Rank-Level Duality of Conformal Blocks

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Abstract

SWARNAVA MUKHOPADHYAY: Rank-Level Duality of Conformal Blocks
(Under the direction of Prakash Belkale)

Classical invariants for representations of one Lie group can often be related to invariants of some other Lie group. Physics suggests that the right objects to consider for these questions are certain refinements of classical invariants known as conformal blocks. Conformal blocks appear in algebraic geometry as spaces of global sections of line bundles on moduli stacks of parabolic bundles on a smooth curve. Rank-level duality connects a conformal block associated to one Lie algebra to a conformal block for a different Lie algebra. In this dissertation we discuss a general approach to rank-level duality questions. The main result of the dissertation is a rank-level duality for $\mathfrak{so}(2r+1)$ conformal blocks on the pointed projective line which was suggested by T. Nakanishi and A. Tsuchiya. As an application of the general techniques developed in the thesis, we prove new symplectic rank-level dualities.

To my parents.

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Introduction

The main goal of this dissertation is to study the rank-level duality of conformal blocks from a general Lie theoretic view point. We consider rank-level dualities arising out of special embeddings of Lie algebras known as conformal embeddings. In this introduction, we put our work in its proper context by explaining the genesis and the recent history of research done on rank-level duality. In Section 0.1, we discuss the motivation and historical background for rank-level duality. We describe the main result obtained in the dissertation in Section 0.2 and in Section 0.3 we discuss the general context of rank-level duality. The main ideas of the proof of rank-level duality are briefly described in Section 0.4. Section 0.5 provides a thumbnail sketch of the general body of the dissertation.

0.1. The context of the problem

It has been known for a long time that invariant theory of GL_r and the intersection theory of Grassmannians are related. This relation gives rise to some interesting isomorphisms between invariants of SL_r and SL_s for some positive integer s. To make it precise recall that the irreducible polynomial representations of GL_r are indexed by r tuples of integers $\lambda = (\lambda^1 \geq \cdots \geq \lambda^r \geq 0) \in \mathbb{Z}^r$. Let V_λ denote the corresponding irreducible GL_r-module.

Consider $\lambda = (\lambda^1 \geq \cdots \geq \lambda^r \geq 0)$ an r tuple of integers such that $\lambda^1 \leq s$. The set of all such λ's is in bijection with $\mathcal{Y}_{r,s}$, the set of all Young diagrams with at most r rows and s columns. For λ, μ, ν in $\mathcal{Y}_{r,s}$ such that $|\lambda| + |\mu| + |\nu| = rs$ we know that

$$\dim_{\mathbb{C}}(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{SL}_r} = \dim_{\mathbb{C}}(V_{\lambda^T} \otimes V_{\mu^T} \otimes V_{\nu^T})^{\mathrm{SL}_s},$$

where $|\lambda|$ denote the number of boxes in the Young diagram of λ and λ^T denotes the transpose of the Young diagram of λ. The above is not only a numerical "strange" duality but the vector spaces are canonically dual to each other (see [**7**]).

Physics suggests that to understand the above kind of relation for other groups the correct objects to consider are certain refinements of the co-invariants known as conformal blocks. Consider a finite dimensional simple complex Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and a non-negative integer ℓ called the level. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an n tuple of dominant weights of $\mathfrak{g}$ of level ℓ. To n distinct points $\vec{p} = (P_1, \ldots, P_n)$ with coordinates $\vec{z} = (z_1, \ldots, z_n)$ on $\mathbb{P}^1$, one associates a finite dimensional vector space $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell, \vec{z})$ known as the space of covacua. The dual of $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}, \ell, \vec{z})$ is called a conformal block and is denoted by $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell, \vec{z})$. We refer the reader to Chapter 1 for more details. More generally, one can define conformal blocks associated to n distinct points on curves of arbitrary genus with at most nodal singularities (see Chapter 1). Conformal blocks form a vector bundle on $\overline{\mathrm{M}}_{g,n}$, the moduli stack of stable n pointed curves of genus g.

Rank-level duality is a duality between conformal blocks associated to two different Lie algebras. In [**27**], T. Nakanishi and A. Tsuchiya proved that on $\mathbb{P}^1$, certain conformal blocks of $\mathfrak{sl}(r)$ at level s are dual to conformal blocks of $\mathfrak{sl}(s)$ at level r. In [**2**], T. Abe proved rank-level duality statements between conformal blocks of type $\mathfrak{sp}(2r)$ at level s and $\mathfrak{sp}(2s)$ at level r. It is important to point out that there are no known relations between the classical invariants for the Lie algebras $\mathfrak{sp}(2r)$ and $\mathfrak{sp}(2s)$.

The rank-level duality of conformal blocks has a geometric perspective under the identification of conformal blocks with the space of non-abelian G-theta functions. This is known as strange duality. The strange duality conjecture for SL_n says that the space of generalized theta functions associated to the pairs (p, q), (q, p) are naturally dual to each other, the duality being induced from the tensor product of vector bundles. This conjecture was proved by P. Belkale (see [**8**] and [**9**]) and also by A. Marian and D. Oprea [**23**]. The symplectic strange duality conjecture in [**6**] was proved by T. Abe (see [**2**]). For a survey of these results we refer e.g. to [**24**], [**29**], [**31**].

0.2. Rank-level duality for odd orthogonal Lie algebras

The paper [**27**] (Section 6, page 368) suggests that one can try to answer similar rank-level duality questions for orthogonal Lie algebras on $\mathbb{P}^1$. Furthermore, it is pointed out in [**27**] that one should only consider the tensor representations, i.e. representations that lift to representations of the special orthogonal group (see Section 6 in [**27**]). In the following we answer the above question for odd orthogonal Lie algebras.

Throughout this dissertation we assume that $r, s \geq 3$. Let $P^0_{2s+1}(\mathfrak{so}(2r+1))$ denote the set of tensor representations of $\mathfrak{so}(2r+1)$ of level $2s+1$. We can realize the set $P^0_{2s+1}(\mathfrak{so}(2r+1))$ as a disjoint union of $\mathcal{Y}_{r,s}$ and $\sigma(\mathcal{Y}_{r,s})$, where σ is an involution $P^0_{2s+1}(\mathfrak{so}(2r+1))$ that corresponds to the action of a diagram automorphism of the affine Lie algebra $\widehat{\mathfrak{so}}(2r+1)$ (see Chapter 2). Our main theorem is the following:

Theorem 0.2.1. *Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathcal{Y}^n_{r,s}$ be an n tuple of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$.*

(1) If $\sum_{i=1}^n |\lambda_i|$ is even, then

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{so}(2r+1), 2s+1, \vec{z}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{so}(2s+1), 2r+1, \vec{z}),$$

where $\vec{z}$ is a tuple of n distinct points on $\mathbb{P}^1$.

(2) If $\sum_{i=1}^n |\lambda_i|$ is odd, then

$$\mathcal{V}_{\vec{\lambda},0}(\mathfrak{so}(2r+1), 2s+1, \vec{z}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{so}(2s+1), 2r+1, \vec{z}),$$

where $\vec{z}$ is a tuple of $(n+1)$ distinct points on $\mathbb{P}^1$.

(3) If $\sum_{i=1}^n |\lambda_i|$ is even, then

$$\mathcal{V}_{\vec{\lambda},\sigma(0)}(\mathfrak{so}(2r+1), 2s+1, \vec{z}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{so}(2s+1), 2r+1, \vec{z}),$$

where $\vec{z}$ is a tuple of $(n+1)$ distinct points on $\mathbb{P}^1$.

Remark 0.2.2. *The above statements are independent of each other.*

0.3. General set up of rank-level duality

We briefly discuss the general context of rank-level duality maps. We closely follow the methods used in [**27**], [**2**] and [**4**] but there are significant differences in key steps.

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ be simple Lie algebras and consider an embedding of Lie algebras $\phi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$. We extend it to a map of affine Lie algebras $\widehat{\phi} : \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \to \widehat{\mathfrak{g}}$. Consider a level one integrable highest weight module $\mathcal{H}_\Lambda(\mathfrak{g})$, and restrict it to $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$. The module $\mathcal{H}_\Lambda(\mathfrak{g})$ decomposes into irreducible integrable $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$-modules of level $\ell = (\ell_1, \ell_2)$ in the following way:

$$\bigoplus_{(\lambda,\mu)\in B(\Lambda)} m^{\Lambda}_{\lambda,\mu} \mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2) \simeq \mathcal{H}_\Lambda(\mathfrak{g}),$$

where $\ell = (\ell_1, \ell_2)$ is the Dynkin multi-index of ϕ and $m^{\Lambda}_{\lambda,\mu}$ is the multiplicity of the component $\mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2)$. In general, the number of components $|B(\Lambda)|$ may be infinite. We only consider those embeddings such that $|B(\Lambda)|$ is finite. These embeddings are known as conformal embeddings (see [**19**] for more details).

Further assume that $m^{\Lambda}_{\lambda,\mu} = 1$ for any level 1 weight Λ. Thus, for an n tuple $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ of level one dominant weights of $\mathfrak{g}$ we have an injective map:

$$\bigotimes_{i=1}^{n} (\mathcal{H}_{\lambda_i}(\mathfrak{g}_1) \otimes \mathcal{H}_{\mu_i}(\mathfrak{g}_2)) \to \bigotimes_{i=1}^{n} \mathcal{H}_{\Lambda_i}(\mathfrak{g}).$$

We consider a tuple of n distinct points $\vec{z}$ on $\mathbb{P}^1$ and taking "coinvariants" we get a map

$$\alpha : \mathcal{V}_{\vec{\lambda}}(\mathfrak{g}_1, \ell_1, \vec{z}) \otimes \mathcal{V}_{\vec{\mu}}(\mathfrak{g}_2, \ell_2, \vec{z}) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{g}, 1, \vec{z}),$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$. We refer the reader to Chapter 1 for more details. If $\dim_{\mathbb{C}}(\mathcal{V}_{\vec{\Lambda}}(\mathfrak{g}, 1, \vec{z})) = 1$, we get a map $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}_1, \ell_1, \vec{z}) \to \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{g}_2, \ell_2, \vec{z})$. This map is known as the rank-level duality map. The above analysis with the embedding $\widehat{\mathfrak{so}}(2r+1) \oplus \widehat{\mathfrak{so}}(2s+1) \to \widehat{\mathfrak{so}}((2r+1)(2s+1))$ gives the maps considered in Theorem 0.2.1. In Chapter 3, we define the rank-level duality for conformal blocks on n pointed nodal curves of arbitrary genus.

Remark 0.3.1. *Note that the conformal blocks in Theorem 0.2.1 can be identified with the space of global sections of a line bundle on the moduli stack of* Spin*-bundles over* $\mathbb{P}^1$ *with parabolic structures on marked points, we have not been able to define the rank-level duality map in Theorem 0.2.1 geometrically.*

0.4. Idea of Proof

We now discuss the main body of the proof of Theorem 0.2.1. This can be broken up into several steps:

0.4.1. Dimension Check. Using the Verlinde formula we show that the dimensions of the source and the target of the conformal blocks in Theorem 0.2.1 are the same. Unlike the case in [**2**] we do not have a bijection between $P_{2s+1}(\mathfrak{so}(2r+1))$ and $P_{2r+1}(\mathfrak{so}(2s+1))$ but we get around the problem by considering bijection of the orbits of $P_{2s+1}(\mathfrak{so}(2r+1))$ and $P_{2r+1}(\mathfrak{so}(2s+1))$ under the involution σ as described in [**28**]. Let $\vec{\lambda} \in \mathcal{Y}^n_{r,s}$ and $\Gamma = \{1, \sigma\}$ is the group of diagram automorphism of $\widehat{\mathfrak{so}}(2r+1)$ acting on $P_{2s+1}(\mathfrak{so}(2r+1))$. The Verlinde formula in this case takes the form

$$\sum_{\mu \in P_{2s+1}(\mathfrak{so}(2r+1))/\Gamma} f(\mu, \vec{\lambda}) |\operatorname{Orb}_\mu|,$$

where $f(\mu, \vec{\lambda})$ is a function, constant on the orbit of μ and $|\operatorname{Orb}_\mu|$ denotes the cardinality of the orbit of μ under the action of Γ. Using a non-trivial trigonometric identity in [**28**] and a generalization of Lemma A.42 in [**10**], we show that $f(\mu, \vec{\lambda})|\operatorname{Orb}_\mu|$ is same for the corresponding orbit for the Lie algebra $\mathfrak{so}(2s+1)$ at level $2r+1$.

0.4.2. Flatness of rank-level duality. The rank-level duality map has constant rank when $\vec{z}$ varies (see [**9**]). The conformal embedding is important in this case as it ensures that the rank-level duality map is flat with respect to the $KZ/Hitchin/WZW$ connection (see [**9**]) on sheaves of vacua over any family of smooth curves.

0.4.3. Degeneration of a smooth family. Let $C_1 \cup C_2$ be a nodal curve, where C_1 and C_2 are isomorphic to $\mathbb{P}^1$ intersecting at one point. A conformal block on $C_1 \cup C_2$ is isomorphic to a direct sum of conformal blocks on the normalization of $C_1 \cup C_2$. This property is known as factorization of conformal blocks. A key ingredient in the proof of rank-level duality in [**2**] is the compatibility of the rank-level duality with factorization. T. Abe uses it to conclude that the rank-level duality map is an isomorphism on certain nodal curves.

This property for nodal curves is no longer true for our present case due to the presence of "non-classical" components (i.e. components that do not appear in the branching of finite dimensional irreducible modules) in the branching of highest weight integrable modules. We refer the reader to Chapter 3 for more details.

We consider a family of smooth curves degenerating to a nodal curve X_0. Instead of looking at the nature of the rank-level duality map on the nodal curve, we study the nature of the rank-level duality map on nearby smooth curves of the nodal curve X_0 under any conformal embedding. We use the "sewing procedure" of [**36**] to understand the decomposition of the rank-level duality map near the nodal curve X_0. The methods used in this step are similar to [**4**]. This degeneration technique and the flatness of the rank-level duality enable us to use induction similar to [**27**] and [**2**] to reduce to the case for one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points. We refer the reader to Chapter 9 for more details.

0.4.4. Minimal Cases. We are now reduced to showing that the rank-level duality maps for one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points are non-zero. Our proof of this step differs significantly from that in [**2**] as we were not able to use any geometry of parabolic vector bundles with a non-degenerate form. This is again due to the presence of non-classical components. Using [**17**], we construct explicit vectors $v_1 \otimes v_2 \otimes v_3$ in the tensor product of three highest weight modules and show by using gauge symmetry (see Chapter 1) that $\Psi(v_1 \otimes v_2 \otimes v_3) \neq 0$. It will be very interesting if one can define the rank-level duality map purely in language of vector bundles with a non-degenerate form.

0.5. Overview of dissertation

A brief description of the remainder of the dissertation is as follows: In Chapter 1, we introduce important notations, conventions, results and definitions. Chapter 2 is devoted to the action of diagram automorphisms on conformal blocks. The rank-level duality map in the general context arising from conformal embeddings is constructed in Chapter 3. In Chapter 4, we study extensions of multi-shift automorphisms under embeddings of Lie algebras. As an application we prove new symplectic rank-level dualities. We discuss the compatibility of the rank-level duality map with factorization in Chapter 5. The branching rules for the conformal embeddings of the orthogonal Lie algebras are described in Chapter 6, and in Chapter 7, we show the equality of the dimensions of the source and the target of the rank-level duality map. Following the works of K. Hasegawa and I. Frenkel, the highest weight vectors of the components that appear in the branching rule are explicitly described in Chapter 8. In Chapter 9, we give a proof of the main result of the dissertation.

CHAPTER 1

Basic definitions in the theory of conformal blocks

In this chapter, we recall some basic definitions from [**36**] in the theory of conformal blocks. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We fix the decomposition of $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ is the set of roots decomposed into a union of $\Delta_+ \cup \Delta_-$ of positive and negative roots. Let $(,)$ denote the Cartan Killing form on $\mathfrak{g}$ normalized such that $(\theta, \theta) = 2$, where θ is the longest root and we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form $(,)$.

1.1. Affine Lie algebras

We define the affine Lie algebra $\widehat{\mathfrak{g}}$ to be

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c,$$

where c belongs to the center of $\widehat{\mathfrak{g}}$ and the Lie bracket is given as follows:

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + (X, Y) \operatorname{Res}_{\xi=0}(gdf).c,$$

where $X, Y \in \mathfrak{g}$ and $f(\xi), g(\xi) \in \mathbb{C}((\xi))$.

Let $X(n) = X \otimes \xi^n$ and $X = X(0)$ for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The finite dimensional Lie algebra $\mathfrak{g}$ can be realized as a subalgebra of $\widehat{\mathfrak{g}}$ under the identification of X with $X(0)$.

1.2. Representation theory of affine Lie algebras

The finite dimensional irreducible modules of $\mathfrak{g}$ are parametrized by the set of dominant integral weights $P_+ \subset \mathfrak{h}^*$. Let V_λ denote the irreducible module of highest weight $\lambda \in P_+$ and v_λ denote the highest weight vector.

We fix a positive integer ℓ which we call the level. The set of dominant integral weights of level ℓ is defined as follows

$$P_\ell(\mathfrak{g}) := \{\lambda \in P_+ | (\lambda, \theta) \leq \ell\}$$

For each $\lambda \in P_\ell(\mathfrak{g})$ there is a unique irreducible integrable highest weight $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda(\mathfrak{g})$ which satisfies the following properties:

(1) $V_\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})$,

(2) The central element c of $\widehat{\mathfrak{g}}$ acts by the scalar ℓ,

(3) Let v_λ denote a highest weight vector in V_λ, then

$$X_\theta(-1)^{\ell-(\theta,\lambda)+1} v_\lambda = 0,$$

where X_θ is a non-zero element in the weight space of $\mathfrak{g}_\theta$. Moreover $\mathcal{H}_\lambda(\mathfrak{g})$ is generated by V_λ over $\widehat{\mathfrak{g}}$ with the above relation. When $\lambda = 0$, the corresponding $\widehat{\mathfrak{g}}$-module $\mathcal{H}_0(\mathfrak{g})$ is known as the vacuum representation.

1.3. Conformal blocks

We fix a n pointed curve C with formal neighborhood $\eta_1, \ldots, \eta_n$ around the n points $\vec{p} = (P_1, \ldots, P_n)$, which satisfies the following properties :

(1) The curve C has at most nodal singularities,

(2) The curve C is smooth at the points $P_1, \ldots, P_n$,

(3) $C - \{P_1, \ldots, P_n\}$ is an affine curve,

(4) A stability condition (equivalent to the finiteness of the automorphisms of the pointed curve),

(5) Isomorphisms $\eta_i : \widehat{\mathcal{O}}_{C,P_i} \simeq \mathbb{C}[[\xi_i]]$ for $i = 1, \ldots, n$.

We denote by $\mathfrak{X} = (C; \vec{p}; \eta_1, \ldots, \eta_n)$ the above data associated to the curve C. We define another Lie algebra

$$\widehat{\mathfrak{g}}_n := \bigoplus_{i=1}^{n} \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi_i)) \oplus \mathbb{C}c,$$

where c belongs to the center of $\widehat{\mathfrak{g}}_n$ and the Lie bracket is given as follows:

$$[\sum_{i=1}^{n} X_i \otimes f_i, \sum_{i=1}^{n} Y_i \otimes g_i] := \sum_{i=1}^{n} [X_i, Y_i] \otimes f_i g_i + \sum_{i=1}^{n} (X_i, Y_i) \operatorname{Res}_{\xi_i=0}(g_i df_i)c.$$

We define the current algebra to be

$$\mathfrak{g}(\mathfrak{X}) := \mathfrak{g} \otimes \Gamma(C - \{P_1, \ldots, P_n\}, \mathcal{O}_C).$$

By local expansion of functions using the chosen coordinates ξ_i we get the following embedding:

$$\mathfrak{g}(\mathfrak{X}) \hookrightarrow \widehat{\mathfrak{g}}_n.$$

Consider an n tuple of weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in P_\ell^n(\mathfrak{g})$. We set $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1}(\mathfrak{g}) \otimes \cdots \otimes \mathcal{H}_{\lambda_n}(\mathfrak{g})$. The algebra $\widehat{\mathfrak{g}}_n$ acts on $\mathcal{H}_{\vec{\lambda}}$. For any $X \in \mathfrak{g}$ and $f \in \mathbb{C}((\xi_i))$, the action of $X \otimes f(\xi_i)$ on the i-th component is given by the following:

$$\rho_i(X \otimes f(\xi_i))|v_1 \otimes \cdots \otimes v_n\rangle = |v_1 \otimes \cdots \otimes (X \otimes f(\xi_i)v_i) \otimes \cdots \otimes v_n\rangle,$$

where $|v_i\rangle \in \mathcal{H}_{\lambda_i}(\mathfrak{g})$ for each i.

Definition 1.3.1. *We define the space of conformal blocks*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}) := \operatorname{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

We define the space of dual conformal blocks, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}) = \mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}$. These are both finite dimensional $\mathbb{C}$-vector spaces which can be defined in families. The dimensions of these vector spaces are given by the Verlinde formula.

The elements of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g})$ (or $\mathcal{H}^*_{\vec{\lambda}}$) will be denoted by $\langle\Psi|$ and those of the dual conformal blocks (or $\mathcal{H}_{\vec{\lambda}}$) by $|\Phi\rangle$. We will denote the natural pairing by $\langle\Psi|\Phi\rangle$.

Remark 1.3.2. *Let $X \in \mathfrak{g}$ and $f \in \Gamma(C - \{P_1, \dots, P_n\}, \mathcal{O}_C)$, then every element of $\langle \Psi | \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g})$ satisfies the following gauge symmetry:*

$$\sum_{i=1}^{n} \langle \Psi | \rho_i(X \otimes f(\xi_i)) \Phi \rangle = 0.$$

1.4. Propagation of vacua

Let P_{n+1} be a new point on the curve C with coordinate η_{n+1} and $\mathfrak{X}'$ denote the new data. We associate the vacuum representation $\mathcal{H}_0$ to the point P_{n+1} and $\vec{\lambda}' = \vec{\lambda} \cup \{\lambda_{n+1} = 0\}$. The "propagation of vacuum" gives an isomorphism

$$f : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}', \mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g})$$

by the formula

$$f(\langle \Psi' |) | \Phi \rangle := \langle \Psi' | \Phi \otimes 0 \rangle,$$

where $|0\rangle$ is a highest weight vector of the representation $\mathcal{H}_0$, $|\phi\rangle \in \mathcal{H}_{\vec{\lambda}}$ and $\langle \Psi' |$ is an arbitrary element of $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}', \mathfrak{g})$.

1.5. Conformal blocks in a family

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\vec{\lambda} \in P^n_{\ell}(\mathfrak{g})$. Consider a family $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_n; \xi_1, \dots, \xi_n)$ of nodal curves on a base $\mathcal{B}$ with sections s_i and formal coordinates ξ_i. In [**36**], a locally free sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ known as the sheaf of conformal blocks is constructed over the base $\mathcal{B}$. The sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ commutes with base change. Similarly one can define another locally free sheaf $\mathcal{V}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ as a quotient of $\mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}_{\vec{\lambda}}$.

Moreover, if $\mathcal{F}$ is a family of smooth projective curves, then the sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ carries a projectively flat connection known as the *KZ/Hitchin/WZW* connection. We refer the reader to [**36**] for more details.

