GEOMETRIC INVARIANT THEORY AND MODULI OF PRINCIPAL BUNDLES

Avery Wilson

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the College of Arts and Sciences.

Chapel Hill
2022

Approved by:
Prakash Belkale
Jiuzu Hong
Shrawan Kumar
Richard Rimanyi
Justin Sawon
ABSTRACT

Avery Wilson: Geometric invariant theory and moduli of principal bundles
(Under the direction of Prakash Belkale)

The thesis contains two separate problems (chapters I & II respectively). In the first problem, we prove finite generation of the conformal blocks algebra associated to a stable curve of genus $g \geq 2$ and a simple Lie algebra of type A or C. This is done by showing that conformal blocks at sufficiently high level can be identified with sections of an ample line bundle on a compactified moduli space of $G$-bundles.

The second problem concerns geometric invariant theory of the flag variety $(G/B)^n$ (for any semisimple group $G$). We give a Schubert calculus description of unstable loci in $(G/B)^n$, and in type A we prove a useful numerical property of the possible codimension one components. As a consequence, we show that one can always assume an unstable locus of codimension $\geq 2$ for the purpose of moduli problems in type A.
Thanks to my mother for inspiring me to pursue a Ph.D., and thanks to my advisor, Prakash, for taking me on as a student. I can’t imagine where I’d be if not for your mentorship and encouragement over the last several years, and I express my sincerest gratitude for everything you both have done for me.
# TABLE OF CONTENTS

1 Compactified moduli of $G$-bundles and conformal blocks ........................................ 1

1.1 Introduction ................................................................................................................. 1

1.1.1 Organization of the paper ....................................................................................... 5

1.2 Background .................................................................................................................. 5

1.2.1 Preliminaries .......................................................................................................... 5

1.2.2 Singular $G$-bundles .............................................................................................. 6

1.2.3 Semistable singular $G$-bundles on nodal curves ....................................................... 7

1.2.4 Good choices of representation ............................................................................. 11

1.3 Notation ..................................................................................................................... 12

1.4 Section extension problem ......................................................................................... 13

1.4.1 Overview ............................................................................................................... 13

1.4.2 Descending $G$-bundles ........................................................................................ 14

1.4.3 Lifting one-parameter families to the Bhosle stack ................................................. 18

1.4.4 Setup for pole calculation ...................................................................................... 19

1.4.5 Pole calculation ..................................................................................................... 20

1.5 Moduli of singular $G$-bundles ................................................................................. 26

1.5.1 Parameter spaces of singular $G$-bundles .............................................................. 26

1.5.2 Semistable tensor fields ........................................................................................ 28

1.5.3 Tensor field associated to a singular $G$-bundle .................................................... 30

1.5.4 Characterization of GIT semistability ................................................................. 30

1.5.5 Semistability for large values of $\delta$ .................................................................... 30

1.5.6 Set-up for finite generation .................................................................................... 31
1.6 Proof of theorem 1.1.1 ................................................................. 32
  1.6.1 Line bundle identities .......................................................... 32
  1.6.2 The injection $H^0(\mathcal{M}, L)^{SL(W)} \to H^0(\text{Bun}_G(C_0), D^f)$ .................................................. 33
  1.6.3 Conclusion of finite generation ............................................. 35
1.7 Compactifications over $\overline{\mathcal{M}}_g$ and conformal blocks ............................................................. 36
  1.7.1 Conformal blocks vector bundles ........................................ 36
  1.7.2 Finite generation of the sheaf of conformal blocks algebras ......... 37
  1.7.3 The family $X \to \overline{\mathcal{M}}_g$ and modular interpretations ......... 39

2 Unstable divisors in flag varieties ................................................. 40
  2.1 Introduction ............................................................................. 40
  2.2 Notation and Setup ................................................................. 44
    2.2.1 Notation ........................................................................... 44
    2.2.2 Parameter spaces ............................................................. 44
  2.3 Review of GIT .......................................................................... 45
    2.3.1 Kempf’s Theory ............................................................... 46
    2.3.2 Hesselink Stratification ...................................................... 47
  2.4 Unstable Loci in $(G/B)^s$ ........................................................ 48
    2.4.1 Levi-movability ............................................................... 50
    2.4.2 Inequalities for subcones of $\Gamma(s, G)$ ............................... 52
  2.5 F-divisors ................................................................................. 52
    2.5.1 F-divisors in $(G/B)^s$ ...................................................... 53
    2.5.2 Proof outline for theorem 2.1.4 .......................................... 54
  2.6 Proof of Theorem 2.1.4 ............................................................. 55
    2.6.1 Pulling Back to the Parameter Space .................................... 55
    2.6.2 Computation in the Parameter Space ................................... 56
  2.7 Applications .............................................................................. 57
  2.8 Examples .................................................................................. 59
CHAPTER 1: COMPACTIFIED MODULI OF $G$-BUNDLES AND CONFORMAL BLOCKS

1.1 Introduction

Let $C_0$ be a stable curve over $\mathbb{C}$ of genus $g \geq 2$, and $G$ a simple, simply connected algebraic group. When $C_0$ is smooth, Ramanathan defined a notion of semistability for $G$-bundles on $C_0$ and constructed a projective moduli space $\mathcal{M}_G(C_0)$ of semistable $G$-bundles [Ram96]. These moduli spaces fit together into a flat, projective family $\mathcal{M}_G \to \mathcal{M}_g$ over the stack of smooth genus $g$ curves. But, the moduli space of $G$-bundles over a singular curve is no longer projective, just as in the case of vector bundles, where one needs to allow vector bundles on singular curves to degenerate into torsion-free sheaves in order to obtain compactifications.

The problem of compactifying moduli of vector bundles on singular curves has a long and rich history and has been studied extensively, for example in [Sesh82], [NS97], [NS99], or [Pa96] (see [BG19], section 11, for a brief historical account). On the other hand, there is no agreed-upon compactification of the moduli of $G$-bundles for singular curves, although there are several out there. For example, Sun constructed a compactification for $G = \text{SL}(r)$ in [Sun01] using certain classes of torsion-free sheaves, and in [F96] Faltings constructed compactifications for orthogonal and symplectic groups using torsion-free sheaves with a symmetric or alternating bilinear form.

Ultimately, one would like to construct, for any $G$, a stack $\mathcal{M}_G \to \mathcal{M}_g$ over the stack of stable curves that satisfies:

1. the morphism $\mathcal{M}_G \to \mathcal{M}_g$ is flat and projective;
2. $\mathcal{M}_G$ corepresents a moduli functor that generalizes the notion of semistable $G$-bundles;
3. the moduli space of semistable $G$-bundles forms a dense open substack of $\mathcal{M}_G$.

So far, no such compactification is known, but there are several very interesting and useful compactifications at hand.

In this paper, we will focus on one particular compactification, which is the moduli space of “singular $G$-bundles” introduced by Schmitt ([Sch05.1], [Sch05.2]). Fixing a faithful representation $G \subset \text{GL}(V)$, a
singular $G$-bundle on $C_0$ consists of a uniform rank $r = \dim V$ torsion-free sheaf $\mathcal{E}$ and a section

$$\tau : C_0 \to \text{Hom}(\mathcal{E}, V \otimes \mathcal{O}_{C_0}) \sslash G,$$

which should be thought of as a degenerate version of a reduction of structure group from $\text{GL}(V)$ to $G$. Every $G$-bundle $E$ gives a singular $G$-bundle $(\mathcal{E}, \tau)$, where $\mathcal{E} = E \times^G V$ and $\tau$ is the natural reduction of structure group. Conversely, if $\mathcal{E}$ is a locally free sheaf and $\tau(C_0) \subset \text{Isom}(\mathcal{E}, V \otimes \mathcal{O}_{C_0})/G$, then we get a $G$-bundle as the pullback

$$\begin{array}{ccc}
E & \longrightarrow & \text{Isom}(\mathcal{E}, V \otimes \mathcal{O}_{C_0}) \\
\downarrow & & \downarrow \\
C_0 & \overset{\tau}{\longrightarrow} & \text{Isom}(\mathcal{E}, V \otimes \mathcal{O}_{C_0})/G.
\end{array}$$

A singular $G$-bundle which gives a $G$-bundle over a dense open subset of $C_0$ is called “honest.”

In the recent paper [MCS20], Schmitt and Muñoz-Castañeda defined a notion of semistability for honest singular $G$-bundles and proved the existence of a projective moduli space of semistable honest singular $G$-bundles. Moreover, they showed that if $V$ has a $G$-invariant, nondegenerate bilinear form, then an honest singular $G$-bundle gives a $G$-bundle over the whole smooth locus of $C_0$ (loc. cit. theorem 3.5).

There is a relative moduli space of singular $G$-bundles over $\mathcal{M}_g$ constructed in [MC20], and, although it does not necessarily meet the criteria for a compactification we described above, it serves as a very useful starting point.

However, there is another very natural candidate for a compactification, which comes from conformal blocks and is of geometric interest, but lacks a modular interpretation. For a $G$-module $V$, let $\mathcal{D}(V)$ be the line bundle on the stack of $G$-bundles $\text{Bun}_G(C_0)$, whose fiber over a $G$-bundle $E$ is the determinant of cohomology

$$\det H^0(C_0, E \times^G V)^* \otimes \det H^1(C_0, E \times^G V)$$

of the vector bundle $E \times^G V$. Let $A(C_0)$ be the section ring

$$A(C_0) = \bigoplus_{m \geq 0} H^0(\text{Bun}_G(C_0), \mathcal{D}(V)^m),$$
which has connections to the algebra of conformal blocks ([BG19], section 9). Specifically, if $d_V$ is the Dynkin index of $V$ (cf. [KNR94]) and $g$ the Lie algebra of $G$, then we have that

$$A(C_0) = \bigoplus_{m \geq 0} V^*_{g,m d_V}$$

is the $d_V$-th Veronese subalgebra of the conformal blocks algebra. By the work of several authors ([F94], [BL94], [KNR94], [LS97]), when $C$ is a smooth curve there is an ample line bundle $\Theta(V)$ on Ramanathan’s moduli space $\mathfrak{M}_G(C)$ and an algebra isomorphism

$$\bigoplus_{m \geq 0} H^0(\mathfrak{M}_G(C), \Theta(V)^m) \cong \bigoplus_{m \geq 0} H^0(\text{Bun}_G(C), D(V)^m).$$

In particular $\mathfrak{M}_G(C) \cong \text{Proj} A(C)$.

Letting $\mathcal{A} = \bigoplus_{m \geq 0} p_* D(V)^m$ for the relative stack of $G$-bundles $p : \text{Bun}_G \to \overline{\mathcal{M}}_g$, the stack $\text{Proj} \mathcal{A}$ seems like quite a nice compactification of $\mathfrak{M}_G$ — it is flat over $\overline{\mathcal{M}}_g$ with normal integral fibers and agrees with Ramanathan’s moduli space over the locus of smooth curves $\mathcal{M}_g$. However, in order to consider $\text{Proj} \mathcal{A} \to \overline{\mathcal{M}}_g$ a compactification, we have to show that $\mathcal{A}$ is a finitely generated sheaf of algebras (otherwise the morphism $\text{Proj} \mathcal{A} \to \overline{\mathcal{M}}_g$ may not actually be projective). Belkale-Gibney ([BG19]) showed that $\mathcal{A}$ is finitely generated in the case $G = \text{SL}(r)$, and Moon-Yoo generalized this to parabolic $\text{SL}(r)$-bundles on pointed curves in [MY20]. However, one does not necessarily expect finite generation to hold in general, as the fact that the spaces $H^0(\text{Bun}_G(C_0), D(V)^m)$ are even finite-dimensional is already surprising, as we only know this through the connection to conformal blocks (finite-dimensionality of conformal blocks is proven in [TUY]).

The aim of this paper is thus two-fold: a) investigate finite generation of $\mathcal{A}$ for other groups $G$, and b) relate $\mathcal{A}$ to the moduli of singular $G$-bundles in hope of finding a modular interpretation for $\text{Proj} \mathcal{A}$. Our main results are as follows.

**Theorem 1.1.1.** Let $V$ be a finite-dimensional representation of a simple, simply connected Lie group of type $A$ or $C$. Then for any stable curve $C_0$ of genus $g \geq 2$ the algebra

$$A(C_0) = \bigoplus_{m \geq 0} H^0(\text{Bun}_G(C_0), D(V)^m)$$

is finitely generated.
Our proof is based on Belkale-Gibney’s, the main point being to show that $A(C_0)$ is isomorphic (in sufficiently high degree) to the section ring of an ample line bundle on a normalized moduli space of singular $G$-bundles (theorem 1.6.4). Technically this isomorphism is only for certain choices of representation $V$, but the choice of representation does not matter for finite generation by equation (1.1). The failure of our method for Lie groups of type B,D is quite interesting, and we wonder if Moon-Yoo’s approach ([MY20]) would work in this case, or if the finite generation result is simply untrue for these groups. See example 1.4.15 for an explicit example that shows why our method fails for type B,D.

In section 1.7, we explain how the proof of theorem 1.1.1 implies finite generation of the sheaf of algebras $A$ on $\overline{M}_g$, and taking $X = \text{Proj} A$ we obtain the following.

**Theorem 1.1.2.** Let $G$ be a simple, simply connected Lie group of type A or C, and let $g \geq 2$. Then there is a flat, relatively projective family $\mathcal{X} \to \overline{M}_g$ such that

1. the fiber over a smooth curve is Ramanathan’s moduli space of semistable $G$-bundles;

2. the fiber over an arbitrary curve is a normalized moduli space of semistable honest singular $G$-bundles (for any choice of representation which is as in section 1.3).

Please note that, in item 2., the moduli space we deal with is possibly a bit smaller than usual, as we are only able to include the singular $G$-bundles which are in the closure of $\text{Bun}_G(C_0)$. It seems important to determine whether $\text{Bun}_G(C_0)$ is dense in the stack of honest singular $G$-bundles (for appropriate choices of representation), but we are so far unable to do so. Also we should clarify that our moduli spaces are not explicitly the normalizations of the usual moduli spaces of singular $G$-bundles, but rather we normalize the “master space” of which the usual moduli space is a GIT quotient (see section 1.5.6 for the exact definition of the moduli spaces).

Since normalizations do not behave well with base change, item 2. of theorem 1.1.2 only applies to fibers over fixed stable curves, i.e. we are not able to give a modular interpretation of $\mathcal{X} = \text{Proj} A$ over a family of stable curves. This would be an interesting problem to resolve in future work. There are other approaches to compactification that perhaps could help with this, for example in [Ba19] Balaji recently constructed a Gieseker-type moduli space over $\text{Spec} \mathbb{C}[[t]]$ that provides a relative compactification compatible with a degeneration of a smooth curve into an irreducible curve with one node. It would be interesting to see how his moduli spaces relate to conformal blocks. It is also possible that $\text{Proj} A$ has no true modular
interpretation, for example the Satake compactification \( A_g^* \) of the moduli space \( A_g \) of principally polarized abelian varieties of dimension \( g \geq 2 \) ([B58], [FC90]) has no known modular interpretation. However, \( A_g^* \) is known to be a “minimal” compactification, in the sense that any other smooth toroidal compactification maps canonically to \( A_g^* \) ([FC90], theorem 2.3). We wonder if the conformal blocks compactification \( \text{Proj} \mathcal{A} \) has such a property.

### 1.1.1 Organization of the paper

Section 1.2 is background on torsion-free sheaves, singular \( G \)-bundles, and their semistability. We show that the definition of semistability is independent of the choice of polarization (proposition 1.2.5).

The rest of the paper is divided into three parts. Section 1.4 is done at the stack level and is devoted to showing that sections of \( D(V) \) over \( \text{Bun}_G(C_0) \) extend to the normalization of the stack of honest singular \( G \)-bundles (technically, we only deal with the closure of \( \text{Bun}_G(C_0) \) in this stack). For this, we have to show that sections on \( \text{Bun}_G(C_0) \) extend over a one-parameter family of \( G \)-bundles degenerating into an honest singular \( G \)-bundle. We show that any such family can be lifted to a family of descending \( G \)-bundles on the normalization of \( C_0 \) (sects. 1.4.2, 1.4.3), and using a factorization of \( H^0(\text{Bun}_G(C_0), D^f) \) (lemma 1.4.10) we are able to show that sections extend by way of an explicit pole calculation (sect. 1.4.5).

Sections 1.5 and 1.6 deal with the actual moduli spaces and go through the geometric invariant theory setup necessary to prove theorem 1.1.1. We outline the construction of a polarized, normalized moduli space \( (X, L) \) of semistable honest singular \( G \)-bundles (sects. 1.5.1-1.5.6) due to Schmitt and Muñoz-Castañeda [MCS20]. After identifying \( L \) (sect. 1.6.1), we establish an injection \( H^0(X, L) \hookrightarrow H^0(\text{Bun}_G(C_0), D^f) \) (section 1.6.2), which is an isomorphism by the above section extension property.

In the last section (sect. 1.7), we show how this implies finite generation of \( \mathcal{A} \) and prove theorem 1.1.2, as well as discuss the connection to conformal blocks.

### 1.2 Background

#### 1.2.1 Preliminaries

A coherent sheaf \( \mathcal{E} \) on a Noetherian scheme \( X \) is **torsion-free** if every nonzero coherent subsheaf \( \mathcal{E}' \subseteq \mathcal{E} \) is supported in dimension \( \dim X \). The **torsion subsheaf** \( T(\mathcal{E}) \subseteq \mathcal{E} \) is the maximal subsheaf supported in dimension \( < \dim X \) (this exists and is coherent), and \( \mathcal{E} \) is torsion-free if and only if \( T(\mathcal{E}) = 0 \). If \( X \) is a
reduced curve over an algebraically closed field, then \( E \) is torsion-free if and only if it has one-dimensional support and has depth one at every closed point of its support.

Suppose \( X \) is projective over a field, and \( L \) is an ample line bundle on \( X \). The Hilbert polynomial of a coherent sheaf \( E \) is the polynomial

\[
P_E(n) = \chi(E \otimes L^n).
\]

We may express

\[
P_E(n) = \sum_{k=1}^{\dim E} a_k(E) \frac{n^k}{k!}
\]

for some coefficients \( a_k(E) \in \mathbb{Z} \), and the rank and degree of \( E \) are defined as

\[
\text{rk} E = \frac{a_d(E)}{a_d(O_X)}, \quad \text{deg} E = a_{d-1}(E) - \text{rk} E \cdot a_{d-1}(O_X),
\]

where \( d = \dim X \) (take \( a_k(E) = 0 \) for \( k > \dim E \)). We will say that \( E \) has uniform rank \( r \) if its restriction to every component of \( X \) is rank \( r \).

### 1.2.2 Singular \( G \)-bundles

Let \( X \) be a projective variety over \( \mathbb{C} \), \( G \) a reductive algebraic group, and \( V \) a rank \( r \) representation of \( G \).

**Definition 1.2.1.** A singular \( G \)-bundle on \( X \) is a pair \((E, \tau)\) consisting of a uniform rank \( r \) torsion-free sheaf \( E \) and a nontrivial algebra homomorphism

\[
\tau : \text{Sym}^*(V \otimes E)^G \to O_X.
\]

*Note that \( \tau \) is given by a section

\[
\tilde{\tau} : X \to \text{Hom}(E, V^* \otimes O_X) \sslash G := \text{Spec} \text{Sym}^*(V \otimes E)^G.
\]

Every \( G \)-bundle \( E \to X \) gives a singular \( G \)-bundle \((E, \tau)\), where \( E = E \times^G V \) and \( \tilde{\tau} \) is the natural reduction of structure group

\[
X = E/G \to (E \times^G \text{GL}(V))/G = \text{Isom}(E, V^* \otimes O_X)/G.
\]
Conversely, given a singular $G$-bundle $(\mathcal{E}, \tau)$ such that $\mathcal{E}$ is locally free and $\overline{\tau}(X) \subseteq \text{Isom}(\mathcal{E}, V^* \otimes \mathcal{O}_X)/G$, we get back a $G$-bundle $E \to X$ as the pullback

$$
\begin{array}{ccc}
E & \longrightarrow & \text{Isom}(\mathcal{E}, V^* \otimes \mathcal{O}_X) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Isom}(\mathcal{E}, V^* \otimes \mathcal{O}_X)/G.
\end{array}
$$

Hence, we will use the term “$G$-bundle” interchangeably for the singular $G$-bundles which give $G$-bundles under the above construction. It will, however, be important for us to distinguish a certain class of “honest” singular $G$-bundles, which are not necessarily $G$-bundles, but have very good properties:

**Definition 1.2.2.** A singular $G$-bundle $(\mathcal{E}, \tau)$ is called an honest singular $G$-bundle if there is a dense open subset $U \subseteq X$ with $\mathcal{E}|_U$ locally free and $\overline{\tau}(U) \subseteq \text{Isom}(\mathcal{E}|_U, V^* \otimes \mathcal{O}_U)/G$.

