A STUDY OF TEACHER TRANSITIONS TO A REFORM-BASED MATHEMATICS CURRICULUM IN AN URBAN SCHOOL: THE INTERACTION OF BELIEFS, KNOWLEDGE, AND CLASSROOM PRACTICES

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ABSTRACT

Wendy S. Bray: A Study of Teacher Transitions to a Reform-Based Mathematics Curriculum in an Urban School: The Interaction of Beliefs, Knowledge, and Classroom Practices
(Under the direction of Mary Ruth Coleman)

This collective case study examines how four third-grade teachers' beliefs and knowledge influenced their ways of supporting and limiting student thinking in their first year using a reform-based mathematics curriculum at an urban school. Of focus is the role teachers' beliefs and knowledge play in supporting and limiting student thinking when student difficulties arise during instruction on multiplication and division. Situated in the growing body of research associated with current reforms in mathematics education, this study is also informed by general education research on urban schools, teacher beliefs, teacher knowledge, and teacher change.

Data sources for case studies on individual teachers include classroom observations, pre-/post-observation interviews, beginning/end-of-year measures of teacher beliefs and knowledge, records of an on-going mathematics professional development project, and student achievement data. Each case study describes teacher's beliefs and knowledge at the beginning and end of the year, presents a case story illuminating the teacher's patterns of response to student difficulties and their relationship to the teacher's beliefs and knowledge, and summarizes data from global measures of teaching. In addition to development of multiple case studies, a simultaneous cross-case analysis was undertaken to illuminate patterns across cases and increase the potential for generalizing beyond the particular cases.

Findings from this study suggest that some aspects of reform-oriented mathematics instruction are more readily adopted than others. While beliefs and knowledge both appear to influence teacher response to student difficulties, certain aspects of instruction seem more greatly influenced by teacher beliefs while others appear more greatly influenced by teacher knowledge. In addition, evidence suggests that teachers' differential classroom experiences during initial use of reform-based mathematics curriculum were related to the degree to which teachers' evolving beliefs and knowledge moved closer to alignment with reform-based mathematics practices. Finally, the urban context of this study was found to influence teachers' transitions to reform-based mathematics teaching practices in a variety of ways.

Study findings have several implications for efforts to support teachers' transitions to reform-based mathematics programs and practices within and outside of urban school settings. These are discussed along with directions for future research.

To my parents,
for always believing in me

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CHAPTER I

INTRODUCTION

Overview of Study

The reform agenda in mathematics education, exemplified by the National Council of Teachers of Mathematics (NCTM) standards documents (1989, 2000), calls for significant changes in the ways teachers orchestrate mathematics learning for students. The NCTM standards contend that elementary school mathematics should emphasize problem-solving, reasoning, and communicating mathematical ideas, which places student thinking and conceptual understanding at the center of the instructional agenda. Teachers are challenged to develop discourse-rich mathematics learning communities where students devise, share, and analyze multiple solution strategies in response to rich mathematics tasks. The new forms of teaching and learning required to meet these standards place significant demands on teachers. Not only are new kinds of mathematics and pedagogical knowledge required (Ball, Lubienski, & Mewborn, 2001; Floden, 1997), but also fundamental changes in beliefs about mathematics and mathematics teaching (Franke, Fennema, & Carpenter, 1997; Thompson, 1992). As a growing number of schools and school districts adopt practices and policies that aim to reflect reform in mathematics education (Pugach & Warger, 1996), many teachers and classrooms are being left behind. The reasons why are as complex as teaching. But they center on the dramatic changes to teachers' mathematics instructional practices envisioned by these policies.

Research on teacher change suggests that teachers' beliefs about and knowledge of subject-matter, teaching, and learning influence the ways they revise their teaching practices in response to reform recommendations (Richardson & Placier, 2001; Spillane & Jennings, 1997). If a teacher's beliefs are in opposition to the pedagogical emphasis of mathematics reform or if teachers do not have adequate knowledge to support reform-based mathematics teaching practices, then the success of policy initiatives will be limited. This has strong implications for the ways reform policies are implemented, particularly at schools serving high numbers of children from poverty where teachers traditionally rely on controlled classroom environments to build basic skills (Anyon, 1981; Knapp, 1995a). Even though the exclusive focus on basic skills is inconsistent with the vision promoted by NCTM, many urban school teachers believe strongly that students' needs demand it. A limited number of professional development projects have been successful in helping urban school teachers to reform their teaching practices with positive outcomes for students (Campbell, 1996; Fuson, Smith, & Lo Cicero, 1997; Hufferd-Ackles, Fuson, & Sherin, 2004; Knapp, 1995b). The question is what more can be done to support teacher transition to a reform-based mathematics curriculum at urban schools as well as to facilitate the transition for all teachers.

Recognizing that teachers undergo change in significantly different ways, researchers have begun to examine relationships between teachers' changing beliefs, knowledge, and classroom practice (Fennema & Franke, 1992; Franke et al., 1997; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998; Richardson & Placier, 2001; Wood, Cobb, & Yackel, 1991; Yackel, Cobb, & Wood, 1998). Based on this small body of research, the relationship among beliefs, knowledge, and practice appears to be an interactive one in which teacher trajectories of change take a variety of paths. Research is needed to explore the interactive nature of this relationship as well as the developmental patterns between all three. Such research could inform efforts to help teachers that differ in their beliefs, knowledge and practice become comfortable following the path of reform-based mathematics in any school setting including challenging urban schools.

Purpose of Study and Research Questions

The purpose of this study is to examine how urban school teachers' beliefs and knowledge influence the ways they support and limit student thinking during their first year using a reform-based mathematics curriculum. The school identified for this study is located in an urban setting and serves a student population characterized by high-poverty and limited English proficiency. By carefully studying a few urban school teachers during a time of change, this study aims to contribute to the growing knowledge of how teachers can be supported in their efforts to transition to reform-oriented mathematics pedagogy, especially in similar urban school settings.

The following broad questions guided this study:

1. In what ways and to what extent do teachers support and limit student thinking during mathematics instruction in their first year implementing a reform-based mathematics curriculum?
2. How do teachers' beliefs influence the ways they incorporate student thinking into their planning for mathematics instruction and on-the-spot instructional decision-making in their first year of implementing a reform-based mathematics curriculum?

3. How does teacher knowledge of mathematics for teaching influence the ways teachers incorporate student thinking in their first year of implementing a reform-based mathematics curriculum?
4. How does the urban context, as defined by the research literature and perceived by teachers and school leaders, influence mathematics instruction in this urban school?

These questions are addressed using data from a collective case study of four third-grade urban school teachers during their first year implementing a reform-based mathematics program.

Focus on Teacher Response to Student Difficulties during Problem Solving

To limit the scope of this study, the research questions posed are examined by focusing on the role teachers' knowledge and beliefs play in supporting and limiting student thinking when student difficulties arise in the context of lessons focused on problem solving. Teaching through problem solving is a central tenant of mathematics reform (NCTM, 2000) and requires a major shift in pedagogical thinking for many teachers (Franke, Fennema, & Carpenter, 1997). Therefore, it was reasoned that a focus on lessons emphasizing problem solving would offer optimal opportunity to study teacher transitions to reform-oriented mathematics pedagogy. The study of teachers' response to student difficulties provides an informative lens for insight into the influence of teachers' knowledge and beliefs because, when teachers encounter students' difficulties, on-the-spot decisions must be made and action taken. Teachers necessarily rely on their existing knowledge and beliefs to decide how to act. Furthermore, since urban schools often serve a high number of students with special academic and behavioral needs, a high incidence of student difficulty in these settings is likely. Therefore, study of teachers' response to student difficulties is especially relevant to urban schools.

Review of the Literature

This study is informed by the growing body of research associated with current reforms in mathematics education as well as research on urban schools, teacher beliefs, and teacher knowledge. This section will begin by describing the vision of mathematics teaching and learning promoted by mathematics-reform with special attention to the role of student thinking. Then the challenges teachers face in revising their classroom practices to reflect this vision will be discussed, followed by a review of additional challenges associated with urban schools. Next, a review of research on teachers' knowledge and beliefs will be provided with focus on their relationship to mathematics teaching practices. Within this discussion, knowledge for teaching multiplication and division will be described, as these are the mathematics topics of focus during classroom observations for this study. Finally, research on teacher change to reform-based mathematics pedagogy will be reviewed.

Mathematics Reform and Student Thinking

Beginning in the late 1980s, the National Council of Teachers of Mathematics (NCTM) proposed and advocated for dramatic changes in the way mathematics instruction is organized and taught in K-12 education. In critique of longstanding traditional practices in the elementary mathematics classroom, NCTM made the following statement in their 1989 publication *Curriculum and Evaluation Standards for School Mathematics*:

> The need for curricular reform in K-4 mathematics is clear. Such reform must address both the content and emphasis of the curriculum as well as approaches to instruction. A long-standing preoccupation with computations and other traditional skills has dominated both *what* mathematics is taught and *the way* mathematics is taught at this level. As a result, the present K-4 curriculum is narrow in scope; fails to foster mathematical insight, reasoning, and problem solving; and emphasizes rote activities. Even more significant is that children begin to lose their belief that learning mathematics is a sense-making experience. They become passive receivers of rules and procedures rather than active participants in creating knowledge (p.15).

In this initial standards document and in a more recent document titled *Principles and Standards for School Mathematics* (2000), NCTM advocates for mathematics curriculum and instruction that emphasizes the interwoven development of conceptual understanding and procedural knowledge. This approach places problem solving and reasoning at the center of the mathematics classroom experience. It also suggests significantly different roles for students and teachers.

Reflecting constructivist views of learning, advocates of reform-based mathematics instruction believe that children actively construct increasingly organized structures of knowledge and personal understanding by reflecting on and reasoning about experiences in relation to their prior knowledge and immediate contexts (von Glasersfeld, 1996). Therefore, students' differential realities and existing knowledge constructions become the starting point for conceptually-focused instruction (Wood, Cobb, & Yackel, 1995). Children develop increasingly sophisticated understandings by constructing relationships among mathematical ideas, extending and applying mathematical knowledge, reflecting on experiences, articulating and defending their thinking, and by making mathematical knowledge their own (Carpenter & Lehrer, 1999).

In order to emphasize conceptual understanding and build on student thinking, teachers must create learning environments that require significant departure from traditional mathematics instructional practices. In general, such classrooms are places where, "…students are encouraged to be curious about mathematical ideas, where they can develop their mathematical intuition and analytical capabilities, where they learn to talk about and with mathematical expertise" (Franke, Kazemi, & Battey, 2007). Several research efforts have illuminated classroom efforts to make students' mathematical thinking a central feature of instruction (Fraivillig, Murphy, & Fuson, 1999; Franke et al., 1997; Hiebert et al., 1997; Kazemi & Stipek, 2001; Stigler, Fernandez, & Yoshida, 1996).

In all classrooms, teachers are the gatekeepers to the kinds of tasks and activities students encounter. Tasks that support student thinking are intentionally selected or designed with specific mathematical goals in mind, and they are viewed as opportunities for students to grapple with mathematics as problem solvers, not rule-followers (Hiebert et al., 1997). In reform-oriented mathematics classrooms, tasks are often situated in real-world contexts that allow students to use their existing knowledge to explore mathematical ideas before they are formally introduced (Carpenter, Fennema, & Franke, 1996). In these classrooms, teachers support student thinking by providing ample opportunity for students to solve mathematics-rich tasks in their own ways (Franke et al., 1997; Stigler et al., 1996) with access to a variety of tools and resources to support mathematical thinking (Hiebert et al., 1997).

In mathematics classrooms that reflect reform principles, teachers are also responsible for establishing a classroom culture and norms in which mathematics instruction builds on students' mathematical thinking (Franke et al., 2007). The social culture of the classroom is such that a variety of mathematical ideas and methods are valued, mistakes are treated as opportunities for learning, and correctness resides in the mathematical argument instead of with the teacher or text (Hiebert et al., 1997; Stipek et al., 1998). Classroom norms emphasize discussion, collaboration, and negotiation as ways of fostering shared meaning among a community of learners (Cobb, Boufi, McClain, & Whitenack, 1997; Gergen, 1995). Students are expected to seek and understand relations among multiple ways of solving mathematical problems, and the classroom discourse is consistently focused on this goal. Students' flawed solution strategies are readily incorporated into class discussion in order to explore the contradictions in student solutions and provide greater insight into the mathematics of focus (Kazemi & Stipek, 2001).

Sociomathematical norms have been introduced as important mathematics-specific norms that constitute what counts as mathematical thinking in the classroom (Yackel & Cobb, 1996). Kazemi and Stipek (2001) have identified a pattern of sociomathematical norms in classrooms that are associated with a high level of mathematical understanding. In these classrooms, it is expected that students explain how they solve problems by providing mathematical arguments along with procedural descriptions. Students are expected to seek and understand relations among multiple strategies, and the classroom discourse is consistently focused on this goal. When students work with a partner or a group, all students are individually accountable for understanding, and they are expected to reach consensus through mathematical argumentation. Finally, when solutions are presented that contain errors, the sociomathematical norm of classroom practice is to explore the contradictions in student solutions and use the error as an opportunity to rethink the problem.

In a study of the relationship between normative patterns of social interaction and children's mathematical thinking, Wood, Williams, and McNeal (2006) found that higher levels of student mathematical thinking are evident in reform-oriented classrooms in which classroom discourse patterns are characterized by an inquiry/argument classroom culture compared to a strategy reporting classroom culture. In both kinds of classrooms, students solve problems in a variety of ways and significant instructional time is dedicated to class discussion of students' varying mathematical solutions. In the strategy reporting classroom culture, however, interactions during class discussions were primarily limited to interaction between the individual students sharing and the teacher, limiting the involvement and learning of the rest of the class. In contrast, the inquiry/argument classroom culture involved class members in asking questions and making judgments about the reasonableness of a method when their peers are presenting mathematical solutions. This greater level of minds-on participation in the classroom discourse led to more sophisticated student thinking. This study illustrates the importance of the teacher's role in facilitating classroom discourse in ways that simultaneously support the learning of students who are sharing their mathematical ideas as well as the other students in the class (Franke et al., 2007).

Other research studies have found positive correlations between factors associated with an instructional emphasis on student thinking and more general measures of mathematics student achievement (Franke et al., 2007). Fennema and her colleagues (1996) conducted a three-year study of first-grade teachers who were attempting to develop teaching practices intentionally grounded in knowledge of student thinking. This study concluded that instructional practices most associated with higher student achievement include a focus on problem-solving, ample opportunity for students to engage in conversation about mathematical ideas, and efforts to adapt instruction to the problem-solving levels of students. Similarly, Gearhart (1999) found that better student achievement outcomes were associated with problem-rich instruction based on students' ways of thinking in which students are encouraged to see the underlying connections among mathematical concepts and symbols.

Clearly, instruction focused on student thinking changes the nature of the teacher's role in the classroom. Fraivillig, Murphy, and Fuson (1999) have devised a pedagogical framework elaborating the teacher's role in advancing children's mathematical thinking through classroom interaction. The three overlapping components comprised in this framework include the teacher's role in eliciting children's solution methods, supporting children's conceptual understanding, and extending children's mathematical thinking. The first component, eliciting children's solution methods, focuses on the strategies teachers use to give students opportunity and encouragement to express their mathematical thinking. Other research has also highlighted the importance of teachers eliciting student thinking in informal and group interactions (Franke et al., 1997; Stigler et al., 1996). Strategies used to elicit student thinking provide a window for the teacher into understanding how children are thinking about mathematics. By skillfully orchestrating class discussions focused on students' many ways to solve a problem, the teacher can simultaneously facilitate group learning and engage in assessment of the thinking of individuals and the group.

The second component of the Fraivillig et al. (1999) framework for advancing children's thinking is supporting children's conceptual understanding within the context of social interaction between the teacher and an individual student or in the context of group discussion. This component focuses on how the teacher might support children in devising and understanding solution methods that are within their current cognitive capabilities without telling them exactly what to do. Strategies that fall within this component aim to support the individual problem solver as well as other students who are listening and learning from a discussion centered on multiple strategies. This distinction highlights the multiple roles the teachers play in supporting student thinking in the classroom setting. A teacher might record students' solution methods on the board as a way of scaffolding the problem solver explaining her solution while at the same time aiming to support the other students who are trying to make sense of the solution. If the problem solver hits a sticking point, the teacher may ask the class to help the student work through it. This action primarily supports the thinking of the student describing a solution, but it also supports the other students in maintaining engagement and considering the problem from the perspective of the given solution.

While the eliciting and supporting components of the Fraivillig et al. (1999) framework elaborate how teachers might facilitate children's understanding of familiar solution methods, the final component focuses on strategies that aim to challenge and extend children's current mathematical thinking. Strategies that aim to extend children's thinking encourage students to press beyond their initial solutions to understand alternative solutions. Students are encouraged to analyze and reflect on patterns, draw generalizations across various student strategies, and to consider relationships among mathematical concepts. This component also includes some strategies that aim to influence students' dispositions toward mathematics, including fostering perseverance and a love of challenge.

Although the vision promoted by the NCTM standards documents and supported by over two decades of research has pointed toward reform that places student thinking at the center of the mathematics classroom experience, the widespread realization of this vision in classrooms across the United States has yet to occur (Ball, 2001). In the next section, I will explore factors that influence the way teachers respond to mathematics-reformers calls for change, especially change related to the role of student thinking.

The Challenge of Reforming Mathematics Teaching

National reform efforts in mathematics education have had widespread impact on general education curriculum policy, with most states revising state curriculum documents to align with the NCTM standards (Pugach & Warger, 1996). Yet many teachers, the front-line implementers of reform-ideas, have not made the dramatic changes to their mathematics instructional practices envisioned by these policies (Spillane & Jennings, 1997). Some would say that this is because teachers are resistant to change (McLaughlin, 1987). However, an alternative view that is increasingly accepted is that teaching is a more complex activity than has been historically acknowledged (Ernest, 1989; Richardson & Placier, 2001). This seems especially true of the current mathematics-reform efforts that call for teachers to emphasize student thinking.

There have been several publications of teacher narratives and university researcher's first-person accounts of attempts to teach in ways consistent with the NCTM standards that highlight the complexity and tension inherent in this work (Ball, 1996, 2001; Heaton, 2000; Lampert, 2001; Schifter, 1996; Schifter & Fosnot, 1993). Challenges that are frequently cited include the difficulty in figuring out what children really understand and trying to determine how to best follow-up on student ideas. When instruction places student thinking at the center, teachers cannot know ahead of time exactly how a lesson will unfold. This kind of teaching presents teachers with the constant challenge of working "on-the-fly" to understand student thinking and to make decisions about which ideas to move forward with and which to leave alone. Student thinking itself is a dynamic phenomena and can never be known for certain.

Sherin (2002a) compares facilitating classroom discussion of student ideas to a balancing act. She describes the constant tension between attempting to support student-centered processes of classroom discourse while, at the same time, facilitating discussion of significant mathematics content. These kinds of discussions also challenge teachers to facilitate classroom discourse such that *students* do the majority of the intellectual work of unpacking mathematical ideas such that they are comprehensible to classmates (Franke et al., 2007). For the teacher, this involves cultivating questioning techniques that focus students' attention on important mathematical ideas while at the same time being careful to avoid funneling the conversation such that the teacher takes on the majority of the intellectual work (Wood, 1998). At the same time, the teacher is responsible for facilitating class discussions such that all students participate in active and productive ways (Williams & Baxter, 1996). These tensions make teaching mathematics more like improvisation than a choreographed dance (Heaton, 2000).

Teachers accustomed to deriving personal teaching efficacy from successful implementation of well-prepared, teacher-centered presentations must be willing to embrace the uncertainty inherent in teaching practices aligned with mathematics-reform (Cooney & Shealy, 1997; Smith, 1996). Furthermore, as teachers transform their classrooms into discourse communities, there are greater opportunities for students to reveal their understandings and misunderstandings. This circumstance makes it likely that teachers will become more aware of what students do not fully understand (Ball, 1996).

Reform-oriented teaching relies on teachers holding the knowledge, skills, and dispositions that support such teaching (Sowder, 2007; Spillane & Zeuli, 1999). In order to interpret and build on student thinking, teachers need strong and flexible knowledge of school mathematics, knowledge of how students learn mathematics, and extensive pedagogical knowledge (Ball et al., 2001). In discussing findings from a study of teachers as they begin to explore multiple solutions for mathematics problems with their students, Silver, Ghousseini, Gosen, Charalambous, and Strawhun (2005) make the following observation:

> If teachers lack a sound, flexible knowledge of mathematics and of children's thinking, they may be more inclined toward managing multiple solutions through a ritualized "show-and-tell" practice, which allows them to avoid the complexity of choosing certain solutions, arranging them in a sequence, and connecting them to extract and highlight important mathematical points (p.298).

These researchers found that teachers became more willing and capable of productively incorporating multiple student-generated solutions into their mathematics instruction as they became increasingly aware of particular pedagogical techniques that could be used to make discussion of student solutions work toward their instructional goals. In addition to needing a robust knowledge base for reform-oriented mathematics teaching, teachers must hold beliefs that support an orientation toward mathematics instruction aligned with mathematics-reform.

Thompson, Phillip, Thompson, and Boyd (1994) draw distinctions between how teachers with a calculational orientation and a conceptual orientation engage students in classroom discourse around student-generated mathematical solutions. When working with contextualized tasks, teachers with a conceptual orientation continually focus students on making sense of numbers in relation to problem contexts and mathematical ideas. In contrast, teachers with a calculational orientation tend to focus on having students report procedures for getting answers without pressing them to justify their procedures in relation to problem contexts. While the teachers studied were implementing specific practices associated with reform-oriented mathematics instruction by engaging students in discussion around student-generated solutions, only the teachers with a conceptual orientation were doing so in a manner consistent with the principles of reform. It is important for teachers to hold beliefs that support an orientation toward mathematics instruction aligned with mathematics-reform in these types of situations, but it becomes essential when teachers must defend the new ways of teaching.

Teachers who engage in reform-based mathematics practices often find themselves having to explain these new ways of teaching to administrators, parents, and students who are accustomed to different kinds of mathematics instruction (Ball, 1996). Cooney (1985) describes a first-year teacher who began the school year with a clear vision of a new way to teach mathematics, but resistance from students and pressures to cover the curriculum convinced him to retreat to more traditional practices. Silver et al. (2005) have also found many teachers are resistant to incorporating particular recommendations of reform due to time constraints and perceived conflict between implementing reform-based practices and being able to cover the curriculum.

In all schools, teachers attempting to adopt reform-based mathematics pedagogy face an array of challenges. Teachers in urban schools face additional challenges. Next, I will briefly review challenges associated with urbanicity.

The Challenge of Urbanicity

Obiakor and Beachum (2005) assert that urban areas have certain characteristics with implications for education: They are densely populated, increasingly populated by ethnically diverse people, and negatively affected by poverty. Drawing on data from several national surveys, Lippman, Burns, and McArthur (1996) report large differences between urban and non-urban schools on a variety of indicators related to student background, school experiences, and student outcomes. Trends indicate that the proportion of urban school students who are living in poverty or have difficulty speaking English is on the rise. Urban students are more likely to be exposed to safety and health risks, and they are less likely to have the family structure, economic security, and stability that are most associated with desirable education outcomes. Urban school students are more likely than their suburban and rural school counterparts to have changed schools frequently, and they are more likely to be absent from school. Compared to their suburban and rural counterparts, teachers in urban schools report having fewer resources, less control over the curriculum, and higher levels of student behavior problems.

Lippman et al. (1996) found many of these trends to be magnified in urban schools that are also characterized by high concentrations of students in poverty. While the challenges presented by urbanicity and poverty are sure to place additional demands on teachers, this report also challenges the perception that urban schools with the highest poverty concentrations are always much worse off than other schools. This study reports high variation among schools, suggesting that some urban schools are successfully meeting the challenges that face them.

In a large-scale study of successful high-poverty schools, Knapp and his colleagues (Knapp, 1995b) found that the more teachers focused on teaching mathematics in ways that emphasize conceptual understanding, the more likely students were to demonstrate proficiency in problem-solving. However, many teachers in urban, high-poverty schools do not provide meaning-centered instruction that emphasizes conceptual understanding (Anyon, 1981; Haberman, 1991). This finding points to an additional challenge for those who promote mathematics-reforms that emphasize student thinking in urban, high-poverty schools.

Many factors appear to influence the ways teachers approach mathematics instruction in their classrooms. As the act of teaching has been recognized as complex, researchers have also begun to better understand the role teachers' cognitions play in instructional decision-making and teacher behavior. As was touched on previously, researchers have come to believe that a teacher's beliefs and knowledge strongly influence how she understands the recommendations of mathematics-reform (Spillane & Zeuli, 1999) and how she engages in mathematics instruction in her classroom (Calderhead, 1996; Ernest, 1989; Fennema & Franke, 1992; Schoenfeld, 1998, 2000; Thompson, 1992). Fennema and Franke (1992) suggest that the relationship between beliefs, knowledge, and practice is an interactive one. They posit that, "Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create a unique set of knowledge that drives classroom behavior" (p.162). The sections that follow will elaborate more fully on how teachers' knowledge and beliefs are thought to influence mathematics teaching.

Knowledge for Teaching Mathematics

There is common agreement that what a teacher is able to do instructionally relies to a great extent on her knowledge base and how she is able to mobilize that knowledge in the act of teaching (Fennema & Franke, 1992; Hill, Sleep, Lewis, & Ball, 2007; Schoenfeld, 1998). Following decades of research on general pedagogical knowledge needed for teaching, Shulman (1986) played an instrumental role in shifting attention to the importance of subject matter knowledge, calling it the missing paradigm in research on teaching. Although there is not yet a consensus on the knowledge needed to teach mathematics (Ball et al., 2001), much headway has been made in exploring the various kinds of knowledge that support teaching (Borko & Putnam, 1996; Ernest, 1989; Fennema & Franke, 1992; Hill et al., 2007; Shulman, 1986, 1987) as well as how that knowledge is mobilized in the act of teaching (Leinhardt, 1993; Schoenfeld, 1998, 2000; Sherin, 2002b).

Borko and Putnam (1996) identify three main categories of knowledge that support teaching: 1) general pedagogical knowledge, 2) subject-matter knowledge, and 3) pedagogical content knowledge. *General pedagogical knowledge* refers to important knowledge about teaching, learners, and learning that transcends particular subject matter domains. This includes knowledge about effective strategies for planning, classroom routines, conducting lessons, and classroom management as well as general knowledge about how children think and learn. *Subject-matter knowledge* refers to knowledge of the discipline of mathematics that is not unique to teaching. This includes knowledge of the important facts, concepts, and procedures as well as knowledge of the concepts underlying the procedures and relationships between important concepts and mathematical ideas.

The third category of knowledge identified by Borko and Putnam, *pedagogical content knowledge,* refers to the unique set of subject matter knowledge used in teaching. As first described by Shulman (1986), pedagogical content knowledge includes, "the ways of representing and formulating the subject that make it comprehensible to others," and, "…an understanding of what makes learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). Also included is the knowledge needed by teachers of the instructional strategies that will help students to reorganize and deepen their understanding of subject matter. Grossman (1990) elaborated on Shulman's initial description of pedagogical content knowledge to include knowledge of curriculum and curricular materials. Other researchers have highlighted the importance of understanding student cognitions in particular mathematics domains (Fennema & Franke, 1992).

Thompson and Thompson's (1994) description of one teacher's work with an individual student on understanding the concept of rates highlights the importance of pedagogical content knowledge. In a one-on-one tutoring session, the teacher focused on procedural, algorithmic aspects of problems posed, assuming that his student's correct answers were evidence that she understood the concepts underlying the procedures. When the student became stuck, the teacher was unable to recognize the source of her difficulty. In this case, the teacher held a strong personal understanding of the concept he was teaching, but his understanding was not flexible enough to consider multiple ways of thinking about rates or the trajectory of his student's mathematical understandings.

Beyond an understanding of the knowledge needed for teaching mathematics is a need to consider how that knowledge is accessed and used in teaching. Teaching is a complex activity that involves negotiating among multiple goals simultaneously in an environment (the classroom) with large information-processing demands (Leinhardt, 1993). As teachers struggle to deal with complexities in subject-matter, some of their attention necessarily moves away from other foci. The complexities of teaching can be reduced if teachers choose to ignore the informational inputs of students (e.g., student thinking) and stick to structured, teacher-centered lesson plans. Conversely, as teachers attempt to focus their cognitive attention on managing a class discussion of the varied mathematical solutions provided by students, they sometimes have difficulty staying focused on mathematics content goals (Williams & Baxter, 1996). Montero-Sieburth (1989) asserts that the complexities of teaching are compounded in the urban school context:

> Today's urban teachers are in a tenuous position. Research evidence has shown that teachers within impoverished urban schools are, at best, so encapsulated and overwhelmed by the demands of their teaching environments that they can barely function. They carry theory around in their heads, but they often do not know how to apply this knowledge in the given context because they are so immersed in practice. Urban teachers in the thick of their routines hardly have the time or energy to reflect on their experiences or their teaching styles (p.337).

It is important to consider the extent to which and how teachers organize their knowledge such that it can be efficiently accessed while teaching.

In the educational psychology literature, it is widely accepted that humans organize their experiences and subsequent knowledge in schemata that help to make sense of their experiences in increasingly efficient ways. In expert-novice studies of teaching, expert teachers have been found to have much more elaborate, interconnected, and flexible cognitive schemata than novices (Borko & Livingston, 1989; Leinhardt, 1993). Leinhardt and her colleagues (Leinhardt, 1993; Leinhardt, Putnam, Stein, & Baxter, 1991) identify some varieties of schema that teachers use to organize their teaching. These include agendas, routines, and curriculum scripts. Schoenfeld (1998; 2000) describes a similar set of constructs that he calls lesson images and action plans.

An a*genda* is the teacher's dynamic "mental notepad" for a lesson (Leinhardt, 1993). Schoenfeld (1998; 2000) discusses a similar construct that he calls a lesson image. He describes the lesson image as, "everything the teacher envisions happening in the lesson – the day's sequence, the forms of interaction with students, what is flexible and what is not, and his or her sense of how the discussion will go" (Schoenfeld, 2000, p.250). The primary function of an agenda or a lesson image is to provide a conceptual roadmap charting the direction of a lesson. It includes the overarching goals and anticipated actions of a lesson and focuses primarily on the non-routine parts of that lesson. In general, experts incorporate more detail into their agendas than novices, and they have a better sense of how a given lesson will play out (Leinhardt, 1993). Expert agendas typically include an image of teacher and student actions, anticipation of student responses, plans for checking student learning at multiple points in the lesson, and subsequent branched plans based on assessment of student understanding. In contrast, novice agendas focus on plans for teacher moves, with little attention to student response, and include few contingency plans. With limited pedagogical content knowledge, especially knowledge of student cognitions, a teacher is limited in her ability to devise an agenda or lesson image that will adequately facilitate a reform-oriented mathematics lesson (Smith, 2000).

Schoenfeld (1998; 2000) identifies *action plans* as mechanisms for accomplishing goals while teaching. Routines and scripts are two types of action plans that are frequently used. *Routines* are social scripts that facilitate management and classroom norms (Leinhardt, 1993). Established routines are necessary in teaching because they help to reduce the teacher's cognitive load and facilitate focus on non-routine aspects of teaching. Routines can meet a diverse array of goals, from distribution of supplies to establishing norms for classroom discourse. Sherin (2002a) describes a teacher who developed a filtering strategy for classroom discussion of mathematical ideas. First she elicited solution strategies from several students; then she made a decision about what to filter out and focus on in order to address established mathematical goals. Through this routine, the teacher was able to utilize student ideas and strategies while ensuring that her class focused on established mathematics goals. Routines serve an essential role in classroom teaching, but they also can be limiting:

> When teachers wish to modify their teaching, they often adopt the large pieces of a new reform (e.g., small group, cooperative teams, problem-centered inquiry, etc.), but they keep the old routines for producing and sharing knowledge. This has two consequences: First, the new system does not work, and they have management problems; second, the class receives mixed implicit messages. The conflict of routines with philosophies or social organization creates serious difficulties (Leinhardt, 1993, p.18).

As teachers transition to reform-based mathematics pedagogy it is important that they intentionally cultivate new routines that support new goals.

Curriculum scripts (Leinhardt et al., 1991) or *scripts* (Schoenfeld, 2000) are action plans that detail content-specific scenarios for ways in which segments of instruction will play out. Scripts can be flexible and interactive with spaces for student actions or they can be more rigid, following a specific progression of teacher-determined ideas. These scripts are derived from a conglomeration of subject-matter knowledge and pedagogical content knowledge that can be easily accessed in the course of a lesson. Sherin (2002b) uses the term *content knowledge complexes* to describe these interwoven pieces of knowledge. Because of the way these aspects of knowledge are interwoven and accessed together, she suggests that they represent larger elements of accessible teacher knowledge that cannot be characterized as solely subject-matter knowledge or pedagogical content knowledge.

Explanations (Leinhardt et al., 1991) or *mini-lectures* (Schoenfeld, 1998) are special cases of scripts that involve packaged information that a teacher may plan to share in the course of a lesson or that the teacher has ready to share in response to a student question. These often involve carefully selected representations that will aid students in understanding the content at hand. Compared to novices, expert teachers are found to have a greater number of contingency explanations that they can turn to if needed (Leinhardt, 1993). Carpenter, Fennema, Peterson, & Carey conducted a study (1988) of how first grade teachers use their pedagogical content knowledge of children's solutions to addition and subtraction word problems during instruction. This study found that most teachers could identify essential elements of problems as well as common student strategies. However, this information did not seem to be organized in ways that supported teachers in making instructional decisions that utilized this knowledge. Given the uncertain nature of classrooms that focus on student thinking, knowledge of alternative explanations and representations that can be accessed at a moment's notice seems especially important.

It is important to keep in mind that teachers do not always have subject-matter and pedagogical content knowledge to support instruction that builds on student thinking. Lehrer and Franke (1992) compare the way a first-grade teacher, Ms. Jackson, approached instruction on addition/subtraction concepts to the way she approached instruction on fractions. Ms. Jackson's measured knowledge of fractions was much less developed than her knowledge of addition and subtraction, and her methods of instruction were also notably different in these two content areas. While teaching addition and subtraction concepts, the central activity of Ms. Jackson's class was problem solving using a variety of problem types. Ms. Jackson orchestrated class discussions in which she intentionally elicited a variety of student strategies and supported students in making sense of the mathematical ideas presented by their classmates. During instruction on fractions concepts, Ms. Jackson directed students' interactions to a much greater extent, and she was much less likely to ask clarifying questions or pose a problem that built on the ideas presented by students.

Warfield (2001) argues that the depth, breadth, and organization of teachers' mathematical knowledge for teaching influences their abilities to attend to and build on the mathematical thinking of children in their classes. Teachers' knowledge of children's thinking and the mathematics they teach influences the extent to which teachers can critically examine their students' thinking to determine if it is mathematically valid. Additionally, limits in teachers' knowledge impact teachers' abilities to pose questions, respond to students' novel ideas and strategies, and press children to extend their thinking.

It is clear that teaching mathematics in ways that honor student thinking requires greater amounts and varied types of knowledge. Furthermore, this knowledge needs to be organized in a way that makes it useful as teachers plan and implement instruction. The content of a teacher's knowledge base as well as its organization appears to facilitate and limit the ways in which teachers are capable of supporting and extending student thinking.

Next a brief discussion of knowledge thought to support reform-oriented teaching of multiplication and division will be presented, since these are the mathematics topics of focus in the data for this research. Rather than providing an exhaustive review of the literature pertinent to knowledge for teaching multiplication and division, the section that follows is designed to selectively preview aspects of knowledge for teaching multiplication and division that are particularly relevant to the classroom lessons observed. Therefore, concepts reviewed in the section that follows are considered central to understanding and interpreting the findings of this study.

Knowledge for Teaching Multiplication and Division

This discussion of knowledge for teaching multiplication and division begins with an overview of problem types and their relationship to students' initial problem solving strategies. Next, students' progression through increasingly efficient strategies for single-digit multiplication and related division problems will be reviewed, with particular attention to things that influence strategy use. This will be followed by a brief discussion of the array model for multiplication and division. Finally, students' strategies and learning trajectory for multidigit multiplication will be described.

There are a variety of situations that can be represented by multiplication and division. These situations can be classified into symmetric and non-symmetric problem types (Greer, 1992). Symmetric problem types include array problems, area problems, and combinations problems. Non-symmetric problem types include problems involving equal groupings of discrete objects, rate problems, and multiplicative comparison problems. It is important for teachers to have a conception of multiplication and division that includes these various problem types for students to encounter multiplication and division problems that encourage different kinds of thinking and involve different kinds of quantities. For instance, students' initial development of multiplication and division understanding is supported by problems involving equal groupings of discrete objects. But rate problems, multiplicative comparison problems, and area problems can be extended to rational numbers. Exposure to these additional problem types helps to lay the foundation for multiplication and division of fractions in later grades (Carpenter, Fennema, Franke, Empson, & Levi, 1999).

Understanding how problem structures underlie children's ways of thinking about mathematics problem types is an important element of knowledge for teaching mathematics in a manner that builds on student thinking (Carpenter et al., 1996; Franke et al., 2007). Multiplication and division instruction often begins with equal-grouping problems. This type of problem involves three quantities: a) the number of groups, b) the number of objects in a group, and c) the total number of objects. The quantities known and missing determine the type of equal-grouping problem. Problems in which the total number of objects is unknown ($a \times b = ?$) can be called *groups-of multiplication*. Problems in which the number of objects in one group is unknown ($a \times ? = c$ or $c \div a = ?$) are typically called *partitive division* problems, corresponding to the social practice of sharing equally. Problems in which the number of groups is unknown ($? \times b = c$ or $c \div b = ?$) are referred to as *measurement division* problems or *quotative division* problems. An example of each of these problem types is presented in Table 1.

Table 1
Equal-Grouping Problem Types

Problem type	Unknown	Example
Groups-of multiplication	Total number of objects	Josh has 4 bags with 6 cookies in each bag. How many cookies does Josh have in total?
Partitive division	Number in one group	Josh has 24 cookies. He wants to put the same number of cookies in each of 4 bags. How many cookies should Josh put in each bag?
Measurement division	Number of groups	Josh is making cookie bags with 6 cookies in a bag. If there are 24 cookies in total, how many bags of 6 cookies can Josh make?

In reform-oriented classrooms that allow children to devise their own ways of solving problems, children's initial efforts at solving equal-grouping problems involve directly modeling the actions and relationships described in the problem situations (Carpenter et al., 1996). In order to directly model groups-of multiplication problems children make the specified number of groups (4 bags) with the specified number of objects in each group (6 cookies) and then count the total number of objects (24 cookies). For partitive division problems, children initially represent the specified number of objects (24 cookies) and groups (4 bags) and then use a guess-and-check strategy to determine the numbers of objects (cookies) that can be divided evenly into each group such that all the objects are used. Later, children develop a more strategic approach to partitive division problems in which they distribute the objects in a systematic manner among the groups. For measurement division problems, children make sets of the number of objects specified for one group (6 cookies) until they have reached the total number of objects (24 cookies). The answer to the measurement division problem is then found by counting the number of sets or groups formed (4 bags).

As is evident from the direct modeling strategies described, children do not initially understand these problem types in terms of multiplication and division as adults do. Consequently, research suggests that instruction that builds on student thinking should begin with students' ways of solving these problems and introduction of concepts and symbolic representation should build on these student-generated solutions (Carpenter et al., 1999). For teachers, it is helpful to have a sense of the learning trajectory students might take as they develop more efficient strategies, including knowledge of the understandings students must acquire to make sense of increasingly sophisticated strategies.

Understanding of multiplication and division that allows children to move beyond direct modeling of situations is linked to children's abilities to simultaneously think of a group of objects as both the number of objects in the group and one group (Fosnot & Dolk, 2001). Called unitizing, students are challenged to simultaneously keep track of two counts: the number of objects and the number of groups of objects. There have been many research-based accounts of how children move from direct modeling with objects (to solve equal-grouping problem types) to increasingly efficient number-based strategies (see, e.g., Carpenter et al., 1999; Fosnot & Dolk, 2001; Kouba, 1989; Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005). Verschaffel, Greer, and De Corte (2007) summarize children's progression through strategies for single-digit multiplication and division as follows:

> Generally speaking, children progress from (material-, fingers-, or paper-based) concrete counting-all strategies, through additive-related calculations (repeated adding and additive doubling), pattern-based (e.g., multiplying $\times$ 9 as by 10 – 1), and derived-fact strategies (e.g., deriving 7×8 from $7 \times 7 = 49$) to a final mastery of learned multiplication products (p.562).

Progression through these strategies varies by problem type, with students using direct modeling strategies for partitive division problems for longer than it takes with groups-of multiplication and measurement division problems (Carpenter et al., 1999). The reason for this is rooted in the relationship between the ways that these problems are directly modeled and their similarity to additive-related calculations. Students who can unitize a group-of multiplication problem involving finding the total number of cookies in 4 bags of 6 cookies, can relate it to the repeated addition strategy 6 + 6 + 6 + 6 or the skip counting strategy 6, 12, 18, 24. In each case, each count represents a group of cookies, with the number of cookies represented by the quantity of the count. These repeated addition and skip counting strategies can also be related to a measurement division problem in which the goal is to figure out how many bags are needed to put 24 cookies in bags such that 6 cookies are in a bag. In this case, the answer is determined by keeping track of the number of sixes it takes to reach 24. For both of these problem types, students use repeated addition or skip counting to represent the number of objects in each group. However, for partitive division problems, the number of objects in each group is the unknown. The distribution action utilized as children are direct modeling partitive division problems does not translate as easily to additive strategies.

As students begin to move beyond direct modeling strategies, strategy use also varies depending on the numbers involved in a problem and students' number-specific computational resources. Drawing on the retrieval-focused literature (Campbell & Graham, 1985), problems with smaller operands (e.g., 2×3) are solved by learned product more quickly than problems with larger operands (e.g., 7×8). However, children appear able to use learned multiplication strategies for problems in which operands are the same (e.g., 6×6) and problems involving 5 as an operand (e.g., 5×8) more quickly than is suggested by their operand-size. Another factor that influences strategy use is the number-specific computational resources students have acquired to operate on the numbers in a problem (Sherin & Fuson, 2005). For instance, students typically learn skip counting sequences for 2, 5, and 10 and use these sequences in problems involving these numbers. However, use of skip counting sequences for problems involving other numbers such as 4 or 7 depend on whether these sequences have been emphasized instructionally.

Implicit or explicit understanding of properties also influences the strategies children use (Fosnot & Dolk, 2001). For example, knowledge of the distributive property allows children to successfully use derived fact strategies, such as solving 8×6 by calculating $(4 \times 6) + (4 \times 6)$. Knowledge of the commutative property of multiplication (e.g., $5 \times 6 = 6 \times 5$) allows children to think about the numbers in problems more flexibly. In a problem involving finding the total number of sodas in five six-packs, a student with knowledge of the commutative property might count by fives to find a solution even though the context of the problem calls for groups of six. However, it should be noted that some researchers have found that, even when students demonstrate knowledge of the commutative property when dealing with symbols on paper, they do not as readily apply this property to make calculations easier when numbers appear in contexts (Ambrose, Baek, & Carpenter, 2003). In reform-oriented classrooms, teachers are responsible for orchestrating discussion of students' strategies for solving multiplication and division problems such that the relationships among strategies and important mathematical ideas, such as the commutative and distributive properties, are illuminated. This involves skillfully posing tasks that lend to use of particular strategies and stimulate discussion of particular ideas.

Some mathematics curriculums, including the reform-based mathematics curriculum utilized by teachers in this study, explicitly introduce arrays to model multiplication and division. An *array* is a rectangular arrangement of objects in rows and columns such that each row has an equal number of objects (Van de Walle, 2007). Arrays are named by stating the number of rows by the number of objects in a row. (See Figure 1.)

Figure 1. *Naming conventions for arrays.*

X X X X X		X X X	
X X X X X	← 3×5 array	X X X	← 5×3 array
X X X X X	3 rows of 5 objects	X X X	5 rows of 3 objects
		X X X	
		X X X	

Arrays are a useful model, as they can be used to illuminate the commutative property of multiplication as well as the relationship between multiplication and division. However, children do not naturally build arrays to represent contextualized problems unless the problem context suggests an array (Carpenter et al., 1999). Therefore, in introducing the array model, teachers are challenged to organize instruction such that children build knowledge of arrays through connections to their own, more natural strategies. Additionally, teachers need to be aware that many students initially have difficulty understanding the array structure, particularly understanding how one square can simultaneously be part of a column and a row (Battista, Clements, Arnoff, Battista, & Van den Borrow, 1998). Consequently, teachers using the array representation must be sensitive to students' developing understanding of arrays and how students are relating the array structure to problems posed.

After children construct meaning of multiplication and division and begin to develop increasingly efficient strategies for working with smaller numbers, instruction turns to a focus on problems involving multidigit calculations. As is the case with calculations with smaller numbers, it is important for teachers to be able to anticipate strategies students will use to solve multidigit problems as well as how students might naturally move through these strategies.

Baek (1998) classifies children's invented solutions to multidigit multiplication problems into four categories[1]: direct modeling, complete number strategies, partitioning number strategies, and compensating strategies. *Direct modeling strategies*, the most basic strategy type, entail modeling each group of objects in a multiplication problem with concrete manipulatives or drawings to count the total number of objects. *Complete number strategies* are based on repeated addition of multiplicands, but do not involve partitioning of the multiplier or multiplicand in any particular way. Doubling strategies that shorten the addition procedure are included in this category. *Partitioning strategies* involve partitioning

[1] Baek's classification is based on children's work with multiplication problems involving groupings of discrete objects, as opposed to rate, multiplicative comparison, or symmetric problem types.

the multiplier and/or the multiplicand into smaller numbers so they can be multiplied more easily. Within this category of problem, Baek distinguishes between partitioning into decade and non-decade numbers (e.g., partitioning 16 into 10 and 6 opposed to 8 and 8). *Compensating strategies* involve adjusting the multiplier and/or multiplicand up or down based on special characteristics of the number combination to make the calculation easier. Then, after major calculations have been completed, students compensate for the initial adjustments to the numbers. For instance, in solving the problem $4 \times 19 = ?$, a student might first find the product of 4×20, and then subtract 4 from this product to compensate for the adjustment to the original problem.

In her research, Baek (1998) found that the students studied progressed through invented multidigit multiplication strategies from direct modeling to complete number to partitioning numbers into non-decade numbers to partitioning numbers into decade numbers. Children's strategies for solving multidigit multiplication problems varied with their conceptual knowledge of addition, base-ten knowledge, knowledge of basic multiplication facts and their relationship to multidigit problems, and properties of the four operations. For instance, student use of partitioning and compensating strategies was dependent on students' knowledge of multiplication facts and the distributive property. Development of understanding of what happens to a number when it is multiplied by a power of ten (e.g., $6 \times 4 = 24$, $6 \times 40 = 240$, $6 \times 400 = 2{,}400$) was found to be essential knowledge to using partitioning strategies involving decade numbers.

Knowledge for teaching multiplication and division in a reform-oriented manner demands that teachers hold deep knowledge of mathematics content, student thinking, and ways of illuminating important mathematical ideas. However, just because a teacher holds certain knowledge does not ensure that she will choose to act on that knowledge. To better understand how teachers choose to use the knowledge they have, I will now turn to a discussion of teachers' beliefs.

Beliefs and Mathematics Teaching

Beliefs and knowledge are not easily distinguishable. Indeed, people often describe their beliefs in terms of things they "know" (Thompson, 1992). However, beliefs generally refer to suppositions, commitments, and ideologies, while knowledge is viewed as factual propositions and understandings (Calderhead, 1996). Teachers' beliefs are important because they influence teachers' perceptions and interpretations of events (Pajares, 1992), and serve as a guiding force in the kinds of actions teacher take (Cooney, Shealy, & Arvold, 1998). Considered in this context, beliefs are the implicit and explicit personal philosophies held by teachers consisting of their conceptions, ideologies and values that shape practice and direct knowledge (Ernest, 1989).

In considering teachers' beliefs, it is important to distinguish between the content of beliefs and the structure of the beliefs system. Before discussing in greater depth the content of teachers' beliefs in relation to mathematics teaching, I will briefly review theory relevant to the structure of beliefs and how they are held.

Green (1971) identified three dimensions of the beliefs system:

> First there is the quasi-logical relation between beliefs. They are primary or derivative. Secondly, there are relations between beliefs having to do with their spatial order or their psychological strength. They are central or peripheral. But there is a third dimension. Beliefs are held in clusters and protected from relationships with other sets of beliefs. Each of these characteristics of belief systems has to do not with the content of our beliefs, but with the way we hold them (p.47-48).

According to Green's theory, beliefs are held in interdependent ways, with some beliefs following other beliefs in a hierarchical manner. For example, if a teacher believes that students learn mathematics from constructing their own strategies, a belief that logically follows is that the teacher's role is to provide opportunities for students to engage in mathematics lessons where they devise their own strategies for solving problems. Green's theory also suggests that beliefs can be held in isolated clusters, making it possible for persons to hold conflicting beliefs.

Additionally, Green's theory (1971) suggests that individuals hold beliefs at varying levels of conviction. This is especially important to consider when examining the relationship between teacher beliefs and instructional practice. The complexity of teaching requires teachers to act in situations where multiple, sometimes conflicting, beliefs are activated at once (Aguirre & Speer, 2000). The action a teacher chooses to take is thought to be, in part, a result of the prioritization of the strength of beliefs. More recently, researchers have also begun to consider instances when a teacher's instructional practice appears to conflict with the teacher's espoused beliefs as being explained by the teacher's prioritization of goals (Leatham, 2006; Philipp, 2007; Skott, 2001).

For instance, Skott (2001) describes a teacher who generally views his role in the classroom as one of supporting students as they solve novel mathematics problems in their own ways in order to allow students to assume responsibility for their own learning. While this teacher's instruction was generally found to reflect these beliefs, Skott describes an instance when this teacher was observed providing heavy computationally-focused guidance to a pair of students such that the complexity of the problem of focus was significantly decreased. When asked to reconcile this action in relation to his beliefs, the teacher asserted that he believed the two struggling students would provide a significant drain on his attention, limiting his ability to attend to the rest of the class, until they felt confident that they had an answer to the problem. In this case, the teacher's priorities related to general management of the class and building students' confidence were deemed of higher priority than the teacher's goal of aiming to provide support such that students assume responsibility for their own learning.

So far, consideration has been given to how teachers hold beliefs in relation to each other and why teachers' actions may sometimes appear to contradict certain beliefs. Now consideration will be given to the content of teachers' mathematics-related beliefs and their implications for mathematics instruction. Calderhead (1996) identifies areas in which teachers are found to hold beliefs relevant to their teaching practice. These include beliefs about subject-matter, learners, and learning as well as beliefs about teaching, the role of the teacher, and teachers' self-related beliefs. Within the mathematics education literature, there is a focus on the importance of teachers' beliefs related to the nature of mathematics and mathematics teaching and learning (Ernest, 1989; Franke et al., 1997; Franke et al., 2007; Thompson, 1992).

According to Thompson (1992), "A teacher's conception of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (p.132). Several researchers have created different terms and frameworks to describe varying conceptions of mathematics (Ernest, 1989; Lerman, 1983; Skemp, 1978; Thompson et al., 1994). Conceptions of mathematics most consistent with current reform efforts view mathematics as a dynamic discipline focused on solving problems by thinking creatively, finding patterns, and reasoning logically. Sometimes referred to as a problem-solving or inquiry-oriented view, this conception emphasizes mathematics as a way of thinking and devising a growing understanding of the world. At the other end of the spectrum, some teachers view mathematics as a static body of knowledge consisting of a collection of “rules without reasons” (Skemp, 1978, p.9). This view of mathematics emphasizes knowledge of established methods for performing mathematics tasks. Teachers’ beliefs about the nature of mathematics have implications for how they will view and approach mathematics teaching (Lerman, 1983).

A teacher’s beliefs or conception of mathematics teaching includes personal philosophies related to the most desirable goals of mathematics instruction, related instructional approaches and emphasis, what counts as mathematical activity, and appropriate roles of teachers and students during classroom instruction (Thompson, 1992). Kuhs and Ball (1986, as discussed in Thompson, 1992) identify four ways teachers view mathematics instruction. A *learner-focused* view concentrates on supporting students’ personal construction of mathematical knowledge and is most related to the problem-solving or inquiry-oriented view of mathematics described earlier. Consistent with a constructivist-orientation to teaching, the teacher’s role is to be a facilitator of student learning by asking probing questions and helping students to uncover misunderstandings and new understandings. This conception of teaching reflects the emphasis of mathematics-reform and places student thinking at the center.

The other three conceptions of mathematics teaching identified by Kuhs and Ball (1986, as discussed in Thompson, 1992) include content-focused with an emphasis on conceptual understanding, content-focused with an emphasis on performance, and classroom-focused. The conception described as *content-focused with an emphasis on conceptual understanding* organizes instruction around the discipline of mathematics instead of student thinking. Mathematics is viewed as a static body of knowledge containing an underlying logic that students must come to understand. In the conception identified as *content-focused with an emphasis on performance*, there is a focus on mastering rules and procedures to get correct answers, but understanding is de-emphasized. Reflecting an instrumentalist view of mathematics, direct instruction is the dominant teaching style and learners are typically passive receivers of knowledge. The final conception of teaching, described as *classroom-focused*, de-emphasizes the nature of mathematics and theories of learning. Instead, it focuses on instruction derived from knowledge about effective classrooms.

These distinctions are useful in understanding the differences among teachers' possible conceptions of teaching mathematics; however, it is important to note that individual teachers are likely to hold a conglomeration of beliefs that cut across multiple models in the Kuhs and Ball framework. Despite the seemingly natural connection between theories of learning and theories of teaching, Thompson (1992) makes the following observation:

> Although it seems reasonable to expect a model of mathematics teaching to be somehow related to or derived from some model of mathematics learning, for most teachers it is unlikely that the two have been developed and articulated into a coherent theory of instruction. Rather, conceptions of teaching and learning tend to be eclectic collections of beliefs and views that appear to be more the result of their years of experience in the classroom than of any type of formal or informal study (p.135).

That beliefs about learning and teaching are not always linked can be explained by Green's claim that beliefs are held in isolated clusters (1971). However, certain beliefs do seem central to teaching practices that give student thinking a central role in instruction.

In a longitudinal study of a teacher development project aimed at helping teachers to develop instructional practices that build on children's mathematical thinking, Fennema et al.

(1996) found that teachers whose mathematics instruction focused on supporting and building on student thinking held a common set of beliefs. These teachers viewed children as coming to their classrooms with mathematical knowledge and the ability to acquire new knowledge by engaging in problem-solving. Related to this belief, teachers believed that students can learn without direct instruction, problem-solving is central to mathematics instruction, and skills and knowledge are interrelated. The second belief that appeared instrumental in classrooms that promoted student thinking was a view that teaching involves listening carefully to students in order to understand their thinking. Also, these teachers believed that understanding of student thinking should inform instruction.

In contrast, Warfield, Wood, and Lehman (2005) found that teachers who believe that only some children can be autonomous learners of mathematics interpret reform recommendations in unintended ways and, consequently, give limited attention to understanding children's thinking. These teachers viewed their students as high kids and low kids, with the low kids needing to be shown how to solve mathematical problems. These teachers interpreted reform recommendations related to partner work and student sharing of mathematical strategies through class discussion as ways for the high kids to show the low kids strategies for finding answers to problems. This is in contrast to a reform-oriented view of class discussion as an opportunity for students to collectively reason about, challenge, and reflect on mathematical thinking through comparison of multiple problem solving strategies (Wood & Turner-Vorbeck, 2001). Warfield et al. indicate that, because the teachers studied believed that children needed to be shown how to solve mathematical problems, they gave minimal attention to trying to figure out how to make their mathematics lessons responsive to students' mathematical thinking.

Cooney and Shealy (1997) point out that teaching practices advocated by mathematics-reform make teachers' classroom experiences more problematic and less predictable and controllable. They ask teachers to accept uncertainty as an on-going reality of classroom life. This is in sharp contrast to more traditionally held views of mathematics that stress rules and order. This "crossing over" to a relativist stance, in which teachers open up to multiple authorities on mathematical knowledge, is central to realizing fundamental change in the way mathematics is taught. Teachers must come to believe that multiple perspectives and flawed solutions are valuable instructionally.

In research examining impediments to teachers adopting reform-oriented teaching practices, Silver et al. (2005) reveal that many of the teachers they studied expressed concern that incorporating discussion of multiple ways of solving mathematical problems, especially flawed solutions, would lead to confusion for students. In a study comparing teachers' handling of mistakes in U.S. and Italian classrooms, Santaga (2005) found that U.S. teachers generally avoid the public discussion of flawed solutions in favor of a focus on correct answers. When mistakes arise in class discussion of mathematics, teachers typically aim to correct the errors quickly and move back to a discussion of correct answers. This manner of addressing students' flawed solutions stands in opposition to a reform-oriented view of students' mistakes as "springboards for inquiry" (Borasi, 1994). Conceived in this manner, students' flawed solutions provide ripe opportunities for students to learn through engagement in genuine problem solving involving analysis of correct and incorrect aspects of solutions in efforts to revise solutions to correct.

Since traditional views of mathematics teaching subscribe to a more instrumental view of the nature of mathematics, changes in teaching practice advocated by mathematics-reform often require teachers to dramatically re-conceptualize the nature of mathematics, teaching, and learning (Cooney & Shealy, 1997; Franke et al., 1997). But even if a teacher holds certain beliefs, it does not mean that she will necessarily act in ways that are consistent with those beliefs. Empson and Junk (2004) report a study in which teachers expressed beliefs about it being a good idea to use student mistakes as opportunities for learning. Despite this professed belief, when asked what they would do in response to a particular teaching scenario, the actions described by several teachers did not reflect the professed belief. Empson and Junk suggest that lack of specific knowledge of children's mathematics may limit teachers' abilities to act on beliefs.

In this example, a hint of the complexity in the relationship of beliefs, knowledge, and practice is visible. Like Empson and Junk, Ernest (1989) suggests that no matter how strongly beliefs are held, necessary knowledge must be on hand to back them up in actions. If the base of knowledge supporting the belief is limited, it is unlikely that actions associated with the particular belief will be realized. Ernest also identifies the social context as an important determinant in the way teachers do their work. He suggests that texts, school norms, expectations from superiors, and external tests yield considerable influence on teachers' prioritization of goals and subsequent practice, regardless of espoused beliefs.

Beliefs Beyond Mathematics

Leatham (2006) cautions researchers against the assumption that mathematics-related beliefs will have the strongest influence over teachers' mathematics teaching. Related to social context, an aspect of beliefs often neglected in the mathematics education literature are teachers' beliefs about children of color and low socio-economic strata (Lubienski & Bowen, 2000; Secada, 1992). The limited number of studies in mathematics education and more general studies of urban and high-poverty schools suggest that teachers' beliefs about the children in their classes play a significant role in their instructional decision-making (Hayward, 1999; Knapp, 1995b; Spillane, 2001; Sztajn, 2003; Walker & Chappell, 1997).

In interviews with teachers and administrators in high-poverty schools, Spillane (2001) found that most educators in these schools believed higher-level learning to be inappropriate for their students. They viewed their students primarily in terms of deficiencies in fundamental knowledge and skills due to their family backgrounds and upbringing. Consequently, these educators saw their role as helping students to master basic skills and not to focus on higher-level thinking. Ms. Brady, a second-grade teacher, contrasted her teaching in a private school with her teaching in a school characterized by poverty. She indicated that she had used teaching practices that were centered on problem-solving and critical thinking in the private school, because those kids already had the basics. In her current situation, she described her teaching practices to be focused on playing catch-up. She shared that she would like to get to higher-level thinking activities, but that basics took up most of her time.

