits inside the encompassing board for any B . For $1 \leq i \leq p + 1 - b$, it can be seen that the length of the i^{th} row μ_i of μ is equal to $|\{j \in B | j < H(i)\}|$. If $B = \{b - 1, b - 2, \dots, 1, 0\}$, then we have $\mu = \xi_{p,b}$. If $B = \{p, p - 1, \dots, p + 1 - b\}$, then μ is the empty shape ϕ . The colors of the board μ are the colors inherited from the encompassing board.

Let u_1, u_2, \dots, u_p be variables. For $1 \leq c \leq p$, define the u -weight of the color $\langle c \rangle$ to be $u^{\langle c \rangle} := u_c$. Given a box $(i, j) \in \mu$, define the *hook* of (i, j) , denoted $hook(i, j)$, to be the set of boxes of μ directly to the right of and of the boxes directly below and including the box (i, j) . Denote the set of hooks of μ by $\mathcal{H}(\mu)$. Define the u -weight monomial of $hook(i, j)$ to be the product of the weights of the colors assigned to the boxes in $hook(i, j)$. Note that when read off from southwest to northeast, the colors assigned to the boxes of a hook form a set of consecutive integers. For a reverse plane partition $R \in rpp(\mu)$, define the u -weight monomial of R to be $u^R := \prod_{(i,j) \in \mu} (u^{c(i,j)})^{R(i,j)}$.

Example 4.2. Fix $p = 8$ and $b = 5$. The encompassing board, along with its colors (written without brackets) is

$$\xi_{8,5} = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & 6 & 7 & 8 \\ \hline 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}.$$

Let $B = \{7, 6, 4, 2, 1\}$. Then $H = \{8, 5, 3, 0\}$. The length of column 1 of μ is equal to $\tau_1 = |\{i \in \{8, 5, 3, 0\} | i > \{7, 6, 4, 2, 1\}(5 + 1 - 1)\}| = |\{i \in \{8, 5, 3, 0\} | i > 1\}| = 3$. The length of column 2 of μ is equal to $\tau_2 = |\{i \in \{8, 5, 3, 0\} | i > \{7, 6, 4, 2, 1\}(5 + 1 - 2)\}| = |\{i \in \{8, 5, 3, 0\} | i > 2\}| = 3$. Similar calculations show that $\tau_3 = 2$, $\tau_4 = 1$, and $\tau_5 = 1$. Thus we have the shape

$$\mu = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}.$$

The following is the shape μ along with the color of each box placed directly in that box:

$$\mu = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & 6 & 7 & 8 \\ \hline 3 & 4 & 5 & & \\ \hline 2 & 3 & & & \\ \hline \end{array}.$$

We have, for example, the u -weight monomial of $hook(2, 1)$ is $u^{hook(2,1)} = u_2u_3u_4u_5$. Now consider the reverse plane partition

$$R = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 1 & 2 & 5 \\ \hline 1 & 2 & 4 & & \\ \hline 2 & 2 & & & \\ \hline \end{array}.$$

The value 2 at location $(3, 2)$ contributes a factor of $(u^{<3>})^2 = u_3^2$ to the u -weight monomial of R . The u -weight monomial of R is equal to $u_2^2u_3^3u_4^2u_5^4u_6u_7^2u_8^5$.

We will use the next lemma to show that the map defined in the next section from certain tableaux to reverse plane partitions on certain shapes is weight preserving. Its proof is obvious.

Lemma 4.3. *For $R_1, R_2 \in rpp(\mu)$, the u -weight monomial of $R_1 + R_2$ is equal to the product of the u -weight monomials of R_1 and R_2 .*

4.4 Template bijection from tableaux to reverse plane partitions

In this section we define a map Θ which we will use as a template for the k parallel bijections in Section 4.6. As in the previous section, fix integers $p \geq 1$ and $1 \leq b \leq p$, a set $B \subset [0, p]$ such that $|B| = b$, and a set $H := [0, p] - B$. Construct the shape μ from B and H

as before. We denote by T_B the column tableau whose set of values is equal to B . Now fix a reverse semistandard column tableau T of length b with values from $[0, p]$ which entrywise dominates T_B . Let t_0, t_1, \dots, t_p be variables and set $u_i := t_{i-1}^{-1}t_i$ for $1 \leq i \leq p$. For a tableau T with values from $[0, p]$, we define the u -weight monomial of T in terms of the u_i variables as $u^T := (t^{T_B})^{-1}t^T$. Here $t^T := \prod_{i=0}^p t_i^{c_i}$, and c_i is the number of times i appears in T .

We now prepare to define a map Θ' from column tableaux T which entrywise dominate T_B to putative reverse plane partitions of 0's and 1's on the shape μ ; this output claim will be confirmed in Lemma 4.5 below. First create a 0-1 filling R' of the encompassing board $\xi_{p,b}$ from the column tableau T as follows: For $(i, j) \in \xi_{p,b}$, set $R'(i, j)$ to 1 if $T(b+1-j, 1) + ((b+1-j) - 1) \geq p+1-i$, and to 0 otherwise. As in Section 1.4, this map should be viewed as going from reverse semistandard column tableaux in the xy -plane to reverse plane partitions in the xz -plane. Now let R be the restriction of the reverse plane partition R' to the shape μ . Then define $\Theta'(T)$ to be the reverse plane partition on μ such that $\Theta'(T; i, j) = R(i, j)$ for all $(i, j) \in \mu$. (Here $\Theta'(T; i, j)$ denotes the value of $\Theta'(T)$ in location (i, j) .) We have the following:

Lemma 4.4. *Fix $p \geq 1$, $1 \leq b \leq p$, and a b -subset $B \subset [0, p]$. Then $\Theta'(T_B)$ is a filling of μ with all entries equal to 0.*

Proof. Define $R := \Theta'(T_B)$. Let $(i, j) \in \mu$. Then the length τ_j of column j of μ is greater than or equal to i . So $\tau_j = |\{l \in H \mid l > B(b+1-j)\}| > i$. Note that $B(b+1-j) = T_B(b+1-j, 1)$. Thus $T_B(b+1-j, 1)$ is smaller than at least the largest i numbers in H . Also $T_B(b+1-j, 1)$ is weakly less than $b+1-j$ numbers in B since $T_B(b+1-j, 1)$ is the $(b+1-j)^{th}$ largest value in T_B . Combining these two, we can see that $T_B(b+1-j, 1)$ is weakly less than at least $(b+1-j) + i$ numbers in $B \cup H = [0, p]$. Hence we have $T_B(b+1-j, 1) \leq p+1 - (b+1-j+i) < p+1 - b + j - i$. Thus by definition $R(i, j) = 0$. ■

From this proof we can see that for $T \geq T_B$, column j of $\Theta'(T)$ contains at least one value equal to 1 if and only if $T(b+1-j, 1) > T_B(b+1-j, 1)$.

We have the following:

Lemma 4.5. *The map Θ' is a weight preserving bijection from column tableaux $T \geq T_B$ to reverse plane partitions on μ with parts bounded by 1.*

Proof. Let $T \geq T_B$ be a column tableau with entries from $[0, p]$. Define $R := \Theta'(T)$. Clearly R is a filling of μ with 0's and 1's. We first show that R is a reverse plane partition on μ . Let $(i, j) \in \mu$ such that $(i + 1, j) \in \mu$. Suppose $R(i, j) = 1$. Then by definition $T(b + 1 - j, 1) + ((b + 1 - j) - 1) \geq p + 1 - i$. Thus $T(b + 1 - j, 1) + ((b + 1 - j) + 1) > p + 1 - b + j - (i + 1)$ and hence $R(i + 1, j) = 1 = R(i, j)$. Now suppose $R(i, j) = 0$. Then we have $R(i + 1, j) \geq R(i, j)$ since R is a filling of μ with 0's and 1's. Thus R weakly increases down the columns. Now consider $(i, j) \in \mu$ such that $(i, j + 1) \in \mu$. Suppose $R(i, j) = 1$. Then $T(b + 1 - j, 1) + ((b + 1 - j) - 1) \geq p + 1 - i$. If $R(i, j + 1) = 0$, then $T(b + 1 - (j + 1), 1) + ((b + 1 - (j + 1)) - 1) < p + 1 - i$. Then we have $T(b + 1 - j, 1) \geq p + 1 - b + j - i$ and $T(b + 1 - j - 1, 1) < p + 2 - b + j - i$. Note that $T(b + 1 - j - 1, 1)$ is the value directly above the value $T(b + 1 - j, 1)$ in T . By the above two inequalities, we see that $T(b + 1 - j - 1, 1) \leq T(b + 1 - j, 1)$. This is a contradiction. Hence $R(i, j + 1) = 1$. Finally, $R(i, j) = 0$ implies $R(i, j) \leq R(i, j + 1)$. Hence R weakly increases in the rows. Thus R is a reverse plane partition on μ .

To show that Θ' is weight preserving, we begin by calculating the u -weight monomial of T , one box at a time. We have

$$u^T = (t^{T_B})^{-1} t^T = \prod_{j=1}^b (t_{T_B(b+1-j,1)})^{-1} (t_{T(b+1-j,1)}).$$

Fix some $1 \leq j \leq \mu_1$. The u -weight contributed by the value $T(b + 1 - j, 1)$ in T is $(t_{T_B(b+1-j,1)})^{-1} (t_{T(b+1-j,1)}) = u_{T_B(b+1-j,1)+1} u_{T_B(b+1-j,1)+2} \cdots u_{T(b+1-j,1)+1}$. Next consider the effect that the value $T(b + 1 - j, 1)$ has on the reverse plane partition $R = \Theta'(T)$. Note that under the Θ' map, the value $T(b + 1 - j, 1)$ only produces the 1's in the j^{th} column of R . Suppose that all the values in column j of R are equal to 0. Then we have $T(b + 1 - j, 1) = T_B(b + 1 - j, 1)$ by the comment after Lemma 4.4. In this case, neither the value $T(b + 1 - j, 1)$ in T nor the entries in column j of R contribute any u_i factor to the u -weight monomial.

Now suppose there is at least one value equal to 1 in column j of R . Let i be the highest row in column j in R which contains a 1. If $T(b+1-j, 1) + ((b+1-j) - 1) > p+1-i$, then $T(b+1-j, 1) + ((b+1-j) - 1) \geq p+1-(i-1)$ and $R(i-1, j) = 1$. This contradicts our assumption on i . If $T(b+1-j, 1) + ((b+1-j) - 1) < p+1-i$, then $R(i, j) = 0$. This is also a contradiction. Hence we must have $T(b+1-j, 1) + ((b+1-j) - 1) = p+1-i$ and thus $T(b+1-j, 1) = p+1-b+j-i$. (We use the preceding argument in the following paragraphs.) Now consider how these 1's in column j contribute to the u -weight of R . The color in the bottom of column j of μ is $\langle B(b+1-j) + 1 \rangle = \langle T_B(b+1-j, 1) + 1 \rangle$ by Lemma 4.12. The color of the box (i, j) is $\langle p+1-b+j-i \rangle = \langle T(b+1-j, 1) \rangle$. Hence the u -weight contributed by the 1's in column j of R is equal to $u_{T_B(b+1-j, 1)+1} u_{T_B(b+1-j, 1)+2} \cdots u_{T(b+1-j, 1)+1}$. This is exactly the u -weight contributed by the value $T(b+1-j, 1)$ which produced these 1's in R . Hence Θ' is weight preserving.

Let $T, S \geq T_B$ be two column tableaux and suppose $\Theta'(T) = \Theta'(S) = R$. Fix $1 \leq j \leq b$. Suppose column j of R doesn't contain a 1. Then we have $T(b+1-j, 1) = T_B(b+1-j, 1)$ and $S(b+1-j, 1) = T_B(b+1-j, 1)$. Suppose, on the other hand, that column j of R contains at least one value equal to 1. Let i be the highest row in column j in R which contains a 1. From the previous paragraph, we have $T(b+1-j, 1) = p+1-b+j-i$ and $S(b+1-j, 1) = p+1-b+j-i$. Hence all the values of S and T are equal. Thus $T = S$ and Θ' is injective.

Finally, fix a reverse plane partition R on μ with parts bounded by 1. Define a column tableau T as follows: Let $1 \leq j \leq b$. If column j of R contains no 1's, define $T(b+1-j, 1) := T_B(b+1-j, 1)$. If column j of R contains at least one value equal to 1, then let i be the highest row in column j in R which contains a 1. In this case, define $T(b+1-j, 1) := p+1-b+j-i$. From the work above, it is clear that $\Theta'(T) = R$ and $T \geq T_B$. Hence Θ' is surjective. Thus Θ' is a weight preserving bijection. ■

Note that the last paragraph of the previous proof gives a method to obtain the column tableau $(\Theta')^{-1}(R)$ from a reverse plane partition R on μ with parts bounded by 1.

The map Θ' is order preserving:

Lemma 4.6. *Let $T, S \geq T_B$ be column tableaux of length b with values from $[0, p]$. Then $T \geq S$ if and only if $\Theta'(T) \geq \Theta'(S)$.*

Proof. Suppose first that $T \geq S$. Fix $1 \leq j \leq \mu_1$. Let $i_{j,T}$ be the highest row in column j in $\Theta'(T)$ which contains a 1. Let $i_{j,S}$ be the highest row in column j in $\Theta'(S)$ which contains a 1. Recall from the previous proof that this implies $T(b+1-j, 1) = p+1-b+j-i_{j,T}$ and $S(b+1-j, 1) = p+1-b+j-i_{j,S}$. The inequality $T(b+1-j, 1) \geq S(b+1-j, 1)$ implies that $i_{j,T} \leq i_{j,S}$. Hence there are more bottom justified 1's in column j of $\Theta'(T)$ than in column j of $\Theta'(S)$. Thus we have $\Theta'(T) \geq \Theta'(S)$. This argument can clearly be reversed to show that $\Theta'(T) \geq \Theta'(S)$ implies $T \geq S$. ■

We now define a new map Θ using the map Θ' . Let $\mathcal{T}_{p,B}$ be the set of reverse semistandard tableaux T such that

1. T contains values from $[0, p]$,
2. T contains one distinct column length $b := |B|$, and
3. T contains exactly one column equal to T_B and this column is its rightmost column.

We then define $\Theta : \mathcal{T}_{p,B} \rightarrow rpp(\mu)$ as follows: Let $T \in \mathcal{T}_{p,B}$ be a tableau with m columns of length b . Recall that T_i denotes the i^{th} column of T . Define $\Theta(T) := \sum_{i=1}^m \Theta'(T_i)$. Then we have the following:

Lemma 4.7. *The map $\Theta : \mathcal{T}_{p,B} \rightarrow rpp(\mu)$ is a weight preserving bijection.*

Proof. Let $T \in \mathcal{T}_{p,B}$ and define $R := \Theta(T)$. Combining the definition of u^R with Lemma 4.3, we see that $u^R = \prod_{i=1}^m u^{\Theta'(T_i)}$. This is equal to $\prod_{i=1}^m u^{T_i} = u^T$ by Lemma 4.5. Hence Θ is weight preserving.

