Aspects of quantum gravity: quantum space-time and black hole thermodynamics.

by
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ABSTRACT

Michele Arzano: Aspects of quantum gravity: quantum space-time and black hole thermodynamics.
(Under the Direction of Prof. Yee Jack Ng)

This work is devoted to the study of certain quantum properties of space-time at the Planck scale and of black holes. We discuss the possibility that in quantum gravity scenarios the symmetry structure of flat space-time might deviate from the classical relativistic picture and lead to broken or deformed Poincaré invariance. The striking feature of these "quantum" space-time models is the possibility that they might have experimentally observable effects. We discuss how a purely kinematical model within these frameworks, besides providing the threshold anomalies needed to explain the existence of above-GZK cosmic rays, can modify the Bachall-Waxman bound on the flux of neutrinos that are expected to be produced together with such cosmic rays.

A relevant characteristic of "quantum" space-time scenarios with modifications of relativistic kinematics is the emergence of a Planck-scale particle localization limit that reflects the presence of the Planck length as an intrinsic spatial resolution limit for regimes in which quantum and gravitational effects are of the same magnitude. We propose a remarkable argument which relates the type of quantum gravity corrections to the Bekenstein-Hawking entropy-area relation for black holes and the form of the Planck-scale particle localization limit. Using this argument we are able to constraint the form of the deformed energy-momentum dispersion relation expected to emerge in the low-energy limit of loop quantum gravity. The same argument is then generalized to quantum gravity frameworks which predict a modifications of Heisenberg's uncertainty relation. We carried on a systematic study of the effects of modified energy-momentum dispersion relation and generalized uncertainty principle for an evaporating black hole ob-

taining also results for Planck-scale modifications of the spectrum of a radiating black-body.

Finally, we extend our study of quantum gravity corrections to the Hawking radiation spectrum by adapting the tunneling picture proposed by Parikh and Wilczek including, in such a way, non-thermal corrections due to back-reaction of the emitted particle. It is also showed that a quantum fluctuating black hole horizon, characterized by a "quantum ergosphere" produces the same type of modification to the emission spectrum expected when higher order quantum gravity corrections to the entropy-area relation are present.

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Chapter 1

Introduction

The beginning of 20th century has witnessed the start of two major revolutions in our understanding of the concepts of space and time and of the microscopic behavior of physical systems. In 1900, Max Planck, motivated by the embarrassing failure of classical electrodynamics and statistical mechanics in explaining the properties of a radiating black-body, proposed a model in which the different modes of the gas of radiation in the cavity of the black body had a discrete or "quantized" distribution. It was the first step of the "quantum" revolution from which quantum mechanics (QM) culminated more than seventy years later with the standard model of particle physics: a very successful description of the microscopic world up to the energy scales probed by current particle accelerators. Five years later, in 1905, Albert Einstein brilliantly resolved the apparent contradiction between Maxwell's theory of electrodynamics and the symmetry structure of classical mechanics introducing the relativity postulates which constitute the basis of the theory of special relativity (SR). Einstein himself in 1916 completed his "relativity" revolution providing, with the theory of general relativity (GR), the most accurate and experimentally successful picture of space-time and gravitation at large scales.

While a unified quantum description seems to be possible for three of the fundamental forces of Nature, electro-magnetic, weak and strong interaction, the very nature of gravity, as described by GR, seems to prevent any straightforward "quantization". Indeed the non-linearity of GR, or in cruder words the facts that gravity "gravitates", together with the fact that the coupling constant of the theory is not dimensionless, conspire to produce, at a perturbative level, a proliferation of divergent quantities which renders impossible any renormalization.

Infinities already emerge in Quantum Electrodynamics (QED), where they can be eliminated with an appropriate renormalization, and they simply reflect the existence of a limited energy range for the validity of the theory. Once a definite energy scale is reached, new physics appears which is appropriately described by a more general theory (Weinberg-Salam model of electroweak interaction in the specific case of QED). In the case of gravity the scale at which new (quantum) effects are expected to become important is set by the Planck energy: $E_p = \sqrt{\frac{c^3\hbar}{G}} \simeq 10^{18}\,GeV$. Associated with this energy are length, time and mass scales: $L_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33}\,cm$, $T_p = \sqrt{\frac{G\hbar}{c^5}} \simeq 10^{-42}\,s$ and $M_p = \sqrt{\frac{c\hbar}{G}} \simeq 10^{-5}\,g$. Moreover in GR the existence of space-time singularities, regions of arbitrarily high curvature, signals the existence of a limit in the predictive power of the theory. Near a singularity, where the curvature can go far below the scale set by the Planck length L_p, a full theory of Quantum Gravity (QG) would be needed to describe the dynamics of space-time.

At a deeper level one can trace the incompatibility of QM and GR back to the fact that each was formulated starting from radically different assumptions: QM assumed a fixed space-time background, while GR was formulated in terms of a dynamical metric field. More precisely, while non-relativistic QM was based on the choice of an external time variable (and later QFT defined positive and negative energy field modes in term of a global time variable), in GR there is no global definition of any special time variable as the theory is invariant under general coordinate transformations.

The present picture of our physical world is then based on a set of theories (GR, QM with QFT and the standard model) which are empirically successful in their own range of validity but that, due to their radically different natures, seem to resist all the efforts to be combined in a new synthesis.

For many years attempts to identify a correct theoretical framework for QG have moved from first principles, as the scales involved in possible manifestations of the quantum nature of gravity are so far from any experimental reach. Nowadays the leading proposals of theories describing quantum gravity are based on the canonical quantization of GR (also known as loop quantum gravity (LQG) [1]) and on the framework of string/M-theory [2] in which classical gravity emerges as a *low energy limit* of a more general model of unified interactions in which elementary particles are excitations of extended objects (strings) (for a review of different approaches to quantum gravity we refer the reader to [3]).

Alternatives to these "direct" approaches to the construction of general QG theories are other, less ambitious, attempts which try to shed some light on particular physical settings in which quantum gravitational effects are expected to play a significant role or which adopt simplified models and obtain results that are of guidance in the path to a more complete QG theory. One of these approches, which has now been very successfully investigated for more than three decades, is the study of "semiclassical" approximations in which quantum matter fields propagate on a curved (classical) space-time background [4, 5]. This is analogous to "pre-QED" semiclassical approximations in which quantum particles were considered interacting with a classical electromagnetic field. Indeed the results obtained in these frameworks turned out to be in perfect agreement with the full theory of QED. One of the most striking predictions obtained from the study of quantum field theory (QFT) on curved spaces is Hawking's discovery [6] of the phenomenon of quantum radiance from black holes. This result, having been re-derived in a variety of different methods (for recent reviews see [7, 8]), is believed to be very fundamental and is expected to provide an important element of guidance in quantum gravity research. Indeed Hawking's discovery justified a previous conjecture by Bekenstein [9] which associated to every black hole an "entropy" proportional to its horizon area. The Bekenstein-Hawking entropy-area relation[1]

$$S_{BH} = \frac{A}{4L_p^2} \tag{1.1}$$

provides a very deep connection between gravity, the quantum realm and the basic thermodynamic concept of entropy. More importantly this relation is expected to be a general prediction of any candidate theory of quantum gravity, and any such theory should also provide a natural explanation of the "degrees of freedom" contributing to the entropy of a quantum black hole.

A different approach to the quantum gravity problem, to which much attention has been given in recent years, tries instead to gain a deeper understanding of the small-scale "quantum" structure of space-time. The main idea is that the "lowest energy" vacuum state of quantum gravity might not simply be flat Minkowski space-time but a more complicated "quantum" geometrical configuration. Indeed various heuristic arguments and "gedanken" experiments (see [10] and references therein) suggest that the Planck length L_p might play the role of a fundamental minimal length scale in quantum gravity, setting a lower limit to *any* distance measurement. The existence of such "operative" limitation on space-time resolution is obviously in contrast with the picture of a classical smooth manifold. Moreover standard relativistic symmetries can not incorporate a fundamental "length" scale which is invariant for different inertial observers.

On one side non-commutative geometry (NCG) [11, 12] provides a possible description of space-time at the Planck scale in which, following the idea that space-time should be "quantized", usual commuting coordinates are replaced by operators with non-trivial commutation relations[2].

On the other hand modifications or "quantum" deformations of the standard

[1]From now on, unless otherwise stated, we will work in units $\hbar = c = 1$.

[2]For a different approach to NCG in which non-commutativity is instead introduced at the level of algebra of functions on a manifold see [13, 14]

Poincaré algebra of symmetries in which the dimensionful deformation parameter appears as a fundamental scale have been studied [15, 16]. These "deformed" symmetry models provide prototypes for the recently proposed doubly (or deformed) relativity scenarios (DSR) [17, 18, 19], modifications of standard special relativity which accommodate a second invariant length (energy) scale reflecting a non-trivial structure of space-time at planckian regimes.

These two approaches are not unrelated and there exist examples of non-commutative space-times which enjoy certain types of deformed symmetries which are also relevant for DSR scenarios [20, 21]. In general there are indications that some particular models of non-commutative space-time can be endowed with deformed relativistic symmetries while in other cases Lorentz invariance is broken by the emergence of a "preferred" orientation defined by the tensor parametrizing the non-commutativity of the coordinates.

A very interesting point to note is that Planck-scale departures from Poincaré invariance of the type encountered in NCG and DSR scenarios are also relevant for their possible phenomenological implications. In fact one of the major breakthrough in quantum gravity research in recent years has been the realization that we may be closer than has been thought in the past to the *observation* of signatures of Planck-scale physics in various experimental settings.

This recent development in "quantum gravity phenomenology" will be the starting point of the present dissertation. After a brief overview of quantum space-time scenarios with departures from classical, flat space, relativistic symmetries at the Planck-scale, in Chapter 2, we will describe the possible effects of space-time quantization on the spectrum of the highest energy neutrinos [24]. We will then discuss, in Chapter 3, a remarkable link between modifications of the special relativistic energy-momentum dispersion relation (MDR) emerging in QG scenarios (with particular emphasis on LQG) and quantum gravity corrections to the entropy area law (1.1) [25]. In Chapter 4, following the strategy outlined in Chapter 3 we will study, more extensively, the relation between Planck-scale modifications of the $E \geq \frac{1}{\delta x}$ relation and its consequences for the spectrum of an evaporating black hole. In this context we will also see how these corrections can affect a black body emission spectrum and will point out some consequences for the Bekenstein entropy bound and the generalized second law of thermodynamics (GSL) [26]. In the following Chapter 5 we discuss how the possible quantum space-time features previously studied can be incorporated in the description of black hole radiation as quantum tunneling and comment on the non-thermal deviations of the radiation spectrum [27]. We also propose a connection between Planck-scale modifications of a black hole emission probability in the tunneling framework and the presence of a "quantum ergosphere" associated with the quantum fluctuating space-time which endows the black hole with a non-trivial horizon structure [28]. Chapter 6 is devoted to a final summary and concluding remarks.

Chapter 2

Planck-scale structure of space-time and some phenomenological implications

2.1 Quantum space-time and the fate of relativistic symmetries in quantum gravity

Departures from standard relativistic symmetries in quantum space-time, intended as Planck-scale discrete or non-commutative space-time, have recently received much attention[3]. It is natural, in fact, to expect that the introducing of additional structure in a flat space-time besides the classical causal Minkowski structure, will accordingly affect its symmetries. This is evident, for example, for Planck-scale discretization of space-time due to the fact that continuous symmetry transformations are clearly at odd with a discrete network of points. One example is LQG which is one of the most developed quantum gravity picture

[3]In the following we will focus principally on departures from Lorentz and Poincaré invariance. However quantum gravity violations of CPT symmetry and their possible phenomenological implications have also been considered in the literature, see e.g. [22, 23]

which introduces the idea of space-time discreteness [29, 30, 31]. LQG does not predict a rigid discrete network of spacetime points, but discretization emerges in a more sophisticated way through the appearance of quantized spectra of areas and volumes while space-time points loose any operational meaning. It appears that even this more advanced form of discretization is incompatible with classical relativistic symmetries; in fact, there is [32, 33] evidence in the LQG literature of Planck-scale departures from Poincaré symmetry (although the issue remains subject to further scrutiny).

Poincaré symmetry, for different reasons, is also often at odds with spacetime non-commutativity. In particular, as we mentioned in the Introduction, studies of non-commutative space-times have shown that the Lie-algebra Poincaré symmetries can be either broken or deformed into a new type of symmetries. It appears that in certain cases the non-commutativity length scale (possibly the Planck length), affects the laws of transformation between inertial observers, and infinitesimal symmetries can be described in terms of the new language of Hopf algebras (quantum groups) [15, 16]. The type of space-time quantization provided by noncommutativity can then be related to a corresponding symmetry quantization in which the concept of Lie-algebra symmetry is replaced by the one of Hopf-algebra symmetry.

In this chapter we will focus on quantum space-time departures from standard relativistic symmetry which lead, in particular, to Planck-scale modifications of the special relativistic energy-momentum dispersion relation. In the next subsections we will discuss MDRs in the context of NCG and LQG[4] and discuss whether or not the emergence of such MDR's is associated with broken or deformed Poincaré symmetry. Then, in section 2.2, we will give a brief overview of experimental phenomena in which our quantum space-time models might play a role and present a detailed derivation of a new, quantum gravity modified, bound

[4]We focus on NCG and LQG scenarios but MDRs also appear in other frameworks, for example "spacetime foam" models (see e.g. [34, 35]).

on the flux of the highest energy cosmic neutrinos.

Non-commutative space-time and MDR

We will focus here on NCG in the particular case of non-commutative flat (Minkowski) space-time[5] in which coordinates commutators are either constant or depend on the coordinate themselves. Two simple examples are "canonical" NCST

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{2.1}$$

and "Lie-algebra" NCST

$$[x_\mu, x_\nu] = iC^\beta_{\mu\nu} x_\beta \,, \tag{2.2}$$

$(\mu, \nu, \beta = 0, 1, 2, 3)$. It is convenient to first focus on the canonical case (2.1). The construction of a QFT on NCSTs of the type (2.1) has been developed rather extensively [37, 38]. While most aspects of these field theories closely resemble their commutative space-time counterparts, a surprising feature that emerges is the so-called "IR/UV mixing" [39, 40]: the high-energy sector of the theory does not decouple from the low-energy sector. Connected with this IR/UV mixing is the type of modified dispersion relations that one encounters in field theory on canonical noncommutative spacetime, which in general take the form

$$m^2 \simeq E^2 - \vec{p}^2 + \frac{\alpha_1}{p^\mu \theta_{\mu\nu} \theta^{\nu\sigma} p_\sigma} + \alpha_2 m^2 \log(p^\mu \theta_{\mu\nu} \theta^{\nu\sigma} p_\sigma) + ... \tag{2.3}$$

where α_1, α_2 are parameters, possibly taking different values for different particles (the dispersion relation is not universal), that depend on various aspects of the field theory, including its field nature and the type of interactions. The fact that this dispersion relation can be singular in the infrared is a result of the IR/UV mixing. The presence of modified dispersion relations in canonical NCST should be expected, since Lorentz symmetry is broken by the tensor $\theta_{\mu\nu}$. This tensor,

[5] Another interesting example of non-commutative Minkowski space is given by Snyder's space-time [36], in which the commutators of the coordinates are expressed in terms of elements of the Lorentz algebra.

infact, plays the role of a background that identifies a preferred class of inertial observers[6]. Different particles can be affected by the presence of this background in different ways, leading to the emergence of different dispersion relations[7].

On the "Lie-algebra" side one example of non-commutative Minkowski space-time that has been extensively studied is κ-Minkowski [16, 21, 42, 43, 44, 45]

$$[x_m, t] = \frac{i}{\kappa} x_m \qquad [x_m, x_l] = 0 \tag{2.4}$$

$(l, m = 1, 2, 3)$. κ-Minkowski is an example of NCST which is invariant under deformed or "quantum" relativistic symmetries expressed by κ-Poincaré Hopf algebra. In fact although it is clear from (2.4) that κ-Minkowski enjoys classical space rotation symmetry it turns out that, in a Hopf-algebra sense (see, e.g., [21]) this space is invariant under non-commutative translations and that boost transformations are necessarily modified. A first hint of this comes from the necessity of a deformed law of composition of momenta, encoded in the so-called "co-product" (a standard structure for a Hopf algebra). One can see this clearly by considering the Fourier tranform. It turns out [43, 46] that in the κ-Minkowski case the correct formulation of the Fourier theory requires a suitable ordering prescription for wave exponentials:

$$: e^{ik^\mu x_\mu} :\equiv e^{ik^i x_i} e^{ik^0 x_0} \tag{2.5}$$

While wave exponentials of the type $e^{ik^\mu x_\mu}$ would not combine in a simple way (as a result of the κ-Minkowski non-commutativity), for the ordered exponential

[6] Very recently, it has been proposed [41] that canonical NCST may, instead, be endowed with a Hopf-algebra type of symmetries represented by the "twisted" Poincaré Hopf algebra.

[7] These remarks apply to canonical NCST as studied in the (often String-Theory inspired) literature, in which $\theta_{\mu\nu}$ is indeed simply a tensor (for a given observer, an antisymmetric matrix of numbers). Earlier studies of canonical NCST however (see e.g. [11]) considered a $\theta_{\mu\nu}$ with richer mathematical properties and nontrivial algebra relations with the spacetime coordinates. In that earlier setup it is not obvious that Lorentz symmetry would be broken: the fate of Lorentz symmetry may depend on the, possibly dynamical, properties attributed to $\theta_{\mu\nu}$.

one finds

$$(: e^{ip^\mu x_\mu} :)(: e^{ik^\nu x_\nu} :) =: e^{i(p\dot{+}k)^\mu x_\mu} : \tag{2.6}$$

The notation $\dot{+}$ introduced in [44] reflects the role of the "co-product" in the composition of momenta:

$$p_\mu \dot{+} k_\mu = \delta_{\mu,0}(p_0 + k_0) + (1 - \delta_{\mu,0})(p_\mu + e^{\frac{p_0}{\kappa}} k_\mu) \ . \tag{2.7}$$

As argued in [17] the non-linearity of the law of composition of momenta might require an absolute (observer-independent) momentum scale with the non-commutativity scale κ playing the role of such scale. This is analogous to the transition from newtonian mechanics to special relativity where upon introducing a non-linear law of composition of velocities one must introduce the absolute observer-independent scale of velocity c. On the basis of (2.7) one is led [16, 43, 46] to the following form of the energy-momentum dispersion relation

$$\left(2\kappa \sinh \frac{m}{2\kappa}\right)^2 = \left(2\kappa \sinh \frac{E}{2\kappa}\right)^2 - e^{\frac{E}{\kappa}} \vec{p}^{\,2} \tag{2.8}$$

which at leading order in $\frac{1}{\kappa}$ takes the form

$$m^2 \simeq E^2 - \vec{p}^{\,2} - \frac{E}{\kappa} \vec{p}^{\,2} \ . \tag{2.9}$$

The precise form of the dispersion relation depends on the choice of ordering prescription for wave exponentials [21] and this point deserves further study. There also appear to be severe obstructions [44, 46] for a satisfactory formulation of a quantum field theory in κ-Minkowski as the techniques that were rather straightforwardly applied for the construction of field theory in canonical non-commutative spacetime do not seem to be applicable in the κ-Minkowski case.

