Singular Milnor Numbers of Non-Isolated Matrix Singularities

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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2010

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Abstract

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(Under the direction of Professor James Damon)

In this dissertation we obtain formulas to describe the local topology of certain non-isolated matrix singularities. We find free divisors in various vector spaces of matrices which include the hypersurface of singular matrices as a component, and use these to express the singular Milnor numbers of matrix singularities in terms of the codimensions of groups of equivalences. On the spaces of symmetric and all $n \times n$ matrices, these free divisors arise through representations of finite dimensional solvable Lie groups; on the space of skew-symmetric matrices, we extend a finite-dimensional representation of a solvable Lie algebra to an infinite-dimensional one.

To my loving family.

Acknowledgements

Many thanks to the members of the Department of Mathematics of the University of North Carolina at Chapel Hill for their instruction, support, and encouragement. I wish to single out my advisor, James Damon, for his exceptional patience and invaluable guidance. His thorough feedback has drastically improved this dissertation. I also wish to thank my committee (Pat Eberlein, Shrawan Kumar, Jon Wahl, Prakash Belkale, and Alexander Varchenko) for their time, enthusiasm, and patience.

Table of Contents

List of Figures

Introduction

In this dissertation, we investigate the topology of matrix singularities. A matrix singularity may be defined by a matrix $(a_{ij}(z))_{i,j=1,\ldots,m}$ of holomorphic functions defined on an open neighborhood of $z_0 \in \mathbb{C}^n$. The matrix may be of a special type, such as symmetric or skew-symmetric (with m even). The set in $\mathbb{C}^n$ where this matrix is singular defines a local hypersurface $\mathscr{V}$ near $z_0 \in \mathbb{C}^n$. Equivalently, we may view this matrix as a holomorphic function f from an open neighborhood of $z_0 \in \mathbb{C}^n$ to a vector space of matrices M. Then $\mathscr{V}$ is formed by $f^{-1}(V)$, where V is the determinantal variety consisting of the singular matrices in M. Matrix singularities typically inherit non-isolated singularities from V.

Since the 1960's, Milnor, Hamm, Greuel, Brieskorn, Arnol'd, and many others have made significant progress in the study of the topology of isolated singularities, especially isolated hypersurface and complete intersection singularities. This is partly due to the fact that for such isolated singularities the Milnor fiber, a smooth fiber of the function defining the singularity in a neighborhood of the singularity, has a simple topology. It is homotopy equivalent to a bouquet of spheres with dimension equal to the dimension of the variety. The number of spheres is called the Milnor number of the isolated singularity. For non-isolated singularities, including most matrix singularities, the Milnor fiber may have nonzero homology in many dimensions and its topology is essentially not computable.

In this work we study the topology of matrix singularities by replacing the Milnor fiber with a "singular Milnor fiber" obtained through a deformation f_t of f, where for $t \neq 0$, f_t is transverse to V. Although the singular Milnor fiber is not smooth, by results of Damon and Mond [**DM91, Dam96a**] it again has the homotopy type of a bouquet of spheres with dimension equal to the dimension of the matrix singularity. The number of spheres is called the "singular Milnor number." In this work we develop a method for computing singular Milnor numbers of matrix singularities by combining the Thom-Mather approach to singularity theory with the representation theory of solvable linear algebraic groups.

Fundamental to our method are a class of hypersurfaces introduced by Saito called "free divisors." These arise as discriminants of versal unfoldings, hyperplane arrangements, and bifurcation sets. If the determinantal variety V was a free divisor, then the singular Milnor number of matrix singularities could be calculated as the codimension of f for a group of equivalences, which in turn is the dimension of a module when considered as a complex vector space. However, V is almost never a free divisor.

Instead, we find free divisors of the form $V \cup W$, where W is also a free divisor. The singular Milnor numbers of nonlinear sections of V, W, and $V \cup W$ are related. This provides an inductive process for computing the singular Milnor numbers of matrix singularities.

To find such free divisors, we study representations of connected complex algebraic groups having an open orbit, particularly when the dimensions of the group and the vector space agree. Mond first observed that such representations may be combined with Saito's criterion to find free divisors. He and Buchweitz ([**BM06**]) found free divisors associated to irreducible representations of certain reductive groups. In contrast, we study representations of solvable groups which have a complete flag of invariant subspaces.

Our motivating families of representations come from real matrix factorization results in numerical linear algebra (LU, Cholesky factorization of symmetric matrices, and a Cholesky-like factorization for skew-symmetric matrices). The complex analogues of these factorizations may be interpreted as statements describing the open orbits of associated representations. We show that the union of all the non-open orbits, the "exceptional orbit variety," is either a free divisor or a weaker non-reduced free* divisor. Although representations having open orbits have been studied before by Sato and Kimura as "prehomogeneous vector spaces," our approach is quite different from theirs.

For representations of solvable linear algebraic groups, the Lie-Kolchin theorem ensures a complete flag of invariant subspaces. We define "block representations," which are representations with a partial flag of distinguished invariant subspaces and associated normal subgroups. The structure of a block representation allows us to make an associated matrix of coefficients block lower triangular. From this form, we are able to identify the exceptional orbit variety explicitly and apply Saito's criterion to conclude that it is a free divisor. Weaker "non-reduced block representations" have exceptional orbit varieties which are only non-reduced free* divisors. Our families of representations yield "towers of representations" whose exceptional orbit varieties often form a "tower of free divisors."

Although computing the singular Milnor numbers of free* divisors is possible, it is complicated by the appearance of "virtual singularities." As the LU factorization and Cholesky-like factorization for skew-symmetric matrices only yield free* divisors, we modify these representations to obtain free divisors. For the modified LU representation, we introduce a different solvable group. For the skew-symmetric matrices, we extend a finite-dimensional representation of a solvable Lie algebra by nonlinear "Pfaffian vector fields" to obtain an infinite-dimensional solvable Lie algebra; we study this representation using the underlying ideas of the block representation, to again obtain free divisors.

Then we apply an inductive process, using the free divisors to obtain formulas which compute the singular Milnor numbers of matrix singularities. We give formulas for spaces of matrices of small size which are valid for source spaces of arbitrarily large dimension n. There appears to be a general inductive process, yet to be determined explicitly, for computing the singular Milnor numbers of matrix singularities.

The dissertation is structured as follows. In Chapter 1, we describe the classical Milnor number for isolated singularities, the singular Milnor number, and several useful formulas which compute the singular Milnor number in terms of the codimensions of groups of equivalences. We also describe the work of Goryunov and Mond [**GM05**] on the Milnor numbers of isolated matrix singularities, which motivated this study. In Chapter 2, we describe representations of complex algebraic Lie groups having open orbits, using three matrix factorizations as motivating examples. In Chapter 3, we describe our theory of block representations. In Chapter 4, we apply the theory of block representations to the examples of Chapter 2. Since two of these only yield free* divisors, in Chapter 5 we develop our modified representations and show that these yield free divisors. We also describe a general extension mechanism for block representations and their free divisors which applies even to certain reductive groups. In Chapter 6, we use our free divisors to obtain formulas for the singular Milnor numbers of matrix singularities. The Appendix contains Lie bracket calculations used in Chapter 5 for the infinite-dimensional solvable Lie algebras.

CHAPTER 1

Background

In this first chapter, we summarize work which is relevant to the problem of studying the singular Milnor numbers of matrix singularities.

Although matrix singularities are generally highly non-isolated, we begin with a discussion in sections 1 and 2 of the classical Milnor fibration and the Milnor number for isolated hypersurface and complete intersection singularities. If $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ defines such a singularity, then the Milnor fiber is a regular fiber of f in a neighborhood of the singular point. In both cases, the Milnor fiber is homotopy equivalent to a bouquet of spheres. The Milnor number is then the number of spheres and can be computed algebraically as the length of one or more modules.

For hypersurfaces or complete intersections with non-isolated singularities, the Milnor fiber is much more complicated because in general it loses connectivity by the dimension of the singular set; in section 3, we sketch what is known in this area. This historical perspective allows comparison with corresponding results for nonlinear sections of free divisors and free complete intersections.

For non-isolated singularities which arise as a nonlinear section $f^{-1}(V)$ of a hypersurface or complete intersection V, a different approach is used to avoid the complexity of the Milnor fiber. We explain in section 4 how a *singular Milnor fiber* is obtained using a nonlinear section of V under a *stabilization* of f. The stabilization is described via transversality to the singular variety V. In a neighborhood of the singularity, the singular Milnor fiber again has the homotopy type of a bouquet of spheres and the number of such spheres is the singular Milnor number.

Unlike the situation for isolated singularities, there is no known universal algebraic formula for the singular Milnor number. In the important cases where V is a free divisor or free complete intersection, there are analogous algebraic formulas as in the isolated singularity case. These formulas are given as extended codimensions for various equivalence groups (in the sense of Thom and Mather). In section 5, we describe these equivalence groups and in section 6, we state several known formulas for the singular Milnor number.

Finally, we discuss in section 8 the notion of matrix singularity, its importance, and work previously done on understanding the Milnor number of isolated matrix singularities. In particular, a paper of Goryunov and Mond, [**GM05**], provided the original motivation for considering the singular Milnor number for non-isolated matrix singularities.

1.1. The Milnor number

1.1.1. The Milnor fiber. In [**Mil68**], Milnor studied the local topology of a holomorphic germ defining a hypersurface (we sometimes use *analytic* as a synonym for holomorphic). Let $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ be the germ of a hypersurface defined by the vanishing of an analytic germ $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$. There are several related locally trivial fibrations associated to this situation, each called "the Milnor fibration." Let $S_\epsilon(x)$ (respectively, $B_\epsilon(x)$) be the sphere (respectively, open ball) of radius ϵ centered at $x \in \mathbb{C}^{n+1}$.

The original fibration studied by Milnor was the map $\phi_1 : S_\epsilon(0) \setminus V \to S^1$ defined by

$$\phi_1(x) = \frac{f(x)}{|f(x)|}.$$

Milnor showed that there exists an $\epsilon_0 > 0$ so that if $0 < \epsilon < \epsilon_0$, then ϕ_1 is a smooth fiber bundle (see Theorem 4.8 of [**Mil68**]). Since S^1 is compact, all of the fibers are homeomorphic (even diffeomorphic).

We may also define two other functions which are simply restrictions of f. Let

$$\phi_2 : f^{-1}(B_\delta(0) \setminus \{0\}) \cap B_\epsilon(0) \to B_\delta(0) \setminus \{0\}$$

and

$$\phi_3 : f^{-1}(B_\delta(0) \setminus \{0\}) \cap \overline{B_\epsilon(0)} \to B_\delta(0) \setminus \{0\}$$

be restrictions of f. As above, there exists a $\epsilon_0 > 0$ so that if $0 < \epsilon < \epsilon_0$, then there exists $0 < \delta \ll \epsilon$ so that ϕ_2 and ϕ_3 are locally trivial fibrations (see [**Dim92**], p. 69).

In fact, the fibers of ϕ_1 and ϕ_2 are diffeomorphic ([**Mil68**], Theorem 5.11), and the generic fiber of ϕ_3 is a manifold with boundary whose interior is a fiber of ϕ_2. Up to homotopy equivalence, the fibers of ϕ_1, ϕ_2, or ϕ_3 are all *the Milnor fiber*. As a generic Milnor fiber is smooth, we can think of the Milnor fiber as a local smoothing of a hypersurface $f^{-1}(0)$ as depicted in Figure 1.1.

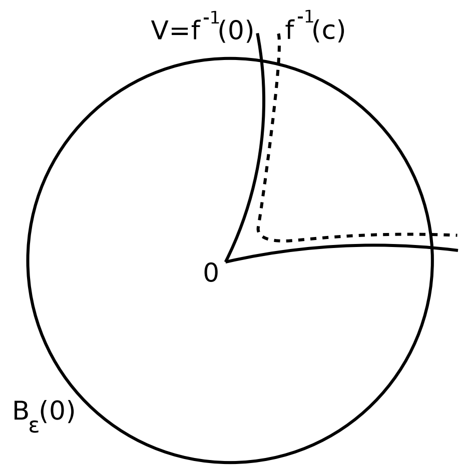

FIGURE 1.1. A diagram of the Milnor fiber.

Milnor showed that the Milnor fiber has the homotopy type of a finite CW complex of real dimension n ([**Mil68**], Theorem 5.1). Additionally, the *link* $S_\epsilon \cap V$ is always $(n-2)$-connected.

1.1.2. Isolated hypersurface singularities. Now assume that the analytic hypersurface germ $(V, 0) \subseteq (\mathbb{C}^{n+1}, 0)$ has at worst an isolated singularity at 0. This means that if $V = f^{-1}(0)$ for a holomorphic germ $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, then the gradient of f does not vanish on some punctured open neighborhood $U \setminus \{0\}$ of 0. In this situation, $(V, 0)$ is called an *isolated hypersurface singularity* (IHS).

REMARK 1.1.1. Whether $V = f^{-1}(0)$ is an IHS depends on f and not merely V considered as a set. If f is a *reduced defining equation*, so that f generates the ideal $I(V)$ of functions vanishing on V, then V is an IHS if and only if V is a manifold on a punctured open neighborhood of 0.

For an IHS $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$, Milnor showed that the Milnor fiber F is $(n-1)$-connected ([**Mil68**], Lemma 6.4). By the Hurewicz Theorem and the earlier result that F has the homotopy type of a finite CW complex of real dimension n, it follows that $H_k(F, \mathbb{Z}) = 0$ for $k \notin \{0, n\}$ and $H_n(F, \mathbb{Z}) \simeq \pi_n(F)$ is free abelian. Using Whitehead's Theorem it is possible to construct a homotopy equivalence between a bouquet of n-spheres and F (see [**Mil68**], Theorem 6.5). Moreover, there is a simple formula for the number of such spheres.

THEOREM 1.1.2 ([**Mil68**], Theorem 7.2 and p.115). *Let $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of an IHS defined by a germ $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$. Then the associated Milnor fiber has the homotopy* *type of a bouquet of μ n-spheres, where*

$$\mu = \dim_{\mathbb{C}} \left(\mathscr{O}_{\mathbb{C}^{n+1},0} / J_f \right), \tag{1.1}$$

$\mathscr{O}_{\mathbb{C}^{n+1},0}$ is the ring of analytic function germs $\mathbb{C}^{n+1}, 0 \to \mathbb{C}$ and J_f is the Jacobian ideal, the ideal of $\mathscr{O}_{\mathbb{C}^{n+1},0}$ generated by $\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}$.

The number μ is called the *Milnor number* of f, or of $(V, 0)$, and is often written $\mu(f)$, $\mu(V)$, or $\mu(V, 0)$.

If $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$ is an analytic germ for which $f^{-1}(0)$ is not an isolated hypersurface singularity, then the quotient module in (1.1) will have infinite dimension over $\mathbb{C}$. As a result, for any germ f we need only calculate (1.1) in order to determine whether $f^{-1}(0)$ is an IHS and, if it is, to find its Milnor number.

Example 1.1.3. For $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^k$, $k \in \mathbb{N}$, the Milnor fiber $B_\epsilon(0) \cap f^{-1}(\delta)$ will consist of the kth roots of δ as long as $0 < |\delta|^{1/k} < \epsilon$. But a set of k points is a bouquet of $\mu = k - 1$ copies of S^0. The ideal J_f is generated by z^{k-1}, and the complex dimension of $R = \mathscr{O}_{\mathbb{C}^1,0}/(z^{k-1})$ is $k - 1$, as R is generated over $\mathbb{C}$ by $1, z^1, \ldots, z^{k-2}$. Note that the IHS in this Example have non-reduced defining equations; this can only occur in $\mathbb{C}^1$.

Example 1.1.4. Fix $p, q \in \mathbb{N}$, and consider the function $f : \mathbb{C}^2 \to \mathbb{C}$ defined by

$$f(x, y) = x^p + y^q.$$

Then $\frac{\partial f}{\partial x} = px^{p-1}$ and $\frac{\partial f}{\partial y} = qy^{q-1}$, so that $J_f = (x^{p-1}, y^{q-1})$. Then $R = \mathscr{O}_{\mathbb{C}^2,0}/(x^{p-1}, y^{q-1})$ is generated over $\mathbb{C}$ by all $x^i y^j$ for $0 \le i \le p - 2$ and $0 \le j \le q - 2$, and it is also clear that this is the minimal number of generators. As a result, the Milnor fiber has the homotopy type of a bouquet of $\mu = (p - 1)(q - 1)$ copies of S^1.

Example 1.1.5. More generally than the previous two examples, we may consider the case where $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a *weighted homogeneous* polynomial of weighted degree d with weights $w_0, \ldots, w_n \in \mathbb{N}$ so that

$$f(t^{w_0} z_0, t^{w_1} z_1, \ldots, t^{w_n} z_n) = t^d f(z_0, \ldots, z_n)$$

for all $z_0, \dots, z_n$ and t. Milnor and Orlik showed [**MO70**] that in this case, the Milnor number depends only on the weights and is given by

$$\mu(f) = \prod_{i=0}^{n} \frac{d - w_i}{w_i}.$$

EXAMPLE 1.1.6. The hypersurface $V \subset \mathbb{C}^{n+1}$ defined by

$$z_0 \cdots z_n = 0$$

is not an IHS for $n > 1$. It is singular on the intersection of any two coordinate hyperplanes. The Jacobian algebra in (1.1) has infinite complex dimension: for example, the elements $1, z_0, z_0^2, z_0^3, \dots$ are all linearly independent.

1.2. Isolated complete intersection singularities

In [**Ham71**], Hamm proved that the class of "isolated complete intersection singularities" also have a Milnor fibration whose fibers are homotopy equivalent to a bouquet of spheres of the same dimension.

DEFINITION 1.2.1. $(V, 0) \subset (\mathbb{C}^p, 0)$ is a *complete intersection germ (of dimension r)* if there exists a complex analytic germ $f : \mathbb{C}^p, 0 \to \mathbb{C}^{p-r}, 0$ so that $f^{-1}(0) = V$ and such that each irreducible component of V has dimension r.

REMARK 1.2.2. We do not require the ideal $I(V)$ defining V to be generated by $f_1, \dots, f_{p-r}$, although the radical of the ideal they generate is $I(V)$. Our definition is sometimes called a "geometric" complete intersection (e.g., [**Loo84**], p.4).

DEFINITION 1.2.3. A complete intersection $(V, 0) \subset (\mathbb{C}^p, 0)$ is an *isolated complete intersection singularity (ICIS)* if the germ's singular set is $\{0\}$, i.e., if $(V, 0)$ is defined by $f : \mathbb{C}^p, 0 \to \mathbb{C}^{p-r}, 0$, then *df* is a submersion at all points of $V \setminus \{0\}$ in some open neighborhood of 0.

EXAMPLE 1.2.4. Any hypersurface is a complete intersection defined by a single function. Any isolated hypersurface singularity is an isolated complete intersection singularity.

EXAMPLE 1.2.5. In $\mathbb{C}^4$, the union of the two planes $\{x_1 = x_2 = 0\}$ and $\{x_3 = x_4 = 0\}$ is not a complete intersection (see, e.g., [**Dim92**], p.81). Not all algebraic sets are complete intersections.

1.2.1. Milnor fibration. Let $(V, 0) \subset (\mathbb{C}^p, 0)$ be an ICIS of dimension r defined by a holomorphic germ $f : \mathbb{C}^p, 0 \to \mathbb{C}^{p-r}, 0$. Let $D(f) \subset \mathbb{C}^{p-r}$ be the *discriminant* of f, the image under f of the set of critical points of f. Let $X = B_\epsilon(0) \cap f^{-1}(B_\delta(0) \setminus D(f))$, an open neighborhood of $0 \in \mathbb{C}^p$, and let $\bar{f} = f|_X : X \to B_\delta(0) \setminus D(f)$. Then Hamm showed

THEOREM 1.2.6 ([**Ham71**], 1.7 or, e.g., [**Loo84**], (5.10)). *There exists an $\epsilon_0 > 0$ so that if $0 < \epsilon < \epsilon_0$, then there exists $0 < \delta \ll \epsilon$ so that $\bar{f}$ is a locally trivial fibration and each fiber of $\bar{f}$ is homotopy equivalent to a bouquet of $(p - r)$-spheres.*

A fiber of $\bar{f}$ is called a *Milnor fiber* of $(V, 0)$, and the number of spheres is called the *Milnor number* of the ICIS $(V, 0)$ and is denoted by $\mu(V)$, $\mu(V, 0)$, or $\mu(f)$.

EXAMPLE 1.2.7. These definitions agree with our earlier definitions for IHS. If $f : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ defines an IHS, then $D(f) \subseteq \{0\}$.

REMARK 1.2.8. Theorem 1.2.6 was proven more generally in the case where f is a function on an ICIS $(W, 0)$ with $(f^{-1}(0) \cap W, 0)$ again an ICIS. This process of building an ICIS was later used to inductively obtain formulas for the Milnor number. This is an important recurring technique.

1.2.2. Formulas for the Milnor number of ICIS. As a result of Hamm's discovery, several workers found formulas to calculate the Milnor number.

There is a general formula for the Milnor number given by Greuel [**BG75, Gre75**] as the length of a relative deRham cohomology module for the fibration (see, e.g., 8.20 of [**Loo84**]). However, it is difficult to compute directly.

There is a more computable formula known as the Lê-Greuel formula. It requires inductively computing the Milnor number of various ICIS of successively increasing codimensions. Let $(V, 0) \subset (\mathbb{C}^p, 0)$ be an ICIS of codimension r defined by $f_1 = \cdots = f_r = 0$. For $i = 1, \ldots, r$, let V_i be defined by $f_1 = \cdots = f_i = 0$. Let $X_i = V_i \cap B_\epsilon(0) \cap f_{i+1}^{-1}(B_\delta(0))$, an open subset of V_i. Let $\bar{f}_{i+1} = f_{i+1}|_{X_i} : X_i \to B_\delta(0)$, a smooth function on $V_i \setminus \{0\}$. We may choose $f_1, \cdots, f_r$

so that each $(f_1,\dots,f_i):\mathbb{C}^p,0\to\mathbb{C}^i,0$ defines an ICIS. Then there is a formula for a "relative Milnor number" for $\bar{f}_{i+1}$ on X_i which is similar to (1.1).

THEOREM 1.2.9 (Lê-Greuel formula; [**Lê 74**] 3.7.1, 3.7.2). *With the above notation,*

$$\mu(V_i)+\mu(V_{i+1})=\dim_{\mathbb{C}}(\mathscr{O}_{\mathbb{C}^p,0}/I_i),$$

where I_i is the ideal generated by $f_1,\dots,f_i$ and the maximal minors of the Jacobian of $(f_1,\dots,f_{i+1}):\mathbb{C}^n\to\mathbb{C}^{i+1}$. Thus

$$\mu(V)=\sum_{i=0}^{r-1}(-1)^{r-i-1}\dim_{\mathbb{C}}(\mathscr{O}_{\mathbb{C}^p,0}/I_i).$$

The proof of this formula by [**Lê 74**] involves taking an unfolding of the function defining the ICIS, showing that a certain sheaf is Cohen-Macaulay, and that the unfolding splits the critical point into Morse singularities which are then counted.

EXAMPLE 1.2.10. For an isolated hypersurface singularity, the above formulas reduce to the formula given by Theorem 1.1.2.

EXAMPLE 1.2.11. Analogous to Example 1.1.5, Greuel-Hamm [**GH78**] found a formula for the Milnor number of a *weighted homogeneous ICIS*. This is an ICIS defined by $f=(f_1,\dots,f_r):\mathbb{C}^p,0\to\mathbb{C}^r,0$, where weights may be assigned to each coordinate so that each f_i is weighted homogeneous of weighted degree d_i. Then the Milnor number may be found using only the weights.

1.3. The topology of non-isolated singularities

The preceding results for the Milnor fiber only applied to isolated singularities. There is a good reason for this: for non-isolated singularities, the Milnor fiber does not necessarily have the homotopy type of a bouquet of spheres. Recall that for an IHS the Milnor fiber of a function $\mathbb{C}^{n+1},0\to\mathbb{C},0$ is $(n-1)$-connected so the homotopy groups below dimension n vanish. For non-isolated singularities, Kato-Matsumoto [**KM75**] showed that the Milnor fiber loses connectivity and hence will generally have non-vanishing homology in a large range of dimensions.

THEOREM 1.3.1 ([**KM75**], Theorem 1). *For $f:\mathbb{C}^{n+1},0\to\mathbb{C},0$ a germ of a holomorphic function, let $\Sigma=\{x|f(x)=0,\operatorname{grad}f(x)=0\}$ be the critical set of $\{f=0\}$, and let s be the dimension of Σ at 0. Then the Milnor fiber F of f at 0 is at least $(n-s-1)$-connected. Moreover, there are examples where this is sharp.*

However, there are large classes of interesting non-isolated singularities. For instance, in this work we are interested in matrix singularities which almost always have non-isolated singularities. There has been much work done trying to develop analogues of the results for IHS and ICIS for non-isolated singularities.

One approach was to continue to require functions to have "isolated singularities" in an appropriate sense, but to allow the spaces on which which they are defined to have singularities. This idea perhaps originated with Lê's proof of the formula for the Milnor number of an isolated complete intersection ([**Lê 74**], Theorem 1.2.9), where he inductively computes the Milnor number for a function defined on an isolated complete intersection. In [**Lê 79**], he showed that the local topology of the intersection of a germ of a complete intersection (perhaps with complicated singularities) with a generic affine hyperplane has the homotopy type of a bouquet of spheres. He subsequently defined [**Lê 87**] the notion of an "isolated singularity" on an analytic space, and commented that when the analytic space is a complete intersection, then the Milnor fiber has the homotopy type of a bouquet of spheres. When the analytic space is not a complete intersection, Tibăr has more recently shown [**Tib96**] that the Milnor fiber is homotopy equivalent to "a bouquet of repeated suspensions of complex links of strata" of X.

Another direction was to attempt to understand the Milnor fiber by gradually increasing the dimensional of the critical set. Siersma [**Sie83, Sie87**] studied the Milnor fiber for germs whose critical set Σ is a 1-dimensional isolated complete intersection and showed that the Milnor fiber has the homotopy type of a bouquet of spheres of at most two different dimensions. He used results of Pellikaan [**Pel85**] to deform the defining function, simplifying the singularities involved. The Dutch group led by Siersma has continued [**Pel90, MS92, dJ90, dJdJ90**] to study this problem. Others [**Zah94, Ném99**] have studied the Milnor fiber for germs whose critical set is a 2-dimensional isolated complete intersection. In later work, [**Sie91**] proves that under certain conditions, certain fibers of the deformation of a function defining a singularity have the homotopy type of a bouquet of spheres.

Mond and Marar [**Mon87, MM89**] noticed that certain weighted homogeneous finitely $\mathscr{A}$-determined germs $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$, the "vanishing Euler characteristic" of images of stabilizations of f_0 equals the $\mathscr{A}_e$-codimension of f_0. The images of such germs $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $2 \leq n < p$ have non-isolated singularities and the images of their stabilizations are called *disentanglements*. This was proven to hold for weighted homogeneous germs by Pellikaan-de Jong (unpublished), by de Jong-van Straten [**dJvS91**] using a deformation theory for normalizations, and later by Mond [**Mon91**]. (Others, notably Houston [**Hou97, Hou02**], have continued to study disentanglements.)

Damon-Mond [**DM91**] discovered that Mond's result for $\mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ had a generalization to finitely $\mathscr{A}$-determined germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $n \geq p$, provided that the image is replaced with the discriminant. Using results of Lê, they proved that the discriminant of the stabilization has the homotopy type of a bouquet of spheres, and they called the number of spheres the *discriminant Milnor number* $\mu_D(f)$. They prove a $\mu_D(f) \geq \mathscr{A}_e$-codim$(f)$-type result, with equality for weighted homogeneous f in Mather's "nice dimensions," using a relation between $\mathscr{A}$-equivalence and the equivalence of nonlinear sections of the discriminant of the stable unfolding of f.

Damon later realized [**Dam96a, Dam96b**] that their methods apply more generally to allow one to define the singular Milnor number for nonlinear sections of complete intersections. In this form, the singular Milnor number is a straightforward generalization of the classical Milnor number for isolated singularities.

1.4. The singular Milnor number

We now generalize the Milnor number of IHS and ICIS to non-isolated singularities of the form $f^{-1}(V)$, for $(V, 0) \subset (\mathbb{C}^p, 0)$ a (non-isolated) hypersurface or complete intersection and $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ a germ transverse to V off $0 \in \mathbb{C}^n$. We define the singular Milnor fiber and state its basic property, that it has the homotopy type of a bouquet of spheres. To state the hypotheses under which the singular Milnor number is defined and to define stabilizations, we describe two notions of transversality to a (possibly) singular $(V, 0)$ using modules of vector fields associated to a germ $(V, 0) \subset (\mathbb{C}^p, 0)$ and its defining function.

1.4.1. Modules of vector fields. Let $(V, 0) \subset (\mathbb{C}^p, 0)$ be a germ of a complex analytic set. Let $I(V) \subset \mathscr{O}_{\mathbb{C}^p,0}$ be the ideal of germs vanishing on V. We may define the module of germs of complex analytic vector fields which are "tangent to V":

$$\operatorname{Derlog}(V) = \{\eta \in \theta_p | \eta(h) \in I(V) \text{ for all } h \in I(V)\},$$

where θ_p is the $\mathscr{O}_{\mathbb{C}^p,0}$-module of germs of complex analytic vector fields on $\mathbb{C}^p$ at 0. In fact, $\operatorname{Derlog}(V)$ extends to a coherent sheaf $\mathit{Derlog}(V)$ consisting of vector fields tangent to the smooth points of V (see [**Sai80**]). $\operatorname{Derlog}(V)$ is the stalk of $\mathit{Derlog}(V)$ at 0.

REMARK 1.4.1. $\operatorname{Derlog}(V)$ is closed under the Lie bracket of vector fields and so is a Lie algebra.

Now consider the case where $V = f^{-1}(0)$ is a hypersurface defined by a reduced germ $f : \mathbb{C}^p, 0 \to \mathbb{C}, 0$. The ideal $I(V)$ contains f. We may define a submodule of $\operatorname{Derlog}(V)$ by

$$\operatorname{Derlog}(f) = \{\eta \in \theta_p | \eta(f) = 0\}.$$

In contrast to elements of $\operatorname{Derlog}(f)$ are *Euler-like vector fields*, $\eta \in \theta_p$ for which $\eta(f) = f$. Euler-like vector fields do not always exist. However, if f is weighted homogeneous of degree d with weights $a_1, \ldots, a_n$, then

$$\frac{1}{d}\sum_{i=1}^{n} a_i z_i \frac{\partial}{\partial z_i}$$

is an Euler-like vector field.

If $V = f^{-1}(0)$ and there is an Euler-like vector field η for f, then we call f a *good defining equation for* V. If V has a good defining equation f with Euler-like vector field η, then

$$\operatorname{Derlog}(V) = \operatorname{Derlog}(f) \oplus \mathscr{O}_{\mathbb{C}^p,0}\{\eta\}$$

by Lemma 3.3, [**DM91**].

If a hypersurface $V = f^{-1}(0)$, $f : \mathbb{C}^p, 0 \to \mathbb{C}, 0$, does not have a good defining equation, then there is a simple procedure to obtain one: consider instead $V \times \mathbb{C}$ defined by $g : \mathbb{C}^p \times \mathbb{C}, (0,0) \to \mathbb{C}, 0$, $g(x,t) = e^t f(x)$. The vector field $\frac{\partial}{\partial t}$ is an Euler-like vector field for g, giving $V \times \mathbb{C}$ a good defining equation. Though we are interested in nonlinear sections $F^{-1}(V)$ of V under $F : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $F^{-1}(V) = (F \times \{0\})^{-1}(V \times \mathbb{C})$ and by Proposition 3.10 of [**DM91**] the singularity-theoretic properties of F as a nonlinear section of V are equivalent to those of $F \times \{0\} : C^n, 0 \to \mathbb{C}^{p+1}, 0$ as a nonlinear section of $V \times \mathbb{C}$. We will thus assume the existence of a good defining equation.

1.4.2. Transversality to singular sets. We may now define several notions of transversality to a singular set; the degree to which these agree are measured by various "codimensions".

Let $(V,0) \subset (\mathbb{C}^p,0)$ be a germ of an analytic set. Let $\eta_1,\ldots,\eta_m$ be a set of generators for $\mathrm{Derlog}\,(V)$. Let $f:\mathbb{C}^n,0 \to \mathbb{C}^p,0$ be an analytic function.

Definition 1.4.2. We shall say that f is *algebraically transverse to* $(V,0)$ *at* $x_0 \in \mathbb{C}^n$ if

$$df_{x_0}(T_{x_0}\mathbb{C}^n) + \langle \mathrm{Derlog}\,(V)\rangle_{f(x_0)} = T_{f(x_0)}\mathbb{C}^p,$$

where for $y \in \mathbb{C}^p$ we define

$$\langle \mathrm{Derlog}\,(V)\rangle_y = \mathbb{C}\{\eta_1(y),\ldots,\eta_m(y)\} \subset T_y\mathbb{C}^p.$$

If there exists a punctured neighborhood of zero $U = B_\epsilon(0)\setminus\{0\} \subset \mathbb{C}^n$ so that f is algebraically transverse to $(V,0)$ at all $x \in U$, then we will say that f is *algebraically transverse to* $(V,0)$ *off* 0, and write

$$f \underset{\mathrm{alg}}{\overset{0}{\pitchfork}} V.$$

Proposition 1.5.1 will show that f is algebraically transverse off 0 to V if and only if f has "finite codimension" for a particular group.

A different notion of transversality comes from a Whitney stratification of V. Let $\mathfrak{X}$ be the canonical Whitney stratification of V. It makes sense to say that f to transverse to all strata of $\mathfrak{X}$ at $x \in \mathbb{C}^n$.

Definition 1.4.3. We will say that f is *geometrically transverse to* $(V,0)$ *off* 0 and write

$$f \underset{\mathrm{geom}}{\overset{0}{\pitchfork}} V$$

if there is a punctured neighborhood $U = B_\epsilon(0)\setminus\{0\} \subset \mathbb{C}^n$ so that f is transverse to all strata of $\mathfrak{X}$ at all $x \in U$.

By Proposition 3.11(i) of [**DM91**], all $\eta \in \mathrm{Derlog}\,(V)$ are tangent to all strata of the canonical Whitney stratification $\mathfrak{X}$ of V. If V has a good defining equation H and $x \in V_i$, a stratum of $\mathfrak{X}$, then we have the inclusions

$$\langle \mathrm{Derlog}\,(H)\rangle_x \subseteq \langle \mathrm{Derlog}\,(V)\rangle_x \subseteq T_xV_i. \tag{1.2}$$

Note that $\langle \text{Derlog}\,(H)\rangle_x$ is independent of the choice of good defining equation by Lemma 2.1 of [**Dam96a**], and that (1.2) implies that algebraic transversality implies geometric transversality. They are not equivalent, however.

We may or may not have equality at each inclusion of (1.2). If the right inclusion is an equality for all points on all strata, then we say that V is *holonomic*; if both inclusions are equalities for all points on all strata, then we say that V is *H-holonomic*.

EXAMPLE 1.4.4. $V = \{0\} \subset \mathbb{C}$ defined by $H(z) = z$ is H-holonomic with $\text{Derlog}\,(H) = \{0\}$. Then a holomorphic germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ defines a hypersurface $f_0^{-1}(V) = f_0^{-1}(0)$ with an isolated singularity at $0 \in \mathbb{C}^n$ if and only if f_0 is geometrically transverse to V off zero. Since V is holonomic, this is equivalent to f_0 being algebraically transverse to V off zero.

If an algebraic set is not H-holonomic, we define a codimension to describe the degree to which this property fails. Call a stratum V_i of the canonical Whitney stratification $\mathfrak{X}$ of V *H-holonomic* if $\langle \text{Derlog}\,(H)\rangle_x = T_x V_i$ for all $x \in V_i$. Define

$$h(V) = \max\{k : \text{all strata } V_i \text{ of codimension} < k \text{ are } H\text{-holonomic}\},$$

the codimension of the largest dimensional stratum which is not H-holonomic. If V is H-holonomic, then write $h(V) = \infty$.

1.4.3. Stabilizations. We may now define the notion of stabilization of a nonlinear section.

Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a holomorphic germ, and let $(V, 0) \subset (\mathbb{C}^p, 0)$ be a germ of a complete intersection of dimension k. Let $f : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to \mathbb{C}^p$ be a 1-parameter holomorphic unfolding of f_0 defined on a neighborhood $U \times T \subset \mathbb{C}^n \times \mathbb{C}$. Let $f_t(x) = f(x,t)$. We define two related notions of a "stabilization" of f_0.

DEFINITION 1.4.5. Assume that f_0 is algebraically transverse off 0 to V. Then we call f a *stabilization* of f_0 with respect to V if for all $t \in T \setminus \{0\}$, f_t is algebraically transverse to V on U.

A stabilization thus has the property that at any $(x,t) \in U \times T$ with $(x,t) \neq (0,0)$,

$$d(f_t)_{(x)}(T_x\mathbb{C}^n) + \langle \text{Derlog}\,(V)\rangle_{f_t(x)} = T_{f_t(x)}\mathbb{C}^p.$$

Definition 1.4.6. Now assume that f_0 is geometrically transverse to V off 0. Then we call f a *geometric stabilization* of f_0 with respect to V if for $t \in T \setminus \{0\}$, f_t is geometrically transverse to V on U.

A geometric stabilization has the property that at any $(x,t) \in U \times T$ with $(x,t) \neq (0,0)$, either $f_t(x) \in V$ or $f_t(x) \in V_i$, a Whitney stratum of V, with

$$d(f_t)_{(x)}(T_x\mathbb{C}^n) + T_{f_t(x)}V_i = T_{f_t(x)}\mathbb{C}^p.$$

Since being algebraically transverse is stronger than being geometrically transverse, every stabilization is also a geometric stabilization. When V is holonomic, the two notions of transversality, and the two notions of stabilization, are equivalent.

Example 1.4.7. Recall from Example 1.4.4 how $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ is geometrically transverse off 0 to $V = \{0\} \subset \mathbb{C}$ if and only if $f_0^{-1}(V)$ has an isolated singularity at 0, i.e., f_0 has an isolated critical point at 0. Then $f_t = f_0 - t$ gives a stabilization of f_0.

Remark 1.4.8. If $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is transverse off $0 \in \mathbb{C}^n$ to $(V,0) \subset (\mathbb{C}^p,0)$ and f_0 has finite $\mathscr{K}_{V,e}$-codimension, then a geometric stabilization of f_0 always exists. For, f_0 has a $\mathscr{K}_V$-versal unfolding $F : \mathbb{C}^n \times \mathbb{C}^k, 0 \to \mathbb{C}^p, 0$ which must be geometrically transverse to V. Since F is transverse to a Whitney stratified set, a parameterized version of Thom's transversality lemma shows that the set of parameters which correspond to maps $\mathbb{C}^n \to \mathbb{C}^p$ geometrically transverse to V is open and dense in $\mathbb{C}^k$. As a result, there is a curve in $\mathbb{C}^k$ which yields a stabilization of f_0. If V is holonomic, then this geometric stabilization is also a stabilization.

1.4.4. The singular Milnor number. We are now able to define the singular Milnor number using the stabilizations defined above.

Theorem 1.4.9 (Theorem 4.6 in [**DM91**], Lemma 7.8 in [**Dam96a**]). *Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a holomorphic germ and let $(V,0) \subset (\mathbb{C}^p,0)$ be a germ of a k-dimensional complete intersection. Suppose that f_0 is geometrically transverse to V off zero. Let $f : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}^p, 0$ be a geometric stabilization of f_0 with respect to V and let $f_t = f(\cdot, t)$. Then there exists an $\epsilon > 0$ and a $0 < \delta \ll \epsilon$ so that if $0 < |t| < \delta$,*

$$f_t^{-1}(V) \cap B_\epsilon(0) \tag{1.3}$$

has the homotopy type of a bouquet of spheres of real dimension $n-p+k$, and the number of spheres is independent of the choice of stabilization, t, and ϵ.

We refer to $f_t^{-1}(V) \cap B_\epsilon(0)$ as the *singular Milnor fiber* of f_0 with respect to V (or of $f_0^{-1}(V)$), while the number of spheres is called the *singular Milnor number* and is written $\mu_V(f_0)$.

Sketch of proof. For a stabilization f_t, let $g = (f_t, t) : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^p \times \mathbb{C}$. Let $X = g^{-1}(V \times \mathbb{C})$ and let $\pi : X \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ be projection on the second term. Under the given hypotheses, X is a complete intersection, it follows from the definition of geometric stabilization that π has an isolated singularity at $(0,0)$ in the sense of Lê (see (2.7) of [**Lê 87**]), and for $0 < |t| \ll \epsilon$, $\pi^{-1}(t) \cap B_\epsilon(0)$ is the singular Milnor fiber. Then by a result of Lê (see (4.4) of [**Lê 87**]), $\pi^{-1}(t) \cap B_\epsilon(0)$ has the homotopy type of a bouquet of spheres of the given dimension. $\square$

Example 1.4.10. In Example 1.4.7 we showed that a stabilization (relative to $\{0\} \subset \mathbb{C}$) of $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ defining an isolated hypersurface singularity was $f_t = f_0 - t$. Then the singular Milnor fiber is

$$f_t^{-1}(V) \cap B_\epsilon(0) = f_0^{-1}(t) \cap B_\epsilon(0)$$

for $0 < |t| \ll \epsilon$, which is the same as the usual Milnor fiber (specifically, ϕ_1 of §1.1). In particular, the singular Milnor number is a generalization of the classical Milnor number for IHS.

Example 1.4.11. More generally, suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{n-r}, 0$ defines an ICIS of dimension r. Then $D(f_0)$, the discriminant of f_0, is an analytic subset of $\mathbb{C}^{n-r}$ of pure dimension $n-r-1$ (see (2.8) of [**Loo84**]). Then we may find $v \in \mathbb{C}^{n-r}$ so that for t small enough, $t \mapsto tv$ intersects $D(f_0)$ only at 0. It follows that $f_t = f_0 - vt$ is a stabilization of f_0 relative to $V = \{0\} \subset \mathbb{C}^{n-r}$. The singular Milnor fiber is

$$f_t^{-1}(V) \cap B_\epsilon(0) = f_0^{-1}(vt) \cap B_\epsilon(0)$$

for $0 < |t| \ll \epsilon$. This is the same as the usual Milnor fiber because $vt \notin D(f_0)$. In particular, the singular Milnor number is a generalization of the classical Milnor number for ICIS.

1.5. Equivalence of germs via groups of diffeomorphisms

The singular Milnor number as defined by Theorem 1.4.9 involves a hypersurface or complete intersection germ $(V, 0) \subset (\mathbb{C}^p, 0)$, a germ f mapping into $(\mathbb{C}^p, 0)$, perturbations of f, and the interaction between V and perturbations of f. A formula for the singular Milnor number must measure the interaction between these analytic sets and a function's perturbations.

This will be accomplished using the Thom-Mather approach to singularity theory which uses the action of groups of germs of diffeomorphisms acting on both germs of mappings and their unfoldings. The codimension of a germ can be viewed as the dimension of a normal section to its orbit in the space of germs. For example, if any "small" perturbation of f lies in the same orbit as f under the group action, then conceptually we may imagine that the orbit of f contains an open neighborhood of f and so the codimension at f of the group action is zero. (In reality, there is no good topology on the space of germs and so there is no actual neighborhood of f. "Nearby" germs to f should be understood in terms of unfoldings of f.)

The formulas of Milnor and Lê-Greuel for the Milnor number of an isolated singularity can be viewed as codimensions for certain equivalence groups. This will also be true for the singular Milnor number.

For an analytic germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, define the $\mathscr{O}_{\mathbb{C}^n,0}$-module

$$\theta(f) = \mathscr{O}_{\mathbb{C}^n,0} \left\{ \frac{\partial}{\partial y_i} \right\}_{i=1}^{p},$$

the module of "vector fields along f," representing the space of possible infinitesimal deformations of f. Elements of $\theta(f)$ are thought of as germs $\eta : \mathbb{C}^n, 0 \to T\mathbb{C}^p, 0$, where $\eta(x) \in T_{f(x)}\mathbb{C}^p$ for all x. $\theta(f)$ may be identified with the tangent space to the space of germs $\mathbb{C}^n, 0 \to \mathbb{C}^t$ without fixing the target.

1.5.1. The contact equivalence group, $\mathscr{K}$. Since the groups we will use are subgroups of the classical contact equivalence group $\mathscr{K}$, introduced in [**Mat68, Tou68**], we first describe this group and its action on the space of analytic germs $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. Although the definition of $\mathscr{K}$ depends on n and p, these numbers are always understood from the context and omitted notationally.

$\mathscr{K}$ consists of the set of germs of analytic diffeomorphisms of $\mathbb{C}^n \times \mathbb{C}^p$ which fix zero and which send graphs of functions to graphs of functions. Specifically, it consists of germs of analytic diffeomorphisms $\Phi : \mathbb{C}^{n+p}, 0 \to \mathbb{C}^{n+p}, 0$ for which there exists a germ of a diffeomorphism $\phi : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ so that if $i : \mathbb{C}^n \to \mathbb{C}^{n+p}$ is the inclusion $i(x) = (x, 0)$ and $\pi : \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^n$ is projection on the first factor, then

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{i} & \mathbb{C}^{n+p} \\ {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\Phi}\big\downarrow \\ \mathbb{C}^n & \xrightarrow{i} & \mathbb{C}^{n+p} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{C}^{n+p} & \xrightarrow{\Phi} & \mathbb{C}^{n+p} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathbb{C}^n & \xrightarrow{\phi} & \mathbb{C}^n \end{array}$$

commute. The first diagram can be understood as the requirement that Φ transforms $\mathbb{C}^n \times \{0\}$ exactly as ϕ transforms $\mathbb{C}^n$, so that Φ sends the graph of the zero function to the graph of the zero function. The second diagram can be understood as the requirement that Φ move fibers of π to fibers of π according to ϕ. As fibers of π represent the possible values that a function can take at a particular point, Φ sends the graph of a germ $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ to the graph of another germ.

An element Φ of $\mathscr{K}$ acts on a germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ via

$$\operatorname{graph}(\Phi \cdot f) = \Phi(\operatorname{graph}(f)).$$

If $\pi_2 : \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^p$ is projection on the second factor and $\operatorname{graph}_f = \operatorname{id}_{\mathbb{C}^n} \times f$ is the function whose image is the graph of f, then since $\Phi \cdot f = \pi_2 \circ \Phi \circ \operatorname{graph}_f$, the action of any $\Phi \in \mathscr{K}$ preserves analyticity.

For a germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, we obtain the *extended tangent space of $\mathscr{K}$ at f*

$$T\mathscr{K}_e \cdot f = \mathscr{O}_{\mathbb{C}^n,0} \left\{ \frac{\partial f}{\partial x_i} \right\}_{i=1}^{n} + f^*(\mathscr{M}_{\mathbb{C}^p,0}) \mathscr{O}_{\mathbb{C}^n,0} \left\{ \frac{\partial}{\partial y_j} \right\}_{j=1}^{p}, \tag{1.4}$$

a submodule of $\theta(f)$. Here, we use coordinates $\{x_i\}$ on $\mathbb{C}^n$ and coordinates $\{y_j\}$ on $\mathbb{C}^p$, $\mathscr{M}_{\mathbb{C}^p,0}$ is the maximal ideal, and $f^* : \mathscr{O}_{\mathbb{C}^p,0} \to \mathscr{O}_{\mathbb{C}^n,0}$ is the ring homomorphism given by composition with f. If $f = (f_1, \ldots, f_p)$, then $\frac{\partial f}{\partial x_i}$ denotes the element $\sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial y_j}$ of $\theta(f)$. Then the *extended $\mathscr{K}$ codimension at f* is

$$\mathscr{K}_e\text{–codim}(f) = \dim_{\mathbb{C}} \left(\theta(f) / T\mathscr{K}_e \cdot f \right),$$

the dimension of the quotient module as a complex vector space. This can be understood as the dimension of the space of nontrivial deformations of f.

1.5.2. Contact equivalence preserving a subvariety, $\mathscr{K}_V$. We may now define a subgroup $\mathscr{K}_V$ of $\mathscr{K}$, which depends on an analytic set $(V,0) \subset (\mathbb{C}^p,0)$. This group was introduced by Damon in [**Dam87**].

Let $\mathscr{K}_V$ denote the subgroup of $\mathscr{K}$ consisting of those germs of diffeomorphisms which preserve $\mathbb{C}^n \times V$:

$$\mathscr{K}_V = \{\Phi \in \mathscr{K} | \Phi(\mathbb{C}^n \times V) \subseteq \mathbb{C}^n \times V\}.$$

$\mathscr{K}_V$ is a "geometrically defined subgroup of $\mathscr{K}$" in the sense of [**Dam84**], so that all the usual theorems of singularity theory apply: finite determinacy, existence of versal unfoldings, etc. Let $\{\eta_j\}_{j=1}^m$ be a set of generators for Derlog(V). Then the *extended $\mathscr{K}_V$ tangent space at* $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is

$$T\mathscr{K}_{V,e} \cdot f = \mathscr{O}_{\mathbb{C}^n,0} \left\{ \frac{\partial f}{\partial x_i} \right\}_{i=1}^n + \mathscr{O}_{\mathbb{C}^n,0} \{\eta_j \circ f\}_{j=1}^m, \tag{1.5}$$

a submodule of $\theta(f)$. Then the *extended $\mathscr{K}_V$ codimension at f* is the dimension of the $\mathscr{K}_V$ *normal space to f*

$$\mathscr{K}_{V,e}\text{-codim}(f) = \dim_{\mathbb{C}} \left(\theta(f)/T\mathscr{K}_{V,e} \cdot f \right).$$

Calculating the extended $\mathscr{K}_V$ codimension of a germ f in fact also checks whether f is algebraically transverse to V off 0.

Proposition 1.5.1 (Proposition 2.2, [**Dam87**]). *For a germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ and a germ $(V,0) \subset (\mathbb{C}^p,0)$ of an analytic set, f has finite $\mathscr{K}_{V,e}$-codimension if and only if f is algebraically transverse to V off zero.*

Note that each of these is equivalent to f being finitely $\mathscr{K}_V$-determined (determined, up to the action of $\mathscr{K}_V$, by a Taylor polynomial of sufficiently large degree) by standard finite determinacy results ([**Dam84**]).

1.5.3. Contact equivalence preserving level sets, $\mathscr{K}_H$. The second group which we will define, $\mathscr{K}_H$, is a subgroup of $\mathscr{K}_V$ when $(V,0) \subset (\mathbb{C}^p,0)$ is a complete intersection germ defined by an analytic germ $H : \mathbb{C}^p, 0 \to \mathbb{C}^r, 0$. Instead of considering diffeomorphisms which preserve $V = H^{-1}(0)$, we consider diffeomorphisms which preserve all level sets of H.

For now, let $H : \mathbb{C}^p, 0 \to \mathbb{C}^r, 0$ be an analytic germ. Let $\pi_2 : \mathbb{C}^n \times \mathbb{C}^p \to \mathbb{C}^p$ be projection on the second factor. Then $\mathscr{K}_H$ is defined by

$$\{\Phi \in \mathscr{K} | H \circ \pi_2 \circ \Phi = H \circ \pi_2\},$$

and the *extended $\mathscr{K}_H$-tangent space at $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$* is the dimension of the $\mathscr{K}_H$ *normal space to* f

$$T\mathscr{K}_{H,e} \cdot f = \mathscr{O}_{\mathbb{C}^n,0} \left\{ \frac{\partial f}{\partial x_i} \right\}_{i=1}^{n} + \mathscr{O}_{\mathbb{C}^n,0} \left\{ \eta_j \circ f \right\}_{j=1}^{m}, \tag{1.6}$$

where now $\{\eta_j\}_{j=1}^m$ is a set of generators for $\operatorname{Derlog}(H)$ (not $\operatorname{Derlog}(V)$). Again, $T\mathscr{K}_{H,e} \cdot f$ is a submodule of $\theta(f)$, and the *extended $\mathscr{K}_H$ codimension at f* is

$$\mathscr{K}_{H,e}\text{-codim}(f) = \dim_{\mathbb{C}} \left(\theta(f) / T\mathscr{K}_{H,e} \cdot f\right).$$

As with $\mathscr{K}_V$, $\mathscr{K}_H$ is a "geometrically defined subgroup of $\mathscr{K}$" in the sense of [**Dam84**], so that the usual theorems of singularity theory apply.

Similar to Proposition 1.5.1, having finite $\mathscr{K}_{H,e}$-codimension is equivalent to a transversality condition.

Proposition 1.5.2 (p.228, [**DM91**]). *For analytic germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ and $H : \mathbb{C}^p, 0 \to \mathbb{C}, 0$, f has finite $\mathscr{K}_{H,e}$-codimension if and only if f is algebraically transverse to $V = H^{-1}(0)$ off zero where we only use generators for* $\operatorname{Derlog}(H)$ *instead of all of* $\operatorname{Derlog}(V)$.

Again, this is equivalent to f being finitely $\mathscr{K}_H$-determined by standard finite determinacy results ([**Dam84**]).

All known formulas for the singular Milnor number involve the $\mathscr{K}_{H,e}$-codimension and apply only when our germs have finite $\mathscr{K}_{H,e}$-codimension. Our formulas will thus hold under the above transversality condition.

Remark 1.5.3. If $V = H^{-1}(0)$ is H-holonomic, then by definition $\langle \operatorname{Derlog}(V) \rangle_v$ and $\langle \operatorname{Derlog}(H) \rangle_v$ agree at any point $v \in V$. Then the two forms of transversality given by Propositions 1.5.1 and 1.5.2 will agree, and f has finite $\mathscr{K}_{V,e}$ codimension if and only if it has finite $\mathscr{K}_{H,e}$ codimension. This happens, for example, when H is reduced and H and f are weighted homogeneous for the same weights (see Lemma 3.4 of [**DM91**]).

1.6. Formulas for the singular Milnor number

By Theorem 1.4.9, the singular Milnor fiber has the homotopy type of a bouquet of spheres of a specific dimension and the number of such spheres is the singular Milnor number. We next describe several important classes of hypersurfaces and complete intersection singularities $(V, 0)$ for which $\mu_V(f_0)$ is given by a computable formula. These are analogous to the formulas of Milnor and Lê-Greuel for isolated singularities.

1.6.1. Free Divisors. The key formula for $(V, 0)$ a hypersurface applies when $(V, 0)$ is a free divisor, a notion originally due to Saito ([**Sai80**]).

Definition 1.6.1. A hypersurface germ $(V, 0) \subset (\mathbb{C}^p, 0)$ is a *free divisor* if $\mathrm{Derlog}\,(V)$ is a free module.

For a free divisor $(V, 0) \subset (\mathbb{C}^p, 0)$, $\mathrm{Derlog}\,(V)$ necessarily has rank p. If additionally V has a good defining equation f, then $\mathrm{Derlog}\,(f)$ is free of rank $p - 1$.

Example 1.6.2. The simplest free divisor is $\{0\} \subset \mathbb{C}$, where $\mathrm{Derlog}\,(\{0\})$ is generated by $z\frac{\partial}{\partial z}$.

Remark 1.6.3. If the germ $(V, 0) \subset (\mathbb{C}^n, 0)$ is a free divisor and we choose a representative V on an open neighborhood U_1 of $0 \in \mathbb{C}^n$, then by coherence of *Derlog*(V) there exists an open neighborhood $U_2 \subset U_1$ of 0 so that (V, v) is a free divisor for any $v \in V \cap U_2$. This is the reason we are so interested in the stalk of *Derlog*(V) at 0.

There are many important classes of examples of free divisors, including

(1) Free hyperplane arrangements, arrangements of hyperplanes which are free divisors, have been studied by Terao [**Ter80a, Ter80b**], Orlik [**OT92**], and others. For example, $f = \prod_{i=1}^n x_i$ defines a free hyperplane arrangement V on $\mathbb{C}^n$ where $\mathrm{Derlog}\,(V)$ is generated by vector fields of the form $x_i\frac{\partial}{\partial x_i}$. The previous example is a special case of this one.

(2) Saito showed [**Sai80**] that the discriminant of the versal unfolding of a function defining an isolated hypersurface singularity is a free divisor. Looijenga [**Loo84**] showed that the same is true for an isolated complete intersection singularity. Van Straten showed [**vS95**] that the discriminant of the versal unfolding of a reduced space curve is a free divisor.

Terao [**Ter83**] and Bruce [**Bru85**] independently showed that the bifurcation set of the versal deformation of an isolated hypersurface singularity is a free divisor. This was extended to finitely determined germs $\mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ for a range of (n, p) by [**Dam98**].

More generally, Damon showed [**Dam98**] that any discriminant or bifurcation set for a $\mathscr{G}$-versal unfolding is a free divisor, where $\mathscr{G}$ is any "geometric subgroup of $\mathscr{A}$ or $\mathscr{K}$" which is "Cohen-Macaulay and generically has Morse-type singularities."

(3) Buchweitz-Mond [**BM06**] showed that the complements of the open orbits of certain rational representations of connected complex algebraic Lie groups are "linear" free divisors.

Definition 1.6.4. Call a free divisor $(V, 0) \subseteq (\mathbb{C}^n, 0)$ *linear* if there exists a set of n linear vector fields $\xi_1, \ldots, \xi_n$ which generate $\mathrm{Derlog}\,(V)$, where a *linear vector field* is homogeneous of degree 0 using the natural grading of vector fields.

Buchweitz and Mond studied examples coming from quiver representations of finite type. Their examples considered and obtained free divisors using irreducible representations of reductive groups.

We shall now discuss the formula for $\mu_V(f_0)$ in the case where $(V, 0) \subset (\mathbb{C}^p, 0)$ is a free divisor with a good defining equation H. We must strengthen the hypotheses of Theorem 1.4.9 by requiring f_0 (and f_t) to satisfy the stronger algebraic transversality conditions and by requiring f_0 to have finite $\mathscr{K}_{H,e}$-codimension. (In fact, Proposition 1.5.2 shows that the finite codimension requirement implies the transversality condition.)

The following Theorem combines several results from the literature. Although the version in [**DM91**] (Theorems 5, 6) requires $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ and H to be weighted homogeneous for the same weights, this restriction may be removed with the same conclusion, provided that $n < h(V)$.

Theorem 1.6.5 ([**DM91**] Theorems 5, 6; [**Dam96a**] Theorem 4.1). *Let $(V, 0) \subset (\mathbb{C}^p, 0)$ be a free divisor with a good defining equation H, let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be an analytic germ which is algebraically transverse to V off 0, and suppose $\mathscr{K}_{H,e}$-$\mathrm{codim}(f_0) < \infty$. If $n < h(V)$,*

$$\mu_V(f_0) = \mathscr{K}_{H,e}\text{-codim}(f_0). \tag{1.7}$$

When V is H-holonomic, then $h(V) = \infty$ and the singular Milnor number is given by the $\mathscr{K}_{H,e}$-codimension. If $n \geq h(V)$, then there are correction terms needed similar to those we will encounter in Theorem 1.6.13.

The proof involves showing that a certain sheaf is Cohen-Macaulay and that the unfolding splits the singularity into Morse singularities which are then counted. This is the same technique used by Lê [**Lê 74**] to prove the Lê-Greuel formula.