To show that Θ is surjective, let $R \in rpp(\mu)$. If R is the reverse plane partition of all 0's, then $\Theta(T_B) = R$ by Lemma 4.4. If R contains a nonzero value, let $m \geq 1$ be the largest value in R . We decompose R as a sum of $m+1$ reverse plane partitions as follows: For $1 \leq l \leq m+1$, define R_l to be the reverse plane partition such that $R_l(i, j) = 1$ if $R(i, j) \geq l$

and 0 otherwise. We have $R_1 \geq R_2 \geq \dots \geq R_m > R_{m+1} = \bar{0}$ where $\bar{0}$ denotes the reverse plane partition of all 0's. It is clear that $R = \sum_{l=1}^{m+1} R_l$. Now define T to be the tableaux with $m+1$ columns such that $T_l = (\Theta')^{-1}(R_l)$ for $1 \leq l \leq m+1$. Then T is a reverse semistandard tableau since Θ' (and hence $(\Theta')^{-1}$) is order preserving by Lemma 4.6. By Lemma 4.4, the rightmost column $(\Theta')^{-1}(R_{m+1}) = (\Theta')^{-1}(\bar{0})$ of T is equal to T_B . Hence $T \in \mathcal{T}_{p,B}$. Then $\Theta(T) = \sum_{l=1}^{m+1} \Theta'(T_l) = \sum_{i=1}^{m+1} R_i = R$. Hence Θ is surjective.

Let $T, S \in \mathcal{T}_{p,B}$ be such that $\Theta(T) = \Theta(S) = R$. Let $m+1 \geq 1$ be the number of columns of T . Again, we have $\Theta'(T_{l-1}) \geq \Theta'(T_l)$ for all $2 \leq l \leq m+1$ since $T_{l-1} \geq T_l$. Recall that $\Theta(T_{m+1}) = \Theta(T_B)$ is the reverse plane partition of all 0's. For $(i, j) \in \mu$, let $\Theta(T; i, j)$ denote the value of $\Theta(T)$ in location (i, j) . Now fix some $(i, j) \in \mu$ such that $\Theta'(T_m; i, j) = 1$. There exists such a location by Lemma 4.4 since $T_m \neq T_B$. Then $\Theta'(T_l; i, j) = 1$ for all $1 \leq l \leq m$. Thus m is the largest value of $\Theta(T) = R$. This is equal to one less than the number of columns in T . The same argument shows that the largest value in $\Theta(S)$ is one less than the number of columns in S . Since $\Theta(T) = \Theta(S)$, the largest value in $\Theta(S)$ is m . Thus S must have $m+1$ columns. Since $\Theta'(T_1) \geq \Theta'(T_2) \geq \dots \geq \Theta'(T_{m+1})$, the set of locations in $\Theta'(T_1)$ with nonzero values must be equal to the set of locations in $\Theta(T)$ with nonzero values. Similarly, the set of locations in $\Theta'(S_1)$ with nonzero values must be equal to the set of locations in $\Theta(S) = \Theta(T)$ with nonzero values. Hence we have $\Theta'(T_1) = \Theta'(S_1)$. By the injectivity of Θ' , we must have $T_1 = S_1$. Now consider the tableaux formed from T and S by removing their first columns T_1 and S_1 . The same argument shows that $T_2 = S_2$. Continuing in this way, we find that $T_l = S_l$ for $1 \leq l \leq m+1$. Thus $T = S$, and hence Θ is injective. ■

4.5 Bijection from labelling tableaux to shrunken labelling tableaux

Now re-fix $n \geq 1$, a subset $Q \subseteq [n]$, and an ordered Q -partition ρ . Fix $1 \leq r \leq k$. We are now ready to begin to construct weight preserving bijections from labelling subtableaux $\mathcal{L}_Q^{(r)}(\rho)$ to multisets of hooks of $\mu^{(r)}$ and then to multisets of certain subsets $\Phi^{(r)}(\rho)$ of inversions of ρ . Let $b_{q_{r+1}-1} > b_{q_{r+1}-2} > \dots > b_1 > b_0$ be the values in B_{r+1} , the union of the first $r+1$ cohorts of ρ . These are the possible values in the labelling subtableaux

$T \in \mathcal{L}_Q^{(r)}(\rho)$. Define a map Ψ_r on these q_{r+1} values by $\Psi_r(b_i) := i$ for $q_{r+1} - 1 \geq i \geq 0$. This is an order preserving bijection onto the set $[0, q_{r+1} - 1]$. Define B to be the set of images of the set of values in $B_r \subset B_{r+1}$ under the map Ψ_r . Set $p := q_{r+1} - 1$. Also use Ψ_r to denote the map which takes a labelling subtableau $T \in \mathcal{L}_Q^{(r)}(\rho)$ and applies the map Ψ_r to each of its values to produce a tableau of the same rectangular shape. This is clearly a bijection from $\mathcal{L}_Q^{(r)}(\rho)$ to $\mathcal{T}_{p,B}$. To emphasize the dependence of $\mathcal{T}_{p,B}$ upon r (and Q and ρ), henceforth we write $\mathcal{T}_{p,B}^{(r)}$. We can write the ρ -weight monomial of a labelling subtableau T in terms of the variables x_i or $z_i = x_{i-1}^{-1}x_i$. We can write the u -weight monomial of a tableau $\Psi_r(T) \in \mathcal{T}_{p,B}^{(r)}$ in either the variables t_i or $u_i = t_{i-1}^{-1}t_i$. As we map b_i to i via the map Ψ_r , we also set $t_i := x_{b_i}$ for $0 \leq i \leq p$. Note that this determines the u_i variables in terms of the z_i variables: $u_i = t_{i-1}^{-1}t_i = x_{b_{i-1}}^{-1}x_{b_i} = z_{b_{i-1}+1}z_{b_{i-1}+2} \dots z_{b_i}$ for $p = q_{r+1} - 1 \geq i \geq 1$. With this specification of the t variables in terms of the x variables, the map Ψ_r is automatically weight preserving. Hence we have:

Lemma 4.8. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. Then for $1 \leq r \leq k$, the map $\Psi_r : \mathcal{L}_Q^{(r)}(\rho) \rightarrow \mathcal{T}_{p,B}^{(r)}$ is a bijection that preserves the z -weights.*

4.6 Bijection from shrunken labelling tableaux to reverse plane partitions

Fix Q , ρ , and $1 \leq r \leq k$ as in the previous section. These determine the integer $p = q_{r+1} - 1$ and the set $B = \Psi_r(B_r)$. Define $\mu^{(r)} := \mu$ to be the shape constructed from B in Section 4.3. We call $\mu^{(r)}$ the r^{th} Hillman-Grassl board for ρ ; it is contained in the encompassing rectangle $\xi_{q_{r+1}-q_r, q_r}$. We defined the u -weight of a reverse plane partition in Section 4.3. To emphasize its dependence upon r , we write Θ_r for the map Θ applied to $\mathcal{T}_{p,B}^{(r)}$. By Lemma 4.7, we have:

Lemma 4.9. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. Then for $1 \leq r \leq k$, the map $\Theta_r : \mathcal{T}_{p,B}^{(r)} \rightarrow rpp(\mu^{(r)})$ is a bijection that preserves the u -weights.*

4.7 The colored Hillman-Grassl algorithm

Fix $d \geq 1$ and let μ be a d -partition. Recall that $\mathcal{H}(\mu)$ is the set of hooks of μ . Denote the set of multisets of hooks of μ by $M(\mathcal{H}(\mu))$. For $R \in rpp(\mu)$, define $|R|$ to be the sum of the entries in R . For $(i, j) \in \mu$, refer to the number of boxes $|hook(i, j)|$ in $hook(i, j)$ as the $(i, j)^{th}$ hook length. For $S \in M(\mathcal{H}(\mu))$ define $|S| := \sum_{hook(i, j) \in S} |hook(i, j)|$.

The Hillman-Grassl algorithm [HG] gave a bijection from reverse plane partitions R on μ to multisets S of hooks of the shape μ such that $|R| = |S|$; see [Sa] for a textbook presentation. We give an overview here.

Let $R \in rpp(\mu)$. The Hillman-Grassl algorithm describes a path of locations in μ : start with the most northeast location (i_1, j_1) of μ such that $R(i_1, j_1) \neq 0$. Extend the path to $(i_1, j_1 - 1)$ if $R(i_1, j_1 - 1) = R(i_1, j_1)$ and to $(i_1 + 1, j_1)$ otherwise. Continue until the path cannot extend via this rule. At that point the path will be at the bottom of some column. The set of locations in this path is a “wiggled” $hook(i, j)$ for some $(i, j) \in \mu$: the diagonals of μ which contain one box in this path are exactly the diagonals which contain one box of $hook(i, j)$. This is the first hook in the multiset of hooks produced by the Hillman-Grassl algorithm. We then decrease the values along the path of locations by 1 and repeat the process until the remaining reverse plane partition consists of only 0’s. This produces a multiset of hooks.

The reverse Hillman-Grassl algorithm builds up a reverse plane partition R from a multiset of hooks on μ . Start with a multiset S of hooks of μ and the reverse plane partition R consisting of all 0’s. We first order the locations of μ : location (i, j) comes before (i', j') if $i < i'$ or if $i = i'$ and $j > j'$. Given the first hook $hook(i, j)$ of S under this order, the reverse Hillman-Grassl algorithm describes how to create the wiggled $hook(i, j)$ on the shape μ . We then increment the corresponding values in R in the locations along this path. Then remove $hook(i, j)$ from S . We proceed in this way until S is empty. We then have the reverse plane partition R created by the reverse Hillman-Grassl algorithm.

Gansner [Ga] described a coloring of the boxes of the shape μ : He defined the weight

monomials of hooks and of reverse plane partitions on μ using essentially the same diagonal colors as we have defined for the shapes $\mu^{(r)}$. (Note that our names for the colors used for the boards $\mu^{(r)}$ differ from Gansner's names; here one only needs one color for each diagonal.) This coloring allowed Gansner to define a colored version of the Hillman-Grassl algorithm. Theorem 3.2 of [Ga] stated that this colored Hillman-Grassl algorithm is a weight preserving bijection.

Now fix Q and an ordered Q -partition ρ . For $1 \leq r \leq k$, let $\mu^{(r)}$ be the r^{th} Hillman-Grassl board, and color the boxes in the diagonals of these shapes as specified in Section 4.3. We define $HG_r : rpp(\mu^{(r)}) \rightarrow M(\mathcal{H}(\mu^{(r)}))$ to be the bijection given by the colored Hillman-Grassl algorithm defined in [Ga].

Restating Theorem 3.2 of [Ga] in our current context gives:

Lemma 4.10. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. Then for $1 \leq r \leq k$, the map $HG_r : rpp(\mu^{(r)}) \rightarrow M(\mathcal{H}(\mu^{(r)}))$ is a bijection that preserves the u -weights.*

4.8 Proof of hook product formula

We can now present a bijective proof of the hook product identity in Theorem 3.4:

Proof of Hook Product Identity in Theorem 3.4. Using the decomposition of $\mathcal{L}_Q(\rho)$ given in Section 4.1, we rewrite the left hand side of the equation in Theorem 3.4 as

$$\sum_{T \in \mathcal{L}_Q(\rho)} z^T = \sum_{T \in \prod_{r=1}^k \mathcal{L}_Q^{(r)}(\rho)} z^T = \prod_{r=1}^k \left(\sum_{T^{(r)} \in \mathcal{L}_Q^{(r)}(\rho)} z^{T^{(r)}} \right).$$

The right hand side of the second identity of Theorem 3.4 is

$$\prod_{r=1}^k \prod_{(i,j) \in \mu^{(r)}} \frac{1}{1 - z^{\text{hook}(i,j)}}.$$

So it will suffice to show that $\sum_{T^{(r)} \in \mathcal{L}_Q^{(r)}(\rho)} z^{T^{(r)}} = \prod_{(i,j) \in \mu^{(r)}} \frac{1}{1 - z^{\text{hook}(i,j)}}$ for all $1 \leq r \leq k$. To do

this, first we write down the standard expansion

$$\prod_{(i,j) \in \mu^{(r)}} \frac{1}{1 - z^{\text{hook}(i,j)}} = \sum_{I \in M(\mathcal{H}(\mu^{(r)}))} z^I.$$

Here $M(\mathcal{H}(\mu^{(r)}))$ is the set of multisets of hooks and for one multiset I we define $z^I :=$

$$\prod_{\text{hook}(i,j) \in I} z^{\text{hook}(i,j)}.$$

Notice that it will now suffice to construct a weight preserving bijection from $\mathcal{L}_Q^{(r)}(\rho)$ to $M(\mathcal{H}(\mu^{(r)}))$. We construct this bijection from the following three weight preserving bijections: from Lemma 4.8 in Section 4.5:

$$\Psi_r : \mathcal{L}_Q^{(r)} \rightarrow \mathcal{T}_{p,B}^{(r)},$$

from Lemma 4.9 in Section 4.6:

$$\Theta_r : \mathcal{T}_{p,B}^{(r)} \rightarrow rpp(\mu^{(r)}),$$

and from Lemma 4.10 in Section 4.7:

$$HG_r : rpp(\mu^{(r)}) \rightarrow M(\mathcal{H}(\mu^{(r)})).$$

After we map to multisets of hooks on $\mu^{(r)}$, the weight monomials are written in terms of the u_i variables. However, we can rewrite these monomials in terms of the z_i variables using the Section 4.5 fact that $u_i = z_{b_{i-1}+1} z_{b_{i-1}+2} \dots z_{b_i}$ for all $1 \leq i \leq p = q_{r+1} - 1$. Composing these three weight preserving bijections we obtain a bijection

$$HG_r \circ \Theta_r \circ \Psi_r : \mathcal{L}_Q^{(r)}(\rho) \rightarrow M(\mathcal{H}(\mu^{(r)}))$$

that preserves the z -weights. ■

4.9 Decomposition of $\Phi(\rho)$

We decompose $\Phi(\rho)$ into k disjoint subsets. Fix $1 \leq r \leq k$. We define $\Phi^{(r)}(\rho)$ to be the set of inversions (ρ_i, ρ_j) such that ρ_j is in the $(r+1)^{st}$ cohort H_{r+1} of ρ . Recall that $|H_{r+1}| = q_{r+1} - q_r$. Clearly we can write the set of inversions $\Phi(\rho)$ as the disjoint union $\bigsqcup_{r=1}^k \Phi^{(r)}(\rho)$. Decomposing $\Phi(\rho)$ in this way will allow us to prove Theorem 3.4.