MDR in LQG

In the modern approach to canonical quantization of GR known as loop quantum gravity various interesting results have been obtained whose understanding, though, is still in a relatively early stage. As we already stressed, LQG predicts an inherently discretized spacetime. There has been much discussion recently, prompted by the studies [32, 33], of the possibility that this discretization might lead to broken Lorentz symmetry and a modified dispersion relation. Arguments presented in [32, 33] suggest that Lorentz symmetry might indeed be broken in LQG. However, recently it has been proposed [47] a mechanism such that LQG would be described at the most fundamental level as a theory that in the flat-spacetime limit admits deformed Lorentz symmetry, in the sense of DSR scenarios [17]. The argument presented in [47] originates from the role that certain quantum symmetry groups (q-deformed algebras) have in the LQG description of space-time with a cosmological constant, and observing that in the flat spacetime limit (the limit of vanishing cosmological constant) these quantum groups might not contract to a classical Lie algebra, but rather contract to a quantum (Hopf) algebra.

All these studies point to the presence of a MDR, although different arguments lead to different intuition for the form of the dispersion relation. A definite result might have to wait for the solution of the well-known classical-limit problem of LQG. However, as we will see in Chapter 3, a compelling argument can be made which relates the form of Planck-scale corrections to a particle's relativistic localization limit and QG corrections to the Bekenstein-Hawking entropy-area relation. This argument can be used to set a stringent constraint on the form of the LQG deformed dispersion relation based on the well established logarithmic form of the LQG correction to the black hole entropy area relation.

2.2 Phenomenology

Over the last few years a growing number of research groups have attempted to tackle the quantum gravity problem with an approach in which non-classical pictures of space-time are being studied with strong emphasis on their observable predictions. Indeed, certain classes of experiments turned out to have extremely high sensitivity to some non-classical features of spacetime. We now even have (see later) some first examples of experimental puzzles whose solution is being sought also within simple ideas involving non-classical pictures of spacetime. The hope in this line of research is that by trial and error, both on the theory side and on the experiment side, one might eventually stumble upon the first few definite (experimental) hints on quantum gravity.

We can observe that the most robust part of the results summarized in the previous subsections is clearly the emergence of a modified dispersion relation. Therefore if one could set up experiments testing directly the dispersion relation the resulting limits would have wide applicability. In principle one could investigate the form of the dispersion relation directly by making simultaneous measurements of energy and space momentum; however, it is easy to see that achieving Planck-scale sensitivity in such a direct test is well beyond our capabilities. Useful test theories on which to base the relevant phenomenology must therefore combine the presence of a MDR with other ingredients. There are three key issues which must be taken into account for such test theories: whether or not in presence of a MDR the relation $v = \frac{dE}{dp}$ between the speed of a particle and its dispersion relation is still valid; the validity of the standard laws of energy-momentum conservation; the formalism to be adopted for the description of dynamics. Unfortunately on these three key points the quantum space-time pictures which are providing motivation for the study of Planck scale modifications of the dispersion relation, which we reviewed in the previous subsections, are not providing much guidance yet. For example, in LQG, while we do have evidence that the dispersion relation should be modified, we do not yet have a clear indication concerning whether the law of energy-momentum conservation should also be modified and we also cannot yet robustly establish whether the relation $v = \frac{dE}{dp}$ should be preserved. Similarly in the analysis of NCSTs we are close to establishing in rather general terms that some modification of the dispersion relation is inevitable, but other aspects of the framework have not yet been clarified.

In the following subsections we will see examples of simple test theory models based on MDRs “at work”, and discuss how it could be possible to detect their signatures in the experimental contexts of cosmological gamma ray bursts (GRB) and ultra high energy cosmic rays (UHECR).

2.2.1 Gamma Ray Bursts

We focus in this section on a possible quantum space-time scenario with a general MDR of the form

$$m^2 = E^2 - \vec{p}^2 + \eta \vec{p}^2 \frac{E^n}{E_p^n} + O\left(\frac{E^n}{E_p^n}\right) \tag{2.10}$$

and in which a particle's velocity is given by the standard $v = \frac{dE}{dp}$ relation. In the energy range $m < E \ll E_p$ such velocity will be approximatively

$$v \simeq 1 - \frac{m^2}{2E^2} + \eta \frac{n+1}{2} \frac{E^n}{E_p^n}\,. \tag{2.11}$$

According to (2.11) two photons emitted simultaneously should reach a far-away detector at different times if they carry different energy. This time-of-arrival difference effect can be significant [49, 50] in the analysis of short duration GRBs that reach us from cosmological distances. For a GRB it is not uncommon that the time travelled before reaching our Earth detectors be of order $T \sim 10^{17} s$. Microbursts within a burst can have very short duration, as short as $10^{-3} s$ (or even $10^{-4} s$), and this means that the photons that compose such a microburst are all emitted at the same time, up to an uncertainty of $10^{-4} s$. Some of the photons in these bursts have energies that extend at least up to the GeV range. For two photons with energy difference of order $\Delta E \sim 1 GeV$ a $\eta \frac{\Delta E}{E_p}$ speed difference over a time of travel of $10^{17} s$ would lead to a difference in times of arrival of order

$$\Delta t \sim \eta T \frac{\Delta E}{E_p} \sim 10^{-2} s\,. \tag{2.12}$$

The sensitivities achievable with the next generation of gamma-ray telescopes, such as GLAST [51, 52], could allow to test very significantly (2.12) in the case

$n = 1$, by possibly pushing the limit on η far below 1 (whereas the effects found in the case $n = 2$, $|\eta| \sim 1$ are too small for GLAST). Whether or not these levels of sensitivity to the Planck-scale effects are actually achieved may depend on progress in understanding other aspects of GRB physics. In fact, the Planck-scale-effect analysis would be severely affected if there were poorly understood at-the-source correlations between energy of the photons and time of emission. Recently, it was emphasized [53] that it appears that one can infer such an energy/time-of-emission correlation from available GRB data. The studies of Planck-scale effects will be therefore confronted with a severe challenge of background/noise removal. At present it is difficult to guess whether this problem can be handled successfully. However a good point can be made observing that our Planck-scale picture predicts that the times of arrival should depend on energy in a way that grows in exactly linear way with the distance of the source. One may hope that, once a large enough sample of gamma-ray bursts (with known source distances) becomes available, one might be able disentangle the Planck-scale propagation effect from the at-the-source background.

2.2.2 UHECR and the the Bahcall-Waxman neutrino bound

Interest in tests of modifications of Lorentz symmetry has increased recently as a result of the realization [54, 55, 56, 57] that these modifications can provide one of the possible solutions of the so-called "cosmic-ray paradox". The spectrum of observed cosmic rays was expected to be affected by a cutoff at the scale $E_{\rm GZK} \sim 5{\cdot}10^{19}$ eV. Cosmic rays emitted with energy higher than $E_{\rm GZK}$ should interact with photons in the cosmic microwave background and lose energy by pion emission, so that their energy should have been reduced to the $E_{\rm GZK}$ level by the time they reach our Earth observatories. However, the AGASA observatory has reported [58] several observations of cosmic rays with energies exceeding the

$E_{\rm GZK}$ limit [59] by nearly one order of magnitude. As other experiments do not see an excess of particles above the GZK limit, this experimental puzzle will only be established when confirmed by larger observatories, such as Auger [60]. Furthermore, numerous other solutions have been discussed in the literature. Still, it is noteworthy that Planck-scale modifications of Lorentz symmetry can raise [54, 55] the threshold energy for pion production in collisions between cosmic rays and microwave photons, and the increase is sufficient to explain away the puzzle associated with the mentioned ultra-high-energy cosmic-ray observations. Bahcall-Waxman [61] have shown that the same particles which we observe as high-energy cosmic rays should also lead to neutrino production at the source. Using the observed cosmic ray fluxes they derive a bound (the Bahcall-Waxman bound) on the flux of high-energy neutrinos that can be revealed in astrophysics observatories. We show here that the same Lorentz symmetry violations that can extend the cosmic ray spectrum also affect the chain of processes that arise in the neutrino production and hence in establishing the Bahcall-Waxman bound [61]. Thus, the departures from Lorentz symmetry that are capable of explaining the "cosmic-ray paradox" inevitably lead to modification of this limit. We will only focus here on one of the chains of processes that are relevant for the Bahcall-Waxman bound: the case in which a proton at the source undergoes photo-pion interactions of the type $p+\gamma \to X+\pi^+$ before escaping the source, then giving rise to neutrino production through the decays $\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$.

The phenomenological model we consider is the simplest one in the literature, which evolved primarily through the studies reported in Refs. [48, 54, 55]. This is a kinematic in which the Planck-scale E_p enters the energy/momentum dispersion relation

$$m^2 = E^2 - \vec{p}^{\,2} + f(E,\vec{p};E_p) \simeq E^2 - \vec{p}^{\,2} + \eta\vec{p}^{\,2}\frac{E^n}{E_p^n} \tag{2.13}$$

while the laws of energy-momentum conservation remain unaffected[8] by the

[8]Our analysis based on (2.13) and standard energy-momentum conservation should be applicable (up to small numerical modifications) to a large class of

Planck scale. η is a dimensionless coefficient which one expects to be roughly of order 1 (but cannot be reliably predicted at the present preliminary level of development of the relevant quantum-gravity models). The power n, which should also be treated as a phenomenological parameter, is a key element of this phenomelogical scenario, since it characterizes the first nonvanishing contribution in a (inverse-)Planck-scale power series of the quantum-gravity-induced correction $f(E, \vec{p}; E_p)$. It is usually expected that $n = 1$ and $n = 2$ are most likely.

In the next subsection we revisit the analysis of the emergence of "threshold anomalies" due to (2.13) in the study of particle production in collision processes. For positive η ($\eta \sim 1$) and $n \leq 2$, according to the Planck-scale effect (2.13) one expects [55] an increase in the threshold energy for pion production in collisions between cosmic rays and microwave photons, and the increase is sufficient to explain away the GZK puzzle raised by the AGASA observations. We also comment on another potentially observable threshold anomaly that concerns electron-positron pair production in photon-photon collisions and emphasize the differences between the case of positive η and the case of negative η.

Next we show that (2.13) also affects significantly the at-the-source processes of the type $p + \gamma \to X + \pi^+$ that are relevant for the Bahcall-Waxman analysis. If $\eta \sim 1$ and $n \leq 2$ (*i.e.* for the same departures from Lorentz symmetry that would explain the cosmic-ray paradox) (2.13) leads to the prediction of a strongly reduced probability for the process $p + \gamma \to X + \pi^+$ to occur before the proton escapes the source. Correspondingly one expects sharply reduced neutrino production, and as a result the "quantum-gravity-modified Bahcall-Waxman bound" should be expected to be many orders of magnitude lower than

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quantum-gravity models which are being considered as possible solutions of the cosmic-ray paradox. One exception is the DSR framework [17], in which one could adopt a dispersion relation of type (2.13) but it would then be necessary to introduce a corresponding modification of the laws of energy-momentum conservation (in order to avoid the emergence of a preferred class of inertial observers [17]). Our conclusions are not applicable to that scheme.

the standard Bahcall-Waxman bound. The opposite effect is found for negative η ($\eta \sim -1$): in that case one would expect the standard Bahcall-Waxman bound to be violated, *i.e.* for negative η one could find a neutrino flux that exceeds the standard Bahcall-Waxman bound.

In the last subsection we show that also the decays $\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ are significantly affected by the Planck-scale effect (2.13). Again the effect goes in the direction of reducing neutrino production for positive η. However we also observe that the dominant quantum-gravity modification of the Bahcall-Waxman bound comes at the level of analysis of processes of the type $p + \gamma \to X + \pi^+$, where a several-order-of-magnitude modification would be expected, whereas the additional modification encountered at the level of the processes $\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ is not as significant.

Previous results on Planck-scale-induced threshold anomalies and the sign of η

We begin by considering the implications of (2.13) for the analysis of processes of the type $1 + \gamma \to 2 + 3$. The key point for us is that, for a given energy E_1 of the particle that collides with the photon, there is of course a minimal energy ϵ_{min} of the photon (γ) in order for the process to be kinematically allowed, and therefore achieve production of the particles 2 and 3. One finds that if η is positive the value of ϵ_{min} predicted according to the Planck-scale effect (2.13) is higher than the corresponding value obtained using ordinary Lorentz symmetry. In the applications that are of interest here the particle that collides with the photon has a very high energy, $E_1 \simeq p_1 \gg m_1$, and its energy is also much larger than the energy of the photon with which it collides $E_1 \gg \epsilon$. This will allow some useful simplifications in the analysis.

Let us start by briefly summarizing the familiar derivation of ϵ_{min} in the ordinary Lorentz-invariant case. At the threshold (no momenta in the CM frame after the collision) energy conservation and momentum conservation become one dimensional:

$$E_1 + \epsilon = E_2 + E_3 \ , \tag{2.14}$$

$$p_1 - q = p_2 + p_3 \ , \tag{2.15}$$

where q is the photon's momentum. The ordinary Lorentz-invariant relations are

$$q = \epsilon \ , \quad E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \ , \tag{2.16}$$

where we have assumed that, since E_1 is large as mentioned, E_2 and E_3 are also large ($E_{2,3} \simeq p_{2,3} \gg m_{2,3}$).

The threshold conditions are usually identified by transforming these laboratory-frame relations into center-of-mass-frame relations and imposing that the center-of-mass energy be equal to $m_2 + m_3$. However, in preparation for the discussion of deformations of Lorentz invariance it is useful to work fully in the context of the laboratory frame. There the threshold condition that characterizes ϵ_{min} can be identified with the requirement that the solutions for E_2 and E_3 as functions of ϵ (with a given value of E_1) that follow from (2.14), (2.15) and (2.16) should be imaginary for $\epsilon < \epsilon_{min}$ and should be real for $\epsilon \geq \epsilon_{min}$. This straightforwardly leads to

$$\epsilon \geq \epsilon_{min} \simeq \frac{(m_2 + m_3)^2 - m_1^2}{4E_1} \ . \tag{2.17}$$

This standard Lorentz-invariant analysis is modified [54, 55] by the deformations codified in (2.13). The key point is that Eq. (2.16) is replaced by

$$\epsilon = q - \eta \frac{q^{n+1}}{2E_p^n} \ , \quad E_i \simeq p_i + \frac{m_i^2}{2p_i} - \eta \frac{p_i^{n+1}}{2E_p^n} \ . \tag{2.18}$$

Combining (2.14), (2.15) and (2.18) one obtains a modified kinematical requirement

$$\epsilon \geq \epsilon_{min} \simeq \frac{(m_2 + m_3)^2 - m_1^2}{4E_1} + \eta \frac{E_1^{n+1}}{4E_p^n} \left(1 - \frac{m_2^{n+1} + m_3^{n+1}}{(m_2 + m_3)^{n+1}} \right) \ . \tag{2.19}$$

where we have included only the leading corrections (terms suppressed by both the smallness of E_p^{-1} and the smallness of ϵ or m were neglected).

The Planck-scale "threshold anomaly" [55] described by (2.19) is relevant for the analysis of the GZK limit in cosmic-ray physics. In fact, the GZK limit essentially corresponds to the maximum energy allowed of a proton in order to travel in the CMBR without undergoing processes of the type $p + \gamma \rightarrow p + \pi$. For a proton of energy $E_1 \sim 5{\cdot}10^{19} eV$ the value of ϵ_{min} obtained from the undeformed equation (2.17) is such that CMBR photons can effectively act as targets for photopion production. But, for $\eta \sim 1$ and $n \leq 2$, the value of ϵ_{min} obtained from the Planck-scale deformed equation (2.19) places CMBR photons below threshold for photo-pion production by protons with energies as high as $E_1 \sim 10^{21} eV$, and would explain [54, 55] observations of cosmic rays above the GZK limit. For negative $\eta \sim -1$ one obtains the opposite result: photopion production should be even more efficient than in the standard case. Therefore negative η is disfavoured by the observations reported by various UHECR observations.

There has also been some interest [54, 55, 62] in the implications of (2.19) for electron-positron pair production in collisions between astrophysical high-energy photons and the photons of the Far Infrared Background. Electron-positron pair production should start to be significant when the high-energy photon has energies of about 10 or 20 TeV. The Planck-scale correction in (2.19) would be significant, though not dominant, at those energies. Observations of TeV photons are becoming more abundant, but the field is still relatively young. Moreover, our knowledge of the Far Infrared Background is presently not as good as our knowledge of the CMBR. Therefore observations of TeV photons do not yet provide a significant insight on the Planck scale physics of interest here. Consistency with those observations only imposes a constraint [62] of the type $|\eta| < 100$, which (since the quantum-gravity intuition favours $|\eta| \sim 1$) is not yet significant from a quantum-gravity perspective.

A similar upper limit ($|\eta| < 100$) is obtained by considering the implications [48, 50] of the deformed dispersion relation for the arrival times of photons with different energies emitted (nearly-)simultaneously from cosmological sources.

In summary, the present situation justifies some interest for the case of positive η, particularly as a possible description of cosmic rays above the GZK limit. The case of negative η is disfavored by various UHECR observations. Additional phenomenological reasons to disfavour negative η have been found in analyses of photon stability (see, *e.g.*, [62]), which is instead not relevant for the positive η case. Moreover, the case of negative η appears to be also troublesome conceptually since it leads to superluminal velocities in a framework, such as the one adopted here, in which the new effects are simply motivated by the idea of a quantum-spacetime medium[9], and therefore do not naturally lead to the expectation of superluminal velocities. Still, as a contribution to this evolving understanding, we will consider the Bahcall-Waxman bound both for positive and negative η.

Planck-scale-induced threshold anomalies and the neutrino bound

A key observation for our analysis comes from the fact that the Planck-scale threshold anomaly described by (2.19) is significant for the Bahcall-Waxman bound for the same reasons that render it significant for the GZK limit in cosmic-ray physics. In fact, both the Bahcall-Waxman bound and the GZK limit involve the analysis of processes of the type $p + \gamma \rightarrow X + \pi$ in which high-energy proton collides with a softer photon. In the case of the Bahcall-Waxman bound one finds that for a proton of energy $E_1 \sim 10^{19} eV$ which is emerging from a source (*e.g.* an AGN), according to the standard kinematical requirement (2.17) the photons in the environment that are eligible for production of charged pions π^+

[9] A deformed dispersion relation is generically expected in a special relativistic theory when a medium is present. The presence of the medium does not alter the principles of special relativity, and superluminal velocities should not be allowed. The situation is different in the context of the approach proposed in [17], in which the deformed dispersion relation is not motivated by the presence of a quantum-spacetime medium but rather by a role for the Planck scale in the relativity principles. In the framework of [17] superluminal velocities would not lead to paradoxical results.

are all the photons with energy $\epsilon \geq \epsilon_{min} \sim 0.01eV$. But, for the Planck-scale scenario of (2.19) with $\eta \sim 1$ and $n = 1$ far fewer photons in the environment, viz. only photons with energy $\epsilon \geq \epsilon_{min} \sim 10^9 eV$, are kinematically eligible for production of charged pions. In a typical source the abundance of photons with $\epsilon \geq 10^9 eV$ is much smaller, by several orders of magnitude, than the abundance of photons with $\epsilon \geq 0.01eV$. Correspondingly the Planck-scale effect predicts a huge reduction in the probability that a charged pion be produced before the proton escapes the source, and in turn this leads (for $\eta > 0$ and $n = 1$) to a decrease in the expected high-energy neutrinos flux by many orders of magnitude below the level set by the Bahcall-Waxman bound.