Similarly, Sztajn (2003) describes a case study of a teacher, Teresa, who works in a school with a significant number of students from low-socioeconomic backgrounds. Although Teresa professes that problem-solving and higher-order thinking skills are important educational outcomes, she believes that basic facts, drill, and practice are at the core of what her students need in order to overcome their backgrounds and become responsible members of society. Sztajn contrasts Teresa with Julie, a teacher in a school serving students from middle-class families. Julie believes that students learn best when they are interested and happy. Consequently, her instruction emphasizes projects and activities that support development of higher-level thinking. At first glance, many would argue that Teresa and Julie's methods of teaching differ due to divergent beliefs about the nature of mathematics and mathematics teaching and learning. However, this would not accurately represent the whole story. Julie reports that she worries less about the basics, because her students have a lot of support for basic skills at home. She states that when she was working in a school with students from lower socio-economic backgrounds, she emphasized basic skills and social skills instead of higher-level thinking.

In another study, comparative analyses of interviews with teachers at an urban school and a suburban school revealed similar patterns of pedagogical goals (Hayward, 1999). At the affluent suburban school, teachers were focused on enabling long and short-term academic success, teaching children to enjoy learning, and helping them become independent learners. At the urban school, teachers were focused on helping students to acquire survival skills to combat problems that they described as "the environment." Teachers believed that it was their responsibility to make up for the environment by directly teaching children the difference between right and wrong and by helping students learn to be compliant.

Although there is evidence of mathematics instruction emphasizing higher-level thinking having positive affects on achievement in urban and high-poverty schools (Campbell, 1996; Fuson et al., 1997; Knapp, 1995b), beliefs to the contrary appear to be reflected in the actions of many teachers in urban schools serving students from low-socioeconomic backgrounds. Knapp (1995a) refers to these pervasive beliefs as the "conventional wisdom" of teaching children in poverty. The conventional wisdom encourages teachers to focus on what students lack and to bolster these deficiencies by providing sequential instruction on discrete skills. The conventional wisdom recommends a teaching style that is fast paced and tightly controlled by the teacher, to maximize time on task and minimize behavior problems. Knapp points out that there is considerable research support for this approach to teaching in high-poverty classrooms. In Brophy and Good's 1986 chapter in *The Handbook of Research on Teaching*, they state the following:

> Interactions between process-product findings and student SES or achievement level indicate that low-SES/low-achieving students need more control and structuring from their teachers: more active instruction and feedback, more redundancy, and smaller steps with higher success rates. This will mean more review, drill, and practice, and thus more lower-level questions. Across the school year, it will mean exposure to less material, but with emphasis on mastery of the material that is taught and on moving students through the curriculum as briskly as they are able to progress (Brophy & Good, 1986, p. 365).

In order for teachers in urban settings to effectively implement reform-based mathematics curriculums in ways that emphasize student thinking, they will need to move beyond this conventional wisdom and adopt beliefs that view their students as competent and capable learners who can take control of their own learning and engage in higher-level thinking.

The Process of Reforming Mathematics Teaching

Several recent studies have identified mechanisms that support teachers in developing the kinds of knowledge, beliefs, and practices associated with mathematics-reform. For instance, researchers have found that increasing teacher knowledge of student thinking through professional development can support teachers in rethinking their beliefs and reforming their instructional practices (Fennema et al., 1996; Kazemi & Franke, 2004). Other research provides support for the positive effects of on-going opportunities to reflect on teaching practice with researchers and other resource partners (Davis, 1997; Wood et al., 1991). Additional studies have found that reform-based mathematics curriculum materials can support growth in teacher knowledge (Empson & Junk, 2004) and change in teaching practice (Remillard, 2000). A finding that is common among several studies of teacher change is that teachers who engage in practical inquiry to better understand student thinking appear to continuously deepen their knowledge and refine their teaching practices (Fennema et al., 1996; Franke et al., 1998; Franke, Carpenter, Levi, & Fennema, 2001; Margolinas, Coulange, & Bessot, 2005; Sowder, 2007; Steinberg, Empson, & Carpenter, 2004). Franke et al. (2001) suggest the kinds of activities that help teachers learn from teaching:

> If teachers can learn to talk to their students about their thinking, puzzle about what the responses tell them about students' understanding, decide how to use this knowledge in planning instruction and interacting with students, and figure out how to learn more about students' thinking, then the teachers' own learning can become generative (p.656).

Finally, it is suggested that a first step in moving toward a different kind of teaching practice is often discontentment with the outcomes of current practices (Smith, 2000).

Other researchers have attempted to develop frameworks that capture descriptions of teacher change as teachers move from traditional to reform-oriented mathematics teaching practices (Brendefur & Frykholm, 2000; Davis, 1997; Franke et al., 1997; Franke et al., 2001; Hufferd-Ackles et al., 2004; Wood et al., 1991). In the remainder of this section, I will describe two of these efforts.

Franke et al. (1997) describe four levels of teacher change based on their study of first-grade teachers involved in the Cognitively Guided Instruction project (Carpenter et al., 1996), a teacher development effort focused on increasing teachers' knowledge of student thinking related to whole number operations. At the first level, teachers believe that students must be taught how to solve problems. Therefore, they do not provide problem-solving opportunities, ask students how they solve problems, or use mathematical thinking in instructional decision-making. At the second level, teachers begin to view children as able to solve problems without direct instruction, and they begin to expand the amount of problem-solving opportunities and the types of problems used. But they often continue to show students how to solve some problems. At the third level, teachers believe it is beneficial for students to solve problems in their own ways, and they provide a variety of problems and opportunities for students to discuss their solutions. At the fourth level, the teacher believes that students' mathematical thinking should influence teacher-student interactions as well as the evolution of the curriculum. Teachers begin to more frequently elicit and build on student thinking and use knowledge of student thinking to make instructional decisions.

In discussing this trajectory of change, Franke et al. (1997) note that individual teachers do not fit neatly into the levels identified. As the process of change is continuous, teachers may exhibit aspects of multiple levels. Interestingly, there was little direct relationship observed between teachers' initial beliefs and practices and change across the study. Some teachers who began the project at level one changed tremendously and moved through all the levels. Other teachers who started further on in the trajectory evidenced only minor changes. Some teachers' beliefs changed before their practices, while others changed after trying out recommended practices, and, for others, beliefs and practices changed concurrently. That being said, when classroom practices changed before beliefs, the changes were consistently found to occur at the lower levels. In order to engage in teaching practices associated with the highest levels, teachers needed to have adopted the beliefs consistent with that level. Although the Franke et al. (1997) study provides a useful way of thinking about teacher change, the generalizability of this study is limited since it focuses on first-grade teachers and their change while teaching whole number concepts after extensive time in professional development building knowledge of the mathematics and children's thinking about these concepts.

Hufferd-Ackles et al. (2004) present a four-level framework to describe changes in teacher actions, student actions, and classroom norms as teachers moved from traditional mathematics teaching to facilitating a *math-talk community*. The overarching shift toward developing a math-talk community is one in which the classroom community becomes increasingly supportive of students acting in leading roles and the discourse shifts from a focus on answers to a focus on student thinking. Hufferd-Ackles et al. initially developed their framework using qualitative data from one teacher's classroom, in which dramatic changes were observed in the course of a single school year. Then the framework was refined using data from teachers at a variety of elementary grades at the same urban-Latino school. This study is especially significant due to the context, because it refutes conventional wisdom and supports the notion that urban classrooms with students that are below grade level in mathematics can learn as a math-talk learning community.

Reflecting traditional mathematics instruction, Level 0 is exemplified by teacher directed instruction with brief, teacher-led question-and-answer sessions focused on answers to problems. At Level 1, teachers begin to pursue mathematical thinking; however, the teacher continues to play a dominant role in the discussion. The key shift is from discussion focused on answers to the inclusion of some questions that focus on student thinking. At Level 2, teachers begin to help students learn to be more involved in discussions of mathematics. Student-to-student talk increases and the teacher physically begins to move to the side or back of the classroom. At Level 3, teachers consistently share mathematical authority with students and place their own focus on learning about student thinking and building on students' mathematical ideas (Hufferd-Ackles et al., 2004).

Within these four levels, Hufferd-Ackles et al. (2004) identify four distinct but related components, suggesting a developmental trajectory for each of four dimensions of a math-talk community. These components include questioning, explaining mathematical thinking, sources of mathematical ideas, and responsibility for learning. Within the *questioning* component, teachers' questioning strategies become increasingly open-ended and focused on student thinking, and teachers begin to facilitate greater amounts of students questioning other students. In turn, students move from only giving short answers and responses to the teacher to asking and answering questions in student-to-student interactions. In the *focus on student thinking* category, the trajectory for the teacher moves from minimal elicitation of student thinking with a focus on answers to elicitation of multiple strategies to an emphasis on encouraging deeper and more thorough thinking. Students move through the trajectory from only giving answers to providing brief and incomplete descriptions of their thinking to defending and justifying answers and thinking with little prompting from the teacher. The third component describes the shift from the teacher being the source of mathematical ideas to students' ideas having influence over the direction of the lesson, and the final component of the trajectory describes the path teacher and students take as students become increasingly responsible for learning and evaluation of others and self.

This study emphasizes the development of the math-talk community as an iterative process to which teachers and students contributed (Hufferd-Ackles et al., 2004). Teacher moves were important for providing opportunities, but student change had to occur for the class to move forward in the trajectory. The authors describe specific teacher actions that one teacher took to facilitate student movement through the levels. First, the teacher began to expect that students share thinking instead of just answers. She supported this transition by modeling language. Next she provided ample scaffolding and probing to support students in developing descriptions of their strategies. Finally, as the math-talk community began to run itself, she remained available to jump in and support students as needed.

Taken together, these two frameworks provide a sense of how teachers move toward mathematics teaching practices that emphasize student thinking. However, neither study explicitly considers the role of teacher knowledge in supporting or limiting student thinking. Although the Franke et al. (1997) study attends to teachers' beliefs, they do not consider the role of beliefs teachers might have about teaching children in urban schools. It seems that the interaction of teachers' knowledge, beliefs, and practices are especially important to consider as teachers begin to use reform-based mathematics curriculum in the urban school setting.

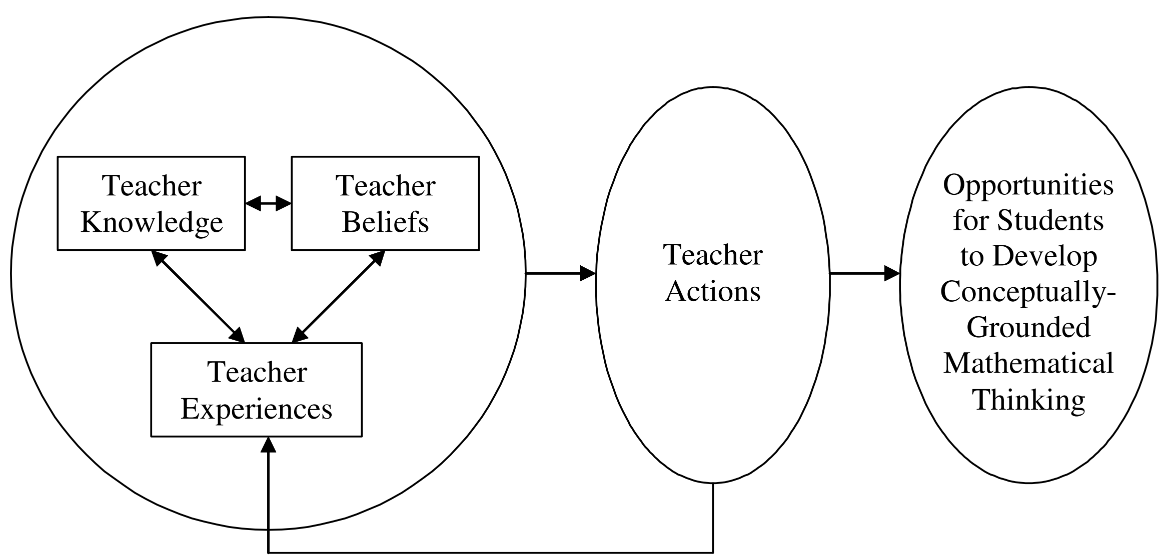
A Synthesis of the Research Literature – My Analytical Framework

This study aims to examine how teachers' beliefs and knowledge influence the ways they support and limit student thinking in their first year of implementing a reform-based mathematics curriculum in an urban school. Reflecting the existing research base, this study views the actions teachers take in relation to opportunities for students to develop conceptually-grounded mathematical thinking as being situated within the larger scope of teaching and learning practices associated with reform-oriented mathematics instruction. This is to acknowledge that teacher actions that support or limit student thinking do not occur in a vacuum. Rather, they are supported and impeded by factors such as lesson design, the culture of the learning community, and student dispositions towards mathematics learning.

Therefore, while focusing on teacher actions related to student thinking during mathematics instruction, these actions will be considered within the broader dimensions of reform-oriented mathematics instruction.

Within this context, teacher actions during mathematics instruction are thought to be facilitated by teachers' knowledge and beliefs, which are shaped by teachers' prior experiences. For this reason, an interactive perspective of teachers' knowledge, beliefs, and experiences provides the underlying analytical framework for this study. This perspective accounts for the simultaneous and interactive influence of teachers' knowledge, beliefs, and experiences on teacher actions during mathematics instruction, which in turn impacts opportunities for students to develop conceptually-grounded mathematical thinking (see Figure 2).

Figure 2. *An interactive perspective of teachers' knowledge, beliefs and experiences.*



Teachers' experiences are included prominently in this framework, as they are thought to be the basis of teachers' knowledge and beliefs (Calderhead, 1996). Additionally, teachers' knowledge and beliefs are thought to serve as the filter through which new experiences are interpreted. It is important to note that a teacher's immediate teaching context is represented in this framework as one type of teacher experience. Through a built-in feedback loop, teacher knowledge, beliefs, and future actions are thought to be influenced by teacher interpretation of immediate context. Each of the three components — teacher knowledge, teacher beliefs, and teacher experiences — can be further elaborated to consist of multiple sub-components.

The knowledge component subsumes multiple types of knowledge for teaching mathematics and attention to the structure of that knowledge. In order for reform-oriented mathematics pedagogy to be fully realized, a teacher must be able to draw on a combination of knowledge of mathematics, pedagogical content knowledge, general pedagogical knowledge, and knowledge of the ways children are likely to think mathematically. Structures that appear to support a teacher's ability to draw on these types of knowledge during instruction include clear and detailed lesson images and a repertoire of routines and scripts that can be accessed during teaching.

The beliefs component of this model is comprised of types of beliefs and attention to the structure of those beliefs. Mathematics-related beliefs that appear to influence teacher actions include beliefs about the nature of mathematics, beliefs about teaching and learning mathematics, and beliefs about student cognitions or how students think about mathematics. Furthermore, teachers have context-specific beliefs about the children they are teaching and the context in which they are teaching. The degree to which certain beliefs influence teacher action is determined, first, by the degree to which a teacher's beliefs reinforce each other resulting in a coherent theory of instruction. Second, the relative strength of beliefs and the teacher's priorities within particular teaching situations are viewed as having significant influence on teacher action.

Finally, the teacher experiences component of the model elaborates the kinds of experiences that influence teacher actions during mathematics instruction. These include the teacher's formal schooling experiences throughout K-12 education and in post-secondary education. It also includes experiences in prior teaching contexts as well as the current teaching context, including interactions with students, colleagues, supervisors, and others. This component emphasizes teacher actions as situated within the particular histories and current realities of the teacher.

CHAPTER II

RESEARCH METHOD

The purpose of this study is to examine how teachers' beliefs and knowledge influence the ways they support and limit student thinking in their first year of implementing a reform-based mathematics program within the context of an urban school. In particular, this study set out to answer four research questions:

1. In what ways and to what extent do teachers support and limit student thinking during mathematics instruction in their first year implementing a reform-based mathematics curriculum?
2. How do teachers' beliefs influence the ways they incorporate student thinking into their planning for mathematics instruction and on-the-spot instructional decision-making in their first year of implementing a reform-based mathematics curriculum?
3. How does teacher knowledge influence the ways teachers incorporate student thinking in their first year of implementing a reform-based mathematics curriculum?
4. How does the urban context, as defined by the research literature and perceived by teachers and school leaders, influence mathematics instruction in this urban school?

In this chapter, I provide a detailed description of the procedures used to address these research questions in a trustworthy manner. The following sections describe: 1) the research design, 2) sample selection procedures, 3) procedures for data collection, 4) data management procedures, and 5) procedures for data analysis.

Research Design

Overview and Justification of Research Design

In order to better understand the influence of teachers' beliefs and knowledge as they begin to use a reform-based mathematics program, a collective case study design (Stake, 2000) utilizing qualitative research methodology was employed. Data were collected on four teacher participants during the 2004-2005 school year from a variety of sources. Interpretive case studies were developed with the individual teacher treated as the unit of analysis, a bounded case. Simultaneously, a cross-case analysis was completed to illuminate patterns across cases and increase the potential for generalizing beyond the particular cases (Merriam, 1998; Yin, 2003).

Qualitative research methodology was determined optimal for this study because it is thought to be especially suited to research that aims to delve into complexities and processes (Marshall & Rossman, 1999). Particular to this study, the interaction of teachers' beliefs, knowledge, and mathematics teaching is highly complex and little understood. The qualitative method facilitates data collection and analysis that is responsive to emerging understandings during the research process. It also allows an array of variables to be considered holistically with attention to the context in which the variables are situated. That was an important consideration in selecting this methodology because the evolution of teachers' beliefs, knowledge, and teaching practices are thought to be intertwined with each other and with the particular contexts in which mathematics teaching occurs (Fennema & Franke, 1992).

Within the qualitative paradigm, this research utilized a collective case study design (Stake, 2000). A collective case study involves the study of multiple cases of a particular phenomenon in order to provide insight into the complexities of the phenomenon. In general, the case study design is thought to be particularly suited to the study of phenomena in which "how" and "why" questions are of focus and the variables of interest cannot be separated from the context in which they exist (Yin, 2003). This study aims to unravel *how* teachers' beliefs and knowledge influence their classroom mathematics teaching as they transition to a reform-based mathematics program in a particular urban teaching context. By focusing on multiple cases, a collective case study design offers the added benefit of increasing the generalizability of findings by providing opportunity to identify similarities and differences across cases that illuminate nuances of the phenomena of interest (Merriam, 1998; Yin, 2003).

While much of the case study research associated with current reforms in mathematics education focus on cases of exemplary teaching, this research intentionally set out to study typical teachers transitioning to the use of a reform-based mathematics curriculum in a challenging urban school setting. Specifically, the research design focused on the beliefs, knowledge, and teaching of urban school teachers who were likely to have limited prior knowledge of and experience with current reforms in mathematics education, but could count on sustained support from professional development. It is believed that teacher change is more likely to occur when teachers have sustained support (Richardson & Placier, 2001), and the teachers at the urban school selected were involved in an on-going program of mathematics professional development during the year of study.

A final design element integral to this study involved my role as a *participant-observer*. Participant observation is a special mode of observation in which the researcher assumes an active role with the potential to influence the case being studied (Yin, 2003). During the year of this study, I served as a mathematics teacher educator for the school in which the case study teachers worked. In designing this research, I attempted to capitalize on the unique opportunities afforded by participant-observation while minimizing the drawbacks associated with this technique.

Benefits associated with participant observation include the opportunity to gain access to case study participants in unique ways and the opportunity to intentionally manipulate aspects of the case (Yin, 2003). In this study, my role as a mathematics teacher educator afforded me the opportunity to build comfortable and trusting relationships with case study teachers and to become personally familiar with their teaching contexts. Furthermore, through this role, I was able to control the contents of mathematics professional development.

Drawbacks associated with participant observation include the potential biases produced by being an insider and conflicts between one's role as insider and researcher (Yin, 2003). During the period of data collection, I attempted to address potential conflicts between my role as a mathematics teacher educator and researcher by being explicit about the role that was dominant in a given situation. For instance, my role as a teacher educator was dominant during professional development meetings and my informal interactions with teachers, while my researcher role was dominant during observations and interviews. Furthermore, I aimed to reduce the influence of bias by engaging in trustworthy research practices and by making the scope of my dual role as researcher and teacher educator transparent to participants as well as consumers of this research. These aspects of the research design will be described in the sections that follow.

Establishing Trustworthiness

Throughout the research process, steps were taken to strengthen the trustworthiness of my research findings (Guba & Lincoln, 1989). In efforts to ensure the credibility of interpretations, this study was designed to allow for prolonged engagement and persistent observation in the research setting. I engaged in persistent monitoring of my own developing constructions by keeping records of my thinking during data collection and analysis. Additionally, I deliberately worked to explicate my own implicit assumptions and consider multiple interpretations of my data. Formal interviews and informal conversations with teachers at multiple points throughout the period of data collection provided a venue for member checks with the research participants, allowing me to test and refine my initial interpretations. Throughout the research process, I utilized peer debriefing to interrogate the basis of developing conjectures and theoretical claims. Through this process of checks, I sought to establish findings that are credible, dependable, and confirmable. In my report of findings, I aim to provide thick, rich description such that others can judge the transferability of my findings and conclusions to their own situations.

Establishing Transparency of Researcher Position

As has already been stated, I served as a mathematics teacher educator at the school in which case study teachers worked during the year of this study. In this role, I worked collaboratively with one other teacher educator[2] to provide on-going mathematics-related teacher development for the teachers who are the focus of this study as well as the other teachers in grades 3 – 5 and support personnel. The overarching goal of the teacher development project was to support teachers in using the newly adopted reform-based mathematics curriculum, *Everyday Mathematics* (Bell et al., 1994), and move toward reform-oriented teaching practices that reflect the vision promoted by the NCTM standards documents (1989, 2000).

In my role as mathematics teacher educator, I spent some time providing in-class support to case study teachers including model teaching, co-teaching, and extra help for particular students[3]. These experiences allowed me to become familiar with each case study teacher's students and allowed the students to become comfortable with my presence in their classrooms. Furthermore, working directly with the case study teachers' students provided opportunity for first-hand insight into the challenges of each teacher's unique teaching context.

During the year of this study, I spent at least part of a day at the school site an average of three days per week to engage in research or teacher development project activities. This level of presence facilitated regular informal interaction with case study participants. It was not unusual for case study participants to stop me in the hallway to tell me about something that happened in their classroom or to ask a question related to an upcoming mathematics lesson. Case study participants also conversed with me about personal and professional topics that were unrelated to mathematics instruction. Through the cumulative effect of my

[2] The teacher educator with whom I worked is a respected mathematics education professor at a university near the school site. We jointly planned and co-led all professional development meetings.

[3] The time spent providing in-class support in case study participants' classrooms varied from four to eight lessons, depending primarily on frequency of teacher requests.

formal and informal interactions with case study participants, I was able to build mutually respectful and trusting relationships. As the year progressed, study participants became increasingly candid regarding their beliefs, struggles, and concerns related to their teaching of mathematics and other topics.

In the context of formal classroom observations, my role was primarily that of researcher. During observations, I generally aimed to take a “fly-on-the-wall” stance. However, there were times when students or teachers would initiate interaction with me related to the lesson being observed. In these instances, I intentionally tried to keep interactions brief and would remind the student/teacher that my role was that of researcher that day. Nonetheless, I would also respond to the issue or question that initiated the interaction. In addition, partway through this study, one case study teacher indicated that she would like me to provide her with critical feedback after formal research observations. Consequently, I began to provide all case study teachers with verbally communicated feedback on formal observations. This feedback was usually provided immediately following the post-observation interview and consisted of noted strengths of the lesson and one or two suggestions for improvement. I also responded to any specific questions that the teachers had about their observed lessons. In this way, the lines between researcher and teacher educator blurred, even in the context of formal research observations.

Sample Selection Procedures

In the following sections, procedures used to select the sample will be described. First, criteria used to select the research site will be presented. Then attention will be given to procedures for selecting case study participants and school leader participants.

Selection of the Research Site

Lincoln Heights Elementary School (a pseudonym) was selected as a site for this research for several reasons. First, it is located within an urban school district that was beginning to use a reform-based mathematics program during the year of this study. This circumstance offered the opportunity to study the interrelationships of teachers' knowledge, beliefs, and mathematics teaching practices as they were encouraged to transition to reform-oriented mathematics teaching. With a student population of approximately 170,000 students, this school district was one of the 15 largest school districts in the United States during the year of this study. Within this school district, Lincoln Heights Elementary was a Title I school that exemplified many of the challenges faced by urban schools. In addition to serving a high number of students living in poverty, over half of the student population at Lincoln Heights was identified with Limited English Proficiency (LEP). Table 2 provides a summary of the composition of the student population at Lincoln Heights and the school district during the year of this study.

Table 2
Composition of the Student Population at Lincoln Heights and the School District

	Enrollment	Race/ethnicity (%)					School Services (%)		
Group	n	Asian	Black	Hisp	White	Other	Fr/rd lunch	Sp Ed	LEP
Lincoln Heights	568	2	12	74	10	2	86	14	59
School district	170,000[a]	4	28	28	38	2	50	15	19

[a]Student enrollment at the school district level is intentionally approximated, in order to protect the identity of the school district.

Additionally, the research literature on school reform suggests that desired change is most likely to occur when teachers have sustained support in developing needed knowledge and skills within their teaching contexts (Richardson & Placier, 2001). During the year of this study, Lincoln Heights teachers were engaged in an on-going program of teacher development aimed at supporting the transition to the new mathematics program. It was reasoned that this circumstance made the likelihood of teachers using the new reform-based mathematics program greater than it might be at schools in which support was more limited. Finally, my role as mathematics teacher educator for this teacher development program provided a context through which I could influence teachers' evolving knowledge and beliefs as well as forge trusting and mutually beneficial relationships with case study participants.

Selection of Case Study Participants

Merriam (1988) asserts that, "…cases should be selected for their power to both maximize and minimize differences in the phenomenon of interest" (p.154). Therefore, I set out to recruit teacher participants from a single grade to facilitate comparison across individual cases. Teachers from a single grade would share common curricular goals as well as a common set of professional development experiences. I invited Lincoln Heights' third-grade teachers to participate in this study because our first professional development meeting suggested that teachers on this team varied with regard to years of teaching experience and comfort with mathematics. Of the six third grade teachers, four agreed to be case study participants. The other two teachers declined participation as case study participants due to the time commitment involved. Characteristics of the four case study teacher participants are presented in Table 3.

Table 3
Characteristics of Case Study Teacher Participants

	Personal characteristics			Education	Teaching experience	
Teacher	Gender	Race/ ethnicity	First language	Highest degree	Years	Grade levels
1	Female	White	English	Bachelors, Elem Ed	0	Gr. 5 (intern)
2	Female	White	English	Bachelors, Elem Ed	20+	Grades K-3
3	Female	Hispanic	Spanish	Bachelors, Elem Ed	10	Grades 1-6
4	Female	Hispanic	Spanish	Masters, Elem Ed	7	Grades 4-5

Reflecting the student population at Lincoln Heights, two of the four teacher participants are Hispanic and learned English as a second language. The other two teachers identify themselves as White and learned English as their first language. All four teachers are female. There is wide variation in the case study teachers' years of teaching experience, from 0 to over 20 years. The grades these teachers have taught also vary. While all of the case study teachers hold college degrees in elementary education, only one of the teachers has completed a Masters degree. Additionally, this is the first year that any of the case study participants have used a reform-based mathematics program as their primary resource for teaching mathematics.

Selection of School Leader Participants

Leaders at the school and school-district levels were selected for their potential to provide insight into the broader context in which case study teachers were transitioning to the new mathematics program. At the building level, the principal and assistant principal were selected for their knowledge of Lincoln Heights' school culture, history, and the school-level context in which implementation of the new mathematics program was situated. At the district-level, the district mathematics coordinator was selected to provide a district perspective on the mathematics-reform initiative. Additionally, the curriculum specialist assigned to oversee high-need schools in the district, including Lincoln Heights, was selected to provide a district perspective on the needs and challenges presented by this group of schools.

Data Collection Procedures

To answer the research questions posed, data were collected from a variety of sources including observations, interviews, survey instruments, and records of the on-going teacher development project. Table 4 provides an overview of the relationship between the research questions and data sources. In the sections that follow, the instruments used and procedures followed during data collection will be described.

Core Classroom Observations

The focus of this research is on how teachers' beliefs and knowledge influence classroom instruction; therefore analyses center on data from 16 core observations of classroom mathematics instruction, 4 for each teacher. Classroom observations concentrate on the teacher's role and decision-making during each teaching episode, particularly the circumstances within which she supports or limits student thinking. During core observations, extensive fieldnotes were taken and audio-recording was used to capture verbatim dialogue between each teacher and her students. The teacher wore a lapel

Table 4
Relationship Between Research Questions and Data Collection

Research questions	Data sources
1. In what ways and to what extent do teachers support and limit student thinking during mathematics instruction in their first year implementing a reform-based mathematics curriculum?	Classroom observations
2. How do teachers' beliefs influence the ways they incorporate student thinking into their planning for mathematics instruction and on-the-spot instructional decision-making in their first year of implementing a reform-based mathematics curriculum?	Classroom observations, including pre-/post- observation interviews Integrating Mathematics and Pedagogy (IMAP) Web-Based Beliefs-Survey (completed at beginning and end of year) Beginning /end-of-year interviews Records of teacher development project
3. How does teacher knowledge influence the ways teachers incorporate student thinking in their first year of implementing a reform-based mathematics curriculum?	Classroom observations, including pre-/post- observation interviews Beginning/end-of-year interviews Records of teacher development project
4. How does the urban context, as defined by the research literature and perceived by teachers and school leaders, influence mathematics instruction in this urban school?	Pre-/post- observation interviews Records of teacher development project Interviews with school and district leaders Aggregated student demographic data Aggregated State Standardized Test (SST) scores

microphone so that conversations with individual students and small groups could be captured during work time, and a limited-range table microphone was used to record teacher and student comments during whole class discussion. Fieldnotes and transcribed audio-recordings were then integrated to form detailed observation transcripts of each lesson. My role during core observations was one of observer as participant (Merriam, 1998). While I aimed to be as unobtrusive as possible, the teacher and students were aware of my role as a researcher in their classrooms and I moved about freely to listen in on teachers' conversations with children and to observe students as they worked. The protocol used to guide core classroom observations is presented in Table 5.

Table 5
Protocol for Core Classroom Observations

Time	Researcher activity
Before observation	Set up audio-recording equipment Note aspects of the physical environment (seating arrangement, mathematics-related displays, etc.) Note number of students present by gender, and estimate racial/ethnic demographics
During observation	Monitor audio-recording equipment Record fieldnotes of observations throughout the lesson. Focus attention on the teacher's "moves" within the lesson. When students are working in groups or individually, focus observations on the teacher and her interactions with students (giving less emphasis to action that is taking place throughout the classroom). Prioritize recording of the following: • Action sequences of the lesson. Identify time-markers for shifts or changes in the lesson (may be natural shifts or the result of an unanticipated event) • Substance of teacher interaction with students, particularly teacher actions that appear to support and limit student thinking • Instances of on-the-spot decision-making • Representations, examples, and explanations of mathematical ideas (Record ideas put on the board or chart paper) • Student participation in the lesson. Who participated and how? • Teacher response to diversity. Does the teacher's interaction with particular students appear to vary in a systematic manner?
After observation	Collect documents relevant to the lesson (i.e., student sheets, lesson plans) Complete *Reformed Teaching Observation Protocol* Prepare for post-observation interview by identifying aspects of the lesson to probe further (i.e., rationale for decisions made during the lesson, interpretation of student strategies, reflections on student thinking, etc.)

To maximize the potential for drawing relationships among observed lessons, core observations were scheduled when the content of instruction was multiplication and division and the focus was on problem solving. Table 6 presents an overview of the four *Everyday Mathematics* lessons targeted for observation. Multiplication and division were selected as

Table 6
Overview of Observed Lessons

Observation	Title and objective
Fall	
Observation 1	Title: Multiplication Arrays
	Objective: To use arrays, multiplication/division diagrams, and number models to represent and solve multiplication number stories
Observation 2	Title: Division Ties to Multiplication
	Objective: To model division number stories with arrays, multiplication/division diagrams, and number models
Spring	
Observation 3	Title: Extended Facts: Multiplication and Division
	Objective: To multiply by multiples of 10, 100, and 1,000; and to divide such multiples by 1-digit numbers
Observation 4	Title: Use Mental Math to Multiply
	Objective: To use mental math to multiply 1-digit numbers by multidigit numbers

Note. Titles and objectives are drawn directly from the *Third Grade Everyday Mathematics Teacher's Lesson Guide* (Bell et al., 1994).

mathematics topics of focus because the *Third Grade Everyday Mathematics Lesson Guide* (Bell et al., 1994) includes three units on multiplication and division. Therefore, it was reasoned that there would be ample opportunity to observe lessons with this topical focus. Two observations occurred as students were developing initial multiplication and division concepts in the Fall. The other two observations occurred as students were working with multiplication contexts involving multidigit numbers in the Spring. All observations focused on lessons in which *Everyday Mathematics* directed teachers to engage students in solving word problems without explicit instruction followed by class discussion of students' solution strategies. These lessons were selected because they strongly reflected the central tenant of reform-mathematics – to teach mathematics through problem solving – and offered optimal opportunity to observe teachers' actions in relation to student thinking. While observations were scheduled based on the particular lessons identified in the *Everyday Mathematics* program, it is important to note that teachers sometimes modified lessons significantly in response to perceived student needs. In these instances, teachers attempted to maintain a focus on the stated objective of the *Everyday Mathematics* lesson.

Immediately following each core observation, I completed the *Reformed Teaching Observation Protocol* (RTOP) (Sawada et al., 2002) in efforts to capture data on a variety of teaching and learning practices associated with reform-oriented mathematics classrooms. The RTOP is a criterion-referenced instrument containing 25 items. Each item is rated on a scale from 0 (not observed) to 4 (very descriptive). This instrument aims to collect data on five sub-scales: 1) lesson design and implementation, 2) the level of significance and abstraction of the content, 3) the processes that students use to manipulate information, 4) the classroom culture as observed through the nature of communicative interactions, and 5) the nature of student-teacher relationships. These scales are designed to assess the degree to which mathematics instruction is reformed, with higher scores reflecting a greater degree of reform. Data collected using the RTOP instrument facilitated consideration of how teacher actions related to supporting and limiting student thinking were situated within a more general set of classroom dimensions associated with reform-oriented classrooms.

The reliability and validity of the RTOP instrument was established by the instrument developers using data from observations of mathematics and science classrooms at the middle school, high school, and college levels (Piburn & Sawada, 2001). Using data from two trained observers on each of 16 classroom observations, estimates of instrument reliability were determined through a best-fit linear regression of one set of observations on the other. The proportion of variance (R-Squared) accounted for by the best-fit line, which also estimates the reliability, was 0.954. The reliabilities (R-Squared) on each of the five subscales of the RTOP were obtained in the same manner and ranged from 0.670 to 0.915. The face validity of the RTOP was established by designing the instrument to reflect national standards documents in mathematics and science, with an overall emphasis on inquiry-orientation to classroom instruction. The construct validity of the RTOP was tested through a correlational analysis in which each of the five RTOP subscales was used to predict the total instrument score, thus testing the coherence of the theoretically-established underlying dimension inquiry-orientation. The R-Squared scores produced through this analysis ranged from 0.769 to 0.967, providing psychometric support of a strong unifying construct underlying the instrument.

Although the scope of this study did not allow for multiple persons to observe and rate classroom instruction using the RTOP instrument, I made every effort to use the RTOP instrument in a reliable manner. Prior to using the RTOP instrument, I studied the RTOP training manual (Sawada et al., 2000), and completed an on-line version of an observer-training workshop, which was available through the RTOP website (Arizona Collaborative for the Excellence in the Preparation of Teachers, 2002). The on-line workshop entailed using the RTOP instrument to rate three videos of classroom instruction and then comparing these ratings to the annotated ratings of an expert rater. I completed this training process two additional times during the year of this study, in efforts to assign RTOP ratings in the manner intended by the instrument developers. Furthermore, in addition to assigning numerical ratings for each of the 25 items after each observed lesson, I created a written justification for each rating. This justification provided a means to check for internal consistency of ratings among teachers and over time, thus increasing the internal reliability of my ratings.

Pre- and Post-Observation Interviews

Semi-structured pre- and post-observation interviews were conducted to enhance my understanding of teachers' instructional thinking before, during, and after core observations. General interview questions were modeled after items from *A Study Package for Examining and Tracking Changes in Teachers' Knowledge*, published by the National Center for Research on Teacher Learning (Kennedy, Ball, & McDiarmid, 1993). The pre-observation interviews took 15-30 minutes, and the post-observation interviews lasted 30-60 minutes. Audio-tape recording and fieldnotes were used to capture the contents of pre- and post-observation interviews.

The pre-observation interview is designed to probe a teacher's lesson image for the lesson to be observed. This includes elaboration of detail regarding the goals and activities of the lesson as well as elaboration of anticipated student response including potential difficulties and teacher plans to address difficulties. In addition, this interview asks teachers to consider ways in which their planning of the lesson was influenced by their particular teaching context. The protocol used to structure the pre-observation interview is presented in Figure 3.

Figure 3. *Pre-observation interview protocol.*

1. Walk me through, step by step, what you are planning to do when I observe your class.
    - Can you give me more detail on what you and the students will be doing in ________ part of the lesson?
    - Why are you planning to do that?
    - How is this lesson related to the rest of your work in mathematics?

2. Where did you get your ideas for this lesson?
    - If the lesson (or part of the lesson) comes from the *Everyday Mathematics* curriculum or another teacher resource, ask: Is the lesson you plan to facilitate different than what is presented in the teacher-resource materials? If yes, how? Why did you decide to make changes?
    - In your planning of this lesson, did you make any decisions to do things in a certain way based on the urban school context in which you teach?

3. Is there anything in particular that you are hoping to have happen in this lesson?
    - How is it that (<u>what was mentioned in the initial answer</u>) will be accomplished?
    - What will it depend on?
    - What might upset your plan?
    - If the teacher has not already discussed learning goals for the lesson: What do you hope students will learn as a result of the way you have planned this lesson? How will that learning be accomplished?

4. How do you anticipate your students will respond to this lesson?
    - What will be easy and what will be difficult?
    - How do you expect students will think about these tasks? What strategies do you think they will use?
    - In what ways will you find out about how students are thinking about the mathematics in this lesson?
    - How will you support students' mathematical thinking in this lesson?

5. What mathematical knowledge is important for you to teach students these concepts?
    - How comfortable do you feel with your knowledge of this mathematics content? Of how students think about these mathematical ideas?
    - Have you done anything specific to prepare to teach these mathematics concepts?

6. Is there anything else you would like me to know before I come to observe this lesson?

The post-observation interview is designed to elicit a teacher's general reaction following the observed lesson as well as elaboration of her instructional thinking at particular points in the lesson. To accomplish the latter, the post-observation interview includes a *lesson walk* in which fieldnotes completed during the observation are used to reconstruct the lesson and provide a context for teachers to discuss instructional decisions made at various points. For instance, teachers were asked to reveal how they decided who to call on during class discussion of students' strategies for solving mathematics problems. In this interview, teachers were also prompted to consider how the urban context of their teaching influenced the observed lesson. The protocol used to structure the post-observation interview is presented in Figure 4.

Figure 4. *Post-observation interview protocol.*

1. How did you feel things went during the observed lesson?
    - How did things compare to what you had expected? Did anything surprise you?
    - Is there anything that you were particularly pleased about? What? Why?
    - Did anything disappoint you? What? Why?
    - What did the lesson tell you about what your students are learning or still need to learn in mathematics?
    - Were there any ways that you (personally) felt challenged during this lesson? Is there anything in particular that you were working on in your teaching? How confident did you feel about the mathematics content involved in this lesson?
    - What are your next steps with this group?

2. Now I would like to walk through your lesson and ask questions about specific parts. (Use fieldnotes to review lesson, inserting questions about particular aspects of the lesson.)

    Typical probes:
    - The selection of tasks/examples/representations
    - Reasons for teacher moves during different parts of the lesson, especially as related to practices thought to support/limit student thinking
    - Impressions of how students were thinking about the various tasks
    - Why the teacher chose to use certain kinds of grouping arrangements (whole class, independent work, partner work)
    - How the teacher decided whom to call on
    - On-the-spot decision-making

3. Finally, I am interested in how teachers adapt their teaching in urban school classrooms. How did the urban school context influence your teaching of this lesson (if at all)?

4. Was this math class typical of what you are doing in math these days?

Following the pre- and post-observation interviews, fieldnotes were used to guide on-going data collection and analyses. Audio-tapes were transcribed and integrated with observation transcripts for use in additional analyses.

Beliefs Survey

Teachers' beliefs for reformed mathematics teaching were measured at the beginning and end of the year using the *Integrating Mathematics and Pedagogy (IMAP) Web-Based Beliefs-Survey* (Ambrose, Phillip, Chauvot, & Clement, 2003). This instrument is designed to indirectly measure teachers' adherence to seven beliefs aligned with reform-mathematics pedagogy using an interactive web-based platform in which teachers comment on a series of classroom-based scenarios presented through video and text. The seven beliefs measured by the IMAP instrument are presented in Figure 5.

Figure 5. *Seven teacher beliefs measured by the IMAP survey.*

Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).

Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.

Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.

Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.

Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.

Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.

Because individuals are thought to hold beliefs with varying degrees of intensity, the scoring rubrics used to analyze the IMAP Web-Based Beliefs-Survey instrument are designed to differentiate among *strong evidence, evidence, weak evidence, and no evidence* for a respondent's holding of each belief (Integrating Mathematics and Pedagogy, 2004). Each target belief is measured using responses from at least two segments of the survey, allowing opportunity for evidence of respondent's beliefs to be revealed in multiple contexts. A notable strength of this survey is that it overcomes many of the limitations associated with Likert scales by situating responses in scenarios and eliciting open-ended responses that allow respondents to emphasize or downplay issues of personal importance (Ambrose, Phillip et al., 2003). For instance, in one scenario respondents view a video of a teaching episode and are then asked to identify what stands out for them before being prompted to specifically identify particular weaknesses of the teaching episode. In turn, those respondents who identify particular weaknesses before being prompted to do so are considered to demonstrate greater evidence of holding the associated belief than those who only note weaknesses after being prompted. The web-based platform used for this survey does not allow respondents to revise previous responses, thus making it possible for respondents' initial reactions to be captured prior to their thinking being influenced by subsequent questions that might more fully reveal the survey designers intentions. The IMAP Web-Based Beliefs-Survey can be viewed in its on-line format at the IMAP web-site (Integrating Mathematics and Pedagogy, 2003).

The reliability and validity of the IMAP Web-Based Beliefs-Survey instrument was established by survey developers in multiple ways (Ambrose, Clement, Phillip, & Chauvot, 2004). First, the survey was reviewed by a panel of 15 mathematics educators, who agreed that the survey segments provided sufficient opportunity to elicit evidence of the target beliefs identified. This expert panel also reviewed pilot data to verify that participants' responses provided information on the target beliefs. The validity of the survey was further established through a pilot study in which 15 undergraduated students enrolled in a mathematics education course completed the survey in its web-based format and then discussed survey segments in greater depth during individual interviews. Furthermore, the 15 participants' beliefs were studied through analysis of class discussion and written work during the semester-long mathematics education course and through periodic interviews that were not directly related to the survey. It was determined that the 15 participants' beliefs as measured by the IMAP instrument were consistent with this interview and observational data, thus providing evidence of instrument validity. Finally, the reliability of the IMAP Web-Based Beliefs-Survey instrument was established through a study in which multiple coders rated 20% of the responses of 159 participants who completed the survey before and after taking part in one of five treatments. The mean interrater reliability for the 17 rubrics associated with the survey was 84%.

In the research of focus in this manuscript, it took the four case-study teachers 60-90 minutes to complete the IMAP Web-Based Beliefs-Survey at the beginning and end-of-year data points.

Teacher Knowledge Interview

To explore each teacher's knowledge of mathematics for reformed teaching, a two-part, semi-structured interview was used at the beginning and end of the school year. A summary of the knowledge interview protocol is provided in Figure 6.

Figure 6. *Summary of knowledge interview protocol.*

Open-ended Interview: Knowledge of Teaching and Learning Multiplication

Probes:

- What are the important concepts or understandings that students need to develop related to multiplication?
- Is there an order to understanding these ideas? How do students develop these understandings?
- Is there anything that children need to be able to do or understand before they are ready to learn about multiplication?
- What makes learning multiplication tricky for some children? What do you have to watch out for?
- How do you think a teacher should go about teaching multiplication?

Classroom Scenario 1: Knowledge of Non-standard Strategies

One of the goals of the Everyday Mathematics curriculum is to get students to solve problems in many different ways. What are some different strategies that children might use to solve the following story problem? (Teacher records strategies.)

> *Kristen and Amber were setting up some chairs for a play. They made 16 rows of chairs. Each row had 8 chairs. How many chairs did Kristen and Amber set up?*

Follow-up Probes:

- How does this strategy work?
- What does a child need to know or be able to do to use this strategy?
- Which of these strategies indicates the most sophisticated understanding? The least sophisticated understanding? Why?

Classroom Scenario 2: Interpretation of Students' Mathematical Strategies[a]

Using student work provided, teachers are asked to interpret how three students solve the following partitive division problem, which results in a fraction:

> *Twenty-four children want to share eight pancakes so that each one gets the same amount. How much pancake can each child have?*

Follow-up Probes:

- What mathematics do you think each child understands based on the strategy used to solve this problem?
- What questions might you ask these students to find out more about their mathematical understanding?

Classroom Scenario 3: Addressing and Avoiding a Common Student Error[b]

Teachers are presented with a representative sample of sixth-grade student work in which the standard U.S. multiplication algorithm is executed without maintaining the place values of the partial products.

Probes:

- What would you do if you noticed that several of your students were doing this?
- As a third-grade teacher, what do you do to ensure that your students do not make these kinds of errors when they get to sixth grade?

Classroom Scenario 4: Interpretation of and Response to a Flawed Solution[c]

Teachers are presented with student work and a student's explanation of an unusual flawed solution to the division problem $144 \div 8$.

Probes:

- How would you respond to this student?
- Where is the students' mistake? What does the mistake suggest about the students' understanding?
- Could the strategy be modified to solve the problem? If yes, how?

[a]The pancake problem and student work used for Classroom Scenario 2 are from Empson (2001).
[b]Classroom Scenario 3 is an interview item from Kennedy, Ball, and McDiarmid (1993).
[c]Classroom Scenario 4 is an interview item from Empson and Junk (2004).

The first part of this interview uses open-ended questions to probe a teacher's knowledge of mathematics for reformed teaching specifically related to multiplication concepts taught in third grade. Teachers are asked to describe key concepts related to multiplication, how children learn these concepts, and how multiplication should be taught. In the remainder of the interview, a teacher's knowledge of mathematics for reformed teaching is further explored as the teacher is presented with specific scenarios set in multiplication and division instructional contexts and prompted to describe how they might respond. Drawing on scenarios developed by Kennedy, Ball, and McDiarmid (1993) and Empson and Junk (2004), teachers are asked to anticipate the varied ways children might solve a particular problem, interpret student work, and respond to students' difficulties.

The teacher knowledge interviews took teachers 75-90 minutes to complete at each data point. Some teachers chose to complete this interview over multiple sessions. Audio-tape recording and fieldnotes were used to capture the contents of the teacher knowledge interviews. Fieldnotes were used to guide on-going data collection and analyses. Audio-tapes were transcribed and integrated with data from fieldnotes for use in additional analyses.

Records of the Teacher Development Project

Throughout the year of this study, records were collected of formal and informal aspects of the teacher development project and collaboration among grade-level teachers and researchers. Data sources include written researcher reflections on professional development meetings, fieldnotes describing in-class support activities, fieldnotes describing observations of mathematics teaching[4], and a log of conversations and emails between teacher educators and third-grade teachers. These data sources were used to further understand teachers' beliefs, knowledge, and instructional practices.

Interviews with School Leaders

Interviews with school and school-district leaders were conducted to better understand the context in which the case study teachers were working. Interviews with the principal and assistant principal at Lincoln Heights were designed to elicit the respondents' perspectives on school history and culture as well as current issues related to mathematics teaching and learning at Lincoln Heights. These interviews were conducted in the Fall semester and took 2-3 hours. Interviews with the district mathematics coordinator and a district-level curriculum specialist were designed to gain a district-level perspective on the transition to the *Everyday Mathematics* program as well as an understanding of the perceived needs and challenges of high-need schools in the school district. These interviews were completed in the Spring semester. The interview with the district mathematics coordinator was completed over two, one hour meetings. The interview with the district-level curriculum specialist took one hour. Sample questions from these interviews are presented in Table 7.

During all interviews with school and school-district leaders, audio-tape and fieldnotes were used to capture the contents of the interviews. Following these interviews, fieldnotes were used to guide on-going data collection and analyses. Audio-tapes were transcribed for use in additional analyses.

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[4] This refers to observations of mathematics teaching in addition to core observations. These observations usually occurred spontaneously, before or after in-class support or an interview. Beyond core observations, each case study teacher's mathematics instruction was observed three to six times.

Table 7

Sample Questions from Interviews with School and School-District Leaders

Group	Sample interview questions
School leaders	School history and culture: • Who are the students that come to this school? • What have been the challenges at this school historically and in the present? • What do you perceive as the current strengths of this school? • How are decisions, particularly about what is taught, made at this school? • What, if any, “pressures” come from the district and state leadership and how they influence what goes on at this school? • In what ways and to what extent do the accountability measures related to the State Standardized Test influence the way things are done at this school? Mathematics-related school improvement efforts: • What initiatives are underway to improve mathematics instruction this year? What is the intent of each? From your vantage point, how are teachers and students responding to these initiatives? • What is your involvement with improving mathematics instruction? How do you see your role? • When you observe a teacher’s mathematics instruction, what are you looking for? What would you consider to be the characteristics of a good mathematics lesson?
School-district leaders	District history in relation to mathematics teaching and learning: • How would you characterize elementary school student achievement in mathematics in this school district, both currently and over the last 10 years? How would you characterize student achievement in urban schools compared to other schools in the district? To what do you attribute this characterization? • What is the level of priority of mathematics instruction in your district currently? What has it been historically (last 10 years)? • What are the district-level expectations for mathematics instruction? To what extent have these remained constant or changed? • What factors (external to the school) influence the way mathematics is taught? How have the factors influencing mathematics instruction in this district changed over the last 10 years? • What current initiatives are underway to improve mathematics instruction? District adoption of *Everyday Mathematics*: • Why was this particular curriculum adopted? • How is this curriculum the same and different from what teachers were using last year? I am especially interested in your perception of the change for teachers in urban schools. • What do you see as the major challenges elementary teachers face in transitioning to the *Everyday Mathematics* program? Are there any challenges that you see as specific to urban schools? • What is the nature of district-level support during this transition year to the *Everyday Mathematics* program? What is the district doing to support teachers? Schools? Administrators?

Student Data

Lincoln Heights' student population has a history of mobility; therefore aggregated data on the demographics of each case study teacher's class were collected from teachers at the end of each quarter. Because the case study teachers' classes remained relatively stable during the year of this study, this study only reports aggregated class demographic data that were collected halfway through the year. Demographic data on the student population at Lincoln Heights were collected from Lincoln Heights' principal and, when possible, were verified through publicly available information on the state department of education website.

At the end of the school year, aggregated data related to students' achievement on the mathematics section of the State Standardized Test[5] (SST) were collected. The SST is a criterion-referenced achievement test in which students receive scores between 1 and 5, with a score of 3 or higher considered passing. The test is administered to students in late-March, at the end of the third quarter of the school year. Aggregated data on each case study teacher's class were collected from Lincoln Heights' principal. Aggregated third-grade SST data for Lincoln Heights, the school-district, and the state were collected from the principal and verified through publicly available information on the state department of education website.

Data Management Procedures

Data collection yielded a combination of products in the form of paper and electronic documents as well as numerous audio-recordings. To organize the voluminous amount of

[5] State Standardized Test (SST) is a generic pseudonym that will be used throughout this manuscript in place of the actual test name.

data and facilitate systematic data analysis, I engaged in the activities listed below:

- All audio-recordings were transcribed, integrated with fieldnotes, and saved as text documents. I personally transcribed audio-recordings in which an understanding of the data was partially reliant on fieldnotes, including all observations, post-observation interviews, and the second part of the Teacher Knowledge interview. Three transcribers supported transcription of interviews that consisted primarily of dialogue between the interviewee and me. I reviewed and made minor revisions to transcripts completed by others, being especially attentive to places in the transcripts where transcribers indicated uncertainty in the verbatim dialogue.
- Next, the transcripts of teachers' comments from pre- and post-observation interviews were nested within observation transcripts. For instance, a teacher's comments from a post-observation interview justifying mathematical strategies emphasized in a particular class discussion were nested within the observation transcript at the point of the lesson segment of focus. This process of nesting teachers' commentary about a lesson within the transcript of real-time instruction facilitated easy access to data revealing teachers' instructional thinking in relation to actual instructional practice being discussed.
- After all data sources had been converted into text documents, they were imported into the software package ATLAS.ti (Muhr, 2004) in preparation for systematic data analysis. This text analysis software package facilitates the iterative process of creating codes from data, refining those codes, and looking for meaningful patterns across sources (Muhr, 1991).

Data Analysis Procedures

The purpose of this study is to illuminate how teachers' beliefs and knowledge influence the way they support and limit student thinking in their first year of implementing a reform-based mathematics curriculum at an urban school. Through a collective case study approach, multiple data sources were used to inform analyses as themes and patterns were generated within case studies of individual teachers as well as through comparative analysis across cases (Merriam, 1998; Yin, 2003). Triangulation among data sources was used to construct and interrogate emerging themes throughout a process of iterative data collection and analysis (Mathison, 1988).

Analysis during Data Collection

In keeping with qualitative research design (Lincoln & Guba, 1985), I began analyzing data as soon as I collected them. Fieldnotes from the observations and interviews were read holistically in efforts to understand how teachers' beliefs and knowledge were related to instructional practices that support and limit student thinking. I was especially attentive to instances in observed lessons when instruction seemed to move away from a teacher's lesson image, as these moments are thought to be especially revealing of a teacher's goals, knowledge, and beliefs (Aguirre & Speer, 2000). One aspect of instruction that emerged through this process as particularly interesting was teachers' response to student difficulties.

The beginning-of-year measures of beliefs and knowledge and data from professional development meetings provided an initial understanding of each case study teacher's beliefs and knowledge. These early understandings served as a lens that shaped my subsequent interviews with and observations of each teacher. As conjectures about the nature of and interaction among teachers' beliefs, knowledge, and instructional practice were formulated, I continuously sought evidence to confirm or disconfirm these conjectures. In particular, the lesson walk portion of post-observation interview provided opportunity for me to directly ask teachers to explain their instructional thinking in relation to specific actions within observed lessons. This, in turn, led to revision of conjectures and development of new conjectures. Additionally, themes that emerged through study of one teacher were intentionally explored in my observations of and interviews with the other teachers.

Analysis after Data Collection

In addition to the informal analysis I engaged in while collecting data, I conducted formal and systematic data analysis after all data were collected, transcribed, and integrated with fieldnotes. In the sections that follow, I present the details of various aspects of my research analyses after data collection was completed, including the purpose and process for conducting each analysis. For the sake of clarity, I discuss these analyses as if they were discrete and linear events. However, in reality, different aspects of the analysis overlapped and some analyses occurred concurrently. In actuality, I experienced the relationships between these analyses as recursive and cumulative, each informing the others.

Analysis of Classroom Instruction

Because the heart of my research questions focus on how knowledge and beliefs influence instruction, my analysis began with trying to understand each teacher's instructional practice. First, I reviewed observation transcripts holistically, making note of emerging themes related to my interest in student thinking. Then I engaged in an iterative decomposition of each core observation transcript, parsing it into chunks or action sequences based on the natural segments of the lesson (Schoenfeld, 1998). Through this process, action sequences of varying grain size were identified. A summary example of lesson decomposition is presented in Figure 7.

Figure 7. *Sample decomposition of lesson into action sequences of varying grain size.*

0.00 (0.20) Teacher introduces mathematics task (provides working instructions only)

(1.10) Students begin working, teacher circulates and observes.

- (2.05) B1 asks teacher question
- (2.30) Teacher elicits explanation of solution strategy from B2
- (2.50) Teacher elicits explanation of solution strategy from G1
- (3.45) Teacher elicits explanation of (flawed) solution strategy from B3 & B4
  - B3 & B4 provide minimal response
  - Teacher reviews problem
  - B3 devises a new strategy, teacher encourages record keeping to keep track of solution
- (4.40) Teacher tells G2 that she needs to show work
- (4.55) Teacher solicits explanation of solution strategy from B5's table

5.00 (6.00) Teacher initiates whole group discussion

- Teacher praises desirable behaviors
- (6.25) Teacher asks G2 to read the problem aloud
- (6.55) Teacher elicits description of solution strategy from a representative of group 1
  - G3 shares group 1 strategy
  - Teacher records strategy on board
- (7.40) Teacher requests another way – B7 volunteers an alternate strategy (group 2 solution)
  - Teacher demonstrates B7's solution strategy
- (8.30) Teacher elicits group 3 solution strategy
  - B5 describes group 3's strategy
  - (9.35) Teacher asks if someone else can explain why this solution works
    - G4 provides a flawed explanation
    - 10.00 Teacher suggests a conceptual way to think about the solution
  - (10.35) G5 says that she doesn't understand how group 3's strategy works
    - (11.00) Interruption by B6, behavior
    - (11.10) Teacher reiterates conceptual explanation of strategy, asking students questions as she goes
- (13.25) B1 comments on group 1's solution and relates to a new solution idea
  - (13.35) Teacher says that G8 had a similar approach and asks her to share
  - (14.20) B2 comments that he likes that strategy, teacher asks him to elaborate
  - (14.45) G5 agrees with B2

Note. Bullet indentation is used to indicate action sequences imbedded within larger action sequences.

In this 15-minute block of instruction, there were three major action sequences: introduction of mathematics task, student work time, and class discussion. Within the student work time and class discussion, action sequences of a finer grain size were identified. For instance, the whole group discussion (starting at 6.00 minutes) is organized around discussion of four student solution strategies to a mathematics task. The discourse related to each of these strategies was identified as a bounded action sequence within the larger action sequence of class discussion. Within an action sequence around a particular solution strategy, there are action sequences of grain sizes that are smaller yet, such as teacher records of student ideas and student response to teacher questions.

During this initial review of observation transcripts, I also coded instances when teachers engaged in actions that appeared to support or limit student thinking. These codes were drawn from my review of the research literature as well as from the data themselves. To illustrate, a sample of codes used to identify teachers' actions supporting and limiting student thinking within the observation transcripts are presented in Table 8.

Table 8
Sample of Codes Used for Teacher Actions that Support and Limit Student Thinking

Supporting actions	Limiting actions
SpptST-ElctMltSltn: Teacher elicits multiple solutions for a problem posed	LmtST-SnglMthd: Teacher directs students to use a particular method to solve a problem.
SpptST-PrblmCntxt: Teacher encourages focus on problem context (to solve contextualized mathematics task)	LmtST-IgnrPrblmCntxt: Teacher encourages student to use a method that minimizes focus on problem context (to solve contextualized mathematics task)
SpptST-SggstRsrc: Teacher suggests use of resources to support problem solving other than direct teacher help (i.e., tools, problem solving chart, conversation with peer)	LmtST-HvyHlp: Teacher provides "heavy help" by directing student step-by-step through problem solving
SpptST-CmprSltn: Teacher prompts students to compare and contrast multiple solution methods	LmtST- LgtmtSltnIncrrct: Teacher treats legitimate solution as incorrect

Through this analysis, I aimed to devise an understanding of how teachers' actions and non-actions supported and limited student thinking. For instance, the action of prompting students to engage in comparison among students' multiple solution strategies was considered supportive of student thinking, while rarely or never engaging in this practice was considered a non-action that served to limit student thinking. As new codes were developed in response to one teacher's instructional practice, observation transcripts from the other teachers' classrooms were reviewed and coded accordingly.

