Essays in Durable Goods Monopolies

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Abstract

BAŞAK ALTAN: Essays in Durable Goods Monopolies.
(Under the direction of Gary Biglaiser.)

This study analyzes a vertically differentiated market for an imperfectly durable good served by a monopolist in an infinite horizon, discrete time game. I characterize Markov perfect equilibria of this game as a function of the common discount rate, the common depreciation rate of the goods, the length of the time period between successive price changes, and the quality levels of the goods. I establish that quality differentiation may alleviate the time inconsistency problem of a durable goods monopolist. In particular, I prove that when the monopolist is not allowed to buy the goods back from previous buyers, the set of parameters supporting the monopoly outcome is larger and the set of parameters supporting the Coase Conjecture is smaller. When the monopolist, however, is allowed to buy the goods back from previous buyers, quality differentiation only affects the off-equilibrium path either by increasing the rate at which a steady state is reached or by expanding the set of steady states supporting the monopoly outcome. This study suggests that when the innate durability of a good is high, the monopolist must commit to not buying the used goods back and produce a lower quality good to maintain his market power.

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Chapter 1

Introduction

Durable goods markets have received much attention since Coase (1972) argued that a durable goods monopolist cannot exercise his monopoly power when the good is perfectly durable. Upon sale of a unit, the monopolist has incentives to sell more by lowering the price of the good. Thus, a rational consumer, anticipating that the monopolist will cut the price as long as it is no less than the marginal cost of the good, prefers deferring consumption so long as the price is not close to the competitive level. Hence, in the absence of the ability to commit to the future prices, the monopoly power of a durable goods monopolist is largely deteriorated, and the monopolist is unable to extract as much consumer surplus as a monopolist selling a perishable good could. Moreover, when the monopolist becomes extremely flexible in adjusting prices, market power vanishes and the competitive outcome is immediately achieved.

The existing theories on durable goods monopolies support the Coase Conjecture by establishing that in Markov perfect equilibria, a durable goods monopolist cannot exercise his market power.[1] Casual empirical evidence, however, suggests that durable goods monopolists charge prices much higher than the marginal cost of production and make considerable profits (e.g. Microsoft, Apple, etc.).

This dissertation studies an unexplored motive, the role of quality differentiation, for solving the commitment problem a durable goods monopolist faces. I consider a market for an imperfectly durable good served by a monopolist in an infinite horizon, discrete time game. In

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[1]See Bulow (1982), Stokey (1981), Gul, Sonnenschein and Wilson (1986), and Sobel (1991).

each period, the monopolist can sell two versions, high and low quality, of the durable good that depreciate stochastically. I analyze whether the simultaneous introduction of vertically differentiated goods enables the monopolist to maintain his market power and to credibly commit to deferring sales to low valuation consumers.

In an interesting study, Deneckere and McAfee (1996) analyze damaged goods in a static market and show that manufacturers may intentionally damage a portion of their goods in order to price discriminate. For example, Intel first produced a fully functioning 486DX processor, then by disabling the math coprocessor produced the 486SX that is inferior to the 486DX. Even though the 486SX was more expensive to produce, the price of the 486SX was lower. In addition to price discrimination, Intel might use the 486SX to maintain his market power. Because, Intel might credibly defer sales of the 486DX to low valuation consumers by selling the 486SX to low valuation consumers instead. The issue that the current study investigates is whether a monopolist will damage a portion of the goods and produce a lower quality version of it in an attempt to mitigate the commitment problem and enhance his market power.

Earlier studies on durable goods monopolies suggest that there are many responses a durable goods monopolist can adopt to restore the profitability, such as reducing the durability of the good (Coase, 1972), leasing (Bulow, 1986), contractual provisions (Butz, 1990), or using an inferior high cost technology (Karp and Perloff, 1996). Such responses, however, are not extensively observed in several durable goods markets. This implies that the commitment problem that Coase conjectures does not decrease profits of durable goods monopolists as much as theory claims. Deneckere and Liang (2008) provide an answer for why casual empirical evidence may contradict with Coase's insight by establishing that a durable goods monopolist can commit to high future prices when the good is sufficiently perishable.

Recent studies extend the seminal analyses on durable goods monopolies by considering new product introductions. By offering new products for sale the monopolist seller can increase the economic depreciation of the initial version of the good and regain his profitability.[2] Even though we have a clearer understanding of why the Coase Conjecture may not hold, our exact

[2]See Levinthal and Purohit (1989), Waldman (1993, 1996), Choi (1994), Fudenberg and Tirole (1998), Lee and Lee (1998), Fishman and Rob (2000), Kumar (2002, 2006), and Anton and Biglaiser (2009).

understanding of Coase's insight remains incomplete, since a glance at durable goods markets suggests that many durable goods are characterized by menus of multiple quality levels and prices: Dell's Inspiron laptop vs its XPS laptop, Mathematica's student version vs its full version, hardcover textbooks vs paperback textbooks, etc.

From a theoretical point of view, this study follows Deneckere and Liang (2008). I extend their single good setting into a setting of a vertically differentiated market to analyze the effect of quality differentiation on the commitment problem a durable goods monopolist faces. Similar to Deneckere and Liang (2008), I establish that there exist three types of Markov perfect equilibria: a Coase Conjecture equilibrium, a monopoly equilibrium, and a reputational equilibrium. *For sufficiently low depreciation rates*, the unique equilibrium is the Coase Conjecture equilibrium. The Coase Conjecture equilibrium has a unique steady state equal to the competitive quantity. At the steady state, the monopolist serves the high quality good to all consumers. *For sufficiently high depreciation rates,* the unique equilibrium is the monopoly equilibrium. The intuition is that when the durable good is sufficiently perishable, replacement sales at high prices compensate the desire to penetrate the market further by lowering the price of at least one of these two versions of the durable good. This equilibrium has two monopoly steady states one of which is equal to the static monopoly quantity. At this steady state, the monopolist only serves the high quality good to the high type consumers. The market at the other monopoly steady state is segmented into two: the monopolist serves the high quality good to the high type consumers and serves the low quality good to the low type consumers. Upon deviation from a monopoly steady state, if the good is sufficiently perishable, the monopolist returns to one of these two steady states from any state. Otherwise, the Coase Conjecture steady state coexists with the monopoly steady states. *For intermediate values of the depreciation rate*, all three types of equilibria exist. In the reputational equilibrium, the monopolist cuts the production of the high quality good to create a reputation of pricing high. The steady state quantity of the high quality good falls short of the monopoly quantity of the high quality good.

When buyers are allowed to trade the good with each other in a perfectly competitive second-hand market that the monopolist is not allowed to enter, I prove that the set of parameters supporting the Coase Conjecture equilibrium is smaller and the set of the parameters supporting the monopoly equilibrium is larger when the monopolist produces low and high quality goods. When the monopolist is, however, allowed to buy back the goods from previous buyers, I prove that quality differentiation does not affect the domain of the parameters supporting each type of equilibrium but affects the off-equilibrium path.

This dissertation establishes that when the monopolist is not allowed to buy back the goods from previous buyers, quality differentiation may enhance market power of a durable goods monopolist and alleviate the commitment problem. In particular, when the innate durability of a good is high, to credibly commit to the monopoly prices of the good the monopolist will produce a lower quality good either by damaging a portion of the goods or by producing the lower quality good from scratch. On the other hand, when the monopolist is allowed to buy back the goods from previous buyers, if the innate durability of a good is sufficiently high, it is less likely to observe quality differentiation. In particular, if the depreciation rate is so low that the Coase Conjecture equilibrium is the unique equilibrium then the monopolist is reluctant to introduce lower quality versions of the good. Moreover, a durable goods monopolist selling a sufficiently perishable good has penetrated the market in an attempt to increase profit, then he may introduce a higher (or a lower) quality version of the good to be able to restore his market power. However, if the monopolist has already been committed to the monopoly price, he will be reluctant to introduce a lower quality version of the good.

The rest of my dissertation proceeds as follows. Chapter 2 discusses the literature. Chapter 3 introduces the model. Chapter 4 characterizes equilibria in a durable goods monopoly when buyers can trade with each other in a perfectly competitive second-hand market. Chapter 5 characterizes equilibria in a durable goods monopoly when the seller can buy back the goods from previous buyers. Chapter 6 concludes. All proofs are relegated to the Appendix.

Chapter 2

Literature Review

The time inconsistency problem a durable goods monopolist faces has been aggressively studied since Coase (1972) conjectured that the sequence of prices (or outputs) of a monopolist selling a perfectly durable good does not maximize his overall profitability. The goal of this section is to briefly survey the literature in an effort to understand the theory of durable goods monopolies and to place this study in the appropriate context.[1]

Early studies analyzing a monopolist selling a perfectly durable good establish that if buyers condition their strategies on payoff relevant part of the histories (that is, if buyers use Markov strategies), Coase's prediction holds. Bulow (1982) studies a durable goods monopolist in a two period model and shows that the optimum price charged by the monopolist is strictly less than the static monopoly price. Intuitively, unless the monopolist credibly precommit to a production plan, consumers anticipate that the monopolist will produce additional units to exploit residual demand which decreases the present value of the durable good. Therefore, since consumers are reluctant to pay the static monopoly price in the first period, in response to the expectations of the consumers the monopolist cuts the price of the durable good. Stokey (1981) extends Bulow's (1982) setting into an infinite horizon setting and proves the existence of an equilibrium that is the limit of the unique equilibrium of the finite version of her model which satisfies the Coase Conjecture. Similarly, Gul, Sonnenschein, and Wilson (1986) show that even though a continuum of subgame perfect equilibria may exist in an infinite horizon

[1]See Waldman (2003) for a detailed literature survey on durable goods monopolies.

model, the Coase Conjecture is verified for Markov strategies. Sobel (1991) extends these initial analyses by considering a market for a perfectly durable good in which demand expands over time. His study also verifies Coase's prediction for Markov strategies. Intuitively, if the monopolist charges the static monopoly price forever, as new consumers enter the market the number of low valuation consumers grows cutting the price then becomes inevitable.

On the contrary to the early studies, Ausubel and Deneckere (1989) show that if buyers condition their strategies on not only payoff relevant part of the histories but the past actions as well, there exist equilibria in which the monopolist creates a reputation and maintains some or all of his market power when the marginal cost of production is no less than the lowest valuation of the buyers. In addition to establishing a reputation, depreciation of a durable good can also help a durable goods monopolist avoid the time inconsistency problem. Bond and Samuelson (1984) show that in a discrete time, infinite horizon game when the good depreciates, replacement sales may deter the monopolist from cutting the price as long as the time period between successive offers of the monopolist is nonzero. However, in the limit, as the time period approaches zero, the competitive outcome is achieved and the Coase Conjecture holds. Karp (1996), on the other hand, by using a continuous time model with replacement sales constructs continuous time equilibria in which the monopolist can earn profits above the competitive level. However, Karp (1996) also shows the existence of an equilibrium that verifies the Coase Conjecture. Following Karp (1996), Deneckere and Liang (2008) characterize the effect of the depreciation rate on the market outcome of a durable goods monopoly when agents use Markov strategies. They conclude that below a certain level of durability, there exists a unique stationary equilibrium in which the monopolist charges the static monopoly price in each period which continues to exist even when the seller becomes highly impatient. Intuitively, when the product depreciates, replacement sales become more profitable than penetrating the market by cutting the price of the good.

A number of recent studies on durable goods monopolies revolves around the issue of introduction of new products with quality improvements. Levinthal and Purohit (1989), Fudenberg and Tirole (1998), and Lee and Lee (1998) study optimal sales strategy of a durable goods monopolist who may introduce an improved version of the good in a two period model. Waldman (1993, 1996), Choi (1994), Fishman and Rob (2000), and Kumar (2002) study the quality and pricing strategy of a durable goods monopolist. These studies develop the idea of planned obsolescence and established that Coase's insight holds. Intuitively, introduction of higher quality products lower the value of used units. Since consumers foresee that the units they have will become obsolete, they refuse to pay the static monopoly price of the good. Hence, the monopolist faces a problem of time inconsistency and the overall profitability is reduced. Kumar (2006) analyzes a discrete time, infinite horizon model in which the monopolist selling a perfectly durable good can vary the quality of the good and shows that when lowest buyer valuation is greater than the marginal cost of production every subgame perfect equilibrium verifies the Coase Conjecture. Anton and Biglaiser (2009) study an exogenous quality growth in an infinite horizon durable goods monopoly model. They show that, for any positive discount factor, the support of Markov perfect equilibrium payoffs ranges from getting all the surplus to getting the single period flow value of each upgrade. Takeyama (2002) studies quality differentiation of a perfectly durable good and the possibility of upgrades in a two period model. Inderst (2008), on the other hand, studies optimal strategy of a durable goods monopolist who can offer perfectly durable goods in different qualities in an infinite horizon model and shows that when the monopolist becomes extremely flexible in adjusting the prices and qualities, he immediately loses his monopoly power and competitive outcome is achieved.

Chapter 3

The Model

Consider a market for an indivisible and imperfectly durable good. The market is served by a monopolist who can produce two versions of the durable good that differ in quality: high quality and low quality goods with quality levels q_H and q_L, respectively. The durable good with quality level q_i may be referred as durable good i. The monopolist offers both versions of the durable good for sale simultaneously at discrete points in time. The monopolist is risk neutral and has discount rate r. The objective of the monopolist is to maximize the present value of his expected profits.

There exists a continuum of infinitely-lived buyers indexed by $b \in [0,1]$. Buyers are segmented into two groups: high valuation buyers and low valuation buyers. Buyer b's reservation price for acquiring one unit of durable good i is represented by

$$f^i(b) = \begin{cases} \overline{\theta} q_i \text{ for } b \in [0, \widehat{b}] \\ \underline{\theta} q_i \text{ for } b \in (\widehat{b}, 1] \end{cases} .$$

All buyers are risk neutral and have the same discount rate r. Buyer b derives a net surplus of $e^{-rt}(f^i(b) - p^i)$ if she purchases durable good i at price p^i at time t. Each buyer wishes to possess at most one unit of the durable good.[1] A buyer is allowed to access the markets as often as she wishes and seeks to maximize the present value of her expected payoffs.

[1] We may assume that the storage costs of the second unit is infinite.

The length of the time period between successive price changes is $z > 0$. Both versions of the durable good depreciate stochastically at the same rate. The probability that a good is still working after a length of time t is $e^{-\lambda t}$. Hence, with probability $\mu \equiv 1 - e^{-\lambda z}$, the good fails between successive price changes. For all versions of the good, the marginal cost of production is assumed to be lower than $\underline{\theta}(q_H - q_L)$. Hence, without loss of generality, it is set to zero.

In Chapter 4, we assume that there exists a perfectly competitive second-hand market in which buyers can trade with each other. In Chapter 5, however, we assume that the monopolist can buy back the goods from previous buyers. Sales occur only at discrete points in time, $t = 0, z, 2z, \ldots, nz, \ldots$ in all markets. The time nz is referred as period n. In each period, the game runs as follows. First, the monopolist sets the price of each version of the good before trade occurs. Then, buyers choose whether or not to hold a good and which version to hold, and trade occurs in all markets simultaneously. After a time interval of z passes, the game repeats itself.

Markov perfect equilibria of this game in which agent strategies only depend on the current state are sought to derive. A strategy of the monopolist specifies the price he charges for each version of the good, and a strategy of a buyer specifies whether or not to hold a durable good and which version to hold. Formally, the game is denoted by $G(z, r, \lambda, q_H, q_L)$. Let σ be a pure strategy for the monopolist where σ determines the price charged for each version of the good by the monopolist in a period as a function of the prices charged in the previous period and the actions chosen by the buyers in the previous period. Let the set of buyers' acceptances of good i in a period be denoted by Q_i. Since either there exists an active perfectly competitive second-hand market or the monopolist is allowed to buy back, strategies of a buyer are independent of her holding status and depend only on the current prices.[2] A buyer's strategy in such equilibria is described by acceptance functions, $V^H(\cdot)$ and $V^L(\cdot)$, where buyer b chooses to hold good i

[2]If there were no perfectly competitive second-hand market, the strategies of the monopolist would depend on the distribution of the current holdings of each version of the good rather than the size of the current stock of goods, and the strategies of a buyer would depend on not only the current prices, but the current holding status as well.

in the current period if and only if the current price of good i satisfies $p^i \leq V^i(b)$.[3] Hence, the set of buyers holding good i after trade is an interval of the form $[0, b^H]$ for the high quality good and $(b^H, b^L]$ for the low quality good. It follows that σ can be represented as a function of the current stock of goods in the market. Hence, the prices of the monopolist in a period are determined by $\sigma : \Omega \times \Omega \rightarrow \mathbb{R}^2_+$ where Ω is the Borel sigma algebra on $[0, 1]$. A strategy of a buyer with valuation parameter $\theta \in \{\overline{\theta}, \underline{\theta}\}$ is denoted by τ_θ where $\tau_\theta : \mathbb{R}^2_+ \rightarrow \{0, 1, 2\}$. Decision 0 indicates that the buyer chooses not to hold any good in the current period. Decision 1 indicates that the buyer chooses to hold the low quality good in the current period. Decision 2 indicates that the buyer chooses to hold the high quality good in the current period. The pure strategy profile $\{\sigma, \tau\}$ generates a stationary path of prices and sales that can be derived recursively. The monopolist is also allowed to mix. Later, however, it is established that the monopolist never randomizes in any period of the game unless it is the initial period. As in Gul, Sonnenschein, and Wilson (1986), the attention is restricted to equilibria in which deviations by sets of measure zero buyers change neither the actions of the monopolist nor the actions of the other buyers. Hence, in such equilibria buyers behave as price takers. From now on, I refer to *Markov perfect equilibrium* simply as *equilibrium*.

[3] If there exists b such that $p^H \leq V^H(b)$ and $p^L \leq V^L(b)$ then b accepts the offer that gives her the highest payoff.

Chapter 4

Perfectly Competitive Second-hand Market

In this chapter, we study vertical product differentiation in a durable goods market with a perfectly competitive second-hand market. Buyers are allowed to trade the good with each other in the second-hand market. The monopolist, however, is not allowed to enter this market.[1]

4.1 Characterization of Dynamic Optimization

In this section, first, we represent the dynamic optimization problems of the buyers and the monopolist. Then, we discuss the characteristics of equilibrium strategies of the monopolist.

The acceptance function $V^i(b)$ must be consistent with buyer b's intertemporal optimization which requires buyer b to be indifferent between purchasing the good today at the price $V^i(b)$ and waiting one period to purchase it. Hence, $V^i(b)$ is derived from

$$f^i(b) - V^i(b) = \rho(f^i(b) - \acute{p}^i) \tag{4.1}$$

where $\rho \equiv \delta(1-\mu)$ and $\acute{p}^i$ is the expected price of good i in the next period. Since $f^i(\cdot)$ is monotone and deviations of measure zero buyers do not affect the equilibrium, $V^i(\cdot)$ is a non-increasing left-continuous function.

Let x^i denote the stock of the durable good i before trade and y^i denote the stock of the

[1] All proofs of this section are relegated to the Appendix A.

durable good i after trade. The value function of the monopolist is $R(x^H, x^L)$ and must satisfy

$$R(x) = \max_{y^i \in [x^i, 1]} \left\{ P^H(y) \left(y^H - x^H \right) + P^L(y)(y^L - x^L) + \delta R((1-\mu) y) \right\} \tag{4.2}$$

where $\delta \equiv e^{-rz}$ is the discount factor and $P^i(\cdot)$ is the price of durable good i. The price of the high quality good must be consistent with the incentive compatibility constraint of the marginal buyer of the high quality good, and the price of the low quality good must be consistent with the participation constraint of the marginal buyer of the low quality good. Hence, when the stock of the goods after trade is (y^H, y^L), the price of the high quality good is

$$P^H(y) = V^H\left(y^H\right) - V^L\left(y^H\right) + P^L(y), \tag{4.3}$$

and the price of the low quality good is

$$P^L(y) = V^L(y^H + y^L) \tag{4.4}$$

where $V^L(y^H) - P^L(y)$ is the payoff of buyer y^H if she purchases the low quality good at the price $P^L(y)$.

Let $T(\cdot)$ denote the argmax correspondence of the objective function. By the generalized theorem of the maximum and the contraction mapping theorem, there exists a unique continuous function $R(\cdot)$, and $T(\cdot)$ is a non-empty and compact valued correspondence.[2] Moreover, the supermodularity of the objective function implies that $T(\cdot)$ is non-decreasing. It follows that there exists at most countable number of points for which $T(\cdot)$ is multi-valued. Even though the monopolist is allowed to use behavioral strategies, we establish that

Proposition 1. *The monopolist does not randomize along any equilibrium path and chooses the minimum of the argmax correspondence with probability one unless it is the initial period.*

Therefore, given a state (x^H, x^L), the equilibrium output choice of the monopolist is $t(x) = \min T(x)$. Moreover, the output function, $t^i(\cdot) : [0, 1-\mu] \times [0, 1-\mu] \to \mathbb{R}_+$, is nondecreasing

[2]See Ausubel and Deneckere (1989) for the generalized theorem of the maximum.

since $T(x)$ is a monotone correspondence.

An equilibrium is represented by $\{P^H(\cdot), P^L(\cdot), t^H(\cdot), t^L(\cdot), R(\cdot)\}$. The structure of a stationary path is as follows. In the initial period, the monopolist selects prices: $P^H(y_0)$ and $P^L(y_0)$. All buyers $b \leq y_0^H$ purchase the high quality good and all buyers $y_0^H < b \leq y_0^H + y_0^L$ purchase the low quality good.[3] At the beginning of the next period, the stock of the high quality good is $x_1^H = (1-\mu)y_0^H$ and the stock of the low quality good is $x_1^L = (1-\mu)y_0^L$. The monopolist selects prices, $P^H(y_1) = P^H(t(x_1))$ and $P^L(y_1) = P^L(t(x_1))$, and all buyers $b \leq y_1^H$ choose to hold the high quality good and all buyers $y_1^H < b \leq y_1^H + y_1^L$ choose to hold the low quality good. This continues until a steady state is reached. Once a steady state is reached, the monopolist continues by selling to replacement demands.

In order to construct equilibria of this game, the solution method introduced by Deneckere and Liang (2008) is followed. First, we prove the existence of a steady state in any equilibrium. Then, we characterize all possible steady states. Finally, we derive all stationary paths that reach a steady state by using backward induction from the steady state.

The analysis of this study focuses on the nontrivial case where $\widehat{b}\ \overline{\theta} > \underline{\theta}$. Otherwise, the static monopoly prices would be $\underline{\theta} q_H$ and $\underline{\theta} q_L$.[4] Since a durable goods monopolist who does not have any commitment power can achieve this outcome, the unique stationary steady state of this game when $\widehat{b\theta} < \underline{\theta}$ is the static monopoly outcome.[5]

4.2 Characterization of Steady States

In this section, first, we establish the existence of a steady state in any equilibrium. Then, we characterize all possible steady states that may coexist.

Let a steady state (y_s^H, y_s^L) be defined as the stock levels of durable goods satisfying $t^H((1-\mu)y_s) = y_s^H$ and $t^L((1-\mu)y_s) = y_s^L$. We establish that any equilibrium has a corresponding

[3]If $T(0,0)$ is multi-valued, the monopolist may select the price randomly from $P(T(0,0))$.

[4]Indeed, any price for the low quality good would be an equilibrium price as long as all buyers purchase the high quality good.

[5]For the hairline case $\widehat{b\theta} = \underline{\theta}$, one can use the limit of $\widehat{b\theta} > \underline{\theta}$.

steady state.

Proposition 2. *Any equilibrium has at least one steady state, and the steady state prices satisfy* $P^H\left(y_s^H, y_s^L\right) = f^H\left(y_s^H\right) - f^L\left(y_s^H\right) + f^L\left(y_s^H + y_s^L\right)$ *and* $P^L\left(y_s^H, y_s^L\right) = f^L\left(y_s^H + y_s^L\right)$.

The economic intuition behind the steady state prices is as follows. At a steady state (y_s^H, y_s^L), the marginal buyer of the high quality good is y_s^H and the marginal buyer of the low quality good is $y_s^H + y_s^L$ in each period. Buyer $y_s^H + y_s^L$ is indifferent between today's and tomorrow's offer for the low quality good when the price of the low quality good is $f^L(y_s^H + y_s^L)$. Similarly, buyer y_s^H is indifferent between today's and tomorrow's offer for the high quality good when the price of the high quality good is $f^H(y_s^H)$. However, when the price of the low quality good is $f^L\left(y_s^H + y_s^L\right)$, buyer y_s^H's net surplus from the low quality good is $f^L(y_s^H) - f^L(y_s^H + y_s^L)$. Hence, in order to sell the high quality good to buyer y^H, the monopolist has to leave an information rent no less than $f^L(y_s^H) - f^L(y_s^H + y_s^L)$ to the high type buyers. Therefore, at the steady state, the price of the high quality good is $f^H(y_s^H) - f^L(y_s^H) + f^L(y_s^H + y_s^L)$ and the price of the low quality good is $f^L(y_s^H + y_s^L)$.

Let us consider a market for *a perfectly durable good* served by a monopolist. The monopolist cannot credibly commit to a static monopoly output since he has an irresistible temptation to cut the price to sell the good to the remaining buyers. Thus, in this setting, the static monopoly output would never be a steady state. Deneckere and Liang (2008) point out that when the good depreciates the monopolist may prefer serving to the replacement demand of the high type buyers at a higher price rather than cutting the price in an attempt to increase sales. They show that there exist three types of steady states: a Coase Conjecture, a monopoly, and a reputational steady state.

When a monopolist produces two versions of a durable good, there exist five possible steady states: $(1,0)$, $(\widehat{b},0)$, $(\widehat{b},1-\widehat{b})$, $(\hat{y}^H,0)$, and $(\check{y}^H, 1-\check{y}^H)$ where $\hat{y}^H, \check{y}^H \in (0,\widehat{b})$. At the steady state $(1,0)$, all buyers hold the high quality good after trade and the monopolist serves their replacement demand in each period. This steady state is called the *Coase Conjecture steady state* and the equilibrium having $(1,0)$ as the unique steady state is called the Coase Conjecture equilibrium. At the steady state $(\widehat{b},0)$, all high type buyers hold the high quality good after trade and the monopolist sells to the replacement demand of the high type buyers for the high quality good in each period. At the steady state $(\widehat{b}, 1-\widehat{b})$, all high type buyers hold the high quality good and all low type buyers hold the low quality good after trade, and the monopolist serves the replacement demands in each period. In the one-period version of this game, due to the assumption that $\widehat{b\theta} > \underline{\theta}$, the monopolist sells the high quality good to the high type buyers at the price $\overline{\theta} q_H$ and sets the price of the low quality good high enough so that none of the buyers purchase it. Hence, $(\widehat{b}, 0)$ is called *the static monopoly steady state* and $(\widehat{b}, 1-\widehat{b})$ is called *the segmented monopoly steady state.* The static monopoly steady state always coexists with the segmented monopoly steady state. Depending on the magnitude of μ, the Coase Conjecture steady state $(1, 0)$ may coexist with the monopoly steady states. The equilibrium with a monopoly steady state is called the monopoly equilibrium. Finally, the states $(\hat{y}^H, 0)$ and $(\check{y}^H, 1-\check{y}^H)$ are called *reputational steady states* and the equilibrium corresponds with them is called the reputational equilibrium. At a reputational steady state, the monopolist limits the production of the high quality good and sells it to some of the high type buyers. The Coase Conjecture steady state $(1, 0)$ always coexist with the reputational steady states. These results are summarized by

Proposition 3. *Let S denote the set of steady states. In any equilibrium one of the followings holds:*

1. $S = \{(1, 0)\}$;
2. $S = \{(\widehat{b}, 1-\widehat{b}), (\widehat{b}, 0)\}$ *or* $S = \{(\widehat{b}, 1-\widehat{b}), (\widehat{b}, 0), (1, 0)\}$ *or* $S = \{(\widehat{b}, 1-\widehat{b}), (1, 0)\}$;
3. $S = \{(\hat{y}^H, 0), (\hat{y}^H, 1-\hat{y}^H), (1, 0)\}$ *or* $S = \{(\check{y}^H, 1-\check{y}^H), (1, 0)\}$ *where* $\check{y}^H, \hat{y}^H \in (0, \widehat{b})$.

The intuition behind Proposition 3 comes from the following two observation. First, given the expectations of the buyers, some states cannot be a steady state as the monopolist can profitably deviate from these states. Second, since the number of the steps in f^i is two, if the marginal buyer of the high quality good is a high type then the marginal buyer of the low quality good can be either a high type or a low type, whereas if the marginal buyer of the high quality good is a low type, then the marginal buyer of the low quality good must be a low type as well. Hence, at most three steady states can coexist in an equilibrium.

4.3 Characterization of Equilibria

In this section, we derive all possible equilibria and establish that the equilibrium of each type is unique. Moreover, we analyze the effects of quality differentiation on each type of equilibrium.

4.3.1 The Coase Conjecture Equilibrium

Consider equilibria with a unique steady state at which all buyers, after trade, hold the high quality good. By conducting backward induction from the Coase Conjecture steady states, we construct the stationary paths for all (x^H, x^L).

First, we describe stationary paths when stock of the high quality good is greater than $(1-\mu)\widehat{b}$. Since the stock of the high quality good has to increase in each period in such equilibria, when $x^H > (1-\mu)\widehat{b}$, the marginal buyers are low types thereafter. The low type buyers, anticipating that the monopolist will saturate the entire market with the high quality good, do not accept any price greater than $\underline{\theta}q_i$ for durable good i.[6] Hence, in this case the stationary paths are defined as follows. When none of the buyers hold the low quality good, the monopolist sets the price of the high quality good as $\underline{\theta}q_H$ and the price of the low quality good as $\underline{\theta}q_L$. All buyers hold the high quality good after trade and the monopolist continues by selling the high quality good to the replacement demand μ at the price $\underline{\theta}q_H$ thereafter. When some buyers hold the low quality good, if the stock of the low quality good is low enough, the monopolist prefers selling the high quality good to buyers who do not currently hold any good. Hence, the monopolist sets the price of the high quality good as $\underline{\theta}q_H$ and the price of the low quality good as $\underline{\theta}q_L$ and continues by selling the high quality good to the replacement demand

[6] If buyer $b \in (\widehat{b}, 1]$ is willing to pay more than $\underline{\theta}q_i$ for durable good i, so is buyer $b \in (\widehat{b}, \widehat{b}+\varepsilon)$, since $V^i(.)$ is non-increasing. This implies that b is expecting to make a capital gain by purchasing it. Namely, buyer b expects that the price of good i will increase next period. However, since neither $(\widehat{b}, 1-\widehat{b})$ nor $(\widehat{b}, 0)$ is a steady state, this is not possible.

μ at the price $\underline{\theta}q_H$. However, if the stock of the low quality good falls above this threshold, the monopolist prefers selling the high quality good to all buyers. Due to an excess supply of the low quality good in the second-hand market, the low quality good will be available for free. Hence, the monopolist sets the price of the high quality good as $\underline{\theta}(q_H - q_L)$.[7] All buyers hold the high quality good after trade, and the monopolist continues by selling the high quality good to the replacement demand μ at the price $\underline{\theta}\Delta q$.

Second, we describe the stationary paths when stock of the high quality good is less than $(1-\mu)\widehat{b}$. There exist four paths depending on the state of the low quality good. On path 1, none of the buyers hold the low quality good. On path 2, the marginal buyer of the low quality good is a high type. On path 3, the marginal buyer of the low quality good is a low type. On path 4, there exists an excess supply of the low quality good. The sequence of states $\{(\overline{x}^H_{k,j}, \overline{x}^L_{k,j})\}_{k=2}^{m_j+1}$ for each path $j = 1, \ldots, 4$ is constructed such that when the state is $(\overline{x}^H_{k,j}, \overline{x}^L_{k,j})$, the monopolist is indifferent between bringing the next period's state to $(\overline{x}^H_{k-1,j}, \overline{x}^L_{k-1,j})$ by charging $(\overline{p}^H_{k-1,j}, \overline{p}^L_{k-1,j})$ and bringing the next period's state to $(\overline{x}^H_{k-2,j}, \overline{x}^L_{k-2,j})$ by charging $(\overline{p}^H_{k-2,j}, \overline{p}^L_{k-2,j})$. The sequence $\{(\overline{p}^H_{k,j}, \overline{p}^L_{k,j})\}_{k=0}^{m_j+1}$ is set such that the incentive compatibility constraint of the marginal buyer of the high quality (4.3) and the participation constraint of the marginal buyer of the low quality good (4.4) hold.

Figure 1 illustrates how states move towards the Coase Conjecture steady state. When none of the buyers hold the low quality good, the movement is as follows. For $y^H \leq y$, the stock of the high quality good after trade in the next period will be $\widehat{b}$. For $y^H > y$, the monopolist will penetrate the entire market with the high quality good in the next period and reach the Coase Conjecture steady state, and continue by serving the high quality good to the replacement demand of all buyers thereafter. When some buyers hold the low quality good, the arrows indicate the direction of movement of the state at any (y^H, y^L) to the Coase Conjecture steady state. Figure 1 also indicates that as the stock of the low quality good increases, the real time that passes to reach the Coase Conjecture steady state increases as well.

[7] For ease of exposition from now on $q_H - q_L$ is referred as Δq.

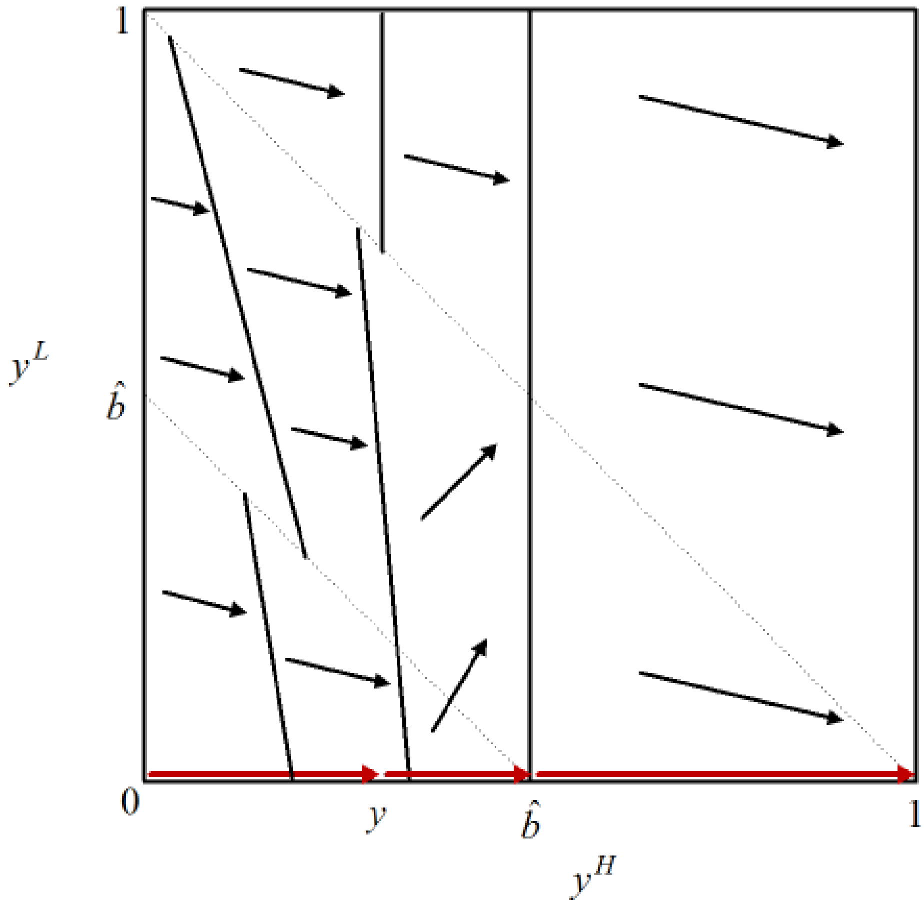

Figure 4.1: The Coase Conjecture Equilibrium

Now, let us illustrate the movement of the high quality good towards the Coase Conjecture steady state with its corresponding prices when some high type buyers hold the low quality good. For expositional purposes, the monopolist is assumed to immediately penetrate the market with the high quality good by charging $\underline{\theta}\Delta q$ for the high quality good when $x^H \geq (1-\mu)\widehat{b}$ and $x^L > 0$. If $x^H \in (\overline{x}^H_{k+2,2}, \overline{x}^H_{k+1,2}]$ for $k \geq 1$, the monopolist sets the price of good i as $\overline{p}^i_{k,2}$ so that after trade the stock of good i will be $\overline{y}^i_{k,2}$. If $x^H \in (\overline{x}^H_{1,2}, 1-\mu]$, the monopolist sets the price of the high quality good as $\underline{\theta}\Delta q$ to penetrate the entire market with the high quality good. Then, all buyers will hold the high quality good and the low quality good will be available for free. Figure 2 illustrates how the price and the stock of the high quality good evolves along path 2. For ease of exposition $\overline{y}^H_{k,2}$ is referred as y_k and refer $\overline{p}^H_{k,2}$ as p_k. The arrows indicate the direction of the movement of a state.

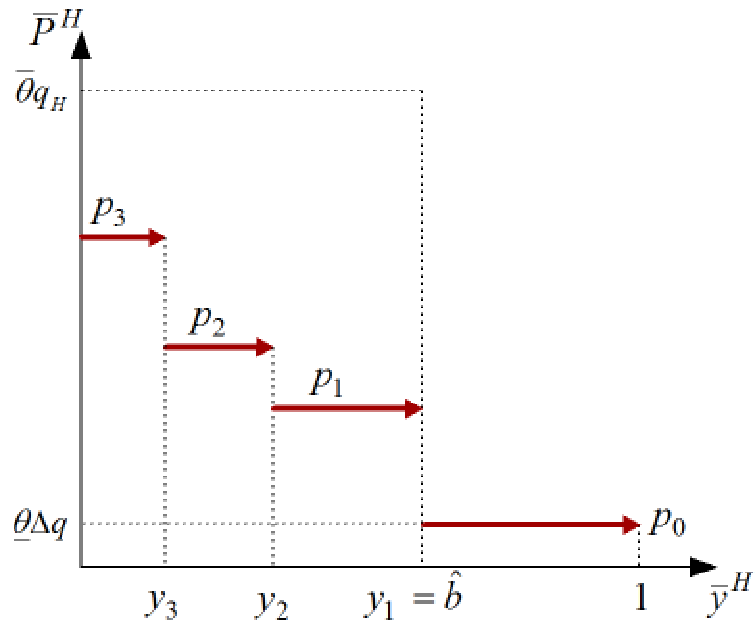

Figure 4.2: Path 2

We establish that existence of a Coase Conjecture equilibrium requires that $\{\overline{x}^H_{k,j}\}_{k=0}^{m_j+1}$ must be strictly decreasing and must satisfy $\overline{x}^H_{m_j+1,j} \leq 0 < \overline{x}^H_{m_j,j}$. The set of parameters supporting this condition is derived and it is proved that the equilibrium is unique.

Theorem 1. *There exists at most one Coase Conjecture equilibrium if and only if $\mu < \overline{\mu}$ for some $\overline{\mu} \in (0,1)$.*

The Coase Conjecture equilibrium does not exist for sufficiently perishable goods. Because, for large μ, rather than fully penetrating the market with the high quality good at the price $\overline{p}^H_{0,j}$, the monopolist would sell to the replacement demands $\mu\overline{y}^H_{1,j}$ and $\mu\overline{y}^L_{1,j}$ at the prices $\overline{p}^H_{1,j}$ and $\overline{p}^L_{1,j}$, respectively.

The differences among the paths with respect to the high quality good is illustrated to analyze how the production of the low quality good affects the stock and the price of the high quality good and to analyze whether the production of the low quality good helps the monopolist maintain his market power. For ease of exposition $\overline{y}^H_{k,j}$ is referred as $y_{k,j}$ and $\overline{p}^H_{k,j}$ is referred as $p_{k,j}$.

First, we study the effect of the low quality good when the initial stock of the low quality good is zero. We must compare path 1 on which none of the buyers hold the low quality good

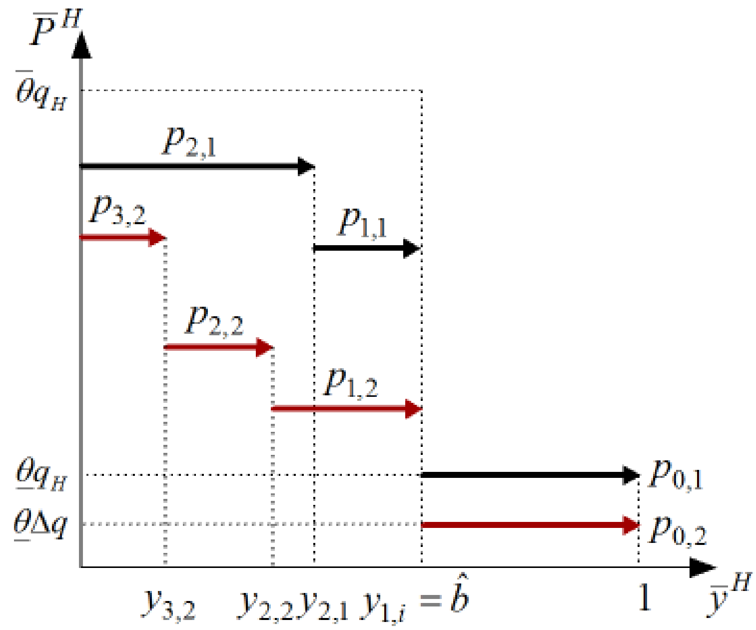

Figure 4.3: Path 1 vs Path 2

with path 2 on which the marginal buyer of the low quality good is a high type. Figure 3 illustrates these two paths. We observe that when some high types hold the low quality good, the price of the high quality good diminishes. For all $x^H < (1-\mu)\widehat{b}$, on path 2 the marginal buyer of the low quality good is a high type. Hence, information rent cannot explain the price difference between these two paths. However, while moving towards the steady state along path 2, once all high type buyers hold the high quality good, the marginal buyer of the low quality good becomes a low type. When some low type buyers hold the low quality good, to make high type buyers hold the high quality good rather than the low quality good, the monopolist has to leave some information rent to the high type buyers. Hence, the high type buyers, anticipating that the monopolist will eventually lower the price of the high quality good, are willing accept a price for the high quality good that is significantly lower than the price when none of the buyers hold the low quality good. In addition to the price difference, we also observe that when some high types hold the low quality good, the time elapses to reach the Coase Conjecture steady state is longer.

Second, we study the effect of the low quality good when some high types hold the low quality good. We must compare path 2 on which the marginal buyer of the low quality good

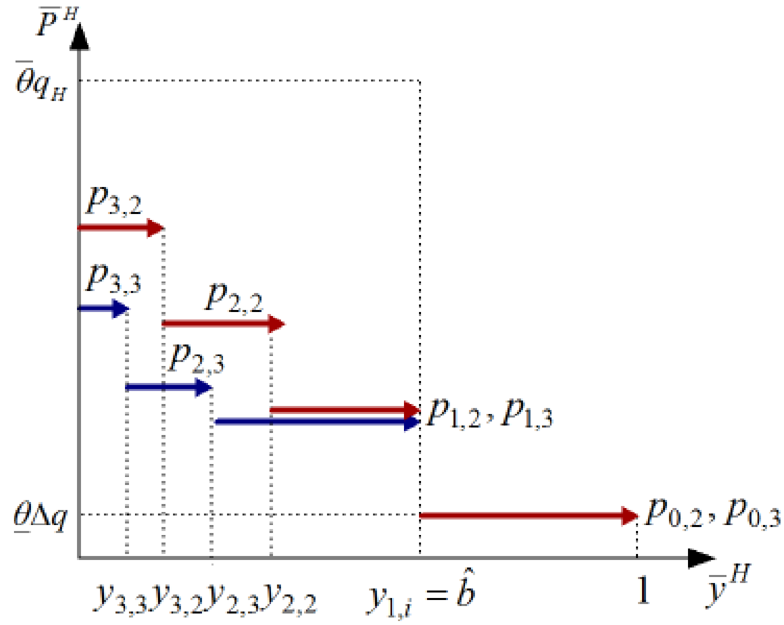

Figure 4.4: Path 2 vs Path 3

is a high type with path 3 on which the marginal buyer of the low quality good is a low type. Figure 4 illustrates these two paths. We, first, observe that the price of the high quality good is lower on path 3. The reason is that on path 3 the monopolist has to leave some information rent to the high type buyers to eliminate their incentives to buy the low quality good rather than the high quality good. In addition to the price difference, we also observe that the time elapses to reach the Coase Conjecture steady state is longer on path 3.

Last, we study the effect of the low quality good when there is an excess supply of the low quality good. In this case, the low quality good is available for free. Since, the monopolist does not make any profits from the low quality good when there is an excess supply of the low quality good, on path 4 the stock of the high quality good does not depend on the stock of the low quality good. Figure 5 illustrates path 1 and path 4. We can see that the stock of the high quality good is the same on these two paths. Moreover, since the low quality good is available for free on path 4, the information rent that the monopolist has to leave to the high type buyers is greater than the information rent on path 3. Hence, on path 4 the price of the high quality good is even lower.

