Diffusion Approximations for Multiscale Stochastic Networks in Heavy Traffic

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Abstract

XIN LIU: Diffusion Approximations for Multiscale Stochastic Networks in Heavy Traffic.
(Under the direction of Amarjit Budhiraja.)

Applications arising from computer, telecommunications, and manufacturing systems lead to many challenging problems in the simulation, stability, control, and design of stochastic models of networks. The networks are usually too complex to be analyzed directly and thus one seeks suitable approximate models. One class of such approximations are diffusion models that can be rigorously justified when networks are operating in heavy traffic, i.e., when the network capacity is roughly balanced with network load.

We study stochastic networks with time varying arrival and service rates and routing structure. Time variations are governed, in addition to the state of the system, by two independent finite state Markov processes $\tilde{X}$ and $\tilde{Y}$. Transition times of $\tilde{X}$ are significantly smaller than the typical interarrival and processing times whereas the reverse is true for the Markov process $\tilde{Y}$. We first establish a diffusion approximation for such multiscale queueing networks in heavy traffic. The result shows that, under appropriate heavy traffic conditions, properly normalized queue length processes converge weakly to a Markov modulated reflected diffusion process. More precisely, the limit process is a reflected diffusion with drift and diffusion coefficients that are functions of the state process, the invariant distribution of $\tilde{X}$ and a finite state Markov process which is independent of the driving Brownian motion. We then study the stability properties of such Markov modulated reflected diffusion processes and establish positive recurrence and geometric ergodicity properties under suitable stability conditions. As consequences, we obtain results on the moment generating function of the invariant probability measure, uniform in time moment estimates and functional central limit results for such processes. We also study relationship between invariant measures of the Markov modulated constrained diffusion processes and that of the underlying queueing network. It is shown that, under suitable heavy traffic and stability conditions, the invariant probability measure of the queueing process converges to that of the corresponding Markov modulated reflected diffusion.

The last part of this dissertation focuses on ergodic control problems for discrete time controlled Markov chains with a locally compact state space and a compact action space under suitable stability, irreducibility and Feller continuity conditions. We introduce a flexible family of controls, called action time sharing (ATS) policies, associated with a given continuous stationary Markov control. It is shown that the long term average cost for such a control policy, for a broad range of one stage cost functions, is the same as that for the associated stationary Markov policy. Through examples we illustrate the use of such ATS policies for parameter estimation and adaptive control problems.

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Contents

List of Notation and Symbols

$I\!\!N$	Set of natural numbers
$I\!\!N_0$	$I\!\!N \cup \{0\}$
$I\!\!R$	Set of real numbers
$I\!\!R_+$	Set of nonnegative real numbers
$I\!\!R^d$	d-dimensional Euclidean space
$I\!\!R^d_+$	$\{(x_1, x_2, \cdots, x_d)' : x_i \in I\!\!R_+\}$
1_A	Indicator function of set A
A^o	Interior of set A
A^c	Compliment of set A
$\bar{A}$	Closure of set A
∂A	Boundary of set A
$\mathrm{dist}(x, A)$	The distance between $x \in I\!\!R^d$ and $A \subset I\!\!R^d$: $\inf\{\lvert x - y\rvert : y \in A\}$
$\mathbb{M}'$	Transpose of matrix $\mathbb{M}$
$\mathbb{M}_i$ or $\mathbb{M}(i)$	The i^{th} row of matrix $\mathbb{M}$
$\mathbb{M}_{ij}$ or $\mathbb{M}(i, j)$	The $(i, j)^{th}$ entry of $\mathbb{M}$
$\mathbb{I}$ or $\mathbb{I}_{K\times K}$	The K-dimensional identity matrix
$\mathbb{M} > 0$	$\mathbb{M}_{ij} > 0$ for all i, j
v_i	The i^{th} component of vector v
$\mathrm{diag}(v)$	The $K \times K$ diagonal matrix whose diagonal entries are given by the components of K-dimensional vector v
$(d_1 \vert d_2 \vert \ldots \vert d_K)$	A matrix with d_i as the i^{th} column, $i = 1, 2, \ldots, K$
$\mathcal{P}(U)$	Collection of all probability measures on space U

$\mathcal{B}(U)$	Borel σ-field on space U
$\mathcal{C}(U)$	The collection of continuous functions from space U to $I\!R$
$\mathcal{C}_0^k(U)$	The collection of functions from space U to $I\!R$ which are k-times continuously differentiable and vanish at infinity
$\mathcal{C}_0^\infty(U)$	The collection of functions from space U to $I\!R$ which are infinitely differentiable and vanish at infinity
$\mathcal{C}_b(U)$	The collection of bounded continuous functions from space U to $I\!R$
$\mathrm{BM}(U)$	The class of bounded measurable functions from space U to $I\!R$
$U^{\otimes L}$	$\{(x_1,\ldots,x_L) : x_i \in U, i = 1,\ldots,L\}$ for space U
$D([0,\infty),V)$	The class of right continuous functions with left limits defined from $[0,\infty)$ to $V(\in \mathcal{B}(I\!R^d))$ with the usual Skorohod topology
$C([0,\infty),V)$	The class of continuous functions from $[0,\infty)$ to $V(\in \mathcal{B}(I\!R^d))$ with the local uniform topology
$\Delta f(t)$	$f(t) - f(t-)$
$\lvert f\rvert$	L_1 norm of function f
$\lvert f\rvert_\infty$	$\sup_{x\in U} \lvert f(x)\rvert$ for $f \in \mathrm{BM}(U)$
$\lvert g\rvert_t^*$	$\sup_{0\le s\le t} \lvert g(s)\rvert$ for $g : [0,\infty) \to I\!R^d$ and $t \ge 0$
$\langle\cdot,\cdot\rangle$	Inner product operator
∇	Gradient operator
∇^2	Hessian matrix
$[X,Y]$	The quadratic covariation of stochastic processes X, Y (also called the bracket process of X, Y)
$X_n \Rightarrow X$	Weak convergence of stochastic process X_n to X
$\mu_n \Rightarrow \mu$	Weak convergence of probability measures μ_n to μ
$A \doteq B$	A is defined by B
$A \equiv B$	A equals B identically

$\|\mu\|$	Total variation norm of a signed measure μ: $\sup_{f:\|f\|\leq 1} \|\mu(f)\|$
$\|\mu\|_f$	f-norm of signed measure μ ($f \geq 1$): $\sup_{f:\|g\|\leq f} \|\mu(g)\|$
ι	The identity map from $[0, \infty)$ to $[0, \infty)$
$c_1, c_2, \ldots$	Generic positive constants, whose values may change from one proof to another
RCLL	Right continuous and having left limit
SRBM	Semimartingale reflected Brownian motion

Chapter 1

Introduction

The dissertation contains two distinct sets of problems. The first concerns stochastic networks with Markov modulated parameters in heavy traffic and their diffusion approximations while the second studies estimation and adaptive ergodic control for discrete time Markov chains.

Stochastic networks is an active research area dealing with problems in simulation, approximation, stability, control, and design of stochastic models of networks, with applications in diverse areas such as computer, telecommunications, and manufacturing systems. One of the most fundamental models in queueing systems is the well known Jackson network ([31]), which considers exponential interarrival and service times and Bernoulli type routing. However, the elegant distributional theory and asymptotic properties of Jackson networks break down when one attempts to incorporate some more realistic features of specific application settings into such a model. For example, when the distributions of the primitives are relaxed to be more general than i.i.d. exponential, or the rates of interarrivals and services are allowed to depend on the state of the system, such generalized Jackson networks become intractable and thus one seeks suitable approximate models. One class of such approximations, of particular interest in the current work, are diffusion models that can be rigorously justified when networks are operating in the heavy traffic regime, i.e., when the network capacity is roughly balanced with network load. Attraction of such approximations primarily lies in the fact that, analogous to the central limit theory, the limit model is described only using a few important parameters of the underlying networks and the complex distributional properties of the primitives are averaged out.

In this dissertation, we study a Markov modulated queueing network, where the arrival and service rates and the routing structure are modulated by Markov processes. We consider two independent finite state continuous time Markov processes $\{\tilde{X}(t) : t \geq 0\}$ and $\{\tilde{Y}(t) : t \geq 0\}$ which can be interpreted as the random environment in which the system is operating. The process $\tilde{X}$ changes state at a much higher rate than the typical interarrival and service times in the system, while the reverse is true for $\tilde{Y}$. The arrival and service rates depend on the state (i.e. queue length) and two background Markov processes $\tilde{X}$ and $\tilde{Y}$. The routing mechanism is governed by $\tilde{X}$. More precisely, the queueing network consists of K service stations each of which has an infinite capacity buffer. We denote the i^{th} station by $P_i, i \in \mathbb{K} \doteq \{1, 2, \ldots, K\}$. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station P_i a customer is routed to another service station or exits the system. The routing mechanism is modulated by $\tilde{X}$ whose state x, at any given instant, determines a $K \times K$ substochastic matrix $\mathbb{P}_x$. Roughly speaking, the conditional probability that a job completed at time instant t at station P_i is routed to station P_j, given $\tilde{X}(t) = x$, equals the $(i, j)^{th}$ entry of the matrix $\mathbb{P}_x$. The goal of the study is

(i) to establish suitable reduced models using techniques from diffusion approximations and heavy traffic theory;

(ii) to develop a comprehensive stability theory for the diffusion approximations of such networks;

(iii) to study the validity of the approximation of the steady state of the queueing network through that of the limit diffusion model.

In Chapter 2, we study diffusion approximations for Markov modulated queueing networks of the form described above, in heavy traffic. In order to formulate a precise heavy traffic condition we consider a sequence of queueing networks, indexed by $n \in \mathbb{N}$. Let Q^n denote the K-dimensional queue length process in the n^{th} network. Roughly speaking, the arrival and service rates in the n^{th} network are $\mathcal{O}(n)$. Markov process governing the routing in the n^{th} network is denoted as X^n. We assume that

$$X^n(t) = X(l_n t) \text{ and } \frac{l_n}{n^{r_0+1}} \to \infty \text{ for some } r_0 > 1/2,$$

where X is a Markov process with values in a finite state space $\mathbb{L}$ and a unique stationary distribution $\{p_x^* : x \in \mathbb{L}\}$. This condition says that the transitions of X^n occur at a faster rate than arrivals and service completions. Denote by Y^n the slowly changing background process modulating the arrival and service rates in the n^{th} network. Y^n is a finite state continuous time Markov process with infinitesimal generator $\mathbb{Q}^n$ which converges to some matrix $\mathbb{Q}$. We are interested in limit theorems for such networks under a diffusion type scaling and appropriate heavy traffic conditions. We consider the pair Markov process $(\widehat{Q}^n, Y^n)$, where $\widehat{Q}^n = Q^n/\sqrt{n}$ is the normalized queue length process. The main result (Theorem 2.3.2) shows that $(\widehat{Q}^n, Y^n)$ converges weakly to a Markov process (Z, Y), where Y is a finite state continuous time Markov process with infinitesimal generator $\mathbb{Q}$, and the process Z is a Markov modulated reflected diffusion process with coefficients depending on (Z, Y), described as follows.

$$Z(t) = \Gamma\left(Z(0) + \int_0^{\cdot} b\left(Z(u), Y(u)\right) du + \int_0^{\cdot} \sigma\left(Z(u), Y(u)\right) dW(u)\right)(t), \; t \geq 0. \quad (1.0.1)$$

Here W is a K-dimensional standard Brownian motion independent of Y and Γ is the Skorohod (reflection) map associated with the reflection matrix $\mathbb{I} - \sum_{x \in \mathbb{L}} p_x^* \mathbb{P}'_x$, where $\mathbb{I}$ is the $K \times K$ identity matrix. The coefficients b and σ are obtained in terms of the averaged (with respect to p^*) arrival/service rates and routing matrices. Thus one finds that, as $n \to \infty$, the effect of the Markov process X^n on the dynamics of $\widehat{Q}^n$ is averaged out with respect to p^*. Such a result, in addition to model simplification, allows for weaker assumptions on the routing matrices and the traffic load in the network. For example, the assumptions permit the routing structure to oscillate between that of a closed network and an open network. Also the network can change between states with traffic intensity greater than one and those with intensity smaller than one. A paper [12] based on this work has been accepted for publication in *Stochastic Processes and Their Applications*.

Chapter 3 is devoted to the study of stability properties of Markov modulated constrained diffusion processes as in (1.0.1). The state process Z is assumed to be constrained to take values in a convex polyhedral cone $G \subset I\!\!R^K$. Denote by $I\!\!H$ the finite state space of Y and $q^* = \{q_j^* : j \in I\!\!H\}$ its unique stationary distribution. We assume that the drift b can be expressed as $b(z,y) = b_1(z,y) + b_2(y)$, $(z,y) \in G \times I\!\!H$, where $b_1 : G \times I\!\!H \to I\!\!R^K$ and $b_2 : I\!\!H \to I\!\!R^K$ are measurable maps. Define $b_2^* = \sum_{j \in I\!\!H} q_j^* b_2(j)$ and $b^*(z,y) = b_1(z,y) + b_2^*$. Under regularity assumption on the associated Skorohod map, we show that, when $b^*(z,y)$ takes values in the δ-interior of a certain cone $\mathcal{C}$ (described in terms of the given directions of constraints – see (3.2.5)) for some $\delta > 0$, the pair Markov process (Z,Y) is positive recurrent and has a unique invariant probability measure. In fact, we establish a significantly stronger result, namely the process (Z,Y) is geometrically ergodic and its invariant distribution has a finite moment generating function in a neighborhood of zero. We also obtain uniform time estimates for polynomial moments (of all orders) of the process and functional central limit results for long time fluctuations of the empirical means around their stationary averages. For the case when $b_1 = 0$, we obtain a sharper result. Namely, if b_2^* is in the interior of $\mathcal{C}$, then (Z,Y) is geometrically ergodic, and if b_2^* is outside of $\mathcal{C}$, (Z, Y) is transient. We also obtain similar stability results for a Markov modulated semimartingale reflected Brownian motion, which can be considered as a special case of Markov modulated constrained diffusion processes with $b_1 = 0$ and constant covariance matrix σ (however, here we make much weaker assumptions on the associated Skorohod problem). With the usual completely-$\mathcal{S}$ assumption on the reflection matrix (Assumption 3.2.6), the stability condition is formulated in terms of certain fluid trajectories (see [24]) associated with the "average drift", where the average is taken with respect to the stationary distribution q^*.

In Chapter 4, we study convergence of invariant measures for the Markov modulated queueing networks in heavy traffic considered in Chapter 2. In view of the complex structure of the original queueing network, it is important for computational purposes that the steady state behavior of the limit diffusion model is a good approximation for that of the underlying queueing system. In this chapter, we provide a rigorous justification for such an approximation and show that under suitable heavy traffic and stability conditions, the invariant probability measure for the Markov modulated queueing network converges to that of the limit diffusion process. For simplicity and since in the heavy traffic the effect of the fast Markov process on the network is averaged out with respect to its stationary distribution, we consider in this chapter an open queueing network with constant routing matrix and arrival and service rates that only depend on the state and the slowly changing Markov process (i.e., the network parameters do not depend on X^n). In the n^{th} network, $\widehat{Q}^n$ and Y^n are the normalized queue length process and the modulating Markov process, respectively. Recall that in Chapter 2 we show that $(\widehat{Q}^n, Y^n)$ converges to (Z, Y) weakly, and in Chapter 3 that under suitable stability condition, (Z, Y) has a unique invariant probability measure. The main result in Chapter 4 (Theorem 4.1.1) shows that, under conditions, $(\widehat{Q}^n, Y^n)$ admits a stationary distribution which converges to that of (Z, Y) as $n \to \infty$.

Finally, Chapter 5 considers a topic in estimation and adaptive ergodic control for Markov chains. Markov Decision processes have been used extensively to model systems that involve both stochastic behavior and control. A common measure of performance in such systems is the long-time average (or ergodic) criterion. Given all relevant parameters, a typical goal is to find a simple (e.g. feedback or deterministic stationary) policy that achieves the optimal value. However, in many practical setting, the information on the underlying parameters of the system is incomplete. The goal of adaptive control is to obtain an optimal policy, when some relevant information concerning the behavior of the system is missing. The classical approach is to design an algorithm which collects information, while at the same time choosing controls, in a way that the chosen controls "approach optimality over time." The paper [2] shows for a finite state controlled Markov processes, that given any Markov policy q, one can construct another policy (the so-called action time sharing (ATS) policy corresponding to q) for which the control decisions can deviate from those dictated by the Markov policy q, and still produce the same long term average cost, as long as certain conditional frequencies converge to the correct values. This flexibility is useful in many situations that require estimation and control under incomplete information. For example, ATS policies can be used to develop variance reduction schemes for ergodic control problems and allow for sampling (namely using controls without regards to the ensuing cost), e.g., for the purpose of collecting auxiliary information. In the current work we are concerned with a setting where the state and action spaces are not (necessarily) countable. Our main objective is to formulate an appropriate definition for an ATS policy which, similar to the countable case, not only leads to long term costs that are identical to those for the corresponding given Markov control, but also allows for flexible implementation well suited for various estimation and adaptive control goals. Using a suitable sequence of "converging partitions" of the state space, we show how ATS policies can be constructed for a given setting and used for estimation of unknown parameters and adaptive control problems while preserving desirable optimality properties. A paper [13] based on this work (joining with A. Budhiraja and A. Schwartz) has been submitted for publication to *SIAM Journal on Control and Optimization*.

Chapter 2

Diffusion approximations for multiscale stochastic networks in heavy traffic

2.1 Introduction

We study stochastic networks in which arrival and service rates, as well as the routing structure change over time. More precisely, we consider a setting in which two independent finite state continuous time Markov processes $\{\tilde{X}(t) : t \geq 0\}$ and $\{\tilde{Y}(t) : t \geq 0\}$ govern the variations in the parameters of the system. These processes can be interpreted as a random environment in which the system is operating. The process $\tilde{X}$ changes states at a much higher rate than the typical inter-arrival and service times in the system, while the reverse is true for $\tilde{Y}$. The variations in the routing mechanism of the network are governed by $\tilde{X}$, whereas the arrival and service rates at various stations depend on the state process (i.e., queue length process) and both $\tilde{X}$ and $\tilde{Y}$. It is shown that, under appropriate heavy traffic conditions, the properly normalized sequence of queue length processes converges weakly to a reflected Markov modulated diffusion process. More precisely, the limit process is a reflected diffusion with drift and diffusion coefficients that are functions of the state process, the invariant distribution of $\tilde{X}$ and a finite state Markov process which is independent of the driving Brownian motion.

Queueing systems studied here can be regarded as generalizations of Jackson networks. The first general result in the study of diffusion approximations for such networks is due to Reiman[40], who considered the case where the arrival and service processes, associated with K processing stations in the network, are mutually independent renewal processes, and the routing mechanism is governed by a fixed $K \times K$ substochastic routing matrix $\mathbb{P}$. The main result in [40] shows that, under a suitable heavy traffic condition, the properly scaled queue length processes converge weakly to a certain reflected Brownian motion. Yamada[50] (also see Mandelbaum and Pats[36] and Kushner[35]) extended Reiman's work to queueing networks with state dependent rates, i.e., a setting where the rates of arrival and service processes depend on the current state of network. In this case the scaling limits are reflected diffusion processes with state dependent drift and diffusion coefficients. In a related work, Chen and Whitt[16] establish heavy-traffic limit theorems for a class of queueing networks with time inhomogeneous service times. Mandelbaum and Pats, in [37], considered open queueing networks with state dependent routing structure. The diffusion limit involves a Skorohod problem with reflection directions that vary as functions of the state process. Queues in random environment have been considered by Choudhury, Mandelbaum et al[17] (see also Chapter 6 of [39]). The authors considered a $G/G/s$ queue with $s > 1$, where the traffic intensity changes according to the state of the environment. The environment process is taken to be a finite state right continuous process (with finitely many jumps over any finite interval), whose states change at rates slower than typical arrival and service rates.

In the model considered in this chapter, the queueing network consists of K service stations each of which has an infinite capacity buffer. We denote the i^{th} station by $P_i, i \in I\!\!K \doteq \{1, 2, \ldots, K\}$. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station P_i, a customer is routed to another service station or exits the system. The arrival and service rates depend on the state of the system and two Markov processes $\tilde{X}$ and $\tilde{Y}$. Transition times of $\tilde{X}$ are fast, while $\tilde{Y}$ changes states slowly relative to the typical arrival and service rates. The routing mechanism is modulated by $\tilde{X}$ whose state x, at any given instant, determines a $K \times K$ substochastic matrix $\mathbb{P}_x$. Roughly speaking, the conditional probability that a job completed at time instant t at station P_i is routed to station P_j, given $\tilde{X}(t) = x$, equals the $(i,j)^{th}$ entry of the matrix $\mathbb{P}_x$.

In order to give a precise mathematical formulation, we introduce a scaling parameter $n \geq 1$ and consider a sequence of queueing networks of the type described above, indexed by n. Let Q^n denote the K-dimensional queue length process in the n^{th} network. Roughly speaking, the arrival and service rates in the n^{th} network are $\mathcal{O}(n)$. Markov process, governing the routing and arrival and service rates in the n^{th} network, is denoted as X^n. Transition times of this process are of the order $\mathcal{O}(1/l_n)$, where $l_n n^{-(1+r_0)} \to \infty$ for some $r_0 > 1/2$ (see (2.3.2)). In addition, we are given another Markov process Y^n with transition times of the order $\mathcal{O}(1)$ that modulates the arrival and service rates in the n^{th} network. More precisely, Y^n is a finite state continuous time Markov process whose infinitesimal generator $\mathbb{Q}^n$ converges to some matrix $\mathbb{Q}$. The goal of this work is to establish limit theorems for networks with such a hierarchy of time scales under appropriate heavy traffic conditions. The heavy traffic conditions used in this work (see Assumption 2.3.1) differ from the usual formulation. Here we do not require (near) traffic balance for each fixed state of the system and background processes. In fact the traffic intensity can change values according to the state of the Markov process X^n and take values both smaller and larger than 1. In this sense, we impose a weaker form of traffic balance condition which is formulated in terms of the equilibrium measure of X^n. In a similar spirit, our assumptions allow for the routing structure to oscillate between that of different (e.g., open and closed) networks (see the example below Assumption 2.3.1). However, we require the network to be open "on the average", where "the average" is taken with respect to the equilibrium measure of X^n.

The main result, Theorem 2.3.2, considers the pair Markov process $(\widehat{Q}^n, Y^n)$, where $\widehat{Q}^n$ is the appropriately normalized queue length process, and shows that $(\widehat{Q}^n, Y^n)$ converges weakly to a Markov process (Z, Y). In fact we will see that the process Y is Markov by itself with a finite state space and generator $\mathbb{Q}$, and Z can be characterized as the solution of a reflected stochastic differential equation with coefficients depend on both Z and Y and a driving Brownian motion that is independent of Y. One of the main steps in the proof of Theorem 2.3.2 is Theorem 2.3.3 which proves the tightness of $(\widehat{Q}^n, Y^n)$ and characterizes weak limit points in terms of a suitable martingale problem (cf. [34]). Proof of Theorem 2.3.3 is given in Section 2.4. The key ingredient in this proof is Lemma 2.4.1 which makes precise the intuitive property that, as $n \to \infty$, the dynamics of $\widehat{Q}^n$ depend on X^n only through its equilibrium distribution.

Multiscale models considered in this work are natural for many network settings (cf. [17]). Consider, for example, a large computer network where one is interested in modeling the traffic behavior of files with moderate size over a long period of time within a small subset of nodes in the system. Denote by $\mathcal{E}$ the collection of all nodes in the network and let $\mathcal{E}_0 \subset \mathcal{E}$ be the subset of nodes of interest. One is interested in building a model for traffic between nodes in $\mathcal{E}_0$ without taking a very precise account of the interactions between such nodes and those in $\mathcal{E} \setminus \mathcal{E}_0$. Alternatively, $\mathcal{E}_0$ may be the entire network (i.e. $\mathcal{E}_0 = \mathcal{E}$) but one would like to consider a reduced model which does not take an explicit and detailed account of small file sized traffic. One approach to such problems is to model the effect of nodes in $\mathcal{E} \setminus \mathcal{E}_0$ (or alternatively of small sized files) at a node $e \in \mathcal{E}_0$ by a rapidly varying channel capacity (at e), with variations governed by an extraneous Markov process. If a large number of nodes in $\mathcal{E} \setminus \mathcal{E}_0$ are connected to e (or the small file sized traffic exhibits temporal variations at rate significantly higher than that of the large file traffic), one expects that the rate at which the channel capacity changes is much higher than the transmission rate of a typical file through e. Rapid changes in channel capacity lead to variations in processing rates and available routing options for moderate sized files processed at nodes in $\mathcal{E}_0$. In addition to such rapid changes, one may have changes in input flows, and processor failure and repair patterns that occur infrequently but need to be accounted for in a treatment of such systems over long periods of time. One is thus led to a traffic model for nodes in $\mathcal{E}_0$ in terms of a Jackson type network where the arrival/service rates and routing probability matrices vary randomly over time according to finite state Markov processes of the form considered in this work.

The chapter is organized as follows. In Section 2.2, we describe the precise network model. Section 2.3 introduces the diffusion scaling considered in this work and formulates the main assumptions that are used. In this section we also present the main result (Theorem 2.3.2) and its proof. The key ingredients for the proof are given through Theorem 2.3.3, Theorem 2.3.1 and Proposition 2.3.2. The first result (Theorem 2.3.3) is proved in Section 2.4 while the last two are relegated to Appendix.

2.2 Network model

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space satisfying the usual conditions, that is $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$ and $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$ for all $t \in [0, \infty)$. Define two continuous time finite state $\{\mathcal{F}_t\}$ Markov processes $\{\tilde{X}(t) : t \in [0, \infty)\}$ and $\{\tilde{Y}(t) : t \in [0, \infty)\}$ on $(\Omega, \mathcal{F}, P)$. Let $\mathbb{L} \doteq \{1, 2, \ldots, L\}$ and $\mathbb{H} \doteq \{1, 2, \ldots, H\}$ be the state spaces of $\tilde{X}$ and $\tilde{Y}$, respectively. We will make the following assumption on $\tilde{X}$.

Assumption 2.2.1. *The Markov process* $\tilde{X}$ *has a unique stationary distribution* $p^* = (p_1^*, \ldots, p_L^*)$*. Furthermore, denoting the transition probability* $P[\tilde{X}(t) = i | \tilde{X}(0) = j]$ *by* $p_{ji}(t)$*, we have, for all* $t \geq 0$*,*

$$\sup_{i,j \in \mathbb{L}} |p_{ji}(t) - p_i^*| \leq a_1 e^{-a_2 t}$$

for some constants $a_1, a_2 \in (0, \infty)$*.*

The queueing network consists of K service stations, denoted as $P_1, \ldots, P_K$, each of which has an infinite capacity buffer. All customers at a station are homogeneous in terms of service requirement and routing decisions (in a sense to be made precise). Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station P_i, a customer is routed to another service station or exits the system. Let $Q_i(t)$ denote the number of customers at station P_i at time t. Then

$$Q_i(t) = Q_i(0) + A_i(t) - D_i(t) + \sum_{j=1}^{K} D_{ji}(t), \quad i \in \mathbb{K}, \tag{2.2.1}$$

where $A_i(t)$ is the number of arrivals from outside at station P_i by time t, $D_i(t)$ is the number of service completions by time t at station P_i, and $D_{ji}(t)$ is the number of jobs that are routed to P_i immediately upon completion at station P_j by time t. We write $A = (A_1, \ldots, A_K)'$, $D = (D_1, \ldots, D_K)'$, and $Q = (Q_1, \ldots, Q_K)'$. The processes A_i and $D_i, i \in \mathbb{K}$, are counting processes defined on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ such that $A_i, D_j, D_{ji} : i, j \in \mathbb{K}$ have no common jumps. It is also assumed that, for certain measurable functions $\lambda_i, \tilde{\alpha}_i : \mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L} \to \mathbb{R}_+$, the processes

$$\begin{aligned} &A_i(\cdot) - \int_0^{\cdot} \lambda_i(Q(u), \tilde{Y}(u), \tilde{X}(u))du, \\ &D_i(\cdot) - \int_0^{\cdot} \tilde{\alpha}_i(Q(u), \tilde{Y}(u), \tilde{X}(u))du \end{aligned} \tag{2.2.2}$$

are locally square integrable $\{\mathcal{F}_t\}$ martingales. Finally, we assume that

$$A, D, \tilde{Y} \text{ have no common jumps.}$$

The functions λ_i and $\tilde{\alpha}_i, i \in I\!\!K$, represent the arrival and service rates. We denote by $I\!\!K_0$ ($I\!\!K_0 \subseteq I\!\!K$) the set of indices of stations which receive arrivals from outside. In particular, $\lambda_i(z, y, x) = 0$ for all $(z, y, x) \in I\!\!R_+^K \times I\!\!H \times I\!\!L$ whenever $i \in I\!\!K \backslash I\!\!K_0$. Reflecting the fact that no service occurs when the buffer is empty, $\tilde{\alpha}_i(z, y, x) = 0$ if $z_i = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_K)'$ and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_K)'$. Additional conditions on λ and $\tilde{\alpha}$ will be introduced in Assumption 2.3.1.

The counting process D_{ji} is given in terms of an auxiliary Markov process $\tilde{X}$ and certain marked point process constructed from D_j. To formulate this precisely, we introduce a collection of routing matrices. For each $x \in I\!\!L$, we are given a nonnegative $K \times K$ substochastic matrix $\mathbb{P}_x$ with zero diagonal entries. We denote the $(i, j)^{th}$ entry of $\mathbb{P}_x$ by p^x_{ij} and $1 - \sum_{j=1}^K p^x_{ij}$ by $p^x_{i,K+1}$. Roughly speaking, upon completion of service at time t at station P_i, given $\tilde{X}(t) = x$, a customer is routed to station P_j with probability p^x_{ij}, $j \in I\!\!K$, or exits the system with probability $p^x_{i,K+1}$.

In order to make this precise, we proceed as follows. For $i \in I\!\!K$, let E_i be the space of K dimensional vectors. Each vector in E_i has 0 or 1 components and 0 at the i^{th} coordinate, and there is at most one 1 component. Let $G_i = E_i^{\bigotimes L}$. Define $\mu_i \in \mathcal{P}(G_i)$ as

$$\mu_i(v) = \prod_{x=1}^{L} \left(\sum_{j=1}^{K} p^x_{ij} 1_{\{v^x_j = 1\}} + p^x_{i,K+1} 1_{\{\sum_{j=1}^K v^x_j = 0\}} \right), \; v = (v^1, \ldots, v^L) \in G_i. \quad (2.2.3)$$

Note that for $v \in G_i$ and $x \in I\!\!L$, v^x is a K dimensional vector in E_i with $\mu_i\{v : v^x_j = 1\} = p^x_{ij}$, where v^x_j is the j^{th} component of v^x. Consequently, the measure μ_i captures the probabilities of routing from station i to other stations in the network for all possible states of the background Markov process $\tilde{X}$. More precisely, upon completion of service at station P_i at time t, the routing of the job is governed by a G_i valued random variable v with probability distribution μ_i such that a customer is routed to station P_j if $v_j^{\tilde{X}(t)} = 1$. Otherwise, if $v_j^{\tilde{X}(t)} = 0$ for every $j \in I\!\!K$, the customer exits the network. Abusing notation, we write v^x as $v(x)$, and denote the j^{th} component of $v(x)$ by $v_j(x)$.

We next introduce a collection of marked point processes. For $i \in I\!\!K$, Let $\{Z_k^i\}_{k\geq 1}$ be an i.i.d. sequence of G_i valued random variables with law μ_i. For $i \in I\!\!K$, let $\{T_k^i\}_{k\geq 1}$ be the transition times of D_i. Consider the marked point process (See Chapter VIII of [7].)

$$S_i(t, B) = \sum_{k\geq 1} 1_{\{Z_k^i \in B\}} 1_{\{T_k^i \leq t\}}, \quad B \in \mathcal{B}(G_i), \ t \geq 0. \tag{2.2.4}$$

We assume that $S_i(t, B)$ is $\{\mathcal{F}_t\}$ adapted and has a $\{\mathcal{F}_t\}$ intensity kernel $\lambda_i(t, B)$ given as

$$\lambda_i(t, B) = \tilde{\alpha}_i(Q(t-), \tilde{Y}(t-), \tilde{X}(t-))\mu_i(B). \tag{2.2.5}$$

Namely, for any bounded predictable map $H(\omega, t, v)$ from $\Omega \times [0, \infty) \times G_i$ to $I\!\!R$,

$$\int_{[0,t]} \int_{G_i} H(s, v) S_i(ds, dv) - \int_{[0,t]} \int_{G_i} H(s, v) \tilde{\alpha}_i(Q(t), \tilde{Y}(t), \tilde{X}(t))\mu_i(dv) ds$$

is a $\{\mathcal{F}_t\}$ martingale.

Note that Z_k^i can be written as $(Z_k^i(1), \ldots, Z_k^i(L))$, where each $Z_k^i(x)$ is a K dimensional vector in E_i. More importantly, $Z_k^i(x)$ defines the routing vector, corresponding to state x in $I\!\!L$, for the k^{th} job completion at station i. More precisely, if this job completion occurs at time instant t, then it is routed to state j if and only if the j^{th} entry of $Z_k^i(\tilde{X}(t))$ is 1.

For $i, j \in I\!\!K$, the process D_{ji} can now be written as follows.

$$D_{ji}(t) = \int_{[0,t]\times G_j} v_i(\tilde{X}(u-))S_j(du, dv). \tag{2.2.6}$$

Note that, for $j \in I\!\!K$, G_j is finite, so the integral on G_j is a finite sum. Combining (4.1.1) and (2.2.6), the evolution of the system state is described by the following equation.

$$Q_i(t) = Q_i(0) + A_i(t) - D_i(t) + \sum_{j=1}^{K} \int_{[0,t]\times G_j} v_i(\tilde{X}(u-))S_j(du, dv), \; i \in I\!\!K. \tag{2.2.7}$$

2.3 Diffusion scaling and main results

In this section, we present a diffusion limit theorem for a suitably scaled version of the queue length process. Consider a sequence of queueing networks of the type described in Section 2.2, indexed by $n \in I\!\!N$. We assume that all networks have the same topology and the same set of routing matrices $\{\mathbb{P}_x : x \in I\!\!L\}$. All the notation introduced in Section 2.2 is carried forward except that we append an n in an appropriate place to denote quantities which depend on n. In particular, on the filtered probability space $(\Omega^n, \mathcal{F}^n, P^n, \{\mathcal{F}^n_t\}_{t\geq 0})$, for $i \in I\!\!K$, A^n_i and D^n_i are counting processes with rates λ^n_i and $\tilde{\alpha}^n_i$ respectively, and Q^n_i denotes the queue length process at station P_i in the n^{th} network. The marked point process S^n_i is defined by (2.2.4) with $\{T^i_k\}$ denoting the transition times of D^n_i. Denote by X^n the $\{\mathcal{F}^n_t\}$ Markov process governing the routing in the n^{th} network. Let $l_n \in (0, \infty)$ be such that for some $r_0 > 1/2$,

$$l_n n^{-(r_0+1)} \to \infty, \quad \text{as } n \to \infty. \tag{2.3.1}$$

We assume that, for $t \geq 0$,

$$X^n(t) = X(l_n t), \tag{2.3.2}$$

where X has the same probability law as the process $\tilde{X}$ introduced in Section 2.2 (in particular, X has the state space $\mathbb{L}$ and satisfies Assumption 2.2.1). This condition, along with Assumption 2.3.1(ii) and (v) introduced below, makes mathematically precise the property that the transition times of X^n are significantly smaller than the typical inter-arrival and service times. The slow process Y^n in the n^{th} network is assumed to be a $\{\mathcal{F}_t^n\}$-Markov process with state space $\mathbb{H}$ and infinitesimal generator $\mathbb{Q}^n$, such that $\mathbb{Q}^n$ converges to some matrix $\mathbb{Q}$ as $n \to \infty$. We assume

$$A^n, D^n, Y^n \text{ have no common jumps.} \tag{2.3.3}$$

With this notation, (2.2.7) holds with $(Q, A, D, S, \tilde{X})$ replaced by $(Q^n, A^n, D^n, S^n, X^n)$. Given a $\mathbb{R}^K$ valued stochastic process $Z^n \doteq (Z_1^n, \ldots, Z_K^n)$, we will denote by $\widehat{Z}^n$ the scaled process which is defined as

$$\widehat{Z}^n(t) \doteq \left(\frac{Z_1^n(t)}{\sqrt{n}}, \ldots, \frac{Z_K^n(t)}{\sqrt{n}} \right), \ t \geq 0. \tag{2.3.4}$$

The main result of this work shows that the scaled queue length process $\widehat{Q}^n$ converges weakly to a certain constrained diffusion process. We now introduce the coefficients in this limit diffusion model. We assume that, for each $i \in \mathbb{K}$, $\tilde{\alpha}_i^n$ restricted to $\mathbb{R}_+^K \backslash \{z \in \mathbb{R}_+^K : z_i = 0\} \times \mathbb{H} \times \mathbb{L}$ can be extended to a function α_i^n defined on $\mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$. We assume α_i^n has the following form

$$\alpha_i^n(z, y, x) = \varpi_i^n(z, y)\theta_i(x), \ (z, y, x) \in \mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}, \tag{2.3.5}$$

where $\varpi_i^n : \mathbb{R}_+^K \times \mathbb{H} \to \mathbb{R}_+$ and $\theta_i : \mathbb{L} \to \mathbb{R}_+ \backslash \{0\}$ are measurable maps. We write $\theta = (\theta_1, \ldots, \theta_K)'$ and $\varpi^n = (\varpi_1^n, \ldots, \varpi_K^n)'$. Then for $(z, y, x) \in \mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$,

$$\alpha^n(z, y, x) \doteq (\alpha_1^n(z, y, x), \ldots, \alpha_K^n(z, y, x))' = \text{diag}(\theta(x))\varpi^n(z, y).$$

Additional conditions on ϖ^n and θ are specified in Assumption 2.3.1. Recall p^* introduced in Assumption 2.2.1. Let $\bar{\theta} = \sum_{x=1}^{L} p_x^* \theta(x)$. In order to define the drift and diffusion coefficients of the limit model, define, for $(z, y, x) \in I\!R_+^K \times I\!H \times I\!L$,

$$\begin{aligned}
\bar{\lambda}^n(z, y) &= \sum_{x=1}^{L} p_x^* \lambda^n(z, y, x), \\
\bar{\alpha}^n(z, y) &= \sum_{x=1}^{L} p_x^* \alpha^n(z, y, x) = \text{diag}(\bar{\theta}) \varpi^n(z, y), \\
\overline{\mathbb{P}} &= \left(\sum_{x=1}^{L} p_x^* \mathbb{P}_x \text{diag}(\theta(x)) \right) \left[\text{diag}(\bar{\theta}) \right]^{-1}.
\end{aligned} \tag{2.3.6}$$

To formulate the heavy traffic conditions, define

$$b^n(z, y) \doteq \frac{1}{\sqrt{n}} \left(\bar{\lambda}^n(z, y) - [\mathbb{I} - \overline{\mathbb{P}}'] \bar{\alpha}^n(z, y) \right), \ (z, y) \in I\!R_+^K \times I\!H. \tag{2.3.7}$$

The following is our main assumption. Parts (viii) and (ix) restate, for convenience, the assumptions made in (2.3.1), (2.3.2) and (5.4.2).

Assumption 2.3.1.

(i) *The spectral radius of* $\overline{\mathbb{P}}$ *is strictly less than* 1.

(ii) *For all* $n \in I\!N$ *and* $(z, y, x) \in I\!R_+^K \times I\!H \times I\!L$*, there exists some* $\kappa_1 \in (0, \infty)$ *such that*

$$|\lambda^n(z, y, x)| \le n\kappa_1, \quad |\alpha^n(z, y, x)| \le n\kappa_1.$$

(iii) *There exists a constant* $\kappa_2 \in (0, \infty)$ *such that*

$$\sup_{(z,y) \in I\!R_+^K \times I\!H} |b^n(z, y)| \le \kappa_2.$$

(iv) *There exists* $b \in \mathcal{C}_b(I\!R_+^K \times I\!H)$ *such that* $b^n(\sqrt{n}z, y) \to b(z, y)$ *uniformly for* (z, y)

in compact subsets of $\mathbb{R}_+^K \times \mathbb{H}$ *as* $n \to \infty$.

(v) *There exist* $\mathbb{R}_+^K$ *valued bounded Lipschitz functions* $\bar\lambda, \bar\alpha$ *defined on* $\mathbb{R}_+^K \times \mathbb{H}$*, such that*

$$\frac{\bar\lambda^n(\sqrt{n}z, y)}{n} \to \bar\lambda(z, y), \quad \frac{\bar\alpha^n(\sqrt{n}z, y)}{n} \to \bar\alpha(z, y)$$

uniformly for (z, y) *in compact subsets of* $\mathbb{R}_+^K \times \mathbb{H}$ *as* $n \to \infty$*. Furthermore,* $\bar\lambda = [\mathbb{I} - \overline{\mathbb{P}}']\bar\alpha$.

(vi) *There exist* $\underline{\lambda}, \underline{\alpha} \in (0, \infty)$ *such that, for any* $n \in \mathbb{N}$ *and* $(z, y) \in \mathbb{R}_+^K \times \mathbb{H}$,

$$\inf_{i \in \mathbb{K}_0} \frac{\bar\lambda_i^n(\sqrt{n}z, y)}{n} \geq \underline{\lambda}, \quad \inf_{i \in \mathbb{K}} \frac{\bar\alpha_i^n(\sqrt{n}z, y)}{n} \geq \underline{\alpha}.$$

(vii) *For each* $i \in \mathbb{K} \backslash \mathbb{K}_0$*, there exists* $j \in \mathbb{K}_0$ *such that* $\overline{\mathbb{P}}_{ji}^m > 0$ *for some* $m \in \mathbb{N}$.

(viii) *The Markov process*

$$X^n(t) = X(l_n t), \ t \geq 0,$$

where X *is a Markov process with the same probability law as* $\tilde{X}$*, and* l_n *satisfies that, for some* $r_0 > 1/2$, $l_n n^{-(r_0+1)} \to \infty$ *as* $n \to \infty$.

(ix) *For* $i \in \mathbb{K}$*, the service rate* α_i^n *defined on* $\mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$ *has the following form*

$$\alpha_i^n(z, y, x) = \varpi_i^n(z, y)\theta_i(x),$$

where $\varpi_i^n : \mathbb{R}_+^K \times \mathbb{H} \to \mathbb{R}_+$ *and* $\theta_i : \mathbb{L} \to \mathbb{R}_+ \backslash \{0\}$ *are measurable maps.*

Part (i) of the assumption says that the network under a suitable averaging is open. This averaging is given in terms of the stationary distribution p^* of the fast Markov process X^n. In particular, the assumption allows for the routing structure to oscillate between that of different (e.g., open and closed) networks. For example, consider the setting of a single server queue where $\mathbb{L} = \{1,2\}$, $\mathbb{P}_1 = 1, \mathbb{P}_2 = 1/2$, $\theta(x) \equiv 1$, and λ, α are constants (see Figure 1). Clearly whenever both p_1^* and p_2^* are

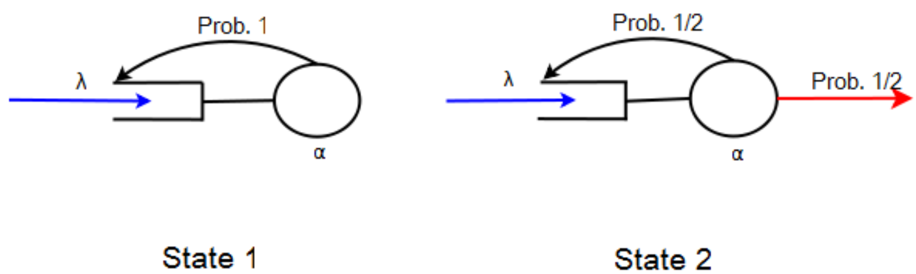

Figure 2.1: The network is closed when X^n is in state 1 and it is open when X^n is in state 2.

strictly positive, Assumption 2.3.1(i) is satisfied. Parts (ii)-(v) of the assumptions, culminating in the requirement that

$$\bar{\lambda} = [\mathbb{I} - \overline{\mathbb{P}}']\bar{\alpha}, \tag{2.3.8}$$

represent the heavy traffic condition for this model. Once more note that the traffic balance condition is formulated in terms of the parameters averaged w.r.t. the stationary distribution of X^n. For example, consider the setting of a single server queue where $\mathbb{K} = \{1,2\}, \mathbb{L} = \{1,2\}, \overline{\mathbb{P}} = 0, \theta(x) \equiv 1, p_1^* = p_2^* = 1/2$, $\lambda_1^n(z,y,x) = \lambda_2^n(z,y,x) = 2nx$, and $\alpha_1^n(z,y,x) = \alpha_2^n(z,y,x) = 3n$, for all $(z,y,x) \in \mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$. Then although, for any given state of the background process X^n, the system is either underloaded or overloaded, the traffic balance condition in the sense of (2.3.8) is satisfied. Parts (vi) and (vii) are nondegeneracy conditions which ensure that the diffusion coefficients in the reflected diffusion limit model are uniformly nondegenerate (cf. (2.3.10)). We note that condition (vii) can be assumed without loss of generality since if it fails for some $i \in I\!\!K \backslash I\!\!K_0$, one can consider a reduced model that is obtained by omitting station P_i.

Assumptions introduced in this section will be made throughout this work and will not be noted explicitly in the statement of the results.