Through these initial stages of data analysis, teacher response to students' difficulties emerged as an interesting aspect of instruction in which there was significant variation among teachers with regard to actions supporting and limiting student thinking. Consequently, teacher response to student difficulties was an aspect of instruction identified for more fine-tuned analysis. First, I identified action sequences in which teachers encountered and responded to (or did not respond to) student difficulties. Next I developed a coding scheme to classify these action sequences based on the nature of the student difficulties encountered by teachers, with attention to the part of lesson in which the difficulty surfaced (i.e., difficulty evidenced through flawed solution presented in whole class discussion). This coding scheme is presented in Table 9. Coded action sequences included the occurrence or identification of the particular student difficulty, the teacher's response, and any text deemed relevant to understanding the context of the teacher's response. In addition, memos were attached to the identified action sequences providing additional information considered relevant to understanding the context in which each action sequence was located.

Table 9
Codes Used to Identify Occurrence of Student Difficulty

Part of lesson	Student difficulty codes and definitions
Student work time	StdDiff-WrkTm-GttngStrtd: The teacher perceives a student/partnership/small group having difficulty getting started on a task.
	StdDiff-WrkTm-Stck: The teacher perceives a student/partnership/small group is stuck while working on a task.
	StdDiff-WrkTm-FlwdSltn: The teacher perceives a student/partnership/small group is devising or has devised a flawed solution strategy.
	StdDiff-WrkTm-IncrrctRspns: The teacher observes a student provide an incorrect response to question posed another student or herself. This code also applies when the teacher treats a correct response as incorrect.
	StdDiff-WrkTm-FlwdCmmnt: The teacher observes a student make a comment that is flawed. This code also applies when a correct comment is treated as flawed by the teacher.
Whole group time	StdDiff-WhGrp-FlwdSltn: A flawed solution is made public to the class (through presentation by a student or teacher OR through a record on the board or white board). This code also applies when a correct solution is treated as flawed by the teacher.
	StdDiff-WhGrp-FlwdExplntn: A student's explanation of a correct solution is flawed. This code also applies when a correct explanation is treated as flawed by the teacher.
	StdDiff-WhGrp-IncrrctRspns: A student provides an incorrect response to a question (or incorrect responses to multiple related questions) posed by the teacher or a student. This code also applies when correct responses are treated as incorrect.
	StdDiff-WhGrp-FlwdCmmnt: A student makes a comment that is incorrect or flawed. This code also applies when a legitimate comment is treated as flawed.

Next the constant comparative method (Glaser & Strauss, 1967) was used to analyze these sets of action sequences involving student difficulties in order to construct an understanding of the typical patterns of response for each teacher. Consideration was given to patterns of response observed in the Fall and Spring lessons separately and together, as it was considered possible that teachers' instructional responses might change over the course of the school year. Action sequences that stood in contrast to typical patterns of response were also identified at this time. As case stories detailing each teacher's ways of responding to student difficulties emerged out of analysis of core observation transcripts, records of other observations of mathematics teaching were reviewed for evidence that supported or stood in contrast to identified response patterns. Throughout this process of data analysis, themes that emerged in one teacher's instruction were explored across the other cases.

Finally, quantitative analysis of the ratings collected using the *Reformed Teaching Observation Protocol* (RTOP) (Sawada et al., 2002) were completed in order to consider how each teacher's patterns of response to student difficulties were situated in a broader set of characteristics associated with reform-oriented mathematics classrooms. First, mean scores on each of the RTOP's 25 items were calculated for Fall and Spring observations. Next, item scores were summed to determine five subscale scores, with each subscale representing a particular aspect of reformed teaching. These subscale scores were then summed to establish a total score between 0 and 100, with higher scores reflecting instruction more closely aligned with principles of teaching and learning associated with current reforms in mathematics education.

Analysis of Teacher Beliefs

Teachers completed the *Integrating Mathematics and Pedagogy (IMAP) Web-based Beliefs Survey* (Ambrose, Phillip et al., 2003) at the beginning and end of the school year. Following survey completion at each data point, a set of rubrics that accompany the survey were used as directed to evaluate seven target beliefs considered central to reform-oriented mathematics teaching (Integrating Mathematics and Pedagogy, 2004). Prior to using these rubrics to evaluate the responses of case study teachers, I completed an extensive set of training exercises, in order to become oriented to the nuances associated with each rubric and to ensure consistency with the survey designers' intentions. Next, each of 17 *segment rubrics* were used to rate the degree to which case study teachers' responses in a given segment provided evidence that each teacher held the particular target belief. Finally, each of seven *beliefs rubrics* (one for each target belief) were used to determine a numeric score indicating evidence of the teachers' adherence to each target belief based on the multiple segment rubrics associated with the given belief. The beliefs rubrics differentiated evidence of a teacher's beliefs using a scale ranging from 0 (no evidence of belief) to 3 (strong evidence of belief). Finally, a descriptive portrait of each teacher's beliefs, as suggested by survey responses and reflected in numeric ratings, was constructed for use in further data analyses.

In addition to the portrait of teachers' beliefs suggested by the IMAP Web-Based Beliefs-Survey, interviews and records of the teacher development project were examined to identify teachers' professed beliefs. Through these sources, additional data on teachers' beliefs about what school mathematics is important, how children learn mathematics, the role of teachers and students during mathematics instruction, and specific instructional practices were identified. These data sources also provided insight into teachers' beliefs about their current teaching contexts and students. Data were coded in accordance with these different kinds of beliefs and organized for use in further analyses. When data were available on a given dimension from the Fall and Spring data points, consideration was given to evidence of change in beliefs over the course of the year.

Analysis of Teacher Knowledge

Beginning and end-of-year responses to the teacher knowledge interview were analyzed in relation to the research literature on knowledge theorized to be supportive of teaching multiplication concepts in a reform-oriented manner. Specifically, the following six facets of teacher knowledge related to the teaching and learning of multiplication concepts were explored:

- Knowledge of big ideas related to multiplication
- Knowledge of student strategies – for problems involving basic facts and multidigit numbers
- Knowledge of student learning trajectories related to developing understanding of multiplication
- Knowledge of common student difficulties
- Knowledge of teaching strategies to support development of conceptual understanding
- Ability to use mathematical knowledge to interpret student work

First, interviews were coded according to these different facets of teacher knowledge. Then interview responses at the beginning and end of the year were analyzed separately, and a summary of teacher knowledge suggested by each interview was constructed. Next, these summaries of teacher knowledge at the beginning and end-of-year data points were analyzed in relation to each other to consider the extent to which there was evidence of change in teacher knowledge.

Analysis of Student Mathematics Achievement

The mathematics achievement of each teacher's class was measured using aggregated student scores on the mathematics section of the State Standardized Test (SST). The aggregated scores for each teacher's class were considered in relation to the aggregated scores of third graders at Lincoln Heights, in the school district, and in the state.

Development of Case Stories

Following separate analyses of teachers' beliefs, knowledge, and instructional practices, case stories were developed to examine the interactive relationship of these variables. In broad strokes, this analysis involved identifying evidence of each teacher's beliefs and knowledge that might explain the teacher's patterns of response when various kinds of student difficulties arose in her classroom. For a given pattern of response to student difficulties, I first reviewed the pre- and post-observation interview data surrounding the action sequences associated with the response pattern. Through this analysis, teacher commentary on instructional decision-making related to the particular action sequences was considered. Next I examined how beliefs and knowledge data collected over the course of the year might be related to the particular pattern of response. Through this process, I sought to establish key beliefs and aspects of knowledge that appeared to drive the particular pattern of instructional practice as well as the interplay between beliefs and knowledge in these instances. In other words, I attempted to discern when a given belief or aspect of knowledge seemed dominant in determining a particular response pattern and when beliefs and knowledge seemed equally in play. The outcome of these analyses was construction of case stories offering a theoretical explanation of how each teacher's ways of responding to student difficulty were related to their beliefs and knowledge. At multiple points during the process of case story development, I presented drafts of the case stories to a peer debriefer who, in turn, interogatted the case stories in relation to my data corpus. Through this process of peer debriefing, I sought to ensure that the case stories were comprehensive and reflective of my data.

Cross-case Analysis

Throughout data analysis, themes that emerged in one teacher's instruction were explored across the other cases. After interpretive case stories were developed for each teacher, focus shifted to examining patterns of similarities and differences across cases. Specifically, four analyses across cases were performed.

First, occurrences of teacher actions that support and limit student thinking were identified across cases. To accomplish this analysis, excerpts from observation transcripts marked with codes used to identify teachers' actions supporting and limiting student thinking were collected across cases by code. Through review of these excerpts, teachers' actions that support and limit student thinking were sorted into two groups: those that were observed often in the classrooms of at least three of the teachers studied and those that were observed less frequently. Then consideration was given to what these sets of teacher actions have in common, and a theoretical explanation suggesting why some practices are more readily utilized than others was devised.

The second cross-case analyses involved reviewing the case stories to discern patterns in how teachers' beliefs and knowledge influenced their response to student difficulties. Through this review, an attempt was made to conceptualize the data across cases by first identifying dimensions of teachers' response to student difficulties that served to capture the response patterns of the four teachers studied. Next, the theoretical explanations for how beliefs and knowledge influenced teachers' actions were examined in relation to each dimension of teacher response. In particular, effort was made to identify aspects of teachers' beliefs and knowledge that appeared to facilitate or limit teachers' actions related to the particular dimension of response to student difficulties. Through this analysis, a theoretical explanation was developed to explain how teacher beliefs and knowledge drove particular teaching practices as well as the aspects of teacher actions related to student difficulties most influenced by teacher beliefs and those most influenced by teacher knowledge.

The third cross-case analysis involved examining change in teachers' beliefs, knowledge, and classroom mathematics teaching during the school year and considering what the data corpus suggests about the factors influencing that change. First, beginning and end of year measures of beliefs and knowledge as well as Fall and Spring scores on the Reformed Teaching Observation Protocol were reviewed to determine the extent to which and the nature of change in teachers' beliefs, knowledge, and teaching. Next, data from interviews, professional development meetings, records of informal conversations, and other sources were reviewed for teachers' commentary related to changes in their beliefs, knowledge, and teaching practice. Also, teachers' classroom experiences, as detailed in the case stories, were taken into account. Through this process, a theoretical explanation of teachers' differing levels of change was constructed with particular consideration of how teachers' classroom experiences influenced change in beliefs and knowledge.

The final cross-case analysis aimed to identify how the urban school context influenced mathematics instruction at the school in which this study was situated. First, consideration was given to the ways in which Lincoln Heights Elementary reflects the research literature on urban schools, particular regarding the challenges these schools face. Next, the data corpus was reviewed for instances when teachers and school leaders provided commentary on how their work was influenced by their particular teaching contexts and the school context. In particular, attention was given to instances when teachers referenced the needs or nature of students *at this school* or *in my class* as shaping mathematics instruction. Also, instances in which teachers pointed to external factors as supporting or limiting their efforts to teach mathematics were identified. Throughout this analysis, consideration was given to how and to what extent particular factors influenced the beliefs, knowledge, and mathematics teaching practices of the four teachers studied. This analysis resulted in construction of a theoretical explanation of the role the urban context played in teachers' transitions to a reform-based mathematics curriculum, particularly its relationship to teachers' beliefs, knowledge, and mathematics teaching.

CHAPTER III

FINDINGS

A year-long collective case study of four third-grade teachers in an urban school was undertaken to address the following research questions:

1. In what ways and to what extent do teachers support and limit student thinking during mathematics instruction in their first year implementing a reform-based mathematics curriculum?
2. How do teachers' beliefs influence the ways they incorporate student thinking into their planning for mathematics instruction and on-the-spot instructional decision-making in their first year of implementing a reform-based mathematics curriculum?
3. How does teacher knowledge influence the ways teachers incorporate student thinking in their first year of implementing a reform-based mathematics curriculum?
4. How does the urban context, as defined by the research literature and perceived by teachers and school leaders, influence mathematics instruction in this urban school?

This chapter presents research findings beginning with a description of Lincoln Heights Elementary School, the school in which the case study teachers work. Next each case study teacher's background and teaching context during the year of this study will be introduced. This will be followed by a case study of each of the four teacher participants. Each case study begins with discussion of the teacher's beliefs and knowledge at the beginning and end of the school year. Then a case story detailing the teacher's patterns of response to student difficulties and their relationship to the teacher's beliefs and knowledge is presented. Finally, findings from broader measures of teaching are offered, allowing the case story to be considered within a more general understanding of the teacher's mathematics instruction and students' learning.

The School Context: Lincoln Heights Elementary

The four teachers of focus in this research taught third grade at Lincoln Heights Elementary (a pseudonym) during the year of this study. In this section, a portrait of the school context in which these teachers worked will be presented to provide a backdrop for consideration of each individual case. First, attention will be given to the school culture, history, and demographics of Lincoln Heights during this research. This will be followed by an account of the mathematics-related new initiatives and mathematics teacher development project that took place at Lincoln Heights during the year of this study.

School Culture, History, and Demographics

Located in a large urban school district in the Southeastern United States, Lincoln Heights consists of a single-level main building and a maze of adjacent portable classrooms. Walking through the open-air hallways before and after school, faculty and staff engage in friendly exchanges with each other, children, and parents. There is a general feeling that this is a place where people will care about you.

Although Lincoln Heights is situated within a residential neighborhood of small houses, most of the students attending Lincoln Heights reside in nearby apartments and rental properties catering to low-income families. Lincoln Heights serves 568 students from kindergarten to grade 5, with 86% of those students qualifying for free or reduced lunch. The campus also houses two pre-kindergarten classes that are not included in the elementary population count. Based on the low socioeconomic status of the student population, Lincoln Heights is a designated Title I school that receives supplementary funding from the federal government.

Approximately three-fourths of the students at Lincoln Heights Elementary are Hispanic and many are recent immigrants to the U.S. from countries in the Caribbean islands or Central and South America. The language needs of Lincoln Heights' student population are significant, with most students having a first language other than English and 59% identified as Limited English Proficient (LEP). Reflecting the majority Hispanic student population, the principal at Lincoln Heights is Hispanic as are many of the teachers. It is common to hear both English and Spanish being spoken by children, parents, faculty, and staff in the hallways and classrooms. At each grade level, there are at least two transitional language classes designated for LEP identified students. In these classes, children receive instruction in both English and Spanish. In addition to the 74% of Lincoln Heights students who are Hispanic, the student population consists of 12% Black students, 10% White students, and 4% from other racial/ethnic designations.

Lincoln Heights is a school with a recent history of school failure, having been designated an "F" school by the state's school grading system five years prior to this study. School grades are determined based on student performance in reading, mathematics, and writing on the State Standardized Test (SST). Schools can raise their school grade in two ways: 1) by increasing the number of students achieving passing scores on the SST, and 2) by increasing the number of students who demonstrate improvement from year to year on the SST.

Since the time Lincoln Heights was designated an “F” school, five years prior to this study, steady improvements have been made and Lincoln Heights has held the grade of “C” for the two years preceeding this research. School administrators attribute recent improvement in school grade to a variety of factors including introduction of a school-wide behavior management system, professional development on working with impoverished and LEP student populations, and implementation of a scripted reading program. Still, approximately 20% of the third grade students from the previous year were retained in third grade during the year of this study because they did not achieve a passing score on the reading section of the SST. Despite the improvement in recent years, there is significant pressure on teachers and administrators to continue to raise student achievement scores on the SST. For instance, this pressure has led administrators to eliminate daily recess for students in tested grades.

In total, there are 81 faculty and staff at Lincoln Heights, including 51 teachers. As a group, the teachers’ years of teaching experience vary widely. Most of the persons that interview for teaching positions at Lincoln Heights have limited or no teaching experience. While Lincoln Heights’ administrators seek to hire experienced teachers with rich curricular knowledge, they are most focused on finding teachers who will provide a caring presence in their students’ lives and who will be open to professional development on how to teach. As evidence, consider the following comment made by one of Lincoln Heights’ administrators:

> [When hiring teachers] the curriculum knowledge is important, but I think that when you are in a school like ours, the other things are more important. Because if you have all the other things – if you have the passion and commitment, we can teach you how to do the reading program.

At Lincoln Heights, curricular decision-making is highly centralized and a great deal of emphasis is placed on mandatory professional development related to school curricular initiatives. Lincoln Heights' teachers generally express appreciation for the many professional development opportunities at the school. However, teachers also report being overwhelmed by the multitude of simultaneous initiatives that they are expected to learn about and implement. Mathematics-related initiatives underway during the year of this study will be described next followed by discussion of the mathematics teacher development project in which teachers participated during the year of this study.

Mathematics-Related New Initiatives

Three major mathematics-related initiatives were implemented for the first time in the year of this study: 1) the *Everyday Mathematics* program (Bell et al., 1994), 2) mini-lessons and assessments tied to state benchmarks, and 3) *SuccessMaker* (Computer Curriculum Corporation, 2002), an individualized computer-based instructional program. Each of these new initiatives will be discussed in turn.

First, Lincoln Heights teachers began to use the newly adopted *Everyday Mathematics* program (Bell et al., 1994) along with most of the other elementary schools in the school-district during the year of this study. According to the district mathematics coordinator, the intent of this new adoption was to encourage teachers to move away from traditional textbook-based mathematics teaching practices toward the reform-based mathematics instructional practices described in the NCTM standards documents (1989, 2000). According to school-district leaders, this change was anticipated to be a challenging one due to the significant shifts in the ways teachers would need to think about mathematics instruction and prepare for daily mathematics lessons. Reflecting these claims, the teachers at Lincoln Heights express that they find *Everyday Mathematics* to be a challenging program to teach. Teachers indicate that they must spend significantly more time preparing to teach *Everyday Mathematics* lessons than with previous programs and that, even then, they are sometimes unsure of how to put into practice the ideas presented in the teacher guide.

The second mathematics-related new initiative during the year of this study was implementation of a program of mini-lessons and assessments tied to state benchmarks for mathematics (and reading). All of the schools in the district serving at-risk student populations, including Lincoln Heights, were required to participate in this program of extra instruction and frequent assessment, using tests provided by the school-district. The mathematics part of this testing included bi-weekly short assessments on select state benchmarks and longer, comprehensive assessments administered three times during the year. Lincoln Heights' teachers were responsible for providing daily mini-lessons to prepare students for these mathematics tests. This district-level initiative was put in place during the year of this study in response to state mandates to devise a data-driven system of continuous school improvement. Therefore, the school-district vision for these tests was that they would help teachers to understand their students' academic needs and make informed instructional decisions that would lead to gains in student achievement. However, the teachers studied viewed these tests and mini-lessons as one more thing to fit in to their limited instructional time. Furthermore, teachers regularly identified test items that they viewed as inappropriate or ambiguous, thus limiting the degree to which they considered test results to be valid.

The third mathematics-related initiative in the year of this study was implementation of the individualized computer-based instructional program *SuccessMaker* (Computer Curriculum Corporation, 2002). Lincoln Heights' administrators purchased this program because they believed it would positively impact student performance on standardized tests. After initial testing to determine each student's instructional needs, *SuccessMaker* provides an individualized program of instruction in which students are prompted to answer mathematics problems that are matched to their identified learning needs. When incorrect responses are made, the program provides instruction that guides students through the process of providing correct responses. For instance, if a student provides an incorrect response to the problem 6×31, the program demonstrates the steps of the standard U.S. multiplication algorithm and then guides the student to apply the steps on a similar problem. During the year of this study, Lincoln Heights' teachers were mandated to have all students spend 30 minutes daily working on the *SuccessMaker* program. This included 15 minutes on the math portion of the program and 15 minutes on the reading section. However, the 30-minute block of time spent on *SuccessMaker* was not supposed to interfere with the 2-hour reading block; therefore this time always fell during instructional time allotted for other subjects, including math.

While these initiatives each individually sought to improve student achievement in mathematics, coordination among initiatives was poor. In many weeks, teachers were expected to focus on one set of mathematics objectives related to the *Everyday Mathematics* program and another to prepare students for the benchmark mini-assessments. The *SuccessMaker* computer-based program also addresses some key mathematics objectives in ways that are in conflict with the approach taken by *Everyday Mathematics*. Most notably, *SuccessMaker* teaches students to perform multidigit computations by learning the procedures of the standard U.S. algorithms while *Everyday Mathematics* encourages use of invented and alternative algorithms. Furthermore, the instructional focus of *SuccessMaker* for some skills is highly procedural. This stands in contrast to the emphasis of mathematics-reform which stresses the importance of developing conceptual understandings to support learning of procedures. As a result, this fragmentation pulled teachers in different directions and their energies often became scattered rather than focused.

Throughout the year, teachers frequently expressed frustration with these competing initiatives because they found it difficult to implement all of them in the ways they were intended. Even though the school district leadership made it clear that effective use of the *Everyday Mathematics* program requires at least one hour of instructional time each day, the time required to fulfill the two other mandated initiatives made that difficult. In practice, time for the other mandated initiatives was routinely taken out of the math hour. Additionally, teachers were regularly pulled out of their classrooms for professional development. The objective of the professional development was to provide teachers with support for implementing mathematics-related initiatives as well as initiatives focused on other curricular areas. But it also took away time from teachers' use of the reform-based curriculum that was already eroded by the other two mandates. Therefore, while the *Everyday Mathematics* program was intended to be a catalyst toward reform-based mathematics teaching, it is important to recognize the extent to which teachers expended their energies just trying to keep up with all that was required of them. There was limited time left for teachers to focus on changing their mathematics teaching. This should be kept in mind later on as case studies describing teachers' beliefs, knowledge, and teaching practices are discussed in relation to reform-based mathematics ideals.

Within this broad context of a school engaged in many initiatives and activities to increase student achievement on state tests, Lincoln Heights' administrators also committed significant resources to put in place a teacher development project to improve mathematics teaching and ease the transition to the new *Everyday Mathematics* program. This mathematics teacher development project will be described next.

Mathematics Teacher Development Project

In anticipation of the difficulty teachers might have implementing the *Everyday Mathematics* program, Lincoln Heights administrators hired three mathematics educators to support teachers in grades Kindergarten through 5 with the transition to *Everyday Mathematics*. One mathematics educator worked with teachers in the primary grades (K-2), while two mathematics educators worked collaboratively with intermediate teachers (3-5)[6]. Across grades, the general format and intent of the teacher development project was the same. Teachers met with mathematics educators along with the other teachers on their grade-level teams for monthly half-day workshops over the course of the year. Each workshop was scheduled for 2.5 hours and included a 10-minute break. Sometimes support personnel, such as special education teachers, would join these sessions. Workshops focused on mathematics knowledge for teaching select grade level topics coming up in the *Everyday Mathematics* program and pedagogical knowledge associated with reform-based mathematics teaching. Workshops also intentionally offered opportunities for teachers to discuss issues and ask

[6] As was previously noted in the Methods chapter, the author was one of the two mathematics teacher educators for intermediate grade teachers.

questions related to lessons currently underway. In addition to these on-going workshops, the mathematics teacher development project included a limited number of in-class support experiences such as model teaching.

The third-grade teaching team, including the four case study teachers, met for workshops nine times during the school year. The focus of each of these workshops is presented in Table 10. Seven of the nine workshops focused on number-concepts including

Table 10
Focus of Third-grade Workshops

Session	Major workshop topics
1	Incorporating games into mathematics instruction Using the number grid (hundreds board) to develop place value concepts
2	Nonstandard strategies for multidigit addition and subtraction and their relationship to place value understanding
3	Having children talk about mathematics: when, why, and how Making sense of *Everyday Mathematics* focus algorithms for multidigit addition and subtraction, with and without base-ten blocks
4	Beginning multiplication concepts and student thinking Benefits and challenges of the array model for multiplication and division
5	The teacher's role in building instruction around student thinking Division problem types: partitive vs. measurement division Students' strategies for solving division problems
6	Classification of 2-D shapes (geometry)
7	Introducing fractions with equal-sharing problems Comparing fractions with physical models and reasoning strategies
8	Derived fact strategies for basic multiplication facts Nonstandard strategies for multidigit multiplication
9	Exploring area concepts with geoboards (measurement)

whole number concepts and operations and rational number concepts. In workshops 6 and 9, select geometry and measurement concepts were of focus. While pedagogical issues were continuously discussed alongside focus on mathematics content, workshops 1, 3, 5, and 7 included particular attention to pedagogical topics. These included instructional use of mathematics games, having students engage in explanation and justification of mathematical ideas, and the teacher's role in reform-oriented mathematics instruction.

Workshops typically began with each teacher sharing her current location in the *Everyday Mathematics* program and airing questions and concerns. In early workshops, this portion of the meeting took much more time than was considered by teacher educators to be ideal. As the year continued, efforts were made to contain this section of the meeting to 20 minutes or less. Next teacher educators led teachers through activities related to the math content and pedagogical foci of the workshop. Typical activities in this part of workshops included engagement in problem solving activities and discussion of personal mathematics strategies, analysis of student work, discussion of print articles and videos of classroom instruction, and presentation of information. Then, when appropriate, workshops ended with making plans for in-class support activities and/or previewing the upcoming lessons in the *Everyday Mathematics* teacher guide.

During the workshop portion dedicated to previewing upcoming lessons, teacher educators aimed to illuminate connections between *Everyday Mathematics* lessons and workshop experiences, unpack how *Everyday Mathematics* lessons might work together to develop particular mathematical ideas, alert teachers to activities in which students might experience difficulty, and offer suggestions for modifying and supplementing lessons. Although teacher educators encouraged teachers to preview lessons in the teacher guide prior to workshop meetings, teachers typically reported that they did not have time to look ahead and that they instead relied on this part of workshop sessions to help them understand what was coming up in the *Everyday Mathematics* curriculum.

As part of the teacher development project, the third-grade team of six teachers was allocated five half-days of in-class support in the first half of the year. Each of these half-day sessions typically involved a mathematics teacher educator engaging in model teaching or co-teaching in each of two or three teachers' classrooms. Although efforts were made to de-brief with teachers after in-class support experiences, these interactions were usually short and often occurred during instructional time as students were working. Decisions regarding which teachers would receive in-class support on a given day were made by the grade-level team at the end of a workshop session. Then, individual teachers designated to receive in-class support worked collaboratively with a mathematics teacher educator to determine the mathematics focus and support format (i.e., model teaching, co-teaching, or other) for in-class support in their classrooms. Most often, teachers requested model teaching related to a mathematics concept perceived difficult for students or model teaching of an upcoming *Everyday Mathematics* lesson with which the teacher felt uncomfortable. As part of the author's reciprocity agreement with case study teachers, additional opportunities for in-class support were provided. Case-study teachers each received an additional 4 – 6 hours of in-class support over the course of the school year.

Teacher feedback throughout the school year and on end-of-year evaluations indicates that teachers found the mathematics teacher development project to have been helpful in supporting their initial use of *Everyday Mathematics*. Teachers expressed that the workshops helped them to feel more confident with their teaching of *Everyday Mathematics* lessons and generally more knowledgeable about elementary school mathematics and how children learn mathematics. In addition, teachers felt that the in-class support helped them to see how the kinds of teaching practices discussed in the workshops might work with their own students. While most of the feedback on the mathematics teacher development project was positive, teachers also expressed concern about being out of their classrooms so frequently for this and other professional development. Teachers expressed that, in the following school year, they would like to continue with mathematics professional development with meetings scheduled after school to minimize time out of their classrooms.

Summary of the School Context

During the year of this study, Lincoln Heights Elementary was an urban school intent on overcoming its history of failure by improving student achievement on state accountability measures. The school had made many positive strides in recent years, which were attributed to improved behavior management, implementation of strategies to support students with Limited English Proficiency, and a school-wide scripted reading program. In turn, the success of these initiatives was attributed to on-going professional development efforts. With this in mind, Lincoln Heights' administrators hired mathematics educators to provide professional development in the form of monthly workshops when the new *Everyday Mathematics* program was adopted. The goal of these workshops was to support teachers' use of this new program and transition to reform-based mathematics teaching practices. But it was undermined somewhat by other initiatives started during the same year that were also intended to improve students' mathematics achievement. Rather than working together in a complementary manner, these initiatives competed for instructional time and focus and made it hard for teachers to keep up with all that was expected of them.

The Case Study Teachers and Their Classes

The four case study teachers began the year of this study with some tentative knowledge of reform-based mathematics teaching practices. Three of the teachers had participated in a series of professional development experiences in the previous school year, which were designed to increase teachers' mathematics content knowledge and introduce teaching strategies consistent with reform-based mathematics. The fourth teacher, a first-year teacher, was exposed to reform-based mathematics teaching practices through coursework in an undergraduate teacher education program. None of the teacher participants had taught mathematics using reform-based mathematics teaching materials as their primary curriculum resource. In fact, the three teacher participants employed at Lincoln Heights the year before the adoption of *Everyday Mathematics* had reported that, in the previous year, they had primarily used a traditional mathematics textbook series and a direct instruction teaching methodology. They did, however, report greater sensitivity to the importance of helping students develop conceptual understanding of mathematics as a result of mathematics professional development experiences the year before. Moreover, they reported an increased use of manipulatives in the classroom as a result of these experiences.

The case study teachers entered the year of this study both excited and apprehensive about the adoption of the reform-based *Everyday Mathematics* program. At the beginning of the school year, the third-grade teachers agreed that they should meet weekly to help each other work through this new math program. However, due to other demands on their time, the third-grade team did not follow through with this plan. Instead, teachers discussed successes and difficulties with *Everyday Mathematics* in the scheduled workshops and spontaneously when opportunities presented themselves. Through informal conversations, Lincoln Heights' third-grade teachers shared instructional ideas, helped each other troubleshoot problems, and provided moral support for each other's experiences with the new mathematics program.

Beyond these commonalities across participants, the four teachers of focus in this study have varied personal backgrounds and professional teaching experiences. Furthermore, there was significant variation in the composition of the four case study teachers' classes. A summary of class demographics by teacher is provided in Table 11.

Table 11
Student Demographics of Case Study Teachers' Classes

		Gender (%)		Race/ethnicity (%)				School services (%)			
Teacher	n	M	F	Blck	Hisp	Wht	Othr	Fr/rd lunch	Gifted	Sp Ed	LEP
Aria	18	50	50	17	78	5	0	89	6	11	6
Jarmin	12	50	50	0	100	0	0	100	0	25	25
Larsano	20	65	35	0	100	0	0	100	0	10	100
Rosena	10	50	50	30	60	10	0	90	0	30	20

In the sections that follow, aspects of each case study teacher's background and current teaching context will be described.

Ms. Aria and her Class

During the year of this study, Ms. Aria is a first-year teacher who had just completed a traditional university-based teacher education program at a university near Lincoln Heights. Before deciding to pursue an elementary education degree, Ms. Aria contemplated majors in mathematics and science, as she considers herself strong in both of these disciplines. As part of her teacher education program, Ms. Aria completed courses on elementary mathematics content and elementary mathematics methods. Ms. Aria had some familiarity with and interest in the reform movement in mathematics education, as evidenced by her attendance at the NCTM annual meeting in the school year prior to this study. With multiple job offers, Ms. Aria chose to teach at Lincoln Heights Elementary because she viewed this job as an opportunity to make a difference in the lives of disadvantaged children. Ms. Aria is White and speaks little Spanish at the start of the school year.

Ms. Aria's class is one of two *regular* sections of third grade at Lincoln Heights during the study year. This designation indicates that Ms. Aria is not teaching one of the classes of retained students or one of the classes for students identified with strong English language needs. However, many of Ms. Aria's students entered third grade with below grade-level skills and, throughout the year, Ms. Aria reports that her class poses significant behavior management challenges. Ms. Aria describes her class as constantly in motion. Students move around in their chairs, tap their pencils, and shuffle papers on their desks. Several children have significant difficulties focusing on their school work when they are expected to work without direct teacher support. Not reflected in Table 11, there are several children in this class who are in the process of special education screening during the year of this study.

Ms. Jarmin and her Class

Ms. Jarmin has been teaching elementary school for over 20 years, the last 10 years at Lincoln Heights. Reflecting her Bachelors degree in Elementary Education from a university in the Midwest United States, Ms. Jarmin's teaching experience has primarily been with general education classes of primary students. But she also has limited experience as a special education teacher. Ms. Jarmin is the designated leader of the third-grade team and is respected for her knowledge of how things work at Lincoln Heights. Ms. Jarmin knows many children and families throughout the school and is often observed helping students in need. She expresses that she is able to connect with the student population at Lincoln Heights because, like many of them, she grew up in a poor family. Ms. Jarmin is a veteran teacher who prides herself on, "being an old dog who can learn new tricks." Although she frankly states that mathematics is an area of personal weakness, Ms. Jarmin is eager to improve her own mathematical knowledge especially as it relates to teaching. Ms. Jarmin is White and speaks limited Spanish during the year of this study.

Ms. Jarmin teaches one of two classes at Lincoln Heights comprised of students who are repeating third grade. While the students in Ms. Jarmin's class were retained because of low scores on the state reading test, only 1 of the 12 students achieved a passing score on the mathematics section of the state test. In general, Ms. Jarmin's students appear to approach mathematical concepts in ways that are comparable to the students in the other third-grade classes. As a group, Ms. Jarmin's students appear to believe that they are not good at school. Throughout the school year, Ms. Jarmin talks about her efforts to convince students that they can be successful learners. All of Ms. Jarmin's students are Hispanic, with one-fourth of the students receiving special education services and one-fourth identified with Limited English Proficiency.

Ms. Larsano and her Class

At the time of this study, Ms. Larsano has been teaching at Lincoln Heights for two years. All of her previous teaching experience and education were in her native Puerto Rico. Ms. Larsano is passionate about helping her students and their families transition to life in the United States. In her classroom, Ms. Larsano is primarily focused on helping her Hispanic students acquire English. This commitment has led Ms. Larsano to begin working on a Masters degree in English for Speakers of Other Languages (ESOL), despite the challenges inherent in being a single parent with school age children. Within the Lincoln Heights school community and on the third-grade team in particular, Ms. Larsano is a social organizer. She regularly makes coffee for her colleagues and coordinates potluck breakfasts and lunches for her team members. In general, Ms. Larsano seems to be well-liked by the faculty and staff at Lincoln Heights and respected for her work with students and their families. In comparing mathematics teaching in Puerto Rico and the U.S., Ms. Larsano asserts that the mathematics at a given grade level is more difficult in the U.S. She explains that her own mathematics education and teaching of mathematics in Puerto Rico have focused on rote memorization and drill. She identifies mathematics as an area of personal weakness and her least favorite subject to teach.

Additionally, Ms. Larsano's first language is Spanish, and she sometimes expresses personal difficulty with quickly understanding new ideas – mathematical or otherwise – when they are presented in English. Reflecting this difficulty, Ms. Larsano is often observed clarifying ideas in Spanish with her Spanish-speaking colleagues in the context of mathematics professional development. In general, Ms. Larsano indicates that she is able to think more easily and fluently in her native Spanish in comparison to English.

Ms. Larsano's class during the year of this study is one of two third-grade sections designated for students who require significant English language support. All of the 20 students in Ms. Larsano's class are identified as having Limited English Proficiency (LEP), with approximately one-fourth of the students at an early stage of learning English. Several of Ms. Larsano's students have only recently arrived in the United States, and their out-of-school lives are in significant transition. In class, Ms. Larsano's students are typically well-behaved and helpful to one another. As Ms. Larsano claims is typical among students with Limited English Proficiency, many of her students appear to take a passive approach to instruction, sometimes tuning out the instructional task at hand.

Ms. Rosena and her Class

At the time of this study, Ms. Rosena has recently moved to the U.S. from her native Puerto Rico and has been teaching at Lincoln Heights for one year. Ms. Rosena's educational background includes a Bachelor's degree from a U.S. university and a Masters degree completed in Puerto Rico. Like Ms. Larsano, Ms. Rosena indicates that thinking in Spanish is more comfortable for her than thinking in English. However, since her post-secondary education in the U.S. and in Puerto Rico was conducted primarily in English, Ms. Rosena has had a great deal of experience learning and conversing in English prior to taking a job at Lincoln Heights. In Puerto Rico, Ms. Rosena spent six years teaching upper-elementary grade students in private schools. At times, she specialized in mathematics and science. Ms.

Rosena finds the academic expectations for students at Lincoln Heights to be lower than that of the private school sector in Puerto Rico. Although Ms. Rosena views her Hispanic background and Spanish-language skills to be an asset at Lincoln Heights, she indicates the desire to work at a school with more curricular freedom. With regard to mathematics, Ms. Rosena reports that math was a weak subject for her when she was in school. However, she asserts that her years of teaching fourth and fifth grade have forced her to improve her understanding of elementary mathematics concepts. Throughout the year of this study, Ms. Rosena expresses confidence in her mathematical knowledge of concepts addressed at the third-grade level.

Ms. Rosena teaches one of two classes at Lincoln Heights comprised of students who are repeating third grade because of low scores on the state reading test. While scores on the mathematics portion of the state test did not factor into retention decisions, none of the 10 students in Ms. Rosena's class achieved a passing score on the state mathematics test. As a group, Ms. Rosena's students often exhibit off-task behaviors and one student frequently acts out in ways that significantly disrupt instruction. Throughout the year, Ms. Rosena talks about her efforts to get her students to see themselves as capable learners and take on greater responsibility for their learning. Of Ms. Rosena's 10 students, 3 qualify for special education services and 2 are identified with Limited English Proficiency.

Now that an overview of the four case study teachers and their classes has been shared, attention will turn to presenting the case of each teacher, starting with Ms. Aria.

The Case of Ms. Aria

Ms. Aria enters the year of this study as a first-year teacher who is excited about having her own classroom. Having just completed a university-based teacher education program, Ms. Aria is the third-grade teacher at Lincoln Heights most knowledgeable of current reforms in mathematics education, and she is determined to provide mathematics learning experiences in her classroom that are different than those she received in her own elementary school years.

Ms. Aria is assigned to teach one of two regular classes of third grade at Lincoln Heights. The designation *regular* indicates that her students have not been retained and their language needs are not severe enough to warrant placement in one of the transition language classes. This designation does not, however, suggest that Ms. Aria's teaching situation is easier than that of the other teachers. In fact, Ms. Aria's class contains a handful of students who are either identified with or in the process of being screened for attention-related issues. In general, this class of third graders keeps Ms. Aria on her toes with a variety of academic and behavioral needs.

The case of Ms. Aria that follows is presented in four sections. In the first two sections, evidence of Ms. Aria's mathematics-related beliefs and knowledge at the beginning and end of the year will be presented. Next, the relationship among Ms. Aria's beliefs, knowledge, and classroom practice will be explored through a case story focused on her patterns of response to student difficulties. Finally, global measures of Ms. Aria's adherence to reform-oriented mathematics teaching and student achievement will be presented, which allows the case story findings to be considered in relation to these broader measures of teaching.

Ms. Aria's Beliefs about Mathematics Teaching and Learning

Ms. Aria's beliefs about mathematics teaching and learning were measured at the beginning and end of the school year using the *IMAP Web-Based Beliefs-Survey* (Ambrose, Phillip et al., 2003). The IMAP instrument requires teachers to respond to instructional scenarios in an open-ended format, allowing respondents to emphasize and downplay issues of personal importance. A series of rubrics are then used to measure survey responses in relation to seven target beliefs considered central to reform-oriented mathematics instruction. Ms. Aria's IMAP results are presented in Table 12, suggesting the degree to which Ms. Aria's survey responses provide evidence that she holds each target belief at the beginning and end of the school year.

Table 12
Ms. Aria's IMAP Web-Based Beliefs-Survey Results

Belief	Beginning of year	End of year
Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).	2	3
Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.	3	3
Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.	3	3
Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.	3	3
Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.	1	3
Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.	1	2
Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.	1	3

Note. 0 = No evidence; 1 = Weak evidence; 2 = Moderate evidence; and 3 = Strong evidence

In the sections that follow, evidence of Ms. Aria's adherence to each reform-oriented belief at the beginning and end of the year will be presented.

Aria: Belief about the Nature of Mathematics (IMAP Belief 1)

At the beginning of the year, there is *moderate evidence* that Ms. Aria views school mathematics as a web of interrelated concepts and procedures. In response to a scenario presenting multiple ways of solving a multidigit addition problem and asking which strategies she would like children to share, Ms. Aria identifies four strategies to be included in a unit on multidigit addition and discusses the mathematical value of each of these strategies. However, when prompted to discuss how she might order these strategies within such a unit, Ms. Aria's justification for order includes only one strategy-to-strategy connection. Therefore, while some belief in the interrelatedness of mathematics concepts and procedures may lead to inclusion of multiple strategies in mathematics instruction, her focus on fostering student understanding by emphasizing the interrelatedness of strategies is likely to be limited.

In contrast, at the end of the year, Ms. Aria describes multiple relationships among the strategies presented and indicates that she would use strategies devised by students to draw connection to other strategies and further develop conceptual understanding. Therefore, at the end of the year, there is *strong evidence* that Ms. Aria holds the belief that school mathematics should be presented as a web of interrelated concepts and procedures.

Aria: Belief about the Distinction between Understanding Concepts and Applying Mathematical Procedures (IMAP Belief 2)

At the beginning and end of the year, there is *strong evidence* that Ms. Aria believes that one's knowledge of how to apply mathematical procedures does not necessarily go with understanding of underlying concepts. Ms. Aria consistently indicates that students can get correct answers to computations using algorithmic procedures without understanding the conceptual basis of the algorithms. For instance, when asked if a student who has solved a multidigit addition problem using the standard U.S algorithm could use and explain particular non-standard strategies, Ms. Aria asserts that there is not enough information to make this judgment because many students complete the standard U.S. algorithm without conceptual understanding.

Aria: Belief about Source of Generative Mathematical Understanding (IMAP Belief 3)

At the beginning and end of the year, there is *strong evidence* that Ms. Aria believes conceptual understanding is more powerful and generative than remembering mathematical procedures. Ms. Aria consistently asserts that students are more likely to be successful with strategies when they have conceptual understanding, indicating that students need to have conceptual understanding in order for learned procedures to stick. For instance, when a student who has received procedurally-focused instruction on dividing fractions is unable to complete a division of fractions problem, Ms. Aria explains, "She was unable to solve the problem because she forgot the algorithm and had no understanding of what she was doing, so she could not rely on any other strategies. Once she forgot the algorithm, she was stuck." Without conceptual understanding, Ms. Aria suggests that students are likely to forget the details of procedures and make errors. Therefore conceptual understanding is considered more generative.

Aria: Belief about the Consequences of Teaching Concepts/Procedures First (IMAP Belief 4)

At the beginning and end of the year, there is *strong evidence* that Ms. Aria believes conceptual understanding must be developed before instructional focus turns to procedures. At both data points, Ms. Aria provides conceptually-based rationale for when and why she would have children share non-traditional strategies in a unit of multidigit addition, and she indicates that procedures learned before concepts would be rote.

While there is strong evidence at both data points that Ms. Aria believes conceptual understanding should precede introduction of specific procedures, Ms. Aria explains the instructional implications of this belief in more detail at the end of the year. When asked to suggest how she would go about teaching fraction division at the end of the year, Ms. Aria states, “I would allow students to divide using drawings and manipulatives, before ever teaching the algorithm. Allowing students to form their own strategies would allow them to have strategies to rely on when solving a problem like this.” At the end of year, Ms. Aria generally advocates having children invent their own strategies for solving problems and using these student-generated strategies to develop conceptual understanding prior to introduction of algorithmic procedures.

Aria: Belief about Children's Problem Solving Capabilities (IMAP Belief 5)

At the beginning of the year, there is *weak evidence* that Ms. Aria believes children can solve problems in novel ways before being taught how to solve such problems. She indicates that children are capable of devising novel ways of solving some kinds of problems but not others. In discussing instruction of fraction division, for instance, Ms. Aria indicates that children need teacher support to devise strategies successfully.

In contrast, at the end of the year, Ms. Aria indicates that she believes students can develop novel solutions to contextualized fraction division problems if they are allowed to approach problems using manipulatives and they are given the space to problem solve. In discussing her own classroom practices, Ms. Aria discusses her commitment to having students develop their own strategies to solve problems. Overall, Ms. Aria's end-of-year survey responses provide *strong evidence* of adherence to the belief that children can solve problems in novel ways before being taught how to solve such problems.

Aria: Belief about Children's Ways of Thinking about Mathematics (IMAP Belief 6)

At the beginning of the year, there is *weak evidence* that Ms. Aria believes that children think about problems differently than adults. Ms. Aria indicates that multiplication and join-change unknown problems could be solved by first graders if manipulatives were available. However, Ms. Aria also indicates that use of word problems sometimes inhibits, rather than facilitates, students' initial understanding of mathematical ideas. Therefore, real-world contexts in the form of written words are not viewed by Ms. Aria as particularly supportive. Furthermore, survey responses do not indicate awareness of the difficulty children have understanding mathematical symbols. Taken together, it appears that Ms. Aria may believe that students think differently than adults at the beginning of the year, but she is not certain of how this plays out in the classroom. Therefore, this belief is likely to have only a moderate influence on her actions.

At the end of the year, there is *moderate evidence* that Ms. Aria believes that children think about problems differently than adults. As at the beginning of the year, Ms. Aria emphasizes the importance of providing children with opportunities to use manipulatives and pictures to explore concepts. In addition, she stresses the difficulty students have with making sense of mathematical symbols and indicates that focus on symbols should be delayed until students have developed models and conceptual frameworks to which symbols can be linked.

As at the beginning of year data point, Ms. Aria suggests that a word problem providing a context for comparing fractions is confusing, and she expresses the preference to use non-contextualized numbers in this situation. However, in her classroom and other interviews, Ms. Aria consistently demonstrates commitment to the belief that real-world contexts support students' initial mathematical understandings. Therefore, Ms. Aria's response to the IMAP survey may not reflect her beliefs about the use of contextualized problems in general.

Aria: Belief about Teacher's Role in Supporting Student Learning (IMAP Belief 7)

At the beginning of the year, there is weak evidence that Ms. Aria believes that the teacher's role in mathematics instruction is to get students to do as much thinking as possible. After viewing a video of a teacher providing explicit step-by-step direction to a student completing a problem solving task, Ms. Aria identified the teacher's strong guidance as one of the strengths of the instructional segment rather than a factor that inhibited learning.

In contrast, at the end of the year, Ms. Aria indicates that the teacher's strong direction during the video is a weakness of the instructional segment. She suggests that it would have been better if the teacher had used questions to prompt student thinking rather than just telling the student what to do. At the end of the year, Ms. Aria generally expresses commitment to having students engage in problem solving with as little teacher intervention as possible. Therefore, there is strong evidence of Ms. Aria's adherence to the belief that teacher interactions with students during mathematics instruction should allow students to do as much thinking as possible.

Summary of Ms. Aria's Beliefs

Ms. Aria's responses to the IMAP survey instrument suggest that some of the beliefs measured by this instrument were aligned with a reform-orientation from the start of the year, while all measured beliefs were aligned with a reform-orientation by the end of the year. In particular, there is evidence of an on-going belief in the importance of developing students' conceptual understanding. Throughout the year, Ms. Aria views conceptual understanding as more powerful and generative than remembering mathematical procedures, and she indicates that procedures learned without conceptual understanding are likely to be forgotten. Ms. Aria also appears to hold the belief that school mathematics should be a web of interrelated concepts and procedures from the start of the year. However, at the end of the year, she is able to draw clearer relationship between this belief and classroom practice, thus indicating stronger adherence to the belief.

While some of Ms. Aria’s beliefs appear to have been aligned with a reform-orientation from the start of the school year, there is evidence that other beliefs changed somewhat over the course of the year. Specifically, Ms. Aria’s beliefs related to children’s capabilities as problem solvers and the teacher’s role in interactions related to the learning of mathematics were aligned with a reform-orientation much more at the end of the year than at the beginning. At the beginning of the year, Ms. Aria’s survey responses suggested only weak adherence to the belief that children are capable of solving problems in novel ways before being taught how to solve such problems. Additionally, at the beginning of the year, strong teacher direction during interactions between teacher and students is considered an instructional strength. In contrast, at the end of the year, Ms. Aria expresses confidence in children’s abilities to solve problems in novel ways, and she views the teacher’s role in supporting student learning as one of encouraging students to do as much thinking as possible during teacher-student interactions. Finally, there is evidence that, at the end of the year, Ms. Aria more strongly holds the belief that children think about mathematics in ways that are different than adults than she did at the beginning of the year.

Ms. Aria’s Knowledge of Mathematics for Teaching

Ms. Aria’s knowledge of mathematics for teaching with a reform-orientation was measured at the beginning and end of the year using data from two parts of the Teacher Knowledge interview. In particular, this interview explores teachers’ knowledge related to the teaching of multiplication and division to third-grade students. First, Ms. Aria’s open-ended discussion of teaching and learning multiplication will be presented. In this part of the interview, teachers were prompted to describe important understandings and common student difficulties related to the learning of multiplication. In addition, interview questions direct teachers to discuss how children develop the important understandings identified and how multiplication should be taught. Next, Ms. Aria's responses to four classroom scenarios will be described. These classroom scenarios probe specific aspects of teachers' knowledge related to the teaching and learning of multiplication and division, specifically teachers' knowledge of nonstandard strategies for multidigit multiplication, knowledge of a common student difficulty underlying the standard U.S. algorithm and strategies for addressing this difficulty, and teachers' abilities to interpret and respond to student work.

Aria: Open-ended Discussion of Teaching and Learning Multiplication

At the beginning of the year, Ms. Aria reports that, through third-grade instruction on multiplication, students should come to understand that multiplication means finding the total of objects when each group has an equal number of objects (e.g., 7×8 means 7 groups of 8). She also wants students to understand that multiplication is closely related to repeated addition. Lastly, Ms. Aria wants her students to begin to build fluency with basic multiplication facts and have knowledge of back-up strategies to rely on when a given fact cannot be recalled. Ms. Aria indicates that the aforementioned learning goals are accomplished by having students initially model real-world multiplication situations with manipulatives. Then these physical models are used as a basis for introducing multiplication notation as well as the link between multiplication and addition. Ms. Aria asserts that, at this phase of development, students are likely to struggle with making sense of symbolic multiplication notation. After students have become comfortable with the meaning of multiplication and its relationship to symbolic notation, students should begin memorization of basic facts starting with the easiest facts, times tables involving 2, 5, and10; then moving to times tables involving 3 and 4; and finally working on the remaining facts. Ms. Aria lists number sense, problem solving skills, and knowledge of addition as prerequisites to multiplication instruction.

In the end-of-year interview, Ms. Aria asserts that third-grade multiplication instruction should develop students' understanding of multiple representations of multiplication including real-world situations, pictures, and symbolic notation. She indicates that third-grade students should also be able to utilize knowledge of emerging, basic facts to solve problems involving difficult facts and larger numbers (e.g., using knowledge of 3×8 to solve 6×8 as well as 3×80 to solve 3×84). First, Ms. Aria suggests that students need to come to understand that multiplication involves multiple equal groups. This is accomplished through making pictures or physical models of multiplication situations and discussing the attributes of these models as well as methods for finding the total number of objects. Through this process, Ms. Aria suggests that students move from counting objects one-by-one to using repeated addition and skip counting strategies. As students internalize these more efficient strategies, Ms. Aria suggests that they move away from needing to use manipulatives and pictures. Through varied experiences and practice, students move toward recall of some of the easier facts and use of derived fact strategies and then recall for the more difficult facts. This leads to using basic facts to solve multiplication situations involving multidigit calculations. In discussing student difficulties, Ms. Aria stresses the difficulty students have understanding symbolic notation for multiplication without being explicitly encouraged to consider it in relation to real-world and visual representations. She asserts that students should have many experiences with multiplication situations and physical models, and symbolic notation should be built out of these experiences. Ms. Aria indicates that knowledge of addition and the ability to move forward and backward through the number system are prerequisites to studying multiplication.

Aria: Knowledge of Non-standard Strategies (Classroom Scenario 1)

At the beginning of the year, Ms. Aria identifies seven strategies that students might use to solve a word problem involving finding the number of chairs in 16 rows with 8 chairs in each row. These are presented in Table 13. Ms. Aria identifies direct modeling strategies as the least sophisticated, with use of a picture being slightly more sophisticated than direct modeling with objects. For both of these strategies, Ms. Aria assumes that students would find the product by counting objects one at a time. Ms. Aria identifies complete number strategies as next in order of sophistication, with counting by 8 on a number grid (hundreds board) being less sophisticated than repeated addition. Ms. Aria reasons that, with the number grid strategy, students can rely on the number grid to help them track their counts to a greater extent than is possible without this tool. Ms. Aria identifies the standard U.S. multiplication algorithm is next in order of sophistication, noting that students must use multiplication knowledge to perform the steps of this procedure. But that they do not necessarily have to understand why or how the algorithm works. Finally, partitioning strategies are identified as the most sophisticated of the strategies listed. Ms. Aria explains that partitioning strategies require students to have a strong understanding of the relationships among multiplication and addition and multiplication and division because students must know how to break numbers apart and put them back together.

Table 13

Strategies Identified by Ms. Aria in Response to Classroom Scenario 1

Data point	Direct modeling	Complete number	Partitioning number	Compensating	Other
Beginning of year	16 rows of 8 squares 16 rows of 8 counters	Counting by 8 on a number grid Repeated addition, 8 + 8 + 8…	$8 \times 8 = 64$, $64 + 64 = 128$ $8 \times 10 = 80$, $8 \text{ x } 6 = 48$, $80 + 48 = 128$		Standard U.S. multiplication algorithm
End of year	16 rows of 8 chairs (pictures of chairs), Counting by ones or counting by groups An array, 16 rows of 8 Xs	Repeated addition 8 + 8 + 8…	$10 \times 8 = 80$, $6 \times 8 = 48$, $80 + 48 = 128$ $2 \times 16 = 32$, $32 + 32 + 32 + 32 = 128$	$20 \times 8 = 160$, $4 \times 8 = 32$, $160 - 32 = 128$	Standard U.S. multiplication algorithm

Note. Analysis utilizes strategy classification presented in Baek (1998).

At the end of the year, Ms. Aria again identifies seven strategies to solve the 16 rows of 8 chairs problem. She again indicates that direct modeling strategies are the least sophisticated, but this time emphasizes the difference between students who need to make a picture of actual chairs as being at a more concrete level than students who can make an array that uses a symbol to represent each chair. She also indicates that a student at the most basic level would solve the problem by counting objects by 1. Ms. Aria identifies strategies that rely primarily on addition as next in order of sophistication. These include repeated addition and counting objects that have been directly modeled in uniform chunks. As at the beginning of the year, Ms. Aria lists the standard U.S. multiplication algorithm as next in order of sophistication. She asserts that most students who use this algorithm do not understand why it works, but that they do rely on knowledge of basic multiplication facts to apply the steps. Finally, Ms. Aria identifies partitioning and compensating strategies as the most sophisticated, suggesting that students who use these strategies are able to move back and forth through the number system flexibly thus demonstrating understanding of multiplication.

At both data points, Ms. Aria identifies direct modeling, complete number, and partitioning strategies that students might use to solve the problem posed. At the end of the year, she also identifies a compensation strategy. At both data points, Ms. Aria identifies cognitive resources students need to use select strategies with understanding and she is able to fluently discuss connections among strategies. In summary, Ms. Aria's response to Classroom Scenario 1 provides evidence of reasonably strong knowledge of nonstandard strategies for multidigit multiplication at both data points, with stronger knowledge at the end of the year as evidenced by inclusion of a compensation strategy and discussion of connections among strategies.

Aria: Interpretation of Students' Mathematical Strategies (Classroom Scenario 2)

At the beginning and end-of-year data points, Ms. Aria is able to quickly and fluently provide reasonable interpretations of three student work samples representing students' strategies for solving the pancake problem (Twenty-four children want to share eight pancakes so that each one gets the same amount. How much pancake can each child have?). She describes plausible theories for how each child approached the task, indicates what the strategies suggest about each child's mathematics understanding, and identifies how she might follow-up with each child in response to their work. For instance, consider the following excerpts of Ms. Aria's beginning-of-year discussion of one student's strategy:

> He decided that he had to break up the children and the pancakes into equal groups. So, from what it looks like, he decided to make 4 groups. So, he broke the 24 children up into 4 groups, so each had six children. And he had the 8 pancakes broken up so he had 2 pancakes in each group. Um, to get this he had to have a pretty sophisticated understanding of multiplication and division, because he has it written here: $4 \times 6 = 24$, $4 \times 2 = 8$. He knew that he was trying to get to a number that would have to divide evenly into both of them. So, he didn't use 3 because 3 doesn't go into 8. But I think he realized that 2 would still be so big. So, he went with 4, which was the greatest common factor. Um, that's pretty sophisticated. But I don't think that he got far enough to give a response from his picture. He didn't…he has 6 children eating 2 pancakes, but he didn't note how much of the pancake each child would get…so, I would probably ask him, for this group of six children and two pancakes, how would you, what would you do with those pancakes? How would you decide how much each child gets? And see if he could just divide it up a little further.

In her discussion of the student's strategy, Ms. Aria describes possible reasoning that the student might have used to initially divide the children and pancakes into four groups. She notes the strategy's close ties to multiplication and division knowledge as well as how the strategy falls short of showing a final solution. Finally, Ms. Aria identifies questions that she might ask the student to further probe his understanding and support him with moving toward a final solution. In summary, Ms. Aria's responses to Classroom Scenario 2 provide evidence of her strong ability to interpret student work from the start of the year.

Aria: Addressing and Avoiding a Common Student Error (Classroom Scenario 3)

In Classroom Scenario 3, teachers are presented with student work in which the standard U.S. multiplication algorithm is executed without maintaining the place values of the partial products. At the beginning and end of the year, Ms. Aria quickly identifies that this student error is grounded in failure to connect this procedure with knowledge of place value. At both data points, Ms. Aria suggests that she would respond to this error by having students focus on understanding what each place represents through a process of breaking down the problem into sub-problems.

At the end-of-year data point, however, Ms. Aria is able to provide a more detailed description of how she would go about illuminating the connection between place value and multidigit multiplication. She discusses teaching the *Everyday Mathematics* partial products algorithm[7] in lieu of the traditional U.S. multiplication algorithm in order to make place values more transparent. At the end of the year, Ms. Aria also suggests that she would prompt students to make magnitude estimates and consider if the results of their computations make sense. Finally, at the end of year, Ms. Aria indicates that she aims to help third-grade students make connections between place value and multidigit operations by infusing transparent place value language into talk about mathematics on a regular basis. For instance, instead of referring to a 4 in the tens place as 4, she asserts that she would identify that its value is 40.

In summary, the content of Ms. Aria's response to Classroom Scenario 3 suggests a strong understanding of the relationship between place value and multidigit operations at the beginning and end of the year. At both data points, there is evidence that Ms. Aria has knowledge of instructional strategies to support student development of conceptual understandings, with comparatively stronger knowledge demonstrated at the end of the year.