As in Section 4.5, let $b_{q_{r+1}-1} > b_{q_{r+1}-2} > \dots > b_1 > b_0$ be the values in B_{r+1} , the set of values in the first $r+1$ cohorts of ρ . As unordered sets, we have $\{b_i\}_{i=0}^{q_{r+1}-1} = \{\rho_i\}_{i=0}^{q_{r+1}-1}$. Let Ψ_r be the map defined on these values in Section 4.5. Again define B to be the set of images of the values of B_r under the map Ψ_r . Define $p := q_{r+1} - 1$. Define $H := [0, p] - B$. It can be seen that H is the set of images of H_{r+1} under the map Ψ_r .

Note that every inversion in $\Phi^{(r)}(\rho)$ is an ordered pair of values from the set B_{r+1} . Since all the values of B_{r+1} appear in the list $b_{q_{r+1}-1} > b_{q_{r+1}-2} > \dots > b_1 > b_0$, each inversion in $\Phi^{(r)}$ can be written as (b_i, b_j) for some $0 \leq i < j \leq q_{r+1} - 1$. We re-use Ψ_r to denote the map defined by $\Psi_r[(b_i, b_j)] := (\Psi_r(b_i), \Psi_r(b_j)) = (i, j)$. This is clearly a bijection. We denote the set of images of $\Phi^{(r)}(\rho)$ under the map Ψ_r by $\overline{\Phi^{(r)}(\rho)}$. For an inversion $(b_i, b_j) \in \Phi^{(r)}(\rho)$, we have $\Psi_r(b_i) = i < j = \Psi_r(b_j)$ and $b_i \in B_r$ and $b_j \in H_{r+1}$. Thus $\overline{\Phi^{(r)}(\rho)} = \{(i, j) | i < j, i \in B, \text{ and } j \in H\}$.

We define the *u-weight monomial* of an ordered pair (i, j) such that $0 \leq i < j \leq n$ to be $u^{(i,j)} := u_{i+1}u_{i+1} \dots u_j$. Recall that we defined the ρ -weight of an inversion (b_i, b_j) to be $z_{b_{i+1}}z_{b_{i+2}} \dots z_{b_j}$. Also recall from Section 4.5 that $t_i := x_{b_i}$ for $0 \leq i \leq p$ and $u_i := z_{b_{i-1}+1}z_{b_{i-1}+1} \dots z_{b_i}$ for $1 \leq i \leq p$. Then it can be seen that the u -weight of an ordered pair $(i, j) \in \overline{\Phi^{(r)}(\rho)}$ is equal to the ρ -weight of its pre-image (b_i, b_j) under the map Ψ_r . We then re-use the Ψ_r to denote the function from multisets of inversions in $\Phi^{(r)}(\rho)$ to multisets of shruken inversions $\overline{\Phi^{(r)}(\rho)}$. We now have the following:

Lemma 4.11. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. Then for $1 \leq r \leq k$, the map $\Psi_r : M(\Phi^{(r)}(\rho)) \rightarrow M(\overline{\Phi^{(r)}(\rho)})$ is a bijection that preserves the z -weights.*

4.10 Bijection from multisets of hooks to multisets of shrunken inversions

Fix Q , ρ , and $1 \leq r \leq k$ as in the previous section. These determine the integer $p = q_{r+1} - 1$ and the set $B = \Psi_r(B_r)$. Fix $1 \leq r \leq k$. Let $\mu^{(r)}$ be the r^{th} Hillman-Grassl board. We define a map Γ'_r from the set $\mathcal{H}(\mu^{(r)})$ of hooks of the shape $\mu^{(r)}$ to the set $\overline{\Phi^{(r)}(\rho)}$. Let $(i, j) \in \mu^{(r)}$. We define a map $\Gamma'_r : \mathcal{H}(\mu^{(r)}) \rightarrow \overline{\Phi^{(r)}(\rho)}$ by $\Gamma'_r(\text{hook}(i, j)) := (B(b+1-j), H(i))$. To show that the map Γ'_r is weight preserving, we use the following lemma:

Lemma 4.12. *Let $p \geq 1$ and $1 \leq b \leq p$ be integers. Fix a b -subset $B \subset [0, p]$ and let $H = [0, p] - B$. Let μ be the shape defined from B and H . Assign the colors as above. For $1 \leq j \leq \mu_1$, the color assigned to the box at the bottom of column j is $\langle B(b+1-j) + 1 \rangle$. For $1 \leq i \leq \tau_1$, the color assigned to the rightmost box of row i is $H(i)$. Hence the u -weight monomial of $\text{hook}(i, j)$ is equal to $u^{(B(b+1-j), H(i))} := u_{B(b+1-j)+1} u_{B(b+1-j)+2} \cdots u_{H(i)}$.*

Proof. Let (h, j) be the bottom box of column j of μ . We need to show $p+1-b+j-h = B(b+1-j) + 1$. Consider the j^{th} smallest value $B(b+1-j)$ of B . If every value of B is less than every value of H , then we have $B(b+1-j) = j-1$. If not, then the value $B(b+1-j)$ is equal to $j-1$ plus the number of values of H less than $B(b+1-j)$. Thus we have $B(b+1-j) = j-1 + |\{l \in H \mid l < B(b+1-j)\}|$. Rewriting this equality we obtain

$$\begin{aligned} B(b+1-j) + 1 &= j + |H| - |\{l \in H \mid l > B(b+1-j)\}| \\ &= j + (p+1-b) - |\{l \in H \mid l > B(b+1-j)\}| = p+1-b+j-\tau_j. \end{aligned}$$

Note here that h is the length of the j^{th} column of μ . Hence we have $h = \tau_j$. Thus the expression above becomes $B(b+1-j) + 1 = p+1-b+j-h$. Thus, the color of the box (h, j) at the bottom of column j is equal to $\langle B(b+1-j) + 1 \rangle$.

Now let (i, h) be the rightmost box of row i . We need to show that $p+1-b+h-i = H(i)$. Consider the i^{th} largest value $H(i)$ of H . If every values of H is larger than every value of B , then we have $H(i) = p+1-i$. If not, then $H(i)$ is equal to $p+1-i$ minus the number of values in B larger than $H(i)$. Thus we have $H(i) = p+1-i - |\{l \in B \mid l > H(i)\}|$. Rewriting

this equality, we obtain

$$H(i) = p + 1 - i - (|B| - |\{l \in B \mid l < H(i)\}|) = p + 1 - i - b + \mu_i.$$

Since (i, h) is the rightmost box in row i , we have $\mu_i = h$. Thus $H(i) = p + 1 - b + h - i$. Hence the color assigned to the rightmost box (i, h) of row i is equal to $\langle H(i) \rangle$.

Since the colors of the boxes in a hook for a sequence of consecutive integers, we can see that the u -weight monomial of $hook(i, j)$ is equal to

$$u^{(B(b+1-j), H(i))} := u_{B(b+1-j)+1} u_{B(b+1-j)+2} \cdots u_{H(i)}.$$

■

We can now prove the following:

Lemma 4.13. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. For all $1 \leq r \leq k$, the map $\Gamma'_r : \mathcal{H}(\mu^{(r)}) \rightarrow \overline{\Phi^{(r)}(\rho)}$ is a bijection that preserves the u -weights.*

Proof. Fix some $(i, j) \in \mu^{(r)}$. We first need to show that $\Gamma'_r(hook(i, j)) \in \overline{\Phi^{(r)}(\rho)}$. Since $(i, j) \in \mu^{(r)}$, we have $\tau_j \geq 1$. Thus $B(b+1-j)$ is smaller than at least i values in H . Since $H(i)$ is the i^{th} largest value of H , we have $B(b+1-j) < H(i)$. Since $B(b+1-j) \in B$ and $H(i) \in H$, we see that $(B(b+1-j), H(i)) \in \overline{\Phi^{(r)}(\rho)}$.

Now clearly the map Γ'_r is injective. To show that Γ'_r is surjective, let $(l, m) \in \overline{\Phi^{(r)}(\rho)}$. Suppose l is the j^{th} largest value in B . Then we have $l = B(b+1-j)$. Since $m \in H$, there exists a $1 \leq i \leq p+1-b$ such that $m = H(i)$. Since $B(b+1-j) = l < m = H(i)$, the i^{th} largest value of H is larger than at least j values in B . Hence $\mu^{(r)}$ has a box at location (i, j) . Thus $hook(i, j) \in \mathcal{H}(\mu^{(r)})$. We have $\Gamma'_r(hook(i, j)) := (B(b+1-j), H(i)) = (l, m)$. Thus Γ'_r is surjective.

By Lemma 4.12, the u -weight monomial of $hook(i, j)$ is $u^{hook(i, j)} = u^{(B(b+1-j), H(i))} = u_{B(b+1-j)+1} z_{B(b+1-j)+2} \cdots u_{H(i)}$. This is exactly equal to the u -weight monomial of the inversion $(B(b+1-j), H(i))$. Hence Γ'_r is weight preserving. ■

The map Γ'_r induces a map Γ_r from the set $M(\mathcal{H}(\mu^{(r)}))$ of multisets of $\mathcal{H}(\mu^{(r)})$ to the set $M(\Phi^{(r)}(\rho))$ of multisets of $\Phi^{(r)}(\rho)$. The weight monomial of a multiset of either type of object is defined to be the product of the weight monomials of the elements in the multiset. It is clear that we then have the following:

Lemma 4.14. *Let $Q \subseteq [n]$ and let ρ be an ordered Q -partition. For all $1 \leq r \leq k$, the map $\Gamma_r : M(\mathcal{H}(\mu^{(r)})) \rightarrow M(\overline{\Phi^{(r)}(\rho)})$ is a bijection that preserves the u -weights.*

4.11 Bijective proof

We complete the bijective proof of our combinatorial interpretation of the K-P identity:

Proof of Identity 1 of Theorem 3.4.

First we rewrite the right hand side of the equation in Theorem 3.4 using the decomposition of $\Phi(\rho)$ as follows:

$$\begin{aligned} \frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})} &= \frac{1}{\prod_{(\rho_i, \rho_j) \in \sqcup_{r=1}^k \Phi^{(r)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})} \\ &= \prod_{r=1}^k \left(\frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(r)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})} \right). \end{aligned}$$

Given our proof of the hook product identity in Section 4.8, at this point it will suffice to show that

$$\frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(r)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})} = \frac{1}{\prod_{(i,j) \in \mu^{(r)}} (1 - z^{\text{hook}(i,j)})}$$

for $1 \leq r \leq k$.

To do this, first we write down the standard expansion

$$\frac{1}{\prod_{(\rho_i, \rho_j) \in \Phi^{(r)}(\rho)} (1 - z_{\rho_i+1} \cdots z_{\rho_j})} = \sum_{I \in M(\Phi^{(r)}(\rho))} z^I.$$

Here $M(\Phi^{(r)}(\rho))$ is the set of multisets of elements of $\Phi^{(r)}(\rho)$ and for one multiset I we define

$$z^I := \prod_{(\rho_i, \rho_j) \in I} z_{\rho_i+1} \cdots z_{\rho_j}.$$
 From Lemma 4.11 in Section 4.9, the map

$$\Psi_r : M(\Phi^{(r)}(\rho)) \rightarrow M(\overline{\Phi^{(r)}(\rho)})$$

is a bijection that preserves the z -weights. And from Lemma 4.14 in Section 4.10, the map

$$\Gamma_r^{-1} : M(\overline{\Phi^{(r)}(\rho)}) \rightarrow M(\mathcal{H}(\mu^{(r)}))$$

is a bijection that preserves the u -weights. Note that we can write the u -weights of both the shrunken inversions and the hooks on $\mu^{(r)}$ in terms of the z_i variables using the Section 4.5 fact that $u_i = z_{b_{i-1}+1} z_{b_{i-1}+2} \cdots z_{b_i}$. Then this bijection also preserves the z -weights. Composing these two maps we obtain a bijection

$$\Gamma_r^{-1} \circ \Psi_r : M(\Phi^{(r)}(\rho)) \rightarrow M(\mathcal{H}(\mu^{(r)}))$$

that preserves the z -weights. Combining the inverse of this bijection with the bijection of Section 4.8, we obtain a z -weight preserving bijection from $\mathcal{L}_Q^{(r)}(\rho)$ to $M(\Phi^{(r)}(\rho))$ for all $1 \leq r \leq k$. ■

5 Lie Theoretic proof of combinatorial K-P identity

Here we prove the first identity of Theorem 3.4 by translating the Kumar-Peterson identity for Type A_n from Lie theory to combinatorics. For an outline of a Lie theoretic proof of the hook product identity in the $Q = \{b\}$ case, consult the paragraph of Section 1.6 that refers to Figures 1 and 2. For the definitions of Lie theoretic terms used in this chapter, refer to [Hum] and [Kum2].

5.1 Transition from Lie theory to combinatorics

We quote results from the appendix of [PW]. There weight spaces of Demazure submodules of irreducible representations of $sl_n(\mathbb{C})$ were described with semistandard tableaux with values from $[n]$. The axis basis, positive roots, and simple reflections were chosen accordingly. Here we consider irreducible representations of $sl_{n+1}(\mathbb{C})$. Hence our tableaux contain values from $[0, n]$, not $[n]$. We also use reverse semistandard tableaux instead of semistandard tableaux because they more naturally give rise to the Hillman-Grassl build-up viewpoint of this thesis. Therefore, when quoting from the appendix of [PW] we need to make those two changes. We choose the Cartan subalgebra H to be the subspace of $sl_{n+1}(\mathbb{C})$ consisting of the diagonal matrices. For $n \geq i \geq 0$, we define $\overline{\phi}_i \in H^*$ to be the linear function that extracts the coefficient of the elementary matrix $E_{n-i, n-i}$. These $n+1$ linearly dependent functionals are used to describe weights. In [PW], the axis basis was chosen to be the basis of linear functions $\{\phi_i\}_{i=0}^n$ such that ϕ_i extracts the coefficient of the matrix $E_{i,i}$. For $n \geq i \geq 0$, we have $\overline{\phi}_{n-i} = \phi_i$. For $n \geq i \geq 1$, we define the positive simple roots to be $\alpha_i = \overline{\phi}_i - \overline{\phi}_{i-1}$. Then the set Φ^+ of positive roots is $\{\overline{\phi}_i - \overline{\phi}_j | n \geq i > j \geq 0\}$ and the set Φ^- of negative roots is $\{-\overline{\phi}_i + \overline{\phi}_j | n \geq i > j \geq 0\}$. Here the Borel subalgebra B is the subalgebra of trace free upper triangular matrices. For $n \geq b \geq 1$, the fundamental weights are $\omega_b = \overline{\phi}_n + \overline{\phi}_{n-1} + \dots + \overline{\phi}_b$. In the axis basis of H^* , the fundamental weight ω_b can be depicted by the $(n+1)$ -tuple

$(1, 1, \dots, 1, 0, 0, \dots, 0)$ which contains $n + 1 - b$ 1's.