Define, for $(z, y) \in I\!\!R_+^K \times I\!\!H$, a $K \times [K + K(K+1)]$ dimensional matrix $\Sigma(z, y)$ as

$$\Sigma(z,y) \doteq (A(z,y), B_1(z,y), \ldots, B_K(z,y)), \tag{2.3.9}$$

where A and $B_i, i \in I\!\!K$, are $K \times K$ and $K \times (K+1)$ matrices given as follows. For $(z, y) \in I\!\!R_+^K \times I\!\!H$,

$$A(z,y) = \text{diag}\left(\sqrt{\bar{\lambda}_1(z,y)}, \ldots, \sqrt{\bar{\lambda}_K(z,y)}\right),$$
$$B_i(z,y) = \left(B_i^0(z,y), B_i^1(z,y), \ldots, B_i^K(z,y)\right),$$

where $B_i^0(z,y) \doteq -\mathbf{1}_i \sqrt{\overline{\mathbb{P}}_{i,k+1} \bar{\alpha}_i(z,y)}$, $B_i^i(z,y) = \mathbf{0}$ and for $j \in I\!\!K$ and $j \neq i$, $B_i^j(z,y) \doteq \mathbf{1}_{ij} \sqrt{\overline{\mathbb{P}}_{ij} \bar{\alpha}_i(z,y)}$. Here $\mathbf{1}_i$ is a K dimensional vector with 1 at the i^{th} coordinate and 0 elsewhere, $\mathbf{0}$ is K dimensional zero vector, and $\mathbf{1}_{ij}$ is a K dimensional vector with -1 at the i^{th}, 1 at the j^{th} coordinates, and 0 elsewhere. It is easy to see that due to Assumption (vi) and (vii), $\Sigma(z,y)\Sigma(z,y)'$ is uniformly nondegenerate (see [9, Appendix]). More precisely, there exists a $\kappa \in (0, \infty)$ such that, for all $\zeta \in I\!\!R^K$ and $(z, y) \in I\!\!R_+^K \times I\!\!H$,

$$\zeta'(\Sigma(z,y)\Sigma(z,y)')\zeta \geq \kappa \zeta' \zeta. \tag{2.3.10}$$

One can then find a Lipschitz function $\sigma : I\!\!R_+^K \times I\!\!H \to I\!\!R^{K \times K}$ (cf. [45, Theorem 5.2.2]) such that $\Sigma(z,y)\Sigma(z,y)' = \sigma(z,y)\sigma(z,y)'$. Note that the uniform boundedness of λ and α (Assumption 2.3.1(v)) implies that $\sigma(z,y)$ is uniformly bounded on $I\!\!R_+^K \times I\!\!H$. We next recall the definition of a Skorohod map associated with the reflection matrix $[\mathbb{I} - \overline{\mathbb{P}}']$. For $i \in I\!\!K$, define $F_i = \{z \in I\!\!R_+^K : z_i = 0\}$. We will call F_i the i^{th} face of

$S \doteq I\!R_+^K$.

Definition 2.3.1. Let $\psi \in D([0,\infty), I\!R^K)$ be such that $\psi(0) \in S$. Then $(\phi, \eta) \in D([0,\infty), I\!R_+^K \times I\!R_+^K)$ solves the Skorohod problem (SP) for ψ with respect to S and $[\mathbb{I} - \overline{\mathbb{P}}']$ if and only if the following hold.

(i) $\phi(t) = \psi(t) + [\mathbb{I} - \overline{\mathbb{P}}']\eta(t) \in S$ for all $t \geq 0$.

(ii) Write $\eta = (\eta_1, \ldots, \eta_K)'$ and $\phi = (\phi_1, \ldots, \phi_K)'$. We have, for $i \in I\!K$, (a) $\eta_i(0) = 0$, (b) η_i is nondecreasing, and (c) η_i can increase only when ϕ is on F_i, that is

$$\int_0^\infty 1_{\{\phi_i(s)>0\}} d\eta_i(s) = 0.$$

Let $D_S([0,\infty), I\!R^K) \doteq \{\psi \in D([0,\infty), I\!R^K) : \psi(0) \in S\}$. Let $D \subset D_S([0,\infty), I\!R^K)$ be the domain on which there is a unique solution to the SP. On D, we define the Skorohod map (SM) Γ associated with the data $(S, [\mathbb{I} - \overline{\mathbb{P}}'])$ as $\Gamma(\psi) = \phi$, if $(\phi, [\mathbb{I} - \overline{\mathbb{P}}']^{-1}(\phi - \psi))$ is the unique solution of the SP posed by ψ. The following result, which is a consequence of Assumption 2.3.1(i) (See [29, 50]), gives the regularity of the Skorohod map defined by $(S, [\mathbb{I} - \overline{\mathbb{P}}'])$.

Proposition 2.3.1. *The Skorohod map is well defined on all of $D_S([0,\infty), I\!R^K)$, that is, $D = D_S([0,\infty), I\!R^K)$, and the SM is Lipschitz continuous in the following sense: There exists a constant $C \in (1,\infty)$ such that, for all $\psi_1, \psi_2 \in D_S([0,\infty), I\!R^K)$,*

$$\sup_{t\geq 0} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| \leq C \sup_{t\geq 0} |\psi_1(t) - \psi_2(t)|.$$

As an immediate corollary of the above proposition we have the following.

Corollary 2.3.1. *For $n \in I\!N$, let $\psi^n \in D$, $\phi^n = \Gamma(\psi^n)$, and $\eta^n = [\mathbb{I} - \overline{\mathbb{P}}']^{-1}(\phi^n - \psi^n)$. Suppose $\{\psi^n : n \geq 1\}$ is C-tight in $D([0,\infty), I\!R^K)$. Then (ψ^n, ϕ^n, η^n) is C-tight in*

$D([0,\infty), \mathbb{R}^K \times \mathbb{R}_+^K \times \mathbb{R}_+^K)$. *Moreover, if* (ψ, ϕ, η) *is any weak limit of* (ψ^n, ϕ^n, η^n), *then* (ϕ, η) *is the unique solution of the SP for* ψ, *that is* $\phi = \Gamma(\psi)$ *and* $\eta = [\mathbb{I} - \overline{\mathbb{P}}']^{-1}(\phi - \psi)$.

We now introduce the diffusion limit model.

Definition 2.3.2. Fix $\nu \in \mathcal{P}(\mathbb{R}_+^K \times \mathbb{H})$. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}_{t\geq 0})$ be a filtered probability space on which are given RCLL adapted processes Z, Y, W which satisfy the following conditions.

(i) W is a K dimensional standard $\{\bar{\mathcal{F}}_t\}$ Brownian motion.

(ii) Y is a $\mathbb{H}$ valued $\{\bar{\mathcal{F}}_t\}$ Markov process with infinitesimal generator $\mathbb{Q}$.

(iii) $\bar{P} \circ (Z(0), Y(0))^{-1} = \nu$.

(iv) The following stochastic integral equation holds. For all $t \geq 0$, a.s.,

$$Z(t) = \Gamma\left(Z(0) + \int_0^{\cdot} b\left(Z(u), Y(u)\right) du + \int_0^{\cdot} \sigma\left(Z(u), Y(u)\right) dW(u)\right)(t). \quad (2.3.11)$$

We denote $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, Z, Y, W)$ satisfying the above properties by Ψ_ν and refer to it as a system with initial condition ν.

The following result, proved in the appendix, is a consequence of Lipschitz properties of σ, b and the Skorohod map Γ.

Theorem 2.3.1. *For each* $\nu \in \mathcal{P}(\mathbb{R}_+^K \times \mathbb{H})$, *there is a system* Ψ_ν *with initial condition* ν. *If* $\Psi_\nu^{(i)} \doteq (\Omega_i, \mathcal{F}_i, P_i, \{\mathcal{F}_t^{(i)}\}, Z^{(i)}, Y^{(i)}, W^{(i)}), i = 1, 2,$ *are two such systems, then*

$$P_1 \circ (Z^{(1)}, Y^{(1)}, W^{(1)})^{-1} = P_2 \circ (Z^{(2)}, Y^{(2)}, W^{(2)})^{-1}.$$

The following is the main result of this chapter.

Theorem 2.3.2. *Let $\Psi_\nu = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, Z, Y, W)$ be a system with initial condition ν. Suppose that the sequence of measures $P^n \circ (\widehat{Q}^n(0), Y^n(0))^{-1}$ converges weakly to ν. Then*

$$P^n \circ (\widehat{Q}^n, Y^n)^{-1} \Rightarrow \bar{P} \circ (Z, Y)^{-1},$$

as $n \to \infty$, in $\mathcal{P}(D([0,\infty), \mathbb{R}_+^K \times \mathbb{H}))$.

To illustrate Theorem 2.3.2, we consider two special cases. In Corollary 2.3.2, the arrival and service rates λ^n, α^n are assumed to only depend on the fast Markov process X^n, while in Corollary 2.3.3, λ^n, α^n are modulated by both X^n and Y^n (but not the queue length process). In both corollaries, the routing mechanism is modulated by X^n as before. We also provide two explicit examples following the corollaries.

Corollary 2.3.2. *Suppose the arrival and service rates only depend on the Markov process X^n, i.e., λ^n and α^n are measurable maps from $\mathbb{L}$ to $\mathbb{R}_+$ satisfying (2.2.2). If $P^n \circ \widehat{Q}^n(0)$ converges to a probability measure $\bar{\nu} \in \mathcal{P}(\mathbb{R}_+^K)$, then $P^n \circ \widehat{Q}^n \Rightarrow \bar{P} \circ Z$, where Z is a reflected diffusion defined on a filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}_{t\geq 0})$ such that*

$$Z(t) = \Gamma\left(Z(0) + b\cdot + \sigma W(\cdot)\right)(t), \ t \geq 0. \tag{2.3.12}$$

Here $Z(0)$ is a random variable with law $\bar{\nu}$ and W is a K dimensional $\{\mathcal{F}_t\}$ Brownian motion. The drift and diffusion coefficients b, σ are constant which can be defined in the same way as $b(z,y), \sigma(z,y)$ with $\lambda^n(z,y,x)$ and $\alpha^n(z,y,x)$ replaced by $\lambda^n(x)$ and $\alpha^n(x)$, respectively.

Suppose now the arrival and service rates λ^n and α^n are modulated by both X^n and Y^n (but not the queue length process). Then λ^n and α^n are measurable maps from $\mathbb{H} \times \mathbb{L}$ to $\mathbb{R}_+$ satisfying (2.2.2). For $(y,x) \in \mathbb{H} \times \mathbb{L}$, we can define $b(x), \sigma(x)$ in the same way as $b(z,y), \sigma(z,y)$ with $\lambda^n(z,y,x)$ and $\alpha^n(z,y,x)$ replaced by $\lambda^n(y,x)$

and $\alpha^n(y,x)$, respectively. For $t \geq 0$, let

$$\tilde{Z}(t) = \Gamma\left(\tilde{Z}(0) + \int_0^{\cdot} b\left(Y(u)\right) du + \int_0^{\cdot} \sigma\left(Y(u)\right) dW(u)\right)(t). \qquad (2.3.13)$$

Define system $\tilde{\Psi}_\nu = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, \tilde{Z}, Y, W)$ as in Definition 2.3.2 with Z replaced by $\tilde{Z}$.

Corollary 2.3.3. *Suppose λ^n and α^n only depend on the Markov processes X^n and Y^n. Let $\tilde{\Psi}_{\tilde{\nu}} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, \tilde{Z}, Y, W)$ be a system with initial condition $\tilde{\nu}$. If $P^n \circ (\widehat{Q}^n(0), Y^n(0))$ converges to a probability measure $\tilde{\nu} \in \mathcal{P}(I\!R_+^K \times I\!H)$, then $P^n \circ (\widehat{Q}^n, Y^n) \Rightarrow \bar{P} \circ (Z, Y)$ as $n \to \infty$.*

Example 2.3.1. *Let*

$$K = 2, \lambda^n(x) = (\frac{4}{3}nx, \frac{4}{3}nx)', \alpha^n = (4n, 4n)', I\!L = \{1,2\}, p^* = (\frac{1}{2}, \frac{1}{2})',$$
$$\mathbb{P}_1 = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}, \mathbb{P}_2 = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix}.$$

Therefore,

$$\bar{\lambda}^n = (2n, 2n)', \bar{\alpha}^n = \alpha^n = (4n, 4n)', \overline{\mathbb{P}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

and

$$\bar{\lambda} = (2,2)', \bar{\alpha} = (4,4)', b^n = (0,0)', \Sigma\Sigma' = \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix}.$$

Then we can find $b = (0,0)'$ and $\sigma = \begin{pmatrix} 2\sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{6} \end{pmatrix}$. Hence the limit process is

$$Z(t) = \Gamma\left(Z(0) + \sigma W(\cdot)\right)(t), \quad t \geq 0.$$

Example 2.3.2. *Let $K, \mathbb{L}, p^*, \mathbb{P}_1, \mathbb{P}_2$ as in the above example, and make $\lambda^n(x,y) = (\frac{2}{3}\sqrt{n}x + 2ny, \frac{2}{3}\sqrt{n}x + 2ny)', \alpha^n(y) = (4ny, 4ny)'$. Therefore,*

$$\bar{\lambda}^n(y) = (\sqrt{n} + 2ny, \sqrt{n} + 2ny)', \bar{\alpha}^n(y) = \alpha^n = (4ny, 4ny)', b^n = (1,1)',$$

$$\bar{\lambda}(y) = (2y, 2y)', \bar{\alpha}(y) = (4y, 4y)', \Sigma\Sigma' = \begin{pmatrix} 8y & -4y \\ -4y & 8y \end{pmatrix}.$$

Then we can find $b = (1,1)', \sigma(y) = \begin{pmatrix} 2\sqrt{2y} & 0 \\ -\sqrt{2y} & \sqrt{6y} \end{pmatrix}$. The limit process can be described as follows.

$$Z(t) = \Gamma\left(Z(0) + b\cdot + \int_0^{\cdot} \sigma(Y(u))W(u)\right)(t),\ t \geq 0.$$

In order to prove Theorem 2.3.2, we will characterize the processes (Z, Y) in terms of a suitable martingale problem. We begin by introducing the generator of this Markov process. For $f_1 \in \mathcal{C}_0^2(\mathbb{R}_+^K)$, $(z,y) \in \mathbb{R}_+^K \times \mathbb{H}$ and $i \in \mathbb{K}$, define

$$Lf_1(z,y) = b(z,y)'\nabla f_1(z) + \frac{1}{2}Tr\left(\Lambda(z,y)\nabla^2 f_1(z)\right),\ D_i f_1(z) = d_i'\nabla f_1(z),$$

where d_i is the i^{th} column of $[\mathbb{I} - \overline{\mathbb{P}}']$, $\Lambda(z,y) = \sigma(z,y)\sigma(z,y)'$, ∇f denotes the gradient vector of f, and $\nabla^2 f$ is the Hessian matrix of f. For $f_2 \in BM(\mathbb{H})$, let $\mathbb{Q}f_2 \in BM(\mathbb{H})$ be defined as

$$\mathbb{Q}f_2(y) \doteq \sum_{j=1}^{H} \mathbb{Q}_{yj} f_2(j), \quad y \in \mathbb{H}.$$

Define $f \doteq f_1 \otimes f_2$ as $f(z,y) = f_1(z)f_2(y), (z,y) \in \mathbb{R}_+^K \times \mathbb{H}$. We denote the class of all such functions by $\mathcal{G}$. For $f \in \mathcal{G}$, let $\mathcal{L}f, \mathcal{D}_i f \in BM(\mathbb{R}_+^K \times \mathbb{H})$ be defined as

$$\mathcal{L}f(z,y) = Lf_1(z,y)f_2(y) + f_1(z)\mathbb{Q}f_2(y),$$
$$\mathcal{D}_i f(z,y) = D_i f_1(z) f_2(y). \tag{2.3.14}$$

Proof of Theorem 2.3.2 is based on the following characterization result in terms of a martingale problem, the proof of which is given in the appendix.

Proposition 2.3.2. *Fix $\nu \in \mathcal{P}(\mathbb{R}_+^K \times \mathbb{H})$. Let (Z, η, Y) be a stochastic process with sample paths in $C([0,\infty) : \mathbb{R}_+^K \times \mathbb{R}_+^K) \times D([0,\infty), \mathbb{H})$ on some filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\})$, such that $\bar{P} \circ (Z(0), Y(0))^{-1} = \nu$, and for all $f \in \mathcal{G}$,*

$$f(Z(\cdot), Y(\cdot)) - f(Z(0), Y(0)) - \int_0^{\cdot} \mathcal{L}f(Z(u), Y(u))\,du - \sum_{i=1}^K \int_0^{\cdot} \mathcal{D}_i f(Z(u), Y(u))\,d\eta_i(u) \tag{2.3.15}$$

is a $\{\bar{\mathcal{F}}_t\}$ martingale. Then there is a K dimensional (standard) Brownian motion W defined on this filtered space such that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, Z, Y, W)$ is a Ψ_ν system with initial condition ν.

The multiplication form of the operators $\mathcal{L}, \mathcal{D}_i$ (see (2.3.14)) is a consequence of the assumption about no common jumps made in (2.3.3). This, in particular, is consistent with the fact that the processes Y and W in Proposition 2.3.2 are independent (see Lemma 2.5.1).

The following theorem, proved in Section 2.4, will be the key step in the proof of Theorem 2.3.2.

Theorem 2.3.3. *Assume that the sequence of measures $P^n \circ (\widehat{Q}^n(0), Y^n(0))^{-1}$ converges weakly to some $\nu \in \mathcal{P}(\mathbb{R}_+^K \times \mathbb{H})$. Then we have the following two results.*

(i) *$(\widehat{Q}^n, Y^n, \eta^n)$ is tight, where, for $i \in \mathbb{K}$,*

$$\eta_i^n = \frac{1}{\sqrt{n}} \int_0^t \bar{\alpha}_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u)) 1_{\{\widehat{Q}_i^n(u)=0\}} du.$$

Furthermore, $(\widehat{Q}^n, \eta^n)$ is C-tight.

(ii) *Let (Z, Y, η) be any weak limit given on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Then, for $f \in \mathcal{G}$, expression (2.3.15) is a $\{\bar{\mathcal{F}}_t\}$ martingale with*

$$\bar{\mathcal{F}}_t = \sigma\left(Z(u), Y(u), \eta(u) : u \le t\right).$$

Combining the above theorem with Proposition 2.3.2 and Theorem 2.3.1, the proof of Theorem 2.3.2 is completed as follows.

Proof of Theorem 2.3.2. Since $P^n \circ (\widehat{Q}^n(0), Y^n(0))$ converges to ν, by Theorem 2.3.3, $(\widehat{Q}^n, Y^n, \eta^n)$ is tight. Let (Z, Y, η) be any (weak) limit point of $(\widehat{Q}^n, Y^n, \eta^n)$ on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Define $\bar{\mathcal{F}}_t = \sigma\left(Z(u), Y(u), \eta(u) : u \le t\right)$. From Theorem 2.3.3 (ii) and Proposition 2.3.2, there is a K dimensional (standard) Brownian motion W on this probability space such that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, Z, Y, W)$ is a Ψ_ν system with initial distribution ν. Finally, weak uniqueness from Theorem 2.3.1 yields that $P^n \circ (\widehat{Q}^n, Y^n)^{-1} \Rightarrow \bar{P} \circ (Z, Y)^{-1}$ as $n \to \infty$. ■

2.4 Proof of Theorem 2.3.3

This section is devoted to the proof of Theorem 2.3.3. In Proposition 2.4.1, we prove Theorem 2.3.3 (i), i.e., the property that $(\widehat{Q}^n, Y^n, \eta^n)$ is tight. Proposition 2.4.2 gives the proof of Theorem 2.3.3 (ii), which characterizes the limit points of $(\widehat{Q}^n, Y^n, \eta^n)$ in terms of a martingale problem. We begin with a lemma which is a key ingredient in the proofs of both propositions.

Lemma 2.4.1. *Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions from $\mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$ to $\mathbb{R}$ such that, for some $\kappa_0 \in (0, \infty)$, $|g_n(z, y, x)| < \kappa_0$ for all $n \in \mathbb{N}$ and $(z, y, x) \in$*

$\mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$. *For* $(z,y) \in \mathbb{R}_+^K \times \mathbb{H}$, *define* $\bar{g}_n(z,y) = \sum_{x=1}^{L} p_x^* g_n(z,y,x)$. *Let*

$$h^n(t) \doteq \sqrt{n} \int_0^t \left[g_n(\widehat{Q}^n(u), Y^n(u), X^n(u)) - \bar{g}_n(\widehat{Q}^n(u), Y^n(u)) \right] du, \;\; t \geq 0.$$

Then h^n *converges to* 0 *in probability, in* $C([0,\infty), \mathbb{R})$, *as* $n \to \infty$.

Proof: For $(z,y,x) \in \mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$, let $g_n^c(z,y,x) = g_n(z,y,x) - \bar{g}_n(z,y)$. Recall the parameter $r_0 > 1/2$, introduced in (2.3.1). Fix $r \in (1/2, r_0)$. Then

$$\begin{aligned} h^n(t) &= \sqrt{n} \int_0^t g_n^c(\widehat{Q}^n(u), Y^n(u), X^n(u)) du \\ &= \frac{1}{n^{\frac{1}{2}+r}} \int_0^{tn^{1+r}} g_n^c\Big(\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big), Y^n\Big(\frac{u}{n^{r+1}}\Big), X^n\Big(\frac{u}{n^{r+1}}\Big)\Big) du. \end{aligned}$$

Let, for $t \in (0,\infty)$,

$$\begin{aligned} h_1^n(t) &\doteq \frac{1}{n^{\frac{1}{2}+r}} \int_0^{\lfloor tn^{1+r} \rfloor} g_n^c\Big(\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big), Y^n\Big(\frac{u}{n^{r+1}}\Big), X^n\Big(\frac{u}{n^{r+1}}\Big)\Big) du, \\ h_2^n(t) &\doteq \frac{1}{n^{\frac{1}{2}+r}} \int_{\lfloor tn^{1+r} \rfloor}^{tn^{1+r}} g_n^c\Big(\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big), Y^n\Big(\frac{u}{n^{r+1}}\Big), X^n\Big(\frac{u}{n^{r+1}}\Big)\Big) du. \end{aligned}$$

Then $h^n = h_1^n + h_2^n$. Since $g_n, n \in \mathbb{N}$, are uniformly bounded, $E\left[\sup_{0 \leq t \leq N} |h_2^n(t)|\right]$ converges to 0 for each $N \in \mathbb{N}$. Consider now h_1^n. Define for $t \in (0,\infty)$,

$$\begin{aligned} h_{11}^n(t) &\doteq \frac{1}{n^{\frac{1}{2}+r}} \sum_{k=1}^{\lfloor tn^{1+r} \rfloor} \int_{k-1}^{k} g_n^c\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), X^n\Big(\frac{u}{n^{r+1}}\Big)\right) du, \\ h_{12}^n(t) &\doteq \frac{1}{n^{\frac{1}{2}+r}} \sum_{k=1}^{\lfloor tn^{1+r} \rfloor} \int_{k-1}^{k} \Bigg[g_n^c\Big(\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big), Y^n\Big(\frac{u}{n^{r+1}}\Big), X^n\Big(\frac{u}{n^{r+1}}\Big)\Big) \\ &\quad - g_n^c\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), X^n\Big(\frac{u}{n^{r+1}}\Big)\right) \Bigg] du. \end{aligned} \tag{2.4.1}$$

For time $t < 0$, we set $\widehat{Q}^n(t) = \widehat{Q}^n(0)$ and $Y^n(t) = Y^n(0)$. Thus $h_1^n = h_{11}^n + h_{12}^n$. Let's first consider h_{12}^n. Note that the expectation of

$$\sup_{k-1\le u\le k}\left|g_n^c\Big(\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big),Y^n\Big(\frac{u}{n^{r+1}}\Big),X^n\Big(\frac{u}{n^{r+1}}\Big)\Big)-g_n^c\Big(\widehat{Q}^n\Big(\frac{k-2}{n^{r+1}}\Big),Y^n\Big(\frac{k-2}{n^{r+1}}\Big),X^n\Big(\frac{u}{n^{r+1}}\Big)\Big)\right|$$

can be bounded by the sum of

$$4\kappa_0 P\left(\sup_{k-1\le u\le k}\left|\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big)-\widehat{Q}^n\Big(\frac{k-2}{n^{r+1}}\Big)\right|>0\right) \tag{2.4.2}$$

and

$$4\kappa_0 P\left(Y^n\Big(\frac{u}{n^{r+1}}\Big)\ne Y^n\Big(\frac{k-2}{n^{r+1}}\Big)\text{ for some } u\in[k-1,k]\right). \tag{2.4.3}$$

By Assumption 2.3.1(ii), $(|\lambda^n(z,y,x)|+|\alpha^n(z,y,x)|)/n<2\kappa_1$. Hence we have, for some $c_1\in(0,\infty)$, the expression in (2.4.2) is bounded, for all k and n, by c_1/n^r. Furthermore, since $\mathbb{Q}^n\to\mathbb{Q}$, there exists some $c_2\in(0,\infty)$, for all k and n, the expression in (2.4.3) is bounded by c_2/n^{r+1}. Combining these estimates, we have for a suitable $c_3\in(0,\infty)$,

$$\begin{aligned}E\left[\sup_{0\le t\le N}|h_{12}^n(t)|\right]&\le\frac{1}{n^{\frac12+r}}\sum_{k=1}^{\lfloor Nn^{1+r}\rfloor}E\left[\sup_{k-1\le u\le k}\left|g_n^c\Big(\widehat{Q}^n\Big(\frac{u}{n^{r+1}}\Big),Y^n\Big(\frac{u}{n^{r+1}}\Big),X^n\Big(\frac{u}{n^{r+1}}\Big)\Big)\right.\right.\\&\qquad\left.\left.-g_n^c\Big(\widehat{Q}^n\Big(\frac{k-2}{n^{r+1}}\Big),Y^n\Big(\frac{k-2}{n^{r+1}}\Big),X^n\Big(\frac{u}{n^{r+1}}\Big)\Big)\right|\right]du\\&\le\frac{c_3N}{n^{r-\frac12}}.\end{aligned}$$

Since $r>1/2$, the last expression converges to 0.

We now consider h_{11}^n. For $f\in BM(\mathbb{L})$, $t\in(0,\infty)$ and $x\in\mathbb{L}$, let $P_n^tf(x)=E[f(X(t))|X(0)=x]$. Fix $(z,y)\in\mathbb{R}_+^K\times\mathbb{H}$. For $\phi\in BM(\mathbb{R}_+^K\times\mathbb{H}\times\mathbb{L})$ we will write $E[\phi(z,y,X(t))|X(0)=x]$ as $P_n^t\phi(z,y,x)$. For each n, let $\hat{g}_n(z,y,\cdot)$ be a solution of the Poisson equation for $g_n^c(z,y,\cdot)$ corresponding to the Markov semigroup $\{P_n^t\}_{t\ge0}$, i.e., for $x \in \mathbb{L}$ and $t \in (0, \infty)$,

$$P_n^t \hat{g}_n(z, y, x) - \hat{g}_n(z, y, x) - \int_0^t P_n^u g_n^c(z, y, x) du = 0 \tag{2.4.4}$$

(cf. [27]). We can assume without loss that $M \doteq \sup_{n,z,y,x} |\hat{g}_n(z, y, x)| < \infty$. By (2.4.4), recalling the relation between X^n and X in (2.3.2), we have that

$$\hat{g}_n(z, y, X^n(t)) - \hat{g}_n(z, y, X^n(0)) - \int_0^t g_n^c(z, y, X^n(u)) du \tag{2.4.5}$$

is a $\{\mathcal{F}_t^n\}$ martingale. Define, for $n \in \mathbb{N}, (z, y) \in \mathbb{R}_+^K \times \mathbb{H}, \vartheta \in (0, \infty)$, and $k \in \mathbb{N}$,

$$M_n^\vartheta(z, y, k) \doteq \hat{g}_n(z, y, \tilde{X}^n(\vartheta k)) - \hat{g}_n(z, y, \tilde{X}^n(\vartheta(k-1))) - \int_{\vartheta(k-1)}^{\vartheta k} g_n^c(z, y, \tilde{X}^n(u)) du. \tag{2.4.6}$$

First, from (2.4.1),

$$\begin{aligned} h_{11}^n(t) &= \frac{1}{n^{\frac{1}{2}+r}} \sum_{k=1}^{\lfloor tn^{1+r} \rfloor} \int_{k-1}^k g_n^c\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), \tilde{X}^n\left(\frac{l_n u}{n^{r+1}}\right)\right) du \\ &= \frac{\sqrt{n}}{l_n} \sum_{k=1}^{\lfloor tn^{1+r} \rfloor} \int_{\frac{l_n(k-1)}{n^{r+1}}}^{\frac{l_n k}{n^{r+1}}} g_n^c\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), \tilde{X}^n(u)\right) du. \end{aligned}$$

Now, combining with (2.4.6), we can rewrite $h_{11}^n(t)$ as

$$\begin{aligned} h_{11}^n(t) = \frac{\sqrt{n}}{l_n} \sum_{k=1}^{\lfloor tn^{1+r} \rfloor} \Bigg[& \hat{g}_n\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), X^n\left(\frac{k}{n^{r+1}}\right)\right) \\ & - \hat{g}_n\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), X^n\left(\frac{k-1}{n^{r+1}}\right)\right) \\ & - M_n^{l_n/n^{r+1}}\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), k\right) \Bigg]. \end{aligned}$$

For $k_1, k_2 \in I\!N$, let

$$\begin{aligned}
S_1^n(k_1,k_2) &\doteq \hat{g}_n\bigg(\widehat{Q}^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), Y^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), X^n\bigg(\frac{k_2}{n^{r+1}}\bigg)\bigg) \\
&\quad - P^{l_n/n^{r+1}}\hat{g}_n\bigg(\widehat{Q}^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), Y^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), X^n\bigg(\frac{k_2-1}{n^{r+1}}\bigg)\bigg), \\
S_2^n(k_1,k_2) &\doteq P^{l_n/n^{r+1}}\hat{g}_n\bigg(\widehat{Q}^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), Y^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), X^n\bigg(\frac{k_2-1}{n^{r+1}}\bigg)\bigg) \\
&\quad - \sum_{x=1}^{L} p_x^* \hat{g}_n\bigg(\widehat{Q}^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), Y^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), x\bigg), \\
S_3^n(k_1,k_2) &\doteq M_n^{l_n/n^{r+1}}\bigg(\widehat{Q}^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), Y^n\bigg(\frac{k_1-2}{n^{r+1}}\bigg), k_2\bigg).
\end{aligned}$$

With this notation,

$$h_{11}^n(t) = \frac{\sqrt{n}}{l_n}\sum_{k=1}^{\lfloor tn^{1+r}\rfloor}\left[S_1^n(k,k) - S_1^n(k,k-1) + S_2^n(k,k) - S_2^n(k,k-1) - S_3^n(k,k)\right].$$

Let $S_0^n(k) = S_1^n(k,k) - S_1^n(k,k-1) - S_3^n(k,k)$. Then $\sum_{k=1}^{\lfloor tn^{1+r}\rfloor} S_0^n(k)$ is a $\{\mathcal{F}^n_{\lfloor tn^{1+r}\rfloor/n^{1+r}}\}$ martingale. By the boundedness of $\hat{g}_n$, we can find a constant c_4 such that

$$\sup_{k,n}|S_0^n(k)| < c_4.$$

By Doob's inequality,

$$\begin{aligned}
E\left[\sup_{0\le t\le N}\left|\frac{\sqrt{n}}{l_n}\sum_{k=1}^{\lfloor tn^{1+r}\rfloor} S_0^n(k)\right|\right]^2 &\le 4E\left[\frac{\sqrt{n}}{l_n}\sum_{k=1}^{\lfloor Nn^{1+r}\rfloor} S_0^n(k)\right]^2 = 4\sum_{k=1}^{\lfloor Nn^{1+r}\rfloor} E\left[\frac{\sqrt{n}}{l_n}S_0^n(k)\right]^2 \\
&\le 4c_4 N\frac{n^{2+r}}{l_n^2}.
\end{aligned}$$

Since $l_n/n^{r_0+1} \to \infty$ and $r < r_0$, the last expression converges to 0. Recalling that X

and $\tilde{X}$ have the same law, we have by Assumption 2.2.1,

$$E\left|S_2^n(k,k)\right| = E\left|\, E\left[\hat{g}_n\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), X^n\left(\frac{k-1}{n^{r+1}}\right)\right) \mid X^n\left(\frac{k-2}{n^{r+1}}\right)\right]\right.$$
$$\left. - \sum_{x=1}^{L} p_x^* \hat{g}_n\left(\widehat{Q}^n\left(\frac{k-2}{n^{r+1}}\right), Y^n\left(\frac{k-2}{n^{r+1}}\right), x\right) \right|$$
$$\leq M \sum_{x=1}^{L} E\left|P\left[X^n\left(\frac{k-1}{n^{r+1}}\right) = x \mid X^n\left(\frac{k-2}{n^{r+1}}\right)\right] - p_x^*\right|$$
$$\leq LMa_1 e^{-a_2 l_n/n^{r+1}}.$$

Similarly, we have $E\left|S_2^n(k,k-1)\right| \leq LMa_1 e^{-a_2 l_n/n^{r+1}}$. Therefore

$$E\left[\sup_{0\leq t\leq N}\left|\frac{\sqrt{n}}{l_n}\sum_{k=1}^{\lfloor tn^{1+r}\rfloor}\left(S_2^n(k,k) - S_2^n(k,k-1)\right)\right|\right] \leq LMa_1 N \frac{n^{\frac{3}{2}+r}}{l_n} e^{-a_2 l_n/n^{r+1}}.$$

Since $l_n/n^{r+1} = n^{r_0-r} l_n/n^{r_0+1}$, $l_n/n^{r_0+1} \to \infty$ and $r_0 > r$, $\frac{n^{\frac{3}{2}+r}}{l_n} e^{-a_2 l_n/n^{r+1}} \leq \sqrt{n} e^{-a_2 n^{r_0-r}}$ for large n. Thus the last expression converges to 0 as $n \to \infty$. Combining the above estimates, we now have $E\left[\sup_{0\leq t\leq N}|h_{11}^n(t)|\right]$ converges to 0. The result follows on noting that $E\left[\sup_{0\leq t\leq N}|h^n(t)|\right] \to 0$ as $n \to \infty$ for all $N \in \mathbb{N}$. ■

Let $\tilde{S}_j^n(du,dv) \doteq S_j^n(du,dv) - \tilde{\alpha}_j^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du\mu(dv)$ and define $\mathbb{R}^K$ valued processes $\widehat{A}^n, \widehat{F}^n, B^n, R^n$ as follows. For $i \in \mathbb{K}$ and $t \geq 0$,

$$\widehat{A}_i^n(t) = \frac{1}{\sqrt{n}}\left(A_i^n(t) - \int_0^t \lambda_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du\right),$$
$$\widehat{F}_i^n(t) = -\frac{1}{\sqrt{n}}\left(D_i^n(t) - \int_0^t \tilde{\alpha}_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du\right)$$
$$+ \frac{1}{\sqrt{n}}\sum_{j=1}^{K}\int_{[0,t]\times G_j} v_i(X^n(u-))\tilde{S}_j^n(du,dv),$$
$$B_i^n(t) = \int_0^t b_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u))du,$$

and

$$R_i^n(t) = \frac{1}{\sqrt{n}}\int_0^t \left(\lambda_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u)) - \bar{\lambda}_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u))\right) du$$
$$+ \frac{1}{\sqrt{n}}\int_0^t \left(\bar{\alpha}_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u))1_{\{\widehat{Q}_i^n(u)>0\}} - \tilde{\alpha}_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))\right) du$$
$$+ \frac{1}{\sqrt{n}}\sum_{j=1}^K \int_{[0,t]\times G_j} v_i(X^n(u))\tilde{\alpha}_j^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du\mu(dv)$$
$$- \frac{1}{\sqrt{n}}\sum_{j=1}^K \int_0^t \overline{\mathbb{P}}_{ji}\bar{\alpha}_j^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u))1_{\{\widehat{Q}_j^n(u)>0\}}du.$$

Noting that $\tilde{\alpha}_i^n(z,y,x) = \alpha_i^n(z,y,x)1_{\{z_i>0\}}$ for all $(z,y,x) \in I\!R_+^K \times I\!H \times I\!L$, and recalling the definition of η^n from Theorem 2.3.3, we have that

$$\widehat{Q}^n(t) = \widehat{Q}^n(0) + \widehat{A}^n(t) + \widehat{F}^n(t) + B^n(t) + R^n(t) + [\mathbb{I} - \overline{\mathbb{P}}']\eta^n(t). \tag{2.4.7}$$

Proposition 2.4.1. *Suppose $\{(\widehat{Q}^n(0), Y^n(0)) : n \geq 1\}$ is tight in $I\!R_+^K \times I\!H$. Then $(\widehat{Q}^n, Y^n, \eta^n)$ is tight in $D([0,\infty), I\!R_+^K \times I\!R_+^K \times I\!H)$. Furthermore, $(\widehat{Q}^n, \eta^n)$ is C-tight.*

Proof: First, we show $(\widehat{Q}^n, \eta^n)$ is C-tight in $D([0,\infty), I\!R_+^K \times I\!R_+^K)$. Let

$$Z^n = \widehat{Q}^n(0) + \widehat{A}^n + \widehat{F}^n + B^n + R^n.$$

Since $\int_0^t 1_{\{\widehat{Q}_i^n(u)>0\}}d\eta_i^n(u) = 0$ for any $t \geq 0$ and $i \in I\!K$, we have, from Definition 2.3.1 that $\widehat{Q}^n = \Gamma(Z^n)$. Thus by Corollary 2.3.1, it suffices to show that Z^n is C-tight in $D([0,\infty), I\!R^K)$. For $i \in I\!K$ and $(z,y,x) \in I\!R_+^K \times I\!H \times I\!L$, let

$$g_i^n(z,y,x) \doteq \frac{1}{n}\left(\lambda_i^n(\sqrt{n}z,y,x) - \tilde{\alpha}_i^n(\sqrt{n}z,y,x) + \sum_{j=1}^K \left(\int_{G_j} v_i(x)\mu(dv)\right)\tilde{\alpha}_j^n(\sqrt{n}z,y,x)\right)$$

Recalling Assumption 2.3.1 (ii) and applying Lemma 2.4.1 to $g_i^n(z,y,x)$, we get R_i^n converge to 0 in probability in $C([0,\infty), I\!R^K)$. Here we have also made use of the relation:

$$\sum_{x\in\mathbb{L}} p_x^* \left(\int_{G_j} v_i(x)\mu(dv)\tilde{\alpha}_j^n(\sqrt{n}z, y, x) \right) = \overline{\mathbb{P}}_{ji}\bar{\alpha}_j^n(\sqrt{n}z, y)1_{\{z_j>0\}}, \quad i, j \in \mathbb{K}.$$

Using Assumption 2.3.1 (iii),

$$|B^n(t) - B^n(s)| \le \kappa_2 |t-s|, \quad \forall\ 0 \le s \le t < \infty.$$

In particular, B^n is C-tight. We now show that $\widehat{A}^n$ and $\widehat{F}^n$ are C-tight. Recall from (2.2.2) that $\widehat{A}^n$ and $\widehat{F}^n$ are $\{\mathcal{F}_t^n\}$ martingales. Since $\{A_i^n, D_j^n : i, j \in \mathbb{K}\}$ have no common jumps and $v_i(x) = 0$ for any $v \in G_i$, we have

$$\begin{aligned}
\langle \widehat{A}_i^n \rangle(t) &= \frac{1}{n}\int_0^t \lambda_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du \\
\langle \widehat{F}_i^n \rangle(t) &= \frac{1}{n}\int_0^t \tilde{\alpha}_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du \\
&\quad + \frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{K} \int_{[0,t]\times G_j} (v_i(X^n(u)))^2 \tilde{\alpha}_j^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u))du\mu(dv)
\end{aligned}$$

Noting that $|v(x)| \le 1$, and recalling Assumption 2.3.1 (ii), we see that $\sum_{i=1}^K \langle \widehat{A}_i^n \rangle$ and $\sum_{i=1}^K \langle \widehat{F}_i^n \rangle$ are C-tight, which yields the tightness of $\widehat{A}^n$ and $\widehat{F}^n$(cf. [32, Theorem VI.4.13]). The C-tightness of $\widehat{A}^n$ and $\widehat{F}^n$ is now immediate on noting that $|\triangle \widehat{A}^n(t)| \le K/\sqrt{n}$ and $|\triangle \widehat{F}^n(t)| \le K/\sqrt{n}$, a.s., for any $t \ge 0$.

Since $\mathbb{Q}^n$ converges to $\mathbb{Q}$, it follows that (cf. [26, Theorem 4.2.5]), if along some subsequence $\{n_k\}_{k\ge1}$, $Y^{n_k}(0)$ converges in distribution to some probability measure ν on $\mathbb{H}$ then Y^{n_k} converges in distribution, in $D([0,\infty) : \mathbb{H})$ to an $\mathbb{H}$ valued Markov process with initial distribution ν and infinitesimal generator $\mathbb{Q}$. In particular, Y^n is tight. The result follows. ■

Denote by $\{e_j\}_{j=1}^K$ the canonical basis in $\mathbb{R}^K$. For $v \in G_j$ and $x \in \mathbb{L}$, let $v(x) =$

$(v^x_1, \ldots, v^x_K)'$. Let $f = f_1 \otimes f_2 \in \mathcal{G}$. From (2.2.7), and recalling that $A^n_i, D^n_j, i, j \in I\!\!K$, have no common jumps, we have, for all $t \geq 0$, a.s. P^n

$$f_1(Q^n(t)) = f_1(Q^n(0)) + \sum_{j=1}^{K} \int_0^t f_1(Q^n(u-) + e_j) - f_1(Q^n(u-))A^n_j(du)$$
$$+ \sum_{j=1}^{K} \int_{[0,t]\times G_j} f_1(Q^n(u-) + v(X^n(u-)) - e_j) - f_1(Q^n(u-))S^n_j(du, dv).$$

Therefore, (cf. (2.2.2)), recalling that $A^n_i, D^n_j, i, j \in I\!\!K$, and Y^n have no common jumps,

$$\begin{aligned}
&f(Q^n(t), Y^n(t)) - f(Q^n(0), Y^n(0)) \\
&- \sum_{j=1}^{K} \int_0^t \lambda^n_j(Q^n(u), Y^n(u), X^n(u)) f_2(Y^n(u))[f_1(Q^n(u) + e_j) - f_1(Q^n(u))]du \\
&- \sum_{j=1}^{K} \int_{[0,t]\times G_j} \tilde{\alpha}^n_j(Q^n(u), Y^n(u), X^n(u)) f_2(Y^n(u)) \\
&\qquad \times [f_1(Q^n(u) + v(X^n(u)) - e_j) - f_1(Q^n(u))]du\mu(dv) \\
&- \int_0^t f_1(Q^n(u))\mathbb{Q}^n f_2(Y^n(u))du
\end{aligned}$$

is a $\{\mathcal{F}^n_t\}$ martingale, where, for $y \in I\!\!H$, $\mathbb{Q}^n f_2(y) = \sum_{j=1}^{H} \mathbb{Q}^n_{yj} f_2(j)$. Equivalently, in terms of the normalized vector $\widehat{Q}^n(t) = Q^n(t)/\sqrt{n}$, we have

$$\begin{aligned}
&f(\widehat{Q}^n(t), Y^n(t)) - f(\widehat{Q}^n(0), Y^n(0)) \\
&- \sum_{j=1}^{K} \int_0^t \lambda^n_j(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u)) f_2(Y^n(u)) \left[f_1\left(\widehat{Q}^n(u) + \frac{e_j}{\sqrt{n}}\right) - f_1(\widehat{Q}^n(u))\right] du \\
&- \sum_{j=1}^{K} \int_{[0,t]\times G_j} \tilde{\alpha}^n_j(\sqrt{n}\widehat{Q}^n(u), Y^n(u), X^n(u)) f_2(Y^n(u)) \\
&\qquad \times \left[f_1\left(\widehat{Q}^n(u) + \frac{v(X^n(u)) - e_j}{\sqrt{n}}\right) - f_1(\widehat{Q}^n(u))\right] du\mu(dv) \\
&- \int_0^t f_1(\widehat{Q}^n(u))\mathbb{Q}^n f_2(Y^n(u))du
\end{aligned}$$

is a $\{\mathcal{F}_t^n\}$ martingale.