Remark 1.5.1. *When the level ℓ becomes unclear we also include it in the notation of conformal blocks. Let $\mathfrak{X}$ be the data associated to a n pointed curve with chosen coordinates. We consider an n tuple of level ℓ weights $\vec{\lambda}$ of the Lie algebra $\mathfrak{g}$. The* *conformal block is denoted by* $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$ *and the dual conformal block is denoted by* $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$.

CHAPTER 2

Action of center on conformal blocks

A diagram automorphism of a Dynkin diagram is a permutation of its nodes that leaves the diagram invariant. For every diagram automorphism, we can construct a finite order automorphism of the Lie algebra associated to the Dynkin diagram. These automorphisms are known as outer automorphisms. In the following, we restrict ourselves to affine Kac-Moody Lie algebras $\widehat{\mathfrak{g}}$ and to those diagram automorphisms which correspond to the center $Z(G)$ of the simply connected group G associated to a finite dimensional simple Lie algebra $\mathfrak{g}$.

2.1. Diagram automorphisms of symmetrizable Kac-Moody algebras.

Consider a symmetrizable generalized Cartan matrix $A = (a_{i,j})$ of size n and let $\mathfrak{g}(A)$ denote the associated Kac-Moody algebra. To a symmetrizable generalized Cartan matrix, one can associate a Dynkin diagram which is a graph on n vertices, see [K] for details. A diagram automorphism of a Dynkin diagram is a graph automorphism, i.e. it is a bijection ω from the set of vertices of the graph to itself such that for $1 \leq i, j \leq n$,

$$a_{\omega i,\omega j} = a_{i,j}.$$

We will now construct an automorphism of the Kac-Moody algebra $\mathfrak{g}(A)$ from a diagram automorphism ω of the Dynkin diagram of $\mathfrak{g}(A)$. We start by defining the action of ω on the generators e_i and f_j in the following way:

$$\omega(e_i) := e_{\omega i} \text{ and } \omega(f_i) := f_{\omega i}$$

Since ω is a Lie algebra automorphism, it implies the following:

$$\omega(\alpha_i^\vee) = \omega\,[e_i, f_i] = [e_{\omega i}, f_{\omega_i}] = \alpha_{\omega i}^\vee,$$

where $\alpha_i^\vee$ are the simple coroots.

In this way, we have constructed an automorphism of the derived algebra $\mathfrak{g}(A)' = [\mathfrak{g}(A), \mathfrak{g}(A)]$. The extension of this action of ω to $\mathfrak{g}(A)$ follows from Lemma 1.3.1 in [**17**]. The automorphism ω of the Lie algebra $\mathfrak{g}(A)$ has the same order as that of the corresponding diagram automorphism. We will refer to these automorphisms as outer automorphisms.

We restrict to the case when $\mathfrak{g}(A)$ is an untwisted affine Lie algebra. Let $\mathfrak{h}(A)'$ be a Cartan subalgebra of $\mathfrak{g}(A)'$. The map ω restricted to the Cartan subalgebra $\mathfrak{h}(A)'$ defines an automorphism of $\mathfrak{h}(A)'$. The adjoint action of ω^* on $\mathfrak{h}(A)'^*$ is given by $\omega^*(\lambda)(x) = \lambda(\omega^{-1}x)$ for $\lambda \in \mathfrak{h}(A)^*$ and $x \in \mathfrak{h}(A)'$. For $0 \leq i \leq n$, let Λ_i be the affine fundamental weight corresponding to the i-th coroot. Then, $\omega^*(\Lambda_i) = \Lambda_{\omega i}$.

2.2. Single-shift automorphisms of $\widehat{\mathfrak{g}}$

Consider a finite dimensional complex simple Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system associated to $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ be a set of positive roots. In this section, we denote by $\langle , \rangle$ the normalized Cartan killing form such that $\langle \theta, \theta \rangle = 2$ for the longest root θ. For each $\alpha \in \Delta$, choose a non-zero element $X_\alpha \in \mathfrak{g}_\alpha$. Then, we have the following:

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta, \\ 0 & \text{if } \alpha+\beta \notin \Delta, \end{cases}$$

where $N_{\alpha,\beta}$ is a non-zero scalar. The coefficients $N_{\alpha,\beta}$ completely determine the multiplication table of $\mathfrak{g}$. However, they depend on the choice of the elements X_α. We refer the reader to [**33**] for a proof of the following proposition:

Proposition 1. *One can choose the elements X_α is such a way so that*

$$[X_\alpha, X_{-\alpha}] = H_\alpha \quad \text{for all } \alpha \in \Delta,$$

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta} \quad \text{for all } \alpha,\ \beta,\ \alpha+\beta \in \Delta,$$

where H_α is the coroot corresponding to α. The basis $\{X_\alpha, X_{-\alpha}, H_\alpha : \alpha \in \Delta_+\}$ is known as Chevalley basis.

To every $\alpha_i \in \Delta_+$, the simple coroot $X_i \in \mathfrak{h}$ is defined by the property $X_i(\alpha_j) = \delta_{ij}$ for $\alpha_j \in \Delta_+$. The lattice generated by $\{X_i : 1 \leq i \leq \operatorname{rank}(\mathfrak{g})\}$ is called the coweight lattice and is denoted by $P^\vee$. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ under the normalized Cartan killing form and let h_α denote the image of α under the identification.

For every $\mu \in P^\vee$, we define an map σ_μ of the Lie algebra $\widehat{\mathfrak{g}}$

$$\begin{aligned} \sigma_\mu(c) &:= c, \\ \sigma_\mu(h(n)) &:= h(n) + \delta_{n,0}\langle \mu, h\rangle.c \quad \text{where } h \in \mathfrak{h} \text{ and } n \in \mathbb{Z}, \\ \sigma_\mu(X_\alpha(n)) &:= X_\alpha(n + \langle \mu, \alpha\rangle). \end{aligned}$$

Proposition 2. *The map*

$$\sigma_\mu : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}},$$

is a Lie algebra automorphism.

Proof. We only need to verify that σ_μ is a Lie algebra homomorphism. It is enough to check that σ_μ respects the following relations:

$$\begin{aligned} \left[H_{\alpha_i}(m), H_{\alpha_j}(n)\right] &= \langle H_{\alpha_i}, H_{\alpha_j}\rangle.m.\delta_{m+n,0}c, \\ [H_{\alpha_i}(m), X_\alpha(n)] &= \alpha(H_{\alpha_i})X_\alpha(m+n), \\ [X_\alpha(m), X_\beta(n)] &= N_{\alpha,\beta}X_{\alpha+\beta}(m+n) \quad \text{if } \alpha+\beta \in \Delta, \\ [X_\alpha(m), X_{-\alpha}(n)] &= H_\alpha(m+n) + \langle X_\alpha, X_{-\alpha}\rangle.m\delta_{m+n,0}c. \end{aligned}$$

It is trivial to see that σ_μ respects the first three relations. We only need to verify that σ_μ respects the last relation. Let us calculate the following:

$$
\begin{aligned}
[\sigma_\mu(X_\alpha(m)), \sigma_\mu(X_{-\alpha}(n))] &= [X_\alpha(m + \langle \mu, \alpha \rangle), X_{-\alpha}(n - \langle \mu, \alpha \rangle)], \\
&= H_\alpha(m+n) + \langle X_\alpha, X_{-\alpha} \rangle .(m + \langle \mu, \alpha \rangle)\delta_{m+n,0}c.
\end{aligned}
$$

If we apply σ_μ to the right hand side of the last relation, we get the following:

$$
\begin{aligned}
\sigma_\mu(H_\alpha(m+n) &+ \langle X_\alpha, X_{-\alpha} \rangle .m\delta_{m+n,0}c) \\
&= H_\alpha(m+n) + (\langle \mu, H_\alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle .m.)\delta_{m+n,0}c, \\
&= H_\alpha(m+n) + (\langle \mu, \langle X_\alpha, X_{-\alpha} \rangle h_\alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle .m.)\delta_{m+n,0}c, \\
&= H_\alpha(m+n) + (\langle X_\alpha, X_{-\alpha} \rangle \langle \mu, h_\alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle .m.)\delta_{m+n,0}c, \\
&= H_\alpha(m+n) + (\langle X_\alpha, X_{-\alpha} \rangle \langle \mu, \alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle .m.)\delta_{m+n,0}c, \\
&= [\sigma_\mu(X_\alpha(m)), \sigma_\mu(X_{-\alpha}(n))].
\end{aligned}
$$

This completes the proof. $\square$

The automorphism σ_μ was studied in [**16**], [**20**] and [**11**] and is called a single-shift automorphism. It is easy to observe that single-shift automorphisms are additive, i.e. for $\mu_1, \mu_2 \in P^\vee$, we have $\sigma_{\mu_1+\mu_2} = \sigma_{\mu_1} \circ \sigma_{\mu_2}$.

2.3. Multi-shift automorphisms

We recall the definition of multi-shift automorphisms following [**12**]. Let us fix a sequence of pairwise distinct complex numbers z_s for $s \in \{1, \cdots, n\}$, the coroot H_α corresponding to the roots α. Let $P^\vee$ and $Q^\vee$ denote the coweight and the coroot lattice of $\mathfrak{g}$ respectively and $\Gamma_n^{\mathfrak{g}} = \{(\mu_1, \ldots, \mu_n) | \mu_i \in P^\vee \text{and} \sum_{i=1}^n \mu_i = 0\}$. Consider a Chevalley basis given by $\{X_{-\alpha}, X_\alpha, H_\alpha : \alpha \in \Delta_+\}$. For $\vec{\mu} \in \Gamma_n^{\mathfrak{g}}$, we define a multi-shift automorphism

$\sigma_{\vec{\mu},t}(\vec{z})$ of $\widehat{\mathfrak{g}}$ as follows:

$$\begin{aligned}
\sigma_{\vec{\mu},t}(\vec{z})(c) &:= (c),\\
\sigma_{\vec{\mu},t}(\vec{z})(h)\otimes f &:= h\otimes f + \left(\sum_{s=1}^{n}\langle h,\mu_s\rangle \operatorname{Res}(\varphi_{t,s}.f)\right).c,\\
\sigma_{\vec{\mu},t}(\vec{z})(X_\alpha\otimes f) &:= X_\alpha\otimes f.\prod_{s=1}^{n}\varphi_{t,s}^{-\alpha(\mu_s)},
\end{aligned}$$

where $f\in\mathbb{C}((\xi))$, $\varphi_{t,s}(\xi)=(\xi+(z_t-z_s))^{-1}, h\in\mathfrak{h}$.

Let us now recall some important properties of the multi-shift automorphisms.

(1) The multi-shift automorphism $\sigma_{\vec{\mu},t}(\vec{z})$ has the same outer automorphism class as the single shift automorphism σ_{μ_t}.

(2) It is shown in [FS] that $\sigma_{\vec{\mu},t}$ is a Lie algebra automorphism of $\widehat{\mathfrak{g}}$ and can be easily extended to an automorphism of $\widehat{\mathfrak{g}}_n$.

(3) Multi-shift automorphisms of $\widehat{\mathfrak{g}}_n$ preserve the current algebra $\mathfrak{g}\otimes\Gamma(\mathbb{P}^1-\vec{p},\mathcal{O}_{\mathbb{P}^1})$, where $\vec{p}=(P_1,\dots,P_n)$ are n distinct points with coordinates $z_1,\dots,z_n$.

Remark 1. *If one of the chosen points P_i is infinity, the formula of the multi-shift automorphism needs a minor modification to accommodate the new coordinate at infinity. This has been considered in* [**12**].

2.4. Action of center on conformal blocks

In [**12**], the conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g},\ell)$ and $\mathcal{V}^{\dagger}_{\vec{\omega}\vec{\lambda}}(\mathfrak{X},\mathfrak{g},\ell)$ are identified via an isomorphism which is flat with respect to the KZ connection, where $\vec{\omega}=(\omega_1,\omega_2,\cdots,\omega_n)\in Z(G)^n$ such that $\prod_{s=1}^n\omega_s=\mathrm{id}$, $\vec{\omega}\vec{\lambda}=(\omega_1^*\lambda_1,\dots,\omega_n^*\lambda_n)$ and ω^* is a permutation of level ℓ weights $P_\ell(\mathfrak{g})$ corresponding to the diagram automorphism ω.

We briefly recall the construction of the above isomorphism in [**12**]. For $\vec{z}\in\mathbb{P}^1$, a positive integer s and $\vec{\mu}\in\Gamma^{\mathfrak{g}}_n$, a multi-shift automorphism $\sigma_{\vec{\mu},s}(\vec{z})$ of $\widehat{\mathfrak{g}}$ is constructed in [**12**]. To implement the action of $\sigma_{\vec{\mu}}(\vec{z})$ on tensor product of highest weight modules, a map from $\Theta_{\vec{\mu}}(\vec{z}):\otimes_{i=1}^n\mathcal{H}_{\lambda_i}\to\otimes_{i=1}^n\mathcal{H}_{\omega_i^*\lambda_i}$ is given, where $\mathcal{H}_\lambda$ is an integrable irreducible highest weight module of weight λ and ω_i is the image of μ_i in $Z(G)$ under the exponential map. Since the automorphism $\sigma_{\vec{\mu}}(\vec{z})$ preserves the current algebra $\mathfrak{g} \otimes \Gamma(\mathbb{P}^1 - \vec{z}, \mathcal{O}_{\mathbb{P}^1})$, the map descends to a map of conformal blocks. It is easy to see that the map is an isomorphism. Moreover the map $\Theta_{\vec{\mu}}(\vec{z})$ is chosen such that the induced map between the conformal blocks is flat with respect to the KZ connection. The above discussion is summarized in the following proposition from [**12**].

PROPOSITION 2.4.1. *Let $\mathfrak{X}$ be the data associated to n distinct points with chosen coordinates on $\mathbb{P}^1$. Then there is an isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) \to \mathcal{V}_{\vec{\omega}\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell).$$

Moreover the isomorphism is flat with respect to the KZ/Hitchin connection.

CHAPTER 3

Conformal subalgebras and rank-level duality

In this chapter, we discuss conformal embeddings of Lie algebras and give a general formulation of the rank-level maps.

3.1. Conformal embedding

Let $\mathfrak{s}$, $\mathfrak{g}$ be two simple Lie algebras and $\phi : \mathfrak{s} \to \mathfrak{g}$ an embedding of Lie algebras. Let $(,)_{\mathfrak{s}}$ and $(,)_{\mathfrak{g}}$ denote the normalized Cartan killing forms such that the the length of the longest root is 2. We define the Dynkin index of ϕ to be the unique integer d_ϕ such satisfying

$$(\phi(x), \phi(y))_{\mathfrak{g}} = d_\phi (x, y)_{\mathfrak{s}}$$

for all $x, y \in \mathfrak{s}$. When $\mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is semisimple, we define the Dynkin multi-index of $\phi = \phi_1 \oplus \phi_2 : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ to be $d_\phi = (d_{\phi_1}, d_{\phi_2})$.

If $\mathfrak{g}$ is simple, we define for any level ℓ and a dominant weight λ of level ℓ the conformal anomaly $c(\mathfrak{g}, \ell)$ and the trace anomaly $\Delta_\lambda(\mathfrak{g})$ as

$$c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{g^* + \ell} \quad \text{and} \quad \Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(g^* + \ell)},$$

where g^* is the dual Coxeter number of $\mathfrak{g}$ and ρ denotes the half sum of positive roots, also known as the Weyl vector. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is semisimple, we define the conformal anomaly and trace anomaly by taking sum of the conformal anomalies over all simple components.

Definition 3.1.1. *Let $\phi = (\phi_1, \phi_2) : \mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be an embedding of Lie algebras with Dynkin multi-index $k = (k_1, k_2)$. We define ϕ to be a conformal embedding $\mathfrak{s}$ in $\mathfrak{g}$ at level ℓ if $c(\mathfrak{g}_1, k_1\ell) + c(\mathfrak{g}_2, k_2\ell) = c(\mathfrak{g}, \ell)$.*

It is shown in [**19**] that the above equality only holds if $\ell = 1$. Many familiar and important embeddings are conformal. For a complete list of conformal embeddings we refer the reader to [**3**]. Next we list two important properties which makes conformal embeddings special.

(1) We recall the following from [**18**]. Since $\mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is semisimple, $\phi : \mathfrak{s} \to \mathfrak{g}$ is a conformal subalgebra if and only if any irreducible $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\Lambda(\mathfrak{g})$ of level one decompose into a finite sum of irreducible $\widehat{\mathfrak{s}}$-modules of level $\ell = (\ell_1, \ell_2)$, where ℓ is the Dynkin multi-index of the embedding ϕ.

(2) If $\phi : \mathfrak{s} \to \mathfrak{g}$ is a conformal embedding, then the action of the Virasoro operators are the same, i.e. for any n the following equality holds

$$L_n^{\mathfrak{s}} = L_n^{\mathfrak{g}} \in \operatorname{End}(\mathcal{H}_\Lambda(\mathfrak{g})),$$

where $L_n^{\mathfrak{s}}$ and $L_n^{\mathfrak{g}}$ are n-th Virasoro operators of $\mathfrak{s}$ and $\mathfrak{g}$ acting at level ℓ and one respectively. We refer the reader to [**18**] for more details.

Example 1. *Consider the following embedding of Lie algebras induced from the tensor product of vector spaces equipped with a symmetric non-degenerate bilinear form. The Dynkin multi-index is* (s, r).

$$\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \subset \mathfrak{so}(4rs).$$

$$\begin{aligned} c(\mathfrak{sp}(2r), s) \quad + \quad & c(\mathfrak{sp}(2s), r) \\ = \quad & \frac{s(2r+1)r}{s+r+1} + \frac{r(2s+1)s}{s+r+1}, \\ = \quad & 2rs, \\ = \quad & \frac{2rs(4rs-1)}{4rs-2+1}, \\ = \quad & c(\mathfrak{so}(4rs), 1). \end{aligned}$$

Example 2. *Consider the embedding of Lie algebras given by the tensor product of vector spaces. The Dykin multi-index of the embedding is* (s, r)*.*

$$\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \subset \mathfrak{sl}(rs).$$

$$\begin{aligned} c(\mathfrak{sl}(r), s) &+ c(\mathfrak{sl}(s), r) \\ &= \frac{(r^2-1)s}{r+s} + \frac{(s^2-1)r}{r+s}, \\ &= \frac{(rs+1)(r+s)}{r+s}, \\ &= rs+1, \\ &= \frac{(rs)^2-1}{rs-1}, \\ &= c(\mathfrak{sl}(rs), 1). \end{aligned}$$

3.2. General context of rank-level duality

Consider a level one integrable highest weight $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\Lambda(\mathfrak{g})$ and restrict it to $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$. The module $\mathcal{H}_\Lambda(\mathfrak{g})$ decomposes into irreducible integrable $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$-modules of level $\ell = (\ell_1, \ell_2)$ as follows:

$$\bigoplus_{(\lambda,\mu)\in B(\Lambda)} m^\Lambda_{\lambda,\mu} \mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2) \simeq \mathcal{H}_\Lambda(\mathfrak{g}),$$

where ℓ is the Dynkin multi-index of ϕ and $m^\Lambda_{\lambda,\mu}$ is the multiplicity of the component $\mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2)$. Since the embedding is conformal, we know that both $|B(\Lambda)|$ and $m^\Lambda_{\lambda,\mu}$ are finite.

We consider only those conformal embeddings such that for every $\Lambda \in P_1(\mathfrak{g})$ and $(\lambda, \mu) \in B(\Lambda)$, the multiplicity $m^\Lambda_{\lambda,\mu} = 1$. Let $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be a n-tuple of level one dominant weights of $\mathfrak{g}$. We consider $\mathcal{H}_{\vec{\Lambda}}(\mathfrak{g})$ and restrict it to $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$. Choose $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ such that $(\lambda_i, \mu_i) \in B(\Lambda_i)$ for all $1 \le i \le n$. We get an injective map

$$\bigotimes_{i=1}^{n}(\mathcal{H}_{\lambda_i}(\mathfrak{g}_1)\otimes\mathcal{H}_{\mu_i}(\mathfrak{g}_2))\to\bigotimes_{i=1}^{n}\mathcal{H}_{\Lambda_i}(\mathfrak{g}).$$

Let $\mathfrak{X}$ denote the data associated to a curve C of genus g with n distinct points $\vec{p}=(P_1,\ldots,P_n)$ with chosen coordinates $\xi_1,\ldots,\xi_n$. Taking coinvariants with respect to $\mathfrak{g}(\mathfrak{X})$, we get the following map:

$$\alpha:\mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g}_1,\ell_1)\otimes\mathcal{V}_{\vec{\mu}}(\mathfrak{X},\mathfrak{g}_2,\ell_2)\to\mathcal{V}_{\vec{\Lambda}}(\mathfrak{X},\mathfrak{g},1),$$

If $\dim_{\mathbb{C}}(\mathcal{V}_{\vec{\Lambda}}(\mathfrak{X},\mathfrak{g},1))=1$, we get a map well defined up to constants

$$\alpha^{\vee}:\mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g}_1,\ell_1)\to\mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{X},\mathfrak{g}_2,\ell_2).$$

This map is known as the rank-level duality map.

DEFINITION 3.2.1. *Let $\vec{\lambda}\in P^n_{\ell_1}(\mathfrak{g}_1)$ and $\vec{\mu}\in P^n_{\ell_2}(\mathfrak{g}_2)$. The pair $(\vec{\lambda},\vec{\mu})$ is called admissible if one can define a rank-level duality map between the corresponding conformal blocks.*

Let $\mathcal{F}=(\pi:\mathcal{C}\to\mathcal{B};s_1,\ldots,s_n;\xi_1,\ldots,\xi_n)$ be a family of nodal curves on a base $\mathcal{B}$ with sections s_i and local coordinates ξ_i. The map α can be easily extended to a map of sheaves

$$\alpha(\mathcal{F}):\mathcal{V}_{\vec{\lambda}}(\mathcal{F},\mathfrak{g}_1,\ell_1)\otimes\mathcal{V}_{\vec{\mu}}(\mathcal{F},\mathfrak{g}_2,\ell_2)\to\mathcal{V}_{\vec{\Lambda}}(\mathcal{F},\mathfrak{g},1).$$

3.3. Properties of rank-level duality

In this section we recall some interesting properties of the rank-level duality maps. The following proposition tells us about the behavior of the rank-level duality map in a smooth family of curves. For a proof we refer the reader to [**9**].

PROPOSITION 3.3.1. *Let $\mathcal{F}=(\pi:\mathcal{C}\to\mathcal{B};s_1,\ldots,s_n;\xi_1,\ldots,\xi_n)$ be a family of smooth projective curves on a base $\mathcal{B}$ with sections s_i and local coordinates ξ_i. Then the rank-level duality map α is projectively flat with respect to the $KZ/Hitchin/WZW$ connection.*

The rank-level duality map commutes with the propagation of vacua. The following has a direct proof.

Proposition 3.3.2. *Let Q be a point on the curve C distinct from $\vec{p} = (P_1, \dots, P_n)$ and $\mathfrak{X}'$ be the data associated to the n pointed curve. Consider $\vec{\lambda}' = (\lambda_1, \dots, \lambda_n, 0)$ and $\vec{\mu}' = (\mu_1, \dots, \mu_n, 0)$. The rank-level duality map $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \to \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2)$ is an isomorphism if and only if the rank-level duality map $\mathcal{V}_{\vec{\lambda}'}(\mathfrak{X}', \mathfrak{g}_1, \ell_1) \to \mathcal{V}^{\dagger}_{\vec{\mu}'}(\mathfrak{X}', \mathfrak{g}_2, \ell_2)$ is an isomorphism.*

CHAPTER 4

Diagram automorphisms and rank-level duality

We show that, under certain conditions, the single-shift and multi-shift automorphisms extend under embeddings of Lie algebras (see Section 4.1). As a corollary we prove new symplectic rank-level dualities (see Section 4.2) on genus zero smooth curves with marked points. We also show that the rank-level dualities (see Section 4.3) for the pair $\mathfrak{sl}(r)$, $\mathfrak{sl}(s)$ in genus 0 arising from conformal embeddings can be obtained from the rank-level duality of [**32**].