Thus, an honest singular $G$-bundle gives a $G$-bundle over a dense open subset of $X$.

### 1.2.3 Semistable singular $G$-bundles on nodal curves

Now suppose $X$ is a connected (possibly reducible) nodal curve with ample line bundle $\mathcal{L}$, and $G$ is semisimple. In [MCS20], Schmitt and Muñoz-Castañeda defined a notion of semistability for honest singular $G$-bundles on nodal curves and showed that there is a projective moduli space of semistable honest singular $G$-bundles. To give the definition of semistability, we first define a reduction of a singular $G$-bundle to a tuple of one-parameter subgroups. Let $\eta_i = \text{Spec} \, K_i$, $1 \leq i \leq t$, be the generic points of the irreducible components of $X$.

**Definition 1.2.3.** ([MCS20], section 3) Let $(\mathcal{E}, \tau)$ be an honest singular $G$-bundle and $E \to U$ the induced $G$-bundle over a dense open subset $U \subseteq X$. For a tuple $\tilde{\lambda} = (\lambda^1, \ldots, \lambda^t)$ of one-parameter subgroups of $G$, define a reduction of $(\mathcal{E}, \tau)$ to $\tilde{\lambda}$ to be a tuple $\tilde{s} = (s_1, \ldots, s_t)$ of points $s_i \in E_{\eta_i}/P(\lambda^i)_{K_i}$, where

$$
P(\lambda^i) = \{ g \in G : \lim_{t \to 0} \lambda^i(t) g \lambda^i(t)^{-1} \text{ exists in } G \}
$$

is the parabolic subgroup associated to $\lambda^i$. 

7
The semistability condition involves weighted filtrations associated to each reduction of the singular $G$-bundle. A weighted filtration of a sheaf $\mathcal{E}$ is a pair $(\mathcal{E}_*, m_*)$ consisting of a filtration by subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{q+1} = \mathcal{E}$$

and a sequence of rational numbers $m_* = (m_1, \ldots, m_q)$. Given an honest singular $G$-bundle $(\mathcal{E}, \tau)$ and a reduction $\bar{s}$ of $(\mathcal{E}, \tau)$ to $\bar{\lambda}$, we can form a weighted filtration $(\mathcal{E}_*, m_*)$ as follows. Let $\bar{P}(\lambda^i) \subseteq \text{GL}(V)$ be the parabolic subgroup for $\lambda^i$ in $\text{GL}(V)$, and let $\lambda^i_1 < \cdots < \lambda^i_{q_i+1}$ be the distinct weights of $\lambda^i$ acting on $V^*$. Via the embedding

$$E/P(\lambda^i) \rightarrow \text{Isom}(\mathcal{E}_U, V^* \otimes \mathcal{O}_U)/\bar{P}(\lambda^i),$$

the sections $s_i$ give partial flags

$$0 = W^i_0 \subset W^i_1 \subset \cdots \subset W^i_{q_i+1} = W^i$$

in $W^i = \mathcal{E}_{\eta_i}$ for $1 \leq i \leq t$. So we have a weighted flag at each fiber $\mathcal{E}_{\eta_i}$, and we take the “direct sum” of these weighted flags to get a weighted flag in $W = \mathcal{E}_{\eta_1} \oplus \cdots \oplus \mathcal{E}_{\eta_t}$. This is done as follows. Let

$$\mu_1 < \cdots < \mu_{q+1}$$

be the distinct values of the $\lambda^*_j$, and for each $1 \leq i \leq t$ and $1 \leq j \leq q + 1$ define $W_j(i) = W^i_k$, where $k$ is maximal such that $\lambda^i_k \leq \mu_j$. Then we have a filtration of $W$ given by

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{q+1} = W,$$

where $W_j = W_j(1) \oplus \cdots \oplus W_j(t)$. Let $\bar{\mathcal{E}}_j = \mathcal{E}/(\mathcal{E} \cap \iota_* W_j)$, and define the filtration $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{q+1} = \mathcal{E}$ by

$$\mathcal{E}_j = \ker[\mathcal{E} \rightarrow \bar{\mathcal{E}}_j/T(\bar{\mathcal{E}}_j)].$$

The weights $m_* = (m_1, \ldots, m_q)$ are defined by $m_j = (\mu_{j+1} - \mu_j)/r$.

**Definition 1.2.4.** ([MCS20], section 3) A singular $G$-bundle $(\mathcal{E}, \tau)$ is semistable if it is honest and, for every nontrivial tuple of one-parameter subgroups $\bar{\lambda}$ (i.e. not all constant) and reduction $\bar{s}$ of $(\mathcal{E}, \tau)$ to $\bar{\lambda}$,
we have
\[ \sum_{j=1}^{q} m_j (\chi(\mathcal{E}) \text{rk}(\mathcal{E}_j) - \chi(\mathcal{E}_j) \text{rk}(\mathcal{E})) \geq 0, \]  
where \((\mathcal{E}_\bullet, m_\bullet)\) is the weighted filtration constructed from \((\tilde{\lambda}, \tilde{s})\) as above. We say \((\mathcal{E}, \tau)\) is stable if strict inequality holds for every such \((\tilde{\lambda}, \tilde{s})\).

A somewhat surprising fact about this type of semistability is that it does not depend on the choice of polarization.

**Proposition 1.2.5.** Semistability for honest singular \(G\)-bundles does not depend on the choice of polarization \(\mathcal{E}\).

**Proof.** Note that the second term \(\chi(\mathcal{E}_j) \text{rk}(\mathcal{E})\) appearing in (1.2) does not depend on the polarization, because \(\mathcal{E}\) is uniform rank, and because Euler characteristics never depend on polarization. Thus, looking only at the first term \(\chi(\mathcal{E}) \text{rk}(\mathcal{E}_j)\), it suffices to show that
\[ \sum_{j=1}^{q} m_j \text{rk}(\mathcal{E}_j) \]  
is independent of polarization.

Let \(X_1, \ldots, X_t\) be the irreducible components of \(X\), and let \(1 \leq j_1 < \cdots < j_{q+1} \leq q + 1\) be the indices such that \(\mu_{j_k} = \lambda^i_{k}\). Then
\[ \text{rk}\mathcal{E}_j|_{X_i} = \dim W_j(i) = \dim W^i_k \]  
for \(j_k \leq j < j_{k+1}\). Hence for each component \(X_i\) we have
\[ \sum_{j=1}^{q} m_j \text{rk}(\mathcal{E}_j|_{X_i}) = \frac{1}{r} \sum_{j=1}^{q} (\mu_{j+1} - \mu_j) \dim W_j(i). \]  
Since the ranks \(\dim W_j(i)\) are the same for the summands indexed by \(j_k \leq j < j_{k+1}\), the sum breaks into several telescoping sums and simplifies to
\[ \frac{1}{r} \left( \sum_{k=1}^{q} (\lambda^i_{k+1} - \lambda^i_k) \dim W^i_k + r(\mu_{q+1} - \lambda^i_{q+1}) \right) = \mu_{q+1} - \frac{1}{r} \sum_{k=1}^{q+1} \lambda^i_k (\dim W^i_k - \dim W^i_{k-1}), \]  
which is just equal to \(\mu_{q+1}\), because \(\lambda^i : \mathbb{G}_m \to G \to \text{GL}(V)\) lands in \(\text{SL}(V)\) if \(G\) is semisimple.
Thus, we have shown that
\[
\sum_{j=1}^{q} m_j (\text{rk} E_j|_{X_i}) = \mu_{q+1}
\]
for each component \(X_i\). To compute (1.3) in terms of (1.4), we will use the following rank formula. For any torsion-free sheaf \(\mathcal{F}\) on the nodal curve \(X\), there is an exact sequence ([Sesh82], section 7.1)
\[
0 \to \mathcal{F} \to \bigoplus_{i=1}^{t} \mathcal{F}|_{X_i} \to \mathcal{T} \to 0
\]
with \(\mathcal{T}\) a torsion sheaf (supported only at points where two components meet). It follows that
\[
\text{rk} \mathcal{F} = \frac{1}{\deg L} \sum_{i=1}^{t} d_i \text{rk} \mathcal{F}|_{X_i},
\]
where \(d_i = \deg L|_{X_i}\). Combining (1.4) and (1.5) we get
\[
\sum_{j=1}^{q} m_j \text{rk}(E_j) = \mu_{q+1}
\]
as well, which proves the proposition. \(\square\)

**Remark 1.2.6.** The semistability inequality (1.2) can therefore also be written as
\[
\frac{\sum_{j=1}^{q} m_j \chi(E_j)}{\mu_{q+1}} \leq \frac{\chi(E)}{r},
\]
or alternatively
\[
\sum_{j=1}^{q+1} \mu_j \chi(E_j/E_{j-1}) \geq 0.
\]
It is interesting to compare this, in the case \(G = \text{SL}(r)\), to the usual definition of semistability for sheaves, which requires
\[
\frac{\chi(E')}{r'} \leq \frac{\chi(E)}{r}
\]
for every nonzero subsheaf \(E' \subseteq E\) with \(r' = \text{rk} E'\). If \(G = \text{SL}(r)\) and \(X\) is irreducible, then semistability of a singular \(G\)-bundle is equivalent to semistability of the underlying sheaf, so we can think of definition 1.2.4 as a generalization of sheaf semistability (perhaps it seems most natural when we write both notions of semistability in the form (1.8)).
1.2.4 Good choices of representation

We continue to assume that $X$ is a projective, connected nodal curve, and $G$ a semisimple group with a faithful representation $G \subset \text{GL}(V)$. For our purposes later on (e.g. prop. 1.4.8), it will be important that the singular $G$-bundles we deal with are not just honest, but even give $G$-bundles over the entire smooth locus of $X$. Schmitt and Muñoz-Castañeda showed that every honest singular $G$-bundle will have this property if the representation $V$ is chosen appropriately.

**Proposition 1.2.7.** ([MCS20], theorem 3.5) Assume that $V$ has a $G$-invariant, nondegenerate bilinear form.\(^1\) Then any honest singular $G$-bundle $(E, \tau)$ with $\deg E = 0$ gives a $G$-bundle over the smooth locus of $X$.

**Proof.** Let us sketch the proof from [MCS20]. We first construct a bilinear form $\mathcal{E} \otimes \mathcal{E} \to \mathcal{O}_{C_0}$ as follows. Consider the morphisms

$$
\text{Hom}(\mathcal{E}, V^* \otimes \mathcal{O}_{C_0}) \xrightarrow{q} \text{Hom}(\mathcal{E}, V^* \otimes \mathcal{O}_{C_0}) \parallel G \xrightarrow{p} C_0 \xleftarrow{p}
$$

Over $H := \text{Hom}(\mathcal{E}, V^* \otimes \mathcal{O}_{C_0}) = \text{Spec} \text{Sym}^*(V \otimes \mathcal{E})$ there is a universal map

$$p^* \text{Sym}^*(V \otimes \mathcal{E}) \to \mathcal{O}_H,$$

whose degree one part gives a map

$$p^* \mathcal{E} \to V^* \otimes \mathcal{O}_H.$$

Let $f : V \to V^*$ be the isomorphism induced by the bilinear form on $V$, and $B^\dagger : V^* \otimes V^* \to \mathbb{C}$ the bilinear form given by $f^{-1}$. Then we have a bilinear form on $p^* \mathcal{E}$,

$$\psi : p^* \mathcal{E} \otimes p^* \mathcal{E} \to V^* \otimes V^* \otimes \mathcal{O}_H \xrightarrow{B^\dagger} \mathcal{O}_H.$$

\(^1\)Every group has such a representation, because if $V$ is any representation, then $V \oplus V^*$ has an invariant, nondegenerate bilinear form.
Note that the universal map $p^* \text{Sym}^r (V \otimes \mathcal{E}) \to \mathcal{O}_H$ is $G$-equivariant, hence so is $\psi$ by $G$-invariance of $B^\dagger$. Since
\[
p^* \mathcal{E} \otimes p^* \mathcal{E} \cong p^*(\mathcal{E} \otimes \mathcal{E}) = q^* \overline{p^*}(\mathcal{E} \otimes \mathcal{E}),\]
we get by adjunction a map
\[
\overline{\psi} : p^*(\mathcal{E} \otimes \mathcal{E}) \to q_* \mathcal{O}_H.
\]
By the equivariance, the image lies in $(q_* \mathcal{O}_H)^G = \mathcal{O}_H\!/G$. Pulling back $\overline{\psi}$ by the section $\overline{\tau} : C_0 \to H\!/G$ gives a bilinear form $\varphi : \mathcal{E} \otimes \mathcal{E} \to \mathcal{O}_{C_0}$.

It easy to check that, for any smooth point $x \in X$, the map $\mathcal{E}|_x \to \mathcal{E}|^*_x$ induced by $\varphi$ factorizes as
\[
\mathcal{E}|_x \xrightarrow{\overline{\tau}_x} V^* \xrightarrow{f^{-1}} V \xrightarrow{\varphi^*_x} \mathcal{E}|^*_x,
\]
where $\overline{\tau}_x \in \text{Hom}(\mathcal{E}|_x, V^*)$ denotes any preimage of $\overline{\tau}(x) \in \text{Hom}(\mathcal{E}|_x, V^*)\!/G$. In particular, $\mathcal{E} \to \mathcal{E}'$ is surjective (on stalks, not just fibers) at any smooth point where $\overline{\tau}(x) \in \text{Isom}(\mathcal{E}|_x, V^*)/G$. Since $\mathcal{E}$ and $\mathcal{E}'$ have the same multirank, this implies that $\mathcal{E} \to \mathcal{E}'$ is an isomorphism over the maximal open subset of the smooth locus where $(\mathcal{E}, \tau)$ gives a $G$-bundle. Then $\mathcal{E} \to \mathcal{E}'$ is injective over a dense open subset, so it is injective everywhere because $\mathcal{E}$ is torsion-free. By [MCS20], appendix, we have $\chi(\mathcal{E}) = \chi(\mathcal{E}')$ for any degree zero torsion-free sheaf on a nodal curve, so $\mathcal{E} \to \mathcal{E}'$ is an isomorphism, and the factorization (1.10) implies that $\overline{\tau}(x) \in \text{Isom}(\mathcal{E}|_x, V^*)/G$ for every smooth point $x \in X$. Thus $(\mathcal{E}, \tau)$ is a $G$-bundle over the entire smooth locus. 

1.3 Notation

The following notation will be fixed for the remainder of the chapter. Let $C_0$ be a stable curve of genus $g \geq 2$ with normalization $\nu : C \to C_0$. Let $S$ be the set of nodes of $C_0$, and for $x \in S$ let $\nu^{-1}(x) = \{x_1, x_2\}$. Let $G$ be a simple, simply-connected algebraic group, and fix a faithful representation $G \subset \text{GL}(V)$ of rank $r$. We will assume that $V$ has a nondegenerate bilinear form preserved by $G$, so that by proposition 1.2.7 every degree zero honest singular $G$-bundle on $C_0$ gives a $G$-bundle on $C_0 \setminus S$. In particular this is true of any honest singular $G$-bundle which is a flat limit of $G$-bundles. For a singular $G$-bundle $(\mathcal{E}, \tau)$ and point $x \in C_0$, we denote by $\overline{\tau}_x \in \text{Hom}(\mathcal{E}|_x, V^*)$ any preimage of $\overline{\tau}(x) \in \text{Hom}(\mathcal{E}|_x, V^*)\!/G$. 


12
1.4 Section extension problem

1.4.1 Overview

We will work with the following stacks.

- \( \text{Bun}_G(C_0) \): the stack of \( G \)-bundles, whose fiber over a scheme \( T \) is the groupoid of \( G \)-bundles on \( C_0 \times T \).

- \( \text{SB}_G(C_0) \): the stack of singular \( G \)-bundles, whose fiber over a scheme \( T \) is the groupoid of singular \( G \)-bundles on \( C_0 \times T \) (we require that the underlying sheaf is flat over \( T \) and its restriction to each fiber over \( T \) is torsion-free, uniform rank \( r \)).

- \( \text{SB}_G^*(C_0) \): the open substack of honest singular \( G \)-bundles (meaning fiberwise honest over the base).

- \( \text{SB}_G^0(C_0) \): the closure of \( \text{Bun}_G(C_0) \) in \( \text{SB}_G^*(C_0) \).

All are algebraic stacks locally of finite type over \( \mathbb{C} \). \( \text{Bun}_G(C_0) \) is smooth and connected ([BF19], proposition 5.1) and forms a dense open substack of \( \text{SB}_G^0(C_0) \).

Let \( D \) be the determinant of cohomology line bundle on the stack of coherent sheaves \( \text{Coh}(C_0) \), whose fiber over a sheaf \( E \) is

\[
\det H^0(C_0, \mathcal{E})^* \otimes \det H^1(C_0, \mathcal{E}).
\]

Let the pullback of \( D \) to \( \text{Bun}_G(C_0) \) along the contraction map

\[
\text{Bun}_G(C_0) \rightarrow \text{Coh}(C_0),
\]

\[
E \mapsto E \times^G V
\]

also be denoted \( D \).

The goal of this section is to show that sections of \( D \) over \( \text{Bun}_G(C_0) \) extend to the normalization of \( \text{SB}_G^0(C_0) \). Recall that an algebraic stack \( S \) is normal if there is a smooth surjection \( U \rightarrow S \) with \( U \) a normal scheme. Any locally Noetherian algebraic stack \( S \) has a normalization \( \bar{S} \rightarrow S \), which is defined by the property that \( \bar{S} \rightarrow S \) is representable and, for any scheme \( T \) and smooth morphism \( T \rightarrow S \), the base change \( \bar{S} \times_S T \rightarrow T \) is the normalization of \( T \).
We will prove:

**Theorem 1.4.1.** Suppose $G$ is a simple Lie group of type A or C, and the representation $V$ is chosen as follows:

(i) if $G = \text{SL}(n)$, choose $V = W \oplus W^*$, where $W = \mathbb{C}^n$ is the standard representation;

(ii) if $G = \text{Sp}(2n)$, choose $V = \mathbb{C}^{2n}$ the standard representation.

Let $\mathcal{Y}$ be the normalization of $\text{SB}_G^0(C_0)$, and let $\mathcal{D}$ still denote the pullback of $\mathcal{D}$ to $\mathcal{Y}$. Then for any $l \geq 0$, the restriction map

$$H^0(\mathcal{Y}, \mathcal{D}^l) \rightarrow H^0(\text{Bun}_G(C_0), \mathcal{D}^l)$$

is an isomorphism.

The plan for the proof is as follows. Let $A = k[[t]], \quad K = k((t))$. We have to show that, for any map

$$f : \text{Spec} \ A \rightarrow \text{SB}_G^0(C_0)$$

sending $\text{Spec} \ K$ into $\text{Bun}_G(C_0)$, the pullback of a section on $\text{Bun}_G(C_0)$ has no pole at $t = 0$. We first recall the definition of descending $G$-bundles from [Sch05.1] and show that any map $f$ as above lifts to the stack of descending $G$-bundles (sect. 1.4.3). Using a factorization of $H^0(\text{Bun}_G(C_0), \mathcal{D}^l)$ (lemma 1.4.10), we are then able to compute the pole explicitly (sect. 1.4.5).

### 1.4.2 Descending $G$-bundles

Bhosle introduced (e.g. [Bh92]) the following method to model torsion-free sheaves on $C_0$ in terms of certain bundles on the normalization $C$, which we call “Bhosle bundles.” A Bhosle bundle is a pair $(\mathcal{F}, \vec{q})$ consisting of a rank $r$ vector bundle $\mathcal{F}$ on $C$ and a collection $\vec{q} = (q_x)_{x \in S}$ of rank $r$ quotients

$$q_x : \mathcal{F}|_{x_1} \oplus \mathcal{F}|_{x_2} \rightarrow Q_x.$$

Every Bhosle bundle yields a torsion-free sheaf

$$\mathcal{E} = \ker[\nu_x \mathcal{F} \rightarrow \bigoplus_{x \in S} (i_x)_* Q_x]$$
on $C_0$, and every torsion-free sheaf on $C_0$ arises this way ([Sun00], lemma 2.1). If the coordinate maps $q_{x_1}, q_{x_2}$ are isomorphisms, then $-q_{x_2}^{-1}q_{x_1}$ gives an identification $\mathcal{F}|_{x_1} \sim \mathcal{F}|_{x_2}$, and $\mathcal{E}$ is the vector bundle resulting from gluing the fibers of $\mathcal{F}$ along this identification. If one of the $q_{x_i}$ is not an isomorphism, then $\mathcal{E}$ is not locally free, but the local structure of $\mathcal{E}$ is still determined by the ranks of the $q_{x_i}$ (see loc. cit.).

The analogous model for $G$-bundles is called a “descending $G$-bundle” (see [Sch05.1], section 4.3, for the definition of families).