EXAMPLE 1.6.6. If $V = \{0\} \subset \mathbb{C}$ is our free divisor, then by Example 1.4.4 $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ is algebraically transverse to V off 0 if and only if $f_0^{-1}(V)$ has an isolated hypersurface singularity at 0. In this case, the singular and classical Milnor fibers agree. Use coordinates $\{x_i\}$ on $\mathbb{C}^n$ and y on $\mathbb{C}$. Since Derlog(V) is free of rank 1, generated by the Euler vector field $y\frac{\partial}{\partial y}$, Derlog$(y)$ only contains the zero vector field. Thus

$$\begin{aligned}
T\mathscr{K}_{H,e} \cdot f_0 &= \mathscr{O}_{\mathbb{C}^n,0}\left\{\frac{\partial f_0}{\partial x_1}\frac{\partial}{\partial y}, \ldots, \frac{\partial f_0}{\partial x_n}\frac{\partial}{\partial y}\right\} + \mathscr{O}_{\mathbb{C}^n,0}\left\{0\frac{\partial}{\partial y}\right\} \\
&= J_{f_0} \cdot \frac{\partial}{\partial y},
\end{aligned}$$

where J_{f_0} is the Jacobian ideal of f_0. Since $\theta(f_0) = \mathscr{O}_{\mathbb{C}^n,0} \cdot \frac{\partial}{\partial y}$, the formulas given by Theorems 1.6.5 and 1.1.2 agree.

EXAMPLE 1.6.7. There is again an expression in terms of the weights for the singular Milnor number when V and f_0 are weighted homogeneous for the same weights (see §6 of [**Dam96a**] or [**Dam95**]).

1.6.2. Almost free divisors. The class of *almost free divisors* is the class of hypersurface singularities obtained as pullbacks of free divisors under maps algebraic transverse in the complement of a point. The singular Milnor number of certain nonlinear sections of almost free divisors may be calculated (under restrictive hypotheses) by reducing to Theorem 1.6.5 (see Corollary 4.2 of [**Dam96a**]). Almost free divisors may be thought of as an analogue of IHS, where we replace local smoothness with local freeness.

1.6.3. Almost free complete intersections. One can similarly study the intersections of almost free divisors. Call two analytic sets $(V_1, 0), (V_2, 0) \subset (\mathbb{C}^p, 0)$ in *algebraic general position off* 0 if, for all $x \in V_1 \cap V_2$, $x \neq 0$, we have $\langle \text{Derlog}(V_1)\rangle_x$ and $\langle \text{Derlog}(V_2)\rangle_x$ in general position.

The class of *almost free complete intersections* include almost free divisors and the intersection of almost free complete intersections which are in algebraic general position off 0 (see Proposition 7.6 of [**Dam96a**]). All almost free complete intersections arise as the pullback of the direct product of free divisors under a map algebraically transverse in the complement of a point. In many ways, these are analogues of ICIS.

There is a formula for the singular Milnor numbers of almost free complete intersections. We state a slight generalization of this result for the intersection of two hypersurfaces which is more suggestive of things to come.

THEOREM 1.6.8 (Theorem 2 of [**Dam96a**]). *For $i = 1, 2$, let $(V_i, 0) \subset (\mathbb{C}^p, 0)$ be hypersurfaces, with $(V_1 \cap V_2, 0)$ a complete intersection. If f is algebraically transverse of* 0 *to V_1, V_2, $V_1 \cup V_2$, and $V_1 \cap V_2$, then*

$$\mu_{V_1 \cap V_2}(f) = \mu_{V_1}(f) + \mu_{V_2}(f) - \mu_{V_1 \cup V_2}(f). \tag{1.8}$$

When V_1, V_2, and $V_1 \cup V_2$ are free divisors, then this Theorem together with Theorem 1.6.5 allows the computation of $\mu_{V_1 \cap V_2}(f)$. Moreover, $f^{-1}(V_1)$, $f^{-1}(V_2)$, and $f^{-1}(V_1 \cup V_2)$ are almost free divisors.

This type of result and its method of proof, using combinatorial results for computing Euler characteristics, motivate our approach in Chapter 6.

1.6.4. An ICIS on an almost free divisor. Let $f_1 : \mathbb{C}^n, 0 \to \mathbb{C}^{p_1}, 0$ define an ICIS $(V_1, 0)$ of dimension $n - p_1$. Let $f_2 : \mathbb{C}^n, 0 \to \mathbb{C}^{p_2}, 0$ be algebraically transverse off 0 to the free divisor $(V, 0) \subset (\mathbb{C}^{p_2}, 0)$, so that $V_2 = f_2^{-1}(V)$ is an almost free divisor. Let $f = (f_1, f_2) : \mathbb{C}^n, 0 \to \mathbb{C}^{p_1+p_2}, 0$. Then

$$f^{-1}(\{0\} \times V) = V_1 \cap V_2, \tag{1.9}$$

and we may wish to compute $\mu_{\{0\} \times V}(f)$. If V_1 and V_2 are in algebraic general position off 0, then $V_1 \cap V_2$ is a complete intersection.

REMARK 1.6.9. A sufficient condition for f_1 to define an ICIS, f_2 to be algebraically transverse off 0 to V, and for V_1 and V_2 to be in algebraic general position off 0 is that f be algebraically transverse off 0 to $(\{0\} \times \mathbb{C}^{p_2} \cup \mathbb{C}^{p_1} \times V)$, (see Lemma 7.2 of [**Dam96a**]).

There exists a formula for the rank of the $(n-p_1)$st relative homology group of the Milnor fiber of $(V_1, 0)$ and the singular Milnor fiber of (1.9). This formula for the "relative singular Milnor number" is a generalization of the Lê-Greuel formula (Theorem 1.2.9).

THEOREM 1.6.10 (Corollary 9.6, (9.1) of [**Dam96a**], or Theorem 4.2 of [**Dam01**]). *Let* V_1*,* V_2*, and* $f : \mathbb{C}^n, 0 \to \mathbb{C}^{p_1+p_2}, 0$ *be as above, with* V_1 *and* V_2 *in algebraic general position off* 0*. Let* $H : \mathbb{C}^{p_2} \to \mathbb{C}$ *be a good defining equation for* V *and let* $\zeta_1, \ldots, \zeta_{p_2-1}$ *be a set of generators for* $\operatorname{Derlog}(H)$*. Let* M *be the quotient* $\mathscr{O}_{V_1,0}$*-module given by*

$$M = (\mathscr{O}_{V_1,0})^{p_1+p_2} \Big/ \left(\mathscr{O}_{V_1,0} \left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \zeta_1 \circ f_2, \ldots, \zeta_{p_2-1} \circ f_2 \right\} \right).$$

There is the following formula involving the Milnor number of the ICIS $(V_1, 0)$ *and the singular Milnor number of* f *for* $\{0\} \times V$*:*

$$\mu(V_1) + \mu_{\{0\}\times V}(f) = \dim_{\mathbb{C}}(M).$$

1.6.5. Free* divisors. We may weaken the notion of free divisor but still compute singular Milnor numbers by using a weaker free* divisor structure.

DEFINITION 1.6.11. For a hypersurface germ $(V, 0) \subset (\mathbb{C}^p, 0)$ with $p' = p + m \geq p$, we may let $V' = V \times \mathbb{C}^m \subset \mathbb{C}^{p'}$. Then a *free* divisor structure for* V *defined on* $\mathbb{C}^{p'}$ is an $\mathscr{O}_{(\mathbb{C}^{p'},0)}$-module $\mathscr{L} \subseteq \operatorname{Derlog}(V')$ such that

(1) $\mathscr{L}$ is a free $\mathscr{O}_{(\mathbb{C}^{p'},0)}$ module of rank p', and
(2) $\operatorname{supp}(\theta_{p'}/\mathscr{L}) = V'$, where supp denotes the support of a sheaf.

For example, when $m = 0$, a free* divisor structure for V is a free submodule $\mathscr{L} \subseteq \operatorname{Derlog}(V)$ of rank p, for which the p generators of $\mathscr{L}$ are linearly dependent exactly on V. However, this may define V with non-reduced structure. A free* divisor structure is non-unique and not necessarily closed under the Lie bracket; when a free* divisor is closed under the Lie bracket, we will make note of it.

EXAMPLE 1.6.12. A germ $(V, 0) \subset (\mathbb{C}^3, 0)$ of the Whitney umbrella is defined by the equation $yw^2 - z^2 = 0$. A Macaulay 2 calculation ([**GS**]) shows that $\operatorname{Derlog}(V)$ has four generators:

$$\xi_1 = 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \qquad \xi_2 = w\frac{\partial}{\partial w} + z\frac{\partial}{\partial z}$$
$$\xi_3 = 2z\frac{\partial}{\partial y} + w^2\frac{\partial}{\partial z} \qquad \xi_4 = z\frac{\partial}{\partial w} + yw\frac{\partial}{\partial z}.$$

$(V,0)$ is not a free divisor. But (see, e.g., [**Dam03**], Example 1.6) the module $\mathscr{L}$ generated by $\xi_1, \xi_4, z\xi_3 + w^2\xi_2$ is a free module of rank 4 for which the generators are linearly dependent only on $(V,0)$; thus $(V,0)$ has $\mathscr{L}$ as a free* structure.

The usefulness of a free* structure $\mathscr{L} \subseteq \operatorname{Derlog}(V)$ for computing the singular Milnor number of a nonlinear section $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ of $(V,0) \subset (\mathbb{C}^p, 0)$ depends on how close $\dim_{\mathbb{C}}(\langle \mathscr{L} \rangle_y)$ is to the dimension of the canonical Whitney stratum containing y, for all $y \in V$. We must calculate a correction term for each *virtual singularity* v that arises when f_0 is geometrically transverse to V at v but f_0 is not algebraically transverse for $\mathscr{L}$, i.e.,

$$d(f_0)_{(v)}(T_v\mathbb{C}^n) + \langle \mathscr{L} \rangle_{f_0(v)} \neq T_{f_0(v)}\mathbb{C}^p.$$

Virtual singularities may arise unavoidably when we find a stabilization of f_0 relative to V.

The correction terms for $\mu_V(f_0)$ are given in terms of "codimensions" of f_0. Let H be a good defining equation for $(V,0)$, with free* structure $\mathscr{L}$. We may then define an *extended* $\mathscr{K}^*_{H,e}$*-codimension* for f_0 just as we do for $\mathscr{K}_H$, except using generators for $\mathscr{L}_0 = \mathscr{L} \cap \operatorname{Derlog}(H)$ instead of $\operatorname{Derlog}(H)$. This may not correspond to the codimension of any group of equivalences.

The version of Theorem 1.6.5 with correction terms becomes

THEOREM 1.6.13 ([**Dam03**], Theorem 5). *Let $(V,0) \subset (\mathbb{C}^p, 0)$, $\mathscr{L}$, and H be as above. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ have finite $\mathscr{K}^*_{H,e}$-codimension. Then for f_t a stabilization of f_0,*

$$\mu_V(f_0) = \mathscr{K}^*_{H,e}\text{-codim}(f_0) - \sum_{x \in S_t} \mathscr{K}^*_{H,e}\text{-codim}(f_t, x),$$

where S_t is the (finite) set of virtual singularities of f_t in $\mathbb{C}^n$ for a sufficiently small t, and where $\mathscr{K}^*_{H,e}$-codim(f_t, x) denotes the $\mathscr{K}^*_{H,e}$ codimension of the germ $f_t : \mathbb{C}^n, x \to \mathbb{C}^p, f_t(x)$.

It can be difficult to use this Theorem to calculate the singular Milnor number because we must determine the set of virtual singularities in a stabilization and calculate a codimension for each. Where possible we will find and use free divisors.

1.7. Matrix singularities

While important classes of highly non-isolated singularities such a discriminants, bifurcation sets, and hyperplane arrangements are free divisors, this is not the case for matrix singularities, our principal objects of consideration. We first define this term and review some work from the literature regarding the classical Milnor numbers of isolated matrix singularities.

1.7.1. Families of matrices. Bruce and Tari in [**BT04**] study parametrized families of square matrices up to equivalence by a group action. They define a group which, when acting on a family $A : \mathbb{C}^n \to \mathrm{Hom}(V, W)$ with $V \simeq W \simeq \mathbb{C}^p$, allows parametrized linear changes of coordinates for both V and W as well as diffeomorphisms of the parameter space $\mathbb{C}^n$. Up to equivalence of such families under this group action, they classify such families until moduli appear (the "simple families").

Similarly, Bruce classifies in [**Bru03**] the simple families of symmetric matrices up to a the action of a group which allows parametrized linear changes of coordinates for the source space (which equals the target space). Haslinger in [**Has01**] has studied the same problem for the space of skew-symmetric matrices.

1.7.2. Matrix singularities. For any of the cases listed above, we shall call an orbit of the group's actions on a family of matrices a *matrix singularity*. Goryunov and Mond [**GM05**] showed that in each case the tangent spaces of the groups used by Bruce, Tari, and Haslinger are equal to the $\mathscr{K}_V$ tangent space, where V is the variety of the singular matrices in each case, and therefore the orbits of the groups on families of matrices agree (see §2 of [**GM05**]). Thus the following is an equivalent definition of matrix singularity.

Definition 1.7.1. Let M be a vector space of matrices, and let $V \subset M$ be the *determinantal variety*, the hypersurface consisting of singular matrices. A *matrix singularity* is an orbit in the space of germs $\mathbb{C}^n, 0 \to M, 0$ under the action of $\mathscr{K}_V$.

Note that the codimension of $\mathrm{Sing}(V)$ in V is 2, respectively 3, respectively 5, for the case of symmetric, respectively general $k \times k$, respectively even skew-symmetric matrices. Hence, so will matrix singularities defined from them when $n \geq \mathrm{codim}(\mathrm{Sing}(V))$. By results of Kato-Matsumoto (Theorem 1.3.1), the regular Milnor fiber may have homology in dimensions greater than or equal to 1, respectively, 2, respectively, 4.

1.7.3. Goryunov-Mond. One possibility is to consider only isolated matrix singularities where $n \leq \mathrm{codim}(\mathrm{Sing}(V))$. This is the approach taken by Goryunov-Mond [**GM05**]. They studied deformations of holomorphic germs of the form $f \circ F$, for fixed $f : \mathbb{C}^n, 0 \to \mathbb{C}, 0$, and

$F : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$ may vary. Their goal is to relate the Milnor number of $f \circ F$ with the deformation theory of F as a nonlinear section of $f^{-1}(0)$. They take free resolutions of the $\mathscr{O}_{\mathbb{C}^n,0}$-modules $\mathscr{O}_{\mathbb{C}^n,0}/J_f$ and $\mathscr{O}_{\mathbb{C}^n,0}/(f) + J_f$, for J_f the ideal generated by the first partials of f, and by homological algebra obtain exact sequences which have zero terms when $f \circ F$ has an isolated singularity at $0 \in \mathbb{C}^n$. By taking the alternating sum of the ranks of the modules involved, they obtain the following Theorem.

Theorem 1.7.2 ([**GM05**], Theorem 1.5). *If $f \circ F$ has isolated singularity, then*

$$\mathscr{K}_{f,e}\text{-codim}(F) = \mu(f \circ F) - \beta_0 + \beta_1,$$

where $\beta_i = \operatorname{rank} \operatorname{Tor}_i^{\mathscr{O}_{\mathbb{C}^n,0}}(\mathscr{O}_{\mathbb{C}^n,0}/J_f, \mathscr{O}_{\mathbb{C}^m,0})$ and $\mu(f \circ F)$ is the Milnor number.

Their main interest is where f is the determinant or Pfaffian function on a vector space of square matrices (symmetric, skew-symmetric, or general $k \times k$). They find explicit free resolutions in the literature for these cases and when m is small are able to make conclusions about β_0 and β_1. For example, when f is the determinant function on the space of symmetric matrices and $m = 2$, they show $\beta_0 = \beta_1 < \infty$, so that $\mathscr{K}_{f,e}$-codim$(F) = \mu(f \circ F)$ (their Corollary 4.4).

1.7.4. Our approach. Our goal in this dissertation is to compute the singular Milnor number of general matrix singularities. Since we allow $n \geq \operatorname{codim}(\operatorname{Sing}(V))$, our matrix singularities may be highly non-isolated. Although the variety V of singular matrices is not a free divisor, motivated by (1.8) we embed it as a component of a free divisor, express $\mu_V(f)$ in terms of other singular Milnor numbers, and use Theorems 1.6.5 and 1.6.10 to compute these auxiliary singular Milnor numbers.

CHAPTER 2

Lie group representations with open orbits

In Example 3 of Chapter 1, we pointed out that certain rational representations of connected complex algebraic Lie groups with open orbits have a free divisor as the complement of their open orbit. Free divisors which arise in this way and are contained in a space of matrices will enable us to calculate the singular Milnor number of certain matrix singularities.

This chapter will serve as an introduction to representations of Lie groups with open orbits. In the first section we recall a number of basic results for a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex algebraic Lie group G with Lie algebra $\mathfrak{g}$. Any element $A \in \mathfrak{g}$ gives rise to a vector field ξ_A on V, and the map $A \mapsto \xi_A$ can be given the structure of a Lie algebra homomorphism. Calculations involving a basis of this finite-dimensional Lie algebra of vector fields allow us to determine if ρ has an open orbit and, if it does, to obtain an explicit description of it. Any open orbit must be Zariski open, and when $\dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(V)$ the complement of the open orbit is a hypersurface. In Chapter 3 we will see that such hypersurfaces are either free divisors or free* divisors.

In section 2 of this chapter, we see how several classical matrix factorization results (LU, Cholesky, and a variant of Cholesky factorization for skew-symmetric matrices) can be viewed as statements about the open orbit of a rational representation of a connected complex algebraic Lie group. These matrix factorizations generally factor a matrix into a product of lower and upper triangular matrices. The original factorization results in the real case are used heavily in computational linear algebra.

The observation that all of the groups in the classical factorizations are solvable leads to a short discussion in section 3 of the properties of solvable algebraic Lie groups, primarily the Lie-Kolchin Theorem. We shall use these properties freely to develop our “block representations” in Chapter 3.

Since representations with open orbits have been studied before as "prehomogeneous vector spaces," we summarize the basic theory in section 2.4. It is interesting to view our theory from this other perspective, a task we will continue in the later portions of Chapters 3 and 5.

2.1. Representations of Lie groups and the Lie algebra of vector fields

Let G be a connected complex algebraic Lie group with Lie algebra $\mathfrak{g}$. Let V be a complex vector space and let $\rho : G \to \mathrm{GL}(V)$ be a rational representation, so that $G \times V \to V$ is a rational map.

For any $A \in \mathfrak{g}$ we may define a holomorphic vector field ξ_A on V by

$$\xi_A(x) = \frac{d}{dt}(\rho(\exp(t \cdot A))(x)) \mid_{t=0} . \tag{2.1}$$

By the naturality of the exponential function (see, e.g., I (3.2) of [**BtD85**]), we may rewrite this as

$$\begin{aligned} \xi_A(x) &= \frac{d}{dt}(\rho(\exp(t \cdot A))(x)) \mid_{t=0} \\ &= \frac{d}{dt}(\exp(t \cdot d\rho_{(e)}(A))(x)) \mid_{t=0} \\ &= d\rho_{(e)}(A)(x), \end{aligned} \tag{2.2}$$

where $d\rho_{(e)}(A) \in \mathrm{End}\,(V)$. Each ξ_A is a "linear" vector field in two senses: first, evaluation of ξ_A at a point is given by evaluating a linear map $V \to V$; and also that, after choosing vector space coordinates, all of the coefficient functions of ξ_A are either 0 or homogeneous of degree 1.

We might expect that $A \mapsto \xi_A$ is a Lie algebra homomorphism. This is true, with the caveat that we must use the negative of the usual Lie bracket[1] for vector fields, an arbitrary choice anyway (see §1.7 of [**Akh95**]). This is a convention we use throughout the rest of this dissertation.

Proposition 2.1.1. *The association $A \mapsto \xi_A$ is a Lie algebra homomorphism.*

Proof. Since $d\rho_{(e)}$ is a linear map, (2.2) indicates that $A \mapsto \xi_A$ is linear.

Fix $y \in V$ and $A, B \in \mathfrak{g}$. The flow related to any ξ_X for $X \in \mathfrak{g}$ is

$$(x, t) \mapsto \rho(\exp(tX))(x),$$

[1]For us, the usual Lie bracket is $[\eta, \xi]_u(f) = \eta(\xi(f)) - \xi(\eta(f))$.

for t small enough. Denoting by $[\cdot,\cdot]_u$ the usual Lie bracket and recalling a formula for the usual Lie bracket of vector fields ([**Spi05**], pages 162, 176), we have

$$\begin{aligned}
[\xi_A,\xi_B]_u(y) &= \frac{d}{dt}\left(\rho(\exp(-\sqrt{t}B))\circ\rho(\exp(-\sqrt{t}A))\circ\rho(\exp(\sqrt{t}B))\circ\rho(\exp(\sqrt{t}B))(y)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\rho\left(\exp(-\sqrt{t}B)\cdot\exp(-\sqrt{t}A)\cdot\exp(\sqrt{t}B)\cdot\exp(\sqrt{t}B)\right)(y)\right)\Big|_{t=0}
\end{aligned}$$

Using the Baker-Campbell-Hausdorff formula several times, we see that

$$\begin{aligned}
\exp(-&\sqrt{t}B)\cdot\exp(-\sqrt{t}A)\cdot\exp(\sqrt{t}B)\cdot\exp(\sqrt{t}B) \\
&= \exp(-\sqrt{t}B-\sqrt{t}A-\frac{1}{2}t[A,B]+\cdots)\exp(\sqrt{t}A+\sqrt{t}B-\frac{1}{2}t[A,B]+\cdots) \\
&= \exp(-\sqrt{t}B-\sqrt{t}A-\frac{1}{2}t[A,B]+\sqrt{t}A+\sqrt{t}B-\frac{1}{2}t[A,B] \\
&\qquad +\frac{1}{2}([-\sqrt{t}A,\sqrt{B}]+[-\sqrt{t}B,\sqrt{t}A])+\cdots) \\
&= \exp(-t[A,B]+\cdots),
\end{aligned}$$

where $\cdots$ is used to denote terms of order greater than one in t. Thus

$$\begin{aligned}
[\xi_A,\xi_B]_u(y) &= \frac{d}{dt}\left(\rho(\exp(-t[A,B]+\cdots))(y)\right)\big|_{t=0} \\
&= d\rho_{(e)}(-[A,B])(y) \\
&= -d\rho_{(e)}([A,B])(y).
\end{aligned}$$

As a result, using the negative of the usual Lie bracket makes $A\mapsto\xi_A$ a Lie algebra homomorphism. $\square$

We thus have a natural commutative diagram

$$\begin{array}{ccccc}
\mathfrak{g} & \xrightarrow{\tilde{\rho}} & \mathfrak{gl}(V) & \xrightarrow{\tilde{i}} & \mathscr{M}\cdot\theta_n \\
{\scriptstyle\exp}\downarrow & & {\scriptstyle\exp}\downarrow & & {\scriptstyle\exp}\downarrow \\
G & \xrightarrow{\rho} & \mathrm{GL}(V) & \xrightarrow{i} & \mathrm{Diff}(V,0)
\end{array} \tag{2.3}$$

where i is the natural inclusion, $\mathrm{Diff}(V,0)$ is the group of germs of diffeomorphisms on V fixing 0, $\mathscr{M}$ is the maximal ideal in $\mathscr{O}_{\mathbb{C}^n,0}$, and $\tilde{\rho}$ and $\tilde{i}$ are the Lie algebra homomorphisms coming from ρ and i, respectively.

Remark 2.1.2. We are interpreting the commutativity of the square on the right in terms of 1-parameter subgroups: For $X\in\mathfrak{gl}(V)$, the curve $\gamma(t)=tX$ in $\mathfrak{gl}(V)$ maps to $\Gamma(t)=(v\mapsto$

$e^{tX} \cdot v)$ under $i \circ \exp$, where we use the matrix exponential. But $\Gamma : \mathbb{R} \to \mathrm{Diff}(V, 0)$ is a group homomorphism with $\Gamma'(0) = (v \mapsto X \cdot v) = \tilde{i}(X)$, justifying the diagram.

The image of $\mathfrak{g}$ under the composition $\tilde{i} \circ \tilde{\rho}$ is a finite dimensional Lie algebra which generates an $\mathscr{O}_{\mathbb{C}^n,0}$-module L, where L is also an infinite-dimensional Lie algebra. When we have a finite collection of vector fields on $\mathbb{C}^n$ which generate a $\mathscr{O}_{\mathbb{C}^n,0}$-module $L \subset \theta_n$ which is also an infinite-dimensional Lie algebra, we will call L a *holomorphic Lie algebra*.

2.1.1. Representations with open orbits. Let $A_1, \ldots, A_n$ be a basis for $\mathfrak{g}$. The vector fields $\xi_{A_1}, \ldots, \xi_{A_n}$ obtained via (2.1) give a great deal of information about the infinitesimal action of G on V. The following basic result will be of repeated use to us (for a proof, see §1.7 of [**Akh95**]).

PROPOSITION 2.1.3. *Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of a connected complex algebraic Lie group. For any $v \in V$, the tangent space to the orbit of v is spanned by $\{\xi_{A_1}(v), \ldots, \xi_{A_n}(v)\}$. The subsets*

$$\begin{aligned} E_k &= \{v \in V | \dim (G \cdot v) < k\} \\ &= \{v \in V | \mathrm{rank}\, \{\xi_{A_1}(v), \ldots, \xi_{A_n}(v)\} < k\} \end{aligned}$$

are analytic. If G has an open orbit Ω in V, then the open orbit is unique, connected, Zariski open, and G acts transitively on Ω; moreover, the complement to Ω is $E_{\dim_{\mathbb{C}}(V)}$.

EXAMPLE 2.1.4. Let G be the group of upper triangular 2×2 matrices whose lower right entry is equal to 1. Consider the representation $\rho : G \to \mathrm{GL}(M(1, 2, \mathbb{C}))$ on the space of 1×2 matrices where

$$\rho(A)(M) = MA^T.$$

Using coordinates $\begin{pmatrix} x & y \end{pmatrix}$ on $M(1, 2, \mathbb{C})$, the vector fields as in the Proposition are $\xi_{E_{1,1}} = -x\frac{\partial}{\partial x}$ and $\xi_{E_{1,2}} = -x\frac{\partial}{\partial y}$ which are linearly dependent whenever $x = 0$. By the Proposition, $E_2 = \{x = 0\}$ is the complement of the open orbit. Note that in this example, $\det(\xi_{E_{1,1}}, \xi_{E_{1,2}}) = -x^2$ does not give a reduced defining equation for E_2.

EXAMPLE 2.1.5. Consider the representation $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(M(2, 2, \mathbb{C}))$ on the space of 2×2 complex matrices, where

$$\rho(A)(M) = AMA^T.$$

The dimension of the group and the vector space agree and $\ker(\rho) = \pm I_2$ is discrete. However, this representation does not have an open orbit. After choosing coordinates, vector fields as in the Proposition may be chosen to be

$$\begin{aligned}
\xi_{E_{1,1}} &= 2x_{11}\frac{\partial}{\partial x_{11}} + x_{12}\frac{\partial}{\partial x_{12}} + x_{21}\frac{\partial}{\partial x_{21}} + 0\frac{\partial}{\partial x_{22}} \\
\xi_{E_{1,2}} &= (x_{12}+x_{21})\frac{\partial}{\partial x_{11}} + x_{22}\frac{\partial}{\partial x_{12}} + x_{22}\frac{\partial}{\partial x_{21}} + 0\frac{\partial}{\partial x_{22}} \\
\xi_{E_{2,1}} &= 0\frac{\partial}{\partial x_{11}} + x_{11}\frac{\partial}{\partial x_{12}} + x_{11}\frac{\partial}{\partial x_{21}} + (x_{12}+x_{21})\frac{\partial}{\partial x_{22}} \\
\xi_{E_{2,2}} &= 0\frac{\partial}{\partial x_{11}} + x_{12}\frac{\partial}{\partial x_{12}} + x_{21}\frac{\partial}{\partial x_{21}} + 2x_{22}\frac{\partial}{\partial x_{22}}.
\end{aligned}$$

There is a relation between these vector fields:

$$-(x_{12}+x_{21})\xi_{E_{1,1}} + 2x_{11}\xi_{E_{1,2}} - 2x_{22}\xi_{E_{2,1}} + (x_{12}+x_{21})\xi_{E_{2,2}} = 0,$$

so that they are linearly dependent. Equivalently, the matrix formed by the coefficients of the vector fields has determinant 0. The rank as in the Proposition is always < 4 and the isotropy subgroup at any point of $M(2,2,\mathbb{C})$ has positive dimension. Thus information about the kernel of a representation is not enough to show it has an open orbit.

An interpretation of this example is that there are an infinite number of (not necessarily symmetric) bilinear forms which are distinct, even allowing a change of basis. Note that ρ is reducible, with $M(2,2,\mathbb{C}) \simeq \mathrm{Sym}_2(\mathbb{C}) \oplus \mathrm{Sk}_2(\mathbb{C})$, and that ρ has open orbits on each component.

It will be useful to test if a particular point lies in the open orbit. A criterion for this is easiest to state by using the following notation.

Notation 2.1.6. For a representation $\psi : H \to \mathrm{GL}(W)$ and $w \in W$, we will use the notation $\psi^w : H \to W$ for the holomorphic function $\psi^w(h) = \psi(h)(w)$.

Note that $\xi_A(v) = d(\psi^v)_{(e)}(A)$. Then for a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex algebraic group,

Corollary 2.1.7. *The orbit of $v \in V$ under ρ is open if and only if $d(\rho^v)_{(e)}$ is surjective.*

Proof. If the orbit of $v \in V$ under ρ is open, then $v \notin E_{\dim_{\mathbb{C}}(V)}$ by Proposition 2.1.3 and so $d(\rho^v)_{(e)}$ is surjective. If $d(\rho^v)_{(e)}$ is surjective, then ρ^v is a submersion at e and a corollary of the implicit function theorem says that the image of ρ^v, the orbit of v, contains an open neighborhood of v. But then the orbit of v is open. $\square$

EXAMPLE 2.1.8. Irreducibility is also not enough to ensure that a representation has an open orbit. For example, the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ for a semisimple connected complex algebraic Lie group G is irreducible. For any $X \in \mathfrak{g}$,

$$\begin{aligned} d(\mathrm{Ad}^X)_{(e)}(X) &= \frac{d}{dt}\left(\mathrm{Ad}(\exp(tX))(X)\right)|_{t=0} \\ &= [X, X] = 0. \end{aligned}$$

Since $\dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(\mathfrak{g})$, each $d(\mathrm{Ad}^X)_{(e)}$ is not surjective. Thus by Corollary 2.1.7, no $X \in \mathfrak{g}$ has an open orbit.

It is possible that the complement $E_{\dim_{\mathbb{C}}(V)}$ of the open orbit is reducible with components of different dimensions.

EXAMPLE 2.1.9. Let $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ act on $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$ by

$$(A, \lambda) \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix}, z \right) = \left(A \begin{pmatrix} x \\ y \end{pmatrix}, \lambda z \right).$$

It is clear that

$$E_3 = \{(x, y, z) | z = 0\} \cup \{(x, y, z) | x = y = 0\}$$

consists of two components of dimensions two and one, respectively.

Notice, however, that this example has a 5-dimensional group acting on a 3-dimensional vector space. If we require the group and its vector space to have the same dimension, then all components of the complement of the open orbit will have the same dimension.

PROPOSITION 2.1.10. *Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of a connected complex algebraic group with $A_1, \ldots, A_n$ a basis for $\mathfrak{g}$. If the representation has an open orbit Ω and $n = \dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(V)$, then we may define an analytic function $h_\rho : V \to \mathbb{C}$ by*

$$h_\rho = \det(\xi_{A_1}, \ldots, \xi_{A_n}), \tag{2.4}$$

in which case the complement

$$\Omega^c = E_n = \{h_\rho = 0\}$$

is a hypersurface.

REMARK 2.1.11. By (2.4) we mean: choose coordinates $x_1, \ldots, x_n$ for V, write each vector field in terms of these coordinates as

$$\xi_j = \sum_{i=1}^{n} \alpha_{ij} \frac{\partial}{\partial x_i},$$

form a matrix M whose (i,j)th entry is the coefficient function α_{ij}, and let h_ρ be the function formed by taking the determinant of M.

PROOF. As each column of M consists of the coefficients of a single vector field, $h_\rho(v) = 0$ if and only if the columns of $M(v)$ are linearly dependent, i.e., rank $\{\xi_{A_1}(v), \ldots, \xi_{A_n}(v)\} < n$. By Proposition 2.1.3, this happens if and only if $v \in E_n = \Omega^c$. Thus Ω^c is defined by the vanishing of a single function and is a hypersurface. $\square$

DEFINITION 2.1.12. We call the analytic set described in Proposition 2.1.10 the *exceptional orbit variety* of the representation.

DEFINITION 2.1.13. We say that a rational representation of a connected complex algebraic Lie group is *equidimensional* when the complex dimensions of the group and the vector space agree and the representation has an open orbit.

In particular, the kernel of an equidimensional representation is discrete.

We are ultimately interested in understanding the structure of the exceptional orbit variety. Each vector field ξ_A is tangent to the exceptional orbit variety.

LEMMA 2.1.14. *Let $\rho : G \to \mathrm{GL}(V)$ be an equidimensional representation, with exceptional orbit variety $\mathscr{E}$. If $A \in \mathfrak{g}$, then $\xi_A \in \mathrm{Derlog}\,(\mathscr{E})$.*

PROOF. The integral flow $\theta : (\mathbb{R} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C}^n, 0)$ for ξ_A is given by $\theta(t,x) = \rho(\exp(tA))(x)$. Since, by definition, $\mathscr{E}$ is the union of all non-open orbits, $\theta(t,x) \in \mathscr{E}$ for all t and $x \in \mathscr{E}$. If h is any analytic function vanishing on $\mathscr{E}$, then $(h \circ \theta)(t,x) = 0$ when $x \in \mathscr{E}$, so that $\xi_A(h)$ is again a function vanishing on $\mathscr{E}$. $\square$

Consequently, if $\mathscr{E}$ is the exceptional orbit variety of an equidimensional representation and L is the holomorphic Lie algebra generated by its finite-dimensional Lie algebra of vector fields, then $L \subseteq \mathrm{Derlog}\,(\mathscr{E})$. We need not have equality as exhibited by Example 2.1.4.

2.2. Matrix factorizations

In this section we will see how several classical matrix factorizations can be seen as statements precisely describing the open orbit of a representation on a space of matrices. We shall state and prove all of our results over $\mathbb{C}$, although the corresponding results over $\mathbb{R}$ are more often used for applied numerical linear algebra.

2.2.1. LU decomposition.

The classical LU decomposition of a real square matrix M factors M into the product of a lower triangular matrix and an upper triangular matrix (see [**Dem97**], Theorem 2.2.4). In numerical linear algebra, it is used to calculate the determinant of a matrix, to find the inverse of a matrix, or to solve a linear system of equations. These operations are all very easy for upper or lower triangular matrices; by computing the LU decomposition it is possible to break a difficult problem into two easy problems.

There is the following LU decomposition for complex matrices.

THEOREM 2.2.1 (Complex LU Decomposition). *Let M be an $n \times n$ complex matrix. Let $M^{(k)}$ denote the upper left $k \times k$ submatrix of M for $k = 1, \ldots, n$. Then $\det(M^{(k)}) \neq 0$ for all $k = 1, \ldots, n$ if and only if there exists a unique invertible lower triangular matrix L and an upper triangular matrix U with ones on the diagonal so that $M = LU$.*

PROOF. The result clearly holds in the 1×1 case. To show that the decomposition exists and is unique, assume there are unique factorizations for $(n-1) \times (n-1)$ matrices whose upper left submatrices are nonsingular. Let $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, with M_1 a $(n-1) \times (n-1)$ matrix. By the induction hypothesis, there exists unique matrices of the form described in the theorem so that $M_1 = L_1 U_1$. We wish to find L_2, L_3 and U_2 so that

$$\begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} L_1 U_1 & L_1 U_2 \\ L_2 U_1 & L_2 U_2 + L_3 \end{pmatrix} \tag{2.5}$$

is equal to M. The unique possibility is to let $U_2 = L_1^{-1} M_2$, $L_2 = M_3 U_1^{-1}$ and $L_3 = M_4 - L_2 U_2$, choices which guarantee that $M = LU$. To see that L_3 is nonzero, it is sufficient to note that U and M are invertible, so that L must be as well. We have thus shown existence and uniqueness of the LU decomposition for $n \times n$ matrices.

To show the other direction, let $M = LU$ be an LU decomposition. For any $k = 1, \ldots, n$, (2.5) implies that $M^{(k)} = L^{(k)}U^{(k)}$. Since the right side is nonsingular, $\det(M^{(k)}) \neq 0$. $\square$

REMARK 2.2.2. The LU decomposition is sometimes defined as having ones on the diagonal of L and allowing nonzero entries on the diagonal of U. We could also define a unique "LDU" decomposition, where both L and U have ones on the diagonal and D is diagonal and invertible.

Theorem 2.2.1 can also be seen as a statement about the orbit of the identity under the action of the group consisting of pairs of lower triangular matrices and upper triangular matrices with ones on the diagonal. Let $L_n(\mathbb{C})$ denote the group of lower triangular matrices and let $N_n(\mathbb{C})$ denote the group of upper triangular matrices with ones on the diagonal. Consider the representation of $L_n(\mathbb{C}) \times N_n(\mathbb{C})$ on the space $M(n, n, \mathbb{C})$ of $n \times n$ matrices by

$$(A, B) \cdot M = AMB^{-1}. \tag{2.6}$$

Then Theorem 2.2.1 can be rephrased as a statement about the open orbit of this representation.

It will be helpful to first give some useful notation.

DEFINITION 2.2.3. By a *generic matrix* for a vector space W of matrices, we mean a matrix S whose entries are holomorphic functions $W \to \mathbb{C}$ so that for any $M \in W$, $S(M) = M$.

The entries of S are usually coordinates on W but may be 0 or a constant multiple of a coordinate.

EXAMPLE 2.2.4. For example, the generic 3×3 skew-symmetric matrix (using the standard coordinates) is

$$\begin{pmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{pmatrix},$$

where $x_{ij}(M) = (M)_{ij}$.

COROLLARY 2.2.5. *The action defined by* (2.6) *of* $L_n(\mathbb{C}) \times N_n(\mathbb{C})$ *on the space* $M(n, n, \mathbb{C})$ *of* $n \times n$ *complex matrices has a Zariski open orbit* Ω*, whose complement consists of those matrices which, for some* $k = 1, \ldots, n$*, the upper left* $k \times k$ *submatrix is singular. If* S *is a generic* $n \times n$

matrix, then Ω^c is the algebraic hypersurface defined by

$$\prod_{k=1}^{n} \det(S^{(k)}) = 0.$$

Thus, we have identified the exceptional orbit variety of this equidimensional representation.

2.2.2. Cholesky factorization of symmetric matrices. The Cholesky factorization is an analogue of LU factorization for symmetric matrices, except that the two triangular matrices are transposes of each other (see, e.g., [**Dem97**] Proposition 2.2.2). In numerical linear algebra, it enables problems involving positive definite real symmetric matrices to be solved in half the time and half the storage space of a problem involving an arbitrary square matrix.

Theorem 2.2.6 (Complex Cholesky factorization). *Let M be an $n \times n$ symmetric matrix. Let $M^{(k)}$ denote the upper left $k \times k$ submatrix of M for $k = 1, \ldots, n$. Then $\det(M^{(k)}) \neq 0$ for all $k = 1, \ldots, n$ if and only if there exists an invertible lower triangular matrix L so that $M = LL^T$. L is unique up to multiplication on the right by a diagonal matrix whose diagonal entries are ± 1.*

Proof. To show that the factorization exists we use the LU decomposition. By Theorem 2.2.1, there exists a lower triangular matrix L_1 and an upper triangular matrix U_1 with ones on the diagonal so that $M = L_1 U_1$. There exists a unique diagonal matrix D consisting of the diagonal entries of L_1 so that $L_1 = L_2 D$, for L_2 a lower triangular matrix with ones on the diagonal. Then $M = L_2 D U_1$, and by Remark 2.2.2, L_2, D, and U_1 are unique. Since M is symmetric,

$$M = M^T = (U_1)^T D^T (L_2)^T = (U_1)^T D (L_2)^T,$$

and the uniqueness of the LDU decomposition implies that $U_1^T = L_2$. We may then find a diagonal matrix E so that $E^2 = D$; such a matrix is unique up to multiplication of the diagonal elements by ± 1. But then

$$M = (L_2 E)(E L_2^T) = (L_2 E)(L_2 E)^T,$$

giving the decomposition.

Given a Cholesky factorization $M = LL^T$, there is a unique diagonal matrix D and a lower triangular matrix L_1 with ones on the diagonal so that $L = L_1 D$. Then

$$M = L_1 D (L_1 D)^T = (L_1 D D) L_1^T$$

is an LU decomposition for M, so that by Theorem 2.2.1, $\det(M^{(k)}) \neq 0$ for $k = 1, \ldots, n$. This also shows that the L chosen for the Cholesky factorization is unique up to multiplication by a diagonal matrix D whose square must be the identity. □

Cholesky factorization can also be seen as a statement about the orbit of the identity matrix under the action of the lower triangular group. Consider the action of $L_n(\mathbb{C})$ on the space $\mathrm{Sym}_n(\mathbb{C})$ of $n \times n$ symmetric matrices by

$$A \cdot S = ASA^T. \tag{2.7}$$

Then Theorem 2.2.6 can be seen as a statement about the open orbit of this group action.

Corollary 2.2.7. *The action defined by* (2.7) *of* $L_n(\mathbb{C})$ *on the space* $\mathrm{Sym}_n(\mathbb{C})$ *of* $n \times n$ *complex symmetric matrices has a Zariski open orbit* Ω*, whose complement is made up of those matrices which, for some* $k = 1, \ldots, n$*, the upper left* $k \times k$ *submatrix is singular. If* S *is a generic symmetric* $n \times n$ *matrix, then* Ω^c *is the algebraic hypersurface defined by*

$$\prod_{k=1}^{n} \det(S^{(k)}) = 0.$$

We have again identified the exceptional orbit variety of this equidimensional representation.

Remark 2.2.8. Although over $\mathbb{C}$ all non-degenerate symmetric bilinear forms can be written as the standard symmetric bilinear form by a change of coordinates, Theorem 2.2.6 and Corollary 2.2.7 indicate that not all of these can be written as the standard symmetric bilinear form if we only allow "lower triangular" changes of coordinates.

Remark 2.2.9. Although the equations which define the algebraic sets in Corollary 2.2.5 and Corollary 2.2.7 appear identical, they are different because their generic matrices are different. For example, in the 2×2 case they are, respectively,

$$x_{11}(x_{22}x_{11} - x_{12}x_{21}) = 0 \qquad \text{and} \qquad x_{11}(x_{22}x_{11} - (x_{12})^2) = 0.$$

2.2.3. Cholesky-type factorization of skew-symmetric matrices. Benner et al. ([**BBF$^+$00**]) found an analogue of the Cholesky factorization which factors a real skew-symmetric matrix into a product of a lower triangular matrix L, a particular skew-symmetric matrix, and L^T. The middle term is needed to make the product skew-symmetric. Define the standard $(2\ell) \times (2\ell)$ skew-symmetric matrix by

$$J_{2\ell} = \begin{pmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{pmatrix} \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the standard $(2\ell+1) \times (2\ell+1)$ skew-symmetric matrix by

$$J_{2\ell+1} = \begin{pmatrix} J_{2\ell} & 0 \\ 0 & 0 \end{pmatrix}.$$

A complicating factor is that any skew-symmetric matrix of odd dimensions is singular, so that a direct analogue of the LU factorization or Cholesky factorization (Theorems 2.2.1,2.2.6) is not possible. The correct statement is essentially that all of the upper left submatrices which can be nonsingular should be so.

We have the following complex analogue of the Cholesky-like factorization for skew-symmetric matrices found by Benner et al. (Theorem 2.2 of [**BBF$^+$00**]).

Theorem 2.2.10 (Cholesky-like factorization for complex skew-symmetric matrices). *For $n = 2\ell$, let V_n be the group of invertible lower triangular matrices with 2×2 blocks of the form*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \lambda \in \mathbb{C}^*$$

along the diagonal. For $n = 2\ell + 1$, let V_n be the group of invertible lower triangular matrices with 2×2 blocks of the above form along the diagonal, except the lower right diagonal entry is 1.

Let M be a skew-symmetric $n \times n$ matrix. Let $M^{(k)}$ denote the upper left $k \times k$ submatrix of M for all $k = 1, \ldots, n$. Then for $n = 2\ell$ or $n = 2\ell + 1$, $\det(M^{(2k)}) \neq 0$ for $k = 1, \ldots, \ell$ if *and only if there exists a* $L \in V_n$ *so that*

$$M = LJ_nL^T.$$

L *is unique up to multiplication on the right by a diagonal matrix* D *in* V_n *whose* 2×2 *diagonal blocks are* $\pm I_2$*.*

PROOF. The result clearly holds in the $n = 1$ and $n = 2$ cases.

We shall first show the existence and uniqueness of the factorization for even n by induction on n. Assume the result holds for $n \times n$ matrices with $n = 2\ell$. We shall show it holds for $(n+2) \times (n+2)$ matrices; let M be such a matrix and write $M = \begin{pmatrix} M_1 & M_2 \\ -M_2^T & M_3 \end{pmatrix}$ for M_3 a 2×2 skew-symmetric matrix.

A calculation shows that

$$\begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix} \begin{pmatrix} J_n & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix}^T = \begin{pmatrix} L_1J_nL_1^T & L_1J_nL_2^T \\ L_2J_nL_1^T & L_2J_nL_2^T + L_3JL_3^T \end{pmatrix}. \tag{2.8}$$

We want (2.8) to equal M; choose L_1 according to the induction hypothesis so that $L_1J_nL_1^T = M_1$. In order for (2.8) to equal M, we need

$$L_2J_nL_1^T = -M_2^T,$$

so we must let $L_2 = -M_2^T(L_1^T)^{-1}J_n^{-1} = M_2^T(L_1^T)^{-1}J_n$.

We also want the lower right entry of (2.8) to equal M_3. Since the space of skew-symmetric 2×2 matrices has dimension 1, we know that $M_3 = \eta J$ and $L_2J_nL_2^T = \mu J$. If we choose $L_3 = \lambda \cdot I$, where I denotes the identity and $\lambda \in \mathbb{C}^*$ is as yet undetermined, then $L_3JL_3^T = \lambda^2 J$. We must therefore have

$$\mu J + \lambda^2 J = (\mu + \lambda^2)J = M_3 = \eta J$$

for (2.8) to be satisfied. We must let $\lambda = \pm\sqrt{\eta - \mu}$, in which case (2.8) is satisfied.

To show that L_3 is invertible, we only need to examine the equation $LJ_{n+2}L^T = M$. Since the right-hand side is invertible by hypothesis and L is invertible if and only if L_3 is invertible, L_3 must be invertible. Moreover, for a given L_1, L_3 is unique up to multiplication by $\pm I_2$ on the right.

Regarding the uniqueness, suppose that we had used instead $L_1' = L_1 D_1$, where D_1 is a diagonal matrix with 2×2 diagonal blocks of the form $\pm I_2$. Then $L_2' = M_2^T (D_1^T L_1^T)^{-1} J_n = M_2^T (L_1^T)^{-1} D_1^{-1} J_n$. The particular form of D_1 means that $D_1 J_n = J_n D_1$ and $D_1 = D_1^{-1}$, so $L_2' = L_2 D_1^{-1} = L_2 D_1$. Since $L_2 J_n L_2^T = L_2' J_n (L_2')^T$ as $D_1^2 = I$, the possible choices for L_3' are the same as the choices for L_3. Thus, from any solution L, we may reach any other solution L' by multiplying on the right

$$\begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} L_1 D_1 & 0 \\ L_2 D_1 & L_3 D_2 \end{pmatrix}$$

by a suitable matrix D as described in the statement of the theorem.

To show the existence of the factorization in the $n = 2\ell + 1$ case, let M be a $(2\ell+1) \times (2\ell+1)$ skew-symmetric matrix satisfying the hypotheses in the theorem. Write M as $M = \begin{pmatrix} M_1 & M_2 \\ -M_2^T & 0 \end{pmatrix}$ for M_1 a $(2\ell) \times (2\ell)$ skew-symmetric matrix and M_2 a column vector. By the result for the even-dimensional case, M_1 has a factorization $M_1 = L_1 J_{2\ell} L_1^T$. Let $L_2 = -M_2^T (J_{2\ell} L_1^T)^{-1}$. Examining the calculation

$$\begin{pmatrix} L_1 & 0 \\ L_2 & 1 \end{pmatrix} \begin{pmatrix} J_{2\ell} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ L_2 & 1 \end{pmatrix}^T = \begin{pmatrix} L_1 J_{2\ell} L_1^T & L_1 J_{2\ell} L_2^T \\ L_2 J_{2\ell} L_1^T & L_2 J_{2\ell} L_2^T \end{pmatrix} = \begin{pmatrix} M_1 & M_2 \\ -M_2^T & L_2 J_{2\ell} L_2^T \end{pmatrix} \tag{2.9}$$

shows that we have almost completed the factorization with no choice in choosing L_2. But $L_2 J_{2\ell} L_2^T$ is a skew-symmetric 1×1 matrix and therefore equals 0. We have thus found a factorization of M; uniqueness follows from the uniqueness in the even case and, given L_1, the uniqueness of L_2.

To show that such a matrix with a factorization must have its upper-left submatrices nonsingular, suppose we had a matrix M and its factorization $M = L J_n L^T$, where $n = 2\ell$ or $n = 2\ell + 1$ and L is as given in the theorem. Let $k = 1, \ldots, \ell$. Then we may rewrite this equation in block form as

$$\begin{pmatrix} M_1 & M_2 \\ -M_2^T & M_3 \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix} \begin{pmatrix} J_{2k} & 0 \\ 0 & J_{n-2k} \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix}^T,$$

where L_1 and M_1 are $2k \times 2k$ matrices. A calculation shows that the upper left entry of the right-hand side will be $L_1 J_{2k} L_1^T$, an invertible matrix which must equal $M_1 = M^{(2k)}$. Thus each $M^{(2k)}$ is nonsingular for $k = 1, \ldots, \ell$. $\square$

As in the previous two cases, this theorem can be seen as a statement about the orbit of J_n under the action of the group V_n (as in the theorem). Let V_n act on the the space $\mathrm{Sk}_n(\mathbb{C})$ of $n \times n$ skew-symmetric matrices by

$$A \cdot M = AMA^T. \tag{2.10}$$

Then Theorem 2.2.10 says that this action has an open orbit.

Corollary 2.2.11. *Choose ℓ so that $n = 2\ell$ or $n = 2\ell + 1$. The action defined by* (2.10) *of V_n on the space* $\mathrm{Sk}_n(\mathbb{C})$ *of $n \times n$ skew-symmetric complex matrices has a Zariski open orbit Ω, whose complement is made up of those matrices M which, for some $k = 1, \ldots, \ell$, the upper left $2k \times 2k$ submatrix is singular. If S is a generic $n \times n$ skew-symmetric matrix, then Ω^c is the algebraic hypersurface defined by*

$$\prod_{k=1}^{\ell} \det(S^{(2k)}) = 0. \tag{2.11}$$

We have identified the exceptional orbit variety of this equidimensional representation. However, (2.11) is not a reduced defining equation for this hypersurface.

Remark 2.2.12. Recalling that the determinant of a square skew-symmetric matrix can always be written as the square of the Pfaffian Pf, we may define the algebraic hypersurface in (2.11) by

$$\prod_{k=1}^{\ell} \mathrm{Pf}(S^{(2k)}) = 0.$$

2.3. Properties of solvable linear algebraic groups

All of the representations we have discussed in Corollaries 2.2.5, 2.2.7, and 2.2.11 have been representations of solvable groups. Neither solvable groups nor their representations have been classified. For us, this is an advantage: there are many examples of equidimensional representations of solvable linear algebraic groups. We begin by pointing out a crucial property of solvable Lie groups: for any rational representation of a solvable connected linear algebraic group there are many invariant subspaces.

THEOREM 2.3.1 (Lie-Kolchin Theorem, e.g., Theorem III.10.5, [**Bor69**]). *Let* $\rho : G \to \mathrm{GL}(V)$ *be a rational representation of a solvable connected complex linear algebraic group* G *on a finite-dimensional vector space. Then* $\rho(G)$ *leaves invariant a complete flag of subspaces*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

where $\dim(V_i) = i$.

We will use invariant subspaces to analyze equidimensional representations.

EXAMPLE 2.3.2. Consider the 2×2 lower triangular matrices acting on the space of symmetric 2×2 matrices by $A \cdot M = AMA^T$. The calculation

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^T = \begin{pmatrix} a^2x & abx + acy \\ abx + acy & b^2x + 2bcy + c^2z \end{pmatrix}$$

indicates that the flag

$$\left\{ \begin{pmatrix} 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \right\}$$

is invariant.

NOTATION 2.3.3. We shall often use notation like $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ to denote a matrix where $*$ can be replaced by any complex number so that the resulting matrix makes sense in context (for example, we may require that the resulting matrix be symmetric). Let 0_k denote the $k \times k$ zero matrix, I_k denote the $k \times k$ identity matrix, and $*_k$ denote any $k \times k$ matrix. We may use these to express a block matrix of a particular type, where now $*$ can be replaced by any submatrix of the appropriate size such that the resulting matrix makes sense in context. For example, the set $\left\{ \begin{pmatrix} I_k & * \\ 0 & I \end{pmatrix} \right\}$ of $n \times n$ matrices is the group consisting of matrices whose entries are all zero except the diagonal entries are ones and the upper right $k \times (n-k)$ submatrix can be anything.

2.4. Prehomogeneous vector spaces

In this section we will review the theory of prehomogeneous vector spaces, originally introduced by Sato, Kimura, and others (see [**Sat90, Kim03**]) for the purpose of studying harmonic analysis. This theory studies all representations with open orbits in general, while we are interested in only the equidimensional case. In particular, the representations corresponding to the various matrix factorizations (LU, Cholesky, Cholesky-type for skew-symmetric matrices) are examples of prehomogeneous vector spaces. For such representations, there is a surprising connection between the components of the exceptional orbit variety of a representation $\rho : G \to \mathrm{GL}(V)$ with an open orbit and the characters of G.

2.4.1. Definition.

DEFINITION 2.4.1. Let V be a complex vector space and let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of a connected complex linear algebraic group G. We call (G, ρ, V) a *prehomogeneous vector space* if G has an open orbit in V.

By Proposition 2.1.3, the open orbit is unique, connected, and Zariski open, and the complement of the open orbit is an algebraic set S defined by a rank condition (our "exceptional orbit variety"). This set is not necessarily irreducible, nor are its irreducible components all of the same dimension, although this occurs with equidimensional representations.

2.4.2. Relative invariants. The structure of S is closely related to the group of characters of G.

DEFINITION 2.4.2. Let f be a rational function on V (defined on $V \setminus S$) which is not identically zero. Then f is a *relative invariant* of (G, ρ, V) if there exists a rational character $\chi : G \to \mathbb{C}^*$ so that for all $v \in V \setminus S$ and $g \in G$ we have

$$f(\rho(g)(v)) = \chi(g)f(v),$$

in which case we write $f \leftrightarrow \chi$.

Relative invariants play an important role in understanding the geometry of prehomogeneous vector spaces. For example, it is immediate from the definition that a relative invariant f has the property that the hypersurface X defined by $f = 0$ is invariant under ρ, so that $X \subset S$.

From the following Proposition, we see that the relative invariants associated to a particular character are unique, up to a constant.

PROPOSITION 2.4.3 (e.g., Proposition 2.2 of [**Kim03**]). *Let (G, ρ, V) be a triplet with $\rho : G \to \mathrm{GL}(V)$, for G a connected complex algebraic group. Then the following are equivalent:*

(1) (G, ρ, V) *is a prehomogeneous vector space.*

(2) *Any relative invariant* f *with* $f \leftrightarrow 1$ *is constant.*

(3) *If* $f \leftrightarrow \chi$ *and* $g \leftrightarrow \chi$*, then there exists a constant* c *with* $f = cg$*.*

The "uniqueness" of (3) of the Proposition goes in the other direction as well. Let f be a relative invariant of a prehomogeneous vector space, and fix $v_0 \in V \setminus S$ with $f(v_0) \neq 0$. Then by the definition of relative invariant,

$$\chi(g) = \frac{f(\rho(g)(v_0))}{f(v_0)}$$

must define the unique character with $f \leftrightarrow \chi$.

Relative invariants of prehomogeneous vector spaces must be homogeneous rational functions.

Example 2.4.4. We continue Example 2.1.9, where $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ acts on $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$ by

$$(A, \lambda) \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix}, z \right) = \left(A \begin{pmatrix} x \\ y \end{pmatrix}, \lambda z \right),$$

and the exceptional orbit variety has two components defined by $x = y = 0$ and $z = 0$. Note that $\chi_1(A, \lambda) = \det(A)$ and $\chi_2(A, \lambda) = \lambda$ are multiplicative characters. The function $f_2(x, y, z) = z$ is clearly a relative invariant for χ_2, so that $f_2 \leftrightarrow \chi_2$.

However, χ_1 does not have a corresponding relative invariant. Suppose that $f_1 \leftrightarrow \chi_1$ is a rational function on $\mathbb{C}^2$. f_1 must be independent of z since χ_1 has no dependence on λ. Pick a nonzero vector $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ where f_1 does not have a pole or a zero. Pick a vector $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$ linearly independent from v_1. For all $\mu \in \mathbb{C}^*$, there exists an $A_\mu \in \mathrm{GL}_2(\mathbb{C})$ so that v_1 and v_2 are eigenvectors with eigenvalues of 1 and μ, respectively. Then

$$f_1(v_1) = f_1(A_\mu v_1) = \det(A_\mu) f_1(v_1) = \mu f_1(v_1),$$

yielding a contradiction.

This example demonstrates that relative invariants are related to only the codimension 1 components of the exceptional orbit variety. For equidimensional representations, all components of the exceptional orbit variety have codimension 1.

Example 2.4.5. Consider the prehomogeneous vector space corresponding to the Cholesky factorization (Corollary 2.2.7), where the group $L_n(\mathbb{C})$ of $n \times n$ lower triangular matrices acts on $\mathrm{Sym}_n(\mathbb{C})$ by $A \cdot M = AMA^T$. For any matrix let $M^{(k)}$ denote its upper left $k \times k$ submatrix. For $k = 1, \ldots, n$, let

$$\chi_k(A) = \det(A^{(k)})^2 \qquad \text{and} \qquad f_k(M) = \det(M^{(k)})$$

be, respectively, a character of $L_n(\mathbb{C})$ and a function on $\mathrm{Sym}_n(\mathbb{C})$. Note that

$$f_k(A \cdot M) = f_k(AMA^T) = \det(A^{(k)}) \det(M^{(k)}) \det((A^T)^{(k)}) = \chi_k(A) f_k(M),$$

so that $f_k \leftrightarrow \chi_k$. By Corollary 2.2.7, each hypersurface defined by $f_k = 0$ is a codimension one component of the exceptional orbit variety.

