Collusion Among Bitcoin Mining Pools

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ABSTRACT

One of the most striking financial developments of the last five years involves the emergence and rapid adoption of digital currencies, with Bitcoin being the most prominent. This paper seeks to determine whether there is evidence of collusion between mining pools (coalitions of individuals that verify transactions for monetary returns). We first constructed a theoretical framework which modeled the mining activity as an infinitely repeated game between two competing pools. By devising payoffs in the form of value functions and applying the one shot deviation principle, we found that a collusive strategy was indeed an equilibrium–if certain conditions held. However, our empirical analysis offered more ambiguous results. Ultimately, our attempt to capture peer effects suggests the relationship between a mining pool and its competitors is negative and non-linear. While this could serve as evidence against collusive behavior, we also postulate alternate explanations that could account for the finding.

I. INTRODUCTION AND MOTIVATION

First introduced in 2008 and finally released in January 2009 by Satoshi Nakamoto, an enigmatic figure whose identity remains a mystery to this day, Bitcoin originated with lofty ambitions. Conceived as a peer-to-peer system of transactions, the cryptocurrency sought to eliminate the need for a central bank and provide patrons with a straightforward means of purchasing goods and services. Though Bitcoin received substantial news coverage from various outlets following its inception, few mainstream pundits considered it more than a temporary novelty. However, perceptions of Bitcoin began to morph in 2014 following a nearly 900% increase in the value of the currency, reaching an all-time high of $1000. While Bitcoin has experienced periods of intense volatility over the course of the last two years, the value of a single bitcoin has remained relatively high and is currently pegged at over $1200 (having surged due to uncertainty in U.S and European politics). With the daily value of Bitcoin transactions consistently exceeding $100 million and reputable retailers such as Microsoft accepting the digital currency, its influence on the global economy is poised to rapidly increase in the coming decades.

My research will focus on whether Bitcoin mining pools are engaging in collusive behavior in order to limit the amount of computational effort they must exert while maintaining profitability. Though the process of mining will be thoroughly explored in the following section, the activity revolves around firms (pools) verifying transactions and receiving newly minted currency as a reward. Over the course of the last five years, the most startling development in the mining community has been the formation of these mining pools and the rapid rate at which they rendered individual miners obsolete. These pools (coalitions of individuals and corporations that benefit from pooled computational power) have dramatically altered the process of mining and now serve as high-tech data centers that can compute mathematical functions billions of times per second.

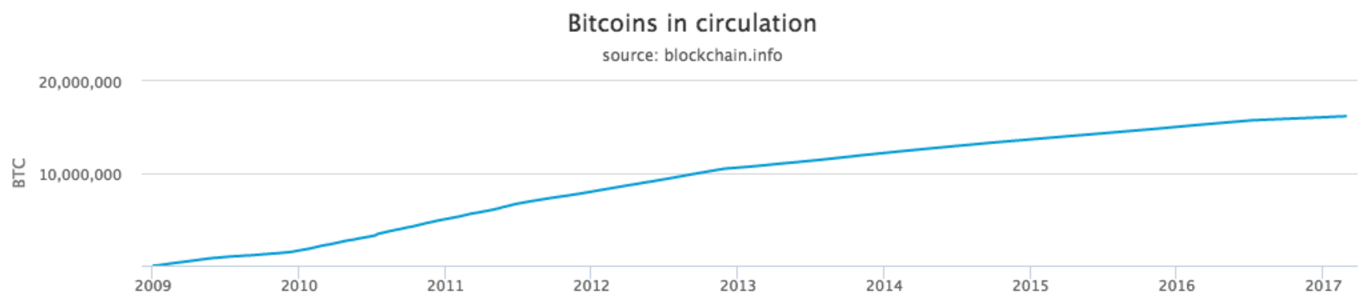
Although such a topic may initially seem extremely esoteric and of little interest to economists, this naive attitude is incorrect. As was previously mentioned, digital currencies are becoming increasingly ubiquitous as consumers recognize the myriad of benefits associated with them (increased security, anonymity, etc), a fact reflected by the large volume of transactions completed daily. While Bitcoin has certainly received the largest share of media attention in recent years, innovative cryptocurrencies have continued to emerge (often modifying the mining process in intriguing ways) and offer consumers numerous ways to engage in this evolving market.

As the two graphs provided above demonstrate, the history of Bitcoin has been characterized by rapid growth and significant volatility. Although the number of Bitcoins in circulation has followed a monotonic and nearly linear trajectory, the same cannot be said of the market price of the currency. This exchange rate, much like stocks traded in the market, is influenced heavily by speculation and has experienced surges and reductions in quick succession. Fol-

Figure 1: Graph depicting the rapid growth of Bitcoin's valuation

Market value of a single Bitcoin since the currency's launch in 2009. Measured in USD.

Figure 2: Graph depicting the number of Bitcoins in circulation



Number of Bitcoins in circulation since the currency's launch in early 2009.

lowing a low and almost constant valuation for over four years, the price of a Bitcoin spiked in late 2013 as a result of increased interest in Chinese markets (ballooning past the $1000 mark), before adapting to more cautious market forces. Only in recent weeks has Bitcoin been able to eclipse these previous heights–attributable, to a large extent, to increased uncertainty in the United States and Europe.

Since cryptocurrencies are a recent phenomenon and require knowledge of fundamental computer science concepts, a scant amount of economics literature has been devoted to this topic. Thus, my research will certainly make a contribution as it can be considered pioneering in the field. Along with venturing into uncharted territory, this paper makes a variety of contributions which future researchers can utilize to develop more advanced and innovative models of the mining activity. For instance, the theoretical model presented in this paper–while quite simplistic and plagued by limitations that will be expanded upon later–captures the peer effects and interdependence among pools by incorporating game theoretic tools. As future literature will no doubt need to account for these concerns, my model could serve as a basis for these works. On top of that, no other economist has made use of actual data made available from the blockchain, rendering the utilization here novel and contributory. However, the most significant contribution of this paper is in regards to the unintuitive and surprising empirical results generated through statistical testing. The negative relationship between the hash rate of a given pool and its competitors was an unexpected result and warrants further study in future work.

As the previous paragraph noted, the application of econometric methods yielded unanticipated results. One of the main findings is the aforementioned negative relationship between the hash rate of a particular pool and that of its peers. While it could be argued this serves as an argument against the existence of collusion between mining pools, the results also suggest its presence by indicating pools reduce their hash rates as a mining period elapses ("slacking off" in order to extend the duration of the round and prevent a large spike in the difficulty level). More discussion of the possible implications and explanations of this phenomenon will be provided in the results section. We also proposed a theoretical model which attempts to capture the interdependence between pools. According to the model, collusion is a subgame equilibrium if certain constraints on firm behavior and levels of computational resources hold.

The remainder of the thesis is organized as follows: in the following section, a more thorough explanation will be given of the underlying algorithm facilitating Bitcoin mining (focusing on the nature of computing hash functions and verifying transactions) to provide a necessary background and understanding for subsequent discussion. Next, we present a theoretical model which draws upon game theoretic techniques and determine if collusion is a sustainable equilibrium for two opponents. We then present an empirical model to test whether data supports the hypothesis of collusion, describe the data source and the means of its acquisition, and provide discussion of the results obtained through regression analysis. Finally, we offer a conclusion that summarizes the main findings of the paper and suggests how future work could expand upon the foundation established here.