When none of the buyers hold the low quality good, the monopolist fully penetrates the

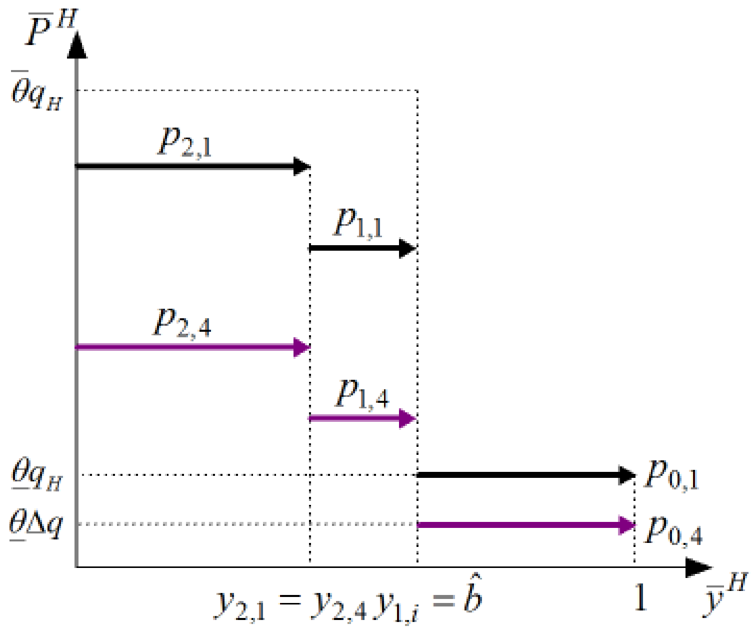

Figure 4.5: Path 1 vs Path 4

market by charging $\underline{\theta}q_H$ for the high quality good. However, when some of the buyers hold the low quality good, the monopolist either fully penetrates the market with the high quality good at the price $\underline{\theta}\Delta q$ or sells the high quality good to some of the low type buyers at the price $\underline{\theta}q_H$ and reaches the steady state gradually. Moreover, we observe from the figures that as the stock of the low quality good increases, so does the real time that passes before the monopolist reaches the Coase Conjecture steady state. It follows that the threshold depreciation rate supporting the Coase Conjecture equilibrium must be smaller when x^L is positive. Therefore,

Corollary 1. *The domain of the parameters consistent with the Coase Conjecture equilibrium is smaller when the monopolist can produce multiple goods that differ in quality.*

The monopolist fully penetrates the entire market with the high quality good in m_j periods on path j. Hence, the real time that elapses until the market is fully penetrated with the high quality good is $m_j z$. We show that m_j has a finite limit $\widehat{m}_j$ regardless of the state of the low quality good. Therefore,

Corollary 2. *In the Coase Conjecture equilibrium, the initial price of the high quality durable good converges to the lowest buyer valuation $\underline{\theta}q_H$ as the length of the time period between* *successive offers approaches zero.*

We show that $\widehat{m}_1 = \widehat{m}_4 = \widehat{m}$ and $\widehat{m} < \widehat{m}_j$ for $j = 2, 3$ where $\widehat{m}$ is the corresponding limit of a durable goods monopolist selling a single version of the good. Additionally, we establish that $\widehat{m}_j$ for $j = 2, 3$ increases with the state of the low quality good and decreases with the quality difference between the versions of the good. Therefore,

Corollary 3. *When the time period between successive price changes is arbitrarily small, as the quality difference between two versions of the durable good increases, the real time that elapses until the market is fully penetrated with the high quality good converges to $\widehat{m}$.*

4.3.2 The Monopoly Equilibrium

Consider equilibria in which the monopolist credibly commits not to selling the high quality good to low type buyers. The monopoly steady states of such equilibria are $(\widehat{b}, 0)$ and $(\widehat{b}, 1-\widehat{b})$.

The necessary conditions for the existence of a monopoly equilibrium are as follows. First, when the state before trade is $((1-\mu)\widehat{b}, 0)$, the monopolist must prefer selling the high quality good to the high type buyers' replacement demand $\mu\widehat{b}$ at the price $\overline{\theta}q_H$ forever to selling the high quality good to buyers who do not hold the high quality good $(1-(1-\mu)\widehat{b})$ at the price $P^H(1,0)$ and continuing by selling the high quality good to all buyers' replacement demand μ at the price $P^H(1,0)$ thereafter. If

$$\frac{\mu\widehat{b\overline{\theta}}q_H}{1-\delta} \geq (1-(1-\mu)\widehat{b})P^H(1,0) + \frac{\delta\mu}{1-\delta}P^H(1,0) \tag{4.5}$$

holds, the monopolist never cuts the price of the high quality good to $P^H(1,0)$ to serve the high quality good to all buyers. Since $P^H(1,0) \leq \underline{\theta}q_H$, (4.5) holds for $\mu \geq \frac{(1-\delta)(1-\widehat{b})\underline{\theta}}{\widehat{b}\Delta\theta - \delta(1-\widehat{b})\underline{\theta}} \equiv \underline{\mu}^{st}$.

Second, when the state before trade is $((1-\mu)\widehat{b}, 0)$, the monopolist must prefer selling the high quality good to the high type buyers' replacement demand $\mu\widehat{b}$ at the price $\overline{\theta}q_H$ forever to selling the high quality good to the high type buyers' replacement demand $\mu\widehat{b}$ at the price $P^H(\widehat{b}, 1-\widehat{b})$ forever and selling the low quality good to the low type buyers $1-\widehat{b}$ at the price $P^L(\widehat{b}, 1-\widehat{b})$ and continuing by selling the low quality good to their replacement demand $\mu(1-\widehat{b})$

at the price $P^L(\widehat{b}, 1-\widehat{b})$ thereafter. If

$$\frac{\mu\widehat{b}\overline{\theta}q_H}{1-\delta} \geq \frac{\mu\widehat{b}}{1-\delta}P^H(\widehat{b}, 1-\widehat{b}) + (1-\widehat{b})P^L(\widehat{b}, 1-\widehat{b}) + \frac{\delta\mu(1-\widehat{b})}{1-\delta}P^L(\widehat{b}, 1-\widehat{b}) \tag{4.6}$$

holds, the monopolist never cuts the price of the low quality good to $P^L(\widehat{b}, 1-\widehat{b})$ in order to serve the low quality good to the low type buyers. Since $P^H(\widehat{b}, 1-\widehat{b}) \leq \overline{\theta}q_H - \overline{\theta}q_L + P^L(\widehat{b}, 1-\widehat{b})$ and $P^L(\widehat{b}, 1-\widehat{b}) \leq \underline{\theta}q_L$, (4.6) holds if $\mu \geq \underline{\mu}^{st}$. Therefore, the monopolist does not deviate from the static monopoly steady state $(\widehat{b}, 0)$ when $\mu \geq \underline{\mu}^{st}$.

Third, when the state before trade is $((1-\mu)\widehat{b}, (1-\mu)(1-\widehat{b}))$, the monopolist must prefer selling the high quality good to the high type buyers' replacement demand $\mu\widehat{b}$ at the price $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ and selling the low quality good to the low type buyers' replacement demand $\mu(1-\widehat{b})$ at the price $\underline{\theta}q_L$ forever to penetrating the entire market with the high quality good by charging $P^H(1, (1-\mu)(1-\widehat{b}))$ and continuing by selling the high quality good to all buyers' replacement demand μ at the price $P^H(1, x^L)$ thereafter. If

$$\frac{\mu\widehat{b}(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L)}{1-\delta} + \frac{\mu(1-\widehat{b})\underline{\theta}q_L}{1-\delta} \geq (1-(1-\mu)\widehat{b})P^H(1, x^L) + \frac{\delta\mu}{1-\delta}P^H(1, x^L) \tag{4.7}$$

holds, the monopolist never cuts the price of the high quality good to $P^H(1, x^L)$ to serve the high quality good to all buyers. Since $P^H(1, x^L) \leq \underline{\theta}\Delta q$ for all $x^L > 0$, (4.7) holds if $\mu \geq \frac{(1-\delta)(1-\widehat{b})\underline{\theta}\Delta q}{\widehat{b}\Delta\theta\Delta q - \delta(1-\widehat{b})\underline{\theta}\Delta q + \underline{\theta}q_L} \equiv \underline{\mu}^{sg}$. Therefore, the monopolist does not deviate from the segmented monopoly steady state $(\widehat{b}, 1-\widehat{b})$ for $\mu \geq \underline{\mu}^{sg}$. Moreover, since $P^H(1, 0) > P^H(1, x^L)$ for all $x^L > 0$, $\underline{\mu}^{sg} < \underline{\mu}^{st}$ must hold.

It is established that the necessary condition for the existence of a monopoly equilibrium, $\mu \geq \underline{\mu}^{sg}$, is also sufficient for the existence and the uniqueness of the equilibrium.

Theorem 2. *There exists at most one monopoly equilibrium iff $\mu \geq \underline{\mu}^{sg}$. The monopoly steady states of such equilibrium are $\{(\widehat{b}, 1-\widehat{b})\}$ for $\underline{\mu}^{sg} \leq \mu < \underline{\mu}^{st}$ and $\{(\widehat{b}, 0), (\widehat{b}, 1-\widehat{b})\}$ for $\mu \geq \underline{\mu}^{st}$.*

In such an equilibrium, for $\mu \geq \underline{\mu}^{st}$ the monopolist initially charges $\overline{\theta}q_H$ for the high quality good and charges a price for the low quality good high enough that none of the buyers purchase it. Hence, from the initial state $(0, 0)$ the monopolist brings the state to $(\widehat{b}, 0)$ by selling the high quality good to all high types. He then continues to charge the static monopoly prices to serve the replacement demand of the high type buyers for the high quality good. For $\underline{\mu}^{sg} \leq \mu < \underline{\mu}^{st}$ the monopolist initially charges $\overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L$ for the high quality good and $\underline{\theta} q_L$ for the low quality good. Hence, from the initial state $(0, 0)$ the monopolist brings the state to $(\widehat{b}, 1 - \widehat{b})$ by selling the high quality good to all high types and selling the low quality good to all low types. He then continues to charge the segmented monopoly prices to serve the replacement demands.

If the monopolist deviates from a monopoly steady state by selling more of the high quality good in an attempt to increase profits, the movement of the states in a monopoly equilibrium is as follows. The sequence of states $\{(\widetilde{x}^H_{k,j}, \widetilde{x}^L_{k,j})\}_{k=0}^{m_j+1}$ is constructed such that when that state is $(\widetilde{x}^H_{k,j}, \widetilde{x}^L_{k,j})$, the monopolist is indifferent between bringing the next period's state to $(\widetilde{x}^H_{k-1,j}, \widetilde{x}^L_{k-1,j})$ by charging $(\widetilde{p}^H_{k-1,j}, \widetilde{p}^L_{k-1,j})$ and staying at $(\widetilde{x}^H_{k,j}, \widetilde{x}^L_{k,j})$ by charging $(\widetilde{p}^H_{k,j}, \widetilde{p}^L_{k,j})$ forever. There exist three paths depending on the state of the low quality good. On all paths, the initial value of the state of the high quality good is $\widetilde{x}^H_{0,j} = (1 - \mu)\widehat{b}$ and its end value is $\widetilde{x}^H_{m_j+1,j} \leq 1 - \mu$.

On path 1, none of the buyers hold the low quality good. For all $x^H \leq \widetilde{x}^H_{m_1+1,1}$, path 1 reaches the static monopoly steady state. On path 2, some buyers hold the low quality good. For all $x^H \leq \widetilde{x}^H_{m_2+1,2}$, path 2 reaches the segmented monopoly steady state. Since the low type buyers anticipate that the price of the low quality good will be eventually equal to $\underline{\theta} q_L$, the monopolist cannot charge more than $\underline{\theta} q_L$ for the low quality good. Hence, $\widetilde{p}^L_{k,2} = \underline{\theta} q_L$ for all k. On path 3, there exists an excess supply of the low quality good. Hence, $\widetilde{p}^L_{k,3} = 0$ for all k. For all $x^H \leq \widetilde{x}^H_{m_3+1,3}$, path 3 reaches the segmented monopoly steady state.

If the good is sufficiently perishable, the monopolist will return to a monopoly steady state from any state of the high quality good above $(1 - \mu)\widehat{b}$. Hence, we have $\widetilde{x}^H_{m_j+1,j} = 1 - \mu$. *Otherwise,* the Coase Conjecture steady state coexists with monopoly steady states. In this case, we have $\widetilde{x}^H_{m_j+1,j} < 1 - \mu$. Hence, when the state is $(\widetilde{x}^H_{m_j+1,j}, \widetilde{x}^L_{m_j+1,j})$, the monopolist is indifferent between bringing the state to $(\widetilde{y}^H_{m_j,j}, \widetilde{y}^L_{m_j,j})$ and fully penetrating the market by selling the high quality good to all buyers and continuing by serving the replacement demand μ for the high quality good thereafter. It follows that the monopolist strictly prefers penetrating

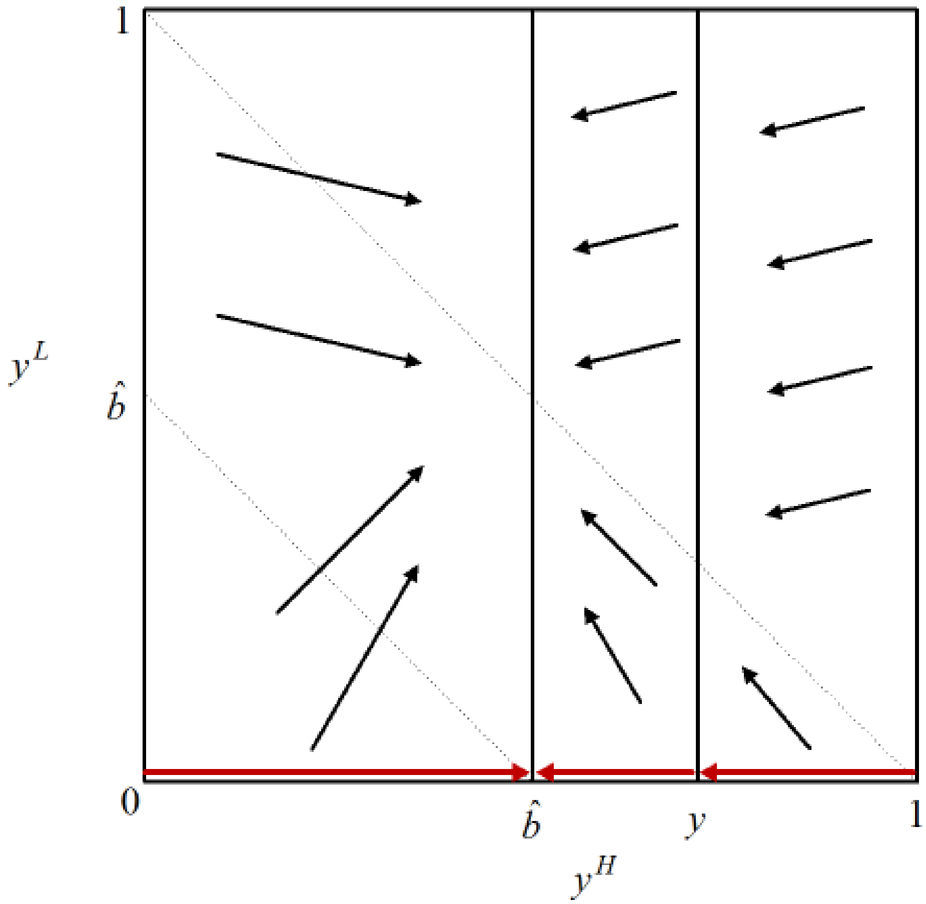

Figure 4.6: The Monopoly Equilibrium I

the market with the high quality good when the state of the high quality good is greater than $\widetilde{x}^H_{m_j+1,j}$.

When the state is moving towards a monopoly steady state, low type buyers purchase the high quality good at a price exceeding their valuation, in order to make capital gains by reselling it in the second-hand market at a later date. If none of the buyers hold the low quality good, the steady state of the market will be $(\widehat{b}, 0)$; otherwise it will be $(\widehat{b}, 1 - \widehat{b})$.

Figure 6 illustrates how states move to a monopoly steady state when the good is sufficiently perishable. The arrows indicate the direction of movement of the state at any (y^H, y^L). When none of the buyers hold the low quality good, the states move towards the static monopoly steady state from any $(y^H, 0)$ as follows. For $y^H \leq y$, the stock of the high quality good after trade in the next period will be $\widehat{b}$ and the monopolist will continue by selling the high quality

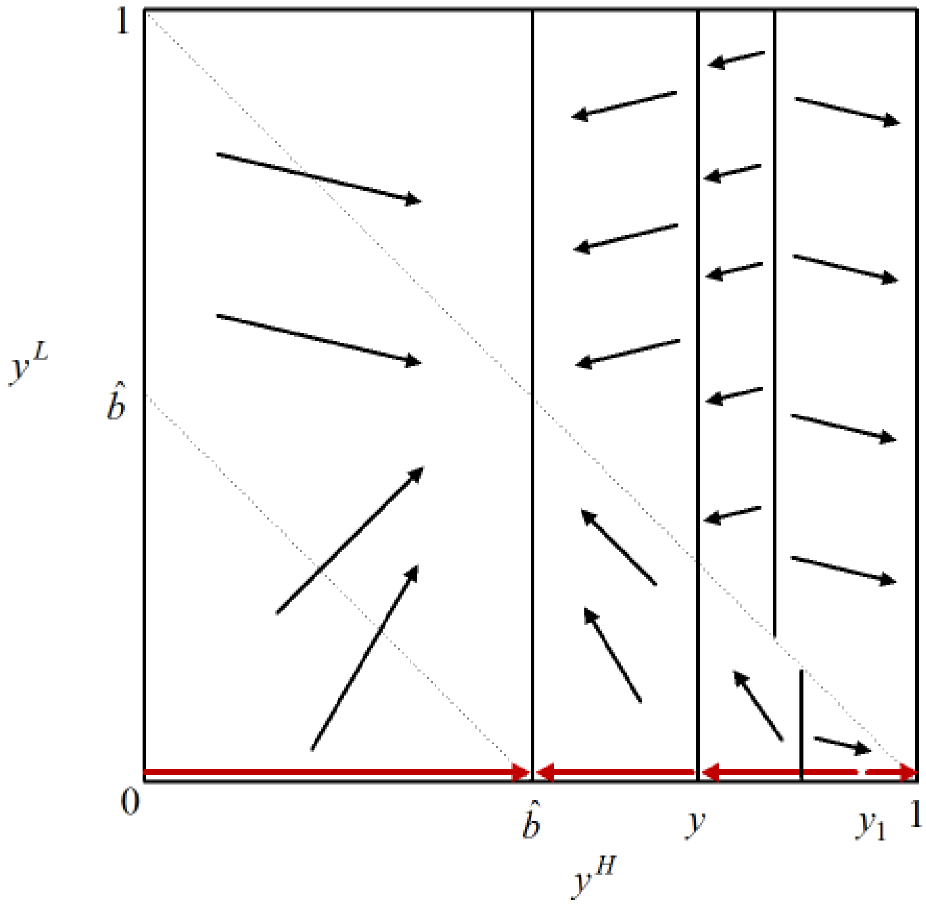

Figure 4.7: The Monopoly Equilibrium II

good to the replacement demand of the high type buyers thereafter. For $y^H > y$, the stock of the high quality good after trade in the next period will be y. When some buyers hold the low quality good, the states move towards the segmented monopoly steady state along the arrows.

Figure 7 illustrates how states move to the a monopoly steady state when the good is not sufficiently perishable. The arrows indicate the direction of movement of the state at any (y^H, y^L). If y^H is low, the states move towards a monopoly steady state, otherwise the states move towards the Coase Conjecture steady state. When none of the buyers hold the low quality good, the states move as follows. For $y^H \leq y$ the stock of the high quality good after trade in the next period will be $\widehat{b}$ and the monopolist will continue by selling the high quality good to replacement demand of the high type buyers thereafter. For $y < y^H \leq y$, the stock of the high quality good after trade in the next period will be y. For $y^H > y$, the stock of the

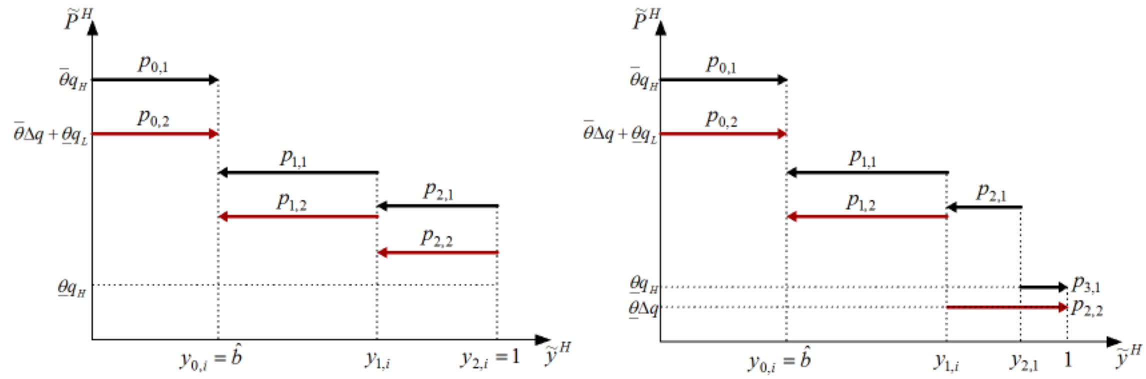

Figure 4.8: The Monopoly Equilibrium

high quality good after trade in the next period will be 1 and the monopolist will continue by selling the high quality good to replacement demand of all buyers thereafter. When some buyers hold the low quality good, the states move towards a steady state along the arrows.

Figure 8 illustrates the movement of the high quality good towards a monopoly steady state with its corresponding prices. For expositional purposes only path 1 and path 2 are studied, and when the monopolist moves towards the Coase Conjecture steady state, the monopolist is assumed to immediately penetrate the market with the high quality good by charging $\underline{\theta}\Delta q$ for the high quality good. For ease of exposition $\widetilde{y}^H_{k,j}$ is referred as $y_{k,j}$ and $\widetilde{p}^H_{k,j}$ is referred as $p_{k,j}$. On the left graph the state moves back to a monopoly steady state from any state, whereas on the right graph the state moves towards the Coase Conjecture steady state when the stock of the high quality good is high. The arrows indicate the direction of the movement. We can observe from these graphs that production of a low quality good lowers the price of the high quality good in a monopoly equilibrium.

Let $\underline{\mu}^s(\delta)$ be the threshold depreciation rate derived in Deneckere and Liang (2008). When a monopolist produces a single version of a durable good, a monopoly equilibrium exists for all $\mu > \underline{\mu}^s(\delta)$. It is established that

Corollary 4. *The threshold depreciation rate supporting the static monopoly steady state* $\underline{\mu}^{st}$ *is a function of* δ *such that* $\underline{\mu}^{st}(\delta) = \underline{\mu}^s(\delta)$ *holds for all* δ*.*

Since the segmented monopoly steady state is supported for $\underline{\mu}^{sg} \leq \mu < \underline{\mu}^{st}$, the set of parameters consistent with the monopoly equilibrium expands when a monopolist produces a low quality good as well as a high quality good.

Now, consider the structure of the stationary path as the time period between successive offers of the monopolist diminishes. For $x^H > (1-\mu)\widehat{b}$, the state either immediately moves to the Coase Conjecture steady state or slowly goes back to a monopoly steady state. It is established that the rate at which a monopoly steady state is reached is independent of the state of the low quality good and that

Corollary 5. *As the length of the time period between successive price changes approaches zero, the state of the high quality good moves towards a monopoly steady state at the rate of* $\dot{x}^H = \lambda x^H \left(1 - \frac{\underline{\theta}}{\overline{\overline{\theta}}}\left(\frac{y^H}{\widehat{b}}\right)^{\frac{\lambda+r}{\lambda}}\right)$ *for* $x^H > (1-\mu)\widehat{b}$.

4.3.3 The Reputational Equilibrium

Consider equilibria in which the monopolist establishes a reputation by cutting the production of the high quality good. The stock of the high quality good at a reputational steady state falls short of the static monopoly output of the high quality good. The reputational steady states of such equilibria are $(\check{y}^H, 1-\check{y}^H)$ and $(\hat{y}^H, 0)$ where $\hat{y}^H$, $\check{y}^H \in (0, \widehat{b})$.

If μ is sufficiently low, from the initial state $(0,0)$, the monopolist will immediately bring the state to $(\check{y}^H, 1-\check{y}^H)$ by charging $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ for the high quality good and $\underline{\theta}q_L$ for the low quality good and continue by selling to the replacement demand $\mu\check{y}^H$ for the high quality good at the price $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ and to the replacement demand $\mu(1-\check{y}^H)$ for the low quality good at the price $\underline{\theta}q_L$ thereafter.

If μ falls above this threshold, from the initial state $(0,0)$, the monopolist will immediately bring the state to $(\hat{y}^H, 0)$ by charging $\overline{\theta}q_H$ for the high quality good and charging a price no less than $\overline{\theta}q_L$ for the low quality good and continue by selling to the replacement demand $\mu\hat{y}^H$ for the high quality good at the price $\overline{\theta}q_H$ thereafter.

If the monopolist penetrates the market by selling more of the high quality good, he loses his reputation for pricing high. Since buyers expect that the future prices will be lower, they are reluctant to pay a high price for the high quality good. Hence, the monopolist has to drastically lower the price of the high quality good, and the state slowly moves to the Coase Conjecture steady state $(1,0)$. Therefore, upon deviation from a reputational steady state by increasing the stock of the high quality good, the game follows the Coase Conjecture path. When the steady state is $(\hat{y}^H, 0)$, if the monopolist penetrates the market by selling more of the low quality good, buyers expect that the monopolist will increase the sales of the low quality good and hence, they do not accept any price for the low quality good significantly greater than $\underline{\theta}q_L$. Hence, the monopolist immediately brings the state to $(\hat{y}^H, 1-\hat{y}^H)$ by charging $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ for the high quality good and charging $\underline{\theta}q_L$ for the low quality good and continues by selling to the replacement demands thereafter.

Theorem 3. *There exists a unique reputational equilibrium if and only if $\underline{\mu}^{sg} < \mu \leq \overline{\mu}$. The reputational steady states of such equilibrium are $\{(\check{y}^H, 1-\check{y}^H)\}$ for $\underline{\mu}^{sg} < \mu \leq \mu'$ and $\{(\hat{y}^H, 0), (\hat{y}^H, 1-\hat{y}^H)\}$ for $\mu' < \mu \leq \overline{\mu}$ where $\mu' \geq \underline{\mu}^{st}$.*

The existence of the reputational equilibrium necessitates existence of the Coase Conjecture equilibrium and the monopoly equilibrium. The intuition behind this is as follows. If the Coase Conjecture equilibrium does not exist, neither does the reputational equilibrium. Because when the Coase Conjecture equilibrium does not exist, the steady state stock of the high quality good falls short of a certain level below which a path that the monopolist follows upon deviation from a reputational steady state cannot be constructed. Moreover, the monopoly equilibrium does not exist when the monopolist cannot resist penetrating the market further in an attempt to increase profits. Since, the monopolist makes less profits by limiting the production of the high quality good, when the monopoly equilibrium does not exist, neither does the reputational equilibrium.

The structure of the reputational equilibrium, when the time period between successive price changes is infinitesimal is studied and it is established that

Corollary 6. *Let $\check{y}^H = \frac{\underline{\theta}}{\overline{\theta}}$ and $\hat{y}^H = \frac{(\lambda+r)\underline{\theta}}{\lambda\overline{\theta}+r\underline{\theta}}$. As the length of the time period between successive price changes approaches zero, the reputational steady state converges to $(\check{y}^H, 1-\check{y}^H)$ for $\mu \in (\underline{\mu}^{sg}, \underline{\mu}^{st}]$, and converges to $(\hat{y}^H, 0)$ for $\underline{\mu}^{st} < \mu \leq \overline{\mu}$.*

4.4 Coexistence of Equilibria: Single Good vs Multiple Goods

In this section, first, we study an example to analyze how production of a low quality product affects the existence and the uniqueness of each type of equilibrium. Then, we characterize the set of parameters supporting each type of equilibrium, when the time period between successive price changes is arbitrarily small.

Let $\underline{\mu}^s(\delta)$ and $\overline{\mu}^s(\delta)$ be the threshold depreciation rates when the monopolist produces a single version of a durable good.[8] It is established that

Proposition 4. *The threshold depreciation rates are functions of δ such that $\underline{\mu}^{sg}(0) < \overline{\mu}(0) < \underline{\mu}^s(0) = \overline{\mu}^s(0)$, and $\lim_{\delta\to 1}\underline{\mu}^{sg}(\delta) = \lim_{\delta\to 1}\underline{\mu}^{st}(\delta) = 0$ and $\lim_{\delta\to 1}\overline{\mu}(\delta) = \lim_{\delta\to 1}\overline{\mu}^s(\delta)$.*

We study an example to identify how production of a low quality good affects the parameters consistent with each type of equilibrium. Consistent with the example in Deneckere and Liang (2008), we assume that $\underline{\theta} = 0.6\,\overline{\theta}$, $\widehat{b} = 0.7$, and $q_H = 3\,q_L$.

First, let us study Figure 9 which illustrates the range of (μ, δ) where each type of equilibrium exists when the monopolist produces only a high quality good. As discussed in detail by Deneckere and Liang (2008), for $\mu < \underline{\mu}^s(\delta)$ the Coase Conjecture equilibrium is the unique equilibrium, for $\underline{\mu}^s(\delta) \leq \mu \leq \overline{\mu}^s(\delta)$ all types of equilibrium coexist, and for $\mu > \overline{\mu}^s(\delta)$, the monopoly equilibrium is the unique equilibrium. The intuition behind this result is as follows. As depreciation factor μ increases, profits from replacement sales increase. Hence, rather than penetrating the market further, the monopolist prefers serving replacement demands.

Second, let us study Figure 10 which illustrates the range of (μ, δ) supporting each type of equilibrium when the monopolist produces a low quality good as well as the high quality good. For expositional purposes, we refer $\underline{\mu}^{sg}(\delta)$ as $\underline{\mu}(\delta)$. We can observe that with multiple goods differ in quality, the range of (μ, δ) consistent the Coase Conjecture equilibrium is smaller and the range of (μ, δ) consistent with the monopoly equilibrium is larger. The economic intuition behind this result is as follows. For $\overline{\mu}(\delta) \leq \mu < \overline{\mu}^s(\delta)$, the Coase Conjecture equilibrium does not exist when the monopolist produces multiple goods, since the depreciation rate is

[8] See Deneckere and Liang (2008) for details.

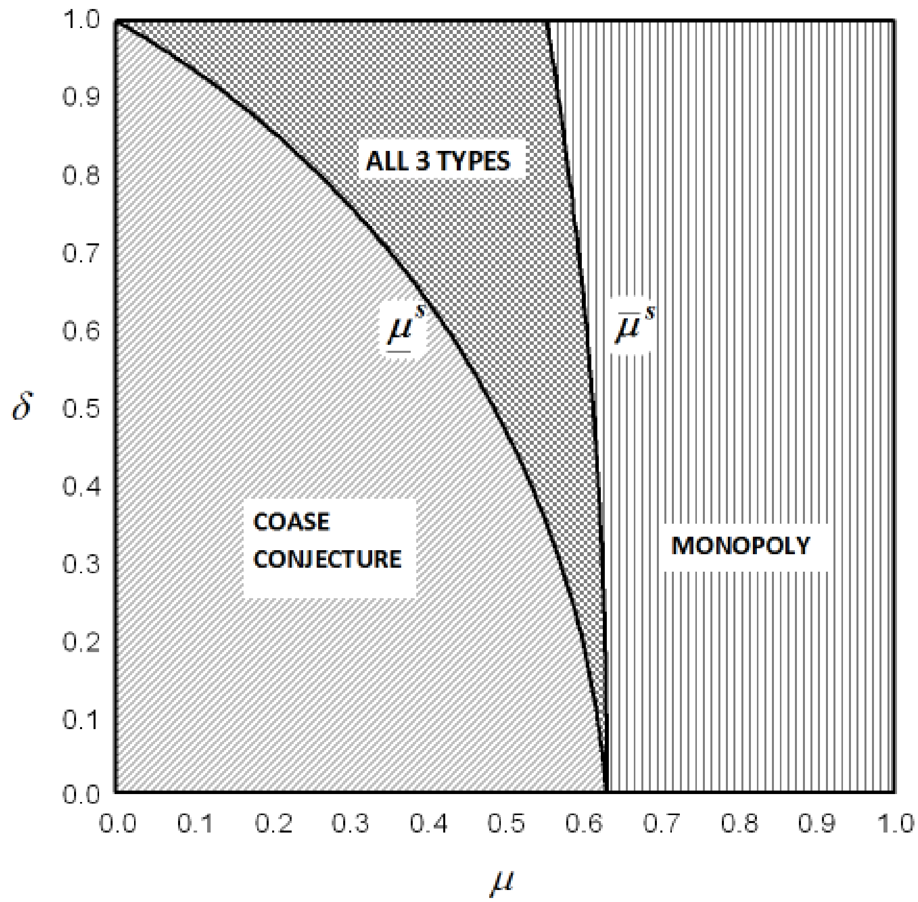

Figure 4.9: Support of Each Type of Equilibrium: Single Good

high enough so that when the state is $(\widehat{b}, 1-\widehat{b})$, the monopolist prefers selling to replacement demands rather than fully penetrating the market with the high quality good. Moreover, for $\underline{\mu}(\delta) < \mu \leq \underline{\mu}^s(\delta)$, a monopoly equilibrium exists when the monopolist produces multiple goods because for a given value of δ, $\mu \in (\underline{\mu}(\delta), \underline{\mu}^s(\delta)]$ supports the segmented monopoly steady state but not the static monopoly steady state.

Figure 10 also helps us identify the structure of equilibria when the monopolist can adjust the prices frequently. As the length of the time period diminishes, (μ, δ) converges to $(0, 1)$ for all $r > 0$ and $\lambda < \infty$. Since $\mu(\delta) = 1 - \delta^{\frac{\lambda}{r}}$, $(\mu(\delta), \delta)$ lies below $(\underline{\mu}(\delta), \delta)$ when $\frac{\lambda}{r}$ is small. Moreover, when the length of the time period between successive offers of the monopolist is arbitrarily small, the cutoff value λ_0 below which the Coase Conjecture equilibrium is the unique

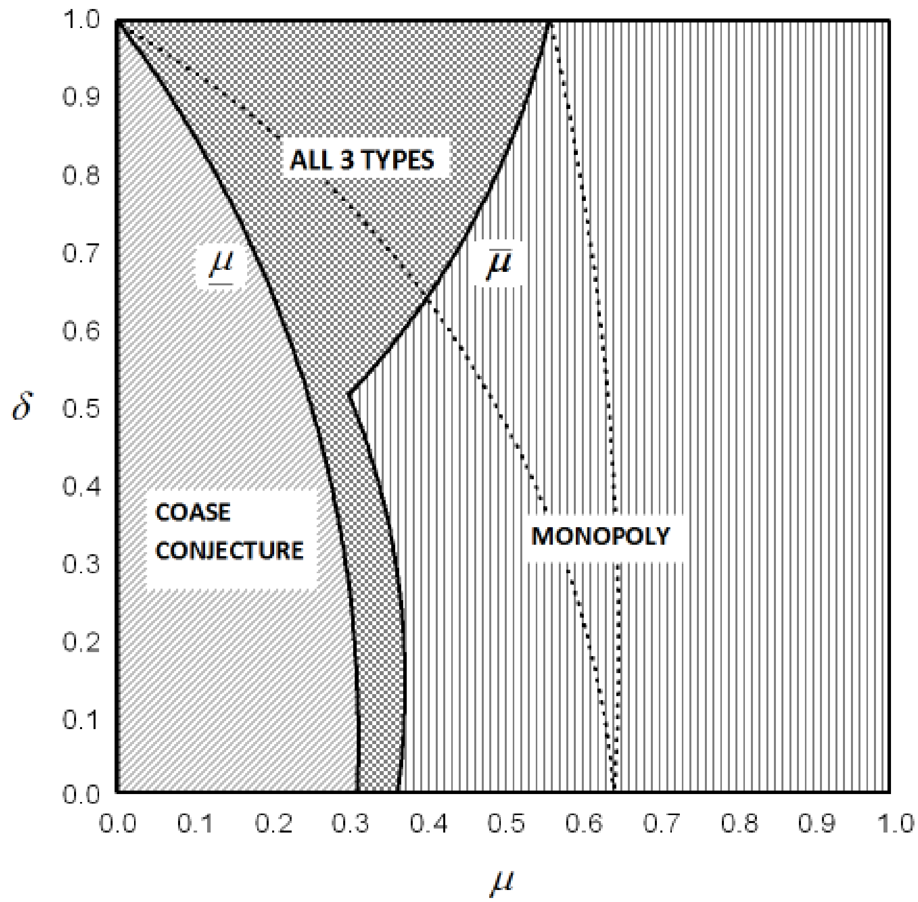

Figure 4.10: Support of Each Type of Equilibrium: Multiple Goods

equilibrium is $\lambda_0 = \frac{r(1-\widehat{b})\underline{\theta}\Delta q}{(\widehat{b\overline{\theta}}-\underline{\theta})\Delta q+\underline{\theta}q_L}$, whereas the corresponding threshold when the monopolist produces a single version of the good is $\lambda_0^s = \frac{r(1-\widehat{b})\underline{\theta}}{(\widehat{b\overline{\theta}}-\underline{\theta})}$. Therefore,

Corollary 7. *(i) Let* $\lambda_0 = \frac{r(1-\widehat{b})\underline{\theta}\Delta q}{(\widehat{b\overline{\theta}}-\underline{\theta})\Delta q+\underline{\theta}q_L}$. *As the length of the time interval between successive offers of the monopolist converges to zero, a Coase Conjecture equilibrium exists for all* $\lambda < \infty$. *When* $\lambda < \lambda_0$, *the Coase Conjecture equilibrium is the unique equilibrium. When* $\lambda \geq \lambda_0$, *all three types of equilibrium coexist.*

(ii) Let $\lambda_0^s = \frac{r(1-\widehat{b})\underline{\theta}}{(\widehat{b\overline{\theta}}-\underline{\theta})}$. *When the monopolist produces single version of the good, as the length of the time interval between successive offers of the monopolist converges to zero, a Coase Conjecture equilibrium exists for all* $\lambda < \infty$. *When* $\lambda < \lambda_0^s$, *the Coase Conjecture equilibrium is the unique equilibrium. When* $\lambda \geq \lambda_0^s$, *all three types of equilibrium coexist.*

Hence, since $\lambda_0 < \lambda_0^s$, it is concluded that for all $z \geq 0$, the monopoly equilibrium is more likely to occur, when the monopolist offers multiple goods differ in quality. For example, when $\underline{\theta} = 0.6\ \overline{\theta}$, $\widehat{b} = 0.7$, $q_H = 3\ q_L$, and $\delta = 0.60$, the threshold depreciation rates are $\underline{\mu} = 0.15$, $\overline{\mu} = 0.41$, and $\underline{\mu}^s = 0.42$, $\overline{\mu}^s = 0.64$. Therefore, if the monopolist only produces the high quality good, the monopoly equilibrium exists when the expected life time is less than 3 years, and is the unique equilibrium when the expected life time is less than 1 year. However, if the monopolist produces the low quality good as well as the high quality good, the monopoly equilibrium exists when the expected life time is less than 6 years, and is the unique equilibrium when it is less than 2 years.

Chapter 5

Buybacks

In this chapter, we study vertical product differentiation in a durable goods market when the monopolist is allowed to buy the goods back from previous buyers.[1]

5.1 Characterization of the Dynamic Optimization

In this section, first, we represent the dynamic optimization problems of the buyers and the monopolist. Then, we discuss the characteristics of equilibrium strategies of the monopolist.

First, let us provide the first result to help us characterize the optimization problem. There are two numbers β^H and β^L summarizing the actions of the previous buyers in a stationary equilibrium.

Proposition 5. *In any stationary equilibrium, for any current price* $p = \left(p^H, p^L\right)$*, there exist cutoff indices* $\beta^H(p)$ *and* $\beta^L(p)$ *with* $\beta^H(p) < \beta^L(p)$ *such that after trade, every buyer with index not exceeding* $\beta^H(p)$ *holds a high quality good, every buyer with index between* $\beta^H(p)$ *and* $\beta^L(p)$ *holds a low quality good, and every buyer with index greater than* $\beta^L(p)$ *holds none of the goods and prefers waiting for the next period's offer.*

Hence, a buyer's strategy in a stationary equilibrium is described by non-increasing left-continuous functions[2], $P^H(\cdot)$ and $P^L(\cdot)$, with buyer b choosing to hold a high quality good (a

[1] All proofs of this section are relegated to the Appendix B.

[2] Since we restrict our attention to stationary equilibria in which bilateral deviations of measure zero buyers does not change actions of the agents, we can assume that acceptance functions are left continuous.

low quality good) in the current period if and only if the current price of the high quality good satisfies $p^H \leq P^H(b)$ $(p^L \leq P^L(b))$.[3]

Let x^i denote the stock of the durable good with quality i before trade and let $V\left(x^H, x^L\right)$ denote the monopolist's net present value of profits. Therefore, we have

$$V\left(x^H, x^L\right) = \max_{\substack{y^H, y^L \geq 0 \\ y^H + y^L \in [0,1]}} \left\{ \begin{array}{c} \wp^H\left(y^H, y^L\right)\left(y^H - x^H\right) + \wp^L\left(y^H, y^L\right)\left(y^L - x^L\right) \\ +\delta V\left(\left(1-\mu\right) y^H, \left(1-\mu\right) y^L\right) \end{array} \right\}$$

where $\delta = e^{-rz}$ represents the discount factor and $\wp^H$ and $\wp^L$ stand for the market price of the high quality good and the market price of the low quality good, respectively. The price of the high quality good satisfies the incentive compatibility constraint of the marginal high type buyer. Hence, we must have

$$\wp^H\left(y^H, y^L\right) \leq P^H\left(y^H\right) - f^L\left(y^H\right) + \wp^L\left(y^H, y^L\right)$$

where $f^L\left(y^H\right) - \wp^L\left(y^H, y^L\right)$ is the surplus of buyer y^H, purchasing the low quality good.

Let $T(\cdot)$ denote the argmax correspondence of the objective function. By the generalized theorem of the maximum (Ausubel and Deneckere (1989)) and the contraction mapping theorem, there exists a unique continuous function $V(\cdot)$, and $T(\cdot)$ is a non-empty and compact valued correspondence. Moreover, supermodularity of the objective function implies that $T(\cdot)$ is non-decreasing. The stock of the high quality good and the stock of the low quality good after trade are represented by $t^H\left(x^H, x^L\right)$ and $t^L\left(x^H, x^L\right)$, respectively, where $\left(t^H\left(x^H, x^L\right), t^L\left(x^H, x^L\right)\right) \in T\left(x^H, x^L\right)$. The acceptance function $P^i(b)$ $i = H, L$ is derived from buyer b's optimization problem. Hence, $P^H(\cdot)$ and $P^L(\cdot)$ should satisfy the arbitrage equations of the buyers,

$$f^H\left(y^H\right) - P^H\left(y^H\right) = \rho\left(f^H\left(y^H\right) - P^H\left(t^H(\cdot)\right)\right)$$

[3] If there exists b such that $p^H \leq P^H(b)$ and $p^L \leq P^L(b)$ then b accepts the offer that gives her the highest payoff.

and

$$f^L\left(y^H+y^L\right) \; - P^L\left(y^H+y^L\right) = \rho\left(f^L\left(y^H+y^L\right) \; - P^L\left(t^H(\cdot)+t^L(\cdot)\right)\right)$$

where $\rho = \delta\left(1-\mu\right)$ and $t^i(\cdot) = t^i\left(\left(1-\mu\right)y^H, \left(1-\mu\right)y^L\right)$, $i = H, L$.

Since $T(\cdot)$ is a non-decreasing compact correspondence, there exist at most countable number of points for which it is multi-valued. The following lemma shows that the monopolist does not gain from randomization.

Proposition 6. *The monopolist does not randomize along any stationary equilibrium path and chooses the minimum element of the argmax correspondence with probability 1 unless it is the initial period.*

Therefore, given the state variables, $\left(x^H, x^L\right)$, the monopolist chooses $\min T\left(x^H, x^L\right)$ unless it is the initial period which implies that the output function $t^i(\cdot) : [0, 1-\mu] \times [0, 1-\mu] \to \mathbb{R}_+$ for all $i = H, L$ is nondecreasing. Since $T(\cdot)$ is upper-hemicontinuous and monotone, $t^i(\cdot)$ is left-continuous as well.