Although each $\chi_k^r = (A \mapsto \det(A^{(k)})^r)$, $r \in \mathbb{N}$, is a multiplicative character of $L_n(\mathbb{C})$, r must be even to be a character of $\rho(L_n(\mathbb{C}))$ as $\ker(\rho) = \{\pm I\}$. Although $\chi_k^{2\ell} = (\chi_k)^\ell \leftrightarrow (f_k)^\ell$, the generator χ_k holds the most interest for us.

2.4.3. Characters. Let $X(G)$ be the group of rational characters on G. If a set of r elements of $X(G)$ generates a free abelian group of rank r, call that set *multiplicatively independent.* It is clear that $X(G)$ is closely related to the structure of the relative invariants: if $f_1 \leftrightarrow \chi_1$ and $f_2 \leftrightarrow \chi_2$, then $f_1 \cdot f_2$ and $\frac{f_1}{f_2}$ are relative invariants with $f_1 \cdot f_2 \leftrightarrow \chi_1 \cdot \chi_2$ and $\frac{f_1}{f_2} \leftrightarrow \frac{\chi_1}{\chi_2}$. The relative invariants, in turn, are related to properties of the complement of the open orbit of ρ.

Theorem 2.4.6 (e.g., Theorem 2.9 of [**Kim03**]). *Let (G, ρ, V) be a prehomogeneous vector space, and let S be the complement of the unique open orbit. Let $S_i = \{x \in V | f_i(x) = 0\}$, $i = 1, \ldots, r$, be the distinct irreducible components of S of codimension one in V. Then the irreducible polynomials $f_1, \ldots, f_r$ are relative invariants and algebraically independent. Moreover, any relative invariant can be uniquely written as*

$$c f_1^{m_1} \cdots f_r^{m_r} \qquad c \in \mathbb{C}^*,\ m_i \in \mathbb{Z}.$$

We will call $f_1, \ldots, f_r$ from the theorem the *basic relative invariants* of (G, ρ, V). Suppose that $f_i \leftrightarrow \chi_i$ for $i = 1, \ldots, r$; then clearly

$$c f_1^{m_1} \cdots f_r^{m_r} \leftrightarrow \chi_1^{m_1} \cdots \chi_r^{m_r}.$$

Let $X_1(G)$ denote the subgroup of $X(G)$ consisting of characters which have corresponding relative invariants. The above observation combined with the uniqueness of the Theorem and of the correspondence between relative invariants and their characters implies that $X_1(G)$ is a free abelian group of rank r generated by $\chi_1, \ldots, \chi_r$.

There is a more surprising interpretation of $X_1(G)$. Let $v \in V$ be in the open orbit, and let G_v be the isotropy subgroup of ρ at v. G_v is independent of v up to conjugation by an element of G. Let G_1 be the subgroup of G generated by G_v and $[G, G]$, the commutator subgroup of G.

Proposition 2.4.7 (e.g., [**Kim03**], Proposition 2.12). *Let (G, ρ, V) be a prehomogeneous vector space, with G_1 defined as above. Then G_1 does not depend on v (the choice of an element of the open orbit), and we have an isomorphism*

$$X_1(G) \simeq X(G/G_1),$$

so that the rank of $X(G/G_1)$ as a free abelian group equals the number of basic relative invariants and the number of irreducible components of S of codimension one in V.

Example 2.4.8. We continue Example 2.4.5 related to the Cholesky factorization. Since $f_1, \ldots, f_n$ are irreducible polynomials and have distinct degrees, they must be basic relative invariants. Although the description of the exceptional orbit variety given by Corollary 2.2.7 indicates that $f_1, \ldots, f_n$ are all of the basic relative invariants, we may also prove this using Theorem 2.4.6.

The isotropy subgroup of $L_n(\mathbb{C})$ at $I \in \mathrm{Sym}_n(\mathbb{C})$ is $G_I = L_n(\mathbb{C}) \cap O_n(\mathbb{C})$, the diagonal matrices whose diagonal entries are ± 1. The commutator of the group of lower triangular matrices is the group of lower triangular matrices with ones on the diagonal. As a result, G_1 is the subgroup of the lower triangular matrices where all diagonal entries are either $+1$ or -1. Then

$$G/G_1 \simeq (\mathbb{C}^*)^n/(\{\pm 1\})^n \simeq (\mathbb{C}^*)^n,$$

so $X(G/G_1)$ has rank n as a free abelian group. Thus $f_1, \ldots, f_n$ is a set of basic relative invariants.

Remark 2.4.9. The terminology used here is similar to other terms commonly used in mathematics. For a prehomogeneous space, the open orbit Ω is a *homogeneous space*, a set on which the group acts transitively. An *invariant*, as in invariant theory, is a function which is constant on the orbits of a group action. Relative invariants weaken the requirement of constancy but add the requirement that the function be rational.

2.4.4. Classification. An obvious first step in an attempt to classify prehomogeneous vector spaces would be limit one's focus to *irreducible prehomogeneous vector spaces* (G, ρ, V), where ρ is irreducible. But there is a general way to obtain a prehomogeneous vector space from another one.

Let $\rho : H \to \mathrm{GL}(V)$ be a finite-dimensional rational representation. Let $\rho^* : H \to \mathrm{GL}(V^*)$ denote the associated contragredient representation, $\rho^*(h)(\phi)(v) = \phi(\rho(h^{-1})(v))$, another rational representation. Let $\Omega_k : \mathrm{GL}_k(\mathbb{C}) \to \mathrm{GL}_k(\mathbb{C})$ be the canonical representation.

THEOREM 2.4.10 (M. Sato, Theorem 7.3 of [**Kim03**]). *Let* $m = \dim(V) > n \geq 1$. *Then the following are equivalent:*

(1) $(H \times \mathrm{GL}_n, \rho \otimes \Omega_n, V \otimes \mathbb{C}^n)$ *is a prehomogeneous vector space.*
(2) $(H \times \mathrm{GL}_{m-n}, \rho^* \otimes \Omega_{m-n}, V^* \otimes \mathbb{C}^{m-n})$ *is a prehomogeneous vector space.*

The idea of the proof of this theorem is that both (1) and (2) induce actions on subsets of the Grassmannian (n-planes in $\mathbb{C}^m$, and $(m-n)$-planes in $\mathbb{C}^{m*}$, respectively) and prehomogeneity is equivalent to these actions having open dense orbits. But the spaces can be identified to show that the conditions are equivalent.

By taking a prehomogeneous vector space (G, ρ, V) and writing G as a direct product involving a general linear group (even GL_1), we can use the Theorem repeatedly to obtain other prehomogeneous vector spaces. Often one can obtain infinitely many examples in this way, although these will almost never be equidimensional.

EXAMPLE 2.4.11. Let $n > 1$, and let $\rho : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ be the prehomogeneous vector space from Example 2.4.5. Define the subgroups $H \subset L_n(\mathbb{C})$, consisting of elements whose upper left entry is 1, and $D \subset L_n(\mathbb{C})$, consisting of multiples of the identity. Then $L_n(\mathbb{C}) \simeq H \times D$, and $L_n(\mathbb{C})/\ker(\rho) \simeq L_n(\mathbb{C})/\{\pm I\} \simeq H \times D/\{\pm 1\}$. By examining ρ and noticing that $D/\{\pm 1\} \simeq \mathrm{GL}_1(\mathbb{C})$, we observe that we have a representation of $H \times \mathrm{GL}_1(\mathbb{C})$ on $\mathrm{Sym}_n(\mathbb{C}) \simeq \mathrm{Sym}_n(\mathbb{C}) \otimes \mathbb{C}^1$ of the type given in Theorem 2.4.10. By the same Theorem, we obtain a prehomogeneous vector space of $H \times \mathrm{GL}_{\binom{n}{2}-1}(\mathbb{C})$ on $\mathrm{Sym}_n(\mathbb{C})^* \otimes \mathbb{C}^{\binom{n}{2}-1}$. However,

$\mathrm{GL}_{\binom{n}{2}-1}(\mathbb{C}) \simeq \mathrm{SL}_{\binom{n}{2}-1}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, and we can use Theorem 2.4.10 to obtain a prehomogeneous vector space of even larger dimension. We can continue this process indefinitely.

Definition 2.4.12. If (1) and (2) in Theorem 2.4.10 are prehomogeneous vector spaces, then we will call them *castling transforms* of each other. If, for two prehomogeneous vector spaces (G, ρ, V) and (G', ρ', V'), one can be reached from the other by a finite number of castling transforms, we will call them *castling equivalent*.

Of all the prehomogeneous vector spaces one can reach through castling, there is a "smallest" one. We will call a prehomogeneous vector space (G, ρ, V) *reduced* if, for any castling transform (G', ρ', V') of (G, ρ, V), $\dim(V) \leq \dim(V')$.

Theorem 2.4.13 (Sato and Kimura [**SK77**]; [**Kim03**], Theorem 7.50). *There exists a list of reduced irreducible prehomogeneous vector spaces such that any irreducible prehomogeneous vector space is castling equivalent to one of them.*

For us, this classification is not particularly useful except as a source of basic examples. We will be mainly interested in reducible equidimensional prehomogeneous vector spaces. In particular, for solvable groups the only irreducible rational representations have dimension 1 by Theorem 2.3.1.

CHAPTER 3

Linear free and free* divisors via representation theory

In this chapter we develop a framework to determine when the exceptional orbit varieties of equidimensional representations are free or free* divisors. Mond first observed that Saito's criterion may be applied to this case. He and Buchweitz [**BM06**] used this criterion to study free divisors arising from irreducible representations of reductive groups corresponding to quivers of finite type. We review this criterion, and then show that all exceptional orbit varieties of equidimensional representations which are not linear free divisors have the structure of a linear free* divisor. Because the representations identified in Chapter 2 are of solvable linear algebraic groups and hence not reducible, we must use an alternate approach.

We introduce the notion of a *block representation*. Such representations have invariant subspaces with certain desirable properties which enable us to inductively analyze our representation by studying a series of related equidimensional representations of smaller dimensions. The exceptional orbit varieties of the smaller representations constitute a portion of the exceptional orbit variety of our original representation, and each of the smaller representations also have the structure of a block representation. We give in Theorem 3.2.14 a sufficient condition for the exceptional orbit variety to be a free divisor, or that a weaker *non-reduced block representation* yields a free* divisor. In our examples, we frequently encounter families of representations which form an infinite *tower* of representations and of free divisors. In Proposition 3.2.18, we show how a block structure for the tower leads to an inductive construction of their free divisors. In Chapter 4 we will use the theory of block representations to study the infinite families of representations involved in the matrix factorizations in Chapter 2 (LU, Cholesky, Cholesky-like factorization for skew-symmetric matrices); these provide examples of inductively defined infinite families of linear free and free* divisors.

Determining whether the exceptional orbit variety of an equidimensional representation is a free or free* divisor requires checking whether a polynomial defining it is square-free. In section

3.3, we prove several statements which provide sufficient conditions for polynomials defining determinantal varieties to be irreducible.

3.1. Linear free and free* divisors from representations

We shall first review Saito's criterion, which ensures that a hypersurface singularity is a free divisor. We use it to explain why certain exceptional orbit varieties are free divisors, and then show that all exceptional orbit varieties of equidimensional representations have at least a free* divisor structure.

3.1.1. Saito's criterion. Let $(V,0) \subset (\mathbb{C}^n, 0)$ be a germ of a hypersurface. In the same paper ([**Sai80**]) in which Saito defines the notion of a free divisor (Definition 1.6.1), he gives two criteria for $(V,0)$ to be a free divisor.

THEOREM 3.1.1 (Saito's criterion, [**Sai80**], Theorems 1.8(ii), 1.9).

(1) *Let $(V,0) \subset (\mathbb{C}^n, 0)$ be defined near 0 by a reduced germ $f : \mathbb{C}^n, 0 \to \mathbb{C}, 0$. Let $\{x_j\}_{j=1}^n$ be local coordinates for $(\mathbb{C}^n, 0)$. Then $(V,0)$ is a free divisor if and only if there exists n elements $\delta^1, \cdots, \delta^n \in \operatorname{Derlog}(V)$ with $\delta^i = \sum_{j=1}^n a_{ij}(z)\frac{\partial}{\partial z_j}$ such that the germ $\det(a_{ij}(z))_{i,j=1,\ldots,n}$ is a unit multiple of f. In this case, $\delta^1, \cdots, \delta^n$ form a free basis for $\operatorname{Derlog}(V)$.*

(2) *Let $\delta^1, \cdots, \delta^n$ be germs of vector fields on $(\mathbb{C}^n, 0)$, with $\delta^i = \sum_{j=1}^n a_{ij}(z)\frac{\partial}{\partial z_j}$, so that $[\delta^i, \delta^j] \in \mathscr{O}_{\mathbb{C}^n,0}\{\delta^1, \cdots, \delta^n\}$ for all i, j. Suppose that $h = \det(a_{ij}(z)) = 0$ defines a reduced hypersurface $(V,0)$. Then $(V,0)$ is a free divisor, each $\delta^i \in \operatorname{Derlog}(V)$, and $\delta^1, \cdots, \delta^n$ form a free basis of $\operatorname{Derlog}(V)$.*

We are primarily interested in applying Saito's criterion to the case where $(V,0)$ is the exceptional orbit variety of an equidimensional representation $\rho : G \to \mathrm{GL}(\mathbb{C}^n)$. Let $A_1, \ldots, A_n$ be a basis of the Lie algebra $\mathfrak{g}$ of G, and let $\xi_{A_1}, \ldots, \xi_{A_n}$ be the associated vector fields on $\mathbb{C}^n$. Consider Theorem 3.1.1 where $\delta^i = \xi_{A_i}$. In this situation, we already know that each $\xi_{A_i} \in \operatorname{Derlog}(V)$ (Lemma 2.1.14), that the module generated by $\xi_{A_1}, \ldots, \xi_{A_n}$ must be closed under the Lie bracket (Proposition 2.1.1), and that $(V,0)$ is the subvariety where $\xi_{A_1}, \ldots, \xi_{A_n}$ are linearly dependent (Proposition 2.1.10). Let $\xi_{A_i} = \sum_{j=1}^n a_{ij}(z)\frac{\partial}{\partial z_j}$ and $h = \det(a_{ij}(z))_{i,j=1,\ldots,n}$. Then by Theorem 3.1.1(1) or (2), $(V,0)$ is a free divisor if h is reduced.

3.1.2. Analogue of Saito's criterion for free* divisors. Since a "free* divisor" is a weaker version of "free divisor," one might expect that there is a criterion for free* divisors analogous to Saito's Criterion. The following is similar to Theorem 3.1.1(1).

PROPOSITION 3.1.2. *Let* $(V,0) \subset (\mathbb{C}^n,0)$ *be a hypersurface germ. If there exists* n *vector fields* $\delta^1,\cdots,\delta^n \in \operatorname{Derlog}(V)$ *with* $\delta^i = \sum_{j=1}^n a_{ij}(z)\frac{\partial}{\partial z_j}$ *such that for* $g = \det(a_{ij}(z)_{i,j=1,\dots,n})$ *we have* $g^{-1}(0) = V$*, then* $\mathscr{L} = \mathscr{O}_{\mathbb{C}^n,0}\{\delta^1,\cdots,\delta^n\}$ *is a free* structure for* V*. In particular,* g *need not be reduced.*

PROOF. $\mathscr{L}$ is clearly a submodule of $\operatorname{Derlog}(V)$. If $\mathscr{L}$ is not free, then there is a relation among the generators $\delta^1,\dots,\delta^n$ of $\mathscr{L}$ of the form

$$\sum_{i=1}^n f_i\delta^i = 0, \tag{3.1}$$

with at least one $f_i \neq 0$. Pick representatives for each f_i and each δ^i on some open neighborhood U of $0 \in \mathbb{C}^n$. Let X be the analytic subset of U where all f_i vanish, whose complement U' is necessarily nonempty and Zariski open in U. Evaluating (3.1) at any $z \in U'$ shows that $\{\delta^1(z),\dots,\delta^n(z)\}$ are linearly dependent, so that $g(z) = 0$ and $z \in V$. Since $g = 0$ on a dense subset of U, g is identically zero, so $V = \mathbb{C}^n$ is not a hypersurface. Thus $\mathscr{L}$ is free.

We must now show that $\theta_n/\mathscr{L}$ is supported on V. Note that $v \in V$ if and only if $g(v) = 0$, if and only if $\delta^1(v),\dots,\delta^n(v)$ are linearly dependent, if and only if $\operatorname{span}\{\delta^1(v),\dots,\delta^n(v)\} \neq T_v\mathbb{C}^n$. If $v \notin V$, there exists an open neighborhood U of v with $U \cap V = \emptyset$. We may write any germ of a vector field ξ at v in terms of $\delta^1,\dots,\delta^n$, which are linearly independent everywhere in U. Thus $v \notin \operatorname{supp}(\theta_n/\mathscr{L})$. If $v \in V$, then $\operatorname{span}\{\delta^1(v),\dots,\delta^n(v)\} \neq T_v\mathbb{C}^n$, so $\langle\mathscr{L}\rangle_v \neq \langle\theta_n\rangle_v = T_v\mathbb{C}^n$. In particular, the stalks $\mathscr{L}_v \neq (\theta_n)_v$ and $v \in \operatorname{supp}(\theta_n/\mathscr{L})$. □

3.1.3. Exceptional orbit varieties as free or free* divisors. We may now use these criteria to show that the exceptional orbit variety of an equidimensional representation is either a free divisor or a free* divisor.

COROLLARY 3.1.3. *Let* $\rho : G \to GL(\mathbb{C}^n)$ *be an equidimensional representation of a connected complex algebraic Lie group, with* V *the complement of the open orbit. Let* $A_1,\dots,A_n$ *be a basis for* $\mathfrak{g}$*, let* $\xi_{A_1},\dots,\xi_{A_n}$ *be the associated vector fields (as defined by* (2.1)*), and let* $f = \det(\xi_{A_1},\dots,\xi_{A_n})$ *(as in Remark 2.1.11). Then* $V = f^{-1}(0)$ *has a free* divisor structure* $\mathscr{L} =$

$\mathscr{O}_{\mathbb{C}^n,0}\{\xi_{A_1},\ldots,\xi_{A_n}\}$ *which is closed under the Lie bracket. If f is square-free, then $(V,0)$ is a free divisor.*

Proof. First, $V = f^{-1}(0)$ by Proposition 2.1.10. Let $\mathscr{L} = \mathscr{O}_{\mathbb{C}^n,0}\{\xi_{A_1},\ldots,\xi_{A_n}\}$. These vector fields are closed under the Lie bracket since $A \mapsto \xi_A$ is a Lie algebra homomorphism by Proposition 2.1.1. Each $\xi_{A_i} \in \operatorname{Derlog}(V)$ by Lemma 2.1.14, so $\mathscr{L} \subseteq \operatorname{Derlog}(V)$ is a submodule, closed under the Lie bracket.

Applying Proposition 3.1.2 shows that V has a free* structure given by $\mathscr{L}$. If f is square-free, then by (2) of Saito's Criterion (Theorem 3.1.1), V is a free divisor and $\mathscr{L} = \operatorname{Derlog}(V)$. $\square$

Example 3.1.4. Let $\rho : \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(M(1,2,\mathbb{C}))$ be defined by $\rho(A,B)(M) = AMB^{-1}$. Use coordinates $\begin{pmatrix} x & y \end{pmatrix}$ for $M(1,2,\mathbb{C})$. The vector fields arising from this representation include

$$\begin{aligned} \xi_{(E_{11},0)} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ \xi_{(0,E_{12})} &= -x\frac{\partial}{\partial y} \\ \text{and} \qquad \xi_{(0,E_{22})} &= -y\frac{\partial}{\partial y}. \end{aligned}$$

Restrict ρ to the 2-dimensional subgroup G whose Lie algebra is spanned by $(E_{11},0)$ and $(0,E_{12})$, similar to the group involved in the LU decomposition (Corollary 2.2.5). Then $f = \det(\xi_{(E_{11},0)}, \xi_{(0,E_{12})}) = -x^2$. As a result, these two vector fields only define a free* divisor structure for the linear subspace $(V,0)$ defined by $x = 0$. While $(V,0)$ is a free divisor, this representation only defines it as a free* divisor. Note that $(V,0)$ is not a linear free divisor, as $\frac{\partial}{\partial y} \in \operatorname{Derlog}(V)$ is not a linear vector field.

Example 3.1.5. If ρ is as in the previous example and G is instead the subgroup of $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ whose Lie algebra is spanned by $(E_{11},0)$ and $(0,E_{22})$, then $f = \det(\xi_{(E_{11},0)}, \xi_{(0,E_{22})}) = -xy$, which defines a free divisor. Restricting a particular representation to different subgroups can give quite different exceptional orbit varieties.

A main challenge in applying this Corollary is to factor f and determine whether it is square-free. Although in principle it is possible to find f by direct computation, in practice it becomes difficult to compute and factor f even when using a computer on relatively small

$\dim(G)$; we will also be interested in infinite families of representations whose dimensions become arbitrarily large. Moreover, this brute-force approach provides no insight the role the Lie algebra structure is playing.

EXAMPLE 3.1.6. Let $\mathrm{D}_n(\mathbb{C})$ denote the group of diagonal $n\times n$ invertible matrices. Consider the representation $\rho : G \to \mathrm{GL}(M(n, n+1, \mathbb{C}))$ of the reductive group $G = \mathrm{GL}_n(\mathbb{C}) \times (\mathrm{D}_{n+1}(\mathbb{C}) \cap \mathrm{SL}_{n+1}(\mathbb{C}))$ on the space of $n \times (n+1)$ complex matrices defined by

$$\rho(A, B)(M) = AMB^{-1}.$$

Using quiver representations, this was shown to be an equidimensional representation whose exceptional orbit variety is a linear free divisor defined by the product of the $n \times n$ minors of the generic $n \times (n+1)$ matrix (Proposition 7.4 of [**BM06**] or Example 5.3 of [**GMNRS09**]).

If we try to show this directly when $n = 2$ by applying Corollary 3.1.3, we find that

$$f = \det \begin{pmatrix} x_{11} & x_{21} & 0 & 0 & -x_{11} & 0 \\ x_{12} & x_{22} & 0 & 0 & 0 & -x_{12} \\ x_{13} & x_{23} & 0 & 0 & x_{13} & x_{13} \\ 0 & 0 & x_{11} & x_{21} & -x_{21} & 0 \\ 0 & 0 & x_{12} & x_{22} & 0 & -x_{22} \\ 0 & 0 & x_{13} & x_{23} & x_{23} & x_{23} \end{pmatrix}$$

defines the exceptional orbit variety of ρ. Although f factors as

$$f = -3(x_{11}x_{22} - x_{12}x_{21})(x_{11}x_{23} - x_{13}x_{21})(x_{12}x_{23} - x_{13}x_{22}),$$

this is not obvious. Direct computation in the $n = 4$ case, which involves taking the determinant of a sparse 20×20 matrix of polynomials, is not feasible even for a computer.

3.2. Block representations

In this section we will describe a new approach to identifying the exceptional orbit variety $\mathscr{E}$ of an equidimensional representation $\rho : G \to \mathrm{GL}(V)$. We shall identify certain special invariant subspaces which allow us to decompose ρ into pieces via quotient representations. Each piece will provide partial information about $\mathscr{E}$. Such a ρ with these invariant subspaces will be called a *block representation*.

3.2.1. Characteristics of the Cholesky-type factorizations. All of the matrix factorizations of Chapter 2 (LU, Cholesky, Cholesky-type factorization for skew-symmetric matrices) share a number of interesting properties when viewed as statements about the open orbit of an equidimensional representation. As these properties motivate our definition of a block representation, we begin by identifying them in the case of the Cholesky factorization.

The Cholesky factorization for symmetric matrices is a statement about a series of related representations. For each $k \in \mathbb{N}$, let $\psi_k : L_k(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_k(\mathbb{C}))$ be the representation of the lower triangular matrices on the space of $k \times k$ symmetric matrices given by

$$\psi_k(A)(M) = AMA^T.$$

Fix $n \in \mathbb{N}$, and consider ψ_n. By Theorem 2.2.6 the exceptional orbit variety of ψ_n includes all of the components of the exceptional orbit varieties of the ψ_k for $1 \leq k < n$. By understanding the relation between ψ_k and ψ_n, we may focus on the differences. Fix $1 \leq k < n$.

In fact, we may recover ψ_k as a quotient representation of ψ_n. Let Z be the vector subspace of $\mathrm{Sym}_n(\mathbb{C})$ consisting of the symmetric matrices whose upper left $k \times k$ submatrix is 0. Then Z is a ψ_n-invariant subspace by the following calculation:

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} 0_k & M_{12} \\ (M_{12})^T & M_{22} \end{pmatrix} \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}^T = \begin{pmatrix} 0_k & * \\ * & * \end{pmatrix}.$$

Consider the quotient representation $\overline{\psi}_n : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})/Z)$ induced from ψ_n. A calculation shows that K, the connected component of the identity of $\ker(\overline{\psi}_n)$, is

$$K = \left\{ \begin{pmatrix} \pm I_k & 0 \\ * & * \end{pmatrix} \in L_n(\mathbb{C}) \right\}.$$

Then $\mathrm{Sym}_n(\mathbb{C})/Z \simeq \mathrm{Sym}_k(\mathbb{C})$, $L_n(\mathbb{C})/K \simeq L_k(\mathbb{C})$, and may be checked that the representation $L_n(\mathbb{C})/K \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})/Z)$ induced from $\overline{\psi}_n$ is isomorphic to ψ_k.

We shall see that $L_n(\mathbb{C})/K \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})/Z)$ is a "simplification" of ψ_n whose exceptional orbit variety will form part of the exceptional orbit variety of ψ_n, while the rest of the exceptional orbit variety of ψ_n can be determined from $\psi_n|_K$.

Note that for both ψ_n and $L_n(\mathbb{C})/K \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})/Z)$ to be equidimensional representations requires $\dim_{\mathbb{C}}(K) = \dim_{\mathbb{C}}(Z)$. This is a strong restriction on acceptable invariant subspaces.

REMARK 3.2.1. Note that if we define the inclusions $i : L_k(\mathbb{C}) \hookrightarrow L_n(\mathbb{C})$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$, and $j : \mathrm{Sym}_k(\mathbb{C}) \hookrightarrow \mathrm{Sym}_n(\mathbb{C})$, $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, then $j(\psi_k(A)(M)) = \psi_n(i(a))(j(M))$ for all $A \in L_n(\mathbb{C})$, $M \in \mathrm{Sym}_n(\mathbb{C})$. Moreover, composing with the quotients gives an isomorphism of representations

$$(L_k(\mathbb{C}), \mathrm{Sym}_k(\mathbb{C})) \hookrightarrow (L_n(\mathbb{C}), \mathrm{Sym}_n(\mathbb{C})) \twoheadrightarrow (L_n(\mathbb{C})/K, \mathrm{Sym}_n(\mathbb{C})/Z).$$

As this behavior is a property of a collection of representations, we shall not require this of a block representation.

3.2.2. Coefficient determinants. Let $\psi : H \to \mathrm{GL}(V)$ be a rational representation of a connected complex algebraic Lie group. We will give a condition for there to exist a function on V which defines the set of v where $d(\psi^v)_{(e)}$ fails to have maximal rank.

PROPOSITION 3.2.2. *Let $W \subset V$ be a ψ-invariant subspace of dimension $\dim_{\mathbb{C}}(H)$ so that the induced representation $H \to \mathrm{GL}(V/W)$ is trivial. Then there exists a polynomial f on V, well-defined up to multiplication by a nonzero constant, so that $f(v) \neq 0$ if and only if $d(\psi^v)_{(e)}$ has maximal rank. If $f \not\equiv 0$, then f is homogeneous of degree $\dim_{\mathbb{C}}(H)$.*

We first give a geometric meaning to the kernel of a quotient representation.

LEMMA 3.2.3. *If $W \subset V$ is an invariant subspace for $\psi : H \to \mathrm{GL}(V)$ and $h \in \ker(H \to \mathrm{GL}(V/W))$, then h leaves all affine subspaces of the form $v + W \subset V$ invariant.*

PROOF. The quotient representation $\overline{\psi} : H \to \mathrm{GL}(V/W)$ is defined by $\overline{\psi}(k)(v + W) = \psi(k)(v) + W$. Since $h \in \ker(\overline{\psi})$, we must have $v + W = \psi(h)(v) + W \in V/W$, or $\psi(h)(v) \in v + W$. □

PROOF OF PROPOSITION 3.2.2. Let $n = \dim_{\mathbb{C}}(H)$, let $\mathfrak{h}$ be the Lie algebra of H, and let $A_1, \ldots, A_n \in \mathfrak{h}$ be a basis of $\mathfrak{h}$. Then $\xi_{A_1}, \ldots, \xi_{A_n}$ are linear vector fields on V. For all $v \in V$, the derivative of $\psi^v : H \to V$ has a matrix representation in terms of these vector fields when we use the appropriate bases:

$$d(\psi^v)_{(e)} = \begin{pmatrix} \xi_{A_1}(v) & \cdots & \xi_{A_n}(v) \end{pmatrix}.$$

Choose a basis $e_1, \ldots, e_n$ of W and extend to a basis $f_1, \ldots, f_k, e_1, \ldots, e_n$ of V. By Lemma 3.2.3, all ξ_{A_i} are tangent to all $v + W$. Thus for each i we may write $\xi_{A_i} = \sum_{j=1}^{n} \alpha_{ji} e_j$ where each α_{ji} is zero or homogeneous of degree 1 on V. (Note the absence of any f_i components.) Let $A = (\alpha_{ji})$, an $n \times n$ matrix of functions on V, and let $f = \det(A)$. Let $A(v)$ denote A evaluated at v. Then with respect to the bases $f_1, \ldots, f_k, e_1, \ldots, e_n$ for V and $A_1, \ldots, A_n$ for $\mathfrak{h}$, we have for all $v \in V$,

$$d(\psi^v)_{(e)} = \begin{pmatrix} 0 \\ A(v) \end{pmatrix}. \tag{3.2}$$

Then $f(v) \neq 0$ if and only if $A(v)$ is invertible, if and only if $d(\psi^v)_{(e)}$ has maximal rank. If $f \not\equiv 0$, then f is homogeneous of degree n, the dimension of A.

Fix the above bases, and consider the $\dim(V) \times \dim(H)$ matrix $d(\psi^{(\cdot)})_{(e)}$ whose entries are functions on V. Let I be the ideal generated by maximal minors of $d(\psi^{(\cdot)})_{(e)}$ in the ring of polynomials on V. By the above argument and the form of (3.2), the only nonzero maximal minor is $\det(A) = f$, so that f generates I. Finally, observe that different choices of bases will give another generator of I. Since I is principal, the generator is unique up to multiplication by a unit. $\square$

Definition 3.2.4. We will call the function f constructed in Proposition 3.2.2 the *relative coefficient determinant* of ψ for the invariant subspace W. If $\dim(H) = \dim(V)$, we shall call f the *coefficient determinant* of ψ.

A relative coefficient determinant may not be irreducible or square-free.

Remark 3.2.5. The coefficient determinant of an equidimensional representation $\rho : G \to \mathrm{GL}(V)$ defines the exceptional orbit variety of ρ. For example, the process given in Proposition 2.1.10 gives a coefficient determinant.

3.2.3. Block representations. We will now introduce the notion of a block representation. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of a connected complex linear algebraic group with $\dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(V)$. Assume that ρ has a sequence of invariant subspaces

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_l = V. \tag{3.3}$$

For each $j = 0, \ldots, l$, let $\rho_j : G \to \mathrm{GL}(V/W_j)$ be the representation induced from ρ, and let K_j be the connected component of the identity of $\ker(\rho_j)$. Since K_j is a normal subgroup of G, we have a sequence of normal subgroups

$$K_0 \subset K_1 \subset \cdots \subset K_l = G.$$

For $j = 0, \ldots, l$, let $\overline{\rho}_j : G/K_j \to \mathrm{GL}(V/W_j)$ be the representation obtained from ρ_j. For $j = 0, \ldots, l-1$, restrict $\overline{\rho}_j$ to obtain $\tau_j : K_{j+1}/K_j \to \mathrm{GL}(V/W_j)$ with invariant subspace W_{j+1}/W_j. By assuming that $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$ for each j, τ_j has a relative coefficient determinant $f_j : V/W_j \to \mathbb{C}$. By composing with the quotient $V \to V/W_j$, we may consider f_j to be a polynomial on V.

Definition 3.2.6. We say that the rational representation ρ is a *candidate block representation (with invariant subspaces $W_0, \ldots, W_l$)* if

(1) There is a sequence of invariant subspaces as in (3.3) with corresponding normal subgroups K_j so that
(2) for each $j = 0, \ldots, l$, $\dim(K_j) = \dim(W_j)$, and
(3) for each $j = 0, \ldots, l-1$ there exists an orbit of τ_j in V/W_j whose generic projection onto W_{j+1}/W_j is Zariski open.

If also

(4) each f_j is reduced and $\{f_j\}$ are relatively prime,

then we will call ρ a *block representation*. If (4) does not hold for a candidate block decomposition ρ, we will call ρ a *non-reduced block representation*.

Note that our assumption that $\dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(V)$ follows from (2).

Remark 3.2.7. If $\rho : G \to \mathrm{GL}(V)$ is an equidimensional representation with an open orbit, then ρ always has a trivial candidate block representation structure with invariant subspaces $\{0\} \subset V$: $\rho_0 = \rho$, $\rho_1 = 0$, $K_0 = \ker(\rho)$, $K_1 = G$, and the open orbit of ρ in $V = V/W_0 = W_1/W_0$ is the same as the open orbit of τ_0. However, this trivial candidate block representation structure provides no information about the exceptional orbit variety of ρ. By contrast, we seek to find block representations in which the relative coefficient determinants are irreducible.

3.2.4. The matrix of a block representation. The exceptional orbit variety of a candidate block representation $\rho : G \to \mathrm{GL}(V)$ is defined by the determinant of a matrix of polynomials (a "coefficient matrix") as in Proposition 2.1.10. In this section, we will show that we may use the block representation structure to make this matrix block lower triangular; the determinant of the diagonal blocks give the relative coefficient determinants.

For $j = 1, \ldots, l$, let $n_j = \dim_{\mathbb{C}}(W_j/W_{j-1}) = \dim_{\mathbb{C}}(K_j/K_{j-1})$. Choose a basis $e_1^{(1)}, \ldots, e_{n_1}^{(1)}$ of W_1. Then for each j, extend the basis of W_{j-1} to a basis of W_j by choosing additional vectors $e_1^{(j)}, \ldots, e_{n_j}^{(j)}$, so that $\left\{ e_i^{(k)} | 1 \leq k \leq j, 1 \leq i \leq n_k \right\}$ is a basis for W_j. Similarly choose a basis for $\mathfrak{k}_1, \ldots, \mathfrak{k}_l$ so that $\left\{ X_i^{(k)} | 1 \leq k \leq j, 1 \leq i \leq n_k \right\}$ is a basis for $\mathfrak{k}_j$, with each $X_i^{(j)} \in \mathfrak{k}_j$.

We may write each of the vector fields $\xi_{X_i^{(j)}}$ on V in terms of the basis of V and use the coefficients to form a matrix A where the rows correspond to the basis for V and the columns to the various vector fields; this matrix will have $\sum_{j=1}^{l} n_j$ rows and $\sum_{j=1}^{l} n_j$ columns, so that we may subdivide A into blocks:

$$\begin{array}{rc} & \begin{matrix} \overbrace{\hphantom{A_{1,1}}}^{\mathfrak{k}_l/\mathfrak{k}_{l-1}} & \hphantom{\cdots} & \overbrace{\hphantom{A_{1,l}}}^{\mathfrak{k}_1/\mathfrak{k}_0} \end{matrix} \\ \begin{matrix} W_l/W_{l-1}\{ \\ \\ W_1/W_0\ \ \{ \end{matrix} & \begin{pmatrix} A_{1,1} & \cdots & A_{1,l} \\ \vdots & \ddots & \vdots \\ A_{l,1} & \cdots & A_{l,l} \end{pmatrix} \end{array} = A, \tag{3.4}$$

where $A_{i,j}$ is a $n_{l-i+1} \times n_{l-j+1}$ matrix. The entry $(A_{i,j})_{p,q}$ is the coefficient of $e_p^{(l-i+1)}$ in $\xi_{X_q^{(l-j+1)}}$. We will call A the *coefficient matrix of the candidate block representation* ρ.

Lemma 3.2.8. *If $1 \leq i < j \leq l$, then $A_{i,j} = 0$, making A a block lower triangular matrix.*

Proof. Consider an arbitrary column corresponding to $X_q^{(l-j+1)}$. Since $X_q^{(l-j+1)} \in \mathfrak{k}_{l-j+1}$, it follows from Lemma 3.2.3 that $\xi_{X_q^{(l-j+1)}}$ is tangent to each affine subset $v + W_{l-j+1}$ of V. Thus $\xi_{X_q^{(l-j+1)}}$ can be written only in terms of our basis for W_{l-j+1}, i.e.,

$$\xi_{X_q^{(l-j+1)}} = \sum_{k=1}^{l-j+1} \sum_{i=1}^{n_k} g_{i,j,k,q} e_i^{(k)}.$$

In particular, the coefficient of any $e_p^{(l-i+1)}$ when $l-i+1 > l-j+1$ is zero. Thus $(A_{i,j})_{p,q} = 0$ whenever $i < j$, i.e., the qth column of all blocks above $A_{j,j}$ are zero. $\square$

Since all of the diagonal blocks of A are square,

$$\det(A) = \prod_{i=1}^{l} \det(A_{i,i}) \tag{3.5}$$

is a partial factorization of the function defining the exceptional orbit variety of ρ.

EXAMPLE 3.2.9. Consider the equidimensional representation corresponding to the Cholesky factorization for 3×3 symmetric matrices. With respect to appropriate bases, the coefficient matrix is

$$A = \begin{pmatrix} 2x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{11} & x_{12} & 0 & 0 & 0 \\ 0 & 2x_{12} & 2x_{22} & 0 & 0 & 0 \\ x_{13} & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & x_{13} & x_{23} & x_{12} & x_{22} & x_{23} \\ 0 & 0 & 0 & 2x_{13} & 2x_{23} & 2x_{33} \end{pmatrix}.$$

A has 3 square diagonal blocks. The product of the determinants of the diagonal blocks defines the same exceptional orbit variety as in Corollary 2.2.7.

We are now able to relate the coefficient determinants to the diagonal blocks of A.

PROPOSITION 3.2.10. *Up to multiplication by a unit, the coefficient determinant f_j of $\tau_j : K_{j+1}/K_j \to \mathrm{GL}(V/W_j)$ equals $\det(A_{l-j,l-j})$, where $A_{l-j,l-j}$ is given in* (3.4).

Before proving this statement, we first prove a Lemma which compares the vector fields on V arising from ρ and on V/W_j arising from $\overline{\rho}_j$. Let $\beta : V \to V/W_j$ and $\alpha : G \to G/K_j$ be the quotient maps.

LEMMA 3.2.11. *For $X \in \mathfrak{g}$,*

$$d\beta_{(v)}(\xi_X(v)) = \xi_{d\alpha_{(e)}(X)}(\beta(v)).$$

PROOF. Note that $\beta(\rho^v(g)) = \rho_j^{\beta(v)}(g) = \overline{\rho}_j^{\beta(v)}(\alpha(g))$ for all $g \in G$, $v \in V$. By differentiating the case where $g = \exp(t \cdot X)$, we obtain the result. $\square$

Notation 3.2.12. In what follows, we will denote the point $\beta(v)$ by $\overline{v}$, the subspace $\beta(W_{j+1})$ by $\overline{W}_{j+1}$, and the affine subset $\beta(v+W_{j+1})$ by $\overline{v}+\overline{W}_{j+1}$.

Proof of 3.2.10. We shall follow the proof of Proposition 3.2.2 which defines f_j. Recall that τ_j leaves $\overline{W}_{j+1}$ invariant. In fact, $\beta(e_1^{(j+1)}),\ldots,\beta(e_{n_{j+1}}^{(j+1)})$ is a basis of $\overline{W}_{j+1}$ and $\left\{\beta(e_i^{(k)})|j<k\leq l, 1\leq i\leq n_k\right\}$ is a basis of V/W_j. Similarly, $d\alpha_{(e)}(X_1^{(j+1)}),\ldots,d\alpha_{(e)}(X_{n_{j+1}}^{(j+1)})$ is a basis of $\mathfrak{k}_{j+1}/\mathfrak{k}_j$. Since each of the vector fields are tangent to all $\overline{v}+\overline{W}_{j+1}$, we may write

$$\xi_{X_i^{(j+1)}}=\sum_{p=1}^{j+1}\sum_{q=1}^{n_p}g_{i,p,q}e_q^{(p)}.$$

But by Lemma 3.2.11,

$$\begin{aligned}\xi_{d\alpha_{(e)}(X_i^{(j+1)})}(\beta(v))&=d\beta_{(v)}(\xi_{X_i^{(j+1)}}(v))\\&=\sum_{p=1}^{j+1}\sum_{q=1}^{n_p}g_{i,p,q}(v)d\beta_{(v)}(e_q^{(p)})\\&=\sum_{q=1}^{n_{j+1}}g_{i,j+1,q}(v)d\beta_{(v)}(e_q^{(j+1)}).\end{aligned}$$

It follows that each $g_{i,j+1,q}$ factors through β, and is zero or a homogeneous polynomial of degree 1 on V/W_j. Then in the notation of the proof of Proposition 3.2.2, $\alpha_{k,i}=g_{i,j+1,k}$. In comparison, $(A_{l-j,l-j})_{k,i}$ is the coefficient of $e_k^{(j+1)}$ in $\xi_{X_i^{(j+1)}}$, $g_{i,j+1,k}$. Thus $A_{l-j,l-j}=(\alpha_{k,i})$, and $f_j=\det(A_{l-j,l-j})$. □

As a corollary of the proof, we obtain

Corollary 3.2.13. *The entries of $A_{l-j,l-j}$ factor through the quotient $V\to V/W_j$ to be polynomials on V/W_j.*

3.2.5. The exceptional orbit variety of block representations. For a candidate block representation, the coefficient determinants define the exceptional orbit variety. We thus have a condition for when a block representation gives a free divisor.

Theorem 3.2.14. *Let ρ be a candidate block representation with coefficient determinants $f_0,\ldots,f_{l-1}$. If ρ is a block representation, then its exceptional orbit variety is a linear free*

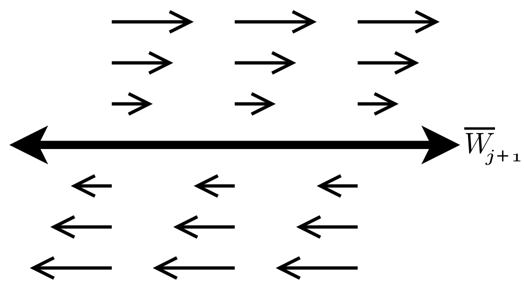

FIGURE 3.1. A diagram of vector fields on V/W_j coming from τ_j. There may not exist a $\overline{v} \in \overline{W}_{j+1}$ whose orbit has the correct dimension. However, the generic projection onto $\overline{W}_{j+1}$ of the orbit of a generic point may be Zariski open.

divisor defined by

$$\prod_{i=0}^{l-1} f_i = 0. \tag{3.6}$$

If ρ is a non-reduced block representation, then its exceptional orbit variety is a linear free divisor defined with non-reduced structure by* (3.6)*.*

PROOF. (3.6) defines the exceptional orbit variety of ρ by Proposition 2.1.10, (3.5), and Proposition 3.2.10. Then apply Corollary 3.1.3. □

For our examples we will carefully choose bases so that the diagonal blocks of the coefficient matrix take a nice form, making the coefficient determinants and the exceptional orbit variety straightforward to find.

3.2.6. Equivalent formulations. We now explore several alternate formulations of condition (3) of Definition 3.2.6. In particular, we will eventually show that it is equivalent to the condition that ρ has an open orbit.

We first give equivalent conditions for property (3) for a particular $0 \le j \le l-1$. Figure 3.1 illustrates the situation.

PROPOSITION 3.2.15. *Assume that properties (1) and (2) of Definition 3.2.6 hold. Let* $0 \le j \le l-1$*. Then the following are equivalent:*

(a) *Property (3) of Definition 3.2.6 holds for τ_j.*

(b) *There exists a point $\overline{v} \in V/W_j$ where $d(\tau_j^{\overline{v}})_{(e)} : \mathfrak{k}_{j+1}/\mathfrak{k}_j \to T_{\overline{v}}(V/W_j)$ has maximal rank.*

(c) $f_j \not\equiv 0$.

(d) *τ_j has an orbit in V/W_j of dimension* $\dim_{\mathbb{C}}(K_{j+1}) - \dim_{\mathbb{C}}(K_j)$*.*

Remark 3.2.16. To prove this proposition, we will use the fact that the statements given in Chapter 2 for rational representations of connected complex algebraic Lie groups (Proposition 2.1.1, Proposition 2.1.3, Corollary 2.1.7, Proposition 2.1.10, and Lemma 2.1.14) in fact hold more generally for any connected complex algebraic Lie group acting rationally on an irreducible complex space. In particular, the statements hold for τ_j acting on an affine subset of the form $\overline{v} + \overline{W}_{j+1} \subset V/W_j$, where $\overline{W}_{j+1} = W_{j+1}/W_j$.

Proof of Proposition 3.2.15. (a) $\Longrightarrow$ (b): Suppose that the orbit O of $\overline{v} \in V/W_j$ under τ_j has the property that its generic projection onto $\overline{W}_{j+1}$ is Zariski open. By Lemma 3.2.3, $O \subset \overline{v} + \overline{W}_{j+1} \subset V/W_j$ and the image of $d(\tau_j^{\overline{v}})_{(e)}$ lies in $T_{\overline{v}}(\overline{v} + \overline{W}_{j+1})$. Choose a projection $\pi : V/W_j \to \overline{W}_{j+1}$ such that $\pi(O)$ is Zariski open and $\ker(\pi)$ is transverse to $\overline{W}_{j+1} \subset V/W_j$. For such a projection, $(\pi|_{\overline{v}+\overline{W}_{j+1}}) : \overline{v} + \overline{W}_{j+1} \to \overline{W}_{j+1}$ is a biholomorphism. Since $\pi(O) = \pi|_{\overline{v}+\overline{W}_{j+1}}(O)$ is Zariski open, O is open in $\overline{v} + \overline{W}_{j+1}$. Applying Corollary 2.1.7 and Remark 3.2.16 shows that the image of $d(\tau_j^{\overline{v}})_{(e)}$ equals $T_{\overline{v}}(\overline{v} + \overline{W}_{j+1})$, i.e., $d(\tau_j^{\overline{v}})_{(e)}$ has maximal rank.

(b) $\Longrightarrow$ (a): Conversely, if $d(\tau_j^{\overline{v}})_{(e)}$ has maximal rank then it follows from Lemma 3.2.3 that the image of $d(\tau_j^{\overline{v}})_{(e)}$ must equal $T_{\overline{v}}(\overline{v} + \overline{W}_{j+1})$. Then by Corollary 2.1.7 and Remark 3.2.16, the orbit O of $\overline{v}$ in $\overline{v} + \overline{W}_{j+1} \subset V/W_j$ under τ_j must be Zariski open in $\overline{v} + \overline{W}_{j+1}$. Picking any projection $\pi : V/W_j \to \overline{W}_{j+1}$ whose kernel is transverse to $\overline{W}_{j+1} \subset V/W_j$ gives a biholomorphism from $\overline{v} + \overline{W}_{j+1} \subset V/W_j$ to $\overline{W}_{j+1}$. Thus $\pi(O) = (\pi|_{\overline{v}+\overline{W}_{j+1}})(O)$ is Zariski open in $\overline{W}_{j+1} \subset V/W_j$.

(b) $\Longleftrightarrow$ (c): These are equivalent by the definition of coefficient determinant: $f_j(v) \neq 0$ if and only if $d(\tau_j^{\overline{v}})_{(e)}$ has maximal rank.

(b) $\Longleftrightarrow$ (d): By Proposition 2.1.3 and Remark 3.2.16, the dimension of the orbit of $\overline{v}$ under τ_j is described by the rank of $d(\tau_j^{\overline{v}})_{(e)}$. $\square$

We are now able to provide several alternate characterizations of a candidate block representation.

Proposition 3.2.17. *Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of a connected complex algebraic Lie group. Assume that (1) and (2) of Definition 3.2.6 hold with invariant subspaces*

$$\{0\} = W_0 \subset \cdots \subset W_l = V.$$

Then the following are equivalent:

(1) *ρ is a candidate block representation, i.e., (3) of Definition 3.2.6 holds.*

(2) *ρ has an open orbit.*

(3) $\prod_{j=0}^{l-1} f_j \not\equiv 0$.

PROOF. 1 $\iff$ 3 by Proposition 3.2.15, parts (a) and (c).

2 $\iff$ 3 follows from Proposition 2.1.10 and (3.5). $\square$

3.2.7. Quotients. We can always form a quotient of a block representation to obtain another one.

PROPOSITION 3.2.18. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a candidate block representation with associated invariant subspaces $\{0\} = W_0 \subset \cdots \subset W_l = V$, and fix $0 < j < l$. Then $\overline{\rho}_j : G/K_j \to \mathrm{GL}(V/W_j)$ is a candidate block representation with invariant subspaces $\{0\} \simeq \overline{W}_j \subset \cdots \subset \overline{W}_l = V/W_j$. With respect to the correct bases, $\overline{\rho}_j$'s coefficient matrix is*

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,l-j} \\ \vdots & \ddots & \vdots \\ A_{l-j,1} & \cdots & A_{l-j,l-j} \end{pmatrix},$$

where A is the coefficient matrix of ρ, subdivided as in (3.4).

PROOF. By assumption, the subspaces $\overline{W}_i$ for $j \le i \le l$ are $\overline{\rho}_j$ invariant. Let $j \le i \le l$. The kernel of $(\overline{\rho}_j)_{i-j} : G/K_j \to \mathrm{GL}((V/W_j)/\overline{W}_i) \simeq \mathrm{GL}(V/W_i)$ is the quotient of the kernel of ρ_i by K_j; thus $\dim(\ker((\overline{\rho}_j)_{i-j})) = \dim(K_i/K_j) = \dim(\overline{W}_i)$. Thus (1) and (2) of Definition 3.2.6 hold.

Now consider the blocks. If $\alpha : G \to G/K_j$ and $\beta : V \to V/W_j$ are the quotient maps, then use

$$\overline{e}_i^{(k)} = \beta(e_i^{(k+j)}) \qquad \text{and} \qquad \overline{X}_i^{(k)} = d\alpha_{(e)}(X_i^{(k+j)})$$

as bases for the subspaces and Lie algebras, respectively. Then Lemma 3.2.11 identifies the relation between vector fields on V arising from ρ and on V/W arising from $\overline{\rho}_j$. As in the proof of Proposition 3.2.10, because we chose bases using the invariant subspaces, $d\beta_{(v)}(\xi_X(v))$ will merely delete some terms from $\xi_X(v)$; the remaining coefficients are already functions on V/W_j by the same argument used in the proof of Proposition 3.2.10 to prove Corollary 3.2.13. $\square$

3.2.8. Towers of block representations. The structure of a block representation does not address the features of the matrix factorizations in Chapter 2 which involve inclusions of representations which may be recovered by quotients (see Remark 3.2.1). We will now define a few terms which capture this behavior.

Definition 3.2.19. A *tower of Lie groups* is a collection of Lie groups with inclusions:

$$\{e\} = G_0 \subset G_1 \subset \cdots .$$

Such a tower has a corresponding *tower of representations* if there is a collection of vector spaces with inclusion maps so that

$$\{0\} = V_0 \subset V_1 \subset \cdots ,$$

where we require that each V_j is a representation of G_j and that $(G_{j-1}, V_{j-1}) \hookrightarrow (G_j, V_j)$ is a homomorphism of representations (i.e., homomorphisms of groups and vector spaces which commute with the group operations).

Now consider when each representation in a tower is a block representation. We shall require the block representations to be compatible.

Definition 3.2.20. A tower of representations of Lie groups is a *tower of (non-reduced) block representations* if, for all $j \geq 0$,

(1) Each V_j is a (non-reduced) block representation of G_j via the subspaces and subgroups

$$\{0\} = W_0^j \subset W_1^j \subset \cdots \subset W_{k-1}^j \subset W_k^j = V_j$$
$$\{e\} = K_0^j \subset K_1^j \subset \cdots \subset K_{k-1}^j \subset K_k^j = G_j,$$

with K_i^j the connected component of the identity of $\ker(G_j \to \mathrm{GL}(V_j/W_i^j))$. Furthermore,

(2) For each j, the composition

$$(G_{j-1}, V_{j-1}) \hookrightarrow (G_j, V_j) \twoheadrightarrow (G_j/K_1^j, V_j/W_1^j)$$

is an isomorphism of representations.

Conceptually, a tower of block representations is a sequence of (non-reduced) block representations where at each step we expand the group and the vector space, and where we can recover previous representations by applying Proposition 3.2.18. For each j, let $\mathscr{E}_j$ be the exceptional orbit variety of (G_j, V_j). In a tower of block representations, the free divisor $\mathscr{E}_{j-1}$ is included as a component of the free divisor $\mathscr{E}_j$: if $\pi : V_j \to V_j/W_1^j \to V_{j-1}$ is the quotient map composed with the inverse of the isomorphism of (2) above, then by Proposition 3.2.18, the free divisor $\pi^{-1}(\mathscr{E}_{j-1})$ constitutes a portion of $\mathscr{E}_j$. The relationship between these free divisors gives useful relations between their singular Milnor numbers.

All of the families of (non-reduced) block representations which we shall consider form towers of (non-reduced) block representations.

Remark 3.2.21. For each j in a tower of (non-reduced) block representations, G_j is isomorphic to the semidirect product of K_1^j and G_{j-1}.

3.3. Irreducibility of polynomial components

In order to show that we have a free divisor, it is necessary to prove that a collection of polynomials are all irreducible and distinct ($f \neq \lambda h$ for some λ). Most of the polynomials of interest have the special property that they take the form of a determinant of a matrix M of polynomials where one variable occurs in only the lower right entry:

$$f = \det \begin{pmatrix} a_{1,1}(x) & \cdots & a_{1,n-1}(x) & a_{1,n}(x) \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}(x) & \cdots & a_{n-1,n-1}(x) & a_{n-1,n}(x) \\ a_{n,1}(x) & \cdots & a_{n,n-1}(x) & y \end{pmatrix}.$$

Not only is $g = \frac{\partial f}{\partial y}$ a function only of x, but g is the determinant of the upper left $(n-1)\times(n-1)$ submatrix of M. In this section we will show that if g is irreducible and $f \neq \lambda g y$, then f is irreducible.

We will first prove the following Proposition.

Proposition 3.3.1. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$. If $f \in R[y]$ and $g \in R$ with $g \neq 0$, $\frac{\partial f}{\partial y} = g$, and $\gcd(f, g) = 1$, then f is irreducible in $R[y]$.*

PROOF. First recall that the *content* of a nonzero $F \in R[y]$, denoted $\mathrm{cont}(F)$, is the greatest common divisor of the coefficients of F written as a polynomial in y (see, e.g., [**Lan02**], IV, §2). The content is well-defined up to multiplication by a unit in R.

The hypotheses imply that f takes the form $f = gy + h$ for $h \in R$. Then

$$1 = \gcd(f, g) = \gcd(gy + h, g) = \gcd(h, g). \tag{3.7}$$

In particular, (3.7) implies that $\mathrm{cont}(f) = 1$.

If f factored as $f = f_1 f_2$ in $R[y]$, then Gauss' Lemma (e.g., [**Lan02**], IV, Theorem 2.1) states that

$$\mathrm{cont}(f) = \mathrm{cont}(f_1)\mathrm{cont}(f_2)$$

in R. Since $\mathrm{cont}(f)$ is a unit, both $\mathrm{cont}(f_1)$ and $\mathrm{cont}(f_2)$ must be units in R. Since $f = gy + h$ has $\deg_y(f) = 1$, one of f_1 or f_2 has $\deg_y = 1$ and the other has $\deg_y = 0$. We may assume without loss of generality that

$$f_1 = ay + b \qquad \text{and} \qquad f_2 = c,$$

where $a, b, c \in R$. But $c = \mathrm{cont}(f_2)$ is a unit in R and thus in $R[y]$, so f is irreducible in $R[y]$. □

If we know the factorization of g, then it is easy to prove that f is irreducible.

COROLLARY 3.3.2. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $f \in R[y]$, with $\frac{\partial f}{\partial y} = g \in R$. If $g = g_1 \cdots g_m$ is a factorization of g into irreducible components and for each $i = 1, \cdots, m$ there exists a point $(x_0, y_0) \in \mathbb{C}^n \times \mathbb{C}$ so that $g_i(x_0) = 0$ and $f(x_0, y_0) \neq 0$, then f is irreducible.*

PROOF. The g_i are the irreducible components of g in R. Evaluation at the points shows that no g_i divides f. Thus g and f have no common factors and $\gcd(f, g) = 1$. By Proposition 3.3.1, f is irreducible. □

For us, g itself is usually irreducible. We state this case separately.

COROLLARY 3.3.3. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $f \in R[y]$, with $\frac{\partial f}{\partial y} = g \in R$. If g is irreducible and there exists a point $(x_0, y_0) \in \mathbb{C}^n \times \mathbb{C}$ so that $g(x_0) = 0$ and $f(x_0, y_0) \neq 0$, then f is irreducible.*

Proof. Apply Corollary 3.3.2 with $m = 1$. $\square$

CHAPTER 4

Representations for Cholesky-like factorizations as block representations

In this chapter we will use the block representation structure developed in the previous chapter to study the representations of Chapter 2, which correspond to matrix factorizations. The exceptional orbit varieties of these representations provide examples of families of free divisors and free* divisors in spaces of matrices.

Since our groups and representations all come from two basic representations, corresponding to change of bases for linear transformations and bilinear forms, we initially study the properties of these basic representations. As we are primarily interested in the case where our group is solvable, we restrict ourselves to "triangular" changes of bases. For each of these basic representations, we identify the main elements for a block representations: the invariant subspaces, the kernels of the quotient representations, the associated vector fields, and the orderings used to obtain a candidate block representation.

We then study the representations of Chapter 2 on spaces of symmetric, general $n \times n$, and skew-symmetric matrices as specializations of these two basic representations. For each case we apply the criteria of Chapter 3 to identify the exceptional orbit variety as either a free divisor or free* divisor.

For the symmetric representation, we also give a classification of the orbits. This will be used in Chapter 6 for the computation of singular Milnor numbers.

Because several exceptional orbit varieties are only free* divisors, in Chapter 5 we shall modify the representations so that their exceptional orbit variety is a free divisor which still contains the variety of singular matrices.