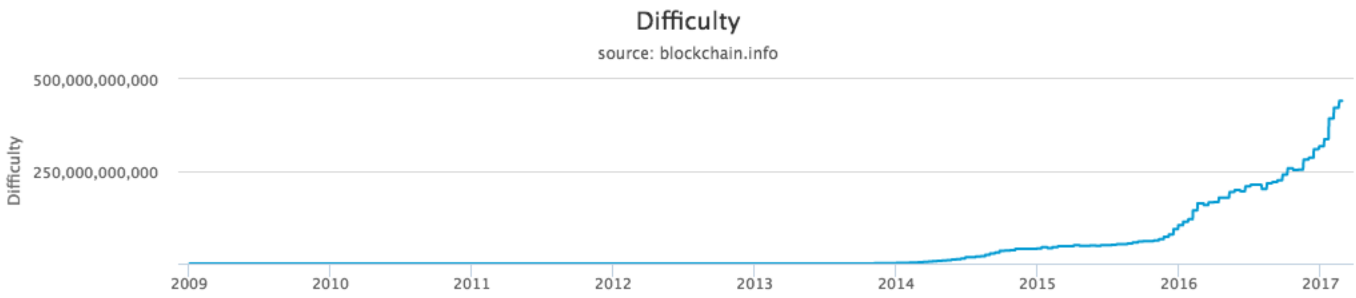
II. Bitcoin Mining Algorithm

While Bitcoin possesses many unique characteristics, none have attracted as much attention or become as recognizable as the process known as Bitcoin mining. As Bitcoin is a decentralized currency with no bureaucracy monitoring its use, the onus of verifying transactions falls to individuals collectively known as miners. As a comprehensive understanding of the nuances surrounding the process is unnecessary for the application of economic analysis, a description of the relevant aspects will now be provided.

When an individual submits a transaction utilizing Bitcoin–that is, they attempt to exchange currency for a good or service–the result is not instantaneous. Since no trusted third party exists that can verify the consumer possesses the necessary funds and then facilitate the movement of bitcoins from one account to another, an alternate means of performing the task is necessary. This dilemma is resolved by perhaps the most significant innovation of Bitcoin: the delegation of this function to the mining community.

The practice of mining originates by compiling a series of transactions (detailing the transfer of bitcoins from one account to another) into a block. Following this, a hash function will be computed by incorporating details about the block, the address of the last successfully added block, and a random component known as a nonce. This hash function–prolific in disciplines such as computer science and cryptography–will simply produce an integer value. If the integer is lower than a specified target value, the block can be added to the blockchain, the official ledger of all transactions. Otherwise, the nonce value is modified and the hash recomputed. In exchange for the service provided, miners are provided with newly minted bitcoins. Following the successful addition of 2016 blocks to the blockchain, the target value is adjusted. Through this adjustment of the target value (commonly referred to as the mining difficulty), the creators of Bitcoin sought to limit the expansion of new currency and establish a means of distribution.

Figure 3: Graph depicting the evolution of the mining difficulty



Mining difficulty since Bitcoin launched in 2009. The creators of Bitcoin release an integer value to denote varying levels of difficulty, with higher numbers indicating adding a block to the blockchain is more difficult. Note that the massive increase in the mining difficulty is likely the result of two concurrent trends: the entry of new mining pools and technological advancements which allowed hash functions to be computed at lower costs.

Along with the reward of new currency, miners also have the potential to receive a small transaction fee for each verification they perform. When engaging in a transaction, consumers have the option of including a small fee that will be provided to miners upon successful addition of a block containing that transaction to the Blockchain. As there is no sales tax associated with Bitcoin, this minuscule fee ($.41 for an average sized transaction) is decided by the individual making a purchase, who may decide to provide no reward to miners. There is, however, a strong negative correlation between the size of the transaction fee and the amount of time for the transaction to be processed.

Since the Bitcoin mining involves such a unique and foreign process, I will address a few concerns that often arise when an individual is first introduced to the activity. For instance, a common inquiry is whether pools have an incentive to create fictitious transactions to bolster transaction volume in an attempt to increase the supply of unverified transactions, thereby allowing currency to be generated an unnatural rate. Although such subversion might have been possible in the initial days of the cryptocurrency, such manipulation is now entirely unnecessary. With the daily transaction value exceeding $100 million and exhibiting an upward trend, a supply of available transactions is constantly available, thereby allowing pools to perform hash functions as often as they desire.

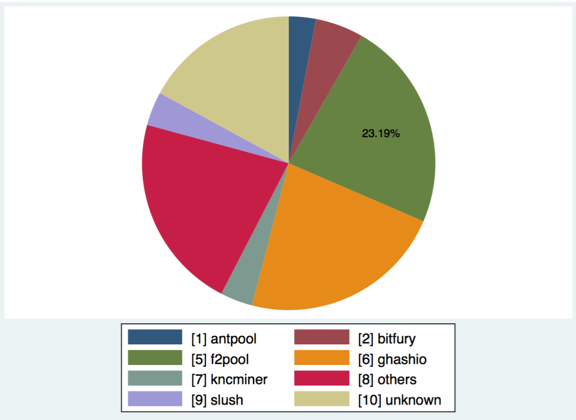
Finally, I will address one of the primary concerns that has arisen in the Bitcoin community in recent years. With the advent of massive mining pools possessing incredible computational power, there exists a plausible scenario in which a pool amasses enough concentration (measured as their proportion of the hash rate) to insidiously manipulate the cryptocurrency. As Halaburda and Sarvary note in their pioneering examination of the ascent of Bitcoin:

> The main role of the proof-of-work is to ensure immutability of the ledger. The solution to the puzzle becomes a part of the block being added to the blockchain. Importantly, the solution depends on the blocks preceding it in the blockchain. Suppose you wanted to go back and change a transaction, for example replacing the recipient of the bitcoins being sent with yourself. This would change one of the past blocks, meaning you would need to redo the proof-of-work for that block to make it a valid addition to the blockchain. Even more important, you would also need to redo the proof-of-work for all the blocks that follow it. You would need 51 percent of the computing power of the whole network to outper- form other miners in order to successfully put fraudulent blocks into the blockchain. Gaining such computational power is very costly, which was the intent in Bitcoin network design [4].

In the excerpt presented above, the authors note that the integrity of the Blockchain would be compromised if a pool accounted for 51% of the hash rate. In this case, the pool could alter transactions and rehash any subsequent blocks (complete the required proof of work) to disguise their criminal activity.

Although the current distribution of mining pool hash rates may alleviate such fears and indicate concern is unwarranted, a more thorough analysis suggests otherwise. As recently as 2014, the mining pool ghash.io possessed the capability to operate at this concentration and potentially compromise the entire currency. Recognizing the immense danger this posed, individuals within ghash.io self-regulated and joined alternate pools. In the two years since, numerous competitors have emerged and the highest concentration of any pool currently stands around 20%.