For $z \in I\!R_+^K$, $j \in I\!K$ and $v \in G_j$, define $f_{z,v}^{n,j} : I\!L \to I\!R$ as $f_{z,v}^{n,j}(x) = f_1\left(z + \frac{v(x)-e_j}{\sqrt{n}}\right)$ and let

$$
\begin{aligned}
\mathcal{L}^n f(z,y) &= \sum_{j=1}^{K} \bar{\lambda}_j^n(\sqrt{n}z,y) f_2(y)\left(f_1(z+e_j/\sqrt{n}) - f_1(z)\right) \\
&\quad + \sum_{j=1}^{K} \varpi_j^n(\sqrt{n}z,y) f_2(y) \int_G \left(\sum_{\tilde{x}=1}^{L} p_{\tilde{x}}^* \theta_j(\tilde{x}) f_{z,v}^{n,j}(\tilde{x}) - \bar{\theta}_j f_1(z)\right)\mu(dv) \\
&\quad + f_1(z)\mathbb{Q}^n f_2(y), \\
\widetilde{\mathcal{L}}^n f(z,y,x) &= \sum_{j=1}^{K} (\lambda_j^n(\sqrt{n}z,y,x) - \bar{\lambda}_j^n(\sqrt{n}z,y)) f_2(y)\left(f_1(z+e_j/\sqrt{n}) - f_1(z)\right) \\
&\quad + \sum_{j=1}^{K} \varpi_j^n(\sqrt{n}z,y) f_2(y) \int_G \left(\theta_j(x) f_{z,v}^{n,j}(x) - \sum_{\tilde{x}=1}^{L} p_{\tilde{x}}^* \theta_j(\tilde{x}) f_{z,v}^{n,j}(\tilde{x})\right)\mu(dv) \\
&\quad + \sum_{j=1}^{K} \varpi_j^n(\sqrt{n}z,y) f_2(y)\left(\bar{\theta}_j - \theta_j(x)\right) f_1(z), \\
\mathcal{D}_i^n f(z,y) &= \sqrt{n} f_2(y) \bar{\theta}_i^{-1} \int_G \left(\bar{\theta}_i f_1(z) - \sum_{\tilde{x}=1}^{L} p_{\tilde{x}}^* \theta_i(\tilde{x}) f_{z,v}^{n,i}(\tilde{x})\right)\mu(dv), \\
\widetilde{\mathcal{D}}_i^n f(z,y,x) &= \sqrt{n} f_2(y) \bar{\theta}_i^{-1} \left[\int_G \left(\sum_{\tilde{x}=1}^{L} p_{\tilde{x}}^* \theta_i(\tilde{x}) f_{z,v}^{n,i}(\tilde{x}) - \theta_i(x) f_{z,v}^{n,i}(x)\right)\mu(dv) \right. \\
&\quad \left. + \left(\theta_i(x) - \bar{\theta}_i\right) f_1(z)\right].
\end{aligned}
$$

The following is an immediate consequence of the above definitions.

Lemma 2.4.2. *Let $f \in \mathcal{G}$. Then*

$$
\begin{aligned}
& f(\widehat{Q}^n(t), Y^n(t)) - f(\widehat{Q}^n(0), Y^n(0)) \\
&\quad - \int_0^t \mathcal{L}^n f(\widehat{Q}^n(u), Y^n(u))du - \sum_{i=1}^{K} \int_0^t \mathcal{D}_i^n f(\widehat{Q}^n(u), Y^n(u)) d\eta_i^n(u) \\
&\quad - \int_0^t \widetilde{\mathcal{L}}^n f(\widehat{Q}^n(u), Y^n(u), X^n(u))du - \sum_{i=1}^{K} \int_0^t \widetilde{\mathcal{D}}_i^n f(\widehat{Q}^n(u), Y^n(u), X^n(u)) d\eta_i^n(u)
\end{aligned}
$$

is a $\{\mathcal{F}_t^n\}$ *martingale.*

Recall that $\Lambda(z,y) = \sigma(z,y)\sigma(z,y)'$ for $(z,y) \in I\!R_+^K \times I\!H$ and d_i is the i^{th} column of $[\mathbb{I} - \overline{\mathbb{P}}']$. Define a $K \times K$ matrix $\Lambda^n(z,y)$ in the same way as $\Lambda(z,y)$ but with $(\bar{\lambda}(z,y), \bar{\alpha}(z,y))$ replaced by $(\bar{\lambda}^n(\sqrt{n}z,y), \bar{\alpha}^n(\sqrt{n}z,y))$.

Lemma 2.4.3. *For each* $f \in \mathcal{G}$ *there are measurable maps* $\xi_i^n : I\!R_+^K \times I\!H \to I\!R, i = 1,2,$ *such that*

$$\begin{aligned}
\mathcal{L}^n f(z,y) &= \frac{1}{\sqrt{n}}\left(\bar{\lambda}^n(\sqrt{n}z,y) - [\mathbb{I} - \overline{\mathbb{P}}']\bar{\alpha}^n(\sqrt{n}z,y)\right)' \nabla f_1(z) f_2(y) \\
&\quad + \frac{1}{2n} Tr\left(\Lambda^n(z,y)\nabla^2 f_1(z)\right) f_2(y) + f_1(z)\mathbb{Q}^n f_2(y) + \xi_1^n(z,y), \\
\mathcal{D}_i^n f(z,y) &= d_i' \nabla f_1(z) f_2(y) + \xi_2^n(z,y),
\end{aligned}$$

and

$$\sup_{(z,y)\in I\!R_+^K \times I\!H} (|\xi_1^n(z,y)| + |\xi_2^n(z,y)|) \to 0 \qquad \text{as } n \to \infty.$$

Proof: Fix $f_1 \in \mathcal{C}_0^2(I\!R_+^K)$, $x \in I\!L$, $j \in I\!K$, and $v \in G_j$. Applying Taylor series expansion for f_1 at z, we have

$$\begin{aligned}
f_1\left(z + \frac{e_j}{\sqrt{n}}\right) &= f_1(z) + \frac{e_j'\nabla f_1(z)}{\sqrt{n}} + \frac{e_j'\nabla^2 f_1(z) e_j}{2n} + \gamma_n(j,z), \\
f_1\left(z + \frac{v(x) - e_j}{\sqrt{n}}\right) &= f_1(z) + \frac{(v(x) - e_j)'\nabla f_1(z)}{\sqrt{n}} + \frac{(v(x) - e_j)'\nabla^2 f_1(z)(v(x) - e_j)}{2n} \\
&\quad + \xi_n(j,z,v,x),
\end{aligned} \tag{2.4.8}$$

where, for some $c_1 \in (0,\infty)$,

$$\sup_{j\in I\!K} \sup_{\substack{v\in G \\ (z,x)\in I\!R_+^K \times I\!L}} (|\gamma_n(j,z)| + |\xi_n(j,z,v,x)|) \le \frac{c_1}{n^{3/2}}. \tag{2.4.9}$$

First we note that $\int_{G_j} v(x)\mu(dv)$ equals $\mathbb{P}_x(j)'$ which is the transport of $\mathbb{P}_x(j)$, where

$\mathbb{P}_x(j)$ is the j^{th} row of $\mathbb{P}_x$. Then, for $f \in \mathcal{G}$ and $(z, y) \in I\!R_+^K \times I\!H$,

$$\mathcal{L}^n f(z,y) = \frac{1}{\sqrt{n}} f_2(y) \left(\bar{\lambda}^n(\sqrt{n}z, y) - \sum_{j=1}^{K} \varpi_j^n(\sqrt{n}z, y) \sum_{x=1}^{L} p_x^* \theta_j(x) \left(e_j - \mathbb{P}_x(j)' \right) \right)' \nabla f_1(z)$$
$$+ \frac{1}{2n} f_2(y) Tr \left(\left(\sum_{x=1}^{L} p_x^* \Lambda^n(x) \right) \nabla^2 f_1(z) \right) + f_1(z) \mathbb{Q}^n f_2(y) + \xi_1^n(z,y),$$
$$\mathcal{D}_i^n f(z,y) = f_2(y) \bar{\theta}_i^{-1} \left(\sum_{x=1}^{L} p_x^* \left(e_i - \mathbb{P}_x(i)' \right) \theta_i(x) \right)' \nabla f_1(z) + \xi_2^n(z,y),$$

where $\sup_{(z,y) \in I\!R_+^K \times I\!H} (|\xi_1^n(z,y)| + |\xi_2^n(z,y)|) \le c_2/\sqrt{n}$ for some $c_2 \in (0, \infty)$. In the above display, for $x \in I\!L$, $\Lambda^n(x)$ is a $K \times K$ matrix defined in the same way as Λ, with $(\bar{\lambda}(z,y), \bar{\alpha}(z,y), \overline{\mathbb{P}})$ replaced by $(\bar{\lambda}^n(\sqrt{n}z, y), \bar{\alpha}^n(\sqrt{n}z, y), \mathbb{P}_x \mathrm{diag}(\theta(x))[\mathrm{diag}(\bar{\theta})]^{-1})$. Finally, it is easily seen that

$$\sum_{x=1}^{L} p_x^* \theta_j(x) \left(e_j - \mathbb{P}_x(j)' \right) = \bar{\theta}_j d_j,$$
$$\sum_{x=1}^{L} p_x^* \Lambda^n(x) = \Lambda^n.$$

The lemma follows. ■

The following elementary lemma (cf. [18, Lemma 2.4]) will be needed in our proof.

Lemma 2.4.4. *Suppose ξ^n converges to ξ in $D([0,\infty), I\!R^K)$ and φ^n converges to φ in $C([0,\infty), I\!R)$ as $n \to \infty$. Further, suppose that φ^n is nonnegative and nondecreasing for each n. Then as $n \to \infty$,*

$$\int_{[0,t)} \xi^n(u) d\varphi^n(u) \to \int_{[0,t)} \xi(u) d\varphi(u).$$

uniformly for all t in any compact subset of $[0, \infty)$.

Proposition 2.4.2. *Assume that the sequence of measures $P^n \circ (\widehat{Q}^n(0), Y^n(0))^{-1}$ converges weakly to some $\nu \in \mathcal{P}(I\!R_+^K \times I\!H)$. Let (Z, Y, η) be any weak limit of $(\widehat{Q}^n, Y^n, \eta^n)$*

given on a probability space $(\Omega, \mathcal{F}, P)$*. Then, for all* $f \in \mathcal{G}$*,*

$$f\left(Z(\cdot),Y(\cdot)\right)-f\left(Z(0),Y(0)\right)-\int_0^{\cdot}\mathcal{L}f\left(Z(u),Y(u)\right)du-\sum_{i=1}^{K}\int_0^{\cdot}\mathcal{D}_i f\left(Z(u),Y(u)\right)d\eta_i(u)$$

is a $\{\mathcal{F}_t\}$ *martingale with* $\mathcal{F}_t \doteq \sigma\left(Z(u),Y(u),\eta(u) : u \le t\right)$*.*

Proof: For $(z,y,x) \in \mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$, let

$$\begin{aligned} f_1^n(z,y,x) \doteq f_2(y)\sum_{j=1}^{K}\Bigg[&\frac{\lambda_j^n(\sqrt{n}z,y,x)}{n}\left(e_j'\nabla f_1(z)+\frac{e_j'\nabla^2 f_1(z)e_j}{2\sqrt{n}}\right)\\ &+\frac{\tilde{\alpha}_j^n(\sqrt{n}z,y,x)}{n}\int_{G_j}\left((v(x)-e_j)'\nabla f_1(z)+\frac{(v(x)-e_j)'\nabla^2 f_1(z)(v(x)-e_j)}{2\sqrt{n}}\right)\mu(dv)\Bigg], \end{aligned}$$

and

$$f_2^n(z,y,x) \doteq f_2(y)\sum_{j=1}^{K}\left[\lambda_j^n(\sqrt{n}z,y,x)\gamma_n(j,z)+\tilde{\alpha}_j^n(\sqrt{n}z,y,x)\int_{G_j}\xi_n(j,z,v,x)\mu(dv)\right].$$

In view of Assumption 2.3.1 (ii) and (2.4.9), f_1^n and f_2^n are uniformly bounded sequences of measurable functions from $\mathbb{R}_+^K \times \mathbb{H} \times \mathbb{L}$ to $\mathbb{R}$. Applying Lemma 2.4.1 with g_n replaced by f_1^n and f_2^n, we have, for $i = 1, 2$,

$$\sqrt{n}\int_0^{\cdot}\left[f_i^n(\widehat{Q}^n(u),Y^n(u),X^n(u))-\bar{f}_i^n(\widehat{Q}^n(u),Y^n(u))\right]du \to 0, \tag{2.4.10}$$

in probability, in $C([0,\infty),\mathbb{R})$. Using (2.4.8), the definitions of $\widetilde{\mathcal{L}}^n$ and $\widetilde{\mathcal{D}}_i^n, i \in \mathbb{K}$, and (2.4.10), we have that

$$\begin{aligned} &\int_0^{\cdot}\widetilde{\mathcal{L}}^n f(\widehat{Q}^n(u),Y^n(u),X^n(u))du+\sum_{i=1}^{K}\int_0^{\cdot}\widetilde{\mathcal{D}}_i^n f(\widehat{Q}^n(u),Y^n(u),X^n(u))d\eta_i^n(u)\\ &\quad=\sqrt{n}\sum_{j=1}^{2}\int_0^{\cdot}\left[f_j^n(\widehat{Q}^n(u),Y^n(u),X^n(u))-\bar{f}_j^n(\widehat{Q}^n(u),Y^n(u))\right]du \to 0, \end{aligned} \tag{2.4.11}$$

in probability in $C([0,\infty),\mathbb{R})$.

Furthermore, by Lemma 2.4.3 and Assumption 2.3.1 (iv) and (v), $\mathcal{L}^n f \to \mathcal{L} f$ and $\mathcal{D}_i^n f \to \mathcal{D}_i f$ uniformly on compact subsets of $\mathbb{R}_+^K \times \mathbb{H}$, and therefore

$$\left(\mathcal{L}^n f(\widehat{Q}^n(\cdot), Y^n(\cdot)), \mathcal{D}_i^n f(\widehat{Q}^n(\cdot), Y^n(\cdot)), \eta_i^n\right) \Rightarrow (\mathcal{L} f(Z(\cdot), Y(\cdot)), \mathcal{D}_i f(Z(\cdot), Y(\cdot)), \eta_i)$$

in $D([0,\infty),\mathbb{R}^2) \times C([0,\infty),\mathbb{R})$. Applying Lemma 2.4.4 with $\varphi^n = \eta_i^n(\cdot)$ and $\xi^n = \mathcal{D}_i^n f(\widehat{Q}^n(\cdot), Y^n(\cdot))$, we have

$$\int_0^\cdot \mathcal{D}_i^n f(\widehat{Q}^n(u), Y^n(u)) d\eta_i^n(u) \Rightarrow \int_0^\cdot \mathcal{D}_i f(Z(u), Y(u)) d\eta_i(u)$$

weakly in $C([0,\infty),\mathbb{R})$. In fact, we have the joint convergence,

$$\begin{aligned}&\left(\widehat{Q}^n, Y^n, \int_0^\cdot \mathcal{L}^n f(\widehat{Q}^n(u), Y^n(u)) du, \int_0^\cdot \mathcal{D}_i^n f(\widehat{Q}^n(u), Y^n(u)) d\eta_i^n(u), i = 1, 2, \ldots, K\right) \\ &\Rightarrow \left(Z, Y, \int_0^\cdot \mathcal{L} f\left(Z(u), Y(u)\right) du, \int_0^\cdot \mathcal{D}_i f\left(Z(u), Y(u)\right) d\eta_i(u), i = 1, 2, \ldots, K\right).\end{aligned}$$

Fix $0 \le s \le t$. For $k \in \mathbb{N}$, $i = 1, \ldots, k$, let $t_i \in \{u \ge 0 : P(Y(u) = Y(u-)) = 1\} \cap [0, s]$, and $h_i \in \mathcal{C}_b(\mathbb{R}_+^K \times \mathbb{H} \times \mathbb{R}_+^K)$. Then for all $f \in \mathcal{G}$,

$$\begin{aligned}&E\left[\left(f\left(Z(t), Y(t)\right) - f\left(Z(s), Y(s)\right) - \int_s^t \mathcal{L} f\left(Z(u), Y(u)\right) du\right.\right. \\ &\qquad \left.\left. - \sum_{i=1}^K \int_s^t \mathcal{D}_i f\left(Z(u), Y(u)\right) d\eta_i(u)\right) \times \prod_{i=1}^k h_i\left(Z(t_i), Y(t_i), \eta(t_i)\right)\right] \\ &= \lim_{n\to\infty} E\left[\left(f(\widehat{Q}^n(t), Y^n(t)) - f(\widehat{Q}^n(s), Y^n(s)) - \int_s^t \mathcal{L}^n f(\widehat{Q}^n(u), Y^n(u)) du\right.\right. \\ &\qquad - \sum_{i=1}^K \int_s^t \mathcal{D}_i^n f(\widehat{Q}^n(u), Y^n(u)) d\eta_i^n(u) - \int_s^t \widetilde{\mathcal{L}}^n f(\widehat{Q}^n(u), Y^n(u), X^n(u)) du \\ &\qquad \left.\left. - \sum_{i=1}^K \int_s^t \widetilde{\mathcal{D}}_i^n f(\widehat{Q}^n(u), Y^n(u), X^n(u)) d\eta_i^n(u)\right) \prod_{i=1}^k h_i(\widehat{Q}^n(t_i), Y^n(t_i), \eta^n(t_i))\right] \\ &= 0,\end{aligned}$$

where the first equality is a consequence of the above joint convergence and (2.4.11), and the second equality is a consequence of Lemma 2.4.2. The result follows. ■

2.5 Appendix

In this section we prove Theorem 2.3.1 and Proposition 2.3.2. Although the proofs are simple modifications of classical arguments, we provide details for the sake of completeness. We begin with the following useful lemma.

Lemma 2.5.1. *Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space on which are given RCLL adapted processes (W, Y) with some initial distribution π, such that W is a K-dimensional standard $\{\mathcal{F}_t\}$ Brownian motion, and Y is a $I\!H$ valued $\{\mathcal{F}_t\}$ Markov process with infinitesimal generator $\mathbb{Q}$. Then W and Y are independent.*

Proof: For $g_1 \in \mathcal{C}_0^2(I\!R^K)$ and $g_2 \in BM(I\!H)$, define $A_1 g_1 \in BM(I\!R^K)$ and $A_2 g_2 \in BM(I\!H)$ as

$$A_1 g_1(x) = \frac{1}{2} Tr(\nabla^2 g_1(x)),\ A_2 g_2(y) = \mathbb{Q} g_2(y) = \sum_{j=1}^{H} \mathbb{Q}_{yj} g_2(j), \quad x \in I\!R^K, y \in I\!H.$$

Let, for $t \geq 0$,

$$\begin{aligned} M_1(t) &= g_1(W(t)) - g_1(W(0)) - \int_0^t A_1 g_1(W(u)) du, \\ M_2(t) &= g_2(Y(t)) - g_2(Y(0)) - \int_0^t A_2 g_2(Y(u)) du. \end{aligned}$$

Then M_1 and M_2 are $\{\mathcal{F}_t\}$ martingales. In particular, M_1 is a continuous martingale, while M_2 is a martingale with sample paths of finite variation on each compact set of $[0,\infty)$. Therefore, $[M_1, M_2] = 0$. By Ito's formula,

$$g_1(W(t))g_2(Y(t)) - g_1(W(0))g_2(Y(0)) - \int_0^t g_1(W(u))A_2 g_2(Y(u))du \\ - \int_0^t g_2(Y(u))A_1 g_1(W(u))du$$

is a $\{\mathcal{F}_t\}$ martingale. Applying Theorem 10.1 of [34] we now have that W and Y are independent. ■

Proof of Theorem 2.3.1: Fix $\nu \in \mathcal{P}(\mathbb{R}_+^K \times \mathbb{H})$ and let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\})$ be a filtered probability space on which are given RCLL adapted processes (W, Y) such that W is a K-dimensional standard $\{\bar{\mathcal{F}}_t\}$ Brownian motion and Y is a $\mathbb{H}$ valued $\{\bar{\mathcal{F}}_t\}$ Markov process with infinitesimal generator $\mathbb{Q}$. Also let $Z(0)$ be a $\bar{\mathcal{F}}_0$ measurable $\mathbb{R}_+^K$ valued random variable such that $(Z(0), Y(0))$ has probability law ν. Using Lipschitz property of b and σ, we have (cf. Theorem 2.1 of Chapter III in [30]) that there is a unique continuous $\{\bar{\mathcal{F}}_t\}$ adapted process Z satisfying the integral equation (2.3.11). Clearly $\Psi_\nu = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\}, Z, Y, W)$ is a system with initial condition ν which proves the first part of the theorem. To prove weak uniqueness, we follow an argument similar to [51] (also see [33, Section 5.3.D.]). Consider two systems $\Psi_\nu^{(i)} \doteq (\Omega_i, \mathcal{F}_i, P_i, \{\mathcal{F}_t^{(i)}\}, Z^{(i)}, Y^{(i)}, W^{(i)})$, $i = 1, 2$, with initial condition ν. Set $R^{(i)}(t) = (Z^{(i)}(0), Y^{(i)}(t))$ and $V^{(i)}(t) = Z^{(i)}(t) - Z^{(i)}(0), t \geq 0$. Consider $(R^{(i)}, W^{(i)}, V^{(i)})$ which induces a measure ρ_i on $(\Theta, \mathcal{B}(\Theta))$, where

$$\Theta \doteq \mathbb{R}_+^K \times D([0,\infty), \mathbb{H}) \times C([0,\infty), \mathbb{R}^K) \times C([0,\infty), \mathbb{R}^K),$$

according to

$$\rho_i(A) \doteq P_i[(R^{(i)}, W^{(i)}, V^{(i)}) \in A], \quad A \in \mathcal{B}(\Theta), i = 1, 2. \tag{2.5.1}$$

Denote by $\theta = (r, w, v)$ a generic element of Θ. The marginal of ρ_i on the w-coordinate is the standard Wiener measure which we denoted by γ_1. Also, using the Markov property of $Y^{(1)}$ and $Y^{(2)}$, we see that $R^{(1)}$ and $R^{(2)}$ have the same probability law. Define $\gamma_2 \in \mathcal{P}(\mathbb{R}_+^K \times D([0,\infty), \mathbb{H}))$ as $\gamma_2 = P_1 \circ (R^{(1)})^{-1} = P_2 \circ (R^{(2)})^{-1}$. Then the marginal of ρ_i on the r-coordinate of θ is γ_2. From Lemma 2.5.1 $W^{(i)}$ is independent of $R^{(i)}, i = 1, 2$, and so $(R^{(1)}, W^{(1)})$ and $(R^{(2)}, W^{(2)})$ have the same probability law. Denoting this common law by γ, we have that $\gamma = \gamma_2 \otimes \gamma_1$. Disintegrate the probability measure ρ_i as $\rho_i(drdwdv) = \nu_i(r, w, dv)\gamma(drdw)$. Here ν_i is the regular conditional probability kernel given as

$$\nu_i : \mathbb{R}_+^K \times D([0,\infty), \mathbb{H}) \times C([0,\infty), \mathbb{R}^K) \times \mathcal{B}(C([0,\infty), \mathbb{R}^K)) \to [0,1], \qquad (2.5.2)$$

which satisfies:

(i) for each $r \in \mathbb{R}_+^K \times D([0,\infty), \mathbb{H})$ and $w \in C([0,\infty), \mathbb{R}^K)$, $\nu_i(r, w, \cdot)$ is a probability measure on $(C([0,\infty), \mathbb{R}^K), \mathcal{B}(C([0,\infty), \mathbb{R}^K)))$,

(ii) for each $F \in \mathcal{B}(C([0,\infty), \mathbb{R}^K))$, the mapping

$$(r, w) \to \nu_i(r, w, F)$$

is $\mathcal{B}(\mathbb{R}_+^K) \otimes \mathcal{B}(D([0,\infty), \mathbb{H})) \otimes \mathcal{B}(C([0,\infty), \mathbb{R}^K))$-measurable, and

(iii) for $F \in \mathcal{B}(C([0,\infty), \mathbb{R}^K)), G \in \mathcal{B}(\mathbb{R}_+^K) \otimes \mathcal{B}(D([0,\infty), \mathbb{H})) \otimes \mathcal{B}(C([0,\infty), \mathbb{R}^K))$,

$$\rho_i(G \times F) = \int_G \nu_i(r, w, F)\gamma(drdw).$$

To bring the two weak solutions $(Z^{(i)}, Y^{(i)}, W^{(i)}), i = 1, 2$, together on the same space, consider the measurable space $(\Xi, \mathcal{J})$, where $\Xi = \Theta \times C([0,\infty), \mathbb{R}^K)$ and $\mathcal{J}$ is the completion of the σ-field $\mathcal{B}(\Theta) \otimes \mathcal{B}(C([0,\infty), \mathbb{R}^K))$ under the probability measure

ν defined as:

$$\nu(A \times B \times C) = \int_A \nu_1(r, w, B)\nu_2(r, w, C)\gamma(drdw), \tag{2.5.3}$$

where $A \in \mathcal{B}(I\!R_+^K) \otimes \mathcal{B}(D([0,\infty), I\!H) \otimes \mathcal{B}(C([0,\infty), I\!R^K))$ and $B, C \in \mathcal{B}(C([0,\infty), I\!R^K))$. In order to endow $(\Xi, \mathcal{J}, \nu)$ with a filtration that satisfies the usual conditions, we take, for $0 \le t < \infty$,

$$\widetilde{\mathcal{J}}_t \doteq \sigma\{(r(s), w(s), v_1(s), v_2(s) : 0 \le s \le t)\},\ \mathcal{J}_t \doteq \sigma(\widetilde{\mathcal{J}}_t \cup \mathcal{N}),$$

where $\mathcal{N}$ is the collection of null sets under measure ν, $\xi \doteq (r, w, v_1, v_2)$ is a generic element of Ξ and $r(s) = (r_1, r_2(s))$, with $r_1 \in I\!R_+^K$ and $r_2 \in D([0,\infty), I\!H)$. Here, with an abuse of terminology, we have denoted the canonical coordinate maps on Ξ by the coordinate themselves.

By (2.5.1), (iii) and (2.5.3), we have

$$\nu[\xi \in \Xi : (r, w, v_i) \in A] = P_i[(R^{(i)}, W^{(i)}, V^{(i)}) \in A],\ A \in \mathcal{B}(\Theta),\ i = 1, 2.$$

Thus for $i = 1, 2$, the distribution induced by $\xi \mapsto (r_1 + v_i, r_2, w)$ under ν is the same as that of $(Z^{(i)}, Y^{(i)}, W^{(i)})$ under P_i, where $(r_1, r_2(t)) = r(t), t \ge 0$. Consequently we have constructed, on the space $(\Xi, \mathcal{J})$, two strong solutions of the integral equation (2.3.11). By pathwise uniqueness of (2.3.11) established earlier in the proof we get

$$\nu[\xi \in \Xi : v_1 = v_2] = 1.$$

Then for $A \in \mathcal{B}(C([0,\infty), I\!R^K)) \otimes \mathcal{B}(D([0,\infty), I\!H)) \otimes \mathcal{B}(C([0,\infty), I\!R^K))$,

$$\begin{aligned} P_1[(Z^{(1)}, Y^{(1)}, W^{(1)}) \in A] &= \nu[\xi \in \Xi : (r_1 + v_1, r_2, w) \in A] = \nu[\xi \in \Xi : (r_1 + v_2, r_2, w) \in A] \\ &= P_2[(Z^{(2)}, Y^{(2)}, W^{(2)}) \in A]. \end{aligned}$$

The result follows. ■

Proof of Proposition 2.3.2: For $t \geq 0$, let

$$U(t) = Z(t) - Z(0) - \int_0^t b(Z(u), Y(u))du - [\mathbb{I} - \overline{\mathbb{P}}']\eta(t).$$

Define $\tau_n = \inf\{t \geq 0 : |Z(t)| \geq n\}$ for each $n \in I\!N$. For $j \in I\!K$, let $f \in \mathcal{G}$ be such that $f(z,y) = z_j$ on $S_n \doteq \{(z,y) \in I\!R_+^K \times I\!H : |z| \leq n\}$. On S_n

$$\mathcal{L}f(z,y) = b_j(z,y), \quad \mathcal{D}_i f(z,y) = d_{ji}, \ i \in I\!K,$$

where d_{ji} is the $(j,i)^{th}$ entry of $[\mathbb{I} - \overline{\mathbb{P}}']$. Thus, for $n \geq 1$,

$$Z_j(\cdot \wedge \tau_n) - Z_j(0) - \int_0^{\cdot \wedge \tau_n} c_j(Z(u), Y(u))du - \sum_{i=1}^K d_{ji}\eta_i(\cdot \wedge \tau_n)$$

is a $\{\bar{\mathcal{F}}_t\}$ martingale. Since $\tau_n \to \infty$ as $n \to \infty$, $U(t)$ is a $\{\bar{\mathcal{F}}_t\}$ continuous local martingale. Next, for $i, j \in I\!K$, let $f \in \mathcal{G}$ be such that $f(z,y) = z_i z_j$ on S_n. Then, on S_n,

$$\mathcal{L}f(z,y) = b_i(z,y)z_j + b_j(z,y)z_i + \Lambda_{ij}(z,y), \quad \mathcal{D}_l f(z,y) = d_{il}z_j + d_{jl}z_i, \ l \in I\!K.$$

Thus the following is a $\{\bar{\mathcal{F}}_t\}$ martingale.

$$\begin{aligned} &Z_i(\cdot \wedge \tau_n)Z_j(\cdot \wedge \tau_n) - Z_i(0)Z_j(0) \\ &- \int_0^{\cdot \wedge \tau_n} \left(b_i(Z(u), Y(u))Z_j(u) + b_j(Z(u), Y(u))Z_i(u) + \Lambda_{ij}(Z(u), Y(u))\right) du \\ &- \sum_{l=1}^K \int_0^{\cdot \wedge \tau_n} \left(d_{il}Z_j(u) + d_{jl}Z_i(u)\right) d\eta_l(u) \end{aligned} \quad (2.5.4)$$

On the other hand, for $t \geq 0$,

$$\begin{aligned} Z_i(t)Z_j(t) - Z_i(0)Z_j(0) &= \int_0^t Z_i(u)dZ_j(u) + \int_0^t Z_j(u)dZ_i(u) + [Z_i, Z_j](t) \\ &= \int_0^t Z_i(u)dU_j(u) + \int_0^t Z_j(u)dU_i(u) + [U_i, U_j](t) \\ &\quad + \int_0^t \left(Z_i(u)b_j(Z(u), Y(u)) + Z_j(u)b_i(Z(u), Y(u))\right) du \\ &\quad + \sum_{l=1}^K \int_0^t \left(d_{jl}Z_i(u) + d_{il}Z_j(u)\right) d\eta_l(u). \end{aligned} \tag{2.5.5}$$

Combining (2.5.4) and (2.5.5), we have

$$\int_0^\cdot Z_i(u)dU_j(u) + \int_0^\cdot Z_j(u)dU_i(u) + [U_i, U_j](\cdot) - \int_0^\cdot \Lambda_{ij}(Z(u), Y(u))du$$

is a $\{\bar{\mathcal{F}}_t\}$ local martingale. Next, since U is $\{\bar{\mathcal{F}}_t\}$ local martingale, we have

$$\int_0^\cdot Z_i(u)dU_j(u) + \int_0^\cdot Z_j(u)dU_i(u)$$

is a $\{\bar{\mathcal{F}}_t\}$ local martingale as well. Combining the above two observations, we have

$$V_{ij}(\cdot) \doteq [U_i, U_j](\cdot) - \int_0^\cdot \Lambda_{ij}(Z(u), Y(u))du$$

is a $\{\bar{\mathcal{F}}_t\}$ local martingale. Since V_{ij} has continuous sample paths of finite variations, $V_{ij}(t) = V_{ij}(0) = 0$. Therefore, we have

$$[U_i, U_j](t) = \langle U_i, U_j \rangle(t) = \int_0^t \Lambda_{ij}(Z(u), Y(u))du, \quad t \geq 0.$$

Consider the process

$$W(t) \doteq \int_0^t \sigma(Z(u), Y(u))' \Lambda^{-1}(Z(u), Y(u)) dU(u).$$

Clearly W is a continuous local martingale and noting that σ is invertible it is easily checked that $\langle W_i, W_j \rangle(t) = \delta_{ij} t, 1 \leq i \leq j \leq K$. Therefore, W is a standard K-dimensional Brownian motion. Also, clearly

$$U(\cdot) = \int_0^{\cdot} \sigma(Z(u), Y(u)) dW(u),$$

and therefore, for $t \geq 0$,

$$Z(t) = Z(0) + \int_0^t b(Z(u), Y(u)) du + \int_0^t \sigma(Z(u), Y(u)) dW(u) + [\mathbb{I} - \overline{\mathbb{P}}'] \eta(t).$$

Since $\int_0^\infty 1_{\{Z_i(u) \neq 0\}} d\eta_i(u) = 0, 1 \leq i \leq K$, we have, from Definition 2.3.1, that Z solves (2.3.11). The result follows. ■

Chapter 3

Stability properties for constrained Markov modulated diffusions

3.1 Introduction

Stability properties of constrained stochastic processes are useful in many applications arising from computing, telecommunications, and manufacturing systems. In this chapter, we study a family of constrained Markov modulated diffusion processes that arise in the heavy traffic analysis of multiclass queueing networks. We establish positive recurrence and geometric ergodicity properties for such processes under suitable stability conditions on a related deterministic dynamical system (see [24], [4] and [10]). Results of this chapter will be used in Chapter 4 to study the convergence of invariant measures for Markov modulated open queueing networks in heavy traffic. It will be shown there that under suitable conditions, the invariant measure for the queueing length process converges weakly to the invariant measure of a constrained Markov modulated diffusion process of the form studied in the current chapter.

We now describe the basic mathematical setting. Let $G \subset I\!R^K$ be a convex polyhedral cone with vertex at the origin given as the intersection of half spaces $G_i, i = 1, \cdots, N$ and denote by n_i and d_i the inward normal and constraint direction associated with G_i. The constrained Markov modulated diffusion process we study is constrained to take values in G and is defined through the equation

$$Z(t) = \Gamma\left(z + \int_0^{\cdot} b(Z(s), Y(s))ds + \int_0^{\cdot} \sigma(Z(s), Y(s))dW(s)\right)(t), \ t \geq 0. \quad (3.1.1)$$

The process (Z, Y) is defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ such that Y is a $\{\mathcal{F}_t\}$ Markov process with state space $I\!H \doteq \{1, 2, \ldots, H\}$, infinitesimal generator $\mathbb{Q}$ and stationary distribution $q^* = \{q_j^* : j \in I\!H\}$, and W is a $\{\mathcal{F}_t\}$ standard Brownian motion independent of Y. We assume the Skorohod map Γ defined by the data $\{(d_i, n_i) : i = 1, 2, \ldots, N\}$ is well posed and Lipschitz continuous (Assumption 3.2.1). We also assume the Lipschitz continuity on σ and b (Assumption 3.2.2) and the boundedness and and uniform nondegeneracy of σ (Assumption 3.2.3). Furthermore, we assume that the drift b can be expressed as

$$b(z, y) = b_1(z, y) + b_2(y), \ (z, y) \in G \times I\!H, \quad (3.1.2)$$

where $b_1 : G \times I\!H \to I\!R^K$ and $b_2 : I\!H \to I\!R^K$ are measurable maps. Define $b_2^* = \sum_{j \in I\!H} q_j^* b_2(j)$ and $b^*(z, y) = b_1(z, y) + b_2^*$. We will also make a suitable stability assumption (Assumption 3.2.5) on the drift vector b^* which ensures that trajectories of a certain deterministic dynamical system are attracted to the origin. This condition says that there exists a $\delta_0 \in (0, \infty)$ and a bounded set $A \subset G$ such that for all $z \in G \backslash A$ and $y \in I\!H$, $b^*(z, y) \in \mathcal{C}(\delta_0)$ where

$$\mathcal{C}(\delta_0) \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial \mathcal{C}) \geq \delta\},$$

and

$$\mathcal{C} \doteq \left\{-\sum_{i=1}^{N} \alpha_i d_i : \alpha_i \geq 0, i \in \{1, \cdots, N\}\right\}.$$

Our main results (Theorem 3.2.2 and Theorem 3.2.4) show that, under the above conditions, (Z, Y) is positive recurrent and has a unique invariant measure. For the case when $b_1 = 0$, one can obtain a sharper result (Theorem 3.2.3). That is, if b_2^* is in the interior of $\mathcal{C}$, then (Z, Y) is positive recurrent, and if b_2^* is not in $\mathcal{C}$, (Z, Y) is transient. Under the same stability condition, we identify an appropriate exponentially growing Lyapunov function V and establish the V-uniform ergodicity of (Z, Y). Consequently, we show finiteness of the moment generating function of the invariant measure in a neighborhood of zero, uniform time estimates for polynomial moments of all orders and functional central limit results (Theorem 3.2.4).

We note that our stability condition on the drift vector field is substantially weaker than the requirement that $b(z, y) \in \mathcal{C}(\delta_0)$ for all $z \in G \setminus A$ and $y \in \mathbb{H}$, and allow for the drift to be "transient" in some states of the Markov process Y. For example, consider vector $b_1, b_2 \in \mathbb{R}^K$ such that $b_1 \in \mathcal{C}^\circ$ and $b_2 \in \mathcal{C}^c$. Then it is well known that if $b(x, y) \equiv b_1$, Z in (3.1.1) will be positive recurrent and if $b(x, y) \equiv b_2$, Z will be transient. Our results show that in a Markov modulated case where, for example, $\mathbb{H} = \{1, 2\}$ and $b(x, y) \equiv b_y$, the pair (Z, Y) will be positive recurrent (in fact geometrically ergodic) if $b^* = q_1^* b_1 + q_2^* b_2 \in \mathcal{C}^\circ$ and transient if $b^* \in \mathcal{C}^c$.

We next consider a Markov modulated semimartingale reflected Brownian motion Z (modulated by the Markov process Y), which can be considered as a special case of constrained Markov modulated diffusion processes introduced above with $b(z, y) = b_2(y)$ and a constant covariance matrix σ. However, here we pose weaker assumption on the constraint vectors $\{(d_i, n_i) : i = 1, 2, \ldots, K\}$, namely, the matrix $(d_1|d_2|\ldots|d_K)$ is completely-$\mathcal{S}$ (see Section 3.2.2). Using a standard argument based on Girsanov's theorem and classical results of [46], one can establish the existence and uniquely characterize the probability law of such a process (see Theorem 3.2.5). Under a suitable stability condition, we prove that (Z, Y) is positive recurrent and has a unique stationary distribution (Theorem 3.2.6). Furthermore, we show that

(Z, Y) is V-uniformly ergodic where the Lyapunov function V grows exponentially, and establish properties (see Theorem 3.2.7) analogous to those in Theorem 3.2.4 for the more general setting of a completely-$\mathcal{S}$ reflected matrix.

The chapter is organized as follows. In Section 3.2, we collect the main results of this chapter. Proofs are given in Section 3.3. In Appendix, we collect proofs of results that are similar to arguments used in existing literature.

3.2 Main results

In this section we collect the main results of this chapter.

3.2.1 Stability properties under a regular Skorohod map

Recall the set G and vectors $\{n_i : i = 1, 2, \ldots, N\}$ from Section 3.1. Denote the set $\{z \in \partial G : \langle z, n_i \rangle = 0\}$ by F_i. With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the direction of constraint associated with the face F_i. At points on ∂G where more than one faces meet, there are more than one allowed directions of constraint. For $z \in \partial G$, define the set of directions of constraint

$$\mathbf{d}(z) = \left\{ d \in I\!\!R^K : d = \sum_{i \in I(z)} \alpha_i d_i, \alpha_i \geq 0, |d| = 1 \right\}, \tag{3.2.1}$$

where $I(z) = \{i \in \{1, 2, \ldots, N\} : \langle z, n_i \rangle = 0\}$. Note that if $I(z) = \{j\}$ for some $j \in \{1, 2, \ldots, N\}$, then $\mathbf{d}(z) = \{d_j\}$.

We now introduce the Skorohod problem (SP) and the Skorohod map (SM) associated with G and $\mathbf{d}$. Define $D_G([0, \infty) : I\!\!R^K) = \left\{ \psi \in D([0, \infty) : I\!\!R^K) : \psi(0) \in G \right\}$. For $\eta \in D([0, \infty) : I\!\!R^K)$, let $|\eta|(T)$ denote the total variation of η on $[0, T]$ with respect to the Euclidean norm on $I\!\!R^K$.

Definition 3.2.1. Let $\psi \in D_G([0,\infty) : I\!R^K)$ be given. Then the pair $(\phi, \eta) \in D([0,\infty) : G) \times D([0,\infty) : I\!R^K)$ solves the SP for ψ with respect to G and $\mathbf{d}$ if and only if $\phi(0) = \psi(0)$ and for all $t \in [0,\infty)$ the following hold:

(i) $\phi(t) = \psi(t) + \eta(t)$, and $\phi(t) \in G$.

(ii) $|\eta|(t) < \infty$, and $|\eta|(t) = \int_{[0,t]} 1_{\{\phi(s) \in \partial G\}} d|\eta|(s)$.

(iii) There exists Borel measurable map $\gamma : [0,\infty) \to I\!R^K$ such that $\gamma(t) \in \mathbf{d}(\phi(t))$ a.e. $d|\eta|$ and $\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s)$.

Let $D \subset D_G([0,\infty), I\!R^K)$ be the domain on which there is a unique solution to the SP. On D we define the SM Γ as $\Gamma(\psi) \doteq \phi$, if $(\phi, \phi - \psi)$ is the unique solution of the SP posed by ψ. We will make the following assumption on the regularity of the SM defined by the data $\{(d_i, n_i) : i \in \{1, 2, \ldots, N\}\}$.

Assumption 3.2.1. *The SM is well defined on all of* $D_G([0,\infty), I\!R^K)$, *that is* $D = D_G([0,\infty), I\!R^K)$, *and the SM is Lipschitz continuous in the following sense. There exists* $\kappa_1 \in (1,\infty)$ *such that for all* $\psi_1, \psi_2 \in D_G([0,\infty), I\!R^K)$:

$$\sup_{t \geq 0} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| \leq \kappa_1 \sup_{t \geq 0} |\psi_1(t) - \psi_2(t)| . \tag{3.2.2}$$

We refer the reader to [22, 23, 29] for sufficient conditions under which the above assumption holds.

We now introduce the Markov process (Z, Y) which will be studied here. The component Y is a Markov process with a finite state space $I\!H \doteq \{1, 2, \ldots, H\}$ endowed with the discrete metric and infinitesimal generator $\mathbb{Q}$, while Z is a constrained diffusion with drift and diffusion coefficients that, in addition to depending on the current state, are modulated through the values of Y. More precisely, the process Z

satisfies an integral equation of the form

$$Z(t) = \Gamma \left(z + \int_0^{\cdot} b\left(Z(s), Y(s)\right) ds + \int_0^{\cdot} \sigma\left(Z(s), Y(s)\right) dW(s) \right)(t) \in G, \qquad (3.2.3)$$

where $z \in G$, W is a standard Wiener process which is independent of Y, and $\sigma : G \times I\!\!H \to I\!\!R^{K \times K}$, $b : G \times I\!\!H \to I\!\!R^K$ are measurable maps. We will make the usual Lipschitz assumption on the coefficients b and σ as follows.

Assumption 3.2.2. *There exists $\kappa_2 \in (0, \infty)$ such that, for all $z_1, z_2 \in G$ and $y \in I\!\!H$,*

$$|\sigma(z_1, y) - \sigma(z_2, y)| + |b(z_1, y) - b(z_2, y)| \le \kappa_2 |z_1 - z_2|.$$

Let $\mathbb{S} \doteq G \times I\!\!H$ and $\Phi \doteq (Z, Y)$. Using the above Lipschitz property along with the regularity assumption on the SM Γ (Assumption 3.2.1), it is easily seen that equation (3.2.3) is well posed. In particular, we have the following.

Theorem 3.2.1. *Under Assumptions 3.2.1 and 3.2.2, there is a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0})$ on which are given a collection of probability measures $\{P_z\}_{z \in \mathbb{S}}$ and $\{\mathcal{F}_t\}$ adapted processes (Z, W, k) and Y with sample paths in $C([0,\infty) : G \times I\!\!R^K \times I\!\!R^K)$ and $D([0,\infty) : I\!\!H)$ respectively, such that $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ is a Feller-Markov family and, for every $\varphi \equiv (z, y) \in \mathbb{S}$, P_φ-a.s., the following hold.*

(i) *W is a K-dimensional standard $\{\mathcal{F}_t\}_{t \ge 0}$ Brownian motion.*

(ii) *For all $t \in (0, \infty)$,*

$$Z(t) = z + \int_0^t b(\Phi(s))ds + \int_0^t \sigma(\Phi(s))dW(s) + k(t), \qquad (3.2.4)$$

and $Z(t) \in G$.

(iii) *For all $t \in (0, \infty)$, $|k|(t) < \infty$ and $|k|(t) = \int_0^t 1_{\{Z(s) \in \partial G\}} d|k|(s)$.*

(iv) *There is a* $I\!R^K$*-valued* $\{\mathcal{F}_t\}$ *progressively measurable process* γ *such that* $\gamma(t) \in \mathbf{d}(Z(t))$ *a.e.* $d|k|$ *and for all* $t \in (0,\infty)$, $k(t) = \int_0^t \gamma(s) d|k|(s)$.

(v) Y *is a* $I\!H$*-valued* $\{\mathcal{F}_t\}$*-Markov process with* $Y(0) = y$ *and infinitesimal generator* $\mathbb{Q}$.

We will denote the Markov family $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ merely as Φ and denote the transition kernel of Φ by P^t_Φ, namely for $\varphi \in \mathbb{S}$ and $A \in \mathcal{B}(\mathbb{S})$, $P^t_\Phi(\varphi, A) = P_\varphi(\Phi(t) \in A)$.

We recall the definitions of positive recurrence and transience of a Markov process.

Definition 3.2.2. The Markov process $\{\Phi(t) : t \geq 0\}$ is said to be positive recurrent if for each $A \in \mathcal{B}(G)$ with positive Lebesgue measure, $j \in I\!H$, and $\varphi \in \mathbb{S}$, we have $E_\varphi(\tau_{A\times\{j\}}) < \infty$, where $\tau_{A\times\{j\}} = \inf\{t \geq 0 : \Phi(t) \in A \times \{j\}\}$ and E_φ denotes the expectation under P_φ.