4.1. The linear transformation representation

We first study linear transformations under triangular changes of bases. Let $L_n(\mathbb{C})$ denote the group of $n \times n$ invertible lower triangular matrices, let $T_m(\mathbb{C})$ denote the group of $m \times m$

$$
\begin{array}{r}
 \\
k'\{ \\
k-k'\{ \\
n-k\{
\end{array}
\begin{array}{c}
\begin{array}{ccc} \overbrace{}^{l} & \overbrace{}^{l'-l} & \overbrace{}^{n-l'} \end{array} \\
\begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}
\end{array}
$$

FIGURE 4.1. The type of matrices which form the subspace $M(n, m, \mathbb{C})_{k,l} \cap M(n, m, \mathbb{C})_{k',l'}$ in ρ's invariant flag. Here we have assumed $l' > l$ and $k > k'$.

upper triangular matrices, and let $M(n, m, \mathbb{C})$ be the space of $n \times m$ complex matrices. Define the representation ρ of $L_n(\mathbb{C}) \times T_m(\mathbb{C})$ on $M(n, m, \mathbb{C})$ by

$$\rho(A, B)(M) = AMB^{-1}. \tag{4.1}$$

We will study ρ's invariant subspaces, the kernels of the induced quotient representations, the vector fields which arise from ρ, and also motivate the orderings of bases which we will use in our block representations.

4.1.1. Invariant Subspaces. Since ρ is a rational representation of a solvable connected complex algebraic Lie group, by the Lie-Kolchin theorem (Theorem 2.3.1) ρ has an invariant flag of subspaces. These invariant subspaces are easy to determine.

Fix coordinates $\{x_{ij}\}$ on $M(n, m, \mathbb{C})$ so that for $M \in M(n, m, \mathbb{C})$ and any (i, j), $x_{ij}(M) = (M)_{ij}$. Let $M(n, m, \mathbb{C})_{k,l}$ denote the subspace of $M(n, m, \mathbb{C})$ consisting of the matrices whose upper left $k \times l$ submatrix is zero.

PROPOSITION 4.1.1. *The representation ρ on $M(n, m, \mathbb{C})$ given by* (4.1) *leaves invariant the subspaces $M(n, m, \mathbb{C})_{k,l}$, for all k, l.*

PROOF. Consider the product

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}M_{11}U_{11}^{-1} & * \\ * & * \end{pmatrix} \tag{4.2}$$

where M_{11} is a $k \times l$ submatrix. Then (4.2) shows that when $M_{11} = 0$, $A_{11}M_{11}U_{11}^{-1} = 0$. □

REMARK 4.1.2. In fact, there exists a ρ-invariant flag where each subspace in the flag takes the form

$$M(n, m, \mathbb{C})_{k,l} \cap M(n, m, \mathbb{C})_{k',l'}$$

(see Figure 4.1). For example, when $n = m = 2$ such a flag is given by

$$\left\{\begin{pmatrix} 0 \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} * & * \\ * & * \end{pmatrix}\right\}.$$

4.1.2. Kernels of quotient representations. A ρ-invariant subspace $W \subset M(n, m, \mathbb{C})$ used in a block representation must have the property that the kernel of the quotient representation $L_n(\mathbb{C}) \times T_m(\mathbb{C}) \to \mathrm{GL}(M(n, m, \mathbb{C})/W)$ induced from ρ has dimension $\dim_{\mathbb{C}}(W)$ (this is (2) of Definition 3.2.6). We therefore study the kernels of these quotient representations to identify the useful invariant subspaces.

For each $0 \leq k \leq n$, $0 \leq l \leq m$, define the quotient representation

$$\rho_{k,l} : L_n(\mathbb{C}) \times T_m(\mathbb{C}) \to \mathrm{GL}(M(n, m, \mathbb{C})/M(n, m, \mathbb{C})_{k,l})$$

induced from ρ.

Proposition 4.1.3. *For $k, l > 0$,*

$$\ker(\rho_{k,l}) = \left\{ \left(\begin{pmatrix} \lambda \cdot I_k & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} \lambda \cdot I_l & * \\ 0 & * \end{pmatrix} \right) \in L_n(\mathbb{C}) \times T_m(\mathbb{C}) \right\}.$$

Proof. By (4.2), such elements lie in $\ker(\rho_{k,l})$.

For the reverse inclusion, a calculation as in (4.2) shows that

$$\left(\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \right) \in \ker(\rho_{k,l})$$

if and only if $A_{11} M_{11} (U_{11})^{-1} = M_{11}$ for all $k \times l$ matrices M_{11}. As a result, we need only show that if L is a $k \times k$ lower triangular matrix and B is a $l \times l$ upper triangular matrix for which $LMB = M$ for all $k \times l$ matrices M, then $L = \lambda I_k$ and $B = \lambda^{-1} I_l$.

When M is an elementary matrix E_{rs}, by assumption we have

$$(LE_{rs}B)_{ij} = \delta_{ri}\delta_{sj} \tag{4.3}$$

for all i, j. Taking into account that L and B are triangular, the left side can be calculated to be

$$\sum_{p=1}^{i}\sum_{q=1}^{j}(L)_{ip}\delta_{rp}\delta_{sq}(B)_{qj} = \begin{cases} (L)_{ir}(B)_{sj} & \text{when } r \leq i \text{ and } s \leq j \\ 0 & \text{otherwise} \end{cases} . \tag{4.4}$$

Combining (4.3) and (4.4) yields that, for $r \leq i$ and $s \leq j$, (4.3)

$$(L)_{ir}(B)_{sj} = \delta_{ri}\delta_{sj}. \tag{4.5}$$

When $r = i$ and $s = j$, (4.5) yields $(L)_{ii}(B)_{jj} = 1$ for all i, j; thus all diagonal elements of L equal a common $\lambda \in \mathbb{C}^*$ and all diagonal elements of B equal $\frac{1}{\lambda}$. When $r < i$ and $s = j$, (4.5) yields $(L)_{ir}(B)_{jj} = 0$, so that $(L)_{ir} = 0$ for all $r < i$; L is thus diagonal. When $r = i$ and $s < j$, (4.5) yields $(L)_{ii}(B)_{sj} = 0$, so that $(B)_{sj} = 0$ for all $s < j$; B is thus diagonal as well. □

Remark 4.1.4. For ρ itself, $\ker(\rho)$ is a 1-dimensional torus, $\{(\lambda I, \lambda I) | \lambda \in \mathbb{C}^*\}$. To obtain an equidimensional representation of the form $\rho|_H$, for H a subgroup of $L_n(\mathbb{C}) \times T_m(\mathbb{C})$, we must choose H so that $\dim_{\mathbb{C}}(H \cap \ker(\rho)) = 0$. For example, the LU decomposition (Theorem 2.2.1) achieves this by requiring the upper triangular matrices to have ones on the diagonal.

4.1.3. Vector fields. We now obtain explicit descriptions of the vector fields coming from ρ. The vector fields $\left\{\frac{\partial}{\partial x_{i,j}} | 1 \leq i \leq n, 1 \leq j \leq m\right\}$ serve as a basis for the vector fields on $M(n, m, \mathbb{C})$. Denote the Lie algebra of $L_n(\mathbb{C}) \times T_m(\mathbb{C})$ by $\mathfrak{g}$. For $(A, B) \in \mathfrak{g}$ and $M \in M(n, m, \mathbb{C})$, the vector field $\xi_{(A,B)}$ associated to (A, B) evaluated at M is given by

$$\begin{aligned} \xi_{(A,B)}(M) &= \frac{d}{dt} \left(\rho(\exp(tA), \exp(tB))(M)\right)|_{t=0} \\ &= \frac{d}{dt} \left(\exp(tA) M \exp(-tB)\right)|_{t=0} \\ &= AM - MB. \end{aligned}$$

In coordinates, if S is the generic $n \times m$ matrix with $(S)_{i,j} = x_{i,j}$, we may write this as

$$\xi_{(A,B)}(S) = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (AS - SB)_{i,j} \frac{\partial}{\partial x_{i,j}}. \tag{4.6}$$

Especially useful will be the case where only one of A or B is nonzero and equal to an elementary matrix.

Proposition 4.1.5. *For the representation ρ given by* (4.1)*, the vector fields associated to the Lie algebra of $L_n(\mathbb{C}) \times T_m(\mathbb{C})$ are*

$$\xi_{(E_{p,q},0)}(S) = \sum_{1\le j\le m} x_{q,j}\frac{\partial}{\partial x_{p,j}} \qquad \text{and} \qquad \xi_{(0,E_{k,l})}(S) = \sum_{1\le i\le n} -x_{i,k}\frac{\partial}{\partial x_{i,l}}.$$

Proof. This is just a calculation starting with (4.6). For $(E_{p,q},0) \in \mathfrak{g}$, we obtain

$$\begin{aligned}
\xi_{(E_{p,q},0)}(S) &= \sum_{\substack{1\le i\le n\\1\le j\le m}} (E_{p,q}S)_{i,j}\frac{\partial}{\partial x_{i,j}}\\
&= \sum_{\substack{1\le i\le n\\1\le j\le m}} \sum_{k=1}^{n}(E_{p,q})_{i,k}(S)_{k,j}\frac{\partial}{\partial x_{i,j}}\\
&= \sum_{\substack{1\le i\le n\\1\le j\le m}} \sum_{k=1}^{n}\delta_{i,p}\delta_{q,k}(S)_{k,j}\frac{\partial}{\partial x_{i,j}}\\
&= \sum_{1\le j\le m} x_{q,j}\frac{\partial}{\partial x_{p,j}}.
\end{aligned}$$

Similarly, when $(0,E_{k,l}) \in \mathfrak{g}$ we obtain

$$\begin{aligned}
\xi_{(0,E_{k,l})}(S) &= \sum_{\substack{1\le i\le n\\1\le j\le m}} (-SE_{k,l})_{i,j}\frac{\partial}{\partial x_{i,j}}\\
&= \sum_{\substack{1\le i\le n\\1\le j\le m}} \sum_{f=1}^{m}(-S)_{i,f}(E_{k,l})_{f,j}\frac{\partial}{\partial x_{i,j}}\\
&= \sum_{\substack{1\le i\le n\\1\le j\le m}} \sum_{f=1}^{m}(-S)_{i,f}\delta_{k,f}\delta_{l,j}\frac{\partial}{\partial x_{i,j}}\\
&= \sum_{1\le i\le n} -x_{i,k}\frac{\partial}{\partial x_{i,l}}.
\end{aligned}$$

□

4.1.4. Orderings. As discussed in §3.2.4, to analyze a block representation we must choose orderings for a basis of $\mathfrak{g}$ and a basis of our vector space. An appropriate choice of bases and orderings will let us determine the structure of the exceptional orbit variety. We shall choose the order of these elements so that the diagonal blocks of the coefficient matrix have easily understood determinants. Surprisingly, we may often choose orderings so that the diagonal blocks are (perhaps transformed) submatrices of the generic $n \times m$ matrix S.

First suppose we choose $M(n,m,\mathbb{C})_{k,l} \subset M(n,m,\mathbb{C})_{k-1,l}$ to be two adjacent invariant subspaces in a block representation. Then $\frac{\partial}{\partial x_{k,1}}, \ldots, \frac{\partial}{\partial x_{k,l}}$ is a basis for a subspace complementary to $M(n,m,\mathbb{C})_{k,l}$ in $M(n,m,\mathbb{C})_{k-1,l}$. Proposition 4.1.3 shows that $(E_{k,1},0),\ldots,(E_{k,k},0) \in \mathfrak{g}$ is a basis for a subspace complementary to the Lie algebra of $\ker(\rho_{k,l})$ in the Lie algebra of $\ker(\rho_{k-1,l})$. By using these ordered bases for such a "vertical expansion," certain diagonal blocks of our block representation take the form of the transpose of a submatrix of S.

PROPOSITION 4.1.6. *The matrix M of vector field coefficients, where the rows represent the coefficients of $\frac{\partial}{\partial x_{k,1}}, \cdots, \frac{\partial}{\partial x_{k,l}}$ and the columns represent the vector fields $\xi_{(E_{k,1},0)}, \cdots, \xi_{(E_{k,k},0)}$, is equal to the transpose of a contiguous submatrix of the generic matrix S:*

$$M = \begin{pmatrix} x_{1,1} & \cdots & x_{k,1} \\ \vdots & \ddots & \vdots \\ x_{1,l} & \cdots & x_{k,l} \end{pmatrix}.$$

PROOF. M is a $l \times k$ matrix where $(M)_{r,s}$ is the coefficient of $\frac{\partial}{\partial x_{k,r}}$ in $\xi_{(E_{k,s},0)}$. By Proposition 4.1.5, $(M)_{r,s} = x_{s,r}$. $\square$

REMARK 4.1.7. For $\rho|_H$, some $(E_{k,i},0)$ may not be in the Lie algebra $\mathfrak{h}$ of H. However, the subgroups H we use will typically have a basis of $\mathfrak{h}$ consisting of elementary matrices, so we may still use the above result by deleting the appropriate columns.

Now consider $M(n,m,\mathbb{C})_{k,l} \subset M(n,m,\mathbb{C})_{k,l-1}$ as two adjacent invariant subspaces in a block representation. Then $\frac{\partial}{\partial x_{1,l}}, \ldots, \frac{\partial}{\partial x_{k,l}}$ and $(0,E_{1,l}),\ldots,(0,E_{l,l}) \in \mathfrak{g}$ are bases for subspaces complementary to $M(n,m,\mathbb{C})_{k,l}$ in $M(n,m,\mathbb{C})_{k,l-1}$ and complementary to the Lie algebra of $\ker(\rho_{k,l})$ in the Lie algebra of $\ker(\rho_{k,l-1})$, respectively. By using these orderings for such a "horizontal expansion," some diagonal blocks of our block representation take the form of the negative of a submatrix of S.

PROPOSITION 4.1.8. *The matrix M of vector field coefficients, where the rows represent the coefficients of $\frac{\partial}{\partial x_{1,l}}, \cdots, \frac{\partial}{\partial x_{k,l}}$ and the columns represent the vector fields $\xi_{(0,E_{1,l})}, \cdots, \xi_{(0,E_{l,l})}$, is* *equal to the negative of a contiguous submatrix of the generic matrix S:*

$$M = \begin{pmatrix} -x_{1,1} & \cdots & -x_{1,l} \\ \vdots & \ddots & \vdots \\ -x_{k,1} & \cdots & -x_{k,l} \end{pmatrix}.$$

Proof. M is a $k \times l$ matrix where $(M)_{r,s}$ is the coefficient of $\frac{\partial}{\partial x_{r,l}}$ in $\xi_{(0,E_{s,l})}$. By Proposition 4.1.5, $(M)_{r,s} = -x_{r,s}$. $\square$

Based on these observations, for each of our block representations we will use a sequence of invariant subspaces of $M(n,m,\mathbb{C})$ where each step takes the form

$$M(n,m,\mathbb{C})_{k,l} \subset M(n,m,\mathbb{C})_{k-1,l} \quad \text{or} \quad M(n,m,\mathbb{C})_{k,l} \subset M(n,m,\mathbb{C})_{k,l-1}. \tag{4.7}$$

We will then use the above bases and orderings to construct the coefficient matrix of our block representation.

4.2. The bilinear form representation

We will now repeat the steps of the previous section for a representation which models bilinear forms under triangular changes of coordinates. Define the representation θ of $L_n(\mathbb{C})$ on $M(n,n,\mathbb{C})$ by

$$\theta(A)(M) = AMA^T. \tag{4.8}$$

We will examine the invariant subspaces of θ, the kernels of the quotient representations, the vector fields arising from the representation, and the orderings we will use for our block representations. The results for θ are quite similar to the results for ρ.

Remark 4.2.1. In fact, θ is related to ρ in a simple way. Consider the injective Lie group homomorphism $\psi : L_n(\mathbb{C}) \to L_n(\mathbb{C}) \times T_n(\mathbb{C}),\ A \mapsto (A, (A^T)^{-1})$. Then $\theta = \rho \circ \psi$. This observation indicates why the properties of θ are similar to those of ρ.

Remark 4.2.2. θ does not have an open orbit for dimensional reasons. However, we will be interested in restricting θ to subgroups of $L_n(\mathbb{C})$ and subspaces of $M(n,n,\mathbb{C})$ so that the restriction has an open orbit. We are particularly interested in restrictions to the θ-invariant subspaces $\mathrm{Sym}_n(\mathbb{C})$, the space of symmetric matrices, and $\mathrm{Sk}_n(\mathbb{C})$, the space of skew-symmetric matrices.

4.2.1. Invariant Subspaces. As in the previous section, fix coordinates $\{x_{ij}\}$ on $M(n,n,\mathbb{C})$ and let $M(n,n,\mathbb{C})_{k,l}$ be the subspace of matrices where the upper left $k \times l$ block is zero.

PROPOSITION 4.2.3. *The representation θ on $M(n,n,\mathbb{C})$ defined by* (4.8) *leaves invariant the subspace $M(n,n,\mathbb{C})_{k,l}$ for all k,l.*

PROOF. This follows from Remark 4.2.1 and Proposition 4.1.1. □

REMARK 4.2.4. Again, intersections of such subspaces provide the building blocks for an invariant flag on $M(n,n,\mathbb{C})$.

4.2.2. Kernels. As above, only some of these subspaces will be useful for block representations, depending on the kernel of an associated quotient representation. For $0 \leq k,l \leq n$, define the quotient representation

$$\theta_{k,l} : L_n(\mathbb{C}) \to \mathrm{GL}(M(n,n,\mathbb{C})/M(n,n,\mathbb{C})_{k,l})$$

induced from θ.

PROPOSITION 4.2.5. *For $k,l > 0$,*

$$\ker(\theta_{k,l}) = \left\{ \begin{pmatrix} \pm I_{\max\{k,l\}} & 0 \\ * & * \end{pmatrix} \in L_n(\mathbb{C}) \right\}.$$

PROOF. By Remark 4.2.1, $\theta_{k,l} = \rho_{k,l} \circ \psi$, so that

$$\ker(\theta_{k,l}) = \psi^{-1}(\ker(\rho_{k,l})).$$

But $A \in \psi^{-1}(ker(\rho_{k,l}))$ if and only if $(A,(A^T)^{-1}) \in \ker(\rho_{k,l})$, and Proposition 4.1.3 identifies $\ker(\rho_{k,l})$. Thus $A \in \ker(\theta_{k,l})$ if and only if there exists a $\lambda \in \mathbb{C}^*$ so that $A^{(k)} = \lambda I_k$ and $((A^T)^{-1})^{(l)} = \lambda I_l$, i.e., $A^{(l)} = \frac{1}{\lambda} I_l$. But $\lambda = \frac{1}{\lambda}$ if and only if $\lambda = \pm 1$. □

Unlike ρ from the previous section, $\ker(\theta) = \{\pm I_n\}$ is discrete.

Proposition 4.2.5 has interesting consequences for the invariant subspaces we will use in our block representations. We use subspaces of the form $M(n,n,\mathbb{C})_{k,k}$ as these are the smallest invariant subspaces with a particular kernel. Thus we use transitions of the form

$$M(n,n,\mathbb{C})_{k+1,k+1} \subset M(n,n,\mathbb{C})_{k,k}$$

at each step of our block representations.

4.2.3. Vector Fields. We now obtain concrete descriptions of the vector fields coming from θ. The vector fields $\left\{\frac{\partial}{\partial x_{i,j}} | 1 \leq i,j \leq n\right\}$ serve as a basis for the vector fields on $M(n,n,\mathbb{C})$. Denote the Lie algebra of $L_n(\mathbb{C})$ by $\mathfrak{g}$. For $A \in \mathfrak{g}$, the vector field ξ_A associated to A evaluated at $M \in M(n,n,\mathbb{C})$ is given by

$$\begin{aligned}
\xi_A(M) &= \frac{d}{dt}\left(\theta(\exp(tA))(M)\right)\big|_{t=0} \\
&= \frac{d}{dt}\left(\exp(tA)(M)(\exp(tA))^T\right)\big|_{t=0} \\
&= \frac{d}{dt}\left(\exp(tA)(M)\exp(tA^T)\right)\big|_{t=0} \\
&= AM + MA^T.
\end{aligned}$$

In coordinates, if S is the generic $n \times n$ matrix, we may write this vector field as

$$\xi_A(S) = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} (AS + SA^T)_{i,j} \frac{\partial}{\partial x_{i,j}}. \tag{4.9}$$

It will be useful to compute this explicitly when A is an elementary matrix.

Proposition 4.2.6. *For the representation θ given by* (4.8)*, the vector fields associated to the Lie algebra of $L_n(\mathbb{C})$ are*

$$\xi_{E_{k,l}}(S) = \sum_{1 \leq j \leq n} x_{l,j} \frac{\partial}{\partial x_{k,j}} + \sum_{1 \leq i \leq n} x_{i,l} \frac{\partial}{\partial x_{i,k}}.$$

Proof. This is a calculation which starts with (4.9) where $A = E_{k,l}$:

$$\begin{aligned}
\xi_{E_{k,l}}(S) &= \sum_{\substack{1\le i\le n\\ 1\le j\le n}} (E_{k,l}S + SE_{l,k})_{i,j} \frac{\partial}{\partial x_{i,j}} \\
&= \sum_{\substack{1\le i\le n\\ 1\le j\le n}} \sum_{p=1}^{n} \left((E_{k,l})_{i,p}(S)_{p,j} + (S)_{i,p}(E_{l,k})_{p,j}\right) \frac{\partial}{\partial x_{i,j}} \\
&= \sum_{\substack{1\le i\le n\\ 1\le j\le n}} \sum_{p=1}^{n} \left(\delta_{k,i}\delta_{l,p}(S)_{p,j} + (S)_{i,p}\delta_{l,p}\delta_{k,j}\right) \frac{\partial}{\partial x_{i,j}} \\
&= \sum_{1\le j\le n} x_{l,j} \frac{\partial}{\partial x_{k,j}} + \sum_{1\le i\le n} x_{i,l} \frac{\partial}{\partial x_{i,k}}.
\end{aligned}$$

□

4.2.4. Orderings. As for ρ, we wish to order our vector fields and coordinates so that we obtain a matrix of coefficients which are (perhaps transformed) submatrices of the generic $n \times n$ matrix S.

Suppose we use the adjacent subspaces

$$M(n, n, \mathbb{C})_{k,k} \subset M(n, n, \mathbb{C})_{k-1,k-1}$$

in a block representation. First, $\frac{\partial}{\partial x_{1,k}}, \ldots, \frac{\partial}{\partial x_{k,k}}$ is part of a basis for a subspace complementary to $M(n, n, \mathbb{C})_{k,k}$ in $M(n, n, \mathbb{C})_{k-1,k-1}$. By Proposition 4.2.5, $E_{k,1}, \ldots, E_{k,k} \in \mathfrak{g}$ is a basis for a subspace complementary to the Lie algebra of $\ker(\theta_{k,k})$ in the Lie algebra of $\ker(\theta_{k-1,k-1})$. Consider the $\frac{\partial}{\partial x_{1,k}}, \ldots, \frac{\partial}{\partial x_{k,k}}$ components of the corresponding vector fields.

Proposition 4.2.7. *The matrix M of vector field coefficients, where the rows represent the coefficients of $\frac{\partial}{\partial x_{1,k}}, \cdots, \frac{\partial}{\partial x_{k,k}}$ and the columns represent the vector fields $\xi_{E_{k,1}}, \cdots, \xi_{E_{k,k}}$, is equal to a submatrix of the generic matrix S, except the last row is different:*

$$M = \begin{pmatrix} x_{1,1} & \cdots & x_{1,k} \\ \vdots & \ddots & \vdots \\ x_{k-1,1} & \cdots & x_{k-1,k} \\ x_{k,1} + x_{1,k} & \cdots & x_{k,k} + x_{k,k} \end{pmatrix}.$$

Proof. Note that M is a $k \times k$ matrix. $(M)_{r,s}$ is the coefficient of $\frac{\partial}{\partial x_{r,k}}$ in $\xi_{E_{k,s}}$. By Proposition 4.2.6, $(M)_{r,s} = \delta_{r,k}x_{s,k} + x_{r,s}$. □

Except for the last row, this is a submatrix of S. Note that the last row will not appear for $\mathrm{Sk}_n(\mathbb{C})$, while for $\mathrm{Sym}_n(\mathbb{C})$ the last row will be a row of S multiplied by 2. In either case, this matrix of coefficients is easily understood provided we use the orderings described above.

4.3. The symmetric and skew-symmetric bilinear form representations

We now study the restriction of θ to the invariant subspaces $\mathrm{Sym}_n(\mathbb{C})$ and $\mathrm{Sk}_n(\mathbb{C})$ and prove results which are analogous to those for θ.

4.3.1. Symmetric bilinear form representation. Let $\psi : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ be the restriction of θ to $\mathrm{Sym}_n(\mathbb{C})$. Define $\mathrm{Sym}_n(\mathbb{C})_{k,k} = \mathrm{Sym}_n(\mathbb{C}) \cap M(n,n,\mathbb{C})_{k,k}$, the space of symmetric matrices whose upper left $k \times k$ submatrix is zero. Proposition 4.2.3 immediately identifies $\mathrm{Sym}_n(\mathbb{C})_{k,k}$ as an invariant subspace.

Proposition 4.3.1. *ψ leaves $\mathrm{Sym}_n(\mathbb{C})_{k,k}$ invariant.*

For each $0 \leq k \leq n$, define the quotient representation $\psi_k : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})/\mathrm{Sym}_n(\mathbb{C})_{k,k})$ induced from ψ. Proposition 4.2.5 is not sufficient to completely identify $\ker(\psi_k)$.

Proposition 4.3.2. *For $k > 0$,*

$$\ker(\psi_k) = \left\{ \begin{pmatrix} \pm I_k & 0 \\ * & * \end{pmatrix} \in L_n(\mathbb{C}) \right\}.$$

We first prove a Lemma.

Lemma 4.3.3. *Suppose that A is a $n \times n$ matrix so that $ASA^T = S$ for all symmetric matrices S. Then $A = \pm I$.*

Proof. Substituting $E_{ij} + E_{ji}$ for S, we may calculate that

$$\begin{aligned}
(A(E_{ij}+E_{ji})A^T)_{kl} &= \sum_{p=1}^{n}\sum_{q=1}^{n}(A)_{kp}(E_{ij}+E_{ji})_{pq}(A^T)_{ql} \\
&= \sum_{p=1}^{n}\sum_{q=1}^{n}(A)_{kp}\delta_{ip}\delta_{jq}(A)_{lq} + \sum_{p=1}^{n}\sum_{q=1}^{n}(A)_{kp}\delta_{jp}\delta_{iq}(A)_{lq} \\
&= (A)_{ki}(A)_{lj} + (A)_{kj}(A)_{li}.
\end{aligned}$$

On the other hand,

$$(A(E_{ij}+E_{ji})A^T)_{kl} = \delta_{ik}\delta_{lj} + \delta_{jk}\delta_{il},$$

so that for all i, j, k, l we have

$$(A)_{ki}(A)_{lj} + (A)_{kj}(A)_{li} = \delta_{ik}\delta_{lj} + \delta_{jk}\delta_{il}. \tag{4.10}$$

When $i = j$ and $k = l$, (4.10) gives

$$(A)_{ki}^2 = \delta_{ik};$$

thus $(A)_{ki} = 0$ if $i \neq k$ and $(A)_{ii} = \pm 1$, making A diagonal. When $k = i \neq j = l$, (4.10) gives

$$(A)_{ii}(A)_{jj} + 0 = \delta_{ii}\delta_{jj} + 0,$$

so that $(A)_{ii}(A)_{jj} = 1$. Since all diagonal entries are ± 1, all diagonal entries are equal. □

Proof of Proposition 4.3.2. Proposition 4.2.5 shows that such elements lie in the kernel. By a computation analogous to (4.2), any element A of the kernel must have the property that $(ASA^T)^{(k)} = A^{(k)}S^{(k)}(A^{(k)})^T = S^{(k)}$ for all S (in particular, all $S^{(k)}$). By Lemma 4.3.3, $A^{(k)} = \pm I_k$. □

Now choose coordinates $\{x_{ij} | 1 \leq i \leq j \leq n\}$ on $\mathrm{Sym}_n(\mathbb{C})$ so that for $M \in \mathrm{Sym}_n(\mathbb{C})$, $x_{ij}(M) = (M)_{ij} = (M)_{ji}$.

Remark 4.3.4. These are not the same coordinates we used on $M(n, n, \mathbb{C})$ in §4.2. Let $\{\overline{x_{ij}}\}$ be those coordinates. Then for $i \leq j$, $x_{ij} = \overline{x_{ij}}|_{\mathrm{Sym}_n(\mathbb{C})} = \overline{x_{ji}}|_{\mathrm{Sym}_n(\mathbb{C})}$ and on $\mathrm{Sym}_n(\mathbb{C})$,

$$\frac{\partial}{\partial x_{ij}} = \begin{cases} \frac{\partial}{\partial \overline{x_{ij}}} + \frac{\partial}{\partial \overline{x_{ji}}} & \text{if } i \neq j \\ \frac{\partial}{\partial \overline{x_{ii}}} & \text{if } i = j \end{cases}.$$

A vector field ξ on $M(n, n, \mathbb{C})$ tangent to $\mathrm{Sym}_n(\mathbb{C})$ has the property that the coefficients of $\frac{\partial}{\partial \overline{x_{ij}}}$ and $\frac{\partial}{\partial \overline{x_{ji}}}$ are equal on $\mathrm{Sym}_n(\mathbb{C})$. If $\xi = \sum_{i,j} f_{ij} \frac{\partial}{\partial \overline{x_{ij}}}$ is such a vector field, we thus have $\xi|_{\mathrm{Sym}_n(\mathbb{C})} = \sum_{i \leq j} f_{ij}|_{\mathrm{Sym}_n(\mathbb{C})} \frac{\partial}{\partial x_{ij}}$.

This observation allows us to easily translate our explicit calculation of the vector fields from $M(n, n, \mathbb{C})$ to $\mathrm{Sym}_n(\mathbb{C})$.

Proposition 4.3.5. *Let S be a generic $n \times n$ symmetric matrix. For the representation $\psi : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ given by restricting θ, the vector field associated to the Lie algebra of $L_n(\mathbb{C})$ is*

$$\xi_{E_{k,l}}(S) = \sum_{k \leq i \leq n} (S)_{li} \frac{\partial}{\partial x_{ki}} + \sum_{1 \leq i \leq k} (S)_{il} \frac{\partial}{\partial x_{ik}}.$$

PROOF. By Remark 4.3.4 and Proposition 4.2.6, we have

$$\xi_{E_{k,l}}(S) = \sum_{k \le i \le n} \overline{x_{li}}|_{\mathrm{Sym}_n(\mathbb{C})} \frac{\partial}{\partial x_{ki}} + \sum_{1 \le i \le k} \overline{x_{il}}|_{\mathrm{Sym}_n(\mathbb{C})} \frac{\partial}{\partial x_{ik}}.$$

But each $\overline{x_{ij}}|_{\mathrm{Sym}_n(\mathbb{C})} = (S)_{ij}$, whether or not $i \le j$. □

Similarly, we have an analogue of Proposition 4.2.7.

PROPOSITION 4.3.6. *The matrix M of vector field coefficients, where the rows represent the coefficients of $\frac{\partial}{\partial x_{1,k}}, \cdots, \frac{\partial}{\partial x_{k,k}}$ and the columns represent the vector fields $\xi_{E_{k,1}}, \ldots, \xi_{E_{k,k}}$, is*

$$M = \begin{pmatrix} x_{1,1} & \cdots & x_{1,k} \\ \vdots & \ddots & \vdots \\ x_{1,k-1} & \cdots & x_{k-1,k} \\ 2x_{1,k} & \cdots & 2x_{k,k} \end{pmatrix}.$$

PROOF. Apply Remark 4.3.4 to Proposition 4.2.7. □

4.3.2. Skew-symmetric bilinear form representation. Define $\sigma : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C}))$ to be the restriction of θ to $\mathrm{Sk}_n(\mathbb{C})$. Define $\mathrm{Sk}_n(\mathbb{C})_{k,k} = \mathrm{Sk}_n(\mathbb{C}) \cap M(n,n,\mathbb{C})_{k,k}$, the space of skew-symmetric matrices whose upper left $k \times k$ submatrix is zero. Proposition 4.2.3 immediately identifies $\mathrm{Sk}_n(\mathbb{C})_{k,k}$ as an invariant subspace.

PROPOSITION 4.3.7. *σ leaves $\mathrm{Sk}_n(\mathbb{C})_{k,k}$ invariant.*

For each $0 \le k \le n$, define the quotient representation $\sigma_k : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C})/\mathrm{Sk}_n(\mathbb{C})_{k,k})$ induced from σ.

PROPOSITION 4.3.8. $\ker(\sigma_1) = L_n(\mathbb{C})$. $\ker(\sigma_2) = \{A \in L_n(\mathbb{C}) | \det(A^{(2)}) = 1\}$. *For $k \ge 3$,*

$$\ker(\sigma_k) = \left\{ \begin{pmatrix} \pm I_k & 0 \\ * & * \end{pmatrix} \in L_n(\mathbb{C}) \right\}.$$

We first prove a Lemma.

LEMMA 4.3.9. *Suppose that A is a $n \times n$ lower triangular matrix so that $ASA^T = S$ for all skew-symmetric matrices S. If $n \ge 3$, then $A = \pm I$. If $n = 2$, then $A \in L_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{C})$. If $n = 1$, then $A \in L_1(\mathbb{C})$.*

Proof. The result for $n = 1$ or $n = 2$ follows from a straightforward calculation.

Assume $n \geq 3$. A calculation analogous to one in the proof of Lemma 4.3.3 shows that, for all i, j, k, l,

$$(A)_{ki}(A)_{lj} - (A)_{kj}(A)_{li} = \delta_{ik}\delta_{lj} - \delta_{jk}\delta_{il}. \tag{4.11}$$

When $i = k \neq j = l$, (4.11) gives

$$(A)_{ii}(A)_{jj} - (A)_{ij}(A)_{ji} = 1 + 0,$$

but either A_{ij} or A_{ji} is zero as A is lower triangular and $i \neq j$. Thus $(A)_{ii}(A)_{jj} = 1$ if $i \neq j$. In particular, the diagonal entries of A are nonzero, $(A)_{ii} = (A_{11})^{-1}$ for $i > 1$, and $(A)_{jj} = (A_{nn})^{-1}$ for $j < n$. Since $n \geq 3$, all diagonal entries are the same and thus all equal to ± 1. (If $n = 2$, we could have $(A)_{11} = \lambda$, $(A)_{22} = \frac{1}{\lambda}$.)

When $i = k < j$, (4.11) gives

$$(A)_{ii}(A)_{lj} + 0 = \delta_{lj} + 0,$$

so that $(A)_{lj} = 0$ if $j > i$ and $j \neq l$. Thus the triangle below the diagonal and to the right of the ith column is zero.

When $i = k > l$, (4.11) gives

$$(A)_{ii}(A)_{lj} + 0 = \delta_{lj} + 0,$$

so that $(A)_{lj} = 0$ if $l < i$ and $l \neq j$. Thus the triangle below the diagonal and above the ith row is zero.

Since $n \geq 3$, all entries below the diagonal are contained in one such triangle, with the exception of $(A)_{n1}$. But evaluating (4.11) with $l = n$, $j = 1$, and $k = i \notin \{1, n\}$ (possible since $n \geq 3$) gives

$$(A)_{ii}(A)_{n1} + 0 = 0 + 0,$$

so $A = \pm I$. □

Proof of Proposition 4.3.8. By a computation analogous to (4.2), $A \in L_n(\mathbb{C})$ lies in the kernel of σ_k if and only if $(ASA^T)^{(k)} = A^{(k)}S^{(k)}(A^{(k)})^T = S^{(k)}$ for all skew-symmetric S (in particular, for all skew-symmetric $S^{(k)}$). Lemma 4.3.9 then identifies $A^{(k)}$. □

Now choose coordinates $\{x_{ij} | 1 \leq i < j \leq n\}$ on $\mathrm{Sk}_n(\mathbb{C})$ so that for $M \in \mathrm{Sk}_n(\mathbb{C})$ and $i < j$, $x_{ij}(M) = (M)_{ij} = -(M)_{ji}$.

REMARK 4.3.10. Just as with the symmetric bilinear form representation, if $\{\overline{x_{ij}} | 1 \leq i, j \leq n\}$ are the coordinates we used for $M(n, n, \mathbb{C})$ in §4.2, then when $1 \leq i < j \leq n$, $x_{ij} = \overline{x_{ij}}|_{\mathrm{Sk}_n(\mathbb{C})} = -\overline{x_{ji}}|_{\mathrm{Sk}_n(\mathbb{C})}$ and on $\mathrm{Sk}_n(\mathbb{C})$,

$$\frac{\partial}{\partial x_{ij}} = \frac{\partial}{\partial \overline{x_{ij}}} - \frac{\partial}{\partial \overline{x_{ji}}}.$$

A vector field ξ on $M(n, n, \mathbb{C})$ tangent to $\mathrm{Sk}_n(\mathbb{C})$ has the property that, on $\mathrm{Sk}_n(\mathbb{C})$, the coefficient of $\frac{\partial}{\partial \overline{x_{ij}}}$ is the negative of the coefficient of $\frac{\partial}{\partial \overline{x_{ji}}}$. If $\xi = \sum_{i,j} f_{ij} \frac{\partial}{\partial \overline{x_{ij}}}$ is such a vector field, we thus have $\xi|_{\mathrm{Sk}_n(\mathbb{C})} = \sum_{i<j} f_{ij}|_{\mathrm{Sk}_n(\mathbb{C})} \frac{\partial}{\partial x_{ij}}$.

Thus we may translate our vector field calculations to $\mathrm{Sk}_n(\mathbb{C})$.

PROPOSITION 4.3.11. *Let S be a generic $n \times n$ skew-symmetric matrix. If $E_{k,l}$ is an element of the Lie algebra of $L_n(\mathbb{C})$, then*

$$\xi_{E_{k,l}}(S) = \sum_{k<i\leq n} (S)_{li} \frac{\partial}{\partial x_{ki}} + \sum_{1\leq i<k} (S)_{il} \frac{\partial}{\partial x_{ik}}.$$

PROOF. By Remark 4.3.10 and Proposition 4.2.6, we have

$$\xi_{E_{k,l}}(S) = \sum_{k<i\leq n} \overline{x_{li}}|_{\mathrm{Sk}_n(\mathbb{C})} \frac{\partial}{\partial x_{ki}} + \sum_{1\leq i<k} \overline{x_{il}}|_{\mathrm{Sk}_n(\mathbb{C})} \frac{\partial}{\partial x_{ik}}.$$

But $(S)_{ij} = \overline{x_{ij}}|_{\mathrm{Sk}_n(\mathbb{C})}$ for all i, j. □

Similarly, we have an analogue of Proposition 4.2.7.

PROPOSITION 4.3.12. *The matrix M of vector field coefficients where the rows represent the coefficients of $\frac{\partial}{\partial x_{1,k}}, \cdots, \frac{\partial}{\partial x_{k-1,k}}$, and the columns represent the vector fields $\xi_{E_{k,1}}, \ldots, \xi_{E_{k,k}}$, is*

$$M = \begin{pmatrix} 0 & x_{1,2} & \cdots & x_{1,k-1} & x_{1,k} \\ -x_{1,2} & 0 & \cdots & x_{2,k-1} & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{1,k-1} & -x_{2,k-1} & \cdots & 0 & x_{k-1,k} \end{pmatrix}.$$

PROOF. Note that M is $(k-1) \times k$. Apply Remark 4.3.10 to Proposition 4.2.7, omitting the last row. □

4.4. The Cholesky representation for symmetric matrices

We now consider the representation which corresponds to the complex Cholesky factorization of symmetric matrices (Corollary 2.2.7). In fact, this is the representation $\psi : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ defined by

$$\psi(A)(M) = AMA^T$$

that we studied in §4.3.1. Though we already know the exceptional orbit variety V of this equidimensional representation, we shall show that ψ is a block representation and V is a free divisor. In particular, our approach shows how the invariant subspaces and Lie algebras contribute to the structure.

4.4.1. Candidate Block Representation. The results we need to develop a block representation were shown in §4.3.1. In particular, we shall use the coordinates $\{x_{ij} | 1 \leq i \leq j \leq n\}$ defined there for $\mathrm{Sym}_n(\mathbb{C})$.

Proposition 4.4.1. *The representation $\psi : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ has the structure of a candidate block representation with n diagonal blocks and subspaces*

$$W_j = \mathrm{Sym}_n(\mathbb{C})_{n-j,n-j}, \qquad j = 0, \ldots, n.$$

For appropriate choices of bases, the jth diagonal block, $1 \leq j \leq n$, is

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,j} \\ \vdots & \ddots & \vdots \\ 2x_{j,1} & \cdots & 2x_{j,j} \end{pmatrix}.$$

Proof. We shall use the notation of Definition 3.2.6. By Proposition 4.3.1, all $\mathrm{Sym}_n(\mathbb{C})_{k,k}$ are ψ-invariant. Thus we have a partial flag

$$\{0\} = W_0 = \mathrm{Sym}_n(\mathbb{C})_{n,n} \subset \cdots \subset \mathrm{Sym}_n(\mathbb{C}) = W_n$$

of invariant subspaces. Since $\dim_{\mathbb{C}}(W_j) = \binom{n+1}{2} - \binom{n-j+1}{2}$, we have $\dim_{\mathbb{C}}(W_{j+1}/W_j) = n - j$. Alternatively, $\{E_{i,n-j} + E_{n-j,i} \in \mathrm{Sym}_n(\mathbb{C}) | 1 \leq i \leq n-j\}$ will complete a basis of W_j to a basis of W_{j+1}.

Now let K_j be the connected component of the identity of the kernel of the quotient representation $L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C})/W_j)$ induced from ψ. This quotient representation is just

ψ_{n-j}, so by Proposition 4.3.2,

$$K_j = \left\{ \begin{pmatrix} I_{n-j} & 0 \\ * & * \end{pmatrix} \in L_n(\mathbb{C}) \right\}.$$

Since $\dim_{\mathbb{C}}(K_j) = \binom{n+1}{2} - \binom{n-j+1}{2}$, $\dim_{\mathbb{C}}(K_{j+1}/K_j) = n-j$. Alternatively, $\{E_{n-j,i} | 1 \le i \le n-j\}$ will complete a basis of the Lie algebra of K_j to a basis of the Lie algebra of K_{j+1}.

Since $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$ for $j = 0, \dots, n$, we have verified (1) and (2) of Definition 3.2.6. Now fix $1 \le j \le n$, and consider the jth diagonal block $A_{j,j}$ of the matrix A of coefficients, as described in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of W_{n-j+1}/W_{n-j} and the Lie algebra of K_{n-j+1}/K_{n-j}, respectively. We may use $\frac{\partial}{\partial x_{1,j}}, \dots, \frac{\partial}{\partial x_{j,j}}$ and $E_{j,1}, \dots, E_{j,j}$ in the Lie algebra, respectively. By Proposition 4.3.6, the jth block is as claimed.

Finally, at $I \in \mathrm{Sym}_n(\mathbb{C})$, each $A_{j,j}$ is nonsingular. Thus each f_{n-j} is nonzero and we have a candidate block representation by Proposition 3.2.17. □

Example 4.4.2. For $n = 3$, the coefficient matrix A obtained using the orderings above is

$$\begin{pmatrix} 2x_{11} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{11} & x_{12} & 0 & 0 & 0 \\ 0 & 2x_{12} & 2x_{22} & 0 & 0 & 0 \\ x_{13} & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & x_{13} & x_{23} & x_{12} & x_{22} & x_{23} \\ 0 & 0 & 0 & 2x_{13} & 2x_{23} & 2x_{33} \end{pmatrix}.$$

The three diagonal blocks are as in the Proposition.

Let f_j, $j = 0, \dots, n-1$, be the coefficient determinants as in §3.2.

Corollary 4.4.3. *Let S be a generic $n \times n$ symmetric matrix. Let $S^{(k)}$ be the upper left $k \times k$ submatrix of S. Then for $j = 0, \dots, n-1$, $f_j = \det(S^{(n-j)})$ up to multiplication by a nonzero constant.*

Proof. f_j is equal to the determinant of $A_{n-j,n-j}$ by Proposition 3.2.10. By Proposition 4.4.1, $A_{n-j,n-j}$ is $S^{(n-j)}$ with the last row multiplied by 2. Thus up to a unit $f_j = \det(S^{(n-j)})$. □

The complement of the open orbit is given by the product of the f_j by Theorem 3.2.14. In particular, we have deduced the complex Cholesky factorization for symmetric matrices (Theorem 2.2.6) in a non-constructive manner.

4.4.2. Irreducibility of polynomials. To further show that the exceptional orbit variety is a free divisor we must show that $\prod_{j=0}^{n-1} f_j$ is square-free. We will show that each f_j is irreducible and then $\deg(f_j) = n - j$ implies that this product is square-free.

COROLLARY 4.4.4. *Each f_j is irreducible, $j = 0, \ldots, n-1$.*

PROOF. To simplify notation, let $g_j = \det(S^{(j)})$ for $j = 1, \ldots, n$, so that $f_j = g_{n-j}$. We shall show that each g_j is irreducible for $j = 1, \ldots, n$. We proceed by induction on j. $g_1 = x_{1,1}$ is irreducible. Now assume that g_j is irreducible. By expanding the determinant which defines g_{j+1} along the last column it is clear that $\frac{\partial g_{j+1}}{\partial x_{j+1,j+1}} = g_j$, and that g_j does not depend on $x_{j+1,j+1}$. Since

$$\begin{pmatrix} I_{j-1} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

is an example of a $(j+1) \times (j+1)$ symmetric matrix where g_j vanishes and g_{j+1} does not, g_{j+1} is irreducible by Corollary 3.3.3. $\square$

4.4.3. A free divisor. Since $f_0, \ldots, f_{n-1}$ are reduced and relatively prime, ρ is a block representation. Applying Theorem 3.2.14 shows that the product defines a free divisor.

THEOREM 4.4.5. *Let S be a generic $n \times n$ symmetric matrix. Let $S^{(k)}$ denote the upper left $k \times k$ submatrix of S. Then the hypersurface in $\mathrm{Sym}_n(\mathbb{C})$ defined by*

$$\prod_{k=1}^{n} \det(S^{(k)}) = 0$$

is a free divisor.

This result was previous obtained by different reasoning by [**GMNRS09**] (their Example 5.1) and [**BM06**] (their Example 2.1).

4.4.4. Tower of symmetric representations. In fact, this example gives a tower of free divisors in the sense of §3.2.8. Let $G_i = L_i(\mathbb{C})$, $V_i = \mathrm{Sym}_i(\mathbb{C})$, and define the representation

$\rho_i : L_i(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_i(\mathbb{C}))$ by $\rho_i(A)(M) = AMA^T$. Then the inclusion of spaces $V_i \hookrightarrow V_{i+1}$, $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$ and of groups $G_i \hookrightarrow G_{i+1}$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, gives an inclusion of representations by a simple calculation, e.g., (4.2).

Each of these representations is a block representation by Proposition 4.4.1, where the last block corresponds to the subspace

$$W_1^i = \left\{ \begin{pmatrix} 0_{i-1} & * \\ * & * \end{pmatrix} \right\} \subset \mathrm{Sym}_i(\mathbb{C})$$

and the Lie algebra of the subgroup

$$K_1^i = \left\{ \begin{pmatrix} I_{i-1} & 0 \\ * & * \end{pmatrix} \right\} \subset L_i(\mathbb{C}).$$

But $L_i(\mathbb{C})/K_1^i \simeq L_{i-1}(\mathbb{C})$ and $W_i/W_1^i \simeq W_{i-1}$, so that the inclusion and then quotient of representations is an isomorphism of representations. Thus we have a tower of free divisors.

4.4.5. Structure of the orbits. For the calculation of singular Milnor numbers in Chapter 6, it will be useful to know that the number of orbits is finite and hence the free divisors of Theorem 4.4.5 are holonomic. In Chapter 6 we will prove the following Theorem and use it to show that the free divisors satisfy the sharper condition that they are H-holonomic.

THEOREM 4.4.6. *Let $L_n(\mathbb{C})$ act on $\mathrm{Sym}_n(\mathbb{C})$ by*

$$L \cdot S = LSL^T.$$

Then for any $S \in \mathrm{Sym}_n(\mathbb{C})$, there is an $L \in L_n(\mathbb{C})$ so that the columns of $L \cdot S$ are either zero or a standard basis vector, so that (by symmetry) the rows and columns are either entirely zero or have a single nonzero entry which is a 1. We will call such a matrix a "normal form" for S.

4.5. The Cholesky-like representation for skew-symmetric matrices

Now consider the representation $\sigma : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C}))$ on the space of $n \times n$ skew-symmetric matrices defined by

$$\sigma(A)(M) = AMA^T.$$

We previously studied this representation in §4.3.2. Since $\dim_{\mathbb{C}}(L_n(\mathbb{C})) > \dim_{\mathbb{C}}(\mathrm{Sk}_n(\mathbb{C}))$, we must restrict σ to various subgroups to obtain equidimensional representations.

We will show that σ restricted to the group used in the Cholesky-type factorization of skew-symmetric matrices (Corollary 2.2.11) has the structure of a non-reduced block representation.

4.5.1. Candidate block representation. The results we need to develop a candidate block representation have already been shown in §4.3.2. In particular, we shall use the coordinates $\{x_{ij} | 1 \leq i < j \leq n\}$ defined there for $\mathrm{Sk}_n(\mathbb{C})$.

Recall that the groups V_n we used for the Cholesky-like factorization of skew-symmetric matrices (Theorem 2.2.10) consist of lower triangular matrices with 2×2 diagonal blocks which are multiples of the identity when n is even, and when n is odd have the same diagonal blocks with the lower right diagonal entry equal to 1. To obtain a block representation we will use a slightly different group which has smaller blocks and the same open orbit (a candidate block representation structure for V_n requires using transitions of the form $\mathrm{Sk}_n(\mathbb{C})_{2(k+1),2(k+1)} \subset \mathrm{Sk}_n(\mathbb{C})_{2k,2k}$; we can do better). Let G_n be like V_n but with 2×2 diagonal blocks of the form $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$. As before, let J_n denote the skew-symmetric matrix with 2×2 diagonal blocks equal to $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with a 0 in the lower right entry if n is odd.

These two groups have the same open orbit under σ.

Proposition 4.5.1. *Let $M \in \mathrm{Sk}_n(\mathbb{C})$. Then there exists a $A \in V_n$ so that $M = AJ_nA^T$ if and only if there exists a $B \in G_n$ so that $M = BJ_nB^T$.*

Proof. First note that

$$\begin{pmatrix} \gamma & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that if $D \in G_n$ is a diagonal matrix with 2×2 diagonal blocks of the form $\begin{pmatrix} \gamma & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$, then

$$DJ_nD = DJ_nD^T = J_n. \tag{4.12}$$

Note also that

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 \end{pmatrix}. \tag{4.13}$$

Fix $M \in \mathrm{Sk}_n(\mathbb{C})$. If there exists $A \in V_n$ with $M = AJ_nA^T$, then let D be a diagonal matrix where the ith diagonal block is $\begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$ if the ith diagonal block of A is $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Then by (4.12) and the hypothesis,

$$M = AJ_nA^T = ADJ_n(AD)^T;$$

by (4.13), $AD \in G_n$.

Conversely, if there exists a $B \in V_n$ with $M = BJ_nB^T$, then let D be a diagonal matrix where the ith diagonal block is $\begin{pmatrix} \gamma & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$ if the ith diagonal block of B is $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\gamma = \sqrt{\lambda}$. Then by (4.12) and the hypothesis,

$$M = BJ_nB^T = BDJ_n(BD)^T;$$

by (4.13), $BD \in V_n$. □

We now show that $\sigma|_{G_n}$ has the structure of a candidate block representation.

Proposition 4.5.2. *The representation $\sigma|_{G_n} : G_n \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C}))$ has the structure of a candidate block representation with $n-1$ diagonal blocks and subspaces*

$$W_j = \mathrm{Sk}_n(\mathbb{C})_{n-j,n-j}, \qquad j = 0, \ldots, n-1.$$

For $1 \le j \le n-1$ and appropriate choices of bases, the jth block is

$$\begin{pmatrix} 0 & x_{1,2} & \cdots & x_{1,j} \\ -x_{1,2} & 0 & \cdots & x_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{1,j} & -x_{2,j} & \cdots & 0 \end{pmatrix} \quad \textit{for } j \textit{ even} \tag{4.14}$$

and

$$(4.15) \qquad \begin{pmatrix} 0 & x_{1,2} & \cdots & x_{1,j-1} & x_{1,j+1} \\ -x_{1,2} & 0 & \cdots & x_{2,j-1} & x_{2,j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{1,j} & -x_{2,j} & \cdots & -x_{j-1,j} & x_{j,j+1} \end{pmatrix} \qquad \textit{for } j \textit{ odd.}$$

PROOF. We shall use the notation of Definition 3.2.6. By Proposition 4.3.7, each $\mathrm{Sk}_n(\mathbb{C})_{k,k}$ is $\sigma|_{G_n}$-invariant. Since $\mathrm{Sk}_n(\mathbb{C}) = \mathrm{Sk}_n(\mathbb{C})_{1,1}$, have a partial flag

$$\{0\} = W_0 = \mathrm{Sk}_n(\mathbb{C})_{n,n} \subset \cdots \subset W_{n-1} = \mathrm{Sk}_n(\mathbb{C})$$

of invariant subspaces. Since $\dim_{\mathbb{C}}(W_j) = \binom{n}{2} - \binom{n-j}{2}$, we have $\dim_{\mathbb{C}}(W_{j+1}/W_j) = n - j - 1$. Alternatively, $\{E_{i,n-j} - E_{n-j,i} \in \mathrm{Sk}_n(\mathbb{C}) | 1 \le i \le n-j-1\}$ will complete a basis of W_j to a basis of W_{j+1}.

Now let K_j be the connected component of the identity of the kernel of the quotient representation $G_n \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C})/W_j)$ induced from $\sigma|_{G_n}$. But the quotient representation is just $\sigma_{n-j}|_{G_n}$, so by Proposition 4.3.8 (considering the $j = n-1$, $j = n-2$ cases separately),

$$K_j = \left\{ \begin{pmatrix} I_{n-j} & 0 \\ * & * \end{pmatrix} \in G_n \right\}.$$

Let $\mathfrak{k}_j$ denote the Lie algebra of K_j. Because of the unusual structure of G_n, a basis for the Lie algebra of $\mathfrak{k}_{j+1}/\mathfrak{k}_j$ will depend on the parity of $n-j$. When $n-j$ is even, a basis is $E_{n-j,1}, \ldots, E_{n-j,n-j-2}, E_{n-j,n-j}$ (this includes the diagonal element), while when $n-j$ is odd, a basis is $E_{n-j,1}, \ldots, E_{n-j,n-j-1}$. In either case, $\dim_{\mathbb{C}}(K_{j+1}/K_j) = \dim_{\mathbb{C}}(\mathfrak{k}_{j+1}/\mathfrak{k}_j) = n-j-1$.

Since $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$, $j = 0, \ldots, n-1$, we have verified (1) and (2) of Definition 3.2.6. Fix $1 \le j \le n-1$, and consider the jth diagonal block $A_{j,j}$ of the matrix A of coefficients, as described in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of W_{n-j+1}/W_{n-j} and $\mathfrak{k}_{n-j+1}/\mathfrak{k}_{n-j}$, respectively. For j odd, we may use $\frac{\partial}{\partial x_{1,j}}, \ldots, \frac{\partial}{\partial x_{j-1,j}}$ and $E_{j,1}, \ldots, E_{j,j-1} \in \mathfrak{k}_{n-j+1}$, respectively. For j even, we may use $\frac{\partial}{\partial x_{1,j}}, \ldots, \frac{\partial}{\partial x_{j-1,j}}$ and $E_{j,1}, \ldots, E_{j,j-2}, E_{j,j} \in \mathfrak{k}_{n-j+1}$, respectively. In either case, the identification of $A_{j,j}$ follows from Proposition 4.3.12, omitting the appropriate column.

Finally, we show that each $A_{j,j}$ has a nonzero determinant. In fact, evaluated at $J_n \in \mathrm{Sk}_n(\mathbb{C})$, all $A_{j,j}$ are nonsingular: for even j, $A_{j,j} = J_j$, while for odd j, $A_{j,j} = \begin{pmatrix} J_{j-2} & 0 \\ 0 & 1 \end{pmatrix}$. Thus each f_{n-j} is nonzero and we have a candidate block representation by Proposition 3.2.17.

□

EXAMPLE 4.5.3. For $n = 4$, the coefficient matrix A obtained using the orderings above is

$$\begin{pmatrix} x_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{12} & 0 & 0 & 0 \\ x_{23} & -x_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{12} & x_{14} \\ x_{24} & 0 & 0 & -x_{12} & 0 & x_{24} \\ 0 & x_{14} & x_{24} & -x_{13} & -x_{23} & x_{34} \end{pmatrix}.$$

The three diagonal blocks are as in the Proposition.

Let f_j, $j = 0, \ldots, n-2$, be the coefficient determinants, as in §3.2.

COROLLARY 4.5.4. *Let S be a generic $n \times n$ skew-symmetric matrix. Let $M^{(k)}$ represent the upper left $k \times k$ submatrix of a matrix M. For $j = 0, \ldots, n-2$,*

$$f_j = \begin{cases} \mathrm{Pf}(S^{(n-j-1)})^2 & \text{for } n-j \text{ odd} \\ \mathrm{Pf}(S^{(n-j)})\mathrm{Pf}(S^{(n-j-2)}) & \text{for } n-j \text{ even} \end{cases},$$

up to multiplication by a nonzero constant.

PROOF. Let $M_{r,s}$ denote M with its rth row and sth column removed.

By Proposition 3.2.10, f_j is the determinant of $A_{n-1-j,n-1-j}$ as there are $n-1$ diagonal blocks. Proposition 4.5.2 identified these diagonal blocks. For $n-j$ odd, $n-j-1$ is even. By (4.14), $f_j = \det(S^{(n-j-1)})$, which equals the square of the corresponding Pfaffian. For $n-j$ even, $n-j-1$ is odd. By (4.15), $f_j = \det((S^{(n-j)})_{n-j,n-j-1})$. Then, it is a classical result (see, for example, [**MM30**], §406-415 or also Proposition 5.2.9) that this determinant factors as a product of the given Pfaffians. □

The complement of the open orbit is defined by the product of the f_j by Theorem 3.2.14. Hence, we have again proven a matrix factorization (equivalent to Theorem 2.2.10) in a non-constructive manner.

4.5.2. Irreducibility of polynomials. In light of Corollary 4.5.4, the product of the f_j will not be square-free. Since we will later need the fact that the Pfaffian of a generic matrix is irreducible, we will prove it now. We may also describe the number of components in the exceptional orbit variety.

Let S be a generic $n \times n$ skew-symmetric matrix. Let $g_j = \mathrm{Pf}(S^{(2j)})$.

COROLLARY 4.5.5. *Each g_j is irreducible.*

PROOF. $g_1 = x_{1,2}$ is clearly irreducible. Now assume that g_j is irreducible for some $j \geq 1$, with $2(j+1) \leq n$ (so that g_{j+1} is defined). By Proposition 5.2.7, $\frac{\partial g_{j+1}}{\partial x_{2j+1,2j+2}} = \pm g_j$. Define a $n \times n$ skew-symmetric matrix by

$$M_j = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & J_{2j} & 0 & \\ -1 & 0 & 0 & \\ & & & 0_{n-2j-2} \end{pmatrix}.$$

Then $g_j(M_j) = 0$, while $g_{j+1}(M_j) \neq 0$, so that g_{j+1} is irreducible by Corollary 3.3.3. $\square$

4.5.3. A free* divisor. Since the f_j are not reduced nor relatively prime, $\sigma|_{G_n}$ is only a non-reduced block representation. Applying Theorem 3.2.14 shows that the product defines a free* divisor.