Aria: Interpretation of and Response to a Student's Flawed Solution (Classroom Scenario 4)

At the beginning and end of the year, Ms. Aria is able to identify the mathematical basis of the error in the flawed student solution to the problem 144 ÷ 8. In this scenario, the

[7] This algorithm involves multiplying each place in one factor with each place in the second factor, and then summing the partial products. For instance, $35 \times 42 = (30 \times 40) + (30 \times 2) + (5 \times 40) + (5 \times 2)$.

student indicates that she can find the answer to the division problem posed by repeated halving of 144 because $2 + 2 + 2 + 2 = 8$. At both data points, Ms. Aria identifies that the student's error is in treating addition, and not multiplication, as the reciprocal operation to division. She explains that the student should have stopped her halving process after dividing by 2 three times, because $2 \times 2 \times 2 = 8$. When prompted to suggest how she would help students to understand this error, Ms. Aria describes how she might model the strategy with manipulatives in both interviews. Consider her discussion below from the beginning of the year:

> I would probably do it with manipulatives. Because if you took a group of 144 and you broke it up into 2 groups of 72, and then you broke those up into 2 groups, you would have 4 groups of 36. And then when you broke each of those up into 2 groups, you would have 8 groups of 18, which is what they were asking for. And if you keep breaking it up to 9s, you would actually have 16 groups. And she only had to get to 8 groups. Um, so I would try to explain that and I would try to show that when she divided it once that was 2 and when she divided it again, that was times 2. And then when she divided again, it was again times 2. So, 2 times 2 times 2 is 8. Not addition.

Ms. Aria describes how manipulatives might be used to model the flawed student strategy and show the points at which eight groups are made. Then she explains how the multiplication number sentence $2 \times 2 \times 2 = 8$ is related to this strategy instead of the student's suggestion of $2 + 2 + 2 + 2 = 8$. At the end of the year, Ms. Aria also suggests that she might also have students explore this strategy further with smaller numbers.

In summary, Ms. Aria's responses at both data points provide strong evidence of her ability to draw on personal mathematical knowledge to understand the mathematical underpinnings of students' flawed solutions. Furthermore, Ms. Aria's responses provide evidence that she has knowledge of teaching strategies that honor student thinking and guide children to see the conceptual basis of their errors.

Summary: Ms. Aria's Knowledge of Mathematics for Teaching

Ms. Aria demonstrates reasonably strong knowledge of strategies students might use to approach multiplication problems at the beginning of the year and strong knowledge at the end of the year. At both data points, Ms. Aria identifies multiple direct modeling strategies, addition-based strategies, and strategies that use known multiplication facts to find the products of more difficult multiplication problems. At the end of the year, Ms. Aria includes a compensation strategy in her list of ways students might solve a multidigit multiplication problem, and she generally discusses strategies in greater detail.

At the beginning of the year, Ms. Aria's discussion of student learning of multiplication demonstrates knowledge of early understandings that students must develop as well as common difficulties encountered by students. However, her knowledge is limited with regard to how students move from being able to model multiplication situations pictorially and with symbols to fluency with facts. At the end of the year, Ms. Aria fills in these gaps as she describes the knowledge students must gain in order to move from one strategy to the next. In particular, Ms. Aria asserts that, as students internalize the relationship between groups of objects and skip counting or repeated addition, they begin to be able to apply these more efficient strategies without the support of objects or pictures. Then these addition-based strategies support internalization of easy multiplication facts, which in turn can be combined to figure out more difficult facts. Overall, interview responses suggest that Ms. Aria's knowledge of student learning of multiplication increased over the school year, with moderate knowledge at the beginning of the year and demonstration of strong knowledge at the end of the year.

At both data points, Ms. Aria proves skillful at using student work to interpret students' mathematical thinking. Furthermore, she demonstrates the ability to identify the mathematical basis of errors in flawed solutions and devise conceptually-grounded teaching strategies to use in response to these student errors. Throughout both interviews, teaching strategies suggested emphasize conceptual understanding and honor students' ways of thinking about the problems posed.

Case Story of Ms. Aria's Response to Student Difficulties

In the case story that follows, facets of Ms. Aria's typical response to student difficulties will be discussed. Specifically, Ms. Aria's routines to foster student independence and self-confidence in the face of difficulty, the substance of sustained support when student difficulties persist, and her response to student difficulties that arise in the context of whole class discussion will be described. Following illustration of each response pattern, consequences for student thinking and the relationship between Ms. Aria's beliefs, knowledge, and mathematics teaching practices will be explored.

Aria Response Pattern 1: Routines to Foster Student Independence and Self-Confidence

At the beginning of the school year, Ms. Aria expresses concern about her students' reluctance to attempt mathematics tasks without significant teacher guidance. As a consequence, she intentionally develops routines that aim to decrease students' dependence on the teacher and increase their self-confidence and abilities as problem solvers. These practices are observed throughout the school year.

A typical math lesson in Ms. Aria's class begins with a math message problem that encourages students to explore a mathematical idea that is central to the lesson. One way that Ms. Aria fosters increased independence is by holding firm to a commitment to limit involvement in students' exploration of the math message problem. In a post-observation interview, she explains:

> We're still working on independence. I think the students rely so much on a teacher telling them whether they are doing it right or doing it wrong. Are they supposed to do addition or subtraction, are they supposed to do multiplication or division…you know? And I really want them, especially on the math message, I want them to be able to do that, independently, and then we are going to talk about it.

When Ms. Aria does interact with a student who is having difficulty getting started with a problem, she generally keeps her interactions brief, with the intention of providing just enough support to get the student started. Her support often includes questioning related to important information in the problem and then a suggestion to draw on resources other than the teacher.

In the following excerpt, Janelle is working on the problem, "There are 15 pennies. Each child receives 4 pennies. How many children are there?" Janelle is sitting at her desk with a pile of cubes at the top of her desk, when Ms. Aria approaches to help her get started with the problem:

> Ms. Aria: What does the problem say? (*Janelle does not respond.*) How many pennies do you have?
> Janelle: Fifteen.
> Ms. Aria: How many [cubes] do you have? (*Janelle begins touching and counting unit cubes. Ms. Aria says each count aloud.*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. So take one away. Talk to Patty about this and what to do next.

After leading Janelle to identify the 15 pennies as the starting place in the problem, Ms. Aria encourages Janelle to use cubes to represent the pennies. Then Ms. Aria directs Janelle to talk with another student about where to go from there. In this brief interaction, Ms. Aria encourages Janelle to utilize resources other than the teacher including physical manipulatives and conversation with a peer. In Fall observations, when students are working on beginning multiplication and division concepts, manipulatives are consistently available and encouraged, especially when a student is having difficulty getting started on a problem. In all of the lessons observed, Ms. Aria structures lessons such that students are either encouraged or required to work cooperatively with their peers. In a post-observation interview, Ms. Aria explains why she encourages her students to work with peers:

> They don't like to be - if they're on their own, then there's a lot of, "I don't understand this; I don't understand this," and you know, trying to get my attention and stuff, so at least if they're in partners, they have somebody to rely on.

Ms. Aria views the students' peers as resources to draw on for support with problem solving.

An additional way that Ms. Aria supports her students in becoming more independent problem solvers is by fostering the belief in students that they are capable of engaging in difficult mathematics and that learning mathematics involves figuring out difficult problems. Several students struggled to make sense of the sharing pennies problem that Janelle (described above) was having difficulty with. Recognizing this, Ms. Aria began the whole group discussion of this problem by making students' difficulties the focus of discussion:

> Ms. Aria: Okay. Who found this problem a little difficult when they first read it? Be honest. You read this and you thought, 'Oh, my goodness, this is really hard.' (*A few students raise their hands. A couple of students vocalize that they found the problem hard.*) Okay. What was hard about it? Kristin, what was hard about it?
>
> Kristin: That there were 15 pennies and that we were trying…
>
> Ms. Aria: Just a second Kristin. Damian, we were trying to listen to you. Would you please listen to the rest of us. Thank you. (*To Kristin*) Go ahead.
>
> Kristin: We put…umm…15 pennies, and then there was 1 left over. And it was hard because we didn't know how many children there were still.
>
> Ms. Aria: Okay, so it was complicated because you didn't know how many children there were. Who thought it was difficult because they didn't know how many children there were? Be honest. It's okay. (*Some children raise their hands.*) Yeah, that's tough. That's tough. Sometimes it's hard when you don't know

> some of the information, it's hard to figure that out. What do we do? What could help us figure out that information? What kinds of strategies did you use?

Ms. Aria first encourages students to recognize that the problem was difficult and to think about why. After acknowledging students' difficulties, she moves on to a discussion of students' sharing how they overcame the difficulties to figure out the problem. In the post-observation interview, Ms. Aria explains that she wants it to be okay in her classroom for students to think that problems are hard. By highlighting that many students find problems difficult, she aims to encourage students to expect and embrace some level of challenge with problem solving tasks. Ms. Aria explains, "I wanted them to know that, sure, it's hard, but it's not impossible…there are hard things about these problems, but this is what we are learning to do."

In summary, Ms. Aria encourages student independence during mathematics instruction by minimizing her interactions with students while they are problem solving. When a student is having difficulty, Ms. Aria makes sure the student understands the task. Then she suggests utilizing supports other than the teacher, such as peer collaboration and manipulatives. In addition to these strategies, Ms. Aria sometimes insists that students attempt problems without teacher support. Finally, she fosters the belief among students that doing mathematics means figuring out how to solve problems for which the solutions are not readily apparent. She encourages students to talk about their difficulties with problems and how they worked through them.

Aria response pattern 1, consequences for student thinking. Ms. Aria's classroom is a place where, over the course of the year, students increasingly take on the role of problem solver during mathematics instruction. While at the beginning of the year students are observed waiting for Ms. Aria's direct support to initiate work on a task, at the end of the year students readily approach mathematics tasks and engage in a variety of self-help strategies when they encounter difficulties. For instance, students are observed getting manipulatives off the shelves and trying to use the Number Grid (e.g., Hundreds Board) and other resources posted on the walls. Students also readily turn to each other during work time to ask questions, compare ideas, and work together. Through these learning practices, students are generating and testing their own mathematical ideas.

In general, students appear to become more comfortable with problem solving tasks in which a solution strategy is not readily apparent. The number of students verbally expressing confidence in their mathematical abilities increases dramatically during the year. Because students learn to be more self-sufficient while problem solving, Ms. Aria is able to focus on observing students and asking them to justify their solutions. Instead of giving the lion's share of her attention to supporting students in the beginning stages of problem solving, she is able to focus much greater attention on pressing students to reflect on how their solutions work and on what they have learned.

Aria response pattern 1, link to beliefs and knowledge. The routines and practices Ms. Aria uses to foster students' independence and self-confidence as problem solvers are related most to her beliefs but also relate to her knowledge. Throughout the year, Ms. Aria's belief in the importance of allowing children to do as much of the thinking as possible without teacher intervention becomes a strong guiding principle in her instruction. This is evidenced in her responses on the IMAP survey as well as her interviews throughout the year. At the end of the year, Ms. Aria articulates her commitment to encouraging students to engage in problem solving with minimal teacher intervention:

> I think it is important for students to have a chance to struggle with problems a little. I think that developing their own ways to solve a variety of problems will help them take ownership over what they are learning. I also think it helps them to become better problem solvers.

Ms. Aria's belief in her students' abilities as problem solvers also grows through the course of the year. At the beginning of the year, Ms. Aria's responses on the IMAP instrument suggest that she believes children require teacher guidance to be successful with novel tasks. In contrast, at the end of the year, her responses indicate that she believes students can be successful solving novel problems as long as they are permitted to use manipulatives, work with peers, and solve the problems in their own ways. This change in beliefs occurs over time as Ms. Aria works with her students and observes what they can accomplish during problem solving with limited intervention.

While Ms. Aria's commitment to encouraging student confidence and independence is clearly tied more closely to her beliefs than her knowledge, one can argue that Ms. Aria's knowledge of her students as well as her pedagogical content knowledge supports her ability to organize mathematics instruction in adherence with these beliefs. Knowledge of students' mathematical abilities supports Ms. Aria in constructing mathematics tasks that are within her students' zone of proximal development. Although Ms. Aria follows the *Everyday Mathematics* program, she often adapts the problems suggested in the Teacher's Edition to align more closely with her perception of students' interests and instructional levels. Drawing on pedagogical content knowledge, Ms. Aria is able to observe students as they work and assess their difficulties. Then she is usually able to provide brief on-the-spot support to get the students moving toward a viable solution strategy. Finally, as is evidenced by the classroom excerpts shared, Ms. Aria has devised a routine for supporting students while minimizing teacher intervention: After helping students to understand the task at hand, she suggests use of tools and supports other than the teacher. This routine is a type of pedagogical knowledge that allows Ms. Aria to implement her beliefs in the classroom.

Aria Response Pattern 2: Substance of Sustained Support when Student Difficulties Persist

While Ms. Aria's first response pattern for helping students get started with a problem is to ask questions about the problem context and suggest resources other than the teacher that might help, the Fall lessons observed include a few instances when students continue to have difficulty moving forward with a task. When this occurs, Ms. Aria engages in sustained dialogue with students to move them forward with the given task.

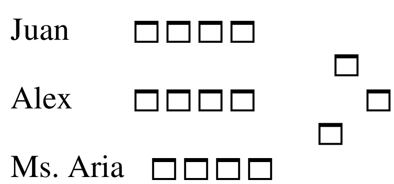
In the instructional excerpt below, Alex and Juan are having trouble getting started on the same measurement division problem cited previously: There are 15 pennies. Each child receives 4 pennies. How many children are there? The excerpt begins with Juan expressing that this problem is different than the problem the class just completed. Although both problems involve division, the focus of the previous problem was on a partitive division context in which cookies were shared fairly among a specified number of children. To solve the sharing cookies problem (partitive division), most students directly modeled the problem with manipulatives or a picture by representing the children and then placing one cookie at a time next to each child until they had distributed all the cookies. Alex and Juan's initial difficulty with the current penny problem is that the number of children is unknown, so they cannot follow the same procedure that was effective at solving the sharing cookies problem. Ms. Aria attempts to move the boys away from a focus on the procedures used on the previous problem to thinking about the context of the current problem conceptually:

> Juan: This is different than that (*indicates the cookie sharing problem on the board involving a partitive division context*).
> Ms. Aria: How is this different?
> Juan: They don't tell how many children there are.
> Ms. Aria: You're right. That is how they are different. (*Alex and Juan reread the penny problem silently.*) Do you want counters? Will they help? (*The students indicate 'yes' with a head nod, and Ms. Aria gets some counters off a nearby table*.) Read the problem.
> Juan: There are 15 pennies. Each child…receives four pennies. How many children are there?
> Ms. Aria: What does that mean? (*points at the word 'receives'*) What does '*receives'* mean?
> Alex: (*Inaudible*)
> Ms. Aria: Yes. It means 'to get'. Each child gets four pennies. (*Pause*) What will you do? (*Pause*) Get your counters. How many pennies do you have?
> Juan: Fifteen.
> Ms. Aria: Okay. Pull out 15 pennies. (*Alex and Juan slowly count out 15 pennies*.) You guys need to decide what to do with those. How many did each child get, Alex?
> Alex: Four.
> Ms. Aria: Four. So, what can we do with those counters? Each child needs four.

As is typical, Ms. Aria's first attempt to help the boys involves questioning aimed at drawing attention to the important information in the problem as well as encouragement to use manipulatives and partner work. At this point, Ms. Aria steps back and observes the boys' progress. Juan pushes the cubes around his desk, and Alex abandons Juan's work in favor of counting out 15 cubes to work with on his own desk. After another 30 seconds, neither boy has made any apparent progress toward a solution. Ms. Aria intervenes a second time by encouraging Juan to think of himself as a child in the problem:

> Ms. Aria: If you were a child in this problem, how many pennies would you get?
> Juan: I don't know
> Ms. Aria: Don't say you don't know. It's in the problem. So, you know it. Okay. (*Juan looks at the problem.*) How many?
> Juan: Four.
> Ms. Aria: Okay, so if you are a child in this problem, you receive four pennies. (*Ms. Aria pushes 4 of the pile of 15 cubes toward Juan.*)

During the time that Ms. Aria has been discussing the problem with Juan, Alex has used 15 counters to make a 5 × 3 array. Recognizing that Alex is no longer focused on the problem context, she attempts to reinitiate partner work by pulling Alex back into her conversation with Juan. They start again with the 15 pennies and act out the problem. Ms. Aria encourages Alex and Juan to think of themselves as children in the problem. She guides the boys to give four counters to Juan and four counters to Alex. When the boys see that they have more counters to distribute, they decide to give four counters to Ms. Aria. Now there are three counters remaining that have not been distributed. Ms. Aria asks questions that encourage Juan and Alex to reflect on the problem context and figure out what to do with the remaining counters.



Ms. Aria: Can you give four pennies to another kid?
Juan: No
Ms. Aria: How come?
Juan: There aren't four.
Ms. Aria: So, there's not enough. These are left over. They are the remainders.

In the post-observation interview, Ms. Aria explains that she decided to use the acting it out strategy with Juan and Alex because it became apparent to her that the problem was too abstract for them. She adds that the acting it out strategy is one that often helps students make a connection to a word problem. Ms. Aria also notes that, in addition to the problem context, she perceived Alex to be struggling with the fact that this problem involves a remainder. She views his attempt to build the 5 × 3 array as an effort to form a solution that avoids a remainder. This is why, at the end of the episode, she focuses her questions on the meaning of the remaining counters.

Ms. Aria aims to provide support that puts her students in the position of doing as much mathematical thinking as possible. This excerpt is an example of Ms. Aria providing more intervention with students than she thinks is ideal. In this case, at the point when she intended to walk away, she perceived that Juan and Alex were still stuck. Consequently, Ms. Aria continued to engage the students in thinking about the context of the problem and how this problem might be directly modeled using an 'acting it out' strategy. At all points, Ms. Aria is focused on guiding the students to devise a solution strategy derived from a conceptual understanding of the problem.

Aria response pattern 2, consequences for student thinking. Ms. Aria's method of providing support that focuses on conceptual understanding coupled with her stance of 'teacher as guide' drives students to develop learning practices that support problem solving. In the Spring semester, students are observed rereading problems, identifying key information, attempting to model problems with pictures or manipulatives, and trying to talk out a problem with a peer much more readily than in the Fall. It is worth noting that there are no instances of Ms. Aria providing sustained support for students solving problems in the Spring lessons observed. As students develop the learning practices and confidence to engage in problem solving, Ms. Aria is able to further minimize the amount of support provided on the front-end of problem solving tasks.

Aria response pattern 2, link to beliefs and knowledge. Ms. Aria's practice of providing sustained support that is conceptually-grounded is related to both her beliefs and knowledge. Ms. Aria's interviews at the beginning and end of the year reveal a strong belief in the importance of students' making sense of mathematical ideas by seeing how they connect to real-world situations:

> I think that students that have trouble with math, a lot of their trouble stems from the fact that they don't see how it applies. They're doing it as an abstract concept…2 plus 2, or 2 plus 6, and they're not applying it to the fact that this means: these two cookies plus my six cookies. And I think that if we make math apply, if we give it some meaning, all those numbers and shapes, if we give them meaning, then students will understand them more deeply. In which case, I think it will help get rid of some of the misconceptions…because the students will be looking for the meaning within the mathematics, they're looking for the reasoning.

In the instructional excerpt shared Ms. Aria could have told students to make groups of four with the cubes. Instead, she focuses them on acting out the problem, which makes sense of it in relation to the context. This focus on context encourages a modeling approach to problem solving as opposed to a procedure-driven focus.

Also supportive of development of conceptual understanding, Ms. Aria believes that her role during problem solving is to act as a guide:

> As far as problem solving goes, I think the teacher is a guide. I think they are there to keep the students interested, to keep the students asking questions and trying strategies, to keep the students from reaching ultimate frustration levels, but I don't think that they are there to give them a strategy or give them an answer. I think that it's more as a, you know, guide. I think the students' role is more hands-on more explorative. They're the ones that are getting into the problem; they're the ones that are trying to figure out strategies.

In order to put her beliefs into practice, Ms. Aria draws on pedagogical content knowledge to support students' thinking. She is aware of the differences between partitive and measurement division problems, and she is sensitive to students' potential difficulties making sense of these different problem types. Ms. Aria is comfortable with asking questions and making suggestions that press students to think about problems conceptually. Finally, she is able to make on-the-spot judgments about the origin of students' difficulties and provide real-time support based on these judgments. In this case, Ms. Aria attributes students' difficulties to the measurement division problem type and the existence of a remainder. Her support involves having students act out the problem situation with counters and focus on the meaning of the three counters remaining.

Aria Response Pattern 3: Response to Student Difficulties during Whole Class Discussion

While Ms. Aria frequently encourages students to help each other devise and make sense of solutions during work time, she only occasionally involves students in helping each other when errors or difficulties arise in the context of whole group discussions. Typically, in the whole group forum, Ms. Aria provides the majority of support to help students move through difficulties. To illustrate, instructional excerpts portraying nuances of this response pattern will be shared. In a Fall example, students are having difficulties answering Ms. Aria's questions related to a mathematics task. In a Spring example, students are having difficulty explaining their (correct) mathematical strategy.

In the Fall instructional excerpt that follows, Ms. Aria is having students think about how their solutions to a measurement division problem might be represented in the form of a number model (number sentence). Symbolic representation of division has been explored in only one lesson prior to this one, so many students appear uneasy with this skill:

> Ms. Aria: So, give me a number model that shows this problem. (*Pause*) What's a number model that shows this problem? (*Pause, a few hands go up*.) Ray.
> Ray: Three times four.
> Ms. Aria: Three times four is what?
> Ray: Twelve.
> Ms. Aria: Twelve, plus your three left over is 15, right? Okay. I want a division problem. Who can tell me a division problem for this? Kristin. (*Pause*) You're alright. Go ahead.
> Kristin: Fifteen times four?

Ms. Aria: Not 15 times 4…remember, we are doing a division problem. Fifteen what?
Kristin: Fifteen plus four?
Ms. Aria: not plus. That's addition. We're doing division.
Kristin: Fifteen…
Ms. Aria: divided…
Kristin: divided by 4…
Ms. Aria: Fifteen divided by four equals three, remainder three. (*Ms. Aria writes on board: 15 ÷ 4 = 3 R3.*)

First, Ray suggests a multiplication number sentence that matches the problem solution. Ms. Aria accepts this solution, but does not record it on the board or involve the other students in justifying whether or not it is reasonable. Next, Ms. Aria specifically requests a division problem to match the word problem. Kristin's initial responses (15 × 4 and 15 + 4) suggest that she is having significant difficulty thinking about this problem in terms of a number sentence. Instead of involving other students in judging the reasonableness of Kristin's suggestions or involving the class in making additional suggestions, Ms. Aria lets Kristin know that these initial attempts are incorrect. Then Ms. Aria ultimately tells Kristin and the rest of the class the correct answer she is seeking. Rather than involving other students in assessing the ideas posed by Ray and Kristin during class discussion, Ms. Aria maintains control of judging the correctness of ideas and, ultimately, formulating a correct number sentence.

Next an example from a Spring lesson is presented. In the instructional excerpt that follows, Thomas and Alvin have been called on to discuss their solution method for the problem, "If I had 5 groups of 24 pencils, how many pencils do I have altogether?"
To solve the problem, Thomas and Alvin have broken the problem into two smaller parts. In order to find the product of 5 × 24, Thomas and Alvin first find the products of 2 × 24 and 3 × 24. Then they add their partial products of 48 and 72 together to get the answer to the

original problem. In the whole group discussion, Thomas and Alvin are having difficulty communicating this strategy to the class:

Ms. Aria: Thomas, how did you guys do it?
Thomas: We did 72 plus 48.
Ms. Aria: How did you get 72? (*To class*) Let's listen to how Thomas got 72. (*Long pause, waiting for attention from class*) You got 72, but how?
Alvin: Because we added umm…like 5 forty….never mind.
Ms. Aria: So, you have five groups of 24 to start with. Right? Thomas, how did you get 72? How many of those groups did you use?
Thomas: Ummm…two.
Ms. Aria: Okay, so two groups of 24 would be…48.
Alvin: We used…
Thomas: I think we used…
Ms. Aria: The rest of you guys should be looking and trying to help them out.
Thomas: Three groups.
Ms. Aria: Three groups. Okay, so you have three groups of 24. So, you took your 24, and instead of doing it 5 times, you did it 3 times and you did it 2 times. Right? (Pause, writes on board) So, then what did you do?
Thomas: We did 72 plus 48, and it equals 120.

Ms. Aria's record on the board:

$3 \times 24 = 72$
$2 \times 24 = 48$
$72 + 48 = 120$

Ms. Aria: Are there any questions about that one? (*No one raises a hand.*)

Throughout the interchange, Ms. Aria keeps her focus on Thomas and Alvin and fills in the details of their solution strategy when they struggle to provide them. Although Ms. Aria tells the other students, "The rest of you guys should be looking and trying to help them out," she never provides opportunity for other students to make conjectures about what Thomas and Alvin might have done.

In summary, Ms. Aria typically provides the majority of support to help students move through difficulties when they arise in the context of whole group discussion. This practice serves to limit the involvement of other students in these instances.

Aria response pattern 3, consequences for student thinking. Because Ms. Aria rarely encourages students to help each other during whole group discussion, few students appear to engage in trying to understand the errors made by their peers or how to correct them. Students seem to learn that they are unlikely to be held accountable for thinking about their peers' difficulties and that the difficulties will be corrected by Ms. Aria if they just wait a little bit. Perhaps because the working out of difficulties usually occurs between Ms. Aria and one or two students, it is typical that many students in the class appear to stop attending to the discussion. Consequently, the opportunity for these students to benefit from grappling with the mathematical ideas being put forward by their peers is limited. When students stop attending to the discussion, Ms. Aria's need to maintain control is heightened and the likelihood that she will relinquish control to allow students to explore mathematical ideas in the whole group discussions is decreased.

Aria response pattern 3, link to beliefs and knowledge. Ms. Aria's practices that inadvertently restrict student participation when errors arise in the context of class discussion stand in conflict with her beliefs and are primarily related to limits in her knowledge. Ms. Aria's interviews throughout the year provide evidence that she believes that whole group discussion is a place for students to develop appreciation and ownership of mathematical ideas. Reflecting this stance, Ms. Aria indicates that it is her vision for students to share control of discussion of mathematical ideas. In an interview at the end of the school year, Ms. Aria articulates this belief and why she finds it so difficult to make it a reality of her mathematics instruction:

> So, from the beginning of the year until now – I really struggled with, instead of leading discussion, letting my kids go a little, giving them a little bit of slack…letting

> them work together, answer each other's questions, come up with questions, and letting them lead a little bit more. And I think that was very difficult, especially as a first-year teacher, trying to give up that control. Because I didn't want to lose control of the class, but at the same time, I didn't like standing up there and saying one thing after the other. I wanted them to be able to explore it a little more.

Ms. Aria identifies an on-going tension between the goal of orchestrating class discussions such that they are driven by students' ideas and the goal of maintaining control of the class.

As is the case for many first year teachers, Ms. Aria finds classroom management to be a persistent challenge. Particularly in Fall lessons, Ms. Aria attributes many of the on-the-spot instructional decisions made during lessons to be primarily driven by classroom management concerns. While Ms. Aria would like student ideas to have a stronger presence in class discussions of mathematics, her classroom management goal holds a higher priority. From a teacher knowledge perspective, Ms. Aria has not developed a set of routines to facilitate students working as a community to grapple with mathematical difficulties that arise in class discussion while also keeping students on-task. Therefore, the reasons for this disconnect between Ms. Aria's beliefs and instructional practices appear to be twofold. First, Ms. Aria is a novice teacher with limited knowledge of classroom management. Second, she has limited knowledge of how to involve multiple students in discussion, especially when student difficulties surface. As Ms. Aria becomes more comfortable with classroom management, she will be able to give more of her mental attention to cultivating instructional practices that reflect her belief in student-led discussions.

Summary of Ms. Aria's Response to Student Difficulty

Three patterns depict Ms. Aria's typical classroom practice in response to student difficulties. First, she intentionally employs teaching strategies to foster student independence and self-confidence in the face of difficulty during problem solving. In particular, Ms. Aria minimizes her interactions with students while they are initially working on mathematics tasks, and she encourages students who are having difficulties to rely on resources other than the teacher such as manipulatives and peer collaboration. Ms. Aria also actively works to foster the belief that doing mathematics involves encountering and working through difficult problems for which solutions are not readily apparent. While these teaching strategies most often prove successful at helping students to move forward with problem solving tasks, there are still occasional instances when Ms. Aria perceives students to need a more sustained level of support to be productive. In these instances of sustained support, Ms. Aria employs teaching strategies that are conceptually-grounded and encourage students to do as much of the mathematical thinking as possible. For instance, she uses questioning to focus students' attention on the context of word problems and encourages students to act out the situations presented by problems.

The third pattern of response to student difficulties observed in Ms. Aria's classroom occurs in the context of whole group discussion of mathematics tasks. When student difficulties arise during whole class discussions, Ms. Aria tends to limit the involvement of class members by restricting the discourse to interaction between herself and the students who have provided incorrect responses or are having difficulty explaining their ideas. During these interactions, Ms. Aria guides the student toward a correct response, sometimes telling him when his response is correct or incorrect. While Ms. Aria often tells the class that they should be paying attention and trying to figure out how to resolve the given difficulty, she rarely invites other students to offer suggestions or evaluate the incorrect ideas put forward by their peers.

As the year moves along, Ms. Aria's students are observed engaging in increasingly productive learning practices during work time allotted for problem solving. In general, students appear to approach problem solving tasks by trying to make sense of them to a much greater extent than was observed at the beginning of the year. Furthermore, students are observed using a variety of self-help strategies when they encounter difficulties, rather than immediately requesting help from Ms. Aria.

While much growth is observed in students' mathematics learning practices in response to difficulties encountered during work time, there is less evidence of the occurrence of meaningful group learning when difficulties arise in the context of whole class discussion. As Ms. Aria guides individual students to revise incorrect answers or work out other kinds of difficulties, many of the students in the class appear to stop attending to the discussion. When this occurs, Ms. Aria feels pressure to exert an even greater level of control over the class discussion in efforts to keep students from becoming unruly. Consequently, students are rarely engaged in working as a community to resolve student difficulties in the context of class discussion. As a result, student thinking related to resolving difficulties that arise in class discussion is limited.

Ms. Aria's ways of responding to students' difficulties are linked to both her beliefs and knowledge. Ms. Aria's beliefs seem most linked to the ways she aims to structure support for students as they approach problem solving tasks. Ms. Aria believes that children benefit from solving problems in their own ways, without significant teacher guidance. Therefore she structures problem solving activities such that students are expected to devise their own ways to approach problems. When student difficulties arise, Ms. Aria limits her direct support of students and instead encourages students to draw on other resources available to them. This practice reflects the belief that, in teacher-student interactions related to mathematics, the teacher's role is to encourage the student to do as much of the thinking as possible. This belief is also in play when Ms. Aria does provide sustained support for students when she perceives they are unable to make progress with less intervention. While these structures for learning are driven by Ms. Aria's beliefs, they also necessarily rely on her knowledge of routines and scripts for facilitating student work time, especially when difficulties arise.

This point is emphasized by contrasting the influence of Ms. Aria's beliefs on teaching practices in response to student difficulties that surface during work time and whole class discussion. While Ms. Aria also believes that whole class discussion should also be a time when students work as a community to overcome challenges and difficulties, in practice she typically limits opportunities for the class to contribute meaningfully when difficulties surface in these discussions. In the case of whole class discussion, Ms. Aria views her goals related to having student-led discussions as conflicting with her goal to maintain control of the class. Since the classroom management goal holds a higher priority, Ms. Aria provides the majority of support to help students move through difficulties that surface in efforts to keep class discussions moving and the class engaged. Whereas Ms. Aria has developed a set of routines and teaching practices that facilitate student-to-student interactions while maintaining control of the class during work time, she has not yet devised a parallel set of strategies to draw on during the whole class discussion time. As Ms. Aria becomes more comfortable with classroom management, it is likely that she will develop a knowledge base that will allow her to more fully put into practice beliefs related student-led discussions without compromising classroom order.

Finally, the substance of Ms. Aria's interactions with students, especially in instances of sustained support, reflects her beliefs and is dependent on her knowledge. There is strong evidence that Ms. Aria believes students need to develop conceptual understanding of mathematics prior to or in conjunction with procedural knowledge. Therefore, Ms. Aria aims to provide students with support that is conceptually-grounded and generative. These beliefs are able to be realized in practice because of Ms. Aria's knowledge of mathematics for teaching. During mathematics instruction, Ms. Aria is able to draw on her knowledge base to make conjectures regarding how a student is thinking about a given problem based on a quick review of the student's written work or what she hears in student-to-student talk. If Ms. Aria suspects a student is having difficulty with a problem, she is able to determine the mathematical basis of the difficulty and tailor her support to address the student's difficulty in a conceptually-supportive manner. In the case of word problems, Ms. Aria has developed scripts to focus student attention on making sense of and modeling the context of the problems. In these ways, Ms. Aria's knowledge base facilitates enactment of her beliefs related to the importance of developing students' conceptual understanding.

Situating Ms. Aria's Case Story in Broader Measures of Teaching

In the previous section, a theoretical explanation was presented suggesting how Ms. Aria's teaching practices in response to student difficulties are linked to her beliefs and knowledge during the year of this study. Teacher response to student difficulties is one of many aspects of reform-based mathematics instruction that contributes to the overall quality of mathematics teaching and learning more broadly defined. Therefore, this section will present results from two more global measures of Ms. Aria's mathematics teaching. First, data collected following each core classroom observation using the *Reformed Teaching Observation Protocol* (RTOP) (Sawada et al., 2002) will be presented. This instrument is designed to measure the degree to which a given mathematics lesson reflects principles and practices associated with reform-based mathematics instruction. Second, aggregated class data from the mathematics section of the State Standardized Test (SST) will be presented in relation to aggregated data at the school, district, and state levels. Taken together, these findings will allow Ms. Aria's patterns of response to student difficulty to be considered within a more general understanding of her mathematics teaching and students' learning.

Aria: Adherence to Reformed Teaching

The degree to which Ms. Aria's mathematics instruction reflects current reforms in mathematics education was measured using the Reformed Teaching Observation Protocol (RTOP) (Sawada et al., 2002). The RTOP is a criterion-referenced instrument containing 25 items, divided into 5 subscales: (1) lesson design and implementation, (2) the level of significance and abstraction of the content, (3) the processes that students use to manipulate information, (4) the classroom culture as observed through the nature of communicative interactions, and (5) the nature of student-teacher relationships. Following each core observation in Ms. Aria's classroom, each of the 25 items on the RTOP was rated on a scale from 0 (not observed) to 4 (very descriptive). Next, sums were calculated for ratings on each subscale as well as the total instrument to assess the degree to which Ms. Aria's mathematics instruction was reformed, with higher scores reflecting a greater degree of reform. Consequently, subscale scores on the RTOP range from 0 – 20, and total instrument scores range from 0 – 100. Aggregated results from core observations of Ms. Aria's mathematics teaching in the Fall and Spring are presented in Table 14.

On the lesson design and implementation subscale, the mean scores for Ms. Aria's instruction are 17.0 in the Fall and 18.0 in the Spring. Throughout the year, lessons observed primarily involve student exploration in which students are encouraged to devise their own methods for solving problems. Instructional strategies consistently respect students' prior knowledge and mathematics concepts are highlighted through discussion of students' various ways of solving problems. Finally, Ms. Aria designs lessons to engage members as a learning community by requiring or encouraging students to collaborate with each other as they work on assigned tasks.

The second RTOP subscale focuses on what the contents of a lesson suggests about a teacher's propositional knowledge, meaning her mathematical knowledge for teaching the particular content of focus in the lesson. On this subscale, Ms. Aria's mean scores are 15.0 in the Fall and 19.0 in the Spring. All lessons observed involve fundamental mathematics concepts and provide evidence that Ms. Aria has a reasonably solid grasp of the mathematics of focus. While all lessons are found to promote conceptual understanding, Spring lessons accomplish this to a much greater extent than Fall lessons. In part, this stems from better inclusion of and connection to symbolic representations in Spring lessons.

The third RTOP subscale focuses on what the contents of a lesson suggests about a teacher's procedural knowledge. For the RTOP instrument, this means the quality of the lesson in terms of an inquiry approach to learning. In comparison to the other subscales, the ratings of Ms. Aria's mathematics instruction are the weakest on this subscale at both data points, with mean scores of 11.0 in the Fall and 14.0 in the Spring. Students are engaged in

Table 14
Ratings of Ms. Aria's Mathematics Teaching on the RTOP

RTOP items by subscale	Fall	Spr
Subscale 1: Lesson design and implementation		
1. The instructional strategies and activities respected students' prior knowledge and preconceptions inherent therein.	4.0	4.0
2. The lesson was designed to engage students as members of a learning community.	3.0	3.0
3. In this lesson, student exploration preceded formal presentation.	4.0	4.0
4. The lesson encouraged students to seek and value alternative modes of investigation and problem solving.	3.5	4.0
5. The focus and direction of the lesson was often determined by ideas originating with students.	2.5	3.0
Subscale 1 score	**17.0**	**18.0**
Subscale 2: Content, propositional knowledge		
6. The lesson involved fundamental concepts of the subject.	3.5	4.0
7. The lesson promoted strongly coherent conceptual understanding.	2.5	4.0
8. The teacher had a solid grasp of the subject matter content inherent in the lesson.	3.5	4.0
9. Elements of abstraction (i.e., symbolic representations, theory building) were encouraged when it was important to do so.	2.5	4.0
10. Connections with other content disciplines and/or real world phenomena were explored and valued.	3.0	3.0
Subscale 2 score	**15.0**	**19.0**
Subscale 3: Content, procedural knowledge		
11. Students used a variety of means (models, drawings, graphs, concrete materials, manipulatives, etc.) to represent phenomena.	3.5	3.0
12. Students made predictions, estimations, and/or hypotheses, and devised means for testing them.	2.0	2.0
13. Students were actively engaged in thought-provoking activity that often involved the critical assessment of procedures.	2.0	3.5
14. Students were reflective about their learning.	2.0	3.0
15. Intellectual rigor, constructive criticism, and challenging of ideas were valued.	1.5	2.5
Subscale 3 score	**11.0**	**14.0**
Subscale 4: Classroom culture, communicative interactions		
16. Students were involved in communication of their ideas to others using a variety of means and media.	3.0	4.0
17. The teacher's questions triggered divergent modes of thinking.	2.5	3.5
18. There was a high proportion of student talk and a significant amount of it occurred between and among students.	3.0	4.0
19. Student questions and comments often determined the focus and direction of classroom discourse.	2.0	3.0
20. There was a climate of respect for what others had to say.	2.0	2.0
Subscale 4 score	**12.5**	**16.5**
Subscale 5: Classroom culture, student/teacher relationships		
21. Active participation of students was encouraged and valued.	3.0	3.0
22. Students were encouraged to generate conjectures, alternative solution strategies, and ways of interpreting evidence.	2.5	3.5
23. In general the teacher was patient with students.	4.0	4.0
24. The teacher acted as a resource person, working to support and enhance student investigations.	3.5	4.0
25. The metaphor "teacher as listener" was very characteristic of this classroom.	4.0	4.0
Subscale 5 score	**17.0**	**18.5**
Total score	**72.5**	**86.0**

thought provoking activity involving the critical assessment of procedures in all lessons. However, in the Spring, students are pressed to justify their mathematical ideas more regularly and students generally appeared to be more reflective about their learning. Rigorous debate of mathematical ideas is limited in the whole class discussion section of all lessons. Yet, students are observed challenging each others' mathematical ideas as they formulate problem solutions during work time, especially in the Spring. Finally, students are encouraged to represent mathematical ideas in multiple ways in all lessons observed.

The fourth RTOP subscale focuses on communicative interactions that are part of the classroom culture. On this subscale, the mean scores for Ms. Aria's instruction are 12.5 in the Fall and 16.5 in the Spring. While students are encouraged to engage in various means of communicating their mathematical ideas in all lessons, a greater percent of students appear to participate in productive communication in the Spring. Throughout the year, students collaborate with each other to generate solutions to mathematics tasks. In the Spring, students are also frequently observed commenting on each others' ideas during whole class discussion. In this way, students' ideas appear to determine the focus and direction of classroom discourse to a greater extent in the Spring. However, throughout the year, Ms. Aria's class has difficulty maintaining a climate of respect during whole group discussions. It is not unusual for Ms. Aria to cut class discussions short because of behavioral concerns.

The last RTOP subscale focuses on the classroom culture in terms of the teacher's role and the roles students are encouraged to take. On this subscale, Ms. Aria's mean scores are 17.0 in the Fall and 18.5 in the Spring. Throughout the year, Ms. Aria acts as a resource person for students during problem solving, while intentionally working to promote greater independence among students. While interacting with students, Ms. Aria typically exhibits patience and spends the majority of time prompting students to explain and justify their mathematical thinking. Active participation of students is valued and encouraged throughout the year. However, teaching strategies to promote active participation are more pervasive during work time than in whole class discussion.

Ms. Aria’s mean scores on the total RTOP instrument are 72.5 in the Fall and 86.0 in the Spring. These scores suggest that Ms. Aria’s mathematics instruction moderately reflects the principles of mathematics reform in the Fall and approaches strong adherence in the Spring.

Aria: Student Achievement

Aggregated student scores on the mathematics section of the State Standardized Test (SST) were used to measure the mathematics achievement of Ms. Aria’s class. The SST is a criterion-referenced achievement test in which students receive scores between 1 and 5, with a score of 3 or higher considered passing. Since students first take the SST in third grade, comparable scores of previous achievement are not available for Ms. Aria’s students. Without this baseline data, only a tentative understanding of the relationship between Ms. Aria’s teaching and her students’ mathematics achievement is possible. Also, caution is warranted when comparing the SST scores of Ms. Aria’s students to the other third-grade classes at Lincoln Heights, as each teacher studied has a class with demographic particularities and a statistically small number of students. With those caveats in mind, SST scores for Ms. Aria’s students are presented in Table 15 along with comparison data for third-grade students at Lincoln Heights, the school district, and the state.

Table 15
Comparison of Aria Student Achievement on SST to School, District, and State

		Achievement level (%)					
Group	Students	1	2	3	4	5	Passing score (%)
Ms. Aria's class	18	17	22	39	22	0	61
Lincoln Heights 3rd grade	88	24	27	33	15	1	49
School District 3rd grade	13,713	18	16	33	24	9	66
State 3rd grade	203,037	15	17	34	25	9	68

Note. Scores of 3 or higher are considered passing.

Of Ms. Aria's 18 students, 61% achieve a passing score of 3 or higher on the mathematics section of the SST. This is in comparison to passing scores achieved by 49% of Lincoln Heights third graders, 66% of the third graders in the school district, and 68% of the third graders in the state. Therefore, the percent of Ms. Aria's students achieving grade-level standards, as measured by the SST, exceeds that of the third-grade students at Lincoln Heights but falls a little bit below the percentages of students achieving this standard at the district and state levels.

Summary of Global Measures of Ms. Aria's Mathematics Instruction

Taken together, results from the *Reformed Teaching Observation Protocol* (RTOP) and the State Standardized Test (SST) suggest that Ms. Aria's classroom is a place where students have the opportunity to learn significant mathematics through instruction that is somewhat reformed. RTOP results suggest that Ms. Aria's mathematics teaching moderately reflects the principles of mathematics reform in the Fall and approaches strong adherence in the Spring. Looking across observations, the kinds of tasks posed and the ways Ms. Aria orchestrates student work time strongly reflect reform recommendations. In particular, students are encouraged to draw on each other as resources during problem solving, and Ms. Aria's interactions with students serve to support and extend students' mathematical thinking rather than directing students to solve problems in particular ways. While students' mathematical ideas are pervasive throughout whole class discussions, most talk during these discussions occurs between Ms. Aria and particular students who are sharing their problem solutions. Only occasionally are students prompted to describe or evaluate other students' problem solutions and student-to-student interaction is generally limited. Perhaps related to these factors, student engagement during whole class discussion time is inconsistent and sometimes quite limited. However, for students who are paying attention, Ms. Aria consistently interprets and discusses student-generated mathematical ideas in conceptually coherent ways that illuminate important mathematical ideas.

Ms. Aria's students' mathematics achievement scores seem to fit with this assessment of mathematics instruction in her classroom. Aggregated SST data indicates that 61% of Ms. Aria's students are performing at or above grade level expectations. Since student attention during whole class discussion is an on-going challenge in Ms. Aria's classroom, it can be expected that some students do not fully benefit from the mathematics learning opportunities available. At the same time, over half of Ms. Aria's students do achieve grade level expectations, suggesting that Ms. Aria's reform-oriented approach to mathematics instruction supported the mathematics learning of much of the class, despite these conditions.

The Case of Ms. Jarmin

Ms. Jarmin begins the year of this study with an attitude of interest in the new mathematics program. Throughout her many years of teaching experience, Ms. Jarmin reports that she has enjoyed trying out all of the new programs and new ways of teaching that have come along. Although Ms. Jarmin does not consider herself to be good at mathematics, she is hopeful that the new *Everyday Mathematics* program will help her make her students' experiences with the subject better than her own.

Ms. Jarmin's class is comprised of twelve students who have been retained in third grade as a result of failing scores on the State Standardized Test (SST). Even though only one-fourth of her students are identified with special education needs, Ms. Jarmin frequently likens her class to a self-contained special education class claiming that students have difficulty retaining what is learned from one day to the next. In part because of their status as retained students, Ms. Jarmin believes her class to have low self-esteem especially with regard to their academic abilities. Ms. Jarmin asserts that her students require a delicate balance of patience, structure, and cajoling to stay focused on learning.

The case of Ms. Jarmin that follows is presented in four sections. In the first two sections, evidence of Ms. Jarmin's mathematics-related beliefs and knowledge at the beginning and end of the year will be presented. Next, the relationship among Ms. Jarmin's beliefs, knowledge, and classroom practice will be explored through a case story focused on her patterns of response to student difficulties. Finally, global measures of Ms. Jarmin's adherence to reform-oriented mathematics teaching and student achievement will be presented, which allows the case story findings to be considered in relation to these broader measures of teaching.

Ms. Jarmin's Beliefs about Mathematics Teaching and Learning

Ms. Jarmin's beliefs about mathematics teaching and learning were measured at the beginning and end of the school year using the *IMAP Web-Based Beliefs-Survey* (Ambrose, Phillip et al., 2003). The IMAP instrument requires teachers to respond to instructional scenarios in an open-ended format, allowing respondents to emphasize or downplay issues of personal importance. A series of rubrics are then used to measure survey responses in relation to seven target beliefs considered central to reform-oriented mathematics instruction. Ms. Jarmin's IMAP results are presented in Table 16, suggesting the degree to which Ms. Jarmin's survey responses provide evidence that she holds each target belief at the beginning and end of the school year.

Table 16
Ms. Jarmin's IMAP Web-Based Beliefs-Survey Results

Belief	Beginning of Year	End of Year
Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).	2	3
Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.	1	2
Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.	2	3
Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.	2	2
Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.	0	2
Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.	1	1
Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.	1	2

Note. 0 = No evidence; 1 = Weak evidence; 2 = Moderate evidence; and 3 = Strong evidence

In the sections that follow, evidence of Ms. Jarmin's adherence to each reform-oriented belief at the beginning and end of the year will be presented.

Jarmin: Belief about the Nature of Mathematics (IMAP Belief 1)

At the beginning of the year, there is *moderate evidence* that Ms. Jarmin holds the belief that mathematics is a web of interrelated concepts and procedures. In response to a scenario presenting multiple ways of solving a multidigit addition problem and asking which strategies she would like children to share, Ms. Jarmin indicates that she would include sharing of all five strategies presented in a unit on multidigit addition. Yet, Ms. Jarmin views this sharing as valuable primarily because it provides opportunity to assess students' understanding of addition concepts. Ms. Jarmin's discussion of the interrelatedness among strategies is limited to consideration of how a base-ten blocks strategy relates to a partial sums strategy. Therefore, survey evidence suggests that while a belief in the interrelatedness of math concepts/procedures may drive Ms. Jarmin to value multiple ways to solve problems, her focus on fostering student understanding by emphasizing the interrelatedness of strategies is likely to be limited.

At the end of the year, Ms. Jarmin's response to the same scenario suggests that she thinks more fully about how the various strategies relate to place value and she uses place value as an organizing idea for discussing student strategies presented. This represents a more sophisticated way of thinking about the mathematics as a web of ideas. Ms. Jarmin also talks about the strategies in terms of what a class of students might learn from a discussion of them rather than what she might glean from students using the strategy (the assessment-only perspective identified at the beginning of the year). This represents a shift in the way she thinks about use of different strategies. At the end of the year, there is *strong evidence* that Ms. Jarmin holds the belief that mathematics is a web in interrelated concepts and procedures because she gives greater consideration to how she might use the interrelatedness of strategies to help students understand mathematical ideas.

Jarmin: Belief about Distinction between Understanding Concepts and Applying Mathematical Procedures (IMAP Belief 2)

At the beginning of the year, there is *weak evidence* that Ms. Jarmin holds the belief that procedural proficiency and conceptual understanding are two different things. Ms. Jarmin observes that students' use of certain strategies demonstrates understanding of particular mathematics concepts more than others. For instance, she suggests that use of the partial sums strategy for multidigit addition indicates understanding of place value concepts while a student using the standard U.S. algorithm may achieve a correct answer without understanding why this algorithm works. When asked to rank order the difficulty of four problems involving fractions, Ms. Jarmin indicates that her rankings are based on the difficulty of the procedures for solving the problems with little attention to the underlying mathematics concepts. Furthermore, Ms. Jarmin states explicitly that, when thinking about mathematical understanding, she is mostly looking to see that students have correct answers. This provides evidence that Ms. Jarmin does not consistently distinguish between students' abilities to carry out mathematical procedures and their understanding of underlying concepts.

At the end of the year, Ms. Jarmin's responses suggest that she more fully recognizes a distinction between performing procedures and conceptual understanding (*moderate* *evidence*). In contrast to the beginning of the year, Ms. Jarmin indicates that she is not thinking about "correct answers" when she evaluates understanding. Additionally, Ms. Jarmin defends her ordering of fraction tasks by referring to the complexity of concepts, although elaboration of ideas on this item is limited. Overall, there is evidence at the end of the year that Ms. Jarmin believes there is a distinction between knowledge of mathematical procedures and understanding of underlying concepts. But limited elaboration in response to survey items suggests the possibility that Ms. Jarmin is not entirely clear on how this distinction plays out with particular mathematical ideas.

Jarmin: Belief about Source of Generative Mathematical Understanding (IMAP Belief 3)

At the beginning of the year, Ms. Jarmin's survey responses provide *moderate evidence* that she believes understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures. Survey responses consistently suggest that mathematical concepts should be taught before or as students learn procedures. In response to one scenario, Ms. Jarmin indicates that a conceptually explicit nonstandard strategy for multidigit subtraction is likely to be easier for students to understand than the standard U.S. subtraction algorithm. However, she also indicates that a higher percentage of students using the standard U.S. algorithm will get correct answers when compared to students using a conceptually explicit nonstandard method. These seemingly contradictory responses hint that, while Ms. Jarmin may view some nonstandard strategies as more conceptually revealing, at the beginning of the year she views the standard algorithm as more reliable.

In contrast, Ms. Jarmin's end-of-year survey indicates belief that a greater percentage of students will be successful with a conceptually explicit, nonstandard method for multidigit subtraction when compared to students using the standard U.S. subtraction algorithm. Additionally, Ms. Jarmin continues to hold the position that students benefit from learning about mathematical concepts before or as they learn mathematical procedures. Overall, Ms. Jarmin's responses at the end of the year provide *strong evidence* of adherence to the belief that understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Jarmin: Belief about Consequences of Teaching Concepts/Procedures First (IMAP Belief 4)

At the beginning and end of the year, there is *moderate evidence* that Ms. Jarmin believes that students are more likely to understand mathematical procedures if they first learn mathematical concepts. For instance, Ms. Jarmin discusses use of manipulatives, pictures, and talking it through to develop students' understanding of division of fractions. At the end of the year, Ms. Jarmin asserts that standard algorithmic procedures for multidigit addition should not be taught until students are exposed to more conceptually explicit methods. At both data points, Ms. Jarmin's responses provide evidence of the belief that instruction on mathematical concepts should precede instruction on procedures, but responses do not go as far as to indicate that students who learn procedures first are less likely to ever gain conceptual understanding.

Jarmin: Belief about Children's Problem Solving Capabilities (IMAP Belief 5)

At the beginning of the year, there is *no evidence* that Ms. Jarmin believes children can solve problems in novel ways before being taught how to solve such problems. For instance, she indicates that she would never ask students to do a problem without teaching them how, because they would be frustrated. Furthermore, she backs her assertion that first grade students would not be capable of completing a certain kind of problem because it is unlikely that they would have been taught yet.

In contrast, at the end of the year, Ms. Jarmin states that she does have students devise solutions to novel problems, with the caveat that she needs to be careful that they have the prior knowledge to be successful. On observing a video clip of a teacher providing step-by-step procedural direction to a student as he solves a problem, Ms. Jarmin's primary reaction assumes that the student must need this kind of direction to be successful. But later she acknowledges that the teacher's intervention leaves little room to assess the student's thinking; therefore it was not possible to know for sure if the student was capable of approaching the problem with less direction. These responses suggest that, at the end of the year, Ms. Jarmin believes children are sometimes capable of solving problems in novel ways before being taught how to solve such problems. But she believes there will also be times when students need strong direction to be successful. Therefore, there is *moderate evidence* of this belief.

Jarmin: Belief about Children's Ways of Thinking about Mathematics (IMAP Belief 6)

At the beginning and end of the year, there is *weak evidence* that Ms. Jarmin believes children think about mathematics in ways that are generally different than adults. Real-world contexts in the form of story problems are not recognized as being facilitative of initial understanding. Ms. Jarmin does not offer ways that first-grade students might successfully solve contextualized word problems with a join-change unknown or multiplication problem structure. In general, Ms. Jarmin's responses used to evaluate this belief are brief at the beginning and end of the year. They give little attention to how children might use their own prior knowledge to make sense of mathematics.

Jarmin: Belief about Teacher's Role in Supporting Student Learning (IMAP Belief 7)

At the beginning of the year, there is only a little evidence (*weak evidence)* that Ms. Jarmin believes that the teacher should allow children to do as much of the thinking as possible during math instruction. Ms. Jarmin praises a teacher's actions in a video clip in which the teacher provides step-by-step direction to a student working on a problem solving task. She views strong teacher direction as a strength of the instructional episode and believes that children need strong direction in order to learn. However, Ms. Jarmin also believes that part of the teacher's role in direction is to get students to talk about what is going on and what they are learning. In this way, she seems to be expecting the students to do some of the thinking after teacher-centered instruction has been provided.

By contrast, in response to the video clip of step-by-step direction at the end of the year, Ms. Jarmin describes the teacher's actions as "doing the thinking for the student," and notes that she does not think the student is likely to learn much from that mode of instruction. At the end of the year, Ms. Jarmin expresses a desire to have children find their own ways to solve problems and not be confined to only solving problems in her ways. But she provides little detail on how this might be accomplished instructionally. All things considered, Ms. Jarmin's responses provide *moderate evidence* that she holds the belief that the teacher should allow children to do as much thinking as possible and hints that she may still feel uncertain about how to adhere to this belief in the classroom.

Summary of Ms. Jarmin's Beliefs

At the beginning of the year, Ms. Jarmin's responses to the IMAP survey instrument suggest that she does not strongly adhere to any of the seven beliefs measured. There is no evidence that Ms. Jarmin believes children can solve novel problems without being taught how to solve such problems. There is little evidence that Ms. Jarmin believes teachers should allow children to do as much thinking as possible during interactions related to the learning of mathematics or that children's ways of thinking about mathematics are different than adults' ways. While Ms. Jarmin's responses indicate value in developing conceptual understanding of mathematics, there is weak evidence that Ms. Jarmin believes that conceptual understanding and procedural knowledge are different things.

By the end of the year, there is evidence that many of Ms. Jarmin's beliefs have moved closer to a reform-orientation. At this point, there is strong evidence that Ms. Jarmin believes school mathematics should promote an interrelated understanding of concepts and procedures and that conceptual understanding is more powerful and generative than simply remembering mathematical procedures. There is moderate evidence that Ms. Jarmin believes teacher interactions with students should allow children to do as much thinking as possible. Furthermore, Ms. Jarmin's responses suggest that she now believes children are capable of solving novel problems without being taught how to solve them in some situations. However, Ms. Jarmin's end-of-year responses continue to indicate that she gives little consideration to the differences in the ways children and adults think about mathematics, particularly with regard to using real-world contexts to support children's mathematical thinking.

Ms. Jarmin's Knowledge of Mathematics for Teaching

Ms. Jarmin's knowledge of mathematics for teaching with a reform-orientation was measured at the beginning and end of the year using data from two parts of the Teacher Knowledge interview. In particular, this interview explores a teacher's knowledge related to the teaching of multiplication and division to third-grade students. First, Ms. Jarmin's open-ended discussion of teaching and learning multiplication will be presented. In this part of the interview, teachers were prompted to describe important understandings and common student difficulties related to the learning of multiplication. In addition, interview questions direct teachers to discuss how children develop the important understandings identified and how multiplication should be taught. Next, Ms. Jarmin's responses to four classroom scenarios will be described. These classroom scenarios probe specific aspects of teacher knowledge related to the teaching and learning of multiplication and division, specifically knowledge of nonstandard strategies for multidigit multiplication, knowledge of a common student difficulty underlying the standard U.S. algorithm and strategies for addressing this difficulty, and teachers' abilities to interpret and respond to student work.

Jarmin: Open-ended Discussion of Teaching and Learning Multiplication

At the beginning of the year, Ms. Jarmin reports that what students need to understand about multiplication in third grade is that it is a short way of doing addition and that a multiplication number sentence has a corresponding physical model (e.g., 2×5 means 2 groups of 5). She indicates that this is accomplished by first having students build physical models of multiplication problems using manipulatives and then by having students talk about their models. During this phase of development, Ms. Jarmin suggests that students sometimes have difficulty making equal groups as they build physical models. Then, after students understand what a physical representation of multiplication is like, instruction should turn to providing ample practice to facilitate memorization of facts. Ms. Jarmin identifies knowledge of basic addition facts and place value as prerequisites to the study of multiplication.

At the end of the year, Ms. Jarmin asserts that third-grade instruction on multiplication should develop students' understanding of strategies for solving multiplication problems, why they get the answer, and how multiplication is related to addition. Ms. Jarmin suggests that first children should be exposed to real-world multiplication situations in the form of word problems, then they should build physical models of these situations. Instruction at this point should emphasize that multiplication involves equal groups. Next students move from finding products for their physical models by counting one-by-one to using the strategies of counting by a number or using repeated addition. Then Ms. Jarmin suggests students are ready to begin solving multiplication problems without the support of physical models. As students begin to have quick recall of a few facts, they can begin to use those facts to figure out harder facts. As instruction turns to multidigit multiplication, Ms. Jarmin indicates that students benefit from first learning to multiply one-digit numbers by multiples of ten (e.g., 4×50 or 4×500). Then students are ready to multiply one-digit numbers by multidigit numbers because they are able to sum partial products. Ms. Jarmin asserts that instruction should involve use of manipulatives, opportunities for students to devise their own strategies, group discussion of strategies, and frequent use of word problems. She also suggests that games should be used as a fun way to practice basic facts.

Jarmin: Knowledge of Non-standard Strategies (Classroom Scenario 1)

At the beginning of the year, Ms. Jarmin identifies five strategies that students might use to solve a word problem involving finding the number of chairs in 16 rows with 8 chairs in each row. These are presented in Table 17. From least to most sophisticated (according to Ms. Jarmin), these strategies include direct modeling and counting by 1 to find the product, direct modeling and counting by 8 to find the product, repeated addition, the standard U.S. algorithm for multidigit multiplication, and the partitioning strategy of multiplying each place and combining partial products. A compensation strategy is not identified. In discussing her rationale for identifying the partitioning strategy as more sophisticated than the standard U.S. algorithm, Ms. Jarmin explains that the standard U.S. algorithm can be performed simply by following memorized steps. In contrast, she suggests that students who use a partitioning strategy demonstrate greater understanding of how multiplication works. Therefore it is a more sophisticated strategy.