The same qualitative picture applies to the case $\eta \sim 1$, $n = 2$, although the effect is somewhat less dramatic because of the large suppression of the effect that is due to the extra power of the Planck scale. In fact, for $\eta \sim 1$, $n = 2$ one finds that the photons in the environment that are energetically enough for the production of charged pions must have energy $\epsilon \geq \epsilon_{min} \sim 1eV$.

Whereas for positive η the Planck-scale effect leads to a lower neutrino bound the reverse is true for negative η. In particular, for $\eta \sim -1$, $n \leq 2$ from (2.19) it follows that photons in the source with energies even below[10] $0.01eV$ are viable targets for the production of charged pions by protons with energy $E_1 \sim 10^{19} eV$. Correspondingly, the Bahcall-Waxman bound would be weakened.

Implications of the Planck-scale for particle decays and the neutrino bound

In the previous subsection we have shown that the Planck-scale effects considered here would affect the production of charged pions before the ultra-high-

[10] Formally in this case (2.19) even admits photon targets with "negative energies". But, of course, considering the approximations we implemented, one can only robustly infer that photons with very low energies can lead to pion production.

energy cosmic-ray proton escapes the source. In this Section we analyze the implications of the same effects for the decay processes $\pi^+ \rightarrow \mu^+ + \nu_\mu$ and $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ which are also relevant for the Bahcall-Waxman bound.

Since we are interested in both a two-body decay, $\pi^+ \rightarrow \mu^+ + \nu_\mu$, and a three-body decay, $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ it is convenient for us to obtain a general result for N-body decays. This will also be a technical contribution to the study of the kinematics governed by (2.13). In fact, the implications of (2.13) for two-body decays have been previously analyzed [63], but for decays in three or more particles there are no previous results in the literature.

We start our analysis of the decay $A \rightarrow 1 + 2 + ... + N$ (A is the generic particle that decays into particles $1, 2, ...N$) with the energy-momentum conservation laws:

$$E_A = E_1 + E_2 + ... + E_N \tag{2.20}$$

$$\vec{p}_A = \vec{p}_1 + \vec{p}_2 + ... + \vec{p}_N \tag{2.21}$$

Denoting with θ_{ij} the angle between the linear momentum of particle i and that of particle j, and denoting with p the modulus of the 3-vector $\vec{p}$, we can use (2.21) to obtain ($i, j = 1, 2, ...N$)

$$p_A^2 = \sum_i p_i^2 + \sum_{i \neq j} p_i p_j \cos\theta_{ij} \tag{2.22}$$

and (2.20) to get

$$E_A^2 = \sum_i E_i^2 + \sum_{i \neq j} E_i E_j \ . \tag{2.23}$$

From (2.22) and (2.23) it follows that

$$E_A^2 - p_A^2 = \sum_i (E_i^2 - p_i^2) + \sum_{i \neq j} (E_i E_j - p_i p_j \cos\theta_{ij}) \ . \tag{2.24}$$

Next we use the deformed dispersion relation (2.13), $E^2 - p^2 = m^2 - \eta E^n p^2 / E_p^n$, to obtain

$$m_A^2 - \eta E_A^2 p_A^2 / E_p^2 = \sum_i (m_i^2 - \eta E_i^n p_i^2 / E_p^n) + \sum_{i \neq j} (E_i E_j - p_i p_j \cos\theta_{ij}) \tag{2.25}$$

For simplicity let us consider separately the cases $n = 1$ and $n = 2$, starting with $n = 1$. It is convenient to rewrite the kinematical condition (2.25), for $n = 1$, in the following way

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} \left(p_i p_j + p_i \frac{m_j^2}{p_j} \right) + \frac{\eta}{E_p} \left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) = \sum_{i \neq j} p_i p_j \cos\theta_{ij} \tag{2.26}$$

where we used again the deformed dispersion relation,

$$E = (p^2 + m^2 - \frac{\eta}{E_p} E p^2)^{\frac{1}{2}} \simeq p - \frac{\eta}{2E_p} p^2 + \frac{m^2}{2p} \ . \tag{2.27}$$

We are neglecting terms of order E_p^{-2} and higher, which are clearly subleading, and we are also neglecting terms of order $E_p^{-1} m^2$ which are negligible compared to terms of order $E_p^{-1} E^2$ since all particles involved in the processes of interest to us have very high momentum.

Using the fact that $\cos\theta_{ij} \leq 1$ for every θ_{ij}, it follows from (2.26) that for the decay to be kinematically allowed a necessary condition is

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} \left(p_i p_j + p_i \frac{m_j^2}{p_j} \right) + \frac{\eta}{E_p} \left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) \leq \sum_{i \neq j} p_i p_j \tag{2.28}$$

or equivalently

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} p_i \frac{m_j^2}{p_j} + \frac{\eta}{E_p} \left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) \leq 0 \tag{2.29}$$

In the analysis of particle-decay processes relations of the type (2.29) impose constraints on the available phase space. For positive η the quantum-gravity effect clearly goes in the direction of reducing the available phase space; in fact, it is easily seen that

$$\left(E_A^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) = \left((\sum_i E_i)^3 - \sum_i E_i^3 - \sum_{i \neq j} E_i E_j^2 \right) > 0 \ . \tag{2.30}$$

The correction is completely negligible as long as $m_A^2 \gg E_A^3/E_p$, but for $m_A^2 \ll E_A^3/E_p$ there is clearly a portion of phase space in which, for positive η, condition

(2.29) is not satisfied. (Think for example of the case $E_1 \sim E_2 \sim ... \sim E_N \sim E_A/N$.) Starting at $E_A \geq (m_A^2 E_p)^{1/3}$ the phase space available for the decay of particle A is gradually reduced as E_A increases. The difference between the standard Lorentz-symmetry prediction for the lifetime and the quantum-gravity-corrected prediction becomes more and more significant as the energy of the decaying particle is increased, and goes in the direction of rendering the particle more stable, i.e., rendering the decay more unlikely. For the relevant decays of pions and muons we expect that the quantum-gravity effect starts being important at pion/muon energies of order $(m_\pi^2 E_p)^{1/3} \sim (m_\mu^2 E_p)^{1/3} \sim 10^{15} eV$. For positive η it is inevitable that at some energy a significant suppression of the decay probability kicks in, while for negative η it is easy to see that there is no effect on the size of the phase space available for the decays. The change in the sign of η turns as usual into a change of sign of the effect, which would go in the direction of extending the phase space available for the decay, but the relevant portion of parameter space (some neighborhood of $E_1 \sim E_2 \sim ... \sim E_N \sim E_A/N$) is already allowed even without the Planck scale effect, so for negative-η effect is not significant. Completely analogous considerations apply to the case $n = 2$. From (2.25), for $n = 2$, one obtains

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} \left(p_i p_j + p_i \frac{m_j^2}{p_j} \right) + \frac{\eta}{E_p^2} \left(E_A^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) = \sum_{i \neq j} p_i p_j \cos\theta_{ij} \, , \tag{2.31}$$

and then, just following the same line of analysis we already adopted for the case $n = 1$, for $n = 2$ one finds that the decay is only allowed if

$$\sum_i m_i^2 - m_A^2 + \sum_{i \neq j} p_i \frac{m_j^2}{p_j} + \frac{\eta}{E_p^2} \left(E_A^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) \leq 0 \, . \tag{2.32}$$

Since

$$\left(E_A^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) = \left((\sum_i E_i)^4 - \sum_i E_i^4 - \sum_{i \neq j} E_i E_j^3 \right) > 0 \, , \tag{2.33}$$

we find again that for positive η the quantum-gravity correction inevitably goes in the direction of reducing the available phase space and therefore rendering the decay more unlikely. In this $n = 2$ case we expect that the quantum-gravity suppression of the decay probability starts being important at pion/muon energies of order $(m_\pi E_p)^{1/2} \sim (m_\mu E_p)^{1/2} \sim 10^{18} eV$. Of course also for $n = 2$ one finds that the case of negative η does not have significant implications, for exactly the same reasons discussed above in considering the $n = 1$ case.

To summarize: we found that Planck-scale effects can have important implications for the neutrino-producing chain of processes $p+\gamma \rightarrow X+\pi^+$, $\pi^+ \rightarrow \mu^+ + \nu_\mu$, $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$, which are relevant for the Bahcall-Waxman bound. We focused on a single simple example of Planck-scale kinematics. Because of the simple kinematical origin of our argument it is reasonable to expect that these more detailed studies will confirm that, for positive η, the quantum-gravity effect leads to a neutrino flux that is many orders of magnitude below the level allowed by the Bahcall-Waxman bound. We have shown here that this is due primarily to a strong suppression of the production of high-energy charged pions by protons at the source. If future observations give us a neutrino flux which is close to the level allowed by the Bahcall-Waxman bound, the type of quantum-gravity physics considered here would be excluded (for positive η). On the other hand, a low neutrino flux will be harder to interpret as, a priori, it is not clear that the Bahcall-Waxman bound should be saturated.

The suppression present for positive η already found full support at the first step in the chain of processes, in the collisions $p + \gamma \rightarrow X + \pi^+$. We felt however that it was appropriate to consider also the processes further down in the chain, in the decays of $\pi^+ \rightarrow \mu^+ + \nu_\mu$, $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$. In fact, it was conceivable that perhaps at that level the production of neutrinos might receive a compensating boost form the quantum-gravity effect which, for positive η, suppresses the likelihood of the process $p+\gamma \rightarrow X+\pi^+$. This turned out not to be the case: for positive η one actually expects a further suppression of neutrino production, since the quantum-gravity effect renders ultrahigh-energy pions and muons more stable. This part of our analysis also provided a technical contribution to the study of the kinematics governed by (2.13), since previously the implications of (2.13) were only known for two-body decays, while here we obtained a generalization to N-body decays for arbitrary N.

We saw that the case of negative η is disfavoured conceptually and starts to be strongly constrained by preliminary observations in astrophysics. We found that it would have striking consequences for the Bahcall-Waxman bound: in the case of negative η the modification of the Bahcall-Waxman bound would amount to violating (raising) the Bahcall-Waxman bound by several orders of magnitude.

The analysis we presented contributes to ongoing work aimed at establishing a web of consequences of the type of Planck-scale kinematics considered here. It is not hard to find several different solutions to a single anomaly in ultrahigh-energy astrophysics, *e.g.* the cosmic-ray paradox, if confirmed by other observatories. However for the type of Planck-scale kinematics considered here, there are several correlated predictions and these together can be used to favor or rule out the scenario. In particular, evidence supporting both a cosmic-ray paradox and an unexpectedly low ultrahigh-energy neutrino flux would fit naturally within the Planck-scale-kinematics scenario (with positive η). More precisely, evidence supporting both a cosmic-ray paradox and an unexpectedly low ultrahigh-energy neutrino flux would favor solutions of the cosmic-ray paradox based on violations of Lorentz symmetry with respect to other proposed solutions of the cosmic-ray paradox. In fact, the correlation we have exposed here between a cosmic-ray paradox and a lowered Bahcall-Waxman bound is a characteristic of models in which the kinematics of the processes $p + \gamma \to X + \pi$ is modified by a violation of Lorentz symmetry. In fact, the processes $p + \gamma \to X + \pi^{\pm,0}$ dominate the GZK threshold and the Bahcall-Waxman limit. An increase in the GZK limit and a lowered Bahcall-Waxman bound are found whenever the violation of Lorentz symmetry causes an increased energy-threshold condition for the processes $p + \gamma \to X + \pi$. Therefore it can distinguish between models with and without this Lorentz-violation effect, but it cannot establish whether the origin of the Lorentz violation is connected with quantum gravity. In particular, it is interesting to consider the suggestion of Coleman and Glashow [56] concerning a specific Lorentz-violation solution of the GZK paradox, which is not motivated by quantum gravity. Coleman and Glashow [56] consider a scheme in which different particles have a different "maximum attainable speed" (essentially a different "speed-of-light constant" for different particles). This can be cast into our formalism with a particle-dependent η and with $n = 0$. It follows from our analysis that any model that resolves the UHECR GZK paradox using the Coleman-Glashow scheme will also lead to a stronger Bahcall-Waxman neutrino bound.

Chapter 3

Planck-scale particle localization limit and black hole entropy

As we stressed in the Introduction and in the previous Chapter, the Planck length, L_p, seems to set the ultimate limit to measurements of distances when both quantum and gravitational effects are taken into account. The simplest heuristic arguments supporting this conclusion states that in order to localize a particle with accuracy $\delta x < L_p$ one needs a probe which, in its interaction with the observed particle, will exchange an energy sufficient to create an event horizon thus preventing the localization procedure from completing successfully.[11]
Historically the first limitation to the spatial localization of a particle was given by Heisenberg's uncertainty relation

$$\delta x \geq 1/\delta p \,, \tag{3.1}$$

valid for non-relativistic quantum mechanics. The generalization to the relativistic realm was proposed by Landau and Peierls [64] and takes the form of the well known relativistic particle localization limit

$$\delta x \geq 1/E \,, \tag{3.2}$$

[11]For a review of the role of a minimal length in quantum gravity and a series of arguments supporting the existence of L_p as a localization limit see [10].

in which E is the energy of the particle.

While in the non-relativistic case there is no absolute limitation to the localization accuracy for a particle of given energy E (the only price to pay for a very high accuracy is a corresponding large uncertainty in the particle's momentum) in relativistic quantum mechanics one must take into account the possibility of particle production which introduces a further, absolute, limitation. Even in the presence of this additional limitation it is still possible to introduce the concept of a *sharply localized* particle in the limit of very large energy. General relativity introduces an obstruction to the idea of sharp localization because of the distortion in the geometry that a very large energy density would produce, eventually leading to gravitational collapse. Indeed the concept itself of classical space-time point, from a quantum localization point of view, loses operative meaning due to the presence of the minimal length L_p.

It is natural to expect that the impossibility of sub-Planckian localization caused by the convergence of quantum and gravitational effects should affect in some way the bound (3.2). In fact, as we will see later in this Chapter, Planck-scale modifications of (3.2) are expected in quantum gravity scenarios in which relativistic kinematics is altered by the presence of a deformed energy-momentum dispersion relation of the type considered in the previous Chapter.

The relativistic particle localization limit (3.2) unexpectedly makes its appearance in another, apparently unrelated, context in which gravity and the quantum interact: the black hole area-entropy relation. The intuition that the entropy of a black hole should be proportional to its (horizon-surface) area, up to corrections that can be neglected when the area A is much larger than the square of the Planck length L_p, has provided an important element of guidance for quantum-gravity research. It is noteworthy that, as shown by Bekenstein [9], this contribution to black hole entropy can be obtained from very simple ingredients. One starts from the GR result [65] that the minimum increase of area when the black hole absorbs a classical particle of energy E and size s is

$\Delta A \simeq 8\pi L_p^2 E s$. Taking into account the quantum properties of particles one can estimate s as roughly given by the position uncertainty $s \sim \delta x \sim 1/E$ leading to the conclusion[9, 67] that the minimum change in the black-hole area must be of order L_p^2, independently of the size of the area. Then using the fact that, also independently of the size of the area, this minimum increase of area should correspond to the minimum ("one bit") change of entropy one easily obtains [9] the proportionality between black-hole entropy and area.

It is remarkable that, in spite of the humble ingredients of this Bekenstein analysis, the entropy-area relation introduced such a valuable constraint for quantum-gravity research. And a rather challenging constraint, since attempts to reproduce the entropy-area-linearity result using directly some quantum properties of black holes were unsuccessful for nearly three decades. But over the last few years both in String Theory and LQG the needed techniques for the analysis of entropy on the basis of quantum properties of black holes were developed. These results [68, 69, 70, 71] now go even beyond the entropy-area-proportionality contribution: they establish that the leading correction should be of log-area type, so that one expects (for $A \gg L_p^2$) an entropy-area relation for black holes of the type

$$S = \frac{A}{4L_p^2} + \rho \ln \frac{A}{L_p^2} + O\left(\frac{L_p^2}{A}\right) \ . \tag{3.3}$$

For the case of loop quantum gravity, on which we will focus here, there is still no consensus on the coefficient of the logarithmic correction, ρ, but it is established [69, 70, 71] that there are no correction terms with stronger-than-logarithimic dependence on the area.

We observe that the availability of results on the log-area correction might provide motivation for reversing the Bekenstein argument: the knowledge of black-hole entropy up to the leading log correction can be used to establish the Planck-scale modifications of the ingredients of the Bekenstein analysis.

In particular, the mentioned role of the relation $E \geq 1/\delta x$ in the Bekenstein analysis appears to provide an opportunity to put under scrutiny some scenarios for the energy-momentum dispersion relation in LQG. Several recent studies have tentatively argued that the LQG dispersion relation might involve a term with a linear dependence on the Planck length, and, as we observe in the next Section this in turn requires a Planck-length modification of the relation $E \geq 1/\delta x$ between the energy and position uncertainty of a particle. However, as we show in Section 3.2, the resulting modification of the $E \geq 1/\delta x$ relation would in turn lead, following the Bekenstein argument, to a contribution to black-hole entropy that goes like the square root of the area. Since such a square-root contribution is, as mentioned, excluded by direct analysis of black-hole entropy in LQG, we conclude that the presence in the energy-momentum dispersion relation of a term with linear dependence on the Planck length is also excluded.

3.1 LQG dispersion relation and its implications for the $E \geq 1/\delta x$ relation

As we stressed in Chapter 2, the possibility of Planck-scale modifications of the dispersion relation has been considered extensively in the recent quantum gravity literature and in particular in LQG [32, 33, 72, 47]. Some LQG calculations [32, 33] provide support for the idea of an energy-momentum dispersion relation that for a particle of high energy would take the approximate form

$$E \simeq p + \frac{m^2}{2p} + \alpha L_p E^2 \ , \tag{3.4}$$

where α is a coefficient of order 1. However, these results must be viewed as preliminary [72, 47] since they essentially consider perturbations of "weave states" [32, 33], rather than perturbations of the ground state of the theory. It is not surprising (and therefore not necessarily insightful) that there would be some states of the theory whose excitations have a modified spectrum. If instead a relation of the type (3.4) were applicable to excitations of the ground state of the theory this would provide a striking characteristic LQG.