THEOREM 4.5.6. *Let S be a generic $n \times n$ skew-symmetric matrix, where $n = 2\ell$ or $n = 2\ell+1$. Then the hypersurface in $\mathrm{Sk}_n(\mathbb{C})$ defined by*

$$\prod_{j=1}^{\ell} \mathrm{Pf}(S^{(2j)}) = 0$$

is a free divisor whose free* structure is generated by the vector fields associated to $\sigma|_{G_n}$.*

In the next Chapter, we will find a free divisor in the space of skew-symmetric matrices which includes the singular skew-symmetric matrices as an irreducible component.

4.6. The LU representation

We now consider the representation which corresponds to the complex LU factorization (Corollary 2.2.5). Let $N_m(\mathbb{C})$ be the group of invertible upper triangular matrices with ones on the diagonal, and let $G = L_n(\mathbb{C}) \times N_m(\mathbb{C})$. Then the representation of interest is $\rho|_G : G \to \mathrm{GL}(M(n, m, \mathbb{C}))$ defined by

$$\rho(A, B)(M) = AMB^{-1},$$

where ρ is the representation studied in §4.1. We shall be interested in the case where $m = n$ (the case addressed by the LU factorization) or $m = n + 1$. We will construct a non-reduced block representation to show that the exceptional orbit variety of $\rho|_G$ is a free* divisor.

4.6.1. Candidate block representation.

The results we need to develop a candidate block representation for $\rho|_G$ were shown in §4.1. For example, we will use the coordinates $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\}$ defined there for $M(n, m, \mathbb{C})$.

Proposition 4.6.1. *The representation $\rho : G \subset L_n(\mathbb{C}) \times T_n(\mathbb{C}) \to \mathrm{GL}(M(n, n, \mathbb{C}))$ has the structure of a candidate block representation with $2n - 1$ diagonal blocks and subspaces*

$$W_j = \begin{cases} M(n, n, \mathbb{C})_{n-\ell, n-\ell} & \text{if } j = 2\ell \\ M(n, n, \mathbb{C})_{n-\ell-1, n-\ell} & \text{if } j = 2\ell + 1 \end{cases}, \qquad j = 0, \ldots, 2n - 1.$$

Let S be a generic $n \times n$ matrix. For appropriate choices of bases, the jth diagonal block, $1 \leq j \leq 2n - 1$, is

$$-S^{(\ell)} \qquad \text{when } j = 2\ell$$

or

$$(S^{(\ell+1)})^T \qquad \text{when } j = 2\ell + 1.$$

Proof. We shall use the notation of Definition 3.2.6. By Proposition 4.1.1, all $M(n, n, \mathbb{C})_{k,l}$ are $\rho|_G$-invariant. Thus we have a partial flag

$$\{0\} = W_0 \subset M(n, n, \mathbb{C})_{n-1,n} \subset \cdots \subset M(n, n, \mathbb{C})_{1,1} \subset M(n, n, \mathbb{C})$$

of $\rho|_G$-invariant subspaces. Note that $\dim_{\mathbb{C}}(W_j) = n^2 - (n-\ell)^2$ for $j = 2\ell$ and $\dim_{\mathbb{C}}(W_j) = n^2 - (n-\ell)(n-\ell-1)$ for $j = 2\ell+1$, so that

$$\dim_{\mathbb{C}}(W_{j+1}/W_j) = \begin{cases} n-\ell & \text{if } j = 2\ell \\ n-\ell-1 & \text{if } j = 2\ell+1 \end{cases}.$$

Alternatively, $\{E_{n-\ell,i} \in M(n,n,\mathbb{C}) | 1 \leq i \leq n-\ell\}$ (respectively, $\{E_{i,n-\ell} \in M(n,n,\mathbb{C}) | 1 \leq i \leq n-\ell-1\}$) completes a basis of W_j to a basis of W_{j+1} when $j = 2\ell$ (respectively, $j = 2\ell+1$).

Let K_j be the connected component of the identity of the kernel of the quotient representation $G \to \mathrm{GL}(M(n,n,\mathbb{C})/W_j)$ induced from $\rho|_G$. But the quotient representation is just $\rho_{n-\ell,n-\ell}|_G$ (respectively, $\rho_{n-\ell-1,n-\ell}|_G$) when $j = 2\ell$ (respectively, $j = 2\ell+1$), so by Proposition 4.1.3,

$$K_j = \begin{cases} \left\{ \left(\begin{pmatrix} I_{n-\ell} & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} I_{n-\ell} & * \\ 0 & * \end{pmatrix} \right) \in G \right\} & \text{when } j = 2\ell \\ \left\{ \left(\begin{pmatrix} I_{n-\ell-1} & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} I_{n-\ell} & * \\ 0 & * \end{pmatrix} \right) \in G \right\} & \text{when } j = 2\ell+1 \end{cases}.$$

Since $\dim_{\mathbb{C}}(K_j) = n^2 - (n-\ell)^2$ for $j = 2\ell$ and $\dim_{\mathbb{C}}(K_j) = n^2 - (n-\ell)(n-\ell-1)$ for $j = 2\ell+1$, we have $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$ for all j.

We have thus verified (1) and (2) of Definition 3.2.6. Now fix $1 \leq j \leq 2n-1$, and consider the jth diagonal block $A_{j,j}$ of the matrix A of coefficients, as in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of $W_{2n-1-j+1}/W_{2n-1-j}$ and of the Lie algebra of $K_{2n-1-j+1}/K_{2n-1-j}$, respectively. For $j = 2\ell$, we may use $\frac{\partial}{\partial x_{1,\ell+1}}, \ldots, \frac{\partial}{\partial x_{\ell,\ell+1}}$ and $(0, E_{1,\ell+1}), \ldots, (0, E_{\ell,\ell+1})$, respectively. Proposition 4.1.8 then identifies $A_{j,j}$ as $-S^{(\ell)}$. For $j = 2\ell+1$, we may use $\frac{\partial}{\partial x_{\ell+1,1}}, \ldots, \frac{\partial}{\partial x_{\ell+1,\ell+1}}$ and $(E_{\ell+1,1}, 0), \ldots, (E_{\ell+1,\ell+1}, 0)$, respectively. Proposition 4.1.6 then identifies $A_{j,j}$ as $(S^{(\ell+1)})^T$.

Finally, note that each $A_{j,j}$ is nonsingular at $I \in M(n,n,\mathbb{C})$. Thus each f_{2n-1-j} is nonzero and we have a candidate block representation by Proposition 3.2.17. □

Example 4.6.2. For $n = 3$, the coefficient matrix A obtained using the orderings above is

$$\begin{pmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{12} & -x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{11} & x_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_{21} & x_{12} & x_{22} & 0 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & 0 & 0 & -x_{11} & -x_{12} & 0 & 0 & 0 \\ 0 & 0 & x_{13} & x_{23} & -x_{21} & -x_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{11} & x_{21} & x_{31} \\ 0 & -x_{31} & 0 & 0 & 0 & 0 & x_{12} & x_{22} & x_{32} \\ 0 & 0 & 0 & 0 & -x_{31} & -x_{32} & x_{13} & x_{23} & x_{33} \end{pmatrix},$$

with five diagonal blocks as given by the Proposition.

Let f_j, $j = 0, \ldots, 2n-2$, be the coefficient determinants as in §3.2.

Corollary 4.6.3. *Let S be the generic $n \times n$ matrix. Let $S^{(k)}$ represent the upper left $k \times k$ submatrix of S. Then for $j = 0, \ldots, 2n-2$,*

$$f_j = \begin{cases} \det((S^{(n-\ell)})^T) & \text{if } j = 2\ell \\ \det(-S^{(n-\ell-1)}) & \text{if } j = 2\ell+1 \end{cases},$$

so that $f_j = \det(S^{(\lfloor (2n-j)/2 \rfloor)})$ up to multiplication by a nonzero constant, where $\lfloor \cdot \rfloor$ is the floor function.

Proof. f_j is equal to the determinant of $A_{2n-1-j,2n-1-j}$ by Proposition 3.2.10. By Proposition 4.6.1, $A_{2n-1-j,2n-1-j}$ is as stated above when $j = 2\ell$ and $j = 2\ell+1$. Up to multiplication by a nonzero constant, neither the transpose nor the negative will affect the determinant. Finally, one may check that for both cases, the formula given in terms of the floor function is correct. $\square$

The complement of the open orbit is defined by the product of the f_j by Theorem 3.2.14. By splitting the product into even and odd terms, using the Corollary, and reindexing, we have

$$\begin{aligned}\prod_{j=0}^{2n-2} f_j &= \left(\prod_{\ell=0}^{n-1} f_{2\ell}\right)\cdot\left(\prod_{\ell=0}^{n-2} f_{2\ell+1}\right)\\ &= \left(\prod_{\ell=0}^{n-1} \det(S^{(n-\ell)})\right)\cdot\left(\prod_{\ell=0}^{n-2} \det(S^{(n-\ell-1)})\right)\\ &= \left(\prod_{k=1}^{n} \det(S^{(k)})\right)\cdot\left(\prod_{k=1}^{n-1} \det(S^{(k)})\right)\\ &= \left(\prod_{k=1}^{n-1} \det(S^{(k)})^2\right)\cdot \det(S^{(n)}).\end{aligned}$$

In particular, this recovers the LU factorization (Theorem 2.2.1) in a non-constructive manner: $A \in M(n,n,\mathbb{C})$ lies in the open orbit (and can be written as $A = BC^T$ for $(B,C) \in G$) if and only if the above product is not zero at A.

4.6.2. Irreducibility of polynomials. We will show that each f_j is irreducible. For ease of notation, let S be the generic $n \times n$ matrix and let $g_k = \det(S^{(k)})$ for $k = 1, \ldots, n$.

Corollary 4.6.4. *Each g_k is irreducible, $k = 1, \ldots, n$.*

Proof. We proceed by induction on k. For $k = 1$, $g_1 = x_{11}$ is irreducible. Assume that g_k is irreducible. By expanding the determinant along the last column we see that $\frac{\partial g_{k+1}}{x_{k+1,k+1}} = g_k$. g_k is irreducible by induction hypothesis and by definition does not depend on $x_{k+1,k+1}$. Since

$$A = \begin{pmatrix} I_{k-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 0_{n-k-1} \end{pmatrix}$$

is an $n \times n$ matrix where $g_k(A) = 0$ and $g_{k+1}(A) \neq 0$, g_{k+1} is irreducible by Corollary 3.3.3. □

4.6.3. A free* divisor. Since $f_0, \ldots, f_{2n-2}$ are reduced but not relatively prime, ρ is a non-reduced block representation. Applying Theorem 3.2.14 shows that the product defines a free* divisor.

THEOREM 4.6.5. *Let S be a generic $n \times n$ matrix. Let $S^{(k)}$ denote the upper left $k \times k$ submatrix of S. Then the hypersurface in $M(n,n,\mathbb{C})$ defined by*

$$\prod_{k=1}^{n} \det(S^{(k)}) = 0$$

is a free divisor whose free* structure is generated by the vector fields associated to the LU representation.*

In the next Chapter we find a family of linear free divisors on $M(n,n,\mathbb{C})$ which includes the singular matrices.

REMARK 4.6.6. Each time the size of the matrices increases by one, we get two additional blocks in the non-reduced block representation for $\rho|_G$. By Proposition 3.2.18 we may remove the last block and obtain a non-reduced block representation for $L_n(\mathbb{C}) \times N_{n+1}(\mathbb{C})$ acting on $M(n,n+1,\mathbb{C})$ by $(A,B) \cdot M = AMB^{-1}$. The exceptional orbit variety is defined by $\prod_{i=1}^{n} \det(S^{(i)}) = 0$, where S is the generic $n \times (n+1)$ matrix. Thus we obtain an LU-like factorization for $n \times (n+1)$ matrices.

More generally, there is an LU-like factorization for $n \times m$ matrices, $n \leq m$, where the group

$$G = \left\{ \left(L, \begin{pmatrix} U & * \\ 0 & I_{m-n} \end{pmatrix} \right) \middle| L \in L_n(\mathbb{C}), U \in N_n(\mathbb{C}) \right\}$$

acts on $M(n,m,\mathbb{C})$ by $(A,B) \cdot M = AMB^{-1}$. The exceptional orbit variety is defined by $\prod_{i=1}^{n} \det(S^{(i)}) = 0$, where S is the generic $n \times m$ matrix. To find a group element which takes M to $\begin{pmatrix} I_n & 0 \end{pmatrix} \in M(n,m,\mathbb{C})$, the LU decomposition is used to determine L and U based on the leftmost n columns of M, and the rest of the group element is determined by column-reducing M. Conversely, if there is such a group element then there is an LU decomposition for the leftmost n columns of M.

Such a representation has a non-reduced block representation: we begin with subspaces

$$\{0\} = M(n,m,\mathbb{C})_{n,m} \subset M(n,m,\mathbb{C})_{n,m-1} \subset \cdots \subset M(n,m,\mathbb{C})_{n,n},$$

and then add the subspaces used for the LU block representation.

CHAPTER 5

Free divisors for the general and skew-symmetric matrices

In the previous Chapter, we showed that the representations corresponding to the LU factorization for general $n \times n$ matrices and the Cholesky-type factorization for skew-symmetric matrices only yield free* divisors (Theorems 4.6.5, 4.5.6). The computation of singular Milnor numbers of nonlinear sections of free* divisors is complicated by virtual singularities. In this Chapter we identify infinite towers of free divisors which arise from modifications of these representations, and will later be used to compute singular Milnor numbers.

In §5.1, we modify the group used for the LU factorization (Corollary 2.2.5) for general $n \times n$ matrices to obtain a block representation whose exceptional orbit variety is a linear free divisor (Theorem 5.1.4). We also show that there is a related block representation on the space of $n \times (n + 1)$ matrices which also yields a linear free divisor.

In contrast, we have been unable to modify the group used for the representation corresponding to the Cholesky-like factorization of skew-symmetric matrices (Corollary 2.2.11) to obtain a representation whose exceptional orbit variety is a linear free divisor. Even for the 4×4 case we conjecture that it is impossible. Instead, we introduce in §5.2 a very different approach by replacing the representation of the solvable group with a representation of an infinite-dimensional "solvable" holomorphic Lie algebra. We obtain this infinite-dimensional Lie algebra by extending a finite-dimensional solvable Lie algebra using generators which are nonlinear Pfaffian vector fields. We identify where these vector fields are linearly dependent in exactly the same way as with the method of block representations, and then apply Saito's criterion. The resulting free divisors (Theorem 5.2.21) are remarkably similar in form to our other free divisors.

In §5.3, we extend the idea of the block representations beyond solvable groups. We prove several criteria for the expansion of an equidimensional representation, we which use to obtain a number of new infinite families of linear free divisors which are a mix between the solvable and reductive.

5.1. A modified LU representation for $n \times n$ and $n \times (n+1)$ matrices

We will now modify the group used for the LU factorization to obtain a block representation, while using the same invariant subspaces as before. Define the subgroup

$$H_n(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| A \in T_{n-1}(\mathbb{C}) \right\}$$

of $T_n(\mathbb{C})$, the upper triangular invertible matrices. Define the representation of $G = L_n(\mathbb{C}) \times H_n(\mathbb{C})$ on $M(n,n,\mathbb{C})$ by

$$(A,B) \cdot M = AMB^{-1}.$$

This is the restriction to G of the representation ρ studied in §4.1. The intersection of the subspace $\left\{ \begin{pmatrix} 0_k & * \\ 0 & * \end{pmatrix} \right\}$ in the Lie algebra of $T_n(\mathbb{C})$ with either the Lie algebra of $H_n(\mathbb{C})$ or the Lie algebra of the group of upper triangular invertible matrices with ones on the diagonal have the same dimensions for all k. We thus might expect $\rho|_G$ also to have a candidate block representation, provided it has an open orbit. We shall next establish that this is the case.

5.1.1. Candidate block representation. The results we need to develop a candidate block representation for $\rho|_G$ were shown in §4.1. We shall use the coordinates $\{x_{ij} | 1 \leq i,j \leq n\}$ defined there for $M(n,n,\mathbb{C})$.

Proposition 5.1.1. *The representation $\rho|_G : G \to \mathrm{GL}(M(n,n,\mathbb{C}))$ has the structure of a candidate block representation with $2n-1$ diagonal blocks and subspaces*

$$W_j = \begin{cases} M(n,n,\mathbb{C})_{n-\ell,n-\ell} & \text{if } j = 2\ell \\ M(n,n,\mathbb{C})_{n-\ell-1,n-\ell} & \text{if } j = 2\ell+1 \end{cases} \qquad j = 0,\ldots,2n-1.$$

Let S be the generic $n \times n$ matrix. Let $S^{(k)}$ denote the upper left $k \times k$ submatrix of S and let $S^{(1,k)}$ denote the upper left $k \times k$ submatrix of S', the matrix obtained by deleting the first column of S. For appropriate choices of bases, the jth diagonal block, $1 \leq j \leq 2n-1$, is given by

$$-S^{(1,\ell)} \qquad \text{when } j = 2\ell$$

and

$$(S^{(\ell+1)})^T \qquad \text{when } j = 2\ell+1.$$

Proof. We shall use the notation of Definition 3.2.6. By Proposition 4.1.1, all $M(n,n,\mathbb{C})_{k,l}$ are $\rho|_G$-invariant. Thus we have a partial flag

$$\{0\} = W_0 \subset M(n,n,\mathbb{C})_{n-1,n} \subset \cdots \subset M(n,n,\mathbb{C})_{1,1} \subset M(n,n,\mathbb{C})$$

of $\rho|_G$-invariant subspaces. Note that $\dim_{\mathbb{C}}(W_j) = n^2 - (n-\ell)^2$ for $j = 2\ell$ and $\dim_{\mathbb{C}}(W_j) = n^2 - (n-\ell)(n-\ell-1)$ for $j = 2\ell+1$, so that

$$\dim_{\mathbb{C}}(W_{j+1}/W_j) = \begin{cases} n-\ell & \text{if } j = 2\ell \\ n-\ell-1 & \text{if } j = 2\ell+1 \end{cases}.$$

Let K_j be the connected component of the identity of the kernel of the quotient representation $G \to \mathrm{GL}(M(n,n,\mathbb{C})/W_j)$ induced from $\rho|_G$. This quotient representation is just $\rho_{n-\ell,n-\ell}|_G$ (respectively, $\rho_{n-\ell-1,n-\ell}|_G$) when $j = 2\ell$ (respectively, $j = 2\ell+1$), so by Proposition 4.1.3,

$$K_j = \begin{cases} \left\{ \left(\begin{pmatrix} I_{n-\ell} & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-\ell-1} & * \\ 0 & 0 & * \end{pmatrix} \right) \in G \right\} & \text{when } j = 2\ell \\ \left\{ \left(\begin{pmatrix} I_{n-\ell-1} & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-\ell-1} & * \\ 0 & 0 & * \end{pmatrix} \right) \in G \right\} & \text{when } j = 2\ell+1 \end{cases}.$$

Since $\dim_{\mathbb{C}}(K_j) = n^2 - (n-\ell)^2$ for $j = 2\ell$ and $\dim_{\mathbb{C}}(K_j) = n^2 - (n-\ell)(n-\ell-1)$ for $j = 2\ell+1$, we have $\dim_{\mathbb{C}}(K_j) = \dim_{\mathbb{C}}(W_j)$ for all j.

We have thus verified (1) and (2) of Definition 3.2.6. Now fix $1 \le j \le 2n-1$, and consider the jth diagonal block $A_{j,j}$ of the matrix A of coefficients, as in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of $W_{2n-1-j+1}/W_{2n-1-j}$ and of the Lie algebra of $K_{2n-1-j+1}/K_{2n-1-j}$, respectively. For $j = 2\ell$, we may use $\frac{\partial}{\partial x_{1,\ell+1}}, \ldots, \frac{\partial}{\partial x_{\ell,\ell+1}}$ and $(0, E_{2,\ell+1}), \ldots, (0, E_{\ell+1,\ell+1})$, respectively. Proposition 4.1.8 then identifies $A_{j,j}$ as $-S^{(1,\ell)}$. For $j = 2\ell+1$, we may use $\frac{\partial}{\partial x_{\ell+1,1}}, \ldots, \frac{\partial}{\partial x_{\ell+1,\ell+1}}$ and $(E_{\ell+1,1}, 0), \ldots, (E_{\ell+1,\ell+1}, 0)$, respectively. Proposition 4.1.6 then identifies $A_{j,j}$ as $(S^{(\ell+1)})^T$.

Finally, for j odd, $A_{j,j}$ is nonsingular at $I \in M(n,n,\mathbb{C})$; for j even, $A_{j,j}$ is nonsingular at $\begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix} \in M(n,n,\mathbb{C})$. Thus each f_{2n-1-j} is nonzero and we have a candidate block representation by Proposition 3.2.17. □

EXAMPLE 5.1.2. For $n = 3$, the coefficient matrix A obtained using the orderings above is

$$\begin{pmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{12} & -x_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{11} & x_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_{22} & x_{12} & x_{22} & 0 & 0 & 0 & 0 & 0 \\ x_{13} & 0 & 0 & 0 & -x_{12} & -x_{13} & 0 & 0 & 0 \\ 0 & 0 & x_{13} & x_{23} & -x_{22} & -x_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{11} & x_{21} & x_{31} \\ 0 & -x_{32} & 0 & 0 & 0 & 0 & x_{12} & x_{22} & x_{32} \\ 0 & 0 & 0 & 0 & -x_{32} & x_{33} & x_{13} & x_{23} & x_{33} \end{pmatrix}.$$

The five diagonal blocks are as in the Proposition.

Let f_j, $j = 0, \ldots, 2n-2$, be the coefficient determinants as in §3.2.

COROLLARY 5.1.3. *Let S be the generic $n \times n$ matrix. Let $S^{(k)}$ denote the upper left $k \times k$ submatrix of S and let $S^{(1,k)}$ denote the upper left $k \times k$ submatrix of S', the matrix obtained by deleting the first column of S. Then for $j = 0, \ldots, 2n-2$,*

$$f_j = \begin{cases} \det(S^{(n-\ell)}) & \textit{if } j = 2\ell \\ \det(S^{(1,n-\ell-1)}) & \textit{if } j = 2\ell+1 \end{cases},$$

up to multiplication by a nonzero constant.

PROOF. f_j is equal to the determinant of $A_{2n-1-j,2n-1-j}$ by Proposition 3.2.10. By Proposition 5.1.1, $A_{2n-1-j,2n-1-j}$ is $(S^{(n-\ell)})^T$ when $j = 2\ell$ and $-S^{(1,n-\ell-1)}$ when $j = 2\ell+1$. Up to multiplication by a nonzero constant, neither the transpose nor the negative affect the determinant. □

5.1.2. Irreducibility of polynomials. The complement of the open orbit is defined by the product of the f_j by Theorem 3.2.14. Each f_j is the determinant of a generic matrix and thus irreducible by Corollary 4.6.4. In fact, these functions are distinct: only each $f_{2\ell-1}$ and $f_{2\ell}$ have the same degree, $n-\ell$, but $f_{2\ell-1}(I_n) = 0$ while $f_{2\ell}(I_n) = 1$.

5.1.3. A free divisor for the $n \times n$ matrices. As each f_j is reduced and $\{f_j\}$ are relatively prime, we have a block representation and thus a free divisor.

THEOREM 5.1.4. *Let S be the generic $n \times n$ matrix. Let $S^{(k)}$ denote the upper left $k \times k$ submatrix of S and let $S^{(1,k)}$ denote the upper left $k \times k$ submatrix of S', the matrix obtained by deleting the first column of S. Then the hypersurface in $M(n, n, \mathbb{C})$ defined by*

$$\left(\prod_{k=1}^{n} \det(S^{(k)})\right) \cdot \left(\prod_{k=1}^{n-1} \det(S^{(1,k)})\right) = 0$$

is a linear free divisor.

PROOF. First we need to show that the product of the f_j's is the function stated in the Theorem. But by Corollary 5.1.3,

$$\begin{aligned}
\prod_{j=0}^{2n-2} f_j &= \left(\prod_{\ell=0}^{n-1} f_{2\ell}\right) \cdot \left(\prod_{\ell=0}^{n-2} f_{2\ell+1}\right) \\
&= \left(\prod_{\ell=0}^{n-1} \det(S^{(n-\ell)})\right) \cdot \left(\prod_{\ell=0}^{n-2} \det(S^{(1,n-\ell-1)})\right) \\
&= \left(\prod_{k=1}^{n} \det(S^{(k)})\right) \cdot \left(\prod_{k=1}^{n-1} \det(S^{(1,k)})\right).
\end{aligned}$$

Since all terms are irreducible and relatively prime, by Theorem 3.2.14 we have a linear free divisor. □

5.1.4. A free divisor for the $n \times (n+1)$ matrices. Each time n increases we add two blocks to the block representation of Proposition 5.1.1. We may quotient the block representation of $L_{n+1}(\mathbb{C}) \times H_{n+1}(\mathbb{C}) \to \mathrm{GL}(M(n+1, n+1, \mathbb{C}))$ defined by $(A, B) \cdot M = AMB^{-1}$ by its last block using Proposition 3.2.18 to obtain a block representation (and thus a free divisor) on $M(n, n+1, \mathbb{C})$. The underlying representation is isomorphic to $L_{n+1}(\mathbb{C}) \times H_n(\mathbb{C})$ acting on $M(n, n+1, \mathbb{C})$ by $(A, B) \cdot M = AMB^{-1}$.

THEOREM 5.1.5. *Let S be a generic $n \times (n+1)$ matrix. Let $S^{(k)}$ denote the upper left $k \times k$ submatrix of S and let $S^{(1,k)}$ denote the upper left $k \times k$ submatrix of S', the matrix obtained by deleting the first column of S. Then the hypersurface in $M(n, n+1, \mathbb{C})$ defined by*

$$\left(\prod_{k=1}^{n} \det(S^{(k)})\right) \cdot \left(\prod_{k=1}^{n} \det(S^{(1,k)})\right) = 0$$

is a linear free divisor.

Proof. The representation $L_{n+1}(\mathbb{C}) \times H_{n+1}(\mathbb{C}) \to \mathrm{GL}(M(n+1,n+1,\mathbb{C}))$, $(A,B)\cdot M = AMB^{-1}$, has the structure of a block representation by Proposition 5.1.1. The last block corresponds to the subspace $M(n+1,n+1,\mathbb{C})_{n,n+1}$ and kernel

$$\left\{ \left(\begin{pmatrix} I_n & 0 \\ * & * \end{pmatrix}, I_{n+1} \right) \in L_{n+1}(\mathbb{C}) \times H_{n+1}(\mathbb{C}) \right\}.$$

The last block's coefficient determinant equals the determinant of the generic $(n+1)\times(n+1)$ matrix, up to multiplication by a nonzero constant.

We may quotient the block representation by Proposition 3.2.18 to obtain a candidate block representation for a representation isomorphic to the representation claimed. The exceptional orbit variety will be the same as that given by Theorem 5.1.4 on $M(n+1,n+1,\mathbb{C})$, except without the determinant of the generic $(n+1)\times(n+1)$ matrix; this is exactly the product claimed. As this product is reduced, we have a block representation, so that the exceptional orbit is a free divisor by Theorem 3.2.14. $\square$

5.1.5. A modified LU factorization for $n \times n$ and $n \times (n+1)$ matrices. That these representations have open orbits implies that we have matrix factorization theorems. Define the $n \times n$ matrix

$$J_{n,n} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 0 & & & \\ 0 & 0 & 1 & 1 & & & \\ 0 & 0 & 0 & 1 & & & \\ \vdots & & & & \ddots & & \vdots \\ & & & & & 1 & 1 \\ 0 & & & & \cdots & 0 & 1 \end{pmatrix},$$

perhaps the simplest matrix where no f_j vanish: each $(J_{n,n})^{(k)}$ is upper triangular with ones on the diagonal, while each $(J_{n,n})^{(1,k)}$ is lower triangular with ones on the diagonal. Thus $J_{n,n}$ is in the open orbit of this representation.

Theorem 5.1.6. *Let M be a $n \times n$ complex matrix. Then $\det(M^{(k)}) \neq 0$ for $k = 1,\ldots,n$ and $\det(M^{(1,k)}) \neq 0$ for $k = 1,\ldots,n-1$ if and only if there exists an $n \times n$ invertible lower triangular matrix L and a $n \times n$ upper triangular matrix U whose first row equals $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ so that $M = LJ_{n,n}U$.*

Proof. Proposition 5.1.1 defines a block representation for a representation on $M(n,n,\mathbb{C})$ whose exceptional orbit variety is defined by the product in Theorem 5.1.4. The open orbit of the representation is the complement of this hypersurface and thus contains $J_{n,n}$. $\square$

We obtain a similar factorization theorem corresponding to the representation on the space of $n \times (n+1)$ matrices. Define the $n \times (n+1)$ matrix

$$J_{n,n+1} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 1 & 1 & 0 & & & & \\ 0 & 0 & 1 & 1 & & & & \\ \vdots & & & & \ddots & & & \vdots \\ & & & & & 1 & 1 & 0 \\ 0 & & & & \cdots & 0 & 1 & 1 \end{pmatrix},$$

an element of the open orbit of this representation.

Theorem 5.1.7. *Let M be a $n \times (n+1)$ complex matrix. Then $\det(M^{(k)}) \neq 0$ for $k = 1, \ldots, n$ and $\det(M^{(1,k)}) \neq 0$ for $k = 1, \ldots, n$ if and only if there exists an $n \times n$ invertible lower triangular matrix L and a $(n+1) \times (n+1)$ upper triangular matrix U whose first row equals $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ so that $M = LJ_{n,n+1}U$.*

Proof. The proof of Theorem 5.1.5 defines a block representation for a representation on $M(n, n+1, \mathbb{C})$ whose exceptional orbit variety is defined by the product in the same Theorem. The open orbit of the representation is the complement of this hypersurface and thus contains $J_{n,n+1}$. $\square$

5.1.6. Structure of the orbits. For each of the modified LU factorizations for $n \times n$ and $n \times (n+1)$ matrices, we would also like to know that the number of orbits is finite. This will imply that the free divisors of Theorems 5.1.4 and 5.1.5 are holonomic. The next Theorem classifies the orbits of our representations. It will be proven and used in Chapter 6 to show that these free divisors satisfy the sharper condition that they are H-holonomic.

We first define what we mean by a "normal form."

Definition 5.1.8. Let $N = \begin{pmatrix} N_1 & N_2 \end{pmatrix}$ be a $n \times m$ matrix with $m = n$ or $n+1$, and N_1 a column vector. Let

$$A = \{i | \text{the } i\text{th row of } N_1 \text{ is nonzero}\} \qquad \text{and} \qquad B = \{i | \text{the } i\text{th row of } N_2 \text{ is nonzero}\}.$$

We say N is a *normal form for the $n \times n$ or $n \times (n+1)$ matrices* if:

(1) All entries of N are either 0 or 1.

(2) Each column and row of N_2 may have at most one nonzero entry.

(3) $|A \setminus B| \leq 1$, i.e., there may be at most one row where N_1 is nonzero and N_2 is zero. Furthermore, if $|A \setminus B| = 1$, then the only element of $A \setminus B$ equals $\max(A)$. Thus, the row where N_1 is nonzero and N_2 is zero must be the last row where N_1 is nonzero.

(4) The rows of N_2 corresponding to $A \cap B$ all take the form e_j^T, for some $1 \leq j \leq m-1$. Suppose that $A \cap B = \{i_1, \ldots, i_l\}$, with $i_1 < \cdots < i_l$, and let the i_jth row of N_2 equal $e_{k_j}^T$. Then we require that

$$k_1 < k_2 < \cdots < k_l,$$

so that these particular rows of N_2 are "increasing".

Example 5.1.9. Here is a listing of the 24 normal forms in the 2×3 case:

N_1	N_2
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note in particular that $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is not a normal form.

Each orbit of this representation contains a normal form.

THEOREM 5.1.10. *Let $G = L_n(\mathbb{C}) \times H_m(\mathbb{C})$ act on $M(n, m, \mathbb{C})$, where $m = n$ or $n + 1$, by*

$$(L, U) \cdot M = LMU^{-1}.$$

Then for all $M \in M(n, m, \mathbb{C})$ there exists an element $g \in G$ so that $g \cdot M$ is a normal form according to Definition 5.1.8

5.2. A free divisor for skew-symmetric matrices

We now pursue a free divisor on the space of $n \times n$ skew-symmetric matrices which, when n is even, includes the Pfaffian zero matrices as a component. To obtain such a free divisor, the determinant of our coefficient matrix cannot have the determinant of a generic skew-symmetric matrix as a factor; such a determinant is either non-reduced or zero. As our previous examples of free divisors incorporate a repeated product of determinants of upper-left submatrices of a generic matrix, any example we obtain in the skew-symmetric case must be substantially different.

Our initial strategy was to find a connected complex algebraic subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ of dimension $\binom{n}{2}$ so that $G \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C}))$, $A \cdot M = AMA^T$, has an open orbit and its exceptional orbit variety is a linear free divisor. Even for $n = 4$, however, we have neither been able to find such a group nor show that one does not exist. We do not even have a strategy for proving that such a subgroup does not exist.

Instead, we use a different approach. We identify a solvable subgroup G of $L_n(\mathbb{C})$, and obtain a representation of its solvable Lie algebra of vector fields on $\mathrm{Sk}_n(\mathbb{C})$. It will have $\dim_{\mathbb{C}}(G) < \dim_{\mathbb{C}}(\mathrm{Sk}_n(\mathbb{C}))$. Then, using *Pfaffian vector fields*, we define an extension to an infinite dimensional holomorphic Lie algebra of vector fields on $\mathrm{Sk}_n(\mathbb{C})$ which has $\dim_{\mathbb{C}}(\mathrm{Sk}_n(\mathbb{C}))$ generators. We then apply Saito's criterion (Theorem 3.1.1) to show that we have a free divisor.

5.2.1. Properties of skew-symmetric matrices and their Pfaffians. We first review a number of useful properties of Pfaffians and determinants of certain submatrices of skew-symmetric matrices. We will always view Pfaffians as polynomial functions on $\mathrm{Sk}_n(\mathbb{C})$ using the standard coordinates.

5.2.1.1. *Pfaffian identities.* It is useful to have a source of identities involving Pfaffians. First we define some notation.

Notation 5.2.1. For a generic $n \times n$ skew-symmetric matrix and any non-empty subset $I \subseteq \{1, \ldots, n\}$, let $S(I)$ denote the Pfaffian of the skew-symmetric submatrix obtained by deleting all rows and columns not indexed by I. By convention, let $S(\emptyset) = 1$.

We denote $S(\{1, \ldots, n\})$ by the standard notation Pf. Note that if $|I|$ is odd, then $S(I) = 0$ as the Pfaffian of a $k \times k$ skew-symmetric matrix with k odd is 0.

Notation 5.2.2. If I's contents are described by a range of variables, then for conciseness we often omit the braces or commas typical of set notation. For example, if $a = 1$, $b = 4$, $c = 6$, $d = 8$, and $I = \{1, 2, 3, 4, 6, 8\} = \{a, \ldots, b, c, d\}$, we write $S(I)$ as $S(a \cdots bcd)$. We will always write these numbers in increasing order. If $e = 3$, we may write $S(a \cdots \hat{e} \cdots bcd)$ for $S(\{1, 2, 4, 6, 8\})$.

Example 5.2.3. On $\mathrm{Sk}_4(\mathbb{C})$,

$$
\begin{aligned}
S(\{1,2\}) &= x_{1,2} \\
S(\{4,3\}) &= x_{3,4} \\
S(\{1,2,3\}) &= 0 \\
S(\{1,2,3,4\}) &= \mathrm{Pf} = x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14}.
\end{aligned}
$$

Dress and Wenzel were able to prove a large collection of useful Pfaffian identities.

Theorem 5.2.4 ([**Wen93**], [**DW95**]). *Fix a space of skew-symmetric matrices* $\mathrm{Sk}_n(\mathbb{C})$*, and let* $M = \{1, \ldots, n\}$*. Let* $I_1, I_2 \subseteq M$ *be subsets of odd cardinality. Let* $i_1, \ldots, i_l \in M$ *be elements with* $i_1 < i_2 < \cdots < i_l$ *and* $\{i_1, i_2, \ldots, i_l\} = I_1 \Delta I_2$ *(symmetric difference). Then we have the following Pfaffian identity:*

$$
\sum_{\tau=1}^{l} (-1)^{\tau} S(I_1 \Delta \{i_\tau\}) S(I_2 \Delta \{i_\tau\}) = 0. \tag{5.1}
$$

Remark 5.2.5. As any i_τ is either an element of I_1 or is not,

$$
|I_1 \Delta \{i_\tau\}| = |I_1| \pm 1.
$$

Thus Theorem 5.2.4 also holds if either I_1 or I_2 has even cardinality, but in that case all terms of (5.1) vanish.

Example 5.2.6. Let $n = 4$, $I_1 = \{1, 2, 3\}$, and $I_2 = \{4\}$. Then (5.1) gives

$$-S(\{2,3\})S(\{1,4\}) + S(\{1,3\})S(\{2,4\}) - S(\{1,2\})S(\{3,4\}) + S(1 \cdots 4) = 0,$$

i.e.,

$$-x_{23}x_{14} + x_{13}x_{24} - x_{12}x_{34} + (x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14}) = 0.$$

5.2.1.2. *Derivative of the Pfaffian.* As a useful application, Theorem 5.2.4 may be used to calculate the derivative of the Pfaffian.

Proposition 5.2.7. *Let $1 \leq r < s \leq n$, so $x_{r,s}$ is a coordinate on $\mathrm{Sk}_n(\mathbb{C})$. Then*

$$\frac{\partial S(1 \cdots n)}{\partial x_{r,s}} = (-1)^{r+s+1} S(1 \cdots \hat{r} \cdots \hat{s} \cdots n).$$

Proof. Let $M = \{1, \ldots, n\}$. Let $I_1 = M \setminus \{s\}$ and $I_2 = \{s\}$, so $I_1 \Delta I_2 = M$. Applying Theorem 5.2.4, we obtain

$$\sum_{1 \leq \tau < s} (-1)^{\tau} S(1 \cdots \hat{\tau} \cdots \hat{s} \cdots n) S(\{\tau, s\}) + (-1)^s S(M) S(\emptyset) + \sum_{s < \tau \leq n} (-1)^{\tau} S(1 \cdots \hat{s} \cdots \hat{\tau} \cdots n) S(\{s, \tau\}) = 0. \tag{5.2}$$

Examining the first sum shows that only the $\tau = r$ term will depend on $x_{r,s}$, while in the second sum no terms will depend on $x_{r,s}$ (we have deleted either the rth row or the sth column). Differentiating (5.2) with respect to $x_{r,s}$ gives

$$(-1)^r S(\cdots \hat{r} \cdots \hat{s} \cdots n) \frac{\partial S(\{r,s\})}{\partial x_{r,s}} + (-1)^s \frac{\partial S(1 \cdots n)}{\partial x_{r,s}} = 0.$$

But $\frac{\partial S(\{r,s\})}{\partial x_{r,s}} = 1$; solving the equation for the derivative of $S(1 \cdots n)$ finishes the proof. $\square$

Remark 5.2.8. We may apply Proposition 5.2.7 more generally using the chain rule: if M is a skew-symmetric $n \times n$ matrix whose entries are functions with $(M)_{k,l} = x_{r,s}$ for some $k < l$, $r < s$, and no other entry of M depends on $x_{r,s}$ (except $(M)_{l,k}$), then

$$\frac{\partial \mathrm{Pf}(M)}{\partial x_{r,s}} = (-1)^{k+l+1} \mathrm{Pf}(N),$$

where N is the skew-symmetric matrix obtained by deleting rows r, s and columns r, s from M.

5.2.1.3. *The determinant of certain submatrices of generic skew-symmetric matrices.* Surprisingly, the determinants of certain submatrices of generic skew-symmetric matrices factor as a reduced product of Pfaffians.

Proposition 5.2.9 (See, e.g., sections 406-415 of [**MM30**]). *Let S be a generic $n \times n$ skew-symmetric matrix. For $1 \leq r, s \leq n$, $r \neq s$, let $S_{r,s}$ denote the $(n-1) \times (n-1)$ submatrix obtained by deleting row r and column s of S. Then*

$$\det(S_{r,s}) = \begin{cases} S(1 \cdots \hat{r} \cdots n) S(1 \cdots \hat{s} \cdots n) & \text{if } n \text{ odd} \\ \mathrm{Pf}(S) S(\{1, \ldots, n\} \setminus \{r, s\}) & \text{if } n \text{ even} \end{cases}.$$

Example 5.2.10. For the generic 4×4 skew-symmetric matrix

$$S = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix},$$

the determinant of $S_{2,3}$ factors as

$$\det \begin{pmatrix} 0 & x_{12} & x_{14} \\ -x_{13} & -x_{23} & x_{34} \\ -x_{14} & -x_{24} & 0 \end{pmatrix} = \mathrm{Pf}(S) S(14).$$

5.2.2. Non-linear free divisors on the space of skew-symmetric matrices. We now develop our free divisors on the space of $n \times n$ skew-symmetric matrices. We shall assume that $n \geq 3$.

5.2.2.1. *Linear vector fields.* We begin by considering the solvable subgroup

$$G = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & L \end{pmatrix} \in L_n(\mathbb{C}) \,\middle|\, L \in L_{n-2}(\mathbb{C}) \right\}$$

of the lower triangular invertible matrices. Let G act on $\mathrm{Sk}_n(\mathbb{C})$ by $A \cdot M = AMA^T$. This is the restriction of $\sigma : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C}))$ from §4.3.2.

Note that $\sigma|_G$ does not have an open orbit because $\dim_{\mathbb{C}}(G) < \dim_{\mathbb{C}}(\mathrm{Sk}_n(\mathbb{C}))$. However, G is solvable, has a partial flag of invariant subspaces, and we may construct a coefficient matrix just as we do for block representations. The vector fields corresponding to the Lie algebras of the kernels of the quotient representations will still be tangent to the corresponding affine subsets. Thus, the resulting coefficient matrix will be "block lower triangular," except the "diagonal blocks" may not be square.

We shall use results and the coordinates $\{x_{ij} | 1 \leq i < j \leq n\}$ for $\mathrm{Sk}_n(\mathbb{C})$ from §4.3.2. Let $\mathfrak{g}$ be the Lie algebra of G.

PROPOSITION 5.2.11. *The representation $\sigma|_G : G \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C}))$ has invariant subspaces*

$$W_j = \begin{cases} \mathrm{Sk}_n(\mathbb{C}) & \text{if } j = n-2 \\ \mathrm{Sk}_n(\mathbb{C})_{n-j,n-j} & \text{otherwise} \end{cases}, \qquad j = 0, \ldots, n-2.$$

By choosing bases for $\mathfrak{g}$ and $\mathrm{Sk}_n(\mathbb{C})$ correctly, the matrix A of coefficients of the corresponding vector fields is "block lower triangular" with $n-2$ diagonal blocks. The first diagonal block is

$$\begin{pmatrix} x_{12} & x_{12} & 0 \\ x_{13} & 0 & x_{13} \\ 0 & x_{23} & x_{23} \end{pmatrix},$$

and for $2 \leq j \leq n-2$, the jth "diagonal block" is the $(j+1) \times j$ matrix

$$\begin{pmatrix} x_{1,3} & x_{1,4} & \cdots & x_{1,j+2} \\ x_{2,3} & x_{2,4} & \cdots & x_{2,j+2} \\ 0 & x_{3,4} & \cdots & x_{3,j+2} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{3,j+1} & -x_{4,j+1} & \cdots & x_{j+1,j+2} \end{pmatrix}. \tag{5.3}$$

PROOF. We shall use the notation of Definition 3.2.6. By Proposition 4.3.7, we have a partial flag

$$\{0\} = W_0 \subset \mathrm{Sk}_n(\mathbb{C})_{n-1,n-1} \subset \cdots \subset \mathrm{Sk}_n(\mathbb{C})_{3,3} \subset \mathrm{Sk}_n(\mathbb{C})$$

of $\sigma|_G$-invariant subspaces.

Let K_j be the connected component of the identity of the kernel of the quotient representation $G \to \mathrm{GL}(\mathrm{Sk}_n(\mathbb{C})/W_j)$ induced from $\sigma|_G$. When $j < n-2$, the quotient representation is just $\sigma_{n-j,n-j}|_G$. Since in this case $n-j \geq 3$, by Proposition 4.3.8,

$$K_j = \left\{ \begin{pmatrix} I_{n-j} & 0 \\ * & * \end{pmatrix} \in G \right\}, \qquad j < n-2.$$

Now fix $1 \leq j \leq n-2$, and consider the jth diagonal block $A_{j,j}$ of the matrix A of coefficients, as in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of $W_{n-2-j+1}/W_{n-2-j}$ and of the Lie algebra of $K_{n-2-j+1}/K_{n-2-j}$, respectively. For $j = 1$, we may use $\frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{23}}$ and $E_{11}, E_{22}, E_{33} \in \mathfrak{g}$, respectively. Then Proposition 4.3.11 shows that $A_{1,1}$ is as claimed. For $2 \leq j \leq n-2$, we may use $\frac{\partial}{\partial x_{1,j+2}}, \ldots, \frac{\partial}{\partial x_{j+1,j+2}}$ and $E_{j+2,3}, \ldots, E_{j+2,j+2}$, respectively, and Proposition 4.3.12 (deleting the left two columns) shows that $A_{j,j}$ is as claimed. □

Example 5.2.12. For the 4×4 skew-symmetric case, the matrix of coefficients given by the Proposition is

$$A = \begin{pmatrix} x_{12} & x_{12} & 0 & 0 & 0 \\ x_{13} & 0 & x_{13} & 0 & 0 \\ 0 & x_{23} & x_{23} & 0 & 0 \\ x_{14} & 0 & 0 & x_{13} & x_{14} \\ 0 & x_{24} & 0 & x_{23} & x_{24} \\ 0 & 0 & x_{34} & 0 & x_{34} \end{pmatrix},$$

where the rows correspond to $\frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{23}}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{24}}, \frac{\partial}{\partial x_{34}}$. Note that if we add a column on the right for a vector field of the form $h\frac{\partial}{\partial x_{34}}$, then the resulting matrix will be block lower triangular with 3 blocks of size 3×3, 2×2, and 1×1.

5.2.2.2. *Nonlinear Pfaffian vector fields.* Besides the $\frac{1}{2}(n^2 - 3n + 6)$ linear vector fields obtained from $\sigma|_G$, we will use $n-3$ nonlinear Pfaffian vector fields to yield $\frac{n(n-1)}{2}$ total vector fields. For each $1 \leq a < b \leq n$, define the Pfaffian vector field

$$\eta_{a,b} = \sum_{b<p<q\leq n} S(\{a, a+1, \ldots, b, p, q\}) \frac{\partial}{\partial x_{p,q}}. \tag{5.4}$$

(Recall that we may write the coefficients as $S(a \cdots bpq)$.)

If $|\{a,\ldots,b\}|$ is odd (equivalently, $b-a$ is even), or $b \geq n-1$, then $\eta_{a,b} \equiv 0$. Define the function

$$\epsilon(b) = \begin{cases} 1 & \text{if } b \text{ is even} \\ 2 & \text{if } b \text{ is odd} \end{cases}.$$

Then the $n-3$ nonlinear Pfaffian vector fields we will use are

$$\eta_{\epsilon(2),2}, \ldots, \eta_{\epsilon(n-2),n-2}, \tag{5.5}$$

having coefficients of degrees $2, 2, 3, 3, \ldots, \lfloor \frac{n}{2} \rfloor$, respectively. (The degree of the coefficients of $\eta_{\epsilon(b),b}$ is $\lfloor \frac{b}{2} \rfloor + 1$.)

5.2.2.3. *Calculation of the bracket.* To use Saito's criterion, we will need to show that the $\mathscr{O}_{\mathrm{Sk}_n(\mathbb{C}),0}$-module M generated by the linear and nonlinear Pfaffian vector fields is closed under the Lie bracket. The bracket of pairs of linear vector fields arising from $\mathfrak{g}$ will yield another linear vector field arising from $\mathfrak{g}$ as $\mathfrak{g}$ is a Lie algebra and $A \mapsto \xi_A$ is a Lie algebra homomorphism. We must worry only about the brackets of a linear vector field and a nonlinear vector field, and the bracket of two nonlinear vector fields.

Proposition 5.2.13. *Let $1 \leq k \leq a \leq n$, $b \in \{1,2\}$, and $b < l \leq n$. If $k < a$, we also require that $b \leq k$. Then*

$$[\xi_{E_{a,k}}, \eta_{b,l}] = \begin{cases} -\eta_{b,l} & \textit{if } k = a \textit{ and } a \in \{b,\ldots,l\} \\ 0 & \textit{otherwise} \end{cases}.$$

Note that the linear vector fields of $\mathfrak{g}$ always satisfy the hypotheses of this Proposition: if $E_{a,k} \in \mathfrak{g}$ and $k < a$, then $k \geq 3$, while $b \in \{1,2\}$. Beyond the hypotheses of the Proposition, the answer is not nearly as simple.

Proposition 5.2.14. *Let $a, b \in \{1,2\}$ and choose $k < l$ so that $a < k \leq n$ and $b < l \leq n$. Then*

$$[\eta_{a,k}, \eta_{b,l}] = -\frac{1}{2}(\delta_{a,b} + l - k - 1) S(\{a,\ldots,k\}) \cdot \eta_{b,l}.$$

The proofs of these statements are not conceptually difficult, but they involve many tedious cases. They make extensive use of the derivative of the Pfaffian (Proposition 5.2.7), and the Dress-Wenzel Pfaffian identity (Theorem 5.2.4). See Appendix A for the proofs.

Proposition 5.2.15. *For the collection of linear vector fields coming from* $\mathfrak{g}$ *and the* $n-3$ *nonlinear vector fields as defined by* (5.4), *the bracket of all pairs of vector fields lie in the module they generate.*

Proof. As described above, the linear vector fields form a Lie algebra. The bracket of two nonlinear vector fields is computed in Proposition 5.2.14. The bracket of a linear and a nonlinear vector field is computed in Proposition 5.2.13. The result of the brackets clearly lie in the module. $\square$

5.2.2.4. *Augmented Coefficient Matrix.* We form a coefficient matrix $\tilde{A}$ by inserting columns into A for each Pfaffian vector field $\eta_{\epsilon(k),k}$, $2 \leq k \leq n-2$.

Proposition 5.2.16. *By ordering columns correctly,* $\tilde{A}$ *is block lower triangular with* $2n-5$ *diagonal blocks. Let* T *denote the matrix obtained by deleting the first two columns of the generic* $n \times n$ *skew-symmetric matrix. Then the determinant of the* j*th diagonal block of* $\tilde{A}$ *is*

$$\begin{cases} x_{12}x_{23}x_{13} & \text{if } j = 1 \\ \det(T^{(\ell+1)}) & \text{if } j = 2\ell \geq 2 \\ S(\epsilon(\ell)\cdots(\ell+3)) & \text{if } j = 2\ell+1 \geq 2 \end{cases},$$

up to multiplication by a nonzero constant.

Proof. Form $\tilde{A}$ by inserting into A a column for $\eta_{\epsilon(b),b}$ immediately to the right of $(A)_{b,b}$, for $b = 2, \ldots, n-2$. This leaves $A_{1,1}$ unchanged, and a calculation shows that the determinant is as claimed.

Otherwise, fix $2 \leq k \leq n-2$. The uppermost nonzero entry in the column of $\tilde{A}$ corresponding to $\eta_{\epsilon(k),k}$ appears in the row corresponding to $\frac{\partial}{\partial x_{k+1,k+2}}$, the last row of $(A)_{k,k}$, with an entry of $S(\epsilon(k)\cdots(k+2))$. Thus by Proposition 5.2.11, the $(k+1)\times(k+1)$ matrix obtained from the $(k+1)\times k$ matrix $A_{k,k}$ with an added column for $\eta_{\epsilon(k),k}$ becomes block lower triangular, with blocks of size $k \times k$ and 1×1 containing $T^{(k+1)}$ and $S(\epsilon(k)\cdots(k+2))$:

$$(5.6) \qquad \begin{pmatrix} x_{1,3} & x_{1,4} & \cdots & x_{1,k+2} & 0 \\ x_{2,3} & x_{2,4} & \cdots & x_{2,k+2} & 0 \\ 0 & x_{3,4} & \cdots & x_{3,k+2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{3,k+1} & -x_{4,k+1} & \cdots & x_{k+1,k+2} & S(\epsilon(k)\cdots(k+2)) \end{pmatrix}.$$

Finally, this kth diagonal block is split into the $j = 2(k-1)$ and $j = 2(k-1)+1$ diagonal blocks of $\tilde{A}$, so that in either case $k = \ell + 1$. Thus the determinants are as claimed. □

EXAMPLE 5.2.17. To Example 5.2.12 we will add the vector field

$$\eta_{12} = S(1\cdots 4)\frac{\partial}{\partial x_{34}}.$$

The resulting $\tilde{A}$ is block lower triangular with determinant

$$\det(\tilde{A}) = x_{12}x_{13}x_{23}(x_{13}x_{24} - x_{14}x_{23})S(1\cdots 4).$$

PROPOSITION 5.2.18. *Let T denote the matrix obtained by deleting the first two columns from the general $n \times n$ skew-symmetric matrix, $n \geq 3$. Then, up to multiplication by a nonzero constant,*

$$(5.7) \qquad \det(\tilde{A}) = \left(\prod_{k=1}^{n-2} \det(T^{(k)})\right) \cdot \left(\prod_{k=2}^{n} S(\epsilon(k)\cdots k)\right).$$

PROOF. We need only use Proposition 5.2.16 and observe that $x_{12}x_{23}x_{13} = S(12) \cdot S(23) \cdot \det(T^{(1)})$. □

5.2.2.5. *Irreducibility of polynomials.* To use Saito's Criterion (Theorem 3.1.1(2)) to show that the product in (5.7) defines a free divisor, we must show that the product is reduced. We showed in Corollary 4.5.5 that the generic Pfaffian is irreducible. We now show that each $\det(T^{(k)})$, as in Proposition 5.2.18, is irreducible.

COROLLARY 5.2.19. *Each $g_k = \det(T^{(k)})$ is irreducible for $k = 1, \ldots, n-2$.*

PROOF. We induct on k. For $k = 1$, $g_1 = x_{13}$ is irreducible. Now assume that g_k is irreducible. It is clear that $\frac{\partial g_{k+1}}{\partial x_{k+1,k+3}} = g_k$ (and $\frac{\partial g_k}{\partial x_{k+1,k+3}} = 0$) because $x_{k+1,k+3}$ is the lower right entry of $T^{(k)}$ and appears nowhere else. Consider the matrix $M \in \mathrm{Sk}_n(\mathbb{C})$ formed by taking the generic skew-symmetric matrix and setting $x_{j,j+1} = 1$ for $j = 2, \ldots, k+1$, $x_{1,k+3} = 1$, and all other variables equal to zero. M takes the form

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & & & & & \\ 0 & -1 & 0 & 1 & & & & \\ \vdots & & -1 & 0 & \ddots & & & \\ 0 & & & \ddots & \ddots & & & \\ -1 & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \end{pmatrix}.$$

Evaluating $T^{(k+1)}$ at M gives a matrix of the form $\begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}$, where L is unit lower triangular. Thus $g_{k+1}(M) = \pm \det(L) \neq 0$. Since the first row of $T^{(k)}$ evaluated at M is zero, $g_k(M) = 0$. Thus g_{k+1} is irreducible by Corollary 3.3.3. $\square$

Proposition 5.2.20. *The product in* (5.7) *is reduced and is factored into irreducible polynomials.*

Proof. We need only show that all of the terms in the product are distinct (not constant multiples of each other), as we have shown they are irreducible in Corollary 4.5.5 and Corollary 5.2.19. $S(1\cdots k)$ and $S(2\cdots(k+1))$ are distinct because the latter depends nontrivially on $x_{k,k+1}$ while the former does not. These are the only Pfaffian terms of the same degree. All of the $T^{(k)}$ terms have distinct degrees.

Moreover, all Pfaffians are distinct from the $T^{(k)}$ terms. For the skew-symmetric matrix $J_n \in \mathrm{Sk}_n(\mathbb{C})$ (as in the Cholesky-like factorization of skew-symmetric matrices, Theorem 2.2.10), all $S(1\cdots(2\ell))$ take the value 1 while $\det(T^{(k)})$ takes the value 0 (the top row of $T^{(k)}$ evaluated at J_n is zero). For $S(2\cdots(2\ell+1))$, we may apply the same reasoning to the skew-symmetric $n \times n$ matrix $\begin{pmatrix} 0 & \\ & J_{n-1} \end{pmatrix}$. $\square$

5.2.2.6. *A free divisor.* We are now able to show that the product in Proposition 5.2.18 defines a free divisor.

Theorem 5.2.21. *The product in* (5.7) *defines a free divisor on* $\mathrm{Sk}_n(\mathbb{C})$ *for all* $n \geq 3$.

Proof. The determinant of the matrix $\tilde{A}$ we formed by taking the coefficients of these vector fields is the product in Proposition 5.2.18. This product is square-free by Proposition 5.2.20. The bracket of the vector fields is contained in the module they generate by Proposition 5.2.15. Applying Saito's Criterion, Theorem 3.1.1(2), proves the Theorem. □

Remark 5.2.22. It is interesting to note that we may use Proposition 5.2.9 to express the hypersurface defined by the product in Proposition 5.2.18 as the zero set of a polynomial made up of determinants of certain minors of S, just as in our other examples. To do this, we must use some factors from Proposition 5.2.18 multiple times, defining the hypersurface with non-reduced structure.

5.3. Extensions of equidimensional representations

In this section we will use our method of block representations to prove several extension results for equidimensional representations and linear free divisors. This approach recovers all of the linear free divisors obtained elsewhere in this dissertation, as well as a number of new linear free divisors which are hybrids between the solvable and reductive.

5.3.1. Symmetric matrices. We first consider extensions on the space of symmetric matrices. Let $\rho_n : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ be the representation defined by

$$\rho_n(A)(M) = AMA^T,$$

the same action we studied in section 4.2. By Sylvester's theorem, ρ_n has an open orbit whose complement is exactly the variety of singular matrices. Thus any restriction of ρ_n to a subgroup and a subspace which gives an equidimensional representation must have the property that the exceptional orbit variety includes the variety of singular matrices. For

Here we take a different approach: suppose that a restriction of ρ_n to a subgroup and a subspace is an equidimensional representation. Then are there corresponding subgroups and subspaces for ρ_{n+1}? How are their exceptional orbit varieties related?