4.1. Extension of shift automorphisms

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ be simple Lie algebras and $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}$ be their Cartan subalgebras. For $i \in \{1,2\}$, let $\phi_i : \mathfrak{g}_i \to \mathfrak{g}$ be an embedding of Lie algebras such that $\phi_i(\mathfrak{h}_i) \subset \mathfrak{h}$. Further let us denote the normalized Cartan Killing form on $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ by $\langle\ ,\ \rangle_{\mathfrak{g}_1}$, $\langle\ ,\ \rangle_{\mathfrak{g}_2}$, $\langle$, and $\rangle_{\mathfrak{g}}$ respectively, and let (ℓ_1, ℓ_2) be the Dynkin multi-index of the embedding $\phi = (\phi_1, \phi_2)$. We can extend the map $\phi = (\phi_1, \phi_2)$ to a map $\widehat{\phi}$ of $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \to \widehat{\mathfrak{g}}$ as follows:

$$\begin{aligned} \widehat{\phi}_1 : \widehat{\mathfrak{g}}_1 &\to \widehat{\mathfrak{g}}, \\ X \otimes f + a.c &\to \phi_1(X) \otimes f + a.\ell_1.c, \end{aligned}$$

where $X \in \mathfrak{g}_1$, $f \in \mathbb{C}((\xi))$ and a is a constant. We similarly map

$$\begin{aligned} \widehat{\phi}_2 : \widehat{\mathfrak{g}}_2 &\to \widehat{\mathfrak{g}}, \\ Y \otimes g + b.c &\to \phi_2(Y) \otimes g + b.\ell_2.c, \end{aligned}$$

where $Y \in \mathfrak{g}_2$, $g \in \mathbb{C}((\xi))$ and b is a constant. We define $\widehat{\phi}$ to be $\widehat{\phi}_1 + \widehat{\phi}_2$. Consider an element $\mu \in P_1^\vee$ such that $\tilde{\mu} = \phi_1(\mu) \in P^\vee$, where $P_1^\vee$ and P denote the coweight lattices of $\mathfrak{g}_1$ and $\mathfrak{g}$ respectively.

Let α be a root in $\mathfrak{g}_1$ with respect to a Cartan subalgebra $\mathfrak{h}_1$ and X_α be a non-zero element of $\mathfrak{g}_1$ in the root space α. If $\phi_1(X_\alpha) = \sum_{i=1}^{\dim \mathfrak{h}} a_i h_i + \sum_{\gamma \in I_\alpha} a_\gamma X_\gamma$, where h_i's be any basis of $\mathfrak{h}$. We have the following lemma:

LEMMA 1. *For all i, we claim that $a_i = 0$.*

PROOF. For any element $h \in \mathfrak{h}_1$, we consider the following Lie bracket.

$$\begin{aligned}
[\phi(h), \phi(X_\alpha)] &= \phi([h, X_\alpha]), \\
&= \phi(\alpha(h) X_\alpha), \\
&= \sum_{i=1}^{\dim \mathfrak{h}} a_i \alpha(h) h_i + \sum_{\gamma \in I_\alpha} a_\gamma \alpha(h) X_\gamma.
\end{aligned}$$

On the other hand $[\phi(h), \phi(X_\alpha)] = \sum_{\gamma \in I_\alpha} a_\gamma \gamma(\phi(h)) X_\gamma$. Comparing, we see that $a_i = 0$ for all i and $\gamma(\phi(h)) = \alpha(h)$ for all h in $\mathfrak{h}_1$. □

Next, we prove the following lemma:

LEMMA 2. *If $\gamma \in I_\alpha$, then for all $h_2 \in \mathfrak{h}_2$*

$$\gamma(\phi(h_2)) = 0.$$

PROOF. For $h_2 \in \mathfrak{h}_2$, we have the following:

$$\begin{aligned}
\phi\,[h_2, X_\alpha] &= [\phi(h_2), \phi(X_\alpha)]\,, \\
&= \sum_{\gamma \in I_\alpha} a_\gamma \gamma(\phi(h_2)) X_\gamma, \\
&= 0.
\end{aligned}$$

Thus comparing, we get $\gamma(\phi(h_2)) = 0$ for all $h_2 \in \mathfrak{h}_2$. □

The following proposition is about the extension of single-shift automorphisms:

Proposition 3. *The single-shift automorphism $\sigma_{\tilde{\mu}}$ restricted to $\widehat{\phi}_1(\widehat{\mathfrak{g}}_1)$ is the automorphism σ_μ. Morever $\sigma_{\tilde{\mu}}$ restricts to identity on $\widehat{\phi}_2(\widehat{\mathfrak{g}}_2)$.*

Proof. Let n be an integer and h, h_1 be elements of $\mathfrak{h}_1$ and $\mathfrak{h}_2$. We need to show the following identities:

$$\sigma_{\tilde{\mu}}(\widehat{\phi}_1(h_1(n))) = \widehat{\phi}_1(\sigma_\mu(h_1(n))),$$

$$\sigma_{\tilde{\mu}}(\widehat{\phi}_1(X_\alpha(n))) = \widehat{\phi}_1(\sigma_\mu(X_\alpha(n))),$$

$$\sigma_{\tilde{\mu}}(\widehat{\phi}_2(h_2(n))) = \widehat{\phi}_2((h_2(n))),$$

$$\sigma_{\tilde{\mu}}(\widehat{\phi}_2(X_\beta(n))) = \widehat{\phi}_2((X_\beta(n))),$$

where α, β are roots of $\mathfrak{g}_1$, $\mathfrak{g}_2$ respectively, and X_α, X_β are non-zero elements in the root space of α, β respectively. Let $h \in \mathfrak{h}_1$, we have the following:

$$\begin{aligned}
\sigma_{\tilde{\mu}}(\widehat{\phi}_1(h(n))) &= \widehat{\phi}_1(h(n)) + \delta_{n,0}.\langle \tilde{\mu}, \phi_1(h)\rangle_{\mathfrak{g}}.c, \\
&= \widehat{\phi}_1(h(n)) + \delta_{n,0}.\langle \phi_1(\mu), \phi_1(h)\rangle_{\mathfrak{g}}.c, \\
&= \widehat{\phi}_1(h(n)) + \delta_{n,0}.\ell_1.\langle \mu, h\rangle_{\mathfrak{g}_1}.c, \\
&= \widehat{\phi}_1(\sigma_\mu(h(n))).
\end{aligned}$$

This completes the proof of the first identity. For the second identity, we use Lemma 1. For any non-zero element X_α in the root space of α, consider the following:

$$\begin{aligned}
\widehat{\phi}_1(\sigma_\mu(X_\alpha(n))) &= \widehat{\phi}_1(X_\alpha(n + \alpha(\mu))), \\
&= \sum_{\gamma \in I_\alpha} a_\gamma X_\gamma(n + \alpha(\mu)), \\
&= \sum_{\gamma \in I_\alpha} a_\gamma X_\gamma(n + \gamma(\phi(\mu))), \\
&= \sigma_{\tilde{\mu}}(\widehat{\phi}_1(X_\alpha(n))).
\end{aligned}$$

To prove the fourth identity we use Lemma 2. For a non-zero element X_β in the root space β, consider the following:

$$\begin{aligned}
\sigma_{\tilde{\mu}}(\widehat{\phi}_2(X_\beta(n))) &= \sigma_{\phi_1(\mu)}(\sum_{\gamma\in I_\beta} X_\gamma(n)),\\
&= \sum_{\gamma\in I_\beta} \sigma_{\phi_1(\mu)}(X_\gamma(n)),\\
&= \sum_{\gamma\in I_\beta} X_\gamma(n+\gamma(\phi(\mu))),\\
&= \sum_{\gamma\in I_\beta} X_\gamma(n),\\
&= \widehat{\phi}_2(X_\beta(n)).
\end{aligned}$$

We are only left to show that $\sigma_{\tilde{\mu}}(\widehat{\phi}_2(h_2(n))) = \widehat{\phi}_2(h_2(n))$ for $h_2 \in \mathfrak{h}_2$, which follows from the following lemma. □

LEMMA 3. $\langle \phi_1(h_1), \phi_2(h_2)\rangle_{\mathfrak{g}} = 0$ *for any element* h_1, h_2 *of* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *respectively.*

PROOF. It is enough to proof the result for all H_β, where β is a root of $\mathfrak{g}_2$. Let $\phi(X_\beta) = \sum_{\lambda\in I_\beta} a_\lambda X_\lambda$ and $\phi(X_{-\beta}) = \sum_{\gamma\in I_{-\beta}} a_\gamma X_\gamma$. Since $\phi_2(\mathfrak{h}_2) \subset \mathfrak{h}$, we get $\phi(H_\beta) = \sum_{\gamma\in I_\beta\cap(-I_{-\beta})} a_\gamma b_{-\gamma} H_\gamma$.

Now $[\phi(h_1), \phi(X_\beta)] = \sum_{\gamma\in I_\beta} a_\gamma \gamma(\phi(h_1)) X_\gamma = 0$. Thus for $\gamma \in I_\beta$, we get $\gamma(\phi(h_1)) = \langle h_\gamma, \phi(h_1)\rangle_{\mathfrak{g}} = 0$ which implies $\langle \phi_1(h_1), H_\gamma\rangle_{\mathfrak{g}} = 0$. Thus, we have the following:

$$\begin{aligned}
\langle \phi_1(h_1), \phi_2(H_\beta)\rangle_{\mathfrak{g}} &= \sum_{\gamma\in I_\beta\cap(-I_{-\beta})} a_\gamma b_{-\gamma} \langle \phi_1(h_1), H_\gamma\rangle_{\mathfrak{g}},\\
&= 0.
\end{aligned}$$

□

Next, we restate the extension proposition for multi-shift automorphisms. The following proposition follows directly from the proof of Proposition 3.

Proposition 4.1.1. *Consider $\vec{\mu} = (\mu_1, \cdots, \mu_n) \in \Gamma_n^{\mathfrak{g}_1}$ such that the n tuple of coweights $\vec{\tilde{\mu}} = \{\phi_1(\mu_1), \cdots, \phi_1(\mu_n)\} \in \Gamma_n^{\mathfrak{g}}$. Consider the multi-shift automorphism from*

$$\sigma_{\vec{\tilde{\mu}},t} : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}.$$

For $x \in \widehat{\mathfrak{g}}_1$, $\sigma_{\vec{\tilde{\mu}},t}(\widehat{\phi_1}(x)) = \widehat{\phi_1}(\sigma_{\vec{\mu},t}(x))$ and for $y \in \widehat{\mathfrak{g}}_2$, $\sigma_{\vec{\tilde{\mu}},t}(\widehat{\phi_2}(y)) = \widehat{\phi_2}(y)$.

The isomorphism $\Theta_{\mu}(\vec{z})$ in [**12**] have the following functorial property under embeddings of Lie algebras. Let G_1, G_2 and G be simply connected Lie groups with simple Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ respectively. Consider a map $\phi : G_1 \times G_2 \to G$ and let $d\phi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be the map of Lie algebras. For any simply connected, simple Lie group G, consider

$$\Gamma(G) = \{(\sigma_1, \ldots, \sigma_n) \in Z(G)^n | \prod_{i=1}^{n} \sigma_i = \mathrm{id}\}.$$

The following proposition follows from Proposition 4.1.1.

Proposition 4.1.2. *Let $\vec{\Sigma} \in \Gamma(G)$, $\vec{\sigma} \in \Gamma(G_1)$ be such that $\phi(\vec{\sigma}) = \vec{\Sigma}$, then the pairing*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$$

is non-degenerate if and only if the following pairing is non-degenerate

$$\mathcal{V}_{\vec{\sigma}\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}_{\vec{\Sigma}\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1).$$

4.2. Branching Rules and new rank-level dualities for $\mathfrak{sp}(2r)$

In this section, we discuss the branching rules of the conformal embedding $\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \to \mathfrak{so}(4rs)$ and prove some new rank-level dualities. Let $\mathcal{Y}_{r,s}$ denote the set of Young diagrams with at most r rows and s columns. For a Young diagram $Y = (a_1, a_2, \cdots, a_r) \in \mathcal{Y}_{r,s}$, we denote by Y^T the Young diagram obtained by exchanging the rows and columns. For a Young diagram $Y \in \mathcal{Y}_{r,s}$, we denote by Y^c the Young diagram given by the conjugate $(s - a_r, s - a_{r-1}, \cdots, s - a_1)$. The Young diagram Y^* is defined to be $(Y^T)^c$. It is easy to see $(Y^T)^c = (Y^c)^T$, where $Y \in \mathcal{Y}_{r,s}$.

4.2.1. The case $\mathfrak{g} = \mathfrak{so}(2N)$ with level 1. Let V be a $2N$ dimensional vector space with a non-degenerate symmetric bilinear form B. The Lie algebra

$$\mathfrak{so}(2N) := \{T : V \to V | B(Tv, w) + B(v, Tw) = 0\}.$$

If we choose a basis $\{e_1, \cdots, e_{2N}\}$ of V such that $B(e_i, e_j) = B(e_{N+i}, e_{N+j}) = 0$ and $B(e_i, e_{N+j}) = \delta_{ij}$, then the Lie algebra $\mathfrak{so}(V)$ is identified with matrices

$$\mathfrak{so}(2N) := \{X \in \mathrm{Mat}_{2N\times 2N} \, | X^t M_{2N} + M_{2N} X = 0\},$$

where M_{2N} is the matrix $\begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$. The Cartan subalgebra is generated by $H_i = E_{i,i} - E_{N+i,N+i}$ and let $\{L_i\} \subset \mathfrak{h}^*$ be the dual basis of $\{H_i\}$'s. A basis of positive roots is given by

$$L_1 - L_2, L_2 - L_3, \cdots, L_{N-1} - L_N, L_{N-1} + L_N.$$

The level one weights of $\mathfrak{so}(4rs)$ are given by the following:

$$P_1(\mathfrak{so}(4rs)) := \{0, \Lambda_1, \alpha, \beta\},$$

where 0 is the vacuum representation, Λ_1 is the first fundamental weight, $\alpha = \frac{1}{2}(L_1 + \cdots + L_N)$ and $\beta = \frac{1}{2}(L_1 + L_2 + \cdots + L_{(N-1)} - L_N)$.

4.2.2. The case $\mathfrak{g} = \mathfrak{sp}(2r)$ with level s. Let V be a $2r$ dimensional vector space with a non-degenerate alternating bilinear form B. The Lie algebra $\mathfrak{sp}(2r)$ is

$$\mathfrak{sp}(2r) := \{T : V \to V | B(T(v), w) + B(v, T(w)) = 0\}.$$

We choose a basis $(v_1, \cdots, v_{2r})$ of V such that

$$(v_i, v_j) = (v_{r+i}, v_{r+j}) = 0, \quad (v_i, v_{r+j}) = -(v_{r+j}, v_i) = \delta_{ij} \text{ for } 1 \leq i, j \leq r.$$

We identify $\mathfrak{sp}(V)$ with

$$\mathfrak{sp}(2r) := \{X \in \mathrm{Mat}_{2r\times 2r} \, | X^t M + MX = 0\},$$

where $M = \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}$.

The Cartan subalgebra of $\mathfrak{sp}(2r)$ is generated by the matrices $H_i = E_{i,i} - E_{r+i,r+i}$. Let $L_i \in \mathfrak{h}^*$ be dual to H_i. The positive simple roots of $\mathfrak{g}$ are $L_i - L_{i+1}$ for $1 \leq i < r$ and $2L_r$. The weight lattice P is $\{a_1L_1 + \cdots a_rL_r | a_i \in \mathbb{Z}\}$. A weight $\lambda = \sum_{i=1}^{r} a_iL_i$ is dominant if $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$. Thus the set of dominant weights of level s is given by

$$P_s(\mathfrak{sp}(2r)) = \{a_1L_1 + \cdots a_rL_r \in P_+ | a_1 \leq s\}.$$

There is a one to one correspondence between $P_s(\mathfrak{sp}(2r))$ and Young diagrams $\mathcal{Y}_{r,s}$. For $\lambda = \sum_{i=1}^{r} a_iL_i \in P_s(\mathfrak{sp}(2r))$, the corresponding Young diagram $(a_1, a_2, \cdots, a_r)$ is denoted by $Y(\lambda)$. The dominant weight of $\mathfrak{sp}(2s)$ of level r, corresponding to the Young diagram $Y(\lambda)^*$ will be denoted by λ^* and that of $Y(\lambda)^T$ by λ^T. It is easy to observe that $\lambda \to \lambda^*$ gives a bijection of $P_s(\mathfrak{sp}(2r))$ with $P_r(\mathfrak{sp}(2s))$. Also $\lambda \to \lambda^T$ gives a bijection of $P_s(\mathfrak{sp}(2r))$ with $P_r(\mathfrak{sp}(2s))$.

We now describe the action of the center of $Sp(2r)$ as diagram automorphisms on $P_s(\mathfrak{sp}(2r))$. Let ω be the outer automorphism that corresponds to the diagram automorphism which sends the i-th vertex to $r-i$-th vertex of the Dynkin diagram of $\widehat{\mathfrak{sp}}(2r)$, where $0 \leq i \leq r$. Then the Young diagram of $\omega^*\lambda$ is given by $Y(\lambda)^c$, where $Y(\lambda)$ is the Young diagram corresponding to $\lambda \in P_s(\mathfrak{sp}(2r))$. The following is proved in [**17**] and describes the branching rules.

Proposition 4. *We have an isomorphism of $(\widehat{\mathfrak{sp}(2r)} \oplus \widehat{\mathfrak{sp}(2s)})$-modules,*

$$\mathcal{H}_\alpha \simeq \bigoplus_{|Y(\lambda)|:even} \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^*},$$

$$\mathcal{H}_0 \simeq \bigoplus_{|Y(\lambda)|:even} \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^T},$$

$$\mathcal{H}_\beta \simeq \bigoplus_{|Y(\lambda)|:odd} \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^*},$$

$$\mathcal{H}_{\Lambda_1} \simeq \bigoplus_{|Y(\lambda)|:odd} \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^T},$$

where $Y(\lambda)$ runs through over the set $\mathcal{Y}_{r,s}$ with $|Y(\lambda)|$ even and odd in the respective cases.

4.2.3. New symplectic rank-level duality. Let us fix n distinct smooth points $\vec{p} = (P_1, \cdots, P_n)$ on the projective line $\mathbb{P}^1$. Let $\vec{z} = (z_1, \ldots, z_n)$ be the local coordinates of $\vec{p}$. We denote the above data by $\mathfrak{X}$. Consider an n tuple of level s weights $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ and $\vec{\lambda}^* := (\lambda_1^*, \cdots, \lambda_n^*)$. As described in Chapter 3, we get a map

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sp}(2r), s) \otimes \mathcal{V}_{\vec{\lambda}^*}(\mathfrak{X}, \mathfrak{sp}(2s), r) \to \mathcal{V}_{\vec{\epsilon}}(\mathfrak{X}, \mathfrak{so}(4rs), 1), \tag{4.1}$$

where $\vec{\epsilon} := ((-1)^{|Y(\lambda_1)|}, \cdots, (-1)^{|Y(\lambda_n)|})$. Here we understand that $+1 = \alpha$ and $-1 = \beta$. If both n and $\sum_{j=1}^n |Y(\lambda_i)|$ are even, then Corollary 3.4 in [**2**] tells us that

$$\dim_{\mathbb{C}} \mathcal{V}_{\vec{\epsilon}}(\mathfrak{X}, \mathfrak{so}(4rs), 1) = 1.$$

Hence the above morphism induces the following rank-level duality map

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sp}(2r), s) \to \mathcal{V}^\dagger_{\vec{\lambda}^*}(\mathfrak{X}, \mathfrak{sp}(2s), r).$$

With the above assumption and notation, the following is the main result in [**2**]:

Proposition 5. *The rank-level duality map*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sp}(2r), s) \to \mathcal{V}^\dagger_{\vec{\lambda}^*}(\mathfrak{X}, \mathfrak{sp}(2s), r),$$

is an isomorphism.

The action of the non-trivial element $\omega \in Z(\mathrm{Sp}(2s))$ on $P_r(\mathfrak{sp}(2s))$ gives us the following:

Lemma 4. *For $\lambda \in \mathcal{Y}_{r,s}$, we get*

$$\omega(\lambda^*) = \lambda^T.$$

From the branching rule described in [**17**], we also get a map

$$\Psi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sp}(2r), s) \otimes \mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{sp}(2s), r) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(4rs), 1),$$

where $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ and Λ_i is the unique level one dominant weight of $\mathfrak{so}(4rs)$ such that λ_i and λ_i^T appear in the branching of Λ_i. Assume that both n and $\sum_{i=1}^n |\lambda_i|$ are even. We choose $\vec{\mu} \in \Gamma_n^{\mathfrak{sp}(2s)}$ such that for $1 \leq i \leq n$, $\mu_i \in P(\mathfrak{sp}(2s))^\vee \backslash Q(\mathfrak{sp}(2s))^\vee$, and use Proposition 5 and Proposition 4.1.2 to get more symplectic rank-level dualities. More precisely we get the following rank-level dualities of conformal blocks.

Proposition 4.2.1. *There is a linear isomorphism of the following spaces:*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sp}(2r), s) \to \mathcal{V}^\dagger_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{sp}(2s), r).$$

4.3. Branching rules and rank-level dualities for $\mathfrak{sl}(r)$

We describe the branching rules of the conformal embedding $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \subset \mathfrak{sl}(rs)$ following [**1**]. Let $P_+(\mathfrak{sl}(r))$ denote the set of dominant integral weights of $\mathfrak{sl}(r)$ and $\Lambda_1, \cdots, \Lambda_{r-1}$ denote the fundamental weights of $\mathfrak{sl}(r)$. If $\lambda = \sum_{i=1}^{r-1} \widetilde{k}_i \Lambda_i$ for non-negative integers $\widetilde{k}_i$, we rewrite λ as $\lambda = \sum_{i=0}^{r-1} \widetilde{k}_i \Lambda_i$, where

$$\sum_{i=0}^{r-1} \widetilde{k}_i = s,$$

where Λ_0 is the affine 0-th fundamental weight.

Let $\widehat{\rho} = g^*(\mathfrak{sl}(r))\Lambda_0 + \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$, where $g^*(\mathfrak{sl}(r))$ is the dual Coxeter number of $\mathfrak{sl}(r)$ and Δ_+ is the set of positive roots respect to a chosen Cartan subalgebra of $\mathfrak{sl}(r)$. We get

$$\lambda + \widehat{\rho} = \sum_{i=0}^{r-1} k_i \Lambda_i,$$

where $k_i = \widetilde{k_i} + 1$ and $\sum_{i=0}^{r-1} k_i = r + s$. The center of $\mathrm{SL}(r)$ is $\mathbb{Z}/r\mathbb{Z}$. The action of $\mathbb{Z}/r\mathbb{Z}$ induced from outer automorphisms on $P_s(\mathfrak{sl}(r))$ is described as follows:

$$\begin{aligned} \mathbb{Z}/r\mathbb{Z} \times P_s(\mathfrak{sl}(r)) &\longrightarrow P_s(\mathfrak{sl}(r)), \\ (\sigma, \Lambda_i) &\longrightarrow \Lambda_{((i+\sigma) mod(r))}. \end{aligned}$$

Let $\Omega_{r,s} = P_s(\mathfrak{sl}(r))/(\mathbb{Z}/r\mathbb{Z})$ be the set of orbits under this action and similarly let $\Omega_{s,r}$ be the orbits of $P_r(\mathfrak{sl}(s))$ under the action of $\mathbb{Z}/s\mathbb{Z}$.