Table 17
Strategies Identified by Ms. Jarmin in Response to Classroom Scenario 1

Data point	Direct modeling	Complete number	Partitioning number	Compensating	Other
Beginning of year	16 rows of 8 Xs (counting by one or eight to find product)	Repeated addition 8 + 8 + 8…	10 x 8 = 80, 6 x 8 = 48, 80 + 48 = 128		Standard U.S. multiplication algorithm
End of year	16 rows of 8 Xs (counting by one or eight to find product)	Repeated addition 8 + 8 + 8… Doubling strategy: Combining 8s to make 16, 16s to make 32, etc	10 x 8 = 80, 6 x 8 = 48, 80 + 48 = 128		16 circles with 8 dots in each A visual model, 16 loops with the numeral 8 in each loop. This is a hybrid strategy.

Note. Analysis utilizes strategy classification presented in Baek (1998).

At the end of the year, Ms. Jarmin identifies eight strategies that children might employ to figure out how many total chairs are in 16 rows of 8 chairs (see Table 17). She identifies visually-based strategies as the least sophisticated, with loops and counters strategies being less sophisticated than modeling the situation with rows of objects (an array). With both of these visually-based strategies, Ms. Jarmin indicates that counting by 8 to find the product is more sophisticated than counting by 1. Complete number strategies, including repeated addition and a combining parts strategy, are identified next in order of sophistication. Finally, Ms. Jarmin identifies the partial products focus algorithm from *Everyday Mathematics* as being the most sophisticated strategy listed. As was the case at the beginning of the year, a compensating strategy is not identified.

While at the beginning of the year Ms. Jarmin struggles to discuss how students' mathematical knowledge is related to the various strategies, at the end of the year she exhibits some increased awareness of the cognitive resources students need to acquire as they develop increasingly sophisticated strategies. For instance, she identifies the point at which students move from counting by 1 to counting by the multiplicand as the point at which students begin to recognize the relationship between multiplication and repeated addition. Also, Ms. Jarmin intentionally omits the standard U.S. algorithm for multidigit multiplication from her list of strategies. When asked about this omission, Ms. Jarmin explains that she believes partial products is much better because it encourages understanding, whereas the traditional algorithm encourages learning a procedure. In summary, Ms. Jarmin's response to Classroom Scenario 1 provides evidence of moderate knowledge of non-standard strategies for multidigit multiplication at both data points, with greater knowledge at the end of the year particularly with regard to knowledge that students need to use specific strategies with understanding.

Jarmin: Interpretation of Students' Mathematical Strategies (Classroom Scenario 2)

In response to Classroom Scenario 2 at the beginning of the year, Ms. Jarmin identifies the correct answer to the pancake problem[8] but struggles to make sense of the three strategies presented through student work. As she initially reviews each student's work, Ms. Jarmin comments that the work does not make sense or that she is unsure why the student would approach the problem in the way presented. After looking at two of the three student strategies, Ms. Jarmin reports, "I feel stupid when I look at these. I'm like, 'I can't understand how you figured this out!'" Through being prompted to continue trying to make sense of the student work, Ms. Jarmin eventually makes sense of each student's solution in relation to the pancake problem context. However, the time and support needed to facilitate

[8] The problem is as follows: Twenty-four children want to share eight pancakes so that each one gets the same amount. How much pancake can each child have?

successful task completion suggests that Ms. Jarmin's ability to interpret students' strategies on-the-spot during instruction is limited at the beginning of the year.

At the end of the year, Ms. Jarmin is able to explain two of the three student strategies presented in this scenario after some time to study them. For these strategies, Ms. Jarmin identifies what she would be listening for in students' explanations of their strategies to determine their mathematical understanding. For the remaining strategy, Ms. Jarmin is able to relate select aspects of the picture to the pancake problem context, but she indicates that she does not understand how number sentences the student has included with the work relate to the problem or the picture. In summary, Ms. Jarmin's response to Classroom Scenario 2 provides evidence of limited ability to interpret students' strategies at the beginning of the year, with comparatively greater skill, albeit some difficulty, at the end of the year.

Jarmin: Addressing and Avoiding a Common Student Error (Classroom Scenario 3)

In this classroom scenario, teachers are presented with student work in which the standard U.S. multiplication algorithm is executed without maintaining the place values of the partial products. Ms. Jarmin's response at the beginning of the year provides evidence that she understands how place value concepts underlie the standard U.S. multiplication algorithm. She explains that the algorithm works because, when you put a zero as a place holder, it maintains place value in the partial products. In discussing how she would help third-grade students avoid errors with multidigit multiplication like the one in the scenario, Ms. Jarmin's approach is primarily teacher-centered and procedural:

> I would just explain over and over and over that the 6 doesn't stand for a 6 and the 4 doesn't stand for a 4. So, you know that the 4 stands for 40, so you always know, whatever you are multiplying by in the tens position, it is going to end in one zero.

> And if you're multiplying by the 6, it's in the hundreds position, so the number is always going to end in two zeros, to hold your place.

Ms. Jarmin's suggestion of how she might respond to sixth graders making these kinds of errors is similar to her approach for third graders.

At the end of the year, Ms. Jarmin states explicitly that students who make errors like the ones in classroom scenario 3 do not have a strong understanding of place value. She suggests that what students need is to have a more developed sense of the size of the product when you multiply large numbers. Ms. Jarmin indicates that, with sixth-grade students making these errors, she would, "…go back to the basics and do some problems with just the zeros, like 600 times 100." She suggests that helping students see the quantities in a visual, hands-on way would be useful, but she is unsure how she could do this. While reflecting on her role helping third-grade students avoid errors like Classroom Scenario 3, Ms. Jarmin expresses her heightened awareness of the importance of developing place value understanding:

> This is the first year that I have seen place value is really, really, really important. I mean, I already knew that. But I have seen it more, like when we do partial products. When you learn [multidigit multiplication] in the rote memory way, you should also know that. But it's really important in partial products to know what that 1 represents, that it is 100.

Ms. Jarmin suggests that she would help third-grade students to understand the relationship between place value and multidigit multiplication by working with base-ten blocks and focusing instruction on the partial products algorithm.

In summary, Ms. Jarmin's response to Classroom Scenario 3 provides evidence that she understands the origin of students' difficulties with the standard U.S. algorithm for multidigit multiplication. However, at both data points, there is evidence that Ms. Jarmin is limited in her knowledge of how to help students overcome the difficulties presented in the scenario.

Jarmin: Interpretation of and Response to a Student's Flawed Solution (Classroom Scenario 4)

At the beginning of the year, Ms. Jarmin is unable to make sense of the flawed strategy presented in Classroom Scenario 4. In this scenario, a student explains that the answer to 144 ÷ 8 = ? can be found by dividing 144 by 2 four times because 2 + 2 + 2 + 2 = 8. Ms. Jarmin initially attempts to make a picture of base-ten blocks in efforts to model the student's thinking. But she quickly abandons this strategy citing that there would be a lot of trading (regrouping) involved and the strategy might not work. Ms. Jarmin identifies that the student would have gotten the answer correct if she had stopped after the third division, but she expresses that she does not know why this is true. When asked how she would respond to a child using this strategy, Ms. Jarmin indicates that she would have the child solve the problem in a different way because, "…this way obviously didn't work."

At the end of the year, Ms. Jarmin is again unable to figure out why the student strategy presented in Classroom Scenario 4 is flawed. In contrast to her response at the beginning of the year, Ms. Jarmin spends more time attempting to make sense of the student's strategy in this interview. She indicates that, in the context of teaching, she would probably have the student use an alternate strategy since she cannot figure out the error. However, in contrast to the beginning of the year, Ms. Jarmin appears much more interested in trying to understand how the strategy is correct and why it does not yield a correct answer. Ms. Jarmin's response to Classroom Scenario 4 at both data points provides evidence of her difficulty interpreting flawed student work and responding in a way that honors student thinking.

Summary of Ms. Jarmin's Knowledge of Mathematics for Teaching

Ms. Jarmin demonstrates moderate knowledge of ways students might approach mathematical problems involving multiplication at the beginning and end of the year, with greater knowledge demonstrated at the end of the year. In particular, her end-of-year interview includes reference to how knowledge of easy multiplication facts can be used to figure out more difficult facts and multidigit multiplication problems. However, at neither data point does Ms. Jarmin identify a compensation strategy as a possible approach to multiplication problems. Furthermore, her inclusion of partitioning strategies is limited to portioning by decade numbers (like the *Everyday Mathematics* partial products strategy) and does not include other ways of partitioning numbers.

While at the beginning of the year, Ms. Jarmin is limited in her ability to explain how multiplication strategies are related to each other and to student thinking, she describes a hypothetical learning trajectory at the end of the year that elaborates what students must know to understand increasingly sophisticated multiplication strategies. This suggests that Ms. Jarmin's knowledge of how students learn multiplication increased significantly over the course of the year. Yet, at both data points, Ms. Jarmin identifies few difficulties that teachers might expect students to encounter as they learn about multiplication. Furthermore, interpretation of unfamiliar student strategies seems to be especially difficult for Ms. Jarmin at the beginning of the year and somewhat difficult at the end of the year. Therefore, even though there is evidence that Ms. Jarmin's knowledge of student thinking increased, her end-of-year interview suggests that she continues to have difficulty anticipating student difficulties and making sense of unfamiliar student strategies.

In discussing how multiplication should be taught, Ms. Jarmin's end-of-year interview emphasizes a greater number of reform-based teaching strategies than at the beginning of the year. In particular, at the end of the year, Ms. Jarmin emphasizes use of real-world situations to support students' initial understanding of multiplication, and she indicates that students should have opportunities to solve problems in their own ways. Neither of these strategies was mentioned at the beginning of the year. Nonetheless, Ms. Jarmin's response to classroom scenarios involving student work suggest that, at both data points, she has difficulty responding to student work in meaningful ways that honor student thinking, especially when students have made errors.

Case Story of Ms. Jarmin's Response to Student Difficulties

In the case story that follows, facets of Ms. Jarmin's typical response to student difficulties will be illuminated. Specifically, I will portray Ms. Jarmin's general routines for responding to students' difficulties, the substance of her response to students' difficulties with solving word problems, and her response to unanticipated student ideas. Following illustration of each response pattern, consequences for student thinking and the relationship between Ms. Jarmin's beliefs, knowledge, and mathematics teaching practices will be discussed.

Jarmin Response Pattern 1: Routines for Responding to Student Difficulties

During mathematics instruction, Ms. Jarmin employs a predictable set of routines when she encounters student difficulties during student work time and in the context of class discussion. After directing students to begin working on a mathematics task, Ms. Jarmin's routine is to begin approaching individual students to converse with them about the task and provide support for devising a viable solution strategy. Of note is the immediacy with which Ms. Jarmin begins her interactions with students, only occasionally standing away from students as they begin working. As the time allotted for students to work on a given task continues, Ms. Jarmin interacts with one student after another, aiming to help as many students as possible achieve a correct solution to the given task. In this way, she provides some degree of help for students, limiting the degree to which they must grapple with the given task on their own.

When Ms. Jarmin encounters a student who is struggling to complete a task or who appears to be moving toward an incorrect answer, she initiates a teaching routine of asking questions and providing prompts until she perceives the student is moving in a productive direction with the problem. In these instances, Ms. Jarmin generally uses teacher-student interaction to address students' difficulties, rarely encouraging students to consult with or seek support from their peers. Questions and prompts used to move student solutions forward seem most focused on moving toward a correct answer.

For instance, in the following student-teacher interaction from a Spring lesson, Ricky is struggling to complete his solution to the problem 9×500. So far, Ricky has written nine 500's on his paper and he has combined pairs of 500s to make 1000s. When Ms. Jarmin approaches, Ricky is trying to figure out what to do with the ninth 500.

Ricky's work:

$$\left.\begin{array}{l}500\\500\end{array}\right\}1000$$
$$\left.\begin{array}{l}500\\500\end{array}\right\}1000$$
$$\left.\begin{array}{l}500\\500\end{array}\right\}1000$$
$$\left.\begin{array}{l}500\\500\end{array}\right\}1000$$
$$500$$

Ricky: Ms. Jarmin, I've got 8…but I don't know, because I have, I can't get…
Ms. Jarmin: Well, okay, you've got 8. That's perfect. That's fine. So, you just need how many more?
Ricky: One more
Ms. Jarmin: One more what? One more thousand?
Ricky: Five-hundred.
Ms. Jarmin: One more 500. So, when you add these (the four 1000s), you just need to add one more 500 to it.

In response to Ricky's uncertainty, Ms. Jarmin assures him that his solution is correct so far and, in the last line, tells him exactly what to do to produce a correct answer. In doing so, Ms. Jarmin makes it unnecessary for Ricky to work through the logic in his solution himself, thus limiting Ricky's opportunity to grapple with the mathematical ideas inherent in the task.

In addition to employing routines that minimize students' struggles during work time, the routines employed for class discussion of mathematics tasks, especially when student difficulties surface, limit the degree to which students are likely to grapple with mathematical ideas. In general, class discussion of mathematics tasks before and after work time is limited to interchanges between individual students and the teacher, with little student-to-student discourse. Teacher-student interactions are highly controlled by Ms. Jarmin, with students primarily answering closed questions about strategies employed. When a student makes an incorrect statement or responds to a question incorrectly, Ms. Jarmin's response script is to restate the same question or ask a new question in efforts to guide the student to give a correct answer. Ms. Jarmin only occasionally invites other students to weigh in with alternate responses or to evaluate the correctness of the given students' response. Consequently, Ms. Jarmin maintains authority over mathematical correctness and provides little incentive for students to grapple with the flawed statements or ideas put forward by their peers.

To illustrate Ms. Jarmin's routine for addressing students' incorrect responses to questions posed, consider the interchange that follows between Ms. Jarmin and Kevin during a class discussion in a Fall lesson. Another student, Samuel, has just finished describing how he made an array representation of the problem, "Hotdog buns are sold 8 to a package. There are 24 hotdog buns. How many packages are there?" (Ms. Jarmin's record of Samuel's strategy is displayed on the overhead.) This excerpt begins with Ms. Jarmin asking Kevin to report the answer to the hotdog word problem.

Samuel's strategy, as recorded by Ms. Jarmin on the overhead:

X X X X X X X X 8
X X X X X X X X 16
X X X X X X X X 24

Ms. Jarmin: So, how many packages are there Kevin, how many packages are there?
Kevin: Eight…I mean…
Ms. Jarmin: How many packages do we have?
Kevin: Eight.
Ms. Jarmin: I would like you to get in your seat Samuel. Please get in your seat. How many buns are in one package Kevin?
Kevin: Twenty-four.
Ms. Jarmin: How many buns are in one package? If I were going to give you a package of buns, how many buns would be in that package?
Kevin: Eight.
Ms. Jarmin: How many?
Kevin: Eight.
Ms. Jarmin: Eight. So, how many packages of eight do we have?

Kevin: Three.
Ms. Jarmin: Three.

Ms. Jarmin's first response to Kevin's reporting the number of packages incorrectly is to ask him the question again, thus providing a cue that the initial answer given was incorrect. Then, when he reports the same answer a second time, Ms. Jarmin poses additional questions to Kevin with the intent of helping him recognize his error and produce the correct answer. Other students are not invited to suggest alternate responses or comment on Kevin's response, even though most of the students in the class are also struggling to make sense of what the objects, rows, and columns of an array represent in relation to division word problems. Ms. Jarmin maintains control over judging the initial correctness of a student response, guiding the student to change his response, and determining when a flawed response has been revised to correct.

In summary, the routines that Ms. Jarmin employs during mathematics instruction often limit the necessity for students to grapple with mathematical ideas, both as students are working on mathematics tasks as well as in the whole class forum. As students work, Ms. Jarmin interacts with students frequently, often providing direction that reduces the challenge of the task. During student work time and in the context of class discussion, Ms. Jarmin positions herself as the primary asker of questions and evaluator of correctness. Students are rarely encouraged to draw on their peers as resources during problem solving or when they encounter difficulties. The implicit message these routines and scripts communicate is that students will likely require teacher support to be successful with mathematics tasks and the teacher is the main source of mathematical knowledge in the classroom.

Jarmin response pattern 1, consequences for student thinking. As the year progresses, the students in Ms. Jarmin's class appear to become accustomed to their teacher's routine of providing one-on-one help during the time allotted for problem solving. As Ms. Jarmin interacts with students one by one, it is typical for some students to spend significant amounts of time sitting idle, only moving forward on a task when Ms. Jarmin interacts with them. Students are only occasionally observed drawing on their peers to support their problem solving efforts, even though Ms. Jarmin indicates that she does not mind students talking to each other about the math problems during work time. As the year progresses, students continue to rely heavily on Ms. Jarmin's direction as they work on problem solving tasks, limiting the degree to which they grapple with mathematical ideas.

In the context of whole group discussion, student engagement in productive learning practices that advance student thinking is limited. Rather than actively listening to ideas put forward by peers and considering whether or not they are reasonable, students regularly appear to have their attention elsewhere. Students seem to have little sense of what they are supposed to be doing during these discussions, besides listening to their peers. If a peer makes an error in his description of a strategy or in response to Ms. Jarmin's questions, it is the classroom norm that Ms. Jarmin will address the problem. Only occasionally are students observed commenting on each other's mathematical strategies or ideas. Consequently, the learning potential associated with class discussion of mathematical ideas is only minimally realized.

Jarmin response pattern 1, link to beliefs and knowledge. Ms. Jarmin's teaching practices that inadvertently minimize students' opportunities to grapple with problem solving tasks seem most related to her beliefs. But they also relate to her knowledge. First, consider Ms. Jarmin's practice of engaging in frequent interaction with students as they work on problem solving tasks. Although Ms. Jarmin believes that problem solving should be a prominent feature of mathematics instruction, she only sometimes encourages students who encounter difficulties with word problems to rely on their own mathematical knowledge to overcome difficulties. Aligned with a traditional orientation to mathematics instruction, Ms. Jarmin views the role of word problems as giving students opportunities to apply previously learned mathematical ideas. Following this way of thinking, Ms. Jarmin expects that students will require guidance to successfully solve word problems when they are incorporated into the introduction of new mathematics topics (as is done in the *Everyday Mathematics* program). As some students develop the practice of waiting for Ms. Jarmin's help to approach mathematics tasks, Ms. Jarmin's belief in the necessity of her on-going interactions with students while they are working on problem solving tasks is reinforced.

Nonetheless, Ms. Jarmin sometimes expresses concern about the amount of help her students require, indicating that it is desirable for students to depend less on her support. In considering why her students are reluctant to work without her direct support, Ms. Jarmin presents several possible causes, including students' particular learning needs and educational histories as students who have been retained:

> I still think some of my students are afraid to try things because they're not sure, and they don't want to be wrong, and I don't know if it's just this class or it's the situation that they've been in that they've been wrong before or whether it's some of their language problems, processing problems, or fear of being wrong.

In general, Ms. Jarmin reconciles her belief in the necessity of providing significant support with the experiences of other teachers who engage in less intervention with the belief that her class of retained students has different academic and emotional needs than students in other classes.

In discussing her practice of maintaining strong control over class discussion, Ms. Jarmin indicates that she would like for students to do more of the talking in discussions around problem solving tasks, but that students have much difficulty providing mathematical explanations as new mathematics concepts are being introduced:

> A lot of times the beginning talk, when I'm first introducing the concept, they're just answering yes and no questions because they actually don't understand anything I'm saying…they don't understand it enough to explain what we're doing, to talk it out.

Ms. Jarmin consistently reports that students need to be taught some things (by the teacher) before they will be capable of explaining ideas and strategies without significant teacher support. She seems to believe that, at first, students must acquire strategies for finding correct answers. Then, through repeated experiences, they will come to better understand and be able to explain how and why the strategies work.

These beliefs also seem in line with Ms. Jarmin's work time practice of providing struggling students with procedural guidance focused on achieving correct answers. From Ms. Jarmin's point of view, using a procedure that yields a correct answer is the first step of understanding. Ms. Jarmin also justifies her practice of providing struggling students with strong direction to achieve correct answers by noting the time constraints of classroom instruction. She states, "…sometimes it would take 20 to 30 minutes for one person to come up with the final answer, and in the real-world we don't always have time for that."

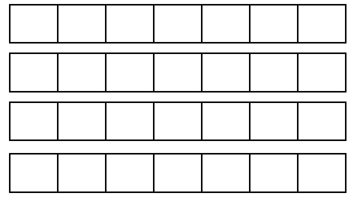
While Ms. Jarmin's teaching practices that minimize the degree to which students grapple with mathematical difficulties seem most related to the beliefs discussed, they are also related to her knowledge for teaching mathematics. In many ways, Ms. Jarmin has adapted teacher-centered instructional practices cultivated over her 20 years of teaching to this newly adopted reform-oriented program. While Ms. Jarmin is willing to follow program recommendations and assign mathematics tasks before students are taught exactly how to solve them, she reverts to familiar routines and scripts when she must respond on-the-spot to perceived student difficulties. These comfortable, teacher-dominated routines and scripts are a kind of knowledge that serves as a barrier to practices that are more likely to facilitate mathematics instruction that builds on student thinking. In order to move away from these practices, Ms. Jarmin will need to cultivate new routines and scripts that intentionally encourage students to grapple with mathematical ideas more fully.

Jarmin Response Pattern 2: Substance of Response to Student Difficulties with Word Problems

As Ms. Jarmin introduces students to multiplication and division concepts, she follows the *Everyday Mathematics* program's recommendation to use word problems in attempts to relate mathematics concepts to real-world situations. In this section, the substance of Ms. Jarmin's typical response to students' difficulties exploring mathematical ideas through word problems will be shared drawing on two classroom excerpts, one from a Fall lesson and the other from the Spring.

In the Fall lesson, Samuel is working on figuring out how many weeks are in 29 days. Ms. Jarmin's attempts to support Samuel are initially characterized by closed questions related to the problem context. When Samuel fails to make sense of the problem in context, Ms. Jarmin's support becomes directive and deemphasizes the problem context. When

Ms. Jarmin joins Samuel at his desk, he has a 4 × 7 array made with snap cubes:



Ms. Jarmin: Okay, so how many weeks are there?

Samuel: Seven…no, 28.

Ms. Jarmin: So, how many days are in a week? Are there four days in a week or are there seven?

Samuel: Seven.

Ms. Jarmin: So, how many groups of seven do we have?

Samuel: Twenty-eight.

Ms. Jarmin: How many rows of seven do we have?

Samuel: Four.

Ms. Jarmin: Four. So, how many weeks would be in 29 days? (*Pointing to each row of counters*) Here is one week, two weeks, three weeks…

Samuel: Twenty-eight.

Ms. Jarmin: Just count them. Count the rows.

Samuel: 1, 2, 3, 4.

Ms. Jarmin: So, how many weeks in 29 days?

Samuel: Four.

Ms. Jarmin: With how many days remaining?

Samuel: One.

Ms. Jarmin: Okay.

Samuel's first incorrect response, that there are 7 or 28 weeks, reflects the common difficulty students have keeping track of the units in a division word problem (Carpenter et al., 1999). Ms. Jarmin's means of follow-up initially reflect the context of the problem, focusing Samuel on what he knows about the number of days in a week. But then, when Samuel is unable to link his knowledge of seven days in a week to the array model he has made, Ms. Jarmin prompts Samuel to identify the number of rows instead of pressing him to understand what each of his counters and each of the rows represents. As Samuel's confusion about units persists, Ms. Jarmin abandons strategies that encourage Samuel to understand the problem, and instead directs him to use a procedure that will yield a correct answer.

In the Spring lesson, students are working on the problem, “If I have 5 groups of 24 pencils, how many pencils do I have altogether?” In the episode below, two students solicit help from Ms. Jarmin to figure out what operation they should use to solve the problem. Instead of encouraging students to imagine (or model) the context of the problem, Ms. Jarmin provides support that encourages development of a major misconception, that the product of a multiplication problem always results in a larger number than the factors:

> Alex: Ms. Jarmin, is this dividing?
> Ms. Jarmin: Well, what do you think it is? We are…
> Sonya: Adding.
> Ms. Jarmin: We want to find out how many altogether, so are we going to be making a bigger number or are we dividing into a smaller number?
> Sonya: Bigger.
> Ms. Jarmin: Bigger. So when we are making a bigger number, what do we do?
> Alex: Add.
> Ms. Jarmin: We could add or what else could we do to make a bigger number?
> Alex: Divide.
> Ms. Jarmin: Divide you make a smaller number.
> Alex: Ohh, times.
> Ms. Jarmin: Times. You can add or times. Or multiply.
> Alex: Plus?
> Ms. Jarmin: Well, well, what do you know in the problem?
> Sonya: Five groups
> Ms. Jarmin: You have five groups. How many is in each group?
> Alex: Twenty-four.
> *Alex and Sonya begin working on their papers for about 20 seconds. Ms. Jarmin observes their work.*
> Ms. Jarmin: So, you were thinking what? You were thinking 5 times what, honey?
> Sonya: 5 times…5 times 24.
> Ms. Jarmin: 5 times 24. So, how could you use that to figure out?

Although Ms. Jarmin does not directly tell students what to do to solve this problem, she teaches the students to rely on a false premise, that multiplication always makes bigger and division always makes smaller. Following this logic, Ms. Jarmin expects the students to identify multiplication as the correct operation, since the answer to the word problem will be larger than both of the numbers in the situation. While this approach works for whole-number computation, it serves as a significant and persistent barrier to students' understanding of multiplication and division of rational numbers. Furthermore, it provides little support for problems that involve more complexity than simple one-step computations. Therefore, this strategy is unlikely to move students toward deeper understanding of mathematical operations or greater problem solving ability.

In summary, Ms. Jarmin often responds to students' difficulties with word problems in ways that are conceptually unsupportive. She encourages students to think about the numbers from a given problem out of context, placing greater focus on procedures and tricks than on making sense of contextualized situations.

Jarmin response pattern 2, consequences for student thinking. Ms. Jarmin's practice of responding to student difficulties in conceptually unsupportive ways limits the degree to which her students develop problem solving practices that will be helpful for a variety of problems. Instead, students are observed attempting to apply procedures learned for one type of problem to other problems for which the procedures are not appropriate. For instance, after working on a series of partitive division problems, several students apply a distributing objects one-by-one action (reflective of partitive division) when prompted to create a physical model with counters for a measurement division problem. In general, students struggle with determining how to approach word problems throughout the year. Confusion over which mathematical operations to use persists, and students are limited in their abilities to build and justify models that reflect particular word problems. Instead of relying on their innate knowledge of real-world contexts to make sense of word problems, students appear to focus on using recently taught mathematical procedures to solve problems, with little attention to whether the procedures make sense.

Jarmin response pattern 2, link to beliefs and knowledge. Ms. Jarmin's practice of responding to student difficulties in conceptually unsupportive ways is related to both her beliefs and knowledge. Beginning and end of year beliefs measures provide only weak evidence that Ms. Jarmin views real-world contexts as being supportive of children's initial thinking about mathematics concepts. In addition, there is weak evidence at the beginning of the year that Ms. Jarmin views conceptual understanding and procedural proficiency as two different things. At the beginning of the year, Ms. Jarmin indicates that she thinks of mathematical understanding as being able to get correct answers. Reflecting these two beliefs, it logically follows that Ms. Jarmin's focus is on providing children with hints and tricks that she knows will yield correct answers. At the end of the year, there is evidence that Ms. Jarmin believes more strongly in the difference between conceptual understanding and procedural knowledge. However, her knowledge of how to promote conceptual understanding is limited.

The hints and tricks that Ms. Jarmin suggests to students throughout the year are grounded in her personal approach to mathematics, which is most often procedurally-focused. In trying to teach in ways that support students' development of conceptual understanding, Ms. Jarmin is challenged to devise on-the-spot explanations and support that illuminate mathematics concepts. She discusses her struggle to provide conceptually-based explanations in an end-of-year interview:

> I do struggle with what to say and how to get it across to the kids without telling them you know; I'm so used to just telling them, "This is the way you do it and don't ask

> why. We're going to try to figure it out with our hands-on, but don't ask me why because I'm not sure." You know, but sometimes it's hard for me - I mean, I know how to get there, but I just don't know how to tell them, how to convey it to them.

This difficulty providing conceptually-based explanations reflects Ms. Jarmin's limited pedagogical content knowledge. Furthermore, Ms. Jarmin's beginning-of-year interviews reveal limited knowledge of the conceptual basis of students' difficulties related to multiplication and division concepts, another pedagogical content knowledge issue.

End-of-year measures of mathematical knowledge provide evidence of modest gains in Ms. Jarmin's knowledge of student strategies for approaching multiplication and division problems. Related to this knowledge growth, there is evidence of some movement toward beliefs that are aligned with a reform-orientation. In particular, Ms. Jarmin expresses greater openness to the idea that children can solve problems in novel ways using a variety of solution strategies, and she indicates that students' initial understandings can be supported by use of contextualized word problems.

Reflecting these changes, there is an outlier instance in a Spring lesson where Ms. Jarmin deliberately uses a particular real-world context to support students' thinking when they struggle. In this case, students are working on figuring out how many meters a bicycle racer travels in eight laps around a 500 meter track. Ms. Jarmin provides each student with an image of a bicycle racing track, and it is part of her lesson image to support students by encouraging them to simulate the laps and connect this experience to a repeated addition solution strategy. What is notable here is that, when Ms. Jarmin thinks deliberately in her lesson image about how context might be used to support student thinking, she is able to provide more conceptually-supportive assistance.

While Ms. Jarmin's thinking about the bicycle racer problem provides some evidence of changed practice, it is not representative of the general response pattern observed in Ms. Jarmin's class in the Fall or Spring lessons. Even though Ms. Jarmin's beliefs have moved closer to a reform-orientation at the end of the year, she continues to struggle with knowing how to support student difficulties in conceptually-grounded ways that will facilitate development of generative problem solving practices.

Jarmin Response Pattern 3: Response to Unanticipated Student Ideas

During the mathematics lessons observed, there are times when students respond to Ms. Jarmin's questions or prompts in ways that are different than the correct response she is anticipating. Sometimes when this occurs, Ms. Jarmin treats the unanticipated response as flawed or incorrect when, in fact, the response is mathematically viable. At times, students' responses that are treated as incorrect have minor errors or are not especially productive, but they are nonetheless legitimate responses to the questions or prompts posed. Two instructional excerpts will be shared to illustrate this facet of Ms. Jarmin's practice, one from a whole class discussion and the other from an interaction with a student during work time.

In a Fall lesson, Ms. Jarmin is using whole class discussion time to get students to consider how they might find the total number of objects in a 4 × 7 array (see illustration) in a way other than counting each object one at a time.

On the board:

X X X X X X X
X X X X X X X
X X X X X X X
X X X X X X X

Ms. Jarmin wants students to notice that they can count the objects more efficiently by using the rows and columns organization of the array structure to count by fours (the columns) or by sevens (the rows). However, more efficient counting methods are not readily apparent to several students. Consider Ms. Jarmin's interchange with Angela during this whole class discussion:

> Ms. Jarmin: Who can think of another way we could count? Another way we could count? Angela, what could we count by?
> Angela: We could count by threes.
> Ms. Jarmin: Well, where do you get threes? Where do you get threes? (*Pause, no response from Angela*) What are the numbers that we are using?
> Angela: Four and seven.
> Ms. Jarmin: So, what do you think…we could do threes, maybe, but I don't know if it would get us there. We might have some left over. We don't want to have any left over. What is another way we could count?
> Angela: Five?
> Ms. Jarmin - By fives? Well, where do you get fives sweetie? (*Pause*) (*Some students are murmuring to each other*.) Arnold, I need your attention up here please. Umm…Arnold. (*Pause*). Angela, look at Ms. Jarmin. We have packages by fours. What is another way we could count?
> Angela - Twos?
> Ms. Jarmin - (*Ms. Jarmin sighs*.) By twos, yes. But, looking at our numbers, sweetie. What could we count by? We've already counted by fours, what else could we count by?
> Angela – Seven.

In the end, Angela produces the response, 7, that Ms. Jarmin is seeking, but not until several of Angela's viable suggestions are dismissed. Angela's suggestions to count by threes, fives, and twos likely reflect the numbers that she feels comfortable counting by (Sherin & Fuson, 2005). Although counting by these numbers would not utilize the array structure, they could yield a correct product in a more efficient way than counting by ones, so Angela's ideas are legitimate. While Ms. Jarmin recognizes that counting by 4 or 7 reflects the structure of the array, it is not surprising that this understanding is not shared by Angela at this early point in working with the array model (Battista et al., 1998). Instead of viewing Angela's suggestions as legitimate and exploring counting by threes, fives, and twos on the 4×7 array in relation to counting by fours or sevens, Ms. Jarmin asks increasingly narrow questions until Angela produces the answer that she is seeking.

In a different Fall lesson focused on division concepts, students are independently working on figuring out how many weeks are in 29 days. In the instructional excerpt that follows, Ms. Jarmin interacts with Rene in efforts to help him understand the problem and devise a solution. When Rene vocalizes that he thinks subtraction can be used to solve the problem, Ms. Jarmin discounts Rene's idea and instead directs him to solve the problem in a different way:

> Ms. Jarmin: Okay, Rene, now I am going to tell you that you are going to take a bigger number, Twenty-nine days, and make it into weeks.
> Rene: Twenty-nine weeks, right?
> Ms. Jarmin: Twenty-nine days and you want to make it into weeks.
> Rene: Twenty-nine days…
> Ms. Jarmin: So, if you had 29 days…
> Rene: Oh, oh, oh…we subtract.
> Ms. Jarmin: You're not subtracting. What are we doing?
> Rene: Ummm
> Ms. Jarmin: We are going to make them into…how many are going to be in each group if we have 29 days and there are 7 days in a week?
> Rene: How many are in each group?
> Ms. Jarmin: Yes. What are the two numbers? What is the small number? How many are going to be in each group?
> Rene: Seven.
> Ms. Jarmin: So, but you only need how many counters? What is the biggest number up there?
> Rene: Twenty-nine.
> Ms. Jarmin: So, you only need 29 counters. 29 is the biggest number up there. And you are going to make 29 counters into groups of…
> Rene: Seven.
> Ms. Jarmin: Seven. So, get 29 counters.
> Rene: These are my 29.

When Rene proposes that the number of weeks in 29 days can be determined using subtraction, Ms. Jarmin tells him that today's mathematics lesson is not on subtraction, thus indicating that subtraction is not a legitimate solution strategy. However, one viable way to solve this problem is to engage in a repeated subtraction strategy, keeping track of the number of sevens that can be subtracted from 29 to determine the number of weeks. Instead of encouraging Rene to move forward with his idea, Ms. Jarmin responds with a series of closed questions and prompts that direct Rene to represent the division problem with cubes similar to the way other problems have been represented during the class period. The strategy of making an array with cubes reflects the strategy that Ms. Jarmin identified in her pre-observation interview when asked to anticipate how students would solve problems.

In summary, Ms. Jarmin sometimes treats students' plausible mathematical ideas as incorrect. This primarily occurs when a student suggests a mathematical strategy or idea that differs from that which Ms. Jarmin has in mind when she poses a task or asks a question. In these instances, Ms. Jarmin typically narrows her questions or prompts until the student produces the strategy or response she has in mind.

Jarmin response pattern 3, consequences for student thinking. Ms. Jarmin's habit of treating viable mathematical ideas as incorrect discourages students from taking risks by formulating and suggesting their own ideas for approaching problems. Instead of focusing their energies on trying to make sense of mathematics tasks, students appear to engage in trying to guess at the answer or strategy Ms. Jarmin is seeking. Ms. Jarmin, in turn, views her students' perceived incorrect responses as evidence of misunderstanding. She only occasionally recognizes the learning potential of her students' ideas when they differ from her own. Consequently, students' opportunities to learn from their own ways of thinking are limited.

Jarmin response pattern 3, link to beliefs and knowledge. Ms. Jarmin's pattern of treating students' legitimate mathematical ideas as incorrect can be linked to her beliefs and knowledge. The beginning and end of year IMAP beliefs survey provides evidence that Ms. Jarmin is minimally sensitive to the differences in the ways adults and children think about mathematics. Ms. Jarmin consistently professes the belief that children learn in a variety of ways, but her instructional actions suggest that she primarily expects students to think about mathematical ideas and models in the same ways that she does. Ms. Jarmin is only occasionally observed persisting to make sense of students' ideas when they are not aligned with her own way of thinking. Instead, she determines that the given student ideas are incorrect or unproductive and guides the student to approach problems in particular ways that are aligned with her (teacher) thinking. When asked in post-observation interviews about instances when students respond in ways that are different than what she anticipated, Ms. Jarmin often responds that she believes the students are guessing, copying from a neighbor, or just making careless mistakes. After student ideas are judged to be incorrect, Ms. Jarmin rarely considers the logic of the responses students provide.

Whereas Ms. Jarmin's beliefs shape her responses when she perceives student answers or solutions are incorrect, her limited ability to think flexibly about mathematics tasks on-the-spot seems most related to her initial judgments that student ideas are not viable. Ms. Jarmin's responses to classroom scenarios posed in the beginning and end-of-year interviews indicate that she has difficulty interpreting student work, with comparatively greater difficulty at the beginning of the year. During the beginning-of-year interview, Ms. Jarmin attempts to give up on the task of interpreting student work multiple times and only perseveres through the task with significant prompting. While at the end-of-year interview

Ms. Jarmin appears more interested in understanding the correct and flawed aspects of student work, she is unable to uncover the reason for a student's error at both data points. Given her difficulties interpreting unfamiliar student solutions in the context of an interview, it is not surprising that Ms. Jarmin fails to recognize the possibilities of an unanticipated strategy or ideas in the context of real-time teaching.

Summary of Ms. Jarmin's Response to Student Difficulty

Three patterns portray Ms. Jarmin's typical classroom practice in response to student difficulties. First, she employs classroom routines and scripts that limit the degree to which students grapple with mathematical ideas. In particular, Ms. Jarmin interacts with students frequently while they are working, offering help that significantly reduces the challenge of tasks. Ms. Jarmin also positions herself as the primary source of help as well as evaluator of correctness. This teacher-as-authority stance limits the opportunity and incentive for students to participate meaningfully in partner work and whole class discussion. Second, the nature of Ms. Jarmin's help when students struggle to make sense of word problems is often conceptually unsupportive. She tends to minimize focus on word problem contexts, instead directing students to follow procedures that will yield correct answers. At times, Ms. Jarmin promotes use of tricks that act as crutches for completing the task at hand and are likely to lead to future confusion. Third, Ms. Jarmin sometimes treats students' viable mathematical ideas as if they are flawed or incorrect. This practice contributes to students abandoning their own sense-making, as the implicit message of this action is that students are only correct when their answers are in sync with Ms. Jarmin's way of thinking.

Taken together, Ms. Jarmin's ways of responding to student difficulties inhibit the opportunities for students to develop generative problem solving practices and dispositions, meaning practices and dispositions that will support students in solving a variety of mathematical problems. Throughout the school year, Ms. Jarmin's students' difficulties with problem solving are persistent. In the face of difficulty, some students develop the habit of waiting for Ms. Jarmin's help rather than attempting to persevere through challenging mathematics problems themselves. Students are only occasionally observed commenting on the mathematical ideas of others, and student engagement in class discussion is limited. In these ways, the degree to which mathematics instruction builds on student thinking is limited.

Ms. Jarmin's ways of responding to students' difficulties are linked to both her beliefs and knowledge. Ms. Jarmin's beliefs seem most related to the ways she structures support for students. Ms. Jarmin expects her students will require strong support to be successful working on and discussing novel mathematics tasks. Therefore, she provides substantial help to students as they work on tasks and exhibits a high level of control over class discussion. At the beginning of the year, there is evidence that Ms. Jarmin believes that her role is to provide strong direction for students by telling them how to solve problems. As the year progresses and Ms. Jarmin questions this belief, she begins to make attempts to support students without telling them exactly what to do. However, she often finds this new way of approaching instruction difficult and expresses frustration about not being able to figure out on-the-spot how to support students in productive ways. In general, Ms. Jarmin's practices of providing conceptually unsupportive assistance and discounting viable mathematical ideas seem most related to the limits of her mathematical knowledge for teaching. In the occasional observed instances when Ms. Jarmin makes a point of using a conceptually supportive strategy or emphasizing a particular mathematical idea, her teaching is more conceptually-grounded even as it continues to be rather teacher-centered. With regard to her practice of discounting viable mathematical ideas, this occurs because the ideas students put forward are outside the set of strategies or ideas Ms. Jarmin thinks about as correct. As Ms. Jarmin's personal knowledge of mathematical strategies becomes stronger, it is reasonable to assume that she will engage students' legitimate mathematical ideas more often and more thoroughly.

Situating Ms. Jarmin's Case Story in Broader Measures of Teaching

In the previous section, a theoretical explanation was presented suggesting how Ms. Jarmin's teaching practices in response to student difficulties are linked to her beliefs and knowledge during the year of this study. Teacher response to student difficulties is one of many aspects of reform-oriented mathematics instruction that contributes to the overall quality of mathematics teaching and learning more broadly defined. Therefore, this section will present results from two more global measures of Ms. Jarmin's mathematics teaching. First, data collected following each core classroom observation using the *Reformed Teaching Observation Protocol* (RTOP) (Sawada et al., 2002) will be presented. This instrument is designed to measure the degree to which a given mathematics lesson reflects principles and practices associated with reform-based mathematics instruction. Second, aggregated class data from the mathematics section of the State Standardized Test (SST) will be presented in relation to aggregated data at the school, district, and state levels. Taken together, these findings will allow Ms. Jarmin's patterns of response to student difficulties to be considered within a more general understanding of her mathematics teaching and students' learning.

Jarmin: Adherence to Reformed Teaching

The degree to which Ms. Jarmin's mathematics instruction reflects current reforms in mathematics education was measured using the Reformed Teaching Observation Protocol (RTOP) (Sawada et al., 2002). The RTOP is a criterion-referenced instrument containing 25 items, divided into five subscales: 1) lesson design and implementation, 2) the level of significance and abstraction of the content, 3) the processes that students use to manipulate information, 4) the classroom culture as observed through the nature of communicative interactions, and 5) the nature of student-teacher relationships. Following each core observation in Ms. Jarmin's classroom, each of the 25 items on the RTOP was rated on a scale from 0 (not observed) to 4 (very descriptive). Next, sums were calculated for ratings on each subscale as well as the total instrument to assess the degree to which Ms. Jarmin's mathematics instruction was reformed, with higher scores reflecting a greater degree of reform. Consequently, subscale scores on the RTOP range from 0 – 20, and total instrument scores range from 0 – 100. Aggregated results from core observations of Ms. Jarmin's mathematics teaching in the Fall and Spring are presented in Table 18.

On the lesson design and implementation subscale, the mean scores for Ms. Jarmin's instruction are 11.0 in the Fall and 15.0 in the Spring. Throughout the year, instructional strategies and mathematics activities observed in Ms. Jarmin's classroom are found to inconsistently respect students' prior knowledge and engage students as a learning community. However, four of the five items comprising this scale contain higher ratings at the Spring observation point compared to the Fall. This reflects greater inclusion of opportunities for students to explore their own ideas during student work time and through class discussion in the Spring.

Table 18
Ratings of Ms. Jarmin's Mathematics Teaching on the RTOP

RTOP items by subscale	Fall	Spr
Subscale 1: Lesson design and implementation		
1. The instructional strategies and activities respected students' prior knowledge and preconceptions inherent therein.	2.5	2.5
2. The lesson was designed to engage students as members of a learning community.	2.0	2.5
3. In this lesson, student exploration preceded formal presentation.	2.5	4.0
4. The lesson encouraged students to seek and value alternative modes of investigation and problem solving.	2.0	3.0
5. The focus and direction of the lesson was often determined by ideas originating with students.	2.0	3.0
Subscale 1 score	**11.0**	**15.0**
Subscale 2: Content, propositional knowledge		
6. The lesson involved fundamental concepts of the subject.	3.0	4.0
7. The lesson promoted strongly coherent conceptual understanding.	1.5	3.0
8. The teacher had a solid grasp of the subject matter content inherent in the lesson.	2.5	2.5
9. Elements of abstraction (i.e., symbolic representations, theory building) were encouraged when it was important to do so.	3.0	3.5
10. Connections with other content disciplines and/or real world phenomena were explored and valued.	2.0	3.0
Subscale 2 score	**12.0**	**16.0**
Subscale 3: Content, procedural knowledge		
11. Students used a variety of means (models, drawings, graphs, concrete materials, manipulatives, etc.) to represent phenomena.	2.5	3.5
12. Students made predictions, estimations, and/or hypotheses, and devised means for testing them.	1.0	1.0
13. Students were actively engaged in thought-provoking activity that often involved the critical assessment of procedures.	2.0	2.0
14. Students were reflective about their learning.	1.5	1.0
15. Intellectual rigor, constructive criticism, and challenging of ideas were valued.	1.5	1.5
Subscale 3 Score	**8.5**	**9.0**
Subscale 4: Classroom culture, communicative interactions		
16. Students were involved in communication of their ideas to others using a variety of means and media.	2.0	2.5
17. The teacher's questions triggered divergent modes of thinking.	1.5	2.0
18. There was a high proportion of student talk and a significant amount of it occurred between and among students.	2.0	2.5
19. Student questions and comments often determined the focus and direction of classroom discourse.	2.0	2.0
20. There was a climate of respect for what others had to say.	2.0	2.5
Subscale 4 score	**9.5**	**11.5**
Subscale 5: Classroom culture, student/teacher relationships		
21. Active participation of students was encouraged and valued.	1.5	2.0
22. Students were encouraged to generate conjectures, alternative solution strategies, and ways of interpreting evidence.	1.5	2.0
23. In general the teacher was patient with students.	2.0	3.0
24. The teacher acted as a resource person, working to support and enhance student investigations.	2.0	2.0
25. The metaphor "teacher as listener" was very characteristic of this classroom.	2.0	2.5
Subscale 5 score	**9.0**	**11.5**
Total score	**50.0**	**63.0**

The second RTOP subscale focuses on what the contents of a lesson suggests about a teacher's propositional knowledge, meaning her mathematical knowledge for teaching the particular content of focus in the lesson. On this subscale, Ms. Jarmin's mean scores are 12.0 in the Fall and 16.0 in the Spring. Mirroring findings from beginning and end-of-year interviews, Ms. Jarmin is found to have a moderate grasp of the subject matter content inherent in lessons observed at both data points. On other items included on this scale, ratings increase from Fall to Spring. Of particular note, Spring lessons are found to more fully involve fundamental concepts of mathematics and promote stronger conceptual understanding.

The third RTOP subscale focuses on what the contents of a lesson suggests about a teacher's procedural knowledge. For the RTOP instrument, this means the quality of the lesson in terms of an inquiry approach to learning. The ratings of Ms. Jarmin's mathematics instruction are the weakest on this subscale at both data points, with mean scores of 8.5 in the Fall and 9.0 in the Spring. Students are only sometimes observed engaging in activities of inquiry such as making predictions and devising means for testing them, critically assessing procedures, and reflecting on learning. The presence of rigorous debate of mathematical ideas is minimal. However, students are encouraged to represent mathematical ideas in multiple ways sometimes in the Fall and with greater frequency in the Spring.

The fourth RTOP subscale focuses on communicative interactions that are part of the classroom culture. On this subscale, mean scores of Ms. Jarmin's instruction are 9.5 in the Fall and 11.5 in the Spring. At both data points, Ms. Jarmin is observed to maintain strong control over the communicative interactions during work time as well as the discourse that occurs during class discussion. Yet, students are sometimes prompted to present their mathematical solutions and Ms. Jarmin asks some questions intended to trigger divergent modes of thinking.

The last RTOP subscale focuses on the classroom culture in terms of the teacher's role and the roles students are encouraged to take. On this subscale, the mean scores of Ms. Jarmin's instruction are 9.0 in the Fall and 11.5 in the Spring. Spring ratings reflect greater opportunity for student participation and generation of alternate solution strategies when compared with the Fall. In the Spring, Ms. Jarmin also exhibits greater patience with the process of allowing students to generate and describe their own ideas.

The mean scores of Ms. Jarmin's mathematics instruction on the total RTOP instrument are 50.0 in the Fall and 63.0 in the Spring. These ratings suggest that Ms. Jarmin's mathematics teaching moderately reflects the principles of reformed instruction, with a greater evidence of reformed practices in the Spring than in the Fall.

Jarmin: Student Achievement

Aggregated student scores on the mathematics section of the State Standardized Test (SST) were used to measure the mathematics achievement of Ms. Jarmin's class. The SST is a criterion-referenced achievement test in which students receive scores between 1 and 5, with a score of 3 or higher considered passing. Since Ms. Jarmin's students are repeating third grade, SST scores from the previous year are available for nine of the ten students in Ms. Jarmin's class[9]. All of these students received a score of 1 on the mathematics section of the SST in the previous year, suggesting that their understanding of third-grade mathematics when they entered Ms. Jarmin's class was extremely limited.

[9] Ms. Jarmin's class experienced some attrition around the time of the SST test. While Ms. Jarmin had 12 students in her class for most of the year, only 10 students were in her class at the time of testing.

Ms. Jarmin’s students’ SST scores in the year of this study are presented in Table 19 along with comparison data for third-grade students at Lincoln Heights, the school district, and the state. However, comparisons between the scores of Ms. Jarmin’s students and the other groups should be made with caution, as the demographic particularities of this class of retained students make it arguably different than the other groups.

Table 19
Comparison of Jarmin Student Achievement on SST to School, District, and State

		Achievement level (%)					
Group	Students	1	2	3	4	5	Passing score (%)
Ms. Jarmin's class	10	30	40	20	10	0	30
Lincoln Heights 3rd grade	88	24	27	33	15	1	49
School District 3rd grade	13,713	18	16	33	24	9	66
State 3rd grade	203,037	15	17	34	25	9	68

Note. Scores of 3 or higher are considered passing.

Of Ms. Jarmin’s ten students, 70% achieve a higher score than the previous year on the mathematics section of the SST, but only 30% achieve a passing score of 3 or higher. This is in comparison to passing scores achieved by 49% of Lincoln Heights third graders, 66% of the third graders in the school district, and 68% of the third graders in the state. Hence, the mathematics achievement of Ms. Jarmin’s class appears to be somewhat lower than that of third graders at Lincoln Heights and considerably lower than third graders at the district and state levels.

Jarmin: Summary of Global Measures of Teaching

Taken together, results from the *Reformed Teaching Observation Protocol* (RTOP) and the State Standardized Test (SST) suggest that Ms. Jarmin's classroom is a place where there is inconsistent adherence to reformed teaching practices and student learning outcomes are mixed. RTOP results suggest that Ms. Jarmin's mathematics teaching moderately reflects the principles of mathematics reform throughout the year, with somewhat greater adherence in the Spring. For instance, Ms. Jarmin's mathematics instruction often integrates aspects of reform such as utilizing multiple representations and sharing student solutions to problems. However, Ms. Jarmin inconsistently employs strategies that encourage students to engage in higher-level thinking around mathematics tasks, and student engagement is noted as limited in all lessons observed. Consequently, Ms. Jarmin's attempts to provide reform-oriented mathematics instruction sometimes fall short of promoting generative understanding for students. The mathematics achievement data for Ms. Jarmin's class lend further evidence to this claim. Aggregated SST scores for Ms. Jarmin's class indicate that most of her students demonstrate growth over the previous year, but only 30% are performing at grade level. This finding suggests that Ms. Jarmin's students' understanding of the mathematics concepts and skills of focus in third grade is limited and underscores the importance of more professional development with a focus on increasing her propositional knowledge as well as teaching strategies to get the full benefit from a reform-based curriculum.

The Case of Ms. Larsano

Ms. Larsano enters the year of this study open to the new ways of teaching associated with the *Everyday Mathematics,* but worried that she and her class of students with Limited

English Proficiency (LEP) will have a hard time adjusting to this new program. Ms. Larsano openly states that she is not good at doing mathematics and that it is her least favorite subject to teach. Furthermore, Ms. Larsano's instructional priorities are heavily weighted toward supporting her students' transitions to life in the United States and mastering the English language. Yet, Ms. Larsano views herself as a team-player and outwardly displays a positive and inquisitive attitude toward the new mathematics program and willingness to experiment with the teaching practices recommended in mathematics professional development.

Ms. Larsano's class is comprised of 18 students, who are grouped together because of their high-level language needs. Since this is a designated transition-language class, Ms. Larsano is mandated to help students learn English while also providing support in her students' native language, which is Spanish. A credit to Ms. Larsano's loving but strict classroom management style, her students are typically well-behaved and the classroom appears to run smoothly. However, within and outside of math time, student engagement and participation is highly variable. Some students appear consistently reluctant to engage in learning without direct teacher support.

The case of Ms. Larsano that follows is presented in four sections. In the first two sections, evidence of Ms. Larsano's mathematics-related beliefs and knowledge at the beginning and end of the year will be presented. Next, the relationship among Ms. Larsano's beliefs, knowledge, and classroom practice will be explored through a case story focused on her patterns of response to student difficulties. Finally, global measures of Ms. Larsano's adherence to reform-based mathematics teaching and student achievement will be presented so that case story findings can be considered in relation to these broader measures of teaching.

Ms. Larsano's Beliefs about Mathematics Teaching and Learning

Ms. Larsano's beliefs about mathematics teaching and learning were measured at the beginning and end of the school year using the *IMAP Web-Based Beliefs-Survey* (Ambrose, Phillip et al., 2003). The IMAP instrument requires teachers to respond to instructional scenarios in an open-ended format, allowing respondents to emphasize or downplay issues of personal importance. A series of rubrics are then used to measure survey responses in relation to seven target beliefs considered central to reform-oriented mathematics instruction. The degree to which Ms. Larsano's survey responses provide evidence that she holds each target belief at the beginning and end of the school year is presented in Table 20.

Table 20
Ms. Larsano's IMAP Web-Based Beliefs-Survey Results

Belief	Beginning of Year	End of Year
Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).	1	1
Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.	0	0
Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.	0	1
Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.	1	2
Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.	0	1
Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.	1	1
Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.	0	0

Note. 0 = No evidence; 1 = Weak evidence; 2 = Moderate evidence; and 3 = Strong evidence

In the sections that follow, evidence of Ms. Larsano's adherence to each reform-oriented belief at the beginning and end of the year will be presented.

Larsano: Belief about the Nature of Mathematics (IMAP Belief 1)

At the beginning and end of the year, there is *weak evidence* that Ms. Larsano views mathematics as a web of interrelated concepts and procedures. In response to a scenario presenting multiple ways of solving a multidigit addition problem and asking which strategies she would like children to share, Ms. Larsano indicates that she would have children share all five strategies at both data points. However, her discussion of why and how she would have students share these strategies does not consider how the strategies are related or supportive of one another. Therefore, while survey evidence suggests that Ms. Larsano sees value in including multiple ways to solve problems in her instruction, her instructional emphasis is unlikely to be on the interrelationships among strategies.

Larsano: Belief about Distinction between Understanding Concepts and Applying Mathematical Procedures (IMAP Belief 2)

Ms. Larsano's beginning and end-of-year responses on the IMAP survey provide *no evidence* that she distinguishes between conceptual understanding and students' abilities to perform procedures and get correct answers. There is no indication in Ms. Larsano's responses that she differentiates between students who perform the steps of standard algorithms with understanding and without understanding. Furthermore, Ms. Larsano indicates that students who carry out standard algorithms accurately demonstrate greater mathematical understanding than students who use other methods.

Larsano: Belief about Source of Generative Mathematical Understanding (IMAP Belief 3)

At the beginning of the year, there is *no evidence* that Ms. Larsano believes understanding mathematical concepts is more powerful and generative than remembering mathematical procedures. Although Ms. Larsano indicates that she thinks teachers should explain why standard procedures work, she asserts that repeated practice of standard procedures is what students most benefit from. When presented with non-standard methods for multidigit subtraction, Ms. Larsano indicates that she thinks these methods require too much thinking and that the known steps of the standard algorithm are better. Overall, at the beginning of the year, Ms. Larsano's responses suggest that she believes procedural knowledge to be more important than conceptual knowledge.

In contrast, Ms. Larsano's responses at the end of the year do not emphasize the importance of paper-and-pencil practice of standard procedures as they did at the beginning of the year. Instead, Ms. Larsano suggests that students need experiences with manipulatives to help them understand why standard procedures work. However, she also indicates that students with good memory can be successful with learning the steps of standard procedures without manipulatives. Furthermore, Ms. Larsano continues to consider the standard U.S. multidigit subtraction algorithm to be an easier and preferable way of subtracting than a non-standard, conceptually-explicit method. Overall, Ms. Larsano's responses at the end of the year provide *weak evidence* of the belief that understanding mathematical concepts is more powerful and generative than remembering mathematical procedures, which represents a small shift toward a reform-orientation.

Larsano: Belief about Consequences of Teaching Concepts/Procedures First (IMAP Belief 4)

At the beginning of the year, there is *weak evidence* that Ms. Larsano believes that students are more likely to understand mathematical procedures if they first learn related mathematics concepts. In discussing instructional practice, Ms. Larsano suggests that students benefit from working with manipulatives and visuals before using number-only procedures. However, Ms. Larsano does not explicitly discuss development of conceptual understanding through use of manipulatives. Rather, she indicates that manipulatives should be used first because they are easier than paper-pencil methods. Also, responses used to evaluate this belief never suggest that there may be negative outcomes if procedures are taught without an understanding of underlying concepts.

Ms. Larsano's end-of-year IMAP responses suggest some movement on this belief toward a reform-orientation. She explicitly states that manipulatives should be used with students to develop conceptual understanding of mathematical procedures. Furthermore, Ms. Larsano indicates that, in a unit on multidigit addition, she would expose students to a manipulative-based method and a non-standard partial sums method before introducing the standard algorithm. However, her reasons for doing this are not clearly tied to development of conceptual understanding. Furthermore, as at the beginning of the year, Ms. Larsano's end-of-year responses do not suggest that there may be negative outcomes for teaching procedures before students understand concepts. Therefore, Ms. Larsano's responses at the end of the year provide *some evidence* of adherence to the belief that students are more likely to understand mathematical procedures if they first learn mathematical concepts.

Larsano: Belief about Children's Problem Solving Capabilities (IMAP Belief 5)

At the beginning of the year, there is *no evidence* that Ms. Larsano believes that children can solve problems in novel ways before being taught how to solve such problems. She indicates that, in her classroom, she does not have students attempt to solve problems without first showing them how. Additionally, Ms. Larsano states the belief that children require examples and teacher support in order to learn how to solve new kinds of problems.

Ms. Larsano's responses at the end of the year suggest limited movement on this belief in the direction of a reform-orientation. At this data point, Ms. Larsano's responses suggest greater openness to incorporating student ideas for solving problems prior to demonstrating particular methods. However, Ms. Larsano consistently inserts herself as a mediator of student ideas. There is no indication that she believes students will be able to successfully solve problems without some level of teacher support. Therefore, at the end-of-year data point, there is *weak evidence* that Ms. Larsano believes children can solve problems in novel ways before being taught how to solve such problems.

Larsano: Belief about Children's Ways of Thinking about Mathematics (IMAP Belief 6)

At the beginning and end of the year, there is *weak evidence* that Ms. Larsano believes children think about mathematics in ways that are generally different than adults. At both data points, Ms. Larsano suggests that students benefit from use of manipulatives and visuals to understand mathematics. However, her responses do not include an indication of the belief that mathematical symbols are difficult for students to understand. IMAP responses also suggest that Ms. Larsano views real-world contexts in the form of word problems as likely to impede rather than facilitate mathematical understanding, because students are likely to get caught up in trying to understand the words instead of the mathematics. It is important to note that Ms. Larsano's responses regarding this issue may be related to her LEP students' difficulties with reading word problems presented in English. Interviews and observations provide evidence that Ms. Larsano often attempts to incorporate real-world contexts to which students can relate through verbal instruction in English and Spanish.