Proposition 5.3.1. *Let* G_n *be a connected complex algebraic subgroup of* $\mathrm{GL}_n(\mathbb{C})$. *Let* $V_n \subseteq \mathrm{Sym}_n(\mathbb{C})$ *be a vector subspace which is* $\rho_n|_{G_n}$*-invariant. Suppose that* $\rho_n|_{G_n} : G_n \to \mathrm{GL}(V_n)$ *is* *an equidimensional representation with coefficient determinant f, so that its exceptional orbit variety is defined (perhaps with non-reduced structure) by $f = 0$. Define the group*

$$G_{n+1} = \left\{ \begin{pmatrix} A & 0 \\ b & c \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \middle| A \in G_n \right\},$$

and the vector space

$$V_{n+1} = \left\{ \begin{pmatrix} M & * \\ * & * \end{pmatrix} \in \mathrm{Sym}_{n+1}(\mathbb{C}) \middle| M \in V_n \right\}.$$

If V_n contains an invertible matrix, then $\rho_{n+1}|_{G_{n+1}} : G_{n+1} \to \mathrm{GL}(V_{n+1})$ is an equidimensional representation whose coefficient determinant is $f \cdot g$, where g is the restriction to V_{n+1} of the determinant function on $\mathrm{Sym}_n(\mathbb{C})$.

Proof. Let $\theta = \rho_{n+1}|_{G_{n+1}} : G_{n+1} \to \mathrm{GL}(V_{n+1})$. First note that G_{n+1} is a connected complex algebraic Lie subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$. By the calculation

$$\begin{pmatrix} A & 0 \\ b & c \end{pmatrix} \begin{pmatrix} M & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix} \begin{pmatrix} A & 0 \\ b & c \end{pmatrix}^T = \begin{pmatrix} AMA^T & AMb^T + AM_{12}c \\ bMA^T + cM_{12}A^T & bMb^T + 2bM_{12}c + c^2M_{22} \end{pmatrix}, \tag{5.8}$$

θ leaves V_{n+1} invariant. Thus θ is a well-defined representation. Moreover, θ is the restriction of a rational map (ρ_{n+1}) to a subvariety (G_{n+1}) and thus rational.

We will show that θ has the structure of a candidate block representation. Let $W_1 = \left\{ \begin{pmatrix} 0_n & * \\ * & * \end{pmatrix} \in V_{n+1} \right\}$. By (5.8), W_1 is invariant under θ. Thus we have a partial flag

$$\{0\} = W_0 \subset W_1 \subset W_2 = V_{n+1}$$

of G_{n+1}-invariant subspaces.

Now let K_j be the connected component of the identity of the kernel of the quotient representation $G_{n+1} \to \mathrm{GL}(V_{n+1}/W_j)$ induced from θ. Clearly $K_2 = G_{n+1}$. For K_1, (5.8) shows that

$$\begin{pmatrix} A & 0 \\ b & c \end{pmatrix} \in \ker(G_{n+1} \to \mathrm{GL}(V_{n+1}/W_1))$$

if and only if $A \in \ker(\rho_n|_{G_n})$. Since $\rho_n|_{G_n}$ is an equidimensional representation, its kernel is discrete. Thus

$$K_1 = \left\{ \begin{pmatrix} I_n & 0 \\ b & c \end{pmatrix} \middle| \, c \neq 0 \right\}.$$

Finally, consider K_0. Since $K_0 \subset K_1$, any element of K_0 must take the form $\begin{pmatrix} I_n & 0 \\ b & c \end{pmatrix}$. By (5.8), an element of K_0 must have $Mb^T + M_{12}c = M_{12}$ for all $M \in V_n$ and all M_{12}. In particular, when $M_{12} = 0$, then $Mb^T = 0$ for all $M \in V_n$. Since V_n contains an invertible matrix, $b = 0$. By (5.8), it follows that $c = \pm 1$. Since K_0 is the connected component of the identity, $K_0 = \{I_{n+1}\}$.

Since $\dim_{\mathbb{C}}(W_j) = \dim_{\mathbb{C}}(K_j)$ for $j = 0, 1, 2$, we have verified (1) and (2) of Definition 3.2.6. Now consider the matrix A of coefficients, as described in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of W_{3-j}/W_{2-j} and the Lie algebra of $\mathfrak{k}_{3-j}/\mathfrak{k}_{2-j}$, respectively. Use any basis of W_2/W_1 and $\mathfrak{k}_2/\mathfrak{k}_1$. For W_1/W_0 and $\mathfrak{k}_1/\mathfrak{k}_0$, use $\frac{\partial}{\partial x_{1,n}}, \ldots, \frac{\partial}{\partial x_{n,n}}$ and $E_{n,1}, \ldots, E_{n,n} \in \mathfrak{k}_1$, respectively. By Proposition 4.3.6, the determinant of $A_{2,2}$ is g, up to multiplication by a nonzero constant. Note that $g \neq 0$: if $A \in V_n$ has nonzero determinant, then $g\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \neq 0$. By Proposition 3.2.18, the determinant of $A_{1,1}$ is the coefficient determinant of $G_{n+1}/K_1 \to \mathrm{GL}(V_{n+1}/W_1)$, but this representation is isomorphic to $\rho_n|_{G_n} : G_n \to \mathrm{GL}(V_n)$. Identify V_{n+1}/W_1 and V_n and consider f to be a function on V_{n+1}. Then by Theorem 3.2.14, θ has the structure of a candidate block representation, $f \cdot g = 0$ defines the exceptional orbit variety, and by definition $f \cdot g$ is the coefficient determinant of θ. $\square$

As a result, we have the following criterion for extending free divisors.

Corollary 5.3.2. *Suppose $\rho_n|_{G_n} : G_n \to \mathrm{GL}(V_n)$ is an equidimensional representation whose coefficient determinant is $f_1 \cdots f_r$. Define $\rho_{n+1}|_{G_{n+1}} : G_{n+1} \to \mathrm{GL}(V_{n+1})$ and $g : V_{n+1} \to \mathbb{C}$ as in Proposition 5.3.1. If $f_1 \cdots f_r = 0$ is a reduced equation defining a linear free divisor on V_n, V_n contains an invertible matrix, and*

$$f_1 \cdots f_r \cdot g = 0 \tag{5.9}$$

is reduced, then (5.9) defines a linear free divisor on V_{n+1}.

Proof. By Proposition 5.3.1 and Theorem 3.2.14, (5.9) defines the exceptional orbit variety of the candidate block representation $\rho_{n+1}|_{G_{n+1}} : G_{n+1} \to \mathrm{GL}(V_{n+1})$. By the same Theorem, since (5.9) is reduced it defines a linear free divisor. $\square$

This Corollary enables us to find a number of new linear free divisors.

Example 5.3.3. Fix $m \in \mathbb{N}$. Define G_m to be the subgroup of diagonal matrices in $\mathrm{GL}_m(\mathbb{C})$ and V_m to be the subspace of $\mathrm{Sym}_m(\mathbb{C})$ consisting of diagonal matrices. Then $\rho_m|_{G_m} : G_m \to \mathrm{GL}(V_m)$ is clearly an equidimensional representation whose exceptional orbit variety is defined by $\prod_{i=1}^m x_{ii} = 0$. This is a free divisor by Corollary 3.1.3.

For all $n \geq m$, define $G_n \subseteq L_n(\mathbb{C})$ and $V_n \subseteq \mathrm{Sym}_n(\mathbb{C})$ as in Proposition 5.3.1. Each V_n contains the identity matrix, so the Proposition shows that each $\rho_n|_{G_n} : G_n \to \mathrm{GL}(V_n)$ is an equidimensional representation. We will show that the exceptional orbit variety of each $\rho_n|_{G_n} : G_n \to \mathrm{GL}(V_n)$ is a free divisor defined (with non-reduced structure) by the product of the determinants of the upper left square submatrices of

$$\begin{pmatrix} x_{1,1} & & & x_{1,m+1} & \cdots & x_{1,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & x_{m,m} & x_{m,m+1} & \cdots & x_{m,n} \\ x_{1,m+1} & \cdots & x_{m,m+1} & x_{m+1,m+1} & \cdots & x_{m+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{m,n} & x_{m+1,n} & \cdots & x_{n,n} \end{pmatrix}. \tag{5.10}$$

First, we will show that the determinant h_n of (5.10) is irreducible when $n > m$. For h_{m+1}, $\frac{\partial h_{m+1}}{\partial x_{m+1,m+1}} = \prod_{i=1}^n x_{i,i}$. Moreover, $x_{i,i} \nmid h_{m+1}$: define $M \in V_{m+1}$ by $(M)_{j,j} = 1$ for $j \neq i$, $(M)_{i,m+1} = (M)_{m+1,i} = 1$, and let all other entries of M be zero. Then $x_{i,i}(M) = 0$ while $h_{m+1}(M) \neq 0$. By Corollary 3.3.2, h_{m+1} is irreducible.

Now assume that h_{n-1} is irreducible for some $n-1 > m$. Then $\frac{\partial h_n}{\partial x_{n,n}} = h_{n-1}$ is irreducible, and for

$$P = \begin{pmatrix} I_{n-2} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-n} \end{pmatrix} \in V_m,$$

we have $h_n(P) \neq 0$ and $h_{n-1}(P) = 0$. By Corollary 3.3.3, h_n is irreducible.

With irreducibility proven, it follows that each

$$x_{11} \cdots x_{mm} \cdot h_{m+1} \cdot h_{m+2} \cdots h_n = 0 \tag{5.11}$$

is reduced. By Corollary 5.3.2, (5.11) defines a linear free divisor on V_n for each $n \geq m$.

Remark 5.3.4. When $m = 1$ in the above Example, we recover the linear free divisors of Theorem 4.4.5.

Example 5.3.5. Consider

$$G_3 = \left\{ \begin{pmatrix} *_1 & 0 \\ 0 & *_2 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \right\} \qquad \text{and} \qquad V_3 = \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \in \mathrm{Sym}_3(\mathbb{C}) \right\}.$$

A calculation using Corollary 3.1.3 shows that $\rho_3|_{G_3} : G_3 \to \mathrm{GL}(V_3)$ is an equidimensional representation whose exceptional orbit variety is the linear free divisor on V_3 defined by

$$(x_{22}x_{33} - x_{23}^2)(2x_{12}x_{13}x_{23} - x_{13}^2 x_{22} - x_{12}^2 x_{33}) = 0.$$

Using Proposition 5.3.1 to build G_4 and V_4, we see that the additional component is irreducible, giving a free divisor on $\mathrm{Sym}_4(\mathbb{C})$. As in the previous Example, we may repeat this process to give a tower of free divisors on V_4, V_5, etc.

Example 5.3.6. Similarly, we could start with the connected subgroup $G_3 \subset \mathrm{GL}_3(\mathbb{C})$ whose Lie algebra is spanned by $E_{11}, E_{21}, E_{22}, E_{31}, E_{33} \in \mathfrak{gl}_3(\mathbb{C})$ and the vector space $V_3 = \left\{ \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \in \mathrm{Sym}_3(\mathbb{C}) \right\}$. Then $\rho_3|_{G_3} : G_3 \to \mathrm{GL}(V_3)$ is an equidimensional representation whose exceptional orbit variety is the linear free divisor defined by

$$x_{12}x_{13}(2x_{12}x_{13}x_{23} - x_{13}^2 x_{22} - x_{12}^2 x_{33}) = 0.$$

Once again, we may expand this linear free divisor to V_4, V_5, etc., obtaining a tower of free divisors.

5.3.2. General $n \times m$ matrices. We can similarly find expansion criteria for representation and linear free divisors on the space of $n \times m$ matrices. Define $\rho_{n,m} : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to$

$\mathrm{GL}(M(n,m,\mathbb{C}))$ by

$$\rho_{n,m}(A,B)(M) = AMB^{-1},$$

the same action we studied in §4.1. This is a rational representation of a connected complex algebraic Lie group. For any $0 \le p \le m$, define the subgroup $L^p_m(\mathbb{C})$ of $\mathrm{GL}_m(\mathbb{C})$ by

$$L^p_m(\mathbb{C}) = \left\{ \begin{pmatrix} *_p & 0 \\ * & * \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\},$$

consisting of block lower triangular matrices.

We consider when we can expand an equidimensional representation from $M(n,m,\mathbb{C})$ to $M(n,m+1,\mathbb{C})$.

Proposition 5.3.7. *Fix n,m with $p=(m+1)-n \ge 0$. In particular, $p \le m$. Let $G_{n,m}$ be a connected complex algebraic subgroup of $\mathrm{GL}_n(\mathbb{C}) \times L^p_m(\mathbb{C})$. Let $V_{n,m} \subseteq M(n,m,\mathbb{C})$ be a vector subspace which is $\rho_{n,m}|_{G_{n,m}}$-invariant. Suppose that $\rho_{n,m}|_{G_{n,m}} : G_{n,m} \to \mathrm{GL}(V_{n,m})$ is an equidimensional representation with coefficient determinant f, so that the exceptional orbit variety is defined (perhaps with non-reduced structure) by $f=0$. Define the group*

$$G_{n,m+1} = \left\{ \left(A, \begin{pmatrix} B & 0 & 0 \\ C & D & E \\ 0 & 0 & F \end{pmatrix} \right) \in \mathrm{GL}_n(\mathbb{C}) \times L^p_{m+1}(\mathbb{C}) \,\middle|\, \left(A, \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \right) \in G_{n,m} \right\},$$

where B is $p \times p$, D is $(n-1)\times(n-1)$, and F is 1×1, and the vector space

$$V_{n,m+1} = \left\{ \begin{pmatrix} M_1 & M_2 \end{pmatrix} \in M(n,m+1,\mathbb{C}) \,\middle|\, M_1 \in V_{n,m} \right\}.$$

If $V_{n,m}$ contains a $n \times m$ matrix whose rightmost $n-1$ columns have full rank, then $\rho_{n,m+1}|_{G_{n,m+1}} : G_{n,m+1} \to \mathrm{GL}(V_{n,m+1})$ is an equidimensional representation whose coefficient determinant is $f \cdot g$, where g is the restriction to $V_{n,m+1}$ of the determinant of the rightmost n columns of $M(n,m+1,\mathbb{C})$.

Proof. Let $\theta = \rho_{n,m+1}|_{G_{n,m+1}} : G_{n,m+1} \to \mathrm{GL}(V_{n,m+1})$. First note that $G_{n,m+1}$ is a connected complex algebraic subgroup of $\mathrm{GL}_n(\mathbb{C}) \times L^p_{m+1}(\mathbb{C})$. We will write elements of $V_{n,m+1}$

as $\begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix}$, where M_1 is $n \times p$, M_2 is $n \times (n-1)$, and M_3 is $n \times 1$. By the calculation

$$\begin{aligned} \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} B & 0 & 0 \\ C & D & E \\ 0 & 0 & F \end{pmatrix}^{-1} &= \\ \Big(A(M_1 B^{-1} - D^{-1} C B^{-1} M_2) \quad A M_2 D^{-1} \quad & A(-M_2 D^{-1} E F^{-1} + M_3 F^{-1}) \Big), \end{aligned} \tag{5.12}$$

θ leaves $V_{n,m+1}$ invariant. Thus θ is a well-defined representation. Moreover, θ is the restriction of the rational map $\rho_{n,m+1}$ to a subvariety and is thus rational.

We will show that θ has the structure of a candidate block representation. Let $W_1 = \left\{ \begin{pmatrix} 0 & 0 & * \end{pmatrix} \in V_{n,m+1} \right\}$. By (5.12), W_1 is invariant under θ. Thus we have a partial flag

$$\{0\} = W_0 \subset W_1 \subset W_2 = V_{n,m+1}$$

of $G_{n,m+1}$-invariant subspaces.

Now let K_j be the connected component of the identity of the kernel of the quotient representation $G_{n,m+1} \to \mathrm{GL}(V_{n,m+1}/W_j)$ induced from θ. Clearly $K_2 = G_{n,m+1}$. For K_1, (5.12) shows that

$$\left(A, \begin{pmatrix} B & 0 & 0 \\ C & D & E \\ 0 & 0 & F \end{pmatrix} \right) \in \ker(G_{n,m+1} \to \mathrm{GL}(V_{n,m+1}/W_1))$$

if and only if

$$\left(A, \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \right) \in \ker(\rho_{n,m}|_{G_{n,m}}).$$

Since $\rho_{n,m}|_{G_{n,m}}$ is an equidimensional representation, its kernel is discrete. Thus

$$K_1 = \left\{ \left(I_n, \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{n-1} & * \\ 0 & 0 & * \end{pmatrix} \right) \in G_{n,m+1} \right\}.$$

Finally, consider K_0. Since $K_0 \subset K_1$, any element must take the form

$$\left(I_n, \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{n-1} & E \\ 0 & 0 & F \end{pmatrix} \right).$$

By (5.12), this element lies in $\ker(\theta)$ if and only if $-M_2E+M_3 = M_3F$ for all $\begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix} \in V_{n,m+1}$. By setting $M_2 = 0$ and $M_3 \neq 0$, we see that $F = 1$. If M_2 has maximal rank, then $M_2E = 0$ if and only if $E = 0$. Thus $K_0 = \{(I_n, I_{m+1})\}$.

Since $\dim_{\mathbb{C}}(W_j) = \dim_{\mathbb{C}}(K_j)$ for $j = 0, 1, 2$, we have verified (1) and (2) of Definition 3.2.6. Now consider the matrix A of coefficients, as described in §3.2.4. $A_{j,j}$'s rows and columns correspond to ordered bases of W_{3-j}/W_{2-j} and the Lie algebra of $\mathfrak{k}_{3-j}/\mathfrak{k}_{2-j}$, respectively. Use any basis of W_2/W_1 and $\mathfrak{k}_2\mathfrak{k}_1$. For W_1/W_0 and $\mathfrak{k}_1/\mathfrak{k}_0$, use $\frac{\partial}{\partial x_{1,m+1}}, \ldots, \frac{\partial}{\partial x_{n,m+1}}$ and $(0, E_{p+1,m+1}), \ldots, (0, E_{m+1,m+1}) \in \mathfrak{k}_1$, respectively. By Proposition 4.1.8, the determinant of $A_{2,2}$ is g, up to multiplication by a nonzero constant. Note that $g \neq 0$: if $(M_1 M_2) \in V_{n,m}$ is such that M_2 has maximal rank, then we can easily choose M_3 so that $g\begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix} = \det\begin{pmatrix} M_2 & M_3 \end{pmatrix} \neq 0$. By Proposition 3.2.18, the determinant of $A_{1,1}$ is the coefficient determinant of $G_{n,m+1}/K_1 \to \mathrm{GL}(V_{n,m+1}/W_1)$, but this representation is isomorphic to $\rho_{n,m}|_{G_{n,m}} : G_{n,m} \to \mathrm{GL}(V_{n,m})$. Identify $V_{n,m+1}/W_1$ and $V_{n,m}$ and consider f to be a function on $V_{n,m}$. Then by Theorem 3.2.14, θ has the structure of a candidate block representation, $f \cdot g = 0$ defines the exceptional orbit variety, and by definition $f \cdot g$ is the coefficient determinant of θ. $\square$

There is a criterion analogous to Corollary 5.3.2 for extending a linear free divisor.

We can also consider the expansion of a representation from $M(n, m, \mathbb{C})$ to $M(n+1, m, \mathbb{C})$. For any $0 \leq q \leq n$, define the subgroup $T_n^q(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$ by

$$T_n^q(\mathbb{C}) = \left\{ \begin{pmatrix} *_q & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\},$$

consisting of block upper triangular matrices.

Corollary 5.3.8. *Fix n, m with $q = (n+1) - m \geq 0$. In particular, $q \leq n$. Let $G_{n,m}$ be a connected complex algebraic subgroup of $T_n^q(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$. Let $V_{n,m} \subseteq M(n, m, \mathbb{C})$ be a vector subspace which is $\rho_{n,m}|_{G_{n,m}}$-invariant. Suppose that $\rho_{n,m}|_{G_{n,m}} : G_{n,m} \to \mathrm{GL}(V_{n,m})$ is* *an equidimensional representation with coefficient determinant f, so that the exceptional orbit variety is defined (perhaps with non-reduced structure) by $f = 0$. Define the group*

$$G_{n+1,m} = \left\{ \left(\begin{pmatrix} A & B & 0 \\ 0 & C & 0 \\ 0 & D & E \end{pmatrix}, F \right) \in T^q_{n+1}(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \middle| \left(\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, F \right) \in G_{n,m} \right\},$$

where A is $q \times q$, C is $(m-1) \times (m-1)$, and E is 1×1, and the vector space

$$V_{n+1,m} = \left\{ \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \in M(n+1, m, \mathbb{C}) \middle| M_1 \in V_{n,m} \right\}.$$

If $V_{n,m}$ contains a $n \times m$ matrix whose bottom $m-1$ rows have full rank, then $\rho_{n+1,m}|_{G_{n+1,m}} : G_{n+1,m} \to \mathrm{GL}(V_{n+1,m})$ is an equidimensional representation whose coefficient determinant is $f \cdot g$, where g is the determinant of the bottom m rows of $M(n+1, m, \mathbb{C})$ restricted to $V_{n+1,m}$.

Proof. Take the transpose of all matrices and groups involved and apply Proposition 5.3.7. □

Example 5.3.9. We can recover the free divisors associated to the modified LU factorizations (Theorems 5.1.4 and 5.1.5). Start with $G_{1,1} = \{(*, 1) \in \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})\}$ and $V_{1,1} = M(1, 1, \mathbb{C})$ and alternately apply Proposition 5.3.7 (where $p = 1$) and Corollary 5.3.8 (where $q = 0$) to move from $M(n, n, \mathbb{C})$ to $M(n, n+1, \mathbb{C})$ to $M(n+1, n+1, \mathbb{C})$. We only need to check that the components are irreducible, accomplished by Corollary 4.6.4, and distinct, demonstrated with some example matrices.

Example 5.3.10. More generally, one can start with a variant of Example 3.1.6, where for a fixed $m \in \mathbb{N}$, $G_{m,m+1} = \mathrm{GL}_m(\mathbb{C}) \times (\{1\} \times D_m(\mathbb{C})) \subset \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_{m+1}(\mathbb{C})$ acts on $M(m, m+1, \mathbb{C})$ by $(A, B) \cdot M = AMB^{-1}$. This is an equidimensional representation whose exceptional orbit variety is a free divisor defined by the product of the maximal minors of the generic $m \times (m+1)$ matrix. Alternately applying Corollary 5.3.8 and Proposition 5.3.7, we obtain equidimensional representations (and free divisors) on $M(n, n, \mathbb{C})$ and $M(n, n+1, \mathbb{C})$ for any $n > m$. Specifically, for $n > m$ and $r = n$ or $n+1$, consider the restriction of $\rho_{n,r}$ to the group

$$\left\{\left(\begin{pmatrix} *_m & 0 \\ * & L \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & * \\ 0 & 0 & U \end{pmatrix}\right) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_p(\mathbb{C}) \,\middle|\, L \in L_{n-m}(\mathbb{C}), D \in D_m(\mathbb{C}), U \in T_{p-m-1}(\mathbb{C})\right\}.$$

Let S be the generic $n \times p$ matrix. Then the exceptional orbit variety of this representation is a linear free divisor defined by the product of the maximal minors of the upper left $m \times (m+1)$ submatrix of S, $\det(S^{(k)})$ for $m < k \leq n$, and $\det(S^{(1,k)})$ for $m < k \leq p-1$. At each step we only need to check irreducibility (Corollary 4.6.4) and that all terms are distinct.

Note that the previous Example comes from the case $m = 1$.

CHAPTER 6

The vanishing topology of matrix singularities

Determinantal varieties of singular matrices are not free divisors. Thus we cannot directly apply any of the results from Chapter 1 to calculate the singular Milnor numbers of nonlinear sections of determinantal varieties.

We present in this Chapter a new method to calculate singular Milnor numbers of nonlinear sections of hypersurfaces which are not free divisors, applying our method to determinantal varieties. We observe that many of the free divisors found in Chapters 4 and 5 are the union of the determinantal variety V and a free divisor W. Although $V \cap W$ is generally not a complete intersection and thus has no associated singular Milnor number, its *vanishing Euler characteristic* is related to the singular Milnor numbers of V, $V \cup W$, and W. We adapt methods from Chapter 1 to write the vanishing Euler characteristic of $V \cap W$ in terms of singular Milnor numbers of nonlinear sections of free divisors and products of $\{0\}$ with free divisors, which may be computed algebraically using Theorems 1.6.5 and 1.6.10, respectively. The resulting formula expresses the singular Milnor number of V in terms of algebraically computable singular Milnor numbers.

We apply this approach to derive formulas for the singular Milnor numbers of nonlinear sections of determinantal varieties on the spaces of 2×2 and 3×3 symmetric matrices, 2×2 general matrices, and 4×4 skew-symmetric matrices. (Nonlinear sections for the 1×1 matrices correspond to isolated hypersurface singularities.) Although the complexity of the calculation increases with the dimensions of the vector space, we see no obstacle for further formulas to be obtained using our method.

To apply the formulas from Chapter 1, we show in sections 6.3.2 and 6.3.4 that the free divisors given by the Cholesky factorization and the modified LU factorization are H-holonomic (see Theorem 6.2.2).

6.1. Vanishing Euler characteristics of unions and intersections

In this section we develop the necessary tools to study relations between various singular Milnor numbers. We begin by defining the vanishing Euler characteristic.

Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a holomorphic germ which has finite $\mathscr{K}_{V_i,e}$-codimension for some finite collection $\{V_i\}$ of complete intersections in $\mathbb{C}^p$. Then there exists a common stabilization f_t of f_0 relative to all V_i (see Remark 1.4.8). We may fix $\epsilon > 0$ and $0 < |t| \ll \epsilon$ so that $f_t^{-1}(V_i) \cap B_\epsilon(0)$ is the singular Milnor fiber for each i. Then for any algebraic set $(W, 0) \subset (\mathbb{C}^p, 0)$, we define the *vanishing Euler characteristic of* $(W, 0)$ by

$$\tilde{\chi}_W(f_0) = \chi(f_t^{-1}(W) \cap B_\epsilon(0)) - 1. \tag{6.1}$$

(This term has also been applied to the absolute value of the right side of (6.1) under additional hypotheses.) Since f_t may not be transverse to W, our vanishing Euler characteristic may depend on the particular stabilization, t, and ϵ chosen above.

Note that the Euler characteristic on the right side of (6.1) is defined: by results of Hamm [**Ham86**], the analytic set $f_t^{-1}(W)$ has the homotopy type of a CW complex of dimension $\leq n - \operatorname{codim}(W)$.

When W is one of the complete intersections V_i that we used to obtain a stabilization, then $f_t^{-1}(W) \cap B_\epsilon(0)$ has the homotopy type of a bouquet of spheres of dimension $n - r$, where $r = \operatorname{codim}(W)$. Directly from the definitions of the Euler characteristic and the singular Milnor number (see Theorem 1.4.9), we have the following simple relationship:

$$\tilde{\chi}_W(f_0) = (-1)^{n-r} \mu_W(f_0). \tag{6.2}$$

We next explain combinatorial relations between the vanishing Euler characteristics of unions and intersections of algebraic sets.

PROPOSITION 6.1.1. *Let $(A_i, 0) \subset (\mathbb{C}^p, 0)$, $1 \leq i \leq k$, be algebraic sets. Then*

(1) $\tilde{\chi}_{\cap_{i=1}^k A_i}(f_0) = \sum_{\emptyset \neq \mathbf{j} \subseteq \{1,\ldots,k\}} (-1)^{|\mathbf{j}|+1} \tilde{\chi}_{\cup_{i \in \mathbf{j}} A_i}(f_0)$ *and*

(2) $\tilde{\chi}_{\cup_{i=1}^k A_i}(f_0) = \sum_{\emptyset \neq \mathbf{j} \subseteq \{1,\ldots,k\}} (-1)^{|\mathbf{j}|+1} \tilde{\chi}_{\cap_{i \in \mathbf{j}} A_i}(f_0)$.

PROOF. Lemma 8.1 of [**Dam96a**] uses the Mayer-Vietoris sequence repeatedly to obtain (1) and (2) for the Euler characteristic. The only thing left to show is that when χ is replaced by $\tilde{\chi}+1$, the constants cancel. The constants cancel by the binomial theorem:

$$\begin{aligned}\sum_{\emptyset\neq\mathbf{j}\subseteq\{1,\dots,k\}}(-1)^{|\mathbf{j}|+1} &= -\sum_{i=1}^{k}\binom{k}{i}(-1)^i(1)^{k-i}\\ &= -\left(((-1)+1)^k-1\right)\\ &= 1.\end{aligned}$$

□

We shall often apply Proposition 6.1.1 when $k=2$, where the relation given may be stated as

$$\tilde{\chi}_{A\cap B}(f_0)=\tilde{\chi}_A(f_0)+\tilde{\chi}_B(f_0)-\tilde{\chi}_{A\cup B}(f_0), \tag{6.3a}$$

$$\tilde{\chi}_{A\cup B}(f_0)=\tilde{\chi}_A(f_0)+\tilde{\chi}_B(f_0)-\tilde{\chi}_{A\cap B}(f_0), \tag{6.3b}$$

$$\text{or}\quad \tilde{\chi}_A(f_0)=\tilde{\chi}_{A\cup B}(f_0)+\tilde{\chi}_{A\cap B}(f_0)-\tilde{\chi}_B(f_0). \tag{6.3c}$$

We will often cite these equations to explain the operation we are performing.

We single out a particularly useful case. After choosing a particular stabilization, t, and ϵ, we begin the derivation of each formula by citing the following Lemma.

Lemma 6.1.2. *Let $(V,0)$ and $(W,0)\subset(\mathbb{C}^p,0)$ be hypersurfaces. Suppose that $f_0:\mathbb{C}^n,0\to\mathbb{C}^p,0$ is algebraically transverse off 0 to V, W, and $V\cup W$. Then*

$$\mu_V(f_0)=\mu_{V\cup W}(f_0)-\mu_W(f_0)+(-1)^{n-1}\tilde{\chi}_{V\cap W}(f_0).$$

Proof. Choose a common stabilization $f_t:\mathbb{C}^n,0\to\mathbb{C}^p$ of f_0 with respect to V, W, and $V\cup W$. Choose ϵ and t so that $f_t^{-1}(V)\cap B_\epsilon(0)$, $f_t^{-1}(W)\cap B_\epsilon(0)$, and $f_t^{-1}(V\cup W)\cap B_\epsilon(0)$ are singular Milnor fibers. In particular, they each have the homotopy type of a bouquet of spheres of dimension $n-1$. Then by (6.3c), we have

$$\tilde{\chi}_V(f_0)=\tilde{\chi}_{V\cup W}(f_0)+\tilde{\chi}_{V\cap W}(f_0)-\tilde{\chi}_W(f_0), \tag{6.4}$$

i.e.,

$$\begin{aligned}\tilde{\chi}(f_t^{-1}(V)\cap B_\epsilon(0)) =&\tilde{\chi}(f_t^{-1}(V\cup W)\cap B_\epsilon(0))+\tilde{\chi}(f_t^{-1}(V\cap W)\cap B_\epsilon(0))\\ &-\tilde{\chi}(f_t^{-1}(W)\cap B_\epsilon(0)).\end{aligned}$$

We may use (6.2) to replace three of the vanishing Euler characteristics of (6.4) with singular Milnor numbers to obtain

$$(6.5) \qquad (-1)^{n-1}\mu_V(f_0) = (-1)^{n-1}\mu_{V\cup W}(f_0) + \tilde{\chi}_{V\cap W}(f_0) - (-1)^{n-1}\mu_W(f_0).$$

Multiplying (6.5) by $(-1)^{n-1}$ and rearranging terms finishes the proof. $\square$

6.2. The singular Milnor numbers of matrix singularities

In this section, we find formulas to compute singular Milnor numbers of nonlinear sections of the variety V of singular matrices in a space of matrices (identified with $\mathbb{C}^p$). We found in Chapters 4 and 5 free divisors W so that $V \cup W$ is again a free divisor. We call this a *free completion* of V. For a germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ which is algebraically transverse off 0 to V, W, and $V \cup W$, Lemma 6.1.2 gives

$$(6.6) \qquad \mu_V(f_0) = \mu_{V\cup W}(f_0) - \mu_W(f_0) + (-1)^{n-1}\tilde{\chi}_{V\cap W}(f_0),$$

provided we use an appropriate stabilization, ϵ, and t. Since $V \cup W$ and W are free divisors, both $\mu_{V\cup W}(f_0)$ and $\mu_W(f_0)$ may be computed algebraically as lengths of modules by Theorem 1.6.5.

The remaining term on the right side of (6.6), $\tilde{\chi}_{V\cap W}(f_0)$, is more difficult to compute. However, $V \cap W$ is typically a union of algebraic sets, and we may write $\tilde{\chi}_{V\cap W}(f_0)$ in terms of other vanishing Euler characteristics using (6.3b) (or more generally, Proposition 6.1.1(2)). For each of the resulting terms, we may either: find a "free completion" and apply (6.3c), write the set as an intersection and apply (6.3a), or write the set as a union and apply (6.3b).

Eventually, we reach a situation where all terms are the vanishing Euler characteristics of either free divisors, or the direct product of $\{0\} \subset \mathbb{C}^{p'} \subset \mathbb{C}^p$ with a free divisor in $\mathbb{C}^{p-p'}$. Under the assumption that f_t is also a stabilization with respect to these sets, the vanishing Euler characteristics are singular Milnor numbers which may be computed algebraically by Theorem 1.6.5 (for sections of free divisors) or Theorem 1.6.10 (for almost free divisors on an ICIS).

Remark 6.2.1. Since the determinantal variety is invariant under various transformations (e.g., unitary transformations of the underlying vector spaces), we can assume by the parametrized transversality theorem that a germ equivalent to a given f_0 is transverse to these auxiliary varieties by composing f_0 with such a transformation. Our formulas may be applied to this modified f_0, and the resulting singular Milnor number will be the singular Milnor number of f_0.

The formulas given by these Theorems simplify a great deal when the free divisors are H-holonomic (for example, the former formula is given by a single $\mathscr{K}_{H,e}$-codimension). It is thus useful to know that a number of free divisors we will use are H-holonomic.

THEOREM 6.2.2. *The following free divisors are H-holonomic.*

(1) *The exceptional orbit variety of the representation $L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$, $A \cdot M = AMA^T$, corresponding to the Cholesky factorization (see Theorem 4.4.5).*
(2) *The exceptional orbit variety of the representation $L_n(\mathbb{C}) \times H_n(\mathbb{C}) \to \mathrm{GL}(M(n,n,\mathbb{C}))$, $(A,B) \cdot M = AMB^{-1}$, corresponding to the modified LU factorization for $n \times n$ matrices (see Theorem 5.1.4).*
(3) *The exceptional orbit variety of the representation $L_n(\mathbb{C}) \times H_{n+1}(\mathbb{C}) \to \mathrm{GL}(M(n,n+1,\mathbb{C}))$, $(A,B) \cdot M = AMB^{-1}$, corresponding to the modified LU factorization for $n \times (n+1)$ matrices (see Theorem 5.1.5).*

We prove this statement in §6.3.2 and §6.3.4.

Our formulas will be written more concisely using the following notation.

NOTATION 6.2.3. For $g_1, \ldots, g_k$ functions from $\mathbb{C}^p$ to $\mathbb{C}$, let $V(g_1, \ldots, g_k)$ be the algebraic set W in $\mathbb{C}^p$ defined (set-theoretically) by $g_1 = \cdots = g_k = 0$. Thus we write

$$\tilde{\chi}_{V(g_1,\ldots,g_k)}(f_0) = \tilde{\chi}_W(f_0),$$

and

$$\mu_{V(g_1,\ldots,g_k)}(f_0) = \mu_W(f_0).$$

6.2.1. 1-dimensional spaces of matrices. Since $\{0\} \subset \mathbb{C}^1$ is an H-holonomic free divisor, Theorem 1.6.5 provides a formula for matrix singularities coming from 1×1 matrices and the 2×2 skew-symmetric matrices, provided that $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ is algebraically transverse off $0 \in \mathbb{C}^n$ to $\{0\}$. In fact, the singular Milnor number here is just the classical Milnor number of an isolated hypersurface singularity (see Example 1.4.10).

6.2.2. 2×2 symmetric matrices. Consider the space of 2×2 symmetric matrices using the coordinates

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

We want to calculate the singular Milnor number associated with nonlinear sections of the determinantal variety $V(ac-b^2)$ using the program described above.

THEOREM 6.2.4. *Suppose $f_0 : \mathbb{C}^n, 0 \to \mathrm{Sym}_2(\mathbb{C}), 0$ is algebraically transverse off 0 to $V(ac-b^2)$, $V(a(ac-b^2))$, and $V(ab)$. Then*

$$\mu_{V(ac-b^2)}(f_0) = \mu_{V(a(ac-b^2))}(f_0) - \mu_{V(a,b)}(f_0) - \mu_{V(a)}(f_0), \tag{6.7}$$

or equivalently,

$$\mu_{V(ac-b^2)}(f_0) = \mu_{V(a(ac-b^2))}(f_0) + \mu_{V(b)}(f_0) - \mu_{V(ab)}(f_0). \tag{6.8}$$

All terms on the right of (6.7) or (6.8) are computable using Theorems 1.6.5 or 1.6.10, and all free divisors are H-holonomic.

PROOF. Let f_t be a common stabilization of f_0 relative to all algebraic sets in (6.7) (or (6.8)). Choose $\epsilon > 0$ and then $t \neq 0$ so that $f_t^{-1}(Z) \cap B_\epsilon(0)$ is the singular Milnor fiber of Z, where Z is any algebraic set in (6.7) (or (6.8)). Then $V(a(ac-b^2))$ is a H-holonomic free completion of $V(ac-b^2)$ by Theorem 6.2.2. By Lemma 6.1.2,

$$\mu_{V(ac-b^2)}(f_0) = \mu_{V(a(ac-b^2))}(f_0) - \mu_{V(a)}(f_0) + (-1)^{n-1}\tilde{\chi}_{V(a,ac-b^2)}(f_0). \tag{6.9}$$

Note that $V(a, ac-b^2) = V(a,b)$. Since this is a complete intersection of codimension 2, by (6.2),

$$\tilde{\chi}_{V(a,b)}(f_0) = (-1)^{n-2}\mu_{V(a,b)}(f_0). \tag{6.10}$$

Substituting (6.10) into (6.9) finishes the proof of (6.7).

For (6.8), note that $V(ab)$, $V(a)$, and $V(b)$ are free divisors and $V(a,b)$ is a free complete intersection. By Theorem 1.6.8,

$$\mu_{V(a,b)}(f_0) = \mu_{V(a)}(f_0) + \mu_{V(b)}(f_0) - \mu_{V(ab)}(f_0). \tag{6.11}$$

Substituting (6.11) into (6.7) gives (6.8). □

EXAMPLE 6.2.5. Consider the map $F : \mathbb{C}^2 \to \mathrm{Sym}_2(\mathbb{C})$ defined by

$$F(x,y) = \begin{pmatrix} x & y \\ y & x^5 \end{pmatrix}.$$

First note that $\mu_{V(ac-b^2)}(F)$ is defined: since $\mathscr{K}_{V(ac-b^2),e}$-codim$(F) < \infty$, F is algebraically transverse off 0 to $V(ac-b^2)$ by Proposition 1.5.1. Since $2 < \mathrm{codim}(\mathrm{Sing}(V(ac-b^2))) = 3$, the hypersurface defined by $\det(F(x,y)) = x^6 - y^2 = 0$ has an isolated singularity at $(0,0)$ and its stabilization will miss $\mathrm{Sing}(V(ac-b^2))$ and will thus be smooth. Hence, the Milnor fiber of this IHS agrees with the singular Milnor fiber of F for $V(ac-b^2)$. Since the Milnor number of this IHS is $(6-1)(2-1) = 5$ by Theorem 1.1.2 and Example 1.1.4, we should find that $\mu_{V(ac-b^2)}(F) = 5$.

Since $\mathscr{K}_{V(a(ac-b^2)),e}$-codim$(F) < \infty$, F is algebraically transverse off 0 to $V(a(ac-b^2))$. Since $(a,b) \circ F = (x,y)$ and $b \circ F = y$ are submersions, F is fully transverse to $V(ab)$ and $V(b)$ and so their singular Milnor numbers are 0. Thus by (6.8) and Theorem 1.6.5,

$$\begin{aligned} \mu_{V(ac-b^2)}(F) &= \mu_{V(a(ac-b^2))}(F) + \mu_{V(b)}(F) - \mu_{V(ab)}(F) \\ &= \mathscr{K}_{a(ac-b^2),e}\text{-codim}(F), \end{aligned}$$

by which we mean the $\mathscr{K}_{H,e}$-codim(F) for H the function defined by $H = a(ac-b^2)$. A calculation shows that $\mu_{V(ac-b^2)}(F) = \mathscr{K}_{a(ac-b^2),e}$-codim$(F) = 5$.

EXAMPLE 6.2.6. Now consider $F : \mathbb{C}^4 \to \mathrm{Sym}_2(\mathbb{C})$ defined by

$$F(u,v,w,x) = \begin{pmatrix} u & u^3+v^2+w^2+x^2 \\ u^3+v^2+w^2+x^2 & v \end{pmatrix}.$$

Note that $g = \det \circ F = uv - (u^3+v^2+w^2+x^2)^2$, so that

$$\begin{aligned} \frac{\partial g}{\partial u} &= v - 6(u^3+v^2+w^2+x^2)u^2 \\ \frac{\partial g}{\partial v} &= u - 4(u^3+v^2+w^2+x^2)v \\ \frac{\partial g}{\partial w} &= -4(u^3+v^2+w^2+x^2)w \\ \text{and} \quad \frac{\partial g}{\partial x} &= -4(u^3+v^2+w^2+x^2)x. \end{aligned}$$

Since the singular set includes $u = v = w^2+x^2 = 0$, $g^{-1}(0)$ does not have an isolated singularity. Nevertheless, $\mu_{V(ac-b^2)}(F)$ is defined. First note that F is algebraically transverse off 0 to $V(ac-b^2)$ since $\mathscr{K}_{V(ac-b^2),e}$-codim$(F) < \infty$. F is fully transverse to $V(a)$ since $a \circ F = u$ is a submersion; thus $\mu_{V(a)}(F) = 0$. Although F is only algebraically transverse off 0 to $V(a,b)$, we can easily compute $\mu_{V(a,b)}(F)$ by finding the Milnor number of the ICIS defined by $(a,b) \circ F = (u, u^3 + v^2 + w^2 + x^2)$. Since this ICIS is also defined by $(u, v^2 + w^2 + x^2)$, $\mu_{V(a,b)}(F) = \mu(V(u, v^2 + w^2 + x^2)) = 1$. By (6.7) and Theorem 1.6.5, we have

$$\begin{aligned}\mu_{V(ac-b^2)}(F) &= \mu_{V(a(ac-b^2))}(F) - \mu_{V(a,b)}(F) - \mu_{V(a)}(F) \\ &= \mathscr{K}_{a(ac-b^2),e}\text{-codim}(F) - 1,\end{aligned}$$

provided this codimension is finite.

It only remains to compute this codimension. Using the Cholesky representation we can show that Derlog $(a(ac-b^2))$ is generated by

$$\eta_1 = 2a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - 4c\frac{\partial}{\partial c} \qquad \text{and} \qquad \eta_2 = a\frac{\partial}{\partial b} + 2b\frac{\partial}{\partial c}.$$

The $\mathscr{O}_{\mathbb{C}^4,0}$-module $T\mathscr{K}_{H,e} \cdot F$ is generated by the following "vector fields along F":

$$\begin{aligned}\frac{\partial F}{\partial u} &= 1\frac{\partial}{\partial a} + 3u^2\frac{\partial}{\partial b} \\ \frac{\partial F}{\partial v} &= 2v\frac{\partial}{\partial b} + 1\frac{\partial}{\partial c} \\ \frac{\partial F}{\partial w} &= 2w\frac{\partial}{\partial b} \\ \frac{\partial F}{\partial x} &= 2x\frac{\partial}{\partial b} \\ \eta_1 \circ F &= 2u\frac{\partial}{\partial a} - (u^3 + v^2 + w^2 + x^2)\frac{\partial}{\partial b} - 4v\frac{\partial}{\partial c} \\ \eta_2 \circ F &= u\frac{\partial}{\partial b} + 2(u^3 + v^2 + w^2 + x^2)\frac{\partial}{\partial c}.\end{aligned}$$

Then $\mathscr{K}_{a(ac-b^2),e}$-codim$(F)$ is the dimension over $\mathbb{C}$ of the quotient module formed by quotienting $\mathscr{O}_{\mathbb{C}^4,0}\left\{1\frac{\partial}{\partial a}, 1\frac{\partial}{\partial b}, 1\frac{\partial}{\partial c}\right\}$ by $T\mathscr{K}_{H,e} \cdot F$. It follows that $\mathscr{K}_{a(ac-b^2),e}$-codim$(F) = 2$ and $\mu_{V(ac-b^2)}(F) = 1$.

Thus, for example,

$$F_t = \begin{pmatrix} u & u^3+v^2+w^2+x^2-t \\ u^3+v^2+w^2+x^2-t & v \end{pmatrix}$$

is a stabilization of F; for appropriate t and ϵ, $V(uv-(u^3+v^2+w^2+x^2-t)^2)\cap B_\epsilon(0)$ has the homotopy type of a single 3-sphere.

6.2.3. 2×2 **general matrices.** Consider $M(2,2,\mathbb{C})$, using coordinates

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We want to calculate the singular Milnor number of nonlinear sections of the determinantal variety $V(ad-bc)$.

THEOREM 6.2.7. *Suppose $f_0 : \mathbb{C}^n, 0 \to M(2,2,\mathbb{C}), 0$ is algebraically transverse off 0 to $V(ad-bc)$, $V(ab(ad-bc))$, $V(abc)$, and $V(abd)$. Then*

$$\begin{aligned}\mu_{V(ad-bc)}(f_0) =& \mu_{V(ab(ad-bc))}(f_0)-\mu_{V(ab)}(f_0)-\mu_{V(a,bc)}(f_0)\\ &-\mu_{V(b,ad)}(f_0)+\mu_{V(a,b)}(f_0).\end{aligned} \tag{6.12}$$

All terms on the right of (6.12) *are computable using Theorems 1.6.5 or 1.6.10, and all free divisors are H-holonomic.*

PROOF. Let f_t be a common stabilization of f_0 relative to all algebraic sets in (6.12). Choose ϵ and t appropriately. Since $V(ab(ad-bc))$ and $V(ab)$ are H-holonomic free divisors by Theorem 6.2.2(2) and (3), we may apply Lemma 6.1.2 to obtain

$$\mu_{V(ad-bc)}(f_0) = \mu_{V(ab(ad-bc))}(f_0)-\mu_{V(ab)}(f_0)+(-1)^{n-1}\tilde{\chi}_{V(ab,ad-bc)}(f_0). \tag{6.13}$$

Since $V(ab,ad-bc) = V(a,bc)\cup V(b,ad)$, by (6.3b) we have

$$\tilde{\chi}_{V(ab,ad-bc)}(f_0) = \tilde{\chi}_{V(a,bc)}(f_0)+\tilde{\chi}_{V(b,ad)}(f_0)-\tilde{\chi}_{V(a,b)}(f_0). \tag{6.14}$$

Applying (6.2) to (6.14) and multiplying by $(-1)^{n-1}$ gives

$$(-1)^{n-1}\tilde{\chi}_{V(ab,ad-bc)}(f_0) = -\mu_{V(a,bc)}(f_0)-\mu_{V(b,ad)}(f_0)+\mu_{V(a,b)}(f_0). \tag{6.15}$$

Substituting (6.15) into (6.12) finishes the proof. $\square$

We may also obtain a formula which uses only H-holonomic free divisors.

COROLLARY 6.2.8. *Under the assumptions of Theorem 6.2.7, we may alternately write*

$$\begin{aligned}\mu_{V(ad-bc)}(f_0) =&\mu_{V(ab(ad-bc))}(f_0) + \mu_{V(bc)}(f_0) - \mu_{V(abc)}(f_0)\\ &+ \mu_{V(ad)}(f_0) - \mu_{V(abd)}(f_0).\end{aligned} \tag{6.16}$$

All terms on the right are the singular Milnor numbers of H-holonomic free divisors, computable using Theorem 1.6.5.

PROOF. Let f_t be a stabilization of f_0 relative to all of the algebraic sets in (6.16). Choose ϵ and t appropriately. By the proof of Theorem 6.2.7,

$$\begin{aligned}\mu_{V(ad-bc)}(f_0) =&\mu_{V(ab(ad-bc))}(f_0) + (-1)^{n-2}\big(\tilde{\chi}_{V(ab)}(f_0) - \tilde{\chi}_{V(a,bc)}(f_0)\\ &- \tilde{\chi}_{V(b,ad)}(f_0) + \tilde{\chi}_{V(a,b)}(f_0)\big).\end{aligned} \tag{6.17}$$

Since $V(a,bc) = V(a) \cap V(bc)$, $V(b,ad) = V(b) \cap V(ad)$, and $V(a,b) = V(a) \cap V(b)$, by (6.3a) we have

$$\begin{aligned}\tilde{\chi}_{V(a,bc)}(f_0) &= \tilde{\chi}_{V(a)}(f_0) + \tilde{\chi}_{V(bc)}(f_0) - \tilde{\chi}_{V(abc)}(f_0)\\ \tilde{\chi}_{V(b,ad)}(f_0) &= \tilde{\chi}_{V(b)}(f_0) + \tilde{\chi}_{V(ad)}(f_0) - \tilde{\chi}_{V(abd)}(f_0)\\ \tilde{\chi}_{V(a,b)}(f_0) &= \tilde{\chi}_{V(a)}(f_0) + \tilde{\chi}_{V(b)}(f_0) - \tilde{\chi}_{V(ab)}(f_0).\end{aligned} \tag{6.18}$$

Substituting (6.18) into (6.17) and cancelling yields

$$\begin{aligned}\mu_{V(ad-bc)}(f_0) =&\mu_{V(ab(ad-bc))}(f_0) + (-1)^{n-2}\big(- \tilde{\chi}_{V(bc)}(f_0) + \tilde{\chi}_{V(abc)}(f_0)\\ &- \tilde{\chi}_{V(ad)}(f_0) + \tilde{\chi}_{V(abd)}(f_0)\big).\end{aligned} \tag{6.19}$$

Since the vanishing Euler characteristics are the vanishing Euler characteristics of hypersurfaces, applying (6.2) to (6.19) and simplifying signs finishes the proof. $\square$

EXAMPLE 6.2.9. Now consider $F : \mathbb{C}^5 \to M(2,2,\mathbb{C})$ defined by

$$F(u,v,w,x,y) = \begin{pmatrix} u & v \\ w & x^3+y^3+u^2+v^2+w^2 \end{pmatrix}.$$

Then $\det \circ F = u(x^3 + y^3 + u^2 + v^2 + w^2) - vw$ defines a hypersurface which does not have an isolated singularity at 0. However, $\mu_{V(ab-cd)}(F)$ is defined. F is algebraically transverse off 0 to $V(ab - cd)$ because $\mathscr{K}_{V(ab-cd),e}\text{-codim}(F) < \infty$. Since $(a, b, c) \circ F = (u, v, w)$ and $(b, c) \circ F = (v, w)$ are submersions, F is fully algebraically transverse to $V(abc)$ and $V(bc)$, with $\mu_{V(bc)}(F) = \mu_{V(abc)}(F) = 0$. A calculation analogous to that in Example 6.2.6 but using the vector fields in Derlog $(ab(ad - bc))$ given by

$$\begin{aligned}
\eta_1 &= a\frac{\partial}{\partial a} - c\frac{\partial}{\partial c} - 2d\frac{\partial}{\partial d} \\
\eta_2 &= b\frac{\partial}{\partial b} - 2c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d} \\
\eta_3 &= a\frac{\partial}{\partial c} + b\frac{\partial}{\partial d}
\end{aligned}$$

shows that $\mathscr{K}_{ab(ad-bc),e}\text{-codim}(F) = 16$. Since $V(ad)$ and $V(abd)$ are coordinate hyperplane arrangements, Derlog (ad) and Derlog (abd) are generated, respectively, by $\{a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}\}$ and $\{a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d}, b\frac{\partial}{\partial b} - d\frac{\partial}{\partial d}, \frac{\partial}{\partial c}\}$. Using these vector fields, a calculation shows that $\mathscr{K}_{ad,e}\text{-codim}(F) = 8$ and $\mathscr{K}_{abd,e}\text{-codim}(F) = 16$.

Thus F is algebraically transverse off 0 to $V(ab(ad-bc))$, $V(ad)$, and $V(abd)$. By Corollary 6.2.8 and Theorem 1.6.5, the singular Milnor number is

$$\begin{aligned}
\mu_{V(ad-bc)}(F) &= \mu_{V(ab(ad-bc))}(F) + \mu_{V(ad)}(F) - \mu_{V(abd)}(F) \\
&= \mathscr{K}_{ab(ad-bc),e}\text{-codim}(F) + \mathscr{K}_{ad,e}\text{-codim}(F) - \mathscr{K}_{abd,e}\text{-codim}(F) \\
&= 16 + 8 - 16 \\
&= 8.
\end{aligned}$$

6.2.4. 3×3 **symmetric matrices.** Consider the space of 3×3 symmetric matrices, using coordinates

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

We want to calculate the singular Milnor numbers of nonlinear sections of the determinantal variety $V(D)$, where

$$D = \det \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = -c^2d + 2bce - ae^2 - b^2f + adf.$$

It will be convenient to also define the functions

$$D_a = \det \begin{pmatrix} 0 & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = -c^2d + 2bce - b^2f$$

$$\text{and} \quad D_f = \det \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{pmatrix} = -c^2d + 2bce - ae^2,$$

on $\mathrm{Sym}_3(\mathbb{C})$. Note that $V(D_a)$ and $V(D_f)$ are diffeomorphic via a diffeomorphism which interchanges the first and third coordinates and so swaps b with e and a with f.

THEOREM 6.2.10. *Suppose* $f_0 : \mathbb{C}^n, 0 \to \mathrm{Sym}_3(\mathbb{C}), 0$ *is algebraically transverse off* 0 *to all of the algebraic sets appearing in* (6.20)*. Then*

$$\begin{aligned} \mu_{V(D)}(f_0) = &\mu_{V(a(ad-b^2)D)}(f_0) - \mu_{V(a,bcD_a)}(f_0) + \mu_{V(a,b,cd)}(f_0) + \mu_{V(a,c,bf)}(f_0) \\ &- \mu_{V(a,b,c)}(f_0) + \mu_{V(a,bc)}(f_0) - \mu_{V(a)}(f_0) + \mu_{V(acD_f)}(f_0) \\ &- \mu_{V(a,bc(2be-cd))}(f_0) - \mu_{V(c,ae)}(f_0) + \mu_{V(a,c)}(f_0) - \mu_{V(ac)}(f_0) \\ &- \mu_{V((ad-b^2)D_f)}(f_0). \end{aligned} \tag{6.20}$$

Except perhaps for $V(bcD_a)$, $V(acD_f)$, *and* $V((ad-b^2)D_f)$, *all of the varieties are* H*-holonomic. If* f_0 *additionally has finite* $\mathscr{K}_{H,e}$*-codimension for* H *defining these three free divisors, then all terms on the right of* (6.20) *are computable using Theorems 1.6.5 or 1.6.10.*

PROOF. Let f_t be a common stabilization of f_0 relative to all algebraic sets in (6.20). Choose ϵ and t appropriately. Then $V(a(ad-b^2)D)$ and $V(a(ad-b^2))$ are H-holonomic free divisors by Theorem 6.2.2(1). By the proof of Lemma 6.1.2,

$$\mu_{V(D)}(f_0) = \mu_{V(a(ad-b^2)D)}(f_0) + (-1)^{n-1}\left(-\tilde{\chi}_{V(a(ad-b^2))}(f_0) + \tilde{\chi}_{V(a(ad-b^2),D)}(f_0)\right). \tag{6.21}$$

We shall find a way to express $\tilde{\chi}_{V(a(ad-b^2),D)}(f_0)$ in terms of computable singular Milnor numbers and one copy of $\tilde{\chi}_{V(a(ad-b^2))}(f_0)$. Since $V(a(ad-b^2),D) = V(a,D) \cup V(ad-b^2,D)$ and $V(a, ad-b^2, D) = V(a,b,cd)$, by (6.3b) we have

$$\begin{aligned} \tilde{\chi}_{V(a(ad-b^2),D)}(f_0) &= \tilde{\chi}_{V(a,D)}(f_0) + \tilde{\chi}_{V(ad-b^2,D)}(f_0) - \tilde{\chi}_{V(a,ad-b^2,D)}(f_0) \\ &= \tilde{\chi}_{V(a,D)}(f_0) + \tilde{\chi}_{V(ad-b^2,D)}(f_0) - \tilde{\chi}_{V(a,b,cd)}(f_0). \end{aligned} \tag{6.22}$$

Since $V(a,b,cd)$ is a product of $\{0\}$ with a free divisor, nonlinear sections of it are "almost free divisors on ICIS." Thus to compute (6.22), we must only find ways of computing $\tilde{\chi}_{V(a,D)}(f_0)$ and $\tilde{\chi}_{V(ad-b^2,D)}(f_0)$.