Figure 4: Pool Concentration in September 2014



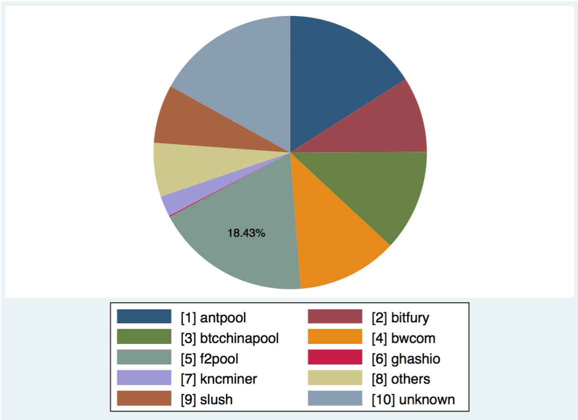
Distribution (measured by proportion of overall hash rate) of mining pool concentration in September 2014. Note that certain pools had not come into existence yet, and thus possess no share of overall hashing.

Figures 4 and 5 portray the concentration of mining pools (measured by their proportion of the overall hash rate) at periods approximately two years apart. As evidenced by the pie charts, this period witnessed the emergence of new pools and a trend toward decentralization among any major pool.

III. Related Literature

Although Bitcoin has, to a large extent, remained unexamined by economists, a recent paper has sought to develop the theoretical framework for determining how the exchange rate of

Figure 5: Pool Concentration in September 2016



Mining distribution in September 2016. The emergence of additional pools appears to have diminished the dominance of any particular pool.

Bitcoin is determined [1]. While such a topic is not directly related to my own, the paper provided guidance in how to specify my theoretical model and enhanced my understanding of the various processes underlying the cryptocurrency. The authors begin by presenting a simplified market (devoid of investors) in which the value of Bitcoin in determined by the laws of supply and demand. In this scenario, a steady state exchange rate emerges in which the ratio of expected transaction volume is equal to the supply of Bitcoins. With this established, they then allow for the introduction of investors and find that these individuals would exit the market once the steady state equilibrium is reached.

Since there is a minimal amount of economics research on the topic of Bitcoin, the remaining relevant papers I have considered simply incorporate game theoretical and time series elements that have been applied to varying industries and situations. These papers, while covering topics completely unrelated to Bitcoin, have offered insight into structuring my theoretical and empirical models.

A primary paper that influenced my theoretical model and attempted to model situations similar to Bitcoin mining explored the desire of firms to ruthlessly clash in the confines of a competitive market [5]. Two players (1 and 2) are presented as engaging in a repeated game in which each is able to exert high or low effort during each round. Ultimately, the collusive equilibrium (exerting low effort) is supported only if the lead of a given firm remains moderate. If a firm possesses a large lead, they have an incentive to choose high effort in an attempt cement their dominance and further crush their opponent. On the other hand, when a lead is small both players have an incentive to deviate from a collusive stratagem: the player with the lead seeks to hold off their opponent, while the lagging opponent observes an opportunity to converge with the leader. Since the specification of my model was closely aligned with Hörner, we will determine if our model supports similar conclusions.

Although not a major focus of my research, another paper sheds insight onto the decision for individuals to transition from single miners into coalitions of miners. Armando Gomes proposes a model which may provide illumination on the topic [3]. In the paper, he describes the process of coalition formation as a repeated game in which players have the option to enter into new contracts at the end of each round. His analysis concludes by demonstrating that the strategy profile of each player results in a Markov perfect equilibrium in which the actions of the players in round t depends only on the state at the end of the previous round (t-1). Moreover, Fonseca and Normann jointly explore the effects of explicit and implicit collusion in markets exhibiting characteristics of oligopolies [2]. Not surprisingly, the theoretical model they construct implies that explicit collusion often leads to higher prices and less defection from collusive behaviors.

Since I intend to model the process of Bitcoin mining as a dynamic game in which mining pools (firms) must account for how their decisions will influence subsequent rounds through the adjustment of the target value, I will need to employ recursive value functions and a Markov equilibrium to capture the dynamics of the situation. In order to develop a more thorough understanding of the theoretical considerations and familiarize myself with standard notation, I have relied primarily on two papers. In the first, Maskin and Tirole discuss Markov strategies and equilibrium in games that exhibit observable actions [8]. The paper, dealing predominately with abstract game theory and complex mathematics, proposes an iterative algorithm for computing a Markov Perfect Equilibrium from a value function that recursively determines the payoff in repeated game of arbitrary length. Additionally, a lauded paper published by Weintraub, Benkard, and Van Roy applies similar techniques to industries characterized by the presence of numerous firms [9]. Rather than computing the Markov equilibrium–which proves extremely difficult for many applications–the researchers introduce what they refer to as an oblivious equilibrium. Utilizing this concept, they propose a method in which the actions of a given firm depend only on the current state of the firm and the long-run averages of the industry; that is, firms dismiss knowledge of the current state of their competitors. This simplification enables equilibria to be computed with significantly greater ease, while still accounting for the existence of states that influence of behavior of participants.

Additionally, I have consulted sources incorporating time series analysis in order to improve my knowledge of the discipline and gain insight into how to structure my empirical model. Mall utilizes techniques that resemble those I will need to employ to effectively estimate the degree of collusion between Bitcoin mining pools [6]. In investigating the effect of foreign investment on Pakistan's growth over a period spanning over three decades, Mall must include the effects of lag in his model to compensate for the endogeneity present between his dependent and independent variables. Although his research topic bears little to resemblance to my own, I too must incorporate lags as a result of the high degree of covariance between my regressors, and can thus use his empirical model as reference when constructing mine.

Finally, I have analyzed a seminal paper by Charles Manski which explores the reflection problem and the various issues that often arise when estimating endogenous variables [7]. Also referred to as the peer or neighborhood effect, the reflection problem consists of firms or individuals incorporating the actions of others in their decision making process. After outlining the differences between endogenous, exogenous, and correlated effects, Manski delivers a staggering assertion: that inference of group influence is not possible unless the researcher possesses prior information that specifies the composition of reference groups. By noting that the "prospects are poor to nil" if such characteristics remain elusive, he concludes that only more advanced theory or the collection of richer data can alleviate these concerns. As my analysis will attempt to capture and estimate a peer effect (how the behavior of a particular mining pool influences their peers), I will need to note the limitations and concerns described by Manski when specifying my empirical model and interpreting regression results.

Overall, the availability of similar research has provided tremendous aid in completing this paper. Although few economists have thoroughly analyzed Bitcoin or any of its attributes, the existing body of work on competitive environments served as a basis for our work. However, certain limitations and caveats remain. Most notably, accurately measuring the peer effect remains a difficult endeavor and is fraught with error. Additionally, the uniqueness of Bitcoin mining (in which the actions of pools in one round will influence the difficulty and computational costs in subsequent rounds) differs from a standard perfectly competitive marketplace, thus rendering some aspects of current literature inapplicable.