Larsano: Belief about Teacher's Role in Supporting Student Learning (IMAP Belief 7)

At the beginning and end of the year, there is *no evidence* that Ms. Larsano believes that teachers should allow children to do as much thinking as possible during mathematics instruction. In response to a video clip of a teacher providing strong, procedural direction for performing a fraction division computation, Ms. Larsano praises the clear, step-by-step approach used by the teacher and considers it one of the strengths of the instructional episode. At neither data point does Ms. Larsano consider that the teacher's instructional guidance may have been excessive and served to limit the student thinking. Furthermore, at both data points, Ms. Larsano suggests that students require teacher support and guidance to learn how to solve problems successfully.

Summary of Ms. Larsano's Beliefs

Ms. Larsano's IMAP survey responses suggest that her beliefs at the beginning and end of the year are minimally aligned with a reform-orientation to mathematics instruction. At both data points, there is weak or no evidence that Ms. Larsano holds six of the seven reform-oriented beliefs measured. In particular, there is no evidence that Ms. Larsano draws significant distinction between applying mathematical procedures and understanding mathematical concepts. In general, Ms. Larsano considers correct answers to be evidence of understanding.

Although Ms. Larsano expresses openness to students using a variety of nonstandard strategies to solve problems, she prefers that students learn standard algorithmic procedures because she believes these procedures to be clear and easy to follow. At the end of the year, there is only weak evidence that Ms. Larsano believes understanding mathematics concepts to be more powerful and generative than remembering mathematical procedures.

Ms. Larsano's survey results also suggest that she believes children have limited capability to solve problems in novel ways before being taught how to solve such problems. Related to this belief, Ms. Larsano's IMAP responses indicate the view that the teacher should provide strong direction during mathematics lessons. Specifically, the teacher should demonstrate and coach students on how to apply procedures that will yield correct answers. Finally, Ms. Larsano's responses indicate that she gives little consideration to the differences in the ways children and adults think about mathematics, particularly with regard to use of symbols as students are learning about mathematics concepts.

Ms. Larsano's Knowledge of Mathematics for Teaching

Ms. Larsano's knowledge of mathematics for teaching with a reform-orientation was measured at the beginning and end of the year using data from the Teacher Knowledge interview. In particular, this interview explores teachers' knowledge related to the teaching of multiplication and division to third-grade students. First, Ms. Larsano's open-ended discussion of teaching and learning multiplication will be presented. In this part of the interview, teachers were prompted to describe important understandings and common student difficulties related to the learning of multiplication. In addition, interview questions direct teachers to discuss how children develop the important understandings identified and how multiplication should be taught. Next, Ms. Larsano's responses to four classroom scenarios will be described. These classroom scenarios probe specific aspects of teachers' knowledge related to the teaching and learning of multiplication and division, specifically teachers' knowledge of nonstandard strategies for multidigit multiplication, knowledge of a common student difficulty underlying the standard U.S. algorithm and strategies for addressing this difficulty, and teachers' abilities to interpret and respond to student work.

Larsano: Open-ended Discussion of Teaching and Learning Multiplication

At the beginning of the year, Ms. Larsano explains that third-grade instruction on multiplication should help students to understand that multiplication is a faster way to add equal groups and should emphasize memorization of the times tables. Ms. Larsano asserts that students should first build physical models of multiplication situations. Then these physical models should be connected to symbolic multiplication notation. After students understand the connection between physical and symbolic models of multiplication, focus should turn to memorization of facts and then memorization of the steps for the standard U.S. multidigit multiplication algorithm. Ms. Larsano indicates that, while learning about multiplication, she anticipates that some students will encounter difficulties with moving away from counting objects one-by-one in physical models and working with large numbers.

Ms. Larsano identifies knowledge of addition facts and place value as prerequisites to learning multiplication.

At the end of the year, Ms. Larsano indicates that third-grade students should understand that multiplication involves situations in which there are equal amounts in each group and finding the total of the groups. Also, students should understand that multiplication is a faster way of doing addition. At the beginning of learning multiplication, Ms. Larsano suggests that students should make physical models of real-world multiplication situations and they should be prompted to identify the number of groups and the number in each group. Next students are ready to learn about multiplication number sentences in relation to physical models and real-world situations. At this point, Ms. Larsano asserts that instruction should move students toward devising increasingly efficient strategies to find products, such as skip counting, repeated addition, doubling strategies, and using known facts to figure out unknown facts. One difficulty that Ms. Larsano notes students have at this stage is with avoiding counting errors. As students become more efficient in finding products, Ms. Larsano states that they need ample practice, in the form of drill and games, to remember facts quickly. After they know some facts, third-grade students are ready to encounter contextualized multidigit multiplication situations involving one-digit by two-digit calculations. Initially, Ms. Larsano indicates that students should be encouraged to devise, share, and discuss their own strategies for solving these problems. Then, after this period of exploration, Ms. Larsano asserts that students should be taught and then asked to memorize the steps of the standard U.S. multidigit multiplication algorithm. Ms. Larsano notes that a difficulty students often have with this process is remembering where to put the numbers at each step. Finally, Ms. Larsano identifies knowledge of addition and the ability to count by some numbers (2s, 5s, 10s) as being prerequisites for the study of multiplication.

Larsano: Knowledge of Non-standard Strategies (Classroom Scenario 1)

At the beginning of the year, Ms. Larsano identifies five strategies that students might use to solve a word problem involving finding the total number of chairs in 16 rows of 8. These are presented in Table 21. Three of the strategies identified can be classified as direct modeling. Of these, Ms. Larsano considers direct modeling with counters or pictures of chairs to be the less sophisticated than direct modeling with tally marks. Ms. Larsano identifies increasingly sophisticated strategies to include repeated addition and then use of the standard U.S. multidigit multiplication algorithm. In her discussion of strategies, Ms. Larsano suggests that it is important for students to understand that the steps of the standard

Table 21
Strategies Identified by Ms. Larsano in Response to Classroom Scenario 1

Data point	Direct modeling	Complete number	Partitioning number	Compensating	Other
Beginning of year	16 rows of 8 chairs (look like chairs)	Repeated addition 8 + 8 + 8…			Standard US algorithm
	16 rows of 8 tally marks				
	16 rows of 8 counters				
End of year	16 rows of 8 chairs	Repeated addition 8 + 8 + 8…	16x4 and 16x4		16 circles with 8 tally marks in each
		Grouping 8s to make 16s, then set-up vertically 16 + 16 + 16…			Standard US algorithm

Note. Analysis utilizes strategy classification presented in Baek (1998).

algorithm provide a quick way to count the objects represented through a direct modeling method, but she does not explain how a visual model of the multiplication problem relates to the steps of the standard algorithm. Furthermore, Ms. Larsano's discussion of the standard algorithm suggests that she is focused on students knowing how to apply the steps of this method and not on their understanding why those steps work. At the beginning of the year, Ms. Larsano does not include partitioning or compensation strategies in her list of ways students might use to solve the problem posed.

At the end of the year, Ms. Larsano identifies six strategies that children might use to figure out how many chairs are in 16 rows of 8 chairs. (See Table 21.) At this data point, Ms. Larsano identifies visually-based strategies, including direct modeling rows of objects and filling circles with tally marks, to be the least sophisticated. She assumes that students using these strategies would most likely find the product by counting objects one-by-one but notes that students who count their models by groups of eight or apply repeated addition would be at a higher level of understanding. Ms. Larsano lists complete number strategies, including repeated addition and a grouping strategy, next in order of sophistication because students using these strategies do not need the visuals to solve the problem. Finally, Ms. Larsano suggests that use of the standard U.S. algorithm or the halving and doubling (partitioning) strategy listed are the most sophisticated, because they both require students to recognize the situation as one that can be solved with multiplication. Between these two strategies, Ms. Larsano considers the partitioning strategy to be more sophisticated than the standard algorithm:

> I think this one (the partitioning strategy) is a little bit harder [than the standard algorithm]… because they have to have prior experience doubling the numbers or halving the number, putting them in half, knowing that 8 is an even number and that

> they can half it. I think that they need more experience or practice doing this one (the partitioning strategy).

When asked what students need to understand to use the standard U.S. multiplication algorithm, Ms. Larsano responds, “The steps to multiply and their facts.” These responses suggest that Ms. Larsano recognizes that students using the non-standard partitioning strategy described require a higher level of understanding; however, Ms. Larsano continues to assert that learning the steps of the standard U.S. multiplication algorithm is the ultimate goal of instruction. As was the case at the beginning of the year, Ms. Larsano does not identify a compensating strategy at the end-of-year data point. Also notable, Ms. Larsano does not identify the partitioning strategy emphasized by *Everyday Mathematics*, which involves partitioning the factors by decade numbers and summing the partial products.

In summary, Ms. Larsano’s response to Classroom Scenario 1 provides evidence of weak knowledge of non-standard strategies for multidigit multiplication at the beginning and end of the year, with evidence of greater variety of strategies at the end of the year. It is particularly notable that, at the end of the year, Ms. Larsano does not discuss the non-standard partial products strategy that is the focus of several lessons in the *Everyday Mathematics* program. At both data points, Ms. Larsano demonstrates limited knowledge of the cognitive resources required for students to use the strategies with understanding. While Ms. Larsano is able to describe a progression from direct modeling to use of complete number strategies, her primary goals related to instruction on multidigit multiplication appear to be for students to recognize when multiplication is the appropriate operation to use and then to be able to apply the standard U.S. multiplication algorithm to achieve a correct result. Therefore, her focus on using multiple strategies to develop mathematical understanding is minimal.

Larsano: Interpretation of Students' Mathematical Strategies (Classroom Scenario 2)

At the beginning and end-of-year data points, Ms. Larsano is able to identify the logic in two of the three student solutions presented for the pancake problem (Twenty-four children want to share eight pancakes so that each one gets the same amount. How much pancake can each child have?). For these solutions, Ms. Larsano is able to describe ways she might follow-up with each student to find out more about their mathematical understanding. However, her ideas for follow-up seem to come with greater ease and more elaboration at the end of the year. For instance, at the beginning of the year, Ms. Larsano suggested that she would follow up on one student's strategy by asking him to explain his model. At the end of the year, her response to how she would learn more about this student's understanding is more detailed:

> I would ask him what does each picture represent. What does the little dots represent…even though that I know, I would want him to tell me. What does the little dots represent and what does the big dots represent. And why did he divide the pancakes into 3 parts each? So, why. And maybe he could explain me a little bit. And how much is each kid going to eat.

This greater degree of detail suggests that Ms. Larsano's pedagogical content knowledge of probes and questions that will elicit information about student understanding of mathematical ideas has increased.

At the beginning and end of the year, Ms. Larsano has difficulty making sense of one of the three student solutions presented in Classroom Scenario 2. Interestingly, the student solution that Ms. Larsano has difficulty interpreting is different at the two data points. However, her response when she is not able to interpret student work quickly is remarkably similar. In these instances, Ms. Larsano indicates that she does not find the students' solutions to be sensible and that she thinks they were likely guessing. She determines that the student does not have a good understanding of the mathematics of focus. These responses suggests that, when Ms. Larsano has difficulty interpreting a student solution in the classroom context, she is likely to spend little time trying to understand the solution and instead assume that the student holds limited understanding.

In summary, Ms. Larsano's response to Classroom Scenario 2 provides evidence of limited ability to interpret student work at the beginning and end of the year, with increased knowledge of strategies for following-up on student strategies at the end of the year.

Larsano: Addressing and Avoiding a Common Student Error (Classroom Scenario 3)

In Classroom Scenario 3, teachers are presented with student work in which the standard U.S. multiplication algorithm is executed without maintaining the place values of the partial products. In efforts to explain the mathematical basis of this student error, Ms. Larsano's responses at the beginning and end of the year indicate that she assumes place value is involved but that she is not entirely clear how. Consider her response when asked to elaborate on the relationship between place value and the standard U.S. multiplication algorithm at the end of the year:

> Well, if they are multiplying, they start in the ones place. So, they are going to multiply first the ones with the ones. So the answer is going to be in the ones place. Then the ones with the tens place, so the answer is going to be in the tens place. Then the ones with the hundreds, so the answer is going to be in the hundreds place. If that works out – this is just coming out of my mind right now – I am assuming that it will work out.

As was the case at the beginning of the year, Ms. Larsano's explanation is limited to a review of the steps of the standard algorithm with attention to the places where numbers should be written down. She does not identify how place value impacts the magnitude of the partial products.

At both data points, Ms. Larsano indicates that she would respond to students making the error presented in Classroom Scenario 3 by having them work problems on grid paper or draw lines to emphasize the places where they should put the numbers. She indicates that she would show students where the numbers should go and have them practice the steps of the procedure until they could do it. When asked how she would respond to a student who asks why the numbers have to be moved over when multiplying by a number in the tens or hundreds place, Ms. Larsano responds that she is not sure. When asked to share what she can do as a third-grade teacher to help students avoid such difficulties, Ms. Larsano again emphasizes the importance of teaching the steps of the standard U.S. multiplication algorithm and teaching students how to line up the numbers.

In summary, Ms. Larsano's responses at the beginning and end of the year suggest that she holds weak personal knowledge of how the standard U.S. algorithm is related to place value. Furthermore, evidence of pedagogical content knowledge of conceptually-supportive strategies for helping students understand and avoid errors in using this algorithm is limited at both data points.

Larsano: Interpretation of and Response to a Student's Flawed Solution (Classroom Scenario 4)

At the beginning and end of the year, Ms. Larsano is unable to determine the mathematical basis of the flawed strategy presented in Classroom Scenario 4. In this scenario, a student explains that the answer to $144 \div 8 = ?$ can be found by dividing 144 by 2 four times because 2 + 2 + 2 + 2 = 8. The teacher is prompted to describe how she might respond to this student. At the beginning of the year, Ms. Larsano notes that the student would have the correct answer if she had stopped after her third step (e.g., after dividing by 2 three times), but she cannot explain why that would be the case. After identifying that the solution is incorrect at the beginning of the year, Ms. Larsano suggests that she would respond by teaching the student to use the standard U.S. long division algorithm. At the end of the year, however, Ms. Larsano takes a different approach. She says that she would try to have the student explain her process again to allow her to catch the error, and then she might have her check the answer by using another strategy. Ms. Larsano's response to Classroom Scenario 4 at both data points provides evidence of her difficulty interpreting flawed student work and responding in ways that honor student thinking.

Summary of Ms. Larsano's Knowledge of Mathematics for Teaching

Ms. Larsano demonstrates limited knowledge of ways students might approach mathematical problems involving multiplication at the beginning and end of the year, with greater knowledge demonstrated at the end of the year. At the beginning of the year, Ms. Larsano's discussion of multiplication strategies is confined to direct modeling, repeated addition, memorization of facts, and use of the standard U.S. multiplication algorithm for multidigit problems. In contrast, at the end of the year, she identifies a limited number of additional nonstandard strategies, including a combining numbers strategy and a partitioning strategy involving halving and doubling. However, in her discussion of ways to approach multidigit multiplication, Ms. Larsano does not discuss at either data point compensation strategies or other partitioning strategies, including the partial products focus algorithm emphasized by *Everyday Mathematics*.

In discussing multiplication instruction, there is evidence that Ms. Larsano holds weak knowledge of reform-minded teaching practices at the beginning of the year and comparatively greater knowledge at the end of the year. At the beginning of the year, Ms. Larsano asserts that students are ready to memorize facts and standard algorithmic procedures after they have an understanding of how physical models are related to multiplication notation. In contrast, at the end of the year, Ms. Larsano places greater emphasis on supporting student understanding by using real-world situations and on allowing students to experiment with a variety of nonstandard strategies prior to focusing on fact memorization and the standard U.S. algorithm for multidigit multiplication. Yet, at both data points, Ms. Larsano considers the end goal of instruction to be memorization of basic facts and the standard U.S. multidigit multiplication algorithm. Evidence suggests that Ms. Larsano's personal conceptual understanding of the standard U.S. multiplication algorithm is weak, and her discussion of instructional practice reflects her own procedurally-based understanding of mathematics.

Finally, Ms. Larsano's ability to interpret student strategies is limited at both data points. If Ms. Larsano is unable to make sense of a student's strategy quickly, she assumes the strategy to be illogical and indicates that she would have the student try something different. This trend suggests that Ms. Larsano is likely to sometimes respond to students' correct and flawed strategies in ways that do not honor student thinking.

Case Story of Ms. Larsano's Response to Student Difficulties

In the case story that follows, facets of Ms. Larsano's response to student difficulties will be presented. Of focus will be Ms. Larsano's response to students' struggles with mathematics tasks during work time, her ways of responding to flawed solutions that are shared in the context of class discussion, and her response when students answer questions incorrectly. Following illustration of each response pattern, consequences for student thinking and the relationship between Ms. Larsano's beliefs, knowledge, and mathematics teaching practice will be discussed.

Larsano Response Pattern 1: Response to Student Difficulties during Work Time

During time allotted for students to work on mathematics tasks, Ms. Larsano sometimes observes and interacts with students, while at other times her attention is focused on things other than students' mathematical activity. Therefore, Ms. Larsano's awareness of student difficulties that arise during work time is inconsistent among and within observed lessons. In the text that follows, Ms. Larsano's response to students' difficulties when she is attending to students' mathematical activity during work time will be discussed. First, consideration will be given to how Ms. Larsano addresses students' difficulties getting started with mathematics problems. Then her response to flawed solutions in the context of student work time will be described.

The students in Ms. Larsano's class have been grouped together intentionally because they have limited proficiency with the English language. All of the students are Spanish-speaking, and Ms. Larsano moves between English and Spanish regularly with individual students. When Ms. Larsano encounters a student who is having difficulty getting started on a mathematics word problem, the most frequent response observed is for her to repeat the problem in Spanish or English, sometimes breaking down the task for the student. Consider, for instance, Ms. Larsano's support for Alicia, who is having difficulty with the problem, "23 candles are arranged with 3 in each row. How many rows are there?" Ms. Larsano addresses Alicia's difficulty getting started on the candle problem by reframing this problem in the following manner:

> You are going to have 23 in total (*pointing at counters*), and you are going to have rows. And each row is going to have 3. Each row is going to have 3 until you have 23. And I am going to find out how many…and you are going to find out how many rows.

At this point, Ms. Larsano moves away from Alicia. In this brief intervention, Ms. Larsano reframes the candle task in such a way that she deemphasizes the problem context and tells Alicia a procedure for finding the answer. This instructional excerpt illustrates Ms. Larsano's tendency to address students' difficulties getting started with word problems by suggesting procedures.

In the moments that follow, Alicia appears to become focused on trying to remember and follow Ms. Larsano's steps rather than on using the problem context to make sense of the mathematics task on her own. At the end of the time allotted for work on the candle task, Alicia has not devised a problem solution. In general, it is not unusual for Ms. Larsano to begin class discussion with only a fraction of the class having devised a personal solution to a given problem. Some students, like Alicia, appear to stay focused on mathematics tasks during work time and have difficulty figuring out what to do. Other students routinely sit idle during work time prior to whole group discussions and do not appear to be attempting the assigned problem. Ms. Larsano often provides verbal encouragement for students to try and figure out problems during these work sessions, but there is not an immediate consequence for students who choose not to attempt the problems. In this way, Ms. Larsano's response to students' difficulties getting started on problems is to allow her students to engage minimally with mathematics problems during work time and instead learn from the solutions presented by their peers during whole class discussion.

In addition to student difficulties getting started with mathematics problems, student difficulties also surface during work time when students devise flawed solutions to problems posed. When Ms. Larsano interacts with students who have flawed solutions in the context of work time, her most typical response is to provide procedural hints for correcting the errors. To illustrate, consider the following interaction between Ms. Larsano and Sean that occurs during initial exploration of multidigit multiplication problems. Sean is working on figuring out how many total M&Ms there are in eight bags of 500. On his paper, Sean has written the false number sentence: $8 \times 500 = 40,000$.

> Ms. Larsano: What is your strategy? Show me. Where did you get that number? Show me. I don't see anything to show me where did you get that number? And why this number if that number is this big?
> Sean: 5 times 8 equals 40. So, it is this (*points at 40,000 on his paper*).
> Ms. Larsano: Check the zeros. I think I see too many. I see 2 here (*points at 500*) and 4 here (*points at 40,000*). I see too many.

In this instance, Ms. Larsano first asks Sean questions about how he came to his answer of 40,000 while also hinting that the answer is incorrect. When Sean explains that he found his answer by thinking about the relationship between 5×8 and the problem of focus, Ms. Larsano explicitly indicates that the answer is incorrect and directs Sean to, "Check the zeros." This procedural response of focusing on counting zeros stands in contrast to considering the reasonableness of the solution or checking a solution using an alternate method. This instructional excerpt exemplifies Ms. Larsano's practice of responding to students' flawed solutions in the context of work time with support that emphasizes procedures for obtaining answers and deemphasizes conceptual understanding.

In summary, Ms. Larsano's response to student difficulties that arise during work time typically involves brief interactions with students in which procedurally-focused support is provided. Additionally, Ms. Larsano responds to students' difficulties by sometimes allowing students to engage minimally with mathematics tasks during work time, trusting that these students will learn how to solve the problems when their peers present solutions during whole class discussion.

Larsano response pattern 1, consequences for student thinking. Ms. Larsano's practice of providing procedurally-focused support for students when they are struggling to make sense of mathematical problems appears to limit the degree to which students develop conceptual understanding and generative problem solving practices. Instead of focusing on a word problem's context to support development of reasonable solutions, some of Ms. Larsano's students are observed attempting to apply recently shared procedures to new problems regardless of whether the procedures make sense for the problems at hand. Students are also observed applying procedures taught incorrectly, often unaware when their final answers are unreasonable. In these ways, solving problems in Ms. Larsano's class becomes more about remembering procedures and less about trying to make sense of the mathematical ideas underlying why particular procedures work. This kind of focus limits the likelihood that conceptual understanding will be developed through doing and discussing problem solving tasks.

In addition, when students are focused on remembering and applying procedures, they generally have little in the way of back-up strategies when correct procedures cannot be recalled. This may, in part, explain why some of Ms. Larsano's students sit idle during time allotted for work on a given task. Without immediate recognition of a procedure to apply, students may find themselves without a way to move forward with a problem. This kind of dilemma results when students focus on trying to apply mathematical procedures without understanding how they work or why they are appropriate for particular mathematics tasks. Nonetheless, when students do not engage in devising personal solutions to mathematics tasks, they miss out on important opportunities to use their existing understandings to construct new mathematical knowledge.

Larsano response pattern 1, link to beliefs and knowledge. Ms. Larsano's ways of responding to student difficulties that arise in the context of work time are related to both her beliefs and knowledge. First, consider Ms. Larsano's tendency to provide procedurally-focused support during work time. IMAP survey responses suggest that, at the beginning and end of the year, Ms. Larsano draws little distinction between procedural knowledge and conceptual understanding. As student difficulties arise, Ms. Larsano often interprets them in terms of difficulty remembering the appropriate procedures rather than considering how the difficulties or mistakes might be related to conceptual understanding. This is the case in the example above when Ms. Larsano directs Sean to check the zeros for his solution to the problem 8×500. It is also the case when Ms. Larsano provides Alicia with support that involves helping her to understand the candle problem by telling her a procedure to obtain a correct answer.

While related to beliefs, Ms. Larsano's practice of providing procedurally-focused support also reflects her limited personal knowledge of mathematics. Ms. Larsano's personal approach to solving mathematical problems involves heavy reliance on remembering standard procedures. When pressed to explain why procedures work, Ms. Larsano struggles to discuss the conceptual underpinnings of procedures throughout the year. For instance, in a post-observation interview, Ms. Larsano is unable to explain why the “count the zeros” rule works in problems like 8×500. Since Ms. Larsano primarily understands mathematics in terms of procedural knowledge, it is not surprising that her support for students emphasizes procedures.

Ms. Larsano is also observed responding to students’ difficulties approaching mathematical problems by allowing them to engage minimally with problems during work time. This practice is primarily explained by Ms. Larsano’s beliefs. At the beginning and end of the school year, Ms. Larsano discusses her belief that most students require explicit teacher direction in order to be successful with mathematics tasks. What follows is shared in an interview at the end of the year:

> I think first, the teacher needs to present it. The teacher needs to give them examples, to explain the concepts, make them aware of what they’re expected to do or what do they need to do or how do they need to do it…because teachers sometimes assume that they bring some concept. We assume that they know some things and sometimes they don’t. So we need to present the concept, present the skill, give them an example, maybe two maybe three, and then let’s try it together and walk them through the process before they are going to do it by themselves.

Although Ms. Larsano follows the *Everyday Mathematics* teacher guide and assigns novel mathematics tasks before providing explicit direction, she is not surprised when some students do not work out personal solutions prior to whole group discussion. She views the whole class discussion as the time when she and the students who have devised solutions will share models of procedures that students without solutions can follow on subsequent, similar problems.

Larsano Response Pattern 2: Response to Flawed Solutions in Whole Class Discussion

Following the recommendations of the *Everyday Mathematics* program, Ms. Larsano allots time during mathematics instruction for students to share solution strategies for select mathematical problems in a whole class discussion format. During these class discussions, Ms. Larsano primarily calls on volunteers to share their problem solutions while also aiming to select students who, at a glance, appear to have different strategies for a given problem. Ms. Larsano never reports intentionally having a student share a flawed solution; however, flawed solutions are presented during whole class discussion in every lesson observed. In this section, Ms. Larsano's primary response to flawed solutions shared in the context of whole class discussion will be explored. This will be followed by discussion of an additional, less robust pattern of response to students' publicly shared flawed solutions.

First, consider an excerpt from a Fall lesson in which Ms. Larsano is leading the class in discussing the problem, "23 candles are arranged with 3 in each row. How many rows are there?" Five students have put their solutions to the candle problem on the board, and Ms. Larsano has identified that Andre's solution does not match the problem context:

Andre's work on the board:

O O O
O O O
O O O
O O O   24 ÷ 8 = 3
O O O
O O O
O O O
O O O

Ms. Larsano: Let's look at this one (Andre's solution) over here. Look at what we have here. We have (*pointing at each column*) 8, 8, 8. What's the total?
A few students: Twenty-four.
Ms. Larsano: What's the total?

Several students: Twenty-four.
Ms. Larsano: Twenty-four. But I started out with how many?
A few students: Twenty-three.
Ms. Larsano: Twenty-three. So, I have how many extra?
A few students: One
Ms. Larsano: One extra. So, I have to take one from here. (*Erases the bottom right circle from Andre's model*.)

Andre's work on the board, as modified by Ms. Larsano:

O	O	O		
O	O	O		
O	O	O		
O	O	O		$24 \div 8 = 3$
O	O	O		
O	O	O		
O	O	O		
O	O			

Ms. Larsano: And now it wouldn't be equal, it wouldn't be equal. We have to change it, because the problem says, "Twenty-three candles, 3 in each row". And I have how many? (*Pointing at each column*) I have 8, I have 8, and I have 7. (*Pause*) You're looking, you're looking, you're seeing it? (*To Andre*) I'm not telling you that you are wrong, I am just explaining to you what you did. So, he put that 24 divided by 8 equals 3. So, he did have 3 groups. The only problem was that he put one more. But it's okay, it's a model.

In this instructional episode, Ms. Larsano first asks the class a series of closed questions to establish that Andre's model contains 24 objects instead of the 23 candles specified in the problem. This exchange is highly controlled by Ms. Larsano and does not invite Andre's peers to evaluate his solution. Then Ms. Larsano revises the model by erasing one of the objects and pointing out that, with only 23 objects, the columns are not equal. Up to this point, Ms. Larsano's actions indicate that having 24 objects represented does not match the problem. But, at this point, she turns to Andre and says, "I am not telling you that you are wrong…" which contradicts previous actions and introduces uncertainty regarding whether Andre's approach is correct. Next Ms. Larsano addresses Andre's number sentence, $24 \div 3 = 8$, and identifies that he has three groups. This is confusing because the 3 in the problem denotes the number of candles in each row, not the number of groups. Before moving on to another student's work, Ms. Larsano tells the students, "But it's okay, it's a model," suggesting that if you make a model of some kind, it might be considered a correct answer. The correct and incorrect aspects of Andre's model and number sentence are never made clear. While the revised model could be used to determine the solution to the rows of candles problem, this is not brought out. Furthermore, at the end of the episode, Andre's number sentence that remains on the board does not match the problem context or the revised model.

Throughout the year, Ms. Larsano's primary response to students' flawed solutions is to attempt to illuminate correct and incorrect aspects of the solutions with the goal of revising the flawed solutions to correct. However, this process of understanding and revising a flawed solution is often ended before a correct solution has been made clear. Along the way, Ms. Larsano sometimes appears to lose track of the logic of the solution and inserts comments or ideas that are mathematically incorrect or confusing. Additionally, Ms. Larsano is reluctant to publicly tell students their solutions are incorrect. This sometimes contributes to significant ambiguity regarding the correctness of a solution. Finally, Ms. Larsano typically maintains tight control over class discussions when flawed solutions are of focus. She does most of the talking and explaining, allowing students to participate primarily by answering closed questions.

Another pattern observed when students share flawed solutions in the context of whole class discussion is that Ms. Larsano sometimes does not address an error in any way. For example, in a Fall lesson, students are focused on the task, "There are 24 cheerleaders in a big parade. Use your counters to represent the cheerleaders." After students use counters to complete the task at their desks, Ms. Larsano calls on several students to draw pictures of their solutions on the board. Kenny draws the 2 x 12 array displayed below. Following a discussion of how to write a number sentence to match other students' arrays of 24 objects, Ms. Larsano requests that Kenny write a number model (number sentence) for the array he put on the board. Kenny approaches the board and writes $6 \times 2 = 24$ beside his array.

O O O O O O O O O O O O
O O O O O O O O O O O O $6 \times 2 = 24$

While Kenny is at the board, Ms. Larsano continues with the lesson. She never comes back to Kenny's array or the incorrect number model that he has written beside it. In the post-observation interview for this lesson, Ms. Larsano confirms that she had not noticed Kenny's error and that she is surprised that he would make such an error. This instance is an example of a pattern observed in Ms. Larsano's teaching of not responding to students' errors because she does not notice them. While this pattern is less robust than the primary response pattern shared, it is considered relevant because there are multiple instances of overlooking students' publicly shared errors in Fall and Spring observations.

In summary, when Ms. Larsano notices flawed solutions that surface in whole class discussion, her intention is to guide the class to analyze and revise the flawed solutions such that students can learn from their peers' errors. However, this intention is often not realized because Ms. Larsano maintains tight control over these discussions and provides explanations that are conceptually unsupportive and fall short of making correct and incorrect aspects of flawed solutions clear. Additionally, there are instances when Ms. Larsano does not notice flawed aspects of solutions that are presented in class discussion and consequently does not respond at all.

Larsano response pattern 2, consequences for student thinking. Ms. Larsano's handling of students' flawed solutions in the context of whole class discussion often contributes to confusion over the correctness of solutions and mathematics concepts in general. When flawed solutions are of focus, it is noted that Ms. Larsano typically maintains tight control of discussion. While students are prompted to respond to particular questions posed by Ms. Larsano, the majority of questions asked are closed questions with specific answers. Consequently, there is limited opportunity for students' ideas in relation to flawed solutions to enter into discussion segments focused on flawed solutions.

When Ms. Larsano loses track of the logic of a solution and shares mathematically confusing or incorrect information, students rarely question her flawed or incorrect assertions. Instead, they appear to follow their teacher's presentation of flawed information, writing in their notebooks and responding to the closed questions posed. Rather than thinking critically about the mathematical ideas put forward during discussions, Ms. Larsano's students appear to accept information stated by their teacher as true.

Finally, it is noted that Ms. Larsano's discussion of particular flawed solutions is often ended before the correct and incorrect aspects of the given solution are made clear. This practice coupled with Ms. Larsano's tendency to overlook flawed solutions displayed publicly on the board appears to contribute to confusion over the correctness of solutions as well as mathematics concepts in general. Overall, opportunities for development of student thinking during class discussion of flawed solutions in Ms. Larsano's class are limited.

Larsano response pattern 2, link to beliefs and knowledge. Throughout the year, Ms. Larsano expresses that she feels challenged and sometimes overwhelmed by the teaching demands of the *Everyday Mathematics* lessons, especially during whole class discussions. Ms. Larsano's handling of students' flawed solutions during whole class discussions seems primarily related to her limited knowledge for teaching mathematics. However, facets of Ms. Larsano's response are tied to her beliefs.

Ms. Larsano consistently has difficulty making sense of students' strategies and errors both on-the-spot during instruction and in interviews and professional development meetings away from the pressures of real-time teaching. In Ms. Larsano's response to Andre's flawed solution, it was her intention to help students see how the response was flawed. However, she was unable to unpack Andre's solution in such a way that correct and incorrect aspects of the solution were illuminated. During this episode, Ms. Larsano lost track of the relationship between the word problem context and the problem solution being discussed. In order to provide a conceptually-grounded discussion of this flawed solution, Ms. Larsano would have needed to maintain mental focus on how to think about the problem in relation to its context, the student's flawed solution, and a potential revised solution as well as the additional challenges of teaching related to orchestrating a whole class discussion. Mentally juggling these many cognitive demands consistently proves difficult for Ms. Larsano.

A related problem is that Ms. Larsano does not generally anticipate the difficulties students are likely to have in a mathematics lesson. Consequently, Ms. Larsano only sometimes focuses her attention on looking for likely student errors. When asked why she did not respond to Kenny's flawed solution (detailed in the instructional excerpt shared), Ms. Larsano expressed surprise that Kenny would write an incorrect number sentence. Ms. Larsano's surprise indicates that she has underestimated the difficulty students have learning to make sense of symbolic representations in relation to real-world and pictorial representations in initial multiplication instruction.

Furthermore, observation of Ms. Larsano's teaching suggests that her pedagogical knowledge for facilitating student discussion of mathematical ideas and debate over their correctness is limited. In the episode focused on Andre's solution, Ms. Larsano dominates the discourse, limiting student participation to responding to closed questions. She seems to intentionally avoid telling the class that Andre's response is incorrect, perhaps because she wants students to make this determination. However, Ms. Larsano is unable to orchestrate the discussion to make it likely to occur.

When responding to students' flawed solutions in the context of whole class discussion, Ms. Larsano's mathematical knowledge for teaching appears to be the primary barrier to teaching practices supportive of student thinking. Yet, her beliefs also shape aspects of her response. First, consider Ms. Larsano's practice of avoiding telling students their answers are incorrect in the public forum of whole class discussion. Ms. Larsano wants to be encouraging and supportive of students when they share ideas publicly. She intentionally avoids actions that she perceives might hurt students' feelings or discourage them from participating in the future. Additionally, Ms. Larsano's practice of maintaining tight control of class discussions when flawed solutions arise is, in part, related to her general belief in the role of the teacher being to provide explanation and direction on how to solve problems. In these ways, Ms. Larsano's beliefs contribute to her ways of responding to students' flawed solutions in the context of whole class discussion.

Larsano Response Pattern 3: Responding to Students' Differing Answers to Teacher Questions

Throughout class discussion of mathematics, it is a class norm for Ms. Larsano to pose closed questions to her class and for her students to respond to these questions through choral response. As would be expected, students sometimes call out multiple answers including correct and incorrect responses. Facets of Ms. Larsano's response in this type of situation will be illustrated through two instructional excerpts, both from Fall lessons.

First consider the following instructional episode in which Ms. Larsano is leading the class to replace the question mark in a multiplication-division diagram with the answer to the problem presented below:

28 pennies are shared equally by 4 children. How many pennies per child is that?

Children	Pennies Per Children	Total
4	?	28

Ms. Larsano: So, over here (*pointing to the "Pennies Per Children" section of the multiplication-division diagram*), I have to say what? How many? How many pennies per child?
Several students: (*Students call out different answers.*) Seven; six.
Ms. Larsano: Six?
Several students: No, seven.

When asked what number should replace the question mark, several students call out suggestions, some suggesting six and others seven. Ms. Larsano then responds by repeating the answer, "Six?" in a questioning voice with a high pitched tone. Although Ms. Larsano does not directly state that six is an incorrect response, her habit of questioning a given answer with a high-pitched tone appears to cue students to the fact that the answer is incorrect. In this instance, students seem to know that, if they initially answered six, they should change their answers to something different.

The next instructional excerpt occurs after students have shared visual models used to solve the problem in which a boy with 36 pencils gives away 3 pencils to as many friends as possible. Students are to determine how many of the boy's friends could get pencils. At this point, Ms. Larsano is leading the class to think about how this problem might be represented in a division number model (number sentence):

> Ms. Larsano: Let's write a number model. We started with…
> Many students: Thirty-six.(*Ms. Larsano writes "36 ÷" on the board*.)
> Ms. Larsano: Then we divided by the number of pencils. How many was that?
> Many students: (*Students call out different answers*) Thirty-six; three; twelve.
> Ms. Larsano: 3. (*Ms. Larsano adds to the board: 36 ÷ 3 =*) So, how many friends?
> A few students - Twelve.

In this instance, students are having difficulty identifying a number sentence that matches the problem they have just finished solving. Ms. Larsano responds to students' apparent confusion over which value to put to the right of the division sign by writing the correct answer on the board. In doing so, she seemingly ignores the incorrect suggestions.

In summary, when students' responses to questions vary, Ms. Larsano sometimes responds by repeating the incorrect answer in a questioning voice using a high pitched tone in the tail of her reply. This practice appears to cue students to change their answers. At other times, Ms. Larsano responds to students' varying ideas by identifying the correct answer. Through these practices, Ms. Larsano maintains tight control of classroom discourse when students suggest differing ideas. Furthermore, she establishes herself as the primary judge of mathematical correctness by consistently resolving differences among student responses.

Larsano response pattern 3, consequences for student thinking. Ms. Larsano's practice of resolving differences among students' responses has multiple consequences for student thinking. First, an opportunity is missed to allow the presence of multiple answers to stimulate student reflection on the logic of answers posed. Teachers who delay resolution of differing responses to questions posed often encourage students to provide justification for their varying answers. This practice serves to promote reflection on problem solving processes and mathematical concepts, which often leads to a higher level of mental activity and deeper understanding. When differing answers arise in Ms. Larsano's class, students look to their teacher to tell them which answer is correct. In relying on her, they engage in minimal reflection on their own mathematical thinking.

Furthermore, Ms. Larsano's practice of resolving differing student responses establishes the teacher as the mathematical authority in the classroom. Consequently, Ms. Larsano's students do not appear to view it as part of their role to evaluate the correctness and incorrectness of their own or other students' answers. This student belief plays out in many aspects of mathematics class activity. For instance, as students work on tasks, they only occasionally look to each other for support. Furthermore, students frequently look to adults to tell them whether or not their answers are correct. In general, students look to Ms. Larsano and other adults for affirmation of correctness rather than relying on their own mathematical reasoning or the insights of their peers.

Larsano response pattern 3, link to beliefs and knowledge. Ms. Larsano's practice of responding to students' differing ideas by establishing mathematical correctness is related to her beliefs and knowledge. As has been discussed previously, Ms. Larsano holds the belief that good teaching involves providing strong, explicit direction. This belief leads Ms. Larsano to maintain tight control over class discussions, positioning herself as arbiter of student ideas. Furthermore, Ms. Larsano believes that her students require fast-paced instruction. Hence, when students call out a mix of correct and incorrect responses to a question, Ms. Larsano aims to resolve that difference as quickly as possible. These beliefs contribute to Ms. Larsano's practice of telling students correct answers.

Yet, at the end of the year, Ms. Larsano expresses a value in inclusion of student ideas in class discussions as she talks about how working with the *Everyday Mathematics* program has changed her teaching:

> We're letting them also give the ideas or give the way or give the strategy, so I think we're a little bit more open to suggestions from the kids than maybe we were before than maybe I was before. I was taught this is what you teach and we learn, and that's it. But now everybody's involved in the learning, and we learn a lot from the kids…I think it pumps up their self-esteem, and it also helps them think. If they're giving ideas, they're thinking of what they're doing. They're giving meaning to what they're doing.

This quote suggests that Ms. Larsano has come to see benefits of students' mathematical ideas being a part of mathematics learning, especially during class discussions. However, organizing discussions to include students' mathematical ideas is a new way of teaching for Ms. Larsano. It is possible that she has not considered the negative consequences of telling students correct answers or how she might respond differently to the situation of differing student answers.

Thus, Ms. Larsano's practice of resolving students' differing answers can also be linked to her limited knowledge base related to leading reform-oriented mathematics discussions. While she has cultivated routines for having students share their multiple strategies, Ms. Larsano has not developed routines that encourage students to work as a community to resolve differing mathematical ideas through discussion and debate. Furthermore, interviews with Ms. Larsano suggest that her pedagogical content knowledge related to students' difficulties with specific mathematics topics and problem solving processes is limited. Consequently, she has difficulty identifying when it would be instructionally beneficial to stress particular mathematical ideas through extended focus on incorrect student answers. In these ways, Ms. Larsano's practice of maintaining control over resolving student difficulties is related to her knowledge.

Summary of Ms. Larsano's Response to Student Difficulty

Three patterns portray Ms. Larsano's typical response to student difficulties during mathematics instruction. First, Ms. Larsano's response to difficulties observed during work time involves providing brief, procedurally-focused suggestions to students and allowing students to enter class discussions without having devised personal solutions. In the context of class discussions, Ms. Larsano responds to flawed solutions that are shared by attempting to lead students to analyze the solutions and demonstrate how they can be revised. But discussion of flawed solutions is typically ended by Ms. Larsano before correct and incorrect aspects of the solutions are made clear. Therefore, the benefits to this approach are not realized. Finally, when students' responses to questions during class discussion differ, Ms. Larsano typically maintains control over resolving differing ideas and establishing mathematical correctness.

Taken together, Ms. Larsano's ways of responding to student difficulties limit the likelihood that students will develop reform-oriented learning practices and dispositions as well as conceptual understanding of mathematics. By encouraging students to apply mathematical procedures without understanding, Ms. Larsano encourages students to view mathematics as a series of unrelated procedures and tricks to memorize. Furthermore, students become reliant on others for mathematical knowledge rather than learning to rely on their own prior knowledge to make sense of mathematics problems. These beliefs are reinforced by Ms. Larsano's practices whenever difficulties surface in class discussions. By maintaining tight control over class discussion and positioning herself as the primary authority of mathematical correctness, Ms. Larsano inadvertently encourages her students to develop a reliance on teacher direction. Additionally, these practices minimize opportunities for students to develop mathematical reasoning and critical reflection. Another problem is that mathematical concepts and the correctness of particular solutions are often left ambiguous at the end of each math period, which leads to student confusion and misunderstanding. These factors contribute to Ms. Larsano's students' difficulties with problem solving tasks throughout the year and their limited engagement during class discussion.

Ms. Larsano's response to students' difficulties appears linked to both her beliefs and knowledge. Ms. Larsano's beliefs seem most related to her expectations for students and the ways she structures learning experiences. In particular, Ms. Larsano's teaching practices seem to be strongly influenced by her belief in teacher-centered instruction coupled with her belief in students' limited capabilities to approach novel problems without teacher support. These beliefs explain Ms. Larsano's low expectations of students as they encounter novel kinds of problems. Since she expects that students will require teacher support to be successful, Ms. Larsano is not surprised or upset when some students do not solve problems prior to group discussion. In whole class discussions, Ms. Larsano maintains tight control of the discourse, in part, because she believes that good teaching involves the teacher providing models of and explanations for how to solve problems correctly. Ms. Larsano's domination of classroom discourse and mathematical correctness is also a reflection of her limited knowledge of reform-oriented teaching strategies that orchestrate mathematics learning around student discussion and debate over their peers' mathematical ideas. Furthermore, Ms. Larsano's limited knowledge of mathematics seems most responsible for instances when her review of flawed solutions includes mathematically incorrect or confusing information.

Ms. Larsano's knowledge and beliefs are also both in play when she provides procedurally-focused support. Ms. Larsano believes that an important part of teaching mathematics is to drill students until they remember mathematical procedures that yield correct answers. Furthermore, Ms. Larsano views mathematical understanding as the ability to get correct answers, suggesting that she makes little distinction between remembering procedures and conceptual understanding. While Ms. Larsano's practice of providing procedurally-focused support is linked to her beliefs, it seems rooted in Ms. Larsano's base of knowledge for teaching mathematics. In interviews and workshops, Ms. Larsano often struggles to explain the mathematical underpinnings of particular procedures that she encourages students to use. At the same time, she views non-standard strategies that are more conceptually explicit than standard algorithms to be confusing. In general, Ms. Larsano experiences personal struggles with understanding mathematics conceptually. Therefore, during on-the-spot teaching situations, the knowledge that Ms. Larsano has available to draw on is primarily procedural.

Situating Ms. Larsano's Case Story in Broader Measures of Teaching

In the previous section, a theoretical explanation was presented suggesting how Ms. Larsano's teaching practices in response to student difficulties are linked to her beliefs and knowledge during the year of this study. Teacher response to student difficulties is one of many aspects of reform-oriented mathematics instruction that contributes to the overall quality of mathematics teaching and learning more broadly defined. Therefore, this section will present results from two more global measures of Ms. Larsano's mathematics teaching. First, data collected following each core classroom observation using the *Reformed Teaching Observation Protocol* (RTOP) (Sawada et al., 2002) will be presented. This instrument is designed to measure the degree to which a given mathematics lesson reflects principles and practices associated with reform-based mathematics instruction. Second, aggregated class data from the mathematics section of the State Standardized Test (SST) will be presented in relation to aggregated data at the school, district, and state levels. Taken together, these findings will allow Ms. Larsano's patterns of response to student difficulty to be considered within a more general understanding of her mathematics teaching and students' learning.

Larsano: Adherence to Reformed Teaching

The degree to which Ms. Larsano's mathematics instruction reflects current reforms in mathematics education was measured using the Reformed Teaching Observation Protocol (RTOP) (Sawada et al., 2002). The RTOP is a criterion-referenced instrument containing 25 items, divided into five subscales: (1) lesson design and implementation, (2) the level of significance and abstraction of the content, (3) the processes that students use to manipulate information, (4) the classroom culture as observed through the nature of communicative interactions, and (5) the nature of student-teacher relationships. Following each core observation in Ms. Larsano's classroom, each of the 25 items on the RTOP was rated on a scale from 0 (not observed) to 4 (very descriptive). Next, sums were calculated for ratings on each subscale as well as the total instrument to assess the degree to which Ms. Larsano's mathematics instruction was reformed, with higher scores reflecting a greater degree of reform. Consequently, subscale scores on the RTOP range from 0 – 20, and total instrument scores range from 0 – 100. Aggregated results from core observations of Ms. Larsano's mathematics teaching in the Fall and Spring are presented in Table 22

On the lesson design and implementation subscale, the mean scores for Ms. Larsano's instruction are 13.0 in the Fall and 11.0 in the Spring. All lessons are designed to allow students to devise problem solutions using their prior knowledge, thus stressing exploration prior to formal presentation. However, it is typical for some of Ms. Larsano's students to sit idle during this exploration time, limiting the degree to which student exploration actually occurs. Within class discussions of students' solutions, Ms. Larsano typically limits student contributions to sharing of solutions and takes on the primary role of providing commentary and analysis of solutions shared. In this way, Ms. Larsano is primarily responsible for determining the focus and direction of lessons in which students are minimally engaged as members of a learning community.

The second RTOP subscale focuses on what the contents of a lesson suggests about a teacher's propositional knowledge, meaning her mathematical knowledge for teaching particular content. On this subscale, Ms. Larsano's mean scores are 12.5 in the Fall and 11.5 in the Spring. All lessons are found to involve fundamental mathematics and draw on

Table 22

Ratings of Ms. Larsano's Mathematics Teaching on the RTOP

RTOP Items by subscale	Fall	Spr
Subscale 1: Lesson design and implementation		
1. The instructional strategies and activities respected students' prior knowledge and preconceptions inherent therein.	3.0	2.0
2. The lesson was designed to engage students as members of a learning community.	2.0	2.0
3. In this lesson, student exploration preceded formal presentation.	3.0	2.0
4. The lesson encouraged students to seek and value alternative modes of investigation and problem solving.	3.0	3.0
5. The focus and direction of the lesson was often determined by ideas originating with students.	2.0	2.0
Subscale 1 score	**13.0**	**11.0**
Subscale 2: Content, propositional knowledge		
6. The lesson involved fundamental concepts of the subject.	3.0	3.0
7. The lesson promoted strongly coherent conceptual understanding.	2.0	2.0
8. The teacher had a solid grasp of the subject matter content inherent in the lesson.	2.5	1.5
9. Elements of abstraction (i.e., symbolic representations, theory building) were encouraged when it was important to do so.	2.0	2.0
10. Connections with other content disciplines and/or real world phenomena were explored and valued.	3.0	3.0
Subscale 2 score	**12.5**	**11.5**
Subscale 3: Content, procedural knowledge		
11. Students used a variety of means (models, drawings, graphs, concrete materials, manipulatives, etc.) to represent phenomena.	3.0	2.5
12. Students made predictions, estimations, and/or hypotheses, and devised means for testing them.	1.5	1.5
13. Students were actively engaged in thought-provoking activity that often involved the critical assessment of procedures.	2.0	2.0
14. Students were reflective about their learning.	1.0	1.5
15. Intellectual rigor, constructive criticism, and challenging of ideas were valued.	1.0	1.0
Subscale 3 score	**8.5**	**8.5**
Subscale 4: Classroom culture, communicative interactions		
16. Students were involved in communication of their ideas to others using a variety of means and media.	2.5	2.0
17. The teacher's questions triggered divergent modes of thinking.	2.5	2.0
18. There was a high proportion of student talk and a significant amount of it occurred between and among students.	2.0	2.0
19. Student questions and comments often determined the focus and direction of classroom discourse.	2.0	2.5
20. There was a climate of respect for what others had to say.	3.0	2.5
Subscale 4 score	**12.0**	**11.0**
Subscale 5: Classroom culture, student/teacher relationships		
21. Active participation of students was encouraged and valued.	2.5	2.0
22. Students were encouraged to generate conjectures, alternative solution strategies, and ways of interpreting evidence.	1.5	2.0
23. In general the teacher was patient with students.	2.0	2.5
24. The teacher acted as a resource person, working to support and enhance student investigations.	2.0	2.0
25. The metaphor "teacher as listener" was very characteristic of this classroom.	2.0	2.5
Subscale 5 score	**10.0**	**11.0**
Total score	**56.0**	**53.0**

connections to real-world phenomena. However, lessons inconsistently promote coherent conceptual understanding of the fundamental mathematics of focus, including understanding of symbolic representations. Finally, Ms. Larsano’s instruction suggests that she holds moderate knowledge of the content of focus in Fall lessons and weak knowledge of the content of focus in Spring lessons.

The third RTOP subscale focuses on what the contents of a lesson suggests about a teacher’s procedural knowledge. For the RTOP instrument, this means the quality of the lesson in terms of an inquiry approach to learning. On this subscale, Ms. Larsano’s instruction rated a mean score of 8.5 in the Fall and Spring. At both data points, Ms. Larsano’s students are collectively observed to use multiple means to represent mathematical ideas. However, students are only sometimes observed engaging in activities of inquiry such as making predictions and devising means for testing them, critically assessing procedures, and reflecting on learning. Furthermore, the presence of student debate of mathematical ideas is minimal.

The fourth RTOP subscale focuses on communicative interactions that are part of the classroom culture. On this subscale, the mean scores for Ms. Larsano’s instruction are 12.0 in the Fall and 11.0 in the Spring. At both data points, Ms. Larsano maintains strong control over the classroom discourse. Mathematics tasks are designed to trigger divergent modes of thinking, but questions posed by Ms. Larsano are primarily low-level, closed questions. During class discussion, students primarily participate by presenting their mathematical solutions and answering teacher questions. While students are typically polite to their classmates during these discussions, there is limited evidence of student engagement. This suggests that students are limited in the degree to which they respect and value the contributions of their peers.

The last RTOP subscale focuses on the classroom culture in terms of the teacher's role and the roles students are encouraged to take. On this subscale, the mean scores for Ms. Larsano's instruction are 10.0 in the Fall and 11.0 in the Spring. Active participation of students is valued by Ms. Larsano but inconsistently encouraged through specific instructional strategies. Furthermore, Ms. Larsano inconsistently acts as a resource person during student investigations and only sometimes encourages students to generate conjectures and alternative solution strategies. Finally, Ms. Larsano sometimes exhibits patience and takes on the role of a listener during instruction. But, at other times, she limits time for students to work on problems or answer questions and gives limited attention to students' ideas.

Ms. Larsano's mean scores on the total RTOP instrument are 56.0 in the Fall and 53.0 in the Spring. These ratings suggest that Ms. Larsano's mathematics instruction moderately reflects the principles of reform at both data points.

Larsano: Student Achievement

Aggregated student scores on the mathematics section of the State Standardized Test (SST) were used to measure the mathematics achievement of Ms. Larsano's students. The SST is a criterion-referenced achievement test in which students receive scores between 1 and 5, with a score of 3 or higher considered passing. Since students first take the SST in third grade, comparable scores of previous mathematics achievement are not available for Ms. Larsano's students. Without baseline data, only a tentative understanding of the relationship between Ms. Larsano's teaching and her students' mathematics achievement is possible. Also, caution is warranted when comparing the SST scores of Ms. Larsano's students to the other third-grade classes at Lincoln Heights, as each teacher studied has a class with demographic particularities and a statistically small number of students. In the case of Ms. Larsano's class of LEP students, it should be known that testing modifications (e.g., the students could request that test items be read to them) were made to compensate for students' English-language needs. With these caveats in mind, SST scores for Ms. Larsano's students are presented in Table 23 along with comparison data for third-grade students at Lincoln Heights, the school district, and the state.

Table 23
Comparison of Larsano Student Achievement on SST to School, District, and State

		Achievement Level (%)					
Group	Students	1	2	3	4	5	Passing Score (%)
Ms. Larsano's Class	18	33	24	24	19	0	43
Lincoln Heights 3rd Grade	88	24	27	33	15	1	49
School District 3rd Grade	13,713	18	16	33	24	9	66
State 3rd Grade	203,037	15	17	34	25	9	68

Note. Scores of 3 or higher are considered passing.

Of the 18 students in Ms. Larsano's class who took the SST, 43% achieve a passing score of 3 or higher on the mathematics section of the SST. This is in comparison to passing scores achieved by 49% of Lincoln Heights third graders, 66% of the third graders in the school district, and 68% of the third graders in the state. Hence, the mathematics achievement of Ms. Larsano's class on this test is slightly lower than that of the third graders at Lincoln Heights and considerably lower than third graders at the district and state levels.

Summary of Global Measures of Ms. Larsano's Mathematics Instruction

Taken together, results from the *Reformed Teaching Observation Protocol* (RTOP) and the State Standardized Test (SST) suggest that Ms. Larsano's classroom is a place where there is inconsistent adherence to reformed teaching practices and mixed student learning outcomes. RTOP results suggest that Ms. Larsano's mathematics teaching moderately reflects the principles of mathematics reform throughout the year. Yet, lessons observed promote limited conceptual understanding and provide evidence that Ms. Larsano's grasp of the mathematics is weak, especially in the Spring when multidigit multiplication is of focus. Consequently, Ms. Larsano's attempts to provide reform-oriented mathematics instruction sometimes falls short of facilitating generative understanding of mathematics for students. The mathematics achievement data for Ms. Larsano's class provides further evidence to support this claim. Aggregated SST scores for Ms. Larsano's class indicate that 43% of her students are performing at or above grade level. This finding suggests that Ms. Larsano's students' understanding of the mathematics concepts and skills of focus in third grade is mixed. And the high percentage of LEP students in the class is only part of the reason why. Ms. Larsano could benefit from more professional development with a focus on increasing her propositional knowledge as well as teaching strategies to get the full benefit from a reform-based curriculum.

The Case of Ms. Rosena

Ms. Rosena begins the year of this study new to third grade, having had most of her teaching experiences at the fourth and fifth grade levels. Ms. Rosena readily talks about her observation that the fifth grade students she taught in the previous year had many gaps in their mathematical understanding. She attributes these problems to inadequate foundation in primary grade mathematics instruction and is hopeful that the new *Everyday Mathematics* program will support her efforts to get across these foundational mathematical ideas to third graders. Although Ms. Rosena focused her recent Masters degree studies on deepening her knowledge for teaching language arts, she expresses interest in and enthusiasm for teaching mathematics. Ms. Rosena did not consider herself to be a strong math student in school, but reports that her teaching assignments in upper elementary grades forced her to revisit and make sense of the important mathematics encountered in the elementary curriculum such that she now considers herself a competent teacher of elementary mathematics.

This year Ms. Rosena is assigned to teach a class of ten students who have been retained in third grade due to failing scores on the State Standardized Test (SST). Perhaps because Ms. Rosena is skillful at classroom management, her class includes a few students who have a history of being particularly disruptive due to behavioral and emotional needs. In addition, two students have on-going English-language needs, which Ms. Rosena is able to address through her fluent Spanish. In general, Ms. Rosena finds her students to be reluctant learners, requiring much prodding and encouragement to engage in instructional tasks.

The case of Ms. Rosena that follows is presented in four sections. In the first two sections, evidence of Ms. Rosena's mathematics-related beliefs and knowledge at the beginning and end of the year will be presented. Next, the relationship among Ms. Rosena's beliefs, knowledge, and classroom practice will be explored through a case story focused on her patterns of response to student difficulties. Finally, global measures of Ms. Rosena's adherence to reform-based mathematics teaching and student achievement will be presented so that case story findings can be considered in relation to these broader measures of teaching.

Ms. Rosena's Beliefs about Mathematics Teaching and Learning

Ms. Rosena's beliefs about mathematics teaching and learning were measured at the beginning and end of the school year using the *IMAP Web-Based Beliefs-Survey* (Ambrose et al., 2003). The IMAP instrument requires teachers to respond to instructional scenarios in an open-ended format, allowing respondents to emphasize and downplay issues of personal importance. A series of rubrics are then used to measure survey responses in relation to seven target beliefs considered central to reform-oriented mathematics instruction. The degree to which Ms. Rosena's survey responses provide evidence that she holds each target belief at the beginning and end of the school year is presented in Table 24. In the sections that follow, evidence of Ms. Rosena's adherence to each reform-oriented belief at the beginning and end of the year will be presented.

Rosena: Belief about the Nature of Mathematics (IMAP Belief 1)

At the beginning and end of the year, Ms. Rosena's responses on the IMAP survey provide *weak evidence* that she holds the belief that mathematics is a web of interrelated concepts and procedures. In response to a scenario presenting multiple ways of solving a multidigit addition problem and asking which strategies she would like students to learn about, Ms. Rosena indicates that she would expose students to all five strategies at both data points. This suggests that Ms. Rosena perceives exploration of multiple strategies to be

Table 24
Ms. Rosena's IMAP Web-Based Beliefs-Survey Results

Belief	Beginning of year	End of year
Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).	1	1[a]
Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.	1	2
Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.	2	3
Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.	1	2[a]
Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.	1	3
Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.	2	2
Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.	0	3

Note. 0 = No evidence; 1 = Weak evidence; 2 = Moderate evidence; and 3 = Strong evidence.
[a] Evidence from classroom observations suggests that the end of year ratings underestimate Ms. Rosena's adherence to beliefs 1 and 4. Classroom observations provide *strong evidence* of these beliefs.

supportive of learning particular mathematical ideas, which implicitly implies a view of mathematics as interrelated. When prompted to justify her suggested ordering of strategies in a unit on multidigit addition, Ms. Rosena describes a progression that appears to start with the strategy she thinks students will grasp most readily to the strategy she thinks students will have the most difficulty understanding:

> Begin with manipulatives, then with easier rounding, then with a more complicated approach to rounding, then the "traditional algorithm" and finally a single strategy that does work with a student after the rest of the class understands the rest.