Let $\lambda = \sum_{i=1}^n a_i \omega_i$ be a dominant integral weight, where $a_i \in \mathbb{N}$ for $1 \leq i \leq n$. We produce an n -partition λ from this Lie dominant integral weight λ . This is the n -partition whose shape consists of a_i columns of length $n + 1 - i$ for $1 \leq i \leq n$. Recall from Section 2.1 that $Q(\lambda) = \{q_1, q_2, \dots, q_k\}$ is equal to the set of column lengths of the n -partition λ . Take for example the adjoint representation, which has highest weight $\lambda = \sum_{i=1}^n \omega_i$. The n -partition λ produced from the highest weight λ is the ‘‘staircase’’ shape which has one column of length i for $n \geq i \geq 1$. We then have $Q(\lambda) = [n]$. Given a dominant integral weight λ , the weights-with-multiplicities of the irreducible representation V_λ of $sl_{n+1}(\mathbb{C})$ can be described by the reverse semistandard tableaux on the shape λ with values from $[0, n]$. In [PW], the weight described by a semistandard tableau T is equal to $\sum_{i=0}^n c_i \phi_i$, where c_i is the number of times i appears in T . For a reverse semistandard tableau T , the weight described by T is equal to $\sum_{i=0}^n c_i \overline{\phi_i}$, where c_i is the number of times i appears in T . Since $\overline{\phi_{n-i}} = \phi_i$, to obtain our $(n + 1)$ -reverse semistandard tableaux from $(n + 1)$ -semistandard tableaux, we subtract the values in the tableaux entrywise from n . Given an irreducible representation V_λ of $sl_{n+1}(\mathbb{C})$, the highest weight λ is depicted by the semistandard tableau with the smallest possible values. When we subtract all the values of the tableau from n , the highest weight λ is depicted by the reverse semistandard tableau with the largest possible values. It can also be seen that this tableau describes the highest weight directly from the definitions without needing to subtract values from a semistandard tableau.

Fix a dominant integral weight λ . For $n \geq i \geq 1$, the simple reflection s_i permutes $\overline{\phi_i}$ and $\overline{\phi_{i-1}}$. These simple reflections generate the Weyl group $W = S_{n+1}$. To precisely index the Demazure submodules of V_λ , first set $J := J_\lambda := \{i \in [n] : s_i \cdot \lambda = \lambda\}$. There is one distinct Demazure module for each coset wW_J in the set of cosets $W^J := W/W_J$. Each such coset has a unique minimal length representative in W ; let W^λ denote the set of these representatives. For some $t \geq 1$ and $i_1, i_2, \dots, i_t \in [n]$, we can write $w \in W^\lambda$ as the product of simple reflections $s_{i_t} \dots s_{i_2} s_{i_1}$. For $w \in W^\lambda$, the set $\Phi(w)$ is defined to be the set $\Phi^+ \cap w(\Phi^-)$.

Fix $w \in W^\lambda$. Here we describe how to produce an ordered Q -partition ρ from w . Let $n \geq i \geq 0$. Define ρ_i by $\overline{\phi_{\rho_i}} := w(\overline{\phi_i})$. Set $\rho := (\rho_n, \rho_{n-1}, \dots, \rho_1, \rho_0)$. From the following it can be seen that ρ becomes the standard form of an ordered Q -partition: For our combinatorial model of w , the simple reflections in the product $w = s_{i_t} \dots s_{i_2} s_{i_1}$ act by value on the $(n+1)$ -tuple $(n, n-1, \dots, 2, 1, 0)$. The positions of the values of this $(n+1)$ -tuple are indexed decreasing from n to 0 from left to right. For $n \geq i \geq 1$, the simple reflection s_i permutes the values i and $i-1$. The identity $e \in W^\lambda$ produces the $(n+1)$ -tuple $(n, n-1, \dots, 2, 1, 0)$. Let λ be the n -partition produced from the weight λ and let $Q(\lambda) = \{q_1, q_2, \dots, q_k\}$. Recall that we defined $q_0 := 0$ and $q_{k+1} := n+1$. For $1 \leq r \leq k$, we place a semicolon into ρ between positions $n+1-q_r$ and $n-q_r$:

$$\rho := (\rho_n, \rho_{n-1}, \dots, \rho_{n+1-q_1}; \rho_{n-q_1}, \dots, \rho_{n+1-q_2}; \rho_{n-q_2}, \dots; \dots; \rho_{n-q_k}, \dots, \rho_0).$$

This separates the set of positions into $k+1$ carrels. The r^{th} carrel from the left consists of the positions $\{n-q_{r-1}, n-1-q_{r-1}, \dots, n+1-q_r\}$. The size of the r^{th} carrel is equal to $q_r - q_{r-1}$. Placing these semicolons also separates the set of values into $k+1$ cohorts corresponding to the $k+1$ carrels. When we mod out by the parabolic subgroup W_J , the positions within each of the $k+1$ carrels become indistinguishable. Thus the shortest length coset representative $w \in W^\lambda$ produces an $(n+1)$ -tuple wherein the values within a cohort decrease from left to right. The resulting $(n+1)$ -tuple is the standard form of an ordered Q -partition ρ . It can be seen that every ordered Q -partition is produced once in this fashion.

In the Section 1.1 generic statement of the Kumar-Peterson identity, the adjusted Demazure characters $y^{-wm\lambda} \mathcal{D}_{m\lambda}(w; y)$ were written in terms of a variable y . We used the y variable to denote a generic coordinatization of these adjusted characters. Now use the variables x_i to coordinatize these adjusted characters with respect to the axis basis $\{\overline{\phi_i}\}_{i=0}^n$: Here we set $x_i := \exp(\overline{\phi_i})$, the formal exponential of $\overline{\phi_i}$. We use the variables z_i to coordinatize these adjusted characters with respect to the simple root basis $\{\alpha_i\}_{i=1}^n$. Here we have $z_i := \exp(\alpha_i)$. We now have two ways to relate the sets of variables x_i and z_i . Since $\alpha_i = \overline{\phi_i} - \overline{\phi_{i-1}}$, for our Lie

theoretic definition of these variables, we have $z_i = \exp(\alpha_i) = \exp(\overline{\phi_i}) \exp^{-1}(\overline{\phi_{i-1}}) = x_{i-1}^{-1} x_i$. Note that this agrees with our combinatorial definition of $z_i := x_{i-1}^{-1} x_i$ made in Section 2.7. We refer to the formal exponential z_i of the simple root α_i as the z -weight of this simple root. We similarly refer to $z_i z_{i+1} \dots z_j$ as the z -weight of the positive root $\alpha_i + \alpha_{i+1} + \dots + \alpha_j$. The term “weight” here is combinatorial terminology and does not refer to the Lie theoretic definition of weight.

5.2 Translation of the left hand side

Fix a dominant integral weight λ and $w \in W^\lambda$. Let λ be the n -partition obtained from the weight λ and let ρ be the ordered Q -partition produced from w . As in [PW], we describe the Demazure polynomial $[\lambda(w; y)]$ with the Demazure tableaux for λ and ρ : Using the Lascoux-Schützenberger-Willis description, we express this polynomial in the axis basis as $\sum_{T \in \mathcal{D}_\lambda(\rho)} x^T$, where x^T is the traditional weight monomial of T . The lowest weight $w \cdot \lambda$ of the Demazure module is described by the λ -key $Y_\lambda(\rho)$ of ρ . Then the adjusted Demazure character $y^{-w \cdot \lambda} [\lambda(w; y)]$ is $(x^{Y_\lambda(\rho)})^{-1} \sum_{T \in \mathcal{D}_\lambda(\rho)} x^T$. This can be re-expressed as $\sum_{T \in \mathcal{D}_\lambda(\rho)} z^T$, where z^T is the ρ -weight monomial of T . By the remark near the end of the preceding section, the x -to- z transition here can be explained combinatorially or Lie theoretically.

Now we need to translate the limit of characters on the left hand side of the Kumar-Peterson identity to combinatorics. From above, for all $m \geq 1$ the adjusted characters $y^{-w \cdot m\lambda} \mathcal{D}_{m\lambda}(w; y)$ coordinatized in the simple root basis are equal to $\sum_{T \in \mathcal{D}_{m\lambda}(\rho)} z^T$. Thus $\lim_{m \rightarrow \infty} y^{-w \cdot m\lambda} \mathcal{D}_{m\lambda}(w; y)$ becomes $\lim_{m \rightarrow \infty} \sum_{T \in \mathcal{D}_{m\lambda}(\rho)} z^T$. To calculate the right hand side, we define the weighted direct system $(\mathcal{D}_{m\lambda}(\rho), \gamma_{i,j}, wt)$ as in Section 2.9. Using Lemma 3.1, we know that the weighted limit $\lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)$ is stable. Thus $\lim_{m \rightarrow \infty} \sum_{T \in \mathcal{D}_{m\lambda}(\rho)} z^T = {}^F \lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)(z)$ by Lemma 2.13. By Theorem 3.3, we can label the equivalence classes of the weighted limit with the labelling tableaux $\mathcal{L}_Q(\rho)$. Thus we can write the original left hand side $\lim_{m \rightarrow \infty} y^{-w \cdot m\lambda} \mathcal{D}_{m\lambda}(w; y)$ of the Kumar-Peterson identity for λ and w in terms

of labelling tableaux as:

$$\lim_{m \rightarrow \infty} \sum_{T \in \mathcal{D}_{m\lambda}(\rho)} z^T = F \lim_{m \rightarrow \infty} (\mathcal{D}_{m\lambda}(\rho), \gamma, wt)(z) = F_{\mathcal{L}_Q(\rho)}(z) := \sum_{T \in \mathcal{L}_Q(\rho)} z^T.$$

We have shown:

Lemma 5.1. *In the simple root basis coordinatization, the left hand side $\lim_{m \rightarrow \infty} y^{-wm\lambda} \mathcal{D}_{m\lambda}(w; y)$ of the Kumar-Peterson identity becomes*

$$\sum_{T \in \mathcal{L}_Q(\rho)} z^T.$$

5.3 Translation of the right hand side

Fix the same λ and w as in the previous section. Again, let ρ be the ordered Q -partition produced for w . We need to show that in the simple root basis coordinatization the Kumar-Peterson product $\prod_{\alpha \in \Phi(w)} (1 - y^\alpha)$ becomes

$$\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} z_{\rho_i+2} \cdots z_{\rho_j}).$$

To prove this, we define a map $\Upsilon : \Phi(\rho) \rightarrow \Phi(w)$ by $\Upsilon((\rho_i, \rho_j)) := -\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}}$. Recall that the ρ -weight of an inversion (ρ_i, ρ_j) was defined in terms of the z_i variables as $z_{\rho_i+1} z_{\rho_i+2} \cdots z_{\rho_j}$. We have the following:

Lemma 5.2. *The map $\Upsilon : \Phi(\rho) \rightarrow \Phi(w)$ is a weight preserving bijection. Hence in the simple root basis coordinatization the right hand side $\prod_{\alpha \in \Phi(w)} (1 - y^\alpha)$ of the Kumar-Peterson identity becomes*

$$\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1} z_{\rho_i+2} \cdots z_{\rho_j}).$$

Proof. Let $(\rho_i, \rho_j) \in \Phi(\rho)$. Then $i > j$ and $\rho_i < \rho_j$. Hence $-\overline{\phi_i} + \overline{\phi_j} \in \Phi^-$ and $-\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}} \in \Phi^+$. Since $w(-\overline{\phi_i} + \overline{\phi_j}) = -\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}}$, we have $-\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}} \in \Phi(w)$. Hence $\Upsilon(\Phi(\rho)) \subseteq \Phi(w)$.

Clearly Υ is injective. To show that Υ is surjective, let $0 \leq i' < j' \leq n$ be such $-\overline{\phi_{i'}} + \overline{\phi_{j'}} \in \Phi(w)$. Find i' and j' within ρ and define $i, j \in [0, n]$ to be such that $i' = \rho_i$ and $j' = \rho_j$. Then $-\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}} \in \Phi^+$. Since $w(-\overline{\phi_i} + \overline{\phi_j}) = -\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}}$ and $-\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}} \in \Phi(w)$ we have $-\overline{\phi_i} + \overline{\phi_j} \in \Phi^-$. Hence we have $i > j$ and $\rho_i < \rho_j$. Thus $(i', j') = (\rho_i, \rho_j)$ is an inversion of ρ . Since $\Upsilon((i', j')) = -\overline{\phi_{i'}} + \overline{\phi_{j'}}$, the map Υ is surjective.

As noted above, the weight monomial of the inversion (ρ_i, ρ_j) is equal to $z_{\rho_i+1}z_{\rho_i+2} \cdots z_{\rho_j}$. On the other hand, the z -weight of the positive root $-\overline{\phi_{\rho_i}} + \overline{\phi_{\rho_j}}$ is $z_{\rho_i+1}z_{\rho_i+2} \cdots z_{\rho_j}$ in the simple root basis coordinatization. Hence the bijection Υ is weight preserving. Using its inverse, we see that the right hand side $\prod_{\alpha \in \Phi(w)} (1 - y^\alpha)$ of the Kumar-Peterson identity becomes

$$\prod_{(\rho_i, \rho_j) \in \Phi(\rho)} (1 - z_{\rho_i+1}z_{\rho_i+2} \cdots z_{\rho_j}). \blacksquare$$

5.4 Lie theoretic proof

Combining Lemma 5.1 (which is a consequence of Theorem 3.3) and Lemma 5.2 we obtain our second proof of the first identity of Theorem 3.4. The fact that this identity can be derived in this fashion was stated as our third main result, Theorem 3.7.

6 Demazure polynomials from Gelfand patterns

6.1 Introduction

In this chapter, we use semistandard tableaux with values from $[n]$ instead of reverse semistandard tableaux with values from $[0, n]$ and permutations $\pi \in S_n^Q$ instead of ordered Q -partitions $\rho \in S_{n+1}^Q$. To obtain a permutation $\pi \in S_n^Q$ from an ordered Q -partition $\rho \in S_{n+1}^Q$, we subtract the values of ρ from n and reorder the positions to increase from left to right from 1 to n .

Fix an integer $n \geq 1$. Fix an n -partition λ and a permutation $\pi \in S_n^Q$. In Section 2.7, we expressed the Demazure polynomial for λ and π as a sum over the set of Demazure tableaux for π on the shape λ . In this chapter, we express the Demazure polynomial for λ and π as a sum over certain Gelfand patterns.