Definition 3.2.3. The Markov process $\{\Phi(t) : t \geq 0\}$ is said to be transient if there exist $A \in \mathcal{B}(G)$ with positive Lebesgue measure, $j \in I\!H$, and $\varphi \in \mathbb{S}$ such that $P_\varphi(\tau_{A\times\{j\}} < \infty) < 1$.

We now introduce additional assumptions that will be needed for the main stability results. The second part of the following assumption (along with Assumption 3.2.4) will ensure irreducibility of the Markov process Φ while the first will be needed in some moment estimates.

Assumption 3.2.3.

(i) *For some* $\kappa_3 \in (0,\infty)$ *and for all* $\varphi \in \mathbb{S}$, $|\sigma(\varphi)| \leq \kappa_3$.

(ii) *There exists* $\kappa_4 \in (0,\infty)$ *such that for all* $\varphi \in \mathbb{S}$ *and* $\zeta \in I\!R^K$, $\zeta'\sigma(\varphi)\sigma'(\varphi)\zeta \geq \kappa_4 \zeta'\zeta$.

We will make the following irreducibility assumption on the finite state Markov process associated with the generator $\mathbb{Q}$. Let $\mathbb{T}_t = \exp(t\mathbb{Q})$, $t \geq 0$.

Assumption 3.2.4. *For every* $t > 0$ *and* $i, j \in I\!H$, $\mathbb{T}_t(i,j) > 0$.

This assumption ensures that the Markov process with the infinitesimal generator $\mathbb{Q}$ has a unique stationary distribution $q^* \equiv \{q_j^*\}_{j \in I\!H}$. We now introduce the main stability assumption on the drift coefficient b. Let

$$\mathcal{C} \doteq \left\{ -\sum_{i=1}^{N} \alpha_i d_i : \alpha_i \geq 0, i \in \{1, \cdots, N\} \right\}. \tag{3.2.5}$$

and

$$\mathcal{C}(\delta) \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \delta\},\ \delta \in (0, \infty). \tag{3.2.6}$$

The cone in (3.2.5) plays a key role in the stability analysis of constrained diffusions (see [8, 4, 11]). For example, it follows from results in [4] that if the drift and diffusion coefficients do not depend on the process Y (i.e., for all $(z, y) \in \mathbb{S}$, $b(z, y) \equiv b(z)$ and $\sigma(z, y) \equiv \sigma(z)$), and for some $\delta_0 > 0$, $b(z) \in \mathcal{C}(\delta_0)$ for all $z \in G$, then the Markov process Z is positive recurrent and consequently has a unique invariant probability measure.

In the current work we will assume that the drift b can be expressed as

$$b(z, y) = b_1(z, y) + b_2(y),\ (z, y) \in \mathbb{S}, \tag{3.2.7}$$

where $b_1 : \mathbb{S} \to I\!R^K$ and $b_2 : I\!H \to I\!R^K$ are measurable maps satisfying Assumption 3.2.5 below.

Assumption 3.2.5. *There exist* $\delta_0 \in (0, \infty)$ *and bounded set* $A \subset G$ *such that for all* $z \in G \backslash A$ *and* $y \in I\!H$, $b^*(z, y) \in \mathcal{C}(\delta_0)$ *where*

$$b^*(z, y) = b_1(z, y) + b_2^* \text{ and } b_2^* = \sum_{j \in I\!H} q_j^* b_2(j),\ (z, y) \in \mathbb{S}. \tag{3.2.8}$$

The following theorem is the first main result of this chapter.

Theorem 3.2.2. *Suppose that Assumptions 3.2.1-3.2.5 hold. Then the Markov process* $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ *is positive recurrent and has a unique invariant probability measure* π.

Remark 3.2.1. In [49], the authors consider a 1-dimensional Markov-modulated reflected Ornstein–Uhlenbeck process $\{Z(t) : t \geq 0\}$ defined as follows.

$$Z(t) = -\int_0^t [\lambda_1(Y(s))Z(s) + \lambda_2(Y(s))]ds + \int_0^t \sigma(Y(s))dB(s) + k(t),\ t \geq 0,$$

where $\{Y(t)\}_{t\geq 0}$ is as in Theorem 3.2.1(v), $\{B(t)\}_{t\geq 0}$ is a standard 1-dimensional Brownian motion, and $\lambda_1, \lambda_2, \sigma$ are all strictly positive functions. The paper shows that (Z, Y) has a unique stationary distribution. Clearly $b(z, y) = -[\lambda_1(y)z + \lambda_2(y)]$ satisfies Assumption 3.2.5 and thus Theorem 3.2.2 in particular covers the setting considered in [49]. In fact, Theorem 3.2.2, in addition to covering the much more general multidimensional setting, shows that the positivity assumption on λ_1, λ_2 can be relaxed to the condition that λ_1, λ_2 are nonnegative and $\lambda_2(j) > 0$ for some $j \in I\!H$.

For the case when $b_1 = 0$, one can obtain a sharper result as follows.

Theorem 3.2.3. *Suppose that* $b_1(\varphi) = 0$ *for all* $\varphi \in \mathbb{S}$. *Also suppose that Assumptions 3.2.1–3.2.4 hold. Then the following hold:*

(i) *If* $b_2^* \in \mathcal{C}^o$, *then* $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ *is positive recurrent.*

(ii) *If* $b_2^* \notin \mathcal{C}$, *then* $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ *is transient.*

In section 3.2.4, we will establish geometric ergodicity of the Markov family $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$. More precisely, the following result will be proved. Let $f : \mathbb{S} \to I\!R$ be a measurable function such that, for some measurable $g : \mathbb{S} \to I\!R$ and for all $\varphi \in \mathbb{S}$, $t \geq 0$,

$$E_\varphi|f(\Phi(t))| + E_\varphi\left[\int_0^t |g(\Phi(s))|ds\right] < \infty,\ E_\varphi[f(\Phi(t))] = f(\varphi) + E_\varphi\left[\int_0^t g(\Phi(s))ds\right].$$

Denote by $\mathcal{D}(\mathcal{A})$ the collection of all such measurable functions f. For a pair (f, g) as above, we write $(f, g) \in \mathcal{A}$, or with an abuse of terminology, $g = \mathcal{A}f$. The (multi-valued) operator $\mathcal{A}$ is referred to as the extended generator of Φ and $\mathcal{D}(\mathcal{A})$ its domain.

Theorem 3.2.4. *Suppose that Assumptions 3.2.1-3.2.5 hold. Then the following properties hold.*

(i) *There exists $\beta_1 \in (0, \infty)$ such that for all measurable $f : \mathbb{S} \to I\!R$ which satisfy $|f(\varphi)| \le e^{\beta_1 |z|}$ for all $\varphi = (z, y) \in \mathbb{S}$,*

$$\int_{\mathbb{S}} |f(\varphi)| \pi(d\varphi) < \infty.$$

In particular, for all $c \in I\!R^K$ with $|c| \le \beta_1$,

$$\int_{\mathbb{S}} e^{\langle c, z \rangle} \pi(dzdy) < \infty.$$

(ii) *There are $\beta_2, \beta_3, b_0 \in (0, \infty)$ such that for f as in (i), the following hold.*

(a) *For all $\varphi = (z, y) \in \mathbb{S}$ and $t \in (0, \infty)$,*

$$|E_\varphi(f(\Phi(t)) - \pi(f)| \le e^{\beta_2(|z|+1)} e^{-b_0 t}.$$

(b) *Defining for $t \ge 0$, $S_t \doteq \int_0^t f(\Phi(u))du$, we have that $f_t^c(\varphi) \doteq E_\varphi(S_t - t\pi(f))$ converges to a finite limit $\hat{f}(\varphi)$ for all $\varphi \in \mathbb{S}$.*

(c) *The convergence in (b) is exponentially fast, i.e.,*

$$|f_t^c(\varphi) - \hat{f}(\varphi)| \le e^{\beta_3(|z|+1)} e^{-b_0 t}$$

for all $\varphi = (z, y) \in \mathbb{S}$ and $t \in (0, \infty)$.

(d) *The function* $\hat{f} \in \mathcal{D}(\mathcal{A})$ *and solves the Poisson equation:* $\mathcal{A}\hat{f}(\varphi) = \pi(f) - f(\varphi), \varphi \in \mathbb{S}$.

(iii) *Let* $f : \mathbb{S} \to I\!R$ *be a measurable function such that, with* β_1 *as in (i),* $f^2(\varphi) \leq e^{\beta_1 |z|}$*, for all* $\varphi = (z, y) \in \mathbb{S}$*. Define for* $t \in [0, 1]$,

$$\xi_n(t) \doteq \frac{1}{\sqrt{n}} \int_0^{nt} (f(\Phi(s)) - \pi(f))\, ds.$$

Let $\hat{f}$ *be as in (ii)(b). Define*

$$\gamma_f^2 \doteq 2 \int \hat{f}(\varphi)(f(\varphi) - \pi(f))\pi(dz).$$

Then $|\gamma_f| < \infty$ *and* ξ_n *converges weakly to* $\gamma_f B$ *in* $C([0, 1], I\!R)$*, where* B *is a* 1 *dimensional standard Brownian motion.*

3.2.2 Markov modulated SRBM

In Section 3.4 we will consider a model with somewhat more restrictive conditions on the domain and the coefficients b and σ but significantly weaker assumptions on the constraint vector field $\mathbf{d}$. Suppose that $G = I\!R_+^K$ and $N = K$. For $i \in I\!K$, let $n_i = e_i$. Then the i^{th} face $F_i = \{z \in G : z_i = 0\}$. Define a $K \times K$ matrix $R = (d_1 | \cdots | d_K)$ and a K-dimensional vector $b_0 \in I\!R^K$. Let σ be a $K \times K$ positive definite matrix. We recall from [46] the definition of a SRBM associated with (G, b_0, σ, R).

Definition 3.2.4. For $z \in G$, an SRBM associated with (G, b_0, σ, R) that starts from z is a continuous, $\{\bar{\mathcal{F}}_t\}$-adapted K-dimensional process $\bar{Z}$, defined on some filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \bar{P})$ such that, $\bar{P}$-a.s., the following hold.

(i) $\bar{Z}(t) = z + b_0 t + \sigma \bar{W}(t) + R\bar{U}(t)$ and $\bar{Z}(t) \in G$ for all $t \geq 0$.

(ii) $\bar{W}$ is a K-dimensional standard $\{\bar{\mathcal{F}}_t\}$ Brownian motion.

(iii) $\bar{U}$ is an $\{\bar{\mathcal{F}}_t\}$-adapted K-dimensional process such that, for $i = 1, \ldots, K$, $\bar{U}_i(0) = 0$, $\bar{U}_i$ is continuous and nondecreasing, and $\bar{U}_i$ can increase only when $\bar{Z}$ is on F_i, *i.e.*, $\int_0^\infty 1_{\{\bar{Z}_i(s)>0\}} d\bar{U}_i(s) = 0$.

An SRBM arises as the diffusion approximation limit for many multiclass queueing networks in heavy traffic (see [47]). The paper [46] shows that if R is completely-$\mathcal{S}$, namely for every $k \times k$ principle submatrix $\tilde{R}$ of R, there is a k-dimensional vector $v_{\tilde{R}}$ such that $v_{\tilde{R}} \geq 0$ and $\tilde{R} v_{\tilde{R}} > 0$, then (weak) existence and uniqueness of SRBM hold. This condition, which is significantly weaker than Assumption 3.2.1 made in Section 3.2.2, is in fact known to be a necessary condition for existence of an SRBM ([41, Theorem 2]). We record this condition below for future reference.

Assumption 3.2.6. *The matrix R is completely-$\mathcal{S}$.*

The following result follows from [46] along with a straightforward argument based on Girsanov's theorem. Fix a measurable map $b_2 : \mathbb{H} \to \mathbb{R}^K$.

Theorem 3.2.5. *Suppose that Assumption 3.2.6 holds. Then there is a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$ on which are given a collection of probability measures $\{P_\varphi\}_{\varphi\in\mathbb{S}}$ and $\{\mathcal{F}_t\}$-adapted processes (Z, W, U) and Y with sample paths in $C([0,\infty) : G \times \mathbb{R}^K \times G)$ and $D([0,\infty) : \mathbb{H})$, respectively, such that for every $\varphi \equiv (z, y) \in \mathbb{S}$, P_φ-a.s., the following hold.*

(i) *W is a K-dimensional standard $\{\mathcal{F}_t\}$ Brownian motion.*

(ii) *For all $t \geq 0$,*

$$Z(t) = z + \int_0^t b_2(Y(s))ds + \sigma W(t) + RU(t), \tag{3.2.9}$$

and $Z(t) \in G$.

(iii) *For each* $i = 1, \ldots, K$, $U_i(0) = 0$, U_i *is continuous, and nondecreasing, and*

$$\int_0^\infty 1_{\{Z_i(s)>0\}} dU_i(s) = 0.$$

(iv) Y *is a* $I\!H$*-valued* $\{\mathcal{F}_t\}$*-Markov process with* $Y(0) = y$ *and infinitesimal generator* $\mathbb{Q}$.

Let $\Phi = (Z, Y)$. *Then* $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ *is a Feller-Markov family.*

We now recall the key stability condition, introduced in [24], for positive recurrence of an SRBM, in terms of the associated "fluid limit" trajectories.

Definition 3.2.5. We say a vector $b_0 \in I\!R^K$ satisfies the DW-stability condition if for all $\phi \in C([0, \infty) : G)$ satisfying the property (F) below, we have $\phi(t) \to 0$ as $t \to \infty$.

$$(\mathrm{F}) \begin{cases} \text{For some } z \in G \text{ and } \eta \in C([0,\infty) : G), \phi(t) = z + b_0 t + R\eta(t) \text{ for all } t \geq 0, \\ \text{where for } i = 1, \ldots, K, \eta_i(0) = 0, \eta_i \text{ is nondecreasing, and } \int_0^\infty 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0. \end{cases}$$

In [24], the authors showed that if Φ is a (G, b_0, σ, R) SRBM, i.e., $b_2(y) = b_0$ for all $y \in I\!H$, and b_0 satisfies the DW-stability condition, then the SRBM is positive recurrent and consequently has a unique invariant probability distribution. In the current work we establish the following result.

Theorem 3.2.6. *Suppose that Assumptions 3.2.4 and 3.2.6 hold and the vector* b_2^* *satisfies the DW-stability condition. Then the family* $(\Phi, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ *is positive recurrent and admits a unique invariant probability measure* π.

In fact, we establish the following geometric ergodicity properties. Analogous result for the constant drift case is established in [10].

Theorem 3.2.7. *Under the assumptions made in Theorem 3.2.6, properties in Theorem 3.2.4 hold for Markov process* Φ.

3.3 Stability properties under a regular Skorohod map

3.3.1 Positive recurrence

In this section we prove Theorem 3.2.2. Assumptions 3.2.1-3.2.5 will be assumed throughout this section. Recall the parameter δ_0 introduced in Assumption 3.2.5.

Let $\upsilon : [0, \infty) \to I\!R^K$ be a measurable map such that,

$$\int_0^t |\upsilon(s)| ds < \infty, \text{ for all } t \geq 0. \tag{3.3.1}$$

For $z \in G$ and υ as above, let

$$\mathbf{z}(t) = \Gamma \left(z + \int_0^t \upsilon(s) ds \right), \; t \geq 0. \tag{3.3.2}$$

For $z \in G$, let $\mathcal{A}(z) \equiv \mathcal{A}(z, \delta_0)$ be the set of all absolutely continuous functions $\mathbf{z}$ defined by (3.3.2) for some $\upsilon : [0, \infty) \to \mathcal{C}(\delta_0)$ that satisfies (3.3.1). Define the "hitting time to the origin" function as follows.

$$T(z) = \sup_{\mathbf{z} \in \mathcal{A}(z)} \inf\{t \in (0, \infty) : \mathbf{z}(t) = 0\}, \; z \in G. \tag{3.3.3}$$

Note that $T(0) = 0$. The following lemma from [4](cf. Lemma 3.1) is a key ingredient in our analysis.

Lemma 3.3.1. *The function T defined by* (3.3.3) *satisfies the following properties.*

(i) *For some* $\Theta_1 \equiv \Theta_1(\delta_0) \in (0,\infty)$, $|T(z_1) - T(z_2)| \le \Theta_1 |z_1 - z_2|$ *for all* $z_1, z_2 \in G$.

(ii) *For some* $\Theta_2 \equiv \Theta_2(\delta_0) \in (0,\infty)$, $\Theta_2|z| \le T(z) \le \Theta_1|z|$, *for all* $z \in G$.

(iii) *Fix* $z \in G$ *and let* $\mathbf{z} \in \mathcal{A}(z)$. *Then for all* $t > 0$, $T(\mathbf{z}(t)) \le (T(z) - t)^+$.

Note that for $(z, y) \in \mathbb{S}$, $b(z, y) - b^*(z, y) = b_2(y) - b_2^*$. Define $b^c(y) = b_2(y) - b_2^*$, $y \in I\!H$. The following lemma is an immediate consequence of Lemma 3.3.1 and the Lipschitz property of Γ. The proof is quite similar to that of Lemma 4.1 of [4], however for completeness we provide the arguments in Appendix. Recall the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{P_\varphi\}_{\varphi \in \mathbb{S}})$ and processes Z, W, Y, Φ introduced in Theorem 3.2.1.

Lemma 3.3.2. *Let* $\Delta > 0$ *and* $u > 0$ *be arbitrary. Fix* $\varphi \in \mathbb{S}$. *Then,* P_φ*-a.s., on the set* $\{\omega : Z(t, \omega) \in G \setminus A$ *for all* $t \in (u, u + \Delta]\}$,

$$T(Z(u + \Delta)) \le (T(Z(u)) - \Delta)^+ + \kappa_1 \Theta_1 \nu_\Delta^u,$$

where Θ_1 *and* κ_1 *are as in Lemma 3.3.1(i) and Assumption 3.2.1 respectively, and*

$$\nu_\Delta^u \doteq \sup_{u \le s \le u+\Delta} \left| \int_u^s b^c(Y(v))dv + \int_u^s \sigma(\Phi(v))dW(v) \right|. \tag{3.3.4}$$

Lemma 3.3.3. *There exists a* $\Theta_3 \in (0,\infty)$ *such that for all* $\alpha, t \in (0,\infty)$ *and* $\varphi \in \mathbb{S}$,

$$E_\varphi(\exp\{\alpha \nu_t^0\}) \le 8 \exp\left\{\Theta_3 \alpha(1 + \alpha + \alpha t)\right\}, \tag{3.3.5}$$

where ν_t^0 *is defined as in* (3.3.3) *with* u, Δ *replaced by* $0, t$*, respectively.*

Proof: By Holder's inequality,

$$\left[E_\varphi\left(\exp\left\{\alpha\sup_{0\le s\le t}\left|\int_0^s b^c(Y(u))du+\int_0^s\sigma(\Phi(u))dW(u)\right|\right\}\right)\right]^2$$
$$\le E_\varphi\left(\exp\left\{2\alpha\sup_{0\le s\le t}\left|\int_0^s b^c(Y(u))du\right|\right\}\right)E_\varphi\left(\exp\left\{2\alpha\sup_{0\le s\le t}\left|\int_0^s\sigma(\Phi(u))dW(u)\right|\right\}\right). \tag{3.3.6}$$

Consider the first expectation on the right hand side of the above inequality. For $f\in \mathrm{BM}(\mathbb{H})$, $s\ge 0$ and $y\in\mathbb{H}$, let $P_Y^s f(y)=E(f(Y(s))|Y(0)=y)$. Let $g(\cdot)$ be a solution of the Poisson equation for $b^c(\cdot)$ corresponding to the Markov semigroup $\{P_Y^s\}_{s\ge 0}$, i.e., for $y\in\mathbb{H}$ and $s\ge 0$,

$$P_Y^s g(y)-g(y)-\int_0^s P_Y^u b^c(y)du=0.$$

Then, under P_φ,

$$M_s\doteq g(Y(s))-g(Y(0))-\int_0^s b^c(Y(u))du \tag{3.3.7}$$

is an $\{\mathcal{F}_s\}$ martingale.

We next show that, for all $s\ge 0$ and $y\ge 0$,

$$P_\varphi(|M_s|\ge y)\le 2\exp\left\{\frac{-2y^2}{(1+v^2)^2(s+1)}\right\}, \tag{3.3.8}$$

where $v=2(|g|_\infty+|b^c|_\infty)<\infty$. For fixed $s\ge 0$, let

$$\xi_k=\begin{cases}M_{k+1}-M_k, & 0\le k\le\lfloor s\rfloor-1,\\ M_s-M_{\lfloor s\rfloor}, & k=\lfloor s\rfloor.\end{cases}$$

Then $M_s=\sum_{i=0}^{\lfloor s\rfloor}\xi_i$ and for $0\le k\le\lfloor s\rfloor$, $E_\varphi(\xi_k|\mathcal{F}_k)=0$ and $|\xi_k|\le v$. Using well known concentration inequalities for martingales with bounded increments (see e.g.

Corollary 2.4.7 in [19]), for $0 \le k \le \lfloor s \rfloor$ and $y \ge 0$,

$$P_\varphi\left(\sum_{i=0}^{k} \xi_i \ge y\sqrt{k+1}\right) \le \exp\left\{-\frac{2y^2}{(1+v^2)^2}\right\}.$$

Therefore,

$$P_\varphi\left(\sum_{i=0}^{\lfloor s \rfloor} \xi_i \ge y\right) \le \exp\left\{-\frac{2y^2}{(1+v^2)^2(\lfloor s \rfloor+1)}\right\} \le \exp\left\{-\frac{2y^2}{(1+v^2)^2(s+1)}\right\}.$$

Similarly,

$$P_\varphi\left(-\sum_{i=0}^{\lfloor s \rfloor} \xi_i \ge y\right) \le \exp\left\{-\frac{2y^2}{(1+v^2)^2(s+1)}\right\}.$$

The inequality in (3.3.8) follows on combining the above two estimates.

Denoting $\frac{2}{(1+v^2)^2}$ by c_1, we have,

$$\begin{aligned} E_\varphi\left(\exp\{2\alpha|M_s|\}\right) &\le \int_0^\infty 2\exp\left\{-\frac{c_1(\log y)^2}{4\alpha^2(s+1)}\right\}dy \\ &= 2\sqrt{\frac{4\pi\alpha^2(s+1)}{c_1}}\exp\left\{\frac{\alpha^2(s+1)}{c_1}\right\} \\ &\le 2\exp\left\{\frac{(1+4\pi)\alpha^2(s+1)}{c_1}\right\}. \end{aligned}$$

An application of Doob's inequality now yields that

$$E_\varphi\left(\exp\left\{2\alpha\sup_{0\le s\le t}|M_s|\right\}\right) \le 4E_\varphi\left(\exp\left\{2\alpha|M_t|\right\}\right) \le 8\exp\left\{\frac{(1+4\pi)\alpha^2(t+1)}{c_1}\right\}.$$

Combining this with (3.3.7), we have that

$$E_\varphi\left(\exp\left\{2\alpha\sup_{0\le s\le t}\left|\int_0^s b^c(Y(u))du\right|\right\}\right) \le 8\exp\left\{4\alpha|g|_\infty + \frac{(1+4\pi)\alpha^2(t+1)}{c_1}\right\}. \tag{3.3.9}$$

Next consider the second expectation on the right side of (3.3.6). Using Assumption

3.2.3 (i), we have by standard estimates (see e.g. Lemma 4.2 of [4])

$$E_\varphi\left(\exp\left\{2\alpha \sup_{0\le s\le t}\left|\int_0^s \sigma(\Phi(u))dW(u)\right|\right\}\right) \le 8\exp\left\{2\alpha^2\kappa_3^2K^2t\right\}. \tag{3.3.10}$$

Using (3.3.9) and (3.3.10), we now have that the left side of (3.3.5) is bounded above by

$$8\exp\left\{2|g|_\infty\alpha + \frac{1+4\pi}{2c_1}\alpha^2 + \left(\frac{1+4\pi}{2c_1} + \kappa_3^2K^2\right)\alpha^2 t\right\}.$$

The result follows. ■

Using the fact that Φ is a $\{\mathcal{F}_t\}$-Markov process and that W is a $\{\mathcal{F}_t\}$-Brownian motion (cf. Lemma 4.3 of [4]) we have the following lemma. Proof is omitted.

Lemma 3.3.4. *Let $\varphi \in \mathbb{S}$ and $\Delta > 0$ be fixed. For $n \in I\!N$, let $\nu_n \equiv \nu_\Delta^{(n-1)\Delta}$, where $\nu_\Delta^{(n-1)\Delta}$ is defined as in* (3.3.4) *with u replaced by $(n-1)\Delta$. Then for any $\alpha \in (0,\infty)$ and $m, n \in I\!N; m \le n$,*

$$E_\varphi\left(\exp\left\{\alpha\sum_{i=m}^n \nu_i\right\}\right) \le \left(8\exp\left\{\Theta_3\alpha(1+\alpha+\alpha\Delta)\right\}\right)^{n-m+1},$$

where Θ_3 is as in Lemma 3.3.3.

Given a compact set $C \subset \mathbb{S}$, let

$$\tau_C \doteq \inf\{t \ge 0 : \Phi(t) \in C\}. \tag{3.3.11}$$

For $M > 0$, let $B_M \doteq \{\varphi \equiv (z,y) \in \mathbb{S} : T(z) \le M\}$ and $C_M = \{\varphi \equiv (z,y) \in \mathbb{S} : |z| \le M\}$.

Theorem 3.3.1. *There exist $L, a \in (0,\infty)$ and $\varsigma \in (0,1)$ such that for any $\varphi \equiv (z,y) \in \mathbb{S}$ and $t \in (0,\infty)$,*

$$P_\varphi(\tau_{B_L} > t) \le \exp\{\varsigma T(z) + (a-\varsigma)L\}\exp(-at). \tag{3.3.12}$$

In particular, for every $M \in (0,\infty)$ and $a_0 < a$,

$$\sup_{\varphi \in C_M} E_\varphi(\exp\{a_0 \tau_{B_L}\}) < \infty.$$

Proof. Proof is similar to that of Theorem 4.1 of [4] so only a sketch is provided. Fix $\varphi = (z,y) \in \mathbb{S}$. Recall the set A from Assumption 3.2.5. Choose $L > 0$ large enough so that $A \times I\!\!H \subset B_L$. Additional restrictions on L will be imposed later in the proof. Let

$$\Omega_n \doteq \{\omega : \tau_{B_L} > nL\} = \left\{\omega : \inf_{0 \le s \le nL} T(Z(s,\omega)) > L\right\}.$$

Then for $\varphi \in \mathbb{S}$, $P_\varphi(\Omega_n) \le P_\varphi(T(Z(nL)) > L)$. By Lemma 3.3.2 we have, for $\omega \in \Omega_n$,

$$T(Z(nL)) \le T(z) - nL + \kappa_1 \Theta_1 \sum_{j=1}^{n} \nu_j,$$

where $\{\nu_j\}$ are defined as in Lemma 3.3.4 with Δ replaced by L. Using Lemma 3.3.4, a calculation similar to that in the proof of Theorem 4.1 of [4] now yields for any $\varsigma \in (0,1)$,

$$P_\varphi(\Omega_n) \le \exp(\varsigma(T(z) - L)) \exp\left\{n\left[\Theta_3 \varsigma \kappa_1 \Theta_1 (1 + \varsigma\kappa_1\Theta_1) + \log 8 + \Theta_3 \varsigma^2 \kappa_1^2 \Theta_1^2 L - \varsigma L\right]\right\}.$$

Take $\varsigma = (2\Theta_3\kappa_1^2\Theta_1^2 + 1)^{-1}$. Choose L sufficiently large so that, in addition to the property $A \times I\!\!H \subset B_L$, we have $L^{-1}[\Theta_3\varsigma\kappa_1\Theta_1(1+\varsigma\kappa_1\Theta_1) + \log 8] < \varsigma/2$. Then

$$L^{-1}\left[\Theta_3\varsigma\kappa_1\Theta_1(1+\varsigma\kappa_1\Theta_1) + \log 8 + \Theta_3\varsigma^2\kappa_1^2\Theta_1^2 L - \varsigma L\right] < \Theta_3\varsigma^2\kappa_1^2\Theta_1^2 - \frac{\varsigma}{2} \doteq -a < 0,$$

and

$$P_\varphi(\Omega_n) \le \exp\{\varsigma(T(z) - L)\} \exp(-nLa).$$

The proof of (3.3.12) now follows from the above estimate, exactly as in the proof of Theorem 4.1 of [4]. Second part of the theorem is an immediate consequence of (3.3.12). ■

The lemma below gives the tightness of the family $\{P_\varphi \circ \Phi(t)^{-1} : \varphi \in C_M, t \geq 0\}$ for any $M > 0$. The proof is similar to Lemma 4.4 of [4]. A sketch is given in Appendix.

Lemma 3.3.5. *There exists $\kappa \in (0, \infty)$ such that for all $M > 0$,*

$$\sup_{\varphi \in C_M} \sup_{t \geq 0} E_\varphi \left(\exp\{\kappa |Z(t)|\} \right) < \infty.$$

The following irreducibility property is used in showing uniqueness of the invariant measure. For $j \in I\!H$, $\varphi \in \mathbb{S}$, $t > 0$, define $m_\varphi^{j;t} \in \mathcal{P}(G)$ as $m_\varphi^{j;t}(E) = P_\varphi(\Phi(t) \in E \times \{j\})$, $E \in \mathcal{B}(G)$.

Lemma 3.3.6. *For every $j \in I\!H$, $\varphi \in \mathbb{S}$ and $t > 0$, $m_\varphi^{j;t}$ is mutually absolutely continuous with respect to the Lebesgue measure λ on G.*

Proof: Without loss of generality we can assume that on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$, introduced in Theorem 3.2.1 we have, for each $\varphi = (z, j) \in G \times I\!H$, probability measures P_z^j under which (i) - (iv) of Theorem 3.2.1 hold, with (3.2.4) replaced by

$$Z(t) = z + \int_0^t b(Z(s), j) ds + \int_0^t \sigma(Z(s), j) dW(s) + k(t), \text{ a.s..}$$

As argued in the proof of Lemma 5.7 of [10],

$$\text{for all } (z, j) \in \mathbb{S}, t > 0, P_z^j \circ Z(t)^{-1} \text{ is mutually absolutely continuous to } \lambda. \quad (3.3.13)$$

Fix $\varphi = (z, i) \in \mathbb{S}$. Denote by $\{\tau_k\}_{k \in I\!N_0}$ the sequence of transition times of the pure jump process Y, namely, $\tau_0 = 0$, $\tau_{k+1} = \inf\{t > \tau_k : Y_t \neq Y_{t-}\}$, $k \in \mathbb{N}_0$. Then P_φ a.s., τ_k is strictly increasing to ∞. Also, the law of $\tau_{k+1}-\tau_k$, conditioned on $\mathcal{F}_{\tau_k}$ (under P_φ) has density $\varphi_{Y_{\tau_k}}$ where for $i \in \mathbb{H}$, φ_i is the Exponential density with rate $\sum_{j:j\neq i} \mathcal{Q}_{ij}$. For $k \geq 0$, let $m_k^\varphi \in \mathcal{P}([0,\infty) \times G \times \mathbb{H})$ be the probability law of $(\tau_k, Z(\tau_k), Y(\tau_k))$. Also for $j \in \mathbb{H}$, define sub-probability measures $m_k^{\varphi,j}$ on $[0,\infty) \times G$ by the relation

$$m_k^{\varphi,j}(E) = m_k^\varphi(E \times \{j\}),\ E \in \mathcal{B}([0,\infty) \times G).$$

Then, for $A \in \mathcal{B}(G)$, $j \in \mathbb{H}$, $t > 0$,

$$\begin{aligned} P_\varphi(Z(t) \in A, Y(t) = j) &= \sum_{k=0}^{\infty} P_\varphi(Z(t) \in A, Y(t) = j, \tau_k < t < \tau_{k+1}) \\ &= \sum_{k=0}^{\infty} \int_{[0,t)\times G} \left(\int_{t-u}^{\infty} \varphi_j(v)dv \right) P_{\tilde{z}}^j(Z(t-u) \in A) m_k^{\varphi,j}(du d\tilde{z}). \end{aligned}$$

From (3.3.13) and the above display, if $\lambda(A) = 0$, then $P_\varphi(Z(t) \in A, Y(t) = j) = 0$. Conversely suppose that $\lambda(A) > 0$. From Assumption 3.2.4, for some $k_0 \in \mathbb{N}_0$, $P(Y(\tau_{k_0})|Y(0) = i) > 0$ and therefore $m_{k_0}^{\varphi,j}([0,t] \times G)$ is nonzero for every $t > 0$. Finally, from the above display and using (3.3.13) once again we obtain

$$P_\varphi(Z(t) \in A, Y(t) = j) \geq \int_{[0,t)\times G} \left(\int_{t-u}^{\infty} \varphi_j(v)dv \right) P_{\tilde{z}}^j(Z(t-u) \in A) m_{k_0}^{\varphi,j}(du d\tilde{z}) > 0.$$

The result follows. ■

Proof of Theorem 3.2.2. Let S be a compact subset of G with a positive Lebesgue measure. For the proof of positive recurrence, it suffices to show that for every $M > 0$ and $j \in \mathbb{H}$,

$$\sup_{\varphi \in C_M} E_\varphi(\tau^{(j)}) < \infty, \tag{3.3.14}$$

where $\tau^{(j)} = \inf\{t \geq 0 : \Phi(t) \in S^j\}$, $S^j = S \times \{j\}$. Let L be as in Theorem 3.3.1.

From Assumptions 3.2.3 and 3.2.4, it follows that

$$p_1 = \inf_{\varphi \in B_L} P_\varphi(\Phi(1) \in S^j) > 0.$$

Since the family $\{P_\varphi \circ \Phi(1)^{-1}, \varphi \in B_L\}$ is tight, there exists $c_1 \in (0, \infty)$ such that

$$\inf_{\varphi \in B_L} P_\varphi(\Phi(1) \in S^j, |Z|_1^* \leq c_1) \geq \frac{p_1}{2}.$$

Arguing as in the proof of Theorem 2.2 of [4], we now have that for all $M > c_1$,

$$\sup_{\varphi \in C_M} E_\varphi(\tau^{(j)}) \leq \sup_{\varphi \in C_M} E_\varphi(\tau_{B_L}) + \frac{2}{p_1}\left(1 + \sup_{\varphi \in C_M} E_\varphi(\tau_{B_L})\right).$$

This, in view of Theorem 3.3.1, proves (3.3.14) and positive recurrence of Φ follows. Existence of a unique invariant probability measure is an immediate consequence of Lemma 3.3.5, the Feller property of Φ, and the irreducibility property in Lemma 3.3.6. ∎

3.3.2 Transience

In this subsection, we prove Theorem 3.2.3. We will assume through this section that Assumption 3.2.1-3.2.4 hold and that $b_1(\varphi) = 0, \varphi \in \mathbb{S}$. Let ι be the identity map from $[0, \infty)$ to $[0, \infty)$. The following lemma is taken from [8] (cf. Lemma 3.1 and Theorem 3.10 therein).

Lemma 3.3.7. *For each $\zeta \in \mathbb{R}^K$, there is a $\tilde{\zeta} \in \mathbb{R}_+^K$ such that $\Gamma(\zeta\iota)(t) = \tilde{\zeta}t$ for all $t \geq 0$. Furthermore, $\tilde{\zeta} \neq 0$ if and only if $\zeta \notin \mathcal{C}$.*

Proof of Theorem 3.2.3: Part (i) is immediate from Theorem 3.2.2. Consider now Part (ii). Since $b_1 \equiv 0$, the process $\Phi = (Z, Y)$ satisfies, for every $\varphi = (z, y) \in \mathbb{S}$,

P_φ-a.s.,

$$Z(t) = \Gamma\left(z + \int_0^{\cdot} b_2(Y(s))ds + \int_0^{\cdot} \sigma(\Phi(s))dW(s)\right)(t),\ t \geq 0.$$

An application of triangle inequality shows that

$$\begin{aligned}\frac{|Z(t)|}{t} &= \frac{1}{t}\left|\Gamma\left(z + \int_0^{\cdot} b_2(Y(s))ds + \int_0^{\cdot} \sigma(\Phi(s))dW(s)\right)(t)\right| \\ &\geq \frac{1}{t}\left|\Gamma\left(b_2^*\iota\right)(t)\right| - \frac{1}{t}\left|\Gamma\left(z + \int_0^{\cdot} b_2(Y(s))ds + \int_0^{\cdot} \sigma(\Phi(s))dW(s)\right)(t) - \Gamma(b_2^*\iota)(t)\right|.\end{aligned}$$

Noting that $b^c(y) = b_2(y) - b_2^*$, we have from the Lipschitz property of Γ and Lemma 3.3.7,

$$\frac{|Z(t)|}{t} \geq |\tilde{\beta}| - \frac{\kappa_1}{t}\sup_{0\leq s\leq t}\left|\int_0^s \sigma(\Phi(u))dW(u)\right| - \frac{\kappa_1}{t}\sup_{0\leq s\leq t}\left|\int_0^s b^c(Y(u))du\right| - \frac{\kappa_1|z|}{t}, \tag{3.3.15}$$

where $\tilde{\beta} = \Gamma(b_2^*\iota)(1)$. Since $b_2^* \notin \mathcal{C}$, from Lemma 3.3.7, $\tilde{\beta} \neq 0$. Let $\tilde{W}_i(t) \doteq \langle e_i, \int_0^t \sigma(\Phi(s))dW(s)\rangle$, where $\{e_i\}_{i=1}^K$ is the standard basis in $\mathbb{R}^K$. Then the quadratic variation of the martingale $\tilde{W}_i$ equals $\langle \tilde{W}_i\rangle_t = \int_0^t e_i' a(\Phi(s))e_i ds$, where $a = \sigma\sigma'$. By Assumption 3.2.3, $\kappa_4 t \leq \langle \tilde{W}_i\rangle_t \leq \kappa_3^2 t$. Using a standard time change argument and the law of iterated logarithm for a scaler Brownian motion, we now have for some $c_1 \in (0,\infty)$,

$$\limsup_{t\to\infty}\frac{1}{t}\sup_{0\leq s\leq t}\left|\int_0^s \sigma(\Phi(u))dW(u)\right| \leq c_1 \limsup_{t\to\infty}\frac{1}{t}\sup_{0\leq s\leq t}\sum_{i=1}^K |\tilde{W}_i(s)| = 0,\ P_\varphi\text{-a.s.}$$

Next consider the martingale $\{M_t : t \geq 0\}$ defined by (3.3.7). From (3.3.8), there exists $c_2 \in (0,\infty)$ such that for all $t, y \in (0,\infty)$ and $\varphi \in \mathbb{S}$,

$$P_\varphi\left(|M_t| \geq y\right) \leq 2\exp\left\{-\frac{c_2 y^2}{t+1}\right\}.$$

Consequently, using Markov's inequality and L^p maximal inequality, for any $\epsilon > 0$,

$$P_\varphi\left(\frac{1}{t}\sup_{0\le s\le t}|M_s| > \epsilon\right) \le \frac{1}{\epsilon^4 t^4}E_\varphi\left(\left(\sup_{0\le s\le t}|M_s|\right)^4\right) \le \frac{4^4}{3^4\epsilon^4 t^4}E_\varphi\left(|M_t|^4\right)$$
$$\le 2\frac{4^4}{3^4\epsilon^4 t^4}\int_0^\infty \exp\left\{-\frac{c_2\sqrt{y}}{t+1}\right\}dy = \frac{4^5(t+1)^2}{3^4 c_2^2\epsilon^4 t^4}.$$

An application of Borel-Cantelli lemma now shows that, P_φ-a.s.

$$\limsup_{t\to\infty}\left(\frac{1}{t}\sup_{0\le s\le t}|M(s)|\right) = 0.$$

Combining this with (3.3.7), we have

$$\limsup_{t\to\infty}\left(\frac{1}{t}\sup_{0\le s\le t}\left|\int_0^s b^c(Y(u))du\right|\right) \le \limsup_{t\to\infty}\left(\frac{1}{t}\sup_{0\le s\le t}|M(s)| + \frac{2|g|_\infty}{t}\right) = 0.$$

Recalling that $\tilde{\beta} \neq 0$, we now have from (3.3.15) that,

$$\liminf_{t\to\infty}\frac{|Z(t)|}{t} > 0, P_\varphi \text{ -a.s.} \tag{3.3.16}$$

Finally, we argue that, for some $\varphi \in \mathbb{S}$,

$$P_\varphi(\tau_{C_1} < \infty) < 1, \tag{3.3.17}$$

where τ_{C_1} is as in (3.3.11) with C_1 defined as below (3.3.11) with M replaced by 1. Suppose that (3.3.17) is false. Then by a straightforward application of the strong Markov property, we have that

$$P_\varphi(\Phi(t_n) \in C_1 \text{ for some sequence } \{t_n\}, \text{ s.t. } t_n \uparrow \infty) = 1.$$

However, this contradicts (3.3.16) and the result follows. ■

3.3.3 Geometric ergodicity

In this section we prove Theorem 3.2.4. Assumption 3.2.1-3.2.5 will be assumed throughout this section. The following drift inequality is at the heart of Theorem 3.2.4.

Lemma 3.3.8. *For some $\varpi > 0$, the ϖ-skeleton chain $\{\check{\Phi}_n \doteq \Phi(n\varpi) : n \in I\!N\}$ satisfies the following drift inequality: There exist $\alpha_0, \beta_0 \in (0,1), \gamma_0 \in (0,\infty)$ and a compact set $S \subset G$ such that*

$$E_\varphi\left(V(\check{\Phi}_1)\right) \le (1-\beta_0)V(\varphi) + \gamma_0 1_{S\times I\!H}(\varphi),\ \varphi \in \mathbb{S}, \tag{3.3.18}$$

where, for $\varphi = (z,y)$, $V(\varphi) \doteq e^{\alpha_0 T(z)}$.

Proof: Define $\sigma_A = \inf\{t \ge 0 : Z(t) \in A\}$. From Lemma 3.3.2, for $\alpha_0, \varpi \in (0,\infty)$,

$$E_\varphi\left(V(\check{\Phi}_1)1_{\sigma_A>\varpi}\right) \le E_\varphi\left(\exp\left\{\alpha_0(T(z)-\varpi)^+ + \alpha_0\kappa_1\Theta_1\nu^0_\varpi\right\}1_{\sigma_A>\varpi}\right), \tag{3.3.19}$$

where ν^0_ϖ is defined by (3.3.4) with Δ and u replaced by ϖ and 0, respectively. Recall $B_\varpi = \{\varphi = (z,y) \in \mathbb{S} : T(z) \le \varpi\}$. Thus for $\varphi \in (B_\varpi)^c$, by Lemma 3.3.3,

$$\begin{aligned}\frac{E_\varphi\left(V(\check{\Phi}_1)1_{\{\sigma_A>\varpi\}}\right)}{V(\varphi)} &\le E_\varphi\left(\exp\left\{\alpha_0\kappa_1\Theta_1\nu^0_\varpi - \alpha_0\varpi\right\}1_{\sigma_A>\varpi}\right)\\ &\le 8\exp\{c_1\alpha_0 + c_2\alpha_0^2 + c_2\alpha_0^2\varpi - \alpha_0\varpi\},\end{aligned}$$

where $c_1 = \kappa_1\Theta_3\Theta_1$ and $c_2 = \Theta_3\kappa_1^2\Theta_1^2$. Now fix α_0 small enough and ϖ large enough so that

$$8\exp\{c_1\alpha_0 + c_2\alpha_0^2 + c_2\alpha_0^2\varpi - \alpha_0\varpi\} \doteq (1-2\beta_0) < 1.$$

Then for $\varphi \in (B_\varpi)^c$,

$$E_\varphi \left(V(\check{\Phi}_1) 1_{\sigma_A > \varpi} \right) \leq (1 - 2\beta_0) V(\varphi).$$

From the strong Markov property of Φ, we see that for all $\varphi \in \mathbb{S}$,

$$E_\varphi \left(V(\check{\Phi}_1) 1_{\sigma_A \leq \varpi} \right) = E_\varphi \left[E_\varphi \left(V(\check{\Phi}_1) | \mathcal{F}_{\sigma_A} \right) 1_{\sigma_A \leq \varpi} \right] = E_\varphi \left[E_{Z(\sigma_A)} \left[V(Z(\varpi - \sigma_A)) \right] 1_{\sigma_A \leq \varpi} \right].$$

Therefore, by Assumptions 3.2.1, 3.2.2 and 3.2.3, there exists $c_1 \in (0, \infty)$ such that

$$E_\varphi \left(V(\check{\Phi}_1) 1_{\sigma_A \leq \varpi} \right) \leq \sup_{\tilde{\varphi} \in A \times I\!H} E_{\tilde{\varphi}} \left(\sup_{0 \leq t \leq \varpi} V(\Phi(t)) \right) \leq c_1.$$

Choose $M > \varpi$ such that $\beta_0 V(\varphi) \geq c_1$ for all $\varphi \in (B_M)^c$. Then on $(B_M)^c$, $E_\varphi \left(V(\check{\Phi}_1) \right) \leq (1 - \beta_0) V(\varphi)$. For $\varphi \in B_M$, $(T(z) - \varpi)^+ \leq M$ and from (3.3.19),

$$\begin{aligned} E_\varphi \left(V(\check{\Phi}_1) \right) &\leq E_\varphi \left(\exp \left\{ \alpha_0 \left[(T(z) - \varpi)^+ + \kappa_1 \Theta_1 \nu^0_\varpi \right] \right\} \right) \\ &\leq 8 \exp \left\{ M\alpha_0 + c_1 \alpha_0 + c_2 \alpha_0^2 + c_2 \alpha_0^2 \varpi \right\} \doteq c_2. \end{aligned}$$

The lemma follows on setting $\gamma_0 = c_2$ and $S = B_M$. ■

For a signed measure μ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ and a measurable function $f : \mathbb{S} \to I\!R$, let $\mu(f) = \int_{\mathbb{S}} f(\varphi)\mu(d\varphi)$ if f is $|\mu|$ integrable. If $f : \mathbb{S} \to (0, \infty)$ is a $|\mu|$ integrable map, we define the f-norm of μ as $\|\mu\|_f \doteq \sup_{|g| \leq f} |\mu(g)|$. We set $\|\mu\|_f = \infty$ if f is not $|\mu|$ integrable. As an immediate consequence of Lemma 3.3.8 and Theorems 14.0.1, 16.0.1 in [43], we have the following theorem. Denote by $\{P^n\}_{n \in I\!N}$ the transition kernel of the chain $\{\check{\Phi}_n : n \in I\!N\}$, namely, for $\varphi \in \mathbb{S}$ and $A \in \mathcal{B}(\mathbb{S})$, $P^n(\varphi, A) = P_\varphi(\check{\Phi}_n \in A)$. From Lemma 3.3.8, it follows that $P^n(\varphi, V) < \infty, \forall n \in I\!N$ and $\varphi \in \mathbb{S}$.