**Definition 1.4.2.** A descending $G$-bundle is a triple $(\mathcal{F}, \sigma, \vec{q})$, where $(\mathcal{F}, \sigma)$ is a $G$-bundle on $C$ (presented as a singular $G$-bundle), and $(\mathcal{F}, \vec{q})$ is a Bhosle bundle such that the image of

$$\operatorname{Sym}^* (V \otimes \mathcal{E})^G \to \nu_* \operatorname{Sym}^* (V \otimes \mathcal{F})^G \xrightarrow{\nu_* \sigma} \nu_* \mathcal{O}_C$$

(1.11)

is contained in $\mathcal{O}_{C_0} \subset \nu_* \mathcal{O}_C$, where $\mathcal{E}$ is the induced torsion-free sheaf.

**Remark 1.4.3.** Note that, even though $\nu_*$ does not commute with $\operatorname{Sym}$, there is a natural map

$$\operatorname{Sym}^* (\nu_* (V \otimes \mathcal{F})) \to \nu_* \operatorname{Sym}^* (V \otimes \mathcal{F}),$$

and (1.11) is obtained from the composition

$$\operatorname{Sym}^* (V \otimes \mathcal{E}) \to \operatorname{Sym}^* (\nu_* (V \otimes \mathcal{F})) \to \nu_* \operatorname{Sym}^* (V \otimes \mathcal{F}).$$

**Notation 1.4.4.** For a descending $G$-bundle $(\mathcal{F}, \sigma, \vec{q})$, we will write $q_{x_i}$ for the restriction of $q_{x}$ to $\mathcal{F}|_{x_i}$, and $\kappa_{x_i} = \det q_{x_i} \circ (\det \vec{\sigma}_{x_i})^{-1}$, $i = 1, 2$.

By design, every descending $G$-bundle $(\mathcal{F}, \sigma, \vec{q})$ on $C$ gives a singular $G$-bundle $(\mathcal{E}, \tau)$ on $C_0$. The assignment $(\mathcal{F}, \sigma, \vec{q}) \mapsto (\mathcal{E}, \tau)$ has the following property (which can be generalized to families).

**Proposition 1.4.5.** Suppose $(\mathcal{F}, \sigma, \vec{q})$ and $(\mathcal{F}', \sigma', \vec{q}')$ are descending $G$-bundles which produce actual $G$-bundles $(\mathcal{E}, \tau)$ and $(\mathcal{E}', \tau')$ on $C_0$. Given an isomorphism $(\mathcal{E}, \tau) \sim (\mathcal{E}', \tau')$, there is a unique isomorphism $(\mathcal{F}, \sigma, \vec{q}) \sim (\mathcal{F}', \sigma', \vec{q}')$ making the following diagram commute:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \nu_* \mathcal{F} & \longrightarrow & \bigoplus_{x \in S} Q_x & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \nu_* \mathcal{F}' & \longrightarrow & \bigoplus_{x \in S} Q'_x & \longrightarrow & 0
\end{array}
$$

15
Proof. Since $\mathcal{E}$ and $\mathcal{E}'$ are locally free, an isomorphism $(\mathcal{E}, \tau) \sim (\mathcal{E}', \tau')$ pulls back to an isomorphism $(\nu^* \mathcal{E}, \nu^* \tau) \sim (\nu^* \mathcal{E}', \nu^* \tau')$ respecting the natural Bhosle structures. Therefore we assume $(\mathcal{E}, \tau) = (\mathcal{E}', \tau')$ and show there is an isomorphism of descending $G$-bundles $\nu^* \mathcal{E} \to \mathcal{F}$. The map $\nu^* \mathcal{E} \to \mathcal{F}$ induced by $\mathcal{E} \to \nu_* \mathcal{F}$ is surjective, because the $q_{x_i}$ are isomorphisms, so it is an isomorphism, because $\nu^* \mathcal{E}$ and $\mathcal{F}$ are vector bundles of the same rank. This isomorphism respects the singular $G$-bundle structures, because the singular $G$-bundle structure on $\mathcal{E}$ is just the restriction of that on $\mathcal{F}$. Therefore we assume $\mathcal{E} = \mathcal{E}'$ and show there is an isomorphism of descending $G$-bundles

$$
\nu^* \mathcal{E} \to \mathcal{F}
$$

that makes the diagram

$$
\begin{array}{ccc}
\nu^* \mathcal{E}|_{x_1} \oplus \nu^* \mathcal{E}|_{x_2} & \longrightarrow & \mathcal{E}|_x \\
\downarrow & & \downarrow \\
\mathcal{F}|_{x_1} \oplus \mathcal{F}|_{x_2} & \longrightarrow & Q_x
\end{array}
$$

commute, thus $\nu^* \mathcal{E} \sim \mathcal{F}$ is an isomorphism of descending $G$-bundles. Uniqueness is easy to see. □

The next proposition gives a version of the descending $G$-bundle condition that is easier to use in practice (see the examples following the proposition). Recall the Schur functor $S^\lambda$ associated to an integer partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$, which associates a subspace $S^\lambda(M) \subseteq M^\otimes |\lambda|$ to each finite-dimensional vector space $M$. A map of vector spaces $f : M \to M'$ induces a map $S^\lambda(f) : S^\lambda(M) \to S^\lambda(M')$.

**Proposition 1.4.6.** A triple $(\mathcal{F}, \sigma, \bar{q})$ consisting of a $G$-bundle $(\mathcal{F}, \sigma)$ on $C$ and a Bhosle bundle $(\mathcal{F}, \bar{q})$ is a descending $G$-bundle if and only if the map

$$
S^\lambda(\ker q_x) \xrightarrow{(S^\lambda(\bar{q}_{x_1}), S^\lambda(\bar{q}_{x_2}))} S^\lambda(\mathcal{F}|_{x_1}) \oplus S^\lambda(\mathcal{F}|_{x_2}) \xrightarrow{S^\lambda(\bar{\sigma}_{x_1}) - S^\lambda(\bar{\sigma}_{x_2})} S^\lambda(V)^* \to (S^\lambda(V)^G)^* \tag{1.12}
$$

is zero for every partition $\lambda$ and node $x \in S$. If all $q_{x_i}$ are isomorphisms, then this is equivalent to the requirement that the gluing functions

$$
-\bar{\sigma}_{x_2} q_{x_2}^{-1} \bar{q}_{x_1} \bar{\sigma}_{x_1}^{-1} : V^* \to V^*
$$

lie in $G \subset \text{GL}(V^*)$. 

16
Proof. Recall that $\mathcal{O}_{C_0}$ is the kernel of the map

$$\nu_4\mathcal{O}_C \to (\nu_4\mathcal{O}_C)_x = \bigoplus_{x \in S} k(x_1) \oplus k(x_2) \to \bigoplus_{x \in S} k(x),$$

where the last map is given by $(a_1, a_2) \mapsto a_1 - a_2$ in each summand. So, it suffices to check the descending $G$-bundle condition at the fiber over each node. The image of $\text{Sym}^*(V \otimes \mathcal{E})$ in $\nu_4\text{Sym}^*(V \otimes \mathcal{F})_x = \text{Sym}^*(V \otimes \mathcal{F}|_{x_1}) \oplus \text{Sym}^*(V \otimes \mathcal{F}|_{x_2})$ is the same as the image of

$$(\text{Sym}^*(\text{pr}_1), \text{Sym}^*(\text{pr}_2)) : \text{Sym}^*(V \otimes \ker q_x) \to \text{Sym}^*(V \otimes \mathcal{F}|_{x_1}) \oplus \text{Sym}^*(V \otimes \mathcal{F}|_{x_2}),$$

so the requirement for descent is that the composition

$$\text{Sym}^*(V \otimes \ker q_x)^G \to \text{Sym}^*(V \otimes \mathcal{F}|_{x_1})^G \oplus \text{Sym}^*(V \otimes \mathcal{F}|_{x_2})^G \to k(x_1) \oplus k(x_2) \to k(x) \quad (1.13)$$

is zero for all nodes $x \in S$, where the first and last map are as above and the middle map is $(\nu_4\sigma)|_x = (\sigma|_{x_1}, \sigma|_{x_2})$. The first part of the proposition then follows from the decomposition

$$\text{Sym}^*(V \otimes M) = \bigoplus_{\lambda, d} S^\lambda(V) \otimes S^\lambda(M).$$

for any vector space $M$.

For the second assertion, let

$$g = -\sigma_{x_2}q_{x_2}^{-1}q_{x_1}^{-1} \in \text{GL}(V^*),$$

so that

$$\ker q = \{(v^*, gv^*) : v^* \in V^*\}$$

as a subspace of $V^* \oplus V^*$ (using the $\sigma_{x_i}$ to identify $\mathcal{F}|_{x_i}$ with $V^*$). By the first part of the proposition, the descent requirement becomes

$$v^*(gv) = v^*(v)$$

for any $v \in V$. 

17
for all \( v \in S^\lambda(V)^G, \ v^* \in S^\lambda(V)^* \). This is the same as saying \( f(g) = f(e) \) for every \( G \)-invariant function \( f \) on \( GL(V^*) \), or rather \( \bar{g} = \bar{e} \) as points of \( GL(V^*)/G \) (note that \( GL(V^*)/G \) is affine since \( G \) is reductive).

\[ \square \]

**Remark 1.4.7.** The proposition is actually quite usable. As shown in the proof, the requirement for descent is that a certain algebra homomorphism

\[ \text{Sym}^*(V \otimes \ker q_x)^G \to \mathbb{C} \oplus \mathbb{C} \]

has image contained in the diagonal subalgebra of \( \mathbb{C} \oplus \mathbb{C} \). So, it suffices to check (1.12) only in degrees \( \lambda \) containing algebra generators for \( \mathbb{C}[\text{End}(V)]^G = \mathbb{C}[\text{End}(V) \mod G] \), where \( G \) acts by \( g \cdot f = f \circ g^{-1} \) on \( \text{End}(V) \). Proposition 1.4.6 can then be used to show that, for the following classical groups \( G \subset \text{GL}(V) \), a descending \( G \)-bundle consists of a \( G \)-bundle \( (\mathcal{F}, \sigma) \) with a Bhosle structure \( \bar{q} \) satisfying the following conditions.

1. For \( G = \text{SL}(V) \), we only need \( \kappa_{x_1} = (-1)^r \kappa_{x_2} \).

2. If \( V \) is a symplectic space and \( G = \text{Sp}(V) \), then the reduction of structure group \( \bar{\sigma} \) gives a symplectic form \( \psi : \wedge^2 \mathcal{F} \to \mathcal{O}_C \), and the descending \( G \)-bundle condition becomes: \( \ker q_x \) is isotropic for the symplectic form

\[ \langle (v_1, v_2), (w_1, w_2) \rangle = \psi_{x_1}(v_1, w_1) - \psi_{x_2}(v_2, w_2) \]

on \( \mathcal{F}|_{x_1} \otimes \mathcal{F}|_{x_2} \).

3. If \( V \) is a quadratic space and \( G = \text{SO}(V) \), then the condition is the same as for symplectic groups, with the addition that \( \kappa_{x_1} = (-1)^r \kappa_{x_2} \).

### 1.4.3 Lifting one-parameter families to the Bhosle stack

Next we will show that any family of singular \( G \)-bundles given by a map \( f \) as in section 1.4.1 lifts to a family of descending \( G \)-bundles.

**Proposition 1.4.8.** Let \( T \) be a smooth curve with a closed point \( 0 \in T \), and let \( (\mathcal{E}, \tau) \) be a family of singular \( G \)-bundles on \( C_0 \times T \) that gives a \( G \)-bundle on the complement of \( S \times 0 \). Then \( (\mathcal{E}, \tau) \) is induced by a family of descending \( G \)-bundles \( (\mathcal{F}, \sigma, \bar{q}) \) on \( C \times T \).
Proof. Let $\mathcal{F} = (\nu^*E)^\vee$, and let $j : U \rightarrow C \times T$ be the inclusion of $U = C \times T - \nu^{-1}(S) \times T$. Note that $\mathcal{F}$ is locally free by [H80], corollary 1.4, and satisfies $j^*F \cong j^*\nu^*E$. Since $C \times T$ is normal, we have $j_*$ $j^*\mathcal{O}_{C \times T} \cong \mathcal{O}_{C \times T}$, hence $j^*\nu^*\tau$ extends to a morphism $\sigma : \text{Sym}^* (V \otimes F)^G \rightarrow \mathcal{O}_{C \times T}$. It is an algebra homomorphism, because the condition to be an algebra homomorphism is that a certain map $\text{Sym}^* (V \otimes F)^G \otimes \text{Sym}^* (V \otimes F)^G \rightarrow \mathcal{O}_{C \times T}$ is zero, which can be checked generically. Since $\sigma$ is nondegenerate in codimension one (nondegenerate meaning $\tilde{\sigma}$ lands in $\text{Isom}(\mathcal{F}, V^* \otimes \mathcal{O}_C)/G$), it follows that $\sigma$ is nondegenerate everywhere, because the degeneracy locus is the divisor where $\det \tilde{\sigma} : \det \mathcal{F} \rightarrow \mathcal{O}$ vanishes. Thus, $(\mathcal{F}, \sigma)$ is a $G$-bundle. To get the quotient maps $\bar{q}$, just take the ones we get from pulling back $E$ over $T - \{0\}$ and extend to $t = 0$ using properness of Grassmannians (the condition on $\ker q_x$ given by proposition 1.4.6 will continue to hold at $t = 0$ since it is defined by the vanishing of a map of vector bundles on $T$).

Now, $(\mathcal{F}, \sigma, \bar{q})$ induces a family of torsion-free singular $G$-bundles $(\mathcal{E}', \tau')$ on $C_0$ which agrees with the original family over $U' = C_0 \times T - S \times 0$. Both $\mathcal{E}$ and $\mathcal{E}'$ are flat families of depth 1 sheaves parametrized by a smooth curve, so they are S2 sheaves on $C_0 \times T$ by [EGAIV, II], 6.3.1. Hence both equal their pushforward from $U'$. To see $\tau = \tau'$, note that the inclusion $\mathcal{O}_{C_0 \times T} \subseteq j_* \mathcal{O}_{U'}$ gives an inclusion

$$\text{Hom}_{C_0 \times T}(\mathcal{M}, \mathcal{O}_{C_0 \times T}) \subseteq \text{Hom}_{C_0 \times T}(\mathcal{M}, j_* \mathcal{O}_{U'}) = \text{Hom}_{U'}(j^* \mathcal{M}, \mathcal{O}_{U'}).$$

for any sheaf $\mathcal{M}$. Since $\tau = \tau'$ over $U'$, the two families coincide and the proposition is proved. \qed

1.4.4 Setup for pole calculation

Recall that $A = k[[t]]$, $K = k((t))$. Let $\text{Bh}_G(\nu)$ be the stack of descending $G$-bundles, and $\text{Bh}_G^0(\nu)$ the open substack where all $q_x$ are isomorphisms. Let $\pi : \text{Bh}_G(\nu) \rightarrow \text{SB}^*_G(C_0)$ be the natural projection, which restricts to an isomorphism

$$\text{Bh}_G^0(\nu) \xrightarrow{\sim} \text{Bun}_G(C_0).$$

By proposition 1.4.8, a map $f : \text{Spec} A \rightarrow \text{SB}^*_G(C_0)$ as in section 1.4.1 lifts to a map $\bar{f} : \text{Spec} A \rightarrow \text{Bh}_G(\nu)$ sending $\text{Spec} K$ into $\text{Bh}_G^0(\nu)$. Thus, to prove theorem 1.4.1 it suffices to show the following.
Situation 1.4.9. For any map \( \bar{f} : \text{Spec } A \rightarrow \text{Bun}_G(\nu) \) sending \( \text{Spec } K \) into \( \text{Bun}_G^0(\nu) \) and any section \( s \) in \( H^0(\text{Bun}_G^0(\nu), \pi^*D^!_I) = H^0(\text{Bun}_G(C_0), D^!_I) \), we must show \( \bar{f}^*s \) has no pole at \( t = 0 \).

In order to compute the pole of \( \bar{f}^*s \), we will use the following factorization of \( H^0(\text{Bun}_G(C_0), D^!_I) \).

Let \( \mathcal{E} \) be the universal family of \( G \)-bundles parametrized by \( \text{Bun}_G(C) \), and for a dominant integral weight \( \lambda \) of \( G \) and point \( x \in C \) let

\[
\mathcal{E}^\lambda_x = \mathcal{E}_x \times_G V^\lambda,
\]

where \( V^\lambda \) is the irreducible representation of \( G \) with highest weight \( \lambda \).

Lemma 1.4.10. ([BG19], lemma 6.4) Let \( d_V \) be the Dynkin index of the \( G \)-module \( V \) (see [KNR94]). Then the pullback map \( p : \text{Bun}_G(C_0) \rightarrow \text{Bun}_G(C) \) induces an isomorphism

\[
\bigoplus_{\lambda} H^0(\text{Bun}_G(C), D^!_I \otimes \bigotimes_{x \in S} \mathcal{E}^\lambda_{x_1} \otimes \mathcal{E}^\lambda_{x_2}) \sim H^0(\text{Bun}_G(C_0), D^!_I),
\]

(1.14)

where the sum is over all functions \( \lambda \) assigning a dominant integral weight \( \lambda_x \) of level \( \leq ld_V \) to each node \( x \in S \), and \( \lambda^*_x \) denotes the highest weight of \((V^{\lambda_x})^*\).

The isomorphism in the lemma has the following formula. Let \( E \rightarrow C_0 \) be a \( G \)-bundle and \( s \) a section in the \( \lambda \)-component on \( \text{Bun}_G(C) \). Picking trivializations of \( F = \nu^*E \) at \( x_1 \) and \( x_2 \) for each \( x \in S \) gives a collection of transition elements \( g_x \in G \), and we may express \( s|_E \) as a sum of terms \( \alpha \otimes \bigotimes_{x \in S} (v_x \otimes v^*_x) \) with \( \alpha \in D(\mathcal{E})^I \), \( v_x \in V^{\lambda_x}, v^*_x \in V^{\lambda^*_x} \), where \( \mathcal{E} = E \times^G V \). Then as a section on \( \text{Bun}_G(C_0) \), \( s|_E \) is the corresponding sum of the terms \( \prod_{x \in S} v^*_x(g_xv_x)\alpha \).

1.4.5 Pole calculation

We will resolve situation 1.4.9 by bounding the pole of \( \bar{f}^*s \) in proposition 1.4.12. Before carrying out the pole calculation, let me point out the following two items. First, if \((\mathcal{F}, \sigma, \tilde{q})\) is a family of descending \( G \)-bundles given by a map \( \bar{f} \) as in situation 1.4.9, then by proposition 1.4.6 the gluing function \( g_x = -\tilde{\sigma}_{x_2}q^{-1}_{x_2}q_{x_1}\tilde{\sigma}^{-1}_{x_1} \) is in \( G(K) \) for each node \( x \in S \). But, the element \( g_x \in G(K) \) is only well-defined up to the left and right action of \( G(A) \), as it depends on a choice of \( G(A) \)-coset representatives of the \( \tilde{\sigma}_{x_i} \in \text{Isom}(\mathcal{F}|_{x_i}, V^* \otimes A)/G_A \). Recall that the double cosets \( G(A)\backslash G(K)\backslash G(A) \) are parametrized by
dominant one-parameter subgroups of a maximal torus of $G$, where an OPS $\varphi$ corresponds to the $K$-point

$$\gamma_{\varphi} : \text{Spec } K \to \text{Spec } \mathbb{C}[t, t^{-1}] = \mathbb{G}_m \xrightarrow{\varphi} G.$$  

Thus, we may always put $\tilde{f}$ into a “normal form,” i.e. pick coset representatives of the $\tilde{\sigma}_x$, such that, for each $x \in S$, we have $g_x = \gamma_{\varphi_x}$ for some dominant OPS $\varphi_x$.

We also have the following identities between $D$ and $D_{Bh}$, where $D_{Bh}$ is the determinant of cohomology line bundle on $B_G(\nu)$. The exact sequence

$$0 \to \mathcal{E} \to \nu_\ast \mathcal{F} \to \bigoplus_{x \in S} Q_x \to 0 \quad (1.15)$$

for a Bhosle bundle $(\mathcal{F}, \tilde{q})$ shows that there is an isomorphism

$$\pi^* D \sim D_{Bh} \otimes \bigotimes_{x \in S} \det Q_x,$$  

where $Q_x$ is the “universal $Q_x$” vector bundle. If $(\mathcal{E}, \tau)$ is a $G$-bundle on $C_0$ (not just a singular $G$-bundle), then the exact sequence

$$0 \to \mathcal{E} \to \nu_\ast \nu^* \mathcal{E} \to \bigoplus_{x \in S} \mathcal{E}_x \to 0 \quad (1.17)$$

gives a canonical isomorphism

$$D(\mathcal{E}) \cong D(\nu^* \mathcal{E}) \otimes \bigotimes_{x \in S} \det \mathcal{E}_x \cong D(\nu^* \mathcal{E}),$$  

where the second map is given by $\det \tilde{\tau} : \det \mathcal{E} \xrightarrow{\sim} \mathcal{O}$. If $(\mathcal{E}, \tau)$ is induced by a Bhosle bundle $(\mathcal{F}, \sigma, \tilde{q})$, then we have a natural isomorphism of exact sequences (1.15) and (1.17) by lemma 1.4.5, and the two identities (1.16) and (1.18) get related by the following lemma.