IV. THEORETICAL MODEL

As a result of the iterative nature of Bitcoin mining, the activity can naturally be modeled as an infinitely repeated game in which pools compete in a tournament like endeavor. Since the process contains a high level of uncertainty and nuance that unnecessarily complicates the underlying model, let us consider a highly simplified model consisting of only two mining pools. Suppose that pool i contains a stock of capital, ${k_t}^i$ and is able to make an investment in an additional capital in period t by scaling their current stock by ${e_t}^i$. For the sake of simplicity, assume the investment decision made by the firm is a binary decision in which

$$e_t^i \in \{1, e\} \tag{1}$$

That is, the pool must decide between making no investment in capital (in which ${e_t}^i$ =1) or increasing their stock by a factor of e, which is greater than 1. Thus, assuming no depreciation of capital occurs, the capital possessed by a pool can be defined as

$$k_t^i = k_{t-1}^i \cdot e_t^i \tag{2}$$

During each period of the repeated game, the two pools compete to successfully include a block within the Blockchain, thereby accruing the reward of new Bitcoins. As a result of the simplifications imposed in the model, the payoff of a pool in a particular round is dependent only on three factor: the capital of that particular pool in round t, the capital of its competitor, and the current difficulty. These three aspects represent a state in an automata (often referred to as a finite state machine) and completely determine the outcome received both pools. Through this mechanism, the simple combination of the three is sufficient to generate the current state and payoff. Thus, the state in period t is represented by the following ordered triple:

$$s_t = (k_t^1, k_t^2, d_t) \tag{3}$$

Given a particular state, s_t, the utility derived by either pool can be easily computed. Assuming a pool experiences a utility of 1 if they succeed in the mining endeavor and a utility of 0 in the case of failure, the utility function $u_i(s_t)$ can be defined by its expectation:

$$u_i(s_t, \mathbf{e}_t) = \frac{k_t^i \cdot e_t^i}{k_t^1 \cdot e_t^1 + k_t^2 \cdot e_t^2 + d_t} \tag{4}$$

Note that the utility merely corresponds to the probability that pool i successfully adds a block to the Blockchain.

Suppose that the rate of failure– that is, the probability that neither pool computes an acceptable hash function– is defined by γ and the adjustment of the difficulty mechanism forces a constant rate. This rate of failure can be specified as follows

$$\gamma = \frac{d_{t+1}}{k_t^1 + k_t^2 + d_{t+1}} \tag{5}$$

Solving for d_{t+1}, we find the law of motion for the difficulty to be

$$d_{t+1} = \frac{\gamma(k_t^1 + k_t^2)}{1-\gamma} \tag{6}$$

We can similarly define the rate of success, ρ (the probability that either pool succeeds in adding a block to the Blockchain in round t)

$$\rho = \frac{k_t^1 + k_t^2}{k_t^1 + k_t^2 + d_t} \tag{7}$$

Two straightforward strategy profiles immediately present themselves when considering this model. In the first—the collusive strategy—the two pools continually abstain from investing in additional capital, remaining satiated with the status quo. Assume that the initial level of difficulty, d_0 satisfies a steady state equilibrium–that is, it is defined by equation 6. In this case, the value function representing the payoff of pool i is:

$$V_i(s_t) = u_i(s_t) + \delta V_i(s_{t+1}) = \sum_{n=0}^{\infty} \delta^n u_i(s_{t+n}) = \tag{8}$$

$$= \sum_{n=0}^{\infty} \delta^n \frac{k_0^i}{k_0^1 + k_0^2 + d_n} = \sum_{n=0}^{\infty} \delta^n \frac{k_0^i}{k_0^1 + k_0^2 + \frac{\gamma}{1-\gamma}(k_0^1 + k_0^2)} =$$

$$= \frac{k_0^i}{(k_0^1 + k_0^2)(1 + \frac{\gamma}{1-\gamma})} \cdot \frac{1}{1-\delta} \tag{9}$$

As demonstrated by the value function, this scenario consists of the state (and thus both levels

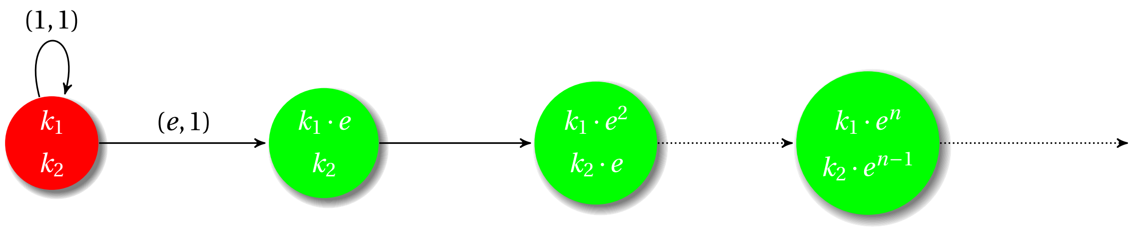


Figure 6: The collusive strategy profile

Pools following the collusive strategy profile will refrain from investing in each round, thereby maintaining their initial stocks of capital and rendering the mining difficulty constant. Once one pool deviates from the strategy and invests, both pools transition to the competitive strategy profile and invest in each subsequent round

of capital and the difficulty) remaining constant in each period. If either firm were to deviate from this strategy by investing in additional capital, the opponent would merely respond by choosing to invest in each subsequent round. Figure 6 depicts this strategy profile.

On the other hand, there could exist a strategy profile—the non-collusive strategy—in which both firms choose to invest in every round, thereby increasing their stock of capital by a factor of e in the transition from period t to t+1. In this scenario, the difficulty path satisfies:

$$d_{t+1} = \frac{\gamma}{1-\gamma}((k_0^1 + k_0^2) \cdot e^{t+1}) \tag{10}$$

The value function would be specified as:

$$V_i(s_t) = u_i(s_t) + \delta V_i((k_t^1 \cdot e, k_t^2 \cdot e, \frac{\gamma(k_t^1 + k_t^2)}{1-\gamma})) = \sum_{n=0}^{\infty} \delta^n u_i(s_{t+n}) = \tag{11}$$

$$= \sum_{n=0}^{\infty} \delta^n \frac{k_0^i \cdot e^{n+1}}{(k_0^1 + k_0^2) \cdot e^{n+1} + d_n} = \sum_{n=0}^{\infty} \delta^n \frac{k_0^i \cdot e^{n+1}}{(k_0^1 + k_0^2) \cdot e^{n+1} + \frac{\gamma}{1-\gamma}(k_0^1 + k_0^2) \cdot e^n} =$$

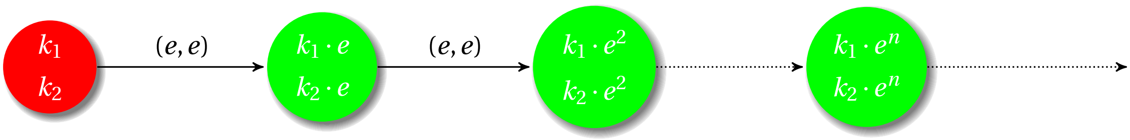


Figure 7: The non-collusive strategy profile

In the competitive (non-collusive) strategy profile, pools invest in each round.

$$= \sum_{n=0}^{\infty} \delta^n \frac{e^n \cdot (k_0^i \cdot e)}{e^n((k_0^1 + k_0^2) \cdot e + \frac{\gamma}{1-\gamma}(k_0^1 + k_0^2))} = \frac{k_0^i \cdot e}{(k_0^1 + k_0^2) \cdot (e + \frac{\gamma}{1-\gamma})} \cdot \frac{1}{1-\delta} \tag{12}$$

With both pools perpetually investing in further capital, observe that the difficulty must consequently adjust in order to retain the constant rate of failure. Therefore, the benefit resulting from investment is reduced, as an increased hash rate is required to compensate for the greater computational challenge. Figure 7 depicts this strategy profile.