A stationary equilibrium is represented by $\left\{P^H(\cdot), P^L(\cdot), R(\cdot), t^H(\cdot), t^L(\cdot)\right\}$. The structure of a stationary path is as follows. In the initial period, the monopolist selects prices: for the high quality good, $\wp^H\left(y_0^H, y_0^L\right)$, and for the low quality good, $\wp^L\left(y_0^H + y_0^L\right)$. All buyers $b \leq y_0^H$ purchase the high quality good and all buyers $y_0^H < b \leq y_0^H + y_0^L$ purchase the low quality good.[4] At the beginning of the next period, a stock of high quality good is $x_1^H = \left(1-\mu\right)y_0^H$ and the stock of low quality good is $x_1^L = \left(1-\mu\right)y_0^L$. In period 2 the game is repeated and all buyers $b \leq y_1^H$ choose to hold the high quality good and all buyers $y_1^H < b \leq y_1^H + y_1^L$ choose to hold the low quality good. This continues until a steady state is reached. The market prices are derived from acceptance functions, and the price of the high quality good should satisfy the incentive compatibility constraint. We therefore have

$$\wp^H\left(y_i^H, y_i^L\right) = P^H\left(y_i^H\right) - f^L\left(y_i^H\right) + \wp^L\left(y_i^H, y_i^L\right)$$

and

[4] If $T\left(0,0\right)$ is multi-valued, the monopolist may select randomly.

$$\wp^L \left(y_i^H + y_i^L\right) = P^L \left(y_i^H + y_i^L\right)$$

where $P^H \left(y_0^H\right) = P^H \left(t^H (0,0)\right)$ and $P^L \left(y_0^H + y_0^L\right) = P^L \left(t^H (0,0) + t^L (0,0)\right)$, and $P^H \left(y_i^H\right) = P^H \left(t^H \left(x_i^H, x_i^L\right)\right)$ and $P^L \left(y_i^H + y_i^L\right) = P^L \left(t^H \left(x_i^H, x_i^L\right) + t^L \left(x_i^H, x_i^L\right)\right)$ for all $i = 1, 2, \ldots$.

To construct the stationary equilibria, we follow the solution method introduced by Deneckere and Liang (2008). First, we prove the existence of a steady state in any stationary equilibrium. Then, we characterize all possible steady states. Finally, we derive the stationary path that reaches a steady state by using backward induction from the steady state.

5.2 Characterization of Steady States

In this section, after proving the existence of a steady state for any stationary equilibrium, we characterize all possible steady states that may coexist.

Let a steady state $\left(y_s^H, y_s^L\right)$ be defined as stock levels of the durable goods after trade satisfying $t^H \left((1-\mu) y_s^H, (1-\mu) y_s^L\right) = y_s^H$ and $t^L \left((1-\mu) y_s^H, (1-\mu) y_s^L\right) = y_s^L$. The following proposition establishes the existence of a steady state by showing that for any stationary equilibrium there exists at least one corresponding steady state.

Proposition 7. *Any stationary equilibrium has at least one steady state. That is, there exists* $\left(y^H, y^H\right)$ *such that*

$$t^H \left((1-\mu) y^H, (1-\mu) y^L\right) = y^H$$

and

$$t^L \left((1-\mu) y^H, (1-\mu) y^L\right) = y^L$$

where $y^H \in (0,1]$, $y^H + y^L \leq 1$. *Moreover, the steady state prices satisfy*

$$\wp^H \left(y^H, y^L\right) = f^H \left(y^H\right) - f^L \left(y^H\right) + f^L \left(y^H + y^L\right)$$

and

$$\wp^L \left(y^H, y^L\right) = f^L \left(y^H + y^L\right).$$

The steady state prices are derived from marginal buyers' arbitrage equations and the incentive compatibility condition. When the steady state stock levels are y_s^H and y_s^L for the high and the low quality goods, respectively, then in each period the marginal buyer of the high quality good is y_s^H and the marginal buyer of the low quality good is $y_s^H + y_s^L$. Buyer y_s^H is indifferent between today's and tomorrow's offer for the high quality good when the price of the high quality good is $f^H\left(y_s^H\right)$. Similarly, buyer $y_s^H + y_s^L$ is indifferent between today's and tomorrow's offer for the low quality good when the price of the low quality good is $f^L\left(y_s^H + y_s^L\right)$. However, when the price of the low quality good is $f^L\left(y_s^H + y_s^L\right)$, buyer y_s^H's net surplus from the low quality good is $f^L\left(y_s^H\right) - f^L\left(y_s^H + y_s^L\right)$. Therefore, to sell the high quality good to buyer y^H, the monopolist has to leave information rent no less than $f^L\left(y_s^H\right) - f^L\left(y_s^H + y_s^L\right)$. Therefore, at the steady state, the price of the high quality good is $f^H\left(y_s^H\right) - f^L\left(y_s^H\right) + f^L\left(y_s^H + y_s^L\right)$ and the price of the low quality good is $f^L\left(y_s^H + y_s^L\right)$.

The analysis of this study focuses on the nontrivial case where $\widehat{b}\,\overline{\theta} > \underline{\theta}$. Otherwise, the static monopoly prices would be $\underline{\theta} q_H$ and $\underline{\theta} q_L$[5], and a durable goods monopolist who does not have any commitment power can achieve this outcome. This result follows from Lemma 3 which shows that neither version's price is less than the lowest reservation price for that version of the durable good.[6]

The seller never charges a price less than $\underline{\theta} q_H$ for the high quality good and a price less than $\underline{\theta} q_L$ for the low quality good.

Proposition 8. *Since the unique stationary steady state of this game when $\widehat{b\overline{\theta}} < \underline{\theta}$ is the static monopoly outcome, we restrict our attention to $\widehat{b\overline{\theta}} > \underline{\theta}$.*[7]

The result that there always exists a steady state associated with any stationary equilibrium helps us conduct backward induction from each steady state in order to construct the corresponding stationary equilibria. To establish it we first need to derive all possible steady states.

[5]Indeed, any price for the low quality good would be an equilibrium price as long as all buyers purchase the high quality good.

[6]This result is also established by Fudenberg, Levine and Tirole's (1985) Lemma 2 for the single good case.

[7]For the hairline case $\widehat{b\overline{\theta}} = \underline{\theta}$, we can use the limit of $\widehat{b\overline{\theta}} > \underline{\theta}$.

Let us consider a market for *a perfectly durable good* served by a monopolist. The monopolist cannot credibly commit to a static monopoly output since he has an irresistible temptation to cut the price to sell to the remaining buyers. Thus, in this setting, the static monopoly output would never be a steady state. Deneckere and Liang (2008) points out that when the good depreciates the monopolist may prefer selling to the replacement demand of the high type buyers at a higher price rather than cutting the price in an attempt to increase the sales. They show that there exist three types of steady states: a Coase Conjecture, a monopoly, and a reputational steady state. If the monopolist moves the market to a state by selling the good to all buyers and continues fulfilling the replacement demand then the state is called the Coase Conjecture steady state. The monopoly steady state is the one in which the monopolist preserves his market power by selling to the high type buyers and fulfilling their replacement demand forever. In the reputational steady state, the monopolist cuts the production and sells the good to some of the high type buyers.

Despite the similarity with the results of Deneckere and Liang (2008), we show that production of a low-end durable good necessitates existence of other steady states. We establish that there exist five possible steady states: $(1,0)$, $\left(\widehat{b},0\right)$, $\left(\widehat{b},1-\widehat{b}\right)$, $(\grave{y}^H,\grave{y}^L)$, $(\grave{y}^H,1-\grave{y}^H)$ where $\grave{y}^H \in \left(0,\widehat{b}\right)$ and $\grave{y}^L \in \left[0,\widehat{b}-\grave{y}^H\right]$. If the steady state is $(1,0)$ then all buyers hold the high quality good after trade and the monopolist serves their replacement demand in each period. We call $(1,0)$ *the Coase Conjecture steady state* and call the equilibrium having $(1,0)$ as the unique steady state the Coase Conjecture equilibrium. We call $\left(\widehat{b},0\right)$ *the standard monopoly steady state.* At such a steady state, after trade all high type buyers hold the high quality good and the monopolist sells to the replacement demand of the high type buyers for the high quality good forever. At the standard monopoly steady state the price of the low quality good is set sufficiently high so that none of the buyers purchases it. We call $\left(\widehat{b},1-\widehat{b}\right)$ *the segmented monopoly steady state.* At the segmented monopoly steady state, after trade all high type buyers hold the high quality good, all low type buyers hold the low quality good, and the monopolist serves the replacement demands in each period. The segmented monopoly steady state $\left(\widehat{b},1-\widehat{b}\right)$ always coexists with the standard monopoly steady state $\left(\widehat{b},0\right)$. Depending on the magnitude of the depreciation rate, μ, we may observe the Coase Conjecture steady state $(1,0)$ with $\left(\widehat{b},0\right)$ and $\left(\widehat{b},1-\widehat{b}\right)$ as well. We call an equilibrium with the standard and the segmented monopoly steady states the monopoly equilibrium. We call $\left(\grave{y}^H,\grave{y}^L\right)$ and $\left(\grave{y}^H,1-\grave{y}^H\right)$ *reputational steady states* and the corresponding equilibrium the reputational equilibrium. At a reputational steady state, the monopolist limits the production of the high quality good and sells it to some of the high type buyers. At the first reputational steady state, after trade some of the high type buyers hold the high quality good and some of them hold the low quality good whereas at the second steady state, while some high type buyers hold the high quality good, the rest of the buyers holds the low quality good. The steady states $\left(\grave{y}^H,1-\grave{y}^H\right)$ and $(1,0)$ always coexist with $\left(\grave{y}^H,\grave{y}^L\right)$. The result is established by the following.

Proposition 9. *Let S denote the set of steady states. In any stationary equilibrium one of the followings hold:*

1. $S=\{(1,0)\}$;
2. $S=\left\{\left(\widehat{b},0\right),\left(\widehat{b},1-\widehat{b}\right)\right\}$ *or* $S=\left\{\left(\widehat{b},0\right),\left(\widehat{b},1-\widehat{b}\right),(1,0)\right\}$;
3. $S=\left\{\left(\grave{y}^H,1-\grave{y}^H\right),(1,0)\right\}$ *or* $S=\left\{\left(\grave{y}^H,\grave{y}^L\right),\left(\grave{y}^H,1-\grave{y}^H\right),(1,0)\right\}$ *where* $\grave{y}^H\in\left(0,\widehat{b}\right)$ *and* $\grave{y}^L\in\left[0,\widehat{b}-\grave{y}^H\right]$.

The intuition behind this comes from the following two casual observation. First, given the expectations of the buyers, some of the states cannot be a steady state as the monopolist can profitably deviate from these states. Additionally, since the number of the steps in f^i is two, at most three steady states can coexist in an equilibrium.[8]

5.3 Characterization of Stationary Equilibria

In this section, we derive stationary equilibria for a given demand curve. Similar to Deneckere and Liang (2008), there exist three types of equilibria: a Coase Conjecture equilibrium, a

[8] If the marginal buyer of the high quality good is high type, then the marginal buyer of the low quality good can be either high type or low type. However, if the marginal buyer of the high quality good is low type, then the marginal buyer of the low quality good must be low type as well.

monopoly equilibrium and a reputational equilibrium. We show the uniqueness of the equilibrium of each type, and analyze the effects of quality differentiation on each type of equilibrium.

5.3.1 The Coase Conjecture Equilibrium

We characterize equilibria with a unique steady state at which all buyers, after trade, hold the high quality good.[9]

We now describe the derivation of the stationary path that leads us to the steady state $(1,0)$ from any state. In such equilibria, since the low type buyers anticipate that the monopolist saturates the entire market with the high quality good eventually, they are reluctant to pay more than their reservation price for the durable goods. Hence, for any buyer $b \in \left(\widehat{b}, 1\right]$, the acceptance price of the high quality good is $\underline{\theta} q_H$ and the acceptance price of the low quality good is $\underline{\theta} q_L$.[10] Thus, the stationary path that reaches $(1,0)$ is immediately defined for all $x^H \geq (1-\mu)\,\widehat{b}$ and all x^L. To construct the stationary path for $x^H < (1-\mu)\,\widehat{b}$, we define a sequence of states $\left\{\left(x_k^H, x_k^L\right)\right\}_{k=2}^{m+1}$ such that when the state is $\left(x_k^H, x_k^L\right)$, the monopolist is indifferent between $\left(y_{k-1}^H, y_{k-1}^L\right)$ with $\left(p_{k-1}^H, p_{k-1}^L\right)$ and $\left(y_{k-2}^H, y_{k-2}^L\right)$ with $\left(p_{k-2}^H, p_{k-2}^L\right)$ where x_k^i represents the stock of the durable good with quality i before trade in period $k+1$ and $y_k^i = \frac{x_k^i}{1-\mu}$ represents the stock of the durable good with quality i after trade in period k, $i = H, L$. Given the state of the market, the seller serves either the high type buyers or all buyers. These two possible cases are represented by $\left\{\left(\overline{x}_k^H, (1-\mu)\,\widehat{b} - \overline{x}_k^H\right)\right\}_{k=2}^{m'+1}$ and $\left\{\left(\widehat{x}_k^H, 1-\mu-\widehat{x}_k^H\right)\right\}_{k=2}^{m+1}$. On the path $\left\{\left(\overline{x}_k^H, (1-\mu)\,\widehat{b} - \overline{x}_k^H\right)\right\}_{k=2}^{m'+1}$, the seller serves only the high type buyers until the steady state is reached. Some of the high type buyers hold the high quality good and the rest of them holds the low quality good. While moving towards the steady state, the monopolist buys the low quality good gradually and sells the high quality good instead. On the path

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[9] One should observe that selling only the low quality good can never be an equilibrium, since the monopolist would be strictly better off by buying all low quality good back and selling the high quality good instead.

[10] If the acceptance price of a buyer $b \in \left(\widehat{b}, 1\right]$ for the good with quality i was greater than $\underline{\theta} q_i$ then the acceptance price of buyer $b \in \left(\widehat{b}, \widehat{b}+\varepsilon\right)$ would be greater than $\underline{\theta} q_i$, since $P^i\left(.\right)$ is non-increasing. This would imply that b is expecting to make a capital gain by purchasing it. Namely, buyer b would expect that the price of the good with quality i will increase next period. However, since neither $\left(\widehat{b}, 1-\widehat{b}\right)$ nor $\left(\widehat{b}, 0\right)$ is a steady state, this is not possible.

$\left\{\left(\widehat{x}_k^H, 1-\mu-\widehat{x}_k^H\right)\right\}_{k=2}^{m+1}$, the seller serves some high type buyers with the high quality good and fulfills the rest of the market with the low quality good. Until the steady state is reached, all low type buyers hold the low quality good and the quality of the good that they hold is upgraded once the steady state is reached. Upon reaching the steady state, the monopolist sells to the replacement demand of all buyers for the high quality good thereafter. The stock of the low quality good determines which path that the monopolist follows. If the stock of the low quality good before trade is sufficiently low then the seller prefers buying all excess low quality good back immediately and serving only the high type buyers until the steady state is reached. Otherwise, in addition to selling the high quality good to some of the high type buyers, the monopolist would like to fulfill the rest of the market with the low quality good.

The market prices depend on which path the monopolist follows. *On the first path*, the marginal buyer of each version of the good is high type. The market prices in period k, $\overline{p}_k^H$ and $\overline{p}_k^L$, are set such that a high type buyer is indifferent between purchasing the good with quality i at $\overline{p}_k^i$ today and waiting one more period to purchase it at $\overline{p}_{k-1}^i$. Hence, in period k, the market prices, $\overline{p}_k^H$ and $\overline{p}_k^L$, are $\overline{\theta} q_H - \rho^k \left(\overline{\theta}-\underline{\theta}\right) q_H$ and $\overline{\theta} q_L - \rho^k \left(\overline{\theta}-\underline{\theta}\right) q_L$, respectively.[11] *On the second path*, the marginal buyer of the high quality good is high type, whereas the marginal buyer of the low quality good is low type. Hence, the price of the high quality good should satisfy the incentive compatibility constraint rather than the indifference condition. In this case, the market prices are $\widehat{p}_k^H = \overline{\theta} q_H - \rho^k \left(\overline{\theta}-\underline{\theta}\right) q_H - \left(1-\rho^k\right)\left(\overline{\theta}-\underline{\theta}\right) q_L$ and $\widehat{p}_k^L = \underline{\theta} q_L$.

The following theorem shows that the necessary and sufficient condition for the existence of the Coase Conjecture equilibrium is the depreciation rate being sufficiently low.[12]

Theorem 4. *There is at most one Coase Conjecture equilibrium. This equilibrium exists if*

[11] Suppose that a buyer is indifferent between purchasing the high quality good today and purchasing it tomorrow and that the same buyer is also indifferent between purchasing the low quality good today and purchasing it tomorrow. Then the buyer is indifferent between these two versions of the good in the current trading period.

[12] The Coase Conjecture equilibrium does not exist when μ is large. The intuition is as follows. When the depreciation rate is large, rather than fulfilling the replacement demand of all buyers for the high quality good at the price of $\underline{\theta} q_H$, the seller would be better off by selling either to the replacement demand of the high type buyers for the high quality good and to the replacement demand of the low type buyers for the low quality good at the market prices $\overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L$ and $\underline{\theta} q_L$, respectively or to the replacement demand of the high type buyers for the high quality good at the market price $\overline{\theta} q_H$.

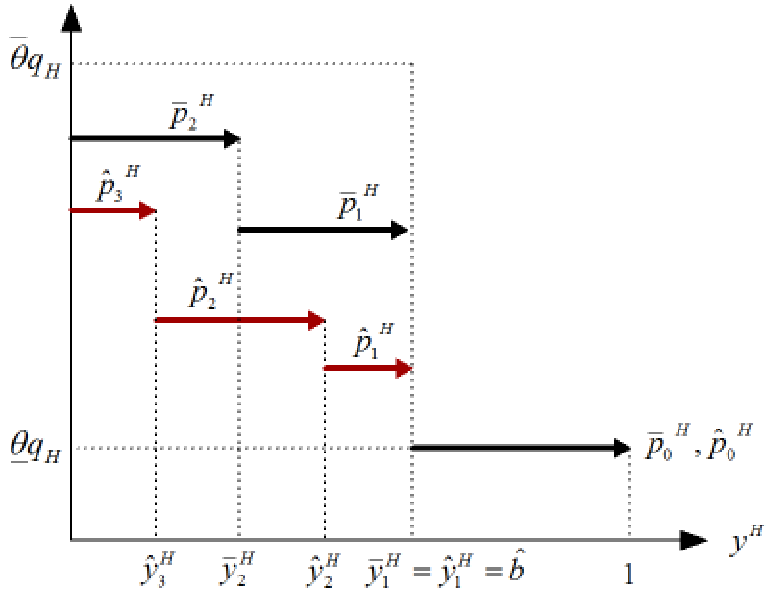

Figure 5.1: The Coase Conjecture Equilibrium

and only if $\mu < \overline{\mu}$, *for some* $\overline{\mu} \in (0, 1)$.

The proof establishes that both $\{\overline{x}_k^H\}$ and $\{\widehat{x}_k^H\}$ are strictly decreasing and there exist m' and m such that $\overline{x}_{m'+1}^H < 0 \leq \overline{x}_{m'}^H$ and $\widehat{x}_{m+1}^H < 0 \leq \widehat{x}_m^H$ when μ falls below $\overline{\mu}$.

The movement of the high quality good with the corresponding prices on these two paths illustrated in Figure 11. We suppose that $m' = 2$ and $m = 3$.[13] Arrows indicate the direction of the movement of the state of the high quality good. On the first path (top path), the monopolist only sells to the high types, whereas on the second path (bottom path), the monopolist sells to low type buyers as well. Hence, the monopolist has to leave some information rent to the high type buyers on the second path. We therefore have $\overline{p}_k^H > \widehat{p}_k^H$. However, as the state gets closer to the steady state (as k decreases), the information rent that a high type buyer gets diminishes. That is, the distance between $\overline{p}_k^H$ and $\widehat{p}_k^H$ diminishes and it vanishes at the steady state. Even though the price of the high quality good decreases on both paths, the rate at which it diminishes on the first path is greater, $\overline{p}_{k+1} - \overline{p}_k < \widehat{p}_{k+1} - \widehat{p}_k$ for all $k < \min(m', m)$. Moreover, we also establish that even though the proportion of the high type buyers holding the high quality good on the second path is greater than the one on the first path, $\widehat{y}_k > \overline{y}_k$ for all

[13] Actually, we have $\lim_{z \to 0^+} m' < \lim_{z \to 0^+} m$.

$k < \min(m', m)$, the rate at which the monopolist penetrates the market with the high quality good on the first path is strictly greater than the rate on the second path. Indeed, the exact relation between these two is $\left(\overline{y}_k - \overline{y}_{k+1}\right) = \left(\widehat{y}_k - \widehat{y}_{k+1}\right) \frac{q_H}{(q_H - q_L)}$ for all $k < \min(m', m)$.

On the equilibrium path, from the initial state $(0, 0)$ the seller serves only the high type buyers until the steady state is reached and fully penetrates the market in m' periods. We show that as the time period between the successive offers of the monopolist diminishes, m' converges its finite limit. This implies that the real time passes before the prices drop to $\underline{\theta} q_H$ and $\underline{\theta} q_L$ vanishes as the time period between successive price changes approaches zero. Hence, the following concludes.

Corollary 8. *In the Coase Conjecture equilibrium, the initial price of the durable good with quality i converges to the lowest buyer valuation $\underline{\theta} q_i$ as the length of time period between two successive offers approaches zero.*

We also show that the limit of m' is strictly less that the limit of the number of the periods required to reach the Coase Conjecture Steady state when the monopolist produces only one version of the good.

Corollary 9. *The rate at which the market prices converge to $\underline{\theta} q_H$ and $\underline{\theta} q_L$ increases as the quality difference between two versions of the durable good increases.*

Therefore, our result suggest that if the good that a monopolist produces is sufficiently durable, he is reluctant to expand the product-line by producing different versions of it in response to the time inconsistency problem he faces and that vertical product differentiation in a market for highly durable goods can only be explained by profit motives.

5.3.2 The Monopoly Equilibrium

We study equilibria in which the monopolist can credibly commit to the static monopoly prices. If this was a one period game, the monopolist would sell the high quality good to the high type buyers at the price $\overline{\theta} q_H$ and set the price of the low quality good high enough so that none of the buyers purchases it.[14] Hence, we consider equilibria in which $\left(\widehat{b}, 0\right)$ is a steady state and characterize the set of parameters so that such equilibria exist.

If the monopoly equilibria exist then, from the initial state $(0,0)$, the monopolist moves to $\left(\widehat{b}, 0\right)$ by charging $\overline{\theta} q_H$ for the high quality good and $\overline{\theta} q_L$ for the low quality good and then he continues fulfilling the replacement demand of the high type buyers for the high quality good $\mu \widehat{b}$ at the price $\overline{\theta} q_H$.[15] However, if the monopolist deviates from $\left(\widehat{b}, 0\right)$ by charging a lower price for the low quality good in an attempt to increase profit by selling the low quality good to the low type buyers then, rather than moving the state back to $\left(\widehat{b}, 0\right)$ by buying the low quality good back or waiting for the low quality good dissipating gradually, the monopolist may immediately bring the state to $\left(\widehat{b}, 1-\widehat{b}\right)$ and continue serving the replacement demands. Hence, there exist two types of monopoly steady state: the standard monopoly steady state $\left(\widehat{b}, 0\right)$ and the segmented monopoly steady state $\left(\widehat{b}, 1-\widehat{b}\right)$. The monopolist can credibly commit to the static monopoly prices at the standard monopoly steady state, whereas at the segmented monopoly steady state the price of the high quality good must be lower than its static monopoly price due to the incentive compatibility constraint.

In a monopoly equilibrium, the Coase Conjecture steady state $(1,0)$ always coexists with the monopoly steady states. Upon deviation from a monopoly steady state, if the stock of the high quality good is sufficiently high and the depreciation rate is sufficiently low, the monopolist will immediately bring the state to $(1,0)$ and continue serving the replacement demand of all buyers for the high quality good.

The existence of a monopoly equilibrium with the standard monopoly steady state requires that when the state before trade is $\left((1-\mu)\widehat{b}, 0\right)$, the monopolist must prefer selling to the replacement demand of the high type buyers for the high quality good to penetrating the market further by cutting the price of the goods. Hence, the monopolist must prefer serving the replacement demand $\mu \widehat{b}$ at the price $\overline{\theta} q_H$ to cutting the price of the high quality good to $P^H(1)$ in order to sell the high quality good to all buyers and continuing by selling to the

[14]It is due to the restriction on the parameters to eliminate the nontrivial case: $\widehat{b}\overline{\theta} > \underline{\theta}$.

[15]We assume that when a buyer is indifferent between the two versions of the durable good, she buys the high quality good.

replacement demand μ at the price $P^H(1)$ thereafter. Therefore, we must have

$$\frac{\mu\widehat{b}\overline{\theta}q_H}{1-\delta} \geq \left(1-(1-\mu)\widehat{b}\right)P^H(1) + \frac{\delta\mu P^H(1)}{1-\delta}.$$

Additionally,when the state before trade is $\left((1-\mu)\widehat{b}, 0\right)$, the monopolist must not cut the price of the low quality good in an attempt to increase profit by selling the low quality good as well. That is, the monopolist must prefer serving the replacement demand $\mu\widehat{b}$ at the price $\overline{\theta}q_H$ to serving the high type buyers' replacement demand for the high quality good forever at the price $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ and selling the low quality good to low type buyers at $P^L(1)$ and continuing by selling to their replacement demand $\mu\left(1-\widehat{b}\right)$ for the low quality good at the price $P^L(1)$ thereafter. Therefore, we must have

$$\frac{\mu\widehat{b}\overline{\theta}q_H}{1-\delta} \geq \frac{\mu\widehat{b}\left(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L\right)}{1-\delta} + \left(1-\widehat{b}\right)P^L(1) + \frac{\delta\mu\left(1-\widehat{b}\right)P^L(1)}{1-\delta}.$$

Similarly, the existence of a monopoly equilibrium with the segmented monopoly steady state requires that when the state is $\left((1-\mu)\widehat{b}, (1-\mu)\left(1-\widehat{b}\right)\right)$, the monopolist must prefer selling to the replacement demands to penetrating the entire market by charging $P^H(1)$ and selling to the replacement demand of all buyers for the high quality good thereafter. Therefore, we must have

$$\frac{\mu\widehat{b}\left(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L\right) + \mu\left(1-\widehat{b}\right)\underline{\theta}q_L}{1-\delta} \geq \left(1-(1-\mu)\widehat{b}\right)P^H(1) - (1-\mu)\left(1-\widehat{b}\right)P^L(1) + \frac{\delta\mu P^H(1)}{1-\delta}.$$

We show that these three inequalities hold if and only if $\mu \geq \underline{\mu}$. The following theorem shows that the condition, $\mu \geq \underline{\mu}$, is also sufficient for the existence the monopoly equilibrium.

Theorem 5. *There exists a unique monopoly equilibrium iff $\mu \geq \underline{\mu}$.*

We construct a sequence of states, $\{\widetilde{x}_k\}_{k=0}^{m'+1}$ where $\widetilde{x}_k = \left(\widetilde{x}_k^H, 0\right)$ with the initial value $\widetilde{x}_0^H = (1-\mu)\widehat{b}$ and the end value $\widetilde{x}_{m'+1}^H \leq (1-\mu)$ such that when the state is $\widetilde{x}_k$, the monopolist is indifferent between bringing the next period's state to $\widetilde{x}_{k-1} < \widetilde{x}_k$ and staying at $\widetilde{x}_k$ forever.

Similarly, we construct a sequence of states $\{\widehat{x}_k\}_{k=0}^{m+1}$ where $\widehat{x}_k = \left(\widehat{x}_k^H, 1-\mu-\widehat{x}_k^H\right)$ with the initial value $\widehat{x}_0^H = (1-\mu)\,\widehat{b}$ and the end value $\widehat{x}_{m+1}^H \leq (1-\mu)$ such that when the state is $\widehat{x}_k$, the monopolist is indifferent between bringing the next period's state to $\widehat{x}_{k-1} < \widehat{x}_k$ and staying at $\widehat{x}_k$ forever. We show that for all $k \leq \min(m', m)$, $\widetilde{x}_k^H = \widehat{x}_k^H$, and that the limit state of the high quality good depends on the depreciation rate. When the good is sufficiently perishable we have $\widetilde{x}_{m'+1}^H = \widehat{x}_{m+1}^H = 1-\mu$, whereas otherwise we have $\widehat{x}_{m+1}^H < \widetilde{x}_{m'+1}^H < 1-\mu$. We call the sequence $\{\widetilde{x}_k\}_{k=0}^{m'+1}$ the standard monopoly path and the sequence $\{\widehat{x}_k\}_{k=0}^{m+1}$ the segmented monopoly path.

On the standard monopoly path, if some buyers hold the low quality good, the monopolist buys all low quality good back from them immediately. Hence, the monopolist prefers incurring a higher cost rather than waiting for the low quality good dissipating gradually. If the monopolist left some of the low quality good in the market, he wouldn't be able to sell as much the high quality good as he could. Since the marginal benefit of selling one unit of the high quality good is greater than the marginal cost of buying one unit of the low quality good back, and as the states get closer to the steady state, the marginal cost of having the low quality good in the market raises, the monopolist strictly prefers buying all low quality good back immediately when he is on the standard monopoly path. On the other hand, if the segmented monopoly steady state is eventually reached, anticipating that the price of the low quality good will be equal to the low type buyers' reservation price $\underline{\theta} q_L$ sooner or later, the low type buyers are not willing to pay more than $\underline{\theta} q_L$ for the low quality good. Hence, when the monopolist is on the segmented monopoly path, he always sells the low quality good to the low type buyers who does not hold the high quality good, and gradually buys the high quality good from the low type buyers and sells the low quality good instead until the steady state $\left(\widehat{b}, 1-\widehat{b}\right)$ is reached.

For any state of the high quality good x^H, there exists a threshold for the state of the low quality good $\overline{x}\left(x^H\right)$. Suppose $x^H \in \left(\widetilde{x}_{k-1}^H, \widetilde{x}_k^H\right] \cap \left(\widehat{x}_{t-1}^H, \widehat{x}_t^H\right]$. If $x^L \leq \overline{x}\left(x^H\right)$ then the monopolist follows the path that reaches the standard monopoly steady state $\left(\widehat{b}, 0\right)$. The market prices will be $\widetilde{p}_k^H$ and $\widetilde{p}_k^L$, and the stock of the high quality and the stock of the low quality good after trade will be $t^H\left(x^H, x^L\right) = \widetilde{y}_{k-1}^H$ and $t^L\left(x^H, x^L\right) = 0$, respectively. If

$x^L > \overline{x}\left(x^H\right)$ then the monopolist follows the path that reaches the segmented monopoly steady state $\left(\widehat{b}, 1-\widehat{b}\right)$. The market prices will be $\widehat{p}_t^H$ and $\widehat{p}_t^L$, and the stock of the high quality and the stock of the low quality good after trade will be $t^H\left(x^H, x^L\right) = \widehat{y}_{t-1}^H$ and $t^L\left(x^H, x^L\right) = 1-\widehat{y}_{t-1}^H$, respectively.

If the good is sufficiently perishable, $\mu \geq \mu'$, then the monopolist will return to a monopoly steady state from any state above $(1-\mu)\widehat{b}$. The buyers in the interval $\left(\widehat{b}, 1\right]$ purchase the high quality good at a value exceeding their reservation prices, in order to make capital gains by reselling it to the monopolist at a later date. If the state of the low quality good is below its threshold value then the steady state of the market will be $\left(\widehat{b}, 0\right)$; otherwise it will be $\left(\widehat{b}, 1-\widehat{b}\right)$. Therefore, for any state $\left(x^H, x^L\right)$ with $x^H \in \left((1-\mu)\widehat{b}, 1\right]$ it will take at most $m+1$ periods to return to a steady state (if $x^L \leq \overline{x}^L\left(x^H\right)$, the steady state will be $\left(\widehat{b}, 0\right)$; otherwise, it will be $\left(\widehat{b}, 1-\widehat{b}\right)$). Upon reaching $\left(\widehat{b}, 0\right)$ the monopolist charges the standard monopoly prices $\overline{\theta}q_H$ and $\overline{\theta}q_L$ and serves the replacement demand of the high type buyers for the high quality good forever, and upon reaching $\left(\widehat{b}, 1-\widehat{b}\right)$ the monopolist charges the segmented monopoly prices $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ and $\underline{\theta}q_L$ and serves the replacement demand of the high type buyers for the high quality good and the replacement demand of the low type buyers for the low quality good forever. When the good is sufficiently perishable, we have $\widetilde{x}_k^H = \widehat{x}_k^H$ and $m' = m$. Even though the stock of the high quality good after trade is the same on both paths, since some of the low type buyers hold the low quality good on the segmented monopoly path, the price of the high quality good should satisfy the incentive compatibility constraint and hence, it is strictly less than its price on the standard monopoly path.

If the good is sufficiently durable, $\underline{\mu} \leq \mu < \mu'$, the Coase Conjecture steady state $(1,0)$ exists with $\left(\widehat{b}, 0\right)$ and $\left(\widehat{b}, 1-\widehat{b}\right)$. Suppose that the state of the high quality good equals $\widetilde{x}_{m'+1}^H$ and that the state of the low quality good falls below the threshold. Then the monopolist is indifferent between moving to $\left(\widetilde{y}_{m'}^H, 0\right)$ and fully penetrating the market by selling the high quality good to all buyers and serving their replacement demand for the high quality good forever. It follows that the monopolist strictly prefers moving to $(1,0)$ and staying there forever when the state of the high quality good is greater than $\widetilde{x}_{m'+1}^H$. Suppose now that the state of the high quality good equals $\widehat{x}_{m+1}^H$ and that the state of the low quality good falls above the threshold. Then the monopolist is indifferent between moving to $\left(\widetilde{y}_m^H, 1-\widetilde{y}_m^H\right)$ and fully penetrating the market by selling the high quality good to all buyers and serving their replacement demand for the high quality good forever. It implies that the monopolist strictly prefers moving to $(1,0)$ and staying there forever when the state of the high quality good is greater than $\widehat{x}_{m+1}^H$. In this case, it is possible to have $m' > m$. When stock of high quality good is high enough, the monopolist brings the state to the Coase Conjecture steady state $(1,0)$ immediately. The set of states of the high quality good supporting the standard monopoly equilibrium is smaller than the set of states supporting the segmented monopoly steady state.

Now, we consider the structure of the equilibrium as the time period between successive offers of the monopolist diminishes. If some of the low type buyers hold the high quality good, the state either immediately moves to the Coase Conjecture steady state or slowly goes back to a monopoly steady state. Let $\widetilde{x}^H = \lim_{z\to 0^+} \widetilde{x}_{m'(z)+1}^H$ and $\widehat{x}^H = \lim_{z\to 0^+}^H \widetilde{x}_{m(z)+1}$. Suppose the stock of the low quality good is sufficiently low (high). Then, if the stock of the high quality good after trade lies in the interval $\left((1-\mu)\,\widehat{b}, \widetilde{x}^H\right]$ $(\left((1-\mu)\,\widehat{b}, \widehat{x}^H\right])$ then it takes real time for the state to move back to $\left(\widehat{b}, 0\right)$ $(\left(\widehat{b}, 1-\widehat{b}\right))$. However, if the stock of the high quality good lies in the interval $\left(\widetilde{x}^H, 1\right]$ $(\left(\widehat{x}^H, 1\right])$, the state immediately moves to $(1,0)$. We show that the quality of the durable goods does not play any role in the rate at which the state moves back to a steady state. Therefore, we have the following.

Corollary 10. *If total stock of durable goods before trade $x^H + x^L$ is less than $\left(y^H\overline{\theta} - \underline{\theta}\right)\dfrac{\lambda + r}{r}$, then*

$$P^i\left(y^H, y^L\right) = \begin{cases} \overline{\theta} q_i & \text{for} \quad y^H \in \left[0, \widehat{b}\right], \text{ all } y^L \\ \overline{\theta} q_i \left(\dfrac{\widehat{b}}{y^H}\right)^{\dfrac{\lambda + r}{\lambda}} & \text{for} \quad y^H \in \left(\widehat{b}, \widetilde{x}^H\right], \text{ all } y^L \\ \underline{\theta} q_i & \text{for} \quad y^H \in \left(\widetilde{x}^H, 1\right], \text{ all } y^L \end{cases}$$

where $i = H, L$. When $x^H < \widehat{b}$, the monopolist sells $\widehat{b} - x^H$ units of high quality good; when $x^H \in \left(\widehat{b}, \widetilde{x}^H\right]$, the monopolist selects $\dot{x}^H = \lambda x^H \left(1 - \dfrac{\underline{\theta}}{\overline{\theta}}\left(\dfrac{y^H}{\widehat{b}}\right)^{\dfrac{\lambda + r}{\lambda}}\right)$; when $x^H > \widetilde{x}^H$, the monopolist sells $1 - x^H$ units of high quality good. Moreover, the monopolist buys all low quality *good back immediately. Otherwise, we have*

$$P^H\left(y^H,y^L\right)=\begin{cases}\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L & \text{for}\quad y^H\in\left[0,\widehat{b}\right],\ \text{all } y^L\\ \overline{\theta}\Delta q\left(\dfrac{\widehat{b}}{y^H}\right)^{\dfrac{\lambda+r}{\lambda}}+\underline{\theta}q_L & \text{for}\quad y^H\in\left(\widehat{b},\widehat{x}^H\right],\ \text{all } y^L\\ \underline{\theta}q_H & \text{for}\quad y^H\in\left(\widehat{x}^H,1\right],\ \text{all } y^L\end{cases}$$

and

$$P^L\left(y^H,y^L\right)=\ \underline{\theta}q_L\quad \text{for}\quad \text{all } y^H \text{ and } y^L.$$

When $x^H<\widehat{b}$, the monopolist sells $\widehat{b}-x^H$ units of high quality good; when $x^H\in\left(\widehat{b},\widehat{x}^H\right]$, the monopolist selects $\dot{x}^H=\lambda x^H\left(1-\dfrac{\underline{\theta}}{\overline{\overline{\theta}}}\left(\dfrac{y^H}{\widehat{b}}\right)^{\dfrac{\lambda+r}{\lambda}}\right)$, and sells low quality good to the rest of the buyers; when $x^H>\widehat{x}^H$ the monopolist sells $1-x^H$ units of high quality good. When $x^H\leq\widehat{x}^H$, the monopolist fulfills the rest of the market with low quality good but buys all low quality good back when $x^H>\widehat{x}^H$. Moreover, $\widehat{x}^H>\widetilde{x}^H$.

We also prove that the steady state path of the monopolist who produces single version, say q_H only, of the durable good coincide with $\left\{\widetilde{x}_k^H\right\}_{k=0}^{m'+1}$ and that as the time horizon between successive offers of the monopolist approaches zero, the limit if the path for single good market is the same as $\widetilde{x}^H$. We therefore conclude that

Corollary 11. *The set of states of the high quality good supporting the monopoly equilibrium expands with the introduction of the low quality good.*

5.3.3 Reputational Equilibrium

We now consider equilibria in which the monopolist establishes a reputation by cutting the production of the high quality good. The steady states of such equilibria are $\left(\grave{y}^H,\grave{y}^L\right)$, $\left(\grave{y}^H,1-\grave{y}^H\right)$ and $(1,0)$ where $\grave{y}^H\in\left(0,\widehat{b}\right)$ and $\grave{y}^L\in\left[0,\widehat{b}-\grave{y}^H\right]$. The stock of the high quality good in a reputational steady state falls short of the static monopoly output. However, whether or not the monopolist limits the production of the low quality good as well depends

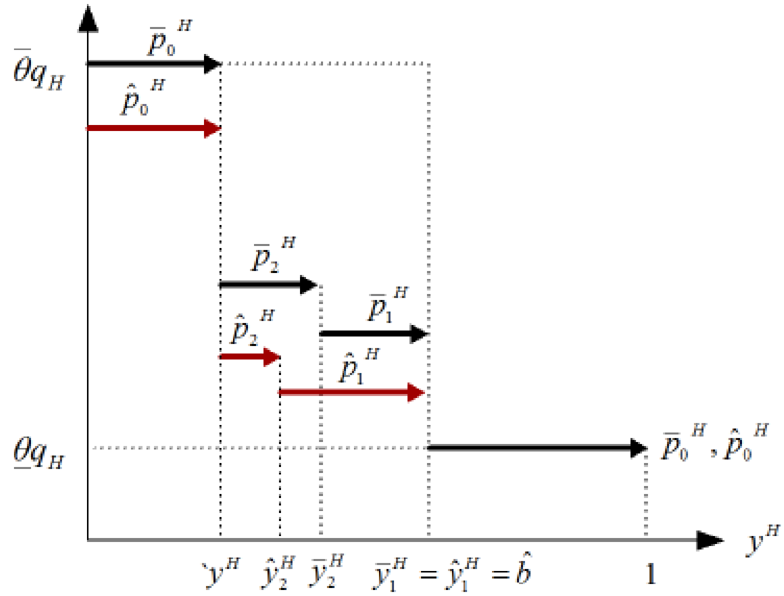

Figure 5.2: The Reputational Equilibrium

on the parameter values. If $\overline{\theta}\left(\grave{y}^H + \grave{y}^L\right) \geq \underline{\theta}$, from the initial state $(0,0)$, the monopolist will immediately move to $\left(\grave{y}^H, \grave{y}^L\right)$ by charging $\overline{\theta}q_H$ for the high quality good and $\overline{\theta}q_L$ for the low quality good and continue selling to the replacement demands. Otherwise, from the initial state $(0,0)$, the monopolist will immediately move to $\left(\grave{y}^H, 1-\grave{y}^H\right)$ by charging $\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ for the high quality good and $\underline{\theta}q_L$ for the low quality good and continue providing for the replacement demands. If the monopolist penetrates the market by selling more of the high quality good, he loses his reputation for pricing high and has to drastically lower the price of the high quality goods. Since buyers expect that the future prices will be lower, they are reluctant to pay a high price for the high quality good and hence, the state slowly moves to the Coase Conjecture steady state $(1,0)$.

The movement of the high quality good's price and state in a reputational equilibrium is illustrated in Figure 12. For all $x^H < (1-\mu)\,\grave{y}^H$, the monopolist immediately brings the state of the high quality good to $\grave{y}^H$. Depending on the parameter values and the stock of the low quality good, the state of low quality good is moved to $\grave{y}^L$ or $1-\grave{y}^H$. For simplicity, we assume that upon deviation from $\grave{y}^H$, it takes two steps to reach the Coase Conjecture steady state, and is characterized by the stationary path derived for the Coase Conjecture equilibrium.

When the steady state is $\left(\grave{y}^H, \grave{y}^L\right)$, the monopolist does not completely lose his reputation if he penetrates the market by selling more of the low quality good. Upon observing a price cut on the low quality good, buyers expect that the monopolist will increase the sales of the low quality good and hence would not accept any price for the low quality good significantly greater than $\underline{\theta} q_L$. Thus, the monopolist must drop the price of the low quality good rather than losing the sale. This implies that, deviation from $\left(\grave{y}^H, \grave{y}^L\right)$ via cutting the price of the low quality good causes the state to immediately move to $\left(\grave{y}^H, 1 - \grave{y}^H\right)$.

Theorem 6. *There exists a unique reputational equilibrium if and only if* $\underline{\mu} < \mu \leq \overline{\mu}$.

Moreover, in the limit, as the time period between successive offers approaches zero, the steady state output levels and the acceptance prices are represented by the following corollary.

Corollary 12. *Let* $\grave{y}^H$ *and* $\grave{y}^L$ *satisfy* $\left(\grave{y}^H q_H + \grave{y}^L q_L\right) = \dfrac{(\lambda + r)\,\underline{\theta} q_H}{\lambda\overline{\theta} + r\underline{\theta}}$. *Then, as the time period between two successive offers approaches zero, the reputational equilibrium acceptance price converges to*

$$P^H\left(y^H, y^L\right) = \begin{cases} \overline{\theta} q_H & \text{for} \quad y^H \in \left[0, \grave{y}^H\right], \text{ and } y^L \in \left[0, \grave{y}^L\right] \\ \left(\overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L\right) & \text{for} \quad y^H \in \left[0, \grave{y}^H\right], \text{ and } y^L \in \left(\grave{y}^L, 1\right] \\ \underline{\theta} q_H & \text{for} \quad y^H \in \left(\grave{y}^H, 1\right], \text{ all } y^L \end{cases}$$

and

$$P^L\left(y^H, y^L\right) = \begin{cases} \overline{\theta} q_L & \text{for} \quad y^H \in \left[0, \grave{y}^H\right], \text{ and } y^L \in \left[0, \grave{y}^L\right] \\ \underline{\theta} q_L & \text{for} \quad y^H \in \left[0, \grave{y}^H\right], \text{ and } y^L \in \left(\grave{y}^L, 1\right] \\ \underline{\theta} q_L & \text{for} \quad y^H \in \left(\grave{y}^H, 1\right], \text{ all } y^L \end{cases}$$

In a reputational equilibrium, as the time period between successive offers is arbitrarily short, the steady state stock of the durable good with the production of a low-end good falls short of the steady state stock of the durable good when the low-end good is not produced. Moreover, the difference between these two output levels increases with respect to the quality of the low-end good. However, the segment of the market that the monopolist serves expands with the production of the low-end good, and increases as the quality of the low-end good increases. Hence, welfare comparison is not trivial.