We define Gelfand patterns in Section 6.2. There we present our definition of a “key pattern”. In Section 6.3 we recall the well known bijection from the set of semistandard tableaux on the shape λ to the set of Gelfand patterns with top row λ . In Lemma 6.6 we prove that a Gelfand pattern is a key pattern if and only if it is the image of a key tableau under this bijection.

To simplify the description of the tableaux used to describe Demazure polynomials, Willis developed [Wi] a scanning method for tableaux which produces their right keys. A semistandard Demazure tableau for π on the shape λ was defined in Section 2.6 to be a tableau on the shape λ whose scanning tableau is entrywise less than or equal to the λ -key of π .

In Section 6.4, we present our Gelfand pattern scanning method. We define a “Demazure pattern” for π with top row λ to be a pattern with top row λ whose “scanning pattern” is greater than or equal to the “ λ -key pattern of π ”. Proposition 6.13 states that the bijection from tableaux to Gelfand patterns given in Section 6.3 commutes with the tableaux

and Gelfand pattern scanning methods. This result takes the most work to prove. As a consequence of Proposition 6.13, Corollary 6.14 states that the bijection from tableaux to Gelfand patterns restricts to become a bijection from the set of Demazure tableaux for π and on the shape λ to the set of Demazure patterns for π with top row λ . Our main result of this section, Theorem 6.15, describes the Demazure polynomial for λ and π as the sum of the weight monomials of the set of Demazure patterns for λ and π . We present proofs of the results in Section 6.5.

6.2 Gelfand patterns

Fix $n \geq 1$. An n -Gelfand pattern is defined to be a set $P = \{P_{i,j} | 1 \leq i \leq n, 1 \leq j \leq n + 1 - i\}$ of nonnegative integers which satisfy $P_{i,j} \geq P_{i+1,j} \geq P_{i,j+1} \geq 0$ whenever the entries are defined. We display these entries as follows:

$$\begin{array}{ccccccc}
 P_{1,1} & & P_{1,2} & & \dots & & P_{1,n} \\
 & & P_{2,1} & & \dots & & P_{2,n-1} \\
 & & & & \ddots & & \\
 & & & & & & P_{n,1}
 \end{array}$$

Let A and B be n -Gelfand patterns. We define $A + B$ by entrywise addition:

$$(A + B)_{i,j} := A_{i,j} + B_{i,j}.$$

We refer to an n -Gelfand pattern simply as a Gelfand pattern. For $1 \leq i \leq n$, define $P_i := (P_{i,1}, \dots, P_{i,n+1-i})$. The partition P_i is the $(n + 1 - i)$ -partition whose parts are equal to the entries of the i^{th} row of P . We further define $P_{n+1,1} := 0$ and $P_{n+1} := \emptyset$. We call the subsets of entries with the same second index *diagonals*; i.e. the j^{th} diagonal consists of the $n + 1 - j$ entries $P_{1,j}, \dots, P_{n+1-j,j}$.

Let λ be an n -partition. Denote by \mathcal{GP}_λ the set of Gelfand patterns P such that $P_1 = \lambda$. We call these patterns *Gelfand patterns with top row λ* .

Example 6.1. Let $n = 5$ and $\lambda = (5, 3, 2, 1, 0)$. The following is one Gelfand pattern with top row λ :

$$P = \begin{array}{cccccc} & & 5 & & 3 & & 2 & & 1 & & 0 \\ & & & & 4 & & 3 & & 2 & & 1 \\ & & & & & & 4 & & 2 & & 1 \\ & & & & & & & & 2 & & 1 \\ & & & & & & & & & & 2 \end{array} .$$

The *weight monomial* of a Gelfand pattern P is defined to be $x^P := \prod_{i=1}^n x_i^{c_i}$, where $c_i := |P_{n+1-i}| - |P_{n+2-i}|$.

Example 6.2. For the Gelfand pattern P in Example 6.1, we have $x^P = x_1^2 x_2^1 x_3^4 x_4^3 x_5^1$.

We define the *highest weight pattern* for λ to be the Gelfand pattern with top row λ such that for $2 \leq i \leq n$, row i is obtained from row $i - 1$ by deleting the rightmost entry. Since the rightmost entry of a row is the smallest entry in that row, the highest weight pattern for λ has the largest possible entries for a Gelfand pattern with top row λ . It can be seen that the weight monomial of the highest weight pattern for λ is equal to $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. We define the *lowest weight pattern* to be the Gelfand pattern with top row λ such that for $2 \leq i \leq n$, row i is obtained from row $i - 1$ by deleting the leftmost entry. The lowest weight pattern for λ has the smallest possible entries for a Gelfand pattern with top row λ . It can be seen that the weight monomial of the lowest weight pattern for λ is equal to $x_1^{\lambda_n} x_2^{\lambda_{n-1}} \dots x_n^{\lambda_1}$.

Let $A, B \in \mathcal{GP}_\lambda$. If for every (i, j) such that $i + j \leq n + 1$, one has $A_{i,j} \geq B_{i,j}$, then we write $A \geq B$. It can be seen that the highest (lowest) weight pattern for λ is the unique maximal (minimal) element of \mathcal{GP}_λ , when this set is ordered by \geq .

We define a Gelfand pattern to be a *key pattern* if for $2 \leq i \leq n$, the multiset of entries in row i forms a submultiset of the multiset of entries in row $i - 1$. It is clear that the highest and lowest weight patterns with top row λ are key patterns. Let $\pi \in S_n^Q$. We define the

λ -key pattern of π , denoted $K_\lambda(\pi)$, row by row as follows: define $K_\lambda(\pi)_1 := \lambda$. Then for $2 \leq i \leq n$, row i of $K_\lambda(\pi)$ is obtained from row $i - 1$ by deleting one of the entries equal to $\lambda_{\pi^{-1}(n+2-i)}$. For tableaux, it is known that the construction of the λ -key of π , denoted $Y_\lambda(\pi)$, defines a bijection from the set of permutations $\pi \in S_n^Q$ to the set of key tableaux on the shape λ . It will be seen in Lemma 6.6 that the bijection from tableaux to Gelfand patterns given in Section 6.3 defines a bijection from the set of λ -keys of π to the set of λ -key patterns of π .

Example 6.3. Fix $n = 5$. Let $\lambda = (5, 3, 2, 1, 0)$ and $\pi = (3, 5, 4, 1, 2)$. To construct $K_\lambda(\pi) \in \mathcal{GP}_\lambda$, we start with $(K_\lambda(\pi))_1 = (5, 3, 2, 1, 0)$. We form $(K_\lambda(\pi))_2$ from $(K_\lambda(\pi))_1$ by deleting the entry equal to $\lambda_{\pi^{-1}(5+2-2)} = \lambda_{\pi^{-1}(5)} = \lambda_2 = 3$. This produces $(K_\lambda(\pi))_2 = (5, 2, 1, 0)$. Continuing in this way, we construct

$$K_\lambda(\pi) = \begin{array}{cccccc} & 5 & & 3 & & 2 & & 1 & & 0 \\ & & & 5 & & 2 & & 1 & & 0 \\ & & & & & 5 & & 1 & & 0 & . \\ & & & & & & & 1 & & 0 \\ & & & & & & & & & 1 \end{array}$$

The λ -key pattern of the identity permutation is the highest weight pattern. It can be seen that the λ -key pattern of the longest permutation $\pi_0 \in S_n^Q$ is the lowest weight pattern.

6.3 Standard bijection from semistandard tableaux to Gelfand patterns

Fix $n \geq 1$ and an n -partition λ . In this section, we recall the usual bijection from semistandard tableaux on the shape λ to Gelfand patterns with top row λ . This bijection is presented in e.g. [HL, Section 3].

Define the map $\Theta : \mathcal{T}_\lambda \rightarrow \mathcal{GP}_\lambda$ as follows: Let $T \in \mathcal{T}_\lambda$. For $1 \leq i \leq n$, define row i of $\Theta(T)$ to be the partition given by the shape of the boxes of T which contain values less than or equal to $n + 1 - i$. The inverse to Θ is found as follows: Let $P \in \mathcal{GP}_\lambda$. To contain $\Theta^{-1}(P)$, form the Young diagram of the n -partition λ . For $n \geq i \geq 1$, place i in the boxes

of the skew shape P_{n+1-i}/P_{n+2-i} formed from the partitions P_{n+1-i} and P_{n+2-i} .

Example 6.4. Let $n = 5$ and $\lambda = (5, 3, 2, 1, 0)$. Let P be the Gelfand pattern in Example 6.1. We have

$$\Theta^{-1}(P) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 5 \\ \hline 2 & 3 & 4 & & \\ \hline 3 & 4 & & & \\ \hline 4 & & & & \\ \hline \end{array} .$$

For instance, if $i = 2$, then the shape defined by the boxes with values no larger than $n + 1 - i = 4$ in $\Theta(P)$ is the shape of the partition $(4, 3, 2, 1)$. This is exactly row $i = 2$ of P .

The bijection Θ has five properties we will need. The first is the well known result that Θ is weight preserving:

Lemma 6.5. *Let $T \in \mathcal{T}_\lambda$. Then $x^T = x^{\Theta(T)}$.*

To prove Lemma 6.5, one observes that for $1 \leq i \leq n$, the number of values equal to i in T is equal to $|\Theta(T)_{n+1-i}| - |\Theta(T)_{n+2-i}|$.

For Lemma 6.6, recall that $Y_\lambda(\pi)$ is the λ -key of π defined in Section 2.4.

Lemma 6.6. *Let $T \in \mathcal{T}_\lambda$. Then T is a key tableau if and only if $\Theta(T)$ is a key pattern. In particular, we have $\Theta(Y_\lambda(\pi)) = K_\lambda(\pi)$.*

Lemma 6.6 makes clear the motivation to define a key pattern as we did in Section 6.2. We prove Lemma 6.6 in Section 6.5.

The next lemma states that the bijection Θ reverses the order when mapping from tableaux to Gelfand patterns:

Lemma 6.7. *Let $A, B \in \mathcal{T}_\lambda$. Then $A \leq B$ if and only if $\Theta(A) \geq \Theta(B)$.*

To prove Lemma 6.7, one notes that for $1 \leq i \leq n$, the shape of the boxes in A with values no larger than $n + 1 - i$ contains the shape of the boxes in B with values no larger than $n + 1 - i$ if and only if $A \leq B$.

Our next lemma gives a description for the individual entries of $\Theta(T)$ in terms of the values of T :

Lemma 6.8. *Let $T \in \mathcal{T}_\lambda$. Then $\Theta(T)_{i,j}$ is equal to the number of values in row j of T that are less than or equal to $n + 1 - i$.*

To prove Lemma 6.8, note that $\Theta(T)_i$ is the shape of the boxes with values that are less than or equal to $n + 1 - i$. Then $\Theta(T)_{i,j}$ is equal to the length of the j^{th} row of the shape $\Theta(T)_i$. Thus $\Theta(T)_{i,j}$ is equal to the number of values in row j of T that are less than or equal to $n + 1 - i$. Using Lemma 6.8, we can see that the difference $\Theta(T)_{i,j} - \Theta(T)_{i+1,j}$ is equal to the number of values in row j of T that are equal to $n + 1 - i$.

For the final lemma of this section, recall that a column tableau C is a tableau with just one column. It can be seen from Lemma 6.8 that $\Theta(C)$ is a Gelfand pattern consisting only of the values 0 and 1. The number of 1's in the top row of $\Theta(C)$ is equal to the number of values in C . The number of 1's in the j^{th} diagonal of $\Theta(C)$ is equal to $n + 1 - C_{j,1}$. We recall a definition stated in [Wi]: Let C be a column tableau and let T be a tableau. Define $C \oplus T$ to be the result of prepending C to the left side of T . We then have the following:

Lemma 6.9. *Let C be a semistandard column tableau and T be a semistandard tableau. Then $\Theta(C \oplus T) = \Theta(C) + \Theta(T)$.*

The proof of Lemma 6.9 is straightforward: From Lemma 6.8, we know that $\Theta(C \oplus T)_{i,j}$ is equal to the number of values in row j of $C \oplus T$ less than or equal to $n + 1 - i$. Then clearly $\Theta(C \oplus T)_{i,j}$ is equal to the sum of $\Theta(C)_{i,j}$ and $\Theta(T)_{i,j}$.

6.4 Scanning method for Gelfand patterns

Throughout this section, fix $n \geq 1$, an n -partition λ and $\pi \in S_n^Q$. In this section, we present our scanning method for Gelfand patterns. Let $P \in \mathcal{GP}_\lambda$. We define the scanning pattern of P , denoted $S_{\mathcal{GP}}(P)$, to be the output of Algorithm 6.10. As a consequence of Proposition 6.13 below, it can be seen that $S_{\mathcal{GP}}(P)$ is a Gelfand pattern and that its top row is λ .

Let $T \in \mathcal{T}_\lambda$ be such that $T = \Theta^{-1}(P)$. We stated the scanning method for semistandard tableaux in 2.5. Let $S_{\mathcal{T}}(T)$ denote the scanning tableau of T . In the process of running

Algorithm 6.10 for the pattern P , we produce Gelfand patterns $B(t)$ for $1 \leq t \leq \lambda_1$. These Gelfand patterns will consist of only 0's and 1's. It will be seen in Lemma 6.22 that the Gelfand pattern $B(t)$ is equal to the image under Θ of column t of $S_{\mathcal{T}}(T)$. To form our scanning pattern, we add up these individual Gelfand patterns $B(t)$.

Algorithm 6.10 (Gelfand pattern scanning method).

Input: $P \in \mathcal{GP}_\lambda$, Output: $S_{\mathcal{GP}}(P) \in \mathcal{GP}_\lambda$.

For $t = 1$ to $t = P_{1,1}$ do:

1. Initialize $B(t)$ to be the n -Gelfand pattern with all its entries equal to 0.
2. Define the Gelfand pattern $P(t)$ by $(P(t))_{i,j} = \max\{P_{i,j} + 1 - t, 0\}$. Note that $P(1) = P$.
3. While $(P(t))_{1,1} > 0$:
 - (a) Scan the entries of the diagonals of $P(t)$ from bottom to top starting with the rightmost diagonal and moving left. Let (i_1, j_1) be the position of the first non-zero scanned entry. Initialize $I(P(t))$ to be the sequence consisting of $P(t)_{i_1, j_1}$ copies of the position (i_1, j_1) . Continue scanning in the same fashion. If an entry larger than all previously scanned entries is scanned in a position (i, j) , let r be the largest previously scanned entry. Then append $P(t)_{i,j} - r$ copies of the position (i, j) to $I(P(t))$ if it is in a weakly higher row than the most recent position appended to $I(P(t))$. Let $I(P(t)) = ((i_1, j_1), \dots, (i_l, j_l))$ be the resulting sequence. We refer to this sequence of positions as a *scanning path*.
 - (b) For $1 \leq i \leq i_l$, increase the entry in position (i, j_1) of $B(t)$ by 1.
 - (c) Decrease $P(t)$ entrywise according to:

$$(P(t))_{i,j} := (P(t))_{i,j} - |\{(h, j) \in I(P(t)) \mid h \geq i\}|.$$

Return $S_{\mathcal{GP}}(P) := \sum_{t=1}^{\lambda_1} B(t)$.