The following map β parametrizes the bijection in the above lemma.

$$\beta : P_s(\mathfrak{sl}(r)) \to P_r(\mathfrak{sl}(s))$$

Set

$$a_j = \sum_{i=j}^{r} k_i, \text{ for } 1 \leq j \leq r \text{ and } k_r = k_0.$$

The sequence $\vec{a} = (a_1, a_2, \cdots, a_r)$ is decreasing. Let $(q_1, q_2, q_3, \cdots, q_s)$ be the complement of $\vec{a}$ in the set $\{1, 2, \cdots, (r+s)\}$ in decreasing order. We define the following sequence:

$$b_j = r + s + q_s - q_{s-j+1} \text{ for } 1 \leq j \leq s.$$

The sequence b_j defined above is also decreasing. The map β is given by the following formula:

$$\beta(a_1, \cdots, a_r) = (b_1, b_2, \cdots, b_s).$$

Thus when λ runs over an orbit of $\Omega_{r,s}$, $\gamma = \sigma.\beta(\lambda)$ runs over an orbit of $\Omega_{s,r}$ if σ runs over $\mathbb{Z}/s\mathbb{Z}$.

The elements λ of $P_s(\mathfrak{sl}(r))$ can be parametrized by Young diagrams $Y(\lambda)$ with at most $r-1$ rows and at most s columns. Let $Y(\lambda)^T$ be the modified transpose of $Y(\lambda)$. If $Y(\lambda)$ has rows of length s, then $Y(\lambda)^T$ is obtained by taking the usual transpose of $Y(\lambda)$ and deleting the columns of length s. We denote by λ^T the dominant integral weight of $\mathfrak{sl}(s)$ of level r that corresponds to $Y(\lambda)^T$. With this notation we recall the following proposition from [**1**]:

Proposition 6. *Let $\lambda \in P_s(\mathfrak{sl}(r))$ and $c(\lambda)$ be the number of columns of $Y(\lambda)$. Suppose $\sigma = c(\lambda) \bmod s$. Then*

$$\sigma.\beta(\lambda) = \lambda^T.$$

The weights of $\mathfrak{sl}(rs)$ at level one are given by

$$P_1(\mathfrak{sl}(rs)) = \{\Lambda_0, \cdots, \Lambda_{rs-1}\}.$$

We identify it with the set $\{1, 2, \cdots, rs-1\}$.

For $\Lambda \in P_1(\mathfrak{sl}(rs))$, let $\mathcal{H}_\Lambda$ be the unique irreducible highest weight integrable $\widehat{\mathfrak{sl}(rs)}$-module of level one. Now we can decompose $\mathcal{H}_\Lambda$ as a $\widehat{\mathfrak{sl}(r)} \oplus \widehat{\mathfrak{sl}(s)}$-module. Let

$$\mathcal{H}_\Lambda \simeq \bigoplus_{B(\Lambda)} m^\Lambda_{\lambda,\gamma} \mathcal{H}_\lambda \otimes \mathcal{H}_\gamma,$$

where $B(\Lambda)$ is as defined in Chapter 3.

Consider the following:

$$\begin{aligned} \sum_{j=1}^{r} a_j &= \sum_{j=1}^{r}\sum_{i=j}^{r} k_i \text{ and } k_r = k_0, \\ &= (\text{sum of length of rows of } Y(\lambda)) + \frac{1}{2}r(r+1) + r\widetilde{k_o}. \end{aligned}$$

For $\lambda \in P_s(\mathfrak{sl}(r))$ and $\sigma \in \mathbb{Z}/(s\mathbb{Z})$, we define

$$\begin{aligned} \delta(\lambda,\sigma) &= \left(\sum_{j=1}^{r} a_j\right) + r\sigma - \frac{1}{2}r(r+1), \\ &= (\text{sum of length of rows of } Y(\lambda)) + \frac{1}{2}r(r+1) + r\widetilde{k_o} + r\sigma - \frac{1}{2}r(r+1), \\ &= (\text{sum of length of rows of } Y(\lambda)) + r\widetilde{k_0} + r\sigma, \\ &= (\text{sum of length of rows of } Y(\lambda)) + r(\widetilde{k_0} + \sigma), \\ &= |Y(\lambda)| + rs + r(\sigma - c(Y(\lambda))). \end{aligned}$$

Next we state the main result that describes $m^\Lambda_{\lambda,\gamma}$. See [**1**] for a proof.

Proposition 7. *Let $\Lambda \in P_1(\mathfrak{sl}(rs))$, $\lambda \in P_s(\mathfrak{sl}(r))$ and $\gamma \in P_r(\mathfrak{sl}(s))$. Then the multiplicity $m^{\Lambda}_{\lambda,\gamma}$ of $\mathcal{H}_\lambda \otimes \mathcal{H}_\gamma$ in $\mathcal{H}_\Lambda$ has the value*

$$\begin{aligned} m^{\Lambda}_{\lambda,\gamma} &= 1 \text{ if } \gamma = \sigma\beta(\lambda), \quad \sigma \in \mathbb{Z}/s\mathbb{Z} \text{ and } \Lambda = \delta(\lambda,\sigma) \operatorname{mod}(rs), \\ m^{\Lambda}_{\lambda,\gamma} &= 0 \text{ otherwise.} \end{aligned}$$

4.3.1. Rank-level duality of $\mathfrak{sl}(r)$. Let us fix n distinct points $\vec{p} = (P_1, \cdots, P_n)$ on the projective line $\mathbb{P}^1$. Let $\vec{z} = (z_1, \ldots, z_n)$ be the local coordinates of $\vec{p}$. We denote the above data by $\mathfrak{X}$. Consider an n tuple $\vec{\Lambda} = (\Lambda_{i_1}, \cdots, \Lambda_{i_n})$ of level one dominant integral weights of $\mathfrak{sl}(rs)$. Let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ and $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ be such that λ_k, γ_k appear in the branching of Λ_{i_k} for $1 \leq k \leq n$. We get a map

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \otimes \mathcal{V}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{sl}(s), r) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{sl}(rs), 1),$$

If rs divides $\sum_{k=1}^{n} i_k$, it is well known that $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{sl}(rs), 1) = 1$. We get the following morphism well defined up to scalars:

$$\Psi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \to \mathcal{V}^{\dagger}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{sl}(s), r).$$

The rest of the section is devoted to the proof that Ψ is an isomorphism. Without loss of generality we can assume that $\sigma_i - c(Y(\lambda_i))$ is non-negative for all i. Let Q_1 be a new point distinct from $P_1, \ldots, P_n$ on $\mathbb{P}^1$ and η_1 be the new coordinate. Let $\widetilde{\mathfrak{X}}$ be the data associated to the points $P_1, \ldots, P_n, Q_1$ on $\mathbb{P}^1$. We have the following proposition:

Proposition 8. *The following are equivalent:*

(1) The rank-level duality map

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \to \mathcal{V}^{\dagger}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{sl}(s), r),$$

is an isomorphism.

(2) The rank-level duality map

$$\mathcal{V}_{\vec{\lambda}\cup 0}(\widetilde{\mathfrak{X}}, \mathfrak{sl}(r), s) \to \mathcal{V}^{\dagger}_{\vec{\gamma}\cup 0}(\widetilde{\mathfrak{X}}, \mathfrak{sl}(s), r),$$

is an isomorphism.

(3) The rank-level duality map

$$\mathcal{V}_{\vec{\lambda}\cup 0}(\widetilde{\mathfrak{X}}, \mathfrak{sl}(r), s) \to \mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup \gamma}(\widetilde{\mathfrak{X}}, \mathfrak{sl}(s), r),$$

is an isomorphism, where $\gamma = r\omega_\sigma$, $\sigma = \sum_{i=1}^n(\sigma_i - c(Y(\lambda_i))) \operatorname{mod}(s)$ *and* ω_σ *is the* σ*-th fundamental weight.*

Proof. The equivalence of (1) and (2) follows from Proposition 3.3.2. The equivalence of (2) and (3) follows directly from Proposition 4.1.2.

□

The proof that Ψ is non-degenerate follows from the following (see Theorem 4.10 [**32**]):

Proposition 9. *The following rank-level duality map is an isomorphism*

$$\mathcal{V}_{\vec{\lambda}\cup 0}(\widetilde{\mathfrak{X}}, \mathfrak{sl}(r), s) \to \mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup \beta}(\widetilde{\mathfrak{X}}, \mathfrak{sl}(s), r),$$

where $\beta = r\omega_\sigma$, $\sigma = \sum_{i=1}^n(\sigma_i - c(Y(\lambda_i))) \operatorname{mod}(s)$ *and* ω_σ *is the* σ*-th fundamental weight.*

Remark 2. *The rank-level duality for* $\mathfrak{sl}(r)$ *was first proved by T. Nakanishi and A. Tsuchiya. Our result shows that the rank-level dualities for the pair* $\mathfrak{sl}(r), \mathfrak{sl}(s)$ *that appears from conformal embeddings can also be obtained from the geometric rank-level dualities of* [**32**]. *An alternative proof of rank-level duality of* $\mathfrak{sl}(r)$ *without using the result of* [**32**] *can be obtained using the same strategy of the proof of the rank-level duality result in* [**25**].

CHAPTER 5

Sewing and compatibility under factorization

In this chapter, we recall the sewing construction from [**36**]. We consider a family of curves degenerating to a curve with one node. We study the compatibility of the rank-level duality map with factorization of the nodal curve following [**4**].

We will use Proposition 5.2.1 to reduce rank-level duality questions on n pointed curves to rank-level duality for certain one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points. Our strategy is inspired by Proposition 5.2 in [**30**]. We refer the reader to Chapter 9 for more details.

5.1. Sewing

First we recall the following lemma from [**36**].

LEMMA 5.1.1. *There exists a bilinear pairing*

$$(\,,\,)_\lambda : \mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger} \to \mathbb{C}$$

unique up to a multiplicative constant such that

$$(X(n)u, v)_\lambda + (u, X(-n)v)_\lambda = 0.$$

for all $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $u \in \mathcal{H}_\lambda$ and $v \in \mathcal{H}_{\lambda^\dagger}$. Moreover, the restriction of the form $(\,,\,)_\lambda$ to $\mathcal{H}_\lambda(m) \times \mathcal{H}_{\lambda^\dagger}(m')$ is zero if $m \neq m'$ and is non-degenerate if $m = m'$.

Since the restriction of the bilinear form $(\,,\,)_\lambda$ to $\mathcal{H}_\lambda(m) \times \mathcal{H}_{\lambda^\dagger}(m)$ is non-degenerate, we obtain an isomorphism of $\mathcal{H}_{\lambda^\dagger}(m)$ with $\mathcal{H}_\lambda(m)^*$. Let $\gamma_\lambda(m)$ be the distinguished element of $\mathcal{H}_\lambda(m) \otimes \mathcal{H}_{\lambda^\dagger}(m)$ given by $(\,,\,)_\lambda$. Let t be a formal variable. Given $\lambda \in P_\ell(\mathfrak{g})$, we construct an element $\widetilde{\gamma}_\lambda = \sum_{m=0}^{\infty} \gamma_\lambda(m) t^m$ of $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^\dagger}[[t]]$.

We are now ready to describe the sewing procedure in [**36**]. Throughout the chapter, let $\mathcal{B} = \operatorname{Spec} \mathbb{C}[[t]]$. We consider a family of curves $\mathcal{X} \to \mathcal{B}$ with n marked points with chosen coordinates such that its special fiber $\mathcal{X}_0$ is a curve X_0 over $\mathbb{C}$ with exactly one node and its generic fiber $\mathcal{X}_t$ is a smooth curve. Consider the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X}, \mathfrak{g})$ for the family of curves $\mathcal{X}$. The sheaf of conformal blocks commutes with base change and the fiber over any point $t \in \mathcal{B}$ coincides with $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_t, \mathfrak{g})$, where $\mathfrak{X}_t$ is the data associated to the curve X_t over the point $t \in \mathcal{B}$.

Let $\widetilde{X}_0$ be the normalization of X_0. For $\lambda \in P_\ell(\mathfrak{g})$, the following isomorphism is constructed in [**36**]

$$\oplus \iota_\lambda : \bigoplus_{\lambda \in P_\ell(\mathfrak{g})} \mathcal{V}^{\dagger}_{\lambda, \lambda^\dagger, \vec{\lambda}}(\widetilde{\mathfrak{X}}, \mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}),$$

where $\widetilde{\mathfrak{X}}$ is the data associated to the $(n+2)$ points of the smooth pointed curve $\widetilde{X}_0$ with chosen coordinates.

In [**36**], a sheaf version of the above isomorphism is also proved. We briefly recall the construction. For every $\lambda \in P_\ell(\mathfrak{g})$ there exists a map

$$s_\lambda : \mathcal{V}^{\dagger}_{\lambda, \lambda^\dagger, \vec{\lambda}}(\widetilde{\mathfrak{X}}, \mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X}, \mathfrak{g}),$$

where $s_\lambda(\psi) = \widetilde{\psi}$ and $\widetilde{\psi}(\widetilde{u}) := \psi(\widetilde{u} \otimes \widetilde{\gamma}_\lambda) \in \mathbb{C}[[t]]$ for any $\widetilde{u} \in \mathcal{H}_{\vec{\lambda}}[[t]]$. This map extends to a map $s_\lambda(t)$ of coherent sheaves of $\mathbb{C}[[t]]$-modules

$$s_\lambda(t) : \mathcal{V}^{\dagger}_{\lambda, \lambda^\dagger, \vec{\lambda}}(\widetilde{\mathfrak{X}}, \mathfrak{g}) \otimes \mathbb{C}[[t]] \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X}, \mathfrak{g}).$$

With the above notation, the following is proved in [**36**]. We also refer the reader to Theorem 6.1 in [**22**].

PROPOSITION 5.1.2. *The map*

$$\oplus s_\lambda(t) : \bigoplus_{\lambda \in P_\ell(\mathfrak{g})} \mathcal{V}^{\dagger}_{\lambda, \lambda^\dagger, \vec{\lambda}}(\widetilde{\mathfrak{X}}, \mathfrak{g}) \otimes \mathbb{C}[[t]] \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X}, \mathfrak{g}).$$

is an isomorphism of locally free sheaves on $\mathcal{B}$*.*

5.2. Factorization and compatibility of rank-level duality

Consider a conformal embedding $\mathfrak{s} \to \mathfrak{g}$. Assume that all level one highest weight integrable modules of $\widehat{\mathfrak{g}}$ decompose with multiplicity one as $\widehat{\mathfrak{s}}$-modules.

Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an n tuple of level one weights of $\mathfrak{g}$ and $\vec{\mu} \in B(\vec{\lambda})$. We get a map $\mathcal{H}_{\vec{\mu}}(\mathfrak{s}) \to \mathcal{H}_{\vec{\lambda}}(\mathfrak{g})$. As discussed in Chapter 3, we get a $\mathbb{C}[[t]]$-linear map

$$\alpha(t) : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X}, \mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathcal{X}, \mathfrak{s}).$$

For $\mu \in B(\lambda)$, we denote by $\alpha_{\lambda,\mu}$ the map induced from branching as discussed in Chapter 3

$$\mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{X}_0, \mathfrak{g}) \to \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\mu}}(\widetilde{X}_0, \mathfrak{s})$$

and the extension of $\alpha_{\lambda,\mu}$ to a $\mathbb{C}[[t]]$-linear map is denoted as follows:

$$\alpha_{\lambda,\mu}(t) : \mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{X}_0, \mathfrak{g}) \otimes \mathbb{C}[[t]] \to \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\mu}}(\widetilde{X}_0, \mathfrak{s}) \otimes \mathbb{C}[[t]].$$

The following proposition from [**4**] describes how $\alpha(t)$ decomposes under factorization.

Proposition 5.2.1. *On $\mathcal{B}$, we have*

$$\alpha(t) \circ s_{\lambda}(t) = \sum_{\mu \in B(\lambda)} t^{n_{\mu}} s_{\mu}(t) \circ \alpha_{\lambda,\mu}(t),$$

where n_{μ} are positive integers given by the formula:

$$n_{\mu} = \Delta_{\mu} - \Delta_{\lambda}.$$

Remark 5.2.2. *The integers n_{μ} are non-zero if the finite dimensional $\mathfrak{s}$-module V_{μ} does not appear in the decomposition of the finite dimensional $\mathfrak{g}$-module V_{λ}.*

CHAPTER 6

Branching rules for conformal embedding of odd orthogonal Lie algebras

In this chapter, we discuss the branching rule for the conformal embedding $\mathfrak{so}(2r+1) \oplus \mathfrak{so}(2s+1) \to \mathfrak{so}((2r+1)(2s+1))$. Our discussions follow closely the discussions in [**17**].

6.1. Representation of $\mathfrak{so}(2r+1)$

Let $E_{i,j}$ be a matrix whose (i,j)-th entry is one and all other entries are zero. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2r+1)$ is generated by diagonal matrices of the form $H_i = E_{i,i} - E_{r+i,r+i}$ for $1 \leq i \leq r$. Let $L_i \in \mathfrak{h}^*$ be defined by $L_i(H_j) = \delta_{i,j}$. The normalized Cartan killing form on $\mathfrak{h}$ is given by $(H_i, H_j) = \delta_{ij}$. Under the identification of $\mathfrak{h}^*$ with $\mathfrak{h}$ using the Cartan Killing form the image of L_i is H_i for all $1 \leq i \leq r$.

We can choose the simple positive roots of $\mathfrak{so}(2r+1)$ to be $\alpha_1 = L_1 - L_2, \alpha_2 = L_2 - L_3, \dots, \alpha_{r-1} = L_{r-1} - L_r, \alpha_r = L_r$. The highest root is $\theta = L_1 + L_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$. The fundamental weights of the B_n are $\omega_i = L_1 + L_2 + \cdots + L_i$ for $1 \leq i < r$ and $\omega_r = \frac{1}{2}(L_1 + L_2 + \cdots + L_r)$.

The dominant integral weights, P_+, of $\mathfrak{so}(2r+1)$ can be written as

$$P_+ = P_+^0 \sqcup P_+^1,$$

where P_+^0 is the set of dominant weights $\lambda = \sum_{i=1}^r a_i \omega_i$ such that a_r is even and $P_+^1 := P_+^0 + \omega_r$. Let $\mathcal{Y}_r$ be the set of Young diagrams with at most r rows and $\mathcal{Y}_{r,s}$ denote the set of Young diagrams with at most r rows and s columns. Then the set P_+^0 is in bijection with $\mathcal{Y}_r$.

Combinatorially any dominant weight λ of P_+ can be written as $Y + t\omega_r$, where $t = \{0, 1\}$ and $Y \in \mathcal{Y}_r$. If $t = 0$, then $\lambda \in P_+^0$ and if $t = 1$, then $\lambda \in P_+^1$.

Let $\lambda = \sum_{i=1}^r a_i\omega_i$ be a dominant integral weight. Then,

$$(\theta, \lambda) = a_1 + 2(a_2 + \cdots + a_{r-1}) + a_r.$$

The set of level $2s + 1$ dominant weights are described below:

$$P_{2s+1}(\mathfrak{so}(2r + 1)) = \{\lambda \in P_+ | a_1 + 2(a_2 + \cdots + a_{r-1}) + a_r \leq 2s + 1\}.$$

6.2. The action of center on weights

An element σ of the center of the group $\mathrm{Spin}(2r + 1)$ acts as an outer automorphism on affine Lie algebra $\widehat{\mathfrak{so}}(2r + 1)$. For details we refer the reader to [**17**]. The action of σ on the $P_{2s+1}(\mathfrak{so}(2r + 1))$ is given by $\sigma(\lambda) = (2s + 1 - (a_1 + 2(a_2 + \cdots + a_{r-1}) + a_r))\omega_1 + a_2\omega_2 + \cdots + a_r\omega_r$. We denote the intersection $P_+^0 \cap P_{2s+1}(\mathfrak{so}(2r+1))$ by $P_{2s+1}^0(\mathfrak{so}(2r+1))$. The following lemma can be proved by direct calculation:

Lemma 6.2.1. *The action of σ preserves the set $P_{2s+1}^0(\mathfrak{so}(2r + 1))$ and $P_{2s+1}^0(\mathfrak{so}(2r + 1)) = \mathcal{Y}_{r,s} \sqcup \sigma(\mathcal{Y}_{r,s})$.*

Following [**28**], we describe the orbits of $P_{2s+1}(\mathfrak{so}(2r + 1))$ under the action of the center. Let $\rho = \sum_{i=1}^r \omega_i$ be the Weyl vector. For $\lambda = \sum_{i=1}^r a_i\omega_i$, the weight $\lambda + \rho = \sum_{i=1}^r t_i\omega_i$, where $t_i = a_i + 1$. Put $u_i = \sum_{j=i}^{r-1} t_j + \frac{t_r}{2}$ for $1 \leq i \leq r$, $u_r = \frac{t_r}{2}$ and $u_{r+1} = 0$.

The set $P_{2s+1}(\mathfrak{so}(2r + 1))$ is identified with the collection of sets $U = (u_1 > u_2 > \cdots > u_r > 0)$ such that

- $u_i \in \frac{1}{2}\mathbb{Z}$.
- $u_i - u_{i+1} \in \mathbb{Z}$.
- $u_1 + u_2 \leq 2(r + s)$.

Let $P_{2s+1}^+(\mathfrak{so}(2r + 1))$ denote the set of weights in $P_{2s+1}(\mathfrak{so}(2r + 1))$ such that $u_i \in \mathbb{Z}$.

Let us set $k = 2(r + s)$, and we rewrite the action of the center Γ on $P_{2s+1}(\mathfrak{so}(2r + 1))$ as exchanging t_1 with $t_0 = k - t_1 - 2t_2 - \cdots - 2t_{r-1} - t_r$; or in other words changing u_1 with

$k-u_1$. We observe that the action of Γ preserves $P^{+}_{2s+1}(\mathfrak{so}(2r+1))$ and $P^{0}_{2s+1}(\mathfrak{so}(2r+1))$. Then, we can identify the orbits of the action of Γ as follows :

$$P_{2s+1}(\mathfrak{so}(2r+1))/\Gamma = \{U = (u_1, \dots, u_r) | \frac{k}{2} \geq u_1 > \cdots > u_r > 0, u_i \in \frac{1}{2}\mathbb{Z}, u_i - u_{i+1} \in \mathbb{Z}\}.$$

and the length of the orbits are given as follows:

- $|\Gamma(U)| = 2$ if $u_1 < \frac{k}{2}$.
- $|\Gamma(U)| = 1$ if $u_1 = \frac{k}{2}$.

For any number a and a set $U = (u_1 > u_2 > \cdots > u_r)$ we denote by $U - a$ and $a - U$, the set $\{u_1 - a > u_2 - a > \cdots > u_r - a\}$ and $\{a - u_r > a - u_{r-1} > \cdots > a - u_1\}$ respectively. Further, the set $\{1, 2, \dots, r+s\}$ is denoted by $[r+s]$. The following two lemmas from [**28**] give a bijection of orbits.