**Lemma 1.4.11.** Let $B$ be a ring. For an exact sequence of finitely generated projective $B$-modules

$$0 \to M_1 \to M_2 \to \cdots \to M_n \to 0,$$

the canonical isomorphism $\det M \sim B$ is functorial with respect to isomorphisms of exact sequences.
Proof. We mean that, for any isomorphism \( f_\bullet : M_\bullet \to N_\bullet \), the following diagram commutes:

\[
\begin{array}{ccc}
\det M_\bullet & \xrightarrow{\det f_\bullet} & \det N_\bullet \\
\downarrow & & \downarrow \\
B & \to & B
\end{array}
\]

This is easy to check for a short exact sequence, and the general case can be done by induction. \( \square \)

Now we are ready for the pole calculation. In the following proposition, note that any free rank one \( A \)-module has a well-defined valuation function – denoted “\( \text{ord} \)” – given by picking an isomorphism to \( A \) (the choice of isomorphism does not affect the valuation). Recall that \( \kappa_{x_1} = \det q_{x_1} \circ (\det \sigma_{x_1})^{-1} \) for a descending \( G \)-bundle \( (\mathcal{F}, \sigma, \bar{q}) \) and node \( x \in S \). Let us also write \( g_x = -\sigma_{x_2}q^{-1}_{x_2}q_{x_1}\sigma_{x_1}^{-1} \) and let \( \varphi_x \) be the dominant coweights such that \( \gamma_{\varphi_x} \in G(A)\backslash G(K)/G(A) \) represents the double coset of \( g_x \) (cf. the discussion at the beginning of the section).

**Proposition 1.4.12.** Let \( s \) be a section in the \( \lambda \)-component of \( H^0(\text{Bun}_G(C_0), \mathcal{D}^l) \), where \( \lambda \) is an assignment of a level \( \leq ld_V \) dominant integral weight \( \lambda_x \) to each node \( x \in S \) (see lemma 1.4.10). Let \( (\mathcal{F}, \sigma, \bar{q}) \) be a family of descending \( G \)-bundles given by a map \( \tilde{f} \) as in situation 1.4.9. Then \( \tilde{f}^* s \) vanishes to order at least

\[
\sum_{x \in S} (l \cdot \text{ord} \kappa_{x_1} + w_0 \lambda_x(\varphi_x))
\]

at \( t = 0 \), where \( w_0 \) is the longest element of the Weyl group of \( G \).

Proof. Let \( (\mathcal{E}, \tau) \) be the family of singular \( G \)-bundles on \( C_0 \) induced by \( (\mathcal{F}, \sigma, \bar{q}) \). As a section on \( \text{Bun}_G(C) \), we have (using lemma 1.4.10)

\[
s_{(\mathcal{F}, \sigma)} = \beta \otimes \bigotimes_{x \in S} v_x \otimes v^*_x
\]

for some \( \beta \in \mathcal{D}(\mathcal{F})^l, v_x \in V^{\lambda_x} \otimes A, v^*_x \in V^{\lambda^*_x} \otimes A \). We need to transfer \( \beta \) back to \( \mathcal{D}(\mathcal{E}_K)^l \) using isomorphism (1.18), then apply isomorphism (1.16) to get an element of \( \mathcal{D}(\mathcal{F}_K)^l \otimes \bigotimes_{x \in S}(\det Q_x)^l \). As \( \nu^* \mathcal{E}_K \) and \( \mathcal{F}_K \) are isomorphic Bhosle bundles, there is an induced isomorphism of exact sequences (1.15) and (1.17). The isomorphism \( \mathcal{E}|_{x \times K} \to Q_x \otimes K \) is the composition

\[
\mathcal{E}|_{x \times K} \sim \nu^* \mathcal{E}|_{x \times K} \sim \mathcal{F}|_{x \times K} \xrightarrow{q_{x_1}} Q_x \otimes K.
\]

22
Consider the diagram

\[
\begin{array}{ccc}
\mathcal{D}(\nu^* \mathcal{E}_K) & \xrightarrow{(\det \tau)^{-1}} & \mathcal{D}(\nu^* \mathcal{E}_K) \otimes \bigotimes_{x \in S} \det \mathcal{E}_{|_{x \times K}} \\
\downarrow & & \downarrow \\
\mathcal{D}(\mathcal{F}_K) & \xrightarrow{(\det \varpi)^{-1}} & \mathcal{D}(\mathcal{F}_K) \otimes \bigotimes_{x \in S} \mathcal{F}_{|_{x_1 \times K}} \xrightarrow{\det \varphi_1} \mathcal{D}(\mathcal{F}_K) \otimes \bigotimes_{x \in S} \det Q_x.
\end{array}
\]

The right-hand square is commutative by lemma 1.4.11 and equation (1.20), and the left-hand square is commutative because the isomorphism $\nu^* \mathcal{E}_K \to \mathcal{F}_K$ respects the singular $G$-bundle structures. Hence, the diagram commutes. The “canonical route” (given by lemma 1.4.10 and isomorphism (1.16)) to transfer $s$ from $\text{Bun}_G(C)$ to $\text{Bl}_G(\nu)$ is to move the element $\beta \in \mathcal{D}(\mathcal{F}_K)^t \cong \mathcal{D}(\nu^* \mathcal{E}_K)^t$ along the top row and down the last column. This is the same as the map along the bottom row, which is just tensoring with $\bigotimes_{x \in S} \kappa_{x_1}$.

Thus,

\[
s|_{(\mathcal{F}, \sigma, q)} = (\prod_{x \in S} v_x^*(\gamma_{\varphi, v}^x))\beta \otimes \bigotimes_{x \in S} \kappa_{x_1}^t
\]

as a rational section on $\text{Bl}_G(\nu)$. Writing $v$ and $v^*$ as sums of weight vectors, we see that the order of $v^*(\gamma_{\varphi} v)$ at $t = 0$ is at least $w_0 \lambda(\varphi)$, because $w_0 \lambda$ is the lowest weight of $V^\lambda$. This proves the proposition.

Proposition 1.4.14 will show that, if $G$ is a type A or C simple Lie group, then the quantity (1.19) is nonnegative, hence $\tilde{f}^*$ has no pole for these groups. For this we need a simple lemma.

**Lemma 1.4.13.** Let $M, N$ be free $A$-modules of the same rank, and $q : M \oplus M \to N$ a surjective $A$-module map such that $q_1, q_2$ are isomorphisms over $K$. Suppose there is an $A$-basis $\{e_i\}$ of $M$ with respect to which $q_2^{-1}q_1$ is a diagonal matrix $\text{diag}(t^{a_1}, \ldots, t^{a_n})$. Then ord det $q_1$ is the sum of the $a_i$ which are non-negative.

**Proof.** We may find a $K$-basis $\{f_i\}$ of $N \otimes_A K$ such that $q_1 : e_i \mapsto f_i$, $q_2 : e_i \mapsto t^{-a_i}f_i$. The elements $f_i$ must lie in the $A$-submodule $N \subseteq N \otimes_A K$. Let $m_i \geq 0$ be maximal such that $f_i' = t^{-m_i} f_i$ remains in $N$. Note $m_i \geq a_i$ because $q_2(e_i) = t^{-a_i}f_i \in N$, so image($q$) consists of $A$-linear combinations of the $f_i'$. Hence the $f_i'$ form a basis of $N$, and we may assume $M = N = A^t$, $q_1 = \text{diag}(t^{m_i})$, $q_2 = \text{diag}(t^{m_i-a_i})$. As $q$ is surjective, for each $i$ we either have $m_i = a_i$ or we have $m_i = 0$ and $a_i < 0$. 

**Proposition 1.4.14.** Suppose $G$ is a simple Lie group of type A or C, and the representation $V$ is chosen as follows:
(i) if \( G = \text{SL}(n) \), choose \( V = W \oplus W^* \), where \( W = \mathbb{C}^n \) is the standard representation;

(ii) if \( G = \text{Sp}(2n) \), choose \( V = \mathbb{C}^{2n} \) the standard representation.

Then the quantity in equation (1.19) is nonnegative.

**Proof.** We will show that, for each node \( x \in S \), we have

\[
 l \cdot \text{ord} \kappa_x + w_0 \lambda_x(\varphi_x) \geq 0.
\]  

(1.21)

Thus, we will drop the subscripts indicating the node, and just consider an arbitrary dominant coweight \( \varphi \) and a dominant weight \( \lambda \) of level \( \leq ld_V \). Assume that the representation \( G \hookrightarrow \text{SL}(V) \) sends the maximal torus of \( G \) into the subgroup of diagonal matrices. Then by lemma 1.4.13 the term \( \text{ord} \kappa \) in (1.21) is equal to the sum of all the nonnegative diagonal entries in the matrix representation of \( \varphi \in g \subset \text{sl}(V) \).

The proof of the proposition is now to just check cases (i) and (ii).

**Case (i):** Give \( V = W \oplus W^* \) the standard basis \( e_1, \ldots, e_n, e_1^*, \ldots, e_n^* \). In this basis we have

\[
 \varphi = \text{diag}(\varphi_1, \ldots, \varphi_n, -\varphi_1, \ldots, -\varphi_n),
\]

for some integers \( \varphi_1 \geq \cdots \geq \varphi_n \) with \( \varphi_1 + \cdots + \varphi_n = 0 \). Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_n = 0) \) be a dominant weight of \( \text{SL}(n) \) at level \( \lambda_1 \leq ld_V = 2l \). Then the inequality (1.21) becomes\(^2\)

\[
 l(|\varphi_1| + \cdots + |\varphi_n|) \geq \lambda_1 \varphi_1 + \cdots + \lambda_n \varphi_n.
\]  

(1.22)

Let \( j \) be the index such that \( \varphi_j \geq 0 \) and \( \varphi_{j+1} < 0 \). Since \( \varphi_1 + \cdots + \varphi_n = 0 \), we have

\[
 l \cdot (|\varphi_1| + \cdots + |\varphi_n|) = 2l \cdot (\varphi_1 + \cdots + \varphi_j) \\
 \geq \lambda_1 (\varphi_1 + \cdots + \varphi_j) \\
 \geq \lambda_1 \varphi_1 + \cdots + \lambda_n \varphi_n.
\]

Thus inequality (1.22) holds.

\(^2\)Note that technically we are replacing \( w_0 \lambda \) in equation (1.21) with \( -\lambda \), but this is fine since \( \lambda \) and \( -w_0 \lambda \) have the same level.
Case (ii): Our conventions will follow [FH91], lecture 16. A dominant coweight of $\mathfrak{sp}(2n)$ is represented by a diagonal matrix $\varphi = \text{diag}(\varphi_1, \ldots, \varphi_n, -\varphi_1, \ldots, -\varphi_n)$ with $\varphi_1 \geq \cdots \geq \varphi_n \geq 0$ integers. A dominant weight is given by a decreasing sequence of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, and the level of such a weight is $(\lambda, \theta) = \lambda_1$. Since $V$ has Dynkin index $d_V = 1$, we assume $\lambda_1 \leq l$. Then inequality (1.21) is

$$l \cdot (\varphi_1 + \cdots + \varphi_n) \geq \lambda_1 \varphi_1 + \cdots + \lambda_n \varphi_n,$$

which clearly holds.

Example 1.4.15. Proposition 1.4.14 fails when $V$ is the standard representation of a type B or D Lie group (corresponding to the Lie algebras $\mathfrak{so}(2n+1)$, $\mathfrak{so}(2n)$ respectively). Note that for these groups the standard representation has Dynkin index $d_V = 2$. A different choice of representation will not help, because, for any other representation, we will get sections of $\mathcal{D}(V)^m$ for a large $m$ that still have poles (in fact we can take $m = 1$, since every representation of a type B or D Lie group has Dynkin index divisible by 2).

Here is a counterexample to proposition 1.4.14 for type $B_n$. Let $l = 1$, and choose $\lambda = 2\omega_1$ for one of the nodes and $\lambda = 0$ for all the others. So $\lambda$ is a weight of level $2 \leq ld_V$ and defines a component

$$H^0(\text{Bun}_G(C), \mathcal{D}(V) \otimes \mathcal{E}_{x_1}^\lambda \otimes \mathcal{E}_{x_2}^{\lambda^*}) \subset H^0(\text{Bun}_G(C_0), \mathcal{D}(V)), \quad (1.23)$$

as in lemma 1.4.10. Let $\varphi = \text{diag}(\varphi_1, \ldots, \varphi_n, -\varphi_1, \ldots, -\varphi_n, 0)$ be a dominant coweight of $\mathfrak{so}(2n+1)$, where $\varphi_1 \geq \cdots \geq \varphi_n \geq 0$ are integers. Then the inequality (1.21) is

$$\varphi_1 + \cdots + \varphi_n \geq 2\varphi_1,$$

which can clearly fail, e.g. if $\varphi = (1, 0, \ldots, 0, -1, 0, \ldots, 0, 0)$. Thus sections in (1.23) may not extend to the normalized stack of singular bundles.

One hope is that perhaps the $\lambda$-component (1.23) vanishes. However, this is not the case. Suppose the normalization is $C = \mathbb{P}^1$. Then $H^0(\text{Bun}_G(C), \mathcal{D}(V) \otimes \mathcal{E}_{x_1}^\lambda \otimes \mathcal{E}_{x_2}^{\lambda^*})$ is identified with the dual of

$$\frac{V_\lambda \otimes V_\lambda^*}{\mathfrak{g} \cdot (V_\lambda \otimes V_\lambda^*) + \text{im} T^3},$$
where $T = f_\theta^{(2)}$ is the lowest root vector of $\mathfrak{g}$ acting on the second tensor factor of $V_\lambda \otimes V_\lambda^*$ (see e.g. [BK16], second proof of lemma 6.5). It is easy to verify directly that, for $\mathfrak{g} = \mathfrak{so}(5)$ and $\lambda = 2\omega_1$, we have $T^3 = 0$ and

$$ \frac{V_\lambda \otimes V_\lambda^*}{\mathfrak{g} \cdot (V_\lambda \otimes V_\lambda^*)} $$

is nonzero (in fact one-dimensional, spanned by any nonzero weight vector of weight zero).

However we can control the pole order in the following way. Recall that $\varphi$ is the “type” of the family parametrized by $\text{Spec} \ k[[t]] \to \text{Bh}_G(\nu)$, i.e. we have $q_2^{-1}q_1 = t^\varphi$ for this family of Bhosle bundles. Unlike $\lambda$, the $\varphi$ does not depend on $l$ in any way, so we have allowed it to be an arbitrary dominant coweight. However, since the boundary of $\text{Bun}_G(\mathbb{C})$ in the Bhosle stack is the divisor $E$ where $\det q_1 = 0$, we should require that $\det q_1$ vanishes to order exactly one at $t = 0$ in order to compute the correct pole order (although it makes no difference for detecting the existence of a pole). By lemma 1.4.13, there are only finitely many $\varphi$ with this property, and it follows that the pole order is bounded above by a number $m$ that scales linearly with $l$. Then we can replace $\mathcal{L} = \mathcal{D}_{\text{Bh}}$ with $\mathcal{L}' = \mathcal{L}(mE)$, and, since sections over $\text{Bun}_G(\mathbb{C})$ extend to $\mathcal{L}'$, we should be able to carry out the rest of the paper in the same way to generalize our results to arbitrary groups. Unfortunately we have not yet had time to carry this out.

1.5 Moduli of singular $G$-bundles

In this section, we summarize the GIT construction of the moduli space of singular $G$-bundles due to Schmitt and Muñoz-Castañeda ([MCS20]), which we will use to prove theorem 1.1.1 in section 1.6. We continue to focus on the case of a nodal curve, but all of the material in this section can be generalized to higher dimensional smooth varieties (see [GLSS19]).

1.5.1 Parameter spaces of singular $G$-bundles

Fix an ample line bundle $\mathcal{L}$ on $C_0$, and let $Q$ be the quot scheme of coherent quotients $q : W \otimes \mathcal{L}^{-n} \to \mathcal{E}$ with Hilbert polynomial $P$, where $W$ is a vector space of rank $P(n)$. Let $\mathcal{E}$ be the universal quotient sheaf over $C_0 \times Q$. To build a parameter space of singular $G$-bundles, we recall the following well-known result.

**Theorem 1.5.1.** ([FGAEExp], theorem 5.8) Let $p : Z \to S$ be a projective morphism to a Noetherian scheme $S$, and let $\mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_Z$-modules. If $\mathcal{G}$ is flat over $S$, then the functor that sends an $S$-scheme $T$ to the set $\text{Hom}_{\mathcal{O}_T}(\mathcal{F}_T, \mathcal{G}_T)$ is representable by a scheme $\text{H}_{\text{Z/S}}(\mathcal{F}, \mathcal{G})$ which is affine and finite type over $S$. 
It is easy to show as a consequence:

**Corollary 1.5.2.** Let $Z \to S$ be a projective morphism to a Noetherian scheme $S$. If $S_\bullet$ is a quasicoherent, finitely generated, graded $\mathcal{O}_Z$-algebra, then the functor that sends a scheme $T$ to the set of $\mathcal{O}_{Z_T}$-algebra homomorphisms $S_T \to \mathcal{O}_T$ is representable by a finite type affine $S$-scheme.

**Proof.** Pick a generating submodule $F \subseteq S_\bullet$ that is coherent, and note that there is a closed subscheme of $\operatorname{H}_{Z/S}(F, \mathcal{O})$ parametrizing morphisms $F \to \mathcal{O}$ such that $\operatorname{Sym}^*F \to \mathcal{O}$ factors through the multiplication map $\operatorname{Sym}^*F \to S_\bullet$. ∎

Using this, we define two parameter schemes of singular $G$-bundles (an affine version and a projective version), as follows.

**Definition 1.5.3.** Define the affine parameter space of singular $G$-bundles $\widetilde{Q}$ to be the affine $\mathbb{Q}$-scheme granted by corollary 1.5.2 in the case $Z = C_0 \times \mathbb{Q}$, $S = \mathbb{Q}$, $S_\bullet = \operatorname{Sym}^*(V \otimes \widetilde{E})^G$. Then $\widetilde{Q}$ represents the functor that sends a scheme $T$ to the set of pairs $(q, \tau)$ consisting of a quotient $(q : W \otimes \mathcal{L}_T^\mathbb{N} \to \mathcal{E}) \in \mathcal{Q}(T)$ and an algebra homomorphism

$$\tau : \operatorname{Sym}^*(V \otimes \mathcal{E})^G \to \mathcal{O}_{C_0 \times T}$$

on the quotient sheaf.

This scheme has a natural projectivization $Q$, which will be our projective parameter space of singular $G$-bundles.

**Proposition 1.5.4.** There is a projective $\mathbb{Q}$-scheme $Q$ parametrizing, over a scheme $T$, the triples $(q, \mathcal{M}, \tau)$ consisting of a quotient $(q : W \otimes \mathcal{L}_T^\mathbb{N} \to \mathcal{E}) \in \mathcal{Q}(T)$, a line bundle $\mathcal{M}$ on $T$, and a morphism of graded $\mathcal{O}_{C_0 \times T}$-algebras

$$\tau : \operatorname{Sym}^*(V \otimes \mathcal{E})^G \to \mathcal{O}_{C_0} \otimes \operatorname{Sym}^*\mathcal{M},$$

which is surjective in large degree. There is a surjective rational map $\widetilde{Q} \to Q$ defined away from the zero section.

**Proof.** The universal quotient and the Reynolds operator induce a closed embedding $\widetilde{Q} \to Q \times \mathbb{A}^{N+1}$, where

$$\mathbb{A}^{N+1} = \bigoplus_{i=0}^d \operatorname{Hom}_C(\operatorname{Sym}^i(V \otimes W), \mathcal{H}^0(\mathcal{L}^\mathbb{N}))$$
for sufficiently large $d > 0$. Since $\pi : Q \times (\mathbb{A}^{N+1} - 0) \to Q \times \mathbb{P}^N$ is a $\mathbb{G}_m$-bundle and $\widetilde{Q}$ is $\mathbb{G}_m$-stable, there exists by descent theory a unique closed subscheme $Q \subseteq Q \times \mathbb{P}^N$ with $\pi^{-1}(Q) = \widetilde{Q} - (Q \times 0)$. Since $\pi^{-1}(Q) \to Q$ is a $\mathbb{G}_m$-bundle, $Q$ represents the stack quotient $[\widetilde{Q} - (Q \times 0)]/\mathbb{G}_m$ and has the universal property that to give a map $T \to Q$ is to give a map $T \to Q$, a $\mathbb{G}_m,T$-torsor $M \to T$, and a morphism of $Q$-schemes $M \to \widetilde{Q} - (Q \times 0)$. This is easily seen to be equivalent to the description of $Q$ given in the proposition.