First consider $\tilde{\chi}_{V(a,D)}(f_0)$. Since $V(a,D) = V(a,D_a)$, we have $\tilde{\chi}_{V(a,D)}(f_0) = \tilde{\chi}_{V(a,D_a)}(f_0)$. We now use a "free completion" by combining $V(a,D_a)$ with $V(a,bc)$; $V(bcD_a)$ is a free divisor by Example 5.3.6. By (6.3c) we have

$$\tilde{\chi}_{V(a,D)}(f_0) = \tilde{\chi}_{V(a,D_a)}(f_0) = \tilde{\chi}_{V(a,bcD_a)}(f_0) + \tilde{\chi}_{V(a,bc,D_a)}(f_0) - \tilde{\chi}_{V(a,bc)}(f_0). \tag{6.23}$$

Since $V(a,bc,D_a) = V(a,b,D_a) \cup V(a,c,D_a) = V(a,b,cd) \cup V(a,c,bf)$, by (6.3b) we have

$$\tilde{\chi}_{V(a,bc,D_a)}(f_0) = \tilde{\chi}_{V(a,b,cd)}(f_0) + \tilde{\chi}_{V(a,c,bf)}(f_0) - \tilde{\chi}_{V(a,b,c)}(f_0). \tag{6.24}$$

Substituting (6.24) into (6.23), we obtain

$$\tilde{\chi}_{V(a,D)}(f_0) = \tilde{\chi}_{V(a,bcD_a)}(f_0) + \tilde{\chi}_{V(a,b,cd)}(f_0) + \tilde{\chi}_{V(a,c,bf)}(f_0) - \tilde{\chi}_{V(a,b,c)}(f_0) - \tilde{\chi}_{V(a,bc)}(f_0). \tag{6.25}$$

We next compute $\tilde{\chi}_{V(ad-b^2,D)}(f_0)$. Since $V(ad-b^2,D) = V(ad-b^2,D_f)$, we have $\tilde{\chi}_{V(ad-b^2,D)}(f_0) = \tilde{\chi}_{V(ad-b^2,D_f)}(f_0)$. By Example 5.3.5, $V((ad-b^2)D_f)$ is a free divisor. Since $V(ad-b^2,D_f) = V(ad-b^2) \cap V(D_f)$, by (6.3a) we have

$$\tilde{\chi}_{V(ad-b^2,D)}(f_0) = \tilde{\chi}_{V(ad-b^2,D_f)}(f_0) = \tilde{\chi}_{V(ad-b^2)}(f_0) + \tilde{\chi}_{V(D_f)}(f_0) - \tilde{\chi}_{V((ad-b^2)D_f)}(f_0). \tag{6.26}$$

We found a free completion of $V(ad-b^2)$ in the proof of Theorem 6.2.4; by (6.3c) we obtain

$$\tilde{\chi}_{V(ad-b^2)}(f_0) = \tilde{\chi}_{V(a(ad-b^2))}(f_0) + \tilde{\chi}_{V(a,b)}(f_0) - \tilde{\chi}_{V(a)}(f_0). \tag{6.27}$$

Returning our attention to (6.26), we now use a free completion of $V(D_f)$ by combining it with $V(ac)$. $V(acD_f)$ can be shown to be a free divisor in a manner similar to Example 5.3.6. By

(6.3c) we have

$$\tilde{\chi}_{V(D_f)}(f_0) = \tilde{\chi}_{V(acD_f)}(f_0) + \tilde{\chi}_{V(ac,D_f)}(f_0) - \tilde{\chi}_{V(ac)}(f_0). \tag{6.28}$$

Now $V(ac, D_f) = V(a, D_f) \cup V(c, D_f) = V(a, c(2be - cd)) \cup V(c, ae)$. By (6.3b) we have

$$\tilde{\chi}_{V(ac,D_f)}(f_0) = \tilde{\chi}_{V(a,c(2be-cd))}(f_0) + \tilde{\chi}_{V(c,ae)}(f_0) - \tilde{\chi}_{V(a,c)}(f_0). \tag{6.29}$$

In turn, we use a "free completion" of $V(a, c(2be-cd))$ by combining it with $V(a, b)$. $V(bc(2be-cd))$ is diffeomorphic to the H-holonomic free divisor given in Theorem 6.2.2(2). By (6.3c), we have

$$\tilde{\chi}_{V(a,c(2be-cd))}(f_0) = \tilde{\chi}_{V(a,bc(2be-cd))}(f_0) + \tilde{\chi}_{V(a,b,cd)}(f_0) - \tilde{\chi}_{V(a,b)}(f_0). \tag{6.30}$$

Substituting (6.30) into (6.29) and the result into (6.28) gives

$$\begin{aligned}\tilde{\chi}_{V(D_f)}(f_0) =& \tilde{\chi}_{V(acD_f)}(f_0) + \tilde{\chi}_{V(a,bc(2be-cd))}(f_0) + \tilde{\chi}_{V(a,b,cd)}(f_0) - \tilde{\chi}_{V(a,b)}(f_0) \\ &+ \tilde{\chi}_{V(c,ae)}(f_0) - \tilde{\chi}_{V(a,c)}(f_0) - \tilde{\chi}_{V(ac)}(f_0).\end{aligned} \tag{6.31}$$

Substituting (6.27) and (6.31) into (6.26) and cancelling terms gives

$$\begin{aligned}\tilde{\chi}_{V(ad-b^2,D)}(f_0) =& \tilde{\chi}_{V(a(ad-b^2))}(f_0) - \tilde{\chi}_{V(a)}(f_0) + \tilde{\chi}_{V(acD_f)}(f_0) + \tilde{\chi}_{V(a,bc(2be-cd))}(f_0) \\ &+ \tilde{\chi}_{V(a,b,cd)}(f_0) + \tilde{\chi}_{V(c,ae)}(f_0) - \tilde{\chi}_{V(a,c)}(f_0) - \tilde{\chi}_{V(ac)}(f_0) \\ &- \tilde{\chi}_{V((ad-b^2)D_f)}(f_0).\end{aligned} \tag{6.32}$$

Finally, substituting (6.25) and (6.32) into (6.22) and cancelling gives

$$\begin{aligned}\tilde{\chi}_{V(a(ad-b^2),D)}(f_0) =& \tilde{\chi}_{V(a,bcD_a)}(f_0) + \tilde{\chi}_{V(a,b,cd)}(f_0) + \tilde{\chi}_{V(a,c,bf)}(f_0) - \tilde{\chi}_{V(a,b,c)}(f_0) \\ &- \tilde{\chi}_{V(a,bc)}(f_0) + \tilde{\chi}_{V(a(ad-b^2))}(f_0) - \tilde{\chi}_{V(a)}(f_0) + \tilde{\chi}_{V(acD_f)}(f_0) \\ &+ \tilde{\chi}_{V(a,bc(2be-cd))}(f_0) + \tilde{\chi}_{V(c,ae)}(f_0) - \tilde{\chi}_{V(a,c)}(f_0) - \tilde{\chi}_{V(ac)}(f_0) \\ &- \tilde{\chi}_{V((ad-b^2)D_f)}(f_0).\end{aligned} \tag{6.33}$$

All of the vanishing Euler characteristics on the right side of (6.33) (except perhaps $V(a(ad-b^2)))$ correspond to singular Milnor numbers by assumption. Applying (6.2) to (6.33), multiplying through by $(-1)^{n-1}$ and simplifying signs, we find that

$$
\begin{aligned}
(-1)^{n-1}\tilde{\chi}_{V(a(ad-b^2),D)}(f_0) = {} & \\
& - \mu_{V(a,bcD_a)}(f_0) + \mu_{V(a,b,cd)}(f_0) + \mu_{V(a,c,bf)}(f_0) - \mu_{V(a,b,c)}(f_0) \\
& + \mu_{V(a,bc)}(f_0) + (-1)^{n-1}\tilde{\chi}_{V(a(ad-b^2))}(f_0) - \mu_{V(a)}(f_0) + \mu_{V(acD_f)}(f_0) \\
& - \mu_{V(a,bc(2be-cd))}(f_0) - \mu_{V(c,ae)}(f_0) + \mu_{V(a,c)}(f_0) - \mu_{V(ac)}(f_0) \\
& - \mu_{V((ad-b^2)D_f)}(f_0).
\end{aligned}
\tag{6.34}
$$

Substituting (6.34) into (6.21) and cancelling terms yields (6.20).

The only matter left to address is that all of the algebraic sets which appear in (6.20) are free divisors or the direct products of $\{0\} \subset \mathbb{C}^{p'}$ with free divisors. The only terms we have not already addressed are free hyperplane arrangements. $\square$

Note that the free divisor $V(bcD_a)$ is not H-holonomic, as the vector fields which form a basis of Derlog (bcD_a) fail to generically have the necessary rank on $V(b,c)$. In a sense, we have chosen the wrong "free completions" in the proof of Theorem 6.2.10; there are likely different free completions which lead to a more computable formula than (6.20). We have not given such a formula because we are not aware of any systematic method (other than classifying the orbits of a group action with a finite number of orbits) for checking whether a given free divisor is H-holonomic.

6.2.5. 4×4 skew-symmetric matrices. On the space of 4×4 skew-symmetric matrices, we shall use the coordinates

$$
\begin{pmatrix}
0 & a & b & c \\
-a & 0 & d & e \\
-b & -d & 0 & f \\
-c & -e & -f & 0
\end{pmatrix}.
$$

We are interested in the singular Milnor numbers of nonlinear sections of the determinantal variety $V(\mathrm{Pf})$ defined by the Pfaffian function

$$\mathrm{Pf} = af - be + cd.$$

THEOREM 6.2.11. *Suppose $f_0 : \mathbb{C}^n, 0 \to \mathrm{Sk}_4(\mathbb{C}), 0$ is algebraically transverse off 0 to all of the algebraic sets appearing in* (6.35). *Then*

$$
\begin{aligned}
\mu_{V(\mathrm{Pf})}(f_0) =& \mu_{V(abd(be-cd)\mathrm{Pf})}(f_0) - \mu_{V(abd(be-cd))}(f_0) - \mu_{V(b,ad(af+cd))}(f_0)\\
&+ \mu_{V(b,a,cd)}(f_0) + 2\mu_{V(b,d,af)}(f_0) - 2\mu_{V(b,a,d)}(f_0)\\
&+ \mu_{V(b,ad)}(f_0) - \mu_{V(d,ab(af-be))}(f_0) + \mu_{V(d,a,be)}(f_0)\\
&+ \mu_{V(d,ab)}(f_0) + \mu_{V(af)}(f_0) + \mu_{V(bd(be-cd))}(f_0)\\
&- \mu_{V(abdf(be-cd))}(f_0) + \mu_{V(b,af)}(f_0) + \mu_{V(d,af)}(f_0).
\end{aligned}
\tag{6.35}
$$

If f_0 has finite $\mathscr{K}_{H,e}$ codimension for appropriate H's, then all terms on the right of (6.35) *are computable using Theorems 1.6.5 or 1.6.10, and all free divisors (except perhaps for $V(abd(be-cd)\mathrm{Pf})$) are H-holonomic.*

PROOF. Let f_t be a common stabilization with respect to the algebraic sets in (6.35). Choose ϵ and t appropriately. A free completion of $V(\mathrm{Pf})$ is $V(abd(be-cd)\mathrm{Pf})$ by Theorem 5.2.21. Note that $V(abd(be-cd))$ is also a free divisor: $V(bd(be-cd))$ is a H-holonomic free divisor by Theorem 6.2.2, and by taking the product-union with $V(a)$ we obtain $V(abd(be-cd))$. By Lemma 6.1.2,

$$
\mu_{V(\mathrm{Pf})}(f_0) = \mu_{V(abd(be-cd)\mathrm{Pf})}(f_0) - \mu_{V(abd(be-cd))}(f_0) + (-1)^{n-1}\tilde{\chi}_{V(abd(be-cd),\mathrm{Pf})}(f_0).
\tag{6.36}
$$

Note that $V(abd(be-cd),\mathrm{Pf}) = V(a,\mathrm{Pf}) \cup V(b,\mathrm{Pf}) \cup V(d,\mathrm{Pf}) \cup V(be-cd,\mathrm{Pf})$. By Proposition 6.1.1(2), we have

$$
\begin{aligned}
\tilde{\chi}_{V(abd(be-cd),\mathrm{Pf})}(f_0) =& \tilde{\chi}_{V(a,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(b,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(d,\mathrm{Pf})}(f_0)\\
&+ \tilde{\chi}_{V(be-cd,\mathrm{Pf})}(f_0) - \tilde{\chi}_{V(a,b,\mathrm{Pf})}(f_0) - \tilde{\chi}_{V(a,d,\mathrm{Pf})}(f_0)\\
&- \tilde{\chi}_{V(a,be-cd,\mathrm{Pf})}(f_0) - \tilde{\chi}_{V(b,d,\mathrm{Pf})}(f_0) - \tilde{\chi}_{V(b,be-cd,\mathrm{Pf})}(f_0)\\
&- \tilde{\chi}_{V(d,be-cd,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(a,b,d,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(a,b,be-cd,\mathrm{Pf})}(f_0)\\
&+ \tilde{\chi}_{V(a,d,be-cd,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(b,d,be-cd,\mathrm{Pf})}(f_0)\\
&- \tilde{\chi}_{V(a,b,d,be-cd,\mathrm{Pf})}(f_0).
\end{aligned}
\tag{6.37}
$$

Many of the expressions in (6.37) define the same algebraic sets in different ways. For instance, $V(a,b,d,be-cd,\mathrm{Pf}) = V(a,b,d) = V(a,b,d,\mathrm{Pf})$, $V(a,\mathrm{Pf}) = V(a,be-cd) = V(a,be-cd,\mathrm{Pf})$,

$V(a,b,\mathrm{Pf}) = V(a,b,cd) = V(a,b,be-cd,\mathrm{Pf})$, $V(a,d,\mathrm{Pf}) = V(a,d,be) = V(a,d,be-cd,\mathrm{Pf})$, and $V(b,d,\mathrm{Pf}) = V(b,d,af) = V(b,d,be-cd,\mathrm{Pf})$. After making these cancellations, (6.37) becomes

$$\begin{aligned}\tilde{\chi}_{V(abd(be-cd),\mathrm{Pf})}(f_0) =&\tilde{\chi}_{V(b,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(d,\mathrm{Pf})}(f_0) + \tilde{\chi}_{V(be-cd,\mathrm{Pf})}(f_0)\\ &- \tilde{\chi}_{V(b,be-cd,\mathrm{Pf})}(f_0) - \tilde{\chi}_{V(d,be-cd,\mathrm{Pf})}(f_0).\end{aligned} \tag{6.38}$$

Simplifying the Pfaffians in (6.38), we obtain

$$\begin{aligned}\tilde{\chi}_{V(abd(be-cd),\mathrm{Pf})}(f_0) =&\tilde{\chi}_{V(b,af+cd)}(f_0) + \tilde{\chi}_{V(d,af-be)}(f_0) + \tilde{\chi}_{V(af,be-cd)}(f_0)\\ &- \tilde{\chi}_{V(b,cd,af)}(f_0) - \tilde{\chi}_{V(d,be,af)}(f_0).\end{aligned} \tag{6.39}$$

We shall find ways to expand the first three terms on the right of (6.39) in such a way that the remaining terms will cancel. When combined with (6.36), this will allow us to calculate $\mu_{V(\mathrm{Pf})}(f_0)$.

For the first term, we use a “free completion” of $V(b,af+cd)$. Since $V(b,af+cd) \cup V(b,ad) = V(b,ad(af+cd))$, by (6.3c), we obtain

$$\tilde{\chi}_{V(b,af+cd)}(f_0) = \tilde{\chi}_{V(b,ad(af+cd))}(f_0) + \tilde{\chi}_{V(b,ad,af+cd)}(f_0) - \tilde{\chi}_{V(b,ad)}(f_0). \tag{6.40}$$

Since $V(b,ad,af+cd) = V(b,a,cd) \cup V(b,d,af)$ by (6.3b) we have

$$\tilde{\chi}_{V(b,ad,af+cd)}(f_0) = \tilde{\chi}_{V(b,a,cd)}(f_0) + \tilde{\chi}_{V(b,d,af)}(f_0) - \tilde{\chi}_{V(b,a,d)}(f_0). \tag{6.41}$$

Substituting (6.41) into (6.40), we obtain

$$\begin{aligned}\tilde{\chi}_{V(b,af+cd)}(f_0) =&\tilde{\chi}_{V(b,ad(af+cd))}(f_0) + \tilde{\chi}_{V(b,a,cd)}(f_0) + \tilde{\chi}_{V(b,d,af)}(f_0)\\ &- \tilde{\chi}_{V(b,a,d)}(f_0) - \tilde{\chi}_{V(b,ad)}(f_0).\end{aligned} \tag{6.42}$$

For the second term, we use a “free completion” of $V(d,af-be)$. Since $V(d,af-be) \cup V(d,ab) = V(d,ab(af-be))$, by (6.3c) we obtain

$$\tilde{\chi}_{V(d,af-be)}(f_0) = \tilde{\chi}_{V(d,ab(af-be))}(f_0) + \tilde{\chi}_{V(d,ab,af-be)}(f_0) - \tilde{\chi}_{V(d,ab)}(f_0). \tag{6.43}$$

Since $V(d,ab,af-be) = V(d,a,be) \cup V(d,b,af)$, by (6.3b) we have

$$\tilde{\chi}_{V(d,ab,af-be)}(f_0) = \tilde{\chi}_{V(d,a,be)}(f_0) + \tilde{\chi}_{V(d,b,af)}(f_0) - \tilde{\chi}_{V(d,a,b)}(f_0). \tag{6.44}$$

Substituting (6.44) into (6.43), we obtain

$$\begin{aligned}\tilde{\chi}_{V(d,af-be)}(f_0) =&\tilde{\chi}_{V(d,ab(af-be))}(f_0) + \tilde{\chi}_{V(d,a,be)}(f_0) + \tilde{\chi}_{V(d,b,af)}(f_0) \\ &- \tilde{\chi}_{V(d,a,b)}(f_0) - \tilde{\chi}_{V(d,ab)}(f_0).\end{aligned} \tag{6.45}$$

For the third term, we use a type of "free completion" of $V(af, be - cd)$. Since $V(af, be - cd) \cup V(af, bd) = V(af, bd(be - cd))$, by (6.3c) we have

$$\tilde{\chi}_{V(af,be-cd)}(f_0) = \tilde{\chi}_{V(af,bd(be-cd))}(f_0) + \tilde{\chi}_{V(af,bd,be-cd)}(f_0) - \tilde{\chi}_{V(af,bd)}(f_0). \tag{6.46}$$

Although no terms in (6.46) are directly computable as singular Milnor numbers, we can easily reduce each to singular Milnor numbers. Since $V(af, bd(be - cd)) = V(af) \cap V(bd(be - cd))$, by (6.3a) we have

$$\tilde{\chi}_{V(af,bd(be-cd))}(f_0) = \tilde{\chi}_{V(af)}(f_0) + \tilde{\chi}_{V(bd(be-cd))}(f_0) - \tilde{\chi}_{V(abdf(be-cd))}(f_0). \tag{6.47}$$

Note that all algebraic sets on the right side of (6.47) are free divisors. Since $V(af, bd, be-cd) = V(b, af, cd) \cup V(d, af, be)$ and $V(af, bd) = V(b, af) \cup V(d, af)$, by (6.3b) we have

$$\begin{aligned}\tilde{\chi}_{V(af,bd,be-cd)}(f_0) =&\tilde{\chi}_{V(b,af,cd)}(f_0) + \tilde{\chi}_{V(d,af,be)}(f_0) - \tilde{\chi}_{V(b,d,af)}(f_0) \\ \text{and} \quad \tilde{\chi}_{V(af,bd)}(f_0) =&\tilde{\chi}_{V(b,af)}(f_0) + \tilde{\chi}_{V(d,af)}(f_0) - \tilde{\chi}_{V(b,d,af)}(f_0).\end{aligned} \tag{6.48}$$

Substituting (6.47) and (6.48) into (6.46) gives us

$$\begin{aligned}\tilde{\chi}_{V(af,be-cd)}(f_0) &= (\tilde{\chi}_{V(af)}(f_0) + \tilde{\chi}_{V(bd(be-cd))}(f_0) - \tilde{\chi}_{V(abdf(be-cd))}(f_0)) \\ &\quad + (\tilde{\chi}_{V(b,af,cd)}(f_0) + \tilde{\chi}_{V(d,af,be)}(f_0) - \tilde{\chi}_{V(b,d,af)}(f_0)) \\ &\quad - (\tilde{\chi}_{V(b,af)}(f_0) + \tilde{\chi}_{V(d,af)}(f_0) - \tilde{\chi}_{V(b,d,af)}(f_0)) \\ &= \tilde{\chi}_{V(af)}(f_0) + \tilde{\chi}_{V(bd(be-cd))}(f_0) - \tilde{\chi}_{V(abdf(be-cd))}(f_0) \\ &\quad + \tilde{\chi}_{V(b,af,cd)}(f_0) + \tilde{\chi}_{V(d,af,be)}(f_0) - \tilde{\chi}_{V(b,af)}(f_0) \\ &\quad - \tilde{\chi}_{V(d,af)}(f_0).\end{aligned} \tag{6.49}$$

Having now found ways to evaluate the first three terms on the right of (6.39), we substitute (6.42), (6.45), and (6.49) into (6.39) and combine like terms to obtain

$$
\begin{aligned}
\tilde{\chi}_{V(abd(be-cd),\mathrm{Pf})}(f_0) =& \tilde{\chi}_{V(b,ad(af+cd))}(f_0) + \tilde{\chi}_{V(b,a,cd)}(f_0) + 2\tilde{\chi}_{V(b,d,af)}(f_0) \\
& - 2\tilde{\chi}_{V(b,a,d)}(f_0) - \tilde{\chi}_{V(b,ad)}(f_0) + \tilde{\chi}_{V(d,ab(af-be))}(f_0) \\
& + \tilde{\chi}_{V(d,a,be)}(f_0) - \tilde{\chi}_{V(d,ab)}(f_0) + \tilde{\chi}_{V(af)}(f_0) \\
& + \tilde{\chi}_{V(bd(be-cd))}(f_0) - \tilde{\chi}_{V(abdf(be-cd))}(f_0) - \tilde{\chi}_{V(b,af)}(f_0) \\
& - \tilde{\chi}_{V(d,af)}(f_0).
\end{aligned}
\tag{6.50}
$$

Applying (6.2) to (6.50), multiplying through by $(-1)^{n-1}$ and simplifying signs gives

$$
\begin{aligned}
(-1)^{n-1}\tilde{\chi}_{V(abd(be-cd),\mathrm{Pf})}(f_0) =& -\mu_{V(b,ad(af+cd))}(f_0) + \mu_{V(b,a,cd)}(f_0) + 2\mu_{V(b,d,af)}(f_0) \\
& - 2\mu_{V(b,a,d)}(f_0) + \mu_{V(b,ad)}(f_0) - \mu_{V(d,ab(af-be))}(f_0) \\
& + \mu_{V(d,a,be)}(f_0) + \mu_{V(d,ab)}(f_0) + \mu_{V(af)}(f_0) \\
& + \mu_{V(bd(be-cd))}(f_0) - \mu_{V(abdf(be-cd))}(f_0) + \mu_{V(b,af)}(f_0) \\
& + \mu_{V(d,af)}(f_0).
\end{aligned}
\tag{6.51}
$$

Substituting (6.51) into (6.36) gives (6.35), as desired. The holonomicity of most of the free divisors (except for $V(abd(be-cd)\mathrm{Pf})$) follows from Theorem 6.2.2(2). □

We have not been able to check whether $V(abd(be-cd)\mathrm{Pf})$ is H-holonomic.

6.3. The Cholesky and modified LU free divisors are H-holonomic

In this section we will show that the free divisors corresponding to the Cholesky factorization of symmetric matrices and the modified LU factorization of $n \times n$ and $n \times (n+1)$ matrices are H-holonomic. For each, we first classify the orbits of the underlying representation $\rho : G \to \mathrm{GL}(V)$. We identify a subgroup H of G which preserves all level sets of a function defining the free divisor. By studying the isotropy subgroups of representatives of each non-open orbit of ρ, we show that H acts transitively on all non-open orbits of ρ. From this it follows that our free divisor is H-holonomic.

6.3.1. Structure of the orbits of the Cholesky factorization. Recall the representation $\rho : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$ associated to the Cholesky factorization for symmetric matrices

(Corollary 2.2.7). We now prove a fact, stated as Theorem 4.4.6, that any symmetric matrix can be put in a "normal form" under this representation. Since there are a finite number of normal forms, the representation has a finite number of orbits.

THEOREM (4.4.6). *Let $L_n(\mathbb{C})$ act on $\mathrm{Sym}_n(\mathbb{C})$ by*

$$L \cdot S = LSL^T.$$

Then for any $S \in \mathrm{Sym}_n(\mathbb{C})$, there is an $L \in L_n(\mathbb{C})$ so that the columns of $L \cdot S$ are either zero or a standard basis vector, so that (by symmetry) the rows and columns are either entirely zero or have a single nonzero entry which is a 1. We will call such a matrix a "normal form" for S.

Note that if S is a normal form, then we may partition $\{e_1, \ldots, e_n\} = X \sqcup Y$ so that $\mathrm{span}(X) = \ker(S)$ and $\mathrm{span}(Y) = \mathrm{image}(S)$. We use this fact several times.

First we prove two lemmas which show that certain elements of $L_n(\mathbb{C})$ are in the isotropy group of certain symmetric matrices.

LEMMA 6.3.1. *Suppose that the jth column of $S \in \mathrm{Sym}_n(\mathbb{C})$ is zero. Then if*

$$A = \begin{pmatrix} I_{j-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & * & I_{n-j} \end{pmatrix},$$

with $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then $ASA^T = S$.

PROOF. For any $T \in M_n(\mathbb{C})$ we have $(T)_{kl} = e_k^T T e_l$. It is easy to check that

$$e_k^T A = \begin{cases} e_k^T & \text{if } k < j \\ \lambda e_j^T & \text{if } k = j \\ e_k^T + (A)_{kj} e_j^T & \text{if } k > j \end{cases}$$

and

$$A^T e_l = \begin{cases} e_l & \text{if } l < j \\ \lambda e_j & \text{if } l = j \\ e_l + (A)_{jl} e_j & \text{if } l > j \end{cases}.$$

Since $Se_j = e_j^T S = 0$, it follows that $e_k^T AS = e_k^T S$ and $SA^T e_l = Se_l$, whence it follows that $e_k^T ASA^T e_l = e_k^T S e_l$ for all k, l. □

LEMMA 6.3.2. *Suppose that $S \in \mathrm{Sym}_n(\mathbb{C})$ has $Se_i = e_j$ and $Se_j = e_i$, for a pair of integers $i < j$. If A is a diagonal matrix*

$$A = \begin{pmatrix} I_{i-1} & & & & \\ & \lambda & & & \\ & & I_{j-i-1} & & \\ & & & \frac{1}{\lambda} & \\ & & & & I_{n-j} \end{pmatrix}$$

with $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then $ASA^T = S$.

PROOF. This is a straightforward calculation. $\square$

PROOF OF THEOREM 4.4.6. The statement clearly holds for 1×1 matrices.

Let A be an $(n-1) \times (n-1)$ matrix, b a $1 \times (n-1)$ row vector, and c a scalar. Then

$$\begin{pmatrix} A & 0 \\ b & c \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix} \begin{pmatrix} A^T & b^T \\ 0 & c \end{pmatrix} = \begin{pmatrix} AS_1A^T & AS_1b^T + AS_2c \\ bS_1A^T + cS_2^TA^T & bS_1b^T + 2cbS_2 + c^2S_3 \end{pmatrix}. \tag{6.52}$$

We proceed by induction on n. Suppose that, by the induction hypothesis, S_1 is already in its normal form.

First consider the case where S_2 lies in the image of S_1, i.e., rank $\begin{pmatrix} S_1 & S_2 \end{pmatrix} = \mathrm{rank}\,(S_1)$. (Note that this case includes the open orbit situation.) Let $A = I$. Choose b_0 so that $S_1b_0^T = S_2$. Let $b = -cb_0$, with $c \in \mathbb{C}^*$ to be determined, so that $S_1b^T = -cS_2$. Then $S_1b^T + S_2c = -cS_2 + S_2c = 0$, guaranteeing that the upper right block of (6.52) will be zero. The lower right block (a scalar) will be

$$\begin{aligned} bS_1b^T + 2cbS_2 + c^2S_3 &= (-cb_0)S_1(-cb_0^T) + 2c(-cb_0)S_2 + c^2S_3 \\ &= c^2(b_0S_1b_0^T - 2b_0S_2 + S_3) \\ &= c^2(S_3 - b_0S_1b_0^T). \end{aligned}$$

Notice that $S_3 - b_0S_1b_0^T = 0$ is equivalent to the last row being in the row span of $\begin{pmatrix} S_1 & S_2 \end{pmatrix}$, i.e., rank $(S) = \mathrm{rank}\begin{pmatrix} S_1 & S_2 \end{pmatrix}$.

If $S_3 - b_0 S_1 b_0^T = 0$, then the lower right block of (6.52) will be zero regardless of c. Thus we may choose $c = 1$, for example. Note that the resulting matrix satisfies the criteria in the theorem.

If $S_3 - b_0 S_1 b_0^T \neq 0$, then we may choose $c \neq 0$ so that the lower right block of (6.52) will be 1; this satisfies the criteria in the theorem.

Now consider the case where S_2 is not in the image of S_1, i.e., rank $\begin{pmatrix} S_1 & S_2 \end{pmatrix} > \operatorname{rank}(S_1)$. Since $S_2 \neq 0$, there is a least i so that $e_i \notin \operatorname{image}(S_1)$ and $e_i^T S_2 \neq 0$ (the ith entry of S_2 is nonzero). By the induction hypothesis, $e_i \in \ker(S_1)$. Write $S_2 = v_1 + \lambda e_i + v_2$, where $v_1 \in \operatorname{span}\{e_1, \cdots, e_{i-1}\}$, $v_2 \in \operatorname{span}\{e_{i+1}, \cdots, e_n\}$, and $\lambda \neq 0$. By the way we chose i, v_1 is in the image of S_1, so there exists a b_1^T with $S_1 b_1^T = v_1$. As $e_i \in \ker(S_1)$, $S_1(b_1^T + \eta e_i) = v_1$ for any $\eta \in \mathbb{C}$.

Let B be the invertible lower triangular matrix with $Be_j = e_j$ for $j \neq i$ and $Be_i = \lambda e_i + v_2$. Let $A = B^{-1}$. Let $b = -b_1 - \eta e_i^T$, where $\eta = \frac{1}{2\lambda}\left(-b_1 S_1 b_1^T - 2\lambda b_1 e_i - 2 b_1 v_2 + S_3\right)$.

Note first that the ith column of S_1 is zero, so that by Lemma 6.3.1, $BS_1B^T = S_1$. Since B is in the isotropy subgroup, A must be also so that $AS_1A^T = S_1$.

The upper right position of (6.52) will be

$$\begin{aligned} A(S_1 b^T + S_2 c) &= B^{-1}(S_1(-b_1^T - \eta e_i) + (v_1 + \lambda e_i + v_2)) \\ &= B^{-1}(-v_1 - 0 + v_1 + \lambda e_i + v_2) \\ &= B^{-1}(\lambda e_i + v_2) \\ &= e_i. \end{aligned}$$

The lower right position of (6.52) will be

$$\begin{aligned} bS_1b^T + 2cbS_2 + c^2 S_3 &= b_1 S_1 b_1^T + 2(-b_1 - \eta e_i^T)(S_1(b_1^T + \eta e_i) + \lambda e_i + v_2) + S_3 \\ &= b_1 S_1 b_1^T + 2(-b_1 - \eta e_i^T)(S_1 b_1^T + \lambda e_i + v_2) + S_3 \\ &= b_1 S_1 b_1^T - 2 b_1 S_1 b_1^T - 2\lambda b_1 e_i - 2 b_1 v_2 - 2\eta e_i^T S_1 b_1^T - 2\eta\lambda e_i^T e_i - 2\eta e_i^T v_2 + S_3. \end{aligned}$$

Note that $e_i^T e_i = 1$, $e_i^T S_1 = 0$, $e_i^T v_2 = 0$, so we may simplify this to

$$\begin{aligned} &= b_1 S_1 b_1^T - 2 b_1 S_1 b_1^T - 2\lambda b_1 e_i - 2 b_1 v_2 - 2\eta\lambda + S_3 \\ &= -b_1 S_1 b_1^T - 2\lambda b_1 e_i - 2 b_1 v_2 - 2\eta\lambda + S_3. \end{aligned}$$

Finally, we have chosen η so that this quantity is zero.

Since the ith row and column of S_1 are zero, the resulting matrix satisfies the criteria of the theorem. $\square$

Corollary 6.3.3. *$L_n(\mathbb{C})$ acting on $\mathrm{Sym}_n(\mathbb{C})$ has a finite number of orbits. If (a_k) is given by the recurrence relation*

$$a_1 = 2, a_2 = 5, \qquad a_k = 2a_{k-1} + (n-1)a_{k-2},$$

then the number of orbits is less than a_n.

Proof. We need only count the number of normal forms. The 1×1 and 2×2 case can be easily verified by hand.

Now consider the kth case, where $S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix}$ is a normal form. First we count the forms where $S_2 = 0$; then either $S_3 = 0$ or $S_3 = 1$, and each of these options gives a bijection between the normal forms in the $k-1 \times k-1$ case and certain $k \times k$ normal forms. If $S_2 \neq 0$, then $S_2 = e_i$ for some $i \in \{1, \cdots, k-1\}$ and $S_3 = 0$. Notice that the ith row and column of S_1 must be zero. Thus eliminating rows i and k and columns i and k gives a bijection between the normal forms in the $k-2 \times k-2$ case and certain $k \times k$ normal forms; but this happens for each i, yielding the above recurrence relation. $\square$

As a first step, we may now show that the exceptional orbit variety of ρ is holonomic.

Corollary 6.3.4. *The free divisors of Theorem 4.4.5 are holonomic.*

Proof. Since this representation has a finite number of orbits (Corollary 6.3.3), the orbits form a Whitney stratification of $\mathrm{Sym}_n(\mathbb{C})$ (see, e.g., [**Dim92**], p.7). Removing the open orbit still yields a Whitney stratification of the exceptional orbit variety V, where if $v \in V_i$, a stratum of V, then

$$\langle \mathrm{Derlog}\,(V) \rangle_v = d(\rho^v)_{(e)}(\mathfrak{g}) = T_x V_i.$$

As a result, V is holonomic. $\square$

6.3.2. The Cholesky free divisors are H-holonomic. We will now show that the free divisors associated to the Cholesky factorization of symmetric matrices are H-holonomic (earlier stated as Theorem 6.2.2(1)). Fix an $n \in \mathbb{N}$, and recall from Theorem 4.4.5 that the free divisor in $\mathrm{Sym}_n(\mathbb{C})$ is the exceptional orbit variety of $\rho : L_n(\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{C}))$, $\rho(A)(M) = AMA^T$. If S is the generic $n \times n$ symmetric matrix and $g_k = \det(S^{(k)})$, then $f = \prod_{k=1}^n g_k$ defines this free divisor. We will identify the subgroup of $L_n(\mathbb{C})$ which generates Derlog (f) and show that this subgroup acts transitively on all non-open orbits of $L_n(\mathbb{C})$.

Let G_n be the subgroup of $L_n(\mathbb{C})$ consisting of matrices A where

$$\prod_{k=1}^{n}\left(\prod_{j=1}^{k}(A)_{jj}\right) = 1. \tag{6.53}$$

Proposition 6.3.5. *G_n is a connected complex algebraic Lie subgroup of $L_n(\mathbb{C})$ which leaves invariant all level sets of f.*

Proof. G_n is clearly a complex algebraic Lie subgroup of $L_n(\mathbb{C})$. We will show that G_n is connected. Let $L_1 \in G_n$ and let $L_2 \in G_n$ be the diagonal matrix consisting of the diagonal entries of L_1. As (6.53) depends only on the diagonal entries, there is a path in G_n connecting L_1 to L_2. Then for $j = 1, \ldots, n-1$, let $\gamma_j : [0,1] \to \mathbb{C} \setminus \{0\}$ be a curve from $(L_2)_{j,j}$ to 1. Since $A_{n,n}$ has an exponent of 1 in (6.53), define $\gamma_n : [0,1] \to \mathbb{C} \setminus \{0\}$ by

$$\prod_{k=1}^{n}\left(\prod_{j=1}^{k}\gamma_j(t)\right) = 1.$$

Then define $\Gamma : [0,1] \to G_n$ by

$$\Gamma(t) = \begin{pmatrix} \gamma_1(t) & 0 & \cdots & 0 \\ 0 & \gamma_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n(t) \end{pmatrix},$$

which gives a path from L_2 to $I \in G_n$. Thus G_n is connected.

We will now show that G_n leaves invariant all level sets of f. Notice that, for any $S \in \mathrm{Sym}_n(\mathbb{C})$ and $A \in L_n(\mathbb{C})$,

$$f(ASA^T) = \prod_{k=1}^{n} g_k(ASA^t) = \prod_{k=1}^{n} g_k(S)(g_k(A))^2 = f(S)\left(\prod_{k=1}^{n}\prod_{j=1}^{k}(A)_{jj}\right)^2,$$

so that G_n moves points of $\mathrm{Sym}_n(\mathbb{C})$ on level sets of f. $\square$

Consequently, the vector fields associated to the Lie algebra of G_n annihilate f and lie in Derlog (f).

Corollary 6.3.6. *Let $S \in \mathrm{Sym}_n(\mathbb{C})$, and let L be a lower triangular matrix as in Theorem 4.4.6 so that $LSL^T = N$ is in normal form. If S is in the exceptional orbit variety, then there exists an invertible diagonal matrix D with $(DL)S(DL)^T = DLSL^T D^T = DND^T = N$, and $DL \in G_n$; thus G_n acts transitively on all non-open orbits of $L_n(\mathbb{C})$.*

Proof. First consider the case where N is diagonal. Since S is in the exceptional orbit variety, N is not the identity so there is some $e_i \in \ker(N)$; the ith column of N is zero. Let

$$(D)_{pp} = \begin{cases} 1 & \text{if } p \neq i \\ \lambda & \text{if } p = i \end{cases},$$

with $\lambda \in \mathbb{C}^*$ to be determined. Then

$$\prod_{q=1}^{n} \prod_{p=1}^{q} (DL)_{pp} = (\lambda)^{n-i+1} \left(\prod_{q=1}^{n} \prod_{p=1}^{q} (L)_{pp} \right);$$

since $n - i + 1 > 0$, we may choose $\lambda \neq 0$ so that this product is 1 and $DL \in G_n$. By Lemma 6.3.1, $DND^T = N$, proving this part of the corollary.

If N is not diagonal, then there exist nonzero entries off the main diagonal and we may find $i < j$ so that $Ne_i = e_j$ and $Ne_j = e_i$. Let

$$(D)_{pp} = \begin{cases} 1 & \text{if } p \neq i \text{ and } p \neq j \\ \lambda & \text{if } p = i \\ 1/\lambda & \text{if } p = j \end{cases},$$

with $\lambda \in \mathbb{C}^*$ to be determined. Note that for $q < i$ and $q > j$, we have $\prod_{p=1}^{q} (DL)_{pp} = \prod_{p=1}^{q} (L)_{pp}$: in the former case, λ's do not appear, while in the latter case, the λ and $\frac{1}{\lambda}$ cancel. There are $j - i$ other cases, so

$$\prod_{q=1}^{n} \prod_{p=1}^{q} (DL)_{pp} = (\lambda)^{j-i} \left(\prod_{q=1}^{n} \prod_{p=1}^{q} (L)_{pp} \right);$$

since $j - i > 0$, we may choose λ so that this product is 1 and $DL \in G_n$. By Lemma 6.3.2, $DND^T = N$, finishing the corollary. $\square$

Proof of Theorem 6.2.2(1). Corollary 6.3.6 shows that on any non-open orbit O of $L_n(\mathbb{C})$ acting on $\mathrm{Sym}_n(\mathbb{C})$, the group G_n acts transitively on O. Thus at any $x \in O$, the span of the vector fields associated to the Lie algebra of G_n is equal to $T_x O$. All of these vector fields lie in Derlog (f) by Proposition 6.3.5, so that

$$\langle \mathrm{Derlog}\,(f) \rangle_x = T_x O.$$

But Corollary 6.3.4 showed that O is a stratum of the Whitney stratification of V. □

6.3.3. Structure of the orbits of the modified LU factorization. We now prove a fact, stated as Theorem 5.1.10, that any $n \times n$ or $n \times (n+1)$ matrix can be put in a normal form under the representation associated to the modified LU factorization (Theorems 5.1.6,5.1.7). Since there are a finite number of normal forms (see Definition 5.1.8), there are a finite number of orbits. This is the first step in showing that the free divisors found in Theorems 5.1.4 and 5.1.5 are H-holonomic.

We will now prove this theorem:

Theorem (5.1.10). *Let $M(n,m,\mathbb{C})$ be the space of $n \times m$ complex matrices, and let G be the subgroup of $L_n(\mathbb{C}) \times T_m(\mathbb{C})$ consisting of all invertible lower triangular matrices and those invertible upper triangular matrices whose first row is $(1,0,\cdots,0)$. Let G act on $M(n,m,\mathbb{C})$ by*

$$(L,U) \cdot S = LSU^{-1}.$$

If either $m = n$ or $m = n+1$, then for all $S \in M(n,m,\mathbb{C})$ there exists an element $g \in G$ so that $g \cdot S$ is a normal form according to Definition 5.1.8

Remark 6.3.7. Compute calculations suggest that the normal forms of Definition 5.1.8 are in bijective correspondence with the orbits. We will only prove, however, that the normal forms map surjectively to orbits.

Proof of Theorem 5.1.10. The statement is clearly true in the 1×1 case, where we have

$$(l_{11})(s_{11})(1) = (l_{11}s_{11}),$$

and the normal forms are (0) and (1). For the 1×2 case, the normal forms are

$$\begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix},$$

and clearly any matrix can be put into one of these forms by the action of G:

$$\begin{pmatrix} l_{11} \end{pmatrix} \begin{pmatrix} s_{11} s_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} l_{11}s_{11} & l_{11}u_{22}s_{12} \end{pmatrix}.$$

It is then enough to show that the $n \times n$ case implies the $n \times (n+1)$ case and the $n \times (n+1)$ case implies the $(n+1) \times (n+1)$ case. We shall tackle these two steps in Propositions 6.3.11 and 6.3.12. □

First, we prove a few lemmas which show that certain elements of our group will lie in the isotropy subgroup of certain points of our spaces.

LEMMA 6.3.8. *Suppose the ith column of the $n \times m$ matrix S is zero. Let*

$$T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & * & * \\ 0 & 0 & I_{m-i} \end{pmatrix},$$

an upper triangular $m \times m$ matrix. Then $ST = S$.

PROOF. This is a straightforward calculation. □

LEMMA 6.3.9. *Suppose that the jth row of the $n \times m$ matrix S is zero. Let*

$$L = \begin{pmatrix} I_{j-1} & 0 & 0 \\ 0 & * & 0 \\ 0 & * & I_{n-j} \end{pmatrix},$$

a lower triangular $n \times n$ matrix. Then $LS = S$.

PROOF. This is just the transpose of Lemma 6.3.8 with different notation. □

LEMMA 6.3.10. *Suppose that the jth row and kth column of the $n \times m$ matrix S are zero, except perhaps for $(S)_{j,k}$. Let $\lambda \in \mathbb{C}^*$ and let*

$$L = \begin{pmatrix} I_{j-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & I_{n-j} \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & I_{m-k} \end{pmatrix},$$

both be diagonal matrices. Then $LSU = S$.

PROOF. LS will only multiply the jth row of S by λ, while LSU will only multiply the kth column of LS by $\frac{1}{\lambda}$, so that $LSU = S$. □

PROPOSITION 6.3.11. *When $n \geq 2$, the $n \times n$ case of Theorem 5.1.10 implies the $n \times (n+1)$ case of Theorem 5.1.10.*

PROOF. Note that

$$\begin{pmatrix} L \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_1 & U_2 \\ 0 & 0 & U_3 \end{pmatrix} = \begin{pmatrix} LS_{11} & LS_{12}U & L(S_{12}U_2 + S_{13}U_3) \end{pmatrix}. \tag{6.54}$$

Here, S_{11} and S_{13} are $n \times 1$ column vectors; U_2 is a $(n-1) \times 1$ column vector; U_3 is a scalar; and L, S_{12}, and U_1 are matrices with S_{12} $n \times (n-1)$, L $n \times n$, and U_1 $(n-1) \times (n-1)$. By our induction hypothesis, we may already assume that $\begin{pmatrix} S_{11} & S_{12} \end{pmatrix}$ is in its normal form.

First consider the case where $S_{13} \in \text{image}(S_{12})$, so $S_{12}b = S_{13}$. Let $L = I$, $U_1 = I$, $U_2 = -b$, and $U_3 = 1$. Examining (6.54) shows that the leftmost column will be S_{11}, the middle block will be S_{12}, and the rightmost column will be

$$\begin{aligned} L(S_{12}U_2 + S_{13}U_3) &= -S_{12}b + S_{13} \\ &= -S_{13} + S_{13} \\ &= 0. \end{aligned}$$

The resulting matrix clearly satisfies Definition 5.1.8.

Next consider the case where $S_{13} \notin \text{image}(S_{12})$. Let v_1 be a column vector with

$$(v_1)_j = \begin{cases} (S_{13})_j & \text{if } e_j \in \text{image}(S_{12}) \\ 0 & \text{otherwise} \end{cases},$$

so that $v_1 = S_{12}b$ for some $n-1 \times 1$ vector b. Since $S_{13} \notin \text{image}(S_{12})$, there exists a smallest i so that $\lambda = (S_{13} - v_1)_i \neq 0$. Finally, let $v_2 = S_{13} - V_1 - \lambda e_i$ so that

$$S_{13} = v_1 + \lambda e_i + v_2.$$

(1) Consider the case where $(S_{11})_i = 0$. Let

$$T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & v_2 & I_{n-i} \end{pmatrix},$$

a lower triangular matrix with $TS_{12} = S_{12}$ and $TS_{11} = S_{11}$ by Lemma 6.3.9 and $Te_i = \lambda e_i + v_2$. Let $L = T^{-1}$, so that $LS_{11} = S_{11}$, $LS_{12} = S_{12}$, and $L(\lambda e_i + v_2) = e_i$. Let $U_1 = I$, $U_2 = -b$, and $U_3 = 1$. Examining (6.54) shows that the leftmost column will be $LS_{11} = S_{11}$, the middle block will be $LS_{12}U_1 = S_{12}$, and the rightmost column will be

$$\begin{aligned} L(S_{12}U_2 + S_{13}U_3) &= L(-S_{12}b + S_{13}) \\ &= L(\lambda e_i + v_2) \\ &= e_i. \end{aligned}$$

Since the ith rows of S_{11} and S_{12} are zero, the resulting matrix is a normal form.

(2) Now consider the case where $(S_{11})_i \neq 0$, and so by the normal form, $(S_{11})_i = 1$. Since the ith row of S_{12} is zero, by our definition of normal form, S_{11} will be zero below the ith row.

(a) If $v_2 = 0$, then let $L = I$, $U_1 = I$, $U_2 = -\frac{1}{\lambda}b$, and $U_3 = \frac{1}{\lambda}$. Examining (6.54) shows that the leftmost column will be S_{11}, the middle block will be S_{12}, and the rightmost column will be

$$\begin{aligned} L(S_{12}U_2 + S_{13}U_3) &= -\frac{1}{\lambda}S_{12}b + \frac{1}{\lambda}S_{13} \\ &= \frac{1}{\lambda}(S_{13} - S_{12}b) \\ &= \frac{1}{\lambda}\lambda e_i \\ &= e_i. \end{aligned}$$

Because S_{11} is zero below the ith row, the resulting matrix is a normal form.

(b) If $v_2 \neq 0$, then there exists a least $j > i$ so that $(\lambda e_i + v_2)_j = (v_2)_j \neq 0$. Let $\eta = (v_2)_j$, and $v_3 = v_2 - \eta e_j$, so that

$$S_{13} = v_1 + \lambda e_i + \eta e_j + v_3.$$

Let $U_1 = I$, $U_2 = -\frac{1}{\lambda} b$, $U_3 = \frac{1}{\lambda}$, and let T be the identity except that the ith column should equal $e_i + \frac{\eta}{\lambda} e_j + \frac{1}{\lambda} v_3$ and the jth column should equal $-\frac{\eta}{\lambda} e_j - \frac{1}{\lambda} v_3$. Then T is invertible, lower triangular, with $TS_{12} = S_{12}$ (analogous to Lemma 6.3.9; the ith and jth rows of S_{12} must be zero), $T(e_i + e_j) = e_i$, and $Te_i = e_i + \frac{\eta}{\lambda} e_j + \frac{1}{\lambda} v_3$. Let $L = T^{-1}$, so that $LS_{12} = S_{12}$, $LS_{11} = S_{11} + e_j$ (note that $(S_{11})_j = 0$), and $L(e_i + \frac{\eta}{\lambda} e_j + \frac{1}{\lambda} v_3) = e_i$.

Examining (6.54) shows that the leftmost column will be $LS_{11} = S_{11} + e_j$, the middle block will be $LS_{12}U_1 = S_{12}$, and the rightmost column will be

$$\begin{aligned} L(S_{12}U_2 + S_{13}U_3) &= L\left(-\frac{1}{\lambda} S_{12} b + S_{13} \frac{1}{\lambda}\right) \\ &= \frac{1}{\lambda} L(S_{13} - S_{12} b) \\ &= \frac{1}{\lambda} L(\lambda e_i + \eta e_j + v_3) \\ &= L(e_i + \frac{\eta}{\lambda} e_j + \frac{1}{\lambda} v_3) \\ &= e_i. \end{aligned}$$

In the resulting matrix $S' = \begin{pmatrix} S'_{11} & S'_{12} & S'_{13} \end{pmatrix}$, the ith row of $(S'_{12}|S'_{13})$ will have the rightmost 1 so that Definition 5.1.8 part 4 (the "increasing" requirement) is met. S'_{11} will now have a 1 in the jth row and the jth row of $\begin{pmatrix} S'_{12} & S'_{13} \end{pmatrix}$ will equal zero, but the last nonzero entry in S'_{11} appears in the jth row. As a result, S' is a normal form.

□

PROPOSITION 6.3.12. . *When $n \geq 1$, the $n \times (n+1)$ case of Theorem 5.1.10 implies the $(n+1) \times (n+1)$ case of Theorem 5.1.10.*

Proof. Note that

$$(6.55)\qquad \begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} L_1S_{11} & L_1S_{12}U \\ L_2S_{11}+L_3S_{21} & L_2S_{12}U+L_3S_{22}U \end{pmatrix},$$

where L_1, S_{12}, and U are $n\times n$ matrices, L_2 and S_{22} are $1\times n$ row vectors, and S_{11} is a $n\times 1$ column vector, and L_3, S_{21} are scalars. By our induction hypothesis, we may already assume that $\begin{pmatrix} S_{11} & S_{12} \end{pmatrix}$ is in its normal form.

Let v_1 be a $1\times n$ row vector with

$$(v_1)_j = \begin{cases} (S_{22})_j & \text{if } e_j^T \in \text{Rowspan}(S_{12}) \\ 0 & \text{otherwise} \end{cases},$$

so that $v_1 = bS_{12}$ for some $1\times n$ row vector b. Fix i to be the smallest number so that $(S_{22}-v_1)_i \neq 0$, and fix $\lambda = (S_{22}-v_1)_i$; if $S_{22}-v_1 = 0$, then let $i=1$ and $\lambda = 0$. Let $v_2 = S_{22}-v_1-\lambda e_i^T$, so that

$$S_{22} = v_1 + \lambda e_i^T + v_2.$$

First consider the case where there exists a j so that $(S_{11})_j \neq 0$ but the jth row of (S_{12}) is zero, so that $e_j^TS_{12} = 0$ and $e_j^TS_{11} = 1$. Because of our definition of normal form, we must force the lower left entry of (6.55) to be zero. Then either $\lambda = 0$ or $\lambda \neq 0$.

(1) If $\lambda = 0$, we also have $v_2 = 0$, and $S_{22} = v_1 = bS_{12}$, so $S_22 \in \text{Rowspan}(S_{12})$. Let $L_1 = U = I$, $L_2 = -b+(bS_{11}-S_{21})e_j^T$, and $L_3 = 1$. It is clear that the upper blocks of (6.55) will be S_{11} and S_{12}, the lower left will be

$$\begin{aligned} L_2S_{11}+L_3S_{21} &= -bS_{11}+(bS_{11}-S_{21})e_j^TS_{11}+S_{21} \\ &= -bS_{11}+bS_{11}-S_{21}+S_{21} \\ &= 0, \end{aligned}$$

while the lower right will be

$$\begin{aligned} L_2S_{12}U+L_3S_{22}U &= -bS_{12}+(bS_{11}-S_{21})e_j^TS_{12}+S_{22} \\ &= -v_1+0+S_{22} \\ &= 0. \end{aligned}$$

Our result will thus be a normal form.

(2) If $\lambda \neq 0$, then let $L_1 = I$, $L_2 = \frac{1}{\lambda}(-b + (bS_{11} - S_{21})e_j^T)$, $L_3 = \frac{1}{\lambda}$. Let $T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & \frac{1}{\lambda}v_2 \\ 0 & 0 & I_{n-i} \end{pmatrix}$ be an upper triangular matrix. Since $e_i \notin \text{Rowspan}(S_{12})$, the ith column of e_i is zero, so that by Lemma 6.3.8, $S_{12}T = S_{12}$. Note also that $e_i^T T = e_i^T + \frac{1}{\lambda}v_2$. If we let $U = T^{-1}$, then U is upper triangular, $S_{12} = S_{12}U$, and $e_i^T = (e_i^T + \frac{1}{\lambda}v_2)U$. Examining (6.55) show that the resulting matrix will have S_{11} as its upper left block, $S_{12}U = S_{12}$ as its upper right block, the lower left block will be

$$\begin{aligned} L_2S_{11} + L_3S_{21} &= \frac{1}{\lambda}(-b + (bS_{11} - S_{21})e_j^T)S_{11} + \frac{1}{\lambda}S_{21} \\ &= \frac{1}{\lambda}((-bS_{11} + (bS_{11} - S_{21}) + S_{21}) \\ &= 0, \end{aligned}$$

and the lower right block will be

$$\begin{aligned} L_2S_{12}U + L_3S_{22}U &= \frac{1}{\lambda}(-b + (bS_{11} - S_{21})e_j^T)S_{12}U + \frac{1}{\lambda}S_{22}U \\ &= \frac{1}{\lambda}\left(-bS_{12} + (bS_{11} - S_{21})0 + S_{22}\right)U \\ &= \frac{1}{\lambda}\left(-v_1 + S_{22}\right)U \\ &= \frac{1}{\lambda}\left(\lambda e_i^T + v_2\right)U \\ &= \left(e_i^T + \frac{1}{\lambda}v_2\right)U \\ &= e_i^T. \end{aligned}$$

Note that the product will be in a normal form, since the ith column of S_{12} is zero.

Alternatively, we must have for all j, if $(S_{11})_j \neq 0$ then the jth row of S_{12} is not zero. In this case, we may end up with the lower left entry of (6.55) being either 0 or 1. As above, we break into cases.

(1) First consider the case where $\lambda = 0$, so that $v_2 = 0$, $S_{22} = v_1 = bS_{12}$, and $S_{22} \in \text{Rowspan}(S_{12})$. Note that $\begin{pmatrix} S_{21} & S_{22} \end{pmatrix} \in \text{Rowspan}\begin{pmatrix} S_{11} & S_{12} \end{pmatrix}$ if and only if $S_{21} - bS_{11} = 0$. Let $L_1 = U = I$,

$$L_3 = \begin{cases} 1 & \text{if } S_{21} - bS_{11} = 0 \\ (S_{21} - bS_{11})^{-1} & \text{if } S_{21} - bS_{11} \neq 0 \end{cases},$$

and $L_2 = -L_3 b$. It is clear that the upper left and upper right blocks of (6.55) will be S_{11} and S_{12}. The lower left block will be

$$\begin{aligned} L_2 S_{11} + L_3 S_{21} &= -L_3 b S_{11} + L_3 S_{21} \\ &= L_3(S_{21} - bS_{11}), \end{aligned}$$

either 0 or 1, and the lower right block will be

$$\begin{aligned} L_2 S_{12} U + L_3 S_{22} U &= -L_3 b S_{12} + L_3 S_{22} \\ &= L_3(S_{22} - bS_{12}) \\ &= 0. \end{aligned}$$

The result is a normal form.

(2) Next consider the case where $\lambda \neq 0$ and for all $j = 1, \ldots, n$, if $e_j^T S_{12} = e_k^T$ with $k > i$, then $(S_{11})_{j1} = 0$. As above, we will pay attention to $S_{21} - bS_{11}$. Let $L_1 = I$,

$$L_3 = \begin{cases} 1 & \text{if } S_{21} - bS_{11} = 0 \\ (S_{21} - bS_{11})^{-1} & \text{if } S_{21} - bS_{11} \neq 0 \end{cases},$$

and $L_2 = -L_3 b$. Furthermore, let $T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \lambda L_3 & L_3 v_2 \\ 0 & 0 & I_{n-i} \end{pmatrix}$, an upper triangular matrix; since the ith column of S_{12} is zero, Lemma 6.3.8 implies that $S_{12}T = S_{12}$. Additionally, $e_i^T T = L_3(\lambda e_i^T + v_2)$. Let $U = T^{-1}$, so that $S_{12} = S_{12}U$ and $e_i^T = L_3(\lambda e_i^T + v_2)U$. Examining the blocks of (6.55) shows that the upper left block will be S_{11}, the upper right block will be $S_{12}U = S_{12}$, the lower left block will be

$$\begin{aligned} L_2 S_{11} + L_3 S_{21} &= -L_3 b S_{11} + L_3 S_{21} \\ &= L_3(S_{21} - bS_{11}), \end{aligned}$$

either 0 or 1, and the lower right block will be

$$
\begin{aligned}
L_2S_{12}U + L_3S_{22}U &= (-L_3bS_{12} + L_3S_{22})\,U \\
&= L_3\,(S_{22} - bS_{12})\,U \\
&= L_3\left(\lambda e_i^T + v_2\right)U \\
&= e_i^T.
\end{aligned}
$$

Because of our hypotheses on S_{11} and S_{12} in this situation, the resulting matrix is a normal form; we are consistent with Definition 5.1.8 part 4, the "increasing" requirement.

(3) Finally, consider the case where $\lambda \neq 0$ and there exists a $j = 1, \ldots, n$ and a $k > i$ with $e_j^T S_{12} = e_k^T$ and $(S_{11})_{j1} \neq 0$. In fact, $e_j^T S_{11} = 1$. In order to satisfy the "increasing" requirement of Definition 5.1.8, we must force the lower left block of (6.55) to be zero.

Let $L_1 = I$, $L_2 = \frac{1}{\lambda}(-b+(bS_{11}-S_{21})e_j^T)$, and $L_3 = \frac{1}{\lambda}$. Let $T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & e_i^T + \frac{1}{\lambda}v_2 + \frac{1}{\lambda}(bS_{11} - S_{21} \\ 0 & 0 & I_{n-i} \end{pmatrix}$ an upper triangular matrix with $S_{12}T = S_{12}$ (by Lemma 6.3.8) and with $e_i^T T = e_i^T + \frac{1}{\lambda}v_2 + \frac{1}{\lambda}(bS_{11} - S_{21})e_k^T = \frac{1}{\lambda}(\lambda e_i^T + v_2 + (bS_{11} - S_{21})e_k^T)$. Note that we have used the fact that $k > i$ to construct an upper triangular T. Let $U = T^{-1}$. Examining (6.55), we see that the upper left block will be S_{11}, the upper right block will be $S_{12}U = S_{12}$, the lower left block will be

$$
\begin{aligned}
L_2S_{11} + L_3S_{21} &= \frac{1}{\lambda}(-b + (bS_{11} - S_{21})e_j^T)S_{11} + \frac{1}{\lambda}S_{21} \\
&= -\frac{1}{\lambda}bS_{11} + \frac{1}{\lambda}(bS_{11} - S_{21})1 + \frac{1}{\lambda}S_{21} \\
&= 0,
\end{aligned}
$$

and the lower right block will be

$$\begin{aligned}
L_2S_{12}U + L_3S_{22}U &= \frac{1}{\lambda}(-b + (bS_{11} - S_{21})e_j^T)S_{12}U + \frac{1}{\lambda}S_{22}U \\
&= \frac{1}{\lambda}\left(-bS_{12} + (bS_{11} - S_{21})e_j^T S_{12} + S_{22}\right)U \\
&= \frac{1}{\lambda}\left(S_{22} - bS_{12} + (bS_{11} - S_{21})e_k^T\right)U \\
&= \frac{1}{\lambda}\left(\lambda e_i^T + v_2 + (bS_{11} - S_{21})e_k^T\right)U \\
&= e_i^T.
\end{aligned}$$

The resulting matrix is clearly a normal form.

□

6.3.4. The modified LU factorization free divisors are H-holonomic. We will now prove Theorem 6.2.2, that the free divisors found in Theorems 5.1.4 and 5.1.5 are H-holonomic.