Several papers have been devoted to the derivation of tighter and tighter experimental limits on coefficients of the α type for LQG (see, *e.g.*, Ref. [73] and references therein). We intend to show here that the linear-in-L_p term can be excluded already on theoretical grounds, because of an inconsistency with the black-hole-entropy results.

Here we start by observing that a modified dispersion relation implies a modification of the relation $E \geq 1/\delta x$ between the energy of a particle and its position uncertainty. We can see this by simply following the familiar derivation [66] of the relation $E \geq 1/\delta x$, substituting, where applicable, the standard special-relativistic dispersion relation with the Planck-scale modified dispersion relation. It is convenient to focus first [66] on the case of a particle of mass M at rest, whose position is being measured by a procedure involving a collision with a photon of energy E_γ and momentum p_γ. In order to measure the particle position with precision δx one should use a photon with momentum uncertainty $\delta p_\gamma \geq 1/\delta x$. Following the standard argument [66], one takes this $\delta p_\gamma \geq 1/\delta x$ relation and converts it into the relation $\delta E_\gamma \geq 1/\delta x$, using the special-relativistic dispersion relation, and then the relation $\delta E_\gamma \geq 1/\delta x$ is converted into the relation $M \geq 1/\delta x$ because the measurement procedure requires[12] $M \geq \delta E_\gamma$. If indeed LQG hosts a Planck-scale-modified dispersion relation of the form (3.4), it is easy to see that, following the same reasoning, one would obtain from $\delta p_\gamma \geq 1/\delta x$ the requirement $M \geq (1/\delta x)[1 + 2\alpha(L_p/\delta x)]$.

These results strictly apply only to the measurement of the position of a particle at rest, but they can be straightforwardly generalized [66] (simply using a boost) to the case of measurement of the position of a particle of energy E. In the case of the standard dispersion relation (without Planck-scale modification) one

[12] One must take into account the fact [66] that the measurement procedure should ensure that the relevant energy uncertainties are not large enough to possibly produce extra copies of the particle whose position one intends to measure.

obtains the familiar $E \geq 1/\delta x$. In the case of (3.4) one instead easily finds that

$$E \geq \frac{1}{\delta x}\left(1 + 2\alpha \frac{L_p}{\delta x}\right) \ . \tag{3.5}$$

3.2 A requirement of consistency with the black-hole entropy analysis

We now intend to show that the linear-in-L_p modification of the relation between the energy of a particle and its position uncertainty, which follows from the corresponding modification of the energy-momentum dispersion relation, should be disallowed in LQG since it leads to a contribution to the black-hole entropy-area relation which has already been excluded in direct black-hole-entropy analyses.

We do this by following the original Bekenstein argument [9]. As done in [9] we take as starting point the general-relativistic result which establishes that the area of a black hole changes according to $\Delta A \geq 8\pi E s$ when a classical particle of energy E and size s is absorbed. In order to describe the absorption of a quantum particle one must describe the size of the particle in terms of the uncertainty in its position [9, 67], $s \sim \delta x$, and take into account a "calibration[13] factor" [74, 75, 76] $(\ln 2)/2\pi$ that connects the $\Delta A \geq 8\pi E s$ classical-particle result with the quantum-particle estimate $\Delta A \geq 4(\ln 2) L_p^2 E \delta x$. Following the

[13]Clearly some calibration is needed in order to adapt the classical-gravity result for absorption of a classical particle to the case of a quantum black hole absorbing a quantum particle. In particular, a calibration should arise in the description of a quantum particle with position uncertainty δx in terms of a classical particle of size s. A direct evaluation of the calibration coefficient within quantum gravity is presently beyond reach; however, several authors (see, *e.g.*, Refs. [74, 75, 76]) have used the independent analysis of black-hole entropy by Hawking [77] to infer indirectly this calibration needed in the Bekenstein argument. We adopt this calibration for consistency with previous literature, but the careful reader will notice that this calibration does not affect our line of analysis (the calibration could be reabsorbed in the free parameter α).

original Bekenstein argument [9] one then enforces the relation $E \geq 1/\delta x$ (and this leads to $\Delta A \geq 4(\ln 2) L_p^2$), but we must take into account the Planck-length modification in (3.5), obtaining

$$\Delta A \geq 4(\ln 2)\left[L_p^2 + 2\frac{\alpha L_p^3}{\delta x}\right] \simeq 4(\ln 2)\left[L_p^2 + 2\frac{\alpha L_p^3}{R_S}\right] \simeq 4(\ln 2)\left[L_p^2 + \frac{\alpha 4\sqrt{\pi} L_p^3}{\sqrt{A}}\right] ,$$

where we also used the fact that in falling in the black hole the particle acquires [75, 78, 79] position uncertainty $\delta x \sim R_S$, where R_S is the Schwarzschild radius (and of course $A = 4\pi R_S^2$).

Next, following again Bekenstein [9], one assumes that the entropy depends only on the area of the black hole, and one uses the fact that according to information theory the minimum increase of entropy should be $\ln 2$, independently of the value of the area:

$$\frac{dS}{dA} \simeq \frac{min(\Delta S)}{min(\Delta A)} \simeq \frac{\ln 2}{4(\ln 2) L_p^2 \left[1 + \alpha 4\sqrt{\pi}\frac{L_p}{\sqrt{A}}\right]} \simeq \left(\frac{1}{4L_p^2} - \frac{\alpha\sqrt{\pi}}{L_p\sqrt{A}}\right) . \tag{3.6}$$

From this one easily obtains (up to an irrelevant constant contribution to entropy):

$$S \simeq \frac{A}{4L_p^2} - 2\alpha\sqrt{\pi}\frac{\sqrt{A}}{L_p} . \tag{3.7}$$

We therefore conclude that when a quantum-gravity theory predicts the presence of a linear-in-L_p contribution to the energy-momentum dispersion relation it should correspondingly predict the presence of $\sqrt{A}$ contribution to black-hole entropy. Since in LQG such a $\sqrt{A}$ contribution to black-hole entropy has already been excluded [69, 70, 71] in direct black-hole entropy studies, we conclude that in LQG the presence of linear-in-L_p contributions to the energy-momentum dispersion relation is excluded.

It is instead plausible that Loop Quantum Gravity might host a dispersion relation of the type

$$E \simeq p + \frac{m^2}{2p} + \tilde{\alpha} L_p^2 E^3 , \tag{3.8}$$

with a quadratic-in-L_p contribution. In fact, the careful reader can easily adapt our analysis to the case of the dispersion relation (3.8), finding that the quadratic-in-L_p contribution to the dispersion relation ultimately leads to a leading correction to the black-hole-entropy formula which is of log-area type, consistently with the indications obtained in direct black-hole entropy studies [69, 70, 71].

Chapter 4

Black hole thermodynamics in quantum space-time

Various arguments suggest that the description of black holes should be an important aspect of a quantum gravity theory, and that some key operatively meaningful (not merely formal) differences between alternative theories should emerge as we establish, within each approach, how the singularity, the evaporation, the "thermodynamics", and the "information paradox" are handled. A similar role could be played, by the analysis of the energy-momentum dispersion relation and the position-momentum uncertainty principle. As we saw in the previous Chapters approaches to the quantum gravity problem lead to different expectations for what concerns the possibility of a MDR, in particular, in LQG and of models based on NG there has been strong interest [32, 33, 47, 42] in some candidate modifications of the energy-momentum dispersion relation. The possibility of a quantum gravity motivated generalized position-momentum uncertainty principle (GUP) has also been considered. GUPs [80, 10] have been studied primarily in the literature on String Theory [81] and on models based on NCG [11]. The form of the energy-momentum dispersion relation and of the position-momentum uncertainty relation can therefore be used to characterize alternative approaches to the quantum-gravity problem.

In the previous Chapter we studied a possible link between the predictions that a quantum gravity theory makes for black hole thermodynamics and the predictions that the same theory makes for the energy-momentum dispersion relation. By establishing the nature of such a link one would, in our opinion, obtain a valuable characterization of the type of internal logical consistency that various aspects of a quantum-gravity theory should satisfy.

In this Chapter, following the strategy outlined in Chapter 3, we attempt to give the first elements of a general analysis of some key characteristics of black-hole physics, as affected by some scenarios for a MDR or a GUP.

Sections 4.2, 4.3 and 4.4, set the stage, by reviewing some results in the MDR and GUP literature and revisiting the point made in Chapter 3 generalizing the link between the log-area terms in the entropy-area relation for black holes and certain formulations of the MDR and the GUP. In Section 4.5 we explore the implications of a MDR and/or a GUP for the Bekenstein entropy bound and for the Generalized Second Law of thermodynamics. We find that the implications are significant and we conjecture that they should also not be negligible in the analysis of other entropy-bound proposals. Section 4.6 considers a role for MDR/GUP modifications in the analysis of the black-body radiation spectrum, and again exposes some significant changes with respect to the standard picture, including the possibility that the characteristic frequency of black-body radiation at given temperature T might have a dependence on T such that in the infinite-temperature limit the characteristic frequency would take a finite (Planckian) value. It is then perhaps not surprising that in the analysis of the black-hole evaporation process, discussed in Section 4.7, we also find some characteristic MDR/GUP-induced new features, such as the possibility that the energy flux emitted by the black hole might diverge when the black-hole mass reaches a certain finite (Planckian) value. In Section 4.8, we comment on one key aspect which might deserve further consideration: for these theories with MDRs and/or GUPs there has been some speculation that the speed of massless particles might be different from the familiar speed-of-light scale value of c. In Sections 4.1-4.7 we assume throughout that c still is the speed of massless particles, but in Section 4.8 we establish how the analysis of black-body radiation would be changed if one implemented some alternatives considered in the literature. In Section 4.9 we compare our analysis with other studies which have considered the implications of a MDR or a GUP for some aspects of black-hole physics.

4.1 MDR, GUP and black hole entropy

4.1.1 MDRs and GUPs in Quantum Gravity

The emergence of MDRs and/or GUPs in quantum gravity, although of course not guaranteed, can be motivated on general grounds, and also finds support in the direct analysis of certain quantum gravity scenarios.

As we discussed in the previous Chapters, in most cases (LQG, NG scenarios, DSR etc.) one is led to consider a dispersion relation of the type[14]

$$\vec{p}^2 = f(E, m; L_p) \simeq E^2 - \mu^2 + \alpha_1 L_p E^3 + \alpha_2 L_p^2 E^4 + O\left(L_p^3 E^5\right) , \qquad (4.1)$$

where f is the function that gives the exact dispersion relation, and on the right-hand side we just assumed the applicability of a Taylor-series expansion for $E \ll 1/L_p$. The coefficients α_i can take different values in different Quantum-Gravity proposals.

The situation concerning the possibility of a GUP is rather similar and can be motivated, on general grounds, by the intuition [80, 10] that quantum gravity

[14] We denote with m, as conventional, the rest energy of the particle. The mass parameter μ on the right-hand side is directly related to the rest energy, but $\mu \neq m$ if the α_i do not all vanish. For example, if $\alpha_1 \neq 0$ but $\alpha_i = 0$ for every $i \geq 2$ one of course obtains $\mu^2 = m^2 + \alpha_1 L_p m^3$. This needed to be clarified since it is relevant for more general analyses of MDRs, but in our study we are always concerned with particles which are either massless or anyway are analyzed at energies such that the mass can be neglected, and therefore both μ and m will never actually enter our key formulas.

might require the introduction of an absolute Planckian limit on the size of the collision region, applicable to high-energy microscopic collision processes. For example, a GUP of the form

$$\delta x \geq \frac{1}{\delta p} + \alpha L_p^2 \delta p + O(L_p^3 \delta p^2) \ , \tag{4.2}$$

which has been derived within String Theory [81], is such that at small δp one finds the standard dependence of δx on δp (δx gets smaller as δp increases) but for large δp the Planckian correction term becomes significant and keeps $\delta x \geq L_p$. In this approach the coefficient α should take a value of roughly the ratio between the square of the string length and the square of the Planck length, but this of course might work out differently in other quantum gravity proposals.

While in the parametrization of (4.1) we included a possible correction term suppressed only by one power of the Planck length, in (4.2) such a linear-in-L_p is assumed not to be present. This reflects the status of the presently-available literature: for the MDR a large number of alternative formulations, including some with the linear-in-L_p term, are being considered, as they find support in different quantum gravity scenarios (and different preliminary results adopting alternative approximation schemes within a given approach), whereas all the discussions of a GUP assume that the leading-order correction should be proportional to the square of L_p.

4.1.2 MDR, GUP and a Planck scale particle localization limit

One can easily repeat the analysis reported in Chapter 3, and show that if our quantum gravity scenario hosts a Planck-scale modification of the dispersion relation of the form (4.1) then clearly the relation between δp_γ and δE_γ should be re-written as follows

$$\delta p_\gamma \simeq \left(1 + \alpha_1 L_p E + 3\left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{8}\right) L_p^2 E^2\right) \delta E_\gamma \tag{4.3}$$

which then leads to the requirement

$$M \geq \frac{1}{\delta x} - \alpha_1 \frac{L_p}{(\delta x)^2} + \left(\frac{11}{8}\alpha_1^2 - \frac{3}{2}\alpha_2\right) \frac{L_p^2}{(\delta x)^3} + O\left(\frac{L_p^3}{(\delta x)^4}\right) . \tag{4.4}$$

This result can be straightforwardly generalized (simply using a boost) to the case of the measurement of the position of a particle of energy E. For the standard case this leads to the $E \geq 1/\delta x$ relation while in the presence of an MDR one easily finds

$$E \geq \frac{1}{\delta x} - \alpha_1 \frac{L_p}{(\delta x)^2} + \left(\frac{11}{8}\alpha_1^2 - \frac{3}{2}\alpha_2\right) \frac{L_p^2}{(\delta x)^3} + O\left(\frac{L_p^3}{(\delta x)^4}\right) . \tag{4.5}$$

While the connection between a MDR and a Planck-scale particle-localization limit is somewhat less obvious, it is not at all surprising that the GUP would give rise to such a particle-localization limit. In fact, as mentioned, the GUP is primarily viewed as a way to introduce a Planckian limit on the size of the collision region, applicable to high-energy microscopic collision processes, and a limitation on the size of collision regions would naturally be expected to lead to a particle-localization limit. Indeed, as the careful reader can easily verify, from the GUP one obtains (following again straightforwardly the familiar line of analysis discussed in Ref. [66]) a modification of the relation $E \geq 1/\delta x$. The modification is of the type $E \geq 1/\delta x + \Delta$, with Δ of order $\alpha L_p^2/\delta x^3$, and originates from the fact that according to the GUP, (4.2), one obtains $\delta p_\gamma \geq 1/\delta x + \lambda_s^2/\delta x^3$ (instead of the original $\delta p_\gamma \geq 1/\delta x$). Using the standard special-relativistic dispersion relation for a photon $p_\gamma = E_\gamma$ the condition on the momentum uncertainty translates to a condition on the energy uncertainty $\delta E_\gamma \geq \frac{1}{\delta x}\left(1 + \alpha \frac{L_p^2}{\delta x^2}\right)$, and ultimately this leads to

$$E \geq \frac{1}{\delta x} + \alpha \frac{L_p^2}{(\delta x)^3} + O\left(\frac{L_p^3}{(\delta x)^4}\right) . \tag{4.6}$$

4.1.3 MDR and black hole entropy

The argument proposed in Chapter 3 suggested that a Planck-scale modification of the particle-localization limit, of the type (4.5) or (4.6), can be used to motivate corrections to the $S = A/(4L_p^2)$ area-entropy relation for black holes. As we saw in the previous Chapter, the Bekenstein argument implicitly assumes (through the $E \geq 1/\delta x$ relation) that the energy-momentum dispersion relation and the position-momentum uncertainty principle take the standard form. Let us now reformulate the argument, still assuming a standard form for the position-momentum uncertainty principle, but introducing a MDR of the type (4.1). The area of a black hole changes according to $\Delta A \geq 8\pi L_p^2 E s$ when a classical particle of energy E and size s is absorbed. Describing the size of the particle in terms of the uncertainty in its position and taking into account the MDR-induced Planck-length modification in (4.5), one obtains

$$\begin{aligned}
\Delta A &\geq 4(\ln 2)\left[L_p^2 - \frac{\alpha_1 L_p^3}{\delta x} - \frac{\left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right) L_p^4}{(\delta x)^2}\right] \\
&\simeq 4(\ln 2)\left[L_p^2 - \frac{\alpha_1 L_p^3}{R_S} - \frac{\left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right) L_p^4}{(R_S)^2}\right] \\
&\simeq 4(\ln 2)\left[L_p^2 - \frac{\alpha_1 2\sqrt{\pi} L_p^3}{\sqrt{A}} - \frac{\left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right) 4\pi L_p^4}{A}\right] .
\end{aligned} \tag{4.7}$$

From (4.8) we derive an area-entropy relation assuming that the entropy of the black hole depends only on its area and that the minimum increase of entropy should be, independently of the value of the area, $\ln 2$:

$$\begin{aligned}
\frac{dS}{dA} &\simeq \frac{min(\Delta S)}{min(\Delta A)} \\
&\simeq \frac{\ln 2}{4(\ln 2) L_p^2 \left[1 - \frac{\alpha_1 2\sqrt{\pi} L_p}{\sqrt{A}} - \frac{\left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right) 4\pi L_p^2}{A}\right]} \\
&\simeq \left(\frac{1}{4L_p^2} + \frac{\alpha_1\sqrt{\pi}}{2L_p\sqrt{A}} + \frac{\left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right)\pi}{A}\right) ,
\end{aligned} \tag{4.8}$$

which gives (up to an irrelevant constant contribution to entropy)

$$S \simeq \frac{A}{4L_p^2} + \alpha_1\sqrt{\pi}\frac{\sqrt{A}}{L_p} + \left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right)\pi \ln \frac{A}{L_p^2} . \tag{4.9}$$

This result of course reproduces the famous linear formula if all coefficients α_i vanish. If the cubic term $\alpha_1 E^3$ is present in the energy-momentum dispersion relation then the leading correction goes like $\sqrt{A}$, whereas if the first nonzero coefficient in the dispersion relation expansion is α_2 the leading correction term goes like $\log A$. Our "improved Bekenstein argument" therefore provides a possible link between the form of the MDR (and of the GUP, as we stress later) and the all-order form of the entropy-area relation for black holes.

We can also use (4.9) to obtain, using the first law of black hole thermodynamics $dS = \frac{dM}{T}$, a Planck-scale-corrected relation between black-hole temperature and mass:

$$T_{BH}^{MDR} \simeq \frac{E_p^2}{8\pi M}\left(1 - \alpha_1 \frac{E_p}{2\sqrt{2}M} - \left(\frac{15}{32}\alpha_1^2 - \frac{3}{8}\alpha_2\right)\frac{E_p^2}{M^2}\right), \tag{4.10}$$

where we also used the familiar relation between black hole area and mass $A = 16\pi M^2$.