Fix $1 \leq t \leq P_{1,1} = \lambda_1$. Note that while running the algorithm, the values of the Gelfand pattern $B(t)$ may not satisfy the inequalities of the values for a Gelfand pattern before we finish Step 3 for this value of t . But as a consequence of Lemma 6.22, it will be seen that the final $B(t)$ meets the inequalities in the definition of Gelfand pattern. The number of scanning paths formed to produce $B(t)$ will be equal to the number of non-zero diagonals that remain in P at the beginning of the t^{th} stage.

In Step 3.a of Algorithm 6.10, we scan the j^{th} diagonal of P from bottom to top for increases in the entries. This is analogous to scanning the j^{th} row of T from left to right for transitions from one value to a larger value.

Example 6.11. We illustrate some of the steps of the scanning method for the Gelfand pattern P from Example 6.1. For $t = 1$, we initialize

$$B(1) = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{array},$$

and

$$P(1) = P = \begin{array}{ccccc} 5 & 3 & 2 & 1 & 0 \\ & 4 & 3 & 2 & 1 \\ & & 4 & 2 & 1 \\ & & & 2 & 1 \\ & & & & 2 \end{array}.$$

Scanning as prescribed, we construct the sequence of positions for the first scanning path: $I(P(1)) = ((2, 4), (2, 3), (2, 2), (1, 1))$. Thus we increase the 0 in position (1, 4) of $B(1)$

by 1 to obtain

$$B(1) = \begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{array} .$$

We then decrease the entries in $P(1)$ according to Step 3.c. This produces the pattern

$$P(1) = \begin{array}{ccccc} 4 & 2 & 1 & 0 & 0 \\ & 4 & 2 & 1 & 0 \\ & & 4 & 2 & 1 \\ & & & 2 & 1 \\ & & & & 2 \end{array} .$$

Scanning again, we calculate $I(P(1)) = ((3, 3), (3, 2), (3, 1), (3, 1))$. We then increment the 0's in positions $(1, 3)$, $(2, 3)$ and $(3, 3)$ of $B(1)$ by 1 to obtain

$$B(1) = \begin{array}{ccccc} 0 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{array} .$$

Decreasing the entries in $P(1)$ produces

$$P(1) = \begin{array}{ccccc} 2 & 1 & 0 & 0 & 0 \\ & 2 & 1 & 0 & 0 \\ & & 2 & 1 & 0 \\ & & & 2 & 1 \\ & & & & 2 \end{array} .$$

We calculate the next scanning path to obtain $I(P(1)) = ((4, 2))$. We increment the 0's in positions $(1, 2)$, $(2, 2)$, $(3, 2)$ and $(4, 2)$ of $B(1)$ to obtain

$$B(1) = \begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 0 \\ & & 0 & 1 & 1 & . \\ & & & 0 & 1 \\ & & & & 0 \end{array}$$

Decreasing the entries in $P(1)$ again, we have

$$P(1) = \begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & . \\ & & & 2 & 0 \\ & & & & 2 \end{array}$$

Calculating the scanning path once more for $t = 1$, we obtain $I(P(1)) = ((5, 1))$. Incrementing the 0's in positions $(1, 1)$, $(2, 1)$, $(3, 1)$, $(4, 1)$ and $(5, 1)$ of $B(1)$ produces

$$B(1) = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & . \\ & & & 1 & 1 \\ & & & & 1 \end{array}$$

Decreasing the entries in $P(1)$, we obtain

$$P(1) = \begin{matrix} & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{matrix} .$$

Since $P(1)_{1,1} = 0$, we have finished calculating $B(1)$.

We set $t = 2$ and initialize

$$B(2) = \begin{matrix} & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{matrix} ,$$

and

$$P(2) = \begin{matrix} & 4 & 2 & 1 & 0 & 0 \\ & & 3 & 2 & 1 & 0 \\ & & & 3 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{matrix} .$$

After calculating all the scanning paths for $t = 2$ and incrementing the entries of $B(2)$, we have

$$B(2) = \begin{matrix} & 1 & 1 & 1 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{matrix} .$$

Now let $\pi = (3, 5, 4, 1, 2)$. Then

$$S_{\mathcal{GP}}(P) = \begin{array}{cccccc} 5 & 3 & 2 & 1 & 0 & \\ & 3 & 2 & 1 & 0 & \\ & & 3 & 2 & 1 & \not\leq \\ & & & 2 & 1 & \\ & & & & 2 & \end{array} \qquad \begin{array}{cccccc} 5 & 3 & 2 & 1 & 0 & \\ & 5 & 2 & 1 & 0 & \\ & & 5 & 1 & 0 & \\ & & & 1 & 0 & \\ & & & & 1 & \end{array}$$

Again, this last Gelfand pattern is $K_\lambda(3, 5, 4, 1, 2)$. Note, for instance, that $S_{\mathcal{GP}}(P)_{3,1} = 3 \not\leq 5 = K_\lambda(3, 5, 4, 1, 2)_{3,1}$. Thus $P \notin \mathcal{GP}_\lambda(3, 5, 4, 1, 2)$.

■

For a key tableau T , it is known that $S_{\mathcal{T}}(T) = T$. Thus it will be seen from Proposition 6.13 that for a key pattern P , we have $S_{\mathcal{GP}}(P) = P$. Then, since the highest weight pattern is the maximal key pattern with top row λ , the highest weight pattern is a Demazure pattern for every permutation $\pi \in S_n^Q$. For tableaux, it is known that the only Demazure tableau for the identity permutation $e = (1, 2, \dots, n-1, n)$ is $Y_\lambda(e)$. Thus it will be seen from Corollary 6.14 that the only Demazure pattern for the identity permutation is $K_\lambda(e)$, which is equal to the highest weight pattern with top row λ . Let $\pi_0 \in S_n^Q$ be the longest permutation. Then $K_\lambda(\pi_0)$ is the lowest weight pattern. Since the lowest weight pattern is the minimal key pattern with top row λ , every Gelfand pattern with top row λ is a Demazure pattern for π_0 .

The following proposition states that the bijection Θ commutes with the scanning methods for tableaux and Gelfand patterns.

Proposition 6.13. *Let $T \in \mathcal{T}_\lambda$. Then $S_{\mathcal{GP}}(\Theta(T)) = \Theta(S_{\mathcal{T}}(T))$.*

Proposition 6.13 allows us to restrict the map Θ to become a weight preserving bijection from the set of Demazure tableaux to the set of Demazure patterns:

Corollary 6.14. *The map Θ restricts to become a weight preserving bijection $\Theta|_{\mathcal{D}_\lambda(\pi)} : \mathcal{D}_\lambda(\pi) \rightarrow \mathcal{GP}_\lambda(\pi)$.*

We know from Section 2.7 that we can write the Demazure polynomial for λ and π as the sum of the weight monomials of the Demazure tableaux for π on the shape λ . Using Corollary 6.14, we can immediately write the Demazure polynomial for λ and π as the sum over Demazure patterns for π with top row λ :

Theorem 6.15. *Let $n \geq 1$. Fix an n -partition λ and $\pi \in S_n^Q$. Then*

$$d_\lambda(\pi; x) = \sum_{P \in \mathcal{GP}_\lambda(\pi)} x^P.$$

6.5 Proofs

Throughout this section, fix an integer $n \geq 1$, an n -partition λ , a semistandard tableau $T \in \mathcal{T}_\lambda$ and a Gelfand pattern $P \in \mathcal{GP}_\lambda$ defined by $P := \Theta(T)$.

Proof of Lemma 6.6. Define $T^{<i>}$ to be the subtableau of T consisting of the values no larger than $n + 1 - i$. Note that $T^{<1>} = T$ and $\text{shape}(T^{<i>}) = P_i$ for $1 \leq i \leq n$.

Suppose T is a key tableau. We want to show that $\Theta(T)$ is a key pattern. Fix $1 \leq i \leq n - 1$. Consider the subtableau $T^{<i>}$. Clearly $T^{<i>}$ is a key tableau. Since $n + 1 - i$ is the largest possible value in $T^{<i>}$, any value equal to $n + 1 - i$ in $T^{<i>}$ must be a column bottom. Also, if there is an value equal to $n + 1 - i$ in a row, then $n + 1 - i$ must be the rightmost value in that row. Thus the rightmost value in $T^{<i>}$ equal to $n + 1 - i$ must be in a southeast corner of $T^{<i>}$. Since $T^{<i>}$ is a key tableau, if there are any values equal to $n + 1 - i$ in $T^{<i>}$, then these values must be the column bottoms from the first column of $T^{<i>}$ to a column whose bottom location is a southeast corner of $T^{<i>}$.

We want to show that the multiset of entries in the partition P_{i+1} forms a submultiset of the multiset of entries of the partition P_i . If $T^{<i>} = \phi$, then P_i is the $n + 1 - i$ partition of all 0's. By the inequalities in the definition of Gelfand pattern, we must have that P_{i+1} is the $n - i$ partition of all 0's. This multiset of entries is a submultiset of the entries in P_i .

If $T^{<i>} \neq \phi$, let $(P_i)_{t_1} > \dots > (P_i)_{t_l} > 0$ be the distinct row lengths of $T^{<i>}$ for some $l \geq 1$ and some $t_1 < t_2 < \dots < t_l$. Let the rightmost value equal to $n + 1 - i$ in $T^{<i>}$ be in a

row of length $(P_i)_{t_k}$ for some $1 \leq k \leq l$. Construct $T^{<i+1>}$ from $T^{<i>}$ by deleting the values equal to $n + 1 - i$. We will delete values equal to $n + 1 - i$ from right to left from $T^{<i>}$ and consider the effect on the partition P_i . We begin by deleting the values equal to $n + 1 - i$ in the lowest row of length $(P_i)_{t_k}$. If the lowest row of length $(P_i)_{t_k}$ is not the bottom row of $T^{<i>}$, then we will reach a column of a greater length whose bottom value is equal to $n + 1 - i$. This value will be in a row lower than row $(P_i)_{t_k}$. Thus deleting the values in the lowest row of length $(P_i)_{t_k}$ shortens that row so that its length becomes $(P_i)_{t_{k+1}}$. Then the effect on the partition P_i is that the rightmost $(P_i)_{t_k}$ in the partition P_i becomes $(P_i)_{t_{k+1}}$. Similarly, for $k \leq s \leq l - 1$, the rightmost entry equal to $(P_i)_{t_s}$ in the partition P_i becomes $(P_i)_{t_{s+1}}$ after deleting the values equal to $n + 1 - i$. The last values we delete are the values equal to $n + 1 - i$ in the lowest row of length $(P_i)_{t_l}$. This has the effect that the rightmost non-zero entry of P_i becomes 0. Finally, the rightmost 0 of the partition P_i is then deleted. Thus the net effect of this deleting process is that $(P_i)_{t_k}$ gets deleted from P_i when proceeding from P_i to P_{i+1} . Thus for $1 \leq i \leq n - 1$, to proceed from P_i to P_{i+1} we delete one entry in P_i . Hence P is a key pattern.

Now suppose that P is a key pattern. Fix $1 \leq i \leq n - 1$. The entries of P_{i+1} form a submultiset of the entries of P_i . That is, the partition P_i can be obtained from P_{i+1} by inserting an entry into P_{i+1} . Let r_i be the entry to be inserted. If P_{i+1} is the partition with all 0's, then $T^{<i+1>} = \phi$. In this case, change the leftmost 0 of P_{i+1} to r_i . Changing this entry to r_i to form P_i has the effect on the tableau $T^{<i+1>}$ of inserting r_i values equal to $n + 1 - i$ in row 1 to form $T^{<i>}$. Hence the values in $T^{<i>}$ equal to $n + 1 - i$ are in the columns from 1 to r_1 .

If P_{i+1} contains a non-zero entry, let $(P_{i+1})_{t_1} > \dots > (P_{i+1})_{t_l} > 0$ be the distinct non-zero entries of $P_{i+1} = \text{shape}(T^{<i+1>})$ for some $l \geq 1$ and some $t_1 < t_2 < \dots < t_l$. Let $k \geq 1$ be minimal such that $r_i \geq (P_{i+1})_{t_k}$. Thus we must insert r_i just to the left of the leftmost entry of P_{i+1} equal to $(P_{i+1})_{t_k}$. We can then view inserting the entry r_i into the partition P_{i+1} as the following sequence of changes to P_{i+1} : Append 0 to the right of P_{i+1} . Then change the leftmost 0 of P_{i+1} into $(P_{i+1})_{t_l}$. Then change the leftmost $(P_{i+1})_{t_l}$ into $(P_{i+1})_{t_{l-1}}$. Continue

this process, changing the leftmost $(P_{i+1})_{t_s}$ into $(P_{i+1})_{t_{s-1}}$ for s from l to $k + 1$. Finally, change the leftmost $(P_{i+1})_{t_k}$ into r_i . Now view the effect of this process of appending a 0 and changing the entries of the partition on the tableau $T^{<i+1>}$: Changing a 0 to $(P_{i+1})_{t_l}$ in the partition appends values equal to $n + 1 - i$ to the bottoms of the leftmost $(P_{i+1})_{t_l}$ columns. Changing the subsequent entries in the partition, we append values equal to $n + 1 - i$ to the bottom of consecutive columns from left to right until get to the column r_i . Thus the value $n + 1 - i$ is in every column from column 1 to column r_i . Since the values equal to $n + 1 - i$ meet this criterion of a key tableau for all $1 \leq i \leq n$, the tableau T is a key tableau.

Fix $\pi \in S_n^Q$. It is routine to check that distinct key patterns have distinct weight monomials and distinct key tableaux have distinct weight monomials. It is also routine to check that $K_\lambda(\pi)$ and $Y_\lambda(\pi)$ have the same weight monomials. Thus, by Lemma 6.5, we have $\Theta(Y_\lambda(\pi)) = K_\lambda(\pi)$. ■

To prove Proposition 6.13, we first state and prove seven lemmas. These lemmas describe the steps of the Gelfand pattern scanning method in terms of the steps of the tableau scanning method.