Next we want a polarization of $Q$. We have already embedded $Q \hookrightarrow Q \times \mathbb{P}^N$, and, for a large enough $m > 0$, we may follow this with Grothendieck’s embedding

$$Q \hookrightarrow \text{Gr}(W \otimes H^0(L^m), f),$$

$$(q : W \otimes L^{-n} \to \mathcal{E}) \mapsto (W \otimes H^0(L^m) \to H^0(\mathcal{E} \otimes L^{m+n}))$$

to get a $\text{GL}(W)$-equivariant embedding

$$Q \hookrightarrow \text{Gr}(W \otimes H^0(L^m), f) \times \mathbb{P}^N,$$  \hspace{1cm} (1.24)

where $\text{Gr}(\cdots)$ is the Grassmannian of quotients. As above, we also have the embedding

$$\widetilde{Q} \hookrightarrow \text{Gr}(W \otimes H^0(L^m), f) \times \mathbb{A}^{N+1},$$

that makes $\widetilde{Q}$ the (partial) affine cone over $Q$ with respect to the embedding (1.24).

**Definition 1.5.5.** Let $L_m(k_1, k_2)$ be the pullback of the $\text{GL}(W)$-linearized line bundle $\mathcal{O}(k_1) \boxtimes \mathcal{O}(k_2)$ under the embedding (1.24).

**1.5.2 Semistable tensor fields**

Semistability for singular $G$-bundles is defined in terms of their associated “tensor fields.” The rough idea is to pick a generating submodule of $\text{Sym}^*(V \otimes \mathcal{E})^G$, use the Reynolds operator to drop the $G$-invariance requirement, and then “homogenize” (we will give the explicit construction in the next sub-
section). The result is a very simple object of the following form, which retains all of the information about the singular $G$-bundle (up to scalars) and allows us to get a much simpler definition of semistability.

**Definition 1.5.6.** A tensor field on a sheaf $\mathcal{F}$ is a nonzero morphism $\varphi : (\mathcal{F}^{\otimes b})^{\oplus c} \to \mathcal{O}_{C_0}$ for some $b, c$.

Gomez-Sols introduced the following definition of semistability for tensor fields. Recall that a weighted filtration of a sheaf $\mathcal{F}$ is a pair $(\mathcal{F}_\bullet, l_\bullet)$ consisting of an increasing filtration

$$\mathcal{F}_\bullet = (0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p+1} = \mathcal{F})$$

by distinct subsheaves and a sequence of positive rational numbers

$$l_\bullet = (l_1, \ldots, l_p).$$

Let $a = \deg \mathcal{L}$. Given a tensor field $(\mathcal{F}, \varphi)$ of $\text{rk} \mathcal{F} = r$ and a weighted filtration $(\mathcal{F}_\bullet, l_\bullet)$, define the vector

$$\lambda(l_\bullet) = \sum_{i=1}^p l_i \xi_{r_i},$$

where $r_i = \text{rk} \mathcal{F}_i$ and

$$\xi_j = (aj - ar, \ldots, aj - ar, aj, \ldots, aj)$$

with $aj - ar$ repeated $aj$ times and $aj$ repeated $ar - aj$ times. Write $\lambda(l_\bullet) = (\lambda_1, \ldots, \lambda_{ar})$ and define

$$\mu(\mathcal{F}_\bullet, l_\bullet, \varphi) = -\min\{\lambda_{ar_{i_1}} + \cdots + \lambda_{ar_{i_b}} : \varphi|_{(\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_b})^{\oplus c}} \neq 0\},$$

where the min is taken over all $i_1, \ldots, i_b \in \{1, \ldots, p+1\}$, possibly nondistinct. (All the ranks are scaled by $a$ since they might not be integers).

**Definition 1.5.7.** ([GS01], definition 1.3) Let $\delta \in \mathbb{Q}_{>0}$. A tensor field $(\mathcal{F}, \varphi)$ is $\delta$-semistable if $\mathcal{F}$ is torsion-free and

$$\sum_{i=1}^p l_i (\text{rk} \mathcal{F}_i \cdot P \mathcal{F} - \text{rk} \mathcal{F} \cdot P \mathcal{F}_i) + \delta \mu(\mathcal{F}_\bullet, l_\bullet, \varphi) \geq 0 \quad (1.25)$$

for every weighted filtration $(\mathcal{F}_\bullet, l_\bullet)$. We say $(\mathcal{F}, \varphi)$ is $\delta$-stable if strict inequality holds for every such $(\mathcal{F}_\bullet, l_\bullet)$. 
1.5.3 Tensor field associated to a singular $G$-bundle

Recall that $\tilde{E}$ is the universal quotient sheaf on the quot scheme $Q$. Picking a $d$ such that $\text{Sym}^*(V \otimes \tilde{E})^G$ is generated in degree $\leq d$, we may associate to every point $(\mathcal{E}, \tau) \in Q$ a tensor field $\varphi_\tau : (\mathcal{E}^\otimes b)^{\otimes c} \to \mathcal{O}_{C_0}$ given by

$$\bigoplus_{\mathbf{a}} (V \otimes \mathcal{E})^{\otimes d!} \to \bigoplus_{\mathbf{a}} \prod_{i=0}^d \text{Sym}^{a_i}(V \otimes \mathcal{E}) R_G \prod_{i=0}^d \text{Sym}^{a_i}(V \otimes \mathcal{E})^G \to \mathcal{O}_{C_0}, \quad (1.26)$$

where $R_G$ is the Reynolds operator and the sum is over $\mathbf{a} = (a_0, \ldots, a_d)$ with $a_1 + 2a_2 + \cdots + da_d = d!$. It is shown in [MC17], sect. 2.2.3 (or [GLSS19], sect. 5) that the assignment $(\mathcal{E}, \tau) \mapsto (\mathcal{E}, \varphi_\tau)$ is injective on $\mathbb{G}_m$-equivalence classes and defines a proper injective morphism from $Q$ into a parameter space of tensor fields. We therefore define:

**Definition 1.5.8.** A point of $Q$ is $\delta$-semistable if the associated tensor field is $\delta$-semistable.

1.5.4 Characterization of GIT semistability

Let $Q(1)$ be the closure of the set of torsion-free, uniform rank $r$ quotient sheaves in $Q$, and let $Q(1) = Q \times_Q Q(1)$. Recall the $\text{SL}(W)$-linearized ample line bundle $L = L_m(k_1, k_2)$ on $Q$. Following Simpson’s approach for semistable sheaves ([Sim94]), Gomez-Sols proved that $L$-semistability and $\delta$-semistability coincide in the following sense.

**Theorem 1.5.9.** ([GS01], theorem 3.6) Assume $m$, $n$ are sufficiently large. There is a number $\alpha$ such that, if $\frac{k_2}{k_1} = \alpha$, then a point $(q, \tau) \in Q(1)$ is $L$-semistable if and only if $(\mathcal{E}, \tau)$ is a $\delta$-semistable singular $G$-bundle and $W \to H^0(\mathcal{E} \otimes L^n)$ is an isomorphism.

1.5.5 Semistability for large values of $\delta$

By theorem 1.5.9, we thus get a projective moduli space of singular $G$-bundles $Q(1) \sslash L \text{SL}(W)$. The downside is that the definition of $\delta$-semistability is somewhat nasty, and the $\delta$-semistable singular $G$-bundles are too large a class of objects, in that we may not get a $G$-bundle over $C_0 - S$ or even over a dense open subset of $C_0$. However, Schmitt and Muñoz-Castañeda have recently shown that, for large values of $\delta$, the moduli space parametrizes only honest singular $G$-bundles which are semistable in the sense of
definition 1.2.4. There are analogous results for smooth varieties of arbitrary dimension, even in positive characteristic ([GLSS19], theorems 5.4.1 and 5.4.4).

**Theorem 1.5.10.** ([MCS20], theorem 3.3) There is a $\delta_0 > 0$ such that, for $\delta > \delta_0$, the following hold:

1. any $\delta$-semistable singular $G$-bundle with Hilbert polynomial $P$ is honest;

2. for any honest singular $G$-bundle with Hilbert polynomial $P$, $\delta$-(semi)stability is equivalent to (semi)stability as in definition 1.2.4.

**1.5.6 Set-up for finite generation**

We can now define the moduli space which will be used to prove theorem 1.1.1 in the next section. We introduce the following schemes related to $Q$:

- $Q^0 \subseteq Q(1)$ the open subset parametrizing torsion-free, honest singular $G$-bundles such that the map $W \to H^0(E \otimes L^n)$ is an isomorphism (in particular $h^1(E \otimes L^n) = 0)$;

- $Q^G \subseteq Q^0$ the open subscheme of $G$-bundles;

- $M$ the normalization of $\overline{Q^G}$ (closure taken in $Q$);

- $M^0 \subseteq M$ the preimage of $Q^0$.

Define $\mathcal{X} = \mathcal{X}_m(k_1, k_2)$ as the GIT quotient

$$M \sslash L_m(k_1, k_2) SL(W)$$

with respect to the following choices:

- $L$ is an ample line bundle on $C_0$;

- $P = P_{C \otimes r}$;

- $\delta > \delta_0$ as in theorem 1.5.10 and $m, n, k_1, k_2$ are chosen as in theorem 1.5.9.

Remember, as mentioned in the introduction, that we are not sure if $\overline{Q^G}$ contains all of the honest singular $G$-bundles! But, it is necessary for us to work with this smaller moduli space, e.g. the key result of section 1.4 – proposition 1.4.8 – only applies to singular $G$-bundles which are in the closure of $Bun_G(C_0)$. 

31
1.6 Proof of theorem 1.1.1

In the previous section, we constructed a polarized moduli space $(\mathcal{X}, L)$, where $L = L_m(k_1, k_2)$ (or a sufficiently large multiple thereof that descends to $\mathcal{X}$). After identifying $L$ with determinant of cohomology, we will establish an injection

$$H^0(\mathcal{X}, L) \hookrightarrow H^0(\text{Bun}_G(C_0), \mathcal{D}(V)^l)$$

for sufficiently divisible $l$, and show that this map is an isomorphism by way of theorem 1.4.1. This will prove theorem 1.1.1.

1.6.1 Line bundle identities

Let $\mathbb{P}^N$ be the projective space from section 1.5.1 such that $Q \hookrightarrow Q \times \mathbb{P}^N$, and recall that $L_m(k_1, k_2)$ is a tensor product of line bundles

$$L_m(k_1, k_2) = \mathcal{O}_Q(k_1) \boxtimes \mathcal{O}_{\mathbb{P}^N}(k_2).$$

For a vector bundle $\mathcal{G}$ on $C_0$, let $\mathcal{D}_\mathcal{G}$ be the line bundle on the stack of coherent sheaves $\text{Coh}(C_0)$ whose fiber over a coherent sheaf $\mathcal{E}$ is the determinant of cohomology of $\mathcal{E} \otimes \mathcal{G}$, i.e.

$$\mathcal{D}_{\mathcal{G}|\mathcal{E}} = \det H^0(\mathcal{E} \otimes \mathcal{G})^* \otimes \det H^1(\mathcal{E} \otimes \mathcal{G}).$$

For $\mathcal{G} = \mathcal{O}_{C_0}$, this line bundle will simply be denoted $\mathcal{D}$. Note that

$$\mathcal{O}_Q(1) = \mathcal{D}_{\mathcal{L}^{1}_{m+n}}$$

as line bundles on $Q$.

**Proposition 1.6.1.** Let $\mathcal{G}$ be a rank $l$ vector bundle on $C_0$ of degree

$$\deg \mathcal{G} = l(g - 1) - P(n)(1 - g + (m + n) \deg \mathcal{L}),$$

32
and assume that \( G \) is a sum of line bundles \( O(\sum c_i p_i) \), with \( p_i \in C_0 \) smooth points. Then, over \( Q^0 \), there is a \( GL(W) \)-equivariant isomorphism

\[
O_Q(P(n)) \cong D_G.
\]

In particular, if \( n \) is divisible by \( g - 1 \), then we may take \( \deg G = 0 \) and obtain \( O_Q(P(n)) \cong D^l \) over \( Q^0 \).

**Proof.** Over \( Q^0 \), there is a \( GL(W) \)-equivariant isomorphism

\[
O_Q(P(n)) = D^{P(n)} \cong D^{-P(n)} \otimes \det W^{-r(m+n) \deg L}
\]

(use [BG19] lemma 4.7, and note that the universal quotient sheaf has trivialized determinant over \( Q^0 \times (C_0 - S) \) by theorem 1.5.10 and our choice of representation (sect. 1.2.4)). On the other hand,

\[
D_G \cong D^l \otimes \det W^{-r \deg G/P(n)}.
\]

Note \( D_{\mathcal{L}^n} \) is trivial over \( Q^0 \), hence so is \( D \) (by loc. cit.), so the line bundles \( O_Q(P(n)) \) and \( D_G \) can only differ by linearization. The difference in linearization is given by a character \( GL(W) \to \mathcal{O}(Q^0)^* \), so we conclude the lemma by noting that a scalar matrix \( t \in GL(W) \) acts the same on both line bundles. \( \square \)

### 1.6.2 The injection \( H^0(M, L)^{SL(W)} \to H^0(Bun_G(C_0), D^l) \)

Recall that \( D \) also denotes the determinant of cohomology line bundle on \( Bun_G(C_0) \) with respect to the contraction map

\[
Bun_G(C_0) \to \text{Coh}(C_0),
\]

\[
E \mapsto E \times^G V.
\]

In this section we prove:

**Proposition 1.6.2.** If \( n, k_1, k_2 \) are sufficiently divisible, then there is an injection

\[
H^0(M, L_{m(k_1, k_2)})^{SL(W)} \to H^0(Bun_G(C_0), D^l),
\]

33
where \( l \) is given by
\[
l = \frac{k_1(1 - g + (m + n) \deg L)}{g - 1}.
\]
(1.27)

The proof of the proposition is as follows. In lemma 1.6.3 below, we show \( Q^G \) is smooth, so that it is an open subscheme of \( M \), hence restriction gives a map
\[
H^0(M, L)^{SL(W)} \to H^0(Q^G, L)^{SL(W)},
\]
where \( \bar{Q}^G \) is the preimage of \( Q^G \) in \( \bar{Q} \). Since \( Q^G = \bar{Q}^G/\mathbb{G}_m \) and the center of \( SL(W) \) acts trivially on sections of \( L \) under the assumptions of the proposition, we have
\[
H^0(Q^G, L)^{SL(W)} = H^0(Q^G, L)^{PGL(W)} = H^0(\bar{Q}^G, L)^{GL(W)}.
\]

The pullback of \( O_{\mathbb{P}^n}(1) \) to \( \bar{Q}^G \) is trivial, so by proposition 1.6.1
\[
H^0(\bar{Q}^G, L)^{GL(W)} = H^0(\bar{Q}^G, D^l)^{GL(W)}.
\]

Note that \( \bar{Q}^G \) is a \( GL(W) \)-bundle over an open substack
\[
Y_{L^n} = [\bar{Q}^G/GL(W)] \subset Bun_G(C_0)
\]
parametrizing \( G \)-bundles \( E \) such that the associated vector bundle \( E = E \times G V \) has \( E \otimes L^n \) globally generated and \( h^1(E \otimes L^n) = 0 \). If \( n \) is sufficiently large, then \( Y_{L^n} \) contains the locus of semistable \( G \)-bundles, so its complement has codimension \( \geq 2 \) (see [LS97], proof of 1.6, or [LaR96], section 3), thus
\[
H^0(\bar{Q}^G, D^l)^{GL(W)} = H^0(Y_{L^n}, D^l) = H^0(Bun_G(C_0), D^l).
\]

It remains to show \( Q^G \) is smooth.

**Lemma 1.6.3.** \( Q^G \) is a smooth, irreducible variety.

**Proof.** Since \( \bar{Q}^G \to Q^G \) is smooth, it suffices to prove the lemma for \( \bar{Q}^G \). But \( \bar{Q}^G \) is a \( GL(W) \)-bundle over the open substack \( Y_{L^n} \subset Bun_G(C_0) \) ([W11], section 4), and \( Bun_G(C_0) \) is smooth over \( \text{Spec } \mathbb{C} \) and connected ([BF19], proposition 5.1), so \( \bar{Q}^G \) is smooth and irreducible. \( \Box \)
1.6.3 Conclusion of finite generation

Using theorem 1.4.1, we will show that the injection we have set up is an isomorphism and conclude finite generation of $\bigoplus_{k \geq 0} H^0(\text{Bun}_G(C_0), D^k)$.

**Theorem 1.6.4.** Let $G$ be either $\text{SL}(n)$ or $\text{Sp}(2n)$, with the representation $V$ chosen as in proposition 1.4.14. Let $L = L_m(k_1, k_2)$, and assume $n, k_1, k_2$ are as in proposition 1.6.2. Then we have an algebra isomorphism

$$\bigoplus_{k \geq 0} H^0(M, L^k)^{\text{SL}(W)} \cong \bigoplus_{k \geq 0} H^0(\text{Bun}_G(C_0), D^{kl}),$$

where $l$ is given by formula (1.27).

**Proof.** As $M \to \overline{Q}^G$ is finite, the $L$-semistable locus in $M$ is the preimage of that in $Q$, in particular it is contained in $M^0$ by theorem 1.5.10. Hence,

$$H^0(M, L^k)^{\text{SL}(W)} = H^0(M^0, L^k)^{\text{SL}(W)}$$

by [NR93], lemma 4.15. Since sections of $D^{kl}$ on $\text{Bun}_G(C_0)$ are identified with sections of $L^k$ on $Q^G$, and since $M$ is normal, we need to show that sections of $L^k$ over $Q^G$ have no pole at $t = 0$ for a map $\text{Spec } k[[t]] \to Q^0 \cap \overline{Q}^G$ sending the generic point into $Q^G$. As $k[[t]]$ is local, any such map factors through $\overline{Q}$, and the pullback of $L$ to $\overline{Q}^0$ is $D^l$ by proposition 1.6.1 (where $\overline{Q}^0 \in \overline{Q}$ is the preimage of $Q^0$), so we reduce to showing that sections of $D^l$ over $\text{Bun}_G(C_0)$ have no pole at $t = 0$ for a map $\text{Spec } k[[t]] \to \text{SB}_G^0(C_0)$ sending the generic point into $\text{Bun}_G(C_0)$. This is what is shown in theorem 1.4.1. \hfill $\square$

**Corollary 1.6.5.** For any stable curve $C_0$ of genus $g \geq 2$, the algebra $\bigoplus_{k \geq 0} H^0(\text{Bun}_G(C_0), D^k)$ is finitely generated.

**Proof.** Since $\bigoplus_{k \geq 0} H^0(\text{Bun}_G(C_0), D^{kl}) \cong \bigoplus_{k \geq 0} H^0(M, L^k)$ is finitely generated, this follows from [BG19], lemma 8.4. \hfill $\square$
1.7 Compactifications over $\overline{M}_g$ and conformal blocks

1.7.1 Conformal blocks vector bundles

Let $\overline{M}_{g,n}$ be the stack of stable $n$-pointed curves of genus $g$, and let $\mathfrak{g}$ be the Lie algebra of the simple, simply connected group $G$. For a positive integer $l$ and dominant integral weights $\vec{\lambda} = (\lambda^1, \ldots, \lambda^n)$ of level $\leq l$, there is a vector bundle $V_{g,l,\vec{\lambda}}$ on $\overline{M}_{g,n}$ called the vector bundle of conformal blocks.

Let us briefly recall its construction from [Fakh12]. Let $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{k}((t))) \oplus \mathbb{k} \cdot c$ be the affine Lie algebra of $\mathfrak{g}$, where $c$ is central and the Lie bracket is defined by

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) \text{Res}(f'g)c.$$  

For each dominant integral weight $\lambda$ of $\mathfrak{g}$ there is an irreducible representation $H_{\lambda}^{\mathfrak{g}}$ of $\widehat{\mathfrak{g}}$, and we put $H_{\vec{\lambda}} = \bigotimes_{i=1}^n H_{\lambda^i}$, which is an irreducible representation of $\widehat{\mathfrak{g}}_n = (\mathfrak{g} \otimes \mathbb{k}((t))^{\oplus n}) \oplus k \cdot c$.