V. Equilibrium Analysis

Let $V^C(\mathbf{k}, d)$ denote the value function when the stock of capital is $\mathbf{k} = (k_1, k_2)$ and the difficulty is set at d, players are using the collusive strategy profile, and no player has deviated. Also, let $V^N(\mathbf{k}, d)$ denote the value function when the stock of capital is $\mathbf{k} = (k_1, k_2)$, players are using the non-collusive strategy profile, and no player has deviated. Assume these value functions adhere to the law of motion of the difficulty previously defined: when the players collude no change occurs and when they invest the difficulty adjusts accordingly. Note that when a player first deviates when the collusive profile is being played, say player 1 deviates, the value of the continuation game is $V^N(k_1 \cdot e, k_2, \frac{\gamma}{1-\gamma}(k_1 \cdot e + k_2))$.

For the collusive strategy profile to be a subgame perfect Nash equilibrium, by the one shot deviation principle, it has to be the case that:

1. No player has an incentive to invest if no players invested before (or if it is the initial

period).

2. If some player invested before, then no player has an an incentive to not invest.

Formally, these incentive constraints are (we focus on player 1, the conditions for player 2 are analogous).

$$V_1^C(k_1, k_2, d) \geq u_1(k_1 \cdot e, k_2, d) + \delta \cdot V_1^N(k_1 \cdot e, k_2, \frac{\gamma}{1-\gamma}(k^1 \cdot e + k^2)) \tag{C1}$$

$$V_1^N(k_1, k_2, d) \geq u_1(k_1, k_2 \cdot e, d) + \delta \cdot V_1^N(k_1, k_2 \cdot e, \frac{\gamma}{1-\gamma}(k^1 + k^2 \cdot e)) \tag{C2}$$

Unfortunately, as a result of neglecting to include a cost function dependent on a pool's given level of capital, condition 2 is always fulfilled, meaning only the competitive strategy is an equilibrium. In order to remedy this problem, we can modify the original utility function as follows:

$$u_i(s_t, \mathbf{e}_t) = \frac{k_t^i \cdot e_t^i}{k_t^1 \cdot e_t^1 + k_t^2 \cdot e_t^2 + d_t} - c \cdot k_t^i \cdot e_t^i \tag{13}$$

This formulation captures the fact that investing in additional capital leads to higher costs (in the form of electricity needed to power greater computational resources) in all subsequent rounds and may serve as a deterrent from repeated investment.

Furthermore, we introduce a more sophisticated strategy profile that that involves the notion of a punishment phase. Under this strategy, which can be thought of as a variation of tit-for-tat, both players will initially behave collusively and avoid investing in additional levels of capital. Once one of the participants deviates from the strategy by investing, the other will initiate a punishment by investing in the subsequent round, effectively eliminating the advantage gained by the opposing pool.

In order for an equilibrium to exist, three conditions must be satisfied according to the one shot deviation principle: First, the utility derived from always colluding must be greater than or equal to the utility obtained by deviating once, initiating a punishment, and

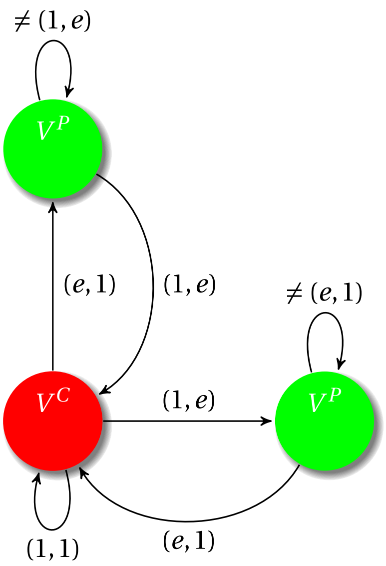


Figure 8: Automata representing the punishment strategy

Pools adhering to this strategy will initiate a punishment if their opponent deviates from collusion and invests in additional capital. As evidenced by the automata, if either player invests while in the collusive state (the red node), the players will transition to a state in which a punishment is enacted. If the deviator accepts the punishment without investing again, collusion will be resumed in the following round.

resuming the collusive strategy with the higher cost. Second, a pool that invests in a given round should find not deviating during the punishment phase must yield at least as much utility as deviating once again and incurring yet another punishment. And third, the player initiating the punishment must find the utility of following through with the punishment (thereby incurring a higher cost in the process) at least as advantageous as allowing this transgression to occur without consequence. These three conditions can be expressed through corresponding value functions.

Again, let $V^C(\mathbf{k}, d)$ denote the value function when the stock of capital is $\mathbf{k} = (k_1, k_2)$ and d represents the mining difficulty, players are using the collusive strategy profile, and no player has deviated. We define $V^P(\mathbf{k}, d)$ to represent the value associated with the punishment phase. Although the following value function is defined for player 1 and assumes player 2 adhered to the collusive strategy in the previous round (that is, player 1 is being punished), an analogous function could be provided for the second pool:

$$V_1^P(\mathbf{k}, d) = \frac{k^1}{k^1 + k^2 \cdot e + d \cdot (k^1 + k^2)} - c \cdot k^1 + \delta V_1^C(k^1, k^2 \cdot e, \frac{\gamma}{1-\gamma}(k^1 + k^2 \cdot e)) \tag{14}$$

If the collusive strategy is indeed a subgame perfect equilibrium, by the one shot deviation principle, the following inequality must hold:

$$V_1^C(k^1, k^2, d) \geq \frac{k^1 \cdot e}{e \cdot k^1 + k^2 + d} - c \cdot k^1 \cdot e + \delta V_1^P(k^1 \cdot e, k^2, \frac{\gamma}{1-\gamma}(k^1 \cdot e + k^2)) \tag{15}$$

This equation corresponds to the first condition outlined above and must clearly hold if constant collusion is an equilibrium.