Lemma 6.2.2. *Let $P_{2r+1}(\mathfrak{so}(2s+1))$ denote the weights of $\mathfrak{so}(2s+1)$ of level $2r+1$. Then there is a bijection between the orbits of $P^{+}_{2s+1}(\mathfrak{so}(2s+1))$ and the orbits of $P^{+}_{2r+1}(\mathfrak{so}(2s+1))$ given by*

$$U = (u_1 > u_2 > \cdots > u_r) \to U^c = (u^c_1 > \cdots > u^c_s),$$

where $U \subset [r+s]$ of cardinality r and U^c is the complement of U in $[r+s]$.

For $\lambda \in P^{0}_{2s+1}(\mathfrak{so}(2r+1))$, we write $\lambda + \rho = \sum_{i=1}^{r}(u'_i - \frac{1}{2})L_i$, where u'_i are all integers. We identify the identify the orbits of $P^{0}_{2s+1}(\mathfrak{so}(2r+1))$ under Γ as subsets $U' = (u'_1 > u'_2 > \cdots > u'_r)$ of $[r+s]$.

Lemma 6.2.3. *There is a bijection between the orbits of $P^{0}_{2s+1}(\mathfrak{so}(2s+1))$ and the orbits of $P^{0}_{2r+1}(\mathfrak{so}(2s+1))$ given by*

$$U' = (u'_1 > u'_2 > \cdots > u'_r) \to ((r+s) + 1 - U'^c) = (u''_1 > \cdots > u''_s),$$

where $U' \subset [r+s]$ of cardinality r and U'^c is the complement of U' in $[r+s]$.

6.3. Branching rules

We now describe the branching rules for the conformal embedding $\mathfrak{so}(2r+1)\oplus\mathfrak{so}(2s+1) \subset \mathfrak{so}((2r+1)(2s+1))$. Let $N = (2r+1)(2s+1) = 2d+1$. The level one highest weights of $\widehat{\mathfrak{so}}(N)$ are 0, ω_1 and ω_d. The following proposition gives the decomposition of level one integrable highest weight modules of weight 0 and ω_1. We refer the reader to [**17**] for a proof.

Proposition 6.3.1. *Let $\mathcal{H}_0(\mathfrak{so}(N))$ and $\mathcal{H}_1(\mathfrak{so}(N))$ denote the highest weight integrable modules of the affine Lie algebra $\widehat{\mathfrak{so}}(2r+1)$ with highest weight 0 and ω_1 respectively. Then the module $\mathcal{H} := \mathcal{H}_0(\mathfrak{so}(N))\oplus\mathcal{H}_1(\mathfrak{so}(N))$ breaks up as a direct sum of highest weight integrable modules of $\widehat{\mathfrak{so}}(2r+1)\oplus\widehat{\mathfrak{so}}(2s+1)$ of the form:*

- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$,
- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$,
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$,
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$,

where $\lambda \in \mathcal{Y}_{r,s}$ and σ is an automorphism associated to the center of $\mathrm{Spin}(2r+1)$. *More over all of the above factors appear with multiplicity one.*

We need to determine which factor in the above decomposition rules comes from $\mathcal{H}_0(\mathfrak{so}(N))$ and which factor comes from $\mathcal{H}_1(\mathfrak{so}(N))$. The following lemma gives the trace anomaly of the level one weights 0 and ω_1 of $\widehat{\mathfrak{so}}(2d+1)$.

Lemma 6.3.2.

$$\Delta_0(\mathfrak{so}(N)) = 0 \quad \Delta_{\omega_1}(\mathfrak{so}(N)) = \frac{1}{2}.$$

In order to determine the components we need to know the trace anomalies for the weight (λ, λ^T).

Lemma 6.3.3. *For $\lambda \in \mathcal{Y}_{r,s}$, we have the following equality*

$$\Delta_\lambda(\mathfrak{so}(2r+1)) + \Delta_{\lambda^T}(\mathfrak{so}(2s+1)) = \frac{1}{2}|\lambda|.$$

Corollary 6.3.4. *Let $\mathcal{H}_0(\mathfrak{so}(N))$ denote the level one highest weight integrable $\widehat{\mathfrak{so}}(N)$-module of weight 0 and $\lambda \in \mathcal{Y}_{r,s}$. Then the following factors appear as the decomposition of $\mathcal{H}_0(\mathfrak{so}(N))$ as $\widehat{\mathfrak{so}}(2r+1) \oplus \widehat{\mathfrak{so}}(2s+1)$-modules.*

- $\mathcal{H}_{\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*
- $\mathcal{H}_{\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*

Corollary 6.3.5. *Let $\mathcal{H}_1(\mathfrak{so}(N))$ denote the level one highest weight integrable $\widehat{\mathfrak{so}}(N)$-module of weight ω_1 and $\lambda \in \mathcal{Y}_{r,s}$. Then the following factors appears as the decomposition of $\mathcal{H}_0(\mathfrak{so}(N))$ as $\widehat{\mathfrak{so}}(2r+1) \oplus \widehat{\mathfrak{so}}(2s+1)$ modules.*

- $\mathcal{H}_{\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*
- $\mathcal{H}_{\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*

Remark 6.3.6. *Unlike the embedding $\mathfrak{sp}(2r) \oplus \mathfrak{sp}(2s) \to \mathfrak{so}(4rs)$, the only components that appear in the decomposition of standard and trivial representations of the finite dimensional Lie algebra $\mathfrak{so}(2d+1)$ into $\mathfrak{so}(2r+1) \oplus \mathfrak{so}(2s+1)$-modules are $\lambda = \omega_1$ and $\lambda = 0$. This is the main obstruction to construct the rank-level duality map in Theorem 0.2.1 geometrically. It is important to study this map geometrically to understand rank-level duality on curves of higher genus. This will be considered in a subsequent project.*

6.4. Rank-level duality map.

In this section, we describe the rank-level duality map using the branching rule. We consider the following weights:

- $\vec{\lambda}_i = (\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{n_1}})$ and $\vec{\lambda}_i^T = (\lambda_{i_1}^T, \lambda_{i_2}^T, \dots, \lambda_{i_{n_1}}^T)$, where $\lambda_{i_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{i_a}|$ is odd for each $1 \le a \le n_1$.
- $\vec{\lambda}_j = (\sigma\lambda_{j_1}, \dots, \sigma\lambda_{j_{n_2}})$ and $\vec{\lambda}_j^T = (\sigma(\lambda_{j_1}^T), \dots, \sigma(\lambda_{j_{n_2}}^T))$, where $\lambda_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{j_a}|$ is odd for all $1 \le a \le n_2$

- $\vec{\lambda}_k = (\sigma\lambda_{k_1}, \ldots, \sigma\lambda_{k_{n_3}})$ and $\vec{\lambda}_k^T = (\lambda_{k_1}^T, \ldots, \lambda_{k_{n_3}}^T)$, where $\lambda_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{k_a}|$ is even for all $1 \leq a \leq n_3$.
- $\vec{\lambda}_l = (\lambda_{l_1}, \lambda_{l_2}, \ldots, \lambda_{l_{n_4}})$ and $\vec{\lambda}_l^T = (\sigma(\lambda_{l_1}^T), \ldots, \sigma(\lambda_{l_{n_4}}^T))$, where $\lambda_{l_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{l_a}|$ is even for each $1 \leq a \leq n_4$.
- $\vec{\beta}_i = (\beta_{i_1}, \ldots, \beta_{i_{m_1}})$ and $\vec{\beta}_i^T = (\beta_{i_1}^T, \beta_{i_2}^T, \ldots, \beta_{i_{m_1}}^T)$, where $\lambda_{i_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{i_a}|$ is even for each $1 \leq a \leq m_1$.
- $\vec{\beta}_j = (\sigma\beta_{j_1}, \ldots, \sigma\beta_{j_{m_2}})$ and $\vec{\beta}_j^T = (\sigma(\beta_{j_1}^T), \ldots, \sigma(\beta_{j_{m_2}}^T))$, where $\beta_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\beta_{j_a}|$ is even for all $1 \leq a \leq m_2$
- $\vec{\beta}_k = (\sigma\beta_{k_1}, \ldots, \sigma\beta_{k_{m_3}})$ and $\vec{\beta}_k^T = (\beta_{k_1}^T, \ldots, \beta_{k_{m_3}}^T)$, where $\lambda_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\beta_{k_a}|$ is odd for all $1 \leq a \leq m_3$.
- $\vec{\beta}_l = (\beta_{l_1}, \beta_{l_2}, \ldots, \beta_{l_{m_4}})$ and $\vec{\beta}_l^T = (\sigma(\beta_{l_1}^T), \ldots, \sigma(\beta_{l_{m_4}}^T))$, where $\beta_{l_a} \in \mathcal{Y}_{r,s}$ such that $|\beta_{l_a}|$ is odd for each $1 \leq a \leq m_4$.

Let $n = \sum_{i=1}^{4}(n_i+m_i)$ be a positive integer, $\vec{\lambda} = \vec{\lambda}_i \cup \vec{\lambda}_j \cup \vec{\lambda}_k \cup \vec{\lambda}_l$, $\vec{\lambda}^T = \vec{\lambda}_i^T \cup \vec{\lambda}_j^T \cup \vec{\lambda}_k^T \cup \vec{\lambda}_l^T$, $\vec{\beta} = \vec{\beta}_i \cup \vec{\beta}_j \cup \vec{\beta}_k \cup \vec{\beta}_l$, $\vec{\beta}^T = \vec{\beta}_i^T \cup \vec{\beta}_j \cup \vec{\beta}_k^T \cup \vec{\beta}_l^T$ and $\mathfrak{X}$ be the data associated to n distinct points on $\mathbb{P}^1$ with chosen coordinates. Then we have the following map between conformal blocks:

$$\alpha : \mathcal{V}_{\vec{\lambda}\cup\vec{\beta}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda}^T\cup\vec{\beta}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\omega}_1\cup\vec{0}}(\mathfrak{X}, \mathfrak{so}(N), 1),$$

where $\vec{\omega}_1 = (\omega_1, \ldots, \omega_1)$ is an $(n_1+n_2+n_3+n_4)$ tuple of ω_1's and $\vec{0} = (0, \ldots, 0)$ be an $(m_1+m_2+m_3+m_4)$ tuple of 0's.

Assume that $(n_1+n_2+n_3+n_4)$ is even, then $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\omega}_1\cup\vec{0}}(\mathfrak{X}, \mathfrak{so}(N), 1) = 1$. Thus, we have the following map:

$$\alpha^{\vee} : \mathcal{V}_{\vec{\lambda}\cup\vec{\beta}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \to \mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\vec{\beta}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \tag{6.1}$$

This map $\alpha^{\vee}$ is called the rank-level duality map. The main result of this thesis is the following:

Theorem 6.4.1. *The rank-level duality map defined above is an isomorphism.*

The rest of the dissertation is devoted to the proof of Theorem 6.4.1. First we observe that by Proposition 4.1.2 and Proposition 3.3.2, we can reduce the statement of Theorem 6.4.1 into the following non-equivalent statements. We will use these to check the equality of dimensions of the source and the target of the following rank-level duality maps.

(1) Let $\sum_{i=1}^{n}|\lambda_i|$ is even.

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \to \mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1).$$

(2) Let $\sum_{i=1}^{n}|\lambda_i|$ is odd.

$$\mathcal{V}_{\vec{\lambda},0}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \to \mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1).$$

(3) Let $\sum_{i=1}^{n}|\lambda_i|$ is even.

$$\mathcal{V}_{\vec{\lambda},\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1, \vec{z}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1).$$

REMARK 6.4.2. *The decomposition of the highest weight integrable module $\mathcal{H}_{\omega_d}$ of $\widehat{\mathfrak{so}}(N)$ is given in* [**17**]. *Furthermore, the decomposition of all level one highest weight integrable modules for the conformal pairs (B_r, D_s) and (D_r, D_s) are given in* [**17**]. *In all of the above cases, the rank-level duality map is not well defined.*

Verlinde formula and equality of dimensions

In this chapter, we give a complete proof of the equality of dimensions (see Section 7.3) of the source and the target of the rank-level duality maps discussed in Chapter 6. Our key tool is the Verlinde formula for the dimensions of the conformal blocks. Another key ingredient in comparing the traces of representations that arise out of the Verlinde formula is a generalization of a lemma from [**10**].

7.1. Dimensions of some conformal blocks

In this section, we calculate the dimensions of some conformal blocks which we use later in the proof of the rank-level duality. Let $\mathfrak{g}$ be any simple Lie algebra and $\mathfrak{s}_\theta$ denote the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ generated by H_θ, $\mathfrak{g}_\theta$ and $\mathfrak{g}_{-\theta}$. A $\mathfrak{g}$-module V of level ℓ decomposes as a direct sum of $\mathfrak{s}_\theta$-modules as follows:

$$V \simeq \oplus_{i=1}^{\ell} V^i,$$

where V^i is a direct sum of $\mathfrak{sl}_2$ modules isomorphic to $\operatorname{Sym}^i \mathbb{C}^2$. We recall the following description of conformal blocks on three pointed $\mathbb{P}^1$ from [**5**].

Proposition 7.1.1. *Let $\mathfrak{X}$ be the data associated to the three pointed $\mathbb{P}^1$ with chosen coordinates and $\lambda, \mu, \nu \in P_\ell(\mathfrak{g})$. Then the conformal block $\mathcal{V}^\dagger_{\lambda,\mu,\nu}(\mathfrak{X}, \mathfrak{g})$ is canonically isomorphic to the space of $\mathfrak{g}$-invariant forms ϕ on $V_\lambda \otimes V_\mu \otimes V_\nu$ such that ϕ restricted to $V^p_\lambda \otimes V^q_\mu \otimes V^r_\nu$ is zero when ever $p + q + r > 2\ell$.*

7.1.1. The case $\mathfrak{g} = \mathfrak{so}(2r+1)$ with level 1. Let $\vec{p} = (P_1, P_2, P_3)$ be three distinct points on $\mathbb{P}^1$ with chosen coordinates and $\mathfrak{X}$ be the associated data. The level one dominant integral weights of $\mathfrak{so}(2r+1)$ are 0, ω_1 and ω_r. Let $\mathcal{V}^\dagger_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{X}, \mathfrak{so}(2r+1))$ denote the conformal blocks on $\mathbb{P}^1$ with three marked points and weights $\lambda_1, \lambda_2, \lambda_3$ at level one. The following are proved in [**13**]:

- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_1,0}(\mathfrak{X}, \mathfrak{so}(2r+1), 1) = 1.$
- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_1,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), 1) = 0.$
- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_1,\omega_r}(\mathfrak{X}, \mathfrak{so}(2r+1), 1) = 0.$
- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_r,\omega_r}(\mathfrak{X}, \mathfrak{so}(2r+1), 1) = 1.$

LEMMA 7.1.2. *Let $P_1, \dots, P_n$ be n distinct points on $\mathbb{P}^1$ with chosen coordinates and $\mathfrak{X}$ be the associated data. Assume that $\vec{\lambda} = (\omega_1, \dots, \omega_1)$. Then,* $\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 1) =$ *1, if n is even, and zero if n is odd.*

PROOF. The proof follows from above and factorization of conformal blocks. □

7.1.2. The case $\mathfrak{g} = \mathfrak{so}(2r+1)$ at level ℓ. We calculate the dimensions of some special conformal blocks on three pointed $\mathbb{P}^1$ at any level ℓ. We first recall the following tensor product decomposition from [**21**]:

PROPOSITION 7.1.3. *Let $\lambda = \sum_{i=1}^{r} a_i\omega_i \in P_+^0$. Then,*

$$V_\lambda \otimes V_{\omega_1} \simeq \oplus_\gamma V_\gamma,$$

where γ is either λ if $a_r \neq 0$ or is obtained from λ by adding or deleting a box from the Young diagram of λ.

We use the above proposition to calculate the dimensions of the following conformal blocks.

PROPOSITION 7.1.4. *Let $\lambda = \sum_{i=1}^{r} a_i\omega_i \in P_+^0$. Assume that $\lambda \in P_\ell(\mathfrak{so}(2r+1))$. Then the dimension of the conformal block $\mathcal{V}^{\dagger}_{\lambda,\gamma,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$ of level ℓ is one, where γ is either λ if $a_r \neq 0$ or is obtained from λ by adding or deleting a box of the the Young diagram of λ, and 0 otherwise.*

PROOF. The otherwise part follows from Proposition 7.1.3. Assume that $a_r \neq 0$ and γ is either λ or obtained from λ by adding or deleting a box. For a $\mathfrak{so}(2r+1)$-equivariant form ϕ on $V_\lambda \otimes V_{\omega_1} \otimes V_\gamma$, it's restriction to $V^1_{\omega_1} \otimes V^{\ell}_{\lambda} \otimes V^{\ell}_{\gamma}$ is zero, since $\mathbb{C}^2 \otimes \mathrm{Sym}^{\ell}\,\mathbb{C}^2$ does not contain $\mathrm{Sym}^{\ell}\,\mathbb{C}^2$ as an $\mathfrak{sl}_2(\mathbb{C})$-submodule. Thus by Proposition 7.1.1, the dimension of $\mathcal{V}^{\dagger}_{\lambda,\gamma,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1), \ell)$ is one. The case when $a_r = 0$ follows similarly.

□

7.2. Verlinde formula

In this section, we recall the Verlinde formula that calculates dimensions of conformal blocks. First we start with the Weyl character formula.

7.2.1. Weyl character formula. Here, we first state a basic matrix identity which is an easy generalization of Lemma A.42 from [**10**]. Suppose $A = (a_{ij})$ is an $(r+s)\times(r+s)$ matrix and $U = (u_1, \dots, u_r)$ and $T = (t_1, \dots, t_r)$ are two sequences of r distinct integers from $\{1, 2, \dots, (r+s)\}$. Let $A_{U,T}$ denote the $(r\times r)$ matrix whose (i,j)-th entry is a_{u_i,t_j} Similarly define the $(s \times s)$ matrix B_{T^c,U^c}, where U^c and T^c are the complements of U and T respectively.

LEMMA 7.2.1. *Let A and B be two $(r+s)\times(r+s)$ matrices whose product is a diagonal matrix D. Suppose the (i,i)-th entry of D is a_i. Let $\pi = (U, U^c)$ and (T, T^c) be permutations of the sequence $(1, \dots, r+s)$, where $|U| = |T| = r$. Then the following identity of determinants holds:*

$$\big(a_{\pi(r+1)} \dots a_{\pi(r+s)}\big) \det A_{U,T} = \operatorname{sgn}(U, U^c) \operatorname{sgn}(T, T^c) \det A \det B_{T^c,U^c}.$$

PROOF. Consider the permutation matrices P, Q^{-1} associated to the permutation (U, U^c) and (T, T^c) respectively. Then,

$$PAQ = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \text{ where } A_{U,T} = A_1$$

and similarly

$$Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \text{ where } B_{T^c,U^c} = B_4.$$

Now

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \times \begin{pmatrix} I_k & B_2 \\ 0 & B_4 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_3 & \Lambda. \end{pmatrix}$$

where Λ is a diagonal matrix whose (i,i)-th entry is $a_{\pi(r+i)}$. Taking determinant of both sides of the above matrix equation we get the desired equality. $\square$

We are now ready to state the Weyl character formula for $\mathfrak{so}(2r+1)$ following [**10**]. Let $\mu \in P^0_{2s+1}(\mathfrak{so}(2r+1))$ and $\mu+\rho = \sum_{i=1}^r u_i L_i$, where u_i is as defined in Chapter 6. Let $\lambda = \sum_{i=1}^r \lambda^i L_i$ be any dominant integral weight of $\mathfrak{so}(2r+1)$ and V_λ be the irreducible highest weight module of $\mathfrak{so}(2r+1)$ with weight λ. Then by the Weyl character formula

$$\mathrm{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{(r+s)}) = \frac{\det\left(\zeta^{u_i(\lambda_j+r-j+\frac{1}{2})} - \zeta^{-u_i(\lambda_j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})} - \zeta^{-u_i(r-j+\frac{1}{2})}\right)},$$

where $\mu+\rho$ is considered as element of $\mathfrak{h}$ under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$, exp is the exponential map from $\mathfrak{so}(2r+1)$ to SO_{2r+1}, $\zeta = \exp\left(\frac{\pi\sqrt{-1}}{r+s}\right)$.

7.2.2. Verlinde Formula. Let us first recall the Verlinde formula in full generality. Let C be a nodal curve of genus g and $P_1,\dots,P_n$ be n distinct smooth points on C and $\mathfrak{X}$ be the associated data. We fix a Lie algebra $\mathfrak{g}$, and $\vec{\lambda} = (\lambda_1,\lambda_2,\dots,\lambda_n)$, an n tuple of dominant integral weights of $\mathfrak{g}$ of level ℓ. We refer the reader to [**5**], [**14**], [**36**] for a proof of the following:

Theorem 7.2.2. *The dimension of the conformal block $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g},\ell)$ is*

$$\{(\ell+g^*)^{\mathrm{rank}\,\mathfrak{g}}|P/Q_{long}|\}^{g-1}\sum_{\mu\in P_\ell(\mathfrak{g})}\mathrm{Tr}_{V_{\vec{\lambda}}}(\exp 2\pi\sqrt{-1}\frac{\mu+\rho}{\ell+g^*})\prod_{\alpha>0}\left|2\sin\pi\frac{(\mu+\rho,\alpha)}{\ell+g^*}\right|^{2-2g},$$

where exp *is the exponential map from $\mathfrak{g}$ to the simply connected Lie group G, Q_{long} is the lattice of long roots and g^* is the dual Coxeter number of $\mathfrak{g}$.*

Let us now specialize to the case $g=0$, $\mathfrak{g}=\mathfrak{so}(2r+1)$, $\ell = 2s+1$ and $\vec{\lambda}=(\lambda_1,\dots,\lambda_n)$ an n tuple of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$. The dual Coxeter number of $\mathfrak{so}(2r+1)$ is $2r-1$ and $\{(\ell+g^*)^{\mathrm{rank}\,\mathfrak{g}}|P/Q_{long}|\} = 4(k)^r$, where $k=2(r+s)$. Then, we can rewrite the Verlinde formula as follows:

$$\sum_{U\in P_{2s+1}(\mathfrak{so}(2r+1))}\prod_{q=1}^n\frac{\det\left(\zeta^{u_i(\lambda_q^j+r-j+\frac{1}{2})} - \zeta^{-u_i(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})} - \zeta^{-u_i(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U)}{4k^r}\right), \tag{7.1}$$

where $\mu+\rho=\sum_{i=1}^{r}u_iL_i$, the set $U=(u_1>u_2>\cdots>u_r)$, $\lambda_q=(\lambda_q^1,\lambda_q^2,\ldots,\lambda_q^r)$ and $\Phi_k(U)$ are as in Section 2 of [**28**]. We recall the definition of $\Phi_k(U)$ in Section 7.5 for completeness.

7.3. Equality of dimensions

LEMMA 7.3.1. *Let σ be the non-trivial element of the center of* $\mathrm{Spin}(2r+1)$*. The element σ acts by diagram automorphism on $P^0_{2s+1}(\mathfrak{so}(2r+1))$. Then,*

$$\mathrm{Tr}_{V_{\vec{\lambda}}}(\exp\pi\sqrt{-1}\frac{\sigma\mu+\rho}{r+s})=\mathrm{Tr}_{V_{\vec{\lambda}}}(\exp\pi\sqrt{-1}\frac{\mu+\rho}{r+s}),$$

where exp *is the exponential map form $\mathfrak{so}(2r+1)$ to the special orthogonal group* $\mathrm{SO}(2r+1)$.