Suppose $\pi : C \rightarrow S$ is a proper flat family of genus $g$ nodal curves parametrized by a smooth $k$-variety $S$, and let $\vec{p} = (p_1, \ldots, p_n)$ be disjoint sections $p_i : S \rightarrow C$ of $\pi$ whose images lie in the smooth locus of $\pi$. Assume that $S = \text{Spec} \ A$ is affine, that $C - (\bigcup_{i=1}^n p_i(S))$ is affine with coordinate ring $B$, and that we are given isomorphisms $\eta_i : \widehat{\mathcal{O}}_{C, p_i(S)} \rightarrow \mathbb{A}[[t]]$. Then the $\eta_i$ make $\mathfrak{g} \otimes_k B$ a Lie subalgebra of $\widehat{\mathfrak{g}}_n \otimes_k A$, and we define

$$V_{g,l,\vec{\lambda}}(C, \vec{p}) = H_{\vec{\lambda}} \otimes_k A/(\mathfrak{g} \otimes_k B) \cdot (H_{\vec{\lambda}} \otimes_k A).$$

The case of a general $S$ can be dealt with by a descent argument ([Fakh12], proposition 2.1, and the discussion following it).

For each conformal blocks bundle $V_{g,l,\vec{\lambda}}$, the sum $\bigoplus_{m \geq 0} V_{g,ml,m,\vec{\lambda}}$ has a natural structure of a flat sheaf of algebras on $\overline{M}_{g,n}$ ([Ma18]). For a marked curve $(C_0, \vec{p})$, let $\text{Parbun}_G(C_0, \vec{p})$ be the stack parametrizing $(E, s_1, \ldots, s_n)$ consisting of a $G$-bundle $E \rightarrow C_0$ and points $s_i \in E_{p_i}/B$ for a fixed Borel subgroup $B \subset G$. Let $p : \text{Parbun}_{G,g,n} \rightarrow \overline{M}_{g,n}$ be the corresponding relative stack. In [BF19], it is shown that for each conformal block $V_{g,l,\vec{\lambda}}$, there is a line bundle $L_{G,l,\vec{\lambda}}$ on $\text{Parbun}_{G,g,n}$ with $p_* L_{G,l,\vec{\lambda}} \cong V_{g,l,\vec{\lambda}}$ (such isomorphisms exist for any family of stable curves, not just globally over $\overline{M}_{g,n}$), and by [BG19] theorem
9.2 this induces an algebra isomorphism

$$\bigoplus_{m \geq 0} H^0(\text{Parbun}_G(C_0, \bar{p}), \mathcal{L}_{G, l, \lambda}(C_0, \bar{p})^m) \cong \bigoplus_{m \geq 0} \mathcal{V}^\vee_{g, ml, m\lambda}(C_0, \bar{p})$$

for all $(C_0, \bar{p}) \in \overline{M}_{g,n}$. The line bundle $\mathcal{L}_{G, l, \lambda}$ is related to determinant of cohomology in the following way. For a representation $V$ of $G$, let $\mathcal{N}_{V, l, \lambda}$ be the line bundle on $\text{Parbun}_G(C_0, \bar{p})$ whose fiber over a parabolic bundle $(E, s_1, \ldots, s_n) \in \text{Parbun}_G(C_0, \bar{p})$ is

- $[\det H^*(E \times^G V) \otimes H^*(V \otimes \mathcal{O}_{C_0})^{-1}]$,
- the fibers of the line bundles $E_{p_i} \times^B \mathcal{C}_{-\lambda_i} \to E_{p_i}/B$ over the elements $s_i$.

If $V$ is irreducible, then $\mathcal{N}_{V, l, \lambda} \cong \mathcal{L}_{G, dV, l, \lambda}$, where $d_V$ is the Dynkin index of $V$ [BF19]. In the case of no marked points ($n = 0$), then even for reducible $V$ we have $\mathcal{N}_{V, l} \cong \mathcal{L}_{G, d}$, where $d$ is the sum of the Dynkin indices of the irreducible summands of $V$, and thus

$$\bigoplus_{m \geq 0} H^0(\text{Bun}_G(C_0), \mathcal{D}(V)^m) \cong \bigoplus_{m \geq 0} \mathcal{V}^\vee_{g, md}(C_0)$$

for any $V$. Therefore, when $g$ is $\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$, corollary 1.6.5 and [BG19] lemma 8.4 give finite generation of the conformal blocks algebra for any stable curve $C_0 \in \overline{M}_g$.

1.7.2 Finite generation of the sheaf of conformal blocks algebras

We would like to show that $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{V}^\vee_{g, m}$ is finitely generated as a sheaf of algebras on $\overline{M}_g$. Recall that $\overline{M}_g$ has a smooth atlas $H_g \to \overline{M}_g$, where $H_g$ is a smooth, irreducible variety [DM69]. To show that $\mathcal{A}$ is finitely generated, it suffices to show that there is a uniform constant $d$ such that, for each closed point $C_0 \in \overline{M}_g$, the fiber $\mathcal{A}(C_0)$ is generated in degree $\leq d$ (for then $\mathcal{A}$ is generated in degree $\leq d$ by Nakayama’s lemma). We will prove this by showing that, for any family of stable curves $T \to \overline{M}_g$ parametrized by a variety $T$, there is a constant $d = d(T)$ such that $\mathcal{A}|_t$ is generated in degree $\leq d$ for all $t \in T$. The proof is by induction on $\dim T$, so that we are free to replace $T$ by a dense open subset.

Let me sketch the adjustments needed for the relative setting. Let $\pi : \mathcal{C} \to T$ be a family of stable curves parametrized by a variety $T$ (we can assume $T$ is smooth and irreducible), and fix a relatively ample line bundle $\mathcal{L}$ on $\mathcal{C}/T$. All of the parameter schemes from section 1.5 have relative versions:
- \(\tilde{Q}_{\mathcal{E}/T}\) and \(Q_{\mathcal{E}/T}\) are defined in the same way, but over a relative quot scheme
\[
Q_{\mathcal{E}/T} = \text{Quot}_{\mathcal{E}/T}(W \otimes \mathcal{L}^{-n}, P);
\]

- \(\tilde{Q}_{\mathcal{E}/T}\) and \(Q_{\mathcal{E}/T}\) again come with closed embeddings into \(Q_{\mathcal{E}/T} \times_T \mathbb{A}^{N+1}_T\) and \(Q_{\mathcal{E}/T} \times_T \mathbb{P}^{N}_T\);

- there is a Grothendieck embedding \(Q_{\mathcal{E}/T} \hookrightarrow Gr \times_T \mathbb{P}^{N}_T\), where
\[
Gr = Gr_T(\pi_*(W \otimes \mathcal{L}^m), f)
\]
is a relative Grassmannian, and we define \(L = L_m(k_1, k_2)\) to be the relatively ample line bundle \(\mathcal{O}_{Gr}(k_1) \otimes \mathcal{O}_{\mathbb{P}^N}(k_2)\) on \(Q_{\mathcal{E}/T}\).

Define \(\delta\)-semistability in the relative case as fiberwise \(\delta\)-semistability over \(T\). The number \(\delta_0\) from theorem 1.5.10 can be chosen uniformly over \(T\) by [MC17], theorems 4.4.17 and 4.4.18 (\(\delta_0\) depends “only on numerical inputs,” the only one of which that depends on the base curve is the degree of the polarization, but we can just use the canonical polarization). By [MC20], theorem 2.16, the numbers \(m, n\) from theorem 1.5.9 can be chosen uniformly over \(T\) (the resulting \(\alpha\) depends only on \(P, m, n, \delta\)). Since \(L\)-semistability coincides with \(L_t\)-semistability on the fibers \(Q_t, t \in T\) ([Sim94], lemma 1.13), the analogue of theorem 1.5.9 holds in the relative setting.

The schemes \(Q^0_{\mathcal{E}/T}, Q^G_{\mathcal{E}/T}, M^\epsilon_{\mathcal{E}/T}\) can all be defined in the analogous way – the only hitch is that the restriction of \(M^\epsilon_{\mathcal{E}/T}\) to a point \(t \in T\) is not the same as \(M^\epsilon_t\) since normalization does not commute with base change. But we are free to replace \(T\) by an open subset, so we only need this to work generically.

**Lemma 1.7.1.** ([BG19], lemma 10.3) Let \(U \subseteq Y\) be schemes over a smooth variety \(S\) over a field of characteristic zero. Assume \(U \to S\) is smooth and the fibers over closed points are irreducible. Also assume \(Y \to S\) is proper. Let \(N\) be the normalization of the closure of \(U\) in \(Y\). Then there is a dense open subset \(V \subseteq S\) such that the fibers of \(N\) over the closed points of \(V\) are normal irreducible varieties.
into a Veronese subalgebra $A^{(N)} = \bigoplus_{k \geq 0} A_{kN}$ as in section 1.6.2, and we know $B|_{t} \rightarrow A^{(N)}|_{t}$ is an isomorphism for all $t \in T$ by theorem 1.6.4. Thus $B \cong A^{(N)}$. Since $B$ is a finitely generated $O_{T}$-algebra, so is $A$ by [BG19] lemma 8.4.

1.7.3 The family $X \rightarrow \overline{M}_{g}$ and modular interpretations

Let $A$ be the conformal blocks algebra on $\overline{M}_{g}$, and $X(C_{0}) = M_{L}/SL(W)$ the moduli space defined in subsection 1.5.6. By theorem 1.6.4,

$$\text{Proj} A(C_{0}) \cong X(C_{0})$$

for every closed point $C_{0} \in \overline{M}_{g}$. The last section showed that $A$ is finitely generated, so taking $X = \text{Proj} A$ we have the following. Note that item 1. is by a theorem of Kumar-Narasimhan-Ramanathan, [KNR94].

**Theorem 1.7.2.** Let $G$ be a simple Lie group of type $A$ or $C$, and let $g \geq 2$. Then there is a flat, relatively projective family $X \rightarrow \overline{M}_{g}$ such that

1. the fiber over a smooth curve is Ramanathan’s moduli space of semistable $G$-bundles;

2. the fiber over an arbitrary curve is a normalized moduli space of semistable honest singular $G$-bundles.

Note that the theorem gives an interpretation of the fibers of $\text{Proj} A \rightarrow \overline{M}_{g}$ over fixed stable curves, but we have not given an interpretation over families of stable curves, since the varieties $X$ may not base change well in the relative setting.
CHAPTER 2: UNSTABLE DIVISORS IN FLAG VARIETIES

2.1 Introduction

Let $G$ be a connected, semisimple complex algebraic group. The tensor decomposition problem asks, “Given irreducible representations $V_{\lambda_1}, \ldots, V_{\lambda_s}$ of $G$, when is the space of invariants

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s})^G$$

nonzero?” In other words, when is $V_{\lambda_s}^* \subseteq V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{s-1}}$. By Borel-Weil theory, this is equivalent to classifying the line bundles on $(G/B)^s$ that have a nonzero $G$-invariant section, where $B \subset G$ is a Borel subgroup.

We will consider a weaker problem (the “saturated tensor problem”), which can be phrased in terms of geometric invariant theory as follows. Given a projective $G$-variety $X$ with a $G$-equivariant ample line bundle $L$, GIT states that there exists a “good quotient” of the semistable locus

$$X^{\text{ss}} = \{ x \in X : \sigma(x) \neq 0 \text{ for some } \sigma \in H^0(X, L^N)^G, N > 0 \}$$

The saturated tensor problem is concerned with finding all line bundles on $(G/B)^s$ which have a nonempty semistable locus (i.e. some multiple of the line bundle has a nonzero invariant section). The equivalent representation theoretic problem is to classify representations $V_{\lambda_1}, \ldots, V_{\lambda_s}$ such that

$$(V_{N\lambda_1} \otimes \cdots \otimes V_{N\lambda_s})^G \neq 0$$

for some $N > 0$. The solutions to the saturated tensor problem span a polyhedral cone

$$\Gamma(s, G) = \{ L \in \text{Pic}_\mathbb{Q}(G/B)^s : H^0((G/B)^s, L^N)^G \neq 0 \text{ for some } N > 0 \}.$$
which has been studied extensively (see [K14] for a survey). A minimal list of inequalities for this cone was given in [BK06], [Ress10], and the extremal rays of the cone were determined in [BelKie18].

In this paper we will study certain divisors in \((G/B)^s\) called “F-divisors,” which play an interesting role in GIT. They are defined as follows.

**Definition 2.1.1.** A \(G\)-invariant and irreducible divisor \(D\) in a normal projective \(G\)-variety \(X\) is an F-divisor if it satisfies (any of) the following equivalent conditions:

1. \(D\) is an irreducible component of the unstable locus of some \(G\)-linearized line bundle on \(X\),

2. \(D\) is contained in the \(\mathcal{O}(D)\)-unstable locus,

3. \(\dim H^0(X, \mathcal{O}(nD))^G = 1\) for all \(n \geq 0\).

Every F-divisor spans an extremal rays of \(\Gamma(s, G)\), though not all extremal rays arise this way ([BelKie18], section 10 for an example).

The goal of this paper is to give a Schubert calculus description of these F-divisors. We show first off that every component of an unstable locus can be described as a locus of points having excess intersection of certain Schubert varieties (this is not new, see [BK06]), and we show that these Schubert positions satisfy a “Levi movability” property reminiscent of [BK06]. We then use this to show that, in type A, the Schubert positions for F-divisors can be chosen to satisfy a certain numerical property, which has consequences in GIT.

To state the main results, let us introduce some notation. Let \(P\) be a standard parabolic subgroup of \(G\) with Levi component \(L\). Let \(W\) be the Weyl group of \(G\), \(W_P\) the Weyl group of \(L\), and \(W_P^\circ\) the set of minimal length coset representatives of \(W/W_P\). For a tuple \(v = (v_1, \ldots, v_s) \in (W_P^\circ)^s\), define the intersection locus

\[
N^P_v = \{(g_1B, \ldots, g_sB) \in (G/B)^s : \bigcap_{j=1}^s g_jX_{v_j}^P \neq \emptyset\},
\]

where \(X_{v_j}^P = \overline{Bv_jP/P} \subseteq G/P\). It is the image of a universal intersection scheme

\[
X^P_v = \{(g_1B, \ldots, g_sB, x) \in (G/B)^s \times G/P : x \in \bigcap_{j=1}^s g_jX_{v_j}^P\}.
\]

In [BK06], it was shown that \(\Gamma(s, G)\) is controlled by inequalities indexed by certain Levi movable Schubert positions, and they introduced a deformed product of cohomology that can be used to test the
Levi movability. However, this assumes that the Schubert varieties intersect with expected dimension zero, i.e. 
\[ \sum_{j=1}^{s} \text{codim} X_{v_j}^P = \text{dim} G/P. \] (2.1)

We extend their definition to higher codimension intersection loci as follows. A tuple of Schubert positions \( v \) is Levi movable if there is a \( G \)-invariant smooth open subset \( U \subseteq N_P^v \) such that

- \( U \) meets the set of Levi points \( \prod_{j=1}^{s} L v_j^{-1} B / B \subset N_P^v \),
- \( \pi : X^P_v \to N_P^v \) is étale over \( U \) (compare also to the notion of “well-covering pair” from \[ Ress10 \]). As observed in \[ BK06 \], for intersection loci arising from GIT the map \( \pi^{-1}(U) \to U \) is an isomorphism, which we call “strongly Levi movable.”

Our first theorem is a decomposition of the unstable locus into certain Levi movable intersection loci. For the statement of the theorem, recall that line bundles on \( G/B \) are parametrized by characters of \( B \), where a character \( \nu : B \to \mathbb{G}_m \) corresponds to the line bundle \( L(\nu) = G \times_B \mathbb{C}_{-\nu} \).

**Theorem 2.1.2.** Let \( L = \mathbb{D}^{s}_{j=1} L(\nu_j) \in \text{Pic}(G/B)^s \). Then the unstable locus of \( L \) is the union of all the intersection loci \( N_P^v \) with \( P \) a maximal parabolic and \( v = (v_1, \ldots, v_s) \in (W^P)^s \) a tuple of strongly Levi movable Schubert positions such that

\[ \sum_{j=1}^{s} \nu_j(v_j x_P) > 0. \]

The proof uses standard techniques from \[ BK06 \], \[ Ress10 \]. As a consequence of theorem 2.1.2, we get inequalities for a sequence of cones

\[ \Gamma_k (s, G) = \{ L \in \text{Pic}_Q(G/B)^s : \text{unstable locus of } L \text{ has codimension } \geq k \}. \]

We should note that these cones (and a set of defining inequalities) were previously studied in \[ ST17 \], though we are not sure how our inequalities compare.

**Corollary 2.1.3.** Let \( L = \mathbb{D}^{s}_{j=1} L(\nu_j) \in \text{Pic}(G/B)^s \), and let \( k \geq 1 \). Then \( L \in \Gamma_k (s, G) \) if and only if \( L \in \Gamma_{k-1} (s, G) \) and for every standard maximal parabolic \( P \) and strongly Levi-movable \( v = (v_1, \ldots, v_s) \in (W^P)^s \) such that

\[ \sum_{j=1}^{s} \text{codim} \Lambda_{v_j}^P = \text{dim} G/P + k - 1, \]
the following inequality is satisfied:
\[ \sum_{j=1}^{s} \nu_j(v_j x_P) \leq 0. \]

When \( k = 1 \), these are the Belkale-Kumar inequalities, which give a minimal list of inequalities defining the cone \( \Gamma(s, G) \subset \text{Pic}_Q(G/B)^s \).

After proving theorem 2.1.2, the remainder of the paper focuses on \( F \)-divisors (codimension one components of unstable loci). For \( G = \text{SL}(n) \), we show that \( F \)-divisors satisfy the following numerical criterion, which has consequences in GIT.

**Theorem 2.1.4.** Suppose \( G = \text{SL}(n) \). Let \( D \) be a \( G \)-invariant and irreducible codimension one subvariety of \( (G/B)^s \). Let \( \mu_1, \ldots, \mu_s \) be the dominant integral weights such that \( O(D) \cong \bigoplus_{j=1}^{s} L(\mu_j) \in \text{Pic}(G/B)^s \).

Then \( D \) is an \( F \)-divisor if and only if \( D = N_P^P \) for some standard maximal parabolic subgroup \( P \) and \( v = (v_1, \ldots, v_s) \in (W_P)^s \) with

\[
\sum_{j=1}^{s} \text{codim} X_{v_j}^P = \dim G/P + 1, \tag{2.2}
\]

\[
\sum_{j=1}^{s} \mu_j(v_j x_P) = 1. \tag{2.3}
\]

We show that, as a consequence, we may always “correct” a line bundle on \( (\text{SL}(n)/B)^s \) to one that has unstable locus of codimension \( \geq 2 \). While we would like the result to extend to other groups, we do not necessarily expect it to. There are a number of GIT phenomena which hold only for \( \text{SL}(n) \), such as the descent of line bundles to GIT quotients (see [Bu16]).

**Corollary 2.1.5.** Assume \( G = \text{SL}(n) \). Let \( \mathcal{L} \) be a line bundle on \( (G/B)^s \) with a nonzero invariant section. Then there is another line bundle \( \mathcal{M} \in \text{Pic}(G/B)^s \) with the same graded ring of invariants

\[
\bigoplus_{m \geq 0} H^0((G/B)^s, \mathcal{L}^m)^G = \bigoplus_{m \geq 0} H^0((G/B)^s, \mathcal{M}^m)^G,
\]

and with semistable locus containing that of \( \mathcal{L} \), such that the unstable locus of \( \mathcal{M} \) has codimension \( \geq 2 \). In particular, there is a well-defined, injective pullback map

\[ \text{Pic}(X/G) \rightarrow \text{Pic} X \]
for any quotient of $X = (G/B)^s$ (note that the quotient is defined even for non-ample $L$ as the quotient of the partial flag variety to which $L$ descends to an ample line bundle).

2.2 Notation and Setup

2.2.1 Notation

Let $G$ be a connected, semisimple, complex algebraic group. Fix a maximal torus $H \subseteq G$ and a Borel subgroup $B$ containing $H$. Let $W = N(H)/H$ be the Weyl group. For a parabolic subgroup $P \subseteq G$, let $L$ denote its Levi component. If $P$ is a standard parabolic, let $W_P = N_L(H)/H$, and let $W^P$ be the set of minimal length coset representatives of $W/W_P$. The Lie algebras of $G, H, B, P, L$ are denoted $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \mathfrak{p}, \mathfrak{l}$ respectively. Let $R \subseteq \mathfrak{h}^*$ be the roots of $G$ with respect to $H$, and $R^+, R^-$ the positive and negative roots. The roots of $L$ with respect to $H$ are denoted $R^L$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subseteq R^+$ be the simple roots, and let $\{x_1, \ldots, x_l\}$ denote the basis of $\mathfrak{h}$ dual to the simple roots. Let $\mathfrak{h}^*_Z = \text{span}_\mathbb{Z}\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ and $\mathfrak{h}^*_Q = \mathfrak{h}^*_Z \otimes \mathbb{Q}$. Similarly, $\mathfrak{h}_Z^+ = \text{span}_\mathbb{Z}\{\omega_1, \ldots, \omega_l\}$ and $\mathfrak{h}_Q^+ = \mathfrak{h}_Z^+ \otimes \mathbb{Q}$. Let $\mathfrak{h}^+$ be the positive Weyl chamber in $\mathfrak{h}_Q^+$. For an integral weight $\mu \in \mathfrak{h}^*_Z$, let $L(\mu) = G \times^B C_{-\mu}$ be the corresponding line bundle on $G/B$, where $C_{-\mu}$ is the one-dimensional $B$-module with character $-\mu$.