Therefore, our results suggest that when the monopolist has a reputation of being tough, and has credibly committed to the static monopoly price, he is reluctant to produce a low-end good. However, if the monopolist has deviated from the equilibrium by cutting the price of the good in an attempt to increase profit, he then wants to introduce a higher quality good to regain his reputation for pricing high.

5.4 Coexistence of Stationary Equilibria

The issue we address in this section is whether or not it is possible to observe more than one type of stationary equilibrium for given exogenous variables: depreciation rate μ, discount factor δ, valuation parameters $\left(\overline{\theta}, \underline{\theta}\right)$, quality parameters (q_H, q_L), proportion of the high type buyers $\widehat{b}$. To establish it, we have the following result.[16]

Proposition 10. *The threshold depreciation rates $\overline{\mu}$ and $\underline{\mu}$ are decreasing in discount factor δ with the same initial values $\overline{\mu}(0) = \underline{\mu}(0) = \dfrac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta}-\underline{\theta}}$ and with the end values of $\lim_{\delta \to 1^-} \overline{\mu}(\delta) > 0$ and $\underline{\mu}(1) = 0$.*

When the monopolist does not count any future profit (i.e. $\delta = 0$), we either observe a Coase Conjecture equilibrium or a monopoly equilibrium depending on the depreciation rate. If $\mu < \dfrac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, the unique equilibrium will be the Coase Conjecture equilibrium. Otherwise, it will be the monopoly equilibrium.

When the monopolist discounts future payoffs (i.e. $\delta > 0$), we observe all types of stationary equilibrium. When $\mu < \underline{\mu}(\delta)$, the Coase Conjecture equilibrium is the unique equilibrium. When $\underline{\mu}(\delta) \leq \mu \leq \overline{\mu}(\delta)$ all types of equilibrium, the Coase Conjecture, the monopoly, and the reputational equilibrium, coexist. When $\mu > \overline{\mu}(\delta)$, the monopoly equilibrium is the unique equilibrium.

This result implies that as long as the durable good depreciates (i.e. $\mu > 0$), a monopoly equilibrium always exists for sufficiently large discount factors δ. Moreover, it also shows that

[16] This auxiliary result is also established by Lemma 3 of Deneckere and Liang (2008).

as the monopolist discounts future less, the domain of (δ, μ) on which a Coase Conjecture equilibrium exists contracts.

Let $\underline{\mu}^S(\delta)$ and $\overline{\mu}^S(\delta)$ denote the threshold depreciation rates of single good model of Deneckere and Liang (2008). We prove that vertical product differentiation does not affect the values of $\underline{\mu}(\delta)$ and $\overline{\mu}(\delta)$.

Corollary 13. *For all $\delta > 0$, $\underline{\mu}(\delta) = \underline{\mu}^S(\delta)$ and $\overline{\mu}(\delta) = \overline{\mu}^S(\delta)$ where $\underline{\mu}^S(\delta)$ and $\overline{\mu}^S(\delta)$ stand for corresponding threshold depreciation rates of single durable good market.*

Since, the threshold depreciation rates are the same as the ones in Deneckere Liang (2008), their result for the existence of stationary equilibria, as the time period between two successive offers gets arbitrarily small remains. The results are as follows.[17] There exists a threshold λ_0 below which the Coase Conjecture equilibrium is the unique equilibrium even when the time period between successive offers is vanished. However, when $\lambda > \lambda_0$, all three types of equilibria coexist.

[17] See Deneckere and Liang (2008) for a detailed analysis.

Chapter 6

Conclusion

In this dissertation, I study the effect of quality differentiation on the commitment problem of a durable goods monopolist. I extend the single good setting of Deneckere and Liang (2008) into a setting of a vertically differentiated market and consider a monopolist selling an imperfectly durable good available in two quality levels in an infinite horizon, discrete time game. I characterize the Markov perfect equilibria as a function of the common discount rate, the common depreciation rate of the goods, the length of the time period between successive price changes, and the quality levels of the goods. Similar to Deneckere and Liang (2008), I establish that there exist three types of Markov perfect equilibria: a Coase Conjecture equilibrium, a monopoly equilibrium, and a reputational equilibrium. For sufficiently low depreciation rates, the unique equilibrium is the Coase Conjecture equilibrium. The Coase Conjecture equilibrium has a unique steady state equal to the competitive quantity. For sufficiently high depreciation rates, the unique equilibrium is the monopoly equilibrium. This equilibrium has two monopoly steady states one of which is equal to the static monopoly quantity. The market at the other monopoly steady state is segmented into two: the monopolist serves the high quality good to the high type buyers and serves the low quality good to the low type buyers. For intermediate values of the depreciation rate, all three types of equilibria exist. In the reputational equilibrium, the monopolist creates a reputation of pricing high by cutting the production of the high quality good. Hence, the reputational steady state quantity of the high quality good falls short of the monopoly quantity of the high quality good. These results survive even when the agents become extremely patient. However, the set of parameters for which the Coase Conjecture equilibrium is unique vanishes. When the length of the time period between successive price changes is arbitrarily close to zero, the Coase Conjecture equilibrium always exists and the monopoly equilibrium exists only if the good is sufficiently perishable.

I prove that the set of parameters supporting the Coase Conjecture equilibrium is smaller and the set of the parameters supporting the monopoly equilibrium is larger when the monopolist, who is not allowed to buy back used goods from previous buyers, can produce a lower quality good and when buyers are allowed to trade the good with each other in a perfectly competitive second-hand market. When the monopolist is, however, allowed to buy back the goods from previous buyers, I prove that quality differentiation does not affect the domain of the parameters supporting each type of equilibrium but affects the off-equilibrium path.

This study establishes that quality differentiation may enhance market power of a durable goods monopolist and alleviate the commitment problem when the monopolist is not allowed to buy back the goods from previous buyers. In particular, when the innate durability of a good is high, to credibly commit to the monopoly prices of the good the monopolist will produce a lower quality good either by damaging a portion of the goods or by producing the lower quality good from scratch. On the other hand, when the monopolist is allowed to buy back the goods from previous buyers, if the innate durability of a good is sufficiently high, it is less likely to observe quality differentiation. In particular, if the depreciation rate is so low that the Coase Conjecture equilibrium is the unique equilibrium then the monopolist is reluctant to introduce lower quality versions of the good. Moreover, a durable goods monopolist selling a sufficiently perishable good has penetrated the market in an attempt to increase profit, then he may introduce a higher (or a lower) quality version of the good to be able to restore his market power. However, if the monopolist has already been committed to the monopoly price, he will be reluctant to introduce a lower quality version of the good.

Appendix A

Appendix for Chapter 4

Proof of Proposition 1. Suppose there exists a history after which the state is (x_0^H, x_0^L) and the monopolist randomizes among the elements of $P(t(x_0))$. Let the expected price of good i be p_2^i. Since $T(\cdot)$ is a sublattice of Y, we have $y_1 \wedge y_2 \in T(x^H, x^L)$ where $y_1, y_2 \in T(x^H, x^L)$. It follows that $p_2^i < p_1^i = P^i(\inf T(x_0))$ and $p_2^{-i} \leq p_1^{-i} = P^{-i}(\inf T(x_0))$ for $i \in \{H, L\}$. Let us define $\widehat{v}_j^H$ by $f^H(y_0^H) - \widehat{v}_j^H = \rho(f^H(y_0^H) - p_j^H)$ and $\widehat{v}_j^L$ by $f^L(y_0^H + y_0^L) - \widehat{v}_j^L = \rho(f^L(y_0^H + y_0^L) - p_j^L)$, $j = 1, 2$ where $y_0^H = \dfrac{x^H}{1-\mu}$ and $y_0^L = \dfrac{x^L}{1-\mu}$. When the monopolist randomizes the price of the high quality and the low quality goods in the previous period cannot be greater than $\widehat{v}_2^H$ and $\widehat{v}_2^L$, respectively. We will show that $P^i(y_0) \geq \widehat{v}_1^i$ and $P^{-i}(y_0) \geq \widehat{v}_1^{-i}$. Let (y_n^H, y_n^L) be defined such that $(y_n^H, y_n^L) \uparrow (y_0^H, y_0^L)$ and $T((1-\mu)y_n)$ is single valued for all n. Since $T(\cdot)$ is a monotone increasing correspondence, we have $P^H(y_0) = \lim_{n\to\infty} P^H(y_n) = \lim_{n\to\infty} \left((1-\rho) f^H\left(y_n^H\right) + \rho P^H\left(T((1-\mu)\, y_n\right)\right) \geq (1-\rho) f^H\left(y_0^H\right) + \rho p_1^H = \widehat{v}_1^H$ and $P^L(y_0) = \lim_{n\to\infty} P^L(y_n) = \lim_{n\to\infty} \left((1-\rho) f^L\left(y_n^H + y_n^L\right) + \rho P^L\left(T((1-\mu)\, y_n\right)\right) \geq (1-\rho) f^L\left(y_0^H + y_0^L\right) + \rho p_1^L = \widehat{v}_1^L$. Hence, we can conclude that the monopolist does not randomize along any equilibrium path and $(t^H(x), t^L(x)) = \inf T(x)$ denotes the monopolist's equilibrium choice.

Proof of Proposition 2. If (y_s^H, y_s^L) is the stock level after trade in a steady state, we must have $V^H(y_s^H) = f^H(y_s^H)$ and $V^L(y_s^H + y_s^L) = f^L(y_s^H + y_s^L)$. This implies that $P^H(y_s^H, y_s^L) = f^H(y_s^H) - f^L(y_s^H) + f^L(y_s^H + y_s^L)$ and $P^L(y_s^H, y_s^L) = f^L(y_s^H + y_s^L)$.

Let us define the sets S^H and S^L as $S^H = \{b^H : V^H(b^H) = f^H(b^H)\}$ and $S^L = \{b^L : V^L(b^L) = f^L(b^L)\}$. Suppose that S^H and S^L are nonempty. Let $(\acute{b}^H, \acute{b}^L)$ be defined as $\acute{b}^H = \sup S^H$ and $\acute{b}^L = \sup S^L$ such that $\acute{b}^H < \acute{b}^L$. First, we show that $(\acute{b}^H, \acute{b}^L - \acute{b}^H)$ is a steady state. Then, we prove that V^i and f^i necessarily cross for all $i \in \{H, L\}$ and that $\acute{b}^H < \acute{b}^L$.

We now prove that $(\acute{b}^H, \acute{b}^L - \acute{b}^H)$ is a steady state. First, we claim that there is no

$b^H > \acute{b}^H$ such that the maximum willingness to pay of buyer b^H for the high quality good is greater $V^H(\acute{b}^H)$. We prove this claim by contradiction. Suppose that there exists $b^H > \acute{b}^H$ with $V^H(b^H) \geq V^H(\acute{b}^H)$. Then, by definition we must have $f^H(b^H) < f^H(\acute{b}^H)$. Since $t^i(\cdot)$ is non-decreasing with respect to both arguments and $\left|t_1^i\right| \geq \left|t_2^i\right|$ we have$V^H(t^H((1-\mu)b^H, (1-\mu)((b'^L - b^H))) \leq V^H(t^H((1-\mu)\acute{b}^H, (1-\mu)((\acute{b}^L - \acute{b}^H)))$. This implies a contradiction, since $V^H(b^H) = (1-\rho)f^H(b^H) + \rho V^H(t^H((1-\mu)b^H, (1-\mu)(b'^L - b^H))) < (1-\rho)f^H(\acute{b}^H) + \rho V^H(t^H((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H))) = V^H(\acute{b}^H)$. Similarly, we now claim that there is no $b^L > \acute{b}^L$ such that the maximum willingness to pay of buyer b^L for the low quality good is greater $V^L(\acute{b}^L)$. Suppose that there is $b^L > \acute{b}^L$ with $V^L(b^L) \geq V^L(\acute{b}^L)$. Then by definition we have $f^L(b^L) < f^L(\acute{b}^L)$. Moreover, by the same reasoning above, we have $V^L((t^H + t^L)((1-\mu)b'^H, (1-\mu)((b^L - b'^H))) \leq V^L((t^H + t^L)((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H)))$. This implies a contradiction, since $V^L(b^L) = (1-\rho)f^L(b^L) + \rho V^L((t^H + t^L)((1-\mu)b'^H, (1-\mu)(b^L - b'^H)))$ $< (1-\rho)f^L(\acute{b}^H) + \rho V^L((t^H + t^L)((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H))) = V^L(\acute{b}^L)$. Therefore, we can conclude that, if the offer is $P^H(\acute{b}^H, \acute{b}^L) = f^H(\acute{b}^H) - f^L(\acute{b}^H) + f^L(\acute{b}^L)$ and $P^L(\acute{b}^H, \acute{b}^L) = f^L(\acute{b}^L)$, all $b \leq \acute{b}^H$ holds the high quality good, all $\acute{b}^H < b \leq \acute{b}^L$ holds the low quality good, and all $b > \acute{b}^L$ rejects. Since $V^H(\acute{b}^H) = (1-\rho)f^H(\acute{b}^H) + \rho V^H(t^H((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H)))$ and $V^H(\acute{b}^H) = f^H(\acute{b}^H)$, we must have $t^H((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H)) = \acute{b}^H$ and since $V^L(\acute{b}^L) = (1-\rho)f^L(\acute{b}^H) + \rho V^L((t^H + t^L)((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H)))$ and $V^L(\acute{b}^L) = f^L(\acute{b}^L)$, we must have $(t^H + t^L)((1-\mu)\acute{b}^H, (1-\mu)(\acute{b}^L - \acute{b}^H)) = \acute{b}^L$ which implies that $t^L((1-\mu)\acute{b}^H, (1-\mu)((\acute{b}^L - \acute{b}^H)) = \acute{b}^L - \acute{b}^H$. Hence, $(\acute{b}^H, \acute{b}^L - \acute{b}^H)$ is a steady state.

Now, we prove that V^i and f^i necessarily cross for all $i \in \{H, L\}$ and that $\acute{b}^H < \acute{b}^L$. First, suppose that there is a stationary equilibrium which does not have any steady states. This implies that S^i for $i \in \{H, L\}$ is empty. We show that if S^H is empty then $V^H(b) < \overline{\theta} q_H$ for $b \in [0, \widehat{b}]$ and $V^H(b) > \underline{\theta} q_H$ for $b \in (\widehat{b}, 1]$. If S^H is empty, then we must have $V^H(b) \neq \overline{\theta} q_H$ for $b \in [0, \widehat{b}]$ and $V^H(b) \neq \underline{\theta} q_H$ for $b \in (\widehat{b}, 1]$. Since the seller never charges a price less than $\underline{\theta} q_H$ for high quality good as long as $p^L > 0$, we cannot have $V^H(b) < \underline{\theta} q_H$. Hence, we must have $V^H(b) > \underline{\theta} q_H$ for $b \in (\widehat{b}, 1]$. Moreover, it is not possible that $V^H(b) > \overline{\theta} q_H$ for some $b \in [0, \widehat{b}]$. If we had $V^H(b) > \overline{\theta} q_H$, it would imply that $V^H(0) > \overline{\theta} q_H$, since $V^H(\cdot)$ is nonincreasing. Given a state, say $(0, x^L)$, the arbitrage equation $V^H(0) = (1-\rho)\overline{\theta} q_H + \rho V^H(t^H(0, x^L))$

implies that $V^H(t^H(0,x^L)) > V^H(0)$ which is a contradiction because $t^H(0,x^L) \geq 0$ and $V^H(0)$ is nonincreasing. Second, suppose that S^L is empty. We show that if S^L is empty then $V^L(b) < \overline{\theta}q_L$ for $b \in [0,\widehat{b}]$ and $V^L(b) > \underline{\theta}q_L$ for $b \in (\widehat{b},1]$. If S^L is empty then $V^L(b) \neq \overline{\theta}q_L$ for $b \in [0,\widehat{b}]$ and $V^L(b) \neq \underline{\theta}q_L$ for $b \in (\widehat{b},1]$. Since the seller never charges a price less than $\underline{\theta}q_L$ for low quality good when $b^L < 1$, we must have $V^L(b) > \underline{\theta}q_L$for $b \in (\widehat{b},1]$. Moreover, it is not possible that $V^L(b) > \overline{\theta}q_L$ for some $b \in [0,\widehat{b}]$. If it was, it would imply that $V^L(0) > \overline{\theta}q_L$, since $V^L(\cdot)$ is nonincreasing. Given a state $(0,b^L)$, the arbitrage equation $V^L(0) = (1-\rho)\overline{\theta}q_L + \rho V^L((t^H+t^L)(0,0))$ implies that $V^L((t^H+t^L)(0,0)) > V^L(0)$ which is a contradiction because $t^i(0,0) \geq 0$ for all $i = H,L$ and $V^L(\cdot)$ is nonincreasing.

We now show that $V^i(b) < \overline{\theta}q_i$ for $b \in [0,\widehat{b}]$ and $V^i(b) > \underline{\theta}q_i$ for $b \in (\widehat{b},1]$ leads to a contradiction with the definition of $t^i(\cdot)$. Suppose that the state is (x^H,x^L) where $x^H \in [0,\widehat{b}]$. Since $\overline{\theta}q_H - V^H(\frac{x^H}{1-\mu}) > 0$, the arbitrage equation $\overline{\theta}q_H - V^H(\frac{x^H}{1-\mu}) = \rho(\overline{\theta}q_H - V^H(t^H(x^H,x^L)))$ implies that $V^H(t^H(x^H,x^L)) < V^H(\frac{x^H}{1-\mu})$. That is, $t^H(x^H,x^L) > \frac{x^H}{1-\mu}$. Suppose now that the state is $(((1-\mu)(\widehat{b}+\epsilon),x^L)$ where $\epsilon \in (0,1-\widehat{b}]$ and $(1-\mu)(\widehat{b}+\epsilon)+x^L \leq 1$. Since $\underline{\theta}q_H - V^H(\widehat{b}+\epsilon) < 0$, the arbitrage equation $\underline{\theta}q_H - V^H(\widehat{b}+\epsilon) = \rho(\underline{\theta}q_H - V^H(t^H((1-\mu)(\widehat{b}+\epsilon),x^L)))$ implies that $V^H(t^H((1-\mu)(\widehat{b}+\epsilon),x^L)) > V^H(\widehat{b}+\epsilon)$. That is, $t^H((1-\mu)(\widehat{b}+\epsilon),x^L) < \widehat{b}+\epsilon$. Therefore, $\lim_{\epsilon\to0^+} t^H((1-\mu)(\widehat{b}+\epsilon),x^L) \leq \widehat{b} < t^H((1-\mu)\widehat{b},x^L)$. Since, $T(\cdot)$ is upper hemi-continuous, $\lim_{\epsilon\to0^+}(t^H((1-\mu)(\widehat{b}+\epsilon),x^L), t^L((1-\mu)(\widehat{b}+\epsilon),x^L)) \in T((1-\mu)\widehat{b},x^L)$. Moreover, since $T(\cdot)$ is a lattice, $\lim_{\epsilon\to0^+}(t^H((1-\mu)(\widehat{b}+\epsilon),x^L), t^L((1-\mu)(\widehat{b}+\epsilon),x^L)) \curlywedge (t^H((1-\mu)\widehat{b},x^L), t^L((1-\mu)\widehat{b},x^L)) \in T(\cdot)$ which contradicts with the definition of $(t^H(\cdot),t^L(\cdot))$. Suppose that the state is (x^H,x^L) where $x^H+x^L \in [0,\widehat{b}]$. Since $\overline{\theta}q_L - V^L(\frac{x^H+x^L}{1-\mu}) > 0$, the arbitrage equation $\overline{\theta}q_L - V^L(\frac{x^H+x^L}{1-\mu}) = \rho(\overline{\theta}q_L - V^L((t^H+t^L)(x^H,x^L)))$ implies that $V^L((t^H+t^L)(x^H,x^L)) < V^L(\frac{x^H+x^L}{1-\mu})$. That is, $(t^H+t^L)(x^H,x^L) > \frac{x^H+x^L}{1-\mu}$. Similarly, suppose the state is $(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H)$ where $\epsilon \in (0,1-\widehat{b}]$. Since $\underline{\theta}q_L - V^L(\widehat{b}+\epsilon) < 0$, the arbitrage equation $\underline{\theta}q_L - V^L(\widehat{b}+\epsilon) = \rho(\underline{\theta}q_L - V^L((t^H+t^L)(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H)))$ implies that $V^L((t^H+t^L)(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H)) > V^L(\widehat{b}+\epsilon)$. That is, $(t^H+t^L)(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H) < \widehat{b}+\epsilon$. So, $\lim_{\epsilon\to0^+}((t^H+t^L)(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H)) \leq \widehat{b} < (t^H+t^L)(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H$. Since t^H is nondecreasing, we have $\lim_{\epsilon\to0^+} t^H(x^H,(1-\mu)(\widehat{b}+\epsilon)-x^H) \geq t^H(x^H,(1-\mu)\widehat{b}-x^H)$. This implies that $\lim\limits_{\epsilon \to 0^+} t^L(x^H, (1-\mu)(\widehat{b}+\epsilon) - x^H) < t^L(x^H, (1-\mu)(\widehat{b} - x^H))$. Since, $T(\cdot)$ is upper hemi-continuous, $\lim\limits_{\epsilon \to 0^+} (t^H(x^H, (1-\mu)(\widehat{b}+\epsilon) - x^H), t^L(x^H, (1-\mu)(\widehat{b}+\epsilon) - x^H)) \in T(x^H, (1-\mu)\widehat{b} - x^H)$. Moreover, since $T(\cdot)$ is a lattice, $\lim\limits_{\epsilon \to 0^+} (t^H(x^H, (1-\mu)(\widehat{b}+\epsilon) - x^H), t^L(x^H, (1-\mu)(\widehat{b}+\epsilon) - x^H))$ $\curlywedge (t^H(x^H, (1-\mu)\widehat{b} - x^H), t^L(x^H, (1-\mu)\widehat{b} - x^H)) \in T(\cdot)$ which contradicts with the definition of $(t^H(\cdot), t^L(\cdot))$.

Lemma 1.

If there exists $\left(\acute{y}^H, \acute{y}^L\right)$ such that (i) $V^H\left(y^H\right) = V^H\left(\acute{y}^H\right)$ and $f^H\left(y^H\right) = f^H\left(\acute{y}^H\right)$, (ii) $V^L\left(y^H + y^L\right) = V^L\left(\acute{y}^H + \acute{y}^L\right)$ and $f^L\left(y^H + y^L\right) = f^L\left(\acute{y}^H + \acute{y}^L\right)$, (iii) $V^H\ \left(y^H\right)\ \left(\acute{y}^H - y^H\right)$ $+\ V^L\ \left(y^H + y^L\right)\left(\acute{y}^L - y^L\right) > 0$ hold for some $\acute{y}^H \geq 0$ and $\acute{y}^L \geq 0$ then $\left(y^H, y^L\right)$ cannot be a steady state.

If these conditions hold, then the monopolist strictly prefers $\left(\acute{y}^H, \acute{y}^L\right)$ to $\left(y^H, y^L\right)$. Therefore $\left(y^H, y^L\right)$ cannot be a steady state since the monopolist deviates from that state with probability one.

Proof of Lemma 1. The proof of this lemma is trivial. Because, it is clear that the monopolist prefers $\left(\acute{y}^H, \acute{y}^L\right)$ to $\left(y^H, y^L\right)$ almost sure.

Proof of Proposition 3. Lemma 1 helps us to establish that some states cannot be a steady state. First, we show that $(\grave{y}^H, 0)$ cannot be a steady state when $\grave{y}^H \in (\widehat{b}, 1)$. Suppose not. Then $V^H(\grave{y}^H) = \underline{\theta} q_H$ and $V^L(\grave{y}^H) = \underline{\theta} q_L$. Since $V^i(\cdot)$ is a non-increasing function, $V^H(b) \leq \underline{\theta} q_H$ and $V^L(b) \leq \underline{\theta} q_L$ for all $b \in (\grave{y}^H, 1]$. So, $V^H(y^H) = \underline{\theta} q^H$ and $V^L(y^H) = \underline{\theta} q_L$ for all $y \in [\grave{y}^H, 1]$. Since the seller would prefer $y^H > \grave{y}^H$, $(\grave{y}^H, 0)$ cannot be a steady state.

Second, for $\grave{y}^H \in (0, \widehat{b}]$, and $\grave{y}^H + \grave{y}^L \in (\widehat{b}, 1)$, $(\grave{y}^H, \grave{y}^L)$ cannot be a steady state. Suppose not. Then $V^H(\grave{y}^H) = \overline{\theta} q_H$ and $V^L(\grave{y}^H + \grave{y}^L) = \underline{\theta} q_L$. Therefore, for all $b > \grave{y}^H + \grave{y}^L$, $V^L(b) \leq \underline{\theta} q_L$ has to hold for $(\grave{y}^H, \grave{y}^L)$ to be a steady state. However, for all $b \in [0, 1]$, we must have $P^L(b) \geq \underline{\theta} q_L$. So, for all $b > \grave{y}^H + \grave{y}^L$, we must have $f^L(b) = \underline{\theta} q_L$. Hence, $(\grave{y}^H, \grave{y}^L)$ cannot be a steady state.

Third, for $\grave{y}^H \in (\widehat{b}, 1)$, and $\grave{y}^H + \grave{y}^L \in (\widehat{b}, 1]$, $(\grave{y}^H, \grave{y}^L)$ cannot be a steady state. Suppose not. Then $V^H(\grave{y}^H) = \underline{\theta} q_H$ and $V^L(\grave{y}^H + \grave{y}^L) = \underline{\theta} q_L$. Therefore, for all $b^H > \grave{y}^H$ and $b^L > \grave{y}^H + \grave{y}^L$, $V^H(b^H) \leq \overline{\theta} q_L$ and $V^L(b^L) \leq \underline{\theta} q_L$ have to hold for $(\grave{y}^H, \grave{y}^L)$ to be a steady state. However, for all $b^H \in [0, 1]$ and $b^L \in [0, 1]$, we must have $V^H(b^H) \geq \overline{\theta} q_L$ and $V^L(b^L) \geq \underline{\theta} q_L$. So, for all $b^H > y^H$ and $b^L > \grave{y}^H + \grave{y}^L$, we have $V^H(b^H) \geq \overline{\theta} q_L$ and $V^L(b^L) = \underline{\theta} q_L$. Hence, we conclude that $(\grave{y}^H, \grave{y}^L)$ cannot be a steady state.

Fourth, for $\grave{y}^H + \grave{y}^L \in (0, \widehat{b}]$ and $\grave{y}^L > 0$, $(\grave{y}^H, \grave{y}^L)$ cannot be a steady state. Suppose not. Then, we must have $V^H(b) < \overline{\theta} q_H$ for $b \in (\grave{y}^H, \widehat{b}]$ and $V^L(b) = \overline{\theta} q_L$ for $b \leq \grave{y}^H + \grave{y}^L$. However, if there exists b' such that $V^H(b') < \overline{\theta} q_H$, it must be $V^L(b') < \overline{\theta} q_L$ which leads to a contradiction.

Proof of (2): Suppose that $(\widehat{b}, 0)$ is a steady state. Then, according to Proposition 2, $P^H(\widehat{b}) = \overline{\theta} q_H$ and $P^L(\widehat{b}) = \overline{\theta} q_L$. Since $V^i(\cdot)$ is non-increasing, we must have $P^i(b) \leq \overline{\theta} q_i$ for all $b \in [0, \widehat{b}]$. So Lemma 1 shows that $(b, 0)$ such that $b < \widehat{b}$ cannot be a steady state. We conclude that the only possible steady states other than $(\widehat{b}, 0)$ are $(\widehat{b}, 1 - \widehat{b})$ and $(1, 0)$. Suppose now that $(\widehat{b}, 1 - \widehat{b})$ is a steady state. Then, $P^H(\widehat{b}) = \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L$ and $P^L(\widehat{b}) = \underline{\theta} q_L$.

Proof of (3): Suppose that $(\grave{y}^H, 0)$ where $\grave{y}^H \in (0, \widehat{b})$ is a steady state. Then $P^H(b) = \overline{\theta} q_H$ and $P^L(b) = \overline{\theta} q_L$ for all $b \in [0, \grave{y}^H]$ and $P^H(b) < \overline{\theta} q_H$ and $P^L(b) < \overline{\theta} q_L$ for all $b \in (\grave{y}^H, \widehat{b}]$. This implies that $(\widehat{b}, .)$ cannot be a steady state. We conclude that the only possible steady states other than $(\grave{y}^H, 0)$ when $\grave{y}^H \in (0, \widehat{b})$ are $(\grave{y}^H, 1 - \grave{y}^H)$ and $(1, 0)$. Suppose that $(\grave{y}^H, \widehat{b} - \grave{y}^H)$ where $\grave{y}^H \in (0, \widehat{b})$ is a steady state. Then $P^H(b^H) = \overline{\theta} q_H$ and $P^L(b^L) = \overline{\theta} q_L$ for all $b^H \in [0, \grave{y}^H]$ and $b^H \in [0, \widehat{b}]$ and $P^H(b^H) < \overline{\theta} q_H$ for all $b^H \in (\grave{y}^H, \widehat{b}]$. This implies that $(\widehat{b}, .)$ cannot be a steady state. We conclude that the only possible steady states other than $(\grave{y}^H, \widehat{b} - \grave{y}^H)$ when $\grave{y}^H \in (0, \widehat{b})$ are $(\grave{y}^H, 1 - \grave{y}^H)$ and $(1, 0)$.

Proof of (1): If there is no steady state in $[0, \widehat{b}]$ then $(1, 0)$ is a steady state.

Proof of Theorem 1. First, let us define the set of states on each path. Let $\overline{x}^j_{k,i}$ be the state of the durable good j in period k on path i. We start with defining the set of states for the high quality good. For $k = 0, 1$, the states are $\overline{x}^H_{0,i} = 1 - \mu$ and $\overline{x}^H_{1,i} = (1 - \mu)\widehat{b}$ on all paths $(i = 1, \ldots, 4)$. For $k = 2$, the state is $\overline{x}^H_{2,i} = \frac{\widehat{b\theta} - \underline{\theta}}{\overline{\theta} - \underline{\theta}}$ on the first and the forth paths $(i = 1, 4)$, and it is $\overline{x}^H_{2,i} = \frac{\left(\widehat{b\theta} - \underline{\theta}\right)(q_H - q_L) + \left(1 - \overline{x}^L_{2,i}\right)\underline{\theta} q_L}{\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)}$ on the second and the third paths ($i = 2, 3$). For $k = 3$, the state is $\overline{x}^H_{3,i} = (1 - \mu)^{-1} \frac{\overline{\theta}}{\rho\left(\overline{\theta} - \underline{\theta}\right)} (\overline{x}^H_{2,i} - (\overline{x}^H_{1,i} - \overline{x}^H_{2,i}))$ on the first and the forth paths $(i = 1, 4)$, and $\overline{x}^H_{3,2} = (1 - \mu)^{-1}(1 + \frac{\underline{\theta} q_L + \overline{\theta}(q_H - q_L)}{\left(\overline{\theta} - \underline{\theta}\right) q_L + \rho\left(\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)\right)}) \overline{x}^H_{2,2} - \frac{\widehat{b\theta}(q_H - q_L) + \left(1 - \overline{x}^L_{3,2}\right)\underline{\theta} q_L}{\left(\overline{\theta} - \underline{\theta}\right) q_L + \rho\left(\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)\right)}$, and $\overline{x}^H_{3,3} = (1 - \mu)^{-1}(1 + \frac{\underline{\theta} q_L + \overline{\theta}(q_H - q_L)}{\rho\left(\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)\right)}) \overline{x}^H_{2,2} - \frac{\widehat{b\theta}(q_H - q_L) + \left(1 - \overline{x}^L_{3,2}\right)\underline{\theta} q_L}{\rho\left(\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)\right)}$. For $k > 3$, the states are $\overline{x}^H_{k,i} = (1 - \mu)^{-1}(\overline{x}^H_{k-1,i} - a_{k,i}(\overline{x}^H_{k-2,i} - \overline{x}^H_{k-1,i}))$ where $a_{k,i} = \frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta} - \underline{\theta}\right)}$ for $i = 1, 4$, $a_{k,2} = \frac{\overline{\theta} q_H}{\rho^{k-3}\left(\left(\overline{\theta} - \underline{\theta}\right) q_L + \rho\left(\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)\right)\right)}$, and $a_{k,3} = \frac{\overline{\theta} q_H}{\rho^{k-2}\left(\underline{\theta} q_L + \left(\overline{\theta} - \underline{\theta}\right)(q_H - q_L)\right)}$.

We now define the set of states for the low quality good. On path 1, the states are $\overline{x}^L_{k,1} = 0$ for all k. On path 2, the states are $\overline{x}^L_{1,2} = (1-\mu)(1-\widehat{b})$ for $k = 1$ and $\overline{x}^L_{k,2} \in (0, (1-\mu)\widehat{b} - \overline{x}^H_{k,2}]$ for all $k \geq 2$. On path 3, states are $\overline{x}^L_{1,3} = (1-\mu)(1-\widehat{b})$ for $k = 1$ and $\overline{x}^L_{k,3} \in ((1-\mu)\widehat{b} - \overline{x}^H_{k,3}, 1-\mu-\overline{x}^H_{k,3}]$ for all $k \geq 2$. On path 4, there is an excess supply of the low quality good, so $\overline{x}^L_{k,4} > 1-\mu-\overline{x}^H_{k,4}$ holds for all k.

Second, let us define $\overline{\mu}$. Let us define the set Λ_i as $\Lambda_i = \{\mu \geq 0 \mid \exists\ m_i < \infty$ s.t. $\overline{x}^H_{m+1,i} < 0 \leq \overline{x}^H_{m,i}$ and $\{\overline{x}^H_{k,i}\}^{m_i}_{k=0}$ is a strictly decreasing sequence$\}$ and let $\overline{\mu}$ be $\min \overline{\mu}_i$, where $\overline{\mu}_i = \sup \Lambda_i$. Since $0 \in \Lambda_i$ and Λ_i is open in $\mathbb{R}_+$for all i, we must have $\overline{\mu} > 0$. Now, we show that if $\mu' \in \Lambda_i$ and $\mu'' < \mu'$ then $\mu'' \in \Lambda_i$. This implies that $\Lambda_i = [0, \overline{\mu}_i)$. The proof for path 1 and path 4 are established in Deneckere and Liang (2008, Theorem 1).[1] To prove that this also holds for path 2 and path 3, we show that $\Delta^H_{k,i} = \overline{x}^H_{k-1,i} - \overline{x}^H_{k,i}$ is decreasing in μ for all $i = 2, 3$. By definition of $\overline{x}^H_{k,i}$, $\frac{d\Delta^H_{k,i}}{d\mu} < 0$ holds for $k \leq 2$, $i = 2, 3$. The rest of the proof comes from induction. As the first step of induction, we show that $\frac{d\overline{x}^H_{3,i}}{d\mu} > 0$ and $\frac{d\Delta^H_{3,i}}{d\mu} < -2(1-\mu)^{-1}\overline{x}^H_{3,i}$. Since $\frac{d\overline{x}^H_{3,i}}{d\mu} = (1-\mu)^{-1}(\overline{x}^H_{3,i} + (1-\mu)^{-1}a_{3,i}\overline{x}^H_{2,i})$ and $a_{3,i} > 1$, we must have $\frac{d\overline{x}^H_{3,i}}{d\mu} > 2(1-\mu)^{-1}\overline{x}^H_{3,i} > 0$. Note that $\frac{d\Delta^H_{3,i}}{d\mu} = -\frac{d\overline{x}^H_{3,i}}{d\mu}$. Thus, $\frac{d\Delta^H_{3,i}}{d\mu} < -2(1-\mu)^{-1}\overline{x}^H_{3,i}$ holds for path $i = 2, 3$. As the second step of induction, we assume $\frac{d\overline{x}^H_{k-1,i}}{d\mu} > 0$ and $\frac{d\Delta^H_{k-1,i}}{d\mu} < -(k-2)(1-\mu)^{-1}\overline{x}^H_{k-1,i}$ hold. We now show that $\frac{d\overline{x}^H_{k,i}}{d\mu} > 0$ and $\frac{d\Delta^H_{k,i}}{d\mu} < -(k-1)(1-\mu)^{-1}\overline{x}^H_{k,i}$ hold as well. Since $\Delta^H_{k,i} = (1-(1-\mu)^{-1})\overline{x}^H_{k-1,i} + (1-\mu)^{-1}a_{k,i}\Delta^H_{k-1,i}$ and $\frac{d\overline{x}^H_{k-1,i}}{d\mu} > 0$, we have $\frac{d\Delta^H_{k,i}}{d\mu} < -(1-\mu)^{-2}\overline{x}^H_{k-1,i} + (1-\mu)^{-2}a_{k,i}\Delta^H_{k-1,i} + (1-\mu)^{-1}\frac{da_{k,i}}{d\mu}\Delta^H_{k-1,i} + (1-\mu)^{-1}a_{k,i}\frac{d\Delta^H_{k-1,i}}{d\mu}$. Moreover, $\frac{da_{k,2}}{d\mu} = a_{k,2}(1-\mu)^{-1}((k-2) - \frac{\overline{\theta}q_H}{(\overline{\theta}-\underline{\theta})q_L + \rho(\underline{\theta}q_L + (\overline{\theta}-\underline{\theta})(q_H - q_L))})$ and $\frac{da_{k,3}}{d\mu} = a_{k,3}(1-\mu)^{-1}(k-2)$ imply that we must have $\frac{d\Delta^H_{k,i}}{d\mu} < -(1-\mu)^{-2}\overline{x}^H_{k-1,i} + (1-\mu)^{-2}a_{k,i}(k-1)\Delta^H_{k-1,i} + (1-\mu)^{-1}a_{k,i}\frac{d\Delta^H_{k-1,i}}{d\mu}$. Additionally, due to the induction, since $\frac{d\Delta^H_{k-1,i}}{d\mu} < -(k-2)(1-\mu)^{-1}\overline{x}^H_{k-1,i}$ and $a_{k,i} > 1$, $\frac{d\Delta^H_{k,i}}{d\mu}$ $< -(k-1)(1-\mu)^{-2}\overline{x}^H_{k-1,i} + (1-\mu)^{-2}a_{k,i}(k-1)\Delta^H_{k-1,i} = -(k-1)(1-\mu)^{-1}\overline{x}^H_{k,i}$ holds.

Now, we prove that a stationary equilibrium exists for $\mu \leq \overline{\mu}$. Let us define $\{P^H(.)$, $P^L(.)$, $t^H(.)$, $t^L(.)$, $R(.)\}$ as follows. First, we define path 1:

[1] These two are identical because the set of states of the high quality good on path 1 and on path 4 is the same as the set of states of a durable good available in only one version.

$$P^H(x^H,0)=\begin{cases} p^H_{m_1,1} & \text{for} \quad x^H\in[0,\overline{y}^H_{m_1,1}] \\ p^H_{k,1} & \text{for} \quad x^H\in(\overline{y}^H_{k+1,1},\overline{y}^H_{k,1}] \end{cases}$$

where $k=0,\ldots,m_1-1$,

$P^L(x^H,0)\geq\overline{\theta}q_L$ for all x^H,

$$t^H\left(x^H,0\right)=\begin{cases} \overline{y}^H_{m_1-1,1} & \text{for} \quad x^H\in[0,\overline{x}^H_{m_1,1}] \\ \overline{y}^H_{k-1,1} & \text{for} \quad x^H\in(\overline{x}^H_{k+1,1},\overline{x}^H_{k,1}] \\ 1 & \text{for} \quad x^H\in(\overline{x}^H_{2,1},1-\mu] \end{cases}$$

where $k=2,\ldots,m_1-1$,

$t^L(x^H,0)=0$ for all x^H,

$$R(x^H,0)=\begin{cases} (\overline{y}^H_{m_1-1,1}-x^H)P^H(\overline{y}^H_{m_1-1,1},0)+\delta R_{m_1-1} & \text{for} \quad x^H\in[0,\overline{x}^H_{m_1,1}] \\ (\overline{y}^H_{k-1,1}-x^H)P^H(\overline{y}^H_{k-1,1},0)+\delta R_{k-1} & \text{for} \quad x^H\in(\overline{x}^H_{k+1,1},\overline{x}^H_{k,1}] \\ (1-x^H)P^H(1,0)+\frac{\delta\mu\underline{\theta}q_H}{1-\delta} & \text{for} \quad x^H\in(\overline{x}^H_{2,1},1-\mu] \end{cases}$$

where $k=2,\ldots,m_1-1$, $R_s\equiv R(\overline{x}^H_{s,1},0)$ and $p^H_{k,1}=(1-\rho^k)\overline{\theta}q_H+\rho^k\underline{\theta}q_H$.

The sequence $\{\overline{x}^H_{k,1}\}^{m_1}_{k=2}$ is defined such that when the state is $(\overline{x}^H_{k,1},0)$ the monopolist is indifferent between selecting $(\overline{y}^H_{k-1,1},0)$ and $(\overline{y}^H_{k-2,1},0)$ where $\overline{y}^H_{k,1}=\frac{\overline{x}^H_{k,1}}{1-\mu}$.

Second, we define path 2:

$$P^H\left(x^H, x^L\right) = \begin{cases} p^H_{m_2,2} & \text{for} \quad x^H \in [0, \overline{y}^H_{m_2,2}(x^L)] \\ p^H_{k,2} & \text{for} \quad x^H \in (\overline{y}^H_{k+1,2}(x^L), \overline{y}^H_{k,2}(x^L)] \end{cases}$$

where $k = 0, \ldots, m_2 - 1$,

$$P^L\left(x^H, x^L\right) = \begin{cases} p^L_{m_2,2} & \text{for} \quad x^H \in [0, \overline{y}^H_{m_2,2}(x^L)] \\ p^L_{k,2} & \text{for} \quad x^H \in (\overline{y}^H_{k+1,2}(x^L), \overline{y}^H_{k,2}(x^L)] \\ 0 & \text{for} \quad x^H \in (\overline{y}^H_{1,2}, \overline{y}^H_{0,2}] \end{cases}$$

where $k = 1, \ldots, m_2 - 1$,

$$t^H(x^H, x^L) = \begin{cases} \overline{y}^H_{m_2-1,2} & \text{for} \quad x^H \in [0, \overline{x}^H_{m_2,2}(x^L)] \\ \overline{y}^H_{k-1,2} & \text{for} \quad x^H \in (\overline{x}^H_{k+1,2}(x^L), \overline{x}^H_{k,2}(x^L)] \\ 1 & \text{for} \quad x^H \in (\overline{x}^H_{2,2}(x^L), 1-\mu] \end{cases}$$

where $k = 2, \ldots, m_2 - 1$,

$t^L(x^H, x^L) = x^L$ for all x^H,

$$R(x^H, x^L) = \begin{cases} (\overline{y}^H_{m_2-1,2} - x^H)P^H(\overline{y}^H_{m_2-1,2}, x^L) + \delta R_{m_2-1,2} & \text{for} \quad x^H \in D_1 \\ (\overline{y}^H_{k-1,1} - x^H)P^H(\overline{y}^H_{k-1,1}, x^L) + \delta R_{k-1,1} & \text{for} \quad x^H \in D_2 \\ (1 - x^H)P^H(1, x^L) + \frac{\delta\mu\underline{\theta}(q_H - q_L)}{1-\delta} & \text{for} \quad x^H \in D_3 \end{cases}$$

where $k = 2, \ldots, m_2 - 1$, $R_{s,i} \equiv R(\overline{x}^H_{s,i}, 0)$, $p^L_{0,2} = 0$,

$p^L_{k,2} = (1 - \rho^{k-1})\overline{\theta} q_L + \rho^{k-1}(1-\rho)\underline{\theta} q_L$ for $k = 1, 2, \ldots$,

$p^H_{k,2} = p^L_{k,2} + (\overline{\theta} - \rho^k(\overline{\theta} - \underline{\theta}))(q_H - q_L)$ for $k = 0, 1, \ldots$,

$D_1 \equiv [0, \overline{x}^H_{m_2,1}(x^L)]$, $D_2 \equiv (\overline{x}^H_{k+1,2}(x^L), \overline{x}^H_{k,2}(x^L)]$, and $D_3 \equiv (\overline{x}^H_{2,2}(x^L), 1-\mu]$

The sequence $\{\overline{x}^H_{k,2}(x^L)\}_{k=2}^{m_2}$ is defined such that when the state is $(\overline{x}^H_{k,2}(x^L), x^L)$ the monopolist is indifferent between selecting $(\overline{y}^H_{k-1,2}(x^L), x^L)$ and $(\overline{y}^H_{k-2,2}(x^L), x^L)$ where $\overline{y}^H_{k,2} = \frac{\overline{x}^H_{k,2}}{1-\mu}$. We now define $x^L_{2,2}(x^L)$ and prove the existence of this path. When the state is $\left(x^H, x^L\right)$, the state of the low quality good in the second period will be $x^L_{2,2}(x^L) = (1-\mu)^s x^L$, $s \in \{\underline{n}, \underline{n}+1, \ldots, \overline{n}\}$. We define $\underline{n}$ as $\underline{n} = \left\lfloor \frac{\ln(\overline{x}^L_{2,2}) - \ln(x^L)}{\ln(1-\mu)} \right\rfloor$ where $\overline{x}^L_{2,2}$ solves $x^H_{2,2}(\overline{x}^L) + \overline{x}^L_{2,2} = (1-\mu)\widehat{b}$, and define $\overline{n}$ as $m((1-\mu)^{\overline{n}} x^L) \geq \overline{n}$ and $m((1-\mu)^{\overline{n}} x^L) < \overline{n}+1$ where $m(x)$ is set such that $\overline{x}^H_{m+1,2} < 0 \leq \overline{x}^H_{m,2}$ with $x^L_{2,2} = x$. Let us define the set Γ_2 as $\Gamma_2 = \{s : x^H \in (\overline{x}^H_{s+2,2}, \overline{x}^H_{s+1,2}]$ s.t $s = \underline{n}, \ldots, \overline{n}\}$ and define k as $k = \min \Gamma_2$. Since $\overline{x}^H_{s,2}(x^L) < \overline{x}^H_{s,2}(\acute{x}^L)$ for $x^L_2 > \acute{x}^L_2$ for all $s = \underline{n}+2, \ldots, \overline{n}+2$, Γ_2 is non-empty.