Consider the space of $n \times m$ matrices, $m \in \{n, n+1\}$. Let S be the generic $n \times m$ matrix. Let

$$f_k = \det(S^{(k)}) \qquad \text{and} \qquad g_l = \det(S^{(1,l)})$$

for $k = 1, \ldots, n$ and $l = 1, \ldots, m-1$. Then the free divisors from the theorems are defined by

$$h = \left(\prod_{k=1}^{n} f_k\right) \cdot \left(\prod_{k=1}^{m-1} g_k\right).$$

Let $G \subset L_n(\mathbb{C}) \times T_m(\mathbb{C})$ be the group consisting of pairs (A, B) of upper triangular and lower triangular matrices, where the first row of B is $\begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}$. Recall that G acts on $M(n, m, \mathbb{C})$ by $(A, B) \cdot M = AMB^{-1}$. Define the subgroup K of G consisting of $(A, B) \in G$ such that

$$\left(\prod_{i=1}^{n}((A)_{ii})^{n+1-i}\right)\left(\prod_{i=1}^{m-1}((A)_{ii})^{m-i}\right)\left(\prod_{i=2}^{n}((B)_{ii})^{-(n+1-i)}\right)\left(\prod_{i=2}^{m}((B)_{ii})^{-(m+1-i)}\right) = 1. \tag{6.56}$$

In the square case where $m = n$, (6.56) is equivalent to

$$\left(\prod_{i=1}^{n}((A)_{ii})^{2(n-i)+1}\right)\left(\prod_{i=2}^{n}\left(\frac{1}{(B)_{ii}}\right)^{2(n-i+1)}\right) = 1. \tag{6.57}$$

In the non-square case where $m = n + 1$, (6.56) is equivalent to

$$\left(\prod_{i=1}^{n} ((A)_{ii})^{2(n-i+1)}\right)\left(\prod_{i=2}^{n+1}\left(\frac{1}{(B)_{ii}}\right)^{2(n-i+1)+1}\right) = 1. \tag{6.58}$$

PROPOSITION 6.3.13. *K is a connected complex algebraic Lie subgroup of G which leaves invariant all level sets of h.*

PROOF. K is clearly a complex algebraic Lie subgroup of G. The proof that K is connected is essentially the same as in the proof of Proposition 6.3.5, due to the fact that (6.57) can be solved for $(A)_{nn}$ and (6.58) can be solved for $(B)_{n+1,n+1}$.

Let $(A, B) \in G$. Since

$$\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}^{-1} = \begin{pmatrix} (B_1)^{-1} & -(B_1)^{-1}B_2(B_3)^{-1} \\ 0 & (B_3)^{-1} \end{pmatrix},$$

$(A, B) \cdot S$ takes the form

$$\begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} (B_1)^{-1} & -(B_1)^{-1}B_2(B_3)^{-1} \\ 0 & (B_3)^{-1} \end{pmatrix} = \begin{pmatrix} L_1S_{11}(B_1)^{-1} & * \\ * & * \end{pmatrix},$$

where L_1, S_{11} and B_1 are all $k \times k$ matrices. Thus

$$f_k(ASB^{-1}) = f_k(S)\left(\prod_{i=1}^{k}(A)_{ii}\right)\left(\prod_{i=2}^{k}((B)_{ii})^{-1}\right).$$

Note that in the $k = 1$ case we interpret the rightmost product as a 1.

Similarly we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (B_{22})^{-1} & -(B_{22})^{-1}B_{23}(B_{33})^{-1} \\ 0 & 0 & (B_{33})^{-1} \end{pmatrix},$$

so $(A,B)\cdot S$ will take the form

$$\begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (B_{22})^{-1} & -(B_{22})^{-1}B_{23}(B_{33})^{-1} \\ 0 & 0 & (B_{33})^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} L_1S_{11} & L_1S_{12}(B_{22})^{-1} & * \\ * & * & * \end{pmatrix},$$

where S_{11} is a column vector and L_1, S_{12}, and B_{22} are $k \times k$ matrices. Therefore,

$$g_k(ASB^{-1}) = g_k(S)\left(\prod_{i=1}^{k}(A)_{ii}\right)\left(\prod_{i=2}^{k+1}((B)_{ii})^{-1}\right).$$

As a result, $h(ASB^{-1})$ equals

$$= \left(\prod_{k=1}^{n} f_k(S)\left(\prod_{i=1}^{k}(A)_{ii}\right)\left(\prod_{i=2}^{k}((B)_{ii})^{-1}\right)\right)\left(\prod_{k=1}^{m-1} g_k(S)\left(\prod_{i=1}^{k}(A)_{ii}\right)\left(\prod_{i=2}^{k+1}((B)_{ii})^{-1}\right)\right)$$
$$= \left(\prod_{k=1}^{n} f_k(S)\prod_{k=1}^{m-1} g_k(S)\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(A)_{ii}\right)\left(\prod_{k=1}^{n}\prod_{i=2}^{k}((B)_{ii})^{-1}\right)\left(\prod_{k=1}^{m-1}\prod_{i=1}^{k}(A)_{ii}\right)\left(\prod_{k=1}^{m-1}\prod_{i=2}^{k+1}((B)_{ii})^{-1}\right)$$
$$= h(S)\left(\prod_{i=1}^{n}((A)_{ii})^{n+1-i}\right)\left(\prod_{i=1}^{m-1}((A)_{ii})^{m-i}\right)\left(\prod_{i=2}^{n}((B)_{ii})^{-(n+1-i)}\right)\left(\prod_{i=2}^{m}((B)_{ii})^{-(m+1-i)}\right).$$

Thus K moves elements of $M(n,m,\mathbb{C})$ along level sets of h. □

In particular, the vector fields associated to the Lie algebra of K lie in Derlog$\,(h)$.

We shall now show that K acts transitively on the non-open orbits of the exceptional orbit variety in each of these cases.

Corollary 6.3.14. *Let $S \in M(n,m,\mathbb{C})$, ($m \in \{n, n+1\}$) and let $(L,U) \in L_n(\mathbb{C}) \times T_m(\mathbb{C})$ (with the first row of U equal to $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$) as in Theorem 5.1.10 so that $LSU^{-1} = N$ is a normal form. If S is in the exceptional orbit variety, then there exists a pair of invertible diagonal matrices D_1, D_2 with $(D_1L)S(D_2U)^{-1} = D_1N(D_2)^{-1} = N$ and $(D_1L, D_2U) \in K$, so that K acts transitively on the non-open orbits of G.*

Proof. First note that the statement is correct in the $n = m = 1$ case. Assume, then, that $m > 1$.

Write $S = \begin{pmatrix} S_{11} & S_{12} \end{pmatrix}$ and $N = \begin{pmatrix} N_{11} & N_{12} \end{pmatrix}$, where S_{11} and N_{11} are column vectors. Note that

$$\begin{pmatrix} M \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} MT_{11} & MT_{12}V \end{pmatrix},$$

where M is lower triangular, V is upper triangular, and T_{11} is a column vector.

First consider the case where there exists an $i \in \{1, \ldots, m-1\}$ with $e_i^T \notin \text{Rowspan}(N_{12})$, so that the ith column of N_{12} is zero. Let $D_1 = I$ and $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & D_3 \end{pmatrix}$, with D_3 equal to the identity, except with $(D_3)_{ii} = \lambda \in \mathbb{C}^*$, a number yet to be determined. From Lemma 6.3.8, we know that $N_{12}D_2^{-1} = N_{12}$, so that $D_1ND_2^{-1} = N$ and $(D_1L)S(D_2U)^{-1} = N$. Considering the left-hand sides of (6.57) and (6.58) with $A = D_1L$ and $B = D_2U$, it is clear that (in either case) we will obtain a nonzero number times $\frac{1}{\lambda}$ raised to a nonzero exponent. We may clearly then choose $\lambda \in \mathbb{C}^*$ so that (6.57) or (6.58) holds so that $(D_1L, D_2U) \in G$.

Otherwise, we will have $e_1^T, \ldots, e_{m-1}^T$ all in Rowspan(N_{12}), so that the $n \times (m-1)$ matrix S_{12} must have maximal rank.

(1) Suppose there exists a $j \in \{1, \ldots, m-1\}$ with $(N_{11})_{j1} = 0$, i.e., N_{11} has a zero row.

  (a) If the jth row of N_{12} is zero (this is possible only in the square case, where N_{12} is $n \times (n-1)$), then let $D_2 = I$ and let D_1 be the identity except with $(D_1)_{jj} = \lambda \in \mathbb{C}^*$, a number yet to be determined. By Lemma 6.3.9, we know that $D_1N_{11} = D_1$ and $D_1N_{12} = N_{12}$, so that $D_1N(D_2)^{-1} = N$ and $(D_1L)S(D_2U)^{-1} = N$. Considering the left-hand side of (6.57) with $A = D_1L$ and $B = D_2U = U$, it is clear that we will have a nonzero number times λ raised to a nonzero exponent. We may then choose $\lambda \in \mathbb{C}^*$ so that (6.57) holds, so $(D_1L, D_2U) \in G$.

  (b) If the jth row of N_{12} is e_k^T, then let D_1 be the matrix which is the identity except for $(D_1)_{jj} = \lambda \in \mathbb{C}^*$ and let $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & D_3 \end{pmatrix}$, with D_3 equal to the identity except with $(D_3)_{kk} = \lambda$, where λ is yet to be determined. Note that $D_1N_{11} = N_{11}$ by Lemma 6.3.9 and $D_1N_{12}D_3^{-1} = N_{12}$ by Lemma 6.3.10. As a result, $D_1ND_2^{-1} = N$, so $(D_1L)S(D_2U)^{-1} = N$. Considering the left-hand side of equations (6.57) and (6.58) with $A = D_1L$ and $B = D_2U$, we see that in either case we shall have a nonzero number times λ raised to a nonzero power times $\frac{1}{\lambda}$ raised to a nonzero power. Since in either cases the sets of exponents do not match (one is odd, one is even), we will not have cancellation and may choose $\lambda \in \mathbb{C}^*$ so that (6.57) or (6.58) are satisfied.

(2) Finally consider the case where N_{11} has all ones. Since N_{12} has maximum rank and N is a normal form, the $n \times (m-1)$ matrix N_{12} must be either $\begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}$ in the $m = n$ case or I_n in the $m = n+1$ case. But in either case, N (and S) lie in the open orbit, which is not considered by the Corollary.

□

Proof of Theorem 6.2.2(2),(3). Corollary 6.3.14 shows that on any non-open orbit O of G acting on $M(n, m, \mathbb{C})$, $m \in \{n, n+1\}$, the group K acts transitively on O. Thus at any $x \in O$, the span of the vector fields associated to the Lie algebra of K is equal to $T_x O$. All of these vector fields lie in $\mathrm{Derlog}\,(f)$ by Proposition 6.3.13, so that

$$\langle \mathrm{Derlog}\,(f) \rangle_x = T_x O.$$

By Theorem 5.1.10, G has a finite number of orbits. Thus O is a stratum of the Whitney stratification of V. □

APPENDIX A

Calculation of Lie brackets of certain vector fields

In this appendix we will calculate the Lie brackets of the vector fields involved in the free divisor on the space of skew-symmetric matrices which we develop in §5.2.2. The bracket of two of the linear vector fields coming from $\mathfrak{g}$, the Lie algebra of G as used in §5.2.2 will of course give another linear vector field coming from $\mathfrak{g}$. Thus we need only calculate the brackets of the linear vector fields with the nonlinear and the nonlinear vector fields with each other.

We will heavily use Proposition 5.2.7 with Remark 5.2.8 (the derivative of the Pfaffian of a skew-symmetric submatrix of a generic skew-symmetric matrix) and especially the Pfaffian identities generated by Theorem 5.2.4. The calculation is made more difficult because of the complicated sign present in the derivative of the Pfaffian; depending on the sign, we either have cancellation or not. Each calculation takes the following form:

(1) Break into a number of cases based on the possible orderings of variables. For example, when calculating $[\eta_{a,k}, \eta_{b,l}]$, where are a, b, k, and l in relation to one another? Knowing this will allow us to find the sign present in the derivative of the Pfaffian.
(2) For each case, we
    (a) Write the bracket as a sum of terms involving derivatives of Pfaffians;
    (b) Simplify the derivative of the Pfaffian, using the correct sign;
    (c) Use the Pfaffian identities to simplify our expression;
    (d) Simplify, often canceling a number of terms.

Though these calculations are simple conceptually, the execution of them is rather tedious.

Remark A.0.1. For the calculations and results in this Appendix only, we use the "usual" Lie bracket.

A.1. Calculation of the bracket of two nonlinear vector fields

We will now prove Proposition 5.2.14, restated here in terms of the usual Lie bracket.

Proposition (5.2.14). *Let $a, b \in \{1, 2\}$ and choose k and l so that $a < k \leq n$, $b < l \leq n$, and $k < l$. Then*

$$[\eta_{a,k}, \eta_{b,l}] = \frac{1}{2}(\delta_{a,b} + l - k - 1)S(\{a, \ldots, k\}) \cdot \eta_{b,l}.$$

For this section it will be useful to let $\alpha(x) = S(\{a \cdots k, x\}\Delta\{1\})$ when $x \notin \{a, \ldots, k\}$, and $\beta(i, x, y) = S((\{b, \cdots, \hat{i}, \cdots, l\} \cup \{x, y\})\Delta\{1\})$.

PROOF. If $\eta_{a,k} = \sum_{p<q} f_{pq} \frac{\partial}{\partial x_{pq}}$ and $\eta_{b,l} = \sum_{p<q} g_{pq} \frac{\partial}{\partial x_{pq}}$, then we can write their bracket in the form

$$[\eta_{ak}, \eta_{bl}] = \sum_{p<q} \left(\left(\sum_{i<j} f_{ij} \frac{\partial g_{pq}}{\partial x_{ij}} \right) - \left(\sum_{i<j} g_{ij} \frac{\partial f_{pq}}{\partial x_{ij}} \right) \right) \frac{\partial}{\partial x_{pq}}.$$

We will consider the coefficient of $\frac{\partial}{\partial x_{pq}}$, and break it into the difference of two quantities, as indicated by the parentheses.

Calculation of the first component. Consider

$$\sum_{i<j} f_{ij} \frac{\partial g_{p,q}}{\partial x_{ij}}. \tag{A.1}$$

We must have $l < p$ in order for $g_{p,q}$ to be nonzero. Thus if $p \leq l$, (A.1) is zero. Assume now that $p > l$, so that $k < l < p < q$. Note that f_{ij} is only nonzero when $i > k$. As a result, (A.1) is equal to

$$\sum_{k<i<j} S(a \cdots kij) \frac{\partial S(b \cdots lpq)}{\partial x_{ij}}. \tag{A.2}$$

To simplify (A.2), we must break the sum into pieces, each consisting of all summands of a particular type. Observe that under the given hypotheses, $b \leq k$.

(1) When $\{i, j\} \not\subseteq \{b \cdots lpq\}$, then the variable x_{ij} does not appear in $S(b \cdots lpq)$, so that the derivative (and each term) is zero.

(2) When $\{i, j\} \subseteq \{b \cdots l\}$, then notice that the only nonzero terms will be where $k < i < j \leq l$. By applying Proposition 5.2.7, we can see that the sum of all terms of this type equals

$$\begin{aligned}
&= \sum_{k<i<j\leq l} S(a \cdots kij) \frac{\partial S(b \cdots lpq)}{\partial x_{ij}} \\
&= \sum_{k<i<j\leq l} S(a \cdots kij)(-1)^{(i-(b-1))+(j-(b-1))+1} S(b \cdots \hat{i} \cdots \hat{j} \cdots lpq) \\
&= \sum_{k<i<j\leq l} (-1)^{i+j+1} S(a \cdots kij) S(b \cdots \hat{i} \cdots \hat{j} \cdots lpq).
\end{aligned}$$

By Lemma A.1.1, this is equal to

$$(1-\delta_{ab})\big((-1)^{k+l}\alpha(p)\beta(b,b,q)+(-1)^{k+l+1}\alpha(q)\beta(b,b,p)\big)$$
$$+\frac{1}{2}\left(l-k+\delta_{ab}-2\right)S(a\cdots k)S(b\cdots lpq)+S(a\cdots kpq)S(b\cdots l).$$

(3) When $i\in\{b\cdots l\}$ and $j=p$, then $k<i\leq l$, $j=p$. Using Proposition 5.2.7, we can see that this portion of the sum can be written as:

$$\begin{aligned}
&=\sum_{k<i\leq l}S(a\cdots kip)\frac{\partial S(b\cdots lpq)}{\partial x_{ip}}\\
&=\sum_{k<i\leq l}S(a\cdots kip)(-1)^{(i-(b-1))+(l-(b-1)+1)+1}S(b\cdots\hat{i}\cdots lq)\\
&=\sum_{k<i\leq l}(-1)^{i+l}S(a\cdots kip)S(b\cdots\hat{i}\cdots lq).
\end{aligned}$$

Applying Lemma A.1.3 shows that this is equal to

$$(-1)^{k+l+1}(1-\delta_{ab})\alpha(p)\beta(b,b,q)+S(a\cdots k)S(b\cdots lpq)-S(a\cdots kpq)S(b\cdots l).$$

(4) When $i\in\{b\cdots l\}$ and $j=q$, then $k<i\leq l$, $j=q$. Using Proposition 5.2.7, we can see that this portion of the sum can be written as:

$$\begin{aligned}
&=\sum_{k<i\leq l}S(a\cdots kiq)\frac{\partial S(b\cdots lpq)}{\partial x_{iq}}\\
&=\sum_{k<i\leq l}S(a\cdots kiq)(-1)^{(i-(b-1))+(l-(b-1)+2)+1}S(b\cdots\hat{i}\cdots lp)\\
&=\sum_{k<i\leq l}(-1)^{i+l+1}S(a\cdots kiq)S(b\cdots\hat{i}\cdots lp).
\end{aligned}$$

Applying Lemma A.1.4 shows that this is equal to

$$(-1)^{k+l}(1-\delta_{ab})\alpha(q)\beta(b,b,p)-S(a\cdots kpq)S(b\cdots l)+S(a\cdots k)S(b\cdots lpq).$$

(5) When $i=p$ and $j=q$, then there is exactly one term which appears:

$$\begin{aligned}
S(a\cdots kpq)\frac{\partial S(b\cdots lpq)}{\partial x_{pq}}&=S(a\cdots kpq)(-1)^{(l-(b-1)+1)+(l-(b-1)+2)+1}S(b\cdots l)\\
&=S(a\cdots kpq)S(b\cdots l).
\end{aligned}$$

Finally, we combine each of these cases: (A.1) is equal to the sum of cases (1)-(5). Thus (A.1) is equal to

$$
\begin{aligned}
&=(1-\delta_{ab})\Big((-1)^{k+l}\alpha(p)\beta(b,b,q)+(-1)^{k+l+1}\alpha(q)\beta(b,b,p)\Big)\\
&\quad+\frac{1}{2}\left(l-k+\delta_{ab}-3\right)S(a\cdots k)S(b\cdots lpq)\\
&\quad+S(a\cdots kpq)S(b\cdots l)+(-1)^{k+l+1}(1-\delta_{ab})\alpha(p)\beta(b,b,q)+S(a\cdots k)S(b\cdots lpq)\\
&\quad-S(a\cdots kpq)S(b\cdots l)+(-1)^{k+l}(1-\delta_{ab})\alpha(q)\beta(b,b,p)-S(a\cdots kpq)S(b\cdots l)\\
&\quad+S(a\cdots k)S(b\cdots lpq)+S(a\cdots kpq)S(b\cdots l)\\
&=(1-\delta_{ab})\Big((-1)^{k+l+1}\alpha(p)\beta(b,b,q)+(-1)^{k+l}\alpha(q)\beta(b,b,p)+(-1)^{k+l}\alpha(p)\beta(b,b,q)\\
&\quad+(-1)^{k+l+1}\alpha(q)\beta(b,b,p)\Big)+\frac{1}{2}\left(l-k+\delta_{ab}-3+4\right)S(a\cdots k)S(b\cdots lpq)\\
&=\frac{1}{2}\left(l-k+\delta_{ab}+1\right)S(a\cdots k)S(b\cdots lpq)
\end{aligned}
\tag{A.3}
$$

Calculation of the second component. The second component of the coefficient of $\frac{\partial}{\partial x_{pq}}$ is

$$\sum_{i<j} g_{ij}\frac{\partial f_{pq}}{\partial x_{ij}}.$$

When $p \le k$, we have $f_{pq}=0$, so that the second component is zero. Thus assume that $p>k$.

The only g_{ij}'s which are nonzero occur when $l<i$, so the sum above is equal to

$$\sum_{l<i<j\le n} g_{ij}\frac{\partial f_{pq}}{\partial x_{ij}}.$$

Substituting the coefficients gives

$$\sum_{l<i<j\le n} S(b\cdots lij)\frac{\partial S(a\cdots kpq)}{\partial x_{ij}}.$$

For all of the summands, $k<l<i<j$ so that the only way $i,j\in\{a,\cdots,k,p,q\}$ is if $i=p$, $j=q$. As we are only interested in the terms appearing in the sum, we must have $l<i=p$. Thus if $p\le l$, the second component will be zero. Assume then that $l<p$. Then the sum of interest will only have one nonzero term, and applying Proposition 5.2.7 will give

$$S(b\cdots lpq)(-1)^{(k-(a-1)+1)+(k-(a-1)+2)+1}S(a\cdots k)=S(b\cdots lpq)S(a\cdots k). \tag{A.4}$$

Conclusion. We must now combine the two components to calculate the coefficient of $\frac{\partial}{\partial x_{pq}}$.

If $p \leq l$, then the first and the second components are zero, so the coefficient of $\frac{\partial}{\partial x_{pq}}$ is zero. If $p > l$, then the coefficient of $\frac{\partial}{\partial x_{pq}}$ will be the difference of (A.3) and (A.4), i.e.,

$$\begin{aligned}\frac{1}{2}(l-k+\delta_{ab}+1)S(a\cdots k)S(b\cdots lpq) - S(b\cdots lpq)S(a\cdots k)\\ = \frac{1}{2}(l-k+\delta_{ab}-1)\,S(a\cdots k)S(b\cdots lpq).\end{aligned}$$

Thus

$$\begin{aligned}[\eta_{a,k},\eta_{b,l}] &= \frac{1}{2}(l-k+\delta_{ab}-1)S(a\cdots k)\sum_{l<p<q\leq n} S(b\cdots lpq)\frac{\partial}{\partial x_{pq}}\\ &= \frac{1}{2}(l-k+\delta_{ab}-1)S(a\cdots k)\cdot \eta_{b,l},\end{aligned}$$

as desired. □

Lemma A.1.1. *If* $\{a,b\}=\{1,2\}$, $k<l<p<q$, *and* $\#\{a,\cdots,k\}$, $\#\{b,\cdots,l\}$ *are even then*

$$\begin{aligned}&\sum_{k<i<j\leq l}(-1)^{i+j+1}S(a\cdots kij)S(b\cdots\hat{i}\cdots\hat{j}\cdots lpq)\\ &\qquad=(1-\delta_{ab})\Big((-1)^{k+l}\alpha(p)\beta(b,b,q)+(-1)^{k+l+1}\alpha(q)\beta(b,b,p)\Big)\\ &\qquad\quad+\frac{1}{2}(l-k+\delta_{ab}-3)\,S(a\cdots k)S(b\cdots lpq)+S(a\cdots kpq)S(b\cdots l).\end{aligned}$$

Proof. Notice that if we replace the requirement that $i<j$ with the requirement that $i\neq j$ we will obtain two copies of every term in the sum. Thus we can rewrite the sum as

$$\frac{1}{2}\sum_{k<i\leq l}(-1)^{i+1}\sum_{k<j\leq l, j\neq i}(-1)^{j}S(\{a,\cdots,k,i,j\})S(\{b,\cdots,\hat{i},\cdots,\hat{j},\cdots,l,p,q\}). \tag{A.5}$$

We will apply the Dress-Wenzel identity to the inside sum. Applying Theorem 5.2.4 with $I_1=\{a\cdots ki\}$, $I_2=\{b\cdots\hat{i}\cdots lpq\}$, we have $I_1\Delta I_2=\{k+1,\cdots,l,p,q\}$ ($\cup\{1\}$ if $a\neq b$), so that

$$\begin{aligned}&(1-\delta_{ab})\alpha(i)\beta(i,p,q)+\sum_{k<j\leq l,j\neq i}(-1)^{j-k}S(a\cdots kij)S(b\cdots\hat{i}\cdots\hat{j}\cdots lpq)\\ &\quad+(-1)^{i-k}S(a\cdots k)S(b\cdots lpq)+(-1)^{l-k+1}S(a\cdots kip)S(b\cdots\hat{i}\cdots lq)\\ &\quad+(-1)^{l-k+2}S(a\cdots kiq)S(b\cdots\hat{i}\cdots lp)=0.\end{aligned} \tag{A.6}$$

Multiplying (A.6) through by $(-1)^k$, solving for the second term, and substituting in (A.5) shows that our sum of interest equals

$$\begin{aligned}\frac{1}{2}\sum_{k<i\leq l}(-1)^{i+1+k+1}\Big(&(1-\delta_{ab})\alpha(i)\beta(i,p,q)+(-1)^{i-k}S(a\cdots k)S(b\cdots lpq)\\ &+(-1)^{l-k+1}S(a\cdots kip)S(b\cdots\hat{i}\cdots lq)+(-1)^{l-k+2}S(a\cdots kiq)S(b\cdots\hat{i}\cdots lp)\Big).\end{aligned}\tag{A.7}$$

Note that after distributing signs, the $S(a\cdots k)S(b\cdots lpq)$ term of (A.7) has no i dependence. Thus we may rewrite (A.7) as

$$\begin{aligned}&\frac{1}{2}(l-k)S(a\cdots k)S(b\cdots lpq)+\frac{1}{2}(1-\delta_{ab})\sum_{k<i\leq l}(-1)^{i+k}\alpha(i)\beta(i,p,q)\\ &\quad+\frac{1}{2}\sum_{k<i\leq l}(-1)^{i+l+1}S(a\cdots kip)S(b\cdots\hat{i}\cdots lq)\\ &\quad+\frac{1}{2}\sum_{k<i\leq l}(-1)^{i+l}S(a\cdots kiq)S(b\cdots\hat{i}\cdots lp).\end{aligned}\tag{A.8}$$

Note that the first sum of (A.8) takes the form handled by Lemma A.1.2, the second sum takes the negative of the form handled by Lemma A.1.3, and the third sum takes the negative of the form handled by Lemma A.1.4. Applying these Lemmas to (A.8) and simplifying, we find that our sum of interest equals

$$\begin{aligned}&(1-\delta_{ab})\Big((-1)^{k+l}\alpha(p)\beta(b,b,q)+(-1)^{k+l+1}\alpha(q)\beta(b,b,p)\Big)\\ &\qquad+\frac{1}{2}\left(l-k+\delta_{ab}-3\right)S(a\cdots k)S(b\cdots lpq)+S(a\cdots kpq)S(b\cdots l),\end{aligned}$$

as desired. □

Lemma A.1.2. *If* $\{a,b\}=\{1,2\}$*,* $k<l<p<q$*, and* $\#\{a,\cdots,k\}$*,* $\#\{b,\cdots,l\}$ *are even then*

$$\begin{aligned}\sum_{k<i\leq l}(-1)^{i+k}\alpha(i)\beta(i,p,q)=&-S(a\cdots k)S(b\cdots lpq)+(-1)^{l-k}\alpha(p)\beta(b,b,q)\\ &+(-1)^{l-k+1}\alpha(q)\beta(b,b,p).\end{aligned}$$

Proof. If $a=1$, $b=2$, then we are interested in

$$\sum_{k<i\leq l}(-1)^{i+k}S(2\cdots ki)S(1\cdots\hat{i}\cdots lpq).\tag{A.9}$$

If we apply Theorem 5.2.4 with $I_1 = \{2, \cdots, k\}$, $I_2 = \{1, \cdots, l, p, q\}$, then $I_1 \Delta I_2 = \{1, k+1, \cdots, l, p, q\}$, so we obtain the identity

$$\begin{aligned}(-1)^0 S(1\cdots k)S(2\cdots lpq) + \sum_{k<i\le l} (-1)^{i-k} S(2\cdots ki)S(1\cdots \hat{i}\cdots lpq) \\ + (-1)^{l-k+1} S(2\cdots kp)S(1\cdots lq) + (-1)^{l-k+2} S(2\cdots kq)S(1\cdots lp) = 0.\end{aligned} \tag{A.10}$$

Multiplying (A.10) through by $(-1)^{2k} = 1$ and solving for the sum shows that the sum in (A.9) equals

$$\begin{aligned}(-1)^{2k+1}\Big(S(1\cdots k)S(2\cdots lpq) + (-1)^{l-k+1} S(2\cdots kp)S(1\cdots lq) \\ + (-1)^{l-k+2} S(2\cdots kq)S(1\cdots lp)\Big),\end{aligned}$$

which also equals the desired quantity.

If $a = 2$, $b = 1$, then we are interested in

$$\sum_{k<i\le l} (-1)^{i+k} S(1\cdots ki)S(2\cdots \hat{i}\cdots lpq). \tag{A.11}$$

If we apply Theorem 5.2.4 with $I_1 = \{1, \cdots, k\}$, $I_2 = \{2, \cdots, l, p, q\}$, then $I_1 \Delta I_2 = \{1, k+1, \cdots, l, p, q\}$, so we obtain the identity

$$\begin{aligned}(-1)^0 S(2\cdots k)S(1\cdots lpq) + \sum_{k<i\le l} (-1)^{i-k} S(1\cdots ki)S(2\cdots \hat{i}\cdots lpq) \\ + (-1)^{l-k+1} S(1\cdots kp)S(2\cdots lq) + (-1)^{l-k+2} S(1\cdots kq)S(2\cdots lp) = 0.\end{aligned} \tag{A.12}$$

Multiplying (A.12) through by $(-1)^{2k} = 1$ and solving for the sum shows that the sum in (A.11) equals

$$\begin{aligned}(-1)^{2k+1}\Big(S(2\cdots k)S(1\cdots lpq) + (-1)^{l-k+1} S(1\cdots kp)S(2\cdots lq) \\ + (-1)^{l-k+2} S(1\cdots kq)S(2\cdots lp)\Big),\end{aligned}$$

which also equals the desired quantity. □

Lemma A.1.3. *If $a, b \in \{1, 2\}$, $k < l < p < q$, and $\#\{a, \cdots, k\}$, $\#\{b, \cdots, l\}$ are even then*

$$\begin{aligned}\sum_{k<i\le l} (-1)^{i+l} S(a\cdots kip)S(b\cdots \hat{i}\cdots lq) =& (-1)^{k+l+1}(1-\delta_{ab})\alpha(p)\beta(b, b, q) \\ &+ S(a\cdots k)S(b\cdots lpq) - S(a\cdots kpq)S(b\cdots l).\end{aligned}$$

PROOF. Applying Theorem 5.2.4 with $I_1 = \{a \cdots kp\}$, $I_2 = \{b \cdots lq\}$, so that $I_1 \Delta I_2 = \{k+1, \cdots, l, p, q\}$ $(\cup\{1\}$ if $a \neq b)$, gives

$$(1-\delta_{ab})\alpha(p)\beta(b,b,q) + \sum_{k<i\leq l} (-1)^{i-k} S(a \cdots kip) S(b \cdots \hat{i} \cdots lq)$$
$$+ (-1)^{l-k+1} S(a \cdots k) S(b \cdots lpq) + (-1)^{l-k+2} S(a \cdots kpq) S(b \cdots l) = 0.$$

Multiplying by $(-1)^{k+l}$ and solving for the sum proves the lemma. □

LEMMA A.1.4. *If $a, b \in \{1,2\}$, $k < l < p < q$, and $\#\{a, \cdots, k\}$, $\#\{b, \cdots, l\}$ are even then*

$$\sum_{k<i\leq l} (-1)^{i+l+1} S(a \cdots kiq) S(b \cdots \hat{i} \cdots lp) = (-1)^{k+l}(1-\delta_{ab})\alpha(q)\beta(b,b,p)$$
$$- S(a \cdots kpq) S(b \cdots l) + S(a \cdots k) S(b \cdots lpq).$$

PROOF. Applying Theorem 5.2.4 with $I_1 = \{a \cdots kq\}$, $I_2 = \{b \cdots lp\}$, so that $I_1 \Delta I_2 = \{k+1, \cdots, l, p, q\}$ $(\cup\{1\}$ if $a \neq b)$, gives

$$(1-\delta_{ab})\alpha q\beta(b,b,p) + \sum_{k<i\leq l} (-1)^{i-k} S(a \cdots kiq) S(b \cdots \hat{i} \cdots lp)$$
$$+ (-1)^{l-k+1} S(a \cdots kpq) S(b \cdots l) + (-1)^{l-k+2} S(a \cdots k) S(b \cdots lpq) = 0.$$

Multiplying by $(-1)^{k+l+1}$ and solving for the sum proves the lemma. □

A.2. Calculation of the bracket of linear and nonlinear vector fields

For the free divisor of interest we use a selection of linear vector fields coming from a Lie algebra $\mathfrak{g}$. The structure of $\mathfrak{g}$ indicates that if $\xi_{E_{a,k}}$ is a linear vector field with $a \neq k$ and $E_{a,k} \in \mathfrak{g}$, then $k \geq 3$. Since $b \in \{1,2\}$, we know that $b < k$. In the following proposition, we have chosen the hypotheses so that we calculate the Lie bracket between all of the linear and nonlinear vector fields of interest. Outside of the hypotheses of the Proposition, the result of the Lie bracket can be quite complicated.

We will now prove Proposition 5.2.13, restated here in terms of the usual Lie bracket.

PROPOSITION (5.2.13). *Let $1 \leq k \leq a \leq n$, $b \in \{1,2\}$, and $b < l \leq n$. If $k < a$, also require that $b \leq k$. Then*

$$[\xi_{E_{a,k}}, \eta_{b,l}] = \begin{cases} \eta_{b,l} & \text{if } k = a \text{ and } a \in \{b, \ldots, l\} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $\#\{b,\dots,l\}$ is odd, then $\eta_{b,l} = 0$ and the statement is clearly correct. Thus assume $\#\{b,\dots,l\}$ is even. By Proposition 4.2.6 and Remark 4.3.10,

$$\xi_{E_{a,k}} = \sum_{i<k} x_{ik}\frac{\partial}{\partial x_{ia}} + \sum_{k<i<a} -x_{ki}\frac{\partial}{\partial x_{ia}} + \sum_{a<i} x_{ki}\frac{\partial}{\partial x_{ai}}$$

and

$$\eta_{bl} = \sum_{l<p<q} S(b\cdots lpq)\frac{\partial}{\partial x_{pq}}.$$

By expanding $\left[\xi_{E_{a,k}}, \eta_{b,l}\right]$ linearly, we find that

$$\begin{aligned}\left[\xi_{E_{a,k}}, \eta_{b,l}\right] =& \sum_{i<k}\sum_{l<p<q}\left[x_{ik}\frac{\partial}{\partial x_{ia}}, S(b\cdots lpq)\frac{\partial}{\partial x_{pq}}\right] - \sum_{k<i<a}\sum_{l<p<q}\left[x_{ki}\frac{\partial}{\partial x_{ia}}, S(b\cdots lpq)\frac{\partial}{\partial x_{pq}}\right] \\ &+ \sum_{a<i}\sum_{l<p<q}\left[x_{ki}\frac{\partial}{\partial x_{ai}}, S(b\cdots lpq)\frac{\partial}{\partial x_{pq}}\right].\end{aligned}$$

Evaluating the individual lie brackets into a difference of two terms and regrouping, we obtain

$$\begin{aligned}\left[\xi_{E_{a,k}}, \eta_{b,l}\right] =& \sum_{l<p<q}\left(\sum_{i<k} x_{ik}\frac{\partial S(b\cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki}\frac{\partial S(b\cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki}\frac{\partial S(b\cdots lpq)}{\partial x_{ai}}\right)\frac{\partial}{\partial x_{pq}} \\ &- \sum_{l<p<q}\left(\sum_{i<k}\frac{\partial x_{ik}}{\partial x_{pq}}S(b\cdots lpq)\frac{\partial}{\partial x_{ia}} - \sum_{k<i<a}\frac{\partial x_{ki}}{\partial x_{pq}}S(b\cdots lpq)\frac{\partial}{\partial x_{ia}}\right. \\ &\left.+ \sum_{a<i}\frac{\partial x_{ki}}{\partial x_{pq}}S(b\cdots lpq)\frac{\partial}{\partial x_{ai}}\right).\end{aligned}$$

It is easy to see in the second collection of terms that the $\frac{\partial x_{ik}}{\partial x_{pq}}$-type terms will be zero unless p and q have specific values; in particular, one of them must equal k. Because the outside sum is over $l < p < q$, the only nonzero terms will occur when $l < k$. Let $\mathrm{If}(l < k)$ be a symbol which is 1 when $l < k$ and 0 otherwise. By introducing this symbol, we may simplify the second collection of terms, pretending that $l < k$:

(A.13)

$$\begin{aligned}&\left[\xi_{E_{a,k}}, \eta_{b,l}\right] = \\ &\quad\sum_{l<p<q}\left(\sum_{b\le i<k} x_{ik}\frac{\partial S(b\cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki}\frac{\partial S(b\cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki}\frac{\partial S(b\cdots lpq)}{\partial x_{ai}}\right)\frac{\partial}{\partial x_{pq}} \\ &\quad+ \mathrm{If}(l<k)\left(\sum_{l<i<k} -S(b\cdots lik)\frac{\partial}{\partial x_{ia}} + \sum_{k<i<a} S(b\cdots lki)\frac{\partial}{\partial x_{ia}} + \sum_{a<i} -S(b\cdots lki)\frac{\partial}{\partial x_{ai}}\right).\end{aligned}$$

Let us call the two collections of terms in (A.13) the "left" component and the "right" component of the Lie bracket.

We now break the calculation of the Lie bracket into a number of cases, depending on the relative values of a, b, k, and l.

First consider the case where $a < b$, so $k \leq a < b < l$. Since a will never appear in $\{b \cdots lpq\}$, all of the derivatives of the left component are zero. The right component is zero because $l \not< k$, making the Lie bracket zero. Since the statement is proven when $a < b$, assume for the remainder of the proof that $b \leq a$.

Since $k \leq a$, we either have $k = a$ or $k < a$.

(1) Consider the case where $k = a$, so we have $b \leq k = a$. Then the left component of (A.13) will have no middle term. We now break into cases, depending on where l lies in relation to b and $k = a$.

  (a) If $a \leq l$, then we have $b \leq k = a \leq l$. Since $l \not< k$, we need only worry about the left component of (A.13). We are now in a specific-enough case to simplify the derivatives in (A.13). Breaking the second inside sum into the three nonzero pieces ($i \in \{b \cdots lpq\}$ in order for the derivative to be nonzero), and then applying Proposition 5.2.7, we obtain

$$\begin{aligned}
&[\xi_{E_{a,k}}, \eta_{b,l}] \\
&\quad = \sum_{l<p<q} \left(\sum_{b \leq i < k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pq}} \\
&\quad = \sum_{l<p<q} \left(\sum_{b \leq i < k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i\leq l} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} + x_{kp} \frac{\partial S(b \cdots lpq)}{\partial x_{ap}} \right. \\
&\qquad\qquad \left. + x_{kq} \frac{\partial S(b \cdots lpq)}{\partial x_{aq}} \right) \frac{\partial}{\partial x_{pq}} \\
&\quad = \sum_{l<p<q} (-1)^{k+1} \left(\sum_{b\leq i<k} (-1)^i x_{ik} S(b \cdots \hat{\imath} \cdots \hat{k} \cdots lpq) + \sum_{a<i\leq l} (-1)^i x_{ki} S(b \cdots \hat{k} \cdots \hat{\imath} \cdots lpq) \right. \\
&\qquad\qquad \left. + (-1)^{l+1} x_{kp} S(b \cdots \hat{k} \cdots lq) + (-1)^l x_{kq} S(b \cdots \hat{k} \cdots lp) \right) \frac{\partial}{\partial x_{pq}}.
\end{aligned}$$

We may now apply Lemma A.2.1 to simplify each term of the sum, obtaining

$$[\xi_{E_{a,k}}, \eta_{b,l}] = \sum_{l<p<q} (-1)^{k+1} \left((-1)^{k+1} S(b \cdots lpq) \right) \frac{\partial}{\partial x_{pq}} = \eta_{b,l}.$$

(b) If $a > l$, then we have $b < l < k = a$. We first consider the left component of (A.13) by remembering that the inside middle sum is an empty sum, noting that the only way $k \in \{b \cdots lpq\}$ is if $p = k$ or $q = k$, and accordingly breaking the outside sum into these two possibilities:

$$\begin{aligned}
\text{Left} =& \sum_{l<p<q} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pq}} \\
=& \sum_{k<q} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots lkq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lkq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{kq}} \\
&+ \sum_{l<p<k} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots lpk)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpk)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pk}}.
\end{aligned}$$

We now break the first internal sums into two components, noting that in the resulting expression the top middle and bottom right sums will be zero, the top right sum will be nonzero only when $i = q$, and the bottom middle sum will be nonzero only when $i = p$:

$$\begin{aligned}
\text{Left} =& \sum_{k<q} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lkq)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lkq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lkq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{kq}} \\
&+ \sum_{l<p<k} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lpk)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lpk)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpk)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pk}} \\
=& \sum_{k<q} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lkq)}{\partial x_{ia}} + 0 + x_{kq} \frac{\partial S(b \cdots lkq)}{\partial x_{aq}} \right) \frac{\partial}{\partial x_{kq}} \\
&+ \sum_{l<p<k} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lpk)}{\partial x_{ia}} + x_{pk} \frac{\partial S(b \cdots lpk)}{\partial x_{pa}} + 0 \right) \frac{\partial}{\partial x_{pk}}.
\end{aligned}$$

We may now apply Proposition 5.2.7 to simplify the derivatives, obtaining

$$\begin{aligned}
\text{Left} =& \sum_{k<q} (-1)^l \left(\sum_{b \le i \le l} (-1)^i x_{ik} S(b \cdots \hat{i} \cdots lq) + (-1)^l x_{kq} S(b \cdots l) \right) \frac{\partial}{\partial x_{kq}} \\
&+ \sum_{l<p<k} (-1)^{l+1} \left(\sum_{b \le i \le l} (-1)^i x_{ik} S(b \cdots \hat{i} \cdots lp) + (-1)^{l+1} x_{pk} S(b \cdots l) \right) \frac{\partial}{\partial x_{pk}}.
\end{aligned}$$

Apply Lemma A.2.1 with $x = p = k$ for the first set of terms and with $x = q = k$ for the second set of terms, to obtain

$$\begin{aligned}
\text{Left} &= \sum_{k<q} (-1)^l \left(-(-1)^{l+1} S(b \cdots lkq) \right) \frac{\partial}{\partial x_{kq}} + \sum_{l<p<k} (-1)^{l+1} \left(-(-1)^l S(b \cdots lpk) \right) \frac{\partial}{\partial x_{pk}} \\
&= \sum_{k<q} S(b \cdots lkq) \frac{\partial}{\partial x_{kq}} + \sum_{l<p<k} S(b \cdots lpk) \frac{\partial}{\partial x_{pk}}.
\end{aligned}$$

The right component of the Lie bracket can be simplified by noting that the middle inner sum is an empty sum:

$$\begin{aligned}
\text{Right} &= \left(\sum_{l<i<k} -S(b \cdots lik) \frac{\partial}{\partial x_{ia}} + \sum_{k<i<a} S(b \cdots lki) \frac{\partial}{\partial x_{ia}} + \sum_{a<i} -S(b \cdots lki) \frac{\partial}{\partial x_{ai}} \right) \\
&= \left(\sum_{l<p<k} -S(b \cdots lpk) \frac{\partial}{\partial x_{pa}} + \sum_{a<q} -S(b \cdots lkq) \frac{\partial}{\partial x_{aq}} \right).
\end{aligned}$$

Since $a = k$, the Right component is just the negative of the Left component, so that the Lie bracket is zero, as expected.

(2) Consider the case where $k < a$, so that we have $b \leq k < a$. We now break into cases, depending on where l lies in relation to b, k, and a.

(a) If $l \geq a$, then we have $b \leq k < a \leq l$. Since $l \not< k$, the Right component of the bracket is zero. We may take (A.13) and break the third inside sum into three pieces which constitute all of the nonzero terms: either $a < i \leq l$, $i = p$, or $i = q$.

$$\begin{aligned}
&[\xi_{E_{a,k}}, \eta_{b,l}] \\
&= \sum_{l<p<q} \left(\sum_{b \leq i<k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pq}} \\
&= \sum_{l<p<q} \left(\sum_{b \leq i<k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i \leq l} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right. \\
&\qquad \left. + x_{kp} \frac{\partial S(b \cdots lpq)}{\partial x_{ap}} + x_{kq} \frac{\partial S(b \cdots lpq)}{\partial x_{aq}} \right) \frac{\partial}{\partial x_{pq}}.
\end{aligned}$$

We may then apply Proposition 5.2.7 to simplify the derivatives, obtaining

$$\begin{aligned}
&\left[\xi_{E_{a,k}}, \eta_{b,l}\right] \\
&\quad = \sum_{l<p<q} (-1)^{a+1} \left(\sum_{b \le i < k} (-1)^i x_{ik} S(b \cdots \hat{i} \cdots \hat{a} \cdots lpq) + \sum_{k<i<a} (-1)^{i+1} x_{ki} S(b \cdots \hat{i} \cdots \hat{a} \cdots lpq) \right. \\
&\qquad + \sum_{a<i\le l} (-1)^i x_{ki} S(b \cdots \hat{a} \cdots \hat{i} \cdots lpq) + (-1)^{l+1} x_{kp} S(b \cdots \hat{a} \cdots lq) \\
&\qquad \left. + (-1)^l x_{kq} S(b \cdots \hat{a} \cdots lp) \right) \frac{\partial}{\partial x_{pq}}.
\end{aligned}$$

Applying Lemma A.2.2, we see that each summand is zero and therefore the Lie bracket is zero.

(b) If $l < a$ and $l \ge k$, then we have $b \le k \le l < a$. Since $l \not< k$, the Right component of the Lie bracket is zero. Breaking up the second internal sum of the Left component of (A.13), we see that

$$\begin{aligned}
&\left[\xi_{E_{a,k}}, \eta_{b,l}\right] \\
&\quad = \sum_{l<p<q} \left(\sum_{b\le i<k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pq}} \\
&\quad = \sum_{l<p<q} \left(\sum_{b\le i<k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i\le l} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{l<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} \right. \\
&\qquad \left. + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pq}}.
\end{aligned}$$

Since $a > l$, the only way $a \in \{b \cdots lpq\}$ is if $p = a$ or $q = a$; the derivative will be zero otherwise. Thus we may break up our sum into

$$\begin{aligned}
&\left[\xi_{E_{a,k}}, \eta_{b,l}\right] \\
&\quad = \sum_{a<q} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} - \sum_{k < i \le l} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} - \sum_{l<i<a} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} \right. \\
&\qquad\qquad \left. + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{aq}} \\
&\quad + \sum_{l<p<a} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} - \sum_{k < i \le l} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} - \sum_{l<i<a} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} \right. \\
&\qquad\qquad \left. + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pa}}.
\end{aligned}$$

Note that the third top sum (and the fourth bottom sum) will vanish as $i \notin \{b \cdots laq\}$. Also see that the only nonzero summand of the fourth top sum will occur when $i = q$, while the only nonzero summand of the third bottom sum will occur when $i = p$. Thus,

$$\begin{aligned}
&\left[\xi_{E_{a,k}}, \eta_{b,l}\right] \\
&\quad = \sum_{a<q} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} - \sum_{k < i \le l} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} + x_{kq} \frac{\partial S(b \cdots laq)}{\partial x_{aq}} \right) \frac{\partial}{\partial x_{aq}} \\
&\quad + \sum_{l<p<a} \left(\sum_{b \le i < k} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} - \sum_{k < i \le l} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} - x_{kp} \frac{\partial S(b \cdots lpa)}{\partial x_{pa}} \right) \frac{\partial}{\partial x_{pa}}.
\end{aligned}$$

Now using Proposition 5.2.7 to simplify the derivatives, we obtain

$$\begin{aligned}
&\left[\xi_{E_{a,k}}, \eta_{b,l}\right] \\
&\quad = \sum_{a<q} (-1)^l \Bigg(\sum_{b \le i < k} (-1)^i x_{ik} S(b \cdots \hat{i} \cdots lq) + \sum_{k<i\le l} (-1)^{i+1} x_{ki} S(b \cdots \hat{i} \cdots lq) \\
&\qquad\qquad + (-1)^l x_{kq} S(b \cdots l) \Bigg) \frac{\partial}{\partial x_{aq}} \\
&\qquad + \sum_{l<p<a} (-1)^{l+1} \Bigg(\sum_{b \le i < k} x_{ik} (-1)^i S(b \cdots \hat{i} \cdots lp) + \sum_{k<i\le l} (-1)^{i+1} x_{ki} S(b \cdots \hat{i} \cdots lp) \\
&\qquad\qquad + (-1)^l x_{kp} S(b \cdots l) \Bigg) \frac{\partial}{\partial x_{pa}}.
\end{aligned}$$

Now apply Lemma A.2.3 with $x = q$ to the top line and Lemma A.2.4 with $x = p$ to the bottom line to see that each summand is zero. Thus the Lie bracket is zero.

(c) If $l < a$ and $l < k$, then since $l > b$ we must have $b < l < k < a$. We shall first simplify the Left component of the bracket. Since $a \notin \{b, \cdots, l\}$, the only way one of the derivatives can be nonzero is if $p = a$ or $q = a$; we break our sum into these situations, as well as breaking the first internal sum into two pieces

$$\begin{aligned}
\text{Left} &= \sum_{l<p<q} \left(\sum_{b\le i<k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pq}} \\
&= \sum_{l<p<q} \Bigg(\sum_{b\le i\le l} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ia}} \\
&\qquad + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpq)}{\partial x_{ai}} \Bigg) \frac{\partial}{\partial x_{pq}} \\
&= \sum_{a<q} \Bigg(\sum_{b\le i\le l} x_{ik} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} \\
&\qquad + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ai}} \Bigg) \frac{\partial}{\partial x_{aq}} \\
&\quad + \sum_{l<p<a} \Bigg(\sum_{b\le i\le l} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} \\
&\qquad + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ai}} \Bigg) \frac{\partial}{\partial x_{pa}}.
\end{aligned}$$

We would like to know where p is in relation to k, so we split the second sum into three pieces, where $p < k$, $p = k$, and $p > k$.

$$\begin{aligned}
\text{Left} =& \sum_{a<q} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} \right. \\
& \qquad \left. - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots laq)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{aq}} \\
& + \sum_{l<p<k} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} \right. \\
& \qquad \left. - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pa}} \\
& + \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lka)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lka)}{\partial x_{ia}} \right. \\
& \qquad \left. - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lka)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lka)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{ka}} \\
& + \sum_{k<p<a} \left(\sum_{b \le i \le l} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} + \sum_{l<i<k} x_{ik} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} \right. \\
& \qquad \left. - \sum_{k<i<a} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ia}} + \sum_{a<i} x_{ki} \frac{\partial S(b \cdots lpa)}{\partial x_{ai}} \right) \frac{\partial}{\partial x_{pa}}.
\end{aligned}$$

On the first pair of rows, the second and third sums will be zero as $i \notin \{b \cdots laq\}$, while the fourth sum will only be nonzero when $i = q$. On the second pair of rows, the third and fourth sums will be zero, while the second term will be nonzero only when $i = p$. On the third pair of rows, the second, third, and fourth terms will vanish. On the fourth pair of rows, the second and fourth terms will be zero, while the third term will be nonzero only when $i = p$. Thus

$$\begin{aligned}\text{Left} = &\sum_{a<q}\left(\sum_{b\le i\le l} x_{ik}\frac{\partial S(b\cdots laq)}{\partial x_{ia}} + x_{kq}\frac{\partial S(b\cdots laq)}{\partial x_{aq}}\right)\frac{\partial}{\partial x_{aq}}\\ &+ \sum_{l<p<k}\left(\sum_{b\le i\le l} x_{ik}\frac{\partial S(b\cdots lpa)}{\partial x_{ia}} + x_{pk}\frac{\partial S(b\cdots lpa)}{\partial x_{pa}}\right)\frac{\partial}{\partial x_{pa}}\\ &+ \left(\sum_{b\le i\le l} x_{ik}\frac{\partial S(b\cdots lka)}{\partial x_{ia}}\right)\frac{\partial}{\partial x_{ka}}\\ &+ \sum_{k<p<a}\left(\sum_{b\le i\le l} x_{ik}\frac{\partial S(b\cdots lpa)}{\partial x_{ia}} - x_{kp}\frac{\partial S(b\cdots lpa)}{\partial x_{pa}}\right)\frac{\partial}{\partial x_{pa}}.\end{aligned}$$

We may now apply Proposition 5.2.7 to simplify the derivative of each term, obtaining

$$\begin{aligned}\text{Left} = &\sum_{a<q}(-1)^l\left(\sum_{b\le i\le l}(-1)^i x_{ik}S(b\cdots\hat{\imath}\cdots lq) + (-1)^l x_{kq}S(b\cdots l)\right)\frac{\partial}{\partial x_{aq}}\\ &+ \sum_{l<p<k}(-1)^{l+1}\left(\sum_{b\le i\le l}(-1)^i x_{ik}S(b\cdots\hat{\imath}\cdots lp) + (-1)^{l+1}x_{pk}S(b\cdots l)\right)\frac{\partial}{\partial x_{pa}}\\ &+ (-1)^{l+1}\left(\sum_{b\le i\le l}(-1)^i x_{ik}S(b\cdots\hat{\imath}\cdots lk)\right)\frac{\partial}{\partial x_{ka}}\\ &+ \sum_{k<p<a}(-1)^{l+1}\left(\sum_{b\le i\le l}(-1)^i x_{ik}S(b\cdots\hat{\imath}\cdots lp) - (-1)^{l+1}x_{kp}S(b\cdots l)\right)\frac{\partial}{\partial x_{pa}}.\end{aligned}$$

Now use Lemma A.2.4 with $x = q$ on the first line, Lemma A.2.5 with $x = p$ on the second line, Lemma A.2.6 on the third line, and Lemma A.2.4 with $x = p$ on the fourth line, to obtain

$$\text{Left} = \sum_{a<q}S(b\cdots lkq)\frac{\partial}{\partial x_{aq}} + \sum_{l<p<k}S(b\cdots lpk)\frac{\partial}{\partial x_{pa}} + 0 + \sum_{k<p<a}-S(b\cdots lkp)\frac{\partial}{\partial x_{pa}}.$$

Examining the Right component

$$\text{Right} = \sum_{l<p<k}-S(b\cdots lpk)\frac{\partial}{\partial x_{pa}} + \sum_{k<p<a}S(b\cdots lkp)\frac{\partial}{\partial x_{pa}} + \sum_{a<q}-S(b\cdots lkq)\frac{\partial}{\partial x_{aq}},$$

we find that Left $= -$Right, so the Lie bracket is zero. □

LEMMA A.2.1. *Let $b \le l < p < q$, and choose x so that $x \in \{b \cdots l, p, q\}$. Then*

$$\sum_{b \le i \le l} (-1)^i S((\{b \cdots lpq\} \setminus \{x\}) \Delta \{i\}) S(\{x\} \Delta \{i\})$$
$$+ (-1)^{l+1} \left(\delta_{xp} S(b \cdots lpq) + (1 - \delta_{xp}) S(\{b \cdots lq\} \setminus \{x\}) S(\{xp\}) \right)$$
$$+ (-1)^{l} \left(\delta_{xq} S(b \cdots lpq) + (1 - \delta_{xq}) S(\{b \cdots lp\} \setminus \{x\}) S(\{xq\}) \right) = 0.$$

PROOF. Use $I_1 = \{b \cdots lpq\} \setminus \{x\}$ and $I_2 = \{x\}$ in Theorem 5.2.4. Then $I_1 \Delta I_2 = \{b \cdots lpq\}$, and multiplying the resulting identity by $(-1)^{b-1}$ gives the result. □

LEMMA A.2.2. *Let $b \le l < p < q$, and let $k, a \in \{b \cdots l\}$ with $k < a$. Then*

$$\sum_{b \le i < k} (-1)^i S(b \cdots \hat{i} \cdots \hat{a} \cdots lpq) S(ik) + \sum_{k < i < a} (-1)^{i+1} S(b \cdots \hat{i} \cdots \hat{a} \cdots lpq) S(ki)$$
$$+ \sum_{a < i \le l} (-1)^i S(b \cdots \hat{a} \cdots \hat{i} \cdots lpq) S(ki) + (-1)^{l+1} S(b \cdots \hat{a} \cdots lq) S(kp)$$
$$+ (-1)^l S(b \cdots \hat{a} \cdots lp) S(kq) = 0.$$

PROOF. Use $I_1 = \{b \cdots \hat{a} \cdots lpq\}$ and $I_2 = \{k\}$ in Theorem 5.2.4. Then the symmetric difference $I_1 \Delta I_2 = \{b \cdots \hat{k} \cdots \hat{a} \cdots lpq\}$, and multiplying the resulting identity by $(-1)^{b-1}$ gives the result. □

LEMMA A.2.3. *Let $b \le l < x$, and let $k \in \{b \cdots l\}$. Then*

$$\sum_{b \le i < k} (-1)^i S(b \cdots \hat{i} \cdots lx) S(ik) + \sum_{k < i \le l} (-1)^{i+1} S(b \cdots \hat{i} \cdots lx) S(ki)$$
$$+ (-1)^l S(b \cdots l) S(kx) = 0.$$

PROOF. Use $I_1 = \{b \cdots lx\}$ and $I_2 = \{k\}$ in Theorem 5.2.4. Then $I_1 \Delta I_2 = \{b \cdots \hat{k} \cdots lx\}$, and multiplying the resulting identity by $(-1)^{b-1}$ gives the result. □

LEMMA A.2.4. *Let $b \le l < x$, and let $l < k < x$. Then*

$$\sum_{b \le i \le l} (-1)^i S(b \cdots \hat{i} \cdots lx) S(ik) + (-1)^{l+1} S(b \cdots lkx) + (-1)^l S(b \cdots l) S(kx) = 0.$$

PROOF. Use $I_1 = \{b \cdots lx\}$ and $I_2 = \{k\}$ in Theorem 5.2.4. Then $I_1 \Delta I_2 = \{b \cdots lkx\}$, and multiplying the resulting identity by $(-1)^{b-1}$ gives the result. □

Lemma A.2.5. *Let $b \leq l < x$, and let $x < k$. Then*

$$\sum_{b \leq i \leq l} (-1)^i S(b \cdots \hat{i} \cdots lx) S(ik) + (-1)^{l+1} S(b \cdots l) S(xk) + (-1)^l S(b \cdots lxk) = 0.$$

Proof. Use $I_1 = \{b \cdots lx\}$ and $I_2 = \{k\}$ in Theorem 5.2.4. Then $I_1 \Delta I_2 = \{b \cdots lxk\}$, and multiplying the resulting identity by $(-1)^{b-1}$ gives the result. $\square$

Lemma A.2.6. *Let $b \leq l < k$. Then*

$$\sum_{b \leq i \leq l} (-1)^i S(b \cdots \hat{i} \cdots lk) S(ik) = 0.$$

Proof. Use $I_1 = \{b \cdots lk\}$ and $I_2 = \{k\}$ in Theorem 5.2.4. Then $I_1 \Delta I_2 = \{b \cdots l\}$, and multiplying the resulting identity by $(-1)^{b-1}$ gives the result. $\square$

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