For $1 \leq t \leq \lambda_1$, let $P(t)$ be defined as in Step 2 of Algorithm 6.10. Define $T(t)$ to be the tableau obtained from T by deleting the first $t - 1$ columns. We then have the following lemma:

Lemma 6.16. *For $1 \leq t \leq \lambda_1$, we have $\Theta(T(t)) = P(t)$.*

Proof. Consider the entry $P_{i,j}$ for some $i + j \leq n + 1$. By Lemma 6.8, this entry $P_{i,j}$ is equal to the number of values in row j of T less than or equal to $n + 1 - i$. Consider the following two cases:

Case 1: Suppose $P_{i,j} \leq t - 1$. Then $P_{i,j} - (t - 1) \leq 0$ and $P(t)_{i,j} := \max\{P_{i,j} - (t - 1), 0\} = 0$. In this case, by Lemma 6.8, there are no more than $t - 1$ values in row j of T that are less than or equal to $n + 1 - i$. Since the smaller values in row j are to the left, when we delete the first $t - 1$ columns of T to form $T(t)$, we remove the values that are less than or equal to $n + 1 - i$. Hence we have 0 values in row j of $T(t)$ less than or equal to $n + 1 - i$. Thus $\Theta(T(t))_{i,j} = 0 = P(t)_{i,j}$.

Case 2: Now suppose $P_{i,j} > t - 1$. Then $P_{i,j} - (t - 1) > 0$ and $P(t)_{i,j} := \max(P_{i,j} - (t - 1), 0) = P_{i,j} - (t - 1)$. In this case, row j of T contains at least t values less than or equal to $n + 1 - i$. Removing the first $t - 1$ columns of T reduces the number of values in row j less than or equal to $n + 1 - i$ by $t - 1$. Thus, by Lemma 6.8, we have $\Theta(T(t))_{i,j} = \Theta(T)_{i,j} - (t - 1) = P_{i,j} - (t - 1) = \max\{P_{i,j} - (t - 1), 0\} =: P(t)_{i,j}$. ■

Now consider scanning the entries of P as prescribed in Step 3.a of Algorithm 6.10. The entries of interest in P are those entries which are greater than any previously scanned entries. In the next two lemmas, we relate these entries of P to the column bottoms of T . For the following two lemmas, fix (i, j) such that $i + j \leq n + 1$. Scanning the entries of P , let r be the largest entry of P scanned before $P_{i,j}$.

Lemma 6.17. *If $P_{i,j} > r$, then there are $P_{i,j} - r$ values in row j of T that are equal to $n + 1 - i$ and are column bottoms.*

Lemma 6.18. *If there is at least one value in row j of T that is equal to $n + 1 - i$ and is a column bottom, then $P_{i,j} > r$.*

Proof of Lemma 6.17. Suppose that $P_{i,j} > r$. By Lemma 6.8, we know that there are $P_{i,j}$ values in row j of T no larger than $n + 1 - i$. We consider two cases:

Case 1: The entry r is first scanned in P in diagonal j . Then the largest entry in diagonal j below row i is in row $i + 1$ and is equal to r . That is, we have $P_{i+1,j} = r$. From Lemma 6.8, the number of values in row j of T that are equal to $n + 1 - i$ is equal to $P_{i,j} - P_{i+1,j} = P_{i,j} - r$. The first $P_{i+1,j} = r$ values in row j of T are the values that are no larger than $n + 1 - (i + 1) = n - i$. Thus the values equal to $n + 1 - i$ in row j must occupy the locations in columns $r + 1$ to $P_{i,j}$. Since in this case, the entry r is first scanned in diagonal j , every entry in a diagonal to the right of diagonal j must be less than r . In particular, we have $P_{1,j+1} < r$. Thus the length of row $j + 1$ of T is less than r . Hence each one of the $P_{i,j} - r$ values in row j of T that is equal to $n + 1 - i$ is a column bottom since there is no value below it.

Case 2: The entry r is first scanned in P in a diagonal to the right of diagonal j . In this case, we have $P_{i+1,j} \leq r$. By Lemma 6.8, row j of T contains $P_{i,j} - P_{i+1,j} \geq P_{i,j} - r$ values

equal to $n + 1 - i$. These values are located in columns $P_{i+1,j} + 1$ to $P_{i,j}$. Since the entry r is scanned in P in a diagonal to the right of the column j , the maximum entry in a diagonal to the right of diagonal j is r . The entry $P_{1,j+1}$ is also equal to the maximum entry in a diagonal to the right of diagonal j . Thus we must have $P_{1,j+1} = r$. Hence row $j + 1$ of T has length r . Since the values equal to $n + 1 - i$ in row j are located in columns $P_{i+1,j} + 1$ to $P_{i,j}$, the number of the values equal to $n + 1 - i$ in row j of T which are not column bottoms is equal to $r - P_{i+1,j} \geq 0$. Hence the number of values in column j of T equal to $n + 1 - i$ which are column bottoms is equal to $(P_{i,j} - P_{i+1,j}) - (P_{i+1,j} - r) = P_{i,j} - r$. ■

Proof of Lemma 6.18. Suppose there is a value in row j of T that is equal to $n + 1 - i$ and is a column bottom. Consider the same cases as in the proof of Lemma 6.17:

Case 1: The entry r is first scanned in P in diagonal j . As above, we have $P_{i+1,j} = r$. By Lemma 6.8, the difference $P_{i,j} - P_{i+1,j} = P_{i,j} - r$ is equal to the number of values in row j of T that are equal to $n + 1 - i$. Since there is at least one such value, we have $P_{i,j} - r > 0$ and $P_{i,j} > r$.

Case 2: The entry r is first scanned in P in a diagonal to the right of diagonal j . From our reasoning in Case 2 of the proof of Lemma 6.17, we have $P_{1,j+1} = r$. That is, row $j + 1$ in T has length r . From Lemma 6.8, we know that $P_{i,j}$ is equal to the number of values in row j that are no larger than $n + 1 - i$. Since there is a value in row j that is equal to $n + 1 - i$, the value in row j and column $P_{i,j}$ must be $n + 1 - i$. This must be the rightmost value equal to $n + 1 - i$ in row j . Since at least one of the values equal to $n + 1 - i$ in row j is a column bottom, this rightmost entry must be a column bottom. Since there is no value below this value in column $P_{i,j}$, the entry $P_{i,j}$ must be greater than the length of row $j + 1$. That is, we have $P_{i,j} > P_{1,j+1} = r$. ■

In the next lemma, we describe the Gelfand pattern scanning path of $P = \Theta(T)$ in terms of the tableau scanning path of T . Note that the process of finding a Gelfand pattern scanning path is stated in Step 3.a of Algorithm 6.10 for the Gelfand pattern $P(t)$. This process can readily be applied to a general Gelfand pattern P .

For $T \in \mathcal{T}_\lambda$, we refer to the sequence of locations in the scanning path of T starting at

the bottom entry of the first column of T simply as the scanning path of T .

Lemma 6.19. *Suppose the tableau scanning path of T consists of $l \geq 1$ locations and for $1 \leq k \leq l$, the k^{th} location in the tableau scanning path of T is in row j_k and contains the value $n + 1 - i_k$ for some $1 \leq j_k \leq n$ and $j_k \leq n + 1 - i_k \leq n$. Then the Gelfand pattern scanning path of P consists of l positions and for $1 \leq k \leq l$, the k^{th} position in the Gelfand pattern scanning path is (i_k, j_k) .*

Proof. We prove this lemma using strong induction on the index of the locations in the tableau scanning path. For some $1 \leq j_1 \leq n$ and $j_1 \leq n + 1 - i_1 \leq n$, let the first location in the tableau scanning path be in row j_1 and contain the value $n + 1 - i_1$. Then j_1 is the bottom row of T . By Lemma 6.8, we know that every entry in P in a diagonal to the right of diagonal j_1 is equal to 0. Further, since T contains a value in row j_1 , there is a non-zero entry in diagonal j_1 of P . Hence the second coordinate of the first position in the Gelfand pattern scanning path of P is j_1 . Since the first location in the tableau scanning path is in row j_1 and contains the value $n + 1 - i_1$, the value $n + 1 - i_1$ is the smallest value in row j_1 of T . Thus, by Lemma 6.8, we have $P_{i_1+1, j_1} = 0$ and $P_{i_1, j_1} > 0$. Hence the first non-zero entry we encounter when scanning the Gelfand pattern entries is in diagonal j_1 and row i_1 . Thus the algorithm initializes the Gelfand pattern scanning path of P with at least one copy of (i_1, j_1) .

Fix $1 \leq k < l$. Suppose that for $1 \leq h \leq k$, the h^{th} location in the tableau scanning path is in row j_h and contains the value $n + 1 - i_h$. Also suppose that the h^{th} position in the Gelfand pattern scanning path is (i_h, j_h) . We need to show that if the $(k + 1)^{\text{st}}$ location in the tableau scanning path is in row j_{k+1} and contains the value $n + 1 - i_{k+1}$, then the $(k + 1)^{\text{st}}$ position in the Gelfand pattern scanning path is (i_{k+1}, j_{k+1}) . Starting at the k^{th} location in the tableau scanning path, we scan the column bottoms to the right to find the first value which is weakly larger than $n + 1 - i_k$. Let the $(k + 1)^{\text{st}}$ location in the tableau scanning path be in row j_{k+1} and contain the value $n + 1 - i_{k+1}$. Consider the following two cases:

Case 1: either $j_{k+1} < j_k$ or $i_{k+1} < i_k$ or both. The $(k + 1)^{\text{st}}$ location in the tableau

scanning path must be a column bottom. Thus, by Lemma 6.18, the entry $P_{i_{k+1}, j_{k+1}}$ is larger than any previously scanned entry. Further, since $n + 1 - i_{k+1} \geq n + 1 - i_k$, we have $i_{k+1} \leq i_k$ and hence $P_{i_{k+1}, j_{k+1}}$ is in a weakly higher row than P_{i_k, j_k} .

Let the $(k + 1)^{st}$ position in the Gelfand pattern scanning path be (i'_{k+1}, j'_{k+1}) . Then $P_{i'_{k+1}, j'_{k+1}}$ is larger than any previously scanned entry and it is in a weakly higher row than P_{i_k, j_k} . For the sake of contradiction, suppose $(i'_{k+1}, j'_{k+1}) \neq (i_{k+1}, j_{k+1})$. Then the entry in position (i'_{k+1}, j'_{k+1}) must be scanned before the entry in position (i_{k+1}, j_{k+1}) . Otherwise, the location (i'_{k+1}, j'_{k+1}) would not be the next position in the scanning path. Since the entry in position (i'_{k+1}, j'_{k+1}) of P is scanned before the entry in position (i_{k+1}, j_{k+1}) , we have either

- (a) $j'_{k+1} = j_{k+1}$ and $i'_{k+1} > i_{k+1}$ or
- (b) $j'_{k+1} > j_{k+1}$.

In Case (a) and Case (b), the entry $P_{i'_{k+1}, j'_{k+1}}$ is larger than any previously scanned entry of P . Thus, by Lemma 6.17, there is a value in row j'_{k+1} of T that is equal to $n + 1 - i'_{k+1}$ and is a column bottom.

In Case (a), we have that the value $n + 1 - i'_{k+1}$ is a column bottom in row $j'_{k+1} = j_{k+1}$. We also have $n + 1 - i'_{k+1} < n + 1 - i_{k+1}$. Since these two values are in the same row $j'_{k+1} = j_{k+1}$ of T , the value $n + 1 - i'_{k+1}$ must be to the left of $n + 1 - i_{k+1}$. That is, the location of the column bottom value $n + 1 - i'_{k+1}$ is to the left of the $(k + 1)^{st}$ location in the tableau scanning path.

The rightmost value equal to $n + 1 - i'_{k+1}$ in row j'_{k+1} is in column $P_{i'_{k+1}, j'_{k+1}}$ of T . Since $P_{i'_{k+1}, j'_{k+1}} > P_{i_k, j_k}$, the value in row j'_{k+1} of T that is equal to $n + 1 - i'_{k+1}$ is in a column to the right of the k^{th} entry in the tableau scanning path. Since $n + 1 - i_k \leq n + 1 - i'_{k+1}$, the tableau scanning path would have included this location in row j'_{k+1} which contains the value $n + 1 - i'_{k+1}$. This is a contradiction since the tableau scanning path skips this location.

In Case (b), the value $n + 1 - i'_{k+1}$ is a column bottom in row j'_{k+1} . Since $j'_{k+1} > j_{k+1}$, the location of this value is in a lower row than the $(k + 1)^{st}$ location in the tableau scanning path. Note that both locations are column bottoms. Thus the location of the value $n + 1 - i'_{k+1}$ in row j'_{k+1} must be in a column to the left of the $(k + 1)^{st}$ location in the tableau scanning

path.

Note that $j'_{k+1} \leq j_k$. Thus the location of this value equal to $n + 1 - i'_{k+1}$ in row j'_{k+1} is in a weakly higher row than the k^{th} location in the tableau scanning path. Further, we have $n + 1 - i'_{k+1} \geq n + 1 - i_k$. Thus the location of this value equal to $n + 1 - i'_{k+1}$ in row j'_{k+1} must be to the right of the k^{th} location in the tableau scanning path. Just as in Case (a), this is a contradiction since the tableau scanning path would have included this location.

Case 2: $j_{k+1} = j_k$ and $i_{k+1} = i_k$. In this case, the $(k+1)^{st}$ location in the tableau scanning path is in the same row as and contains a value equal to the value in the k^{th} location of the tableau scanning path of T . Since (i_k, j_k) is the k^{th} position of the scanning path of P , the entry P_{i_k, j_k} must be larger than any previously scanned entry. Let r be the largest entry of P scanned before P_{i_k, j_k} . By Lemma 6.17, there are $P_{i_k, j_k} - r$ values in row j_k of T that are equal to $n + 1 - i_k$ which are column bottoms. It is clear that if one of the values equal to $n + 1 - i_k$ in row j_k is in a location of the tableau scanning path, then every such value that is a column bottom is also in a location of the tableau scanning path. That is, the number of locations in the tableau scanning path in row j_k that contain $n + 1 - i_k$ is equal to the number of such locations which are column bottoms. From above, the number of such locations is $P_{i, j} - r$. This is exactly the number of locations in the Gelfand scanning path equal to (i_k, j_k) as given in Step 3.a of Algorithm 6.10.