Third, we define path 3:

$$P^H(x^H,x^L)=\begin{cases} p^H_{m_2,3} & \text{for} \quad x^H\in[0,\overline{y}^H_{m_3,3}(x^L)] \\ p^H_{k,3} & \text{for} \quad x^H\in(\overline{y}^H_{k+1,3}(x^L),\overline{y}^H_{k,3}(x^L)] \end{cases}$$

where $k=0,\dots,m_3-1$,

$$P^L(x^H,x^L)=\begin{cases} p^L_{m_3,3} & \text{for} \quad x^H\in[0,\overline{y}^H_{m_3,3}(x^L)] \\ p^L_{k,3} & \text{for} \quad x^H\in(\overline{y}^H_{k+1,3}(x^L),\overline{y}^H_{k,3}(x^L)] \\ 0 & \text{for} \quad x^H\in(\overline{y}^H_{1,3},\overline{y}^H_{0,3}] \end{cases}$$

where $k=1,\dots,m_3-1$,

$$t^H(x^H,x^L)=\begin{cases} \overline{y}^H_{m_3-1,3} & \text{for} \quad x^H\in[0,\overline{x}^H_{m_3,3}(x^L)] \\ \overline{y}^H_{k-1,3} & \text{for} \quad x^H\in(\overline{x}^H_{k+1,3}(x^L),\overline{x}^H_{k,3}(x^L)] \\ 1 & \text{for} \quad x^H\in\left(\overline{x}^H_{2,3}\left(x^L\right),1-\mu\right] \end{cases}$$

where $k=1,\dots,m_3-2$,

$t^L(x^H,x^L)=x^L$ for all x^H,

$$R(x^H,x^L)=\begin{cases} (\overline{y}^H_{m_2-1,2}-x^H)P^H(\overline{y}^H_{m_2-1,2},x^L)+\delta R_{m_2-1,2} & \text{for} \quad x^H\in D_1 \\ (\overline{y}^H_{k-1,1}-x^H)P^H(\overline{y}^H_{k-1,1},x^L)+\delta R_{k-1,1} & \text{for} \quad x^H\in D_2 \\ (1-x^H)P^H(1,x^L)+\frac{\delta\mu\underline{\theta}(q_H-q_L)}{1-\delta} & \text{for} \quad x^H\in D_3 \end{cases}$$

where $k=2,\dots,m_3-1$, $R_{s,i}\equiv R(\overline{x}^H_{s,i},0)$, $p^L_{k,3}=(1-\rho^k)\underline{\theta}q_L$,

$p^H_{k,3}=p^L_{k,3}+(\overline{\theta}-\rho^k(\overline{\theta}-\underline{\theta}))(q_H-q_L)$, $D_1\equiv[0,\overline{x}^H_{m_2,1}(x^L)]$,

$D_2\equiv(\overline{x}^H_{k+1,2}(x^L),\overline{x}^H_{k,2}(x^L)]$, and $D_3\equiv(\overline{x}^H_{2,2}(x^L),1-\mu]$.

The sequence $\{\overline{x}^H_{k,3}(x^L)\}_{k=2}^{m_3}$ is defined such that when the state is $(\overline{x}^H_{k,3}(x^L)),x^L))$ the monopolist is indifferent between selecting $(\overline{y}^H_{k-1,3}(x^L),x^L)$ and $(\overline{y}^H_{k-2,3}(x^L),x^L)$ where $\overline{y}^H_{k,3}=\frac{\overline{x}^H_{k,3}}{1-\mu}$. We now define $x^L_{2,3}(x^L)$ and prove the existence of this path. When the state is (x^H,x^L), the state of the low quality good in the second period is $x^L_{2,3}=(1-\mu)^s x^L$, $s\in\{0,1,\dots,\underline{n}\}$ where $\underline{n}'$ is defined as $\underline{n}'=\left\lceil\frac{\ln(\overline{x}^L_{2,3})-\ln(x^L)}{\ln(1-\mu)}\right\rceil$. Let us define the set Γ_3 as $\Gamma_3=\{s: x^H\in(\overline{x}^H_{s+3,3},\overline{x}^H_{s+2,3}]$ s.t $s=0,\dots,\underline{n}'\}$ and define k as $k=\max\Gamma_3$. To prove that path 3 exists we show that Γ_3 is a non-empty set. For $s=2,\dots,\underline{n}'+2$, we have $\overline{x}^H_{s,3}(x^L)<\overline{x}^H_{s,3}(\acute{x}^L)$ for $x^L_2>\acute{x}^L_2$. To finish the proof we need to show that $\overline{x}^H_{\underline{n}'+2,3}(\overline{x}^L)<\overline{x}^H_{\underline{n}+2,2}(\overline{x}^L)$. Note that $n-1\le\underline{n}'\le\underline{n}$. Hence, we show that $\overline{x}^H_{s,3}(\overline{x}^L)<\overline{x}^H_{s,2}(\overline{x}^L)$ holds by induction. Since $\overline{x}^j_{2,2}=\overline{x}^j_{2,3}=\overline{x}^j_2$ for $j=H,L$, by definition we have $\overline{x}^H_{3,3}<\overline{x}^H_{3,2}$. Now, we show that $\overline{x}^H_{4,3}<\overline{x}^H_{4,2}$ and $\Delta^H_{4,2}<\Delta^H_{4,3}$ hold. Since we have $a_{4,3}>a_{4,2}$ and $a_{4,2}\overline{x}^H_{3,2}>a_{4,3}\overline{x}^H_{3,3}$ for

$\mu < \overline{\mu}$, the difference equations, $\overline{x}^H_{4,2} - \overline{x}^H_{4,3} = (1-\mu)^{-1}((\overline{x}^H_{3,2} - \overline{x}^H_{3,3}) + (a_{4,2}\overline{x}^H_{3,2} - a_{4,3}\overline{x}^H_{3,3}))$ and $\Delta^H_{4,3} - \Delta^H_{4,2} = \overline{x}^H_2(a_{4,3} - a_{4,2}) + (a_{4,2}\overline{x}^H_{3,2} - a_{4,3}\overline{x}^H_{3,3})$, are both positive. Then we assume that $\overline{x}^H_{k-1,3} < \overline{x}^H_{k-1,2}$ and $\Delta^H_{k-1,2} < \Delta^H_{k-1,3}$ hold and show that $\overline{x}^H_{k,3} < \overline{x}^H_{k,2}$ and $\Delta^H_{k,2} < \Delta^H_{k,3}$. Since $a_{k,3} > a_{k,2}$ and $\Delta^H_{k-1,2} < \Delta^H_{k-1,3}$, we have $\overline{x}^H_{k,2} - \overline{x}^H_{k,3} = (1-\mu)^{-1}((\overline{x}^H_{k-1,2} - \overline{x}^H_{k-1,3}) + (a_{k,3}\Delta^H_{k-1,3} - a_{k,2}\Delta^H_{k-1,2})) > 0$. Moreover, since $\Delta^H_{k,i} = (1-(1-\mu)^{-1})\overline{x}^H_{k-1,i} + a_{k,i}\Delta^H_{k-1,i}$, we have $\Delta^H_{k,3} - \Delta^H_{k,2} = (1-(1-\mu)^{-1})(\overline{x}^H_{k-1,3} - \overline{x}^H_{k-1,2}) + a_{k,3}\Delta^H_{k-1,3} - a_{k,2}\Delta^H_{k-1,2} > 0$.

Last, we define path 4:

$$P^H(x^H, x^L) = \begin{cases} p^H_{m_4,4} & \text{for} \quad x^H \in [0, \overline{y}^H_{m_4,4}] \\ p^H_{k,4} & \text{for} \quad x^H \in (\overline{y}^H_{k+1,4}, \overline{y}^H_{k,4}] \end{cases}$$

where $k = 0, \ldots, m_4 - 1$,

$P^L(x^H, x^L) = 0$ for all x^H,

$$t^H(x^H, x^L) = \begin{cases} \overline{y}^H_{m_4-1,4} & \text{for} \quad x^H \in [0, \overline{x}^H_{m_4,4}] \\ \overline{y}^H_{k-1,4} & \text{for} \quad x^H \in (\overline{x}^H_{k+1,4}, \overline{x}^H_{k,4}] \\ 1 & \text{for} \quad x^H \in (\overline{x}^H_{2,4}, 1-\mu] \end{cases}$$

where $k = 2, \ldots, m_4 - 1$,

$t^L(x^H, x^L) = x^L$ for all x^H,

$$R(x^H, x^L) = \begin{cases} (\overline{y}^H_{m_4-1,4} - x^H)P^H(\overline{y}^H_{m_4-1,4}, x^L) + \delta R_{m_1-1} & \text{for} \quad x^H \in D_1 \\ (\overline{y}^H_{k-1,4} - x^H)P^H(\overline{y}^H_{k-1,4}, x^L) + \delta R_{k-1} & \text{for} \quad x^H \in D_2 \\ (1-x^H)P^H(1, x^L) + \frac{\delta\mu\underline{\theta}(q_H - q_L)}{1-\delta} & \text{for} \quad x^H \in D_3 \end{cases}$$

where $k = 2, \ldots, m_4 - 1$, $R_s \equiv R(\overline{x}^H_{s,4}, 0)$,

$p^H_{k,4} = (1-\rho^k)\overline{\theta}(q_H - q_L) + \rho^k\underline{\theta}(q_H - q_L)$,

$D_1 \equiv [0, \overline{x}^H_{m_4,4}]$, $D_2 \equiv (\overline{x}^H_{k+1,4}, \overline{x}^H_{k,4}]$, and $D_3 \equiv (\overline{x}^H_{2,4}, 1-\mu]$.

The sequence $\{\overline{x}^H_{k,4}\}^{m_4}_{k=2}$ is defined such that when the state is $(\overline{x}^H_{k,4}, x^L)$ the monopolist is indifferent between selecting $(\overline{y}^H_{k-1,4}, x^L)$ and $(\overline{y}^H_{k-2,4}, x^L)$ where $\overline{y}^H_{k,4} = \frac{\overline{x}^H_{k,4}}{1-\mu}$.

Now, we verify that the path defined above is a solution to the system. Let's define $\Pi(y, x) = P^H(y)(y^H - x^H) + P^L(y)(y^L - x^L) + \delta R((1-\mu)y)$. First, we show that the monopolist never increases the state of the low quality good. When the state is (x^H_k, x^L_k) the monopolist is indifferent between $(\frac{x_{k-1}^H}{1-\mu}, \frac{x_{k-1}^L}{1-\mu})$ and $(\frac{x_{k-2}^H}{1-\mu}, \frac{x_{k-2}^L}{1-\mu})$ when

$$\begin{aligned}
&(1-\mu)(x_k^H(p_{k-1}^H - p_{k-2}^H) + x_k^L(p_{k-1}^L - p_{k-2}^L)) \\
&= x_{k-1}^H(p_{k-1}^H - \rho p_{k-3}^H) + x_{k-1}^L(p_{k-1}^L - \rho p_{k-3}^L) - x_{k-2}^H(p_{k-2}^H - \rho p_{k-3}^H) - x_{k-2}^L(p_{k-2}^L - \rho p_{k-3}^L)
\end{aligned}$$

holds for $k = 3, \ldots$. Moreover, when the state is $\left(x_2^H, x_2^L\right)$ the monopolist is indifferent between $(\widehat{b}, 1-\widehat{b})$ and $(1, x_2^L)$ if

$$x_2^H(p_1^H - p_0^H) + x_2^L p_1^L = \widehat{b}p_1^H - p_0^H + (1-\widehat{b})p_1^L + \rho(1-\widehat{b})p_0^H$$

Thus, we have $\frac{dx_k^H}{dx_k^L} = -\frac{p_{k-1}^L - p_{k-2}^L}{p_{k-1}^H - p_{k-2}^H}$ for all k. Then, it follows that on both path 2 and path 3 we have $\frac{d\Pi}{dx_K^L} < 0$ for $k = 3, \ldots$. However, for $k = 2$ even though $\frac{d\Pi}{dx_2^L} < 0$ holds for path 3, $\frac{d\Pi}{dx_2^L} < 0$ holds for path 2 if $\underline{\theta} > (\overline{\theta} - \underline{\theta})$ which holds on $\bigcap_{\forall i} \Gamma_i$. This implies that when the state is $(\overline{x}_k^H, \overline{x}_k^L)$ the monopolist is indifferent between moving the state to $(\overline{x}_{k-1}^H, \overline{x}_{k-1}^L)$ and to $(\overline{x}_{k-2}^H, \overline{x}_{k-2}^L)$ where $\overline{x}_{k-1}^L = \overline{x}_{k-2}^L = (1-\mu)\overline{x}_k^L$. We therefore have $\Pi(\overline{y}_{k-1}, \overline{x}_k) = \Pi(\overline{y}_{k-2}, \overline{x}_k)$. Let $h_k(x) = \Pi(\overline{y}_{k-1}, x) - \Pi(\overline{y}_{k-2}, x)$. Then we have $h_k(\overline{x}_k) = 0$ and $\dfrac{dh_k(x)}{dx} = -P^H(\overline{y}_{k-1}) + P^H(\overline{y}_{k-2}) < 0$. It implies that when $\overline{x}_k^L(x^L) = \frac{x_2^L}{(1-\mu)^{k-2}}$, for $x < \overline{x}_k^H(x^L)$ we have $\overline{y}_{k-1} \succ \overline{y}_{k-2}$, and for $x > \overline{x}_k^H(x^L)$ we have $\overline{y}_{k-1} \prec \overline{y}_{k-2}$. For a given x^L, $\Pi(y, x)$ is strictly increasing in y^H on any of the intervals $[0, \overline{y}_{m_i,i}^H(x_2^L)]$,$(\overline{y}_{m_i,i}^H(x_2^L), \overline{y}_{m_i-1,i}^H(x_2^L)]$, $\ldots$, $(\overline{y}_1^H, 1]$. It follows that $t(x)$ is a solution to the problem. Since buyers' arbitrage equation is also satisfied, the set $\{P^H(.),P^L(.),t^H(.),t^L(.),R(.)\}$ is a stationary set for $\mu \leq \overline{\mu}$.

To prove the uniqueness of the solution, we consider any stationary equilibrium with unique steady state $(1, 0)$, and let $\{P_0^H(.),\ P_0^L(.),\ t_0^H(.),\ t_0^L(.),\ R_0(.)\}$ be the associated stationary set. We now prove that $\{P_0^H(.),\ P_0^L(.),\ t_0^H(.),\ t_0^L(.),\ R_0(.)\}$ is equal to $\{P^H(.),\ P^L(.),\ t^H(.),\ t^L(.),\ R(.)\}$.

First, we will show that $P_0^H(y) = P^H(y)$ for all $y^H \in (\widehat{b}, 1]$. Since neither $(\widehat{b}, 0)$ nor $(\widehat{b}, 1-\widehat{b})$ is a steady state, we must have $V^H(\widehat{b}) < \overline{\theta}q_H$. Suppose now that there exists $\epsilon > 0$ such that $V^H(b) > \underline{\theta}q_H$ for all $b \in (\widehat{b}, \widehat{b}+\epsilon)$. This yields a contradiction, since we have $V^H(\widehat{b}) < \overline{\theta}q_H$. Therefore, $V^H(b) = \underline{\theta}q_H$ for all $b \in (\widehat{b}, 1]$. It follows that for all $y^H \in (\widehat{b}, 1]$, we have $P^H(y^H,0) = \underline{\theta}q_H$, and for $\widehat{b} < y^H + y^L \leq 1$, we have $P^H(y^H,y^L) = \underline{\theta}q_H$ and for $y^H + y^L > 1$, we have $P^H(y^H,y^L) = \underline{\theta}(q_H - q_L)$.

Second, we show that $P_0^L(y) = P^L(y)$ for all $y^H \in (\widehat{b},1]$. The same argument can be applied here. Since $V^L(\widehat{b}) < \overline{\theta}q_L$ and there exists no $\epsilon > 0$ such that $V^L(b) > \underline{\theta}q_L$ for all $b \in (\widehat{b},\widehat{b}+\epsilon)$, we have $V^L(b) = \underline{\theta}q_L$ for all $b \in (\widehat{b},1]$. It follows that, for all (y^H,y^L) such that $\widehat{b} < y^H + y^L \leq 1$, we have $P^L(y^H,y^L) = \underline{\theta}q_L$ and for $y^H + y^L > 1$, we have $P^L(y^H,y^L) = 0$.

Then, we prove that $t_0^H(\overline{x}_1^H,x^L) = 1$, and $t_0^L(\overline{x}_1^H,x^L) = x^L$. Let's apply the arbitrage equation for the high quality good to $\widehat{b}$. We then have $\overline{\theta}q_H - V^H(\widehat{b}) = \rho(\overline{\theta}q_H - V^H(t_0^H(\overline{x}_1^H,x^L)))$. Since $V^H(\widehat{b}) < \overline{\theta}q_H$, it follows that $V^H(\widehat{b}) > V^H(t_0^H(\overline{x}_1^H,x^L))$. Since $V^H(.)$ is non-increasing, $t_0^H\left(\overline{x}_1^H,x^L\right) > \widehat{b}$ must hold. If we apply the arbitrage equation for the low quality good to $\widehat{b} + y^L$ where $y^L = \frac{x^L}{1-\mu}$, we will have $t_0^H(\overline{x}_1^H,0) + t_0^L(\overline{x}_1^H,0) > \widehat{b}$, and for $x^L > 0$, $t_0^H(\overline{x}_1^H,x^L) + t_0^L(\overline{x}_1^H,x^L) \geq \widehat{b} + y^L$. Given the price structure described above, the monopolist chooses $t_0^H(\overline{x}_1^H,x^L) = 1$, and $t_0^L(\overline{x}_1^H,x^L) = x^L$. We now show that there exists x_2^H such that $t_0^H(x_2^H,x^L) = 1$, and $t_0^L(x_2^H,x^L) = x^L$ and that $x_2^H = \overline{x}_2^H$. Due to left continuity of t_0^H there exists $\epsilon > 0$ such that $t_0^H(x^H,x^L) = 1$ for all $x^H \in (\overline{x}_1^H - \epsilon,1]$ and x^L. This implies that, due to the arbitrage equations and incentive compatibility constraints,

$$P_0^j\left(y^H,y^L\right) = \begin{cases} p_{1,1}^j & \text{for} \quad y^H \in (\overline{y}_1^H - \frac{\epsilon_1}{1-\mu},\overline{y}_1^H],\, y^L = 0 \\ p_{1,2}^j & \text{for} \quad y^H \in (\overline{y}_1^H - \frac{\epsilon_2}{1-\mu},\overline{y}_1^H],\, y^H + y^L \leq \overline{y}_1^H \\ p_{1,4}^j & \text{for} \quad y^H \in (\overline{y}_1^H - \frac{\epsilon_3}{1-\mu},\overline{y}_1^H],\, \overline{y}_1^H < y^H + y^L \leq 1 \\ p_{1,4}^j & \text{for} \quad y^H \in (\overline{y}_1^H - \frac{\epsilon_4}{1-\mu},\overline{y}_1^H],\, y^H + y^L > 1 \end{cases} \text{for } j = H,L.$$

Let's define $x_{2,i}^H$ as $x_{2,i}^H = \inf\{x^H : t_0^H(x^H,x_i^L) = 1, \forall x_i^L$ where $x_1^L = 0$, $x_2^L \in [0,\overline{x}_1^H - x^H]$, $x_3^L \in (\overline{x}_1^H - x^H, 1-\mu-\overline{x}_1^H]$, $x_4^L > 1-\mu-\overline{x}_1^H\}$. To observe that $x_{2,i}^H = \overline{x}_{2,i}^H$, recall that. $\Pi(\overline{y}_1,x) < \Pi(\overline{y}_0,x)$ for $x^H > \overline{x}_2^H$ and $\Pi(\overline{y}_1,x) > \Pi(\overline{y}_0,x)$ for $x^H < \overline{x}_2^H$. Since $\Pi_0(\overline{y}_1,x) = \Pi(\overline{y}_1,x)$ and $\Pi_0(\overline{y}_0,x) = \Pi(\overline{y}_0,x)$ we must have $x_{2,i}^H = \overline{x}_{2,i}^H$.

We now assume that the sequence holds for $k = t$, we will show that it holds for $k = t+1$ as well. That is we will show that $P_0^H(y^H,y^L) = p_{k+1}^H$, $P_0^L(y^H,y^L) = p_{k+1}^L$ for $y^H \in (\overline{y}_{k+2}^H(x^L),\overline{y}_{k+1}^H(x^L)]$, and $t_0^H(x^H,x^L) = y_k^H(x^L)$, $t_0^L(x^H,x^L) = x^L$ for $x^H \in (\overline{x}_{k+2}^H(x^L),\overline{x}_{k+1}^H(x^L)]$ for all $x^L = (1-\mu)y^L$. To see this, first observe that for $b < \widehat{b}$, we have $V^H(b) < \overline{\theta}q_H$, and for $b < \widehat{b}$ we have $V^L(b) < \overline{\theta}q_L$, for $b \in (\widehat{b},1]$ we have $V^L(b) < \underline{\theta}q_L$, and for $b <$

1 we have $V^L(b) = 0$. If we apply the arbitrage equation for the high quality good at $\overline{y}_k^H(x^L)$ and the arbitrage equation for the low quality good at $\overline{y}_k^H(x^L) + y^L$, we will have $t_0^H(\overline{x}_k^H(x^L), x^L) > \overline{y}_k^H(x^L)$ and $t_0^H(\overline{x}_k^H(x^L), x^L) + t_0^L(\overline{x}_k^H(x^L), x^L) > \overline{y}_k^H(x^L) + y^L$. We know that $\overline{x}_{k+1}^H(x^L) = \inf\{x^H : t_0^H(x^H, x^L) = \overline{y}_{k-1}^H(x^L)\}$. Moreover, as discussed above, since $\frac{d\overline{y}_k^H(y^L)}{dy^L} < 0$ and $\frac{d\Pi}{dy^H} > \frac{d\Pi}{dy^L}$, the seller keeps the low quality good as low as possible. Hence, we must have $t_0^H(\overline{x}_{k+1}^H(x^L), x^L) = \overline{y}_k^H(x^L)$, $t_0^L(\overline{x}_{k+1}^H(x^L), x^L) = x^L$. Similar to the previous discussion, we use left continuity of $t_0^H(\cdot)$ to define $x_{k+2}^H(x^L) = \inf\{x^H : t_0^H(x^H, x^L) = \overline{y}_k^H(x^L)\}$ and conclude that $x_{k+2}^H(x^L) = \overline{x}_{k+2}^H(x^L)$.

Proof of Corollary 1. For the existence of the Coase Conjecture equilibrium in a vertically differentiated market we must have $q_H - q_L > q_L$ and $\underline{\theta} > \overline{\theta} - \underline{\theta}$, and $\mu \in \bigcap_{\forall s} \Gamma_s$, $s = 1, \ldots, 4$. Since $\Gamma_i \subset \Gamma_j$ for all $i = 2, 3$ and $j = 1, 4$ and Γ_j is equivalent to the set Λ defined in Deneckere and Liang which guarantees existence of the Coase Conjecture equilibrium, the threshold depreciation rate for the Coase Conjecture equilibrium to exist in vertically differentiated market is strictly less than the threshold depreciation rate in a market in which only one version of the good is sold.

Proof of Corollary 2. By definition of $\overline{x}_{k,i}^H$, $\overline{x}_{k,i}^H = (1-\mu)^{-1}(\overline{x}_{k-1,i}^H - a_{k,i}(\overline{x}_{k-2,i}^H - \overline{x}_{k-1,i}^H))$ for $k \geq 4$, we have $\lim_{z\to 0} \overline{x}_{k,i}^H = \lim_{z\to 0^+}(\overline{x}_{k-1,i}^H - a_{k,i}(\overline{x}_{k-2,i}^H - \overline{x}_{k-1,i}^H))$. Hence, $\lim_{z\to 0^+}(\overline{x}_{k-1,i}^H - \overline{x}_{k,i}^H) > \lim_{z\to 0}(\overline{x}_{k-2,i}^H - \overline{x}_{k-1,i}^H)$. By using induction, $\lim_{z\to 0^+}(\overline{x}_{k-1,i}^H - \overline{x}_{k,i}^H) > \lim_{z\to 0^+}(\overline{x}_{k-2,i}^H - \overline{x}_{k-1,i}^H) > \lim_{z\to 0^+}(\overline{x}_{2,i}^H - \overline{x}_{3,i}^H) > \widehat{b} - \overline{x}_{2,i}^H$. For $i = 1, 4$ we have $\widehat{b} - \overline{x}_{2,i}^H = \frac{(1-\widehat{b})\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, and for $i = 2, 3$ we have $\widehat{b} - \overline{x}_{2,i}^H = \frac{(1-\widehat{b})\underline{\theta}}{\overline{\theta}-\underline{\theta}} = \frac{\widehat{b}\left(\underline{\theta}q_L + (\overline{\theta}-\underline{\theta})(q_H - q_L)\right)}{(1-\widehat{b})\underline{\theta}(q_H-q_L) - (1-\widehat{b})\underline{\theta}q_L + \overline{x}_2^L\underline{\theta}q_L}$. Therefore, for $i = 1, 4$ it will take at most $m_i \leq \frac{\widehat{b}(\overline{\theta}-\underline{\theta})}{(1-\widehat{b})\underline{\theta}} + 1$ steps and for $i = 2, 3$ it will take at most $m_i \leq \frac{\widehat{b}\left((1-\widehat{b})\underline{\theta}(q_H-q_L) - (1-\widehat{b})\underline{\theta}q_L + \overline{x}_2^L\underline{\theta}q_L\right)}{\widehat{b}\left(\underline{\theta}q_L + (\overline{\theta}-\underline{\theta})(q_H-q_L)\right)} + 1$ steps to sell to all buyers.

Proof of Corollary 3. Let m represents the number of the interactions between the monopolist and the buyers in market sold one version of the good. As z diminishes both m and m_i for $i = 1, 4$ converges to $\frac{\widehat{b}(\overline{\theta}-\underline{\theta})}{(1-\widehat{b})\underline{\theta}} + 1$, whereas m_i for $i = 2, 3$ converges to

$$\frac{\widehat{b}\left(\left(1-\widehat{b}\right)\underline{\theta}\left(q_H - q_L\right) - \left(1-\widehat{b}\right)\underline{\theta}q_L + \overline{x}_2^L\underline{\theta}q_L\right)}{\widehat{b}\left(\underline{\theta}q_L + \left(\overline{\theta}-\underline{\theta}\right)\left(q_H - q_L\right)\right)} + 1.$$

Since $\frac{\widehat{b}\left(\left(1-\widehat{b}\right)\underline{\theta}(q_H-q_L)-\left(1-\widehat{b}\right)\underline{\theta}q_L+\overline{x}_2^L\underline{\theta}q_L\right)}{\widehat{b}\left(\underline{\theta}q_L+\left(\overline{\theta}-\underline{\theta}\right)(q_H-q_L)\right)} > \frac{\widehat{b}\left(\overline{\theta}-\underline{\theta}\right)}{\left(1-\widehat{b}\right)\underline{\theta}}$ for all $\overline{x}_2^L$, the first part of the proof is finished. To prove the second part of the corollary, we should observe that for a give q_H, $\frac{dm_i}{dq_L} > 0$ for $i = 2, 3$ and $\lim_{q_L \to 0^+} m_i = \frac{\widehat{b}\left(\overline{\theta}-\underline{\theta}\right)}{\left(1-\widehat{b}\right)\underline{\theta}} + 1$ for $i = 2, 3$.

Proof of Theorem 2. Let $\widetilde{x}_{k,i}^j$ be the state of the durable good j in period k on path i. There exists three paths depending on the state of the low quality good. First, we define the set of states for the high quality good. For all i, let $\widetilde{x}_{0,i}^H = (1-\mu)\widehat{b}$, and let $\widetilde{x}_{k,i}^H = \frac{\underline{\theta}+\rho^{k-1}\Delta\theta}{\underline{\theta}+(1-\mu)\rho^{k-1}\Delta\theta}\widetilde{x}_{k-1,i}^H$. Second, we define the set of states for the low quality good. For all $k \geq 0$, let $\widetilde{x}_{k,1}^L = 0$, let $\widetilde{x}_{k,2}^L = 1-\mu-\widetilde{x}_{k,2}^H$ and let $\widetilde{x}_{k,3}^L$ be such that $\widetilde{x}_{k,3}^H + \widetilde{x}_{k,3}^L > 1-\mu$. Since, the state of the high quality good is independent of the path, from now on we refer to the state of the high quality good in period k on path i as $\widetilde{x}_k^H$. Observe that $\widetilde{x}_0^H < \widetilde{x}_1^H < \widetilde{y}_0^H < \widetilde{x}_2^H < \widetilde{y}_1^H < \widetilde{x}_3^H < \cdots$ where $\widetilde{y}_k^j = \frac{\widetilde{x}_k^j}{1-\mu}$ for $\mu \geq \underline{\mu}^{sg}$.

We now define $\{\widetilde{p}_{k,i}^j\}_{k=0}^{\infty}$. On path 1, the price of the high quality good is $\widetilde{p}_{k,1}^H = \underline{\theta}q_H + \rho^k\Delta\theta q_H$ and the price of the low quality good is set such that none of the buyers would purchase it. On path 2, the price of the high quality good is $\widetilde{p}_{k,2}^H = \underline{\theta}q_H + \rho^k\Delta\theta\Delta q$ and the price of the low quality good is $\widetilde{p}_{k,2}^L = \underline{\theta}q_L$. On path 3, the price of the high quality good is $\widetilde{p}_{k,3}^H = \underline{\theta}\Delta q + \rho^k\Delta\theta\Delta q$ and the price of the low quality good is $\widetilde{p}_{k,2}^L = 0$.

Now, we prove that a stationary equilibrium exists for $\mu \geq \underline{\mu}^{sg}$. Let us define $\{P^H(.)$, $P^L(.)$, $t^H(.)$, $t^L(.)$, $R(.)\}$. If $\widetilde{x}_\infty^H = \lim_{k\to\infty}\widetilde{x}_k^H \geq 1-\mu$, then set $m = \sup\{k : \widetilde{x}_k^H < 1-\mu\}$ and define $\widetilde{x}_{m+1}^H = 1-\mu$. For a given state (x_H, x_L), we define the stationary path as follows.

$$P^H\left(x^H, x^L\right) = \begin{cases} \overline{\theta}q_H & \text{for} \quad x^H \in D_1 \text{ and } x^L = 0 \\ \overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L & \text{for} \quad x^H \in D_1 \text{ and } x^L \in D_{x^H} \\ \overline{\theta}q_H - \overline{\theta}q_L & \text{for} \quad x^H \in D_1 \text{ and } x^L \in D'_{x^H} \\ \widetilde{p}_{k,1}^H & \text{for} \quad x^H \in D_2 \text{ and } x^L = 0 \\ \widetilde{p}_{k,2}^H & \text{for} \quad x^H \in D_2 \text{ and } x^L \in D_{x^H} \\ \widetilde{p}_{k,3}^H & \text{for} \quad x^H \in D_2 \text{ and } x^L \in D'_{x^H} \end{cases}$$

where $k = 1, \ldots, m-1$,

$$P^L\left(x^H,x^L\right)=\begin{cases}\overline{\theta}q_L & \text{for} \quad x^H\in D_1 \text{ and } x^L=0\\ \widetilde{p}^L_{k,1} & \text{for} \quad x^H\in D_2 \text{ and } x^L=0\\ \underline{\theta}q_L & \text{for} \quad \text{all } x^H \text{ and } x^L\in D_{x^H}\\ 0 & \text{for} \quad \text{all } x^H \text{ and } x^L\in D'_{x^H}\end{cases}$$

where $k=1,\ldots,m+1$,

$$t^H(x^H,x^L)=\begin{cases}\widehat{b} & \text{for} \quad x^H\in D_3\\ \widetilde{y}^H_{k-1} & \text{for} \quad x^H\in D_4\end{cases}$$

where $k=2,\ldots,m+1$,

$$t^L(x^H,x^L)=\begin{cases}0 & \text{for} \quad \text{all } x^H \text{ and } x^L=0\\ 1-t^H(x^H,x^L) & \text{for} \quad \text{all } x^H \text{ and } x^L\in D_{x^H}\\ x^L & \text{for} \quad \text{all } x^H \text{ and } x^L\in D'_{x^H}\end{cases}$$

$$R(x^H,x^L)=\begin{cases}(\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\overline{\theta}q_H & \text{for} \quad x^H\in D_3 \text{ and } x^L=0\\ (\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))(\overline{\theta}\Delta q+\underline{\theta}q_L)+C_1 & \text{for} \quad x^H\in D_3 \text{ and } x^L\in D_{x^H}\\ (\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\overline{\theta}\Delta q & \text{for} \quad x^H\in D_3 \text{ and } x^L\in D'_{x^H}\\ (\frac{\mu\widetilde{y}^H_{k-1}}{1-\delta}+(\widetilde{x}^H_{k-1}-x^H))\widetilde{p}^H_{k-1,1} & \text{for} \quad x^H\in D_4 \text{ and } x^L=0\\ (\frac{\mu\widetilde{y}^H_{k-1}}{1-\delta}+(\widetilde{x}^H_{k-1}-x^H))\widetilde{p}^H_{k-1,2}+C_2 & \text{for} \quad x^H\in D_4 \text{ and } x^L\in C_{x^H}\\ (\frac{\mu\widetilde{y}^H_{k-1}}{1-\delta}+(\widetilde{x}^H_{k-1}-x^H))\widetilde{p}^H_{k-1,3} & \text{for} \quad x^H\in D_4 \text{ and } x^L\in C_{x^H}\end{cases}$$

where $k=2,\ldots,m-1$, $C_1=(\frac{\delta\mu(1-\widehat{b})}{1-\delta}+(1-\widehat{b}-x^L))\underline{\theta}q_L$,

$C_2=(\frac{\mu(1-\widetilde{y}^H_{k-1})}{1-\delta}+(1-\mu-\widetilde{x}^H_{k-1}-x^L))\widetilde{p}^L_{k-1,2}$, $C_{x^H}\ \equiv[0,1-x^H]$,

$D_1\equiv[0,\widehat{b}]$, $D_3\equiv[0,\widetilde{x}^H_1]$, $D_4\equiv(\widetilde{x}^H_{k-1},\widetilde{x}^H_k]$, $D_2\equiv(\widetilde{y}^H_{k-1},\widetilde{y}^H_k]$,

and $D_{x^H}\ \equiv[0,1-\mu-x^H]$.

If $\widetilde{x}^H_\infty=\lim_{k\to\infty}\widetilde{x}^H_{k,i}<1-\mu$. For all $k\geq 1$ define $\widetilde{R}(\cdot)$ as

$$\widetilde{R}_{k-1}(x^H \mid x^L) = \begin{cases} (\frac{\mu \widetilde{y}^H_{k-1}}{1-\delta} + S_1 & \text{for} \quad x^H \in D_4,\, x^L = 0 \\ (\frac{\mu \widetilde{y}^H_{k-1}}{1-\delta} + S_2 + C_3 & \text{for} \quad x^H \in D_4,\, x^L \in C_{x^H} \\ (\frac{\mu \widetilde{y}^H_{k-1}}{1-\delta} + S_3 & \text{for} \quad x^H \in D_4,\, x^L \in C_{x^H} \end{cases}$$

where $C_3 = (\frac{\mu(1-\widetilde{y}^H_{k-1})}{1-\delta} + (1-\mu-(1-\mu)\widetilde{y}^H_{k-1} - x^L))\widetilde{p}^L_{k-1,2}$,

$S_1 \equiv ((1-\mu)\widetilde{y}^H_{k-1} - x^H))\widetilde{p}^H_{k-1,1}$, $S_2 \equiv ((1-\mu)\widetilde{y}^H_{k-1} - x^H))\widetilde{p}^H_{k-1,2}$,

$S_3((1-\mu)\widetilde{y}^H_{k-1} - x^H))\widetilde{p}^H_{k-1,3}$, $C_{x^H} \ \equiv [0, 1-x^H]$,

and $D_4 \equiv (\widetilde{x}^H_{k-1}, \widetilde{x}^H_k]$.

Define $\acute{x}^H_i = \max\{x^H \in [(1-\mu)\widehat{b}, \widetilde{x}^H_\infty] : \widetilde{R}_{k-1}(x^H) \geq R^c(x^H)\}$, where $R^c(x^H \mid x^L) = (\frac{\delta\mu}{1-\delta} + (1-x^H))p^c$ and $p^c = \begin{cases} \underline{\theta} q_H & \text{for} \quad x^L = 0 \\ \underline{\theta}\Delta q & \text{for} \quad x^L > 0 \end{cases}$. Since $\widetilde{R}_{k-1}(. \mid x^L)$ decreases as x^L increases, we have $\acute{x}^H_1 > \acute{x}^H_2 > \acute{x}^H_3$. Let m_i be such that $\acute{x}^H_i \in (\widetilde{x}^H_{m_i}, \widetilde{x}^H_{m_i+1}]$. Since $\widetilde{R}_{k-1}(\widetilde{x}^H_\infty \mid 0) - R^c(\widetilde{x}^H_\infty \mid 0) < 0$ and by definition of $\overline{\mu}$, m_1 exists, so do m_2 and m_3. Now, let $\widetilde{x}^H_{m_i+1,i} \equiv \acute{x}^H_i$ and define the stationary path as follows.

$$P^H\left(x^H, x^L\right) = \begin{cases} \overline{\theta} q_H & \text{for} \quad x^H \in D_1 \text{ and } x^L = 0 \\ \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L & \text{for} \quad x^H \in D_1 \text{ and } x^L \in D_{x^H} \\ \overline{\theta} q_H - \overline{\theta} q_L & \text{for} \quad x^H \in D_1 \text{ and } x^L \in D'_{x^H} \\ \widetilde{p}^H_{k,1} & \text{for} \quad x^H \in D_{2,1} \text{ and } x^L = 0 \\ \widetilde{p}^H_{k,2} & \text{for} \quad x^H \in D_{2,2} \text{ and } x^L \in D_{x^H} \\ \widetilde{p}^H_{k,3} & \text{for} \quad x^H \in D_{2,3} \text{ and } x^L \in D'_{x^H} \\ \underline{\theta} q_H & \text{for} \quad x^H \in D_5 \text{ and } x^L = 0 \\ \underline{\theta}\Delta q & \text{for} \quad x^H \in D_5 \text{ and all } x^L,\, i = 2, 3 \end{cases}$$

where $k_i = 0, \ldots, m_i + 1$,

$$P^L\left(x^H, x^L\right) = \begin{cases} \overline{\theta} q_H & \text{for} \quad x^H \in D_1 \text{ and } x^L = 0 \\ \widetilde{p}^L_{k,1} & \text{for} \quad x^H \in D_2 \text{ and } x^L = 0 \\ \underline{\theta} q_L & \text{for} \quad \text{all } x^H \text{ and } x^L \in D_{x^H} \\ 0 & \text{for} \quad \text{all } x^H \text{ and } x^L \in D'_{x^H} \end{cases}$$

where $k = 1, \ldots, m_1 + 1$,

$$t^H(x^H,x^L)=\begin{cases}\widehat{b} & \text{for} \quad x^H\in D_3\\ \widetilde{y}^H_{k-1} & \text{for} \quad x^H\in D_4\\ 1 & \text{for} \quad x^H\in D_6\end{cases}$$

where $k=2,\ldots,m+1$,

$$t^L(x^H,x^L)=\begin{cases}0 & \text{for} \quad \text{all } x^H \text{ and } x^L=0\\ 1-x^H & \text{for} \quad \text{all } x^H \text{ and } x^L\in D_{x^H}\\ x^L & \text{for} \quad \text{all } x^H \text{ and } x^L\in D'_{x^H}\end{cases}$$

$$R(x^H,x^L)=\begin{cases}(\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\overline{\theta}q_H & \text{for} \quad x^H\in D_3 \text{ and } x^L=0\\ (\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))(\overline{\theta}\Delta q+\underline{\theta}q_L)+C_1 & \text{for} \quad x^H\in D_3 \text{ and } x^L\in D_{x^H}\\ (\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\overline{\theta}\Delta q & \text{for} \quad x^H\in D_3 \text{ and } x^L\in D'_{x^H}\\ (\frac{\mu\widetilde{y}^H_{k-1}}{1-\delta}+(\widetilde{x}^H_{k-1}-x^H))\widetilde{p}^H_{k-1,1} & \text{for} \quad x^H\in D_{4,1} \text{ and } x^L=0\\ (\frac{\mu\widetilde{y}^H_{k-1}}{1-\delta}+(\widetilde{x}^H_{k-1}-x^H))\widetilde{p}^H_{k-1,2}+C_2 & \text{for} \quad x^H\in D_{4,2} \text{ and } x^L\in C_{x^H}\\ (\frac{\mu\widetilde{y}^H_{k-1}}{1-\delta}+(\widetilde{x}^H_{k-1}-x^H))\widetilde{p}^H_{k-1,3} & \text{for} \quad x^H\in D_{4,3} \text{ and } x^L\in C_{x^H}\\ (\frac{\delta\mu}{1-\delta}+(1-x^H))\underline{\theta}q_H & \text{for} \quad x^H\in D_6 \text{ and } x^L=0\\ (\frac{\delta\mu}{1-\delta}+(1-x^H))\underline{\theta}\Delta q & \text{for} \quad x^H\in D_6 \text{ and all } x^L.\end{cases}.$$

where $k_i=0,\ldots,m_i+1$, $D_5\equiv(\widetilde{y}^H_{m_i+1,1},1]$, $D_6\equiv(\widetilde{x}^H_{m_1+1},1]$,

and all other variables are defined as above.

We now show that the stationary path defined above is a solution. We consider the case where $\widetilde{x}^H_\infty<1-\mu$. First, we show that when the state of the low quality good is zero, the monopolist prefers setting the price of the low quality good high enough so that none of the buyers purchase the low quality good. Suppose that $x^H\in(\widetilde{x}^H_{k-1},\widetilde{x}^H_k]$. For all $\mu>\overline{\mu}^{st}$, we have $\widetilde{R}_1((1-\mu)\widehat{b}\mid 0)>\widetilde{R}_1((1-\mu)\widehat{b}\mid(1-\mu)(1-\widehat{b}))$. If there exists x^H_k such that $\widetilde{R}_{k-1}(x^H_k\mid 0)=\widetilde{R}_{k-1}(x^H_k\mid 1-\mu-x^H_k)$ then $\widetilde{R}_{k-1}(x^H_k\mid 0)\leq R_c(x^H_k\mid 0)$. Hence, $t^L\left(x^H,0\right)=0$.

Second, we show that for $x^L\in(0,1-\mu-x^H]$, $t^L(x^H,x^L)=1-t^H(x^H,x^L)$. In period k, when the state is (x^H_k,x^L_k) and $x^H_k+x^L_k\leq 1-\mu$, the monopolist is indifferent between staying at (x^H_k,x^L_k) forever and bringing the state to (x^H_{k-1},x^L_{k-1}) if $x^H_k(\underline{\theta}q_H+(1-\mu)\rho^{k-1}\Delta\theta\Delta q)+y^L_k\underline{\theta}q_L=x^H_{k-1}(\underline{\theta}q_H+\rho^{k-1}\Delta\theta\Delta q)+x^L_{k-1}\underline{\theta}q_L$ holds. Let $\Pi_2(x^H_k,x^L_k)=\frac{\mu\widetilde{x}^H_k}{(1-\delta)(1-\mu)}\widetilde{p}^H_{k,2}+\frac{\mu\widetilde{x}^L_k}{(1-\delta)(1-\mu)}\widetilde{p}^L_{k,2}$. Since $\frac{dx^H_k}{dx^L_k}=-\frac{\underline{\theta}q_L}{\underline{\theta}q_H+(1-\mu)\rho^{k-1}\Delta\theta\Delta q}$, we have $\frac{d\Pi_2(x^H_k,x^L_k)}{dx^L_k}>0$. Hence, $x^L_k=1-\mu-x^H_k$ must hold. For a given (x^H,x^L), since $\widetilde{R}$ is increasing in y^H on any of the intervals $[0,\widetilde{y}^H_0]$, $(\widetilde{y}^H_0,\widetilde{y}^H_1]$,…,$(\widetilde{y}^H_{m_i+1},1]$, we have $t^H \in \{\widetilde{y}_0^H, \widetilde{y}_1^H, \ldots, \widetilde{y}_{m_i+1}^H, 1\}$.