PROOF. Let $\mu=\sum_{i=1}^{r}a_i\omega_i\in P_{2s+1}(\mathfrak{so}(2r+1))$. Then the weight $\sigma(\mu)$ is given by the formula $(2s+1-2(a_1+\cdots+a_r)+a_1+a_r)\omega_1+\sum_{i=2}^{r}a_i\omega_i$. We calculate the following weight:

$$\begin{aligned}\sigma(\mu)+\rho &= (2s+2-2(a_1+\cdots+a_r)+a_1+a_r)\omega_1+\sum_{i=2}^{r}(a_i+1)\omega_i,\\ &= ((2s+1)-(a_1+\cdots+a_{r-1})-\frac{a_r}{2}+\frac{2r-1}{2})L_1+\\ &\quad((a_2+a_3+\cdots+a_{r-1})+(r-2)+\frac{a_r+1}{2})L_2+\cdots+\frac{a_r+1}{2}L_r.\end{aligned}$$

Let w be an element of the Weyl group of $\mathfrak{so}(2r+1)$ which sends $L_1\to -L_1$. Then,

$$\begin{aligned}w.(\sigma\mu+\rho) &= (a_1+a_2+\cdots+\frac{a_r}{2}-(2s+1)-(2r-1)+r-\frac{1}{2})L_1+\\ &\quad((a_2+a_3+\cdots+a_{r-1})+(r-2)+\frac{a_r+1}{2})L_2+\cdots+\frac{a_r+1}{2}L_r,\\ &= \mu+\rho-2(r+s)L_1.\end{aligned}$$

Thus, we get the following identity:

$$\exp(2\pi\sqrt{-1}\frac{w.(\sigma\mu+\rho)}{2(r+s)}) \quad = \quad \exp(2\pi\sqrt{-1}\frac{\mu+\rho}{2(r+s)}).$$

$\square$

Remark 3. *We refer the reader to* [**6**] *for a general discussion of the action of the center of the simply connected group G on $P_\ell(\mathfrak{g})$. The action of the center for all classical Lie algebras is also given in* [**28**].

Consider $\mu \in P^+_{2s+1}(\mathfrak{so}(2r+1))$ and $\mu' \in P^0_{2s+1}(\mathfrak{so}(2r+1))$. Let $\mu+\rho = \sum_{i=1}^r u_i L_i$ and $\mu'+\rho' = \sum_{i=1}^r (u'_i - \frac{1}{2})L_i$. Consider the sets $U = (u_1 > u_2 > \cdots > u_r)$, and $U' = (u'_1 > \cdots > u'_r)$ and let $[U]$ and $[U']$ denote the class of μ, μ' in $P^+_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and $P^0_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ respectively.

Without loss of generality, we can assume that $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$ has non-trivial invariants since this is a necessary condition for the conformal block to be non-zero. Since the function Φ_k is invariant under the action of center, by Lemma 7.3.1 we can rewrite the Verlinde formula in 7.1 as the sum of the following terms:

(1)

$$\sum_{[U]\in P^+_{2s+1}(\mathfrak{so}(2r+1))/\Gamma} |\operatorname{Orb}_U| \prod_{q=1}^{n} \frac{\det\left(\zeta^{u_i(\lambda_q^j+r-j+\frac{1}{2})} - \zeta^{-u_i(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})} - \zeta^{-u_i(r-j+\frac{1}{2})}\right)} \left(\frac{\Phi_k(U)}{4k^r}\right),$$

(2)

$$\sum_{[U']\in P^0_{2s+1}(\mathfrak{so}(2r+1))/\Gamma} |\operatorname{Orb}_{U'}| \prod_{q=1}^{n} \frac{\det\left(\zeta^{(u'_i-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})} - \zeta^{-(u'_j-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u'_i-\frac{1}{2})(r-j+\frac{1}{2})} - \zeta^{-(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}\right)} \Big(\frac{\Phi_k(U'-\frac{1}{2})}{4k^r}),$$

where $|\operatorname{Orb}_U|$, $|\operatorname{Orb}_{U'}|$ denote the length of the orbits of μ and μ' under the action of Γ on $P^+_{2s+1}(\mathfrak{so}(2r+1))$ and $P^0_{2s+1}(\mathfrak{so}(2r+1))$. The sets $P^+_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and $P^0_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ denote the orbits of $P^+_{2s+1}(\mathfrak{so}(2r+1))$ and $P^0_{2s+1}(\mathfrak{so}(2r+1))$ under the action of Γ respectively.

7.3.1. Final Step of Dimension check. Let us recall the following two lemmas from [**28**]. We refer the reader to [**28**], Corollary 1.7 and Corollary 1.8 for a proof.

LEMMA 7.3.2. *For a positive integer a, let V and V^c be complementary subsets of $\{1,\dots,a-1\}$. Then,*

$$\frac{(2a)^{|V|}}{\Phi_{2a}(V)}=\frac{2(2a)^{|V^c\cup\{a\}|}}{\Phi_{2a}(V^c\cup\{a\})}.$$

LEMMA 7.3.3. *Let $V'\subset S=\{\frac{1}{2},\dots a-\frac{1}{2}\}$ and V'^c be the complement. Then, we have:*

$$\frac{(2a)^{|V'|}}{\Phi_{2a}(V')}=\frac{(2a)^{V'^c}}{\Phi_{2a}(a-V'^c)}.$$

Let $\lambda_i\in\mathcal{Y}_{r,s}$ such that $\sum_{i=1}^n|\lambda_i|$ is even and $\mathfrak{X}$ be the data associated to n distinct points on $\mathbb{P}^1$ with chosen coordinates. Denote by $\vec{\lambda}$, an n tuple of weights $(\lambda_1,\dots,\lambda_n)$ and $\vec{\lambda}^T$ the n tuple of weights $(\lambda_1^T,\dots,\lambda_n^T)$. Consider the conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$.

PROPOSITION 7.3.4. *If $\sum_{i=1}^n|\lambda|$ is even, then the following equality of dimensions holds:*

$$\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)=\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1).$$

PROOF. By Lemma 6.2.2, it is enough to show that the following equalities hold

$$\begin{aligned}
&|\operatorname{Orb}_U|\prod_{q=1}^n\frac{\det\left(\zeta^{u_i(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-u_i(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})}-\zeta^{-u_i(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U)}{4k^r}\right)\\
&=|\operatorname{Orb}_{U^c}|\prod_{q=1}^n\frac{\det\left(\zeta^{u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i^c(r-j+\frac{1}{2})}-\zeta^{-u_i^c(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U^c)}{4k^s}\right),
\end{aligned}$$

where $U = \{u_1 > \cdots > u_r)\} \in P^{+}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and $r+s \in U$ and u_i^c is same as in Section 6.2.

$$\begin{aligned}
&|\operatorname{Orb}_{U'}|\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u'_i-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-(u'_i-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U'-\frac{1}{2})}{4k^r}\right)\\
=\ &|\operatorname{Orb}_{((r+s+1)-U'^c)}|\times\\
&\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u''_i-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-(u''_i-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u''_i-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u''_i-\frac{1}{2})(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k((r+s+\frac{1}{2})-U'^c)}{4k^s}\right),
\end{aligned}$$

where $U' = \{u'_1 > u'_2 > \cdots > u'_r\} \in P^0_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and u''_i is same as defined in Section 6.2. Now by Lemma 7.3.2 and Lemma 7.3.3 we know that

$$|\operatorname{Orb}_U|\left(\frac{\Phi_k(U)}{4k^r}\right) = |\operatorname{Orb}_{U^c}|\left(\frac{\Phi_k(U^c)}{4k^s}\right).$$

$$\left(\frac{\Phi_k(U'-\frac{1}{2})}{4k^r}\right) = \left(\frac{\Phi_k((r+s+\frac{1}{2})-U'^c)}{4k^s}\right).$$

We are reduced to showing the following identity of determinants for the pair (U, U^c).

$$\prod_{q=1}^{n}\frac{\det\left(\zeta^{u_i(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-u_i(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})}-\zeta^{-u_i(r-j+\frac{1}{2})}\right)} = \prod_{q=1}^{n}\frac{\det\left(\zeta^{u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i^c(r-j+\frac{1}{2})}-\zeta^{-u_i^c(r-j+\frac{1}{2})}\right)}.$$

This follows directly from Lemma 7.4.2. We also need to show the following equality of determinants for the pair (U', U'^c), where $\lambda_q^T = ((\lambda_q^T)^1 \geq \cdots \geq (\lambda_q^T)^s)$.

$$\begin{aligned}
&\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u'_i-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-(u'_i-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}\right)}\\
&=\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u''_i-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-(u''_i-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u''_i-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u''_i-\frac{1}{2})(r-j+\frac{1}{2})}\right)},
\end{aligned}$$

This also follows from Lemma 7.4.5. □

With the same notation and assumptions as in Proposition 7.3.4, we have the following proposition.

Proposition 7.3.5. *If $\sum_{i=1}^{n}|\lambda|$ is even, then*

$$\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}\cup\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) = \dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1).$$

Proof. The proof follows from the proof of Proposition 7.3.4, Lemma 7.4.6 and Lemma 7.4.7 $\square$

For each $1 \leq i \leq n$, let $\lambda_i \in \mathcal{Y}_{r,s}$ be such that $\sum_{i=1}^{n}|\lambda_i|$ is odd. Let $\mathfrak{X}$ be the data associated to the n distinct points on $\mathbb{P}^1$ with chosen coordinates. Denote by $\vec{\lambda}$ the n tuple of weights $(\lambda_1, \dots, \lambda_n)$ and $\vec{\lambda}^T$ the n tuple of weights $(\lambda_1^T, \dots, \lambda_n^T)$. Consider the conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}\cup 0}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1)$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1)$. Then, we have the following equality of dimensions:

Proposition 7.3.6.

$$\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}\cup 0}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) = \dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\sigma(0)}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1).$$

Remark 7.3.7. *These equalities of the dimensions of the conformal blocks give rise to some new interesting relations between the fusion ring ([**34**]) of* $\mathrm{SO}(2r+1)$ *at level* $2s+1$ *with the fusion ring of* $\mathrm{SO}(2s+1)$ *at level* $2r+1$.

7.4. Key lemmas

Lemma 7.4.1. *Let $\xi = \exp(\frac{\pi\sqrt{-1}}{2(r+s)})$. Consider the matrix W whose (i,j)-th entry is the complex number $(\xi^{i(2j-1)} - \xi^{-i(2j-1)})$. Then,*

$$WW^T = \begin{pmatrix} c & & & \\ & \ddots & & \\ & & c & \\ & & & 2c \end{pmatrix},$$

where $c = -2(r+s)$.

Let U be a partition of $\{1, \ldots, r+s\}$ such that $r+s \in U$ and $|U| = r$. Let P be the permutation matrix associated to the permutation (U, U^c).

$$PWW^TP^{-1} = \begin{pmatrix} 2c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{pmatrix}.$$

Let $A = W$ and $B = W^T$ and U, T as in Lemma 7.2.1. Then, we have the following:

$$c^s \det A_{U,T} = \operatorname{sgn}(U, U^c) \operatorname{sgn}(T, T^c) \det A \det B_{T^c, U^c}. \tag{7.1}$$

Let $[r+s]$ denote the set $\{1, 2, \ldots, r+s\}$. We define the following sets:

(1) Consider $\lambda = (\lambda^1 \geq \lambda^2 \geq \cdots \geq \lambda^r) \in \mathcal{Y}_{r,s}$. We define $\alpha^i = \lambda^i + r + 1 - i$ and $[\alpha] = \{\alpha^1 > \alpha^2 > \cdots > \alpha^r\}$.

(2) Consider the complement $[\beta] = (\beta^1 > \beta^2 > \cdots > \beta^s)$ of $[\alpha]$ in $[r+s]$. We define another set $[\gamma] = (\gamma^1 > \gamma^2 > \cdots > \gamma^s)$ where $\gamma^i = ((r+s) - (\beta^{(s+1-i)} - \frac{1}{2}))$.

(3) Let $T = (t_1 > t_2 > \cdots > t_r)$ where $t_i = r+1-i$; $T' = (t'_1 > t'_2 > \cdots > t'_s)$ where $t'_i = s+1-i$ and $T^c = (t^c_1 > t^c_2 > \cdots > t^c_s)$ is the complement of T in $[r+s]$.

(4) $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r such that $r+s \in U$ and $U^c = (u^c_1 > u^c_2 > \cdots > u^c_s)$ be the complement of U in $[r+s]$.

Then, for $\lambda \in \mathcal{Y}_{r,s}$, we can write

$$\begin{aligned}
\operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) &= \frac{\det(\zeta^{u_i(\alpha^j - \frac{1}{2})} - \zeta^{-u_i(\alpha^j - \frac{1}{2})})}{\det(\zeta^{u_i(t_j - \frac{1}{2})} - \zeta^{-u_i(t_j - \frac{1}{2})})}, \\
&= \frac{\det(\xi^{u_i(2\alpha^j - 1)} - \xi^{-u_i(2\alpha^j - 1)})}{\det(\xi^{u_i(2t_j - 1)} - \xi^{-u_i(2t_j - 1)})},
\end{aligned}$$

where $\mu + \rho = \sum_{i=1}^r u_i L_i$.

For $\lambda^T \in \mathcal{Y}_{r,s}$, $\mu' + \rho' = \sum_{i=1}^{s} u_i^c L_i$ and ρ' the Weyl vector of $\mathfrak{so}(2s+1)$, we can write

$$\begin{aligned}
\mathrm{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}) &= \frac{\det(\zeta^{u_i^c(\gamma^j)} - \zeta^{-u_i^c(\gamma^j)})}{\det(\zeta^{u_i^c(t'_j-\frac{1}{2})} - \zeta^{-u_i^c(t'_j-\frac{1}{2})})}, \\
&= \frac{\det(\zeta^{u_i^c((r+s)-(\beta^j-\frac{1}{2}))} - \zeta^{-u_i^c((r+s)-(\beta^j-\frac{1}{2}))})}{\det(\zeta^{u_i^c((r+s)-(t_j^c-\frac{1}{2}))} - \zeta^{-u_i^c((r+s)-(t_j^c-\frac{1}{2}))})}, \\
&= \frac{\det(\zeta^{-u_i^c(r+s)}(\zeta^{u_i^c(\beta^j-\frac{1}{2})} - \zeta^{-u_i^c(\beta^j-\frac{1}{2})}))}{\det(\zeta^{-u_i^c(r+s)}(\zeta^{u_i^c(t_j^c-\frac{1}{2})} - \zeta^{-u_i^c(t_j^c-\frac{1}{2})}))}, \\
&= \frac{\det(\zeta^{u_i^c(\beta^j-\frac{1}{2})} - \zeta^{-u_i^c(\beta^j-\frac{1}{2})})}{\det(\zeta^{u_i^c(t_j^c-\frac{1}{2})} - \zeta^{-u_i^c(t_j^c-\frac{1}{2})})}, \\
&= \frac{\det(\xi^{u_i^c(2\beta^j-1)} - \xi^{-u_i^c(2\beta^j-1)})}{\det(\xi^{u_i^c(2t_j^c-1)} - \xi^{-u_i^c(2t_j^c-1)})}.
\end{aligned}$$

By applying Equation 7.1, we get the following:

Lemma 7.4.2.

$$\mathrm{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) = \frac{\mathrm{sgn}([\alpha],[\beta])}{\mathrm{sgn}(T,T^c)} \mathrm{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}).$$

The following can be checked by a direct calculation:

Lemma 7.4.3.

$$\mathrm{sgn}([\alpha],[\beta]) = (-1)^{\frac{r(r-1)}{2}+\frac{s(s-1)}{2}+|\lambda|}.$$

$$\mathrm{sgn}(T,T^c) = (-1)^{\frac{r(r-1)}{2}+\frac{s(s-1)}{2}}.$$

Thus we have the following equality:

$$\mathrm{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) = (-1)^{|\lambda|} \mathrm{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}).$$

Let $\xi = \exp \frac{\pi\sqrt{-1}}{4(r+s)}$. Then the following equality holds for any integers a and b:

$$\xi^{(2(r+s)-(2a-1))(2(r+s)-(2b-1))} = (-1)^{(a+b)}\xi^{(2a-1)(2b-1)}.$$

LEMMA 7.4.4. *Let $\xi = \exp(\frac{\pi\sqrt{-1}}{4(r+s)})$. Consider the matrix W whose (i,j)-th entry is the complex number $(\xi^{(2i-1)(2j-1)} - \xi^{-(2i-1)(2j-1)})$. Then the following holds:*

$$WW^T = \begin{pmatrix} c & & & \\ & \ddots & & \\ & & c & \\ & & & c \end{pmatrix},$$

where $c = -2(r+s)$.

Let U be a partition of $\{1, \ldots, r+s\}$ such that $|U| = r$. Let $A = W$, $B = W^T$ and U, T as in Lemma 7.2.1. Then,

$$c^s \det A_{U,T} = \operatorname{sgn}(U, U^c) \operatorname{sgn}(T, T^c) \det A \det B_{T^c, U^c}. \tag{7.2}$$

Let $U' = (u'_1 > u'_2 > \cdots > u'_r)$ be a subset of $[r+s]$ of cardinality r, $U'^c = (u'^c_1 > \cdots > u'^c_s)$ be the complement of U' in $[r+s]$ and $\mu + \rho = \sum_{i=1}^r (u'_i - \frac{1}{2}) L_i$. Then, for $\lambda \in \mathcal{Y}_{r,s}$, we can write

$$\begin{aligned} \operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) &= \frac{\det(\zeta^{(u'_i - \frac{1}{2})(\alpha^j - \frac{1}{2})} - \zeta^{-(u'_i - \frac{1}{2})(\alpha^j - \frac{1}{2})})}{\det(\zeta^{(u'_i - \frac{1}{2})(t_j - \frac{1}{2})} - \zeta^{-(u'_i - \frac{1}{2})(t_j - \frac{1}{2})})}, \\ &= \frac{\det(\xi^{(2u'_i - 1)(2\alpha^j - 1)} - \xi^{-(2u'_i - 1)(2\alpha^j - 1)})}{\det(\xi^{(2u'_i - 1)(2t_j - 1)} - \xi^{-(2u'_i - 1)(2t_j - 1)})}. \end{aligned}$$

For $\lambda^T \in \mathcal{Y}_{r,s}$, $\mu' + \rho' = \sum_{i=1}^s ((r+s+\frac{1}{2}) - u'^c_i) L_i$ and ρ' be the Weyl vector of $\mathfrak{so}(2s+1)$, we can write the following:

$$\begin{aligned} \operatorname{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}) &= \frac{\det(\zeta^{((r+s)-(u'^c_i - \frac{1}{2}))((r+s)-(\beta^j - \frac{1}{2}))} - \zeta^{((r+s)-(u'^c_i - \frac{1}{2}))((r+s)-(\beta^j - \frac{1}{2}))})}{\det(\zeta^{((r+s)-(u'^c_i - \frac{1}{2}))((r+s)-(t^c_j - \frac{1}{2}))} - \zeta^{((r+s)-(u'^c_i - \frac{1}{2}))((r+s)-(t^c_j - \frac{1}{2}))})}, \\ &= \frac{(-1)^{\sum_{i=1}^s (u'^c_i + \beta_i)}}{(-1)^{\sum_{i=1}^s (u'^c_i + t^c_i)}} \frac{\det(\xi^{(2u'^c_i - 1)(2\beta^j - 1)} - \xi^{-(2u'^c_i - 1)(2\beta^j - 1)})}{\det(\xi^{(2u'^c_i - 1)(2t^c_j - 1)} - \xi^{-(2u'^c_i - 1)(2t^c_j - 1)})}, \\ &= \frac{(-1)^{\sum_{i=1}^s (\beta_i)}}{(-1)^{\sum_{i=1}^s (t^c_i)}} \frac{\det(\xi^{(2u'^c_i - 1)(2\beta^j - 1)} - \xi^{-(2u'^c_i - 1)(2\beta^j - 1)})}{\det(\xi^{(2u'^c_i - 1)(2t^c_j - 1)} - \xi^{-(2u'^c_i - 1)(2t^c_j - 1)})}, \\ &= (-1)^{|\lambda|} \frac{\det(\xi^{(2u'^c_i - 1)(2\beta^j - 1)} - \xi^{-(2u'^c_i - 1)(2\beta^j - 1)})}{\det(\xi^{(2u'^c_i - 1)(2t^c_j - 1)} - \xi^{-(2u'^c_i - 1)(2t^c_j - 1)})}. \end{aligned}$$

From Equation 7.2, we get the following lemma:

LEMMA 7.4.5.

$$\mathrm{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) = \mathrm{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}).$$

7.4.1. Some Trace calculations. Let $\zeta = \exp(\frac{\pi\sqrt{-1}}{r+s})$ and $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r. Then, we have the following

$$\begin{aligned}
\zeta^{(u_i-\frac{1}{2})((2r+1)+s-\frac{1}{2})} - \zeta^{-(u_i-\frac{1}{2})((2r+1)+s-\frac{1}{2})} &= \zeta^{(u_i-\frac{1}{2})(2(r+s)-(s-\frac{1}{2}))} - \zeta^{-(u_i-\frac{1}{2})(2(r+s)-(s-\frac{1}{2}))}, \\
&= -\big(\zeta^{-(u_i-\frac{1}{2})(s-\frac{1}{2})} - \zeta^{(u_i-\frac{1}{2})(s-\frac{1}{2})}\big), \\
&= \zeta^{(u_i-\frac{1}{2})(s-\frac{1}{2})} - \zeta^{-(u_i-\frac{1}{2})(s-\frac{1}{2})}.
\end{aligned}$$

The above calculation and the Weyl character formula gives us the following lemma.

LEMMA 7.4.6. *Consider the dominant weight $\lambda = (2s+1)\omega_1$ of $\mathfrak{so}(2r+1)$ of level $2s+1$. Let $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r and $\mu+\rho = \sum_{i=1}^{r}(u_i - \frac{1}{2})L_i$. Then,*

$$\mathrm{Tr}_\lambda(\exp(\pi\sqrt{-1}\frac{\mu+\rho}{r+s})) = 1.$$

Let $\zeta = \exp(\frac{\pi\sqrt{-1}}{r+s})$ and $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r. Then, we have the following:

$$\begin{aligned}
\zeta^{u_i((2r+1)+s-\frac{1}{2})} - \zeta^{-u_i((2r+1)+s-\frac{1}{2})} &= \zeta^{u_i(2(r+s)-(s-\frac{1}{2}))} - \zeta^{-u_i(2(r+s)-(s-\frac{1}{2}))}, \\
&= \zeta^{-u_i(s-\frac{1}{2})} - \zeta^{u_i(s-\frac{1}{2})}, \\
&= -\big(\zeta^{u_i(s-\frac{1}{2})} - \zeta^{-u_i(s-\frac{1}{2})}\big).
\end{aligned}$$

The proof of the next lemma also follows from the above calculation and the Weyl character formula gives us the following lemma.

LEMMA 7.4.7. *Consider the dominant weight $\lambda = (2s+1)\omega_1$ of $\mathfrak{so}(2r+1)$ of level $2s+1$. Let $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r and*

$\mu+\rho=\sum_{i=1}^{r} u_i L_i$. *Then,*

$$\mathrm{Tr}_\lambda(\exp(\pi\sqrt{-1}\frac{\mu+\rho}{r+s}))=-1.$$

7.5. Some trigonometric functions

We recall from [**28**] a family of trigonometric functions which has surprising identities. These identities are fundamental to the reciprocity laws of the Verlinde formula in [**28**].