Theorem 3.3.2. *The invariant measure π satisfies $\pi(V) < \infty$. Furthermore, the ϖ-skeleton chain $\{\check{\Phi}_n\}$ is V-uniformly ergodic, i.e., there exist $\rho_0 \in (0, 1)$ and $B_0 \in$*

$(0,\infty)$ *such that for all* $\varphi \in \mathbb{S}$,

$$\|P^n(\varphi,\cdot) - \pi\|_V \le B_0 \rho_0^n V(\varphi). \tag{3.3.20}$$

Proof of Theorem 3.2.4:

(i) This is immediate from Corollary 3.3.2 and Lemma 3.3.1(ii), on taking $\beta_1 \le \Theta_2 \alpha_0$, where Θ_2 and α_0 are as in Lemma 3.3.1(ii) and Lemma 3.3.8, respectively.

(ii) (a). For a map ν from $\mathbb{S}$ to the space of signed measures on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, let

$$\|\nu\|^V \doteq \sup_{\varphi\in\mathbb{S}} \frac{\|\nu(\varphi)\|_V}{V(\varphi)}.$$

Recall that P_Φ^t denote the transition kernel of Φ. Denoting the signed measure $P_\Phi^t(\varphi,\cdot) - \pi(\cdot)$ as $\tilde{P}^t(\varphi)$, we have from Corollary 3.3.2, $\|\tilde{P}^{n\varpi}\|^V \le B_0\rho_0^n$. Fix $t \in (0,\infty)$ and let $n_0 \in I\!N$ be such that $t \in [n_0\varpi, (n_0+1)\varpi)$. It is easy to check that

$$\begin{aligned}\|\tilde{P}^t\|^V \le \|\tilde{P}^{n_0\varpi}\|^V \|\tilde{P}^{t-n_0\varpi}\|^V &\le B_0\rho^{n_0} \sup_{0\le r\le\varpi} \|\tilde{P}^r\|^V \\ &\le B_0\rho^{n_0} \sup_{\varphi\in\mathbb{S}} \sup_{0\le r\le\varpi} \frac{E_\varphi[V(\Phi(r))] + \pi(V)}{V(\varphi)}.\end{aligned}$$

From Lemma 3.3.2 and Lemma 3.3.3, we have for some $\tilde{B}_0 \equiv \tilde{B}_0(\varpi) \in (0,\infty)$,

$$\sup_{0\le r\le\varpi} E_\varphi[V(\Phi(r))] \le \tilde{B}_0 V(\varphi). \tag{3.3.21}$$

Let $\tilde{\rho} \doteq \rho_0^{1/\varpi}$ and $\tilde{B} \doteq B_0(\tilde{B}_0 + \pi(V))/\rho_0$. Then $\|\tilde{P}^t\|^V \le \tilde{B}\tilde{\rho}^t$. This proves (a).

(b) & (c). By (a), for all $\varphi \in \mathbb{S}$, $f_t^c(\varphi) = E_\varphi(S_t - t\pi(f))$ is well defined. We observe that for $0 \le t < T < \infty$,

$$|f_t^c(\varphi) - f_T^c(\varphi)| \le \int_t^T |P_\Phi^s(\varphi, f) - \pi(f)|\, ds \le B_1 V(\varphi)\left(\tilde{\rho}^t - \tilde{\rho}^T\right). \qquad (3.3.22)$$

where $B_1 = -\tilde{B}/\log\tilde{\rho}$. Noting that $|f_t^c(\varphi) - f_T^c(\varphi)| \to 0$ as $t, T \to \infty$, $\lim_{t\to\infty} f_t^c(\varphi)$ exists. In particular, denoting the limit by $\hat{f}(\varphi)$ and letting $t = 0$ and $T \to \infty$ in (3.3.22), we have for all $\varphi \in \mathbb{S}$,

$$|\hat{f}(\varphi)| \le V(\varphi) B_1. \qquad (3.3.23)$$

Then fixing t and letting $T \to \infty$, we have from (3.3.22), that

$$\left| f_t^c(\varphi) - \hat{f}(\varphi) \right| \le V(\varphi) B_1 \tilde{\rho}^t \le B_1 e^{\Theta_1 \alpha_0 |z|} \tilde{\rho}^t.$$

This proves (b)&(c).

(d). From (3.3.23), for $t > 0$, $E_\varphi|\hat{f}(\Phi(t))| \le B_1 E_\varphi(V(\Phi(t)))$. Also

$$\int_0^t E_\varphi|\pi(f) - f(\Phi(s))| ds < \int_0^t E_\varphi|f(\Phi(s))| ds + t\pi(f) \le \int_0^t E_\varphi[V(\Phi(s))] ds + t\pi(f).$$

Similar to (3.3.21), we have for all $t \ge 0$, $\sup_{0 \le s \le t} E_\varphi[V(\Phi(s))] < \infty$. Consequently,

$$E_\varphi|\hat{f}(\Phi(t))| + \int_0^t E_\varphi|\pi(f) - f(\Phi(s))| ds < \infty.$$

Also note that

$$\begin{aligned} E_\varphi(\hat{f}(\Phi(t))) &= \int_0^\infty E_\varphi(P_\Phi^s(\Phi(t), f) - \pi(f)) ds = \int_0^\infty [P_\Phi^{s+t}(\varphi, f) - \pi(f)] ds \\ &= \int_t^\infty [P_\Phi^s(\varphi, f) - \pi(f)] ds = \hat{f}(\varphi) - \int_0^t [P_\Phi^s(\varphi, f) - \pi(f)] ds \\ &= \hat{f}(\varphi) + \int_0^t E_\varphi[\pi(f) - f(\Phi(s))] ds. \end{aligned}$$

This proves (d).

(iii) The proof is an immediate consequence of [27, Theorem 4.4].

■

3.4 Markov modulated SRBM

This section is devoted to proofs of Theorems 3.2.6 and 3.2.7. We use the notation introduced in Section 3.2.2. In particular, throughout this section, $G = I\!R_+^K$, $N = K$, and for $i = 1, 2, \ldots, K$, $G_i = \{z \in I\!R^K : \langle z, e_i \rangle \geq 0\}$. Also, $R = (d_1|\cdots|d_K)$ and σ is a $K \times K$ positive definite matrix as in Section 2.2.

The proof of Theorem 3.2.6 is similar to the proof of Theorem 2.6 in [24]. However, we give details for completeness. First, we introduce the Lyapunov function F, which was constructed in [24]. Recall the DW-stability condition introduced in Definition 3.2.5.

Theorem 3.4.1. *Suppose that b_2^* satisfies the DW-stability condition. Then there exists a continuous map $F : \mathbb{R}^K \to I\!R$ such that the following hold.*

(i) $F \in C^2(\mathbb{R}^K \setminus \{0\})$.

(ii) *Given $\epsilon \in (0, \infty)$, there exists an $M \in (0, \infty)$ such that, for all $\tilde{z} \in I\!R^K$ and $|\tilde{z}| \geq M$, $|\nabla^2 F(\tilde{z})| \leq \epsilon$.*

(iii) *There exists $c \in (0, \infty)$ such that*

  (a) *for all $\tilde{z} \in G \setminus \{0\}$, $\langle \nabla F(\tilde{z}), b_2^* \rangle \leq -c$,*

  (b) *for all $\tilde{z} \in \partial G \setminus \{0\}$ and $d \in \mathbf{d}(\tilde{z})$, $\langle \nabla F(\tilde{z}), d \rangle \leq -c$.*

(iv) *F is radially homogeneous, i.e., $F(\zeta \tilde{z}) = \zeta F(\tilde{z})$ for all $\zeta \geq 0$ and $\tilde{z} \in I\!R^K$.*

(v) ∇F *is uniformly bounded on* $G \setminus \{0\}$. *We denote*

$$\Lambda \doteq \sup_{\tilde{z} \in G \setminus \{0\}} |\nabla F(\tilde{z})| < \infty.$$

(vi) *There exist* $b_1, b_2 \in (0, \infty)$ *such that, for all* $\tilde{z} \in G$, $b_1|\tilde{z}| \leq F(\tilde{z}) \leq b_2|\tilde{z}|$.

With an abuse of notation, we set $\nabla F(0) = 0$ and $\nabla^2 F(0) = 0$. Fix $\varphi = (z, y) \in \mathbb{S}$ and recall the martingale $\{M_t : t \geq 0\}$ introduced in (3.3.7). Denote

$$\Upsilon(t) \doteq Z(t) - g(Y(t)) + g(Y(0)), \ t \geq 0. \tag{3.4.1}$$

Then from (3.2.9) and (3.3.7), for all $t \geq 0$, P_φ-a.s., $\Upsilon(t) = z + b^* t - M_t + \sigma W(t) + RU(t)$. By Ito's formula, we have that

$$\begin{aligned} F(\Upsilon(t)) = F(z) + \int_0^t \left(\frac{1}{2} tr \left[\nabla^2 F(\Upsilon(s)) \sigma \sigma' \right] + \langle \nabla F(\Upsilon(s)), b^* \rangle \right) ds \\ + \int_0^t \langle \nabla F(\Upsilon(s)), \sigma dW(s) \rangle - \int_{0+}^t \langle \nabla F(\Upsilon(s-)), dM_s \rangle \\ + \sum_{i=1}^K \int_0^t \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) + \mathcal{R}_t, \end{aligned} \tag{3.4.2}$$

where

$$\mathcal{R}_t = \sum_{0 < s \leq t} [F(\Upsilon(s)) - F(\Upsilon(s-)) + \langle \nabla F(\Upsilon(s-)), g(Y(s)) - g(Y(s-)) \rangle]. \tag{3.4.3}$$

Proof of Theorem 3.2.6: Given $\epsilon > 0$, let $r > 2|g|_\infty$ be large enough such that $|\nabla^2 F(\tilde{z})| \leq \epsilon$ whenever $\tilde{z} \in I\!\!R^K$ and $|\tilde{z}| \geq r - 2|g|_\infty$. An appropriate choice of ϵ will be made later in the proof. Define $\tilde{\tau}_r = \inf\{t \geq 0 : |Z(t)| \leq r\}$. Fix $\varphi \equiv (z, y) \in \mathbb{S}$. We first assume $|z| > r$. Using the Lagrange remainder form of Taylor's expansion and Theorem 3.4.1(ii), we have

$$\begin{aligned}
&\mathcal{R}_{t\wedge\tilde{\tau}_r} \\
&= \sum_{0<s\le t\wedge\tilde{\tau}_r} \big[(g(Y(s)) - g(Y(s-)))'\nabla^2 F(\varsigma_1\Upsilon(s) + (1-\varsigma_1)\Upsilon(s-))[g(Y(s)) - g(Y(s-))]\big] \\
&\le \epsilon \sum_{0<s\le t\wedge\tilde{\tau}_r} |g(Y(s)) - g(Y(s-))|^2,
\end{aligned}$$

where $\varsigma_1 \equiv \varsigma_1(s,\omega) \in [0,1]$. Taking expectation, we have for some $c_1 \in (0,\infty)$,

$$E_\varphi(\mathcal{R}_{t\wedge\tilde{\tau}_r}) \le c_1\epsilon E_\varphi(t\wedge\tilde{\tau}_r),\ \forall t \ge 0. \tag{3.4.4}$$

Next by Theorem 3.4.1(ii) and (iii)(a) and for $0 \le s \le t\wedge\tilde{\tau}_r$, there exists $\varsigma_2 \equiv \varsigma_2(s,\omega) \in (0,1)$ such that

$$\begin{aligned}
\langle \nabla F(\Upsilon(s)), b_2^*\rangle &= \langle \nabla F(Z(s)), b_2^*\rangle + \langle \nabla F(\Upsilon(s)) - \nabla F(Z(s)), b_2^*\rangle \\
&\le -c + \langle \nabla^2 F(\varsigma_2\Upsilon(s) + (1-\varsigma_2)Z_s)(g(Y(s)) - g(Y(0))), b_2^*\rangle \\
&\le -c + 2\epsilon|g|_\infty|b_2^*|.
\end{aligned} \tag{3.4.5}$$

Similarly, by Theorem 3.4.1(ii) and (iii)(b), for $0 \le s \le t\wedge\tilde{\tau}_r$ and some $\varsigma_3 \equiv \varsigma_3(s,\omega) \in (0,1)$, whenever $Z(s) \in F_i$,

$$\begin{aligned}
\langle \nabla F(\Upsilon(s)), d_i\rangle &= \langle \nabla F(Z(s)), d_i\rangle + \langle \nabla F(\Upsilon(s)) - \nabla F(Z(s)), d_i\rangle \\
&\le \langle \nabla^2 F(\varsigma_3\Upsilon(s) + (1-\varsigma_3)Z_s)(g(Y(s)) - g(Y(0))), d_i\rangle \\
&\le 2\epsilon|g|_\infty.
\end{aligned} \tag{3.4.6}$$

From Theorem 3.4.1(ii) and (3.4.5), there exists $c_2 \in (0,\infty)$, such that for all $t > 0$

$$\frac{1}{2}tr\left[\nabla^2 F(\Upsilon(t\wedge\tilde{\tau}_r))\sigma\sigma'\right] + \langle \nabla F(\Upsilon(t\wedge\tilde{\tau}_r)), b_2^*\rangle \le c_2\epsilon - c. \tag{3.4.7}$$

By (3.4.6), we have

$$\sum_{i=1}^{K}\int_0^{t\wedge\tilde{\tau}_r}\langle\nabla F(\Upsilon(s)),d_i\rangle dU_i(s)\le 2\epsilon|g|_\infty\sum_{i=1}^{K}U_i(t\wedge\tau_r). \tag{3.4.8}$$

By Theorem 3.2 in [18], there exists $h\in\mathcal{C}_b^2(G)$ such that for $i\in I\!K$ and $z\in F_i$, $\langle\nabla h(z),d_i\rangle\ge 1$. Applying Ito's formula,

$$\begin{aligned}E_\varphi(h(Z(t\wedge\tau_r)))=h(z)+E_\varphi&\left(\int_0^{t\wedge\tilde{\tau}_r}\langle\nabla h(Z(s)),b(Y(s))\rangle+\frac{1}{2}tr[\nabla^2h(Z(s))\sigma\sigma']ds\right)\\&+\sum_{i=1}^{K}E_\varphi\left(\int_0^{t\wedge\tilde{\tau}_r}\langle\nabla h(Z(s)),d_i\rangle dU_i(s)\right).\end{aligned}$$

Thus we have for $c_3\in(0,\infty)$,

$$\begin{aligned}&\sum_{i=1}^{K}E_\varphi(U_i(t\wedge\tilde{\tau}_r))\le\sum_{i=1}^{K}E_\varphi\left(\int_0^{t\wedge\tilde{\tau}_r}\langle\nabla h(Z(s)),d_i\rangle dU_i(s)\right)\\&\le E_\varphi(|h(Z(t))|)+|h(z)|+E_\varphi\left(\int_0^{t\wedge\tilde{\tau}_r}|\langle\nabla h(Z(s)),b(Y(s))\rangle|+\frac{1}{2}tr[\nabla^2h(Z(s))\sigma\sigma']ds\right)\\&\le c_3(1+E_\varphi(t\wedge\tilde{\tau}_r)).\end{aligned} \tag{3.4.9}$$

Using (3.4.8) and (3.4.9), we now have

$$E_\varphi\left(\sum_{i=1}^{K}\int_0^{t\wedge\tilde{\tau}_r}\langle\nabla F(\Upsilon(s)),d_i\rangle dU_i(s)\right)\le 2\epsilon|g|_\infty c_3(1+E_\varphi(t\wedge\tilde{\tau}_r)). \tag{3.4.10}$$

We note that the constants c_1,c_2, and c_3 only depend on bounds of σ,g,h, and b_2^*. In particular, they are independent of ϵ and t. Combining (3.4.4), (3.4.7), (3.4.10) and applying (3.4.2), we have

$$E_\varphi(F(\Upsilon(t\wedge\tilde{\tau}_r)))-F(z)\le 2\epsilon|g|_\infty c_3+[\epsilon(2|g|_\infty c_3+c_1+c_2)-c]E_\varphi(t\wedge\tilde{\tau}_r). \tag{3.4.11}$$

Again applying the Lagrange remainder form of Taylor's expansion and Theorem

3.4.1(ii), (v) and (vi), there exists $\varsigma_4 \in (0,1)$ such that for $0 \le s \le t \wedge \tilde{\tau}_r$,

$$\begin{aligned} F(\Upsilon(s)) &= F(Z(s)) + (g(Y(s)) - g(Y(0)))'\nabla F(Z(s)) \\ &\quad + \frac{1}{2}(g(Y(s)) - g(Y(0)))'\nabla^2 F(\varsigma_4\Upsilon(s) + (1-\varsigma_4)Z(s))(g(Y(s)) - g(Y(0))) \\ &\ge F(Z(s)) - 2|g|_\infty\Lambda - 2|g|_\infty^2\epsilon \\ &\ge -2|g|_\infty\Lambda - 2|g|_\infty^2\epsilon. \end{aligned}$$

Choosing

$$\epsilon = \frac{c}{2(2|g|_\infty c_3 + c_1 + c_2)},$$

we have

$$E_\varphi(t \wedge \tilde{\tau}_r) \le \frac{2}{c}(F(z) + 2\epsilon|g|_\infty c_3 + 2|g|_\infty\Lambda + 2|g|_\infty^2\epsilon) < \infty.$$

Letting $t \to \infty$, we have $E_\varphi(\tilde{\tau}_r) < \infty$. If $|z| \le r$, $E_\varphi(\tilde{\tau}_r) < \infty$ holds automatically. Therefore, $E_\varphi(\tilde{\tau}_r) < \infty$ for all $\varphi \in \mathbb{S}$. The rest of the argument is as in the proof of Theorem 2.6 in [24]. Details are left to the reader. ■

We next establish geometric ergodicity for Φ. We begin with some preliminary estimates. Arguments similar to those used in Lemmas 3.3.3 and 3.3.4 yield the following result. Proof is omitted.

Lemma 3.4.1. *Let $\varphi \in \mathbb{S}$ and $\Delta > 0$ be fixed. For $n \in I\!N$, let $\tilde{\nu}_n$ be defined as follows:*

$$\tilde{\nu}_n \doteq \sup_{(n-1)\Delta \le s \le n\Delta} \left| \int_{(n-1)\Delta}^{s} \langle \nabla F(\Upsilon(s)), \sigma dW(s)\rangle + \int_{(n-1)\Delta}^{s} \langle \nabla F(\Upsilon(s)), dM_s\rangle \right|. \tag{3.4.12}$$

Then there exists $\Theta_4 \in (0,\infty)$ such that, for any $\varphi \in \mathbb{S}, \alpha \in (0,\infty)$ and $m, n \in I\!N; m \le n$,

$$E_\varphi\left(\exp\left\{\alpha \sum_{i=m}^{n} \tilde{\nu}_i\right\}\right) \le \left(4\exp\left\{\Theta_4\alpha^2(1+\Delta)\right\}\right)^{n-m+1}.$$

For $r \in (0,\infty)$, define

$$C(r) = \{\tilde{z} \in G : F(\tilde{z}) \le r\}, \tau_r \doteq \inf\{t \ge 0 : Z(t) \in C(r)\}. \tag{3.4.13}$$

Lemma 3.4.2. *There exist $r_0, \beta, \gamma_1, \gamma_2 \in (0,\infty)$ such that, for all $\varphi = (z,y) \in G$,*

$$E_\varphi\left(\exp\{\beta\tau\}\right) \le \gamma_1 \exp\{\gamma_2 |z|\},$$

where $\tau = \tau_{r_0}$ and τ_{r_0} is defined as in (3.4.13) *with r replaced by r_0.*

Proof: By Theorem 3.4.1(ii), given $\epsilon > 0$, there exists a $r > 2|g|_\infty$ such that $|\nabla^2 F(\tilde{z})| \le \epsilon$ whenever $\tilde{z} \in \mathbb{R}^K$ and $|\tilde{z}| \ge r - 2|g|_\infty$. By Theorem 3.4.1(vi), we can choose r_0 such that $\{\tilde{z} \in G : F(\tilde{z}) \le r_0\} \supset \{\tilde{z} \in G : |\tilde{z}| \le r\}$. Let $\varsigma \in (0,1)$ and $\varphi = (z,y) \in \mathbb{S}$ with $|z| > r$. Similar to the arguments for (3.4.9) and using Lemma 4.2 in [4], there exist $c_1, c_2, c_3 \in (0,\infty)$ such that for all $t \ge 0$,

$$\begin{aligned}
E_\varphi\left(\exp\left\{\varsigma \sum_{i=1}^K U_i(t)\right\}\right) &\le E_\varphi\left(\exp\left\{\varsigma \sum_{i=1}^K \int_0^t \langle \nabla h(Z(s)), d_i\rangle dU_i(s)\right\}\right) \\
&\le E_\varphi\Bigg(\exp\Bigg\{\varsigma|h(Z(t))| + \varsigma|h(z)| + \varsigma\left|\int_0^t \langle \nabla h(Z(s)), \sigma dW(s)\rangle\right| \\
&\quad + \varsigma \int_0^t \left|\langle \nabla h(Z(s)), b(Y(s))\rangle + \frac{1}{2} tr[\nabla^2 h(Z(s))\sigma\sigma']\right| ds\Bigg\}\Bigg) \qquad (3.4.14) \\
&\le \exp\{c_1\varsigma + c_2\varsigma t + c_3\varsigma^2 t\}.
\end{aligned}$$

Fix $\Delta \in (0,\infty)$ and $n \in \mathbb{N}$. Using (3.4.2), we have

$$\begin{aligned}
F(\Upsilon(n\Delta)) = F(\Upsilon((n-1)\Delta)) &+ \int_{(n-1)\Delta}^{n\Delta} \left(\frac{1}{2} tr\left[\nabla^2 F(\Upsilon(s))\sigma\sigma'\right] + \langle \nabla F(\Upsilon(s)), b^*\rangle\right) ds \\
&+ \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\Upsilon(s)), \sigma dW(s)\rangle + \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\Upsilon(s-)), dM_s\rangle \\
&+ \sum_{i=1}^K \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\Upsilon(s)), d_i\rangle dU_i(s) + \mathcal{R}_{n\Delta} - \mathcal{R}_{(n-1)\Delta}.
\end{aligned}$$

For $m \in I\!N_0$ and $m \geq n$, define $A_m \doteq \{\omega \in \Omega : \inf_{0 \leq s \leq m\Delta} F(Z(s)) > r_0\}$. From (3.4.7), we have on A_m for some $c_4 \in (0, \infty)$,

$$\int_{(n-1)\Delta}^{n\Delta} \left(\frac{1}{2} tr \left[\nabla^2 F(\Upsilon(s)) \sigma \sigma' \right] + \langle \nabla F(\Upsilon(s)), b^* \rangle \right) ds \leq (c_4 \epsilon - c)\Delta,$$

where c_4 is independent of the choice of ϵ, Δ and n. From (3.4.14) and (3.4.6), on A_m, for $t \leq m\Delta$,

$$E_\varphi \left(\exp \left\{ \varsigma \sum_{i=1}^{K} \int_0^t \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) \right\} \right) \leq \exp\{2\epsilon |g|_\infty (c_1 \varsigma + c_2 \varsigma t + c_3 \varsigma^2 t)\}.$$

Also note that on A_m and $t \leq m\Delta$, for some $c_5 \in (0, \infty)$ (independent of $\varsigma, t, \epsilon, \Delta, t, m$),

$$E_\varphi(\exp\{\varsigma \mathcal{R}_t\}) \leq E \left(\exp \left\{ \varsigma \epsilon \sum_{0 < s \leq t} |g(Y(s)) - g(Y(s-))|^2 \right\} \right) \leq \exp\{c_5 \epsilon \varsigma t\}.$$

Therefore, on the set A_m and for $n \leq m$,

$$F(\Upsilon(n\Delta)) \leq F(\Upsilon((n-1)\Delta)) + (c_4\epsilon - c)\Delta + \sum_{i=1}^{K} \int_{(n-1)\Delta}^{n\Delta} \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s)$$
$$+ \mathcal{R}_{n\Delta} - \mathcal{R}_{(n-1)\Delta} + \tilde{\nu}_n,$$

where $\tilde{\nu}_n$ is as in (3.4.12). Therefore, on set A_m and for $n \leq m$, we have

$$F(\Upsilon(n\Delta)) \leq F(z) + (c_4\epsilon - c)n\Delta + \sum_{i=1}^{K} \int_0^{n\Delta} \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) + \mathcal{R}_{n\Delta} + \sum_{i=1}^{n} \tilde{\nu}_i.$$

Noting that, on A_m, $F(\Upsilon(n\Delta)) > r_0$, we have

$$P_\varphi(A_m)$$
$$\leq P_\varphi \left(\sum_{i=1}^{K} \int_0^{n\Delta} \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) + \mathcal{R}_{n\Delta} + \sum_{i=1}^{n} \tilde{\nu}_i > r_0 - (c_4\epsilon - c)n\Delta - F(z) \right)$$

$$\leq E_\varphi \left(\exp\left\{ \varsigma \left(\sum_{i=1}^{K} \int_0^{n\Delta} \langle \nabla F(\Upsilon(s)), d_i \rangle dU_i(s) + \mathcal{R}_{n\Delta} + \sum_{i=1}^{n} \tilde{\nu}_i \right) \right\} \right)$$
$$\times \exp\{-\varsigma(r_0 - (c_4\epsilon - c)n\Delta - F(z))\}$$
$$\leq \exp\left\{ n\Delta \left[2|g|_\infty c_1 \frac{\epsilon\varsigma}{n\Delta} + (2|g|_\infty c_2 + c_5)\epsilon\varsigma + 8|g|_\infty c_3 \epsilon \varsigma^2 + 2\Theta_4 \frac{\varsigma^2}{\Delta} + 2\Theta_4 \varsigma^2 \right.\right.$$
$$\left.\left. + \frac{\log 2}{\Delta} + c_4\epsilon\varsigma - r_0 \frac{\varsigma}{n\Delta} - \varsigma c \right] + \varsigma F(z) \right\}.$$

Let ϵ and ς be small enough and Δ large enough, so that

$$2|g|_\infty c_1 \frac{\epsilon\varsigma}{n\Delta} + (2|g|_\infty c_2 + c_5)\epsilon\varsigma + 8|g|_\infty c_3\epsilon\varsigma^2 + \Theta_4 \frac{\varsigma}{\Delta} + 2\Theta_4 \frac{\varsigma^2}{\Delta} + 2\Theta_4\varsigma^2 + \frac{\log 8}{\Delta} + c_4\epsilon\varsigma$$
$$- r_0 \frac{\varsigma}{n\Delta} - \varsigma c \equiv -\eta < 0.$$

For $t \in (0, \infty)$, let $n_0 \in I\!N_0$ such that $t \in [n_0\Delta, (n_0+1)\Delta)$. Then

$$P_\varphi(\tau > t) \leq P_\varphi(A_{n_0}) \leq \exp\{-\eta t + \Delta\eta + \varsigma F(z)\}.$$

The result follows. ■

As an immediate consequence of the above lemma, we have the following. For $\theta_0 \in (0, \infty)$ and a compact set $S \subset G$, define a stopping time $\tau_S(\theta_0) = \inf\{t \geq \theta_0 : Z(t) \in S\}$.

Lemma 3.4.3. *Fix $\theta_0 \in (0, \infty)$ and let β, γ_2, r_0 be as in Lemma 3.4.2 and $C \equiv C(r_0)$ be as in* (3.4.13) *with r replaced by r_0. Then there exists $\gamma_3 \in (0, \infty)$ such that for $\varphi = (z, y) \in \mathbb{S}$,*

$$E_\varphi(\exp\{\beta\tau_C(\theta_0)\}) \leq \gamma_3 \exp\{\gamma_2 |z|\}.$$

Proof: An application of the strong Markov property yields

$$E_\varphi(\exp\{\beta\tau_C(\theta_0)\}) = \exp\{\beta\theta_0\} E_\varphi\left(E_\varphi(\exp\{\beta(\tau_C(\theta_0) - \theta_0)\} | \mathcal{F}_{\theta_0})\right)$$
$$= \exp\{\beta\theta_0\} E_\varphi\left(E_{\Phi(\theta_0)} \exp\{\beta\tau\}\right).$$

By Lemma 3.4.2,

$$E_{\varphi}\left(E_{\Phi(\theta_0)}\exp\{\beta\tau\}\right) \leq \gamma_1 E_{\varphi}\left(\exp\{\gamma_2|Z(\theta_0)|\}\right). \qquad (3.4.15)$$

Using the oscillation estimate from [48](also see [5]), we have the following result: There exists $c_1 \in (0,\infty)$ such that for all $\varphi = (z,y) \in \mathbb{S}$ and $0 \leq t_1 < t_2 < \infty$, P_{φ}-a.s.,

$$\sup_{t_1\leq s\leq t\leq t_2} |Z(t) - Z(s)| \leq c_1 \left(\sup_{t_1\leq s\leq t\leq t_2} |W(t) - W(s)| + (t_2 - t_1)\right). \qquad (3.4.16)$$

Combining the estimates in (3.4.15) and (3.4.16), we have

$$E_{\varphi}\left(E_{\Phi(\theta_0)}\left(\exp\{\beta\tau\}\right)\right) \leq \gamma_1 \exp\{\gamma_2(z + c_1\theta_0)\}E_{\varphi}\left(\exp\left\{c_1\gamma_2 \sup_{0\leq s\leq\theta_0} |W(s)|\right\}\right).$$

The result follows. ■

A key step in the proof of geometric ergodicity is the following result from [20]. For θ_0, β, and C as in Lemma 3.4.3, let

$$V_0(\varphi) \doteq \frac{E_{\varphi}(\exp\{\beta\tau_C(\theta_0)\}) - 1}{\beta} + 1.$$

Define, for $\theta > 0$,

$$V_{\theta}(\varphi) = \mathfrak{R}_{\theta}V_0(\varphi) \doteq \int_0^{\infty} E_{\varphi}[V_0(\Phi(t))]\theta\exp\{-\theta t\}dt.$$

By Lemma 4.3 (a), Theorem 6.2 (b), and Theorem 5.1 (a) in [20], we have the following result.

Theorem 3.4.2. *For all $\theta > 0$, $\mathcal{A}V_{\theta} = \theta(V_{\theta} - V_0)$, where $\mathcal{A}$ is the extended generator of Φ introduced below Theorem 3.2.3. Furthermore, there exist $\kappa_0, h_0 \in (0,\infty)$ such* *that for all* $\varphi \in \mathbb{S}$,

$$\mathcal{A}V_\theta(\varphi) \le -\kappa_0 V_\theta(\varphi) + h_0 1_{C\times I\!H}(\varphi).$$

The following lemma is proved exactly as Lemma 4.8 of [10]. Proof is omitted.

Lemma 3.4.4. *There exist* $a_1, a_2, A_1, A_2 \in (0,\infty)$ *such that for all* $\varphi = (z,y) \in \mathbb{S}$,

$$a_1 e^{a_2|z|} \le V_0(\varphi) \le A_1 e^{A_2|z|}. \tag{3.4.17}$$

Furthermore, there exists a constant $\tilde{\theta} \in (0,\infty)$ *such that for every* $\theta \in (\tilde{\theta},\infty)$ *there are* $\tilde{a}_1, \tilde{a}_2, \tilde{A}_1, \tilde{A}_2 \in (0,\infty)$ *such that for* $\varphi = (z,y) \in \mathbb{S}$,

$$\tilde{a}_1 e^{\tilde{a}_2|z|} \le V_\theta(\varphi) \le \tilde{A}_1 e^{\tilde{A}_2|z|}. \tag{3.4.18}$$

We will fix $\theta \in (\tilde{\theta},\infty)$ and denote $V \equiv V_\theta$. Then the following corollary is an immediate consequence of Theorem 3.4.2.

Corollary 3.4.1. V *is in* $\mathcal{D}(\mathcal{A})$ *and* $\mathcal{A}V = \theta(V - V_0)$. *Furthermore, there exist* $\kappa_0, h_0 \in (0,\infty)$ *such that for all* $\varphi \in \mathbb{S}$,

$$\mathcal{A}V(\varphi) \le -\kappa_0 V(\varphi) + h_0 1_{C\times I\!H}(\varphi).$$

Corollary 3.4.2. *Let* π *be the unique invariant measure of* Φ. *Then* $\pi(V) < \infty$.

Proof: By Corollary 3.4.1 and Theorem 5.1(d) of [20], for any $T > 0$, there exist $\varsigma_0 \in (0,1)$ and $h_1 > 0$ such that for all $\varphi \in \mathbb{S}$, $E_\varphi(V(\Phi(T))) \le \varsigma_0 V(\varphi) + h_1 1_{C\times I\!H}(\varphi)$. Integrating both sides with respect to π, we have $\pi(V) \le h_1 \pi(C \times I\!H)/(1-\varsigma_0) < \infty$. ■

As a consequence of the above corollaries and Theorem 5.2(c) of [20], we have the following geometric ergodicity result.

Theorem 3.4.3. *The Markov process $\Phi \equiv (Z, Y)$ is V-uniformly ergodic, i.e., there exist constants $B_0 \in (0, \infty), \rho_0 \in (0, 1)$ such that for all $t \in (0, \infty)$ and $\varphi \in \mathbb{S}$,*

$$\|P^t(\varphi, \cdot) - \pi\|_V \leq B_0 \rho_0^t V(\varphi).$$

Proof of Theorem 3.2.7: Part (i) of the theorem is immediate from Corollary 3.4.2 and Lemma 3.4.4. Part (ii)(a) is a consequence of Theorem 3.4.3 and Lemma 3.4.4. The rest of the proof is same as that for Theorem 3.2.4. ■

3.5 Appendix

Proof of Lemma 3.3.2: For $t \geq 0$, let $\tilde{Z}(t) = Z(t + u)$. Then P_φ-a.s.,

$$\tilde{Z}(t) = \Gamma\left(Z(u) + \int_0^{\cdot} b(\Phi(s+u))ds + \int_0^{\cdot} \sigma(\Phi(s+u))dW_u(s)\right)(t),$$

where $W_u(s) = W(s + u) - W(u)$. Let

$$\bar{Z}(t) = \Gamma\left(Z(u) + \int_0^{\cdot} b^*(\Phi(s+u))ds\right)(t),$$

where b^* is as defined in Assumption 3.2.5. By Lipschitz property of Γ (Assumption 3.2.1),

$$\begin{aligned}\sup_{0\leq t\leq\Delta}\left|\tilde{Z}(t) - \bar{Z}(t)\right| &\leq \kappa_1 \sup_{0\leq t\leq\Delta}\left|\int_0^t b^c(Y(s+u))ds + \int_0^t \sigma(\Phi(s+u))dW_u(s)\right| \\ &= \kappa_1 \nu_\Delta^u.\end{aligned}$$

Recalling the assumption on b_2^* (Assumption 3.2.5), we have applying Lemma 3.3.1, that on the set $\{\omega : Z(t,\omega) \in G \setminus A \text{ for all } t \in (u, u+\Delta)\}$

$$T(Z(u+\Delta)) = T(\tilde{Z}(\Delta)) \le T(\bar{Z}(\Delta)) + \kappa_1\Theta_1\nu_\Delta^u \le (T(Z(u)) - \Delta)^+ + \kappa_1\Theta_1\nu_\Delta^u.$$

■

Proof of Lemma 3.3.5 (sketch): By arguing as in the proof of Lemma 4.4 of [4], we can show, for $\Delta > 0$ and ν_n defined as in Lemma 3.3.4, there exists $M_0 \in (0,\infty)$ such that

$$\begin{aligned} P_\varphi\left(T(Z(n\Delta) \ge M_0)\right) &\le \sum_{l=1}^n P_\varphi\left(2\kappa_1\Theta_1 \sum_{j=l}^n \nu_j \ge M_0 + (n-l-1)\Delta - T(z)\right) \\ &\le \frac{\exp\{\alpha(T(z)+\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^n \frac{E_\varphi\left(\exp\{2\alpha\kappa_1\Theta_1\sum_{j=l}^n \nu_j\}\right)}{\exp\{\alpha(n-l)\Delta\}}, \end{aligned}$$

where $\alpha \in (0,\infty)$ is arbitrary. From Lemma 3.3.4 we now have

$$\begin{aligned} &P_\varphi\left(T(Z(n\Delta)) \ge M_0\right) \\ &\le \frac{\exp\{\alpha(T(z)+\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^n \frac{(8\exp\{2\Theta_3\kappa_1\Theta_1\alpha(1+2\kappa_1\Theta_1\alpha+2\kappa_1\Theta_1\alpha\Delta)\})^{n-l+1}}{\exp\{\alpha(n-l)\Delta\}} \end{aligned}$$

$$\le \frac{\exp\{\alpha(T(z)+2\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^n \exp\left\{\log 8 + 2\Theta_3\kappa_1\Theta_1\alpha(1+2\kappa_1\Theta_1\alpha+2\kappa_1\Theta_1\alpha\Delta) - \alpha\Delta\right\}^{n-l+1}.$$

As in the proof of Theorem 3.3.1, we can choose α and Δ so that

$$\log 8 + 2\Theta_3\kappa_1\Theta_1\alpha(1+2\kappa_1\Theta_1\alpha+2\kappa_1\Theta_1\alpha\Delta) - \alpha\Delta = -\bar{\theta} < 0.$$

An application of Lemma 3.3.1 yields that for every $\kappa \in (0, \alpha\Theta_2)$ and $M > 0$,

$$\sup_{|\varphi|\le M} \sup_{n\in\mathbb{N}} E_\varphi(e^{\kappa|\Phi(n\Delta)|}) < \infty.$$

The result follows from the above estimate, using the Lipschitz property of Γ, in a straightforward manner (see Lemma 4.4 of [4]). ■

Proof of Lemma 3.4.1 (sketch): By the strong Markov property of Φ, it suffices to show

$$E_\varphi(\exp\{\alpha\tilde{\nu}_1\}) \leq 8\exp\{\Theta_4\alpha^2(1+\Delta)\}.$$

By Holder's inequality,

$$\begin{aligned}
&\left[E_\varphi\left(\exp\left\{\alpha \sup_{0\leq t\leq\Delta}\left|\int_0^t \langle\nabla F(\Upsilon(s)),\sigma dW(s)\rangle + \int_0^t \langle\nabla F(\Upsilon(s)), dM_s\rangle\right|\right\}\right)\right]^2 \\
&\leq E_\varphi\left(\exp\left\{2\alpha \sup_{0\leq t\leq\Delta}\left|\int_0^t \langle\nabla F(\Upsilon(s)),\sigma dW(s)\rangle\right|\right\}\right) \\
&\quad\times E_\varphi\left(\exp\left\{2\alpha \sup_{0\leq t\leq\Delta}\left|\int_0^t \langle\nabla F(\Upsilon(s)), dM_s\rangle\right|\right\}\right).
\end{aligned}$$

We first note that using Lagrange remainder form of Taylor expansion and for $t \geq 0$,

$$\nabla F(\Upsilon(t)) = \nabla F(Z(t)) + \nabla^2 F(\varsigma\Upsilon(t) + (1-\varsigma)Z(t))(g(Y(t)) - g(Y(0))),$$

where $\varsigma \equiv \varsigma(s,\omega) \in (0,1)$. From Theorem 3.4.1 (ii) and (v), there exists some $c_1 \in (0,\infty)$ for all $t \geq 0$, $|\nabla F(\Upsilon(t))| \leq c_1$. We have by standard estimates (see e.g. Lemma 4.2 of [4]) for some $c_2 \in (0,\infty)$,

$$E_\varphi\left(\exp\left\{2\alpha \sup_{0\leq t\leq\Delta}\left|\int_0^t \langle\nabla F(\Upsilon(s)), dW(s)\rangle\right|\right\}\right) \leq 2\exp\left\{c_2\alpha^2\Delta\right\}. \tag{3.5.1}$$

Applying arguments similar to those between (3.3.8) and (3.3.9) in the proof of Lemma 3.3.3, there exist $c_3 \in (0,\infty)$ such that

$$E_\varphi\left(\exp\left\{2\alpha \sup_{0\leq s\leq t}\left|\int_0^t \langle\nabla F(\Upsilon(s)), dM_s\rangle\right|\right\}\right) \leq 8\exp\left\{c_3\alpha^2(1+\Delta)\right\}.$$

Result follows on combining the above estimates. ■

Chapter 4

Convergence of invariant measures for Markov modulated open queueing networks in heavy traffic

4.1 Introduction and main result

Frequently queueing networks of interest are too complex to be analyzed directly, so one would like to use the steady state behavior of the limit diffusion model to approximate that of the underlying queueing system. In this chapter, we justify such an approximation procedure by studying convergence of the invariant measures for the Markov modulated queueing networks considered in Chapter 2 in heavy traffic. For simplicity, we consider an open queueing network with constant routing matrix and arrival and service rates that only depend on the state and the slowly changing Markov process (i.e., the network parameters do not depend on X^n). In the n^{th} network, $\widehat{Q}^n$ and Y^n will denote the normalized queue length process and the modulating Markov process, respectively. The main result of this chapter (Theorem 4.1.1) shows that, under suitable heavy traffic and stability conditions, $(\widehat{Q}^n, Y^n)$ admits a stationary distribution which converges to that of (Z, Y), where (Z, Y) is as in Chapter 3, as $n \to \infty$.

We now recall the basic network description. Consider a sequence of open queueing networks with the following structure. Each network has K service stations each of which has an infinite capacity buffer. We denote the i^{th} station by $P_i, i \in I\!K \doteq \{1, 2, \ldots, K\}$. All customers/jobs at a station are "homogeneous" in terms of service requirement and routing decisions. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at station P_i a customer is routed to some other service station or exits the system. The external arrival processes and service processes are assumed to depend on the state of the system and an auxiliary finite state Markov process. The routing mechanism is governed by a $K \times K$ substochastic matrix $\overline{\mathbb{P}}$. Roughly speaking, the conditional probability that a job completed at station P_i is routed to station P_j equals the $(i, j)^{th}$ entry of the matrix $\overline{\mathbb{P}}$. The above formal description is made precise in what follows.

In the n^{th} network, the Markov process modulating the arrival and service rates is denoted as $\{Y^n(t) : t \geq 0\}$. We assume that Y^n has a finite state space $I\!H$ and infinitesimal generator $\mathbb{Q}^n$ which converges to some matrix $\mathbb{Q}$. Let $Q_i^n(t)$ denote the number of customers at station P_i at time t. Then the evolution of Q^n can be described by the following equation

$$Q_i^n(t) = Q_i^n(0) + A_i^n(t) - D_i^n(t) + \sum_{j=1}^{K} D_{ji}^n(t), \quad i \in I\!K. \tag{4.1.1}$$

Here $A_i^n(t)$ is the number of arrivals from outside at station P_i by time t, $D_i^n(t)$ is the number of service completions by time t at station P_i, and $D_{ji}^n(t)$ is the number of jobs that are routed to P_i immediately upon completion at station P_j by time t. Letting $D_{i0}^n(t)$ be the number of customers by time t who leave the network after service at P_i, we have

$$D_i^n(t) = \sum_{j=0}^{K} D_{ij}^n(t). \tag{4.1.2}$$

The dependance of arrival and processing rates on the system state and Y^n is modeled by requiring that A_i^n and $D_{ij}^n, 1 \le i \le K, 0 \le j \le K$, are counting processes given on a suitable filtered probability space $(\Omega^n, \mathcal{F}^n, P^n, \{\mathcal{F}_t^n\})$ such that for some measurable functions $\lambda_i^n, \tilde{\alpha}_i^n : I\!R_+^K \times I\!H \to I\!R_+$, the processes

$$\begin{aligned}\tilde{A}_i^n(\cdot) &\equiv A_i^n(\cdot) - \int_0^{\cdot} \lambda_i^n(Q^n(u), Y^n(u))du, \\ \tilde{D}_{ij}^n(\cdot) &\equiv D_{ij}^n(\cdot) - \int_0^{\cdot} \overline{\mathbb{P}}_{ij}\tilde{\alpha}_i^n(Q^n(u), Y^n(u))du\end{aligned} \tag{4.1.3}$$

are locally square integrable $\{\mathcal{F}_t^n\}$ martingales. Here $\overline{\mathbb{P}}_{i0} = 1 - \sum_{j=1}^K \overline{\mathbb{P}}_{ij}$. We assume that processes A_i^n and $D_{ij}^n, 1 \le i \le K, 0 \le j \le K$, and Y^n have no common jumps. We also require that Y^n is a $\{\mathcal{F}_t^n\}$ Markov process. The functions λ_i^n and $\tilde{\alpha}_i^n, i \in I\!K$, represent the arrival and service rates. We denote by $I\!K_0$ ($I\!K_0 \subseteq I\!K$) the set of indices of stations which receive arrivals from outside. In particular, $\lambda_i^n(x, y) = 0$ for all $(x, y) \in I\!R_+^K \times I\!H$ whenever $i \in I\!K \backslash I\!K_0$. Reflecting the fact that no service occurs when the buffer is empty, $\tilde{\alpha}_i^n(x, y) = 0$ if $x_i = 0$. Let $\lambda^n = (\lambda_1^n, \ldots, \lambda_K^n)'$. We assume that, for each $i \in I\!K$, $\tilde{\alpha}_i^n$ restricted to $\left(I\!R_+^K \backslash \{z \in I\!R_+^K : z_i = 0\}\right) \times I\!H$ can be extended to a function α_i^n defined on $I\!R_+^K \times I\!H$ (that satisfies additional properties as specified below), and write $\alpha^n = (\alpha_1^n, \ldots, \alpha_K^n)'$. Let

$$b^n \doteq \frac{\lambda^n - [\mathbb{I} - \overline{\mathbb{P}}']\alpha^n}{\sqrt{n}}.$$

We introduce the main assumptions on model parameters, which are similar to Assumption 2.3.1(i)-(vi).