Third, we construct the sequence of states such that when the state is $(\widetilde{x}_k^H, 0)$, we have $\widetilde{R}_{k-1}(\widetilde{x}_k^H \mid 0) = \widetilde{R}_k(\widetilde{x}_k^H \mid 0)$, hence $(\widetilde{y}_{k-1}^H, 0) \sim (\widetilde{y}_k^H, 0)$. When the state is $(\widetilde{x}_k^H, 1-\mu-\widetilde{x}_k^H)$, we have $\widetilde{R}_{k-1}(\widetilde{x}_k^H \mid 1-\mu-\widetilde{x}_k^H) = \widetilde{R}_k(\widetilde{x}_k^H \mid 1-\mu-\widetilde{x}_k^H)$, hence $(\widetilde{y}_{k-1}^H, 1-\widetilde{y}_{k-1}^H) \sim (\widetilde{y}_k^H, 1-\widetilde{y}_k^H)$. When the state is $(\widetilde{x}_k^H, x^L)$ where $\widetilde{x}_k^H + x^L > 1-\mu$, we have $\widetilde{R}_{k-1}(\widetilde{x}_k^H \mid x^L) = \widetilde{R}_k(\widetilde{x}_k^H \mid x^L)$, hence $(\widetilde{y}_{k-1}^H, x^L) \sim (\widetilde{y}_k^H, x^L)$. Now, let $h_{m_i+1}(x^H \mid x^L) = R_c(x^H \mid x^L) - \widetilde{R}_{m_1}(x^H \mid x^L)$, and for $k \leq m_i$ let $h_k(x^H \mid x^L) = \widetilde{R}_k(x^H \mid x^L) - \widetilde{R}_{k-1}(x^H \mid x^L)$. By definition of $\widetilde{x}_k^H$, $h_k(\widetilde{x}_k^H \mid 0) = 0$ for all k. Moreover, we have $\frac{dh_{m_1+1}}{dx^H} = -(\underline{\theta} q_H - \widetilde{p}_{m_1,1}^H) > 0$ and $\frac{dh_k}{dx^H} = -(\widetilde{p}_{k,1}^H - \widetilde{p}_{k-1,1}^H) > 0$. Hence, $t^H(x^H, 0)$ is a solution. To see that $t^H(x^H, 1-\mu-x^H)$ is a solution as well, we should observe that for all k, $h_k(\widetilde{x}_k^H \mid x^L) = 0$ where $x^L \geq 1-\mu-\widetilde{x}_k^H$ and $\frac{dh_{m_i+1}}{dx^H} = -(\underline{\theta}\Delta q - \widetilde{p}_{m_i,2}^H) > 0$ and $\frac{dh_k}{dx^H} = -(\widetilde{p}_{k,i}^H - \widetilde{p}_{k-1,i}^H) > 0$ for $i = 2, 3$. Therefore, we conclude that the path we derived is a solution to the optimization problem.

To prove the uniqueness of the solution, we consider any stationary equilibrium with a steady state $(\widehat{b}, 0)$ with the associated stationary set $\{P_0^H(.), P_0^L(.), t_0^H(.), t_0^L(.), R_0(.)\}$, and show that $\{P_0^H(.), P_0^L(.), t_0^H(.), t_0^L(.), R_0(.)\}$ is equal to $\{P^H(.), P^L(.), t^H(.), t^L(.), R(.)\}$.

Let's define $\check{v}_1^H = \sup_{b>\widehat{b}} V_0^H(b)$. First, suppose that $\check{v}_1^H = \underline{\theta} q_H$. We start with proving $\check{v}_1^L \leq \underline{\theta} q_L$ where $\check{v}_1^L = \sup_{b>\widehat{b}} V_0^L(b)$. Suppose that there exists $b > \widehat{b}$ such that $V_0^L(b) > \underline{\theta} q_L$.

If we apply the arbitrage equation to b, for the low quality good we have $\underline{\theta} q_L - V_0^L(b) = \rho(\underline{\theta} q_L - V_0^L(t^H(\cdot) + t^L(\cdot)))$. Since $\underline{\theta} q_L - V_0^L(b) < 0$, we must have $t^H(\cdot) + t^L(\cdot) < b$. If we apply the arbitrage equation to b, for the high quality good, $\underline{\theta} q_H - V_0^H(b) = \rho(\underline{\theta} q_H - V_0^H(t^H(\cdot)))$. Since $V_0^H(b) = \underline{\theta} q_H$, we must have $t^H(\cdot) > b$, which leads to a contradiction. Hence, we must have $t^H(x^H, x^L) = 1$ and $t^L(x^H, x^L) = x^L$ for $x^H > (1-\mu)\widehat{b}$ and

$$P_0^H\left(x^H, x^L\right) = \begin{cases} \overline{\theta} q_H & \text{if } x^H \leq \widehat{b} \text{ and } x^H + x^L \leq \widehat{b} \\ \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L & \text{if } x^H \leq \widehat{b} \text{ and } \widehat{b} < x^H + x^L \leq 1 \\ \overline{\theta} q_H - \overline{\theta} q_L & \text{if } x^H \leq \widehat{b} \text{ and } x^H + x^L > 1 \\ \underline{\theta} q_H & \text{if } x^H > \widehat{b} \text{ and } \widehat{b} < x^H + x^L \leq 1 \\ \underline{\theta} \Delta q & \text{if } x^H > \widehat{b} \text{ and } x^H + x^L > 1 \end{cases},$$

$$P_0^L\left(x^H,x^L\right)=\begin{cases}\overline{\theta}q_L & \text{if} \quad x^H\le\widehat{b} \text{ and } x^H+x^L\le\widehat{b}\\ \underline{\theta}q_L & \text{if} \quad x^H\le\widehat{b} \text{ and } \widehat{b}<x^H+x^L\le 1\\ 0 & \text{if} \quad x^H\le\widehat{b} \text{ and } x^H+x^L>1\\ \underline{\theta}q_L & \text{if} \quad x^H>\widehat{b} \text{ and } \widehat{b}<x^H+x^L\le 1\\ 0 & \text{if} \quad x^H>\widehat{b} \text{ and } x^H+x^L>1\end{cases},$$

$$t_0^H\left(x^H,x^L\right)=\begin{cases}\widehat{b} & \text{if} \quad x^H\le(1-\mu)\widehat{b}\\ 1 & \text{if} \quad x^H>(1-\mu)\widehat{b}\end{cases},$$

$t_0^L\left(x^H,x^L\right)=x^L$ for all x^H,x^L.

Hence, $\{P_0^H,P_0^L,t_0^H,t_0^L,R_0\}$ is uniquely determined.

Second, suppose that $\check{v}_1^H>\underline{\theta}q_H$. We show that $\check{p}_1^H=\begin{cases}\widetilde{p}_{1,1}^H & \text{if} \quad x^L=0\\ \widetilde{p}_{1,2}^H & \text{if} \quad x^H+x^L\le 1\\ \widetilde{p}_{1,3}^H & \text{if} \quad x^H+x^L>1\end{cases}$. First, we must prove that there exists $\epsilon>0$ such that $t_0^H((1-\mu)y^H,x^L)=\widehat{b}$ for all $y^H\in(\widehat{b},\widehat{b}+\epsilon]$. To see this, observe that $t_0^H((1-\mu)y^H,(1-\mu)y^L)<y^H$ for all y^H such that $V_0^H(y^H)>\underline{\theta}q_H$. If we had $t_0^H((1-\mu)y^H,(1-\mu)y^L)\ge y^H$, we would have $V_0^H(t_0^H((1-\mu)y^H,(1-\mu)y^L))\le V_0^H(y^H)$ that would yield $V_0^H(y^H)=(1-\rho)\underline{\theta}q_H+\rho V_0^H(t_0^H((1-\mu)y^H,(1-\mu)y^L)\le(1-\rho)\underline{\theta}q_H+\rho V_0^H(y^H)<\underline{\theta}q_H$ which contradicts with the initial assumption.

Let $\check{x}_1^H(x^L)=\max\{x^H\le 1-\mu \mid t_0^H(x^H,x^L)=\widehat{b}\}$. First, if $\check{x}_1^H(x^L)=1-\mu$ then $\{P_0^H,P_0^L,t_0^H,t_0^L,R_0\}$ is uniquely determined as follows

$$P_0^H\left(x^H,x^L\right)=\begin{cases}\overline{\theta}q_H & \text{if} \quad x^H\le\widehat{b} \text{ and } x^L=0\\ \overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L & \text{if} \quad x^H\le\widehat{b} \text{ and } x^H+x^L\le 1 \text{ and } x^L>0\\ \overline{\theta}q_H-\overline{\theta}q_L & \text{if} \quad x^H\le\widehat{b} \text{ and } x^H+x^L>1\\ \check{p}_{1,1}^H & \text{if} \quad x^H>\widehat{b} \text{ and } x^L=0\\ \check{p}_{1,2}^H & \text{if} \quad x^H>\widehat{b} \text{ and } x^H+x^L<1 \text{and } x^L>0\\ \check{p}_{1,3}^H & \text{if} \quad x^H>\widehat{b} \text{ and } x^H+x^L>1\end{cases},$$

$$P_0^L\left(x^H,x^L\right)=\begin{cases}\check{p}_{1,1}^L & \text{if} \quad x^L=0\\ \underline{\theta}q_L & \text{if} \quad x^L>0 \text{ and } x^H+x^L\le 1\\ 0 & \text{if} \quad x^H>\widehat{b} \text{ and } x^H+x^L>1\end{cases},$$

$t_0^H\left(x^H,x^L\right)=\widehat{b}$ for all x^H,x^L,

$$t_0^L\left(x^H,x^L\right)=\begin{cases}1-\widehat{b} & \text{if} \quad x^H\le\widehat{b} \text{ and } x^H+x^L\le\widehat{b}\\ x^L & \text{otherwise}\end{cases}.$$

Second, let us consider the case where $\check{x}_1^H(x^L)<1-\mu$. Define $\check{p}_2^H=\sup\{P_0^H(b)\mid b>\frac{\check{x}_1^H(x^L)}{1-\mu}\}$. If $\check{p}_2^H=\underline{\theta}q_H$ then as discussed above we have

$$P_0^H\left(x^H,x^L\right)=\begin{cases}\underline{\theta}q_H & \text{if} \quad x^H\in(\frac{\check{x}_1^H(x^L)}{1-\mu},1] \text{ and } x^H+x^L\le 1\\ \underline{\theta}(q_H-q_L) & \text{if} \quad x^H\in(\frac{\check{x}_1^H(x^L)}{1-\mu},1] \text{ and } x^H+x^L>1\end{cases},$$

$$P_0^L\left(x^H,x^L\right)=\begin{cases}\underline{\theta}q_L & \text{if} \quad x^H+x^L\le 1\\ 0 & \text{if} \quad x^H+x^L>1\end{cases},$$

$t_0^H\left(x^H,x^L\right)=1$ if $x^H\in(\frac{\check{x}_1^H(x^L)}{1-\mu},1]$,

$t_0^L\left(x^H,x^L\right)=x^L$ if $x^H\in(\frac{\check{x}_1^H(x^L)}{1-\mu},1]$.

It follows that $R_0(x^H,x^L)=\begin{cases}R_C(x^H,x^L) & \text{if} \quad x^H\in(\check{x}_1^H(x^L),1-\mu]\\ \tilde{R}(x^H,x^L) & \text{if} \quad x^H+x^L>1-\mu\end{cases}$. By the continuity of R_0 at $x^H=\check{x}_1^H$ since $\tilde{R}(\check{x}_1^H(x^L),x^L)=R_0((\check{x}_1^H(x^L),x^L)$, we must have $\acute{x}_1^H=\acute{x}_2^H=\acute{x}_3^H$.

Suppose now that $\check{p}_2^H>\underline{\theta}q_H$. Then by the same reasoning, there must exits $\varepsilon>0$ such that $t_0^H(x^H,x^L)=\frac{\check{x}_1^H(x^L)}{1-\mu}$, for all $x^H\in(\check{x}_1^H(x^L),\check{x}_1^H(x^L)+\varepsilon]$. Hence, $\check{p}_2^H=\tilde{p}_2^H$. Now we show that $\check{x}_1^H(x^L)=\tilde{x}_1^H(x^L)$.

Let us define

$$V_0(x^H,x^L)=\begin{cases}\pi(\widehat{b},0\mid x^H,0)-\frac{\mu\tilde{p}_{1,1}^H x^H}{(1-\mu)(1-\delta)} & \text{if} \quad x^L=0\\ \pi(\widehat{b},1-\widehat{b}\mid x^H,x^L)-\frac{\mu\tilde{p}_{1,2}^H x^H+\mu\tilde{p}_{1,2}^L x^L}{(1-\mu)(1-\delta)} & \text{if} \quad 0<x^L\le 1-x^H\\ \pi(\widehat{b},x^L\mid x^H,x^L)-\frac{\mu\tilde{p}_{1,3}^H x^H}{(1-\mu)(1-\delta)} & \text{if} \quad x^L>1-x^H\end{cases}.$$

Since $t_0^H(x^H,x^L)=\widehat{b}$ for all $x^H\le\check{x}_1^H(x^L)$ and $t_0^H(x^H,x^L)=\check{x}_1^H(x^L)$ for all $x^H\in(\check{x}_1^H(x^L),\check{x}_1^H(x^L)+\varepsilon]$, we must have $V_0(\check{x}_1^H(x^L),x^L)=0$. Since $V_0(\tilde{x}_1^H(x^L),x^L)=0$ and $\frac{\partial}{\partial x^H}V_0(x^H,x^L)<0$, we must have $\check{x}_1^H(x^L)=\tilde{x}_1^H(x^L)$.

We must apply the same argument inductively. Given $\check{x}_k^H(x^L)=\max\{x^H:t_0^H(x^H,x^L)=\check{x}_{k-1}^H((1-\mu)x^L)\}$, it must be that $\check{x}_k^H(x^L)=\tilde{x}_k^H(x^L)$. If $\check{x}_k^H(x^L)=1-\mu$, then $\tilde{x}_k^H(x^L)=1-\mu$ and $m=k-1$. If $\check{x}_k^H(x^L)<1-\mu$ and $\check{p}_{k+1}^H=\underline{\theta}q_H$ then $\check{x}_k^H(x^L)=\acute{x}=\tilde{x}_k^H(x^L)$ and $m=k-1$. If $\check{x}_k^H(x^L)<1-\mu$ and $\check{p}_{k+1}^H>\underline{\theta}q_H$ then $m>k-1$ and $\check{x}_{k+1}^H(x^L)=\max\{x^H:t_0^H(x^H,x^L)=$

$\check{x}_k^H((1-\mu)x^L)\}$.

Proof of Corollary 4. Since $\widetilde{x}_{k,1}^H = \widetilde{x}_k$ and $\widetilde{p}_{k,1}^H = \widetilde{p}_k$ where $(\widetilde{x}_k, \widetilde{p}_k)$ denotes the path when the monopolist produces one version of the good, we must have $\underline{\mu}^{st}(\delta) = \underline{\mu}^s(\delta)$.

Proof of Corollary 5. On path 1, $\frac{P^H(\widetilde{y}_{k,1}^H,0)-P^H(\widetilde{y}_{k-1,1}^H,0)}{\widetilde{y}_{k,1}^H-\widetilde{y}_{k-1,1}^H} = \frac{-(1-\rho)(\widetilde{p}_{k-1,1}^H-\mu(\widetilde{p}_{k-1,1}^H-\underline{\theta}q_H))}{\mu\widetilde{y}_{k-1,1}^H}$. As z approaches 0, since $\widetilde{y}_{k,1}^H - \widetilde{y}_{k-1,1}^H$ converges to zero, we have $P_1^H(y,0)\,y = -\frac{\lambda+r}{\lambda}P^H(y,0)$. The solution of the differential equation is $P^H(y,0) = c_1 y^{-\frac{\lambda+r}{\lambda}}$. Since $P^H\left(\widehat{b},0\right) = \overline{\theta}q_H$, we have $c_1 = \overline{\theta}q_H\widehat{b}^{\frac{\lambda+r}{\lambda}}$. Hence, $P^H(y,0) = \overline{\theta}q_H\left(\frac{\widehat{b}}{y}\right)^{\frac{\lambda+r}{\lambda}}$. As z approaches zero, $\frac{\widetilde{y}_{k,1}^H-\widetilde{y}_{k-1,1}^H}{z} = \widetilde{y}_{k-1,1}^H\frac{\widetilde{p}_{k-1,1}^H-\underline{\theta}q_H}{-z\left(\widetilde{p}_{k-1,1}^H-\underline{\theta}q_H\right)+\frac{z}{\mu}\widetilde{p}_{k-1,1}^H}$ converges to $\dot{y}_1 = \lambda y_1\left(1-\frac{\underline{\theta}}{\overline{\theta}}\left(\frac{\widehat{b}}{y}\right)^{-\frac{\lambda+r}{\lambda}}\right)$.

On path 2, $\frac{P^H(\widetilde{y}_{k,2}^H,1-\widetilde{y}_{k,2}^H)-P^H(\widetilde{y}_{k-1,2}^H,1-\widetilde{y}_{k-1,2}^H)}{\widetilde{y}_{k,2}^H-\widetilde{y}_{k-1,2}^H} = \frac{-(1-\rho)(\left(\widetilde{p}_{k-1,2}^H-\underline{\theta}q_L\right)-\mu(\widetilde{p}_{k-1,2}^H-\underline{\theta}q_H))}{\mu\widetilde{y}_{k-1,2}^H}$. As z approaches 0, since $\widetilde{y}_{k,2}^H - \widetilde{y}_{k-1,2}^H$ converges to zero, we have $\frac{dP^H(y,1-y)}{dy}y = -\frac{\lambda+r}{\lambda}\,(P^H(y,1-y)$ $-\,\underline{\theta}q_L)$. The solution of the differential equation is $P^H(y,1-y) = c_2 y^{-\frac{\lambda+r}{\lambda}} + \underline{\theta}q_L$. Since $P^H\left(\widehat{b},1-\widehat{b}\right) = \overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$, we have $c_2 = \overline{\theta}\Delta q\widehat{b}^{\frac{\lambda+r}{\lambda}}$. Hence, $P^H(y,1-y) = \overline{\theta}q_H\left(\frac{\widehat{b}}{y}\right)^{\frac{\lambda+r}{\lambda}}$ $+\,\underline{\theta}q_L$. As z approaches zero, $\frac{\widetilde{y}_{k,2}^H-\widetilde{y}_{k-1,2}^H}{z} = \widetilde{y}_{k-1,2}^H\frac{\widetilde{p}_{k-1,2}^H-\underline{\theta}q_H}{-z\left(\widetilde{p}_{k-1,2}^H-\underline{\theta}q_H\right)+\frac{z}{\mu}\left(\widetilde{p}_{k-1,2}^H-\underline{\theta}q_L\right)}$ converges to $\dot{y}_2 = \lambda y_2\left(1-\frac{\underline{\theta}}{\overline{\theta}}\left(\frac{\widehat{b}}{y}\right)^{-\frac{\lambda+r}{\lambda}}\right)$.

On path 3, $\frac{P^H(\widetilde{y}_{k,3}^H,y^L)-P^H(\widetilde{y}_{k-1,3}^H,y^L)}{\widetilde{y}_{k,3}^H-\widetilde{y}_{k-1,3}^H} = \frac{-(1-\rho)(\widetilde{p}_{k-1,3}^H-\mu(\widetilde{p}_{k-1,3}^H-\underline{\theta}\Delta q))}{\mu\widetilde{y}_{k-1,3}^H}$. As z approaches 0, since $\widetilde{y}_{k,3}^H - \widetilde{y}_{k-1,3}^H$ converges to zero, we have $P_1^H(y,.)\,y = -\frac{\lambda+r}{\lambda}P^H(y,.)$. The solution of the differential equation is $P^H(y,.) = c_3 y^{-\frac{\lambda+r}{\lambda}}$. Since $P^H\left(\widehat{b},y^L\right) = \overline{\theta}q_H - \overline{\theta}q_L$, we have $c_3 = \overline{\theta}\Delta q\widehat{b}^{\frac{\lambda+r}{\lambda}}$. Hence, $P^H(y,.) = \overline{\theta}\Delta q\left(\frac{\widehat{b}}{y}\right)^{\frac{\lambda+r}{\lambda}}$. As z approaches zero, $\frac{\widetilde{y}_{k,3}^H-\widetilde{y}_{k-1,3}^H}{z} = \widetilde{y}_{k-1,3}^H\frac{\widetilde{p}_{k-1,3}^H-\underline{\theta}\Delta q}{-z\left(\widetilde{p}_{k-1,3}^H-\underline{\theta}\Delta q\right)+\frac{z}{\mu}\widetilde{p}_{k-1,3}^H}$ converges to $\dot{y}_3 = \lambda y_3\left(1-\frac{\underline{\theta}}{\overline{\theta}}\left(\frac{\widehat{b}}{y}\right)^{-\frac{\lambda+r}{\lambda}}\right)$.

Proof of Theorem 3. As stated in the proof of Proposition 3, if $(\check{y}^H, 1-\check{y}^H)$ is a steady state, we must have

$$V^H(b) = \begin{cases} = \overline{\theta}q_H & \text{for} \quad b \leq \check{y}^H \\ < \overline{\theta}q_H & \text{for} \quad b \in (\check{y}^H, \widehat{b}] \\ = \underline{\theta}q_H & \text{for} \quad b \in (\widehat{b}, 1] \end{cases} \tag{A.1}$$

and

$$V^L(b) = \begin{cases} < \overline{\theta}q_L & \text{for} \quad b \leq \widehat{b} \\ = \underline{\theta}q_L & \text{for} \quad b \in (\widehat{b}, 1] \end{cases}, \tag{A.2}$$

and if $(\hat{y}^H, 0)$ is a steady state, we must have

$$V^H(b) = \begin{cases} = \overline{\theta} q_H & \text{for} \quad b \leq \hat{y}^H \\ < \overline{\theta} q_H & \text{for} \quad b \in (\hat{y}^H, \widehat{b}] \\ = \underline{\theta} q_H & \text{for} \quad b \in (\widehat{b}, 1] \end{cases} \tag{A.3}$$

and

$$V^L(b) = \begin{cases} = \overline{\theta} q_L & \text{for} \quad b \leq \hat{y}^H \\ < \overline{\theta} q_L & \text{for} \quad b \in (\hat{y}^H, \widehat{b}] \\ = \underline{\theta} q_L & \text{for} \quad b \in (\widehat{b}, 1] \end{cases} . \tag{A.4}$$

Let us establish that the existence of a reputational equilibrium implies $\mu \in (\underline{\mu}, \overline{\mu}]$. First, we show that $\mu > \underline{\mu}$. Suppose that the steady state is $(\check{y}^H, 1-\check{y}^H)$. Due to (A.1) and (A.2), for all $y^H \leq \check{y}^H$ and for all y^L, we have $t^H((1-\mu)y^H, (1-\mu)y^L) = \check{y}^H$ and $t^L((1-\mu)y^H, (1-\mu)y^L) = 1-\check{y}^H$ and hence we have $P^H(y^H, y^L) = \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L$ and $P^L(y^H, y^L) = \underline{\theta} q_L$. This implies that $R((1-\mu)\check{y}^H, (1-\mu)(1-\check{y}^H)) = \frac{\mu \check{y}^H (\overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L)}{1-\delta} + \frac{\mu(1-\check{y}^H)\underline{\theta} q_L}{1-\delta}$. By the continuity of $R(\cdot)$, $\check{y}^H$ solves $\psi(y) = 0$ where $\psi(y) = \frac{\mu y(\overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L)}{1-\delta} + \frac{\mu(1-y)\underline{\theta} q_L}{1-\delta} - R^C((1-\mu)y, (1-\mu)(1-y))$. Since $\psi(\widehat{b}) = 0$ when $\mu = \underline{\mu}$, we must have $\psi(\widehat{b}) < 0$ for all $\mu < \underline{\mu}$. Since, $\psi(\cdot)$ is strictly increasing, the existence of $\check{y}^H < \widehat{b}$ requires that $\mu > \underline{\mu}$. Suppose now that the steady state is $(\hat{y}^H, 0)$. Due to (A.3) and (A.4), for all $y^H \leq \hat{y}^H$, we have $t^H((1-\mu)y^H, 0) = \hat{y}^H$ and $t^L((1-\mu)y^H, 0) = 1 - \hat{y}^H$ and hence we have $P^H(y^H, 0) = \overline{\theta} q_H$ and $P^L(y^H, 0) = \overline{\theta} q_L$. This implies that $R((1-\mu)\hat{y}^H, 0) = \frac{\mu \hat{y}^H \overline{\theta} q_H}{1-\delta}$. By the continuity of $R(\cdot)$, $\hat{y}^H$ solves $\varsigma(y) = 0$ where $\varsigma(y) = \frac{\mu y^H \overline{\theta} q_H}{1-\delta} - R^C((1-\mu)y, 0)$. Since $\varsigma(\widehat{b}) = 0$ when $\mu = \mu^{st}$, we must have $\varsigma(\widehat{b}) < 0$ for all $\mu < \mu^{st}$. Since, $\varsigma(\cdot)$ is strictly increasing, the existence of $\hat{y}^H < \widehat{b}$ requires that $\mu > \mu^{st}$. By definition of μ^{st} and $\underline{\mu}$, $\mu^{st} > \underline{\mu}$.

Second, we show that $\mu < \overline{\mu}$. The proof follows from a contradiction. Suppose that for some $\mu > \overline{\mu}$, a reputational equilibrium exists. Let us define

$$\overline{K}_1 = \min\{k : \overline{x}^H_{j,1} < \overline{x}^H_{j-1,1} \text{ for all } j \leq k\}.$$

Since the Coase Conjecture equilibrium does not exist for $\mu > \overline{\mu}$, we have $\overline{x}^H_{\overline{K}_1,1} > 0$. Furthermore, since $\overline{x}^H_{\overline{K}_1+1,1} \geq \overline{x}^H_{\overline{K}_1,1}$, when the initial state is $(\overline{x}^H_{\overline{K}_1,1}, 0)$ the monopolist prefers selling to $(\overline{y}^H_{\overline{K}_1+1,1}, 0)$ to selling to $(\overline{y}^H_{\overline{K}_1,1}, 0)$. This implies that

$$\mu p^H_{\overline{K}_1,1} \overline{y}^H_{\overline{K}_1,1} + \delta R^C(\overline{x}^H_{\overline{K}_1,1}, x^L) > R^C(\overline{x}^H_{\overline{K}_1,1}, x^L).$$

Hence, $\varsigma(\overline{y}^H_{\overline{K}_1,1}) > 0$ and $\hat{y}^H < \overline{y}^H_{\overline{K}_1,1}$. However, constructing the sequence for y^H below $\overline{y}^H_{\overline{K}_1,1}$ is not possible. Similarly, let's define

$$\overline{K}_3 = \min\{k : \overline{x}^H_{j,3} < \overline{x}^H_{j-1,3} \text{ for all } j \leq k \text{ and } x^L = 1 - \overline{y}^H_{\overline{K}_1,3}\}.$$

Since the Coase Conjecture equilibrium does not exist for $\mu > \overline{\mu}$, we have $\overline{x}^H_{\overline{K}_3,3} > 0$. Furthermore, since $\overline{x}^H_{\overline{K}_3+1,3} \geq \overline{x}^H_{\overline{K}_3,3}$, when the initial state is $(\overline{x}^H_{\overline{K}_3,3}, 1 - \mu - \overline{x}^H_{\overline{K}_3,3})$ the monopolist prefers selling to $(\overline{y}^H_{\overline{K}_3+1,3}, 1 - \overline{y}^H_{\overline{K}_1,3})$ to selling to $(\overline{y}^H_{\overline{K}_3,3}, 1 - \overline{y}^H_{\overline{K}_1,3})$. This implies that

$$\mu p^H_{\overline{K}_3,3} \overline{y}^H_{\overline{K}_3,3} + \mu p^L_{\overline{K}_3,3}(1 - \overline{y}^H_{\overline{K}_3,3}) + \delta R^C(\overline{x}^H_{\overline{K}_3,3}, 1 - \mu - \overline{x}^H_{\overline{K}_3,3}) > R^C(\overline{x}^H_{\overline{K}_3,3}, 1 - \mu - \overline{x}^H_{\overline{K}_3,3}).$$

Hence, $\psi(\overline{y}^H_{\overline{K}_3,3}) > 0$ and $\check{y}^H < \overline{y}^H_{\overline{K}_3,3}$. However, constructing the sequence for y^H below $\overline{y}^H_{\overline{K}_3,3}$ is not possible. Hence, we must have $\mu < \overline{\mu}$.

We now establish that a reputational equilibrium exists for any $\mu \in (\underline{\mu}, \overline{\mu}]$. Let $\Delta := \varsigma - \psi$. Then, $\Delta(y) := \frac{\mu(\overline{\theta}y - \underline{\theta})q_L}{1-\delta} - (R^C((1-\mu)y, 0) - R^C((1-\mu)y, (1-\mu)(1-y)))$. We have $\Delta(0) < 0$ and $\Delta(1) > 0$. Moreover, $\psi(\hat{y}^H) > 0$. For $\mu > \underline{\mu}$, we have $\psi(\widehat{b}) > 0$. Since $\psi(\widehat{b}) > 0$. Since $\psi(0) = \frac{\mu\underline{\theta}q_L}{1-\delta} - R^C(0, 1-\mu) < 0$, and $\psi(\cdot)$ is strictly increasing there exists a unique $\check{y}^H \in (0, \widehat{b})$ such that $\psi(\check{y}^H) = 0$. Similarly, for $\mu > \mu^{st}$, we have $\varsigma(\widehat{b}) > 0$. Since $\varsigma(0) = -R^C(0,0) < 0$, and $\varsigma(\cdot)$ is strictly increasing, there exists a unique $\hat{y}^H \in (0, \widehat{b})$ such that $\varsigma(\hat{y}^H) = 0$. Therefore, for $\mu \in (\underline{\mu}, \mu^{st}]$, the steady state of a reputational equilibrium is $(\check{y}^H, 1 - \check{y}^H)$. $P^H(y^H, y^L) = \overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L$ for all $y^H < \check{y}^H$ and all y^L, and $P^L(y^H, y^L) = \underline{\theta}q_L$ for all y^H and y^L. Hence, $R(x^H, x^L) = (\check{y}^H - x^H)(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L) + (1 - \check{y}^H - x^L)\underline{\theta}q_L + \delta(\frac{\mu\check{y}^H(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L)}{1-\delta} + \frac{\mu(1-\check{y}^H)\underline{\theta}q_L}{1-\delta})$ for $x^H \in [0, (1-\mu)\check{y}^H]$ and for $x^L \in [0, (1-\mu)(1-\check{y}^H)]$, and the path is defined by the Coase Conjecture path elsewhere. For $\mu \in (\mu^{st}, \overline{\mu}]$, if $\overline{\theta}\hat{y}^H > \underline{\theta}$, the reputational steady state is

$(\hat{y}^H, 0)$, otherwise it is $(\check{y}^H, 1 - \check{y}^H)$.

Now, we establish that $\underline{\mu} < \overline{\mu}$. Since $\underline{\mu} < \mu^{st}$, to establish it we show that $\mu^{st} < \overline{\mu}$ by proving $\{\overline{x}_{k,1}\}$ is strictly decreasing and m is finite for $\mu \leq \underline{\mu}$. First, we show that $\overline{x}_{2,1} < \overline{x}_{1,1}$ for any $\mu \leq \underline{\mu}$. Since $h_2(\cdot)$ is decreasing and $h_2(\overline{x}_2^H) = 0$, to establish the proof, we show that $h_2(\overline{x}_{1,1}^H) < 0$. We have

$$\begin{aligned} h_2(\overline{x}_{1,1}) = \Pi(\overline{y}_{1,1}, \overline{x}_{1,1}) - \Pi(\overline{y}_{0,1}, \overline{x}_{1,1}) &= P^H(\widehat{b}, 0)\mu\widehat{b} + \delta\Pi(\overline{y}_{0,1}, \overline{x}_{1,1}) - \Pi(\overline{y}_{0,1}, \overline{x}_{1,1}) \\ &= P^H(\overline{y}_1^H, 0)\mu\widehat{b} - (1-\delta)\Pi(\overline{y}_{0,1}, \overline{x}_{1,1}). \end{aligned}$$

Since $\Pi(\overline{y}_{0,1}, \overline{x}_{1,1}) \geq \frac{\mu\widehat{b\overline{\theta}}q_H}{1-\delta}$, we have

$$h_2(\overline{x}_{1,1}) \leq P^H(\widehat{b}, 0)\mu\widehat{b} - \mu\widehat{b\overline{\theta}}q_H = \mu\widehat{b}(P^H(\widehat{b}, 0) - \overline{\theta}q_H) < 0$$

Second, we show that there exist $\Delta > 0$ such that $\overline{x}_{3,1}^H < \overline{x}_{2,1}^H - \Delta$. Since $h_3(\cdot)$ is decreasing and $h_3(\overline{x}_{3,1}^H) = 0$, to establish the proof, we show that $h_3(\overline{x}_{2,1}^H) < 0$. We have $h_3(\overline{x}_{2,1}^H)$ $= \Pi(\overline{y}_{2,1}; \overline{x}_{2,1}) - \Pi(\overline{y}_{1,1}; \overline{x}_{2,1}) = P^H(\overline{y}_{2,1})\mu\overline{y}_{2,1}^H - (1-\delta)\Pi(\overline{y}_{1,1}; \overline{x}_{2,1})$. Since $\Pi(\overline{y}_{1,1}; \overline{x}_{2,1}) =$ $\Pi(\overline{y}_{0,1}; \overline{x}_{2,1})$, we have $\Pi(\overline{y}_{1,1}; \overline{x}_{2,1}) = (\overline{x}_{1,1}^H - \overline{x}_{2,1}^H)\underline{\theta}q_H + \Pi(\overline{y}_{0,1}; \overline{x}_{1,1})$. Therefore, $h_3(\overline{x}_{2,1}^H) <$ $P^H(\overline{y}_{2,1})\frac{\mu\overline{x}_{2,1}^H}{1-\delta} - (1-\delta)(\overline{x}_1^H - \overline{x}_2^H)\underline{\theta}q_H - \frac{\mu\widehat{b\overline{\theta}}q_H}{1-\delta} = (P^H(\overline{y}_{2,1}) - \overline{\theta}q_H)\frac{\mu\overline{x}_{2,1}^H}{1-\delta} - (1-\delta)(\overline{x}_{1,1}^H - \overline{x}_{2,1}^H)\underline{\theta}q_H < 0$. Therefore, $\overline{x}_{2,1}^H - \overline{x}_{3,1}^H = \frac{h_3(\overline{x}_{2,1}^H)}{h_3'(x)} = \frac{(1-\delta)((\overline{x}_1^H - \overline{x}_2^H)\underline{\theta}q_H)}{-P^H(\overline{y}_{2,1}) + P^H(\overline{y}_{1,1})} = \Delta > 0$

Third, we suppose that there exist Δ such that $x_k^H < x_{k-1}^H - \Delta$ for $k \geq 3$ and show that it also holds for $k+1$. Since the sequence is decreasing until $k-1$, we have $\Pi(\overline{y}_{k-1,1}, \overline{x}_{k,1}) \geq$ $\Pi(\overline{y}_0, \overline{x}_k) = (1 - \overline{x}_{k,1})\underline{\theta}q_H + \delta R(1-\mu) = \Pi(\overline{y}_{0,1}, \overline{x}_{1,1}) + (\overline{x}_{1,1}^H - \overline{x}_{k,1}^H)\underline{\theta}q_H \geq \dfrac{\mu\widehat{b\overline{\theta}}q_H}{1-\delta} + (\overline{x}_1^H - \overline{x}_k^H)\underline{\theta}q_H$. By the same argument stated above $h_{k+1,1}(\overline{x}_{k,1}^H) = \Pi(\overline{y}_{k,1}, \overline{x}_{k,1}) - \Pi(\overline{y}_{k-1,1}, \overline{x}_{k,1}) < -(1-$ $\delta)(\overline{x}_{1,1}^H - \overline{x}_{k,1}^H)\underline{\theta}q_H < 0$. Therefore, since $h_{k+1,1}\left(\overline{x}_{k+1,1}^H\right) = 0$ and $h_{k+1,1}'(.) = -P^H(\overline{y}_{k,1}) +$ $P^H(\overline{y}_{k-1,1})$, we have $\overline{x}_{k,1}^H - \overline{x}_{k+1,1}^H = \frac{h_{k+1,1}\left(\overline{x}_{k,1}^H\right)}{h_{k+1,1}'(.)} > \frac{(1-\delta)\left(\overline{x}_{1,1}^H - \overline{x}_{k,1}^H\right)\underline{\theta}q_H}{P^H(\overline{y}_{k,1}) - P^H(\overline{y}_{k-1,1})}$.

Proof of Corollary 6. Consider $\left(\hat{y}^H, 0\right)$. We set $\hat{y}^H$ such that $\zeta\left(\hat{y}^H, 0\right) = 0$. Then $\lim_{z\to 0^+}\zeta\left(y^H, 0\right) = \lim_{z\to 0^+}\left(\frac{\mu\overline{\theta}q_H y^H}{1-\delta} - ((1-(1-\mu)y^H)\underline{\theta}q_H + \frac{\delta\mu\underline{\theta}q_H}{1-\delta})\right)$. Therefore, at $\left(\hat{y}^H, 0\right)$ we have $\hat{y}^H q_H = \lim_{z\to 0^+}\frac{(1-\delta)\underline{\theta}q_H + \delta\mu\underline{\theta}q_H}{\mu\overline{\theta} + (1-\delta)(1-\mu)\underline{\theta}} = \frac{(\lambda+r)\underline{\theta}q_H}{\lambda\overline{\theta} + r\underline{\theta}}$. Since $\overline{\theta}\hat{y}^H = \frac{(\lambda+r)\underline{\theta}\overline{\theta}}{\lambda\overline{\theta} + r\underline{\theta}} > \underline{\theta}$, the seller moves to $\left(\hat{y}^H, 0\right)$ from the initial state $(0,0)$ when $\underline{\mu}^{st} < \mu < \overline{\mu}$.

Now, consider $\left(\check{y}^H, 1-\check{y}^H\right)$. We set $\check{y}^H$ such that $\psi(\check{y}^H, 1-\check{y}^H) = 0$. Then $\lim_{z\to 0^+} \psi$ $(y^H,\, 1-y^H) = \lim_{z\to 0^+}(\frac{\mu(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L)y^H}{1-\delta} + \frac{\mu\underline{\theta}q_L y^L}{1-\delta} - ((1-(1-\mu)y^H)\underline{\theta}q_H + \frac{\delta\mu\underline{\theta}q_H}{1-\delta}))$.

Proof of Proposition 4. Trivial.

Proof of Corollary 7. Trivial.

Appendix B

Appendix for Chapter 5

Proof of Proposition 5. The proof is established by the following two claims. First, we claim that if buyer b buys good i rather than waiting one period then $b' < b$ would prefer buying good i to waiting one period as well. The Buyer b s payoff when she accepts p_t^i is $f^i(b) - p_t^i + \delta V_b(H_t, a)$. When she rejects p_t^i, her payoff is $\delta V_b(H_t, r)$. Since the good may stop functioning with probability $\mu, V_b(H_t, a) = \mu V_b(H_t, r)$. Therefore, buyer b accepts p_t^i if $f^i(b) - p_t^i \geq \rho V_b(H_t, r)$. Suppose that $b' < b$. Show that $f^i(b') - p_t^i \geq \rho V_{b'}(H_t, r)$.Suppose b adopts b' s strategy. Then b s payoff would be less that b' s payoff. That is, $f^i(b) - \widetilde{p}^i - \rho V_b(H_t, r) \leq f^i(b') - \widetilde{p}^i - \rho V_{b'}(H_t, r)$. This implies that

$$\rho\left(V_{b'}(H_t, r) - V_b(H_t, r)\right) \leq f^i\left(b'\right) - f^i(b).$$

If we rewrite the expression we get

$$p_t^i \leq f^i(b) - \rho V_b(H_t, r) \leq f^i\left(b'\right) - \rho V_{b'}(H_t, r)$$

Second, we claim that If buyer b prefers high quality good to low quality good then $b' < b$ would prefer high quality good as well. Buyer b s payoff when she accepts high quality good is $f^H(b) - p_t^H + \delta\mu V_b(H_t, r)$. Her payoff when she accepts low quality good is $f^L(b) - p_t^L + \delta\mu V_b(H_t, r)$. Therefore, buyer b prefers high quality good to quality one if $f^H(b) - f^L(b) \geq p_t^H - p_t^L$. Since for all $b' < b$, $f^H(b') - f^L(b') \geq f^H(b) - f^L(b)$, b' prefers high quality good as well.

For all $p_t^H - p_t^L$, one of the followings is true. All consumers may buy low quality good: $f^H(0) - f^L(0) \leq p_t^H - p_t^L$, all consumers may buy high quality good: $f^H(1) - f^L(1) \geq p_t^H - p_t^L$, or there may exists $\widetilde{b}$ such that for all $b \leq \widetilde{b}$ $f^H(b) - f^L(b) \leq p_t^H - p_t^L$ and for all $b > \widetilde{b}$

$f^H(b) - f^L(b) > p_t^H - p_t^L$.

Proof of Proposition 6. Any stationary equilibrium has at least one steady state. That is, there exists (y^H, y^H) such that

$$t^H\left((1-\mu)\, y^H, (1-\mu)\, y^L\right) = y^H$$

and

$$t^L\left((1-\mu)\, y^H, (1-\mu)\, y^L\right) = y^L$$

where $y^H \in (0,1]$, $y^H + y^L \leq 1$. Moreover, the steady state prices satisfy

$$\wp^H\left(y^H, y^L\right) = f^H\left(y^H\right) - f^L\left(y^H\right) + f^L\left(y^H + y^L\right)$$

and

$$\wp^L\left(y^H, y^L\right) = f^L\left(y^H + y^L\right).$$

If $\left(y^H, y^L\right)$ are the stocks after trade in a steady state. Then $P^H\left(y^H\right) = f^H\left(y^H\right)$ and $P^L\left(y^H + y^L\right) = f^L\left(y^H + y^L\right)$. This implies that

$$\wp^H\left(y^H, y^L\right) = f^H\left(y_H\right) - f^L\left(y_H\right) + f^L\left(y_H + y_L\right)$$

and

$$\wp^L\left(y^H, y^L\right) = f^L\left(y_H + y_L\right).$$

Let's define the sets S^H and $S^L\left(y^H\right)$ as

$$S^H = \left\{y^H : P^H\left(y^H\right) = f^H\left(y^H\right)\right\}$$

and

$$S^L\left(y^H\right) = \left\{y^L : P^L\left(y^H + y^L\right) = f^L\left(y^H + y^L\right)\right\}.$$

Suppose that S^H and $S^L\left(\max S^H\right)$ are nonempty. Let $\left(\overline{y}^H, \overline{y}^L\right)$ be defined as

$$\overline{y}^H = \max S^H$$

and

$$\overline{y}^L = \max S^L\left(\overline{y}^H\right).$$

First we will show that $\left(\overline{y}^H, \overline{y}^L\right)$ is a steady state. To do that, we will prove that $\overline{y}^H$ and $\overline{y}^L$ are the maximum states to have acceptance prices $P^H\left(\overline{y}^H\right)$ and $P^L\left(\overline{y}^H + \overline{y}^L\right)$, respectively. If there existed $\widetilde{y}^H > \overline{y}^H$ with $P^H\left(\widetilde{y}^H\right) = P^H\left(\overline{y}^H\right)$ from the definition of $\overline{y}^H$ we must have $f^H\left(\widetilde{y}^H\right) < f^H\left(\overline{y}^H\right)$. Since $t^H\left(x^H, x^L\right)$ is non-decreasing with respect to x^H for a given x^L and since $P^H(\cdot)$ is non-increasing we have $P^H(t^H((1-\mu)\widetilde{y}^H, (1-\mu)y^L)) \leq P^H(t^H((1-\mu)\overline{y}^H, (1-\mu)y^L))$ which implies a contradiction, since $P^H\left(\widetilde{y}^H\right) = (1-\rho)f^H\left(\widetilde{y}^H\right) + \rho P^H(t^H((1-\mu)\widetilde{y}^H, (1-\mu)y^L)) < (1-\rho)f^H(\overline{y}^H) + \rho P^H(t^H((1-\mu)\overline{y}^H, (1-\mu)y^L)) = P^H\left(\overline{y}^H\right)$.