Some all-order results for MDR modifications of black-hole entropy

In the previous pages we discussed a possible relation between MDR and log corrections to the entropy-area relation. Since the log-area term is a leading-order term it was appropriate to work within a power-series expansion of the MDR. Moreover, the mentioned results from quantum-gravity research (primarily from LQG and approaches based on NG) that provide motivation for a Planck-scale modification of the dispersion relation in most cases are obtained within analyses that only have access to the first terms in a power-series expansion of the dispersion relation. Still for some aspects of our analysis it will be useful to contemplate some illustrative examples of all-order dispersion relations. The careful reader can easily verify that once a given energy-momentum dispersion relation $E = f_{disp}(p)$ is adopted the steps of the calculation reported in the preceding subsection can be followed rather straightforwardly, obtaining

$$\frac{dS}{dA} \simeq \frac{min(\Delta S)}{min(\Delta A)} \simeq \frac{1}{2L_p^2}\sqrt{\frac{\pi}{A}}\frac{1}{f_{disp}\left(\sqrt{\frac{4\pi}{A}}\right)} \tag{4.11}$$

and

$$T_{BH} \simeq \frac{1}{4\pi} f_{disp}\left(\frac{E_p^2}{2M}\right) . \tag{4.12}$$

As illustrative examples of "all-order MDRs" we consider the following three cases:

$$\cosh(E/E_p) - \cosh(m/E_p) - \frac{p^2}{2E_p^2} e^{E/E_p} = 0, \qquad (4.13)$$

$$\frac{E^2}{(1-E/E_p)^2} - \frac{p^2}{(1-E/E_p)^2} - m^2 = 0, \qquad (4.14)$$

$$\cosh(\sqrt{2}E/E_p) - \cosh(\sqrt{2}m/E_p) - \frac{p^2}{E_p^2} \cosh(\sqrt{2}E/E_p) = 0, \qquad (4.15)$$

(4.13) has already been considered in the previous literature [42, 17, 20, 87], particularly as a possible description of particle propagation in κ-Minkowski noncommutative spacetime. It provides an example in which the coefficient of the linear-in-L_p term is nonvanishing: $\alpha_1 = -1/2$. And it is noteworthy that according to (4.13) there is a maximum momentum for fundamental particles: from (4.13) it follows that for $E \to \infty$ one has $p \to E_p$.

The case (4.15) has not been previously considered in the literature. It provides for our purposes a valuable illustrative example since, as in the case of (4.13), it would lead to a maximum momentum ($p \to E_p$ for $E \to \infty$) but, contrary to the case of (4.13), it corresponds to $\alpha_1 = 0$ (whereas $\alpha_2 = -5/18$). This is therefore an example with the maximum-momentum feature and such that one would expect the leading corrections to the entropy-area relation to be logarithmic.

The case of (4.14) has already been considered in the literature for other reasons [18], and it provides us an opportunity to illustrate some consequences of a scenario in which both α_1 and α_2 vanish, but still there are some Planck-scale modifications of the energy-momentum dispersion relation. And it is noteworthy that (4.14) can be implemented in such a way that [18] the Planck scale provides the maximum value of both momentum and energy.

For the cases with dispersion relations (4.13) or (4.15), since $E \to \infty$ for $p \to E_p$, the formulas derived above would lead to the conclusion that the black hole temperature diverges at some finite (nonzero!) value of the black-hole mass $M_{min} = E_p/2$. We would then assume that this M_{min} is the minimum allowed mass for a black hole, and that the standard description of the evaporation process should not be applicable beyond this small value of mass.

In cases in which one introduces both a maximum momentum and a maximum energy while keeping the form of the dispersion relation largely unaffected[15], as done in some applications of (4.14), one would expect (since the energy has a maximum Planckian value, $E_{Max} = E_P$) that the temperature should be bounded to be lower than the Planck scale, $T_{Max} \sim E_p$, and that the minimum allowed value of black-hole mass should be also Plankian, since it should be the value of mass such that temperature reaches is maximum allowed value.

4.1.4 GUP and black hole entropy

In the previous subsection we focused on scenarios in which the energy-momentum dispersion relation is modified but the position-momentum uncertainty principle preserves the Heisenberg form. Clearly the key ingredient of our analysis is the presence of a correction term Δ in the particle-localization-limit relation $E \geq 1/\delta x + \Delta$. As stressed in subsection 4.1.2, both a MDR and a GUP can introduce such a correction term in the particle-localization limit, and therefore, as we want to discuss explicitly in this subsection, also in presence of a GUP one should expect corrections to the entropy-area black-hole formula and to the formula that relates the mass and the temperature of a black hole.

We start the section by considering scenarios in which the position-momentum uncertainty principle is Planck-scale modified, while the energy-momentum dispersion relation preserves its special-relativistic form. Then we comment on the more general case, in which one might be dealing with both a MDR and a GUP.

Let us start by noting here again for convenience the particle-localization limit

[15] Whenever the mass m can be ignored (*i.e.* for massless particles and high-energy particles with finite mass) the dispersion relation (4.14) is indistinguishable from the standard special-relativistic one.

that one obtains assuming a GUP of the form (4.2) and a standard (special-relativistic) energy-momentum dispersion relation:

$$E \geq \frac{1}{\delta x} + \alpha \frac{L_p^2}{(\delta x)^3} + O\left(\frac{L_p^3}{(\delta x)^4}\right) . \tag{4.16}$$

Following the same strategy of analysis adopted in the previous section, one finds that the Bekenstein argument, when taking into account this localization limit (4.16), leads to the conclusion that the minimum increase of black-hole area upon absorption of a particle of energy E is given by

$$\Delta A \geq 4(\ln 2)\left[L_p^2 + \frac{\alpha L_p^4}{(\delta x)^2}\right] \simeq 4(\ln 2)\left[L_p^2 + \frac{\alpha L_p^4}{(R_S)^2}\right] \simeq 4(\ln 2)\left[L_p^2 + \frac{\alpha 4\pi L_p^4}{A}\right] .$$

From this it follows that the entropy-area relation should take the form

$$S \simeq \frac{A}{4L_p^2} - \alpha\pi \ln \frac{A}{L_p^2} , \tag{4.17}$$

and the formula relating the temperature and the mass of the black hole should take the form

$$T_{BH}^{GUP} \simeq \frac{E_p^2}{8\pi M}\left(1 + \alpha \frac{E_p^2}{8M^2}\right) . \tag{4.18}$$

Combining MDR and GUP in the analysis of black-hole entropy

We have argued that both a MDR and a GUP are possible features of a quantum-gravity theory that would affect black-hole termodynamics. Actually, as the careful reader must have noticed, the line of analysis we are advocating is composed of two steps. First we notice that the "particle-localization limit" in its standard form, $E \geq 1/\delta x$, is derived on the basis of two key assumptions, the validity of the Heisenberg position-momentum uncertainty principle and the validity of the special-relativistic energy-momentum dispersion relation, and that by modifying the uncertainty principle and/or the dispersion relation one gets a modified particle-localization limit of the type $E \geq 1/\delta x + \Delta_{\delta x, L_p}$. Then we observe that a key assumption of the Bekenstein argument for the derivation of black-hole entropy is the validity of the standard particle-localization limit

$E \geq 1/\delta x$. With a MDR and/or a GUP one gets a modified particle-localization limit, which in turn leads to a modification of the black-hole area-entropy relationship.

It is worth mentioning that the modifications induced by a MDR and a GUP may (at least in part) cancel out at the level of the area-entropy equation. In order to stress the importance of this possibility let us consider the information presently available on the LQG approach: (i) several LQG studies have argued in favour of a MDR with non-vanishing α_1 (leading Planck-scale correction to the dispersion relation that goes linearly with L_p), (ii) there is no mention of a GUP in the LQG literature, (iii) several LQG studies have argued in favour of an entropy-area relationship in which the leading correction, beyond the linear term, is of log-area type. According to the perspective on the derivation of black-hole entropy that we are advocating one would find these three ingredients to be logically incompatible: if the MDR has nonvanishing α_1 and the position-momentum uncertainty principle is not Planck-scale modified then in the entropy-area relationship the leading correction, beyond the linear term, should have $\sqrt{area}$ dependence. Does this mean that LQG is a logically inconsistent framework? Of course, it does not. It simply means that some of the relevant preliminary results must be further investigated. It may well be that, as the LQG approach is understood more deeply, it turns out that the α_1 coefficient in the MDR vanishes. Or else we might discover that in LQG the α_1 coefficient in the MDR takes a nonzero value, but there is a corresponding linear-in-L_p term in the GUP with just the right coefficient to give an overall vanishing coefficient to the $\sqrt{area}$ term in the entropy-area relation.

Our perspective on the derivation of black-hole entropy provides a logical link between different aspects of a quantum-gravity theory and may be used most fruitfully when, as in the case of LQG, the formalism is very rich and some of the results obtained within that formalism are of preliminary nature. Even before being able to derive more robust results we may uncover that the presently-available preliminary results are not providing us with a logically-consistent picture, and this in turn will give us additional motivation for investigating more carefully those preliminary results.

It is also worth mentioning that on the string-theory side our perspective on the derivation of black-hole entropy provides no evidence of a logical inconsistency among the results so far obtained in that framework. The string-theory literature indicates that the entropy-area relationship should involve a leading correction, beyond the linear term, of log-area type, and provides strong evidence of a GUP of the type (4.2), while the results so far obtained do not indicate the need to modify the dispersion relation in string theory. These three ingredients provide a logically-consistent scenario within our perspective on the derivation of black-hole entropy. As shown above, with a GUP of the type (4.2) and with an unmodified (still special-relativistic) dispersion relation one is indeed led to an entropy-area relationship in which the leading correction, beyond the linear term, is of log-area type.

4.2 Implications for the Bekenstein entropy bound and Generalized Second Law

It is natural at this point, after having shown that a MDR and a GUP can affect the black-hole entropy-area and mass-temperature relationships, to wonder whether other aspects of black-hole thermodynamics are also affected, and whether the overall picture preserves the elegance/appeal of the original scheme, based on standard uncertainty principle and dispersion relation. In this section we investigate the validity of the Generalized Second Law (GSL) of thermodynamics and the implications for the Bekenstein entropy bound. In order to work within a definite scenario we assume here a MDR (while we implicitly assume that the uncertainty principle takes its standard form).

The GSL [88] asserts that the second law of thermodynamics is still valid in presence of collapsed matter. Given the entropy of the black hole, as described by the area-entropy relation, the GSL requires that the total entropy of a system composed of a black hole and ordinary matter never decreases. This means that the following inequality holds for all physical processes

$$S_{BH} + S_{mat} \geq 0\,. \tag{4.19}$$

It was observed [89] that in principle (using the so-called "Geroch process") one could violate the GSL if objects of fixed size R and energy E could have arbitrarily large entropy S. This led Bekenstein to propose a "entropy bound"

$$S_{mat} \leq 2\pi ER \tag{4.20}$$

for an arbitrary system of energy E and effective radius R. The fact that the GSL implies the Bekenstein bound and vice versa has long been debated and is still actively debated. However the Bekenstein bound turns out to hold for a variety of systems in flat Minkowski space and can be derived as weak-gravity limit of the popular "Generalized Covariant Entropy Bound" [90].

A remarkable feature of the Bekenstein bound is that, in spite of being motivated by considerations rooted in the gravitational realm, it does not involve the Planck scale (or equivalently Newton's constant). The absence of the Planck scale is less puzzling in light of the observation that the bound can be derived even without advocating gravity in any way: it is sufficient [75] to analyze some implications of the particle-localization limit $E \geq \frac{1}{\delta x}$. This alternative derivation requires considering a matter system with energy E, in which self-gravitation effects can be neglected, that occupies a region in flat spacetime with radius R smaller than the gravitational radius $R_G \equiv 2L_p^2 E$. The standard particle-localization limit, when generalized to this type of systems, sets a minimum value for the energy of a quantum in a region of spatial radius R

$$\epsilon(R) \geq \frac{1}{R}\,. \tag{4.21}$$

The maximum number of quanta that we can have in the region is then given by

$$N_{max} \simeq \frac{E}{\epsilon(R)} = ER\,. \tag{4.22}$$

If we consider the simple case of a system for which the maximal number of microstates for N particles is given by $\Omega(N) = 2^N$ then the entropy of the system $S = \log\Omega(N)$ is bounded by the inequality

$$S_{mat} \leq (\log 2)ER\,, \tag{4.23}$$

which is indeed consistent with the Bekenstein bound (up to another "calibration factor" $\eta = \frac{2\pi}{\log 2}$).

We briefly reviewed this derivation of the Bekenstein bound especially in order to stress the role played by the particle-localization limit $E \geq \frac{1}{\delta x}$. It is then obvious that the modifications of the particle-localization limit induced by a MDR (and/or a GUP) would affect the Bekenstein bound. As shown earlier, within our parametrization of the MDR[16], one obtains a particle localization limit of the form

$$\epsilon(R) \geq \frac{1}{R}\left(1 - \alpha_1\frac{L_p}{R} - \left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right)\frac{L_p^2}{R^2} + O\left(\frac{L_p^3}{R^3}\right)\right) \tag{4.24}$$

which gives

$$S_{mat} \leq 2\pi ER\left(1 + \alpha_1\frac{L_p}{R} + \left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right)\frac{L_p^2}{R^2} + O\left(\frac{L_p^3}{R^3}\right)\right)\,. \tag{4.25}$$

This MDR-modified Bekenstein bound fits very naturally with our corresponding formula, (4.9), for the entropy-area relation; in fact, the two results combine to provide us with a picture which is still consistent with the GSL. According to (4.25) when a matter system of energy E falls into the black hole, this corresponds to a negative change of entropy which has absolute value not greater than

$$max(|\Delta S_{mat}|) \simeq 2\pi ER\left(1 + \alpha_1\frac{L_p}{R} + \left(\frac{3}{2}\alpha_2 - \frac{11}{8}\alpha_1^2\right)\frac{L_p^2}{R^2} + O\left(\frac{L_p^3}{R^3}\right)\right) \tag{4.26}$$

[16]For an analogous modification of the Bekenstein bound coming from the GUP see Ref. [75].

and correspondingly, according to (4.9), the black hole entropy increases at least by

$$min(\Delta S_{BH}) \simeq 2\pi ER\left(1+\alpha_1\frac{L_p}{R}+\left(\frac{3}{2}\alpha_2-\frac{11}{8}\alpha_1^2\right)\frac{L_p^2}{R^2}+O\left(\frac{L_p^3}{R^3}\right)\right) \tag{4.27}$$

Thus the MDR-induced corrections to S_{BH} and S_{mat} cancel exactly at the level of the inequality relevant for the GSL. The GSL stills holds, even in presence of a modified particle-localization limit.

4.3 Corrections to black-body radiation spectrum

In preparation for some observations on black-hole evaporation, to which we devote Section 4.4, we now want to investigate the implications of a MDR and/or a GUP for the black-body radiation spectrum.

4.3.1 MDR and black-body spectrum in leading order

Let us start by considering photons in a cubical box with edges of length L (and volume $V = L^3$). The wavelengths of the photons are subject to the boundary condition $\frac{1}{\lambda} = \frac{n}{2L}$, where n is a positive integer. This condition implies, assuming that the de Broglie relation is left unchanged, that the photons have (space-)momenta that take values $p = \frac{n}{2L}$. Thus momentum space is divided into cells of volume $V_p = \left(\frac{1}{2L}\right)^3 = \frac{1}{8V}$. From this it follows that the number of modes with momentum in the interval $[p, p+dp]$ is given by

$$g(p)dp = 8\pi V p^2 dp\,. \tag{4.28}$$

Assuming a MDR of the type parametrized in (4.1) one then finds that ($m = 0$ for photons)

$$p \simeq E\left(1+\frac{\alpha_1}{2}L_pE+\left(\frac{\alpha_2}{2}-\frac{\alpha_1^2}{8}\right)L_p^2E^2\right) \tag{4.29}$$

and

$$dp \simeq \left(1 + \alpha_1 L_p E + \left(\frac{3}{2}\alpha_2 - \frac{3}{8}\alpha_1^2\right) L_p^2 E^2\right) dE \tag{4.30}$$

Using this in (4.28) one obtains

$$g(E)dE = 8\pi V \left(1 + 2\alpha_1 L_p E + 5\left(\frac{1}{2}\alpha_2 + \frac{1}{8}\alpha_1^2\right) L_p^2 E^2\right) E^2 dE \tag{4.31}$$

which in terms of the frequency ν takes the form

$$g(\nu)d\nu = 8\pi V \left(1 + 2\alpha_1 L_p \nu + 5\left(\frac{1}{2}\alpha_2 + \frac{1}{8}\alpha_1^2\right) L_p^2 \nu^2\right) \nu^2 d\nu\,. \tag{4.32}$$

In order to obtain the MDR-modified energy density of a black body at temperature T we must now use (4.32) and rely on the statistical arguments which show that in a system of bosons at temperature T the average energy per oscillator is given by

$$\bar{E} = \frac{\nu}{e^{\frac{\nu}{T}} - 1}\ . \tag{4.33}$$

Thus the energy density at a given temperature T, for the frequency interval $[\nu, \nu + d\nu]$, is

$$u_\nu(T)d\nu = 8\pi \left(1 + 2\alpha_1 L_p \nu + 5\left(\frac{1}{2}\alpha_2 + \frac{1}{8}\alpha_1^2\right) L_p^2 E^2\right) \frac{\nu^3 d\nu}{e^{\frac{\nu}{T}} - 1}\,. \tag{4.34}$$

and integrating this formula we get the MDR-modified energy density of a black body at temperature T

$$u(T) = \frac{8\pi^5}{15} T^4 + 384\pi\zeta(5)\alpha_1 L_p T^5 + 5\left(\frac{1}{2}\alpha_2 + \frac{1}{8}\alpha_1^2\right) \frac{160\pi^7}{63} L_p^2 T^6 \tag{4.35}$$

The MDR introduces corrections of the type T^{4+n}/E_P^n to the Stefan-Boltzmann law. Moreover, the maximum value of the integrand in (4.34), as a function of ν, is clearly also shifted: the MDR also introduces a modification of Wien's law. Of course, using the low-energy expansion (4.1) of the dispersion relation we only get a reliable picture at temperatures safely below the Planck scale, but the presence of correction terms of the type T^{4+n}/E_P^n clearly suggests that the MDR-modified description leads to departures from the Stefan-Boltzmann law that can become very significant as the temperature approaches the Planck scale. We intend to show this explicitly by considering an example of all-order MDR formula.