Assumption 4.1.1.

(i) *The spectral radius of* $\overline{\mathbb{P}}$ *is strictly less than* 1.

(ii) *There exist some* $\underline{\theta}_1, \bar{\theta}_1 \in (0, \infty)$ *such that, for all* $n \ge 1, i \in I\!K_0, j \in I\!K$ *and* $(z, y) \in I\!R_+^K \times I\!H$, $n\underline{\theta}_1 \le |\lambda_i^n(z, y)| \le n\bar{\theta}_1$, $n\underline{\theta}_1 \le |\alpha_j^n(z, y)| \le n\bar{\theta}_1$.

(iii) *For some* $\theta_2 \in (0,\infty)$, $\sup_{(z,y)\in \mathbb{R}^K_+\times \mathbb{H}} |b^n(z,y)| \le \theta_2$.

(iv) *There exists a bounded Lipschitz map* $b : \mathbb{R}^K_+\times\mathbb{H} \to \mathbb{R}^K$ *such that* $b^n(\sqrt{n}z, y) \to b(z,y)$ *uniformly on* $\mathbb{R}^K_+ \times \mathbb{H}$ *as* $n\to\infty$.

(v) *There exist* $\mathbb{R}^K_+$*-valued bounded Lipschitz functions* λ, α *defined on* $\mathbb{R}^K_+ \times \mathbb{H}$, *such that*

$$\frac{\lambda^n(\sqrt{n}z, y)}{n} \to \lambda(z,y),\ \frac{\alpha^n(\sqrt{n}z, y)}{n} \to \alpha(z,y)$$

uniformly for (z,y) *in compact subsets of* $\mathbb{R}^K_+ \times \mathbb{H}$ *as* $n\to\infty$. *Furthermore,* $\lambda = [\mathbb{I} - \overline{\mathbb{P}}']\alpha$.

(vi) *For each* $i \in \mathbb{K}\backslash\mathbb{K}_0$, *there exists* $j \in \mathbb{K}_0$ *such that* $\overline{\mathbb{P}}^m_{ji} > 0$ *for some* $m \in \mathbb{N}$.

For $t \ge 0$, let

$$\widehat{Q}^n(t) = \frac{Q^n(t)}{\sqrt{n}}.$$

From Theorem 2.3.2 in Chapter 2, it follows that under Assumption 4.1.1, as $n \to \infty$, $(\widehat{Q}^n, Y^n)$ converges weakly to a Markov process (Z,Y), where (Z,Y) is as in Definition 2.3.2. In particular, Y is a Markov process with infinitesimal generator $\mathbb{Q}$ and Z is a reflected diffusion process with state dependent and Markov modulated coefficients, which can be described as follows.

$$Z(t) = \Gamma\left(z + \int_0^\cdot b(Z(s),Y(s))ds + \int_0^\cdot \sigma(Z(s),Y(s))dW(s)\right)(t),\ t\ge 0.$$

The drift b is as in Assumption 4.1.1(iv) and the diffusion coefficient σ is constructed as between (2.3.9) and (2.3.10). Note that b and σ satisfy Assumptions 3.2.2 and 3.2.3. Denote $\Phi^n \equiv (\widehat{Q}^n, Y^n), \Phi \equiv (Z,Y)$ and $\varphi \equiv (z,y)$. The following is the main result of the chapter.

Theorem 4.1.1. *Suppose that Assumptions 4.1.1 and 3.2.4 hold and that* b *can be expressed as in* (3.2.7) *in terms of functions* b_1 *and* b_2 *that satisfy Assumption 3.2.5.*

Then there exists $N \in I\!\!N$ such that for any $n \geq N$, the Markov process Φ^n admits a stationary distribution. Let π_n be an arbitrary stationary distribution of Φ^n. Then $\pi_n \Rightarrow \pi$ as $n \to \infty$, where π is as in Theorem 3.2.2.

In the following, we provide an explicit example, where assumptions of the above theorem hold.

Example 4.1.1. *Let $K = 2, I\!\!H = \{1, 2\}$, and $\overline{\mathbb{P}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{pmatrix}$. The arrival and service rate λ^n and α^n are defined as follows. For $z = (z_1, z_2) \in I\!\!R^2_+$ and $y \in I\!\!H$,*

$$\lambda^n(z, y) = \left(\sqrt{n}(e^{-z_1/\sqrt{n}} + 4) + ny, \sqrt{n}(e^{-z_2/\sqrt{n}} + 4) + 2ny\right)',$$
$$\alpha^n(y) = \left(\frac{24}{5}\sqrt{n}y + 2ny, \frac{27}{5}\sqrt{n}y + 3ny\right)'.$$

Therefore,

$$b^n(z, y) = \left(e^{-z_1/\sqrt{n}} + 4 - 3y, e^{-z_2/\sqrt{n}} + 4 - 3y\right)',$$

and

$$b(z, y) = b^n(\sqrt{n}z, y) = (e^{-z_1} + 4 - 3y, e^{-z_2} + 4 - 3y)'. \tag{4.1.4}$$

Let $q^* = (\frac{1}{4}, \frac{3}{4})$. We can construct a Markov process Y^n, which has state space $I\!\!H$ and convergent infinitesimal generator, such that it converges to a Markov process Y with stationary distribution q^*. With the above model parameters, we have from Theorem 2.3.2 that $(\widehat{Q}^n, Y^n) \Rightarrow (Z, Y)$, where Z is defined as in (2.3.11) with drift b defined as in (4.1.4) and diffusion coefficient σ constructed as between (2.3.9) and (2.3.10).

We note that the constraint directions for Z are $d_1 = (1, -\frac{1}{2})'$ and $d_2 = (-\frac{1}{3}, 1)'$ and therefore the cone $\mathcal{C} = \{-\alpha_1 d_1 - \alpha_2 d_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}$. We observe that, for $z \in I\!\!R^2_+$,

$$b(z, 1) = \left(e^{-z_1} + 1, e^{-z_2} + 1\right) \in \mathcal{C}^c,\ b(z, 2) = \left(e^{-z_1} - 2, e^{-z_2} - 2\right) \in \mathcal{C}^o,$$

and the "average" drift

$$b^*(z) = \left(e^{-z_1} - \frac{5}{4}, e^{-z_2} - \frac{5}{4}\right)' \in \mathcal{C}^o.$$

In fact, for all $0 < \delta_0 < \frac{1}{4}$, *we have for all* $z \in I\!R^2_+$, $b^*(z) \in \mathcal{C}(\delta_0)$. *By Theorem 3.2.2,* (Z, Y) *is positive recurrent and has a unique invariant measure* π. *Finally, from Theorem 4.1.1,* $(\widehat{Q}^n, Y^n)$ *admits an invariant probability measure* π^n *and* $\pi^n \Rightarrow \pi$ *as* $n \to \infty$.

4.2 Proof of Theorem 4.1.1

In this section we prove Theorem 4.1.1. Recall the processes $Q^n, \tilde{A}^n, \tilde{D}^n$ defined in (4.1.1) and (4.1.3). Define $I\!R^K$ valued stochastic processes M^n, B^n, η^n as follows. For $i \in I\!K$ and $t \geq 0$,

$$\begin{aligned} M_i^n(t) &= \frac{1}{\sqrt{n}}\left(\widetilde{A}_i^n(t) - \sum_{j=0}^{K} \widetilde{D}_{ij}^n(t) + \sum_{j=1}^{K} \widetilde{D}_{ji}(t)\right), \\ B_i^n(t) &= \int_0^t b_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u))du, \\ \eta_i^n(t) &= \frac{1}{\sqrt{n}} \int_0^t \alpha_i^n(\sqrt{n}\widehat{Q}^n(u), Y^n(u))1_{\{\widehat{Q}_i^n(u)=0\}}du. \end{aligned} \tag{4.2.1}$$

Noting that $\tilde{\alpha}_i^n(z, y) = \alpha_i^n(z, y)1_{\{z_i>0\}}$ for all $(z, y) \in I\!R^K_+ \times I\!H$, we have from (4.1.1) that

$$\widehat{Q}^n(t) = \widehat{Q}^n(0) + M^n(t) + B^n(t) + [\mathbb{I} - \overline{\mathbb{P}}']\eta^n(t). \tag{4.2.2}$$

With this notation, equation (4.2.2) can be written as

$$\widehat{Q}^n(t) = \Gamma\left(\widehat{Q}^n(0) + M^n(\cdot) + B^n(\cdot)\right)(t), \tag{4.2.3}$$

where Γ is the Skorohod map with reflection matrix $\mathbb{I} - \overline{\mathbb{P}}'$. As noted in Section 4.1, due to Assumption 4.1.1(i), Γ is Lipschitz continuous, namely Assumption 3.2.1 is satisfied. The following stability result is a key step. Denote by $D_n = \{\varphi = (z, y) \in \mathbb{S} : nz \in I\!N_0\}$.

Proposition 4.2.1. *There exist $N_1 \in I\!N$ and $t_0 \in (0, \infty)$ such that for all $t \geq t_0$,*

$$\lim_{|z|\to\infty} \frac{\sup_{n\geq N_1} E_{\varphi}\left(\left|\widehat{Q}^n(t|z|)\right|^2\right)}{|z|^2} = 0, \ \varphi = (z, y) \in D_n.$$

Proof: Fix $\varphi = (z, y) \in D_n$ such that $z \in G \setminus A$. Let $M > 0$ be large enough such that $G_M \doteq \{z \in G : |z| < M\} \supseteq \bar{A}$. Suppressing n in the notation, define a sequence of stopping times $\{\sigma_k\}_{k \in I\!N_0}$ as $\sigma_0 = 0$,

$$\sigma_{2k+1} = \inf\{t \geq \sigma_{2k} : \widehat{Q}^n(t) \in A\}, \sigma_{2k+2} = \inf\{t \geq \sigma_{2k+1} : \widehat{Q}^n(t) \notin G_M\}, \ k \in I\!N_0.$$

If $t \in [\sigma_{2k+1}, \sigma_{2k+2}]$ for some $k \in I\!N_0$, then

$$|\widehat{Q}^n(t)| \leq M + 1. \tag{4.2.4}$$

Suppose now $t \in [\sigma_{2k}, \sigma_{2k+1})$ for some $k \in I\!N_0$. Then

$$\widehat{Q}^n(t) = \Gamma\left(\widehat{Q}^n(\sigma_{2k}) + M^n(\cdot + \sigma_{2k}) - M^n(\sigma_{2k}) + B^n(\cdot + \sigma_{2k}) - B^n(\sigma_{2k})\right)(t - \sigma_{2k}).$$

From convergence of $\mathbb{Q}^n$ to $\mathbb{Q}$, it follows that for some $n_0 \in I\!N$, the Markov process Y^n has a unique invariant measure q^n, whenever $n \geq n_0$. Furthermore, $q^n \to q^*$ as $n \to \infty$. We will assume without loss of generality that $n \geq n_0$. Define for $s \geq 0$,

$$\breve{X}^n(s) = \Gamma\left(\widehat{Q}^n(\sigma_{2k}) + B^n_*(\cdot + \sigma_{2k}) - B^n_*(\sigma_{2k})\right)(s - \sigma_{2k}),$$

where

$$B_*^n(s) = \int_0^s b_*^n\left(\sqrt{n}\widehat{Q}^n(v), Y^n(v)\right) dv,$$

and

$$b_*^n(z,y) = b^n(z,y) - b_2(y) + \sum_{y \in I\!H} b_2(y) q^n(y).$$

Note that $B^n = B_*^n + B_c^n$, where

$$B_c^n(s) = \int_0^s b_c^n\left(Y^n(v)\right) dv, \text{ and } b_c^n(y) = b_2(y) - \sum_{y \in I\!L} b_2(y) q^n(y).$$

Lipschitz property of Γ yields that

$$|\widehat{Q}^n(t) - \breve{X}^n(t)| \le 2\kappa_1 \sup_{0 \le s \le t} |M^n(s) + B_c^n(s)|.$$

Using Assumption 2.3.1 (iv) and the property $q^n \to q^*$, we see that as $n \to \infty$, $b_*^n(\sqrt{n}z, y) \to b_1^*(z,y)$ uniformly on $\mathbb{S}$. Using Assumption 3.2.5 we now have that for some $n_1 \in I\!N$ and $n \ge n_1$, $b_*^n(\sqrt{n}z, y) \in \mathcal{C}(\delta_0/2)$ for all $(z,y) \in G/A \times I\!L$. Thus

$$\Gamma\left(\int_{\sigma_{2k}}^{\sigma_{2k}+\cdot} b_*^n(\sqrt{n}\widehat{Q}^n(v \wedge \sigma_{2k+1}), Y^n(v \wedge \sigma_{2k+1}))du\right)(\cdot - \sigma_{2k}) \in \mathcal{A}(0, \delta_0/2),$$

where $\mathcal{A}$ is as defined below (3.3.2). Applying Lemma 3.3.1 (ii),(iii), we now have that for all $\sigma_{2k} \le s < \sigma_{2k+1}$,

$$\begin{aligned}&\Gamma(B_*^n(\cdot + \sigma_{2k}) - B_*^n(\sigma_{2k}))(s - \sigma_{2k})\\ &= \Gamma\left(\int_{\sigma_{2k}}^{\sigma_{2k}+\cdot} b_*^n(\sqrt{n}\widehat{Q}^n(v \wedge \sigma_{2k+1}), Y^n(v \wedge \sigma_{2k+1}))du\right)(s - \sigma_{2k}) = 0.\end{aligned}$$

Thus using Assumption 3.2.1, we have that if $k > 0$,

$$|\breve{X}^n(t)| \le \kappa_1 |\widehat{Q}^n(\sigma_{2k})| \le \kappa_1 (M+1), \ \forall t \in [\sigma_{2k}, \sigma_{2k+1}).$$

A similar argument, using Lemma 3.3.1(i), shows that in the case $k = 0$, i.e. $t \in [\sigma_0, \sigma_1)$ and $t \geq \Theta_1(\delta_0/2)|x|$, $|\breve{X}^n(t)| = 0, P_z$-a.s., where $\Theta_1(\delta_0/2)$ is as in Lemma 3.3.1. Combining the above estimates, for all $t \geq \Theta_1|x|$,

$$\left|\widehat{Q}^n(t)\right| \leq 2\kappa_1 \sup_{0\leq s\leq t} |M^n(s)| + 2\kappa_1 \sup_{0\leq s\leq t} |B_c^n(s)| + \kappa_1(M+1). \tag{4.2.5}$$

By martingale properties of processes in (4.1.3), Doob's inequality and Assumption 4.1.1(ii), we have that for some $c_1 \in (0, \infty)$,

$$\begin{aligned} &E_\varphi \left(\sup_{0\leq s\leq t} |M^n(s)| \right)^2 \leq 4 \sum_{i=1}^{K} E_\varphi(|M_i^n(t)|^2) \\ &\leq \frac{4}{n} \sum_{i=1}^{K} E_\varphi \left(\int_0^t \lambda_i^n \left(\sqrt{n}\widehat{Q}^n(v), Y^n(v) \right) + 2 \sum_{j=1}^{K} \alpha_j^n \left(\sqrt{n}\widehat{Q}^n(v), Y^n(v) \right) dv \right) \\ &\leq c_1 t. \end{aligned} \tag{4.2.6}$$

Next we consider $E_\varphi \left(\sup_{0\leq s\leq t} |B_c^n(s)| \right)^2$. Let $g^n(\cdot)$ be a solution of the Poisson equation for $b_c^n(\cdot)$ corresponding to the Markov semigroup $\{P_n^s\}$ of Y^n. Then

$$M_s^n \doteq g^n(Y^n(s)) - g^n(Y^n(0)) - \int_0^s b_c^n(Y^n(v))dv$$

is a $\{\mathcal{F}_s^n\}$ martingale and $\Theta \doteq \sup_n |g^n|_\infty < \infty$. Therefore, another application of Doob's inequality yields

$$E_\varphi \left(\sup_{0\leq s\leq t} |B_c^n(s)| \right)^2 = E_\varphi \left(\sup_{0\leq s\leq t} |g^n(Y^n(s)) - g^n(y) - M_s^n| \right)^2 \leq 8\Theta^2 + 8E_\varphi \left(|M_t^n|^2 \right)$$

Analogous to (3.3.8), we have for some $c_2 \in (0, \infty)$ and $n_2 \in I\!N$,

$$\sup_{n\geq n_2} P_\varphi \left(|M_t^n|^2 \geq x \right) \leq 2 \exp \left\{ -\frac{c_2 x}{t+1} \right\}.$$

Therefore,

$$E_\varphi\left(\sup_{0\le s\le t}|B_c^n(s)|\right)^2 \le 4\Theta^2 + 4\int_0^\infty 2\exp\left\{-\frac{c_2 x}{t+1}\right\}dx \le 4\Theta^2 + \frac{8(t+1)}{c_2}. \tag{4.2.7}$$

Combing (4.2.5), (4.2.6) and (4.2.7), we have for some $c_3 \in (0,\infty)$ and all $t \ge \Theta_1$ and $n \ge \max(n_0, n_1, n_2)$,

$$E_\varphi\left(\left|\widehat{Q}^n(t|z|)\right|^2\right) \le c_3(1+t|z|),\ \varphi = (z,y) \in D_n.$$

The lemma now follows on setting $t_0 = \Theta_1$ and $N_1 = \max(n_0, n_1, n_2)$. ■

The following proposition yields the tightness of

$$\left\{P_\varphi^n \circ \Phi^n(t)^{-1} : \varphi = (z,y), |z| \le M, t \ge 0, n \ge N\right\}$$

for all $M > 0$ and N sufficiently large. Proof is similar to that of Lemma 3.3.5. For completeness, a sketch is given in Appendix.

Proposition 4.2.2. *There exist $N_2 \in I\!N$ and $\check{\kappa} \in (0,\infty)$ such that for $M > 0$,*

$$\sup_{n\ge N_2}\ \sup_{\varphi\in C_M\cap D_n}\ \sup_{t\ge 0} E_\varphi\left(e^{\check{\kappa}\left|\widehat{Q}^n(t)\right|}\right) < \infty,$$

where C_M is defined as below (3.3.11).

The following two propositions will be needed in the proof of Theorem 4.1.1. Proof of the next proposition is identical to that of Proposition 4.2 of [11] and thus is omitted. For $\varrho \in (0,\infty)$ and a compact set $\mathrm{F} \subset \mathbb{S}$, let

$$\tau_{\mathrm{F}}^n(\varrho) \doteq \inf\{t \ge \varrho : \Phi^n(t) \in O\}. \tag{4.2.8}$$

Proposition 4.2.3. *Let $f : \mathbb{S} \to I\!\!R_+$ be a measurable map. Define for $\varrho \in (0, \infty)$,*

$$G_n(\varphi) = E_\varphi \left(\int_0^{\tau^n_{\mathrm{F}}(\varrho)} f(\Phi^n(t))dt \right), \; \varphi \in D_n.$$

Assume

$$\sup_n \sup_{\varphi \in C_M \cap D_n} G_n(\varphi) < \infty \text{ for every } M > 0. \tag{4.2.9}$$

Then there exists a $\bar{\kappa} \in (0, \infty)$ such that, for all $n \in I\!\!N, t \in [\varrho, \infty)$ and $z \in D_n$,

$$\frac{1}{t} E_\varphi \left(G_n(\Phi^n(t)) \right) + \frac{1}{t} \int_0^t E_\varphi \left(f(\Phi^n(s)) \right) ds \leq \frac{1}{t} G_n(\varphi) + \bar{\kappa}.$$

By Proposition 4.2.1, there exists $\Lambda \in (0, \infty)$ such that for $|z| \geq \Lambda$, $\varphi = (z, y) \in D_n$, and $n \geq N_1$,

$$E_\varphi \left(\left| \widehat{Q}^n(t_0|z|) \right|^2 \right) \leq \frac{1}{2} |z|^2,$$

where t_0 and N_1 are as in Proposition 4.2.1. The following proposition is proved exactly as Proposition 4.2 of [9] and thus the proof is omitted.

Proposition 4.2.4. *There exists $N_3 \in I\!\!N$ and $c_0 \in (0, \infty)$ such that for all $n \geq N_3$ and $\varphi \in D_n$,*

$$\sup_{n \geq N_3} E_\varphi \left(\int_0^{\tau^n} \left(1 + \left| \widehat{Q}^n(t) \right| \right) dt \right) \leq c_0 \left(1 + |z|^2 \right),$$

where $\tau^n = \tau^n_{C_\Lambda}(t_0 \Lambda)$ (see (4.2.8)), t_0, Λ are introduced above and C_Λ is defined below (3.3.11) with M replaced by Λ.

Proof of Theorem 4.1.1: From Proposition 5.2, it follows that for all $n \geq N_2$, Z^n has an invariant probability measure on D_n. Denote by $\{\pi_n\}_{n \geq N}$ one such sequence of invariant measures, where $N = \max(N_2, N_3)$ and N_2, N_3 are as in Propositions 4.2.2 and 4.2.4, respectively. Since π is the unique invariant measure of the Feller-Markov process $(\Phi, \{\overline{\mathbb{P}}_\varphi\}_{\varphi \in \mathbb{S}})$, we have from Theorem 2.3.2 that it suffices to establish the tightness of the family $\{\pi_n\}$ (regarded as a sequence of probability measures on $\mathbb{S}$). We apply Proposition 4.2.3 with $f(z) = 1 + |x|, z = (x, y) \in \mathbb{S}$ and $\varrho = t_0\Lambda, \mathrm{F} = C_\Lambda$, where t_0 and Γ are as in Proposition 4.2.4. Note that condition (4.2.9) in Proposition 4.2.3 is satisfied as a consequence of Proposition 4.2.4. To prove the desired tightness we only need to show that, for all $n \geq N$, $\langle \pi_n, f\rangle \leq c_1 < \infty$. Note that for any nonnegative, real measurable function ψ on $\mathbb{S}$ and $n \geq N$,

$$\int_{D_n} E_z\left(\psi(Z^n(t))\right)\pi_n(dz) = \langle \pi_n, \psi\rangle. \tag{4.2.10}$$

Fix $k \in I\!N$ and $t \in (\varrho, \infty)$. Let for $z \in D_n$,

$$\Phi_n(z) \doteq \frac{1}{t}G_n(z) - \frac{1}{t}E_z\left(G_n(Z^n(t))\right).$$

By (4.2.10), $\int_{D_n}\Phi_n(z)\pi_n(dz) = 0$. From Proposition 4.2.3,

$$0 = \int_{D_n}\Phi_n(z)\pi_n(dz) \geq \int_{D_n}\left(\frac{1}{t}\int_0^t E_z(f(Z^n(s)))ds - \bar{\kappa}\right)\pi_n(dz).$$

Recalling (4.2.10), we have that $\langle \pi_n, f\rangle \leq \bar{\kappa}$. The result follows. ■

4.3 Appendix

Proof of Proposition 4.2.2 (sketch): Define, for $j \in I\!N$,

$$\nu_j^n = \sup_{(j-1)\Delta \leq s \leq j\Delta}\left|M^n(s)) - M^n((j-1)\Delta) + B_c^n(s) - B_c^n((j-1)\Delta)\right|.$$

Along the lines of proof of Lemma 4.4 in [4], we have that for all $q \in I\!N$,

$$T(\widehat{Q}^n(q\Delta)) \;\leq\; T(z) + 2\Delta + \sum_{j=1}^{q}(2\kappa_1\Theta_1\nu_j^n - \Delta)$$

$$\leq \quad T(z) + 2\Delta + \max_{1\leq l\leq q} \sum_{j=l}^{q} (2\kappa_1 \Theta_1 \nu_j^n - \Delta).$$

Hence for $\alpha, M_0 \in (0,\infty)$,

$$P_\varphi\left(T(\widehat{Q}^n(q\Delta)) \geq M_0)\right) \leq \sum_{l=1}^{q} P_\varphi\left(2\kappa_1\Theta_1 \sum_{j=l}^{q} \nu_j^n \geq M_0 + (q-l-1)\Delta - T(z)\right)$$
$$\leq \exp\{\varsigma(T(z)+\Delta - M_0)\} \sum_{l=1}^{q} \frac{E_\varphi\left(\exp\{2\varsigma\kappa_1\Theta_1 \sum_{j=l}^{q} \nu_j^n\}\right)}{\exp\{\varsigma(q-l)\Delta\}}.$$

Let $c_1 \doteq 2\kappa_1\Theta_1$. We claim that there exist constants $\varsigma_0, \Delta_0, \eta \in (0,\infty)$ and $N \in I\!N$ such that

$$\sup_{n\geq N} \sup_{j\in I\!N} e^{-\varsigma_0\Delta_0} E_\varphi\left(e^{c_1\varsigma_0\nu_j^n} \mid \mathcal{F}^n_{(j-1)\Delta_0}\right) \leq e^{-\eta\Delta_0}, \tag{4.3.1}$$

where $\mathcal{F}^n_t$ introduced below (4.1.2). Suppose, for now, that the claim holds. Then by the Markov properties of Φ^n and Y^n, we have that, for $n \geq N$ and $q \in I\!N$,

$$P_\varphi\left(T(\widehat{Q}^n(q\Delta_0) \geq M_0)\right) \leq \exp\{\varsigma_0(T(z)+\Delta_0 - M_0)\} \sum_{l=1}^{q} \exp\{-\varsigma_0(q-l+1)\eta\Delta_0\}$$
$$\leq \frac{\exp\{\varsigma_0(T(z)+\Delta_0 - M_0)\}}{1-\exp\{-\eta\Delta_0\}}.$$

Consequently, there exists constant $\kappa_1 \in (0,\infty)$ such that for all $M \in (0,\infty)$,

$$\sup_{n\geq N} \sup_{t\geq 0} \sup_{|\varphi|\leq M} E_\varphi(e^{\kappa_1|\widehat{Q}^n(q\Delta_0)|}) < \infty.$$

The result now follows by a standard argument, using the Lipschitz property of Γ. Finally we prove the claim in (4.3.1). Note that

$$\nu_j^n \leq \sup_{(j-1)\Delta\leq s\leq j\Delta} |M^n(s)) - M^n((j-1)\Delta)| + \sup_{(j-1)\Delta\leq s\leq j\Delta} |B_c^n(s) - B_c^n((j-1)\Delta)|. \tag{4.3.2}$$

Following the proof of Lemma 3.3.3 (see arguments between (3.3.7) and (3.3.9)), we can find $c_2 \in (0,\infty)$ for all $j \in I\!N$, $\varsigma, \Delta \in (0,\infty)$,

$$E_z\left(\exp\left\{\alpha \sup_{(j-1)\Delta \le s \le j\Delta} |B_c^n(s) - B_c^n((j-1)\Delta)|\right\} \Big| \mathcal{F}^n_{(j-1)\Delta}\right) \le 8\exp\{c_2\varsigma(1+\varsigma+\varsigma\Delta)\}.$$

Furthermore, following the proof of Proposition 3.2 in [9] (see arguments below (7.4) therein). We can find $c_3 \in (0,\infty)$ such that for $j \in I\!N$, $\varsigma, \Delta \in (0,\infty)$,

$$E_z\left(\exp\left\{\alpha \sup_{(j-1)\Delta \le s \le j\Delta} |M^n(s)) - M^n((j-1)\Delta)|\right\} \Big| \mathcal{F}^n_{(j-1)\Delta}\right) \le 8\exp\{c_3\varsigma^2\Delta\}.$$

By Holder's inequality, for $j \in I\!N$, $\varsigma, \Delta \in (0,\infty)$,

$$\exp\{-\varsigma\Delta\}E_z\left(\exp\{\varsigma c_1\nu_j^n\}|\mathcal{F}^n_{(j-1)\Delta}\right) \le 8\exp\{c_1c_2\varsigma + 2c_1^2c_2\varsigma^2 + 2(c_1^2c_2 + c_1^2c_3)\varsigma^2\Delta - \varsigma\Delta\}.$$

Finally, choose appropriate (small) ς_0 and (large) Δ_0 such that

$$\log 8 + c_1c_2\varsigma_0 + 2c_1^2c_2\varsigma_0^2 + 2(c_1^2c_2 + c_1^2c_3)\varsigma_0^2\Delta - \varsigma_0\Delta = -\eta\Delta_0,$$

for some $\eta \in (0,\infty)$. The claim follows. ■

Chapter 5

Action time sharing policies for ergodic control of Markov chains

5.1 Introduction

Markov Decision processes are used extensively as the simplest models that involve both stochastic behavior and control [42]. A common measure of performance is the long-time average (or ergodic) criterion. Given all relevant parameters, a typical goal is to find a simple (e.g. feedback, or deterministic stationary) policy that achieves the optimal value.

The goal of adaptive control is to obtain an optimal policy, when some relevant information concerning the behavior of the system is missing. The relevant information needs to be obtained while controls are chosen at each step. The classical approach is to design an algorithm which collects information, while at the same time choosing controls, in such a way that sufficient information is collected for making good control decisions, in the sense that the chosen controls "approach optimality over time." Existing results include general solutions for the case of countable state space, and specify an estimation and a control scheme (see [6, 38] and references therein). For a more refined criterion of optimality for the adaptive case see [1, 14]. A different approach to this issue, including PAC criteria, can be found in the large literature on Reinforcement learning, e.g. [15]. For results on adaptive control in the non-countable setting we refer the reader to [28, 21, 25] and references therein: these deal with the classical setup and with parameterized models.

We are concerned with a more elementary question, namely: What are the basic controlled objects that determine the cost? Since the objective function (see (5.2.2)) is defined as a Cesaro limit, we can expect that a similar Cesaro definition of the choice of controls would suffice to determine the cost. Indeed, [2] shows the following, for the case of countable state and action spaces. Let q be a stationary Markov control, namely it is a map from the state space $\mathbb{X}$ to the space $\mathcal{P}(\mathbb{A})$ of probability measures on the action space $\mathbb{A}$. Together with an initial distribution μ on $\mathbb{X}$ and a transition probability kernel $\mathcal{Q} : \mathbb{X} \times \mathbb{A} \times \mathcal{B}(\mathbb{X}) \to [0,1]$, such a Markov control determines a probability measure $\mathbb{P}^q_\mu$ on the infinite product space $\Omega = (\mathbb{X} \times \mathbb{A})^{\otimes\infty}$ by the relation

$$
\begin{aligned}
&\mathbb{P}^q_\mu\left((X_0, A_0) \in E_0, (X_1, A_1) \in E_1, \cdots, (X_k, A_k) \in E_k\right) \\
&= \int_{E_0}\int_{E_1}\cdots\int_{E_k} q(x_k, da_k)\mathcal{Q}(x_{k-1}, a_{k-1}, dx_k)\cdots q(x_1, da_1)\mathcal{Q}(x_0, a_0, dx_1)q(x_0, da_0)\mu(dx_0), \\
&\qquad\qquad\qquad\qquad E_0, E_1, \cdots, E_k \in \mathcal{B}(\mathbb{X} \times \mathbb{A}),\ k \in I\!N_0,
\end{aligned}
$$

where $(X_k, A_k)_{k\in I\!N_0}$ is the canonical coordinate sequence on Ω. Defining a general admissible control policy requires additional notation and thus a precise description is postponed to Section 5.2. Roughly speaking, such a policy is defined in terms of a non-anticipative sequence $\{\pi_t\}_{t\in I\!N_0}$ of $\mathcal{P}(\mathbb{A})$ valued random variables and, through a formula similar to the above display, describes a probability measure $\mathbb{P}^\pi_\mu$ on Ω. An admissible control policy π is called an ATS policy for a stationary Markov control q if the conditional frequencies:

$$
f_T(a \mid x) = \frac{\sum_{t=0}^{T-1} 1\{X_t = x, A_t = a\}}{\sum_{t=0}^{T-1} 1\{X_t = x\}} \to q(x)(a) \equiv q(a \mid x), \tag{5.1.1}
$$

for all $(x,a) \in \mathbb{X}\times\mathbb{A}$, $\mathbb{P}^\pi_\mu$, a.e. The paper [2] shows that for such a π, for any bounded one stage cost function, the costs (5.2.2) under $\mathbb{P}^q_\mu$ and under $\mathbb{P}^\pi_\mu$ are the same. Such a result says that the control decisions can deviate from those dictated by the Markov policy q, and still produce the same long term average cost, as long as the conditional frequencies converge to the correct values. This flexibility is useful in many situations, some of which will be described towards the end of this Introduction.

In the current work we are concerned with a setting where the state and action spaces are not (necessarily) countable. Our main objective is to formulate an appropriate definition for an ATS policy which, similar to the countable case, on the one hand leads to long term costs that are identical to those for the corresponding Markov control, while on the other hand allows for flexible implementation well suited for various estimation and adaptive control goals. Clearly, conditional frequencies of the form in (5.1.1) are not suitable when $q(x, \cdot)$ and $\mathcal{Q}((x, a), \cdot)$ are not discrete measures. In Section 5.3 (Definition 5.3.1) we propose a definition of an ATS policy given in terms of suitable conditional frequencies over a sequence of "converging partitions" of the state space $\mathbb{X}$. We show in Theorem 5.3.1 that, under suitable stability, irreducibility and Feller continuity conditions (Assumptions 5.2.1, 5.2.2 and 5.2.3) occupation measures for state and action sequences, under an ATS policy given as in Definition 5.3.1, converge a.s. to the same (deterministic) measure as under the corresponding Markov control. Such a result in particular shows that long term costs for a broad family of one stage cost functions, under the two control policies, coincide.

We now comment on the usefulness of such a result. To see the flexibility that ATS policies offer let's first consider the countable setting. Consider the elementary model where $\mathbb{X}$ is a singleton and $\mathbb{A}$ is a finite set. A Markov control in this setting is just a single probability measure on $\mathbb{A}$ and the long term cost for a typical one stage cost function $c : \mathbb{A} \to [0, \infty)$ under q, by the strong law of large numbers is

$c_q = \int_{\mathbb{A}} c(a)q(da)$. Also, the corresponding asymptotic mean square error:

$$\lim_{N\to\infty} \mathbb{E}^q \left(\frac{1}{N} \sum_{k=0}^{N-1} c(A_k) - \int_{\mathbb{A}} c(a)q(da) \right)^2 = \frac{\sigma^2}{N},$$

where $\sigma^2 = \int_{\mathbb{A}} (c(a) - c_q)^2 q(da)$. It is easy to see that one can construct an ATS policy π for the Markov control q (cf. Lemma 5.4.1) under which, for some $\alpha(c) \in (0, \infty)$

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} c(A_k) - \int_{\mathbb{A}} c(a)q(da) \right| \leq \frac{\alpha(c)}{N},\ \mathbb{P}^\pi \text{ a.e.}$$

and thus the asymptotic mean square error under π is $\frac{\alpha^2(c)}{N^2}$.

The above simple example illustrates how ATS policies can be used to develop variance reduction schemes for ergodic control problems. Additionally, ATS policies provide much flexibility for sampling (namely using controls without regards to the ensuing cost), for example for the purpose of collecting information. This could be information which is related to the main optimization objective, but could also be other information which is of interest. Consider, for example, the following elementary setting. Suppose that $\mathbb{X} = \{-1, 0, 1\}$ and $\mathbb{A} = \{a, b\}$. Suppose that the one stage cost function is given as

$$c(\pm 1, a) = c(\pm 1, b) = 0,\ c(0, a) = 1,\ c(0, b) = 2$$

and the transition probability kernel is defined as

$$\mathcal{Q}((0,a),\cdot) = \frac{1}{2}\delta_{\{1\}}(\cdot) + \frac{1}{2}\delta_{\{-1\}}(\cdot); \qquad \mathcal{Q}((0,b),\cdot) = \beta\delta_{\{1\}}(\cdot) + (1-\beta)\delta_{\{-1\}}(\cdot);$$
$$\mathcal{Q}((x,a),\cdot) = \frac{1}{2}\delta_{\{-x\}}(\cdot) + \frac{1}{2}\delta_{\{0\}}(\cdot); \qquad \mathcal{Q}((x,b),\cdot) = (1-\gamma)\delta_{\{-x\}}(\cdot) + \gamma\delta_{\{0\}}(\cdot),\ x = \pm 1,$$

where $0 < \beta, \gamma < 1$. If our goal is the minimize to average cost, then we prefer to stay at states ± 1, and so we should use a at all states if $\gamma > \frac{1}{2}$, but use b at states ± 1 if $\gamma < \frac{1}{2}$. Thus, from the optimization point of view, we need to find γ but β is irrelevant.

Since the probability that $X_t = 1$ is bounded below for $t > 1$, consider the following estimation procedure for γ. Action b will be used at time t if $X_t = \pm 1$ and in addition $t = 10^n$ for some integer n. Let

$$\hat{\gamma}_t = \frac{\sum_{m=1}^{n} 1\{X_{10^m} = \pm 1, X_{10^m+1} = 0\}}{\sum_{m=1}^{n} 1\{X_{10^m} = \pm 1\}} \quad \text{for } 10^n < t \leq 10^{n+1} \ . \tag{5.1.2}$$

This estimator converges to γ a.s. under any policy that is consistent with the above requirement for time instants $t = 10^n$, $n \in I\!N$. In particular, we can now choose the following policy: At $t \neq 10^n$ use b iff $X_t = \pm 1$ and $\hat{\gamma}_t < \frac{1}{2}$. It then follows that there is some (random) time so that, at all later times, the optimal policy is used. It is easy to check that the above recipe defines an implementable ATS policy for the (unknown) optimal stationary policy and is thus optimal as well. Furthermore, we can modify the above policy slightly to define a new ATS policy that also delivers estimates for β, with no effect on the cost. This is done similarly to above: use action b at state 0 at time t if $X_t = 0$ and in addition $t = 10^n$ for some integer n. An estimator as in (5.1.2) will be consistent. Since the number of time points where b is used increases logarithmically it is easy to see that the limits in (5.1.1) are not affected, and consequently the limiting cost does not change.

The above example illustrates the use of ATS policies for estimation and adaptive control for a rather elementary setting. However similar ideas are applicable for general state and action space models as well. In Section 5.4.2 we show how ATS policies introduced in Section 5.3 of this work can be used for estimation of unknown model parameters and in Section 5.4.3 we describe how they can be used for adaptive control problems as well.

The rest of the chapter is organized as follows. In Section 5.2 we begin with some preliminary definitions and the main assumptions on the controlled dynamics. Section 5.3 introduces the definition of an ATS policy through a sequence of "converging partitions" of the state space. The section also presents the main convergence result for occupation measures associated with an ATS policy. Finally, in Section 5.4 we describe how ATS policies can be constructed and used in settings with incomplete model information.

5.2 Definitions and assumptions

The following notation will be used. For two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$, the space of $\mathcal{F}_1/\mathcal{F}_2$ measurable maps from Ω_1 to Ω_2 will be denoted as $\mathcal{M}(\Omega_1, \mathcal{F}_1 : \Omega_2, \mathcal{F}_2)$. When $(\Omega_2, \mathcal{F}_2) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we will merely write $\mathcal{M}(\Omega_1, \mathcal{F}_1)$ and if $\mathcal{F}_1, \mathcal{F}_2$ are clear from the context, we will write $\mathcal{M}(\Omega_1 : \Omega_2)$ and $\mathcal{M}(\Omega_1)$, respectively. The space of all probability measures on a measurable space $(\Omega, \mathcal{F})$ will be denoted by $\mathcal{P}(\Omega, \mathcal{F})$ or $\mathcal{P}(\Omega)$, when clear from the context. Borel sigma fields on a metric space $\mathcal{T}$ will be denoted by $\mathcal{B}(\mathcal{T})$. If $(\Omega, \mathcal{F}) = (\mathcal{T}, \mathcal{B}(\mathcal{T}))$ for some complete and separable metric (Polish) space $\mathcal{T}$, we will endow $\mathcal{P}(\Omega) \equiv \mathcal{P}(\mathcal{T})$ with the the topology of weak convergence. We recall the definition of Bounded-Lipschitz norm on $\mathcal{P}(\mathcal{T})$ for a Polish space $\mathcal{T}$. Let

$$\mathcal{C}_1(\mathcal{T}) = \left\{ \psi : \mathcal{T} \to \mathbb{R} : \sup_{t,t' \in \mathcal{T}, t \neq t'} \left(|\psi(t)| + \frac{|\psi(t) - \psi(t')|}{d(t,t')} \right) \leq 1 \right\},$$

where d is the metric given on $\mathcal{T}$. For $\nu_1, \nu_2 \in \mathcal{P}(\mathcal{T})$ denote

$$\|\nu_1 - \nu_2\|_{\text{BL}} = \sup_{\psi \in \mathcal{C}_1(\mathcal{T})} \left| \int \psi d\nu_1 - \int \psi d\nu_2 \right|.$$

This norm metrizes the topology of weak convergence making $\mathcal{P}(\mathcal{T})$ a Polish space. Throughout we will consider $\mathcal{P}(\mathcal{T})$ with this metric. The class of real valued continuous and bounded functions on a metric space $\mathcal{T}$ will be denoted by $C_b(\mathcal{T})$. $C_{\text{buc}}(\mathcal{T})$ will denote the subset of $C_b(\mathcal{T})$ consisting of all uniformly continuous functions. A class $\mathcal{S} \subset C_b(\mathcal{T})$ is called separating in $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ if whenever $\mu, \nu \in \mathcal{P}(\mathcal{T})$ and $\int f d\mu = \int f d\nu$ for all $f \in \mathcal{S}$, then $\mu = \nu$. Since $\mathcal{T}$ is Polish, one can find a countable collection in $C_{\text{buc}}(\mathcal{T})$ that is separating and we shall use the notation $\mathcal{S}(\mathcal{T})$ to denote such a class. It is easy to check that if $\mathcal{T}_1, \mathcal{T}_2$ are Polish spaces then $\{f \otimes g : f \in \mathcal{S}(\mathcal{T}_1), g \in \mathcal{S}(\mathcal{T}_2)\}$ is separating in $(\mathcal{T}_1 \times \mathcal{T}_2, \mathcal{B}(\mathcal{T}_1) \otimes \mathcal{B}(\mathcal{T}_2))$. Given a subset C of a metric space $\mathcal{T}$ with a distance d, we define $\text{diam}(C) = \sup\{d(x, y) : x, y \in C\}$.

We will consider a controlled stochastic dynamical system in discrete time (i.e. parametrized by the discrete index set $\mathbb{N}_0 \doteq \{0, 1, 2, \ldots\}$) with state space $\mathbb{X}$ that is a complete and separable locally compact space. A Polish space $\mathbb{A}$ will represent the control (or action) space. For each $x \in \mathbb{X}$ we are given a compact set $\mathbb{U}(x) \subset \mathbb{A}$ representing the set of admissible actions when the system is in state $x \in \mathbb{X}$. We assume that $\mathbb{K} = \{(x, a) : x \in \mathbb{X}, a \in \mathbb{U}(x)\}$ is a measurable subset of $\mathbb{X} \times \mathbb{A}$. The dynamics of the controlled Markov chain is described in terms of a transition kernel

$$\mathcal{Q} : \mathbb{K} \times \mathcal{B}(\mathbb{X}) \to [0, 1]$$

satisfying:

(i) For all $(x, a) \in \mathbb{K}$, $\mathcal{Q}((x, a), \cdot) \equiv \mathcal{Q}(\cdot \mid (x, a))$ is in $\mathcal{P}(\mathbb{X})$ and;

(ii) for every $C \in \mathcal{B}(\mathbb{X})$, $\mathcal{Q}(\cdot, C) \in \mathcal{M}(\mathbb{K})$.