2.2.2 Parameter spaces

For $w \in W$, let $G^P_w = BwP/P$ be the corresponding Schubert cell in $G/P$, and $X^P_w = \overline{BwP/P}$ the Schubert variety. Let $X^P_w$ be the fiber bundle $G \times_B X^P_w$ over $G/B$. Note that

$$X^P_w \cong \{(gB, x) \in G/B \times G/P : x \in gX^P_w\}.$$  

For an $s$-tuple of Weyl group elements $w = (w_1, \ldots, w_s) \in W^s$, define the universal intersection scheme

$$X^P_w = \{(g_1B, \ldots, g_sB, x) \in (G/B)^s \times G/P : x \in \bigcap_{j=1}^s g_jC_{w_j}\}.$$  

It can be constructed as the fiber product of the projection

$$X^P_{w_1} \times \cdots \times X^P_{w_s} \to (G/P)^s$$
with the diagonal embedding $G/P \to (G/P)^\ast$.

2.3 Review of GIT

Let $G$ be an algebraic group acting on a variety $X$, and $L$ a $G$-linearized line bundle on $X$. Define the $L$-semistable locus as the open subset

$$X^{ss} = \{ x \in X : \sigma(x) \neq 0 \text{ for some } \sigma \in H^0(X, L^n)^G, n > 0 \}.$$ 

Its complement is the $L$-unstable locus $X^{us}$. In order to construct quotients, one usually requires in the definition of semistability that $X_\sigma = \{ x' \in X : \sigma(x') \neq 0 \}$ be affine, but we will not do so.

For each $x \in X$ and one-parameter subgroup $\lambda : \mathbb{G}_m \to S$ (OPS, for short) such that $\lim_{t \to 0} \lambda(t)x$ exists, define the Hilbert-Mumford number $\mu_L(x, \lambda)$ as follows. Let $\lambda_x : \mathbb{G}_m \to X, t \mapsto \lambda(t)x$, which by assumption extends to a morphism $\overline{\lambda}_x : \mathbb{A}^1 \to X$. Pick any nonzero vector in the fiber $L|_{x_0}$, and use it to create a $\mathbb{G}_m$-invariant section of $\overline{\lambda}_x^*L$ over $\mathbb{G}_m$. Then $\mu_L(x, \lambda)$ is defined as the order of vanishing of this section at $0 \in \mathbb{A}^1$. Equivalently, this is the $\mathbb{G}_m$-weight of the fiber $L_{|x_0}$, where $x_0 = \lim_{t \to 0} \lambda(t)x$.

If $L$ is very ample, giving an equivariant closed embedding $X \hookrightarrow \mathbb{P}(V)$ for some $G$-module $V$, then this number may also be computed as follows. Let $e_1, \ldots, e_k$ be a basis of $V$ consisting of weight vectors of $\lambda$, say $\lambda(t)e_i = t^{\lambda_i}e_i$. If $x = [x_1e_1 + \cdots + x_ke_k]$, then

$$\mu_L(x, \lambda) = \max\{-\lambda_i : x_i \neq 0\}.$$ 

The following properties of $\mu_L(x, \lambda)$ are easy to check.

**Proposition 2.3.1.** For any $x \in X$ and OPS $\lambda$ such that $\lim_{t \to 0} \lambda(t)x$ exists:

1. if $x \in X^{ss}$, then $\mu_L(x, \lambda) \geq 0$;

2. $\mu_L(gx, g\lambda g^{-1}) = \mu_L(x, \lambda)$ for any $g \in S$;

3. if $Y$ is another $G$-variety and $f : Y \to X$ an equivariant morphism, then, for any $y \in Y$ such that $\lim_{t \to 0} \lambda(t)y$ exists, we have $\mu_f^*L(y, \lambda) = \mu_L(f(y), \lambda)$;

4. for any $G$-linearized line bundles $L_1, L_2$ on $X$, we have $\mu_{L_1 \otimes L_2}(x, \lambda) = \mu_{L_1}(x, \lambda) + \mu_{L_2}(x, \lambda)$. 

45
If $L$ is ample and $G$ is reductive, then in fact the Hilbert-Mumford function completely determines semistability. This is the celebrated Hilbert-Mumford criterion.

**Theorem 2.3.2.** ([MFK], theorem 2.1) If $G$ is a reductive group acting on a projective variety $X$, and $L$ is a linearized ample line bundle on $X$, then

$$X^{ss} = \{ x \in X : \mu^L(x, \lambda) \geq 0 \text{ for all } \lambda : \mathbb{G}_m \to G \}.$$  

### 2.3.1 Kempf's Theory

Now assume that $X$ is an irreducible projective variety acted on by a reductive group $G$, $\mathbb{H}$ is a maximal torus in $G$, and $L$ is a $G$-linearized semi-ample line bundle on $X$ (traditionally $L$ is ample, but see [Ress10] for the proof of generalization to semi-ample). Kempf associated to each unstable point a set of rational OPS, which are “most responsible” for its instability. To define this, let $Y(G)$ be the set of OPS of $G$, and let $M(G) = (Y(G) \times \mathbb{Z}_{>0})/\sim$ where $\sim$ is the equivalence relation given by $(\lambda, a) \sim (\mu, b)$ if $\mu^a = \lambda^b$. Also fix a $G$-invariant norm on $M(G)$, by which we mean a $G$-invariant function $q : M(G) \to \mathbb{Q}$ such that there is an inner product $(\cdot, \cdot)$ on $M(\mathbb{H})$ with $q(\beta) = (\beta, \beta)$ for any $\beta \in M(\mathbb{H})$. Extend the Hilbert-Mumford function to $M(G)$ by $\mu^L(x, \beta) = \mu^L(x, \lambda)/a$ for $\beta = (\lambda, a) \in M(G)$, and for $x \in X$ set

$$q^*(x) = \inf\{ q(\beta) : \beta \in M(G), \mu^L(x, \beta) \leq -1 \}.$$  

The “Kempf’s optimal class” of $x \in X$ is defined as

$$\Lambda(x) = \{ \beta \in M(G) : q(\beta) = q^*(x), \mu^L(x, \beta) \leq -1 \},$$  

and an element of $\Lambda(x)$ will be called a “Kempf’s OPS.” When necessary, we may write $\Lambda_G(x)$ to specify the group.

Kempf proved the following properties of these optimal classes. Here,

$$\mathbb{P}(\beta) = \{ g \in G : \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } G \},$$

denotes the parabolic subgroup associated to $\beta = (\lambda, a) \in M(G)$.
Theorem 2.3.3. ([Kempf78]) If \( x \in X^{ss} \), then \( \Lambda(x) \) is nonempty, and \( \Lambda(x) \) is a single \( \mathbb{P}(\beta) \)-orbit in \( M(\mathbb{G}) \) for any \( \beta \in \Lambda(x) \). In particular, \( \mathbb{P}(\beta) \) does not depend on the choice of \( \beta \in \Lambda(x) \).

2.3.2 Hesselink Stratification

Following Kempf’s theory, Hesselink introduced a stratification of the unstable locus based on optimal classes. This is a finite stratification of \( X \) into disjoint, locally closed, \( \mathbb{G} \)-invariant subvarieties, where the open stratum is the semistable locus and the unstable points are grouped by conjugacy of optimal class.

The assumptions on \( X, \mathbb{G}, \mathbb{H}, L \) stay the same. For each rational OPS \( \beta = (\lambda, a) \) of \( \mathbb{H} \), define

\[
Y^{ss}_\beta = \{ x \in X : \beta/q(\beta) \in \Lambda_G(x) \}.
\]

Let \( Z^{ss}_\beta \) be the set of \( \lambda \)-fixed points of \( Y^{ss}_\beta \), and define

\[
S_\beta = \mathbb{G} \cdot Y^{ss}_\beta.
\]

When necessary, we will write \( S^L_\beta \) to specify the line bundle. We call \( Y^{ss}_\beta \) the “blade” and \( S_\beta \) the “stratum.” Each of the sets \( Y^{ss}_\beta, Z^{ss}_\beta, S_\beta \) is a locally closed (but not necessarily connected) subset of \( X \), which we give the reduced scheme structure.

Theorem 2.3.4. ([Hess79], [Kir84]) Let \( \mathfrak{h}_+ \) be a positive Weyl chamber in \( M(\mathbb{H}) \). Then there is a stratification

\[
X = X^{ss} \sqcup \bigcup_{\beta \in \mathfrak{h}_+} S_\beta
\]

into locally closed, \( \mathbb{G} \)-invariant subvarieties of \( X \), finitely many of which are nonempty. Moreover,

1. if \( \lambda \) is an OPS which is a positive multiple of \( \beta \), then \( Z^{ss}_\beta \) is open in the fixed point set \( X^{\lambda} \), and

\[
Y^{ss}_\beta = \{ x \in X : \lim_{t \to 0} \lambda(t)x \in Z^{ss}_\beta \};
\]

2. for any \( x \in Y^{ss}_\beta \), we have

\[
\mathbb{P}(\beta) = \{ g \in \mathbb{G} : gx \in Y^{ss}_\beta \};
\]

3. there is a finite, bijective morphism \( \mathbb{G} \times^{\mathbb{P}(\beta)} Y^{ss}_\beta \to S_\beta \), which is an isomorphism if \( S_\beta \) is normal;
4. if $X$ is nonsingular and $L$ is ample, then $S_\beta$ is nonsingular.

**Remark 2.3.5.** For any equivariant morphism $\pi : X \to Y$ and semi-ample linearization $M$ on $Y$, we have $\pi^{-1}(S_\beta^M) = S_\beta^{\pi^*M}$ as sets. If $\pi$ is a smooth morphism, $Y$ is nonsingular, and $M$ is ample, then this is a scheme-theoretic equality as well, because in this case $\pi^{-1}(S_\beta^M)$ is smooth and therefore must agree with the natural reduced structure on $S_\beta^{\pi^*M}$. Since any effective line bundle on $(G/B)^s$ is the pullback of an ample line bundle on a partial flag variety, it follows that the strata in $(G/B)^s$ are nonsingular.

### 2.4 Unstable Loci in $(G/B)^s$

In this section, we look at components of unstable loci in $(G/B)^s$ of arbitrary codimension. We will prove theorem 2.1.2 and corollary 2.4.6 stated in the introduction. All of the notation is as in section 2.2.1.

For a standard parabolic $P \subset G$ and $v = (w_1, \ldots, v_s) \in W^s$, define the Schubert intersection locus

$$N^P_v = \{(g_1 B, \ldots, g_s B \in (G/B)^s : \cap_{j=1}^s g_j X^P_{v_j} \neq \emptyset)\}$$

It is the scheme theoretical-image of $\mathcal{X}_v \to (G/B)^s$. We will show every component of an unstable locus in $(G/B)^s$ is an $N^P_v$, and the data $(v, P)$ can be chosen to have several nice properties. The starting point is the following result of Leeb-Millson. For an OPS $\lambda : \mathbb{G}_m \to G$, let $\dot{\lambda} = \frac{d}{dt} \lambda(t)|_{t=1} \in \mathfrak{g}$, and for a standard maximal parabolic $P$ define $x_P$ to be the simple coweight $x_i$ such that $\alpha_i$ vanishes on $L$.

**Theorem 2.4.1.** ([BK06], theorem 26) Let $L = L(v_1) \boxtimes \cdots \boxtimes L(v_s) \in \text{Pic} (G/B)^s$, and suppose $\beta = (\lambda, a)$ is a Kempf’s OPS of an $L$-unstable point $x = (g_1 B, \ldots, g_s B)$. Pick $f \in G$ such that $\gamma = f \dot{\lambda} f^{-1}$ has $\dot{\gamma} \in \mathfrak{h}_+$, and define $v_j \in W^{P(\gamma)}$ by $f P(\gamma) \in g_j C^P_{v_j}$ for $1 \leq j \leq s$. Then for any maximal parabolic $P \supset P(\gamma)$ we have

1. $\cap_{j=1}^s g_j C^P_{v_j} = \{f P\}$,

2. $\sum_{j=1}^s \nu_j(v_j x_P) > 0$.

We also recall the following (which is a special case of a more general formula for Hilbert-Mumford numbers in $(G/B)^s$ – see [BK06], lemma 14).
Lemma 2.4.2. ([BK06], lemma 14) For any line bundle \( \mathcal{L} = L(v_1) \otimes \cdots \otimes L(v_s) \in \text{Pic}(G/B)^s \), standard maximal parabolic \( P \), Schubert positions \( v_1, \ldots, v_s \in W^P \), and point \( x \in \prod_{j=1}^s P v_j^{-1} B/B \), we have

\[
\mu^\mathcal{L}(x, t^{x_P}) = -\sum_{j=1}^s \nu_j(v_j x_P),
\]

where \( t^{x_P} \) is the unique OPS of \( H \) with \( \frac{d}{dt} t^{x_P}\big|_{t=1} = x_P \).

It is clear from theorem 2.4.1 and lemma 2.4.2 that every component of an unstable locus in \( (G/B)^s \) is an intersection locus \( N^P_v \) for some standard maximal parabolic \( P \) and \( v = (v_1, \ldots, v_s) \) satisfying inequality 2. of theorem 2.4.1, because these \( N^P_v \) form a finite collection of irreducible subsets whose union is the whole unstable locus. The next result shows that the data \((P, v)\) can be chosen to have several nice properties.

Proposition 2.4.3. Suppose \( Z \) is a component of the unstable locus of \( \mathcal{L} = L(v_1) \otimes \cdots \otimes L(v_s) \in \text{Pic}(G/B)^s \). Then we may find a standard maximal parabolic \( P \), Schubert positions \( v = (v_1, \ldots, v_s) \in (W^P)^s \), and a dense open subvariety \( U \subseteq D \) with the following properties:

(i) \( Z = N^P_v \);

(ii) \( \sum_{j=1}^s \text{codim} X_{v_j} = \dim G/P + \text{codim} Z \);

(iii) \( \sum_{j=1}^s \nu_j(v_j x_P) > 0 \);

(iv) \( \bigcap_{j=1}^s g_j C v_j \) is a point for any \( x = (g_1 B, \ldots, g_s B) \in U \);

(v) \( U \) is smooth and \( G \)-invariant;

(vi) there is an OPS \( \lambda \) with \( P(\lambda) \subseteq P \) such that if \( x = (g_1 B, \ldots, g_s B) \in U \) and \( \bigcap_{j=1}^s g_j C v_j = \{ f P \} \), then \( f \lambda f^{-1} \in \Lambda(x) \) and \( \lim_{t \to 0} f \lambda(t) f^{-1} x \in U \).

Proof. Stratify the \( \mathcal{L} \)-unstable locus as in theorem 2.3.4. Let \( S_\beta = \sqcup S_{\beta,i} \) be the connected components of the strata. Then there is a unique \( S_{\beta,i} \) which is a dense open subset of \( Z \). Let \( Y^{ss}_{\beta,i} = S_{\beta,i} \cap Y^{ss}_\beta \) be the corresponding irreducible component of \( Y^{ss}_\beta \) (recall \( S_\beta \cong G \times P(\beta) Y^{ss}_\beta \)). Let \( \lambda \) be an OPS which is a positive multiple of \( \beta \), and choose a maximal parabolic \( P \supseteq P(\lambda) \). Since \( Y^{ss}_{\beta,i} \) is irreducible, there are unique Schubert positions \( \overline{v}_1, \ldots, \overline{v}_s \in W^{P(\beta)} \) such that \( \prod_{j=1}^s P(\lambda) \overline{v}_j^{-1} B/B \) meets \( Y^{ss}_{\beta,i} \) in a dense open subvariety \( V \subseteq Y^{ss}_{\beta,i} \). Let \( v = (v_1, \ldots, v_s) \in (W^P)^s \) be the minimal length coset representatives of the \( \overline{v}_j \) in \( W/W_P \),
and take \( U = GV \), which is clearly open in \( S_{\beta,i} \cong G \times P(\beta) Y_{\beta,i}^{ss} \). The properties (iii)-(vi) of \( V \) follow from theorems 2.3.4 and 2.4.1. Since
\[
\sum_{j=1}^s v_j (v_j x_P) > 0,
\]
we know \( N^P_\nu \) is contained in the unstable locus. Since it contains a dense subset of the component \( Z \), we have \( Z = N^P_\nu \). Finally (ii) follows from the fact that \( \mathcal{X}_\nu \to N^P_\nu = Z \) is dominant and generically one-to-one.

\[\square\]

### 2.4.1 Levi-movability

The Schubert positions from proposition 2.4.3 have an additional “Levi movability” property reminiscent of [BK06].

**Definition 2.4.4.** A tuple \( \nu = (v_1, \ldots, v_s) \in (W^P)^s \) is Levi movable if there is a smooth open subset \( U \subseteq N^P_\nu \), which meets the set of Levi points \( \prod_{j=1}^s L v_j^{-1} B / B \), such that \( \pi : \mathcal{X}_\nu \to N^P_\nu \) is étale over \( U \). We say \( \nu \) is strongly Levi movable if in addition \( \pi^{-1}(U) \to U \) is an isomorphism.

**Proposition 2.4.5.** Let \( Z \) be a component of an unstable locus in \( (G/B)^s \), and let \( N^P_\nu \) be the description of \( Z \) found in proposition 2.4.3. Then \( \nu \) is strongly Levi movable.

**Proof.** Keep the notation from the proof of the previous proposition. We first show that \( \pi : \mathcal{X}_\nu^P \to Z \) is unramified at a Levi point by analyzing the ramification of the map \( G \times P(\beta) Y_{\beta}^{ss} \to S_\beta \) Note that \( G \times P(\beta) Y_{\beta}^{ss} \) is isomorphic to
\[
M = \{ (g P(\beta), x) \in G / P(\beta) \times (G / B)^s : g^{-1} x \in Y^{ss}_\beta \}.
\]
and the morphism \( G \times P(\beta) Y_{\beta}^{ss} \to S_\beta \) corresponds to the second projection \( p : M \to S_\beta \). Let \( U \subseteq S_\beta \) be the special open subvariety described in proposition 2.4.3, and pick \( z \in Z^{ss} \cap U \), which we may write as \( z = (l_1 v_1^{-1} B, \ldots, l_s v_s^{-1} B) \) for some \( l_1, \ldots, l_s \in L(\beta) \), the Levi factor of \( P(\beta) \). We will identify the kernel of the differential map \( dp_\bar{z} : T_{\bar{z}} M \to T_{\bar{z}} S_\beta \) at the point \( \bar{z} = (e P(\beta), z) \in M \).

Let \( p(\beta), l(\beta) \) be the Lie algebras of \( P(\beta), L(\beta) \). Then \( T_{\bar{z}} M \) consists of all
\[
(a + p(\beta), \xi) \in (g / p(\beta)) \oplus T_{\bar{z}}(G / B)^s
\]
with \( \xi - df_e(a) \in T_{z}(G_{\beta}^{ss}) \), where

\[
f : G \to (G/B)^{s}, \quad g \mapsto gz.
\]

Suppose \((a + p(\beta), \xi) \in \ker dp_{\beta}\). Since \(dp_{\beta}(a + p(\beta), \xi) = \xi\), we have \(\xi = 0\) and \(df_e(a) \in T_{z}(G_{\beta}^{ss})\). By a result of Bialynicki-Birula ([Bia73], section 4), we have \(T_{z}(G_{\beta}^{ss}) = (T_{z}(G/B)^{s})_{\geq 0}\), where \((T_{z}(G/B)^{s})_{\geq 0}\) is the sum of the weight spaces of \(\lambda\) with nonnegative eigenvalue. With \(\lambda\) acting adjointly on \(G\) and by left multiplication on \((G/B)^{s}\), the morphism \(f\) is \(\lambda\)-equivariant, as is \(df_e : g \mapsto T_{z}(G_{\beta}^{ss})\). Since \(p(\beta) = g_{\geq 0}\), we may assume \(a\) is a sum of negative weight vectors of \(\lambda\), so \(df_e(a) = 0\) by the equivariance. In summary,

\[
\ker dp_{\beta} \cong (\ker df_e + p(\beta))/p(\beta).
\]

The kernel of \(df_e\) is

\[
\bigcap_{j=1}^{s} \text{ad}_{l_{j}\gamma_{j}^{-1}}(b),
\]

and its image in \(g/p(\beta)\) is

\[
\bigcap_{j=1}^{s} \left( \bigoplus_{\gamma \in \gamma_{j}^{-1}(R^{+}) \cap (R_{+} \setminus R_{(\beta)})} \left( \text{ad}_{l_{j}}(g_{\gamma}) + p(\beta) \right)/p(\beta) \right),
\]

where \(g_{\gamma} \subset g\) is a root space. This is exactly the intersection of tangent spaces (see e.g. [BKR12])

\[
\bigcap_{j=1}^{s} T_{eP(\beta)}l_{j}^{-1}X_{v_{j}}^{P(\beta)}, \quad (2.4)
\]

which surjects onto \(\bigcap_{j=1}^{s} T_{eP(\beta)}l_{j}^{-1}X_{v_{j}}^{P}\) under the projection \(g/p(\beta) \to g/p\). This latter space is the tangent space to the fiber \(\pi^{-1}(z)\) at \(eP\). Since \(G \times P(\beta) Y_{\beta}^{ss} \to S_{\beta}\) is an isomorphism by theorem 2.3.4, the tangent space (2.4) is zero, hence \(\pi\) is unramified at \((z, eP)\).