In addition to the aforementioned condition, it must also be the case that a player being punished chooses to resume the collusive strategy profile rather than deviating for a second time. Assume that pool 2 deviated in the prior round and is in the process of being punished by pool 1. Again, the one shot deviation principle can be applied to yield the following inequality:

$$V_2^P(k^1, k^2 \cdot e, d) \geq \frac{k^2 \cdot e^2}{e \cdot (k^1 + k^2 \cdot e) + d} - c \cdot k^2 \cdot e^2 + \delta V_2^P(k^1 \cdot e, k^2 \cdot e^2, \frac{\gamma}{1-\gamma}(k^1 \cdot e + k^2 \cdot e^2)) \tag{16}$$

Finally, the player executing the punishment must have enough of an incentive to follow through with the investment, in spite of the higher cost they will then incur in each subsequent round. The value function presented represents the scenario in which player two deviated from the collusive strategy in the previous round and player one is responding.

$$\frac{k^1 \cdot e}{(k^1 + k^2) \cdot e + d} - c \cdot e \cdot k^1 + \delta V_1^C(k^1 \cdot e, k^2 \cdot e, \frac{\gamma}{1-\gamma} \cdot e(k^1 + k^2)) \geq V_1^C(k^1, k^2 \cdot e, d) \tag{17}$$

The aforementioned inequalities were then solved using Mathematica. Due to the complexity of the equations and large number of variables, we assumed e was equal to 1.5 and γ was .1. With these assumptions and other constraints imposed, the system yielded more favorable results than our previous endeavor. The collusive strategy (forgoing investment in all periods) became an equilibrium if certain numerical constraints held for the stock of capital of the pools, the discount factor (δ), and the cost constant c. While the values necessary for this equilibrium will not be presented as a result of the the high level of intricacy and breadth, the economic intuition and reasoning will.

A primary finding is that the pools must be relatively close in capital stock for collusion to hold as an equilibrium. Such a constraint is likely due to the multiplicative nature of the capital investment. Since a pool is able to scale their current inventory (rather than merely add a constant amount), the increase in capital is determined by the current level possessed. Therefore, if a pool has a large enough lead, the increased probability of successfully adding a block to the blockchain may outweigh the higher cost burden associated with this increased level of capital. Such a finding both affirms and disagrees with Horner's analysis of firms in competitive situations, in which he assets that effort is exerted either when a lead is large (to further dominate the opponent and force them out of the market) or a lead is small (to regain dominance) [5]. Although we found that collusion breaks down if a large lead exits, failure to invest happens when a lead is relatively small.

Additionally, the conditions are ripe for equilibrium when the discount factor is close to 1. Intuitively, this result (which suggests that pools value future profits nearly as much as current profits) coincides with economic theory, as pools with such preferences would be unwilling to sacrifice long-term rewards for short term gains.

Lastly, solving the system of equations yields another insight specific to the model. If a pool is guaranteed to punish the other for making an investment in additional capital, the cost constant c must be greater than zero–to prevent the competitive strategy from once again dominating– but still extremely small. This finding likely arises from the fact that a rival pool must be willing to follow through with the punishment, and a large increase in their costs would discourage such action. It should be noted that this conclusion is somewhat unrealistic, as a high enough cost could serve to prevent any rational actor from investing, and stems from the specification of the model.

As a final note, it be should observed that this formulation would be unable to support the competitive equilibrium that existed before the introduction of a cost function. Since perpetual investment in more capital would drive the cost of electricity to infinity, any advantage gained over one's opponent would be nullified by the immense expense required to keep servers running and hash functions computing. However, the model does suggest that under certain conditions–specific combinations of initial stocks of capital, the discount factor, and cost constant–collusion would break down and pools would have an incentive to invest at least once.

VI. Empirical Model

In order to test whether mining pools exhibit collusive behavior, our empirical analysis attempts to capture the direction of the peer effect between a pool and its competitors. In the context of this model, a positive effect would provide a more unambiguous indicator of collusion, as the pools would be increasing and decreasing their hash rates in conjunction with one another. In addition, the effect of increasing the duration of the round can convey evidence of collusion. If collusion were occurring, we would expect to witness pools reducing their hash rates ("slacking off," in a sense) on each additional day in the round in order to prolong the period and prevent a large spike in difficulty in the subsequent round.

To test this hypothesis, we can specify the following ordinary least-squares (OLS) equation that accounts for year, month, and day fixed effects in an effort to handle the heteroscedasticity present. This is represented by FE(d,m,y) in the regressions presented below. We assume standard assumptions on the error term: no colinearity between it and the regressors, normally distributed, and a mean of 0.

$$ln(hashrate)_{i,t,d} = \alpha + \beta_0 \cdot ln(hashrateothers)_{(-i),t,d} + \beta_1 \cdot ln(transfee)_{d-1} + FE(d,m,y)_{i,t,d} + \epsilon_{i,t,d} \tag{18}$$

In the context of the model, note the three different subscripts which may be present. An i subscript denotes a particular pool (antpool, f2pool, etc.) and applies only to the current hash rate. A t subscript is used to indicate the round in which the observation takes place while a d denotes the particular day. As such, the linear regression detailed above presents the hash rate of pool i during period t and on day t as a function of the hash rate of its peers on the same day and the transaction fee paid to all mining pools on the previous day.

We then expand the model to include lagged values from the prior two periods (while still incorporating fixed effects for year, month and day):

$$\begin{split} ln(hashrate)_{i,t,d} = \alpha + \beta_0 \cdot ln(aggother)_{(-i),t-1} + \beta_1 \cdot ln(aggother)_{(-i),t-2} + \beta_2 \cdot ln(transfee)_{d-1} + \\ FE(d,m,y)_{i,t,d} + \epsilon_{i,t,d} \end{split} \tag{19}$$

While the meaning of subscripts remains the same, note that aggother refers to the aggregate hash rate of other pools for the entire round.

We finally allow for a non-linear relationship between the hash rate of the a pool and the lagged rates of competitors, still taking into account fixed effects:

$$ln(hashrate)_{i,t,d} = \alpha + \beta_0 \cdot ln(aggother)_{(-i),t-1} + \beta_1 \cdot ln(aggother)^2_{(-i),t-1} + \beta_2 \cdot ln(aggother)^3_{(-i),t-1} + \beta_3 \cdot ln(transfee)_{d-1} + FE(d,m,y)_{i,t,d} + \epsilon_{i,t,d} \tag{20}$$

In this specification, we examine the linear, quadratic, and cubic effect of the aggregate hash rate in the prior round.

VII. Data

The data I will be using in my analysis comes primarily from two websites which aggregate data from the official blockchain: blockchain.info and bitcoinity.org. The first site contains data on the following metrics: the mining difficulty, the number and value of transactions made with Bitcoin, the total transaction fees distributed to all pools, and an aggregate measure of the hash rate. In this data set, the mining difficulty is represented as an integer where a higher value corresponds to higher difficulty in the traditional sense (the exertion of more work is required). The second site provides data on the hash rate of individual pools and unknown entities. I have since merged the data sets and have daily observations for each metric over a two year period (September 2014 - September 2016). I have also computed the duration into each round for each observation and generated such a variable in the data set.

In order to incorporate lagged values in my empirical model, I have also computed these values for certain metrics. While this proved trivial for the total transaction fee, as a simple Stata command was able to generate lags for the prior two rounds, calculating lags for the previous aggregate hash rates was a more difficult endeavor. In order to accomplish this task, a change in the difficulty between two consecutive days was used to identify the end of one period and the beginning of the subsequent round. An enumeration of the rounds allowed for the summing of all values within that period and enabled lags to be programmatically computed.