By induction, the claim of this lemma is true for the first k locations in the tableau scanning path. For some $1 \leq b \leq k$, let the first location in the tableau scanning path in row j_k that contains the value $n + 1 - i_k$ be the b^{th} location in the tableau scanning path. Then the first position in the Gelfand pattern scanning path equal to (i_k, j_k) is the b^{th} position. Thus if the $(k + 1)^{st}$ location of the tableau scanning path is in row $j_{k+1} = j_k$ and contains the value $n + 1 - i_{k+1} = n + 1 - i_k$, then the $(k + 1)^{st}$ position in the Gelfand pattern scanning path is $(i_k, j_k) = (i_{k+1}, j_{k+1})$.

Finally, we need to prove that the tableau and Gelfand patterns scanning paths have the same number of locations and positions respectively. From above, we know that the number of positions in the Gelfand pattern scanning path is at least as large as the number l of

locations in the tableau scanning path. Suppose the Gelfand pattern scanning path contains at least one more position (i_{l+1}, j_{l+1}) . Then, by Lemma 6.17, there would be a value in row j_{l+1} of T that is equal to $n + 1 - i_{l+1}$ and is a column bottom. Since $P_{i_{l+1}, j_{l+1}} > P_{i_l, j_l}$, the location of this value of T would be to the right of the l^{th} location in the tableau scanning path. Since we also have $n + 1 - i_{l+1} \geq n + 1 - i_l$, the scanning path in T would have extended to this location. This is a contradiction. Thus the Gelfand pattern scanning path contains l locations. ■

Fix an n -partition μ whose Young diagram consists of one column. To state Lemma 6.20, we define a *precolumn* to be a partial filling of the Young diagram of μ with integers from $[n]$. Let C be a precolumn on the shape μ . We refer to the locations in C which have not been filled with an integer from $[n]$ as *empty locations*. For $1 \leq i \leq |\mu|$, we restrict the values in the nonempty locations of C by $C_{i,1} \geq i$.

We then extend the domain of the map Θ to the set of precolumns: Let C be a precolumn on the shape μ . Define $\Theta(C)$ as follows: For $1 \leq j \leq |\mu|$, if the location $(j, 1)$ of C is nonempty, then place $n + 1 - C_{j,1}$ top justified 1's in diagonal j of $\Theta(C)$; set the rest of the entries in diagonal j equal to 0. If the location $(j, 1)$ of C is empty, then set every entry in diagonal j of $\Theta(C)$ equal to 0. For $|\mu| + 1 \leq j \leq n$, set every entry in diagonal j of $\Theta(C)$ equal to 0. This extension of the domain of Θ clearly requires an extension of the codomain as well. The image of a precolumn will not meet the inequalities in the definition of Gelfand pattern if an empty location is above a nonempty location. In this case, there will be a 0 to the left of a 1 in row 1 of $\Theta(C)$. It can be seen from Lemma 6.8 that this extension of the definition of Θ agrees with the original definition of Θ where they are both defined: on column tableau.

Let $1 \leq j \leq n$ be an integer and let C be a precolumn on the shape μ with an empty location $(j, 1)$. Let $j \leq a \leq n$ and $j \leq d \leq n$ be integers. We notate the result of inserting the value a into the empty location $(j, 1)$ of C by $(a)_j \xrightarrow{\mathcal{T}} C$. For a pattern P , we notate the result of incrementing the entries by 1 in positions (i, j) such that $1 \leq i \leq d$ by $(d, j) \xrightarrow{\mathcal{GP}} P$. For some $1 \leq t \leq \lambda_1$, let $I(P(t)) = ((i_1, j_1), \dots, (i_l, j_l))$ be the Gelfand pattern scanning

path of $P(t)$. Then Step 3.b of Algorithm 6.10 could be written using this notation as $(i, j_1) \xrightarrow{\mathcal{GP}} B(t)$. We then have the following:

Lemma 6.20. *Fix an n -partition μ such that the shape μ consists of one column. Let $1 \leq j \leq |\mu|$ be an integer and let C be a precolumn on the shape μ with an empty location $(j, 1)$. Fix $j \leq n + 1 - i \leq n$. Then $\Theta((n + 1 - i)_j \xrightarrow{\mathcal{T}} C) = (i, j) \xrightarrow{\mathcal{GP}} \Theta(C)$.*

Proof. Consider the effect of inserting the value $n + 1 - i$ into row j of C on the pattern $\Theta(C)$. Clearly only diagonal j of $\Theta(C)$ is affected. From the definition of Θ above, inserting the value $n + 1 - i$ into row j of C increments the $n + 1 - (n + 1 - i) = i$ top justified 0's in diagonal j of $\Theta(C)$. This pattern obtained from $\Theta(C)$ by incrementing these 0's is exactly $(i, j) \xrightarrow{\mathcal{GP}} \Theta(C)$. ■

For the next lemma, define P' to be the Gelfand pattern constructed by decreasing the entries of P according to Step 3.c of Algorithm 4.1. That is, define P' entrywise by $(P')_{i,j} := (P)_{i,j} - |\{(h, j) \in I(P) | h \geq i\}|$. Let T' be the tableau obtained from T by removing the values in the locations of the tableau scanning path of T . We then have the following:

Lemma 6.21. *For T' and P' defined as above, we have $\Theta(T') = P'$.*

Proof. For some $i + j \leq n + 1$, consider the entry $\Theta(T')_{i,j}$ in the Gelfand pattern $\Theta(T')$. By Lemma 6.8, the entry $\Theta(T')_{i,j}$ is equal to the number of values in row j of T' no larger than $n + 1 - i$. We can write this entry $\Theta(T')_{i,j}$ as [the number of values in row j of T no larger than $n + 1 - i$] minus [the number of locations in the scanning path of T in row j whose values are no larger than $n + 1 - i$]. For some $k \geq 1$, suppose the k^{th} location in the scanning path of T is in row j and contains the value $n + 1 - h \leq n + 1 - i$ for some $i \leq h \leq n$. By Lemma 5.4, the k^{th} position in the scanning path of P is then (h, j) with $i \leq h \leq n$. Thus the number of locations in the scanning path of T in row j whose values are no larger than $n + 1 - i$ is equal to $|\{(h, j) \in I(P) | h \geq i\}|$. Thus we can write $\Theta(T')_{i,j} = \Theta(T)_{i,j} - |\{(h, j) \in I(P) | h \geq i\}| = P_{i,j} - |\{(h, j) \in I(P) | h \geq i\}| =: (P')_{i,j}$. Hence we have $\Theta(T') = P'$. ■

Fix some $1 \leq t \leq \lambda_1$. Let $B(t)$ be the pattern produced by running Steps 1-3 of Algorithm 6.10 for t . Recall that $S_{\mathcal{T}}(T)_t$ denotes column t of $S_{\mathcal{T}}(T)$. For the final lemma before we prove Proposition 6.13, we show that the image of $S_{\mathcal{T}}(T)_t$ under the map Θ is the Gelfand pattern $B(t)$:

Lemma 6.22. *For $1 \leq t \leq \lambda_1$, we have*

$$\Theta(S_{\mathcal{T}}(T)_t) = B(t).$$

Proof. To calculate column t of the scanning tableau $S_{\mathcal{T}}(T)$, we first form the Young diagram $shape(T(t))$ to contain the values of $S_{\mathcal{T}}(T)_t$. We then calculate the values of $S_{\mathcal{T}}(T)_t$ and insert them into this Young diagram. Thus at each stage in the tableau scanning method for this particular t , the partially filled Young diagram is a precolumn. For the purposes of this proof, we will refer to these precolumns as $S_{\mathcal{T}}(T)_t$. Thus, at the outset, we refer to the newly formed Young diagram $shape(T(t))$ as $S_{\mathcal{T}}(T)_t$. After we have filled all the locations of the Young diagram we will have the final $S_{\mathcal{T}}(T)_t$.

To prepare to calculate the Gelfand pattern $B(t)$, the algorithm initializes $B(t)$ to be the Gelfand pattern of all 0's. Thus we have $\Theta(S_{\mathcal{T}}(T)_t) = B(t)$ when we start running both algorithms for this t value. By Lemma 6.16, we have $\Theta(T(t)) = P(t)$ at the outset. We begin by calculating the Gelfand pattern scanning path of $P(t)$ and the tableau scanning path of $T(t)$ using each method. For some $i_1 + j_1 \leq n + 1$, let the first location in the tableau scanning path of $T(t)$ be in row j_1 and contain the value $n + 1 - i_1$. Note that j_1 is the number of rows in $T(t)$. For some $l \geq 1$ and $i_l + j_l \leq n + 1$, let the final location in the tableau scanning path of $T(t)$ be in row j_l and contain the value $n + 1 - i_l$. Then by Lemma 6.19, the first position in the Gelfand pattern scanning path of $P(t)$ is (i_1, j_1) and the final position is (i_l, j_l) . For the tableau scanning method, we insert the value $n + 1 - i_l$ into the empty location $(j_1, 1)$ of the precolumn $S_{\mathcal{T}}(T)_t$. That is, we form $S_{\mathcal{T}}(T)_t := (n + 1 - i_l)_{j_1} \xrightarrow{\mathcal{T}} S_{\mathcal{T}}(T)_t$. For the Gelfand pattern scanning method, we now increment the entries of $B(t)$ to form $B(t) := (i_l, j_1) \xrightarrow{\mathcal{GP}} B(t)$. By Lemma 6.20, for the newly defined $B(t)$ and $S_{\mathcal{T}}(T)_t$, we have

$$\Theta(S_{\mathcal{T}}(T)_t) = B(t).$$

For the tableau scanning method, we now redefine $T(t)$ to be the tableau obtained from $T(t)$ by removing the locations in the scanning path of $T(t)$, which begins with the location $(j_1, 1)$. Similarly, for the the Gelfand pattern scanning method, we decrease the entries of $P(t)$ according to Step 3.c. Note that the j_1 diagonal becomes all 0's. By Lemma 6.21, we have $\Theta(T(t)) = P(t)$ after we remove the values of $T(t)$ and decrease the entries of $P(t)$.

For the Gelfand pattern scanning method, if $P(t)_{1,1} > 0$, we calculate the next Gelfand pattern scanning path of $P(t)$. Note that $T(t)$ is the null tableau if and only if $P(t)_{1,1} = 0$. So if $P(t)_{1,1} > 0$, we can calculate the next scanning path of the non-null tableau $T(t)$. We repeat the procedure above until $P(t)_{1,1} = 0$. This will happen after p iterations, where p is the number of rows in $T(t)$. Once we finish running Step 3 for this fixed value of t we have $\Theta(S_{\mathcal{T}}(T)_t) = B(t)$. ■

We can now use Lemma 6.22 to prove Proposition 6.13:

Proof of Proposition 6.13. In the tableau scanning method, we calculate $S_{\mathcal{T}}(T)$ column by column. That is, we calculate $S_{\mathcal{T}}(T)_t$ for $1 \leq t \leq \lambda_1$. We can then form $S_{\mathcal{T}}(T)$ as $\bigoplus_{t=1}^{\lambda_1} S_{\mathcal{T}}(T)_t$ by starting with column $S_{\mathcal{T}}(T)_{\lambda_1}$ on the right and prepending columns to the left for $t = \lambda_1 - 1$ to $t = 1$. We know from Lemma 6.22 that $\Theta(S_{\mathcal{T}}(T)_t) = B(t)$ for $1 \leq t \leq \lambda_1$. Then by successive applications of Lemma 6.9 and applying the definition of $S_{\mathcal{GP}}(P)$ in Algorithm 6.10, we have

$$\Theta(S_{\mathcal{T}}(T)) = \Theta \left(\bigoplus_{t=1}^{\lambda_1} S_{\mathcal{T}}(T)_t \right) = \sum_{t=1}^{\lambda_1} B(t) =: S_{\mathcal{GP}}(P) = S_{\mathcal{GP}}(\Theta(T)).$$

■

It is now a straightforward exercise to prove Corollary 6.14:

Proof of Corollary 6.14. We first need to show that the image of a Demazure tableaux under the map $\Theta|_{\mathcal{D}_{\lambda}(\pi)}$ is a Demazure pattern: Let $T \in \mathcal{D}_{\lambda}(\pi)$. By definition, we have $S_{\mathcal{T}}(T) \leq Y_{\lambda}(\pi)$. Note that both $S_{\mathcal{T}}(T)$ and $Y_{\lambda}(\pi)$ are Demazure tableaux for π on the shape λ so we can apply the map $\Theta|_{\mathcal{D}_{\lambda}(\pi)}$ to both of them. Also note that Lemma 6.7 applies to

the map $\Theta|_{\mathcal{D}_\lambda(\pi)}$. Taking the image under $\Theta|_{\mathcal{D}_\lambda(\pi)}$ of both sides, we obtain $\Theta|_{\mathcal{D}_\lambda(\pi)}(S_{\mathcal{T}}(T)) \geq \Theta|_{\mathcal{D}_\lambda(\pi)}(Y_\lambda(\pi))$ by Lemma 6.7. By Proposition 6.13, the left side becomes $\Theta|_{\mathcal{D}_\lambda(\pi)}(S_{\mathcal{T}}(T)) = S_{\mathcal{GP}}(\Theta|_{\mathcal{D}_\lambda(\pi)}(T))$. By Lemma 6.6, the right side becomes $\Theta|_{\mathcal{D}_\lambda(\pi)}(Y_\lambda(\pi)) = K_\lambda(\pi)$. Thus $S_{\mathcal{GP}}(\Theta|_{\mathcal{D}_\lambda(\pi)}(T)) \geq K_\lambda(\pi)$. Hence $\Theta|_{\mathcal{D}_\lambda(\pi)}(T)$ is a Demazure pattern.

The map $\Theta|_{\mathcal{D}_\lambda(\pi)}$ is injective since Θ is injective. Let $P \in \mathcal{GP}_\lambda(\pi)$. An argument similar to the one above shows that $\Theta^{-1}(P) \in \mathcal{D}_\lambda(\pi)$. Since $\Theta|_{\mathcal{D}_\lambda(\pi)}(\Theta^{-1}(P)) = P$, we see that $\Theta|_{\mathcal{D}_\lambda(\pi)}$ is surjective. Finally, by Lemma 3.2, the map $\Theta|_{\mathcal{D}_\lambda(\pi)}$ is weight preserving.

Hence $\Theta|_{\mathcal{D}_\lambda(\pi)} : \mathcal{D}_\lambda(\pi) \rightarrow \mathcal{GP}_\lambda(\pi)$ is a weight preserving bijection. ■

Finally, Theorem 6.15 is immediate from Corollary 6.14:

Proof of Theorem 6.15. From Section 2.7, we know that $d_\lambda(\pi; x) = \sum_{T \in \mathcal{D}_\lambda(\pi)} x^T$. Applying Corollary 6.14 and $x^T = x^{\Theta(T)} = x^P$ from Lemma 6.5, we obtain

$$d_\lambda(\pi; x) = \sum_{P \in \mathcal{GP}_\lambda(\pi)} x^P.$$

■

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