To show strong Levi movability, it suffices to note the following. Suppose \(\pi : X \to Y\) is a proper, surjective morphism of irreducible varieties of the same dimension. Suppose there is a nonsingular point \(y \in Y\) such that \(\pi^{-1}(y) = \{x\}\) is a point and \(\pi\) is unramified at \(x\). Then there is a smooth open neighborhood \(U\) of \(y\) such that \(\pi^{-1}(U) \to U\) is an isomorphism. Proof of claim: take \(U \subseteq Y\) to be the smooth locus minus the branch locus; then \(\pi^{-1}(U) \to U\) is finite étale of degree one, which is an isomorphism.  \(\Box\)
2.4.2 Inequalities for subcones of $\Gamma(s, G)$

Recall that $\Gamma(s, G)$ is the polyhedral cone

$$\Gamma(s, G) = \{ \mathcal{L} \in \text{Pic}_Q(G/B)^s : H^0((G/B)^s, \mathcal{L}^n) \neq 0 \text{ for some } n > 0 \}$$

in $\text{Pic}_Q(G/B)^s = \text{Pic}(G/B)^s \otimes \mathbb{Q}$. We define a sequence of subcones

$$\Gamma(s, G) = \Gamma_1(s, G) \supseteq \Gamma_2(s, G) \supseteq \ldots$$

where

$$\Gamma_k(s, G) = \{ \mathcal{L} \in \text{Pic}_Q(G/B)^s : \text{unstable locus of } \mathcal{L} \text{ has codimension } \geq k \}.$$ 

The following is an immediate consequence of propositions 2.4.3 and 2.4.5.

**Corollary 2.4.6.** Let $\mathcal{L} = \oplus_{j=1}^s L(\nu_j) \in \text{Pic}(G/B)^s$, and let $k \geq 1$. Then $\mathcal{L} \in \Gamma_k(s, G)$ if and only if $\mathcal{L} \in \Gamma_{k-1}(s, G)$ and for every standard maximal parabolic $P$ and strongly Levi movable tuple $\mathbf{v} = (v_1, \ldots, v_s) \in (W_P)^s$ such that

$$\sum_{j=1}^s \text{codim}_{\nu_j}^{P} = \dim G/P + k - 1,$$

the following inequality is satisfied:

$$\sum_{j=1}^s \nu_j(v_j x_P) \leq 0.$$ 

2.5 F-divisors

The remainder of the paper focuses on F-divisors and the proof of theorem 2.1.4.

**Definition 2.5.1.** Let $X$ be a projective $G$-variety. An F-divisor in $X$ is a $G$-invariant and irreducible codimension one closed subvariety $D \subset X$ such that $\dim H^0(X, \mathcal{O}(nD))^G = 1$ for all $n \geq 0$.

The importance of F-divisors in GIT is as follows.

**Proposition 2.5.2.** Let $G$ be a reductive group acting on a normal, irreducible projective $G$-variety $X$. Let $D \subset X$ be an $G$-invariant and irreducible codimension one subvariety. Then the following are equivalent:

1. $D$ is an irreducible component of the unstable locus of some $G$-linearized line bundle $L$ on $X$,
2. $D$ is contained in the $\mathcal{O}(D)$-unstable locus,

3. $D$ is an $F$-divisor, i.e. $\dim H^0(X, \mathcal{O}(nD))^G = 1$ for all $n \geq 0$.

**Proof.** It is clear that (iii) $\implies$ (ii) $\implies$ (i), so we will show (i) $\implies$ (iii). Assume (i). For all $n \geq 0$, we have

$$H^0(X, L(nD))^G \subseteq H^0(X \setminus D, L)^G = H^0(X, L)^G,$$

where the second equality is by [NR93], lemma 4.15. The map $|L| \times |nD| \to |L(nD)|$ has finite fibers (there are only finitely many ways to write a divisor as a sum of two effective divisors), so

$$\dim H^0(X, L(nD))^G \geq \dim H^0(X, L)^G + \dim H^0(X, \mathcal{O}(nD))^G - 1,$$

and combining this with the above we obtain

$$\dim H^0(X, \mathcal{O}(nD))^G \leq 1.$$

Since $1 \in H^0(X, \mathcal{O}(nD))^G$ is a nonzero invariant section, the inequality is an equality.

2.5.1 $F$-divisors in $(G/B)^s$

We know that every $F$-divisor in $(G/B)^s$ is an intersection locus $D = N^P_v$ for some standard maximal parabolic $P$ and Schubert positions $v = (v_1, \ldots, v_s) \in (W^P)^s$. By propositions 2.4.3 and 2.4.5, we may also assume that for generic “Levi points” $z = (l_1v_1^{-1}B, \ldots, l_sv_s^{-1}B) \in D$ with $l_j \in L$ we have:

(i) $D$ is nonsingular at $z$,

(ii) $X_v \to D$ is unramified at $(z, eP)$ (by the Levi-movability),

(iii) $\cap_{j=1}^s l_jA_{v_j} = \{eP\}$ is a point.

Express $\mathcal{O}(D) \cong L(\mu_1) \boxtimes \cdots \boxtimes L(\mu_s)$ for some dominant integral weights $\mu_1, \ldots, \mu_s$. A formula for these weights was given in [BelKie18].$^1$ Recall that for Weyl group elements $u, u' \in W$ and a root $\beta \in R$,

---

$^1$Technically this is a formula for the pushforward class $\pi_* [X_v] \in A^1(G/B)^s \cong \text{Pic}(G/B)^s$, which in general will be some multiple of $\mathcal{O}(D)$, but since $\pi : X_v \to (G/B)^s$ is generically one-to-one, we can apply the formula without any concern.
the notation \( u \xrightarrow{\beta} u' \) means that \( u' = s_\beta u \) with \( l(u') = l(u) + 1 \). We will write \( u \xrightarrow{\beta} u' \in W^P \) if furthermore both \( u, u' \in W^P \).

**Theorem 2.5.3.** ([BelKie18], theorem 1.7) The weights \( \mu_j \) for \( O(D) \) are given by \( \mu_j = \sum_{i=1}^l c_{ij} \omega_i \), where

- if \( v_j \rightarrow s_{\alpha_i} v_j \in W^P \), then \( c_{ij} \) is the (possibly zero) intersection number such that

\[
[X_{v_1}] \cdots [X_{s_{\alpha_i} v_j}] \cdots [X_{v_s}] = c_{ij} \mathbb{p} \in H^*(G/P, \mathbb{Z}),
\]

- \( c_{ij} = 0 \) otherwise.

### 2.5.2 Proof outline for theorem 2.1.4

For the remainder of the paper, we assume \( G = \text{SL}(n) \). Theorem 2.1.4 says that an irreducible, \( G \)-invariant divisor \( D \subset (G/B)^* \) is an \( F \)-divisor if and only if \( D = N^P \) for some maximal parabolic \( P \) and \( v = (v_1, \ldots, v_s) \in (W^P)^s \) such that

\[
\sum_{j=1}^s \text{codim} X_{v_j}^P = \dim G/P + 1, \quad (2.5)
\]
\[
\sum_{j=1}^s \mu_j(v_j x_P) = 1, \quad (2.6)
\]

where \( \mu_j \) are the weights such that \( O(D) \cong \oplus_{j=1}^s L(\mu_j) \), and \((j_0, \alpha)\) is any pair such that \( \mu_{j_0}(\alpha^\vee) \neq 0 \).

One direction of the theorem is clear: if \( D = N^P \) satisfying (2.5) and (2.6), then \( D \) is contained in the \( O(D) \)-unstable locus by lemma 2.4.2, so \( D \) is an \( F \)-divisor by proposition 2.5.2. For the converse, we will fix an \( F \)-divisor \( D \) and give it the strongly Levi movable description \( D = N^P \) from proposition 2.4.3. Express \( O(D) \cong \oplus_{j=1}^s L(\mu_j) \in \text{Pic}(G/B)^s \). Pick an index \( 1 \leq j_0 \leq s \) and a simple root \( \alpha_i \) such that the intersection number \( c_{ij} \) appearing in theorem 2.5.3 is nonzero. Let

\[
w = (v_1, \ldots, s_{\alpha_i} v_{j_0}, \ldots, v_s) \in (W^P)^s.
\]

By lemma 2.4.2, the quantity (2.6) that we need to compute is the Hilbert-Mumford number \( \mu^{O(D)}(z, t x_P) \) at a point \( z = (l_1 v_1^{-1} B, \ldots, l_s v_s^{-1} B) \in (G/B)^s \) with \( l_j \in L \).

We will compute this Hilbert-Mumford number as follows.
1. First note that, since $G = SL(n)$, the Schubert positions $w$ are Levi movable (in type A, anything with nonzero intersection number is Levi movable, see [BK06]).

2. Let $\tilde{D}$ be the divisor $\mathcal{X}_v \subset \mathcal{X}_w$ (section 2.2.2). Using Levi movability of $w$, we show that the pullback of $D$ agrees with $\tilde{D}$ on a neighborhood of $(z, eP) \in \tilde{D}$.

3. By the functoriality of Hilbert-Mumford numbers, we may then compute the Hilbert-Mumford number for $\tilde{D}$ instead of $D$. This is done by a direct computation, which reduces to a single Schubert variety.

2.6 Proof of Theorem 2.1.4

2.6.1 Pulling Back to the Parameter Space

Let $\tilde{D}$ denote the divisor $\mathcal{X}_v \subset \mathcal{X}_w$.

**Proposition 2.6.1.** For sufficiently general $x \in D$,

$$
\mu^{\mathcal{O}(D)}(x, ft^{x_P} f^{-1}) = \mu^{\mathcal{O}(\tilde{D})}(\tilde{x}, ft^{x_P} f^{-1}),
$$

where $\tilde{x} = (x, fP)$ is the unique point of $\mathcal{X}_v^P$ lying over $x$.

**Proof.** Let $w$ be the weakened Schubert positions as in the previous section. Let $R$ be the ramification locus of $\pi : \mathcal{X}_w^P \to (G/B)^s$. We will show that $\pi^* \mathcal{O}(D) = \mathcal{O}(\tilde{D})$ on a neighborhood of a $t^{x_P}$-fixed point $\tilde{x}$. Since $w$ is Levi-movable (recall $G = SL(n)$ in this section), we may find a point of $\mathcal{X}_w^P \setminus R$ of the form $\tilde{y} = (l_1 w_1^{-1} B, \ldots, l_s w_s^{-1} B, eP)$ with $l_j \in L$. If $\tilde{x} \in \mathcal{X}_v^P$ is defined by $\tilde{x} = (l_1 v_1^{-1} B, \ldots, l_s v_s^{-1} B, eP)$, then using [BKR12], lemma 7.3, and [BelKie18], lemma 4.1, we have that $\pi$ is unramified at $\tilde{x}$ (the tangent space to the intersection of Schubert varieties for $\tilde{x}$ is the same as for $\tilde{y}$).

Being unramified is an open condition on $\tilde{x}$, so we may also assume that $x = \pi(\tilde{x})$ is a smooth point of $D$. Since $\pi^{-1}(D) \setminus R \to D$ is étale, $\tilde{x}$ is also a smooth point of $\pi^{-1}(D)$. Hence, $\pi^{-1}(D) = \tilde{D}$ on a neighborhood of $\tilde{x}$. Although the Hilbert-Mumford number has to be computed by taking a limit that could pass outside of this neighborhood, the point $\tilde{x}$ is already fixed by $t^{x_P}$, so equality (2.7) follows. $\square$
2.6.2 Computation in the Parameter Space

By proposition 2.6.1, we have now reduced the proof of theorem 2.1.4 to computing

\[ \mu^{O(\tilde{D})}(\tilde{x}, t^{x_P}) \]

for a point \( \tilde{x} = (l_1 v_1^{-1} B, \ldots, l_s v_s^{-1} B, e P) \in X^P_w \), where \( l_1, \ldots, l_s \in L \) (the choice of \( \tilde{x} \) will not matter).

Let \( E \) denote the divisor \( X^{P}_{v_{j_0}} \subset X^{P}_{w_{j_0}} \) (see section 2.2.2 for notation). Since \( \tilde{D} \) is the pullback of \( E \) along the projection \( p : X^P_w \to X^P_{w_{j_0}} \), we have

\[ \mu^{O(\tilde{D})}(\tilde{x}, t^{x_P}) = \mu^{O(E)}(p(\tilde{x}), t^{x_P}). \]

So, from now on we just write \( v = v_{j_0}, \ w = w_{j_0} \), and our goal is to compute

\[ \mu^{O(E)}(y, t^{x_P}) \tag{2.8} \]

for a point \( y = (l v^{-1} B, e P) \in C_v \), where \( l \in L \).

Consider the curve \( S \subset X_w \) which is the image of \( i : \mathbb{P}^1 \to X_w \),

\[
\begin{cases}
  a \mapsto (eB, \exp(a E_\alpha) w P) \text{ for } a \in \mathbb{A}^1 \\
  \infty \mapsto (e, v P).
\end{cases}
\]

Then \( S \) meets \( E \) with multiplicity one at \( z = \infty \) (e.g. using the functorial properties of \( S \) and \( E \) as in [Bel06], appendix), and pulling back to \( S \cong \mathbb{P}^1 \) we reduce the computation of (2.8) to computing

\[ \mu^{O(z)}(z, \lambda), \]

where \( z = \infty \in \mathbb{P}^1 \) and \( \lambda(t) = vt^{x_P}v^{-1} \) (note that we have to shift our OPS since the limit point is \((eB, v P)\) instead of \((v^{-1} B, e P)\)).
Let us compute the $\mathbb{G}_m$-action on $\mathbb{P}^1$. We have

$$
\lambda(t) \cdot [a : 1] = vt^x v^{-1} \exp(aE_{\alpha}) w P
$$

$$
= vt^x \exp(aE_{v^{-1}\alpha}) v^{-1} w P
$$

$$
= v \exp(\text{ad}_{t} x P) v^{-1} w P
$$

$$
= v \exp(atv^{-1}\alpha(xP)v^{-1}wP)
$$

$$
= \exp(atv^{-1}\alpha(xP)E_{\alpha}) w P,
$$

so $\mathbb{G}_m$ is acting on $\mathbb{P}^1$ by

$$
t \cdot [a_1 : a_2] = [tv^{-1}\alpha(xP)a_1 : a_2].
$$

Suppose the linearization of $\mathcal{O}(z)$ corresponds to the following lift of the action to $\mathbb{C}^2$:

$$
t \cdot e_1 = t^{m_1} e_1, \quad t \cdot e_2 = t^{m_2} e_2,
$$

where

$$
m_1 - m_2 = v^{-1}\alpha(xP).
$$

For $y = 0 \in \mathbb{P}^1$, we must have

$$
0 = \mu(y, \lambda) = -m_2,
$$

(because if $\mu(y, \lambda) > 0$, then $y$ would be stable, so its stabilizer would be finite). Hence $m_2 = 0$, $m_1 = v^{-1}\alpha(xP)$, and

$$
\mu(z, \lambda) = -m_1 = -v^{-1}\alpha(xP).
$$

for $z = \infty \in \mathbb{P}^1$. This number is always $-1$ if $G = \text{SL}(n)$, so this completes the proof of theorem 2.1.4

2.7 Applications

The application is to prove 2.1.5 from the introduction.

**Lemma 2.7.1.** Let $G = \text{SL}(n)$, and suppose $D$ is a codimension one component of the unstable locus of a line bundle $\mathcal{L} \in \text{Pic}(G/B)^\times$. Then invariant sections of $\mathcal{L}^m$ vanish to order $\geq m$ on $D$ for every $m \geq 0$. 

57
Proof. As shorthand, write
\[ M^L(v, P) = \sum_{j=1}^s \nu_j(v_j x_P), \]
for a standard maximal parabolic \( P, v = (v_1, \ldots, v_s) \in W^s, \) and \( L = \oplus_{j=1}^s L(v_j) \in \text{Pic}(G/B)^s. \) By theorem 2.4.3, we may express \( D = N^P_v \) with
\[ M^L(v, P) = l > 0. \]

If \( s \in H^0((G/B)^s, L^m)^G \) vanishes to order \( k \) on \( D, \) then \( s \in H^0((G/B)^s, L^m(-kD))^G, \) and is nonvanishing along \( D \) as a section of \( L^m(-kD). \) Hence \( M^{L^m(-kD)}(v, P) \leq 0. \) By theorem 2.1.4, we know \( M^{O(D)}(v, P) = 1, \) so
\[ M^{L^m(-kD)}(v, P) = lm - k, \]
thus \( k \geq lm \geq m. \)

Now we can prove the corollary 2.1.5 stated in the introduction.

Proof of corollary 2.1.5. Suppose \( D \) is a divisorial component of the unstable locus of \( L \in \text{Pic}(G/B)^s. \)
Let \( M_k = L(-kD). \) By the lemma,
\[ H^0((G/B)^s, M^m_1)^G = H^0((G/B)^s, L^m)^G. \]

If \( D \) is still unstable for \( M_1, \) then invariant sections of \( M^m_1 \) vanish to order \( \geq m \) on \( D, \) so
\[ H^0((G/B)^s, M^m_2)^G = H^0((G/B)^s, M^m_1)^G = H^0((G/B)^s, L^m)^G \]
for all \( m \geq 0. \) Then invariant sections of \( L^m \) must vanish to order \( \geq 2m \) along \( D. \) Continuing like this, we eventually find \( k \geq 1 \) such that the line bundle \( M = L(-kD) \) has
\[ \bigoplus_{m \geq 0} H^0((G/B)^s, M^m)^G = \bigoplus_{m \geq 0} H^0((G/B)^s, L^m)^G, \]
and such that \( D \) contains \( M \)-semistable points. The \( M \)-semistable locus contains the \( L \)-semistable locus, since \( L = M \) off of \( D, \) and we can repeat the above for any other divisors in the \( M \)-unstable locus until
we have an unstable locus of codimension $\geq 2$. The second statement about the injective pullback map of Picard groups follows immediately, since $\text{Pic}(X//G) \to \text{Pic}_G(X^{ss})$ is always injective ([KHV89], proposition 4.2), and now any line bundle extends uniquely from $X^{ss}$.

2.8 Examples

Let $G = \text{SL}(9)$, $s = 3$, and $P = P_3$. The minimal length coset representatives $W^P$ correspond to Schubert positions $\{i_1, i_2, i_3\} \subseteq [9]$ for the Grassmannian $\text{Gr}(3, 9)$, and the dominant integral weights of $G$ correspond to sequences of integers $(\lambda_1 \geq \cdots \geq \lambda_9 = 0)$. Let $D = N_{\nu}^P$ with

$$\nu = (\{2, 6, 9\}, \{3, 6, 9\}, \{3, 6, 9\}) \in W^P.$$ 

Using [Bel17], theorem 1.10 and remark 3.5, we have

$$\mathcal{O}(D) \cong L(\mu_1) \boxtimes L(\mu_2) \boxtimes L(\mu_3)$$

where

$$\mu_1 = (5, 5, 2, 2, 2, 0, 0, 0) = 3\omega_2 + 2\omega_6$$

$$\mu_2 = \mu_3 = (4, 4, 2, 2, 2, 2, 0, 0, 0) = 2\omega_3 + 2\omega_6.$$ 

Technically, this is the formula for the pushforward class $\pi_*[X_\nu] \in A^1(G/B)^s$, but since $(\mu_1, \mu_2, \mu_3)$ is indivisible in $(h_2^*)^3$, we know this is the correct formula.

We have

$$\sum_{j=1}^s \mu_j(v_j x_P) = 1,$$

which tells us that $D$ is an unstable divisor and confirms theorem 2.1.4 for this example. Clearly, $\mathcal{O}(D)$ cannot lie on a type I ray, since no $\mathbb{Q}$-multiple of $\mathcal{O}(D)$ which is in $(h_2^*)^3$ can have a one as a coefficient in the fundamental weight basis. As a further verification, we checked in Sage that

$$\text{dim } H^0((G/B)^s, \mathcal{O}(D))^G = 1,$$
which implies \( \dim H^0((G/B)^s, \mathcal{O}(nD))^G = 1 \) for all \( n \geq 0 \) by Fulton’s conjecture.


[Bu16] N. Bushek, Some conditions for descent of line bundles to GIT quotients $(G/B \times G/B \times G/B) \g G$ (2016), arXiv:1611.09818.