As our analysis is primarily focused on capturing peer effects and interpreting directional causality, we chose to use logarithm values instead of raw values. This transformation was applied to both hash rates and the transaction fee. The inclusion of log values allows regression coefficients to be interpreted as a percent change instead of unit change.

The data was then organized as a panel data set with observations reported for each of the 10 separate entities present in the data. Although each pool has 726 observations, many experience missing values either due to coming into existence in the midst of the data set or other, unknown reasons. Summary statistics for relevant variables have been provided below. Note that the total value of the transaction fee is the same for each pool on a given day and the difficulty is constant for all over the round. On the other hand, each pool has a unique hash rate for each day. This explains the discrepancy between the number of observations for the variables in Table 1.

Table 1: Summary Statistics of Data

Variable Name	Unique Observations	Mean	Std. dev.	Min	Max
Hash Rate (Billions/second)	6877	73710.740	82082.110	1729.892	542083.700
Transaction Fee (USD)	726	13421.950	13484.160	2429.127	158063.200
Mining Difficulty (in billions)	53	94.241	64.536	27.800	220.756

Summary statistics of three primary variables: the hash rate of each individual pool (measured in billions of hashes per second), the total transaction fee received by all pools, and the mining difficulty as determined by the Bitcoin mining algorithm. Data spans the two year period between September 2014 and September 2016.

The absence of additional data (as observed by the small number of rows in the above table) signifies one of the primary limitations of the empirical analysis. With the composition of mining pools remaining elusive, we were unable to acquire data on factors specific to individual pools (number of members, amount of computational resources, etc). Moreover, although the day, month, and year fixed effects included in our empirical model attempt to compensate for variations over time, an accurate measure of available computational potential and innovation would have useful to include. The sparsity of the previously specified models is thus necessitated by the availability (or lack thereof) of data and may impact–through left out variable bias–the results presented in the following section.

VIII. Results

The results we discovered were, at least initially, surprising and unintuitive. Rather than suggesting that pools are engaging in clear signs of collusion (adjusting their hash rates in the same direction) or are operating completely independently of one another (in which case we would expect insignificant peer effects), the regressions found a negative relationship between the hash rate of a particular pool and that of its competitors. In the tables that follow, we present the estimated coefficients and t statistics of the regressors, as well as the same information for the day fixed effects (since these values were also being analyzed to identify collusion). All regressions were performed using Stata, a statistical software program. Asterisks have been included to denote standard levels of confidence in statistical significant. A single asterisk signifies significance at a confidence level of 95%, two at 99%, and three at 99.9%.

We first ran the regression specified by equation 18, in which the hash rate of a particular pool is modeled as a function of the hash rate of peers and the total transaction fee earned from the prior day.

As is evident by the above results, both the hash rate of competing pools and the previous days aggregate transaction fee are statistically significant (at the 95% confidence level). The results suggest the following: a particular pool experiences a 5.01% reduction in their hash rate for each 1% increase in the hash rate of its peers and a .64% increase for a 1% increase in the prior days total transaction fee. Additionally, there is a significantly negative day fixed effect for 7 out of the possible 14 values.

Table 2: Fixed Effects Without Lagged Hash Rates
Observations: 6869

Variable Name	Coefficient	T Statistic	Standard Error
$ln(hashrateothers)_{t,d}$	-2.957	-4.810^{***}	.614
$ln(transfee)_{t-1}$	.418	3.880^{***}	.108
Day=2	-.062	-1.350	.018
Day=3	-.081	-1.654	.028
Day=4	-.054	-1.193	.021
Day=5	-.010	-.292	.020
Day=6	-.075	-1.674	.026
Day=7	.083	-1.711	.017
Day=8	-.174	-3.735^{***}	.022
Day=9	-.13	-2.88^{**}	.041
Day=10	-.115	-2.311^{*}	.025
Day=11	-.174	-3.688^{***}	.025
Day=12	-.112	-2.482^{*}	.023
Day=13	-.146	-3.020^{**}	.024
Day=14	-.262	-4.681^{***}	.040
Day=15	-.096	-.750	.091

Results of running the regression specified in equation 18. The natural log of the hashrate was included as the dependent variable. The hashrate of others and lag of the total transaction fee were independent variables. The day variables included in the table represent fixed effects.

We next performed the regression specified by equation 19. Recall that this modeled the hash rate of a particular pool as a function of the lagged aggregate hash rates in the the prior two rounds and the transaction fee paid out in the previous period.

The results in Table 2 are consistent with the findings of Table 1, although the magnitude of estimated effects vary. According to the estimated coefficients, a 1% increase in the prior rounds aggregate hash rate results in a 1.18% reduction in a pools current hash rate and a 1% increase in the transaction fee causes a .27% increase in the hash rate. Six out of the possible fourteen day fixed effects are significant. Note that the fixed effect for day 15 is positive; likely explained by the fact that once the 14 day threshold has been crossed, the mining difficulty will diminish–thereby reducing the incentive to prolong the current period. Also, observe that the aggregate hash rate from two rounds prior is insignificant, suggesting that pools

Table 3: Fixed Effects With Lagged Hash Rates
Observations: 6869

Variable Name	Coefficient	T Statistic	Standard Error
$ln(agghash)_{t-1}$	-1.186	-2.780^{**}	.426
$ln(agghash)_{t-2}$	-.523	-1.300	.403
$ln(transfee)_{t-1}$	.287	2.650^{**}	.108
Day=2	-.039	-2.260^{*}	.017
Day=3	-.043	-1.580	.027
Day=4	-.032	-1.560	.032
Day=5	.008	.360	.021
Day=6	-.041	-1.670	.024
Day=7	-.017	-1.080	.015
Day=8	-.085	-4.290^{***}	.020
Day=9	-.063	-1.670	.038
Day=10	-.038	-1.560	.024
Day=11	-.089	-4.140^{***}	.021
Day=12	-.024	-1.040	.023
Day=13	-.047	-2.100^{*}	.022
Day=14	-.053	-1.860	.029
Day=15	.198	2.060^{*}	.096

Results of the regression specified in equation 19. The natural log of the hashrate is the dependent variable. Two lags of the aggregate hashrate and a single lag of the transaction fee serve as the regressors. The day values specified correspond to fixed effects.

predominately base their decisions on only the previous period.