4.3.2 Some all-order results for MDR modifications of black-body spectrum

Let us therefore derive once again the modified Stefan-Boltzmann law, now assuming, as illustrative example of an all-order MDR formula, the validity of the dispersion relation (4.15). Clearly the number of modes in momentum space is still given by

$$g(p)dp = 8\pi V p^2 dp \,, \tag{4.36}$$

but now

$$p^2 = E_p^2 \left(1 - \frac{1}{\cosh(\sqrt{2}E/E_p)}\right) \tag{4.37}$$

and this implies that the number of modes for given energy is given by

$$g(E)dE = 16\pi V E_p^2 \sinh^2\left(\frac{E/E_p}{\sqrt{2}}\right)\cosh\left(\frac{E/E_p}{\sqrt{2}}\right)\frac{1}{\cosh^{5/2}\left(\sqrt{2}E/E_p\right)}dE \tag{4.38}$$

i.e. the number of modes for given frequency is

$$g(\nu)d\nu = 16\pi V E_p^2 \sinh^2\left(\frac{\nu/E_p}{\sqrt{2}}\right)\cosh\left(\frac{\nu/E_p}{\sqrt{2}}\right)\frac{1}{\cosh^{5/2}\left(\sqrt{2}\nu/E_p\right)}d\nu \quad . \tag{4.39}$$

Then the modified Stefan-Boltzmann law is given, in integral form, by

$$u(T) = \frac{1}{V}\int_0^\infty \frac{g(\nu)}{e^{\frac{\nu}{T}} - 1}\nu d\nu \ , \tag{4.40}$$

where $g(\nu)$ is the one of (4.39).

It is useful to consider some limiting forms of the integration in (4.40). Clearly, since (4.15) is consistent with (4.1) for $\alpha_1 = 0$ and $\alpha_2 = -5/18$, in the limit $T/E_p \ll 1$ the integration (4.40) gives a result that reproduces (4.35) for $\alpha_1 = 0$ and $\alpha_2 = -5/18$. But, now that we are dealing with an all-order formula, besides considering the case $T/E_p \ll 1$ we can also investigate the opposite limit $T/E_p \gg 1$, finding

$$u(T) = 16\pi E_p^4 \left\{\frac{T}{E_p}C_1 - \frac{1}{2}C_2 - \frac{E_p}{T}C_3 + O(E_p^2/T^2)\right\} \tag{4.41}$$

where

$$C_1 = \int_0^\infty \sinh^2(x/\sqrt{2})\frac{\cosh(x/\sqrt{2})}{\cosh^{5/2}(\sqrt{2}x)}dx = \frac{1}{6}, \tag{4.42}$$

$$C_2 = \int_0^\infty x\sinh^2(x/\sqrt{2})\frac{\cosh(x/\sqrt{2})}{\cosh^{5/2}(\sqrt{2}x)}dx \simeq 0.22, \tag{4.43}$$

$$C_3 = \int_0^\infty x^2\sinh^2(x/\sqrt{2})\frac{\cosh(x/\sqrt{2})}{\cosh^{5/2}(\sqrt{2}x)}dx \simeq 0.41, \tag{4.44}$$

This means that the MDR (4.15) leads to a modification of the Stefan-Boltzmann law which at the Planck scale is very significant: for $T \gg E_p$ one finds that u depends linearly on T, rather than with the fourth power.

It is of particular interest to establish what is the relationship between the "characteristic frequency" (and characteristic wavelength) of the black-body spectrum and temperature. In the standard description of a black body the characteristic frequency grows linearly with the temperature. In order to verify whether this is still the case in our MDR-modified scenario we can take the derivative of $u_\nu(T)$ with respect to ν, so that we can identify the value of frequency for which the energy density (and the radiated flux) reaches a maximum. This leads to the following equation that must be satisfied by the characteristic frequency $\bar{\nu}$:

$$\left(e^{\frac{\bar{\nu}}{T}} - 1\right)\left(g(\bar{\nu}) + g'(\bar{\nu})\bar{\nu}\right) - \frac{e^{\frac{\bar{\nu}}{T}}}{T}g(\bar{\nu})\bar{\nu} = 0 \ . \tag{4.45}$$

For $T \ll E_p$ of course this reproduces the type of small modification of Wien's law, which we already noticed in the previous section. The fact that we are now considering a scenario with a given all-order MDR formula allows us to examine the dependence of the characteristic frequency on temperature even when the temperature reaches and eventually exceeds the Planck scale. And we find that for $T/E_p \gg 1$ the characteristic frequency becomes essentially independent of temperature. No matter how high the temperature goes the characteristic frequency never exceeds the following finite value:

$$\bar{\nu} \simeq E_p\frac{\cosh^{-1}[(1+\sqrt{41})/4]}{\sqrt{2}} \simeq 0.87E_p \tag{4.46}$$

This means that at low temperatures any increase of temperature causes a corresponding increase in characteristic frequency of the black-body spectrum, but gradually a saturation mechanism takes over and even in the infinite-temperature limit the characteristic frequency is still finite, and given by the Planck scale (up to a coefficient of order 1). This occurs with the dispersion relation (4.15), *i.e.* in a scenario with a minimum value of wavelength but no maximum value of frequency. An analogous result for the case of the dispersion relation (4.14), which leads to both a minimum value of wavelength and a maximum value of frequency, would have not been surprising: if the framework introduces from the beginning a maximum Planckian value of frequency, then of course also the characteristic frequency of black-body radiation would be "sub-Planckian". But in analyzing the case of (4.15) we found that the presence of a minimum wavelength at the fundamental level is sufficient for the emergence of a maximum Planckian value of the characteristic frequency of black-body radiation, as shown explicitly by Eq. (4.46).

4.3.3 Black-body spectrum with GUP

In the previous two subsections the key point was that a MDR leads to a modified formula for the density of modes in a given (infinitesimal) frequency interval, $g(\nu)d\nu$. If instead we now assume that the dispersion relation takes its standard special-relativistic form, but there is a GUP, it is not *a priori* obvious that the black-body spectrum is affected. One does indeed obtain a modified black-body spectrum if it is assumed that the GUP should also be reflected in a corresponding modification of the de Broglie relation,

$$\lambda \simeq \frac{1}{p}\left(1+\alpha L_p^2 p^2\right) \tag{4.47}$$

and

$$E \simeq \nu\left(1+\alpha L_p^2 \nu^2\right) . \tag{4.48}$$

For oscillators in a box the number of modes in an infinitesimal frequency interval would still be described by the standard formula

$$g(\nu)d\nu = 8\pi V \nu^2 d\nu \,, \tag{4.49}$$

but, as a result of (4.48), the average energy per oscillator would be given by

$$\bar{E} = \frac{\nu}{e^{\frac{\nu}{T}} - 1}\left(1 + \alpha L_p^2 \nu^2 \left(1 - \frac{\frac{\nu}{T}}{1 - e^{-\frac{\nu}{T}}}\right)\right) \,. \tag{4.50}$$

Combining (4.48) and (4.50) one finds

$$u_\nu(T)d\nu = 8\pi \left(1 + \alpha L_p^2 \nu^2 \left(1 - \frac{\frac{\nu}{T}}{1 - e^{-\frac{\nu}{T}}}\right)\right) \frac{\nu^3 d\nu}{e^{\frac{\nu}{T}} - 1} \,. \tag{4.51}$$

and the modified Stefan-Boltzmann law takes the form

$$u(T) = \frac{8\pi^5}{15} T^4 + \frac{8\pi^6}{9} \alpha L_p^2 T^6 \,. \tag{4.52}$$

The $L_p^2 T^6$ correction term is just one of the $L_p^n T^{4+n}$ correction terms on which we already commented in the context of the MDR modifications of black-body radiation.

4.4 Black hole evaporation

In this section we use some of the results obtained in the previous sections in a description of the black-hole evaporation process. The key ingredients are the relation between the black-hole temperature and mass and the relation between the black-hole temperature and the energy density emitted by the black hole. We neglect possible non-thermal corrections due to back-reaction effects, that will be extensively discussed in the next Chapter, and we therefore treat the radiation emitted by the black-hole as black-body radiation.

4.4.1 MDR and Black hole evaporation

At temperature T the intensity I of the radiation emitted by a black hole of area A is given by

$$I(T) = A\, u(T) \,. \tag{4.53}$$

Using energy conservation one can write

$$\frac{dM}{dt} = -A\,u, \tag{4.54}$$

and assuming a MDR of the type $E = f_{disp}(p)$, in light of our result (4.12), one finds

$$\frac{dM}{dt} = -16\pi \frac{M^2}{E_p^4} u \left(\frac{1}{4\pi} f_{disp}\left(\frac{E_p^2}{2M}\right)\right) \tag{4.55}$$

When $M \gg E_p$ (so that a power-series expansion of $f_{disp}(E_p^2/2M)$ is meaningful) this takes the form

$$\frac{dM}{dt} = \; = \; -k_0 \frac{E_p^8}{M^2} - k_1 \alpha_1 \frac{E_p^9}{M^3} - (k_{21}\alpha_1^2 + k_{22}\alpha_2)\frac{E_p^{10}}{M^4} + O(E_p^5/M^5) \tag{4.56}$$

where $k_0 = \frac{\pi^2}{480}$, $k_1 = k_0 \frac{90\zeta(5)-\pi^5}{\pi^5}$, $k_{21} = k_0 \frac{502\pi^5 - 75600\zeta(5)}{672\pi^5}$ and $k_{22} = -k_0 \frac{211}{672\pi^5}$

This power-series analysis allows to conclude that a MDR can affect the speed of evaporation of a black hole. For example, in the case of the dispersion relation (4.13) the evaporation process is retarded with respect to the standard case, whereas in the case of (4.15) the evaporation process is accelerated.

With a given all-order MDR formula one can obtain of course even more detailed information than available using the power-series expansion. In particular, let us look at the case of the dispersion relation (4.15) and analyze the stage of the evaporation process when the mass of the black hole is of the order of the Planck scale. For $M \sim E_P$ we can approximate the MDR (4.15) as follows

$$E \simeq \frac{E_p}{\sqrt{2}} \ln\left(\frac{2}{1-(p/E_p)^2}\right) \tag{4.57}$$

and then one finds

$$\frac{dM}{dt} \simeq -(16\pi)^2 M^2 \left\{ \frac{C_1}{4\pi\sqrt{2}} \ln\left(\frac{2}{1-(\frac{E_p}{2M})^2}\right) - \frac{1}{2}C_2 \right\} \; . \tag{4.58}$$

This shows that, in the case of the MDR (4.15), the energy flux emitted by the black hole would formally diverge as the black-hole mass approaches $E_p/2$. This is mainly a consequence of the fact that the black-hole temperature diverges when $M \to E_p/2$. In the standard description of black-hole evaporation these divergences occur as $M \to 0$.

4.4.2 GUP and Black hole evaporation

The observations reported in the previous subsection for the case of a MDR (with unmodified energy-momentum uncertainty relation) can be easily adapted to the complementary situation with a GUP and a standard (unmodified) dispersion relation. One must however assume, as already stressed in Subsection 4.3.3, that the GUP is reflected in a corresponding modification of the de Broglie relation ($\lambda \simeq (1 + \alpha L_p^2 p^2)/p$). In this hypothesis one easily finds that the black hole should lose its mass at a rate given by

$$\frac{dM}{dt} = -A\,u\left(\frac{E_p^2}{2M}\right) = -16\pi\frac{8\pi^5}{15}\left(T\left(\frac{E_p^2}{2M}\right)\right)^4 + \frac{8\pi^6}{9}\alpha L_p^2\left(T\left(\frac{E_p^2}{2M}\right)\right)^6 . \quad (4.59)$$

Expanding for $M/E_p \gg 1$ we obtain

$$\frac{dM}{dt} \simeq -16\pi\frac{E_p^4}{M^2}\left(\tilde{k}_0 + \alpha\tilde{k}_1\frac{E_p^2}{M^2}\right) , \quad (4.60)$$

with $\tilde{k}_0 = \frac{\pi}{7680}$ and $\tilde{k}_1 = \frac{1}{294912} + \frac{\pi}{15360}$. Clearly the modifications to the black hole evaporation formula obtained in the GUP scenario are qualitatively the same as in the MDR scenario with $\alpha_1 = 0$.

4.5 A possible dependence on the speed law for photons

Throughout our analysis we have implicitly assumed that the law $v_\gamma = 1$ describing the speed of photons is not affected by the MDR and/or the GUP. The possibility of modifications of the speed law for photons has been however considered rather extensively, particularly in the MDR literature. While several authors have argued that the law $v_\gamma = 1$ should not be modified even in presence of an MDR (see, *e.g.*, Refs. [91, 92, 93] and references therein), one also finds support in the literature for the proposal (see, *e.g.*, Ref. [87] and references therein) of the law $v_\gamma = [dE/dp]_{m=0} = [df_{disp}(p)/dp]_{m=0}$ and the proposal (see, *e.g.*, Ref. [18] and references therein) of the law $v_\gamma = p/E$.

For our analysis a key point is that if, instead of $v_\gamma = 1$, one took $v_\gamma = [dE/dp]_{m=0}$ or $v_\gamma = p/E$ then the speed of photons would acquire an energy dependence which should be taken into account in some aspects of our derivations. We postpone to future studies this more general analysis, but in order to explore the type of modifications which could be induced by such an energy dependence of the speed of photons we do intend to consider here the description of black-body radiation with the dispersion relation (4.15), assuming that the speed of photons is governed by either $v_\gamma = [dE/dp]_{m=0}$ or $v_\gamma = p/E$.

We focus on the emitted "flux density"

$$I_\nu = A\, u_\nu\, v_\gamma(\nu) \tag{4.61}$$

where A is the area of the radiating surface and u_ν is the energy density at a given frequency. Taking $v_\gamma = p/E$, from (4.15) it follows that

$$v_\gamma(\nu) = \frac{p}{E} = \frac{E_p}{E}\sqrt{1 - \frac{1}{\cosh\left(\frac{\sqrt{2}E}{E_p}\right)}}\ . \tag{4.62}$$

From this it would then follow that the energy flux density is given by

$$I_\nu(T) = 4\pi A\sqrt{2}E_p^3 \frac{1}{e^{\nu/T} - 1}\frac{\sinh(\sqrt{2}E/E_p)}{\cosh^3(\sqrt{2}E/E_p)}\left[\cosh(\sqrt{2}E/E_p) - 1\right]\ . \tag{4.63}$$

This suggests that, although there are some small quantitative differences, the qualitative features of black-body radiation with the dispersion relation (4.15) are largely independent of the choice between $v_\gamma = 1$ and $v_\gamma = p/E$. In particular, from (4.63) with one finds that the typical frequency of the photons contributing to the energy flux saturates at

$$\bar{\nu} \simeq 0.76 E_p, \tag{4.64}$$

which is not much different from the typical frequency found for the case $v_\gamma = 1$. The analysis of the total emitted energy ($\int_0^\infty I_\nu(T)d\nu$) also leads to rather small differences between the choices $v_\gamma = 1$ and $v_\gamma = p/E$. In particular from (4.63) one finds

$$I/A = \frac{8}{15}\pi^5 T^4 \left\{ 1 + C_1 \left(\frac{T}{E_p}\right)^2 + C_2 \left(\frac{T}{E_p}\right)^4 + O\left(\frac{T}{E_p}\right)^6 \right\}, \tag{4.65}$$

in the limit $T/E_p \ll 1$, and

$$I/A = E_p^4 \left\{ \widetilde{C}_1 \frac{T}{E_p} + \widetilde{C}_2 + \widetilde{C}_3 \frac{E_p}{T} + O\left(\frac{E_p}{T}\right)^2 \right\}, \tag{4.66}$$

in the limit $T/E_p \gg 1$, where $C_1 = -\frac{100\pi^2}{21}$, $C_2 = \frac{164\pi^4}{5}$ and $\widetilde{C}_1 = 5.57, \widetilde{C}_1 = -\pi$ and $\widetilde{C}_3 = 0.79$.

If instead one adopts the law $v_\gamma = [dE/dp]_{m=0}$, still assuming (4.15), one obtains

$$v_\gamma(\nu) = \frac{dE}{dp} = \frac{\cosh^{\frac{3}{2}} \frac{\sqrt{2}E}{E_p}}{\cosh \frac{E}{\sqrt{2}E_p}} \tag{4.67}$$

and then the flux density takes the form

$$I_\nu = 16\pi A E_p^2 \, \nu \frac{\sinh^2 \frac{\nu}{\sqrt{2}E_p}}{\left(e^{\frac{\nu}{T}} - 1\right) \cosh \frac{\sqrt{2}\nu}{E_p}} \,. \tag{4.68}$$

From this one easily verifies that the effects induced by the Planck-scale deformation in the case $v_\gamma = [dE/dp]_{m=0}$ are essentially of the same type encountered in the cases $v_\gamma = 1$ and $v_\gamma = p/E$, but the quantitative differences between the case $v_\gamma = [dE/dp]_{m=0}$ and the other two cases are more significant then the ones between the cases $v_\gamma = 1$ and $v_\gamma = p/E$. As mentioned, in absence of the Planck-scale effects the typical frequency of the photons contributing to the energy flux grows linearly with the temperature, while in the cases in which the Planck-scale effects of (4.15) are introduced with $v_\gamma = 1$ or $v_\gamma = p/E$ the typical frequency saturates at a Planckian value. If the same Planck-scale effects are introduced with $v_\gamma = [dE/dp]_{m=0}$, as implicitly codified in (4.68), one finds that the growth of the typical frequency with temperature also slows down significantly at high temperatures but it does not completely saturate: at high temperatures the typical frequency grows logarithmically with the temperature.

In summary the choice of the speed law does not appear to affect the core features of the analysis, but it appears that it could in some cases introduce some significant quantitative differences.

4.6 Comparison with previous analyses

To our knowledge, the one we reported here, in spite of its preliminary nature, is at this point the most composite effort of exploration of the implications of a MDR and/or a GUP in black-hole thermodynamics. But parts of the overall picture we attempted to provide had been investigated previously, and it seems appropriate to comment briefly on this previous related studies.

Closest in spirit to our perspective are the studies of the implications of the GUP for black-hole thermodynamics reported in Refs. [83] and [75]. Whereas for us (4.2) is to be handled prudently, as it could possibly be only an approximate form of a more complicated all-order-in-L_p formula, in Refs. [83, 75] the formula (4.2) is taken as the exact form of the GUP, thereby leading to a corresponding form of the entropy-area relation. Perhaps more importantly Refs. [83, 75] assume that the GUP would not affect the black-body spectrum and in particular a standard expression for Stefan's law is used even in Planckian regimes. There was no investigation of MDRs in Refs. [83, 75].

An attempt to describe Hawking radiation in presence of a MDR was reported in Ref. [82]. There the problem is approached from the field-theoretic perspective, considering possible modification of the field equations coming from the MDR. No explicit formula for the corrections to the Hawking spectrum and to the entropy-area relation was obtained in Ref. [82].

Ref. [84] investigates how a general form of the GUP could modify the volume element of phase space, and therefore the black-body-radiation formula, using the Hamiltonian formulation in terms of Poisson brackets.