Consider a positive integer k and let

$$f_k(r)=4\sin^2(\frac{r\pi}{k}).$$

Given a finite set $U=\{u_1,\ldots,u_r\}$ of rational numbers, we consider the following functions defined in Section 1 of [**28**] (where an empty product is deemed to be 1):

$$\begin{aligned}
\mathcal{P}_k(U) &= \prod_{1\le i<j\le r}\Big(f_k(u_i-u_j)f_k(u_i+u_j)\Big),\\
\mathcal{N}_k(U) &= \prod_{i=1}^{r} f_k(u_i),\\
\Phi_k(U) &= \mathcal{P}_k(U)\mathcal{N}_k(U).
\end{aligned}$$

We use the function $\Phi_k(U)$ to rewrite the Verlinde formula. The identities of $\Phi_k(U)$ are among the key ingredients in the proof of the equality of the dimensions as discussed in Section 7.3.

Highest Weight Vectors

In this chapter, we briefly summarize the construction of the highest weight integrable modules $\mathcal{H}_0(\mathfrak{so}(2r+1))$ and $\mathcal{H}_1(\mathfrak{so}(2r+1))$. We use this to explicitly describe the highest weight vectors (see Section 8.2) of the components that appear in the branching. Our discussions closely follow the discussions in [**17**].

8.1. Spin modules

We first recall the definition of the Clifford algebra. Let W be a vector space (not necessarily finite dimensional) with a non-degenerate bilinear form $\{,\}$.

DEFINITION 8.1.1. *We define the* Clifford Algebra *associated to W and $\{,\}$ to be*

$$C(W) := T(W)/I,$$

where $T(W)$ is the tensor algebra of W and I is the two sided ideal generated by elements of the form $v \otimes w + w \otimes v - \{v, w\}$.

8.1.1. Spin Module of $C(W)$. Suppose there exists an isotropic decomposition $W = W^+ \oplus W^-$, i.e. $\{W^\pm, W^\pm\} = 0$ and $\{,\}$ restricted to $W^+ \oplus W^-$ is non-degenerate. Then the exterior algebra $\bigwedge W^-$ can be viewed as a $\bigwedge W^-$-module by taking wedge product on the left. This gives rise to the structure of an irreducible $C(W)$-module on $\bigwedge W^-$ by defining

$$w^+.1 = 0,$$

for all $w^+ \in W^+$ and $1 \in \bigwedge W^-$.

Next if $W = W' \oplus \mathbb{C}e$ is an orthogonal direct sum with $\{e, e\} = 1$ and W' has an isotropic decomposition of the form $W^+ \oplus W^-$(we refer this as quasi-isotropic decomposition of W). Then the $C(W')$-module $\bigwedge W^-$ described above becomes an irreducible $C(W)$-module by defining

$$\sqrt{2}e.v := \pm(-1)^p v \quad \text{for} \quad v \in \bigwedge^p W^-.$$

Any element of W^-(respectively W^+) is called a creation operator (respectively annihilation operator).

8.1.2. Root Spaces and basis of $\mathfrak{so}(2r+1)$. Consider a finite dimensional vector space W_r of dimension $2r+1$ with a non-degenerate symmetric bilinear form $\{,\}$. Let $\{e_i\}_{i=-r}^{r}$ be an orthonormal basis of W_r. For $j>0$, we set

$$\phi^j = \frac{1}{\sqrt{2}}(e_j + \sqrt{-1}e_{-j});\quad \phi^{-j} = \frac{1}{\sqrt{2}}(e_j - \sqrt{-1}e_{-j}) \text{ and } \phi^0 = e_0.$$

Let $\phi^1,\dots,\phi^r,\phi^0,\phi^{-r},\dots,\phi^{-1}$ be the chosen ordered basis of W_r. For any i,j, we define $E^i_j(\phi^k) := \delta_{k,j}\phi^i$.

We identify the Lie algebra $\mathfrak{so}(2r+1)(W_r)$ with $\mathfrak{so}(2r+1)$ as follows:

$$\mathfrak{so}(2r+1) := \{A \in \mathfrak{sl}(2r+1) | A^T J + JA = 0\},$$

where J is the following $(2r+1)\times(2r+1)$ matrix:

$$J = \begin{array}{c c} & \begin{array}{ccccccc} 1 & \dots & r & 0 & -r & \dots & -1 \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ r \\ 0 \\ -r \\ \vdots \\ -1 \end{array} & \begin{pmatrix} & & & & & & 1 \\ & 0 & & & & 1 & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & 1 & & & & 0 & \\ 1 & & & & & & \end{pmatrix} \end{array}.$$

We put $B^i_j = E^i_j - E^{-j}_{-i}$ and take the Cartan subalgebra $\mathfrak{h}$ to be the subalgebra of diagonal matrices. Clearly, $\mathfrak{h} = \oplus_{j=1}^{r}\mathbb{C}B^j_j$. The corresponding dual basis of $\mathfrak{h}^*$ is L_j, where $L_j(B^k_k) = \delta_{j,k}$. The simple positive roots $\{\alpha_i\}_{i=1}^{r}$ of $\mathfrak{so}(2r+1)$ are given by $L_1 - L_2, \dots, L_{r-1} - L_r, L_r$. The root spaces of $\mathfrak{so}(2r+1)$ are of the form $\mathfrak{g}_{L_i \pm L_j} = \mathbb{C}B^i_{\mp j}$ and $\mathfrak{g}_{L_i} = \mathbb{C}B^i_0$.

REMARK 8.1.2. *The basis of the vector space W_r chosen here is different than the basis in* [**10**]. *In this section, we prefer this basis because the branching formulas that we describe in the next section become simpler.*

8.1.3. Spin module $\bigwedge W_r^{\mathbb{Z}+\frac{1}{2},-}$ of $\widehat{\mathfrak{so}}(2r+1)$. Consider as before W_r to be a $2r+1$ dimensional complex vector space with a non-degenerate symmetric bilinear form $\{,\}$. Let

$$W_r^{\pm} = \oplus_{i=1}^{r}\mathbb{C}\phi^{\pm}.$$

A quasi-isotropic decomposition of W_r given by the following:

$$W_r = W_r^{+} \oplus W_r^{-} \oplus \mathbb{C}\phi^{0}.$$

We define a new vector space $W_r^{\mathbb{Z}+\frac{1}{2}}$ with an inner product $\{,\}$ as follows:

$$W_r^{\mathbb{Z}+\frac{1}{2}} := W_r \otimes t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}] \ \text{ with } \ \{w_1(a), w_2(b)\} = \{w_1, w_2\}\delta_{a+b,0},$$

where $w_1, w_2 \in W_r$; $a, b \in \mathbb{Z}+\frac{1}{2}$ and $w_1(a) = w_1 \otimes t^a$. We choose a quasi-isotropic decomposition of $W_r^{\mathbb{Z}+\frac{1}{2}}$ given as follows:

$$W_r^{\mathbb{Z}+\frac{1}{2}} = W_r^{\mathbb{Z}+\frac{1}{2},+} \oplus W_r^{\mathbb{Z}+\frac{1}{2},-},$$

where $W_r^{\mathbb{Z}+\frac{1}{2},\pm} := W_r \otimes t^{\pm\frac{1}{2}}\mathbb{C}[t^{\pm 1}]$. We define the normal order ${}^{o}_{o}\quad{}^{o}_{o}$ for $w_1(a), w_2(b) \in W_r^{\mathbb{Z}+\frac{1}{2}}$ by the following formula:

$${}^{o}_{o}w_1(a)w_2(b){}^{o}_{o} = \begin{cases} -w_2(b)w_1(a) & \text{if } a > 0 > b, \\ \frac{1}{2}(w_1(a)w_2(b) - w_2(b)w_1(a)) & \text{if } a = b = 0, \\ w_1(a)w_2(b) & \text{otherwise.} \end{cases}$$

We now describe the action of $\widehat{\mathfrak{so}}(2r+1)$ on $\bigwedge W_r^{\mathbb{Z}+\frac{1}{2},-}$ and explicitly describe the level one $\widehat{\mathfrak{so}}(2r+1)$-modules $\mathcal{H}_0(\mathfrak{so}(2r+1))$ and $\mathcal{H}_1(\mathfrak{so}(2r+1))$. For a proof, we refer the reader to [**13**].

PROPOSITION 8.1.3. *The following map is a Lie algebra monomorphism:*

$$\begin{array}{rcl}\widehat{\mathfrak{so}}(2r+1) & \to & \mathrm{End}(\bigwedge W_r^{\mathbb{Z}+\frac{1}{2},-}),\\ B^i_j(m) & \to & \sum\limits_{a+b=m} {}^0_0\phi^i(a)\phi^{-j}(b)^0_0,\\ c & \to & \mathrm{id}\,.\end{array}$$

PROPOSITION 8.1.4. *Suppose $r \geq 1$, then the following are isomorphic as $\widehat{\mathfrak{so}}(2r+1)$-modules:*

(1) $\bigwedge^{even}(W_r^{\mathbb{Z}+\frac{1}{2},-}) \simeq \mathcal{H}_0(\mathfrak{so}(2r+1))$,
(2) $\bigwedge^{odd}(W_r^{\mathbb{Z}+\frac{1}{2},-}) \simeq \mathcal{H}_1(\mathfrak{so}(2r+1))$.

The highest weight vectors are given by 1 *and* $\phi^1(-\frac{1}{2}).1$ *respectively.*

8.2. Highest weight vectors

Let W_s be a $2s+1$ dimensional vector space over $\mathbb{C}$ with a non-degenerate bilinear form $\{,\}$, and let $\{e_p\}_{p=1}^s$ be an orthonormal basis of W_s. Let $\phi^1, \dots, \phi^s, \phi^0, \phi^{-s}, \dots, \phi^{-1}$ be an ordered isotropic basis of W_s. The tensor product of $W_d = W_r \otimes W_s$ carries a non-degenerate symmetric bilinear form $\{,\}$ given by the product of the forms on W_r and W_s. Clearly the elements $e_{j,p} := e_j \otimes e_p$ for $-r \leq j, \leq r,\ -s \leq p \leq s\}$ form an orthonormal basis of W_d. By $(j,p) > 0$, we mean $j > 0$ or $j = 0, p > 0$ and put

$$\phi^{j,p} = \frac{1}{\sqrt{2}}(e_{j,p} - \sqrt{-1}e_{-j,-p}); \quad \phi^{-j,-p} = \frac{1}{\sqrt{2}}(e_{j,p} + \sqrt{-1}e_{-j,-p})$$

for $(j,p) > 0$. The form $\{,\}$ on W_d is given by the formula

$$\{\phi^{j,p}, \phi^{-k,-q}\} = \delta_{j,k}\delta_{p,q}, \quad \text{for} \quad -r \leq j,k \leq r; -s \leq p,q \leq s.$$

Let as before $W_d^{\pm} = \bigoplus_{(j,p)>0} \mathbb{C}\phi^{\pm j, \pm p}$ and $\phi^{0,0} = e_{0,0}$. The quasi-isotropic decomposition of W_d is given as follows:

$$W_d = W_d^+ \oplus W_d^- \oplus \mathbb{C}\phi^{0,0}.$$

We define the operators $E^{j,p}_{k,q}$ by $E^{j,p}_{k,q}(\phi^{i,l}) = \delta_{i,k}\delta_{l,q}\phi^{j,p}$. We put

$$B^{j,p}_{k,q} = E^{j,p}_{k,q} - E^{-k,-q}_{-j,-p}.$$

Consider the Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{so}(2d+1)$ to be the subalgebra generated by the diagonal matrices. Clearly $\mathfrak{H} = \oplus_{(j,p)>0}\mathbb{C}B^{j,p}_{j,p}$. Let $\{L_{j,p}\}$ for $(j,p) > 0$ be a dual basis. Thus $\mathfrak{H}^* = \oplus_{(j,p)>0}\mathbb{C}L_{j,p}$.

8.2.1. Highest weight vectors as wedge product. To every Young diagram in $\mathcal{Y}_{r,s}$, we associate an $(2r+1) \times (2r+1)$ matrix as follows. First, to every Young diagram λ we associate a $(r \times s)$ matrix $Y(\lambda)$ as follows:

$$Y(\lambda)_{i,j} = \begin{cases} 0 & \text{if } \lambda \text{ has a box in the } (i,j)\text{-th position,} \\ 1 & \text{otherwise.} \end{cases}$$

Finally to $Y(\lambda)$, we associate the following matrix:

$$\widetilde{Y}(\lambda) = \begin{array}{c} \\ 1 \\ \vdots \\ r \\ 0 \\ -r \\ \vdots \\ -1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & \dots & s & 0 & -s & \dots & -1 \end{array} \\ \left(\begin{array}{ccccccc} & & & 1 & 1 & \dots & 1 \\ & Y(\lambda) & & \vdots & \vdots & \dots & \vdots \\ & & & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right) \end{array}.$$

For $\lambda \in \mathcal{Y}_{r,s}$, let $\widetilde{Y}(\lambda)$ be the image of $Y(\lambda)$. We define the following operations on $\widetilde{Y}(\lambda)$ which produces a new matrix:

$$\sigma^L(\widetilde{Y}(\lambda))_{j,p} := \widetilde{Y}(\lambda)_{j,p} - \delta_{j,1}\delta_{\widetilde{Y}_{1,|p|},1}, \tag{8.1}$$

$$\sigma^R(\widetilde{Y}(\lambda))_{j,p} := \widetilde{Y}(\lambda)_{j,p} - \delta_{p,1}\delta_{\widetilde{Y}_{|j|,1},1}. \tag{8.2}$$

The following proposition in [**17**] gives the highest weight vectors for the branching rules described in Section 6.3.

PROPOSITION 8.2.1. *The vector* $\big(\bigwedge_{\widetilde{y}_{j,p}=0}\phi^{j,p}(-\frac{1}{2})\big).1$ *well defined up to a sign for each of the matrices* $\widetilde{Y}(\lambda)$, $\sigma^L(\widetilde{Y}(\lambda))$, $\sigma^R(\widetilde{Y}(\lambda))$, $\sigma^L(\sigma^R(\widetilde{Y}(\lambda)))$ *gives a highest weight vector of the components with highest weight* (λ,λ^T), $(\sigma(\lambda),\lambda^T)$, $(\lambda,\sigma(\lambda^T))$ *and* $(\sigma(\lambda),\sigma(\lambda^T))$

Next, we describe the highest vectors for some of the components in the "Kac-Moody" form. We use this explicit descriptions to prove the basic cases of the rank-level duality.

8.2.2. Highest weight vectors in Kac-Moody form. Let $\lambda,\lambda'\in\mathcal{Y}_{r,s}$ and assume that λ is obtained from $\lambda'\in\mathcal{Y}_{r,s}$ by adding two boxes. In terms of the matrices described in Section 8.2.1, $Y(\lambda)$ is obtained from $Y(\lambda')$ by changing 1 to 0 in exactly two places of $Y(\lambda')$, say at (a,b) and (c,d). Assume that $(a,b)<(c,d)$ under the lexicographic ordering.

REMARK 8.2.2. *Let* V_λ *be the finite dimensional rep* $\mathfrak{g}$*-module inside* $\mathcal{H}_\lambda(\mathfrak{g})$*, where* $\mathfrak{g}$ *is a finite-dimensional semisimple Lie algebra. Every finite dimensional irreducible representation of* $\mathfrak{g}$ *has a lowest weight vector* v^λ*. This vector is a highest weight vector for the affine Lie algebra* $\widehat{\mathfrak{g}}$ *if we had chosen the opposite Borel as the Borel for* $\mathfrak{g}$*. We call the vector* v^λ *as the opposite highest weight vector of* $\mathcal{H}_\lambda(\mathfrak{g})$*.*

The following proposition describes the highest weight vectors in the "Kac-Moody" form, i.e. as elements of universal enveloping of $\widehat{\mathfrak{so}}(2d+1)$ acting on the highest weight vectors of $\mathcal{H}_0(\mathfrak{so}(2d+1))$ and $\mathcal{H}_1(\mathfrak{so}(2d+1))$.

PROPOSITION 8.2.3. *Let* λ *and* λ' *be as before. Then the following holds:*

(1) If $v_{\lambda'}\in\mathrm{End}(\bigwedge W_d^{\mathbb{Z}+\frac{1}{2},-})$ *is the highest weight vector of the component* $\mathcal{H}_{\lambda'}(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda'^T}(\mathfrak{so}(2s+1))$*, then the highest weight vector* v_λ *of the component* $\mathcal{H}_\lambda(\mathfrak{so}(2r+1))\otimes\mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$ *is given by the following:*

$$v_\lambda = B^{a,b}_{-c,-d}(-1).v_{\lambda'}.$$

(2) If $v^{\lambda'}$ is the opposite highest weight vector of $\mathcal{H}_{\lambda'}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda'^T}(\mathfrak{so}(2s+1))$, then the opposite highest weight vector v^{λ} of the component $\mathcal{H}_{\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$ is:

$$v^{\lambda} = B_{c,d}^{-a,-b}(-1).v^{\lambda'}.$$

PROOF. The proof of the above easily follows from Proposition 8.2.1 and Proposition 8.1.3. □

REMARK 8.2.4. *There is no uniqueness in building a Young diagram λ starting from the empty Young diagram. So there is no uniqueness in the expressions of the highest weight vectors described in Proposition 8.2.3.*

Proof of rank-level duality

In this chapter, we give a proof of Theorem 6.4.1. The main steps of the proof are summarized below.

9.1. Key steps

The strategy of the proof of Theorem 6.4.1 closely follows [**2**] and [**27**] but has some significant differences in the individual steps.

9.1.1. Step I. We study the degeneration of the rank-level duality map on $\mathbb{P}^1$ with n marked points. We use proposition 9.4.1 to reduce to the case for conformal blocks on $\mathbb{P}^1$ with three marked points and the representation attached to one of the marked points in ω_1. The details of this step are explained in Section 9.4.

9.1.2. Step II. We are now reduced to proving rank-level dualities for admissible pairs of the form $((\omega_1, \lambda_2, \lambda_3), (\omega_1, \beta_1, \beta_2))$. We use Proposition 7.1.4 to determine which conformal blocks on $\mathbb{P}^1$ with three marked points with representations of the form $(\omega_1, \lambda_2, \lambda_3)$ are non-zero.

9.1.3. Step III. We use Proposition 4.1.2 and further reduce to proving the rank-duality for three pointed curves for admissible pairs of the following forms:

(1) $(\omega_1, \lambda_2, \lambda_3), (\omega_1, \lambda_2^T, \lambda_3^T)$, where $\lambda_2, \lambda_3 \in \mathcal{Y}_{r,s}$ and λ_2 is obtained from λ_3 either by adding or deleting a box. The rank-level duality for these cases are proved in Section 9.2.

(2) $(\omega_1, \lambda, \lambda), (\omega_1, \lambda^T, \sigma(\lambda^T))$, where $\lambda \in \mathcal{Y}_{r,s}$ and $(\lambda, L_r) \neq 0$. These rank-level dualities are proved in Section 9.3.

9.2. The minimal three point cases

In this section, we prove rank-level dualities for some special one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points. We use these cases to prove the rank-level duality isomorphism in the general case.

The finite dimensional irreducible $\mathfrak{so}(2d+1)$ module V_{ω_1} can be realized inside $\bigwedge^{odd} W_d^{\mathbb{Z}+\frac{1}{2},-}$ as linear span of vectors of the form $\phi^{i,j}(-\frac{1}{2})$. On V_{ω_1} there is a canonical $\mathfrak{so}(2d+1)$ invariant bilinear form Q given by the following formula:

$$Q(\phi^{j,p}(-\frac{1}{2}), \phi^{-k,-q}(-\frac{1}{2})) = \delta_{j,k}\delta_{p,q}.$$

For notational convenience we write $\phi^{i,j}(-\frac{1}{2})$ as $v^{i,j}$.

Throughout this section, we will assume that $(P_1, P_2, P_3) = (1, 0, \infty)$ with coordinates $\xi_1 = z-1$, $\xi_2 = z$ and $\xi_3 = \frac{1}{z}$, where z is a global coordinate on $\mathbb{C}$. We denote by $\mathfrak{X}$ the associated data. Let $\lambda_2, \lambda_3 \in \mathcal{Y}_{r,s}$, $\vec{\lambda} = (\omega_1, \lambda_2, \lambda_3)$, $\vec{\lambda}^T = (\omega_1, \lambda_2^T, \lambda_3^T)$, $\vec{\Lambda} = (\omega_1, \omega_1, 0)$ and λ_2 is obtained from λ_3 is adding or deleting a box.

Remark 9.2.1. *The following strategy is influenced by the proof of Proposition 6.3 in* [**2**].

Let us summarize our main steps to prove these minimal cases. Let $\langle \Psi' | \in \mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$ be a non-zero element. It is enough to produce $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\vec{\lambda}^T}(\mathfrak{so}(2s+1))$ such that

$$\langle \Psi' | \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \rangle \neq 0.$$

9.2.1. Step I. We always choose $|\Phi_2\rangle$(respectively $|\Phi_3\rangle$) to be the highest (respectively opposite highest) weight vector of the module with highest weight (λ_2, λ_2^T) (respectively (λ_3, λ_3^T)).

9.2.2. Step II. If λ_3 is obtained from λ_2 by adding a box in the (a,b)-th coordinate, then we choose $|\Phi_1\rangle$ to be $v^{a,b}$. If λ_2 is obtained from λ_3 by adding a box in the (a,b)-th coordinate, then we choose $|\Phi_1\rangle$ to be $v^{-a,-b}$. With this choice, it is clear that the $\mathfrak{H}$-weight of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle$ is zero.

9.2.3. Step III. We use induction on $\max(|\lambda_2|, |\lambda_3|)$. The base cases of induction are proved in Section 9.2.5. Assume that $|\lambda_2| = |\lambda_3| + 1$. Let $\lambda_2' \in \mathcal{Y}_{r,s}$ be such that

(1) λ_2 is obtained by adding two boxes from λ_2',

(2) λ_3 is obtained by adding a box to λ_2'. (The other case $|\lambda_3| = |\lambda_2| + 1$ is handled similarly.)

9.2.4. Step IV. We use gauge symmetry (see Chapter 1) to reduce to the case for the admissible pair $((\omega_1, \lambda_2', \lambda_3), (\omega_1, \lambda_2'^T, \lambda_3^T))$. This is done in Proposition 9.2.6. Now $\max(|\lambda_2'|, |\lambda_3|) < |\lambda_2|$. The other case is handled similarly. Hence we are done by induction.

Remark 9.2.2. *The minimal cases here are similar to the minimal cases in* [**2**]. *In the case of symplectic rank-level duality, T. Abe identified the rank-level duality map with the symplectic strange duality map and used the geometry of parabolic bundles with a symplectic form to show that the rank-level duality maps are non-zero. As remarked earlier, we were not able to describe the map in Theorem 6.4.1 geometrically. However the steps described above can be used to tackle minimal cases in* [**2**].

9.2.5. The base cases for induction. We think of $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$ and let z be a global coordinate of $\mathbb{C}$. We will assume that $(P_1, P_2, P_3) = (1, 0, \infty)$ with coordinates $\xi_1 = z - 1$, $\xi_2 = z$ and $\xi_3 = \frac{1}{z}$ respectively, and denote by $\mathfrak{X}$ the associated data. Further we let $\vec{\Lambda} = (\omega_1, \omega_1, 0)$.

Lemma 9.2.3. *Let $\vec{\lambda} = (\omega_1, \omega_1, 0)$. Then the following map*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1),$$

is non-zero.