If there existed $\widetilde{y}^L > \overline{y}^L$ with $P^L\left(\overline{y}^H + \widetilde{y}^L\right) = P^L\left(\overline{y}^H + \overline{y}^L\right)$ from the definition of $\overline{y}^H$ and $\overline{y}^L$ we must have $P^L\left(\overline{y}^H + \widetilde{y}^L\right) < P^L\left(\overline{y}^H + \overline{y}^L\right)$. Since $t^L\left(x^H, x^L\right)$ is non-decreasing with respect to x^L for a given x^H and since $P^L(\cdot)$ is non-increasing we have $P^L(t^H((1-\mu)\overline{y}^H, (1-\mu)\widetilde{y}^L) + t^L((1-\mu)\overline{y}^H, (1-\mu)\widetilde{y}^L)) \leq P^L(t^H((1-\mu)\overline{y}^H, (1-\mu)\overline{y}^L) + t^L((1-\mu)\overline{y}^H, (1-\mu)\overline{y}^L))$ which implies a contradiction. $P^L\left(\overline{y}^H + \widetilde{y}^L\right) = (1-\rho)f^L(\overline{y}^H + \widetilde{y}^L) + \rho P^L(t^H((1-\mu)\overline{y}^H, (1-\mu)y^L) + t^L((1-\mu)\overline{y}^H, (1-\mu)\widetilde{y}^L)) < (1-\rho)\, f^L(\overline{y}^H + \overline{y}^L) + \rho\, P^L(t^H((1-\mu)\overline{y}^H, (1-\mu)\overline{y}^L) + t^L((1-\mu)\overline{y}^H, (1-\mu)\overline{y}^L)) = P^L(\overline{y}^H + \overline{y}^L)$. So, following the offers $P^H\left(\overline{y}^H\right) - f^L\left(\overline{y}^H\right) + P^L\left(\overline{y}^H + \overline{y}^L\right)$ for high quality good and $P^L\left(y^H + y^L\right)$ for low quality good, all $b \leq \overline{y}^H$ accept high quality good and all $\overline{y}^H < b \leq \overline{y}^H + \overline{y}^L$ accept low quality good and all $b > \overline{y}^H + \overline{y}^L$ reject the offers of the monopolist. So, we derive from the arbitrage equations for the high and quality goods $P^H\left(\overline{y}^H\right) = f^H\left(\overline{y}^H\right)$, and $P^L\left(\overline{y}^H + \overline{y}^L\right) = f^L\left(\overline{y}^H + \overline{y}^L\right)$ respectively.

Suppose that there is a stationary equilibrium which does not have any steady states. This implies that either S^H or $S^L$is empty. First, suppose S^H is empty and that $P^L(b)$ is a nonincreasing function. We will show that $P^H(b) < \overline{\theta} q_H$ for $b \in \left[0, \widehat{b}\right]$ and $P^H(b) > \underline{\theta} q_H$

for $b \in \left(\widehat{b}, 1\right]$. If S^H is empty then $P^H(b) \neq \overline{\theta} q_H$ for $b \in \left[0, \widehat{b}\right]$ and $P^H(b) \neq \underline{\theta} q_H$ for $b \in \left(\widehat{b}, 1\right]$. We know from Lemma 2 that the seller never charges a price less than $\underline{\theta} q_H$ for high quality good. This implies that $P^H(b) > \underline{\theta} q_H$ for $b \in \left(\widehat{b}, 1\right]$. Moreover, it is not possible that $P^H(b) > \overline{\theta} q_H$ for some $b \in \left[0, \widehat{b}\right]$. If it was, it would imply that $P^H(0) > \overline{\theta} q_H$, since $P^H(\cdot)$ is nonincreasing. Given a state $\left(0, \acute{y}^L\right)$, the arbitrage equation $P^H(0) = (1-\rho)\overline{\theta} q_H + \rho P^H\left(t^H\left(0, (1-\mu)\acute{y}^L\right)\right)$ implies that $P^H\left(t^H\left(0, (1-\mu)\acute{y}^L\right)\right) > P^H(0)$ which is a contradiction because $t^H\left(0, (1-\mu)\acute{y}^L\right) \geq 0$ and $P^H(\cdot)$ is a nonincreasing function. Now, suppose S^L is empty and that $P^H(b)$ is a nonincreasing function. We will show that $P^L(b) < \overline{\theta} q_L$ for $b \in \left[0, \widehat{b}\right]$ and $P^L(b) > \underline{\theta} q_L$ for $b \in \left(\widehat{b}, 1\right]$. If S^L is empty then $P^L(b) \neq \overline{\theta} q_L$ for $b \in \left[0, \widehat{b}\right]$ and $P^L(b) \neq \underline{\theta} q_L$ for $b \in \left(\widehat{b}, 1\right]$. We know from Lemma 2 that the seller never charges a price less than $\underline{\theta} q_L$ for low quality good. This implies that $P^L(b) > \underline{\theta} q_L$for $b \in \left(\widehat{b}, 1\right]$. Moreover, it is not possible that $P^L(b) > \overline{\theta} q_L$ for some $b \in \left[0, \widehat{b}\right]$. If it was, it would imply that $P^L(0) > \overline{\theta} q_L$, since $P^L(\cdot)$ is nonincreasing. Given a state $\left(0, \acute{y}^L\right)$, the arbitrage equation $P^L(0) = (1-\rho)\overline{\theta} q_L + \rho P^L\left(t^H(0,0) + t^L(0,0)\right)$ implies that $P^L\left(t^H(0,0) + t^L(0,0)\right) > P^L(0)$ which is a contradiction because $t^j(0,0) \geq 0$ for all $j = H, L$ and $P^L(\cdot)$ is a nonincreasing function.

Suppose $P^H(b) < \overline{\theta} q_H$ for $b \in [0, \widehat{b}]$ and $P^H(b) > \underline{\theta} q_H$ for $b \in (\widehat{b}, 1]$ and that $P^L(b)$ is a nonincreasing function. Given a state $\left(y^H, \acute{y}^L\right)$ where $y^H \in [0, \widehat{b}]$, since $\overline{\theta} q_H - P^H\left(y^H\right) > 0$, the arbitrage equation $\overline{\theta} q_H - P^H\left(y^H\right) = \rho \overline{\theta} q_H - P^H(t^H((1-\mu)y^H, (1-\mu)\acute{y}^L)))$ implies that $P^H(t^H((1-\mu)y^H, (1-\mu)\acute{y}^L)) < P^H(y^H)$. That is, $t^H((1-\mu)y^H, (1-\mu)\acute{y}^L) > y^H$. Similarly, given a state $(\widehat{b} + \epsilon, \acute{y}^L)$ where $\epsilon \in (0, 1 - \widehat{b}]$, since $\underline{\theta} q_H - P^H(\widehat{b} + \epsilon) < 0$, the arbitrage equation $\underline{\theta} q_H - P^H(\widehat{b} + \epsilon) = \rho(\underline{\theta} q_H - P^H(t^H((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L)))$ implies that $P^H(t^H((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L)) > P^H(\widehat{b} + \epsilon)$. That is, $t^H((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L) < \widehat{b} + \epsilon$. So, $\lim\limits_{\epsilon \to 0} t^H((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L) \leq \widehat{b} < t^H((1-\mu)\widehat{b}, (1-\mu)\acute{y}^L)$. Since, $T(\cdot)$ is upper hemi-continuous, $(\lim\limits_{\epsilon \to 0} t^H((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L), \lim\limits_{\epsilon \to 0} t^L((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L)) \in T((1-\mu)\widehat{b}, (1-\mu)\acute{y}^L)$. Moreover, since $T(\cdot)$ is a lattice, meet of $(\lim\limits_{\epsilon \to 0} t^H((1-\mu)(\widehat{b}+\epsilon), (1-\mu)\acute{y}^L)$, $\lim\limits_{\epsilon \to 0} t^L((1-\mu)(\widehat{b} + \epsilon), (1-\mu)\acute{y}^L))$ and $(t^H((1-\mu)\widehat{b}, (1-\mu)\acute{y}^L), t^L((1-\mu)\widehat{b}, ((1-\mu))\acute{y}^L))$ is in $T(\cdot)$ as well which contradicts with the definition of $(t^H(\cdot), t^L(\cdot))$.

Suppose $P^L(b) < \overline{\theta} q_L$ for $b \in [0, \widehat{b}]$ and $P^L(b) > \underline{\theta} q_L$ for $b \in (\widehat{b}, 1]$ and that $P^H(b)$ is a nonincreasing function. Given a state $\left(\acute{y}^H, y^L\right)$ where $\acute{y}^H+y^L \in [0,\widehat{b}]$, since $\overline{\theta}q_L - P^L(\acute{y}^H+y^L) > 0$, the arbitrage equation $\overline{\theta}q_L - P^L(\acute{y}^H + y^L) = \rho(\overline{\theta}q_L - P^L(t^H((1-\mu)\acute{y}^H, (1-\mu)y^L) + t^L((1-\mu)\acute{y}^H, (1-\mu)y^L)))$ implies that $P^L(t^H((1-\mu)\acute{y}^H, (1-\mu)y^L) + t^L((1-\mu)\acute{y}^H, (1-\mu)y^L)) < P^L(\acute{y}^H + y^L)$. That is, $t^H((((1-\mu)\acute{y}^H, ((1-\mu)y^L) + t^L((((1-\mu)\acute{y}^H, ((1-\mu)y^L) > \acute{y}^H + y^L$. Similarly, given a state $(\acute{y}^H, \widehat{b}+\epsilon-\acute{y}^H)$ where $\epsilon \in (0, 1-\widehat{b}]$, since $\underline{\theta}q_L - P^L(\widehat{b}+\epsilon) < 0$, the arbitrage equation $\underline{\theta}q_L - P^L(\widehat{b}+\epsilon) = \rho(\underline{\theta}q_L - P^L(t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) + t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H))))$ implies that $P^L(t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) + t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H))) > P^L(\widehat{b}+\epsilon)$. That is, $t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) + t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) < \widehat{b}+\epsilon$. So, $\lim\limits_{\epsilon\to 0} t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) + \lim\limits_{\epsilon\to 0} t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) \leq \widehat{b} < t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}-\acute{y}^H)) + t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}-\acute{y}^H))$. Since t^H is increasing with respect to each variable, we have $\lim\limits_{\epsilon\to 0} t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) > t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}-\acute{y}^H))$. This implies that $\lim\limits_{\epsilon\to 0} t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)) < t^L((1-\mu)\acute{y}^H, ((1-\mu))\left(\widehat{b}-\acute{y}^H\right))$. Since, $T(\cdot)$ is upper hemi-continuous, $(\lim\limits_{\epsilon\to 0} t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)), \lim\limits_{\epsilon\to 0} t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H))) \in T((1-\mu)\acute{y}^H, (1-\mu)\widehat{b}-\acute{y}^H)$. Moreover, since $T(\cdot)$ is a lattice, meet of $(\lim\limits_{\epsilon\to 0} t^H((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)), \lim\limits_{\epsilon\to 0} t^L((1-\mu)\acute{y}^H, (1-\mu)(\widehat{b}+\epsilon-\acute{y}^H)))$ and $(t^H((1-\mu)\acute{y}^H, (1-\mu)\widehat{b}-\acute{y}^H), t^L((1-\mu)\acute{y}^H, (1-\mu)\widehat{b}-\acute{y}^H))$ is in $T(\cdot)$ as well which contradicts with the definition of $\left(t^H(\cdot), t^L(\cdot)\right)$.

Proof of Proposition 7. Monopolist's equilibrium valuation must be non-negative. Moreover, equilibrium surplus cannot exceed $\overline{\theta}q_H$. Therefore monopolist's equilibrium valuation cannot exceed $\overline{\theta}q_H$. Moreover, from Lemma 1 we know that buyer's equilibrium valuation is nonincreasing and has modulus of continuity no greater than 1. That is,

$$P^H\left(b'\right) - P^H\left(b\right) \leq f^H\left(b'\right) - f^H\left(b\right).$$

This implies that

$$P^H\left(b'\right) \leq P^H\left(b\right) + f^H\left(b'\right) - f^H\left(b\right).$$

Since buyers' valuation is nonincreasing and does not exceed $\overline{\theta}q_H$, it must be that $P^H\left(1\right) \leq \overline{\theta}q_H$. So above inequality and the upper bound on valuations imply that $P^H\left(0\right) \leq P^H\left(1\right) +$

$f^H(0) - f^H(1) \leq 2\overline{\theta}q_H - \underline{\theta}q_H$. By using the above inequality one more time we get $P^H(0) \leq P^H(1) + f^H(0) - f^H(1)$. After rearranging the inequality, we get $P^H(1) \geq P^H(0) - (f^H(0) - f^H(1))$. That is, $P^H(1) \geq f^H(1) - f^H(0)$. Therefore, $P^H(1) \geq \underline{\theta}q_H - \overline{\theta}q_H$. Therefore all buyers accept any price below $\underline{\theta}q_H - \overline{\theta}q_H$. So the monopolist never charges such prices. Knowing that the lowest possible price is $\underline{\theta}q_H - \overline{\theta}q_H$ all buyers accept prices such that $\underline{\theta}q_H - p \geq \rho\left(\underline{\theta}q_H - \left(\underline{\theta}q_H - \overline{\theta}q_H\right)\right)$. So any price below $\underline{\theta}q_H - \rho\overline{\theta}q_H$ is accepted by all buyers. Therefore, for all positive n, all prices below $\underline{\theta}q_H - \rho^n\overline{\theta}q_H$ are accepted by all buyers. As n goes to infinity the limit price converges to $\underline{\theta}q_H$. Therefore the seller never charges less than $\underline{\theta}q_H$.The idea for the low quality good is the same is the same.

Lemma 2.

The $\arg\max$ correspondence $T\left(x^H, x^L\right)$ of the objective function is a sublattice of $[0,1] \times [0,1]$.

Proof of Lemma 2. To prove the lemma, we show that the objective function is supermodular. We claim that the objective function has increasing differences. Since the objective function has increasing difference on $X \times Y$ where $X = [0, 1-\mu) \times [0, 1-\mu)$ and $Y = [0,1] \times [0,1]$, The objective function is supermodular on $X \times Y$.[1] Therefore, the objective function is supermodular on Y for a given $\left(x^H, x^L\right)$. This implies that argmax correspondence of the objective function is a sublattice of Y.[2]

Lemma 3.

If there exists $\left(\acute{y}^H, \acute{y}^L\right)$ such that

i. $P^H\left(y^H\right) = P^H\left(\acute{y}^H\right)$ and $f^H\left(y^H\right) = f^H\left(\acute{y}^H\right)$,

ii. $P^L\left(y^H + y^L\right) = P^L\left(\acute{y}^H + \acute{y}^L\right)$ and $f^L\left(y^H + y^L\right) = f^L\left(\acute{y}^H + \acute{y}^L\right)$,

iii. $P^H\left(y^H\right)\left(\acute{y}^H - y^H\right) + P^L\left(y^H + y^L\right)\left(\acute{y}^L - y^L\right) > 0$

hold for some $\acute{y}^H \geq 0$ and $\acute{y}^L \geq 0$ then $\left(y^H, y^L\right)$ cannot be a steady state.

Proof of Lemma 3. If the conditions hold then the monopolist strictly prefers $\left(\acute{y}^H, \acute{y}^L\right)$ to $\left(y^H, y^L\right)$. Therefore $\left(y^H, y^L\right)$ cannot be a steady state since the monopolist deviates from that

[1] See Topkis Corollary 2.6.1, pg 45.

[2] See Topkis Theorem 2.7.1 pg 66.

state with probability 1.

Proof of Proposition 8. First we present implications of the previous lemma.

1– We show that $\left(\grave{y}^H,0\right)$ where $\grave{y}^H \in \left(\widehat{b},1\right)$ cannot be a steady state. Suppose not. Then $P^H\left(\grave{y}^H\right)=\underline{\theta}q_H$ and $P^L\left(\grave{y}^H\right)=\underline{\theta}q_L$. Since $P(\cdot)$ is a non-increasing function, $P^H\left(b\right)\leq\underline{\theta}q_H$ and $P^L\left(b\right)\leq\underline{\theta}q_L$ for all $b\in\left(\grave{y}^H,1\right]$. So, $P^H\left(y^H\right)=\underline{\theta}q^H$ and $P^L\left(y^H\right)=\underline{\theta}q_L$ for all $y\in\left[\grave{y}^H,1\right]$. Since the seller would prefer $y^H>\grave{y}^H$, $\left(\grave{y}^H,0\right)$ cannot be a steady state.

2– We show that $\left(\grave{y}^H,\grave{y}^L\right)$ where $\grave{y}^H\in\left(0,\widehat{b}\right]$,and $\grave{y}^H+\grave{y}^L\in\left(\widehat{b},1\right)$ cannot be a steady state. Suppose not. Then $P^H\left(\grave{y}^H\right)=\overline{\theta}q_H$ and $P^L\left(\grave{y}^H+\grave{y}^L\right)=\underline{\theta}q_L$. Therefore, for all $b>\grave{y}^H+\grave{y}^L$, $P^L\left(b\right)\leq\underline{\theta}q_L$ has to hold for $\left(\grave{y}^H,\grave{y}^L\right)$ to be a steady state. But Lemma 3 states that for all $b\in[0,1]$, $P^L\left(b\right)\geq\underline{\theta}q_L$.So, for all $b>\grave{y}^H+\grave{y}^L$, $f^L\left(b\right)=\underline{\theta}q_L$.Lemma 4 concludes that $\left(\grave{y}^H,\grave{y}^L\right)$ cannot be a steady state.

3– We show that $\left(\grave{y}^H,\grave{y}^L\right)$ where $\grave{y}^H\in\left(\widehat{b},1\right)$, and $\grave{y}^H+\grave{y}^L\in\left(\widehat{b},1\right]$ cannot be a steady state. Suppose not. Then $P^H\left(\grave{y}^H\right)=\underline{\theta}q_H$ and $P^L\left(\grave{y}^H+\grave{y}^L\right)=\underline{\theta}q_L$. Therefore, for all $b^H>\grave{y}^H$ and $b^L>\grave{y}^H+\grave{y}^L$, $P^H\left(b^H\right)\leq\overline{\theta}q_L$ and $P^L\left(b^L\right)\leq\underline{\theta}q_L$ have to hold for $\left(\grave{y}^H,\grave{y}^L\right)$ to be a steady state. But Lemma 3 states that for all $b^H\in[0,1]$ and $b^L\in[0,1]$, $P^H\left(b^H\right)\geq\overline{\theta}q_L$ and $P^L\left(b^L\right)\geq\underline{\theta}q_L$. So, for all $b^H>y^H$ and $b^L>\grave{y}^H+\grave{y}^L$, we have $P^H\left(b^H\right)\geq\overline{\theta}q_L$ and $P^L\left(b^L\right)=\underline{\theta}q_L$.Lemma 4 concludes that $\left(\grave{y}^H,\grave{y}^L\right)$ cannot be a steady state.

Proof of (2): Suppose that $\left(\widehat{b},0\right)$ is a steady state. Then, according to Proposition 2, $P^H\left(\widehat{b}\right)=\overline{\theta}q_H$ and $P^L\left(\widehat{b}\right)=\overline{\theta}q_L$. Since $P^i(\cdot)$ is decreasing and $P^i\left(b\right)\leq\overline{\theta}q_i$ for all $b\in\left[0,\widehat{b}\right]$. So Lemma 4 shows that $(b,0)$ such that $b<\widehat{b}$ cannot be a steady state. We conclude that the only possible steady states other than $\left(\widehat{b},0\right)$ are $\left(\widehat{b},1-\widehat{b}\right)$ and $(1,0)$.

Proof of (3): Suppose that $\left(\grave{y}^H,0\right)$ where $\grave{y}^H\in\left(0,\widehat{b}\right)$ is a steady state. Then $P^H\left(b\right)=\overline{\theta}q_H$ and $P^L\left(b\right)=\overline{\theta}q_L$ for all $b\in\left[0,\grave{y}^H\right]$ and $P^H\left(b\right)<\overline{\theta}q_H$ and $P^L\left(b\right)<\overline{\theta}q_L$ for all $b\in\left(\grave{y}^H,\widehat{b}\right]$. This implies that $\left(\widehat{b},.\right)$ cannot be a steady state. We conclude that the only possible steady states other than $\left(\grave{y}^H,0\right)$ when $\grave{y}^H\in\left(0,\widehat{b}\right)$ are $\left(\grave{y}^H,1-\grave{y}^H\right)$ and $(1,0)$. Suppose that $\left(\grave{y}^H,\widehat{b}-\grave{y}^H\right)$ where $\grave{y}^H\in\left(0,\widehat{b}\right)$ is a steady state. Then $P^H\left(b^H\right)=\overline{\theta}q_H$ and $P^L\left(b^L\right)=\overline{\theta}q_L$ for all $b^H\in\left[0,\grave{y}^H\right]$ and $b^H\in\left[0,\widehat{b}\right]$ and $P^H\left(b^H\right)<\overline{\theta}q_H$ for all $b^H\in\left(\grave{y}^H,\widehat{b}\right]$. This implies that $\left(\widehat{b},.\right)$ cannot be a steady state. We conclude that the only possible steady states other than $\left(\grave{y}^H,\widehat{b}-\grave{y}^H\right)$ when $\grave{y}^H\in\left(0,\widehat{b}\right)$ are $\left(\grave{y}^H,1-\grave{y}^H\right)$ and $(1,0)$. Suppose that $\left(\grave{y}^H,\grave{y}^L\right)$ where

$\grave{y}^H \in \left(0, \widehat{b}\right)$ and $\grave{y}^H + \grave{y}^L \in \left(0, \widehat{b}\right)$ is a steady state. Then $P^H\left(b^H\right) = \overline{\theta} q_H$ and $P^L\left(b^L\right) = \overline{\theta} q_L$ for all $b^H \in \left[0, \grave{y}^H\right]$ and $b^H \in \left[0, \grave{y}^H + \grave{y}^L\right]$ and $P^H\left(b^H\right) < \overline{\theta} q_H$ and $P^L\left(b^L\right) < \overline{\theta} q_H$ for all $b^H \in \left(\grave{y}^H, \widehat{b}\right]$ and $b^L \in \left(\grave{y}^H + \grave{y}^L, \widehat{b}\right]$. We conclude that the only possible steady states other than $\left(\grave{y}^H, \grave{y}^L\right)$ when $\grave{y}^H \in \left(0, \widehat{b}\right)$ and $\grave{y}^H + \grave{y}^L \in \left(0, \widehat{b}\right)$ are $\left(\grave{y}^H, 1 - \grave{y}^H\right)$ and $(1, 0)$.

Proof of (1): If there is no steady state in $\left[0, \widehat{b}\right]$ then $(1, 0)$ is a steady state.

Proof of Theorem 4. The proof has two parts. First, we show that there exists an equilibrium in which $(1, 0)$ is the unique steady state. Then, in the second part, we prove the uniqueness of the equilibrium.

Existence:

First, let's define $\{\overline{x}_k\}_{k=0}^{m'}$ and $\{\widehat{x}_k\}_{k=0}^{m}$ where $\overline{x}_k = (\overline{x}_k^H,\ (1-\mu)\widehat{b} - \overline{x}_k^H)$ and $\widehat{x}_k = (\widehat{x}_k^H,\ 1 - \mu - \widehat{x}_k^H)$. The initial values are $\overline{x}_0^H = \widehat{x}_0^H = 1 - \mu$, $\overline{x}_1^H = \widehat{x}_1^H = (1-\mu)\widehat{b}$,

$$\overline{x}_2^H = \widehat{x}_2^H \frac{q_H}{(q_H - q_L)} - (1-\mu)\widehat{b} \frac{q_L}{(q_H - q_L)}$$

where $\widehat{x}_2^H = \dfrac{\widehat{b\theta} - \theta}{\overline{\theta} - \underline{\theta}}$. The rest is defined as

$$\overline{x}_k^H = (1-\mu)^{-1}\left(\overline{x}_{k-1}^H - \left(\overline{x}_{k-2}^H - \overline{x}_{k-1}^H\right)\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta} - \underline{\theta}\right)}\right) + \mu\widehat{b}\frac{q_L}{(q_H - q_L)}$$

and

$$\widehat{x}_k^H = (1-\mu)^{-1}\left(\widehat{x}_{k-1}^H - \left(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H\right)\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta} - \underline{\theta}\right)}\right).$$

Second, we derive the support of μ such that the state of high quality good is decreasing (i.e. $\overline{x}_{k-1}^H > \overline{x}_k^H$ and $\widehat{x}_{k-1}^H > \widehat{x}_k^H$) and that there exist $\overline{x}_{m'}^H$ and $\widehat{x}_m^H$ such that $\overline{x}_{m'+1} < 0 \leq \overline{x}_{m'}$ and $\widehat{x}_{m+1} < 0 \leq \widehat{x}_m$ hold.

We now claim that if $\widehat{x}_{k-1}^H > \widehat{x}_k^H$ then $\overline{x}_{k-1}^H > \overline{x}_k^H$ holds and that $m' < m$. To prove the claim we show that $\overline{x}_k^H - \overline{x}_{k+1}^H = \left(\widehat{x}_k^H - \widehat{x}_{k+1}^H\right)\dfrac{q_H}{(q_H - q_L)}$ and $\overline{x}_k^H = \widehat{x}_k^H\dfrac{q_H}{(q_H - q_L)} - (1-\mu)\widehat{b}\dfrac{q_L}{(q_H - q_L)}$ for $k = 2, \ldots, m'$. By definition we have $\overline{x}_i^H - \overline{x}_{i+1}^H = (\widehat{x}_i^H - \widehat{x}_{i+1}^H)\dfrac{q_H}{(q_H - q_L)}$ and $\overline{x}_i^H = \widehat{x}_i^H\dfrac{q_H}{(q_H - q_L)} - (1-\mu)\widehat{b}\dfrac{q_L}{(q_H - q_L)}$ for $i = 1, 2$. The proof comes from induction. Assume that $\overline{x}_i^H - \overline{x}_{i+1}^H = (\widehat{x}_i^H - \widehat{x}_{i+1}^H)\dfrac{q_H}{(q_H - q_L)}$ and $\overline{x}_i^H = \widehat{x}_i^H\dfrac{q_H}{(q_H - q_L)} - (1-\mu)\widehat{b}\dfrac{q_L}{(q_H - q_L)}$ hold for $i = k-1$. We now show that it also holds for $i = k$. Due to the assumption on $\left(\overline{x}_{k-2}^H - \overline{x}_{k-1}^H\right)$ and $\overline{x}_{k-1}^H$, we have $\overline{x}_k^H = (1-\mu)^{-1}\left((\widehat{x}_{k-1}^H \frac{q_H}{(q_H - q_L)} - (1-\mu)\,\widehat{b}\frac{q_L}{(q_H - q_L)}\right) - \left(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H\right)\frac{q_H}{(q_H - q_L)}\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)}) + \mu\widehat{b}\frac{q_L}{(q_H - q_L)}$. After simplifications $\overline{x}_k^H$ is pinned down to $\overline{x}_k^H = (1-\mu)^{-1}(\widehat{x}_{k-1}^H - (\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H)\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)})\frac{q_H}{(q_H - q_L)} - (1-\mu)\,\widehat{b}\frac{q_L}{(q_H - q_L)}$. By definition of $\widehat{x}_k^H$, we have $\overline{x}_k^H = \widehat{x}_k^H \frac{q_H}{(q_H - q_L)} - (1-\mu)\,\widehat{b}\frac{q_L}{(q_H - q_L)}$. This implies that $\overline{x}_k^H - \overline{x}_{k+1}^H = \left(\widehat{x}_k^H - \widehat{x}_{k+1}^H\right)\frac{q_H}{(q_H - q_L)}$ which finishes the proof. Let's define set Λ' and set Λ as

$$\Lambda' = \left\{\mu \geq 0 \mid \exists\ m' < \infty \text{ s.t. } \overline{x}_{m'+1}^H < 0 \leq \overline{x}_{m'}^H \text{ and } \left\{\overline{x}_k^H\right\}_{k=0}^{m'} \text{ is decreasing}\right\}$$

and

$$\Lambda = \left\{\mu \geq 0 \mid \exists\ m < \infty \text{ s.t. } \widehat{x}_{m+1}^H < 0 \leq \widehat{x}_m^H \text{ and } \left\{\widehat{x}_k^H\right\}_{k=0}^{m} \text{ is decreasing}\right\}.$$

Since $\overline{x}_k^H - \overline{x}_{k+1}^H > \widehat{x}_k^H - \widehat{x}_{k+1}^H$, we have $\Lambda \subset \Lambda'$ and $m' < m$. [3] Therefore, Λ represents the domain of the depreciation rate. Let $\overline{\mu} = \sup \Lambda$. Since $0 \in \Lambda$ and Λ is open in $\mathbb{R}^+$, we have $\overline{\mu} > 0$. Note that $\widehat{x}_2^H \leq \widehat{x}_1^H$, for all $\mu \leq \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\widehat{b}\left(\overline{\theta}-\underline{\theta}\right)}$ whereas, $\widehat{x}_3^H \leq \widehat{x}_2^H$ for all $\mu < \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\widehat{b}\left(\overline{\theta}-\underline{\theta}\right)}$. We now show that if $\grave{\mu} \in \Lambda$ and $\mu < \grave{\mu}$, then $\mu \in \Lambda$. To show that we need to prove that $\left(\widehat{x}_{k-1}^H - \widehat{x}_k^H\right)$ is decreasing in μ. We have $\widehat{x}_0^H < \widehat{x}_1^H < \widehat{x}_2^H < \widehat{x}_3^H$ for all $\mu \leq \overline{\mu}$. To finish the proof, we need to show that $\left(\widehat{x}_{k-1}^H - \widehat{x}_k^H\right)$ is decreasing with respect to μ as well. Let's define $\widehat{x}_k^H$ as

$$\widehat{x}_k^H = (1-\mu)^{-1}\left(\widehat{x}_{k-1}^H - \left(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H\right)\widehat{a}_{k-2}\right)$$

where $\widehat{a}_k = \frac{\overline{\theta}}{\rho^k\left(\overline{\theta}-\underline{\theta}\right)}$. The proof comes from induction.

Step 1: $\frac{d\widehat{x}_3^H}{d\mu} > 0$ and $\frac{d\left(\widehat{x}_2^H - \widehat{x}_3^H\right)}{d\mu} < -2\left(1-\mu\right)^{-1}\widehat{x}_3^H$.

Step 2: Assume that $\frac{d\widehat{x}_{k-1}^H}{d\mu} > 0$ and $\frac{d\left(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H\right)}{d\mu} < -\left(k-2\right)\left(1-\mu\right)^{-1}\widehat{x}_{k-1}^H$.

Step 3: Show that $\frac{d\widehat{x}_k^H}{d\mu} > 0$ and $\frac{d\left(\widehat{x}_{k-1}^H - \widehat{x}_k^H\right)}{d\mu} < -\left(k-1\right)\left(1-\mu\right)^{-1}\widehat{x}_k^H$. Note that $\widehat{x}_{k-1}^H - \widehat{x}_k^H = (1-(1-\mu)^{-1})\widehat{x}_{k-1}^H + (1-\mu)^{-1}a_{k-2}(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H)$. So, $\frac{d\left(\widehat{x}_{k-1}^H - \widehat{x}_k^H\right)}{d\mu} < -(1-$

[3]Since m' is the same as m of DL (2008), the number of periods required to fulfill the market with product differentiation is striclty less than single good setting of this model.

$\mu)^{-2}\widehat{x}_{k-1}+(k-1)(1-\mu)^{-2}\widehat{a}_{k-2}(\widehat{x}^H_{k-2}-\widehat{x}^H_{k-1})+(1-\mu)^{-1}\widehat{a}_{k-2}\frac{d\left(\widehat{x}^H_{k-2}-\widehat{x}^H_{k-1}\right)}{d\mu} \le -(1-\mu)^{-2}(\widehat{x}^H_{k-1}-$ $(k-1)^{-2}a_{k-2}(\widehat{x}^H_{k-2}-\widehat{x}^H_{k-1})+(k-2)\widehat{a}_{k-2}\widehat{x}^H_{k-1}) < -(1-\mu)^{-2}(k-1)(\widehat{x}^H_{k-1}-\widehat{a}_{k-2}(\widehat{x}^H_{k-2}-\widehat{x}^H_{k-1}))$ $< -(1-\mu)^{-1}(k-1)\widehat{x}^H_k$ where the third inequality follows from $\widehat{a}_{k-2} > 1$.

Third, we prove that for $\mu \in \Lambda$ there exists a stationary equilibrium with unique steady state $(1,0)$. The pentad $\left\{P^H(\cdot), P^L(\cdot), t^H(\cdot), t^L(\cdot), R(\cdot)\right\}$ is defined as follows.

$$P(\cdot) = \begin{cases} \left(\overline{p}^H_m, \overline{p}^L_m\right) & \text{for} \quad x^H \in D_1,\ x^L \in C_1 \\ \left(\widehat{p}^H_m, \underline{\theta} q_L\right) & \text{for} \quad x^H \in D_2,\ x^L \in C_1' \\ \left(\overline{p}^H_k, \overline{p}^L_k\right) & \text{for} \quad x^H \in D_3,\ x^L \in C_1,\ k \in S \\ \left(\widehat{p}^H_k, \underline{\theta} q_L\right) & \text{for} \quad x^H \in D_4,\ x^L \in C_1',\ k \in S' \end{cases}$$

where $D_1 \equiv [0, \overline{y}^H_m]$, $D_2 \equiv [0, \widehat{y}^H_{m'}]$, $D_3 \equiv (\overline{y}^H_{k+1}, \overline{y}^H_k]$, $D_4 \equiv (\widehat{y}^H_{k+1}, \widehat{y}^H_k]$, $C_1 \equiv [0, \widehat{b} - x^H]$, $S \equiv \{m-1, ..., 0\}$, $S' \equiv \{m'-1, ..., 0\}$.

$$t(\cdot) = \begin{cases} \left(\overline{y}^H_{m-1}, \widehat{b} - \overline{y}^H_{m-1}\right) & \text{for} \quad x^H \in D_1,\ x^L \in C_1 \\ (\widehat{y}^H_{m'-1}, 1 - \widehat{y}^H_{m'-1}) & \text{for} \quad x^H \in D_1,\ x^L \in C_1' \\ \left(\overline{y}^H_{m-1}, \widehat{b} - \overline{y}^H_{m-1}\right) & \text{for} \quad x^H \in D_2,\ x^L \in C_2 \\ (\widehat{y}^H_{k-1}, 1 - \widehat{y}^H_{k-1}) & \text{for} \quad x^H \in D_2,\ x^L \in C_2' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_1,\ x^L \in C_3 \\ (\widehat{y}^H_{m'-1}, 1 - \widehat{y}^H_{m'-1}) & \text{for} \quad x^H \in D_1,\ x^L \in C_3' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_3,\ x^L \in C_3 \\ (\widehat{y}^H_{k'-1}, 1 - \widehat{y}^H_{k'-1}) & \text{for} \quad x^H \in D_3,\ x^L \in C_3' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_4,\ x^L \in C_3 \\ (\widehat{y}^H_{m'-1}, 1 - \widehat{y}^H_{m'-1}) & \text{for} \quad x^H \in D_4,\ x^L \in C_3' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_4,\ x^L \in C_4 \\ (\widehat{y}^H_{k'-1}, 1 - \widehat{y}^H_{k'-1}) & \text{for} \quad x^H \in D_4,\ x^L \in C_4' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_5,\ x^L \in C_4 \\ (1,0) & \text{for} \quad x^H \in D_5,\ x^L \in C_4' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_6,\ x^L \in C_5 \\ (\widehat{y}^H_{k'-1}, 1 - \widehat{y}^H_{k'-1}) & \text{for} \quad x^H \in D_6,\ x^L \in C_5' \\ \left(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1}\right) & \text{for} \quad x^H \in D_7,\ x^L \in C_5 \\ (1,0) & \text{for} \quad x^H \in D_7,\ x^L \in C_6' \end{cases}$$

where $k,\ t = m-1,\dots 2,\ k' = m'-1,\dots 2$, and $D_1 \equiv [0,\overline{x}^H_m] \cap [0,\widetilde{x}^H_{m'}]$, $D_2 \equiv [0,\overline{x}^H_m] \cap (\widetilde{x}^H_{k+1},\widetilde{x}^H_k]$, $D_3 \equiv [0,\overline{x}^H_m] \cap (\widetilde{x}^H_{k'+1},\widetilde{x}^H_{k'}]$, $D_4 \equiv (\overline{x}^H_{t+1},\overline{x}^H_t] \cap (\widetilde{x}^H_{k'+1},\widetilde{x}^H_{k'}]$, $D_5 \equiv (\overline{x}^H_{t+1},\overline{x}^H_t] \cap (\widetilde{x}^H_2, 1-\mu]$, $D_6 \equiv (\overline{x}^H_2, 1-\mu] \cap (\widetilde{x}^H_{k'+1},\widetilde{x}^H_{k'}]$, $D_7 \equiv (\overline{x}^H_2, 1-\mu] \cap (\widetilde{x}^H_2, 1-\mu]$, $C_1 \equiv [0, t\left(\overline{x}^H_m, x^H\right)]$ $\cap T(x'\left(\overline{y}^H_{m-1},\widetilde{y}^H_{m'-1}\right), x^H)$, $C_2 \equiv [0, t\left(\overline{x}^H_m, x^H\right)] \cap T(x'\left(\overline{y}^H_{m-1},\widetilde{y}^H_{k-1}\right), x^H)$, $C_3 \equiv [t\left(\overline{x}^H_{k+1}, x^H\right),$ $t\left(\overline{x}^H_k, x^H\right)] \cap T(x'\left(\overline{y}^H_{k-1},\widetilde{y}^H_{m'-1}\right), x^H)$, $C_4 \equiv [t\left(\overline{x}^H_{k+1}, x^H\right), t\left(\overline{x}^H_k, x^H\right)] \cap T(x'\left(\overline{y}^H_{k-1}, 1\right), x^H)$, $C_5 \equiv [t\left(\overline{x}^H_{k+1}, x^H\right), t\left(\overline{x}^H_k, x^H\right)] \cap T(x'\left(\overline{y}^H_{k-1}, 1\right), x^H)$. Moreover, we have

$$
\begin{aligned}
R\left(x^H, x^L\right) = \left(t^H\left(x^H, x^L\right) - x^H\right) P^H\left(t^H\left(x^H, x^L\right), t^L\left(x^H, x^L\right)\right) \\
+ \left(t^L\left(x^H, x^L\right) - x^L\right) P^L\left(t^H\left(x^H, x^L\right), t^L\left(x^H, x^L\right)\right) \\
+ \delta R\left((1-\mu)\, t^H\left(x^H, x^L\right), (1-\mu)\, t^L\left(x^H, x^L\right)\right)
\end{aligned}
$$

where $\overline{p}^H_k = \overline{\theta} q_H - \rho^k(\overline{\theta} q_H - \underline{\theta} q_H)$, $\overline{p}^L_k = \overline{\theta} q_L - \rho^k(\overline{\theta} q_L - \underline{\theta} q_L)$, and $\widehat{p}^H_k = \overline{\theta} q_H - (\overline{\theta} - \underline{\theta})(\rho^k q_H + (1-\rho^k) q_L)$ for all $k = 0, ..., m$. We define $\overline{y}^S_k$ as $\overline{y}^S_k = \dfrac{\overline{x}^S_k}{1-\mu}$, $S = H, L$. Moreover, the threshold values on x^L are defined as

$$
t\left(\overline{x}^H_k, x^H\right) = \overline{x}^H_k \frac{q_H - q_L}{q_L} + (1-\mu)\, \widehat{b} - x^H \frac{q_H}{q_L}
$$

and

$$
T\left(x'\left(\overline{y}^H_k, \widehat{y}^H_{k'}\right), x^H\right) = A\left(k\right) + \overline{y}^H_k B\left(k,k\right) - \widehat{y}^H_{k'} B\left(k',k\right) - x^H C\left(k',k\right) + D\left(k\right) \Delta R\left(\overline{y}^H_k, \widehat{y}^H_{k'}\right)
$$

where the functions of k and k' are:

$A\left(k\right) = \dfrac{\widehat{b\theta} - \underline{\theta} - \rho^k \widehat{b} \Delta\theta}{\left(1-\rho^k\right)\Delta\theta}$, $B\left(t,k\right) = \dfrac{\left(\overline{\theta} - \rho^t \Delta\theta\right)\Delta q}{\left(1-\rho^k\right)\Delta\theta q_L}$, $C\left(t,k\right) = \dfrac{\left(\rho^t - \rho^k\right) q_H + \left(1-\rho^t\right) q_L}{\left(1-\rho^k\right) q_L}$, $D\left(k\right) = \dfrac{\delta}{\left(1-\rho^k\right)\Delta\theta q_L}$, and $\Delta R\left(\overline{y}^H_k, \widehat{y}^H_{k'}\right) = R((1-\mu)\overline{y}^H_k, (1-\mu)(\widehat{b} - \overline{y}^H_k)) - R((1-\mu)\widehat{y}^H_{k'}, (1-\mu)(1-\widehat{y}^H_{k'}))$.

Let's define $\overline{\Pi}((\overline{y}^H, \widehat{b} - \overline{y}^H); (x^H, x^L)) = (\overline{y}^H - x^H) P^H(\overline{y}^H, \widehat{b} - \overline{y}^H) + (\widehat{b} - \overline{y}^H - x^L) P^L(\overline{y}^H, \widehat{b} - \overline{y}^H) + \delta R((1-\mu)\overline{y}, (1-\mu)(\widehat{b} - \overline{y}^H))$ and $\widehat{\Pi}((\widehat{y}^H, 1-\widehat{y}^H); (x^H, x^L)) = (\widehat{y}^H - x^H) P^H(\widehat{y}^H, 1-\widehat{y}^H) + (1-\widehat{y}^H - x^L) P^L(\widehat{y}^H, 1-\widehat{y}^H) + \delta R((1-\mu)\widehat{y}, (1-\mu)(1-\widehat{y}^H))$. The state variable $\overline{x}^H_k$ is set such that

$$\overline{\Pi}\left(\left(\overline{y}_{k-1}^{H},\widehat{b}-\overline{y}_{k-1}^{H}\right);\left(\overline{x}_{k}^{H},(1-\mu)\,\widehat{b}-\overline{x}_{k}^{H}\right)\right)=\overline{\Pi}\left(\left(\overline{y}_{k-2}^{H},\widehat{b}-\overline{y}_{k-2}^{H}\right);\left(\overline{x}_{k}^{H},(1-\mu)\,\widehat{b}-\overline{x}_{k}^{H}\right)\right).$$

Let's define

$$\overline{h}_{k}\left(x^{H},x^{L}\right)=\overline{\Pi}\left(\left(\overline{y}_{k-1}^{H},\widehat{b}-\overline{y}_{k-1}^{H}\right);\left(x^{H},x^{L}\right)\right)-\overline{\Pi}\left(\left(\overline{y}_{k-2}^{H},\widehat{b}-\overline{y}_{k-2}^{H}\right);\left(x^{H},x^{L}\right)\right)$$

then by definition of $\overline{x}_{k}^{H}$ we have $\overline{h}_{k}\left(\left(\overline{x}_{k}^{H},(1-\mu)\,\widehat{b}-\overline{x}_{k}^{H}\right)\right)=0$. Since

$$\frac{\partial\overline{h}_{k}\left(x^{H},x^{L}\right)}{\partial x^{H}}=-P^{H}\left(\overline{y}_{k-1}^{H},\widehat{b}-\overline{y}_{k-1}^{H}\right)+P^{H}\left(\overline{y}_{k-2}^{H},\widehat{b}-\overline{y}_{k-2}^{H}\right)<0,$$

for $x^{H}<\overline{x}_{k}^{H}$ we have

$$\overline{h}_{k}\left(x^{H},(1-\mu)\,\widehat{b}-\overline{x}_{k}^{H}\right)>0$$

and for $x^{H}>\overline{x}_{k}^{H}$ we have,

$$\overline{h}_{k}\left(x^{H},(1-\mu)\,\widehat{b}-\overline{x}_{k}^{H}\right)<0.$$

Now, suppose that $x^{H}\in\left(\overline{x}_{k+1}^{H},\overline{x}_{k}^{H}\right]$. Since $\dfrac{\partial\overline{h}_{k}\left(x^{H},x^{L}\right)}{\partial x^{L}}=-P^{L}(\overline{y}_{k-1}^{H},\widehat{b}-\overline{y}_{k-1}^{H})+P^{L}(\overline{y}_{k-2}^{H},\widehat{b}-\overline{y}_{k-2}^{H})<0$, $\exists\ \grave{x}^{L}>0$ such that $\overline{h}_{k}\left(x^{H},\grave{x}^{L}\right)=0$ for $x^{H}\in\left(\overline{x}_{k+1}^{H},\overline{x}_{k}^{H}\right]$. At $\grave{x}_{L}$ we have

$$\frac{\partial\overline{h}_{k}\left(x^{H},\grave{x}^{L}\right)}{\partial x^{H}}\left(\overline{x}_{k}^{H}-x^{H}\right)=\frac{\partial\overline{h}_{k}\left(x^{H},\grave{x}^{L}\right)}{\partial x^{L}}\left(\grave{x}^{L}-\left((1-\mu)\,\widehat{b}-\overline{x}_{k}^{H}\right)\right).$$

Thus, for all $x^{L}>\overline{x}_{k}^{H}\dfrac{\Delta q}{q_{L}}+(1-\mu)\,\widehat{b}-x^{H}\dfrac{q_{H}}{q_{L}}$ we have

$$\overline{h}_{k}\left(x^{H},x^{L}\right)<0$$

and for all $x^{L}<\overline{x}_{k}^{H}\dfrac{\Delta q}{q_{L}}+(1-\mu)\,\widehat{b}-x^{H}\dfrac{q_{H}}{q_{L}}$ we have

$$\overline{h}_{k}\left(x^{H},x^{L}\right)>0.$$

Therefore, since $\overline{\Pi}\left(\left(y^{H},y^{L}\right),\left(x^{H},x^{L}\right)\right)$ is strictly increasing with respect to y^{H} and y^{L} on any of the intervals defined above, given $x^H \in \left(\overline{x}^H_{k+1}, \overline{x}^H_k\right]$, for all $x^L \in (t(\overline{x}^H_{k+1}, x^H), t(\overline{x}^H_k, x^H)]$ where $t(\overline{x}^H_k, x^H) = \overline{x}^H_k \frac{q_H - q_L}{q_L} + (1-\mu)\widehat{b} - x^H \frac{q_H}{q_L}$, the optimum choice of the seller among $\{(\overline{y}^H_0, \widehat{b} - \overline{y}^H_0), (\overline{y}^H_1, \widehat{b} - \overline{y}^H_1), \ldots, (\overline{y}^H_m, \widehat{b} - \overline{y}^H_m)\}$ is $(\overline{y}^H_{k-1}, \widehat{b} - \overline{y}^H_{k-1})$.