Roughly speaking, denoting the state and control processes by $(X_t)_{t \in \mathbb{N}_0}, (A_t)_{t \in \mathbb{N}_0}$, respectively, $\mathcal{Q}(C \mid (x, a))$ represents the conditional probability of $\{X_1 \in C\}$ given that $\{X_0 = x, A_0 = a\}$. A convenient way to give a precise formulation of the controlled system is through canonical sample spaces (cf. [3]), as follows. Let $\Omega =$

$(\mathbb{X} \times \mathbb{A})^{\otimes\infty}$ and denote by $\mathcal{F}$ the Borel σ field on Ω corresponding to the product topology. Define sequences $\{X_t\}_{t\in\mathbb{N}_0}$, $\{A_t\}_{t\in\mathbb{N}_0}$ of $\mathbb{X}$ and $\mathbb{A}$ valued measurable maps, respectively, on $(\Omega, \mathcal{F})$ as follows:

$$X_t(\omega) = x_t;\ A_t(\omega) = a_t,\ \text{where } \omega = (x_0, a_0, \cdots x_t, a_t, \cdots),\ t \in \mathbb{N}_0.$$

We also introduce the sequence of *History maps*, $\{H_t\}_{t\in\mathbb{N}_0}$, $H_t : \Omega \to \mathbb{H}_t$, where

$$\mathbb{H}_t = (\mathbb{X} \times \mathbb{A})^{\otimes(t-1)} \times \mathbb{X},\ t \in \mathbb{N};\ \mathbb{H}_0 = \mathbb{X}$$

as $H_t(\omega) = (x_0, a_0, \cdots x_{t-1}, a_{t-1}, x_t)$. Let

$$\bar{\mathcal{H}}_t = (\mathcal{B}(\mathbb{X} \times \mathbb{A}))^{\otimes(t-1)} \otimes \mathcal{B}(\mathbb{X}),\ \text{and } \mathcal{H}_t = \sigma(H_t) = H_t^{-1}(\bar{\mathcal{H}}_t).$$

Note that $\mathcal{F} = \bigvee_{t=0}^{\infty} \mathcal{H}_t$.

By a controlled system we will mean a probability measure on $(\Omega, \mathcal{F})$ that is described in terms of an admissible control policy which is defined as follows.

Definition 5.2.1 (Admissible Control Policy)**.** A sequence $\pi = \{\pi_t\}_{t\in\mathbb{N}_0}$ of kernels, $\pi_t : \mathbb{H}_t \times \mathcal{B}(\mathbb{A}) \to [0, 1]$ satisfying for all $t \in \mathbb{N}_0$:

(i) $\pi_t(h_t, \cdot) \equiv \pi_t(\cdot \mid h_t)$ is in $\mathcal{P}(\mathbb{A})$, for all $h_t \in \mathbb{H}_t$;

(ii) $\pi_t(\cdot, D) \in \mathcal{M}(\mathbb{H}_t, \bar{\mathcal{H}}_t)$, for all $D \in \mathcal{B}(\mathbb{A})$;

(iii) $\pi_t(h_t, \mathbb{U}(x_t)) = 1$, for all $h_t = (x_0, a_0, \cdots x_{t-1}, a_{t-1}, x_t) \in \mathbb{H}_t$,

is called an admissible (control) policy.

The set of all admissible policies is denoted by Π. Given $\mu \in \mathcal{P}(\mathbb{X})$ and $\pi \in \Pi$, there is a unique probability measure $\mathbb{P}^\pi_\mu$ on $(\Omega, \mathcal{F})$ satisfying:

- $\mathbb{P}^\pi_\mu(X_0 \in C) = \mu(C)$, $C \in \mathcal{B}(\mathbb{X})$,

- $\mathbb{P}^\pi_\mu(A_t \in D \mid \mathcal{H}_t)(\omega) = \pi_t(D \mid H_t(\omega))$, $\mathbb{P}^\pi_\mu$ a.s.,

- $\mathbb{P}^\pi_\mu((X_t(\omega), A_t(\omega)) \in \mathbb{K}) = 1$ for all $t \in I\!N_0$.

- $\mathbb{P}^\pi_\mu(X_{t+1} \in C \mid H_t, A_t)(\omega) = \mathcal{Q}(C \mid X_t(\omega), A_t(\omega))$, $\mathbb{P}^\pi_\mu$ a.s.

The measure $\mathbb{P}^\pi_\mu$ represents a controlled system with initial distribution μ and an admissible control policy $\pi \in \Pi$. The corresponding expectation operator will be denoted by $\mathbb{E}^\pi_\mu$. If $\mu = \delta_x$, we will write $\mathbb{P}^\pi_\mu$ and $\mathbb{E}^\pi_\mu$ as $\mathbb{P}^\pi_x$ and $\mathbb{E}^\pi_x$, respectively.

A family of admissible policies that are particularly useful are the so-called *stationary Markov policies.* These correspond to those $\pi \in \Pi$ for which there is a measurable map $q : \mathbb{X} \to \mathcal{P}(\mathbb{A})$ such that $\pi_t(h_t, \cdot) = q(x_t)(\cdot)$ for every $h_t = (x_0, a_0, \cdots x_{t-1}, a_{t-1}, x_t) \in \mathbb{H}_t$. The class of all such policies is denoted by Π_{SM} and frequently we will identify a policy $\pi \in \Pi_{\text{SM}}$ with the associated map q. Note that for every $\mu \in \mathcal{P}(\mathbb{X})$ and $\pi \equiv q \in \Pi_{\text{SM}}$, $(X_t)_{t \in \mathbb{N}_0}$ is a Markov chain under $\mathbb{P}^\pi_\mu$ with transition probability kernel

$$\varrho_q(x, C) = \int_{\mathbb{A}} \mathcal{Q}((x, a), C) q(x, da), \ (x, C) \in \mathbb{X} \times \mathcal{B}(\mathbb{X}). \tag{5.2.1}$$

If $q \in \Pi_{\text{SM}}$ is such that the map $x \mapsto q(x)$ is continuous (from $\mathbb{X}$ to $\mathcal{P}(\mathbb{A})$), we will refer to q as a *continuous stationary Markov policy* and denote the class of all such policies by Π_{SMC}. Occasionally, for $x \in \mathbb{X}$, we will write $q(x)(\cdot)$ as $q(\cdot \mid x)$.

The next step in the formulation of a control problem is the introduction of the cost function that one will like to optimize. Here we are interested in a criterion that is designed for system optimization over a long time horizon. This criterion – usually referred to as the pathwise cost per unit time, or long time average cost – is given in terms of a measurable map $c : \mathbb{K} \to \mathbb{R}_+$, called the *one stage cost function*, as

$$J_S = \limsup_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} c(X_t, A_t), \tag{5.2.2}$$

where the right side above is a $\bar{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ valued random variable on $(\Omega, \mathcal{F})$. Under suitable conditions one can show that there is a $\pi^* \in \Pi$ and $V \in [0,\infty)$ such that, for all $\mu \in \mathcal{P}(X)$, $\mathbb{P}^{\pi^*}_{\mu}(J_S = V) = 1$ and for all $\pi \in \Pi$, $\mathbb{P}^{\pi}_{\mu}(J_S \geq V) = 1$. Such a π^* is then an optimal control for the problem. One typically finds that π^* can be taken to be an element of Π_{SM} (i.e. a stationary Markov policy). For precise conditions under which the above statements hold we refer the reader to Section 6 of [3]. In this work we are not interested in the optimization of a particular one stage cost function but rather in the study of control policies that perform well over a broad family of cost functions. In that regard the following occupation measure plays a key role.

For $N \in \mathbb{N}$, define a $\mathcal{P}(\mathbb{X}\times\mathbb{A})$ valued random variable, Φ_N as

$$\Phi_N(\omega)(F) = \frac{1}{N}\sum_{t=0}^{N-1} 1_F(X_t(\omega), A_t(\omega)),\ F \in \mathcal{B}(\mathbb{X}\times\mathbb{A}),\ \omega\in\Omega.$$

We will make the following assumptions. The first two can be regarded as blanket stability conditions while the third is the weak Feller property.

Assumption 5.2.1. *For each $\mu \in \mathcal{P}(\mathbb{X})$ and $\pi \in \Pi$, the sequence of probability measures $\{\Phi_N(\omega), N \in \mathbb{N}\}$ is tight, for $\mathbb{P}^{\pi}_{\mu}$ a.e. ω.*

If $\mathbb{X}$ and $\mathbb{A}$ are compact, the above assumption holds trivially. More generally, one can formulate conditions in terms of suitable Lyapunov functions that ensure the above almost sure tightness property. Recall that for every $\mu \in \mathcal{P}(\mathbb{X})$ and $\pi \equiv q \in \Pi_{\text{SM}}$, $(X_t)_{t\in\mathbb{N}_0}$ is a Markov chain under $\mathbb{P}^{\pi}_{\mu}$ with transition probability kernel defined by (5.2.1).

Assumption 5.2.2. *For each $q \in \Pi_{SM}$, the Markov chain with transition kernel ϱ_q has a unique invariant probability measure denoted as λ_q.*

Remark 5.2.1. Note that if $q, \tilde{q} \in \Pi_{\text{SM}}$ and $q(x) = \tilde{q}(x)$ for λ_q a.e. x, then $\lambda_q = \lambda_{\tilde{q}}$. Indeed, for $C \in \mathcal{B}(\mathbb{X})$

$$\begin{aligned}\lambda_q(C) &= \int_{\mathbb{X}} \varrho_q(x, C)\lambda_q(dx) = \int_{\mathbb{X}\times\mathbb{A}} \mathcal{Q}((x,a), C)q(x, da)\lambda_q(dx) \\ &= \int_{\mathbb{X}\times\mathbb{A}} \mathcal{Q}((x,a), C)\tilde{q}(x, da)\lambda_q(dx) = \int_{\mathbb{X}} \varrho_{\tilde{q}}(x, C)\lambda_q(dx).\end{aligned}$$

Thus λ_q is an invariant probability measure for the Markov chain with transition kernel $\varrho_{\tilde{q}}$ and consequently, from Assumption 5.2.2, $\lambda_q = \lambda_{\tilde{q}}$.

Assumption 5.2.3. *For every $f \in C_b(\mathbb{X})$, the function $(x, a) \mapsto \int_{\mathbb{X}} f(\tilde{x})\mathcal{Q}((x,a), d\tilde{x})$ is in $C_b(\mathbb{X} \times \mathbb{A})$.*

Assumptions 5.2.1 – 5.2.3 will hold throughout this work and thus will not be noted explicitly in the statement of results.

5.3 Action time sharing policies

For the rest of this work we will consider a $q \in \Pi_{\text{SMC}}$ which leads to close to optimal performance for the controlled system. Indeed, as remarked earlier, under suitable conditions on the one stage cost function, the transition kernel $\mathcal{Q}$ and spaces $(\mathbb{X}, \mathbb{A})$, one can show that an optimal control can be found in the family Π_{SM}. Under further smoothness and non-degeneracy conditions one can obtain a sequence of controls in Π_{SMC} such that the associated costs converge to that for the optimal control; in particular for every $\epsilon > 0$, we can find a ϵ-optimal control that belongs to Π_{SMC}. In applications one often encounters controls which are continuous except across some "boundary" surfaces: these may be, for example, regions where some queue is empty. Such discontinuities may be handled by re-defining the metric so that these surfaces become "isolated." However, in order to focus on the main issues, we shall not pursue this extension here. Although we will not appeal to the (near) optimality properties in our proofs, the control q considered above can be regarded as such an ϵ-optimal control. Our main goal is to construct, for a given $q \in \Pi_{\text{SMC}}$, a family of control policies that allow for much more flexibility in implementation than q and lead to the same cost value (as that for q) for a broad range of one stage cost functions.

Define $\theta_q \in \mathcal{P}(\mathbb{X} \times \mathbb{A})$ as

$$\theta_q(C \times D) = \int_C q(x)(D)\lambda_q(dx),\ C \in \mathcal{B}(\mathbb{X}),\ D \in \mathcal{B}(\mathbb{A}).$$

An immediate consequence of assumptions made in Section 5.2 is the following lemma. The result can be deduced from a more general result given in Section 5.4.2 (Lemma 5.4.2) and thus the proof is omitted.

Lemma 5.3.1. *For each $\mu \in \mathcal{P}(\mathbb{X})$ the sequence of probability measures $\{\Phi_N(\omega), N \in \mathbb{N}\}$ converges weakly, as $N \to \infty$, to θ_q, for $\mathbb{P}^q_\mu$ a.e. ω.*

Lemma 5.3.1 in particular says that, if the one stage cost function $c \in C_b(\mathbb{X} \times \mathbb{A})$, then the pathwise cost per unit time associated with q, namely J_S (see (5.2.2)), in fact exists as a limit and equals $\int_{\mathbb{X}\times\mathbb{A}} c(x,a)\theta_q(dxda)$, $\mathbb{P}^q_\mu$ a.e.

We now introduce a family of control policies that are quite flexible and are also well suited for estimation of unknown parameters and for broader information collection purposes, referred to as *action time sharing* (ATS) control policies. An ATS policy associated with q will be such that the corresponding pathwise cost per unit time is the same as that for q. Such a policy is defined in terms of a sequence of measurable partitions $\{\Lambda_k\}_{k\geq 1}$ of the state space $\mathbb{X}$:

$$\Lambda_k = \{B_{kl}\}_{l=1}^{\tau(k)},\ \mathbb{X} = \bigcup_{l=1}^{\tau(k)} B_{kl},\ B_{kl} \cap B_{kl'} = \emptyset \text{ if } l \neq l' \tag{5.3.1}$$

such that $|\Lambda_k| = \sup_{l\in R(k)} \operatorname{diam}(B_{kl}) \to 0$ as $k \to \infty$, where $R(k) = \{1, \cdots, \tau(k)\}$.

By convention, when $\tau(k) = \infty$, $R(k) = I\!N$. We refer to $\{\Lambda_k\}_{k\geq 1}$ as a sequence of *converging partitions*. Associated with such a sequence, consider a sequence of random kernels $\{p_k\}_{k\geq 1}$,

$$p_k : \Omega \times \mathbb{X} \times \mathcal{B}(\mathbb{A}) \to [0,1]$$

defined as follows: For $(\omega, x, D) \in \Omega \times \mathbb{X} \times \mathcal{B}(\mathbb{A})$ and $k \in I\!N$, fix l so that $x \in B_{kl}$. Then set

$$p_k(\omega, x, D) \equiv p_k^{\omega}(D \mid x) = \begin{cases} \frac{\sum_{j=0}^{k-1} 1_D(A_j(\omega)) 1_{B_{kl}}(X_j(\omega))}{\sum_{j=0}^{k-1} 1_{B_{kl}}(X_j(\omega))} & \text{if } \sum_{j=0}^{k-1} 1_{B_{kl}}(X_j(\omega)) \neq 0 \\ 1_{\{a_0(x) \in D\}} & \text{if } \sum_{j=0}^{k-1} 1_{B_{kl}}(X_j(\omega)) = 0 \end{cases} \tag{5.3.2}$$

where $a_0 : \mathbb{X} \to \mathbb{A}$ is an arbitrary fixed measurable function such that $a_0(x) \in \mathbb{U}(x)$ for all $x \in \mathbb{X}$.

Definition 5.3.1. Given $\mu \in \mathcal{P}(\mathbb{X})$, a policy $\pi \in \Pi$ is called an action time sharing (ATS) policy for q corresponding to the initial condition μ if for $\mathbb{P}_{\mu}^{\pi}$ a.e. ω, there is a sequence of converging partitions $\{\Lambda_k(\omega)\}_{k\geq 1}$, such that for every compact set $K \subset \mathbb{X}$

$$\sup_{x \in K} \|p_k^{\omega}(\cdot \mid x) - q(\cdot \mid x)\|_{\text{BL}} \to 0, \text{ as } k \to \infty. \tag{5.3.3}$$

We denote the collection of all ATS policies for q, corresponding to the initial condition μ, by $\Pi_{\text{ATS}}(q, \mu)$.

The following is the main result of this section.

Theorem 5.3.1. *Let $\mu \in \mathcal{P}(\mathbb{X})$. Fix $\pi \in \Pi_{ATS}(q, \mu)$. Then, as $k \to \infty$, $\Phi_k(\omega) \to \theta_q$ for $\mathbb{P}_{\mu}^{\pi}$ a.e. ω.*

Proof. From Assumption 5.2.1 we can find $\mathcal{N}_1 \in \mathcal{F}$ such that $\mathbb{P}_{\mu}^{\pi}(\mathcal{N}_1) = 0$ and for all $\omega \in \mathcal{N}_1^c$, $\{\Phi_n(\omega)\}_{n\geq 1}$ is tight. For $f \in \mathcal{S}(\mathbb{X})$, define

$$M_n^f = \sum_{j=0}^{n-1} \left[\int_{\mathbb{X}} f(\tilde{x}) \mathcal{Q}((X_j, A_j), d\tilde{x}) - f(X_{j+1}) \right].$$

Then, under $\mathbb{P}_\mu^\pi$, $\{M_n^f\}$ is a martingale with bounded increments and so by the strong law of large numbers for such martingales (see eg. [44, Theorem VII.5.4]), $\frac{1}{n} M_n^f \to 0$, a.s. $\mathbb{P}_\mu^\pi$. Let $\mathcal{N}_2 \in \mathcal{F}$ be such that $\mathbb{P}_\mu^\pi(\mathcal{N}_2) = 0$ and

$$\text{for all } \omega \in \mathcal{N}_2^c, \text{ and all } f \in \mathcal{S}(\mathbb{X}), \frac{1}{n} M_n^f(\omega) \to 0, \text{ as } n \to \infty. \tag{5.3.4}$$

Since $\mathbb{X}$ is locally compact, we can find a sequence $\{K_n\}_{n\geq 1}$ of compact subsets of $\mathbb{X}$ such that

$$K_n^o \subset K_n \subset K_{n+1}^o, \text{ and } \cup_{n\geq 1} K_n = \mathbb{X}.$$

Since $\pi \in \Pi_{\text{ATS}}(q)$, we can find a $\mathcal{N}_3 \in \mathcal{F}$ such that $\mathbb{P}_\mu^\pi(\mathcal{N}_3) = 0$ and, for each $\omega \in \mathcal{N}_3^c$, a sequence $\{\Lambda_k(\omega)\}_{k\geq 1}$ of converging partitions for which, as $k \to \infty$,

$$\sup_{x\in K_n} \left| \int_{\mathbb{A}} g(a) p_k^\omega(da \mid x) - \int_{\mathbb{A}} g(a) q(da \mid x) \right| \to 0, \text{ for every } n \geq 1 \text{ and } g \in \mathcal{S}(\mathbb{A}), \tag{5.3.5}$$

where p_k is defined through (5.3.2). Now let $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ and fix $\omega \in \mathcal{N}^c$. Choose a subsequence $\{n_k\}$ along which $\Phi_{n_k}(\omega)$ converges to some $\Phi(\omega) \in \mathcal{P}(\mathbb{X} \times \mathbb{A})$. Suppressing ω in notation, the measure Φ can be disintegrated as follows: For some $\gamma \in \mathcal{P}(\mathbb{X})$ and a transition probability kernel $\hat{p} : \mathbb{X} \times \mathcal{B}(\mathbb{A}) \to [0, 1]$, $\Phi(dx\, da) = \hat{p}(x, da)\gamma(dx)$, namely

$$\Phi(C \times D) = \int_C \hat{p}(x, D)\gamma(dx), \text{ for all } C \in \mathcal{B}(\mathbb{X}), D \in \mathcal{B}(\mathbb{A}). \tag{5.3.6}$$

Note that $\hat{p}(\cdot \mid x) \equiv \hat{p}(x, \cdot) \in \Pi_{\text{SM}}$. We claim that

$$\gamma = \lambda_{\hat{p}}. \tag{5.3.7}$$

To prove the claim it suffices, in view of Assumption 5.2.2, to show that for all $f \in \mathcal{S}(\mathbb{X})$

$$\int_{\mathbb{X}\times\mathbb{A}} \left(\int_{\mathbb{X}} f(\tilde{x}) \mathcal{Q}((x,a), d\tilde{x}) \right) \hat{p}(da \mid x)\gamma(dx) = \int_{\mathbb{X}} f(x)\gamma(dx). \tag{5.3.8}$$

Note that the right side of (5.3.8) equals the limit (as $k \to \infty$) of $\frac{1}{n_k}\sum_{j=0}^{n_k-1} f(X_j(\omega))$, while left side equals (using Assumption 5.2.3) the limit of

$$\frac{1}{n_k}\sum_{j=0}^{n_k-1} \int_{\mathbb{X}} f(\tilde{x}) \mathcal{Q}((X_j, A_j), d\tilde{x}).$$

Also, from (5.3.4), the difference of the above two quantities approaches 0 as $k \to \infty$. This proves (5.3.8) and thus (5.3.7) follows. To complete the proof of the theorem we will now show that for every $f \in \mathcal{S}(\mathbb{X})$ and $g \in \mathcal{S}(\mathbb{A})$

$$\int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)q(da \mid x)\lambda_{\hat{p}}(dx) = \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)\hat{p}(da \mid x)\lambda_{\hat{p}}(dx). \tag{5.3.9}$$

This will prove that $q(\cdot \mid x) = \hat{p}(\cdot \mid x)$, a.e. x $[\lambda_{\hat{p}}]$, and consequently, from Remark 5.2.1, $\lambda_{\hat{p}} = \lambda_q$. Now fix a $(f,g) \in \mathcal{S}(\mathbb{X}) \times \mathcal{S}(\mathbb{A})$. We first show that

$$\lim_{k\to\infty} \left| \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)p_{n_k}(da \mid x)\Phi^{(1)}_{n_k}(dx) - \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)q(da \mid x)\lambda_{\hat{p}}(dx) \right| = 0, \tag{5.3.10}$$

where $\Phi^{(1)}_{n_k}$ is the first marginal of Φ_{n_k}. Let

$$\phi_k(x) = \int_{\mathbb{A}} g(a)p_{n_k}(da \mid x),\ \phi(x) = \int_{\mathbb{A}} g(a)q(da \mid x),\ x \in \mathbb{X}.$$

Since $\omega \in \mathcal{N}_3^c$, we have (see (5.3.5)) that for every compact K in $\mathbb{X}$

$$\sup_{x \in K} |\phi_k(x) - \phi(x)| \to 0, \text{ as } k \to \infty.$$

Also, from (5.3.7)

$$\Phi_{n_k}^{(1)} \to \gamma = \lambda_{\hat{p}}.$$

Since $q \in \Pi_{\text{SMC}}$, $\phi \in C_b(\mathbb{X})$ and thus combining the above two displays, we have, as $k \to \infty$,

$$\int_{\mathbb{X}} f(x)\phi_k(x)\Phi_{n_k}^{(1)}(dx) \to \int_{\mathbb{X}} f(x)\phi(x)\lambda_{\hat{p}}(dx).$$

This proves (5.3.10). We now show

$$\lim_{k\to\infty} \left| \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)p_{n_k}(da \mid x)\Phi_{n_k}^{(1)}(dx) - \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)\Phi_{n_k}(da\, dx) \right| = 0. \quad (5.3.11)$$

Suppressing ω from the notation, suppose that $\Lambda_k(\omega) \equiv \Lambda_k$ is given as in (5.3.1). Along with the sequence $\{\Lambda_k\}_{k\geq 1}$ we consider a sequence of sets

$$\mathbb{X}_k = \{x_{k1}, \cdots x_{k\tau(k)}\} \subset \mathbb{X},\ k \geq 1 \quad (5.3.12)$$

such that $x_{kl} \in B_{kl}$ for all $l = 1, \cdots \tau(k)$. We will refer to x_{kl} as the *center* of the set B_{kl}. Define, for $k \geq 1$, $b_k : \mathbb{X} \to \mathbb{X}$ as

$$b_k(x) = \sum_{l=1}^{\tau(k)} x_{kl} 1_{B_{kl}}(x),\ x \in \mathbb{X}.$$

Fix $\epsilon > 0$. Since f is uniformly continuous and $|\Lambda_n| \to 0$ as $n \to \infty$, we can find $n_0 \in \mathbb{N}$ such that

$$\sup_{l \in R(n)} \sup_{x,y \in B_{nl}} |f(x) - f(y)| < \epsilon,\ \text{for all } n \geq n_0. \quad (5.3.13)$$

Fix $k_0 \in I\!N$ such that $n_k \geq n_0$ whenever $k \geq k_0$. For $k \geq k_0$

$$\int_{\mathbb{A}} g(a) p_{n_k}(da \mid x) = \frac{\sum_{j=0}^{n_k-1} g(A_j) 1_{\{b_{n_k}(x)\}}(b_{n_k}(X_j))}{\sum_{j=0}^{n_k-1} 1_{\{b_{n_k}(x)\}}(b_{n_k}(X_j))}.$$

This shows that

$$\int_{\mathbb{X}\times\mathbb{A}} f(x) g(a) p_{n_k}(da \mid x) \Phi^1_{n_k}(dx) = \frac{1}{n_k} \sum_{i=0}^{n_k-1} f(X_i) \frac{\sum_{j=0}^{n_k-1} g(A_j) 1_{\{b_{n_k}(X_i)\}}(b_{n_k}(X_j))}{\sum_{j=0}^{n_k-1} 1_{\{b_{n_k}(X_i)\}}(b_{n_k}(X_j))}.$$

Note that $b_{n_k}(X_j) = b_{n_k}(X_i)$ if and only if X_j and X_i are in the same $B_{n_k l}$ and in that case, whenever $k \geq k_0$, $|f(X_i) - f(X_j)| \leq \epsilon$. Using this observation the right side of the above display can be written as

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \frac{\sum_{j=0}^{n_k-1} f(X_j) g(A_j) 1_{\{b_{n_k}(X_i)\}}(b_{n_k}(X_j))}{\sum_{j=0}^{n_k-1} 1_{\{b_{n_k}(X_i)\}}(b_{n_k}(X_j))} + \varpi(k),$$

where $|\varpi(k)| \leq \epsilon \sup_{a\in\mathbb{A}} |g(a)|$ for $k \geq k_0$. The first term in the display can be written as

$$\begin{aligned}
& \frac{1}{n_k} \sum_{i=0}^{n_k-1} \sum_{l=1}^{\tau(n_k)} 1_{B_{n_k l}}(X_i) \frac{\sum_{j=0}^{n_k-1} f(X_j) g(A_j) 1_{\{x_{n_k l}\}}(b_{n_k}(X_j))}{\sum_{j=0}^{n_k-1} 1_{\{x_{n_k l}\}}(b_{n_k}(X_j))} \\
= \; & \frac{1}{n_k} \sum_{l=1}^{\tau(n_k)} \left(\sum_{i=0}^{n_k-1} 1_{B_{n_k l}}(X_i) \right) \frac{\sum_{j=0}^{n_k-1} f(X_j) g(A_j) 1_{\{x_{n_k l}\}}(b_{n_k}(X_j))}{\sum_{j=0}^{n_k-1} 1_{B_{n_k l}}(X_j)} \\
= \; & \frac{1}{n_k} \sum_{l=1}^{\tau(n_k)} \sum_{j=0}^{n_k-1} f(X_j) g(A_j) 1_{\{x_{n_k l}\}}(b_{n_k}(X_j)) \\
= \; & \frac{1}{n_k} \sum_{j=0}^{n_k-1} f(X_j) g(A_j) \\
= \; & \int_{\mathbb{X}\times\mathbb{A}} f(x) g(a) \Phi_{n_k}(da\, dx).
\end{aligned}$$

Thus for $k \geq k_0$

$$\left| \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)p_{n_k}(da \mid x)\Phi^1_{n_k}(dx) - \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)\Phi_{n_k}(da\, dx) \right| \leq |\varpi(k)| \leq \epsilon \sup_{a\in\mathbb{A}} |g(a)|.$$

Since $\epsilon > 0$ is arbitrary, this proves (5.3.11). Combining (5.3.10) and (5.3.11) we have

$$\lim_{k\to\infty} \left| \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)\Phi_{n_k}(da\, dx) - \int_{\mathbb{X}\times\mathbb{A}} f(x)g(a)q(da \mid x)\lambda_{\hat{p}}(dx) \right| = 0.$$

The above display, along with (5.3.6) and (5.3.7) yields (5.3.9) and, as noted above (5.3.9), shows that $q(\cdot \mid x) = \hat{p}(\cdot \mid x)$, a.e. x $[\lambda_{\hat{p}}]$, and $\lambda_{\hat{p}} = \lambda_q$. Thus $\Phi = \theta_q$ and the result follows. ■

As a immediate corollary of the above theorem and Lemma 5.3.1 we have to following result on the convergence of costs. The result says that for a broad family of one stage cost functions, the pathwise cost per unit time for q is same as that for any $\pi \in \Pi_{\text{ATS}}(q)$.

Corollary 5.3.1. *Let $\mu \in \mathcal{P}(\mathbb{X})$ and $\pi \in \Pi_{ATS}(q, \mu)$. Then for any $c \in C_b(\mathbb{X} \times \mathbb{A})$, J_S defined by* (5.2.2) *in fact exists as a limit and equals $\int c(x, a)\theta_q(dxda)$, both, a.e. $\mathbb{P}^\pi_\mu$ and $\mathbb{P}^q_\mu$.*

5.4 Construction of ATS policies

In this section we will give a basic construction for a $\pi \in \Pi_{\text{ATS}}(q, \mu)$ for an arbitrary $q \in \Pi_{\text{SMC}}$ and $\mu \in \mathcal{P}(\mathbb{X})$. We will then describe how this construction can be modified in a simple manner to define control policies that are well suited for estimation and information collection purposes while producing the same value for the pathwise cost per unit time. To keep the presentation simple we assume that $\mathbb{U}(x) = \mathbb{A}$ and that $\mathbb{A}$ is a compact metric space. We will further make the following recurrence assumption.

Assumption 5.4.1. *For every $\pi \in \Pi$, $\mu \in \mathcal{P}(\mathbb{X})$, and $C \in \mathcal{B}(\mathbb{X})$ with a nonempty interior,*

$$\mathbb{P}_\mu^\pi(X_t \in C, \text{ for some } t \in \mathbb{N}) = 1.$$

The above assumption will hold throughout this section.

We begin with the following lemma. Let

$$\Theta = \{\vartheta \in \mathcal{P}(\mathbb{A}) : \vartheta \text{ is supported on finitely many points .}\} \tag{5.4.1}$$

For $\vartheta \in \Theta$, denote by $S(\vartheta)$ the support of ϑ.

Lemma 5.4.1. *There is a $\Psi \equiv (\Psi_1, \cdots) : \Theta \to \mathbb{A}^\infty$ such that for every $\vartheta \in \Theta$: (i) $\Psi_i(\vartheta) \in S(\vartheta)$, $i \geq 1$; (ii) The probability measure $m_n(\vartheta) = \frac{1}{n}\sum_{i=1}^n \delta_{\Psi_i(\vartheta)}$ satisfies*

$$||m_n(\vartheta) - \vartheta||_{BL} \leq \frac{4\ \#(S(\vartheta))}{n}$$

where $\#(S(\vartheta))$ is the cardinality of $S(\vartheta)$.

Proof. Fix $\vartheta \in \Theta$. Then ϑ can be written as

$$\vartheta = \sum_{j=1}^{l} p_j \delta_{a_j},$$

where $l = \#(S(\vartheta)) \in \mathbb{N}$, $a_j \in \mathbb{A}$, $p_j \in (0, 1]$, and $\sum_{j=1}^l p_j = 1$.

Define, for $m \in \mathbb{N}$ and $j = 1, \ldots, l$,

$$k_j(m) = \lfloor mlp_j \rfloor, \text{ and } \alpha(m) = \sum_{j=1}^{l} k_j(m).$$

Set $\alpha(0) = 0$. It is easily seen that

$$(m-1)l \leq \alpha(m) \leq ml, \tag{5.4.2}$$

and so $\alpha(m) \to \infty$ as $m \to \infty$.

We now define a sequence $\{\psi_j\}_{j=1}^{\infty}$ with values in $\mathbb{A}$, such that, for each $m \geq 1$ and $r = 1, \ldots, l$,

$$\#\{j \in \{1, \ldots, \alpha(m)\} : \psi_j = a_r\} = k_r(m). \tag{5.4.3}$$

One can define $\{\psi_j\}_{j=1}^{\infty}$ inductively as follows.

Consider $m = 1$. Define

$$\psi_j = a_r \text{ whenever } \sum_{i=1}^{r-1} k_i(1) < j \leq \sum_{i=1}^{r} k_i(1), r = 1, \cdots l.$$

This defines $\{\psi_j\}_{j=1}^{\alpha(1)}$. Suppose now $\{\psi_j\}_{j=1}^{\alpha(N)}$ has been defined such that (5.4.3) holds with $m = N$. Assume without loss of generality that $\alpha(N+1) > \alpha(N)$. We now define $\{\psi_j\}_{j=\alpha(N)+1}^{\alpha(N+1)}$. Note that $k_r(N+1) \geq k_r(N)$. Let $b_r(N+1) = k_r(N+1) - k_r(N)$, and set

$$\psi_j = a_r \text{ whenever } \alpha(N) + \sum_{i=1}^{r-1} b_i(N+1) < j \leq \alpha(N) + \sum_{i=1}^{r} b_i(N+1), r = 1, \cdots l.$$

This completes the definition of $\{\psi_j\}_{j=1}^{\alpha(N+1)}$.

Define $\Psi_j(\vartheta) = \psi_j, j \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that $\alpha(N) \leq n \leq \alpha(N+1)$ for some $N \in \mathbb{N}$. If $N = 0$,

$$\|m_n(\vartheta) - \vartheta\|_{\text{BL}} = \left\| \frac{1}{n} \sum_{j=1}^{n} \delta_{\psi_j} - \sum_{i=1}^{l} p_i \delta_{a_i} \right\|_{\text{BL}} = \sup_{f \in \mathcal{C}_1(\mathbb{A})} \left| \frac{1}{n} \sum_{j=1}^{n} f(\psi_j) - \sum_{i=1}^{l} p_i f(a_i) \right| \leq 2 \leq \frac{2l}{n},$$

where the last inequality follows from (5.4.2). Consider now the case $N \geq 1$. Then

$$\|m_n(\vartheta) - \vartheta\|_{\text{BL}} = \left\| \frac{1}{n} \sum_{j=1}^{n} \delta_{\psi_j} - \sum_{i=1}^{l} p_i \delta_{a_i} \right\|_{\text{BL}}$$

$$\leq \sup_{f\in\mathcal{C}_1(\mathbb{A})} \left(\left| \frac{1}{n}\sum_{i=1}^{l} k_i(N) f(a_i) - \sum_{i=1}^{l} p_i f(a_i) \right| + \left| \frac{1}{n} \sum_{j=\alpha(N)+1}^{n} f(\psi_j) \right| \right)$$

$$\leq \sum_{i=1}^{l} \left| \frac{k_i(N) - np_i}{n} \right| + \left| \frac{\alpha(N+1) - \alpha(N)}{n} \right|. \qquad (5.4.4)$$

By (5.4.2),

$$\alpha(N+1) - \alpha(N) \leq (N+1)l - (N-1)l = 2l.$$

Also, for $j = 1, \cdots l$,

$$k_j(N) - np_j \leq Nlp_j - np_j \leq Nlp_j - \alpha(N)p_j \leq Nlp_j - (N-1)lp_j \leq lp_j,$$

and

$$k_j(N) - np_j \geq Nlp_j - 1 - \alpha(N+1)p_j \geq Nlp_j - 1 - (N+1)lp_j = -1 - lp_j.$$

Using the above estimate in (5.4.4) we now have

$$\|m_n(\vartheta) - \vartheta\|_{\text{BL}} \leq \frac{4l}{n}.$$

The lemma follows. ■

5.4.1 A basic construction

Fix $q \in \Pi_{\text{SMC}}$ and $\mu \in \mathcal{P}(\mathbb{X})$. We now give a pathwise construction of a $\pi \in \Pi_{\text{ATS}}(q, \mu)$. Let $\{\tilde{\Lambda}_k\}_{k\geq 1}$ be a sequence of measurable partitions of $\mathbb{X}$:

$$\tilde{\Lambda}_k = \{\tilde{B}_{kl}\}_{l=1}^{\tilde{\tau}(k)},\ \mathbb{X} = \bigcup_{l=1}^{\tilde{\tau}(k)} \tilde{B}_{kl},\ \tilde{B}_{kl} \cap \tilde{B}_{kl'} = \emptyset \text{ if } l \neq l' \qquad (5.4.5)$$

such that $|\tilde{\Lambda}_k| = \sup_{l \in \tilde{R}(k)} \operatorname{diam}(\tilde{B}_{kl}) \to 0$ as $k \to \infty$, where $\tilde{R}(k) = \{1, \cdots, \tilde{\tau}(k)\}$. Each $\tilde{B}_{kl}$ is required to have a nonempty interior. Also, we assume that the sequence $\tilde{\Lambda}_k$ is nested, namely, for every $k \geq 1$ and $l \in \tilde{R}(k+1)$, there is a $l' \in \tilde{R}(k)$ such that $\tilde{B}_{(k+1)l} \subset \tilde{B}_{kl'}$. We also assume that for any compact $K \subset \mathbb{X}$ and $k \geq 1$,

$$\#\{l : \tilde{B}_{kl} \bigcap K \neq \emptyset\} < \infty.$$

Associated with the sequence $\{\tilde{\Lambda}_k\}$, we define sets $\{\tilde{\mathbb{X}}_k\}$ and maps $\{\tilde{b}_k\}$ analogous to as below (5.3.11). Namely, for $k \geq 1$

$$\tilde{\mathbb{X}}_k = \{\tilde{x}_{k1}, \cdots \tilde{x}_{k\tilde{\tau}(k)}\} \subset \mathbb{X}, \tag{5.4.6}$$

is such that $\tilde{x}_{kl} \in \tilde{B}_{kl}$ for all $l \in \tilde{R}(k)$ and $\tilde{b}_k : \mathbb{X} \to \mathbb{X}$ is given as

$$\tilde{b}_k(x) = \sum_{l \in \tilde{R}(k)} \tilde{x}_{kl} 1_{\tilde{B}_{kl}}(x),\ x \in \mathbb{X}.$$

As before, $\tilde{x}_{kl}$ is called the *center* of the set $\tilde{B}_{kl}$. Since $x \mapsto q(\cdot \mid x)$ is a continuous map from $\mathbb{X}$ to $\mathcal{P}(\mathbb{A})$, we have that for every compact $K \subset \mathbb{X}$

$$\sup_{x \in K} ||q(\cdot \mid x) - q(\cdot \mid \tilde{b}_k(x))||_{\mathrm{BL}} \to 0,\ \text{as } k \to \infty. \tag{5.4.7}$$

Next let $\{\Lambda'_k\}_{k \geq 1}$ be a sequence of measurable partitions of $\mathbb{A}$:

$$\Lambda'_k = \{F_{km}\}_{m=1}^{\ell(k)},\ \mathbb{A} = \bigcup_{m=1}^{\ell(k)} F_{km},\ F_{km} \cap F_{km'} = \emptyset \text{ if } m \neq m' \tag{5.4.8}$$

such that $\ell(k) < \infty$ for all k and $|\Lambda'_k| \to 0$ as $k \to \infty$. Define a sequence of finite sets $\mathbb{A}_k = \{a_{k1}, \cdots, a_{k\ell(k)}\}$ such that $a_{km} \in F_{km}$ for all $m = 1, \cdots \ell(k)$. Let, for $k \geq 1$,

$b'_k : \mathbb{A} \to \mathbb{A}_k$ be defined as

$$b'_k(a) = \sum_{m=1}^{\ell(k)} a_{km} 1_{\tilde{F}_{km}}(a), \; a \in \mathbb{A}.$$

We will now construct a sequence of $\mathbb{X} \times \mathbb{A}$ valued random variables $Z \equiv (\bar{X}_t, \bar{A}_t)_{t \in \mathbb{N}_0}$ on a suitable probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that $\bar{X}_0$ has probability law μ and the probability law of Z corresponds to a controlled system associated with a policy $\pi \in \Pi_{\text{ATS}}(q, \mu)$. More precisely, denoting the measure induced by Z on $(\Omega, \mathcal{F})$, by $\mathbb{P}^*$ (i.e. $\mathbb{P}^* = \bar{\mathbb{P}} \circ Z^{-1}$), we will obtain an admissible control policy $\pi = \{\pi_t\}_{t \in I\!N_0}$ by disintegrating, for $t \in I\!N_0$, the measure $\bar{\mathbb{P}}_t = \mathbb{P}^* \circ (H_t, A_t)^{-1} \in \mathcal{P}(\mathbb{H}_t \times \mathbb{A})$, as

$$\bar{\mathbb{P}}_t(dh, da) = \pi_t(h, da) \, \mathbb{P}^* \circ H_t^{-1}(dh). \tag{5.4.9}$$

Note that with π defined through the above equation, we have that the controlled system $\mathbb{P}^\pi_\mu = \mathbb{P}^*$. The construction of $(\bar{X}_t, \bar{A}_t)_{t \in \mathbb{N}_0}$ will be carried out in a recursive fashion such that

$$\mathbb{P}(\bar{X}_{t+1} \in C \mid (\bar{X}_j, \bar{A}_j), j \le t) = \mathcal{Q}((\bar{X}_t, \bar{A}_t), C), \; C \in \mathcal{B}(\mathbb{X}), \; t \in I\!N_0.$$

The recursive construction of the sequence $(\bar{A}_t)$ is described in what follows.

Let $\{K_n\}_{n \ge 1}$ be the sequence of compact sets in $\mathbb{X}$ introduced in Section 5.3. Let, for $r \ge 1$, by relabeling sets if needed,

$$\tilde{\Lambda}^0_r = \{\tilde{B}_{r1}, \cdots \tilde{B}_{rj(r)}\} \subset \tilde{\Lambda}_r$$

be the finite collection of sets such that $\tilde{B}_{rm} \in \tilde{\Lambda}^0_r$ if and only if $\tilde{B}_{rm} \bigcap K_r$ is non-empty. For $m = 1, \cdots, j(r)$, define $q^{r,m} \in \mathcal{P}(\mathbb{A})$ as $q^{r,m} = q(\cdot \mid x_{rm})$. Define, for

$r \geq 1$, $\eta_r : \mathcal{P}(\mathbb{A}) \to \mathcal{P}(\mathbb{A}_r)$ as

$$\eta_r(\vartheta) = \sum_{j=1}^{\ell(r)} \delta_{a_{rj}} \vartheta(F_{rj}),\ \vartheta \in \mathcal{P}(\mathbb{A}).$$

Note that

$$\sup_{\vartheta \in \mathcal{P}(\mathbb{A})} ||\eta_r(\vartheta) - \vartheta||_{\mathrm{BL}} \leq |\Lambda'_r| \to 0, \text{ as } r \to \infty. \tag{5.4.10}$$

Set $\tilde{q}^{r,m} = \eta_r(q^{r,m})$, $r \geq 1$. Note that $\tilde{q}^{r,m} \in \Theta$ (cf. 5.4.1) for all $r \in \mathbb{N}$, $m \leq j(r)$. Denote

$$\Psi(\tilde{q}^{r,m}) = (e^r[m,1], e^r[m,2], \cdots).$$

Note that by definition of Ψ, $e^r[m,i] \in \mathbb{A}_r$ for all $i, r \in \mathbb{N}$, $m \leq j(r)$. Furthermore, from Lemma 5.4.1, for every $N \geq 1$,

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \delta_{e^r[m,i]} - \tilde{q}^{r,m} \right\|_{\mathrm{BL}} \leq \frac{4\ell(r)}{N}.$$

The sequences $\Psi(\tilde{q}^{r,m})$, $m \leq j(r)$, $r \in I\!N$, will form the basic building blocks for the sequence $(\bar{A}_t)_{t \in I\!N_0}$. Let $\{\varepsilon_r\}_{r \geq 1}$ be a sequence of positive reals such that $\varepsilon_r \downarrow 0$ as $r \to \infty$.

Construction of Z. We are now ready to specify the sequence $(\bar{X}_t, \bar{A}_t)$ on a suitable probability space. The definition of the probability space will be implicit in the construction and a detailed description will be omitted. Let $\bar{X}_0$ be a $\mathbb{X}$ valued random variable with probability law μ.

We now define, recursively in r, sequences $\{\xi^r_k, s^r_k, \zeta^r_k, (i^r[m,k])_{m=1,\cdots j(r)}\}_{k \geq 0}$, $r \geq 1$, as follows.

Case $r = 1$: Define $\xi^r_0 = \bar{X}_0$ and let

$$
\begin{aligned}
&i^r[m,0] = 1_{\tilde{B}_{r,m}}(\xi_0^r),\ m = 1,\cdots j(r),\\
&m_0^r = \sum_{m=1}^{j(r)} m 1_{\tilde{B}_{r,m}}(\xi_0^r),\ s_0^r = i^r[m_0^r, 0],\ \text{and}\ \zeta_0^r = e^r[m_0^r, s_0^r].
\end{aligned}
\tag{5.4.11}
$$

Note that $s_0^r = 1$. Having defined $\{\xi_k^r, s_k^r, \zeta_k^r, (i^r[m,k])_{m=1,\cdots j(r)}\}$ for $k \le k_0$, define $\xi_{k_0+1}^r$ through the relation

$$
\bar{\mathbb{P}}(\xi_{k_0+1}^r \in C \mid \mathcal{G}_{k_0}^r) = \mathcal{Q}((\xi_{k_0}^r, \zeta_{k_0}^r), C),\ C \in \mathcal{B}(\mathbb{X}),
\tag{5.4.12}
$$

where $\mathcal{G}_{k_0}^r = \sigma\{(\xi_j^r, \zeta_j^r) : j \le k_0\}$, and set

$$
i^r[m, k_0+1] = i^r[m, k_0] + 1_{\tilde{B}_{r,m}}(\xi_{k_0+1}^r),\ m = 1,\cdots j(r),\ m_{k_0+1}^r = \sum_{m=1}^{j(r)} m 1_{\tilde{B}_{r,m}}(\xi_{k_0+1}^r),
\tag{5.4.13}
$$

and

$$
s_{k_0+1}^r = i^r[m_{k_0+1}^r, k_0+1],\ \zeta_{k_0+1}^r = e^r[m_{k_0+1}^r, s_{k_0+1}^r].
\tag{5.4.14}
$$

This completes the definition for $\{\xi_k^r, s_k^r, \zeta_k^r, (i^r[m,k])_{m=1,\cdots j(r)}\}$ for $r = 1$ and $k \in \mathbb{N}_0$.