While this justification hints at mathematical concepts underlying the different strategies, Ms. Rosena does not demonstrate an interconnected view of mathematics by explicitly identifying connections among strategies.

The lack of focus on strategy-to-strategy connections in Ms. Rosena's survey response is not reflective of her instructional practice. Classroom observation reveals that Ms. Rosena does make the relationships among mathematical strategies a regular focus of class discussions. Therefore, Ms. Rosena's adherence to the belief that mathematics is an interrelated web of concepts and procedures appears to be underestimated by the IMAP instrument, especially at the end of the year.

Rosena: Belief about Distinction between Understanding Concepts and Applying Mathematical Procedures (IMAP Belief 2)

At the beginning of the year, there is *weak evidence* that Ms. Rosena holds the belief that one's knowledge of how to apply mathematical procedures does not necessarily go with understanding of underlying concepts. In response to instances of students using traditional U.S. algorithms, Ms. Rosena sometimes notes that students who use these algorithms may do this by rote. However, when asked to reveal what she is thinking about when judging understanding, Ms. Rosena indicates that she is thinking about whether a student has gotten a correct answer.

At the end of the year, Ms. Rosena's responses provide *moderate evidence* that she believes in a distinction between conceptual understanding and being able to apply mathematical procedures. At this data point, Ms. Rosena indicates clearly and consistently that many students use standard algorithms without understanding. Additionally, she suggests that students should only be permitted to use standard algorithms if they can explain how they work. However, as at the beginning of the year, Ms. Rosena continues to think about understanding as the ability to get correct answers. Taken together, these responses suggest that Ms. Rosena is sensitive to the possibility that students may not understand standard procedures, but this sensitivity would not necessarily extend to non-standard mathematical procedures.

Rosena: Belief about Source of Generative Mathematical Understanding (IMAP Belief 3)

At the beginning of the year, there is *moderate evidence* that Ms. Rosena holds the belief that understanding mathematics concepts is more powerful and more generative than remembering mathematical procedures. She clearly identifies that understanding concepts is important, but she also considers it important for her students to learn standard computational procedures because they are efficient. In response to a video clip of a teacher providing step-by-step procedural instruction on the fraction division algorithm, Ms. Rosena asserts that the student recipient of the instruction should successfully learn division of fractions given enough practice. Then, after observing a follow-up video in which the student is unsuccessful with a similar fraction division problem, Ms. Rosena indicates that the student would benefit from conceptual development with manipulatives. Overall, at the beginning of the year, Ms. Rosena appears to view conceptual understanding as more generative than procedural knowledge alone, but she also places a high value on knowing standard procedures with or without underlying conceptual understanding.

In contrast, at the end of the year, Ms. Rosena's responses indicate considerably less concern with whether or not students master standard algorithms. Instead, she insists that what is important is that students are able to explain why the standard or non-standard procedures they use to solve problems work. In comparing the standard U.S. subtraction algorithm and a non-standard subtraction method that is more conceptually explicit, Ms.

Rosena notes that students who use the standard procedure are likely to make more errors because the conceptual-basis of the procedure is not as clear. Taken together, Ms. Rosena's responses at the end of the year provide *strong evidence* that she believes understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Rosena: Belief about Consequences of Teaching Concepts/Procedures First (IMAP Belief 4)

At the beginning of the year, there is *weak evidence* that Ms. Rosena holds the beliefs that students are more likely to understand mathematical procedures if they first understand related mathematical concepts. Ms. Rosena consistently indicates a preference for using manipulatives during the early stages of instruction on a mathematical topic. However, she does not explicitly connect the use of manipulatives to development of conceptual understandings.

At the end of the year, Ms. Rosena's survey responses provide increased evidence (*moderate evidence*) of adherence to the belief that focusing on development of concepts prior to procedures increases the likelihood that students will understand procedures. Ms. Rosena indicates that she would order sharing of strategies in a unit on multidigit addition from most concrete (manipulatives) to most abstract (standard algorithm), and she justifies her inclusion of each strategy by noting its conceptual value. This response represents a more explicit connection to the role of conceptual understanding than was present at the beginning of the year.

At both data points, Ms. Rosena's responses to the scenarios used to evaluate this belief are sparse. Classroom observations reveal that Ms. Rosena places significant focus on development of conceptual understanding prior to focus on procedures, especially in Spring lessons. Therefore, Ms. Rosena's IMAP responses appear to underestimate her belief in the importance of teaching concepts prior to focus on procedures, especially at the end of the year.

Rosena: Belief about Children's Problem Solving Capabilities (IMAP Belief 5)

At the beginning of the year, there is *weak evidence* that Ms. Rosena holds the belief that children can solve problems in novel ways before being taught how to solve such problems. She indicates in some responses that, given tools like manipulatives, some students can solve some novel problems without instruction. However, she also indicates that most children will need teacher guidance to be able to solve mathematical problems. For instance, Ms. Rosena indicates that most first grade students would require teacher instruction to solve a contextualized multiplication problem.

In contrast, at the end of the year, Ms. Rosena asserts that children can devise novel solution strategies in response to problems, as long as they have conceptual tools available and their teachers give them the freedom to think about problems in their own ways. When asked if first graders could solve a contextualized multiplication problem at this data point, Ms. Rosena asserts that they could as long as they were permitted to use manipulatives or draw a picture of the situation. At the end of the year, Ms. Rosena's survey responses provide *strong evidence* that she believes children are capable of solving problems without being taught how to solve them.

Rosena: Belief about Children's Ways of Thinking about Mathematics (IMAP Belief 6)

At the beginning and end of the year, there is *moderate evidence* that Ms. Rosena believes that the ways children think about mathematics are generally different from the ways adults think about mathematics. For instance, at both data points, Ms. Rosena asserts that manipulatives and real-world contexts are supportive of students' initial thinking about mathematical topics. After viewing a video clip of a teacher providing step-by-step procedural instruction on fraction division at the beginning of the year, Ms. Rosena indicates that the teaching is too mechanical. She suggests that, instead of instruction focused on the mechanics of fraction division, the student would benefit from use of visuals to show how and why the fraction division procedure works. While Ms. Rosena's discussion of ways students can be supported in developing initial understanding of mathematics concepts suggests that she believes students think differently than adults, at neither data point does she explicitly mention the difficulty students have making sense of symbolic representations. Because her responses overlook this important difference between the mathematical thinking of children and adults, her overall adherence to this belief is not considered strong.

Rosena: Belief about Teacher's Role in Supporting Student Learning (IMAP Belief 7)

At the beginning of the year, there is no evidence that Ms. Rosena believes that the teacher should allow children to do as much thinking as possible during mathematics instruction. In her own classroom practice, Ms. Rosena explains that she always shows students exactly how to solve mathematical problems and then guides students to solve problems by following her model. She indicates the belief that, if she did not provide explicit instruction on how to solve problems, most students would not know where to start.

In contrast, Ms. Rosena's IMAP responses at the end of the year suggest a complete turn-around in this belief. In discussing why she now asks students to solve problems before providing explicit instruction on how to solve them, Ms. Rosena states the following:

> I now know better. I can see how I was influencing their thinking when I was giving them the way to solve problems. After I tried asking them to come up with a way to solve the problem, I was able to do different things: evaluate how much they really know, what would be the problems I will have with the lessons ahead, and have them understand that there are other ways besides theirs, to come up with the solution.

At the end of the year, Ms. Rosena indicates that teachers should avoid presenting particular problem solving strategies until students have been given the freedom to solve problems in their own ways. Therefore, Ms. Rosena's responses at the end of the year provide *strong evidence* that she believes that the teacher's role is to orchestrate learning such that students do as much mathematical thinking as possible.

Summary of Ms. Rosena's Beliefs

At the beginning of the year, Ms. Rosena's responses on the IMAP web-based survey suggest that she does not strongly adhere to any of the seven reform-oriented beliefs measured. There is no evidence that Ms. Rosena believes teachers should allow children to do as much thinking as possible during mathematics instruction, and there is only weak evidence that she believes children can devise novel solutions to mathematics problems without being shown how to solve them. Ms. Rosena asserts that conceptual understanding helps students to understand mathematics more than simply remembering procedures. She values use of manipulatives and real-world contexts to support children's initial thinking about mathematics topics. However, Ms. Rosena also places significant value in student mastery of standard algorithms and indicates that she is mainly looking for students to achieve correct answers when she thinks about "understanding."

By the end of the year, there is evidence that Ms. Rosena's beliefs have shifted significantly toward a reform-orientation. Most notably, at the end of the year, there is strong evidence that Ms. Rosena believes students are capable of solving problems in novel ways without explicit teacher instruction on how to solve such problems. Reflecting this change, Ms. Rosena indicates that the teacher's role is one of supporting students by allowing them to do as much thinking as possible during interactions related to mathematics instruction. In particular, Ms. Rosena asserts that students should be given opportunities to devise their own ways of solving problems before being shown particular strategies by the teacher. Furthermore, there is greater evidence at the end of the year that Ms. Rosena believes that understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Ms. Rosena's Knowledge of Mathematics for Teaching

Ms. Rosena's knowledge of mathematics for teaching with a reform-orientation was measured at the beginning and end of the year using data from two parts of the Teacher Knowledge interview. In particular, this interview explores teachers' knowledge related to the teaching of multiplication and division to third-grade students. First, Ms. Rosena's open-ended discussion of teaching and learning multiplication will be presented. In this part of the interview, teachers were prompted to describe important understandings and common student difficulties related to the learning of multiplication. In addition, interview questions direct teachers to discuss how children develop the important understandings identified and how multiplication should be taught. Next, Ms. Rosena's responses to four classroom scenarios will be described. These classroom scenarios probe specific aspects of teachers' knowledge related to the teaching and learning of multiplication and division, specifically teachers' knowledge of nonstandard strategies for multidigit multiplication, knowledge of a common student difficulty underlying the standard U.S. algorithm and strategies for addressing this difficulty, and teachers' abilities to interpret and respond to student work.

Rosena: Open-ended Discussion of Teaching and Learning Multiplication

At the beginning of the year, Ms. Rosena reports that third-grade instruction on multiplication should help students to see a picture for a given fact in their mind's eye, and then it should support students with memorizing the basic multiplication facts. She asserts that this is accomplished instructionally by allowing ample time for students to work with a variety of manipulatives to build physical models of multiplication. Then instruction should turn to practice of multiplication facts through drill and games. Ms. Rosena suggests that students' difficulties with multiplication most often result from attempts to memorize facts before being able to "see" a picture of multiplication. She repeatedly stresses the importance of giving students the time to "mess around" with materials to internalize the conceptual picture of multiplication. Ms. Rosena identifies knowledge of addition as a prerequisite to learning multiplication.

At the end of the year, Ms. Rosena identifies several mathematical ideas that students should come to understand as they study multiplication in third grade: that multiplication is a group of things repeated a certain number of times, the relationship between multiplication and addition, the relationship between multiplication and division, and the relationship between real-world situations and visual and symbolic representations of multiplication. In addition, Ms. Rosena indicates that it is important for students to develop strategies for solving multiplication problems that move beyond direct modeling and repeated addition, but the only alternative strategy suggested is recall.

In order to develop students' understanding of multiplication, Ms. Rosena asserts that students should first work to make physical models of real-world multiplication situations to which they can relate. Then the idea of multiplication and multiplication notation should be developed through a focus on these situations and models. As students use their models to find products, Ms. Rosena suggests that instruction should encourage students to move beyond counting by ones to using more efficient count-by and repeated addition strategies. As students become more comfortable with these strategies, they can begin to solve problems without manipulatives because they are able to "see" a multiplication picture in their heads. At the same time, Ms. Rosena indicates that students should be exposed to division situations, and multiplication should be stressed as a viable strategy for solving division problems. Ms. Rosena indicates that the next step is for students to engage in ample practice, through drill and games, to memorize the basic multiplication facts. As students begin to recall some of their multiplication facts, they are ready to begin work on multidigit multiplication, using the facts they know to find solutions to these problems with larger numbers. Ms. Rosena indicates that students' primary difficulties with multiplication instruction involve moving from thinking of multiplicative situations in terms of multiplication rather than addition. Finally, Ms. Rosena identifies knowledge of addition as a prerequisite to the study of multiplication.

Rosena: Knowledge of Non-standard Strategies (Classroom Scenario 1)

At the beginning of the year, Ms. Rosena identifies six strategies that students might use to solve a word problem involving finding the number of chairs in 16 rows of 8 chairs. These strategies are presented in Table 25. Of the six strategies listed by Ms. Rosena, four are variations of direct modeling. Ms. Rosena also lists a repeated addition strategy and the standard U.S. algorithm for multidigit multiplication. At the beginning of the year, Ms. Rosena does not include a partitioning or compensating strategy in her list of strategy possibilities. However, some of her direct modeling strategies are clearly related to partitioning strategies. Additionally, Ms. Rosena demonstrates knowledge of a particular partitioning strategy in response to Classroom Scenario 3 in this interview.

In discussing students' progression through the strategies listed at the beginning of the year, Ms. Rosena asserts that students are first able to model problems with manipulatives or pictures. Then she asserts that students initially find products by counting objects one at a time. Next Ms. Rosena indicates that students begin to find quicker ways to count their models, such as repeated addition or by making models that involve some grouping. For instance, to find the total of 16 rows of 8 chairs, they might make two sections of 8 rows of 8 chairs. Finally, Ms. Rosena identifies use of the standard U.S. multiplication algorithm as the most sophisticated approach that students learn after they have a sense of what multiplication "looks like."

At the end of the year, Ms. Rosena identifies five strategies that students might use to approach the word problem involving finding the total number of chairs in 16 rows of 8 chairs. At this data point, Ms. Rosena identifies one direct modeling strategy, the standard U.S. multiplication algorithm, and three partitioning strategies. While Ms. Rosena does not

Table 25
Strategies Identified by Ms. Rosena in Response to Classroom Scenario 1

Data point	Direct modeling	Complete number	Partitioning number	Compensating	Other
Beginning of year	16 rows of 8 circles 8 rows of 16 circles 2 “sections” of 8 rows of 8 circles 4 “sections” of 4 rows of 8	Repeated addition, 8 + 8 + 8…			Standard US multiplication algorithm
End of year	16 rows of 8 dots		8 × 8 = 64, 64 + 64 = 128 8 × 4 = 32, 32 + 32 = 64, 64 + 64 = 128 10 × 8 = 80, 6 × 8 = 48, 80 + 48 = 128		Standard US multiplication algorithm

Note. Analysis utilizes strategy classification presented in Baek (1998).

include any complete number strategies at this data point, she demonstrates flexible understanding of how the more mathematically sophisticated partitioning strategies can be used by students. This shows much growth in knowledge from the beginning of the year, when no partitioning strategies were included in her response. Also, a repeated addition strategy (from the complete number category) is included in Ms. Rosena’s discussion of students’ progression through strategies at this data point, so there is evidence that she is aware of this strategy type. As was the case at the beginning of the year, Ms. Rosena does not identify a compensating strategy at this data point.

As at the beginning of the year, Ms. Rosena asserts that students initially use direct modeling with objects or pictures to represent multiplication situations and they count objects by one to determine the totals. Next students begin to count objects more efficiently, using repeated addition or groupings. Ms. Rosena suggests that this leads to use of "halving and doubling" strategies (a type of partitioning strategy) in which students realize they can count half of the group and then double their result to find the total product. After students have exposure to this strategy, Ms. Rosena indicates that students begin to consider other partitioning strategies that break groups apart in different ways. As students become able to use partitioning strategies without the support of visual models, Ms. Rosena suggests that they are ready to make sense of how the standard U.S. multiplication algorithm is similar to partitioning strategies focused on partitioning factors into their expanded form.

In summary, Ms. Rosena's response to Classroom Scenario 1 provides evidence of moderate knowledge of non-standard strategies for multidigit multiplication at both data points, with greater knowledge of partitioning strategies demonstrated at the end of the year. Also, Ms. Rosena's end-of-year interview suggests that her knowledge of how students move from using models to number-only procedures with understanding, including the standard U.S. algorithm, has increased significantly. Furthermore, Ms. Rosena demonstrates strong understanding of the interrelationships among the strategies listed at the end of the year.

Rosena: Interpretation of Students' Mathematical Strategies (Classroom Scenario 2)

At the beginning and end-of-year data points, Ms. Rosena is able to quickly interpret the three student work samples representing students' strategies for solving the pancake problem (Twenty-four children want to share eight pancakes so that each one gets the same amount. How much pancake can each child have?). She describes plausible theories for how each child approached the task, indicates what the strategies suggest about each child's mathematics understanding, and identifies questions she might ask each student in response to their work. For instance, after a brief glance at one student's strategy at the beginning of the year, Ms. Rosena makes the following assessment:

> Nicholas divided the pancakes into threes. I guess these are the pancakes. Yea, because there are eight pancakes here. I don't know why he put the 24 people here without doing any kind of grouping, I just think he just drew them. He might be able to solve this problem if he counts every part of each pancake. If he gets 24 parts, then he might solve the problem. Now, I don't know if he would say that every person gets a third of a pancake. I don't know if he would actually be able to use that kind of vocabulary.

When prompted to identify questions she would ask this student, Ms. Rosena indicates that she would ask Nicholas to describe how he decided to make the parts of his representation, and she would ask him how much pancake each child gets. In summary, Ms. Rosena's response to Classroom Scenario 2 provides evidence of strong ability to interpret students' mathematical strategies at both data points.

Rosena: Addressing and Avoiding a Common Student Error (Classroom Scenario 3)

In this classroom scenario, teachers are presented with student work in which the standard U.S. multiplication algorithm is executed without maintaining the place values of the partial products. At the beginning and end of the year, Ms. Rosena quickly identifies that this student error suggests a lack of recognition regarding how place value affects this algorithm. She suggests that a way to address this problem is to teach students the partial products algorithm, which makes the role of place value in multidigit multiplication more explicit. At the beginning of the year, Ms. Rosena discloses that when she taught fifth grade, her students made the kind of error presented in the scenario and that teaching the partial products approach proved helpful.

When asked what third-grade teachers should do to keep students from making these kinds of errors, Ms. Rosena's responses at the beginning and end of the year vary. At the beginning of the year, she says that she is not sure how she could help students avoid this problem as a third-grade teacher. At the end of the year, however, Ms. Rosena responds adamantly. She identifies the key to helping students avoid place value errors when performing standard algorithms is to be continuously attentive to developing students' understanding of place value within and outside mathematical operations. Reflecting on her work with various math programs, Ms. Rosena makes the following statement:

> That is a mistake that I think a lot of [textbook] series did is that they taught place value at the very beginning of the year and never touched on it again or taught it again or applied it to other units and exercises. Even though they really don't talk about place value [in *Everyday Mathematics*], they are using exercises that incorporate place value throughout the book, throughout the year.

Through her work with *Everyday Mathematics*, Ms. Rosena appears to have developed a vision of how place value can be emphasized alongside her teaching of mathematical operations.

In summary, Ms. Rosena's response to Classroom Scenario 3 provides evidence that she understands the mathematical basis of students' difficulties with the standard U.S. multiplication algorithm at both data points. Furthermore, at both data points, Ms. Rosena has ideas about how to use the partial products method to illuminate the relationship between place value in relation to multidigit multiplication. While at the beginning of the year Ms. Rosena struggles with the question of how she can help students avoid the errors presented in this scenario, at the end of the year she suggests that place value can be woven throughout study of mathematical operations.

Rosena: Interpretation of and Response to a Student's Flawed Solution (Classroom Scenario 4)

At the beginning and end of the year, Ms. Rosena is unable to determine the mathematical basis of the flawed strategy presented in classroom scenario 4. In this scenario, a student explains that the answer to $144 \div 8 = ?$ can be found by dividing 144 by 2 four times because $2 + 2 + 2 + 2 = 8$. When prompted to discuss how she would respond to this student, Ms. Rosena indicates that she is not sure at the beginning of the year. In contrast, at the end of the year, Ms. Rosena suggests that she would encourage the student to compare her (flawed) solution with a classmate, hoping that this would help the student discover her own error. Then, if the student did not revise her solution, Ms. Rosena suggests that she would lead the class to think about the (flawed) solution during group discussion. In summary, Ms. Rosena's responses to this classroom scenario provide evidence that uncovering the mathematical basis of flawed solutions may be difficult for her. Nonetheless, Ms. Rosena's end-of-year response suggests a greater knowledge of pedagogical strategies that she could use when a student has a flawed solution.

Summary of Ms. Rosena's Knowledge of Mathematics for Teaching

Ms. Rosena demonstrates moderate knowledge of students' strategies for multidigit multiplication at the beginning and end of the year, with greater knowledge of partitioning strategies demonstrated at the end of the year. At neither data point does Ms. Rosena identify a compensating strategy. In describing student learning trajectories related to multiplication, Ms. Rosena displays a more thorough understanding of how students develop understanding of increasingly sophisticated strategies at the end of the year. In particular, she describes the cognitive resources students must develop to move from using visual models to working with symbols. Overall, Ms. Rosena's personal knowledge of multiplication concepts and strategies increases over the course of the year, with evidence of stronger and more interconnected knowledge at the end of the year than at the beginning.

At the beginning and end of the year, Ms. Rosena is skillful at making sense of students' novel strategies for solving an equal-sharing division problem that results in a fraction. She is able to quickly make conjectures regarding how students approached the problem, suggest what their strategies indicate about their mathematical understanding, and identify questions she might ask the students to follow-up on her conjectures.

While Ms. Rosena's ability to interpret correct non-standard solutions appears strong at both data points, evidence suggests that her ability to make sense of and respond to students' flawed solutions is more conditional. When presented with student work displaying a common mistake with the standard U.S. multidigit multiplication algorithm, Ms. Rosena is able to describe the mathematical basis of the mistake and suggest ways to support students in developing conceptually-grounded conceptions. However, only at the end of the year is Ms. Rosena able to describe how, as a third-grade teacher, she can help students to avoid making the kind of error in the scenario. In response to a scenario presenting an unfamiliar flawed solution to a division problem, Ms. Rosena is unable to determine the mathematical basis of the error at both data points. Furthermore, at the beginning of the year she is unsure of how she would respond to this student in her classroom. In contrast, by the end of the year, Ms. Rosena is confident that in the classroom context she could support the student in figuring out the error by encouraging peer collaboration and by discussing the (flawed) solution with the class.

Case Story of Ms. Rosena's Response to Student Difficulties

In the case story that follows, three facets of Ms. Rosena's typical response to student difficulties will be discussed. Specifically, Ms. Rosena's treatment of flawed solutions for problem solving tasks, her response to student difficulties with word problems during whole class discussion, and her strategies for supporting struggling students during work time will be described. Following illustration of each response pattern, consequences for student thinking and the relationship between Ms. Rosena's beliefs, knowledge, and mathematics teaching practices will be explored.

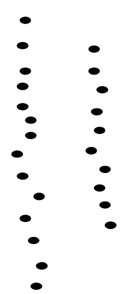
Rosena Response Pattern 1: Treatment of Flawed Solutions for Problem Solving Tasks

Ms. Rosena's primary method of addressing students' flawed solutions for problem solving tasks is consistent through all observations. Although Ms. Rosena's post-observation interviews reveal that she identifies students' flawed solutions as they are working on tasks, she often delays responding to errors until students are sharing task solutions in a whole group discussion. Within the whole group discussion time, Ms. Rosena regularly includes discussion of flawed solutions alongside discussion of correct solutions.

The following classroom episode is representative of Ms. Rosena's treatment of flawed solutions in the context of a whole group discussion. This episode takes place during a Fall lesson focused on introducing arrays. After some exploration, the class engages in a lengthy discussion around solutions to the task, "There are 24 children in the class. Use the counters to represent the children. Arrange the counters to show them in equal rows." Each student devises a solution with counters and makes a drawing of the solution on a white board. Then each student's white board is exchanged with that of another student. Ms. Rosena tells students to take a look at the partner's white board and try to understand how the partner was thinking about the problem. After a few moments, Ms. Rosena says, "Thumbs up if you understand. I'm not saying if its right or wrong, if you understand what the person did, thumbs up." Although a few students indicate that they understand their partner's solution, several students indicate that they are not sure what their partner did. Some of the students further indicate that they think the partner's solution is incorrect. Next each student's solution is discussed in turn, with ample attention given to those solutions that are incorrect.

The episode that follows presents the whole class discussion of Jeremy's solution, which is one of three incorrect solutions discussed. Jeremy's white board shows 2 columns of dots, with 14 dots in the first column and 10 dots in the second column. The dots are placed such that the columns are lined up but the rows are not clear. (See illustration below.) The discussion of Jeremy's solution begins with Kamal, the recipient of Jeremy's white board, talking about his understanding of Jeremy's work.

Jeremy's work:



Ms. Rosena: Kamal, you did not understand what Jeremy did?
Kamal: Yeah.
Ms. Rosena: Okay. Can you tell me or ask him to explain to the class what he did.
Kamal: Because he didn't have an equal row…
Ms. Rosena: He didn't have equal rows?
A few students: Yeah; no.
Ms. Rosena: Where do you see…why do you say that he doesn't have equal rows? I want Christy and Tanya to pay attention to this.

Kamal: The first one, he has I don't know how much. But the second one he has…

Ms. Rosena: Okay. Explain that once more.

Kamal: The first one, I don't know how much he got in it.

Ms. Rosena: Yeah, me neither. Because the first one is up here and the second one down here. So, in the first one (*pointing at the first row*) there is only one, I think. (*Pause*) Jeremy, can you explain to the class what you did here. But, before you do that, are there 24 dots in here?

Kamal: Yeah

Ms. Rosena: So, he does have 24. Jeremy, can you explain to the class how you came up with this arrangement.

Jeremy: Well, I put one row, I mean one column of 14 dots and another column of 10.

Ms. Rosena: Okay, so this 14? (*Indicates the first column, Jeremy nods*) And this is 10? (*Indicates the second column, Jeremy nods*) And that is 24. (*Pause*) But my question was to show the 24 children in equal rows. What does this mean to you - <u>equal</u> rows? What does this mean?

Jeremy: Like, if you have two… (*other children chiming in answers*)

Ms. Rosena: Shhh.

Jeremy: If you have two, you need to put two blacks on the other side. (*Other students are agreeing.*)

Ms. Rosena: If you have two on this row *(indicates first row*), you must have two on the second row, you must have two on the next row, etcetera. Did you do that?

Jeremy: No.

Ms. Rosena: Okay.

Jeremy: But if I have 14 and 14, that would be…28.

Ms. Rosena: If you have 14 and 14, that would be 28. Okay. Is there a way that you can arrange this to make 24? To have equal rows making 24 in total?

At this point, Ms. Rosena directs the class to think about how many rows of 2 children make 24 children total. One student tentatively suggests 12 rows. Ms. Rosena replies, "Let's see," and guides the class to count by twos as she draws a model on the board. After the class is convinced that 24 children can be arranged in 12 rows of 2, Ms. Rosena remarks, "So Jeremy, if you wanted to do two in each row, you needed to have 12 rows."

In this classroom episode, Ms. Rosena first has Kamal try to make sense of Jeremy's work, thus giving the class the opportunity to study the work too. Then Ms. Rosena guides Kamal and the rest of the class to understand the correct aspects of Jeremy's flawed solution (that there are 24 children represented) as well as the error (that they are not placed in equal rows). Next Ms. Rosena interrogates Jeremy's understanding of the concept "equal" as it relates to this situation. Through doing this, Ms. Rosena emphasizes the mathematical idea of equal groups that is central to understanding multiplication and division. Finally, Ms. Rosena shows Jeremy, with the help of Kamal, how to formulate a correct solution using pieces of Jeremy's initial solution. Namely, they work to figure out how many rows of 2 are needed to make 24.

In summary, Ms. Rosena's primary approach to addressing students' flawed solutions is to make these flawed solutions a focus of whole class discussion. In whole class discussion, Ms. Rosena guides students to understand and evaluate each other's solutions, and through this process, errors are identified. After an error is identified, Ms. Rosena leads students to see the underlying logic or misconception in the students' flawed solution. Often, correct pieces of a given solution are also identified. Finally, Ms. Rosena focuses the class on figuring out how to revise the solution to a correct one.

Rosena response pattern 1, consequences for student thinking. As the year progresses, Ms. Rosena's students appear to increasingly view mistakes as a part of the learning process. They do not seem uncomfortable or unhappy when their flawed solutions are shared. Quite the opposite, students appear motivated to understand correct and incorrect solutions. Especially in the Spring observations, there is a notable sense of collaboration among the students. It is not uncommon for students to comment on another student's solution strategy without being prompted by the teacher. When asked about this increase in participation and student engagement, Ms. Rosena explains that she attributes the change to students' increased confidence and mathematical understanding:

> The feeling of "I get this." I understand how to do it. And I don't think they are focused that much on getting it right, but on how to get it. The process itself, and not

> the result. They have realized that they can go and find an answer in so many different ways and still find the right answer. Previously, the books, and the curriculum, and the teacher – us – we were focused on one, maybe two, ways of getting the answer. And if this student didn't understand that way, he got stuck and was not going to get involved. I don't see that happening now.

As further evidence of Ms. Rosena's claim of stronger student understanding, students are observed making significantly fewer errors in Spring math lessons when compared with the Fall. This is an especially notable finding considering that this is a class of retained students, with a history of school failure. Ms. Rosena's practice of making students' errors a focus of instruction appears to be a contributing factor to this positive change.

Rosena response pattern 1, link to beliefs and knowledge. Ms. Rosena's intentional practice of making students' flawed solutions a focus of whole class discussion is most related to her beliefs. Ms. Rosena's reasonably robust knowledge of mathematics for teaching also supports her ability to orchestrate discussion of flawed solutions in ways that emphasize conceptual understanding and honor student thinking. At the end of the year, Ms. Rosena explains that her treatment of students' flawed solutions is a deliberate teaching practice reflecting beliefs colored by her own school experience:

> When I was in school, I was not good in math. So, the teachers were always showing off the people who did the right thing and were right. But they never, for example, they would never take my way of solving the problem and explain to me why it was wrong. So, I never really quite got how to do it right. So, I think that having them share how did they come up with an answer, even when it is wrong…I think that makes it so I can help them understand what it is they were doing wrong so they can make it right in the future. I think that we are focusing on what students are doing right and sometimes not enough on what they are doing wrong.

Ms. Rosena believes deeply that students' errors need to be confronted directly and, after the errors are identified, that they should be revised into correct solutions. Post-observation interviews reveal that Ms. Rosena focuses on particular flawed solutions in the whole group discussion because she believes the mathematical lesson of the error is beneficial for the given student and the rest of the class. She takes the view that there is much for all of her students to learn from their own mistakes as well as the mistakes made by peers.

Contributing to her belief in the value of studying errors, Ms. Rosena views students' flawed solutions as partially correct solutions with underlying logic. This is in contrast to a view of flawed solutions as the result of careless mistakes or unfounded guesses. This belief drives Ms. Rosena to dig into students' incorrect solution strategies to understand where they went wrong, and as in the case of Jeremy, if the error is grounded in a misunderstanding of a fundamental mathematics concept. Although pre-observation interviews reveal that Ms. Rosena's ability to anticipate the difficulties students will have is limited, her interactions with students during instruction demonstrate that she can draw on her personal mathematical knowledge in real-time to make sense of students' difficulties. Ms. Rosena's ability to do this stems from knowledge of key mathematical ideas related to multiplication and division as well as the pedagogical knowledge of how to ask questions that probe students' mathematical thinking.

As Ms. Rosena observes her students successfully completing tasks without explicit instruction, her beliefs regarding students' capabilities and her role in instruction begin to shift. By the end of the year, there is strong evidence that Ms. Rosena believes students are capable of solving instructionally-appropriate novel problems as long as they have access to conceptual tools like manipulatives and the freedom to think about the problems in their own ways. Furthermore, Ms. Rosena expresses her newly formed commitment to having students solve problems without explicit direction. The catalyst for Ms. Rosena's change in beliefs is her observations of her students being successful with solving novel problems.

Rosena Response Pattern 2: Response to Student Difficulties with Word Problems in Whole Class Discussion

When flawed solutions to a word problem become the focus of class discussion, one way that Ms. Rosena supports her students in understanding and revising the flawed solution is by directing students' attention on the problem context. To illustrate this point, consider a classroom episode in which students are working on the partitive division problem, "28 pennies are shared equally by 4 children. How many pennies per child is that?" Ms. Rosena observes multiple students in the class attempting to solve this problem by counting out 28 counters and making groups of four until all of the counters are used. This strategy reflects a measurement division approach to dividing 28 by 4, but it does not match this partitive division context. Ms. Rosena determines that students are approaching the problem procedurally, without attention to context. She decides to use the whole class discussion time to press her students to consider the problem context in relation to Linda's flawed solution:

> Ms. Rosena: Okay, the first thing that Linda did…I am going to share this with the class, because many of you started doing exactly the same thing. She made groups of four in each. Okay…for example (*walks toward the board*)…This is what she did. (*Draws on board, drawing below*) 1, 2, 3, 4…1, 2, 3, 4…1, 2, 3, 4…1, 2, 3, 4…1, 2, 3, 4…1, 2, 3, 4…1, 2, 3, 4. 28.
>
> *Ms. Rosena's drawing of Linda's work*:
>
> O O   O O   O O   O O   O O   O O   O O
> O O   O O   O O   O O   O O   O O   O O
>
> Raul: (*Calling out*) What?
> Ms. Rosena: You do have 28 pennies here.
> Mariluz: But not four kids.
> Ms. Rosena: There are four in each.
> Jeremy: But, its almost like…
> Ms. Rosena: Four comes from this problem.
> Raul: There's a big problem.

Ms. Rosena: What?
Raul: There's a problem.
Ms. Rosena: What's the problem?
Raul: Well, you didn't have four children.
Mariluz: You should have put five in each one.
Kamal: You didn't have 16 children.
Ms. Rosena: Are there 16 children here?
Kamal: No. You have seven children.
Ms. Rosena: (*pointing at the drawing on the board*) How many children do I have here?
Several students: Seven.
Ms. Rosena: Each group…you're telling me that each group represents a child?
Several students: Yes; Yeah
Ms. Rosena: Okay. (*Speaking to Linda*) Is that what you were thinking, that each group is one kid? (*Linda indicates yes with a head nod.*) But there are only four kids [in the problem], Linda. (*Pause*) …So, we need to concentrate on four kids only. (*Ms. Rosena puts a partition between the groups to partition off four groups.*)

On the board:

```
O O   O O   O O   O O | O O   O O   O O
O O   O O   O O   O O | O O   O O   O O
```

Ms. Rosena: Now what are we going to do with these [pennies on the right]?
Kamal: Throw them out.
A few students: Nooo…
Ms. Rosena: If I throw them out, I won't have 28 pennies anymore. That is what I was trying to tell Tanya.
Tom: You need to put those (*pointing to the right*), with, those with, umm, the others (*pointing to the left*).

As the classroom episode continues, a student piggybacks on Tom's comment by suggesting that they put three pennies from the right of the partition with each of the four groups on the left of the partition, making four groups of seven pennies.

Throughout this interaction with students, Ms. Rosena emphasizes the problem context. First she plays devil's advocate to make the point that while the numbers from the problem, 28 and 4, are both represented in Linda's drawing, the drawing does not reflect the problem context of 28 pennies divided among four children. Then she asks students to make suggestions about what Linda's drawing might represent, providing a basis for moving forward with a revised solution. When Kamal suggests that a solution to the dilemma is to get rid of three groups of pennies, Ms. Rosena rejects that suggestion on the basis that it does not match the problem context of 28 pennies.

In summary, when flawed solutions to word problems are the focus of class discussion, Ms. Rosena often helps her students to make sense of the problems by pressing them to consider the relationship between the flawed solution and the problem context.

Rosena response pattern 2, consequences for student thinking. The primary consequence of Ms. Rosena's instructional practice of emphasizing context is that her students come to make sense of problem solving situations in light of their context. Throughout the year, students become increasingly proficient at identifying important information in a problem situation, modeling problem situations, and interpreting models developed by other students. When working with computation problems that are not in context, students sometimes invent a context to illustrate how to go about devising a solution.

Rosena response pattern 2, link to beliefs and knowledge. Ms. Rosena's practice of drawing student attention to problem context during class discussion of flawed solutions to word problems is primarily related to her beliefs, but is supported by her knowledge. In discussing use of real-world problems at the beginning of the school year, Ms. Rosena explains that, "It's easier to understand [mathematics] concepts when you are pulling from the real world." Aligning with advocates of mathematics reform, Ms. Rosena believes that real-world situations to which her students can relate can serve as an instructional bridge between lived experiences and understanding of mathematical concepts. Therefore, she presses her students to make sense of the contexts of word problems and other contextualized tasks as a way to move students closer to understanding mathematics. In part, Ms. Rosena's emphasis on pressing students to understand the context of word problems is a reaction to her own experiences in school:

> I remember when I was growing up I never got word problems. I always got them wrong. I never understood them, and very little emphasis was placed on them…I think word problems are important, especially if they are related to things they know…when you see 3 times 5 equals something, and you don't add any meaning to it except three groups of five, it doesn't, I don't know, it doesn't really make too much sense to me as [compared to] when you have it in a real situation. Three rows of five chairs, oh, okay, now I understand…

Ms. Rosena strongly holds the belief that mathematical ideas are more meaningful to students when students understand them in relation to real-world situations.

Although Ms. Rosena's practice of focusing students' attention on problem context is more clearly a reflection of her beliefs than knowledge, Ms. Rosena's knowledge of mathematics for teaching supports her efforts to emphasize context effectively. First, Ms. Rosena's personal ways of understanding mathematics include relating mathematical ideas to real-world contexts. Consequently, she is able to readily do this in the context of real-time instruction. Second, Ms. Rosena's pedagogical content knowledge facilitates her ability to make sense of a student's flawed solution and determine if the basis of the error is related to a lack of attention to context. Then Ms. Rosena is able to draw on her pedagogical knowledge to formulate questions that guide students to consider and revise the flawed solution in relation to the problem context.

In summary, Ms. Rosena's beliefs seem most related to her pattern of emphasizing problem contexts in efforts to support revision of flawed solutions to word problems in whole class discussion. And her knowledge of mathematics for teaching is instrumental in facilitating effective use of this strategy.

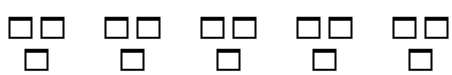
Rosena Response Pattern 3: Strategies for Supporting Struggling Students During Work Time

As has been previously discussed, Ms. Rosena primarily interacts with students around their mathematical difficulties during the class discussion phase of instruction. As students initially work on mathematics tasks, Ms. Rosena is most often observed standing away from her students, keeping a watchful eye on what is going on. When she observes students sitting idle or engaged in other off-task behavior, she provides quick and clear directives demanding that the students work on the task at hand. Ms. Rosena views this general practice as a way to encourage student autonomy and as a means of classroom management:

> When I go around the room, especially with this group, and I stand by a student, that student stops doing what he's doing or she's doing…and he's always looking up for reinforcement. Like Kamal, for example, he doesn't like to work by himself. If I am standing away from him, he will start working. But if I am standing next to him, he will constantly look to me, "Is this right? Is this right?" He is trying to get me to say that he is doing good…I think that by being in the front or in the back or on the side…not walking around…I can catch things more easily and they know I am watching. I've seen if I stand by a student and start working with them, some others will start looking around and stop working.

Although Ms. Rosena most typically avoids interactions with students during their initial work on a task, there are times when individuals in the class seem unable to move forward with devising a solution to a mathematics task and Ms. Rosena determines that help is warranted. Ms. Rosena's approach to addressing students' difficulties in these instances differs at the Fall and Spring observation points. Her response patterns for addressing persistent student difficulties in the Fall will be described next followed by discussion of response patterns observed in the Spring.

In the following excerpt from a Fall lesson, Ms. Rosena approaches a student, Tanya, who is sitting idle several seconds after her classmates have begun working on a mathematics task. The class is working on the partitive division problem, “15 pennies are shared equally by 4 children. How many pennies per child is that?” In this interchange between Ms. Rosena and Tanya, Ms. Rosena directs Tanya to solve the problem is a particular way, rather than asking questions to help Tanya devise her own strategy:

> Ms. Rosena: Use these [cubes] 4, 5, and 5. You have 15 in total. Now you know that we have four children, right? So, make four groups. (*Tanya places 4 cubes on the white board.*) 1, 2, 3, 4. Now divide these equally among that. One for you, one for you, one for you, one for you. (*Tanya begins distributing the cubes one at a time to each group.*) Start always with the same one, okay? (*Tanya continues distributing cubes.*) Okay, well, take these too.
>
> *Tanya’s model of the problem:*
>
> 
>
> Ms. Rosena: How many does each one have?
> Tanya: Three.
> Ms. Rosena: Okay, and three leftover, right? Fifteen pennies and four kids, three left over. Almost there. Where are the kids? Show me the kids. Circle the groups to represent the kids. (*Tanya circles one set of three cubes, which are sitting on a white board.*) One child. That’s one child. That’s another child. How many circles are you supposed to have?
> Tanya: Five.
> Ms. Rosena: Why five? Does it say five up there [in the problem]?
> Tanya: Four.
> Ms. Rosena: So, circle one more and don’t do anything with those [the last set of three cubes that are the left overs].

Ms. Rosena first tells Tanya how to go about modeling the problem. Although Ms. Rosena does draw Tanya’s attention to the context of the problem, her directions are primarily procedural in nature. After Tanya has successfully modeled the problem, Tanya’s answers to Ms. Rosena's questions reveal that she does not fully understand the model. Instead of pressing Tanya to make sense of the model, Ms. Rosena again provides procedural directions.

While the number of times in which Ms. Rosena engages in sustained dialogue with students during work time in Fall lessons is limited, her typical routines for addressing students' difficulties in these instances involves providing strong direction that is procedural in nature. This kind of support tends to decrease the cognitive demand of the problem solving task for students.

In the Spring observations, Ms. Rosena's teaching practice of providing strong direction to individuals having difficulty during work time is replaced by alternative teaching strategies. In fact, the Spring observations contain no instances of Ms. Rosena providing direct support to individuals that decreases the cognitive demand of tasks posed. This change can be attributed in part to the fact that, by this point in the year, Ms. Rosena's students have adopted learning practices consistent with trying to make sense of problems. Students are observed re-reading problems, making note of key information in problems, and trying to make visual models to support their thinking. Nonetheless, there are still instances when individual students appear unsure of how to move forward with a mathematics task and Ms. Rosena determines that teacher support is warranted.

One strategy Ms. Rosena uses to address individual students' difficulties in Spring observations is to provide additional guidance to the whole class related to understanding a particular task. Consider the following classroom episode in which students are working on the problem, "Could 6 harp seals weigh more than 1 ton? Could they weigh less than 1 ton?" To solve this problem, students must use a chart that provides a range of weights for different animals. The chart indicates that harp seals weigh 200 – 396 pounds. In order to successfully solve this problem, students also need to know the relationship between pounds and tons. When Ms. Rosena observes individual students having difficulty getting started, she responds by asking the whole class questions about the relationship between pounds and tons:

> Ms. Rosena: Do you remember what a ton was?
> A few students: Two-hundred pounds.
> Ms. Rosena: Two-*hundred* pounds?
> Raul: Two-thousand.
> Other students: Two-thousand.
> Ms. Rosena: Two-thousand what?
> Several students: Pounds.

In this instance, Ms. Rosena observes individual students struggling with the task, and she conjectures that their struggles may be related to difficulty recalling the relationship between pounds and tons. Through a brief series of questions with the whole class, Ms. Rosena facilitates the process of allowing students to help each other overcome difficulties. When Ms. Rosena becomes satisfied that her students have the information necessary to formulate a solution, she steps back and observes the group.

Another strategy Ms. Rosena uses in Spring lessons to address students' difficulties is to express her own confidence in their abilities to solve the mathematics task of focus using what they already know. To illustrate this point, consider the classroom excerpt below in which Mariluz is having difficulty getting started on the following problem[10]: 6[80]. At first blush, 6[80] may seem like a simple computation task; however, this problem is situated in the first lesson students have had in which they have been asked to solve multidigit multiplication problems. Prior to work on the problem 6[80], students have worked on only one other multidigit multiplication problem, which was presented in a real-world context and without explicit direction on how to solve it.

[10] *Everyday Mathematics* uses this bracket notation for multiplication.

As the class begins working, Mariluz initiates an interchange with Ms. Rosena to solicit help. Through the interchange, Ms. Rosena encourages Mariluz to use what she knows to think about the task rather than providing explicit direction on how to solve the problem:

> Mariluz: Is [6 times 80] the same thing as 6 times 8?
> Ms. Rosena: No. Is 80 the same as 8?
> A few students: No.
> Ms. Rosena: Is 8 kids the same as 80 kids?
> A few students: No.
> Ms. Rosena: No.
> Mariluz: Well, I don't know what to do then.
> Ms. Rosena: Yes. Mariluz, using what we just discussed…you can do it.

After this interchange Mariluz sits idle for a few moments and then begins to doodle on her white board. Ms. Rosena watches Mariluz from a distance until she begins working on a viable solution strategy.

It is worth noting that Mariluz failed to devise a sensible solution strategy on the problem posed prior to 6[80]. In the Fall, Ms. Rosena would likely have provided Mariluz with strong guidance to ensure successful completion of the current task. At this point in the Spring, she encourages Mariluz to use what she learned from class discussion of the first problem instead. Furthermore, she expresses confidence in Mariluz's ability to use what she has learned to devise a solution.

In summary, Ms. Rosena intentionally stands away from students as they work on mathematics tasks, both to encourage autonomy and so she can catch and correct off-task behavior quickly. However, in all lessons observed, there are instances where Ms. Rosena perceives individual students to need support to move forward with the assigned problem solving task. In situations like this, Ms. Rosena's instructional strategies for supporting students' difficulties with mathematics tasks during work time change from Fall to Spring. While in the Fall Ms. Rosena typically provides procedural directives to help a student through a task, her approach in the Spring involves alternative means of support. Specifically, she poses questions to the whole class to provide hints and clarify pertinent details of tasks and she expresses confidence in students' abilities to figure out how to solve problems.

Rosena response pattern 3, consequences for student thinking. Ms. Rosena's practice of standing away from students as they begin working on mathematics tasks while also holding them accountable for on-task behavior proves an effective strategy for getting students to attempt problem solving tasks. At first, this approach to student work time yields many flawed solutions. However, as these flawed solutions are dissected and revised in class discussions, students become more skillful in their problem solving and produce fewer flawed solutions. As the year continues, students also become increasingly autonomous problem solvers.

While Ms. Rosena most typically stands away from students during work time, there are a few instances in each Fall and Spring lesson when she employs strategies to help students who are struggling. Ms. Rosena's Fall practice of responding to some students' difficulties during work time by providing strong, procedural direction serves to limit student thinking, as students can achieve correct solutions by simply following teacher directions. As the year progresses, Ms. Rosena adopts practices in which students are helped to move forward with problem solving tasks in ways that do not reduce the cognitive demands of tasks. As a result, students are more consistently encouraged to think about problems in their own ways. Additionally, Ms. Rosena's practice of telling students that she thinks they are capable of solving problems posed appears to be effective at convincing some students to grapple with uncertainty and persevere through difficulties that they might not have otherwise.

In general, Ms. Rosena's students appear to become increasingly confident in and proud of their abilities to think mathematically. As students become more confident and capable problem solvers, student engagement during mathematics instruction increases dramatically.

Rosena response pattern 3, link to beliefs and knowledge. The change in Ms. Rosena's teaching practice, away from providing struggling students with explicit procedural direction, appears related to a shift in Ms. Rosena's beliefs about students' capabilities as well as developing knowledge of how to provide support without taking over students' thinking. When discussing her reasons for providing strong direction to individual students in the Fall, Ms. Rosena cites students' individual needs and a general belief that some of the retained students in her class need procedural cues to successfully complete tasks. Despite her beginning-of-year belief that some students need explicit instruction, Ms. Rosena follows the recommendations of the *Everyday Mathematics* program and directs her students to attempt tasks without explicit instruction. As Ms. Rosena is surprised by her students' abilities to successfully complete tasks without explicit instruction, her beliefs regarding her students' capabilities begin to shift. As the year continues, Ms. Rosena seems to hold greater expectations for all of her students, including those students who were initially thought to require explicit instruction on procedures. In contrast to her initial beliefs, there is evidence at the end of the year that Ms. Rosena believes all of her students are capable of solving instructionally-appropriate novel problems as long as long as they have access to conceptual tools like manipulatives and the freedom to think about the problems in their own ways.

This shift in beliefs about children's capabilities appears to initiate a change in Ms. Rosena's beliefs about her role as the teacher during mathematics lessons. On the beginning-of-year IMAP, Ms. Rosena's responses provide no evidence that she believes the teacher should allow children to do as much of the thinking as possible when working on mathematics tasks. In contrast, on the end-of-year IMAP, Ms. Rosena asserts that she is committed to having students solve problems without explicit direction. At the end of the year, Ms. Rosena views the practice of pressing students to devise their own ways to solve novel problems as one that allows her insight into students' thinking as well as richer and more varied learning opportunities for students.

The catalyst for Ms. Rosena's change in beliefs is her observations of her students being successful with solving novel problems. Ms. Rosena's initial attempts to have students engage in novel problems without explicit instruction occurs in response to the *Everyday Mathematics* program coupled with professional development support for developing an instructional lesson image. Therefore, Ms. Rosena's change in teaching practice also reflects an expanded knowledge base regarding ways to conduct mathematics lessons and ways to address students' initial struggles with mathematics tasks.

Summary of Ms. Rosena's Response to Student Difficulties

Three patterns portray Ms. Rosena's teaching practice in response to student difficulties. Her primary approach to addressing students' difficulties involves making students' flawed solutions a focus of whole class discussion. Through these discussions, Ms. Rosena guides students to make sense of correct and incorrect aspects of the flawed solutions, while highlighting important mathematical ideas. Through this process, flawed solutions are typically revised to correct solutions. Second, when flawed solutions to word problems are the focus of class discussion, Ms. Rosena often helps students to make sense of a given problem by pressing them to consider the relationship between the flawed solution and the problem context. Finally, during the time allotted for students to work on mathematics tasks, Ms. Rosena generally stands away from the class in efforts to encourage autonomous problem solving and to hold students accountable to working on tasks. If student difficulties persist, in the Fall lessons, Ms. Rosena is observed providing strong, procedural direction to individual students. However, by the Spring, Ms. Rosena intentionally avoids this practice and instead employs alternate strategies to address students' difficulties.

As the year progresses, Ms. Rosena's ways of responding to student difficulties in combination with other teaching practices help students to become increasingly confident and productive problem solvers. Whereas in the Fall observations Ms. Rosena finds it necessary to provide constant behavioral cues to keep students working on problem solving tasks, students exhibit greater persistence and knowledge of problem solving strategies when difficulties arise in the Spring semester. Additionally, student engagement in whole class discussion of problem solving tasks increases over the course of the year. Students appear eager to have their solutions discussed regardless of whether they are correct or incorrect. When a flawed solution is shared, students readily try to understand the logic of the solution and how to revise it to correct. By the Spring, it is the classroom norm for students to offer evaluative comments and suggestions about the mathematical strategies put forward by their peers. As a result, these discussions provide an effective venue for students to construct and test important mathematical ideas in ways that are connected to their own developing mathematical understandings. An additional consequence of Ms. Rosena's ways of responding to student difficulties is that her students appear to believe that errors and mistakes are a natural part of the learning process and that there is much to be learned from them.

Ms. Rosena's ways of responding to students' difficulties are linked to both her beliefs and knowledge. Ms. Rosena's beliefs seem most linked to the ways she aims to structure support for students' difficulties during student work time and class discussions of problem solving tasks. During student work time, Ms. Rosena generally stands away from students in order to promote student autonomy and maintain on-task behavior. However, in the Fall lessons, Ms. Rosena is observed responding to some student difficulties during work time by providing strong, procedural direction. Justifying this practice, Ms. Rosena explains that she believes some of her high-need, retained students will only be successful if she tells them what to do. By the Spring observations, Ms. Rosena has revised her beliefs to considering all students capable of successfully devising personal solutions to novel problems. Reflecting this change, Ms. Rosena is observed responding to student difficulties by offering less directive support and encouraging students to persist through difficulties. As Ms. Rosena comes to believe her students are capable of solving novel problems, she also comes to believe that the teacher's role is one of orchestrating support during problem solving such that students do as much thinking as possible. In the context of whole class discussion, Ms. Rosena's belief in the learning potential of students' mistakes leads her to make flawed solutions a focus of these discussions. Ms. Rosena's strategy of emphasizing word problem contexts in response to students' difficulties reflects her belief that real-world situations can provide a bridge between students' lived experiences and understanding mathematics concepts.

While these ways of responding to student difficulties are driven by Ms. Rosena's beliefs, their successful implementation relies on Ms. Rosena's knowledge for teaching mathematics. Ms. Rosena proves able to make sense of the correct and incorrect aspects of her students' flawed solutions in real-time such that she is able to guide the class discussion toward illuminating important mathematical ideas. Related to her practice of emphasizing the context of word problems, Ms. Rosena is able to quickly determine if the basis of an error is rooted in a lack of attention to context. Then she is able to draw on her pedagogical content knowledge to formulate questions that press students to build solutions in relation to problem contexts. Furthermore, Ms. Rosena has developed routines for facilitating students' problem solving efforts with minimal intervention and for having students examine and evaluate each other's mathematical solutions before and during classroom discussions. In these ways, Ms. Rosena's knowledge base makes it possible for her to realize instructional goals driven by reform-oriented beliefs.

Situating Ms. Rosena's Case Story in Broader Measures of Teaching

In the previous section, a theoretical explanation was presented suggesting how Ms. Rosena's teaching practices in response to student difficulties are linked to her beliefs and knowledge during the year of this study. Teacher response to student difficulties is one of many aspects of reform-oriented mathematics instruction that contributes to the overall quality of mathematics teaching and learning more broadly defined. Therefore, this section will present results from two more global measures of Ms. Rosena's mathematics teaching. First, data collected following each core classroom observation using the *Reformed Teaching*

Observation Protocol (RTOP) (Sawada et al., 2002) will be presented. This instrument is designed to measure the degree to which a given mathematics lesson reflects principles and practices associated with reform-based mathematics instruction. Second, aggregated class data from the mathematics section of the State Standardized Test (SST) will be presented in relation to aggregated data at the school, district, and state levels. Taken together, these findings will allow Ms. Rosena's patterns of response to student difficulty to be considered within a more general understanding of her mathematics teaching and students' learning.

Rosena: Adherence to Reformed Teaching

The degree to which Ms. Rosena's mathematics instruction reflects current reforms in mathematics education was measured using the Reformed Teaching Observation Protocol (RTOP) (Sawada et al., 2002). The RTOP is a criterion-referenced instrument containing 25 items, divided into five subscales: (1) lesson design and implementation, (2) the level of significance and abstraction of the content, (3) the processes that students use to manipulate information, (4) the classroom culture as observed through the nature of communicative interactions, and (5) the nature of student-teacher relationships. Following each core observation in Ms. Rosena's classroom, each of the 25 items on the RTOP was rated on a scale from 0 (not observed) to 4 (very descriptive). Next, sums were calculated for ratings on each subscale as well as the total instrument to assess the degree to which Ms. Rosena's mathematics instruction was reformed, with higher scores reflecting a greater degree of reform. Consequently, subscale scores on the RTOP range from 0 – 20, and total instrument scores range from 0 – 100. Aggregated results from core observations of Ms. Rosena's mathematics teaching in the Fall and Spring are presented in Table 26.

Table 26

Ratings of Ms. Rosena's Mathematics Teaching on the RTOP

RTOP items by subscale	Fall	Spr
Subscale 1: Lesson design and implementation		
1. The instructional strategies and activities respected students' prior knowledge and preconceptions inherent therein.	3.0	4.0
2. The lesson was designed to engage students as members of a learning community.	2.5	3.5
3. In this lesson, student exploration preceded formal presentation.	4.0	4.0
4. The lesson encouraged students to seek and value alternative modes of investigation and problem solving.	3.0	4.0
5. The focus and direction of the lesson was often determined by ideas originating with students.	3.0	3.0
Subscale 1 score	**15.5**	**18.5**
Subscale 2: Content, propositional knowledge		
6. The lesson involved fundamental concepts of the subject.	3.0	4.0
7. The lesson promoted strongly coherent conceptual understanding.	2.5	3.5
8. The teacher had a solid grasp of the subject matter content inherent in the lesson.	3.0	4.0
9. Elements of abstraction (i.e., symbolic representations, theory building) were encouraged when it was important to do so.	2.5	4.0
10. Connections with other content disciplines and/or real world phenomena were explored and valued.	4.0	3.5
Subscale 2 score	**15.0**	**19.0**
Subscale 3: Content, procedural knowledge		
11. Students used a variety of means (models, drawings, graphs, concrete materials, manipulatives, etc.) to represent phenomena.	4.0	3.0
12. Students made predictions, estimations, and/or hypotheses, and devised means for testing them.	2.0	2.0
13. Students were actively engaged in thought-provoking activity that often involved the critical assessment of procedures.	2.0	3.0
14. Students were reflective about their learning.	2.0	2.0
15. Intellectual rigor, constructive criticism, and challenging of ideas were valued.	2.0	2.5
Subscale 3 score	**12.0**	**12.5**
Subscale 4: Classroom culture, communicative interactions		
16. Students were involved in communication of their ideas to others using a variety of means and media.	3.0	3.5
17. The teacher's questions triggered divergent modes of thinking.	2.0	3.5
18. There was a high proportion of student talk and a significant amount of it occurred between and among students.	2.0	2.5
19. Student questions and comments often determined the focus and direction of classroom discourse.	2.0	3.0
20. There was a climate of respect for what others had to say.	1.5	3.0
Subscale 4 score	**10.5**	**15.5**
Subscale 5: Classroom culture, student/teacher relationships		
21. Active participation of students was encouraged and valued.	2.5	3.5
22. Students were encouraged to generate conjectures, alternative solution strategies, and ways of interpreting evidence.	3.0	3.0
23. In general the teacher was patient with students.	2.5	3.0
24. The teacher acted as a resource person, working to support and enhance student investigations.	3.0	3.5
25. The metaphor "teacher as listener" was very characteristic of this classroom.	2.5	3.0
Subscale 5 score	**13.5**	**16.0**
Total score	**66.5**	**81.5**

On the lesson design and implementation subscale, the mean scores for Ms. Rosena's mathematics instruction are 15.5 in the Fall and 18.5 in the Spring. In all observed lessons, student exploration proceeds formal presentation of mathematical ideas and students' ideas determine the direction of some aspects of the lessons. Instruction is designed to respect students' prior knowledge and encourage students to seek and value multiple ways to approach problems in all lessons, with strong adherence to these principles in Spring lessons. Finally, lessons are designed to engage students as a community in all lessons, with Spring lessons realizing this goal to a greater extent.

The second RTOP subscale focuses on what the contents of a lesson suggests about a teacher's propositional knowledge, meaning her mathematical knowledge for teaching particular content. On this subscale, Ms. Rosena's mean scores are 15.0 in the Fall and 19.0 in the Spring. All observed lessons illuminate fundamental mathematics concepts and explore connections with real-world phenomena, although Spring lessons promote somewhat stronger conceptual understanding than those observed in the Fall. Furthermore, Spring lessons illuminate connections between mathematical ideas and their symbolic representations more consistently and coherently than Fall lessons.

The third RTOP subscale focuses on what the contents of a lesson suggests about a teacher's procedural knowledge. For the RTOP instrument, this means the quality of the lesson in terms of an inquiry approach to learning. The scores for Ms. Rosena's instruction on this subscale are similar at both data points, with 12.0 in the Fall and 12.5 in the Spring. Students are only sometimes engaged in activities of inquiry such as making predictions and devising means for testing them. Students are encouraged to engage in critical assessment of procedures and challenging of ideas in all lessons, but these characteristics are more descriptive of classroom norms in the Spring lessons than the Fall. In all lessons, students are encouraged to represent mathematical ideas in multiple ways.