In Ref. [85] an analysis of black-body radiation is carried out in presence of a

MDR of the type emerging from a proposed "semiclassical limit" of LQG, which is analogous to the "leading order" MDR (4.1) we studied in some parts of this paper. The results reported there are consistent with the power-series formulas for Stefan's and Wien's law which we derived. The features we exposed in considering some illustrative examples of all-order MDRs, were not discussed in Ref. [85]. Also the entropy-area relation and the aspects of black-hole evaporation which we considered here were not part of the analysis reported in Ref. [85], and Ref. [85] did not consider the possibility of a GUP.

Ref. [86] is closest in spirit to the part of our analysis where we focused on the black-body radiation spectrum, as affected by a MDR. Although the formal setup differs in several points, the results are roughly consistent with ours, including the possibility of "saturation" of the characteristic frequency at $T \gg E_p$. There was however no investigation of the entropy-area relation and the Generalized Second law in Ref. [86], and Ref. [86] also did not consider the possibility of a GUP.

In summary: the technical difficulties that are encountered in most approaches to the quantum-gravity problem usually only allow one to grasp a few disconnected aspects of the physical picture that the theories could provide. And in some approaches even the few "physical" results that are obtained, are only derived within approximation schemes whose reliability is not fully established. We have argued that in this situation it might be particularly valuable to establish a few logical links connecting some apparently unrelated aspects of the physical picture. And we showed that such a link can be found between some aspects of quantum-gravity research which have attracted strong interest in recent times, a link providing a connection between results on modified energy-momentum dispersion relations and/or modified position-momentum uncertainty principles and results on the thermodynamics of black holes. We have provided a description of log corrections to the entropy-area law for black holes that is based on the availability of a MDR and/or a GUP.

In exploring other aspects of black-hole thermodynamics as affected by MDRs and GUPs we stumbled upon a few noteworthy points. We found that the Generalized Second Law of thermodynamics might be robust enough to survive the introduction of these Planck-scale effects. We found that a MDR introducing a minimum value for wavelengths (even when no maximum value for frequencies is introduced) could lead to a description of black-body radiation in which the characteristic frequency of the radiation never exceeds a finite Planckian value (described in Eq. (4.46)). This in turn also affects black-hole evaporation in such a way that the temperature diverges already when the mass of the black hole decreases to a Planck-scale value (instead of diverging only in the zero-mass limit as usually assumed).

A key test for our line of analysis will come from future improved analyses within the LQG approach. According to the perspective we adopted some preliminary results on the emergence of modifications of the dispersion relation that depend linearly on the Planck length (at low energies) would be incompatible with the LQG results on log corrections to the entropy-area relation for black holes. We predict that improved analyses of the LQG approach should lead to the emergence of a picture that is instead compatible with the conceptual link we are proposing.

As stressed in Section 4.5 one aspect of our analysis in which we took a rather conservative attitude (in comparison with the possibilities considered in the literature) is the one concerning the description of the speed of photons, which we assumed to be still frequency independent. We do not expect major obstacles for a generalization of our analysis with the inclusion of the possibility of a frequency-dependent speed of photons, and the preliminary investigation reported in Section 4.5 suggests that some of the core features that emerged from our analysis are only moderately affected by the choice of law for the speed of photons.

Chapter 5

Tunneling through the quantum horizon

Classical black holes are perfect absorbers: they accrete their (irreducible) mass and no fraction of it can escape as there are no classical allowed trajectories crossing the horizon on the way out. This particular behavior suggests an interpretation of the black hole area as an entropy-like quantity. The key point of Bekenstein's argument is the inclusion of the quantum mechanical properties of a particle crossing the black-hole horizon. For this reason the entropy-area relation can be viewed as a first step towards the understanding of the "quantum" behavior of black holes. Hawking's discovery [6] that quantum fields on a Schwarzschild background do indeed predict a thermal flux of particles *away* from the horizon confirmed that the black hole entropy/area is in all senses a thermodynamic quantity and it is legitimate to define a temperature that corresponds to a "physical" temperature associated with the radiation.

It is interesting to note how the inclusion of quantum effects allows, for particles in a Schwarzschild geometry, to propagate through classically forbidden regions. This suggests that it should be possible to describe the black hole emission process, in a semiclassical fashion, as quantum tunneling. Parikh and Wilczek [95] (see also [96, 97]) showed how such a description of black hole radiance is possible if one considers the emission as a transition between states with the same energy. In this way the lowering of the mass of the black hole during the process and the related change in the radius set the barrier through which the particle tunnels. The resulting probability of emission differs from the standard Boltzmann factor by a corrective term which depends on the ratio of the energy of the emitted particle and the mass of the hole. The appearance of the correction causes the emission spectrum to be non-thermal. This reflects the fact that in order to describe transitions in which the energy of the emitted particle-black hole system does not change one must take into account the particle's self-gravitation. In the limit when the energy of the emitted particle is small compared with the mass of the black hole the emission spectrum becomes thermal and Hawking's result is recovered.

In the tunneling picture the Bekenstein-Hawking entropy-area relation can be deduced from the form of the emission probability. In fact for a generic system undergoing a quantum transition the emission probability is proportional [96, 97] to a phase space factor depending on the initial and final entropy of the system. A phase space factor given by the exponential of the difference between the Bekenstein-Hawking entropy[17] $S_{BH} = \frac{A}{4} = 4\pi M^2$ associated with the black hole after and before the emission corresponds exactly to the Parikh-Wilczek result for the tunneling probability[18].

The derivation of Parikh and Wilczek gives a dynamical description of black hole radiance in terms of the semiclassical tunneling of a shell propagating in a Schwarzschild metric. The metric "knows" of the particle's energy through the phenomenon of back reaction but its role is just that of a *classical* background

[17] In this Chapter, for convenience, we will work in $k = \hbar = c = G = 1$ units, until, in Section 5.2 we will return to $k = \hbar = c = 1$ units to keep track of the Planck-scale suppressed terms.

[18] The same result for the emission probability is obtained, using different techniques, in [98]. In the same work the authors discuss the universal validity of the formula $\Gamma \sim e^{(-\Delta A/4)}$ for a quantum emission from *every* type of event horizon.

space-time. It is interesting to ask then if it is possible to have a complementary derivation of black hole radiance in which space-time itself with its "quantum" properties drives the emission process. In [102] York provided such a description in terms of zero point quantum fluctuations of the black hole metric. In the model he proposed such fluctuations, governed by the uncertainty principle, are responsible for the appearance of a "quantum ergosphere". If one associates the irreducible mass of the quantum ergosphere to the mean thermal energy of a Planckian oscillator at a given temperature the result is that, for the lowest modes of oscillation, the temperature of the heat bath is approximatively given by Hawking's formula.

In this Chapter we proceed a step further and modify the Parikh-Wilczek tunneling picture including Planck-scale corrections for the near-horizon emission process. The type of modification we consider is directly related to the logarithmic corrective term appearing in the Bekenstein-Hawking entropy-area relation discussed in Chapter 3 and 4. We will also show how, within the tunneling framework, the presence of a quantum ergosphere can be related to the appearance of a logarithmic correction. This provides a link between quantum gravity microscopic description of black holes and the origin of the quantum fluctuations responsible for the formation of the quantum ergosphere.

5.1 Hawking radiation as tunneling

5.1.1 Motion of a self-gravitating spherical shell in a Schwarzschild geometry

We summarize here the results of [103] where the corrections to the geodesic motion of a spherical shell due to self-gravitation in a Schwarzschild geometry were calculated. We start by writing the metric for a general spherically symmetric system in ADM form

$$ds^2 = -N_t(t,r)^2 dt^2 + L(t,r)^2[dr + N_r(t,r)dt]^2 + R(t,r)^2 d\Omega^2 \,. \tag{5.1}$$

The action for the black hole plus the emitted shell system is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,\mathcal{R} - m\int dt\sqrt{(\hat{N}^t)^2(\hat{L}^r)^2(\dot{\hat{r}} + \hat{N}^r)^2} + \text{boundary terms}\,, \tag{5.2}$$

where $\hat{r}$ is the shell radius and the other quantities under "ˆ" are evaluated at the shell through $\hat{g}_{\mu\nu} = g_{\mu\nu}(\hat{t},\hat{r})$. The above action can be written in Hamiltonian form where all the canonically conjugate momenta appear. Since the system has only one effective degree of freedom the idea is to solve the constraints of the theory in order to eliminate the dependence of the action from all the momenta but the one conjugate to the shell radius. This remaining degree of freedom can be expressed in terms of the total mass/energy of the system and it is obviously related to the position of the shell. In the approach followed in [103], the total (ADM) mass is allowed to vary with time while the mass of the hole is kept fixed. This time dependence accommodates the dynamics of the system and allows energy conservation to hold at any time of the process.

Once the constraints are solved and the expression for the conjugate momenta are substituted into the action one integrates over the gravitational degrees of freedom to obtain an effective action. Furthermore one specializes to the case of a massless particle ($m = 0$) and fixes the gauge appropriately ($L = 1$ $R = r$). This choice of the gauge corresponds to a particular set of coordinates for the line element (Painleve' coordinates) which is particularly useful to study across horizon phenomena being non-singular at the horizon and having Euclidean constant time slices (for more details see [105]). The effective action for a massless gravitating spherical shell is then

$$S = \int dt \left(p_c \dot{\hat{r}} - M_+ \right) \,, \tag{5.3}$$

where p_c is the momentum canonically conjugate to $\hat{r}$, the radial position of the shell, and M_+ is the total mass of the shell-hole system which plays the role of the Hamiltonian. In terms of the black hole mass M and the shell energy E we have $M_+ = M + E$. Details of the lengthy derivation can be found in [103]. The trajectories which extremize this action are the null geodesics of the metric

$$ds^2 = -[N_t(r; M+E)dt]^2 + [dr + N_r(r; M+E)dt]^2 + r^2 d\Omega^2 \,, \tag{5.4}$$

for which

$$\frac{dr}{dt} = N_t(r; M+E) - N_r(r; M+E) \,. \tag{5.5}$$

5.1.2 The KKW model in a nutshell

In [103] and [104], Keski-Vakkuri, Kraus and Wilczek showed how corrections of the type considered above can affect the emission spectrum of a black hole. They consider the Bogoliubov coefficients, $\alpha_{kk'}$ and $\beta_{kk'}$, connecting the positive and negative frequency modes of an asymptotic observer and one freely falling through the horizon. The (semiclassical) WKB approximation is then used to express the mode solutions for the observer crossing the horizon. Such an approximation is valid since the black hole emission is dominated by modes that have very small wavelengths close to the horizon and undergo a large red-shift when propagating away from it. Once this approximation is taken into account one finds [104]

$$|\alpha_{kk'}| = e^{-Im \int_{r_+(0)}^{r_f} p_+(r)dr} \quad ; \quad |\beta_{kk'}| = e^{-Im \int_{r_-(0)}^{r_f} p_-(r)dr} \,, \tag{5.6}$$

where $r_\pm$ and $p_\pm$ are the trajectories and momenta of positive and negative energy modes propagating in and out of the horizon and r_f is located outside the horizon. From Hamilton's equation $\dot{r} = \frac{\partial H}{\partial p} = \frac{\partial E}{\partial p}$, with the Hamiltonian given by $H = M + E$, one can express $p_\pm$ using (5.5)

$$p_\pm(r) = \int_0^{\pm\omega_k} \frac{dE}{N_t(r; M+E) - N_r(r; M+E)} \,. \tag{5.7}$$

It can be shown [104] that $\alpha_{kk'} = 1$. For $\beta_{kk'}$ one instead finds

$$\text{Im} \int_{r_-(0)}^{r_f} p_-(r)dr = -\pi \int_0^{-\omega_k} \frac{dE}{\kappa(M+E)} \,. \tag{5.8}$$

In order to calculate the amplitude for particle production in the KKW model, it is assumed that the emission in the low energy regime is uncorrelated so that the amplitude in terms of $|\beta_{kk'}/\alpha_{kk'}|^2$ reads

$$\rho(\omega_k) = \frac{|\beta_{kk'}/\alpha_{kk'}|^2}{1 - |\beta_{kk'}/\alpha_{kk'}|^2} \,. \tag{5.9}$$

Instead when ω_k is comparable with the mass of the black hole and at most one quantum can be emitted

$$\rho(\omega_k) \simeq \left|\frac{\beta_{kk'}}{\alpha_{kk'}}\right|^2 \,. \tag{5.10}$$

Using the first law of black-hole thermodynamics $dM = \frac{\kappa(M)}{2\pi} dS$, one can evaluate (5.8). For the first case when ω_k is small compared to M, we obtain the usual emission amplitude governed by the Hawking temperature. For large ω_k instead one has

$$\rho(\omega_k) \simeq \exp\left[S(M - \omega_k) - S(M)\right] \,. \tag{5.11}$$

Substituting the standard Bekenstein-Hawking expression for the black hole entropy in the previous equation leads to a non-thermal correction, quadratic in ω_k, to the typical Boltzmann factor of the emission probability.

5.1.3 Quantum tunneling and non-thermal spectrum

The results of [103] and [104] can be recast in an elegant and simple form if one describes the emission of a particle as a tunneling process [95, 96]. This is done by considering the geometrical optics limit[19] so that one can treat the wave-packets near the horizon as effective particles. The emission of each of these particles is seen as a tunneling through a barrier set by the energy of the particles itself. This simple argument makes it possible to avoid the machinery of Bogoliubov coefficients and also shows how energy conservation is naturally

[19] This limit is valid for the same reason that allowed us to use the WKB approximation in the analysis of the previous section, i.e. the fact that the emitted wave packets are arbitrarily blue-shifted close to the horizon.

preserved during the emission process [97].

The authors consider now an explicit expression for the line element (5.4) obtained from the expressions of N_t and N_r given by the constraint equations [103]

$$N_t = \pm 1 \;\; ; \;\; N_r = \pm\sqrt{\frac{2M_+}{r}} \; . \tag{5.12}$$

In the model proposed in [95] the total mass of the system is kept fixed while the hole mass is allowed to vary. This means that the mass parameter M_+ is now $M_+ = M - E$. One then has the following expression for a spherical shell moving along a radial null geodesic

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M-E)}{r}} \, . \tag{5.13}$$

In the WKB approximation the tunneling probability is a function of the imaginary part of the particle's action

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} \; . \tag{5.14}$$

If we consider the emission of a spherical shell we have

$$\mathrm{Im}\,S = \mathrm{Im}\int_{r_{in}}^{r_{fin}} p_r dr \, , \tag{5.15}$$

where r_{in} and r_{fin} are just inside and outside the barrier through which the particle is tunneling. We can now see the key point: the expression for $\mathrm{Im}\,S$ is the same as the one for the Bogoliubov coefficient $\beta_{kk'}$ used in the KKW analysis. The same coefficient characterizes the emission probability in the field theoretical model. The advantage of considering the particle-tunneling picture is that the correction we get now is present at all energy regimes even though it becomes dominant only in the high energy regime. To calculate $\mathrm{Im}\,S$ we use once again Hamilton's equation, $\dot{r} = \frac{\partial H}{\partial p}$,

$$\mathrm{Im}\,S = \mathrm{Im}\int_{r_{in}}^{r_{fin}} p_r dr = \mathrm{Im}\int_{r_{in}}^{r_{fin}} \int_{M}^{M-E} \frac{dH'}{\dot{r}} dr \, . \tag{5.16}$$

The Hamiltonian is $H' = M - E'$, so the imaginary part of the action reads

$$\mathrm{Im}\,S = -\mathrm{Im}\int_{r_{in}}^{r_{fin}} \int_{0}^{E} \frac{dE'}{\dot{r}} dr \, . \tag{5.17}$$

Using (5.13) and integrating first over r one easily obtains

$$\Gamma \sim \exp\left(-8\pi ME\left(1-\frac{E}{2M}\right)\right) , \tag{5.18}$$

which, provided the usual Bekenstein-Hawking formula $S_{BH} = A/4 = 4\pi M^2$ is valid, corresponds to the KKW result

$$\Gamma \sim \exp\left[S_{BH}(M-E) - S_{BH}(M)\right] . \tag{5.19}$$

If one integrates (5.17) first over the energies it is easily seen that in order to get (5.18) we must have $r_{in} = M$ and $r_{out} = M - E$. So according to what one would expect from energy conservation, the tunneling barrier is set by the shrinking of the black hole horizon with a change in the radius set by the energy of the emitted particle itself.

An interesting aspect to analyze is whether or not the non-thermal correction obtained leads to statistical correlations between the probabilities of emission of quanta with different energies. This would allow for information to be encoded in the emitted radiation. Consider for example the probability of emission of a quantum of energy $E = E_1 + E_2$ and two quanta of energies E_1 and E_2. The function

$$\chi(E_1+E_2; E_1, E_2) = \log(\Gamma(E_1+E_2)) - \log(\Gamma(E_1)\Gamma(E_2)) \tag{5.20}$$

measures the statistical correlation between the two probabilities. It is zero when the probabilities are independent (or "uncorrelated") as e.g. for a thermal emission spectrum like the one of a radiating black body. Using (5.18) it is easy to verify [97] that for the non-thermal correction due to back-reaction effects $\chi(E_1 + E_2; E_1, E_2) = 0$. One concludes that back-reaction effects alone do not provide a straightforward way in which information can emerge from the horizon. There might well be other processes that would allow the information of a pure quantum state to be recovered after its gravitational collapse but one would have to resort to other mechanisms [20] .

5.2 The Parikh-Wilczek tunneling picture revisited

In this section we discuss a modification of the above argument that takes into account the presence of Quantum Gravity-induced corrections. Let us first notice that the probability of emission of a shell with energy E, in the presence of back-reaction effects, put in the form (5.19) is highly suggestive. It is what one would expect from a quantum mechanical calculation of a transition rate where, up to a factor containing the square of the amplitude of the process,

$$\Gamma \sim \frac{e^{S_{fin}}}{e^{S_{in}}} = \exp\left(\Delta S\right) \ . \tag{5.21}$$

In other words the emission probability is proportional to a phase space factor which depends on the initial and final entropy of the system. The entropy is directly related to the number of micro-states available to the system itself.
This observation calls for an immediate generalization. As we discussed in the previous chapters calculations of the black hole entropy in several quantum gravity scenarios [69, 70, 71, 68, 99, 100, 101], besides reproducing the familiar linear relation between area and entropy obtained a leading order "quantum" correction with a logarithmic[21] dependence on the area

$$S_{QG} = \frac{A}{4L_p^2} + \alpha \ln \frac{A}{L_p^2} + O\left(\frac{L_p^2}{A}\right) \ . \tag{5.22}$$

Now consider the emission of a particle of energy E from the black hole. One might expect that a derivation of the emission probability in a quantum gravity

[20]As proposed in [97] correlations might appear when back-reaction effects are considered in transient phases of the black hole emission.