Set $\varrho_0 = 0$ and define, for $r = 1$,

$$
\alpha_r = \varepsilon_r^{-1}\left(2\varrho_{r-1} + 4\left(\ell(r) + \ell(r+1)\right)\right),
\tag{5.4.15}
$$

$$
\sigma_r = \inf\{k : i^r[m,k] \ge \alpha_r \text{ for all } m = 1,\cdots j(r)\},
\tag{5.4.16}
$$

$$
\varrho_r = \varrho_{r-1} + \sigma_r
\tag{5.4.17}
$$

Case $r > 1$: Let

$$
(\xi_0^r, \zeta_0^r) = (\xi_{\sigma_{r-1}}^{r-1} \zeta_{\sigma_{r-1}}^{r-1}),\ i^r[m,0] = 0, m = 1,\cdots j(r).
$$

Definition of $\{\xi_k^r, \zeta_k^r, s_k^r, (i^r[m,k])_{m=1,\cdots j(r)}\}_{k\geq 1}$ and $(\alpha_r, \sigma_r, \varrho_r)$, for $r > 1$, is given recursively, exactly as above through (5.4.12) – (5.4.17).

Finally, the sequence $(\bar{X}_k, \bar{A}_k)$ is now constructed on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ that supports the random variables $\{\xi_k^r, \zeta_k^r, s_k^r, (i^r[m,k])_{m=1,\cdots j(r)}\}_{k\geq 0}$, $(\alpha_r, \sigma_r, \varrho_r)$, $r \in I\!N$, by piecing together the sequence $(\xi_k^r, \zeta_k^r; k, r \in I\!N_0)$ as follows,

$$(\bar{X}_k, \bar{A}_k) = (\xi_{k-\varrho_r}^{r+1}, \zeta_{k-\varrho_r}^{r+1}), \text{ whenever } \varrho_r \leq k < \varrho_{r+1}, \; r \in \mathbb{N}_0.$$

Recall from (5.4.9) the definition of π and $\mathbb{P}_\mu^\pi$ corresponding to the sequence $(\bar{X}_k, \bar{A}_k)_{k\in\mathbb{N}_0}$. We now show that π constructed in the above fashion is an ATS policy for q with initial condition μ.

Theorem 5.4.1. *The policy $\pi \in \Pi$ constructed above is in $\Pi_{ATS}(q,\mu)$.*

Proof. From Assumption 5.4.1 it follows that, with $\bar{\Omega}_0 = \{\omega \in \bar{\Omega} : \varrho_r(\omega) < \infty \text{ for all } r \geq 1\}$, $\bar{\mathbb{P}}(\bar{\Omega}_0) = 1$. Define for $\omega \in \bar{\Omega}_0$, $(x, D) \in \mathbb{X} \times \mathcal{B}(\mathbb{A})$, $\bar{p}_k(\omega, x, D) \equiv \bar{p}_k^\omega(D \mid x)$ by the right side of (5.3.2), replacing (A_j, X_j) there by $(\bar{A}_j, \bar{X}_j)$ and $\{\Lambda_k(\omega)\}$ (suppressing ω from notation throughout) defined as follows: For $k \geq 1$,

$$\Lambda_k = \tilde{\Lambda}_\beta \text{ if } \varrho_\beta < k \leq \varrho_{\beta+1}, \beta = 0, 1, \cdots$$

where $\tilde{\Lambda}_0$ is taken to be $\tilde{\Lambda}_1$. In order to prove the result, it suffices to show that for all $\omega \in \bar{\Omega}_0$ and compact $K \subset \mathbb{X}$

$$\sup_{x\in K} \|\bar{p}_k^\omega(\cdot \mid x) - q(\cdot \mid x)\|_{\text{BL}} \to 0, \text{ as } k \to \infty. \tag{5.4.18}$$

Fix now a compact set $K \subset \mathbb{X}$ and $\epsilon \in (0,1)$. Using (5.4.7) and (5.4.10), choose

r_0 large enough so that for all $r \geq r_0$, $K \subset K_r$,

$$\sup_{x \in K} \|q(\cdot \mid x) - q(\cdot \mid \tilde{b}_r(x))\|_{\text{BL}} \leq \epsilon \tag{5.4.19}$$

and

$$\sup_{\vartheta \in \mathcal{P}(\mathbb{A})} \|\vartheta - \eta_r(\vartheta)\|_{\text{BL}} \leq \epsilon. \tag{5.4.20}$$

We introduce some additional notation. For $t \geq 1$ and $l = 1, \cdots j(t)$, let

$$n_{tl}(m_1, m_2) = \#\{\bar{X}_j \in \tilde{B}_{tl} : m_1 \leq j < m_2\}, \ 0 \leq m_1 \leq m_2 < \infty$$

and for such m_1, m_2, let $\mu_{tl}[m_1, m_2] \in \mathcal{P}(\mathbb{A})$ be defined as follows: For $D \in \mathcal{B}(\mathbb{A})$,

$$\mu_{tl}[m_1, m_2](D) = \begin{cases} n_{tl}(m_1, m_2)^{-1} \sum_{j=m_1}^{m_2-1} 1_D(\bar{A}_j) 1_{\tilde{B}_{tl}}(\bar{X}_j), & \text{if } n_{tl}(m_1, m_2) > 0, \\ \\ \delta_{a_0}(D), & \text{otherwise,} \end{cases}$$

where a_0 is some fixed element of $\mathbb{A}$.

Fix $\beta_0 \geq r_0 + 1$ and consider $k > \varrho_{\beta_0}$. Let $\beta \in \mathbb{N}$, $\beta \geq \beta_0$ be such that $\varrho_\beta < k \leq \varrho_{\beta+1}$. We will now estimate the quantity on the left side of (5.4.18) for such a k. Fix $x \in K$ and let $i \in \{1, \cdots j(\beta)\}$ be such that $x \in \tilde{B}_{\beta i}$. Since $B_{ki} = \tilde{B}_{\beta i}$ for $\varrho_\beta < k \leq \varrho_{\beta+1}$, we can write

$$\bar{p}_k(\cdot \mid x) = n^{-1}(n_1\nu_1 + n_2\nu_2 + n_3\nu_3), \tag{5.4.21}$$

where

$$\nu_1 = \mu_{\beta i}[0, \varrho_{\beta-1}], \ \nu_2 = \mu_{\beta i}[\varrho_{\beta-1}, \varrho_\beta], \ \nu_3 = \mu_{\beta i}[\varrho_\beta, k]$$

and

$$n_1 = n_{\beta i}(0, \varrho_{\beta-1}),\ n_2 = n_{\beta i}(\varrho_{\beta-1}, \varrho_\beta),\ n_3 = n_{\beta i}(\varrho_\beta, k),\ n = n_1 + n_2 + n_3.$$

Recall that the sequence $\{\tilde{\Lambda}_k\}$ is nested. Denote the sets in $\tilde{\Lambda}_{\beta+1}$ that are contained in $\tilde{B}_{\beta i}$ as $G_1, G_2, \cdots G_\gamma$ and denote the corresponding centers by $g_1, \cdots, g_\gamma$. Let, for $t = 1, \cdots \gamma$, $m_t = \#\{X_j \in G_t : \varrho_\beta \le j < k\}$. Then

$$\sum_{t=1}^{\gamma} m_t = n_3 \text{ and } \nu_3 = n_3^{-1} \sum_{t=1}^{\gamma} m_t \nu_{3t}, \tag{5.4.22}$$

where

$$\nu_{3t}(D) = \begin{cases} m_t^{-1} \sum_{j=\varrho_\beta}^{k-1} 1_D(\bar{A}_j) 1_{G_t}(\bar{X}_j), & \text{if } m_t > 0, \\ \\ \delta_{a_0}(D), & \text{otherwise.} \end{cases}$$

Then, whenever $m_t \neq 0$,

$$\|\nu_{3t} - \eta_{\beta+1}(q(\cdot \mid g_t))\|_{\mathrm{BL}} \le \frac{4\ell(\beta+1)}{m_t}. \tag{5.4.23}$$

Also, since $\beta > r_0$, from (5.4.20), whenever $n_3 > 0$,

$$\|\tilde{\nu}_3 - n_3^{-1} \sum_{t=1}^{\gamma} m_t q(\cdot \mid g_t)\|_{\mathrm{BL}} \le \epsilon$$

where $\tilde{\nu}_3 = n_3^{-1} \sum_{t=1}^{\gamma} m_t \eta_{\beta+1}(q(\cdot \mid g_t))$, and from (5.4.19)

$$\|n_3^{-1} \sum_{t=1}^{\gamma} m_t q(\cdot \mid g_t) - q(\cdot \mid x)\|_{\mathrm{BL}} \le 2\epsilon.$$

Thus, whenever $n_3 > 0$, $\|\tilde{\nu}_3 - q(\cdot \mid x)\|_{\mathrm{BL}} \le 3\epsilon$. Letting $\tilde{\nu}_1 = \tilde{\nu}_2 = \eta_\beta(q(\cdot \mid \tilde{b}_\beta(x)))$, we have by this estimate and (5.4.19), (5.4.20) that

$$\|q(\cdot \mid x) - n^{-1}(n_1\tilde{\nu}_1 + n_2\tilde{\nu}_2 + n_3\tilde{\nu}_3)\|_{\text{BL}} \le 3\epsilon.$$

Also, from Lemma 5.4.1,

$$\|\nu_2 - \tilde{\nu}_2\|_{\text{BL}} \le \frac{4\ell(\beta)}{n_2}$$

and if $n_3 \neq 0$, from (5.4.23), (5.4.22) and Lemma 5.4.1, we have

$$\|\nu_3 - \tilde{\nu}_3\|_{\text{BL}} \le \frac{4\ell(\beta+1)}{n_3}.$$

Combining the above three displays with (5.4.21) and the trivial estimate $\|\nu_1 - \tilde{\nu}_1\|_{\text{BL}} \le 2$, we have

$$\begin{aligned} \|\bar{p}_k(\cdot \mid x) - q(\cdot \mid x)\|_{\text{BL}} &\le 3\epsilon + n^{-1}\left(2n_1 + 4\left(\ell(\beta) + \ell(\beta+1)\right)\right) \\ &\le 3\epsilon + \varepsilon_\beta. \end{aligned}$$

where the last inequality follows on observing that $n_1 \le \varrho_{\beta-1}$, $n \ge n_2 \ge \alpha_\beta$ and using (5.4.15). Since $x \in K$ and $\epsilon > 0$ are arbitrary and $\beta \to \infty$ as $k \to \infty$, the result follows. ■

5.4.2 ATS policies for simultaneous estimation and optimization

Consider a setting where one has a (near) optimal $q \in \Pi_{\text{SMC}}$ for pathwise cost per unit time associated with some one stage cost function $c \in C_b(\mathbb{X} \times \mathbb{A})$. However, in addition to cost optimization one has a secondary objective of estimating some unknown parameter in the model. Consistent estimation may require using actions that are not optimal. For example, analogous to the example discussed in the introduction, it could be that under the policy q, estimation is impossible because transitions do not depend at all on the parameter that need to be estimated and thus one needs to deviate from the optimal q in order to gain information on the parameter. ATS policies provide a framework that allows one to introduce such deviations without "paying a price" in terms of the optimization problem. In this section we describe the construction of ATS policies for one such estimation problem.

Let $q \in \Pi_{\text{SMC}}$ be as in Section 5.4.1 and $c \in C_b(\mathbb{X} \times \mathbb{A})$. Suppose we are given another $q_0 \in \Pi_{\text{SMC}}$ and one would like to obtain consistent estimators for

$$\mathcal{J}_f = \int_{\mathbb{X}\times\mathbb{A}} f(x,a)\theta_{q_0}(dxda), \ f \in C_b(\mathbb{X} \times \mathbb{A}),$$

while achieving the pathwise cost per unit time $\int_{\mathbb{X}\times\mathbb{A}} c(x,a)\theta_q(dxda)$. We will show below that by an appropriate modification of the ATS policy constructed in Section 5.4.1 one can achieve both goals. We begin by introducing a strengthening of Assumption 5.2.1.

Let $\{\jmath_k\}_{k\in\mathbb{N}}$ be a sequence of $\{\mathcal{H}_t\}_{t\in\mathbb{N}_0}$- stopping times given on $(\Omega, \mathcal{F})$ such that

$$\jmath_k < \jmath_k + m_k \leq \jmath_{k+1}, \text{ for all } \omega \in \Omega$$

for some $m_k \in \mathbb{N}$, $k \geq 1$. Write $\varpi = (\jmath_k, m_k)_{k\in\mathbb{N}}$ and let $\mathbb{T}$ be the family of all such sequences. For $\varpi = (\jmath_k, m_k)_{k\in\mathbb{N}} \in \mathbb{T}$, and $N \geq 1$, let $\Phi_N[\varpi]$ be a measurable map from $\Omega \to \mathcal{P}(\mathbb{X} \times \mathbb{A})$ defined as

$$\Phi^{\omega}_N[\varpi](F) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{m_k}\sum_{j=\jmath_k}^{\jmath_k+m_k-1} 1_F(X_j, A_j), \ F \in \mathcal{B}(\mathbb{X} \times \mathbb{A}), \ \omega \in \Omega.$$

We will make the following assumption.

Assumption 5.4.2. *For all $\varpi \in \mathbb{T}$, $\mu \in \mathcal{P}(\mathbb{X})$ and $\pi \in \Pi$, $\{\Phi_N^\omega[\varpi] : N \in I\!N\}$ is tight for $\mathcal{P}_\mu^\pi$ a.e. ω.*

We note that the assumption is trivially satisfied if $\mathbb{X}$ and $\mathbb{A}$ are compact spaces. More generally, blanket stability conditions in terms of a suitable Lyapunov function can be formulated under which Assumption 5.4.2 holds.

An immediate consequence of the above assumption and other assumptions from Section 5.2 is the following.

Lemma 5.4.2. *Let $\varpi = (\jmath_k, m_k)_{k \in I\!N} \in \mathbb{T}$ be such that $m_k \to \infty$ as $k \to \infty$. Let $\mu \in \mathcal{P}(\mathbb{X})$, $\pi \in \Pi$ and $q_0 \in \Pi_{SMC}$ be such that for all $k \geq 1$ and $j \in \{0, 1, \cdots m_k - 1\}$*

$$\mathbb{P}_\mu^\pi \left((A_{\jmath_k+j}, X_{\jmath_k+j+1}) \in D \times C \mid \mathcal{H}_{\jmath_k+j} \right) = \int_D \mathcal{Q} \left((X_{\jmath_k+j}, a), C \right) q_0(X_{\jmath_k+j}, da),$$

for all $D \times C \in \mathcal{B}(\mathbb{A} \times \mathbb{X})$, a.e. $\mathbb{P}_\mu^\pi$. Then, as $N \to \infty$,

$$\|\Phi_N^\omega[\varpi] - \theta_{q_0}\|_{BL} \to 0, \quad a.e.\ \omega\ [\mathbb{P}_\mu^\pi].$$

Proof. For $f \in \mathcal{S}(\mathbb{X})$, let $\psi_f(x) \doteq \int_{\mathbb{X}} f(y) \varrho_{q_0}(x, dy), x \in \mathbb{X}$. Then, suppressing ω in notation, we have

$$\begin{aligned}
\Psi_f^N &\doteq \left| \int_{\mathbb{X}\times\mathbb{A}} f(x) \Phi_N[\varpi](dxda) - \int_{\mathbb{X}\times\mathbb{A}} \psi_f(x) \Phi_N[\varpi](dxda) \right| \\
&= \left| \frac{1}{N} \sum_{k=1}^N \frac{1}{m_k} \sum_{j=\jmath_k}^{\jmath_k+m_k-1} (f(X_j) - \psi_f(X_j)) \right| \\
&= \left| \frac{1}{N} \sum_{k=1}^N \frac{1}{m_k} \sum_{j=\jmath_k+1}^{\jmath_k+m_k-1} (f(X_j) - \psi_f(X_{j-1})) + \frac{1}{N} \sum_{k=1}^N \frac{f(X_{\jmath_k})}{m_k} - \frac{1}{N} \sum_{k=1}^N \frac{\psi_f(X_{\jmath_k+m_k-1})}{m_k} \right| \\
&\leq \left| \frac{1}{N} \sum_{k=1}^N \frac{1}{m_k} \sum_{j=0}^{m_k-2} (f(X_{\jmath_k+j+1}) - \psi_f(X_{\jmath_k+j})) \right| + \frac{1}{N} \sum_{k=1}^N \frac{2|f|_\infty}{m_k}.
\end{aligned} \tag{5.4.24}$$

Note that for all $k \geq 1$ and $j \in \{0, 1, \ldots, m_k - 2\}$,

$$\begin{aligned}\psi_f(X_{\jmath_k+j}) &= \int_{\mathbb{X}} f(y)\varrho_{q_0}(X_{\jmath_k+j}, dy) \\ &= \int_{\mathbb{X}\times\mathbb{A}} f(y)\mathcal{Q}((X_{\jmath_k+j}, a), dy)q_0(X_{\jmath_k+j}, da) \\ &= \mathbb{E}^{\pi}_{\mu}[f(X_{\jmath_k+j+1})|\mathcal{H}_{\jmath_k+j}] \text{ a.e. } \mathbb{P}^{\pi}_{\mu}.\end{aligned}$$

By the strong law of large numbers for martingales (cf. [44, Theorem VII.5.4]) and the assumption that $m_k \to \infty$ as $k \to \infty$, $\Psi^N_f \to 0$ as $N \to \infty$ a.e. $\mathbb{P}^{\pi}_{\mu}$.

Next, for $g \in \mathcal{S}(\mathbb{X})$ and $h \in \mathcal{S}(\mathbb{A})$, let $\phi_{(g,h)}(x) \doteq g(x)\int_{\mathbb{A}} h(a)q_0(x, da)$. Then

$$\begin{aligned}\Phi^N_{(g,h)} &\doteq \left|\int_{\mathbb{X}\times\mathbb{A}} g(x)h(a)\Phi_N[\varpi](dxda) - \int_{\mathbb{X}\times\mathbb{A}} \phi_{(g,h)}(x)\Phi_N[\varpi](dxda)\right| \\ &= \left|\frac{1}{N}\sum_{k=1}^{N}\frac{1}{m_k}\sum_{j=0}^{m_k-1}\big(g(X_{\jmath_k+j})h(A_{\jmath_k+j}) - \phi_{(g,h)}(X_{\jmath_k+j})\big)\right|\end{aligned} \tag{5.4.25}$$

For all $k \geq 1$ and $j \in \{0, 1, \ldots, m_k - 1\}$,

$$\begin{aligned}\phi_{(g,h)}(X_{\jmath_k+j}) &= g(X_{\jmath_k+j})\int_{\mathbb{A}} h(a)q_0(X_{\jmath_k+j}, da) \\ &= \mathbb{E}^{\pi}_{\mu}[g(X_{\jmath_k+j})h(A_{\jmath_k+j})|\mathcal{H}_{\jmath_k+j}] \text{ a.e. } \mathbb{P}^{\pi}_{\mu}.\end{aligned}$$

Again from the strong law of large numbers for martingales and the fact that $m_k \to \infty$ as $k \to \infty$, we have $\Phi^N_{(g,h)} \to 0$ as $N \to \infty$ a.e. $\mathbb{P}^{\pi}_{\mu}$.

From Assumption 4.2, and the above two conclusions, we can find $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}^{\pi}_{\mu}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, $\{\Phi^{\omega}_N[\varpi] : N \in \mathbb{N}\}$ is tight and $\Psi^N_f(\omega) \to 0, \Phi^N_{(g,h)}(\omega) \to 0$ as $N \to \infty$, for all $f \in \mathcal{S}(\mathbb{X})$ and all $(g, h) \in \mathcal{S}(\mathbb{X}) \times \mathcal{S}(\mathbb{A})$. Fix such $\omega \in \Omega_0$ and let $\{N_k : k \in \mathbb{N}\}$ be some subsequence along which $\Phi^{\omega}_{N_k}[\varpi]$ converges weakly to some $\Phi \in \mathcal{P}(\mathbb{X} \times \mathbb{A})$. We now show that $\Phi = \theta_{q_0}$. The continuity of q_0

implies that $\psi_f \in \mathcal{C}_b(\mathbb{X})$ for $f \in \mathcal{C}_b(\mathbb{X})$, and so

$$\int_{\mathbb{X}\times\mathbb{A}} f(x)\Phi^{\omega}_{N_k}[\varpi](dxda) - \int_{\mathbb{X}\times\mathbb{A}} \psi_f(x)\Phi^{\omega}_{N_k}[\varpi](dxda) \to \int_{\mathbb{X}} f(x)\Phi^{(1)}(dx) - \int_{\mathbb{X}} \psi_f(x)\Phi^{(1)}(dx)$$

where $\Phi^{(1)}$ is, as before, the first marginal of Φ. Therefore, for any $f \in \mathcal{S}(\mathbb{X})$,

$$\int_{\mathbb{X}} f(x)\Phi^{(1)}(dx) = \int_{\mathbb{X}} \psi_f(x)\Phi^{(1)}(dx) = \int_{\mathbb{X}}\int_{\mathbb{X}} f(y)\varrho_{q_0}(x,dy)\Phi^{(1)}(dx).$$

By Assumption 5.2.2, $\Phi^{(1)} = \lambda_{q_0}$. Similarly, for $g \in \mathcal{S}(\mathbb{X})$ and $h \in \mathcal{S}(\mathbb{A})$,

$$\begin{aligned}&\int_{\mathbb{X}\times\mathbb{A}} g(x)h(a)\Phi^{\omega}_{N}[\varpi](dxda) - \int_{\mathbb{X}\times\mathbb{A}} \phi_{(g,h)}(x)\Phi^{\omega}_{N}[\varpi](dxda)\\ &\to \int_{\mathbb{X}\times\mathbb{A}} g(x)h(a)\Phi(dxda) - \int_{\mathbb{X}} \phi_{(g,h)}(x)\Phi^{(1)}(dx).\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{X}\times\mathbb{A}} g(x)h(a)\Phi(dxda) &= \int_{\mathbb{X}} \phi_{(g,h)}(x)\Phi^{(1)}(dx) = \int_{\mathbb{X}} \phi_{(g,h)}(x)\lambda_{q_0}(dx)\\ &= \int_{\mathbb{X}\times\mathbb{A}} g(x)h(a)\theta_{q_0}(dxda).\end{aligned}$$

Recalling that $\{g \otimes h : g \in \mathcal{S}(\mathbb{X}), h \in \mathcal{S}(\mathbb{A})\}$ is separating in $(\mathbb{X}\times\mathbb{A}, \mathcal{B}(\mathbb{X})\otimes\mathcal{B}(\mathbb{A}))$, we have $\Phi = \theta_{q_0}$. Consequently, $\Phi^{\omega}_{N}[\varpi]$ converges weakly to θ_{q_0} as $N \to \infty$ a.e. ω $[\mathbb{P}^{\pi}_{\mu}]$. ■

Similar to Section 5.4.1, we will now construct a sequence of $\mathbb{X}\times\mathbb{A}$ valued random variables $Z \equiv (\bar{X}_t, \bar{A}_t)_{t\in\mathbb{N}_0}$ on a suitable probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that: (i) $\bar{X}_0$ has probability law μ, (ii) the probability law of Z corresponds to a controlled system associated with a policy $\pi \in \Pi_{\text{ATS}}(q,\mu)$, and (iii) consistent estimation of $\mathcal{J}_f$ can be achieved using the sequence Z. The sequence will be obtained by piecing together suitable sequences $(\xi^r_k, \zeta^r_k; k, r \in I\!N_0)$, $(\bar{\xi}^r_k, \bar{\zeta}^r_k; k, r \in I\!N_0)$ of $\mathbb{X}\times\mathbb{A}$ valued random variables. To construct these sequences we proceed recursively in r. Let

$m_r = -\log(\varepsilon_r)$ for $r \in I\!N$, where $\{\varepsilon_r\}_{r \in I\!N}$ is as in Section 5.4.1, and set $m_0 = 0$.

Case $r = 1$**:** Define $\{\xi^r_k, s^r_k, \zeta^r_k, (i^r[m,k])_{m=1,\cdots j(r)}\}$, $\alpha_r, \sigma_r, \varrho_r$ for $r = 1$ and $k \in \mathbb{N}_0$ exactly as in Section 5.4.1. For $k = 0, 1, \cdots m_r$, define $\mathbb{X} \times \mathbb{A}$ valued random variables, $(\bar{\xi}^r_k, \bar{\zeta}^r_k)$ recursively in k, by setting $(\bar{\xi}^r_0, \bar{\zeta}^r_0) = (\xi^r_{\varrho_r}, \zeta^r_{\varrho_r})$ and through the following two equations

$$\bar{\mathbb{P}}\left(\bar{\xi}^r_k \in C \mid \hat{\mathcal{G}}^r_{k-1}\right) = \mathcal{Q}\left((\bar{\xi}^r_{k-1}, \bar{\zeta}^r_{k-1}), C\right), \tag{5.4.26}$$

$$\bar{\mathbb{P}}\left(\bar{\zeta}^r_k \in D \mid \hat{\mathcal{G}}^r_{k-1}, \bar{\xi}^r_k\right) = q_0(\bar{\xi}^r_k, D), \tag{5.4.27}$$

where for $k = 0, 1, \cdots m_r - 1$, $\hat{\mathcal{G}}^r_k = \mathcal{G}^r_{\varrho_r} \vee \sigma\{(\bar{\xi}^r_j, \bar{\zeta}^r_j), j = 0, 1, \cdots k\}$ and $\mathcal{G}^r_k$ is as in Section 5.4.1.

Case $r > 1$**:** Definition of $\{\xi^r_k, \zeta^r_k, s^r_k, (i^r[m,k])_{m=1,\cdots j(r)}\}_{k \geq 0}$ and σ_r, α_r for $r > 1$, is given exactly as in Section 5.4.1 through (5.4.11) – (5.4.16), in a recursive fashion, but with ϱ_r defined as

$$\varrho_r = \sigma_r + \varrho_{r-1} + m_{r-1} \tag{5.4.28}$$

and by setting

$$(\xi^r_0, \zeta^r_0) = (\bar{\xi}^{r-1}_{m_{r-1}}, \bar{\zeta}^{r-1}_{m_{r-1}}),\ i^r[m,0] = 0.$$

The sequence $(\bar{\xi}^r_k, \bar{\zeta}^r_k)$, for $k = 0, 1, \cdots m_r$, is defined exactly as for the case $r = 1$ through equations (5.4.26)-(5.4.27) (and by setting $(\bar{\xi}^r_0, \bar{\zeta}^r_0) = (\xi^r_{\varrho_r}, \zeta^r_{\varrho_r})$).

Finally, the sequence $(\bar{X}_k, \bar{A}_k)$ is now constructed as follows. Recall that $\varrho_0 = 0$.

$$(\bar{X}_k, \bar{A}_k) = \begin{cases} (\bar{\xi}^r_{k-\varrho_r}, \bar{\zeta}^r_{k-\varrho_r}), & \text{whenever } \varrho_r \leq k < \varrho_r + m_r,\ r \in \mathbb{N}. \\ \\ (\xi^{r+1}_{k-\varrho_r-m_r}, \zeta^{r+1}_{k-\varrho_r-m_r}), & \text{whenever } \varrho_r + m_r \leq k < \varrho_{r+1},\ r \in \mathbb{N}_0. \end{cases} \tag{5.4.29}$$

The above sequence yields a $\pi \in \Pi$ and $\mathbb{P}^\pi_\mu \in \mathcal{P}(\Omega)$ as before. Consistent estimators for $\mathcal{J}_f$, $f \in C_b(\mathbb{X} \times \mathbb{A})$ can now be obtained as follows. Define on $(\bar{\Omega}, \bar{\mathcal{F}})$, a sequence of $\mathcal{P}(\mathbb{X}\times\mathbb{A})$ valued random variables, $\tilde{\Phi}_N$, $N \in I\!N$, as follows.

$$\tilde{\Phi}^{\omega}_N(F) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{m_k}\sum_{j=\varrho_k}^{\varrho_k+m_k-1} 1_F(\bar{X}_j,\bar{A}_j),\ F \in \mathcal{B}(\mathbb{X}\times\mathbb{A}),\ \omega\in\bar{\Omega}.$$

The following is the main result of this section. The second part of the theorem says that for every $f \in C_b(\mathbb{X}\times\mathbb{A})$, $\int_{\mathbb{X}\times\mathbb{A}} f(x,a)\tilde{\Phi}_N(dxda)$ is an (a.e.) consistent estimator for $\mathcal{J}_f$. The proof is very similar to that of Theorem 5.4.1 and so only a sketch will be provided.

Theorem 5.4.2. *The policy constructed above is in* $\Pi_{ATS}(q,\mu)$. *Furthermore, as* $N\to\infty$

$$\|\tilde{\Phi}^{\omega}_N - \theta_{q_0}\|_{BL} \to 0, \quad a.e.\ \bar{\mathbb{P}}.$$

Proof. As in Theorem 5.4.1, we define $\bar{\Omega}_0$, $\bar{p}^{\omega}_k(D|x)$, Λ_k, $n_{ti}(m_1,m_2)$, and $\mu_{ti}(m_1,m_2)$. To show $\pi\in\Pi_{\text{ATS}}(q,\mu)$, it suffices to show that for all $\omega\in\bar{\Omega}_0$ and compact $K\subset\mathbb{X}$, (5.4.18) holds. Fix such a ω and K. As in Theorem 5.4.1, we can find $r_0 \in I\!N$, such that (5.4.19) and (5.4.20) hold for all $r\ge r_0$ and $K\subset K_r$. Fix $\beta_0 \ge r_0+1$ and let $\beta\in\mathbb{N}, \beta\ge\beta_0$ such that $\varrho_\beta < k \le \varrho_{\beta+1}$. Also fix $x\in K$ and let $i\in\{1,\ldots,j(\beta)\}$ be such that $x\in\tilde{B}_{\beta i}$. Similar to (5.4.21), we can write

$$\bar{p}_k(\cdot|x) = \begin{cases} l^{-1}(l_1\tau_1+l_2\tau_2+l_3\tau_3+l_4\tau_4), & \text{whenever } \varrho_\beta\le k<\varrho_\beta+m_\beta. \\ \\ \check{l}^{-1}(l_1\tau_1+l_2\tau_2+l_3\tau_3+l_5\tau_5+l_6\tau_6), & \text{whenever } \varrho_\beta+m_\beta\le k<\varrho_{\beta+1}. \end{cases} \tag{5.4.30}$$

Here

$$\tau_1=\mu_{\beta,i}[0,\varrho_{\beta-1}],\ \tau_2=\mu_{\beta,i}[\varrho_{\beta-1},\varrho_{\beta-1}+m_{\beta-1}],\ \tau_3=\mu_{\beta,i}[\varrho_{\beta-1}+m_{\beta-1},\varrho_\beta],$$

$$\tau_4=\mu_{\beta,i}[\varrho_\beta,k],\ \tau_5=\mu_{\beta,i}[\varrho_\beta,\varrho_\beta+m_\beta],\ \tau_6=\mu_{\beta,i}[\varrho_\beta+m_\beta,k]$$

and

$$
\begin{aligned}
&l_1 = n_{\beta i}(0, \varrho_{\beta-1}),\ l_2 = n_{\beta i}(\varrho_{\beta-1}, \varrho_{\beta-1} + m_{\beta-1}),\ l_3 = n_{\beta i}(\varrho_{\beta-1} + m_{\beta-1}, \varrho_\beta),\\
&l_4 = n_{\beta i}(\varrho_\beta, k),\ l_5 = n_{\beta i}(\varrho_\beta, \varrho_\beta + m_\beta),\ l_6 = n_{\beta i}(\varrho_\beta + m_\beta, k),\\
&l = l_1 + l_2 + l_3 + l_4,\ \check{l} = l_1 + l_2 + l_3 + l_5 + l_6.
\end{aligned}
$$

Analogous to $\tilde{\nu}_3$ in Theorem 5.4.1, we can define a $\tilde{\tau}_6 \in \mathcal{P}(\mathbb{A})$ such that, if $n_6 > 0$,

$$
\|\tilde{\tau}_6 - q(\cdot|x)\|_{\text{BL}} \le 3\epsilon,\ \|\tau_6 - \tilde{\tau}_6\|_{\text{BL}} \le \frac{4\ell(\beta+1)}{l_6}.
$$

Now let $\tilde{\tau}_1 = \tilde{\tau}_2 = \tilde{\tau}_3 = \tilde{\tau}_4 = \tilde{\tau}_5 = \eta_\beta(q(\cdot|\tilde{b}_\beta(x)))$. Then, by our choice of r_0,

$$
\begin{aligned}
&\|q(\cdot|x) - l^{-1}(l_1\tilde{\tau}_1 + l_2\tilde{\tau}_2 + l_3\tilde{\tau}_3 + l_4\tilde{\tau}_4)\|_{\text{BL}} \le 2\epsilon \text{ when } \varrho_\beta \le k < \varrho_\beta + m_\beta.\\
&\|q(\cdot|x) - \check{l}^{-1}(l_1\tilde{\tau}_1 + l_2\tilde{\tau}_2 + l_3\tilde{\tau}_3 + l_5\tilde{\tau}_5 + l_6\tilde{\tau}_6)\|_{\text{BL}} \le 3\epsilon \text{ when } \varrho_\beta + m_\beta \le k < \varrho_{\beta+1}.
\end{aligned}
\tag{5.4.31}
$$

Also note that

$$
\|\tau_3 - \tilde{\tau}_3\|_{BL} \le \frac{4\ell(\beta)}{l_3}.
$$

When $\varrho_\beta \le k < \varrho_\beta + m_\beta$, we have

$$
\begin{aligned}
\|\bar{p}_k(\cdot|x) - q(\cdot|x)\|_{BL} &\le 2\epsilon + l^{-1}(2l_1 + 2l_2 + 4\ell(\beta) + 2l_4)\\
&\le 2\epsilon + \alpha_\beta^{-1}(2\varrho_{\beta-1} + 2m_{\beta-1} + 4\ell(\beta) + 2m_\beta)\\
&\le 2\epsilon + \varepsilon_\beta + \frac{4m_\beta}{\alpha_\beta}.
\end{aligned}
\tag{5.4.32}
$$

When $\varrho_\beta + m_\beta \le k < \varrho_{\beta+1}$,

$$
\begin{aligned}
\|\bar{p}_k(\cdot|x) - q(\cdot|x)\|_{BL} &\le 3\epsilon + \check{l}^{-1}(2l_1 + 2l_2 + 4\ell(\beta) + 2l_5 + 4\ell(\beta+1))\\
&\le 3\epsilon + \alpha_\beta^{-1}(2\varrho_{\beta-1} + 2m_{\beta-1} + 4\ell(\beta) + 2m_\beta + 4\ell(\beta+1))\\
&\le 3\epsilon + \varepsilon_\beta + \frac{4m_\beta}{\alpha_\beta}.
\end{aligned}
$$

Recalling that $m_\beta = -\log(\varepsilon_\beta)$ and that $\alpha_\beta \geq \varepsilon_\beta^{-1}$, $\frac{4m_\beta}{\alpha_\beta} \to 0$ as $\beta \to \infty$. Since $\beta \to \infty$ as $k \to \infty$, we have that $\|\bar{p}_k(\cdot|x) - q(\cdot|x)\|_{BL} \to 0$ as $k \to \infty$ and therefore $\pi \in \Pi_{\text{ATS}}(q, \mu)$. Finally, the second part of the theorem is an immediate consequence of Lemma 5.4.2. ■

5.4.3 Adaptive control

In this section we consider a setting where the (near) optimal $q \in \Pi_{\text{SMC}}$ is not known but there are available sampling schemes that allow for consistent estimation of q. The goal is then to estimate q dynamically and use the estimators of q to construct a control policy for which the associated pathwise cost per unit time coincides with that for q.

In order to give a precise formulation, suppose that $q \in \Pi_{\text{SMC}}$ is given as

$$q(\cdot \mid x) = \mathfrak{q}(\cdot \mid \kappa_0, x), \tag{5.4.33}$$

where κ_0 is an unknown parameter taking values in some compact metric space Γ. We assume that the map $(\kappa, x) \mapsto \mathfrak{q}(\cdot \mid \kappa, x)$, from $\Gamma \times \mathbb{X} \to \mathcal{P}(\mathbb{A})$, is a continuous function. Also suppose that there is a $q_0 \in \Pi_{\text{SMC}}$ and a continuous function $G : \mathcal{P}(\mathbb{X} \times \mathbb{A}) \to \Gamma$ such that

$$G(\theta_{q_0}) = \kappa_0.$$

This relationship, in view of Lemma 5.3.1, says that as $N \to \infty$, $G(\Phi_N)$ is an (a.e.) consistent estimator for κ_0, under $\mathbb{P}_\mu^{q_0}$ for all $\mu \in \mathcal{P}(\mathbb{X})$. However the corresponding pathwise cost is $\int_{\mathbb{X}\times\mathbb{A}} c(x,a)\theta_{q_0}(dxda)$ ($\mathbb{P}_\mu^{q_0}$ a.e.) and thus although the policy q_0 achieves the goal of parameter estimation, it does not meet the criterion of cost (near) optimization. In order to meet both objectives we will now construct a policy π which uses dynamic estimators for κ_0 (and consequently for q) for control decisions and is such that it is an ATS policy for q corresponding to the initial condition μ.

Let $\{\tilde{\Lambda}_k, \tilde{\mathbb{X}}_k, \tilde{b}_k, \Lambda'_k, \mathbb{A}_k, b'_k, \tilde{\Lambda}^0_k, \eta_k\}_{k\in I\!N}$ be as in Section 5.4.1. Let m_r be as in Section 5.4.2. As in Section 5.4.2 we begin by introducing sequences $(\xi^r_k, \zeta^r_k; k, r \in I\!N_0)$, $(\bar{\xi}^r_k, \bar{\zeta}^r_k; r \in I\!N_0, k = 1, \cdots m_r)$ of $\mathbb{X} \times \mathbb{A}$ valued random variables, recursively in r. We will use notation and constructions from Sections 5.4.1 and 5.4.2.

Case $r = 1$**:** Set $\hat{q}_r = q_0$. For $m = 1, \cdots j(r)$, define

$$\hat{q}^{r,m}_r(\cdot) = \hat{q}_r(\cdot \mid x_{rm}),\ \tilde{q}^{r,m}_r = \eta_r(\hat{q}^{r,m}_r),\ m = 1, \cdots j(r). \tag{5.4.34}$$

Abusing notation from Section 5.4.1, denote

$$\Psi(\tilde{q}^{r,m}_r) = (e^r[m,1], e^r[m,2], \cdots). \tag{5.4.35}$$

With this new definition of $e^r[m,i]$, the definition of $\{\xi^r_k, s^r_k, \zeta^r_k, (i^r[m,k])_{m=1,\cdots j(r)}\}$, for $r = 1$ and $k \in I\!N_0$ is given exactly as in Section 5.4.1, through equations (5.4.11) – (5.4.14). Also define $\alpha_r, \sigma_r, \varrho_r$ through equations (5.4.15) – (5.4.17) (with $\varrho_0 = 0$). Next, for $t = 0, 1, \cdots, m_r$, define $\mathbb{X} \times \mathbb{A}$ valued random variables $(\bar{\xi}^r_t, \bar{\zeta}^r_t)$, $t = 0, 1, \cdots m_r$, recursively in t, by (5.4.26) – (5.4.27) (and by setting $(\bar{\xi}^r_0, \bar{\zeta}^r_0) = (\xi^r_{\varrho_r}, \zeta^r_{\varrho_r})$). Define a $\mathcal{P}(\mathbb{X} \times \mathbb{A})$ valued random variable $\tilde{\Phi}_r$ by the relation

$$\tilde{\Phi}_r(F) = \frac{1}{m_r}\sum_{t=1}^{m_r} 1_F(\bar{\xi}^r_t, \bar{\zeta}^r_t),\ F \in \mathcal{B}(\mathbb{X} \times \mathbb{A}).$$

and let $\kappa_r = G(\tilde{\Phi}_r)$.

Case $r > 1$**:** Set $\hat{q}_r(\cdot \mid x) = q(\cdot \mid \kappa_{r-1}, x)$, $x \in \mathbb{X}$. Define for $m = 1, \cdots j(r)$, $\hat{q}^{r,m}_r$ and $\tilde{q}^{r,m}_r$, , through (5.4.34); and $e^r[m,i]$, $i \in I\!N$, through (5.4.35). With this definition of $e^r[m,i]$, the definition of $\{\xi^r_k, s^r_k, \zeta^r_k, (i^r[m,k])_{m=1,\cdots j(r)}\}$, for $k \in I\!N_0$ and $\alpha_r, \sigma_r, \varrho_r$ is given as in Sections 5.4.1 and 5.4.2, through equations (5.4.11) – (5.4.16) and (5.4.28). The sequence $(\bar{\xi}^r_k, \bar{\zeta}^r_k)$, for $k = 0, 1, \cdots m_r$, is defined exactly as for the case $r = 1$ through equations (5.4.26)-(5.4.27) (and by setting $(\bar{\xi}^r_0, \bar{\zeta}^r_0) = (\xi^r_{\varrho_r}, \zeta^r_{\varrho_r})$). To complete the recursion we define

$$\tilde{\Phi}_r(F) = \frac{1}{M_{r-1} + m_r}\left(M_{r-1}\tilde{\Phi}_{r-1}(F) + \sum_{t=1}^{m_r} 1_F(\bar{\xi}^r_t, \bar{\zeta}^r_t)\right),\ F \in \mathcal{B}(\mathbb{X} \times \mathbb{A}),$$

where $M_{r-1} = \sum_{t=1}^{r-1} m_t$, and let $\kappa_r = G(\tilde{\Phi}_r)$.

The definition of the sequence $(\bar{X}_k, \bar{A}_k)$ is now given through (5.4.30). This sequence yields a $\pi \in \Pi$ and $\mathbb{P}^\pi_\mu \in \mathcal{P}(\Omega)$ as before.

The following is the main result of the section. Assumption 5.4.2 will be taken to hold. The proof is similar to that of Theorems 5.4.1 and 5.4.2 and so only a sketch will be provided.

Theorem 5.4.3. *The policy constructed above is in $\Pi_{ATS}(q, \mu)$. Furthermore, for every compact K in $\mathbb{X}$, as $r \to \infty$*

$$\sup_{x \in K} \|\hat{q}_r(\cdot \mid x) - q(\cdot \mid x)\|_{BL} \to 0,$$

a.e. $\bar{\mathbb{P}}$.

Proof. We use the same notation and definitions as in the proof of Theorem 5.4.2. First, we show that, for every compact set $K \subset \mathbb{X}$,

$$\sup_{x \in K} \|\hat{q}_r(\cdot \mid x) - q(\cdot \mid x)\|_{\mathrm{BL}} \to 0 \text{ a.e. } \bar{\mathbb{P}}. \tag{5.4.36}$$

By Theorem 5.3.1, $\tilde{\Phi}_r$ converges weakly to θ_{q_0} a.e. $\bar{\mathbb{P}}$. Since G is continuous, $\kappa_r = G(\tilde{\Phi}_r) \to G(\theta_{q_0}) = \kappa_0$ as $r \to \infty$, a.e. $\bar{\mathbb{P}}$. Note that

$$\|\hat{q}_r(\cdot \mid x) - q(\cdot \mid x)\|_{\mathrm{BL}} = \|\mathfrak{q}(\cdot \mid \kappa_r, x) - \mathfrak{q}(\cdot \mid \kappa, \tilde{b}_r(x))\|_{\mathrm{BL}}.$$

Equation (5.4.36) is now an immediate consequence of the continuity of the map $(\kappa, x) \mapsto q(\cdot \mid \kappa, x)$.

For $\epsilon > 0$, choose r_0 such that all $r > r_0, K \subset K_r$, (5.4.20) holds,

$$\sup_{(x,\kappa)\in K\times\Gamma} \|q(\cdot \mid \kappa, x) - q(\cdot \mid \kappa, \tilde{b}_r(x))\|_{\text{BL}} \leq \epsilon, \tag{5.4.37}$$

and

$$\sup_{x\in K} \|\hat{q}_r(\cdot \mid x) - q(\cdot \mid x)\|_{\text{BL}} \leq \epsilon. \tag{5.4.38}$$

Fix $\beta_0 > r_0 + 1$ and let $\beta \in \mathbb{N}$, $\beta \geq \beta_0$ be such that $\varrho_\beta < k \leq \varrho_{\beta+1}$.

Let $l, \check{l}, l_i, \tau_i, i = 1, 2, \ldots, 6$, and $\bar{p}_k(\cdot|x)$ be the same as in the proof of Theorem 5.4.2. In particular, we have that (5.4.30) holds. Let $\tilde{\tau}_1 = \tilde{\tau}_2 = \tilde{\tau}_3 = \tilde{\tau}_4 = \tilde{\tau}_5 = \eta_\beta(\hat{q}_\beta(\cdot|\tilde{b}_\beta(x)))$. Construct $\tilde{\tau}_6$ in the same way as in Theorem 5.4.2 with $q(\cdot|\tilde{x})$ replaced by $\hat{q}_{\beta+1}(\cdot|\tilde{x})$ for $\tilde{x} \in \mathbb{X}$. Using (5.4.37) and (5.4.38) it is now easily checked that (5.4.31) holds with 2ϵ and 3ϵ, replaced by 3ϵ and 4ϵ respectively. Also note that, if $l_6 > 0$,

$$\|\tau_3 - \tilde{\tau}_3\| \leq \frac{4\ell(\beta)}{l_3}, \quad \|\tau_6 - \tilde{\tau}_6\| \leq \frac{4\ell(\beta+1)}{l_6}.$$

Rest of the proof now follows as for Theorem 5.4.2. ∎

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