Proof. Let $\langle \Psi' | \in \mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$ be a non-zero element and $\langle \Psi |$ is the image of $\langle \Psi' |$ under the propagation of vacua. It is enough to produce $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\vec{\lambda}^T}(\mathfrak{so}(2s+1))$ such that

$$\langle \Psi' | \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \rangle \neq 0.$$

We choose $|\Phi_1\rangle = v^{-1,-1}$, $|\Phi_2\rangle = v^{-1,-1}$ and $|\Phi_3\rangle = 1$. By propagation of vacua, we get

$$\begin{aligned}\langle \Psi' | v^{-1,-1} \otimes v^{1,1} \otimes 1\rangle &= \langle \Psi | v^{-1,-1} \otimes v^{1,1}\rangle, \\ &= Q(v^{-1,-1}, v^{1,1}), \\ &= 1.\end{aligned}$$

□

LEMMA 9.2.4. *Let* $\vec{\lambda} = (\omega_1, \omega_1, 2\omega_1)$ *or* $(\omega_1, \omega_1, \omega_2)$. *Then the following map*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1),$$

is non-zero.

PROOF. First let $\vec{\lambda} = (\omega_1, \omega_1, 2\omega_1)$. We choose $|\Phi_3\rangle$ to be the opposite highest weight vector of the module $\mathcal{H}_{2\omega_1}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\omega_2}(\mathfrak{so}(2s+1))$, $|\Phi_2\rangle = v^{1,1}$. We choose $|\Phi_1\rangle$ such that the $\mathfrak{H}$-weight of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle$ is zero. In this case, $|\Phi_1\rangle = v^{1,2}$. By gauge symmetry, we get the following:

$$\begin{aligned}&\langle \Psi' | v^{1,2} \otimes v^{1,1} \otimes B_{1,2}^{-1,-1}(-1).1\rangle \\ &= \langle \Psi' | v^{1,2} \otimes v^{1,1} \otimes B_{1,2}^{-1,-1}(\frac{1}{\xi_3}).1\rangle, \\ &= -\langle \Psi' | B_{1,2}^{-1,-1} v^{1,2} \otimes v^{1,1} \otimes 1\rangle - \langle \Psi' | v^{1,2} \otimes B_{1,2}^{-1,-1}(1).v^{1,1} \otimes 1\rangle, \\ &= -\langle \Psi' | v^{-1,-1} \otimes v^{1,1} \otimes 1\rangle \quad [\text{Since } B_{1,2}^{-1,-1}(1).v^{1,1} = 0], \\ &\neq 0. \quad [\text{By Lemma 9.2.3}]\end{aligned}$$

The case $\vec{\lambda} = (\omega_1, \omega_1, \omega_2)$ follows similarly.

□

LEMMA 9.2.5. *Let* $\vec{\lambda} = (\omega_1, \omega_1 + \omega_2, 2\omega_1)$ *or* $(\omega_1, \omega_1 + \omega_2, \omega_2)$. *Then the following map:*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1),$$

is non-zero.

Proof. Consider $\lambda_2' = \omega_1$ and $\lambda_2 = \omega_1 + \omega_2$ and let λ_2 is obtained from λ_2' by adding two boxes in the $(1,2)$ and $(2,1)$ coordinate. Thus by Proposition 8.2.3, we get

$$v_{\lambda_2} = B^{1,2}_{-2,-1}(-1)v^{1,1}.$$

As in Lemma 9.2.4, the vector $|\Phi_3\rangle = B^{-1,-1}_{1,2}(-1).1$. We choose $|\Phi_2\rangle = v_{\lambda_2}$ and $|\Phi_1\rangle$ such that the $\mathfrak{H}$-weight of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_2\rangle$ is zero. In this case $|\Phi_1\rangle = v^{-2,-1}$. By gauge symmetry, we get the following:

$$\begin{aligned}
&\langle \Psi' | v^{-2,-1} \otimes B^{1,2}_{-2,-1}(-1)v^{1,1} \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&\quad = -\langle \Psi' | B^{1,2}_{-2,-1} v^{-2,-1} \otimes v^{1,1} \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&\qquad\quad - \langle \Psi' | v^{-2,-1} \otimes v^{1,1} \otimes B^{1,2}_{-2,-1}(1) B^{-1,-1}_{1,2}(-1).1\rangle, \\
&\quad = -\langle \Psi' | B^{1,2}_{-2,-1} v^{-2,-1} \otimes v^{1,1} \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&\qquad\quad - \langle \Psi' | v^{-2,-1} \otimes v^{1,1} \otimes B^{-1,-1}_{1,2}(-1) B^{1,2}_{-2,-1}(1).1\rangle \\
&\qquad\quad - \langle \Psi' | v^{-2,-1} \otimes v^{1,1} \otimes [B^{1,2}_{-2,-1}(1), B^{-1,-1}_{1,2}(-1)].1\rangle, \\
&\quad = -\langle \Psi' | v^{1,2} \otimes v^{1,1} \otimes B^{-1,-1}_{1,2}(-1).1\rangle, \\
&\quad \neq 0. \quad \text{[By Lemma 9.2.4]}
\end{aligned}$$

□

9.2.6. The inductive step.

Proposition 9.2.6. *Let $|\lambda_2| = |\lambda_3| + 1$ and λ_2 be obtained from λ_3 by adding a box in the (c,d)-th coordinate. Further, assume that λ_3 is obtained from λ_2' by adding a box in the (a,b)-th coordinate. Then the rank-level duality isomorphism for the admissible pair $((\omega_1, \lambda_2', \lambda_3), (\omega_1, \lambda_2'^T, \lambda_3^T))$ implies rank-level duality isomorphism for the admissible pair $((\omega_1, \lambda_2, \lambda_3), (\omega_1, \lambda_2^T, \lambda_3^T))$.*

Proof. Without loss of generality assume that $(a,b) < (c,d)$. Consider a non-zero element $\langle \Psi' | \in \mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$. We choose $|\Phi_1\rangle = v^{-a,-b}$, $|\Phi_2\rangle = B^{a,b}_{-c,-d}(-1)v_{\lambda_2'}$ and $|\Phi_3\rangle$ to be the opposite highest weight vector of the component with highest weight (λ_3, λ_3^T). Then, we have the following:

$$
\begin{aligned}
&\langle \Psi' | \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \rangle \\
&= \langle \Psi' | v^{-a,-b} \otimes B^{a,b}_{-c,-d}(-1)v_{\lambda_2'} \otimes \Phi_3 \rangle, \\
&= -\langle \Psi' | B^{a,b}_{-c,-d} v^{-a,-b} \otimes v_{\lambda_2'} \otimes \Phi_3 \rangle \\
&\qquad -\langle \Psi' | v^{-a,-b} \otimes v_{\lambda_2'} \otimes B^{a,b}_{-c,-d}(1)\Phi_3 \rangle, \\
&= \langle \Psi' | v^{c,d} \otimes v_{\lambda_2'} \otimes \Phi_3 \rangle. \ \text{(By Lemma 9.2.7 below)}
\end{aligned}
$$

The last expression is exactly the one that we consider to prove the rank-level duality for the admissible pair $((\omega_1, \lambda_2', \lambda_3), (\omega_1, \lambda_2'^T, \lambda_3^T))$. Hence we are done. $\square$

Lemma 9.2.7. *With the above notation, we have the following:*

$$B^{a,b}_{-c,-d}(1)|\Phi_3\rangle = 0.$$

Proof. Since $|\lambda_3|$ is even, the opposite highest weight vector $|\Phi_3\rangle$ can be chosen to be of the form $B^{-a,-b}_{e,f}(-1)v$. Moreover v has the form $\prod_{\alpha \in I} X_{-\alpha}(-1).1$ such that $(L_{a,b}, \alpha) = 0$, where I is a subset of positive root of $\mathfrak{so}(2d+1)$ and $X_{-\alpha}$ is a non-zero element in the weight space of the negative root $-\alpha$.

$$\begin{aligned}
B^{a,b}_{-c,-d}(1)|\Phi_3\rangle &= B^{a,b}_{-c,-d}(1)B^{-a,-b}_{e,f}(-1)v, \\
&= B^{-a,-b}_{e,f}(-1)B^{a,b}_{-c,-d}(1)v + [B^{a,b}_{-c,-d}(1), B^{-a,-b}_{e,f}(-1)]v, \\
&= B^{-a,-b}_{e,f}(-1)B^{a,b}_{-c,-d}(1)\prod_{\alpha\in I} X_{-\alpha}(-1).1 + [B^{a,b}_{-c,-d}, B^{-a,-b}_{e,f}]\prod_{\alpha\in I} X_{-\alpha}(-1).1, \\
&= B^{-a,-b}_{e,f}(-1)\big(\prod_{\alpha\in I} X_{-\alpha}(-1)\big)B^{a,b}_{-c,-d}(1).1 \\
&\quad +\big(\prod_{\alpha\in I} X_{-\alpha}(-1)\big)[B^{a,b}_{-c,-d}, B^{-a,-b}_{e,f}].1, \\
&= 0.
\end{aligned}$$

Hence the lemma follows.

$\square$

The following proposition has a similar proof to Proposition 9.2.6 and tackles the case $|\lambda_3| = |\lambda_2| + 1$.

Proposition 9.2.8. *Let $|\lambda_3| = |\lambda_2|+1$ and λ_3 is obtained from λ_2 by adding a box in the (c,d)-th coordinate. Further, assume that λ_2 is obtained from λ_3' by adding a box in the (a,b)-th coordinate. Then the rank-level duality isomorphism for the admissible pair $((\omega_1, \lambda_2, \lambda_3'), (\omega_1, \lambda_2^T, \lambda_3'^T))$ implies rank-level duality isomorphism for the admissible pair $((\omega_1, \lambda_2, \lambda_3), (\omega_1, \lambda_2^T, \lambda_3^T))$.*

9.3. The remaining three point cases

As before, we will assume that $(P_1, P_2, P_3) = (1, 0, \infty)$ with coordinates $\xi_1 = z-1$, $\xi_2 = z$ and $\xi_3 = \frac{1}{z}$. We denote by $\mathfrak{X}$ the associated data. Let $\vec{\lambda} = (\omega_1, \lambda, \lambda)$, $\vec{\Lambda} = (\omega_1, \omega_1, 0)$, where $\lambda \in \mathcal{Y}_{r,s}$ such that $(\lambda, L_r) \neq 0$. The proof of the next proposition follows the same pattern as the proof of Proposition 9.2.6. We give a proof of the first part of the proposition for completeness.

Proposition 9.3.1. *The following maps are non-zero:*

(1) Let $|\lambda|$ be odd and $\vec{\lambda}^T=(\omega_1,\lambda^T,\sigma(\lambda^T))$.

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\vec{\Lambda}}(\mathfrak{X},\mathfrak{so}(2d+1),1).$$

(2) Let $|\lambda|$ be even and $\vec{\lambda}^T=(\omega_1,\sigma(\lambda^T),\lambda^T)$.

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\otimes\mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)\to\mathcal{V}_{\vec{\Lambda}}(\mathfrak{X},\mathfrak{so}(2d+1),1).$$

Proof. Let $\lambda'\in\mathcal{Y}_{r,s}$ be such that $\sigma(\lambda)$ is obtained by adding boxes in $(0,1)$ and (r,a) to λ' and λ is obtained by adding a box in the (r,a)-th position. Since $|\lambda|$ is odd, the module with highest weight $(\lambda,\sigma(\lambda^T))$ appears in the branching of $\mathcal{H}_0(\mathfrak{so}(2d+1))$. By Proposition 8.2.1, the opposite highest weight vector is given by $B^{0,-1}_{r,a}(-1)v^{\lambda'}$, where $v^{\lambda'}$ is the opposite highest weight vector of the irreducible module with highest weight (λ',λ'^T).

As before, we choose $|\Phi_3\rangle$ to be the opposite highest weight vector of the module with highest weight $(\lambda,\sigma(\lambda^T))$. We set $|\Phi_2\rangle$ to be the highest weight vector v_λ and $|\Phi_1\rangle$ to be such that the $\mathfrak{H}$-weight of $|\Phi_1\otimes\Phi_2\otimes\Phi_3\rangle$ is zero. In this case $|\Phi_1\rangle$ is $v^{0,1}$.

Let $\langle\Psi'|\in\mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X},\mathfrak{g},1)$ be a non-zero element. We use gauge symmetry as before to get the following:

$$\begin{aligned}
&\langle\Psi'|\Phi_1\otimes\Phi_2\otimes\Phi_3\rangle\\
&=\langle\Psi'|v^{0,1}\otimes v_\lambda\otimes B^{0,-1}_{r,a}(-1)v^{\lambda'}\rangle,\\
&=-\langle\Psi'|B^{0,-1}_{r,a}(-1)v^{0,1}\otimes v_\lambda\otimes v^{\lambda'}\rangle\\
&\qquad-\langle\Psi'|v^{0,1}\otimes B^{0,-1}_{r,a}(1)v_\lambda\otimes v^{\lambda'}\rangle,\\
&=\langle\Psi'|v^{-r,-a}\otimes v_\lambda\otimes v^{\lambda'}\rangle.\quad\text{(By Lemma similar to 9.2.7)}
\end{aligned}$$

Now we know that $\langle\Psi'|v^{-r,-a}\otimes v_\lambda\otimes v^{\lambda'}\rangle\neq0$, since rank-level duality holds for the admissible pair $((\omega_1,\lambda,\lambda'),(\omega_1,\lambda^T,\lambda'^T))$. This completes the proof.

$\square$

9.4. The proof in the general case

In this section, we finish the proof of Theorem 6.4.1. We now formulate and prove a key degeneration result using the compatibility of rank-level duality and factorization discussed earlier. Let $\vec{\lambda}_1, \vec{\lambda}_2$ be n_1, n_2 tuples of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$. Consider an n tuple $\vec{\lambda} = (\vec{\lambda}_1, \vec{\lambda}_2)$ of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$. Similarly, consider $\vec{\mu} = (\vec{\mu}_1, \vec{\mu}_2)$ an $(n_1 + n_2)$ tuple of weights in $P^0_{2r+1}(\mathfrak{so}(2r+1))$ such that $(\vec{\lambda}, \vec{\mu})$ is an admissible pair.

Proposition 9.4.1. *With the above notation, the following statements are equivalent:*

(1) The rank-level duality map for the admissible pair $(\vec{\lambda}, \vec{\mu})$ is an isomorphism for conformal blocks on $\mathbb{P}^1$ with n marked points.

(2) The following rank-level duality maps for the admissible pairs are all isomorphic.

- *The rank-level duality maps are isomorphisms for all admissible pairs of the form $(\vec{\lambda}_1 \cup \lambda, \vec{\mu}_1 \cup \mu)$ for conformal blocks on $\mathbb{P}^1$ with $(n_1 + 1)$ marked points.*
- *The rank-level duality maps are isomorphisms for all the admissible pairs of the form $(\lambda \cup \vec{\lambda}_2, \mu \cup \vec{\mu}_2)$ for conformal blocks on $\mathbb{P}^1$ with $(n_2 + 1)$ marked points.*

We first start with a lemma. We give a proof of Proposition 9.4.1 using this lemma and Proposition 5. Let $\mathcal{B} = \operatorname{Spec} \mathbb{C}[[t]]$. Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector bundles on $\mathcal{B}$ of same rank and let $\mathcal{L}$ be a line bundle on $\mathcal{B}$. Consider a bilinear map $f : \mathcal{V} \otimes \mathcal{W} \to \mathcal{L}$. Assume that on $\mathcal{B}$, there are isomorphisms

$$\begin{aligned} \oplus s_i : \quad \mathcal{V} \to \quad & \bigoplus_{i \in I} \mathcal{V}_i, \\ \oplus t_j : \quad \mathcal{W} \to \quad & \bigoplus_{j \in I} \mathcal{W}_j. \end{aligned}$$

Further assume that $\mathcal{V}_i$ and $\mathcal{W}_i$ have the same rank. Let $f_{i,j}$ be maps from $\mathcal{V}_i \otimes \mathcal{W}_j \to \mathcal{L}$ such that $f_{i,j} = 0$ for $i \neq j$ and $f = \sum_{i \in I} t^{m_i}(f_{i,i} \circ (s_i \otimes t_i))$. The following lemma is easy to prove.

LEMMA 5. *The map f is non-degenerate on $\mathcal{B}^* = \mathcal{B} \setminus \{t=0\}$ if and only if for all $i \in I$ the maps $f_{i,i}$'s are non-degenerate.*

We now return to the proof of Proposition 9.4.1. Let $\mathcal{X} \to \mathcal{B}$ be a family of curves of genus 0 such that the generic fiber is a smooth curve and the special fiber $\mathcal{X}_0$ is a nodal curve. In our case, we let $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{L}$ be locally free sheaves $\mathcal{V}_{\vec{\lambda}}(\mathcal{X}, \mathfrak{so}(2r+1), 2s+1)$ and $\mathcal{V}_{\vec{\mu}}(\mathcal{X}, \mathfrak{so}(2s+1), 2r+1)$ and $\mathcal{V}_{\vec{\Lambda}}(\mathcal{X}, \mathfrak{so}(2d+1), 1)$ respectively, where $\vec{\lambda}$ and $\vec{\mu}$ as in Proposition 9.4.1 and $\vec{\Lambda} \in (P_1^0(\mathfrak{so}(2d+1))^n$ be such that $(\vec{\lambda}, \vec{\mu}) \in B(\vec{\Lambda})$.

We consider $\mathcal{V}_i$'s to be a locally free sheaves of the form

$$\mathcal{V}_{\vec{\lambda}_1 \cup \lambda}(\mathfrak{X}_1, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\lambda \cup \vec{\lambda}_2}(\mathfrak{X}_2, \mathfrak{so}(2r+1), 2s+1) \otimes \mathbb{C}[[t]],$$

where $\lambda \in P^0_{2s+1}(\mathfrak{so}(2r+1))$, $\mathfrak{X}_1$, $\mathfrak{X}_2$ be the data associated to disjoint copies of $\mathbb{P}^1$ (which are obtained from normalization of $\mathcal{X}_0$) with n_1, n_2 points respectively. Similarly, we let $\mathcal{W}_j$'s to be a locally free sheaves of the form

$$\mathcal{V}_{\vec{\mu}_1 \cup \mu}(\mathfrak{X}_1, \mathfrak{so}(2s+1), 2r+1) \otimes \mathcal{V}_{\mu \cup \vec{\mu}_2}(\mathfrak{X}_2, \mathfrak{so}(2s+1), 2r+1) \otimes \mathbb{C}[[t]],$$

where $\mu \in P^0_{2r+1}(\mathfrak{so}(2s+1))$.

Since there are bijections (the bijections depend on the factorization of $\mathcal{V}_{\vec{\Lambda}}(\mathcal{X}, \mathfrak{so}(2d+1), 1)$ into n_1 and n_2 parts) between $P^0_{2s+1}(\mathfrak{so}(2r+1))$ and $P^0_{2r+1}(\mathfrak{so}(2s+1))$, we can choose the indexing set I in Lemma 5 to be $Y_{r,s} \sqcup \sigma(Y_{r,s})$. It is also important to point out that $f_{i,j} = 0$ for $i \neq j$ is guaranteed by the fact that given $\lambda \in P^0_{2s+1}(\mathfrak{so}(2r+1))$, $\Lambda \in P^0_1(\mathfrak{so}(2d+1))$, there exists exactly one $\mu \in P^0_{2r+1}(\mathfrak{so}(2s+1))$ such that $(\lambda, \mu) \in B(\Lambda)$. The proof of Proposition 9.4.1 now follows from Proposition 5.2.1, Lemma 5 and Proposition 3.3.1.

REMARK 9.4.2. *The situation in Proposition 9.4.1 should be compared to Proposition 5.2 in* [**30**].

An immediate corollary of the Proposition 9.4.1 is the following:

Corollary 9.4.3. *If the rank-level duality holds for $\mathbb{P}^1$ with three marked points, then it holds for $\mathbb{P}^1$ with an arbitrary number of marked points.*

By Proposition 4.1.2, we can further reduce to prove the rank-level duality for an admissible pair of the form $((\lambda_1, \lambda, \lambda_2), (\lambda_1^T, \beta, \lambda_2^T))$, where $\lambda_1, \lambda_2 \in \mathcal{Y}_{r,s}$, $\lambda \in P^0_{2r+1}(\mathfrak{so}(2r+1))$ and $\beta \in P^0_{2s+1}(\mathfrak{so}(2s+1))$. Let $\vec{\lambda} = (\omega_1, \dots, \omega_1, \lambda, \lambda_2)$ and $\vec{\mu} = (\omega_1, \dots, \omega_1, \beta, \lambda_2^T)$, the number of ω_1's is $|\lambda_1|$. Clearly the pair $(\vec{\lambda}, \vec{\mu})$ is admissible. The following corollary is a direct consequence of Proposition 9.4.1 and Lemma 9.4.5.

Corollary 9.4.4. *Let $\lambda_1, \lambda_2 \in \mathcal{Y}_{r,s}$. If the rank-level duality is an isomorphism for any $\mathbb{P}^1$ with $|\lambda_1| + 2$ marked points for the admissible pair $\vec{\lambda} = (\omega_1, \dots, \omega_1, \lambda, \lambda_2)$ and $\vec{\mu} = (\omega_1, \dots, \omega_1, \beta, \lambda_2^T)$, then the rank-level duality on $\mathbb{P}^1$ is also an isomorphism for the admissible pair $((\lambda_1, \lambda, \lambda_2), (\lambda_1^T, \beta, \lambda_2^T))$.*

Lemma 9.4.5. *Let $\lambda \in \mathcal{Y}_{r,s}$, and $\vec{\lambda} = (\lambda, \omega_1, \dots, \omega_1)$, where the number of ω_1 is $|\lambda|$, then*

$$\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \neq 0.$$

Proof. The proof follows directly by factorization of fusion coefficients and induction on $|\lambda|$. $\square$

9.4.1. Reduction to the one dimensional cases. In the previous section, we reduced Theorem 6.4.1 for admissible pairs of the form $\vec{\lambda} = (\omega_1, \dots, \omega_1, \lambda, \lambda_2)$ and $\vec{\mu} = (\omega_1, \dots, \omega_1, \beta, \lambda_2^T)$, where $\lambda \in P^0_{2s+1}(\mathfrak{so}(2r+1))$, the number of ω_1's are $|\lambda_1|$, $\lambda_2 \in \mathcal{Y}_{r,s}$ and $\beta \in P^0_{2r+1}(\mathfrak{so}(2s+1))$. The following lemma shows that we can further reduce to the case for certain one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points.

Lemma 9.4.6. *Let $\lambda_1, \lambda_2 \in P^0_{2s+1}(\mathfrak{so}(2r+1))$ and $\beta_1, \beta_2 \in P^0_{2r+1}(\mathfrak{so}(2s+1))$. If the rank-level duality holds for admissible pairs of the form $((\lambda_1, \omega_1, \lambda_2), (\beta_1, \omega_1, \beta_2))$, then the rank-level duality holds for admissible pairs on $\mathbb{P}^1$ with arbitrary number of marked points.*

Proof. The proof follows from Proposition 9.4.1. □

We use Proposition 4.1.2 and Proposition 7.1.2 to further reduce to the following admissible pairs for certain one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points.

(1) $(\omega_1, \lambda_2, \lambda_3), (\omega, \lambda_2^T, \lambda_3^T)$, where $\lambda_2, \lambda_3 \in \mathcal{Y}_{r,s}$ and λ_2 is obtained by λ_3 either by adding or deleting a box.

(2) $(\omega_1, \lambda, \lambda), (\omega_1, \lambda^T, \sigma(\lambda^T))$, where $\lambda \in \mathcal{Y}_{r,s}$ and $(\lambda, L_r) \neq 0$.

The rank-level duality in these cases has been proved in Section 9.2 and Section 9.3. This completes the proof of Theorem 6.4.1.

References

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