[21]We now switch from $k = \hbar = c = G = 1$ units of the previous sections to $k = \hbar = c = 1$ to keep track of the Planck-scale suppressed terms.

framework in presence of back-reaction would lead to an expression analogous to (5.21) with the usual Bekenstein-Hawking entropy $S_{BH} = \frac{A}{4L_p^2}$ replaced by (5.22), i.e.

$$\Gamma \sim \exp\left(S_{QG}(M-E) - S_{QG}(M)\right) . \tag{5.23}$$

The previous expression written in explicit form reads

$$\Gamma(E) \sim \exp\left(\Delta S_{QG}\right) = \left(1 - \frac{E}{M}\right)^{2\alpha} \exp\left(-8\pi GME\left(1 - \frac{E}{2M}\right)\right) . \tag{5.24}$$

The exponential in this equation shows the same type of non-thermal deviation found in [95]. In this case, however, an additional factor depending on the ratio of the energy of the emitted quantum and the mass of the black hole is present. We would now like to know whether or not, in our case, the emission probabilities for different modes are statistically correlated. Using (5.24), we have, for a first emission of energy E_1,

$$\ln[\Gamma(E_1)] = -8\pi GME_1\left(1 - \frac{E_1}{2M}\right) \; + \; 2\alpha \ln\left[\frac{M-E_1}{M}\right] . \tag{5.25}$$

Then a second emission of energy E_2 gives us

$$\ln[\Gamma(E_2)] = -8\pi G(M-E_1)E_2\left(1 - \frac{E_2}{2(M-E_1)}\right) \; + \; 2\alpha \ln\left[\frac{M-E_1-E_2}{M-E_1}\right] . \tag{5.26}$$

Alternatively, a single emission of the same total energy yields

$$\ln[\Gamma(E_1+E_2)] = -8\pi GM(E_1+E_2)\left(1 - \frac{E_1+E_2}{2M}\right) \; + \; 2\alpha \ln\left[\frac{M-E_1-E_2}{M}\right] . \tag{5.27}$$

It is now easily verified that the vanishing of the correlations, or

$$\chi(E_1+E_2; E_1, E_2) = 0 \; , \tag{5.28}$$

is still in effect at least to this logarithmic order. In fact, after just a few iterations, one should readily be convinced that this outcome will persist up to any perturbative order of the quantum-corrected entropy [cf, equation (5.22)]. Hence, it does not appear that the inclusion of such correlations can account for the mode correlations after all.

On the other hand, it is interesting to note that the emission of three or more quanta could still induce a non-zero correlation even for the tree-level calculation. By which we mean that, for the sequential emission of (e.g.) E_1, E_2 and $E_1 + E_2$, then $S(M - E_1) - S(M) + S(M - E_1 - E_2) - S(M - E_1) \neq S(M - 2E_1 - 2E_2) - S(M - E_1 - E_2)$. And so there still appears to be viability, on some level, for the notion that information leaks out in the tunneling process. Let us also notice how the appearance of the Quantum Gravity suppression term in (5.24) can cause $\Gamma(E) \to 0$ when the energy of the emitted quantum approaches the mass of the black hole. However, this suppression can only take place when $\alpha > 0$; whereas a negative value of α will, conversely, cause the probability to diverge as the same limit is approached! [22]

5.2.1 Tunneling in the presence of near-horizon Planck-scale effects

We discuss now a possible modification of the Parikh-Wilczek derivation, in the presence of Planck-scale effects, which could give rise to an emission spectrum of the type (5.24). As we already observed in the previous sections, the black hole radiation spectrum seen from an observer at infinity is dominated by modes that propagate from "near" the horizon where they have arbitrarily high frequencies and their wavelengths can easily go below the Planck length [106, 107]. It turns out then that a key assumption in all the derivations of the Hawking radiation is that the quantum state near the horizon looks, to a freely falling observer, like the Minkowski vacuum. In other words Lorentz symmetry should hold up to extremely short scales or very large boosts. It is plausible then that

[22] Let us, however, point out one possible loophole: higher-order corrections may conspire to induce a suppression that is stronger than this divergence.

the motion of our particle tunneling through the horizon might be affected by Planck scale modifications of relativistic kinematics associated to the presence of a MDR. One would expect that an analysis analogous to the ones of the previous sections with opportune modifications should lead to a result of the form (5.19) with S_{BH} replaced by S_{QG}.

Once again we consider the emission of a spherical shell and compute the tunneling amplitude (5.14) through (5.15)

$$\mathrm{Im}\, S = \mathrm{Im} \int_{r_{in}}^{r_{fin}} p_r dr = \mathrm{Im} \int_{r_{in}}^{r_{fin}} \int_0^H \frac{dH'}{\dot{r}} dr = -\mathrm{Im} \int_{r_{in}}^{r_{fin}} \int_0^E \frac{dE'}{\dot{r}} dr \ . \tag{5.29}$$

where we used the fact that for the Hamiltonian $H = M - E$. Now we proceed to evaluate the integral without using an explicit form for the null geodesic of the spherical shell in terms of its energy. In fact, near the horizon, where our integral is being evaluated, one has

$$N_t(r; M) - N_r(r; M) \simeq (r - R)\, \kappa(M) + O((r - R)^2) \tag{5.30}$$

where R is the Schwarzschild radius and $\kappa(M)$ is the horizon surface gravity. Taking into account self-gravitation effects, $\dot{r}$ can be approximated by

$$\dot{r} \simeq (r - R)\, \kappa(M - E) + O((r - R)^2)\,. \tag{5.31}$$

We can then write

$$\mathrm{Im}\, S = -\mathrm{Im} \int_{r_{in}}^{r_{fin}} \int_0^E \frac{dE'}{(r - R)\, \kappa(M - E')} dr \ . \tag{5.32}$$

Integrating over r, using the Feynman prescription[23] for the pole on the real axis $r = R$, we get

$$\mathrm{Im}\, S = -\pi \int_0^E \frac{dE'}{\kappa(M - E')} \ . \tag{5.33}$$

The surface gravity appearing in the above integral carries quantum gravity corrections coming from modifications of near horizon physics related to Planck-scale departures from Lorentz invariance. As shown explicitly in Chapter 4 these

[23] The pole is moved in the lower half plane as in [95].

modifications are such that they reproduce via the first law of black hole thermodynamics, $dE' = dM' = \frac{\kappa(M)}{2\pi} dS$, the quantum gravity corrected entropy-area law (5.22). Using the first law, (5.33) becomes

$$\operatorname{Im} S = -\frac{1}{2} \int_{S_{QG}(M)}^{S_{QG}(M-E)} dS = \frac{1}{2}[S_{QG}(M) - S_{QG}(M-E)] \qquad (5.34)$$

which leads to a probability of emission

$$\Gamma(E) \sim \exp\left(-2\mathrm{Im}S\right) = \left(1 - \frac{E}{M}\right)^{2\alpha} \exp\left(-8\pi GME\left(1 - \frac{E}{2M}\right)\right) \qquad (5.35)$$

analogous to (5.24).

5.2.2 The quantum ergosphere

In the previous sections a key step toward the tunneling description was the inclusion of back reaction effects for the propagation shell at the classical level. The origin of the "quantum ergosphere" can be also traced back to a calculation of back reaction effects. In this case one studies the response of the metric to the energy momentum tensor associated with the quantum fluctuations near the horizon responsible for the black hole emission process. An estimate [108, 109] of this effect can be given in terms of the black hole luminosity, which for a Hawking flux is given by $L_H = \frac{B}{M^2}$, with B a barrier factor depending on the grey body absorption and the radiated species.

The quantum-induced energy leakage from the black hole [109] produces a splitting between the timelike limit surface (TLS) (on which $\dot{r} = 0$ for radial null geodesics, with r the circumferential radius) and the event horizon (EH), approximately identified [109] with the locus of "unaccelerated" ($\ddot{r} = 0$) photons. This splitting, which is essentially a back reaction effect, leads to the creation of a quantum ergosphere associated with the geometrically well defined difference of areas $\delta A_{QE} = A_{TLS} - A_{EH}$. The important point to note is that if one considers the explicit form of δA_{QE} it is easy to see that this does not go to zero when $L_H \to 0$ (and consequently when the Hawking temperature $T_H \to 0$) as it would be expected. This reveals an intrinsic "quantum" nature of the ergosphere and indeed it turns out that δA_{QE} goes to zero only in the limit $\hbar \to 0$, in which case one recovers the classical Schwarzschild structure. This fact suggests that for quantum black holes, zero point fluctuations of the metric might play an active role in near horizon phenomena, *in primis* in the Hawking effect.

The above arguments served as a starting point for York's description of black hole radiance. In [102] he proposes a model of fluctuating metric whose oscillation amplitudes are determined by the uncertainty principle. A quantum ergosphere is formed for each mode of oscillation with an irreducible mass defined by the difference between the mean irreducible masses associated with the EH and TLS. In order to estimate Hawking's temperature York conjectured that this irreducible mass corresponds to the mean thermal energy of a quantum oscillator in a heat bath at a given temperature. The frequencies of oscillation are then determined by the lowest gravitational quasinormal modes of the black hole. The temperature obtained in this way agrees in good approximation with Hawking's result.

York's model provides an example of how it is possible to "switch on" a quantum ergosphere introducing appropriate quantum effects, namely, quantum oscillations around the classical Schwarzschild metric. More generally one would expect that the presence of a quantum ergosphere would play a role in the phenomenon of black hole radiance whenever quantum properties of the geometry are taken into account. Along these lines it is reasonable to assume that quantum effects on the horizon within a particular quantum gravity framework, without the introduction of an *ad hoc* model for the quantum fluctuations of the metric, will be effectively described in terms of a quantum ergosphere. In the following section we will see this conjecture at work in the context of the previously discussed tunneling framework.

5.2.3 A tunnel through the quantum ergosphere

In this section we will see how it is possible to give a complementary description of the tunneling process with quantum corrections that instead of focusing on Planck-scale effects on the tunneling particle will take into account the quantum properties of space-time itself. Our goal is to show how an emission probability of the type (5.24) can be obtained if one takes into account the possibility that a "quantum" background space-time can alter the geometry near the horizon. In the spirit of York we will assume that zero-point quantum fluctuations of the metric produce a splitting between the timelike limit surface and the event horizon. This would lead to the formation of a quantum ergosphere characterized by the area difference $\delta A_{QE} = \bar{A}_{TLS} - \bar{A}_{EH}$ (where $\bar{A}_{TLS}$ and $\bar{A}_{EH}$ are the mean areas associated with the fluctuating TLS and EH). As in the previous sections, in order to derive the tunneling amplitude, we have to evaluate the integral

$$\mathrm{Im}\, S = \mathrm{Im} \int_{r_{in}}^{r_{fin}} p_r dr = \mathrm{Im} \int_{r_{in}}^{r_{fin}} \int_0^H \frac{dH'}{\dot{r}} dr = -\mathrm{Im} \int_{r_{in}}^{r_{fin}} \int_0^E \frac{dE'}{\dot{r}} dr \ , \tag{5.36}$$

but now taking into account the presence of the quantum ergosphere. Let us focus on the propagation of a classical shell in a Schwarzschild geometry. When no back reaction effects nor quantum gravity corrections are present the geodesic (5.13) is simply

$$\dot{r} = \pm 1 - \sqrt{\frac{2GM}{r}} \,, \tag{5.37}$$

$\dot{r} = 0$ at $r = 2GM$ where the TLS and EH coincide (the apparent horizon (AH) for spherically symmetric configurations coincides with the TLS)[24]. To evaluate the effects of this shifting on (5.37) we consider the mean irreducible masses associated with the TLS and EH

$$\bar{M}_{TLS} = \left(\frac{\bar{A}_{TLS}}{16\pi} \right)^{1/2} , \quad \bar{M}_{EH} = \left(\frac{\bar{A}_{EH}}{16\pi} \right)^{1/2} \tag{5.38}$$

[24]The radial coordinate r is, just like in standard Schwarzschild coordinates and in the coordinate set used in [102, 108], the circumferential radius.

Following [102] we assume that $\bar{M}_{TLS}$ and $\bar{M}_{EH}$ will differ from the standard value of M by a term of order E_p^2/M

$$\bar{M}_{TLS} = M + \tilde{\alpha}\frac{E_p^2}{M} \tag{5.39}$$

$$\bar{M}_{EH} = M + \tilde{\beta}\frac{E_p^2}{M} \tag{5.40}$$

with $\tilde{\alpha} > \tilde{\beta}$. There will be an irreducible mass associated with the quantum ergosphere $M_{QE} = \bar{M}_{TLS} - \bar{M}_{EH}$ which can be seen as a measure of the zero point energy associated with quantum fluctuations of the geometry. We assume that a non-vanishing M_{QE} will cause a shift in the pole of the integrand in ImS. To see this let us recall that, as stressed in Section 5.1.3, the tunneling barrier is set by the energy of the black hole before and after the emission of the shell. This is obtained using only the information about the radial location of the TLS contained in the integral (5.17). We realize then that the position of the TLS is what really determines the emission probability in the tunneling framework. As an estimate of the shift in the pole we will assume that in the expression for the radial null geodesic (5.37) the mass associated with the TLS will be given by the mean value $\bar{M}_{TLS}$. Equation (5.37) then becomes

$$\dot{r} = \pm 1 - \sqrt{\frac{2G\left(M + \tilde{\alpha}\frac{E_p^2}{M}\right)}{r}} \tag{5.41}$$

As a next step we attempt to introduce back reaction effects due to the energy of the propagating shell. In doing so let us recall that, in the absence of a quantum ergosphere, a self-gravitating massless shell of energy E, in its geodetic motion, "sees" an effective black hole mass $M - E$, i.e. in the shell's geodesic equation (5.37) M is replaced by $M-E$. Our assumption is that an analogous replacement will be required in (5.41) in order to take into account the back reaction of the shell. The geodesic would then read

$$\dot{r} = \pm 1 - \sqrt{\frac{2G\left((M-E) + \tilde{\alpha}\frac{E_p^2}{(M-E)}\right)}{r}}\,. \tag{5.42}$$

Equipped with this expression we now turn to the calculation of the transition amplitude *à la* Parikh-Wilczek. Substituting (5.42) (with a plus sign for an outgoing shell) in (5.36) and integrating over r using the usual Feynman prescription we have

$$\mathrm{Im}\, S = 4\pi \int_0^E G(M-E') \left(1 + \tilde{\alpha} \frac{E_p^2}{(M-E')^2}\right) dE' \ . \tag{5.43}$$

Doing the integral over the energy and substituting in (5.14) we obtain for the emission probability

$$\Gamma \sim \exp\left(-2\mathrm{Im}S\right) = \left(1 - \frac{E}{M}\right)^{8\pi\tilde{\alpha}} \exp\left(-8\pi GME\left(1 - \frac{E}{2M}\right)\right) \tag{5.44}$$

which is corresponds to (5.24) provided $\alpha = 4\pi\tilde{\alpha}$.

Let us note that in our derivation the main consequence of the introduction of a quantum ergosphere is to modify the tunneling rate in such a way that a leading order logarithmic correction to the black hole entropy-area law is reproduced. The quantum ergosphere, in this context, has a genuine quantum space-time nature i.e. we expect its presence to be a prediction of a yet-to-be found theory of quantum gravity. Following York's original idea we also expect that a more refined model of a quantum ergosphere, emerging from a consistent quantum gravity framework, should also allow one to derive the Hawking effect in terms of the black hole quasi-normal modes when there are no quantum matter fields propagating in the hole geometry, as in [102].

Chapter 6

Conclusion

We started by considering the possibility that quantum gravity might exhibit a low-energy, flat-spacetime, limit which differs from our current, classical, picture at the Planck scale. This scenario is indeed realized, as we reviewed in Chapter 2, in several quantum gravity frameworks including approaches based on non-commutative geometry and loop quantum gravity.

One remarkable aspect of these departures from classical relativistic symmetries is the possibility that the deformed kinematics they introduce might lead to experimentally testable consequences. We focused in particular on scenarios that predict Planck-scale suppressed modifications of the relativistic energy-momentum dispersion relation which may lead to a violation of Lorentz symmetry or be associated to "quantum" deformations of relativistic symmetries. We saw, in Chapter 2, how observations of GRBs and UHECR might carry signatures of such quantum space-time models. In particular we focused on the threshold anomalies that a specific kinematical framework based on a MDR might induce in the chain of production of very high energy neutrinos associated with UHECRs. We showed that different choices in the parameters of the model lead to different modifications of the bound , proposed by Bahcall and Waxman, on the flux of such high energy neutrinos.

In Chapter 3 we pointed out how a characteristic feature of quantum space-time

scenarios, such as the presence of a length scale, the Planck length L_p, setting a lower bound on the accuracy of distance measurements, necessarily affects the relativistic particle localization limit $E \geq 1/\delta x$. This observation offered the opportunity to investigate a link between the non-trivial flat-space limit of a quantum gravity theory and the, genuinely quantum-gravitational, relation between the area of a black hole and its entropy. In particular we were able to constraint the form of MDR that seems to emerge in preliminary works in LQG, looking at the logarithmic corrections, obtained from direct micro-states counting in LQG, to the semiclassical Bekenstein-Hawking result $S = \frac{A}{4L_p^2}$.
Following this link we discussed, in Chapter 4, the possible consequences that Planck-scale modifications of the $E \geq 1/\delta x$ relation, emerging in quantum gravity scenarios with MDR and/or GUP, might have for the thermodynamic behavior of black holes and their evaporation process. The analysis led also to study how the spectrum of a radiating black body is affected in such quantum space-time scenarios and the consequences that this Planck-scale modified thermodynamics might have for the generalized second law.
Finally in, Chapter 5, we discussed how it is possible to combine quantum gravity effects and back-reaction using the description of Hawking radiation as tunneling proposed by Parikh and Wilczek. We first showed how Planck-scale effects can be incorporated in the Parikh-Wilczek tunneling picture and analyzed the possibility that statistical correlation might appear in the non-thermal, quantum corrected, spectrum obtained. We then adapted the derivation of Parikh and Wilczek in order to include effects due to quantum fluctuations of the black hole horizon which lead to the formation of a "quantum ergosphere". We saw how the "quantum corrected" emission probability contains an additional factor analogous to the corrective factor produced by the inclusion of the quantum gravity logarithmic correction to the entropy-area relation. This analogy suggests that the quantum ergosphere, seen as an indelible signature of quantum gravity on a black hole metric, affects the near horizon geometry of the black hole leading

to the emergence of a logarithmic correction in the entropy-area law. Reversing the view, the argument we presented might support the idea that leading order (logarithmic) quantum corrections to the black hole entropy are related to the presence of zero-point quantum fluctuations of the metric.

Departures from Lorentz/Poincaré symmetry at the Planck scale, with their potential of being experimentally falsifiable, provide, in our opinion, a valuable starting point to guide us toward the understanding of the full quantum description of gravity. In this dissertation we have also shown how possible hints on the quantum structure of space-time can emerge from a microscopic quantum-gravity description of black holes. We believe there is still a lot to learn from the analysis of the interplay between the quantum behavior of black holes and the symmetry structure of space-time at the Planck scale and that future advances will help us to set experimental and logical consistency boundaries to a field that until few years ago was open to the wildest theoretical speculations.

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