The fourth RTOP subscale focuses on communicative interactions that are part of the classroom culture. On this subscale, the mean scores for Ms. Rosena's instruction are 10.5 in the Fall and 15.5 in the Spring. In all observed lessons, students are involved in communicating their mathematical ideas to others in multiple ways. While students participate in the classroom discourse throughout the year, a greater level of student determination of the direction of the discourse is observed in the Spring. This parallels a change in the classroom climate, toward one in which students respect what others have to say to a greater extent. Finally, Ms. Rosena poses questions that trigger divergent thinking more frequently in the Spring when compared to the Fall.

The last RTOP subscale focuses on the classroom culture in terms of the teacher's role and the roles students are encouraged to take. On this subscale, the mean scores for Ms. Rosena's instruction are 13.5 in the Fall and 16.0 in the Spring. Throughout the year, Ms. Rosena deliberately engages in teaching strategies to support students as they work on mathematics tasks and discuss student solutions. Furthermore, Ms. Rosena is regularly observed encouraging students to generate conjectures and interpret solutions shared by their peers. That being said, there are times in Fall lessons when Ms. Rosena dominates class discussion, allowing students to take a more passive role. Therefore, items related to active participation and the characteristic of "teacher as listener" are more characteristic of Spring lessons.

Ms. Rosena's mean scores on the total RTOP instrument are 66.5 in the Fall and 81.5 in the Spring. These ratings suggest that Ms. Rosena's mathematics instruction moderately reflects the principles of reform in the Fall and approaches strong adherence in the Spring.

Rosena: Student Achievement

Aggregated student scores on the mathematics section of the State Standardized Test (SST) were used to measure the mathematics achievement of Ms. Rosena's class. The SST is a criterion-referenced achievement test in which students receive scores between 1 and 5, with a score of 3 or higher considered passing. Since Ms. Rosena's students are repeating third grade, SST scores from the previous year are available for the students in Ms. Rosena's class[11]. Six students received a score of 1 on the mathematics section of the SST in the previous year, and three students received a score of 2. This suggests that these students entered Ms. Rosena's class with limited knowledge of third-grade mathematics.

Ms. Rosena's students' SST scores in the year of this study are presented in Table 27 along with comparison data for third-grade students at Lincoln Heights, the school district, and the state. However, comparisons between the scores of Ms. Rosena's class and the other groups, including the classes of the other teachers' studied, should be made with caution. Each teacher studied has a class with demographic particularities and a statistically small number of students. The fact that Ms. Rosena's class is comprised entirely of students who have been retained makes it arguably different than these other groups.

[11] Ms. Rosena lost 1 of her 10 students shortly before the SST testing, so data are presented for the 9 students tested.

Table 27
Comparison of Rosena Student Achievement on SST to School, District, and State

		Achievement level (%)					
Group	Students	1	2	3	4	5	Passing score (%)
Ms. Rosena's class	9	11	11	56	22	0	78
Lincoln Heights 3rd grade	88	24	27	33	15	1	49
School District 3rd grade	13,713	18	16	33	24	9	66
State 3rd grade	203,037	15	17	34	25	9	68

Note. Scores of 3 or higher are considered passing.

Seven of the nine students in Ms. Rosena’s class who took the SST achieved a higher score on the mathematics test than the previous year, with six students increasing their scores by two or more levels. A passing score of 3 or higher was achieved by 78% of Ms. Rosena’s students. This is in comparison to passing scores achieved by 49% of Lincoln Heights third graders, 66% of the third graders in the school district, and 68% of the third graders in the state. Therefore, the percent of Ms. Rosena’s students achieving grade-level standards, as measured by the SST, exceeds that of the third-grade students at Lincoln Heights as well as in the school district and state.

Summary of Global Measures of Ms. Rosena’s Teaching

Taken together, results from the *Reformed Teaching Observation Protocol* (RTOP) and the State Standardized Test (SST) suggest that Ms. Rosena’s classroom is a place where students learn significant mathematics through instruction that is somewhat reformed. RTOP results suggest that Ms. Rosena’s mathematics teaching moderately reflects the principles of mathematics reform in the Fall and approaches strong adherence in the Spring. Looking across subscales of the RTOP, the lesson design and implementation subscale as well as the propositional knowledge subscale are rated higher than the other subscales in the Fall and the Spring. These scales reflect Ms. Rosena's ability to use her knowledge of mathematics to facilitate classroom experiences that respect students' prior knowledge and build on students' mathematical ideas. Therefore, even as Ms. Rosena is grappling with reform-oriented practices such as creating a classroom culture in which students guide the discourse, she is able to interpret and build on students' mathematical ideas in ways that promote understanding. Ms. Rosena's students' mathematics achievement scores lend further evidence to this assertion. Aggregated SST data indicates that most of Ms. Rosena's students (78%) are performing at or above grade level expectations. This finding suggests that Ms. Rosena's class is one in which there are significant opportunities for mathematics learning.

CHAPTER IV

CROSS-CASE ANALYSIS & DISCUSSION

This research set out to explore how teachers' knowledge and beliefs influence the ways they support and limit student thinking during initial use of a reform-based mathematics curriculum in an urban school setting. To accomplish this, a collective case study was conducted of four, third-grade teachers in an urban school that focused on their ways of responding to student difficulties during mathematics instruction on multiplication and division topics. The following broad questions guided this research:

1. In what ways and to what extent do teachers support and limit student thinking during mathematics instruction in their first year implementing a reform-based mathematics curriculum?
2. How do teachers' beliefs influence the ways they incorporate student thinking into their planning for mathematics instruction and on-the-spot instructional decision-making in their first year of implementing a reform-based mathematics curriculum?
3. How does teacher knowledge influence the ways teachers incorporate student thinking in their first year of implementing a reform-based mathematics curriculum?
4. How does the urban context, as defined by the research literature and perceived by teachers and school leaders, influence mathematics instruction in this urban school?

This final chapter begins with a discussion of findings across cases. Through this discussion, research findings will be synthesized and the four research questions used to guide this study will be answered. Next, implications of this research will be discussed including recommendations for advocates of mathematics reform. Then study limitations will be identified along with suggestions for future research. Finally, summary conclusions will be offered.

Discussion of Findings across Cases

My discussion of findings across cases will be presented in four parts. First, attention will be given to how and to what extent the teachers studied engaged in teaching practices that supported and limited student thinking during their first year working with a reform-based curriculum. Then consideration will be given to what study findings suggest about the influence of teachers' beliefs and knowledge on teaching practices related to student thinking. This will be accomplished by identifying dimensions of teachers' response to student difficulties and describing the relationship between each dimension, teacher beliefs, and teacher knowledge. Next, the relationship among teachers' evolving beliefs, knowledge, and mathematics instructional practices will be discussed. Finally, consideration will be given to what this research suggests about how the urban context influences teacher transitions to reform-based mathematics teaching practices. Through this cross-case analysis, the research questions that guided this study will be answered.

Teacher Actions that Support and Limit Student Thinking

This study set out to consider the ways and extent to which teachers support and limit student thinking in their first year implementing a reform-based mathematics program in an urban school. Reflecting the current research literature (Franke et al., 2007; Sowder, 2007), findings from this study suggest that some aspects of reform-oriented teaching are more readily adopted than others. Table 28 summarizes the frequency with which the group of teachers studied was observed to engage in particular teaching practices thought to support and limit student thinking.

Table 28
Summary of Teaching Strategies Observed to Support and Limit Student Thinking

Frequency	Support	Limit
Teachers often	Assign contextualized mathematical problems that can be solved in multiple ways	Lead class discussions such that interactions occur primarily between teacher and individual students
	Focus on a small number of problems in a given class period	Focus class discussion on show and tell of student strategies (and not evaluation and comparison of strategies)
	Direct students to solve problems in ways that make sense to them	
	Make manipulatives and other tools available for student use	
	Elicit presentation of multiple problem solutions during class discussion.	
	Elicit description of problem solving process from individual students	
	Maintain record of solution strategies on board/overhead during class discussion	
Teachers sometimes	Encourage student collaboration during problem solving or after problem solving	Use closed questions and directives to guide students who struggle, limiting *student* thinking in the problem solving process
	Ask students to justify strategies employed (during work time and class discussion)	Suggest use of procedures and tricks when students struggle with word problems, rather than focusing students on making sense of problem situations
	Intentionally organize order of strategy sharing to highlight relationships among strategies	Lead students to report and discuss strategies in a manner consistent with teacher thinking rather than student thinking
	Ask clarifying questions and provide instant replays of strategies during class discussion	Position the teacher as the primary source of help and authority for mathematical correctness (rather than using the class community to provide support and establish mathematical correctness)
	Use class discussion to illuminate relationships among student generated solutions and mathematical concepts	Treat students' viable mathematical ideas as if they are incorrect
	Orchestrate class discussion such that students participate as a learning community by analyzing, evaluating, and commenting on each others' ideas	

Evidence suggests that the combination of reform-based mathematics curriculum materials and professional development related to use of these materials supported teachers in changing their mathematics instruction to ways that are aligned with reform recommendations. Across classrooms, all mathematics lessons observed focused on a limited number of problems, with initial problems being set in real-world contexts to facilitate informal exploration of mathematical concepts. Students were given time to devise their own ways to solve problems prior to being shown particular solution strategies. Also, teachers regularly incorporated class discussion of students' problem solving strategies into their mathematics lessons. To summarize, all teachers studied were found to structure mathematics lessons, including task assignments, in some ways aligned with reformers' recommendations for mathematics instruction.

While structural changes to mathematics instruction represent important steps toward reform (Franke et al., 1997; Hufferd-Ackles et al., 2004), these changes alone are unlikely to yield the kinds of student learning envisioned by reformers (Franke et al., 2007; Stipek et al., 1998). As other researchers have found (Spillane & Zeuli, 1999), the teachers in this study experienced limited success with cultivating a learning community characterized by rigorous analysis and debate of mathematical ideas. Across classrooms, there was inconsistency in the degree to which teachers held students accountable for attempting problems posed and promoted student-to-student collaboration during problem solving. At times, some teachers were found to limit student thinking during problem solving by directing students to follow particular procedures, sometimes without attention to the conceptual basis of the procedures. There were also instances of teachers limiting students' mathematical thinking by treating their viable mathematical ideas as incorrect.

While teachers regularly convened class discussions around student-generated solutions, these discussions were only sometimes organized to deliberately illuminate particular mathematical ideas. During discussions, teachers varied in the degree to which they asked students to justify strategies used. Teachers also varied with the degree to which their questions and explanations supported the class as a whole in understanding the strategies presented by individual students. Additionally, class discussions in three of the four classrooms studied most often consisted of interactions between the teacher and particular students as they presented their solutions. As has been found by other researchers (Spillane & Zeuli, 1999), the teachers in this study found it particularly difficult to employ strategies to engage the class as a whole in analyzing and evaluating mathematical solutions presented during class discussion. These factors served to limit student attention to and engagement in the ideas shared by peers, thus decreasing the learning potential of class discussions.

The first research question posed is, “In what ways and to what extent do teachers support and limit student thinking during mathematics instruction in their first year implementing a reform-based mathematics curriculum?” In short, teachers consistently supported student thinking by structuring lessons to afford students opportunities to engage in problem solving and describe their mathematical solutions in the whole class forum. There was variation among classrooms in the extent to which teachers provided support that illuminated key mathematical concepts and encouraged development of generative problem solving practices. In addition, teachers’ actions varied in the degree to which they promoted student autonomy and student collaboration. Teachers particularly struggled to facilitate mathematically productive student-to-student interactions during class discussion of problem solving tasks.

It is the thesis of this research that variations in teacher actions that support and limit student thinking are related to teachers' belief and knowledge. This proposition will be elaborated in the next section through illustration of the ways teachers' beliefs and knowledge were found to be related to teachers' response to student difficulties.

The Influence of Beliefs and Knowledge on Teachers' Response to Student Difficulties

This study examines the influence of teachers' beliefs and knowledge on mathematics teaching practices by focusing on teacher response to student difficulties during instruction on multiplication and division. Looking across cases, five dimensions of teacher response to student difficulties emerge. Teachers' ways of responding to student difficulties vary in the extent to which they encourage student autonomy, focus on problem context, emphasize mathematics concepts, explore flawed solutions in whole class discussion, and engage students as a community of learners. These dimensions of teacher response to student difficulties and their relationship to teacher beliefs, teacher knowledge, and student thinking are summarized in Table 29. Each of these dimensions will be discussed in turn, with a focus on how they are linked to teachers' knowledge and beliefs as well as their relationship to student thinking.

Dimension 1: Encouragement of Student Autonomy

First, teachers vary in the extent to which their ways of responding to student difficulties encourage student autonomy. In the context of this research, *student autonomy* refers to student dispositions and learning practices that support productive engagement in

Table 29
Summary of the Relationship Among Teacher Response to Student Difficulties, Teacher Beliefs, Teacher Knowledge, and Student Thinking

Dimension of response to student difficulties	Link to teacher beliefs & knowledge	Consequences for student thinking
Teachers vary in the extent to which their ways of responding to student difficulties encourage student autonomy.	*Beliefs*. As teachers begin to view students as capable of solving novel problems without explicit instruction, they are more likely to respond to student difficulties in ways that discourage dependence on the teacher and encourage autonomous learning practices. *Knowledge*. Practices employed are dependent on a teacher's knowledge of teaching practices that facilitate student autonomy and peer collaboration as well as the ability to exercise these practices in response to students' difficulties.	Students become increasingly productive problem solvers in classrooms where they learn strategies to persevere through difficulties encountered during problem solving and where they are held accountable for engaging in problem solving with limited or no teacher intervention.
When students struggle with word problems, teachers vary in the extent to which they utilize a focus on problem context to support student thinking.	*Beliefs.* Teachers who emphasize problem context view word problems as a way for students to explore mathematical ideas before they are formally introduced. *Knowledge*. Teachers who personally rely on procedural understanding of mathematics are more likely to strip context and provide procedural hints when students are struggling to make sense of a word problem. Teachers with greater conceptual understanding of mathematics tend to encourage students to use the problem context to build a model of the problem situation.	When children are taught to approach problem solving by making sense of problem situations, they are encouraged to rely on their own intuitive, analytical modeling skills. In turn, symbols and formal procedures can be learned in relationship to students' own ways of making sense of problems.
Teachers differ in the extent to which their ways of responding to students' difficulties emphasize mathematics concepts.	*Beliefs.* Teachers who emphasize mathematics concepts are more likely to believe that understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures. *Knowledge*. Teachers with stronger knowledge of school mathematics are more likely to emphasize key mathematics concepts as they respond to students' difficulties. Teachers with weaker knowledge are more likely to focus on procedures for obtaining correct answers without illuminating important mathematics concepts.	Children who develop conceptual understanding of mathematical ideas can use their knowledge more flexibly than children who rely on procedural knowledge alone. Conceptual understanding also helps students make connections among mathematical ideas.

Table 29 (Continued)
Summary of the Relationship Among Teacher Response to Student Difficulties, Teacher Beliefs, Teacher Knowledge, and Student Thinking

Teachers differ in the extent to which they intentionally make flawed solutions a focus of whole class discussion.	*Beliefs*. Some teachers intentionally avoid discussion of flawed solutions because they believe students will be embarrassed to have their mistakes shared publicly. Teachers are also concerned that public display of flawed solutions might further confuse children with fragile understanding. *Knowledge.* Engaging students in productive analysis of flawed solutions requires teachers to draw on their knowledge base in real time to ensure that important mathematical ideas are brought out and misconceptions are confronted.	When class discussion involves collaborative analysis and revision of flawed solutions, students are encouraged to think about underlying mathematics concepts. Additionally, intellectual risk taking is promoted because mistakes are viewed as opportunities for learning.
Teachers differ in the extent to which they engage students as a community of learners.	*Beliefs.* Teachers who believe students have limited capabilities to engage in problem solving without teacher support also tend to believe that students are limited in their capacity to support each others' learning. Additionally, teachers who view the purpose of class discussion as showing ways to get answers (a procedural focus) are less likely to press students – individually or as a community of learners – to discuss the conceptual underpinnings of mathematical ideas. *Knowledge.* Engaging students as a community of learners draws on knowledge of routines and scripts to promote productive student collaboration and mathematical debate in class discussion. Further, this practice relies on a teacher's ability to use her mathematical knowledge to interpret student contributions and steer the dialogue in productive directions.	When classroom norms emphasize discussion, collaboration, and negotiation as ways of fostering shared meaning among a community of learners, students view mathematical authority as residing in mathematical arguments rather than with the teacher or text. Consequently, students' energies are focused on making sense of mathematics in relation to their own developing constructions.

assigned mathematics tasks with limited or no teacher intervention. Ms. Aria and Ms. Rosena intentionally encourage student autonomy by limiting their interactions with students during initial problem solving, encouraging use of resources other than the teacher, and holding students accountable for attempting problems prior to class discussion. In these classes, students learn strategies to persevere through difficulties encountered during problem solving, and they become increasingly productive problem solvers. In contrast, Ms. Jarmin provides one-on-one assistance to students throughout the time allotted for problem solving, and Ms. Larsano does not hold students accountable for developing personal solutions to problems prior to class discussion. In both of these classes, some students regularly wait for the teacher or other students to provide direction on how to solve a given problem, thus limiting personal engagement in problem solving.

Evidence suggests that teachers' practices that encourage student autonomy are related to teachers' beliefs about their students' capabilities to solve novel problems without teacher intervention as well as pedagogical knowledge of routines to minimize students' dependence on the teacher. As Ms. Aria and Ms. Rosena entertain the idea introduced in professional development that their students might be capable of solving novel problems without explicit instruction, they develop and experiment with routines to encourage greater student autonomy and peer-to-peer support. As they observe their students successfully solving novel problems with limited teacher intervention, their beliefs change to viewing students as more capable than they initially thought. In turn, these teachers become more resolute in their commitment to limiting intervention during initial problem solving and continue to cultivate instructional routines that reflect this belief.

In Ms. Jarmin and Ms. Larsano's classes, routines to discourage teacher dependence and encourage student autonomy are not established. Consequently, some students develop the practice of limiting personal engagement with problem solving and instead regularly wait for the teacher or other students to show them how to solve problems. In general, students in these classes are comparatively less successful with problem solving. As a result, these teachers' experiences with their students reinforce the belief that strong teacher direction is necessary for their students to be successful with problem solving. In discussing the necessity of strong teacher intervention, Ms. Jarmin and Ms. Larsano often point out that the students in their classes have special learning needs, suggesting that *other students* may be more capable of engaging in problem solving with greater autonomy. This finding echoes other research indicating that teachers question the capability of low-achieving students to devise their own solutions to problems without direct teacher guidance (Knapp, 1995a; Sztajn, 2003; Warfield et al., 2005).

Dimension 2: Focus on Problem Context

A second way in which teachers' patterns of response to students' difficulties vary is in the extent to which they utilize a focus on problem context to support student thinking about word problems. When children are taught to approach problem solving by making sense of problem situations, they are encouraged to rely on their own intuitive, analytical modeling skills (Carpenter et al., 1999). In turn, symbols and formal procedures can be learned in relationship to students' own ways of making sense of problems. In Ms. Aria and Ms. Rosena's classes, student thinking is supported as students are encouraged to pay close attention to problem context to figure out how to approach a problem or to analyze where a solution went wrong. In Ms. Jarmin and Ms. Larsano's classes, attention to problem context is less consistent. Often these teachers help students obtain answers in ways that deemphasize problem context and, instead, encourage a procedural focus. For instance, Ms. Jarmin tells a student to, "count the rows," to get a correct answer instead of encouraging the student to figure out what the rows represent in the problem.

Teachers' focus on problem context in response to student difficulties seems most related to teachers' personal ways of understanding mathematics. All teachers studied attempted to use problem context to support student thinking some of the time. As students' difficulties persist, teachers who personally rely on procedural understanding of mathematics were more likely to strip context and provide procedural hints when students struggled to make sense of word problems. Conversely, teachers with greater conceptual understanding of mathematics were more committed to encouraging students to use problem contexts to make sense of problems by modeling problem situations. There is some evidence that these practices may also be related to teachers' beliefs. Teachers who consistently emphasize problem context view word problems as a way for students to explore mathematical ideas before they are formally introduced. This belief is in contrast to a more traditional view of word problems, as a way for students to apply mathematics concepts that have already been learned.

Dimension 3: Emphasis on Mathematics Concepts

A third way in which teachers' response patterns vary is in the extent to which teachers emphasize mathematics concepts as they address students' difficulties. Student thinking is supported when mathematics concepts are emphasized, because students who develop conceptual understanding of mathematical ideas can use their knowledge more flexibly than children who rely on procedural knowledge alone (Carpenter & Lehrer, 1999). Conceptual understanding also serves as a basis for making connections among mathematical ideas. In the instructional episodes shared, Ms. Rosena stresses the concept of equal groups and Ms. Aria emphasizes the meaning of remainder as they support students' revision of flawed solutions. In Ms. Larsano and Ms. Jarmin's classes, students' difficulties are addressed in ways that are less likely to illuminate key mathematics concepts. Sometimes these teachers focus on the mechanics of procedures for obtaining correct answers. At other times, teacher attempts to provide conceptual support are confusing and end before correct answers and underlying concepts are clear.

The likelihood that teachers will respond to student difficulties in ways that emphasize key mathematics concepts seems most related to their knowledge of mathematics and their pedagogical content knowledge. Teachers who emphasize key mathematics concepts in response to student difficulties often include the mathematical ideas they want to stress in their lesson image. Additionally, they have either developed premeditated scripts to respond to anticipated student errors or are able to develop clear, conceptually-based explanations on-the-spot. In contrast, teachers with a weaker base of mathematical knowledge are less likely to identify key mathematical ideas as part of their lesson image, and they have difficulty formulating conceptually-based explanations when difficulties arise. As a result, teachers with weaker mathematical knowledge often revert to a focus on procedures, which more closely aligns with their personal ways of understanding mathematics. There is also evidence that teachers who emphasize mathematics concepts in response to student difficulties are more likely to believe that understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Dimension 4: Intentional Use of Flawed Solutions in Whole Class Discussion

A fourth way that teachers' patterns of response to student difficulties vary is in the extent to which students' flawed solutions are intentionally used to support student learning in whole class discussion. Kazemi and Stipek (2001) identify the practice of having students analyze flawed solutions in whole class discussion as a sociomathematical norm associated with classrooms in which students have high levels of mathematical understanding. When class discussion involves collaborative analysis and revision of flawed solutions, students are encouraged to think about underlying mathematics concepts (Borasi, 1994). Additionally, intellectual risk taking is promoted because mistakes are viewed as opportunities for learning. Of the four teachers studied, Ms. Rosena is the only teacher who intentionally integrates discussion of flawed solutions into whole class discussion. The other teachers aim to orchestrate class discussion such that the focus is on multiple correct solutions or they call on volunteers, with limited awareness of the strategies that will be shared.

Reflecting findings of other studies (Santagata, 2005; Silver et al., 2005), some teacher participants in this study were reluctant to focus on flawed solutions in whole group discussions. This reluctance appears most closely related to teachers' beliefs. Teachers worry that students will be embarrassed if their mistakes are shared publicly and that the display of flawed solutions might confuse children with fragile mathematical understanding. Consequently, teachers aim to address students' difficulties prior to whole class discussion, or they assume that student difficulties will be resolved as correct problem solutions are shared. While it appears that teachers' beliefs primarily influence whether they choose to include analysis of flawed solutions in whole class discussion, there is evidence that teacher knowledge influences the quality of such discussions. Engaging students in productive analysis of flawed solutions in the whole class forum requires teachers to draw on their knowledge base in real time to ensure that important mathematical ideas surface and misconceptions are confronted. Williams and Baxter (1996) found that, as teachers attempt to focus their cognitive attention on managing a class discussion of multiple solutions, they sometimes have difficulty staying focused on content goals. There is evidence of this point in the classroom episode featuring Ms. Larsano's handling of a flawed solution in class discussion. In this episode, the discussion is ended before the correct and incorrect aspects of the flawed solution are clear.

Dimension 5: Engagement of Students as a Community of Learners

A fifth way that teachers' patterns of response to student difficulties vary is in the extent to which they engage students as a community of learners. All four teachers studied indicate that their students are allowed to interact with each other during time allotted to work on mathematics tasks. However, only in Ms. Aria and Ms. Rosena's classes are lessons structured to deliberately involve student collaboration during work time. When student difficulties surfaced in the context of whole class discussion, only one of the four teachers – Ms. Rosena – utilizes the reform-based practice of engaging students as members of a learning community to address difficulties. Especially in Spring lessons, Ms. Rosena is observed prompting students to analyze and evaluate their peer's mathematical thinking, whether a student's mathematical ideas are correct or flawed. If a student's mathematical thinking is thought to be flawed, Ms. Rosena guides students to recognize the source of the error and then asks students to suggest ways to resolve the error. In contrast, when student difficulties surface during class discussion in the other three classrooms, discourse typically follows traditional interaction patterns characterized by the teacher maintaining tight control of discourse and limiting the extent to which students work together to analyze their peers' mathematical ideas, judge mathematical correctness, and resolve mathematical errors (Franke et al., 2007).

These different ways of responding to student difficulty fit well with the distinction Wood, Williams, and McNeal (2006) make between reform-oriented classrooms in which discourse patterns are characterized by a strategy reporting classroom culture and an inquiry/argument classroom culture. Similar to the findings of Wood et al., the current research finds the class discussion of student difficulties in Ms. Rosena's classroom, which come the closest to establishing an inquiry/argument classroom culture, to be the most productive of those observed.

The extent to which teachers utilize practices that engage students as members of a learning community seem most related to teachers' knowledge but also related to their beliefs. Knowledge of routines and scripts to initiate and facilitate student collaboration are necessary to engage students as a learning community. This point is highlighted by the contrast in Ms. Aria's instructional strategies to promote a community of learners during work time and the limited practices observed to stimulate student-to-student interaction during class discussion. Ms. Aria's aim is to engage students as a learning community throughout lessons, but she has not cultivated the teaching strategies to do this effectively during class discussion, particularly when student difficulties arise. For Ms. Aria, much of her cognitive effort during class discussions is spent on classroom management issues, limiting the degree to which she can focus on other goals. Furthermore, the ability to orchestrate a class discussion characterized by an inquiry/argumentation classroom culture (Wood et al., 2006) also relies on a teacher's ability to interpret and assess student contributions on-the-spot in order to steer the discourse productively. Some teachers may be unable to facilitate student inquiry into each others' strategies during these discussions because they are expending their cognitive effort on simply making sense of and responding to student contributions. In general, class discussions characterized by student inquiry and argumentation are more complex than discussions focused on strategy reporting. This greater complexity demands skillful application of deep and flexible teacher knowledge.

While a teacher's knowledge appears to facilitate or limit a teacher's ability to engage students as a community of learners, a teacher's inclination to have students collaborate appears related to a teacher's beliefs about students' capabilities. Ms. Rosena and Ms. Aria were found to hold the belief that students were capable of solving novel problems with limited support. Related to this belief, there is evidence that these teachers believe their students are capable of making positive contributions to the learning of others. On the other hand, Ms. Larsano and Ms. Jarmin viewed students as less capable and they established themselves as the primary source of knowledge in the classroom. Also related, these teachers may have different beliefs about the purpose of class discussion. Thompson, Phillip, Thompson, and Boyd (1994) found that teachers with a calculational orientation viewed class discussion as a place primarily for children to learn ways to get answers. Like Ms. Larsano and Ms. Jarmin, these teachers focused on having students elaborate mathematical procedures without pressing them to justify the conceptual underpinnings of their procedures. If a teacher is not focused on using class discussion to focus on conceptual understanding, the potential content of class debate of mathematical ideas is limited.

Summary of the Influence of Teachers' Beliefs and Knowledge on Response to Student Difficulties

The second and third research questions guiding this study ask respectively, "How do teachers' beliefs influence the ways they incorporate student thinking into their planning for mathematics instruction and on-the-spot decision-making in their first year of implementing a reform-based mathematics curriculum?" and, "How does teacher knowledge influence the ways teachers incorporate student thinking in their first year of implementing a reform-based mathematics curriculum?" To answer these questions, five dimensions of teacher response to student difficulties have been identified and discussed in relation to the particular aspects of beliefs and knowledge found to influence teacher actions. Specifically, teachers' beliefs and knowledge were found to influence the degree to which their ways of responding to student difficulties encourage student autonomy, focus on problem context, emphasize mathematics concepts, intentionally incorporate flawed solutions in whole class discussion, and engage students as a community of learners.

Through analyses of the relationship among teachers' response to student difficulties, teacher beliefs, and teacher knowledge, this study provides evidence of an interactive relationship between beliefs and knowledge in shaping classroom mathematics instruction. While beliefs and knowledge appear to simultaneously influence teacher actions, this analysis of teachers' response to student difficulties suggests that certain aspects of mathematics teaching are more greatly influenced by teacher beliefs while others are more greatly influenced by teacher knowledge.

Teacher knowledge seems to be the primary determinant of the particular kinds of support that teachers provide in the face of student difficulties. Teachers with stronger mathematical knowledge are more likely to address student difficulties in ways that connect to students' mathematical thinking and emphasize conceptual understanding. Furthermore, teachers are more likely to support student thinking when they have cultivated routines and scripts that deliberately aim to increase student autonomy and promote a learning community approach to resolving difficulties.

Teacher beliefs, on the other hand, appear highly related to the roles teachers take on for themselves and the roles they ascribe for students in their classrooms. Teachers who believe students need to be told how to solve mathematical problems have low expectations for students' abilities to persevere through novel problem solving tasks. They are not surprised when students have difficulties during problem solving, and they view good teaching as directly helping students to resolve these difficulties. Because students are viewed as having limited capabilities, classroom norms limit the degree to which students are encouraged to utilize peers as sources of mathematical support and knowledge. In contrast, teachers who view students as capable of devising their own solutions to problem solving tasks establish high expectations for students and institute practices that encourage students to view their peers as resources in the face of difficulty.

In this section, a focus on examining teacher response to student difficulties was used to elaborate the interactive relationship between teacher beliefs and knowledge and their influence on mathematics teaching. In the next section, focus will shift to considering how teachers' evolving beliefs and knowledge are influenced by their classroom experiences.

Relationship among Teachers' Evolving Beliefs, Knowledge, and Mathematics Instruction

In addition to suggesting ways that beliefs and knowledge influence instructional practice, this study provides evidence that instructional practice can influence evolving beliefs and knowledge. In this section, consideration will be given to how classroom experiences during initial use of reform-based mathematics curriculum appear to influence teachers' evolving beliefs and knowledge.

A surprising finding of this study is that the combination of professional development and reform-based curriculum materials led teachers to experiment with teaching practices that were not initially aligned with their beliefs. According to the teachers, professional development helped them to envision reform-based teaching practices and the curriculum materials gave them the tools to experiment with the practices in their classrooms. However, the teachers did not commit to sustained change in mathematics teaching practices until seeing how the reform-based ideas played out in their own classrooms. At the end of the year, Ms. Rosena articulates why positive experiences in the classroom are essential to adopting teaching practices learned about in professional development:

> In the professional development, yes, you're sitting there, but you're listening, and of course, you are doing some of the exercises, but until you come to the classroom and put that into practice, you won't really get it. You won't really understand what the impact is going to be with the students, and that's when it really happens. Because…I have been to professional development. They teach all these wonderful things. They help you with all these wonderful ideas. And when you come to the classroom, if you don't see them work, they're out the door.

Ms. Rosena's assertion that teachers wait to determine the worth of ideas encountered in professional development until they experiment with the ideas in their classrooms is echoed in the end-of-year interviews of all four teachers studied. In turn, the degree to which teachers perceive the reform-based mathematics curriculum and instructional strategies to "work" in their own classrooms appears related to the degree to which they revised their beliefs.

The teachers who, by the end of the year, developed beliefs most closely aligned with reform were also the teachers who perceived their students to become increasingly responsive to and successful with the reform-based teaching strategies employed in their classrooms. Teacher beliefs that evidenced the greatest change related to the capabilities of students and beliefs about the kinds of support a teacher should provide. These teachers came to view their students as capable of engaging in problem solving with limited intervention. They also came to believe that, while interacting with students, teachers should encourage students to do as much mathematical thinking as possible. The greater level of success experienced by these teachers is found to be related to their comparatively strong knowledge of mathematics for teaching, when compared to the other teachers studied. Simply stated, these teachers were able to more readily "see" ways to connect to and capitalize on student thinking during instruction. This perception of instructional success initiated a commitment to further cultivate reform-based teaching practices as well as adoption of beliefs aligned with a reform-orientation.

Teachers with weaker mathematical knowledge experienced a greater degree of struggle in their implementation of reform-based pedagogy. For instance, when students experienced difficulties, these teachers often fell back on familiar practices of directing students to use procedures. In general, analyses have demonstrated that students' difficulties in these classes were often addressed in ways that did not move toward greater mathematical understanding. As might be expected, these teachers found the reform-based mathematics practices encouraged by professional development to yield mixed success with their students. Additionally, the mathematics achievement scores of students in these classes provide an external indicator of limited success. Although these teachers' beliefs evidenced change toward reform-orientation in some ways, the shifts were not as dramatic, with evidence of beliefs consistent with a reform-orientation being inconsistent or limited at the end of the year.

In addition to the opportunity for classroom experiences to influence beliefs, this study adds to the growing body of research that suggests classroom experiences have the potential to increase teachers' knowledge for teaching mathematics (Sowder, 2007). All four teachers involved in this study indicate that, through work in their classrooms, they became more knowledgeable of various nonstandard strategies and students' ways of thinking about particular mathematics content. However, analysis of classroom observation and interview data suggest that Ms. Aria and Ms. Rosena more fully capitalized on opportunities to learn in their classrooms when compared to the other teachers.

First, classroom observations and interviews suggest that Ms. Aria and Ms. Rosena were more deliberate than the other teachers in paying close attention to student thinking and considering its implications. During classroom instruction, these teachers more frequently pressed students to justify their correct and incorrect strategies, which created a window for insight into student thinking. Post-observation interviews reveal that these teachers reflected more fully on students' strategies, especially incorrect strategies. Whereas Ms. Jarmin and Ms. Larsano were most likely to chalk up students' incorrect strategies to being baseless, Ms. Aria and Ms. Rosena made effort to understand the logic behind students' errors as well as their instructional implications. In this example, teacher beliefs are also in play, since teachers appear to vary in the extent to which they believe students' solutions to have internal logic.

Second, Ms. Aria and Ms. Rosena were both observed to intentionally cultivate teaching practices aligned with particular instructional goals. For instance, both of these teachers experimented with instructional strategies to deliberately encourage greater student autonomy. While the other teachers also indicated that they would like students to be more autonomous, evidence suggests they gave less consideration to how particular instructional practices supported and inhibited this goal. Furthermore, Ms. Jarmin and Ms. Larsano sometimes had difficulty providing rationales for particular teaching practices. When asked to discuss their reasons for acting in certain ways, these teachers' responses were sometimes limited to, "Because the teacher guide says to do that." In general, the research findings of this study affirm findings of other researchers that teacher reflection seems an integral part of reforming teaching practice (Sowder, 2007).

In summary, new knowledge – like that gained from the professional development experiences and curriculum materials – appears to have stimulated teacher experimentation with new kinds of mathematics teaching practices associated with reform, including practices that were not initially aligned with the teachers' beliefs. As teachers engaged in experimentation, they experienced differing levels of success. This variation appears most related to each teacher's knowledge, with teachers holding greater knowledge experiencing higher levels of success. In turn, the degree to which teachers changed their beliefs to align with a reform-orientation seems related to each teacher's perception of success with the new strategies. Additionally, classroom teaching experiences appear to offer opportunities for teachers to increase their knowledge for teaching mathematics. But these opportunities for learning appear to be mediated by the degree to which teachers attend to and reflect on student thinking and classroom practice.

The Influence of the Urban School Context

Lincoln Heights Elementary was found to provide an urban school context for teachers to engage in reforming mathematics instruction because of its location in an urban setting and its student population characterized by a high level of non-White students living in poverty and struggling to overcome Limited English Proficiency (Obiakor & Beachum, 2005). The students served by each teacher studied presented academic, behavioral, and emotional challenges that would likely not be as prevalent in schools serving middle- or upper-class students. Ms. Jarmin and Ms. Rosena taught classes comprised solely of retained students. These teachers' class sizes were intentionally kept small because students in these classes were viewed by teachers and school leaders to have particularly high levels of need – academically, behaviorally, and emotionally. Ms. Larsano taught a transition language class of students identified with strong English language needs. Students in this class were typically in the midst of significant life changes, many having recently moved to the United States. Finally, the students in Ms. Aria's *regular* class posed significant behavior management challenges. While encountering behavior management challenges is not uncommon for beginning teachers, there was general agreement among teachers and school leaders that the challenges would likely be less if Ms. Aria had taken a job in a non-urban school.

The particular challenges associated with each teacher's class varied, but across classes teachers perceived their students to be less prepared to successfully engage in reform-based mathematics practices than students in non-urban school settings. In general, the teachers judged all or many of their students to be low performing academically. This judgment was regularly validated by scores on mathematics tests mandated by the school district and school administration. Students' status as low-performing students was sometimes used by teachers to explain students' difficulties engaging in reform-based mathematics instruction. This was especially characteristic of Ms. Jarmin. Consider Ms. Jarmin's explanation of why her class experienced difficulty with a lesson in the Fall semester:

> You know you usually you have some high, medium and low, and I do have high, medium, and low. But they are all lower than normal. I don't have any gifted… you usually have some children who always get it, and you have the kids that you know get it with a little instruction, and then you always have some who can explain it to the rest of the class. And I don't always have that.

Rather than considering the mathematical basis of students' difficulties, teachers in this urban school were sometimes found to attribute difficulties to students' general abilities and skills as "low" learners.

Ms. Larsano also made claims about the special needs of her students. She sometimes indicated that she did not believe her students would think about mathematics in the ways suggested in the *Everyday Mathematics* teacher guide. In general, evidence suggests that Ms. Larsano believes ESOL students to need instruction that is characterized by strong teacher direction and ample positive reinforcement. Toward the end of the year, Ms. Larsano talked about why teachers on her team found the mathematics teaching practices learned about in professional development to play out differently with the different groups of students:

> I know you could give me the ideas, but that doesn't mean those ideas are going to work with my kids, you know my personal kids, and some of the things maybe we tried, and it didn't work out with this group, or I couldn't do it this way, and for the other teacher it did work. So it depends on the kids and on the groups.

Like Ms. Jarmin, Ms. Larsano primarily attributes difficulties experienced to the group of students. This perspective leaves open the possibility that other groups of students may be more successful with the suggested pedagogy. However, it limits the degree to which teachers consider the role of their own teaching actions in driving instructional outcomes.

As a group, the teachers studied attributed their difficulties establishing classroom norms supportive of reform-based mathematics teaching practices, in part, to students' backgrounds and previous instructional experiences. For instance, consider Ms. Aria's commentary on her difficulties establishing classroom norms toward the end of the school year:

> Students come to us - many of them - not being respected at home in a way that we expect them to respect each other in the classroom. So they don't necessarily have the social norms already that students might have at a different kind of school. I experienced that in my student teaching. The students listened to each other and were interested. We have worked on that all year here, and we are better, but there are still students who don't listen and we are always working on it.

This notion that students have difficulties engaging in classroom discussions as a community of learners because of their personal backgrounds was echoed in the comments made by multiple teachers. Indeed, the urban context may present teachers with greater challenges with regard to establishing classroom norms for student-to-student work than other instructional settings. However, at times, some teachers exclusively focused on students' backgrounds and limited skills to explain instructional difficulties, failing to consider how their own teaching actions may also have contributed to the difficulties at hand.

These examples suggest that there is a danger at urban schools of teachers attributing instructional challenges encountered completely to the personal characteristics and academic backgrounds of students in their classes. While these factors certainly present challenges, teachers must focus on their own sense of agency and ability to affect change in their classrooms by cultivating particular instructional practices. This is especially true with regard to reform-based mathematics practices. Reform-oriented mathematics instruction is complex and challenging to implement in all instructional settings (Franke et al., 2007). In all settings, but especially urban settings, teachers must exhibit patience and persistence in their efforts to learn about and cultivate reform-based mathematics teaching practices.

Lincoln Heights was also found to be characteristic of urban schools in that administrators reported having difficulty hiring and retaining knowledgeable teachers (Lippman et al., 1996). Because of this challenge, they sought to hire teachers with positive personal characteristics – but not necessarily curricular expertise – who were open to receiving *training* to improve their teaching. Perhaps related to the level of teaching expertise among faculty, Lincoln Heights administrators maintained a high level of control over the curriculum. Often, decisions were made for teachers to utilize curriculum materials that would decrease instructional decisions made by teachers. For instance, teachers spent two hours daily implementing a scripted reading program. Related to mathematics, the *SuccessMaker* computer-based instruction program was instituted to identify and remediate students' mathematics learning needs. In general this program was viewed by the administration as an important tool for helping students to achieve higher scores on statewide achievement tests. At times, teachers expressed that their administrators seemed to believe that time on *SuccessMaker* was going to have a greater impact on student learning than any other learning experiences a teacher might provide.

Additionally, the direct instruction pedagogy of these programs stands in opposition to the constructivist underpinnings of reform-based mathematics programs. Reform-based curricula and teaching practices emphasize the importance of the teacher as a decision-maker and encourage teachers to develop pedagogy that is characterized by inquiry. At the end of the year, Ms. Aria explained how the predominant pedagogy of direct instruction impacted her efforts to engage in reform-based mathematics teaching practices:

> At the beginning it was so hard keeping control of the class that I found myself, you know, in a teacher-led discussion, a lot of direct instruction, which also comes from the fact that our reading program is direct instruction, so you get used to it really fast, even if you're an exploration person, you get used to it very fast, and the kids are used to direct instruction, which I think even more than the teachers is how these kids all came to me, and they work so well in direct instruction, and then when you put them in another situation, everything, you know, goes crazy.

Ms. Aria is the only one of the four teachers studied who explicitly described a tension between the push to use direct instruction for reading and an inquiry-approach for mathematics. However, it seems reasonable that the circumstance of mandated curriculums emphasizing conflicting pedagogical approaches sends conflicting messages to teachers about how children learn. Also, research suggests that teachers in high-poverty schools are particularly inclined to believe their students need explicit instruction on discrete skills (Knapp, 1995a), which is aligned with a direct-instruction approach. Therefore, the implicit conflicting messages sent to teachers may serve as a further barrier to adoption of reform-oriented beliefs and teaching practices in urban school settings.

Finally, at the time of this study, Lincoln Heights was a school working to overcome a history of school failure as determined by highly publicized standardized tests of student achievement. The consequences of these tests appear to weigh heavily on instructional decisions made by school leaders and teachers alike. It is a finding of this study that, because Lincoln Height's has a history of school failure and a student population identified as at-risk by the school district, school leaders mandate additional requirements for teachers and students beyond that expected of other schools. At Lincoln Heights, this includes regular testing in mathematics and reading mandated by school leaders at the district and school level. In the classroom, administrators required that students spend 30 minutes daily on the computer-based instructional program *SuccessMaker* with the hope that this program would address students' personal learning needs and increase scores on the state tests. In practice, time spent preparing for and taking tests and on the *SuccessMaker* program takes away from the time teachers spend teaching mathematics with the *Everyday Mathematics* program. Although these various mathematics initiatives all aim to improve students' mathematics achievement, they send different messages to teachers about the kinds of pedagogy and instructional outcomes that are valued. In turn, this is thought to diminish the attention teachers give to reforming their mathematics instructional practices to reflect reform-based principles.

Furthermore, because the reading achievement test is particularly high-stakes for students[12], the two-hour block of time allotted for reading instruction at Lincoln Heights is given instructional priority. In practice, this means that time allotted for mathematics (and all subjects other than reading) is where the burden of these various initiatives falls. In addition to testing and *SuccessMaker* time coming out of these subjects, non-reading time is used for visits to the library, for students to be pulled for special services, and for occasional assemblies. As a result, teachers frequently indicate that their mathematics instructional time is fragmented, and they often do not have the minimum one-hour instructional block required for teaching the *Everyday Mathematics* program. Consider Ms. Rosena's discussion of the challenge of working at Lincoln Heights:

> They (the administrators) have the reading time as a sacred time, and they don't want any interruptions to occur during reading, but they ignore everything else and I understand why the big focus on reading, but then you have math, and you see that in the test scores. You see that math is not really that great compared to reading. So what about math? What about writing? What about the rest of - social studies or

[12] Third-grade students who do not pass the reading portion of the SST are slated for retention.

> science? I think it's a challenge for not only math but for the other classes, the constant interruption.

Because of the pressures of state testing exacerbated by Lincoln Heights' history of school failure, teachers at this school have more demands on their time than teachers in other instructional settings. These demands translate into less than adequate instructional time for implementing the reform-based *Everyday Mathematics* program. As a result, teachers find themselves perpetually behind the pacing guide and trying to cut corners in ways that are likely to negatively impact students' mathematics learning.

While the urban context appears to serve as a barrier to teachers' transitions to reform-based mathematics teaching practices, there are also some benefits of this context. First, administrators at Lincoln Heights were able to use their Title I funds to put in place a year-long program of professional development to support teachers' transitions to the *Everyday Mathematics* program. Evidence suggests that this professional development effort was successful with initiating changes in teaching practice that would have been unlikely to occur otherwise. Second, a barrier to adoption of reform-based mathematics teaching in some school settings is strong resistance from parents (Ball, 1996). While teachers at Lincoln Heights felt many pressures, they did not generally feel constrained instructionally by the demands of parents. In this way, teachers were allowed a bit more freedom to work through the challenges of transitioning to new teaching practices.

The fourth guiding research question of this study asks, "How does the urban context, as defined by the research literature and perceived by teachers and school leaders, influence mathematics instruction in this urban school?" Study findings suggest that the urban context influenced teachers' transitions to reform-based mathematics teaching practices in a variety of ways. The student population presented greater academic, behavioral, and emotional challenges than might be expected in a non-urban setting. Teachers tended to view their students as having certain deficits, and these deficits were often used to explain instances when reform-based practices were unsuccessful. This may have limited the degree to which teachers looked to their own teaching practices as factors in students' difficulties. In addition, teachers at Lincoln Heights were mandated to implement a variety of programs and initiatives that had significantly different pedagogical emphases than the inquiry-focused pedagogy of mathematics-reform and lessened their focus on transitioning to the new program. The conflicting messages regarding how students learn as well as school priorities and values may have served as barriers to reforming mathematics teaching. However, the opportunity for professional development afforded by Title I funds appears to have had a positive impact on teachers' efforts to use reform-based mathematics curricula. Certainly the professional development activities offset some of the problems faced in an urban context by focusing teachers on strategies for transitioning to a reform-based mathematics program throughout the year.

Study Implications

The findings of this study have several implications for efforts to support teachers' transitions to reform-based mathematics pedagogy. Implications for mathematics educators, curriculum developers, and school leaders are summarized in Table 30.

Surprisingly, teachers studied were found willing to initially engage in practices that were not aligned with their beliefs, but were suggested by the reform-based curriculum and professional development leaders. Professional development activities that helped teachers to envision reform-oriented teaching practices, such as video cases, strongly influenced

Table 30
Summary of Study Implications for Supporting Teacher Transitions to Reform-Based Mathematics Pedagogy

Study implications	Change agent: Curriculum developers	Teacher educators	School leaders
Provide strong support during initial implementation of reform-based mathematics curricula.		X	X
Increase teachers' mathematics content knowledge and pedagogical content knowledge that supports conceptually-grounded mathematics instruction on specific mathematics topics.		X	
Increase teachers' knowledge of and ability to anticipate common student difficulties related to specific mathematics topics.	X	X	
Support teachers with envisioning how flawed solutions can be used productively in class discussion of mathematics.	X	X	
Help teachers acquire knowledge of routines to support facilitation of mathematically productive discussions that engage students as a community of learners.		X	
Increase availability of professional development materials (video and text cases) that provide examples of exemplary reform-based mathematics teaching in diverse and challenging urban school settings.	X		
Focus case-based professional development for urban school teachers on cases that are situated in similar urban school settings.		X	
Engage teachers in critical examination of the nuances of reform-based mathematics teaching practices, especially practices that resemble reform recommendations but fall short of reflecting substantive aspects of reform.		X	X
Supply teachers with reform-based curriculum materials and tools to support students' mathematics learning.			X
Ensure that teachers have adequate instructional time dedicated to mathematics instruction utilizing reform-based mathematics teaching materials.			X
Support school leaders in becoming knowledgeable of and committed to reform-based mathematics pedagogy.		X	X
Limit the number of new initiatives teachers are expected to implement when they are first learning about and experimenting with reform-based mathematics pedagogy.			X
Coordinate curricular initiatives related to mathematics and across academic disciplines to be complementary rather than conflicting.	X		X

teachers' ways of using the new curriculum materials. During this period of experimentation, teachers judged the reform-based mathematics teaching ideas based on their experiences with their students. The degree to which teachers' beliefs aligned with a reform-orientation at the end of the year was related to their level of success with using the reform-based mathematics program. One implication of this study is that strong support during initial implementation of reform-based mathematics programs seems warranted, as this initial period of implementation provides a window in which teachers seem open to change. Additionally, efforts to increase teachers' mathematical knowledge for teaching are essential, as teachers' abilities to successfully teach in conceptually-supportive ways that are connected to student thinking appear highly related to this knowledge base and consequently to teachers' success with reform-based materials.

This study suggests that the teachers studied were particularly limited in their abilities to predict the kinds of difficulties students might have with tasks assigned. Just as curriculum materials and professional development aim to help teachers anticipate various correct ways students commonly approach problems in reform-oriented classrooms, teachers would benefit from guidance related to common student difficulties as well as how to respond to difficulties in ways that connect to student thinking.

An additional implication of this study is that teachers particularly need support with envisioning how students' flawed solutions can be productively used as springboards for inquiry (Borasi, 1994), especially in the context of whole class discussions. If teachers can envision how this practice might work and how it might positively support student learning, they may be willing to give it a try. In particular, video and text-based cases of teachers' different ways of handling student difficulties, like the classroom episodes presented, might be useful content for teachers to unpack and critique in the context of professional development.

Furthermore, this study found that teachers who primarily determined the contents of class discussions by calling on volunteers to share solutions were observed to have difficulty using students' solutions, particularly flawed ones, to make mathematical points. Other teachers, who were more intentional in their efforts to select and order class discussion of student-generated solutions such that particular mathematical ideas could be highlighted, appeared more successful in emphasizing important mathematical ideas in ways that supported student learning. Therefore, one promising avenue for professional development is to help teachers acquire knowledge of routines to support facilitation of meaningful mathematics discussions, as such routines are likely to help teachers to organize and access their content knowledge and pedagogical content knowledge more productively.

Silver et al. (2005) identify five components of pedagogical practices related to meaningful discussion of student-generated solutions that teachers can learn, practice, and improve. Prior to the lesson, teachers can think about solution methods and errors that students are likely to employ. As students work on tasks, teachers can learn to pay close attention to students' solution methods. In addition, they can use this time to select particular solutions to be shared in class discussion and determine the order for sharing such that important mathematical ideas can be optimally emphasized. Finally, during class discussion, teachers can develop student understanding by comparing and contrasting the various solutions. Professional development efforts could be organized to deliberately model and make explicit these components of teaching practice for facilitating meaningful mathematical discussions.

Findings from this study also suggest that some teachers in urban schools believe their students to have limited capability to engage in reform-based mathematics learning practices, such as problem solving with limited support and contributing productively to inquiry-oriented discussions of mathematics tasks. And teachers that held this belief typically attributed students' difficulties with particular aspects of reform-based mathematics instruction to personal backgrounds and academic aptitude rather than considering the contribution of controllable teaching practices. For this reason, it seems especially important for professional development with urban school teachers to emphasize examples of reform-based mathematics teaching situated in urban school settings with diverse and challenging student populations. While exemplary examples of reform-based mathematics teaching in urban settings are needed, there is also a need for teachers to engage in critical reflection around practices that only structurally resemble reform recommendations but fall short of reflecting important substantive aspects of reform. To this end, text and video cases in urban school settings could be used to help teachers critically examine nuances of reform-based mathematics teaching practices, particularly how differential teaching practices are likely to yield different student outcomes.

Finally, this study joins other research (Sowder, 2007) in emphasizing the critical role school leaders play in making conditions favorable for teacher transitions to reform-based mathematics pedagogy. School leaders can make certain that teachers have material support, including reform-based curricula and mathematical tools to support student learning. They can protect instructional time, ensuring that teachers have at least the minimum amount of recommended time designated for instruction centering on reform-based mathematics teaching. Additionally, they can organize opportunities for professional development to support implementation of reform-based mathematics programs. Through participation in professional development alongside teachers, school leaders can also become knowledgeable of reform-based mathematics teaching practices and demonstrate a commitment to such practices. Furthermore, school leaders can limit the number of new initiatives teachers are expected to implement when they are first learning about reform-based mathematics pedagogy. This would serve to establish clear priorities and allow teachers to focus their attention on cultivating teaching practices consistent with reform recommendations. Lastly, school leaders can support teachers' adoption of reform-based mathematics pedagogy by coordinating curricular initiatives within mathematics and across academic disciplines that are complementary rather than conflicting. While leaders in all schools are instrumental to successful adoption of reform-based mathematics pedagogy, instructional leadership for change in urban schools seems especially important due to the unique challenges and pressures faced by these schools.

Study Limitations and Directions for Future Research

The research findings presented provide insight into the interaction of teachers' knowledge, beliefs, and mathematics teaching practices during their first year of implementing a reform-based curriculum. However, this study has several limitations.

First, this study is limited in scope. It focuses on four teachers in a particular school context primarily when the focus of instruction is on multiplication and division. Future research should consider more fully if teachers' response patterns to student difficulties and associated teacher knowledge and beliefs hold across different instructional settings and other mathematics topics. Furthermore, an aspect of the teachers' school context in this study was involvement in a sustained program of professional development that provided a moderate level of support for change. Findings suggests that teachers may not have experimented with certain teaching practices had the professional development effort not been in place. Study of teachers with varying kinds of support is warranted to fully explain this question. In addition, some of the urban school teachers of focus in this study attributed instances of difficulty with reform-based teaching to students' deficits. This question of attribution and its implications would be useful to explore in a larger number of urban school teachers as well as in different types of instructional settings.

Second, this study focused on examining teachers' knowledge and beliefs as they manifested in teacher response patterns to student difficulties. While this focus on teacher response to student difficulties allowed for detailed analysis of the relationship between teacher knowledge, beliefs, and practice, there are other aspects of reform-oriented teaching for which it would be helpful to more fully understand the interaction of these variables. For instance, this data set might be used to consider how teachers' beliefs and knowledge influenced their facilitation of whole class discussion.

Third, this study makes claims about how teachers' actions facilitated or limited student thinking. However, classroom observations and aggregated student achievement scores on a standardized test of general mathematics achievement are the only sources of data used in this study to discern student learning. In a research project with greater scope, it would be helpful to include a component more fully dedicated to determining what students are learning from particular instances of instruction.

Fourth, the process of data collection and analysis for this research was carried out by a single person. While every effort was made to collect and analyze data in a consistent and unbiased manner, a research project of greater scope would be strengthened by coordinating the observations and interpretations of multiple researchers. This limitation was especially apparent in the context of classroom observations, in which the complexity of the classroom environment limited my ability to focus on all that might have been relevant to the focus on my study. It would have also have been helpful for multiple raters to have rated classroom observations using the Reformed Teaching Observation Protocol (RTOP) and teachers' survey responses to the Integrating Mathematics and Pedagogy (IMAP) Web-Based Beliefs-Survey, as this would have strengthened the credibility of findings made from data collected using these instruments.

Related to the particular context of the urban school in which this study was situated, it was suggested by one of the teachers that a barrier to implementation of the inquiry-oriented practices associated with reform-based mathematics teaching was the school-wide focus on using a direct instruction approach for reading. Future research might consider the extent to which teachers in various elementary school settings are expected to use pedagogies across disciplines that reflect conflicting learning theories. Additionally, there is a need to better understand if and how the circumstance of conflicting pedagogical recommendations within and across disciplines impacts teaching. In hindsight, it would have been helpful in this study to have more fully considered teachers' general knowledge and beliefs of child development and how children learn.

Finally, the analytical framework for this study posits that teacher actions are influenced primarily by the interaction of teachers' beliefs and knowledge. However, at times during this research, these constructs did not seem adequate to fully explain teacher behavior. For instance, one teacher readily indicated in interviews that she believed having students work with a partner or small group could have a positive affect on student learning. Yet, she did not regularly engage her students in cooperative learning. When asked to explain why, the teacher revealed that she preferred to have a quiet classroom without a lot of commotion. This preference appeared to override her belief in the value of group work. Schoenfeld (2007) suggests that, instead of focusing solely on beliefs, researchers might consider the broader construct of teachers' *orientations* toward teaching. Research aiming to understand teachers' orientations would include consideration of teachers' beliefs as well as other constructs that might influence decision-making such as preferences and values. Schoenfeld also emphasizes the importance of understanding teachers' goals in efforts to make sense of teacher actions. Future research might be designed to more deliberately consider teachers' actions in light of teacher knowledge as well as these additional promising constructs.

Conclusions

This study was designed to illuminate how particular aspects of teachers' knowledge and beliefs influenced their ways of supporting and limiting student thinking during initial use of a reform-based mathematics curriculum in an urban school. Focus on teachers' response to student difficulties proved a useful lens for examining the interaction of teacher knowledge, beliefs, and classroom practice as instances of student difficulty pressed teachers to rely on their existing knowledge and beliefs to act in the moment. Study findings suggest that, while teachers' ways of responding to student difficulties were clearly linked to both their knowledge and beliefs, some aspects of teacher response were more strongly linked to their knowledge whereas other response patterns were more strongly linked to beliefs. Additionally, reform-based mathematics curriculum materials and professional development were found to stimulate teacher experimentation with reform-oriented practices that were not initially aligned with teachers' beliefs. The degree of subsequent change in beliefs toward a reform-orientation appears to have been moderated by teachers' perceptions of their classroom experiences during this period of experimentation with reform-based recommendations. In addition, the urban context of this study appears to have presented a number of barriers to reforming mathematics teaching that would be present to a lesser extent at non-urban schools. The urban context was also found to influence the way teachers viewed their students and their own experiences experimenting with reform-based mathematics curricula, sometimes allowing teachers to attribute difficulties to uncontrollable student characteristics rather than controllable instructional factors.

This study highlights the complexity of reform-based mathematics teaching that builds on student thinking and reinforces the assertion that it is difficult for teachers to make the kinds of changes envisioned by reformers. In order to realize the vision of widespread, systematic reform in mathematics education, teachers require sustained support on multiple fronts. Teachers need material support in the form of reform-based mathematics curricula and the tools for supporting student learning as well as adequate instructional time and arrangements to utilize these materials. Teachers need time, intellectual space, and human support to critically examine traditional mathematics teaching practices and assumptions about student learning in order to inspire recognition of the need for alternative mathematics teaching practices and open the door to change in beliefs. To implement reform-based mathematics pedagogy well, many teachers require support with expanding and deepening their knowledge of school mathematics, knowledge of children's mathematical thinking, and knowledge of routines and scripts to enact such pedagogy. Additionally, many teachers need school-level support to customize reform-based teaching pedagogy to their particular teaching contexts, troubleshoot difficulties, and reflect on their classroom experiences with this new kind of teaching. While progress has been made in understanding the complexity of teachers' work engaging in reform-based mathematics teaching that builds on student thinking, there is much left to be learned about how to support and sustain teacher transitions to this pedagogical approach.

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