Finally, we present the results of the equation 20, in which non-linear lag values of the prior aggregate hash rate are included. Note that the magnitude of the coefficients for these lags has no valid interpretation and the sign (positive or negative) is the only interpretable insight offered. The results presented in this table suggest that the relationship between the prior aggregate hash rate and the hash rate of given pool is indeed governed by a non-linear relationship. While the simple linear term retains a negative coefficient, the squared lag has a positive coefficient and the cubic's is negative. All three terms are significant. Such a finding suggests that the slope of the curve will vary at differing levels of peer hash rates and renders incorporating only a linear relationship inadequate. The prior days value of transaction fees continues to have a positive relationship with the hash rate and 8 days have significantly

Table 4: Fixed Effects With Non-linear Lags
Observations: 6621

Variable Name	Coefficient	T Statistic	Standard Error
$ln(agghash)_{t-1}$	-591.878	-2.670^{**}	221.633
$ln(agghash)^2_{t-1}$	43.252	2.610^{**}	16.555
$ln(agghash)^3_{t-1}$	-1.056	-2.560^{*}	.412
$ln(transfee)_{t-1}$	.141	1.880	.075
Day=2	-.043	-2.450^{*}	.018
Day=3	-.049	-1.730	.028
Day=4	-.053	-2.260^{*}	.233
Day=5	-.017	-.750	.022
Day=6	-.055	-2.150^{*}	.026
Day=7	-.030	-1.680	.018
Day=8	-.101	-4.600^{***}	.022
Day=9	-.085	-1.940	.044
Day=10	-.070	-2.290^{*}	.030
Day=11	-.114	-3.950^{***}	.029
Day=12	-.053	-2.040^{*}	.026
Day=13	-.088	-3.030^{**}	.029
Day=14	-.058	-1.940	.030
Day=15	.140	1.650	.085

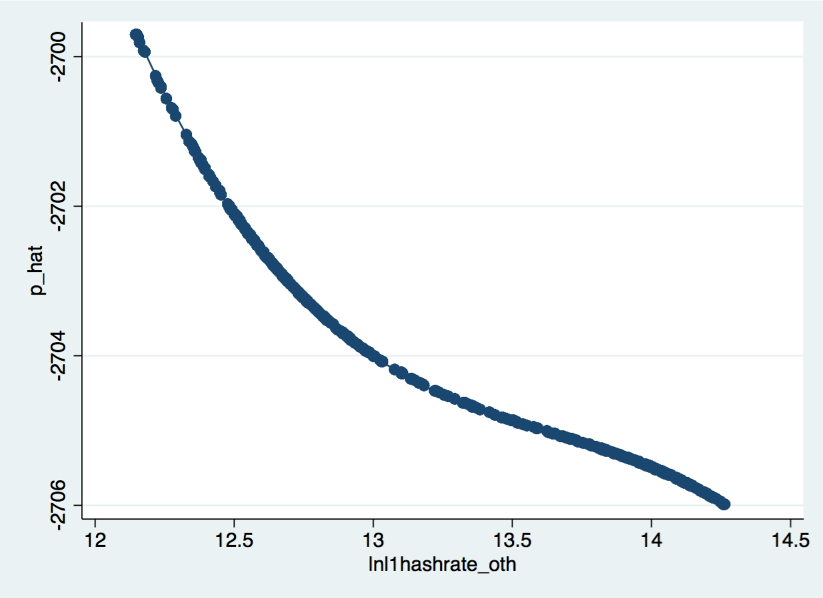
Results of the regression specified in equation 20. Natural log of the hash rate is the dependent variable while linear, quadratic, and cubic functions of hash rate lags and the transaction fee are the regressors.

negative fixed effects.

In totality, the regression results yielded surprising and somewhat contradictory indications. While the day fixed effects would serve as evidence that pools are behaving in a collusive manner (since they appear to reduce their hash rate as the round proceeds), the negative relationship between a particular pools hash rate and that of its competitors suggests the opposite is true. Though we can't dismiss the possibility that the benefits of collusion are being ignored in favor of ruthless competition, we now propose two alternate explanations that would explain the reported results.

Firstly, there exists the possibility that pools are engaging in collusion through the use of a more subtle strategy than was expected. Instead of increasing and decreasing their hash rates

Figure 9: Non-Linear Relationship between pool's hash rate and that of competitors



According to the regression results, the relationship between the hash rate of a particular pool and its competitors is characterized by a non-linear relationship. This suggests the effect of competitors altering their hash rates on a given pool's hash rate will vary depending on the current level of the aggregate hash rate.

in unison, the outcome could imply pools are effectively "trading off" profitable rounds. In such a scenario, other pulls would reduce their hash rates to allow one (or perhaps a subset of all pools) to experience a more prosperous round. Once a given round ends, another pool would then swap into the desired role. While the implementation of this strategy would likely require more explicit communication between pools, it would explain the negative relationship discovered in the data.

On the other hand, one could also argue that the relationship is a result of individual miners often transitioning between pools. Such an explanation would account for the negative relationship between the hash rate of a pool and its peers (since the increase in competitors's hash rates would be facilitated, in part, by miners abandoning their current pool). This fluidity could be motivated by a few different factors. First off, miners could be seeking to maximize their individual monetary rewards by joining a pool where their contribution would be greater (pools share profits according to the share of computational power provided). Another possibility involves the self-regulation that has been described previously. If a pool were to acquire a 51% market share, the entire blockchain would effectively be compromised since the pool could add erroneous blocks and edit the history of the ledger. As this is of grave concern to miners, they could intentionally be leaving a pool that becomes too concentrated. However, this explanation warrants a caveat. Since the most concentrated pool currently accounts for roughly a fifth of all hashing, reaching the 51% level is not a realistic feat for any pool in the near future.

As a final note, observe the final explanation, and the results in general, suggest there are constraints on the total amount of hashing that can occur in the market since a tradeoff appears to occur between competing pools. While one could, in theory, argue in favor of virtually unconstrained conditions–since pools have the ability to acquire a greater amount of computational resources and continually update their existing material–such an assertion is unrealistic. As was expressed in the theoretical model, adding to the stock of computational power will increase costs in the form of electricity bills and eventually render the mining activity profitable. Further constraints may include the slowing introduction of new miners (since profits are heavily dispersed throughout pools) and diminishing computational innovation realized in recent years.

IX. Conclusion

Overall, our analysis of the existence of collusion among mining pools has yielded interesting results that lay the groundwork for future work. Since the topic of Bitcoin has largely been ignored by academic economists, both the theoretical and empirical models presented can serve as the foundation of further research. While both models suffer from flaws, additional refinement could be made to expand the comprehensiveness and accuracy in capturing peer effects.

The theoretical model we constructed incorporated game theoretic elements in order to capture the interdependent relationship between pools. In the repeated game structure defined, we allowed the level of capital of a given pool to be directly related to its ability to successful include a block in the blockchain. By defining corresponding utility and value functions, we were able to compute closed form solutions for the utility obtained over the entire mining duration. These expressions then allowed us to determine which strategies served as subgame perfect Nash equilibriums through the utilization of the one-shot deviation principle. While this simple model proved sufficient for our purposes, it is worth noting the shortcomings and aspects in need of improvement. Further work could seek to the add the following enhancements: allow for an arbitrary number of pools, account for additional factors (transaction fees, desire to avoid too much concentration, etc.), along with a variety of others.

Moreover, the empirical model and statistical results presented are among the first (if not the first) to analyze the hash rates of pool and attempt to capture peer effects. While we were surprised by the negative relationship discovered in the data, this could be the result of self-regulation or a more complex form of collusion. Further work could expand on the model and apply more advanced econometric techniques to develop a more comprehensive understanding of the factors influencing the hashing of mining pools.

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