The state variable $\widehat{x}^H_k$ is set such that

$$\widehat{\Pi}\left(\left(\widehat{y}^H_{k-1}, 1-\widehat{y}^H_{k-1}\right); \left(\widehat{x}^H_k, 1-\mu-\widehat{x}^H_k\right)\right) = \widehat{\Pi}\left(\left(\widehat{y}^H_{k-2}, 1-\widehat{y}^H_{k-2}\right); \left(\widehat{x}^H_k, 1-\mu-\widehat{x}^H_k\right)\right).$$

Let's define

$$\widehat{h}_k\left(x^H, x^L\right) = \widehat{\Pi}\left(\left(\widehat{y}^H_{k-1}, 1-\widehat{y}^H_{k-1}\right); \left(x^H, x^L\right)\right) - \widehat{\Pi}\left(\left(\widehat{y}^H_{k-2}, 1-\widehat{y}^H_{k-2}\right); \left(x^H, x^L\right)\right)$$

then $\widehat{h}_k\left(\left(\widehat{y}^H_k, 1-\mu-\widehat{y}^H_k\right)\right) = 0$.

Since $\frac{\partial \widehat{h}_k\left(x^H, x^L\right)}{\partial x^H} = -P^H\left(\widehat{y}^H_{k-1}, 1-\widehat{y}^H_{k-1}\right) + P^H\left(\widehat{y}^H_{k-2}, 1-\widehat{y}^H_{k-2}\right) < 0$, for $x^H < \widehat{x}^H_k$,

$$\widehat{h}_k\left(x^H, x^L\right) > 0$$

and for $x^H > \widehat{x}^H_k$,

$$\widehat{h}_k\left(x^H, x^L\right) < 0.$$

Since, $\frac{\partial \widehat{h}_k\left(x^H, x^L\right)}{\partial x^L} = 0$, the seller would like to choose highest possible x^L.Therefore, given $x^H \in \left(\widehat{x}^H_{k+1}, \widehat{x}^H_k\right]$, for all x^L the optimum choice of the seller among $\{(\widehat{y}^H_0, 1-\widehat{y}^H_0), (\widehat{y}^H_1, 1-\widehat{y}^H_1), \ldots, (\widehat{y}^H_m, 1-\widehat{y}^H_m)\}$ is $(\widehat{y}^H_{k-1}, 1-\widehat{y}^H_{k-1})$.

Let's define

$$d_k\left(x^H, x^L\right) = \overline{\Pi}\left(\left(\overline{y}^H_k, \widehat{b} - \overline{y}^H_k\right); \left(x^H, x^L\right)\right) - \widehat{\Pi}\left(\left(\widehat{y}^H_{k'}, 1-\widehat{y}^H_{k'}\right); \left(x^H, x^L\right)\right).$$

Suppose $x^H \in \left(\overline{x}^H_{k+1}, \overline{x}^H_k\right] \cap \left(\widehat{x}^H_{k'+1}, \widehat{x}^H_{k'}\right]$, then there exists x^L where $d_k\left(x^H, x^L\right) = 0$. Moreover, $\frac{\partial d_k\left(x^H, x^L\right)}{\partial x^L} = -\left(1-\rho^k\right)\left(\overline{\theta} - \underline{\theta}\right) q_L < 0$. It follows that for a given $x^H \in \left(\overline{x}^H_{k+1}, \overline{x}^H_k\right] \cap \left(\widehat{x}^H_{k'+1}, \widehat{x}^H_{k'}\right]$ if $x^L < T\left(x'\left(\overline{y}^H_k, \widehat{y}^H_{k'}\right), x^H\right)$ then

$$d_k\left(x^H, x^L\right) > 0$$

and if $x^L > T\left(x'\left(\overline{y}^H_{k-1}, \widetilde{y}^H_{k'-1}\right), x^H\right)$ then

$$d_k\left(x^H, x^L\right) < 0.$$

Since $\overline{\Pi}((\overline{y}^H, \widehat{b} - \overline{y}^H); (x^H, x^L))$ is strictly increasing in $\overline{y}^H$ on any of the intervals $[0, \overline{y}_m]$, $(\overline{y}_m, \overline{y}_{m-1}]$, $\ldots$, $(\overline{y}_1, 1]$ and $\widehat{\Pi}((\widehat{y}^H, 1 - \widehat{y}^H); (x^H, x^L))$ is strictly increasing in $\widehat{y}^H$ on any of the intervals $[0, \widehat{y}_{m'}], (\widehat{y}_m, \widehat{y}_{m-1}], \ldots, (\widehat{y}_1, 1]$. This implies that $(t^H(x^H, x^L),\ t^L(x^H, x^L))$ is the smallest solution of the monopolist's optimization problem. Buyer's arbitrage equations are also satisfied. It follows that $\{P^H(\cdot),\ P^L(\cdot),\ t^H(\cdot),\ t^L(\cdot),\ R(\cdot)\}$ is a stationary triplet for $\mu \in \Lambda$.

Uniqueness:

We now consider any stationary equilibrium whose unique steady state is $(1, 0)$. Let $\{P_0^H(\cdot),$ $P_0^L(\cdot),\ t_0^H(\cdot),\ t_0^L(\cdot),\ R_0(\cdot)\}$ be the stationary triplet of that equilibrium. We show that $\{P^H(\cdot),$ $P^L(\cdot),\ t^H(\cdot),\ t^L(\cdot),\ R(\cdot)\} = \{P_0^H(\cdot), P_0^L(\cdot), t_0^H(\cdot), t_0^L(\cdot), R_0(\cdot)\}$.

First, we show that $t_0^H(\overline{x}_1^H, (1-\mu)\widehat{b} - \overline{x}_1^H) = 1$ and $t_0^L(\overline{x}_1^H, (1-\mu)\widehat{b} - \overline{x}_1^H) = 0$. Since $(\widehat{b}, 0)$ and $(\widehat{b}, 1 - \widehat{b})$ are not steady states, we must have $P^H\left(\widehat{b}\right) < \overline{\theta} q_H,\ P^L\left(\widehat{b}\right) < \overline{\theta} q_L$. Moreover, since $(1, 0)$ is a steady state, we must have $P^H(1) = \underline{\theta} q_H,\ P^L(1) = \underline{\theta} q_L$. Now, consider the arbitrage equations at $\left(\widehat{b}, 0\right)$. Due to the arbitrage equation for high quality good $\overline{\theta} q_H - P_0^H\left(\widehat{b}\right) = \rho(\overline{\theta} q_H - P_0^H(t_0^H((1-\mu)\widehat{b}, 0)))$, we have $P_0^H(t_0^H((1-\mu)\widehat{b}, 0)) < P_0^H(\widehat{b})$. Since $P_0^H(\cdot)$ is non-increasing, the stock of high quality good increases at $\widehat{b}$: $t_0^H((1-\mu)\widehat{b}, 0) > \widehat{b}$. Similarly, due to the arbitrage equation for low quality good $\overline{\theta} q_L - P_0^L(\widehat{b}) = \rho(\overline{\theta} q_L - P_0^L(t_0^H((1-\mu)\widehat{b}, 0) + t_0^L((1-\mu)\widehat{b}, 0)))$, we have $P_0^H(t_0^H((1-\mu)\widehat{b}, 0) + t_0^L((1-\mu)\widehat{b}, 0)) < P_0^L(\widehat{b})$. Since $P_0^L(\cdot)$ is non-increasing, total stock of durable good increases at $\widehat{b}$ is $t_0^H((1-\mu)\widehat{b}, 0) + t_0^L((1-\mu)\widehat{b}, 0) > \widehat{b}$. Moreover, at any $b \in [0, 1]$, marginal benefit of high quality good is greater than marginal cost of low quality good. Thus, we have $t_0^H\left((1-\mu)\widehat{b}, 0\right) = 1$ and $t_0^L\left((1-\mu)\widehat{b}, 0\right) = 0$.

Second, we show that $t_0^H\left(\widehat{x}_1^H, 1 - \mu - \widehat{x}_1^H\right) = 1$ and $t_0^L\left(\widehat{x}_1^H, 1 - \mu - \widehat{x}_1^H\right) = 0$. We know that we must have $P^H\left(\widehat{b}\right) < \overline{\theta} q_H,\ P^L\left(\widehat{b}\right) < \overline{\theta} q_L$ and $P^H(1) = \underline{\theta} q_H,\ P^L(1) = \underline{\theta} q_L$. Due to the arbitrage equation for the high quality good $\overline{\theta} q_H - P_0^H(\widehat{b}) = \rho(\overline{\theta} q_H - P_0^H(t_0^H((1-\mu)\widehat{b}, (1-\mu)(1-\widehat{b}))))$, we have $P_0^H(t_0^H((1-\mu)\widehat{b}, (1-\mu)(1-\widehat{b}))) < P_0^H(\widehat{b})$. Since $P_0^H(\cdot)$ is non-increasing, the stock of high quality good increases at $\widehat{b}$: $t_0^H\left((1-\mu)\widehat{b}, (1-\mu)\left(1 - \widehat{b}\right)\right) > \widehat{b}$. Similarly, due to the arbitrage equation for low quality good $\underline{\theta}q_L - P_0^L(1) = \rho(\underline{\theta}q_L - P_0^H(t_0^H((1-\mu)\widehat{b},(1-\mu)(1-\widehat{b})+t_0^L((1-\mu)\widehat{b},(1-\mu)(1-\widehat{b})))$, we have $P_0^H((t_0^H(1-\mu)\widehat{b},(1-\mu)(1-\widehat{b}))+t_0^L(1-\mu)\widehat{b},(1-\mu)(1-\widehat{b})) < P_0^L(1)$. Since $P_0^L(\cdot)$ is non-increasing, total stock of durable good increases at $\widehat{b}$, we have $t_0^H\left((1-\mu)\widehat{b},(1-\mu)\left(1-\widehat{b}\right)\right) + t_0^L\left((1-\mu)\widehat{b},(1-\mu)\left(1-\widehat{b}\right)\right) \geq 1$. Therefore, we must have $t_0^H\left((1-\mu)\widehat{b},(1-\mu)\left(1-\widehat{b}\right)\right) + t_0^L\left((1-\mu)\widehat{b},(1-\mu)\left(1-\widehat{b}\right)\right) = 1$ Moreover, at any b, marginal benefit of high quality good is greater than marginal cost of low quality good. Thus, we have $t_0^H\left((1-\mu)\widehat{b},(1-\mu)\left(1-\widehat{b}\right)\right) = 1$ and $t_0^L\left((1-\mu)\widehat{b},(1-\mu)\left(1-\widehat{b}\right)\right) = 0$.

Third, we show that for all $x^H \in (\overline{x}_1^H - \epsilon, \overline{x}_1^H]$ we have $t_0^H(x^H,(1-\mu)\widehat{b}-x^H) = 1$ and $t_0^L(x^H,(1-\mu)\widehat{b}-x^H) = 0$. By the left continuity of $t_0^H(x^H,x^L)$, $\exists\, \epsilon > 0$ such that $t_0^H(x^H,(1-\mu)\widehat{b}-x^H) = 1$ and $t_0^L(x^H,(1-\mu)\widehat{b}-x^H = 0$ for all $x^H \in (\overline{x}_1^H - \epsilon, \overline{x}_1^H]$. The arbitrage equations for $y^H \in (\overline{y}_1^H - \dfrac{\epsilon}{1-\mu}, \overline{y}_1^H]$ and $y^L \leq \widehat{b} - y^H$ are

for high quality good: $f^H\left(y^H\right) - P_0^H\left(y^H\right) = \rho\left(f^H\left(y^H\right) - P_0^H(1)\right)$,

for low quality good: $f^L\left(y^H+y^L\right) - P_0^L\left(y^H+y^L\right) = \rho\left(f^L\left(y^H\right) - P_0^H(1)\right)$.

Since $y^H \leq \widehat{b}$ and $y^H+y^L \leq \widehat{b}$, the arbitrage equations imply that $P_0^H\left(y^H\right) = \overline{\theta}q_H - \rho\left(\overline{\theta}-\underline{\theta}\right)q_H$ and $P_0^L\left(y^H+y^L\right) = \overline{\theta}q_L - \rho\left(\overline{\theta}-\underline{\theta}\right)q_L$ where $y^H+y^L \leq \widehat{b}$.

Fourth, we show that for all $x^H \in (\widehat{x}_1^H - \epsilon, \widehat{x}_1^H]$, and $x^L \in ((1-\mu)\widehat{b}-x^H-\gamma,(1-\mu)\widehat{b}-x^H]$, $t_0^H\left(x^H,x^L\right) = 1$ and $t_0^L\left(x^H,x^L\right) = 0$. By the left continuity of $t_0^L\left(x^H,.\right)$, $\exists\, \gamma > 0$ such that $t_0^H\left(x^H,x^L\right) = 1$ and $t_0^L\left(x^H,x^L\right) = 0$ for all $x^H \in (\widehat{x}_1^H - \epsilon, \widehat{x}_1^H]$ and all $x^L \in ((1-\mu)\widehat{b}-x^H-\gamma,((1-\mu))\widehat{b}-x^H]$. The arbitrage equations for $y^H \in (\widehat{y}_1^H - \dfrac{\epsilon}{1-\mu}, \widehat{y}_1^H]$ and $y^L \in ((1-\mu)\widehat{b}-y^H-\dfrac{\gamma}{1-\mu},(1-\mu)\widehat{b}-y^H]$ are

for high quality good: $f^H\left(y^H\right) - P_0^H\left(y^H\right) = \rho\left(f^H\left(y^H\right) - P_0^H(1)\right)$

for low quality good: $f^L\left(y^H+y^L\right) - P_0^L\left(y^H+y^L\right) = \rho\left(f^L\left(y^H\right) - P_0^H(1)\right)$. Since $y^H \leq \widehat{b}$ and $y^H+y^L \leq \widehat{b}$, the arbitrage equations imply that $P_0^H\left(y^H\right) = \overline{\theta}q_H - \rho\left(\overline{\theta}-\underline{\theta}\right)q_H$ and $P_0^L\left(y^H+y^L\right) = \overline{\theta}q_L - \rho\left(\overline{\theta}-\underline{\theta}\right)q_L$ where $y^H+y^L \leq \widehat{b}$.

Fifth, we show that $\inf\left(\overline{x}_1^H - \epsilon\right) = \overline{x}_2^H$.Let's define $\grave{x}_2^H = \inf\{x^H : t_0^H\left(x^H,x^L\right) = 1, x^L \in ((1-\mu)\widehat{b}-x^H-\gamma,(1-\mu)\widehat{b}-x^H]\}$. We know that when $x^L = (1-\mu)\widehat{b}-\overline{x}_2^H$, if $x^H < \overline{x}_2^H$ then $\Pi\left(\overline{y}_1;\left(x^H,x^L\right)\right) > \Pi\left(\overline{y}_0;\left(x^H,x^L\right)\right)$. Otherwise, if $x^H > \overline{x}_2^H$ then $\Pi\left(\overline{y}_1;\left(x^H,x^L\right)\right) < \Pi\left(\overline{y}_0;\left(x^H,x^L\right)\right)$. Since $\Pi_0(\overline{y}_1;(x^H,x^L)) = \Pi(\overline{y}_1;(x^H,x^L))$ and $\Pi_0(\overline{y}_0;(x^H,x^L)) = \Pi\ (\overline{y}_0;(x^H,x^L))$, we must have $\grave{x}_2^H = \overline{x}_2^H$.

Sixth, we show that $\inf((1-\mu)\widehat{b}-\overline{x}^H-\gamma)=t(\overline{x}_2^H,x^H)$. Let's define $\grave{x}_2^L=\inf\{x^L: t_0^L(x^H,x^L)=0,\ x^H\in(\overline{x}_2^H,1-\mu]\}$. We know that for a given x^H, if $x^L<t(\overline{x}_2^H,x^H)$ then $\Pi(\overline{y}_1;(x^H,x^L))>\Pi(\overline{y}_0;(x^H,x^L))$; whereas if $x^L>t(\overline{x}_2^H,x^H)$ then $\Pi(\overline{y}_1;(x^H,x^L))<\Pi(\overline{y}_0;(x^H,x^L))$. Since $\Pi_0(\overline{y}_1;(x^H,x^L))=\Pi(\overline{y}_1;(x^H,x^L))$ and $\Pi_0(\overline{y}_0;(x))=\Pi\ (\overline{y}_0;\ (x))$, we must have $\grave{x}_2^L=t(\overline{x}_2^H,x^H)$.

Seventh, suppose $t^H\left(x^H,x^L\right)=\overline{y}_{k-1}^H$ and $t^L\left(x^H,x^L\right)=\widehat{b}-\overline{y}_{k-1}^H$ for $x^H\in\left(\overline{x}_2^H,1-\mu\right]$ and $x^L\in(t(\overline{x}_{k+1}^H,x^H),\ t(\overline{x}_k^H,x^H)]$ for $k=1,2,\ldots,i$. Show that it holds for $i+1$. We have $t^H\left(x^H,x^L\right)=\overline{y}_{i-1}^H$ and $t^L\left(x^H,x^L\right)=\widehat{b}-\overline{y}_{i-1}^H$ for $x^H\in\left(\overline{x}_2^H,1-\mu\right]$ and $x^L\in(t(\overline{x}_{i+1}^H,x^H),\ t(\overline{x}_i^H,x^H)]$. Now, we prove that $\{P^H(\cdot),\ P^L(\cdot),\ t^H(\cdot),\ t^L(\cdot),\ R(\cdot))\}=\{P_0^H(\cdot),\ P_0^L(\cdot),\ t_0^H(\cdot),\ t_0^L(\cdot),\ R_0(\cdot)\}$ holds for $i+1$ for $x^H\in(\overline{x}_2^H,1-\mu]$. We must have $P_0^H\left(\overline{y}_{i+1}^H,\overline{y}_{i+1}^L\right)<\overline{\theta}q_H$ and $P_0^L\left(\overline{y}_{i+1}^H,\overline{y}_{i+1}^L\right)<\overline{\theta}q_L$. This implies that we have $P_0^H\left(\overline{y}_{i+1}^H\right)>P_0^H\left(t_0^H\left(\overline{x}_{i+1}^H,\overline{x}_{i+1}^L\right)\right)$ and $P_0^L\left(\overline{y}_{i+1}^H+\overline{y}_{i+1}^L\right)>P_0^L\left(t_0^H\left(\overline{x}_{i+1}^H,\overline{x}_{i+1}^L\right)+t_0^L\left(\overline{x}_{i+1}^H,\overline{x}_{i+1}^L\right)\right)$. Since $P_0^H(\cdot)$ and $P_0^L(\cdot)$ are non-increasing, stock of high quality good and total stock increase when $\overline{x}_{i+1}^H=(1-\mu)\,\overline{y}_{i+1}^H$ and $\overline{x}_{i+1}^L=(1-\mu)\,\overline{y}_{i+1}^L$. That is, $t_0^H\left(\overline{x}_{i+1}^H,\overline{x}_{i+1}^L\right)\geq\overline{y}_{i+1}^H$ and $t_0^H\left(x_{i+1}^H,x_{i+1}^L\right)+t_0^L\left(x_{i+1}^H,x_{i+1}^L\right)\geq\overline{y}_{i+1}^H+\overline{y}_{i+1}^L$. Since $\overline{x}_{i+1}^H$ and $\overline{x}_{i+1}^L$ are defined as

$$\overline{x}_{i+1}^H=\inf\left\{x^H: t_0\left(x^H,x^L\right)=\overline{y}_{i-1}^H \text{ where } x^L=(1-\mu)\,\widehat{b}-x^H\right\}$$

and

$$\overline{x}_{i+1}^L=\inf\left\{x^L: t_0\left(x^H,x^L\right)=\widehat{b}-\overline{y}_{i-1}^H \text{ where } x^H\in\left(\overline{x}_2^H,1-\mu\right]\right\},$$

this imply that $\lim_{\epsilon\to 0}t_0^H\left(\overline{x}_{i+1}^H-\epsilon,\overline{x}_{i+1}^L\right)=\overline{y}_i^H$ and $\lim_{\epsilon\to 0}t_0^L\left(\overline{x}_{i+1}^H,\overline{x}_{i+1}^L-\epsilon\right)=\widehat{b}-\overline{y}_i^H$. Now, we can use left continuity of $t^i(\cdot)$ functions to define

$$\grave{x}_{i+2}^H=\inf\left\{x^H: t_0\left(x^H,x^L\right)=\overline{y}_i^H \text{ where } x^L=(1-\mu)\,\widehat{b}-x^H\right\}$$

and

$$\grave{x}_{i+2}^L=\inf\left\{x^L: t_0\left(x^H,x^L\right)=\overline{y}_i^L \text{ where } x^H\in\left(\overline{x}_2^H,1-\mu\right]\right\}.$$

Now, to finish the proof we need to show that $\grave{x}_{i+2}^H=\overline{x}_{i+2}^H$ and $\grave{x}_{i+2}^L=\overline{x}_{i+2}^L$.We know that when $x^L = (1-\mu)\widehat{b} - \overline{x}^H_{i+2}$ if $x^H < \overline{x}^H_{i+2}$ then $\Pi\left(\overline{y}_{i+1}; \left(x^H, x^L\right)\right) > \Pi\left(\overline{y}_i; \left(x^H, x^L\right)\right)$; otherwise, if $x^H > \overline{x}^H_{i+2}$ then $\Pi\left(\overline{y}_{i+1}; \left(x^H, x^L\right)\right) < \Pi\left(\overline{y}_i; \left(x^H, x^L\right)\right)$. Since $\Pi_0\left(\overline{y}_{i+1}; \left(x^H, x^L\right)\right) = \Pi\left(\overline{y}_{i+1}; \left(x^H, x^L\right)\right)$ and $\Pi_0\left(\overline{y}_i; \left(x^H, x^L\right)\right) = \Pi\left(\overline{y}_i; \left(x^H, x^L\right)\right)$ then we must have $\grave{x}^H_{i+2} = \overline{x}^H_{i+2}$.

Similarly, for a given x^H, if $x^L < t\left(\overline{x}^H_{i+2}, x^H\right)$ then $\Pi\left(\overline{y}_{i+1}; \left(x^H, x^L\right)\right) > \Pi\left(\overline{y}_i; \left(x^H, x^L\right)\right)$; whereas if $x^L > t\left(\overline{x}^H_{i+2}, x^H\right)$ then $\Pi\left(\overline{y}_{i+1}; \left(x^H, x^L\right)\right) < \Pi\left(\overline{y}_i; \left(x^H, x^L\right)\right)$. Since Π_0 $(\overline{y}_{i+1}; (x^H, x^L)) = \Pi(\overline{y}_{i+1}; (x^H, x^L))$ and $\Pi_0(\overline{y}_i; (x^H, x^L)) = \Pi(\overline{y}_i; (x^H, x^L))$, we must have $\grave{x}^L_{i+2} = t\left(\overline{x}^H_{i+2}, x^H\right)$.

Eighth,we show that for all $x^H \in \left(\widehat{x}^H_1 - \epsilon, \widehat{x}^H_1\right]$, we have $t^H_0\left(x^H, 1-\mu-x^H\right) = 1$ and $t^L_0\left(x^H, 1-\mu-x^H\right) = 0$. By the left continuity of $t^H_0\left(x^H, x^L\right)$, $\exists\, \epsilon > 0$ such that $t^H_0(x^H, 1-\mu-x^H) = 1$ and $t^L_0\left(x^H, 1-\mu-x^H\right) = 0$ for all $x^H \in \left(\widehat{x}^H_1 - \epsilon, \widehat{x}^H_1\right]$. Since $y^H \leq \widehat{b}$ and $y^H + y^L > \widehat{b}$, the arbitrage equations *for high quality good* $f^H\left(y^H\right) - P^H_0\left(y^H\right) = \rho\left(f^H\left(y^H\right) - P^H_0(1)\right)$ and *for low quality good* $f^L(y^H+y^L) - P^L_0(y^H+y^L) = \rho(f^L(y^H) - P^H_0(1))$ imply that $P^H_0\left(y^H\right) = \overline{\theta}q_H - \rho\left(\overline{\theta} - \underline{\theta}\right)q_H$ and $P^L_0\left(y^H+y^L\right) = \underline{\theta}q_L$ where $y^H + y^L > \widehat{b}$. Now, consider buyer $b \in \left(\widehat{y}^H_1 - \dfrac{\epsilon}{1-\mu}, \widehat{y}^H_1\right]$. Buyer b strictly prefers low quality good for all $p^H \geq \overline{\theta}q_H - \left(\overline{\theta} - \underline{\theta}\right)\left(\rho q_H + (1-\rho)q_L\right)$. The monopolist has to leave some rent to high type buyers when $y^H + y^L > \widehat{b}$. Hence, when $y^H \in \left(\widehat{y}^H_1 - \dfrac{\epsilon}{1-\mu}, \widehat{y}^H_1\right]$ and $y^L > \widehat{b} - y^H$

$$P^H\left(y^H\right) = \overline{\theta}q_H - \left(\overline{\theta} - \underline{\theta}\right)\left(\rho q_H + (1-\rho)q_L\right)$$

and

$$P^L\left(y^L\right) = \underline{\theta}q_L.$$

Ninth, we show that for all $x^H \in \left(\widehat{x}^H_1 - \epsilon, \widehat{x}^H_1\right]$, and $x^L \in \left(1-\mu-\widehat{x}^H - \gamma, 1-\mu-\widehat{x}^H_1\right]$, we have $t^H_0\left(x^H, x^L\right) = 1$ and $t^L_0\left(x^H, x^L\right) = 0$. By the left continuity of $t^L_0\left(x^H, .\right)$ $\exists\, \gamma > 0$ such that $t^H_0\left(x^H, x^L\right) = 1$ and $t^L_0\left(x^H, x^L\right) = 0$ for all $x^H \in \left(\widehat{x}^H_1 - \epsilon, \widehat{x}^H_1\right]$ and all $x^L \in \left(1-\mu-x^H-\gamma, 1-\mu-x^H\right]$. Since $y^H \leq \widehat{b}$ and $y^L \in (1-y^H-\dfrac{\gamma}{1-\mu}, 1-y^H]$ and $y^H + y^L > \widehat{b}$, the arbitrage equations *for high quality good* $f^H\left(y^H\right) - P^H_0\left(y^H\right) = \rho\left(f^H\left(y^H\right) - P^H_0(1)\right)$ and *for low quality good:* $f^L\left(y^H+y^L\right) - P^L_0\left(y^H+y^L\right) = \rho\left(f^L\left(y^H\right) - P^H_0(1)\right)$ imply that $P^H_0\left(y^H\right) = \overline{\theta}q_H - \rho\left(\overline{\theta} - \underline{\theta}\right)q_H$ and $P^L_0\left(y^H+y^L\right) = \underline{\theta}q_L$ where $y^H + y^L > \widehat{b}$. Hence, due to the incentive compatibility constraint, when $y^H \in (\widetilde{y}_1^H - \dfrac{\epsilon}{1-\mu}, \widetilde{y}_1^H]$ and $y^L \in (1-y^H-\dfrac{\gamma}{1-\mu}, 1-y^H]$ we have

$$P^H\left(y^H\right) = \overline{\theta} q_H - \left(\overline{\theta} - \underline{\theta}\right)\left(\rho q_H + (1-\rho)\, q_L\right)$$

and

$$P^L\left(y^L\right) = \underline{\theta} q_L.$$

Tenth, we show that $\inf\left(\widetilde{x}_1^H - \epsilon\right) = \widetilde{x}_2^H$. Let's define $\acute{x}_2^H = \inf\{x^H : t_0^H\left(x^H, x^L\right) = 1,$ $x^L \in (1-\mu-x^H-\gamma,\ 1-\mu-x^H]\}$. We know that when $x^L = 1-\mu-\widetilde{x}_2^H$, if $x^H < \widetilde{x}_2^H$ then $\Pi\left(\widetilde{y}_1; \left(x^H, x^L\right)\right) > \Pi\left(\widetilde{y}_0; \left(x^H, x^L\right)\right)$; otherwise if $x^H > \widetilde{x}_2^H$ then $\Pi\left(\widetilde{y}_1; \left(x^H, x^L\right)\right) < \Pi\left(\widetilde{y}_0; \left(x^H, x^L\right)\right)$. Since $\Pi_0\left(\widetilde{y}_1; \left(x^H, x^L\right)\right) = \Pi(\widetilde{y}_1; (x^H, x^L))$ and $\Pi_0(\widetilde{y}_0;\ (x^H, x^L)) = \Pi(\widetilde{y}_0;$ $(x^H, x^L))$, we must have $\acute{x}_2^H = \widetilde{x}_2^H$.

Eleventh, we show that $\inf\left(1-\mu-\widehat{x}^H-\gamma\right) = 0$. Let's define $\acute{x}_2^L = \inf\{x^L : t_0^L(x^H, x^L) =$ $0,\ x^H \in (\widehat{x}_2^H, 1-\mu]\}$. Since $\Pi\left(\widehat{y}_1; \left(x^H, x^L\right)\right) < \Pi\left(\widehat{y}_0; \left(x^H, x^L\right)\right)$, *for all* x^L and for $x^H \in$ $\left(\widehat{x}_2^H, 1-\mu\right]$, we must have $\acute{x}_2^L = 0$.

Twelfth, suppose that $x^H \in \left(\overline{x}_2^H, 1-\mu\right] \cap \left(\widetilde{x}_2^H, 1-\mu\right]$ and that $x^L \in (t(\overline{x}_{k+1}^H, x^H), t(\overline{x}_k^H, x^H)]$. We now prove the existence of $T\left(\grave{x}\left(\overline{y}_{k-1}^H, 1\right), x^H\right)$.First, we should observe that there exists a threshold on x^L, let's call is X', such that *for all* $x^L < X'$, $\overline{\Pi}_0((\overline{y}_{k-1}^H, \widehat{b} - \overline{y}_{k-1}^H);\ (x^H, x^L)) >$ $\widehat{\Pi}_0((1,0); (x^H, x^L))$ and *for all* $x^L > X'$, $\overline{\Pi}_0((\overline{y}_{k-1}^H, \widehat{b} - \overline{y}_{k-1}^H); (x^H, x^L)) < \widehat{\Pi}_0((1,0);\ (x^H, x^L))$. Now, we show that $X' = T\left(\grave{x}\left(\overline{x}_k^H, x^H\right)\right)$. For a given $x^H \in \left(\overline{x}_2^H, 1-\mu\right] \cap \left(\widetilde{x}_2^H, 1-\mu\right]$,*if* $x^L < T\left(\grave{x}\left(\overline{x}_k^H, x^H\right)\right)$ then $\overline{\Pi}\left(\left(\overline{y}_{k-1}^H, \widehat{b} - \overline{y}_{k-1}^H\right); \left(x^H, x^L\right)\right) > \widetilde{\Pi}\left(1,0\right); \left(x^H, x^L\right)$; otherwise *if* $x^L > T\left(\grave{x}\left(\overline{x}_k^H, x^H\right)\right)$ then $\overline{\Pi}((\overline{y}_{k-1}^H, \widehat{b} - \overline{y}_{k-1}^H); (x^H, x^L)) < \widetilde{\Pi}((1,0);\ (x^H, x^L))$. Since we have $\overline{\Pi}_0((\overline{y}_{k-1}^H, \widehat{b} - \overline{y}_{k-1}^H);\ (x^H, x^L)) = \overline{\Pi}((\overline{y}_{k-1}^H, \widehat{b} - \overline{y}_{k-1}^H); (x^H, x^L))$, and $\widehat{\Pi}_0((1,0);\ (x^H, x^L))$ $= \widehat{\Pi}((1,0);\ (x^H, x^L))$, we must have $X' = T\left(\grave{x}\left(\overline{x}_k^H, x^H\right)\right)$.

Finally, suppose the sequence holds for $k = 1, 2, \ldots, i$. Show that it holds for $i+1$ as well.

Proof of Corollary 8.

See the proof of Corollary 2.

Proof of Corollary 9. By definition of $\overline{x}_k^H$ and $\widehat{x}_k^H$, for $k \geq 3$; $\lim_{z\to 0} \overline{x}_k^H = \lim_{z\to 0^+}(\overline{x}_{k-1}^H -$ $\left(\overline{x}_{k-2}^H - \overline{x}_{k-1}^H\right) \dfrac{\overline{\theta}}{(\overline{\theta}-\underline{\theta})})$ and similarly $\lim_{z\to 0^+} \widehat{x}_k^H = \lim_{z\to 0}(\widehat{x}_{k-1}^H -$ $\left(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H\right) \dfrac{\overline{\theta}}{(\overline{\theta}-\underline{\theta})})$.

Moreover, $\lim_{z\to 0^+} \overline{x}_2^H = \dfrac{\widehat{b\theta} - \underline{\theta}}{\overline{\theta} - \underline{\theta}} \dfrac{q_H}{(q_H - q_L)} - \widehat{b} \dfrac{q_L}{(q_H - q_L)} < \widehat{x}_2^H$.

By induction,

$$\lim_{z\to 0^+} \left(\overline{x}_{k-1}^H - \overline{x}_k^H\right) > \lim_{z\to 0} \left(\overline{x}_{k-2}^H - \overline{x}_{k-1}^H\right) > \widehat{b} - \lim_{z\to 0} \overline{x}_2^H = \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta} - \underline{\theta}} \frac{q_H}{(q_H - q_L)}$$

and

$$\lim_{z\to 0} \left(\widehat{x}_{k-1}^H - \widehat{x}_k^H\right) > \lim_{z\to 0} \left(\widehat{x}_{k-2}^H - \widehat{x}_{k-1}^H\right) > \widehat{b} - \widehat{x}_2^H = \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta} - \underline{\theta}}.$$

Therefore, when the seller serves only high type buyers on the path to the steady state, it takes at most $m' \le \dfrac{\widehat{b}\left(\overline{\theta} - \underline{\theta}\right)}{\left(1-\widehat{b}\right)\underline{\theta}} \dfrac{(q_H - q_L)}{q_H} + 1$ steps to sell the high quality good to all consumers. However, when the seller serves both buyers, it takes at most $m \le \dfrac{\widehat{b}\left(\overline{\theta} - \underline{\theta}\right)}{\left(1-\widehat{b}\right)\underline{\theta}} + 1$ steps to sell the high quality good to all consumers.

Proof of Theorem 5. The proof has two parts. First, we show that there exists an equilibrium in which $\left(\widehat{b}, 0\right)$ is a steady state. Then we prove that the equilibrium derived in the first part is unique.

Existence:

First, we derive a path of states that reaches a steady state. The path $\{\widetilde{x}_k\}_{k=1}^{\infty}$ is for the standard monopoly steady state $\left(\widehat{b}, 0\right)$, and the path $\{\widehat{x}_k\}_{k=1}^{\infty}$ is for segmented monopoly steady state $\left(\widehat{b}, 1-\widehat{b}\right)$. Let's iteratively define the sequence $\{\widetilde{x}_k\}_{k=1}^{\infty}$ where $\widetilde{x}_k = \left(\widetilde{x}_k^H, 0\right)$ as $\widetilde{x}_k^H = \dfrac{\widetilde{p}_{k-1}^H \widetilde{x}_{k-1}^H}{\mu \underline{\theta} q_H + (1-\mu)\widetilde{p}_{k-1}^H}$ with the initial value $\widetilde{x}_0^H = (1-\mu)\widehat{b}$. We set $\widetilde{x}_k^H$ such that when the state is $\left(\widetilde{x}_k^H, 0\right)$ the monopolist is indifferent between staying at $\left(\widetilde{x}_k^H, 0\right)$ forever and moving the state to $\left(\widetilde{x}_{k-1}^H, 0\right)$. The price of high quality good $\widetilde{p}_k^H$ and low quality good $\widetilde{p}_k^L$ are derived from buyers' arbitrage equations. The prices are set such that marginal buyers are indifferent between today and tomorrow. The prices are defined as $\widetilde{p}_k^H = \left(1-\rho^k\right)\underline{\theta} q_H + \rho^k \overline{\theta} q_H$ and $\widetilde{p}_k^L = \left(1-\rho^k\right)\underline{\theta} q_L + \rho^k \overline{\theta} q_L$. Similarly, we iteratively define the sequence $\{\widehat{x}_k\}_{k=1}^{\infty}$ where $\widehat{x}_k = \left(\widehat{x}_k^H, 1-\mu-\widehat{x}_k^H\right)$ as $\widehat{x}_k^H = \dfrac{\left(\widehat{p}_{k-1}^H - \underline{\theta} q_L\right)\widehat{x}_{k-1}^H}{\mu \underline{\theta} q_H - \underline{\theta} q_L + (1-\mu)\widehat{p}_{k-1}^H}$ with the initial value $\widehat{x}_0^H = (1-\mu)\widehat{b}$. The prices that make buyers be indifferent between today and tomorrow are $\widehat{p}_k^H = \underline{\theta} q_H + \rho^k \Delta\theta \Delta q$ and $\widehat{p}_k^L = \underline{\theta} q_L$. We should note that $\widetilde{x}_0^H < \widetilde{x}_1^H < \widetilde{x}_2^H < \ldots$ and $\widehat{x}_0^H < \widehat{x}_1^H < \widehat{x}_2^H < \ldots$.

Which sequence that the monopolist follows depends on the amount of low quality good in the market. If the low quality good is low enough, the monopolist prefers buying all low quality good back and following $\{\widetilde{x}_k\}_{k=1}^{\infty}$ thereafter. However, if the amount of low quality good in the market is sufficiently high then he would follow $\{\overline{x}_k\}_{k=1}^{\infty}$. The steady that the monopolist will reach not only depend on the amount low quality good but the limit value of $\widetilde{x}_k^H$ and $\widehat{x}_k^H$ as well. If the limit value is less than the market size then when the amount of high quality good is sufficiently high, the monopolist fulfills the market immediately and reaches the Coase Conjecture steady state $(1, 0)$.

If $x_{\infty}^H = \lim_{k\to\infty} x_k^H \geq 1-\mu$, then set $m = \sup\left\{k : x_k^H < 1-\mu\right\}$ and define $x_{m+1} = 1-\mu$. Let $y_k^H = \dfrac{x_k^H}{1-\mu}$. The pentad $\left\{P^H(\cdot), P^L(\cdot), t^H(\cdot), t^L(\cdot), R(\cdot)\right\}$ is defined as follows.

$$P^H\left(x^H, x^L\right) = \begin{cases} \overline{\theta} q_H & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ \widetilde{p}_k^H & \forall x^H \in \left(y_{k-1}^H, y_k^H\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \widehat{p}_k^H & \forall x^H \in \left(y_{k-1}^H, y_k^H\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \end{cases}$$

$$P^L\left(x^H, x^L\right) = \begin{cases} \overline{\theta} q_H & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta} q_L & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ \widetilde{p}_k^L & \forall x^H \in \left(y_{k-1}^H, y_k^H\right],\ k \in S \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta} q_L & \forall x^H \in \left(y_{k-1}^H, y_k^H\right],\ k \in S \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \end{cases}$$

$$t^H\left(x^H, x^L\right) = \begin{cases} \widehat{b} & \forall x^H \in \left[0, x_1^H\right] \text{ and } \forall x^L \\ y_{k-1}^H & \forall x^H \in \left(x_{k-1}^H, x_k^H\right],\ k \in \{2, \ldots, m+1\} \end{cases}$$

$$t^L\left(x^H, x^L\right) = \begin{cases} 0 & \forall x^H \in \left[0, x_1^H\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ 1-\widehat{b} & \forall x^H \in \left[0, x_1^H\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ 0 & \forall x^H \in \left(x_{k-1}^H, x_k^H\right],\ k \in S' \text{and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ 1-y_{k-1}^H & \forall x^H \in \left(x_{k-1}^H, x_k^H\right],\ k \in S' \text{and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \end{cases}$$

$$R\left(x^H,x^L\right)=\begin{cases}(\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\overline{\theta}q_H-\overline{\theta}q_Lx^L\\(\frac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\left(\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L\right)+C_1\\(\frac{\mu y_{k-1}^H}{1-\delta}-(x^H-x_{k-1}^H))\widetilde{p}_{k-1}^H-x^L\widetilde{p}_{k-1}^L\\(\frac{\mu y_{k-1}^H}{1-\delta}-\left(x^H-x_{k-1}^H\right))\widehat{p}_{k-1}^H+C_2\end{cases}$$

where $S\equiv\{1,\ldots,m+1\}$, $S'\equiv\{2,\ldots,m+1\}$, $C_1\equiv(\frac{\delta\mu(1-\widehat{b})}{1-\delta}+(1-\widehat{b}-x^L))\underline{\theta}q_L$, $C_2\equiv(\frac{\mu\left(1-y_{k-1}^H\right)}{1-\delta}-\left(x^L-\left(1-\mu-x_{k-1}^H\right)\right))\underline{\theta}q_L$.

If $x_\infty^H=\lim_{k\to\infty}x_k^H<1-\mu$ then let's define

$$\widetilde{R}\left(x^H,x^L\right)=\left(\frac{\mu\widetilde{x}_{k-1}^H}{(1-\delta)\left(1-\mu\right)}-\left(x^H-\widetilde{x}_{k-1}^H\right)\right)\widetilde{p}_{k-1}^H-x^L\widetilde{p}_{k-1}^L,$$

$$\widehat{R}\left(x^H,x^L\right)=(\frac{\mu\widehat{x}_{k-1}^H}{(1-\delta)\left(1-\mu\right)}-(x^H-\widehat{x}_{k-1}^H))\widehat{p}_{k-1}^H+(\frac{\mu(1-\mu-\widehat{x}_{k-1}^H)}{(1-\delta)\left(1-\mu\right)}-(x^L-1-\mu-\widehat{x}_{k-1}^H)))\underline{\theta}q_L$$

for $x^H\in\left(x_{k-1}^H,x_k^H\right]$ and $k\geq1$. Set

$$\widetilde{x}^*=\max\left\{x^H\in\left((1-\mu)\,\widehat{b},\widetilde{x}_\infty\right]:\widetilde{R}\left(x^H,x^L\right)\geq R_1\left(x^H,x^L\right)\right\}$$

and

$$\widehat{x}^*=\max\left\{x^H\in\left((1-\mu)\,\widehat{b},\widehat{x}_\infty\right]:\widehat{R}\left(x^H,x^L\right)\geq R_1\left(x^H,x^L\right)\right\}$$

where

$$R_1\left(x^H,x^L\right)=\left(\frac{\delta\mu}{(1-\delta)}+\left(1-x^H\right)\right)\underline{\theta}q_H-x^L\underline{\theta}q_L.$$

Let m and m' be such that $\widetilde{x}^*\in\left(x_m^H,x_{m+1}^H\right]$ and $\widehat{x}^*\in\left(x_{m'}^H,x_{m'+1}^H\right]$.

Now, we prove the existence of m and m':

Since $\widetilde{x}_\infty^H<1-\mu$, we have

$$\widetilde{R}\left(\widetilde{x}_\infty^H,x^L\right)-R_1\left(\widetilde{x}_\infty^H,x^L\right)=\left(\frac{\mu\widetilde{x}_\infty^H}{(1-\delta)\left(1-\mu\right)}-\left(\frac{\delta\mu}{(1-\delta)}+\left(1-\widetilde{x}_\infty^H\right)\right)\right)\underline{\theta}q_H<0.$$

This implies that $\widetilde{x}^*<\widetilde{x}_\infty^H$. Furthermore, due to definition of $\underline{\mu}$, $\widetilde{x}^*\geq(1-\mu)\,\widehat{b}$. Similarly,

$\widehat{x}^* < \widehat{x}^H_\infty$, since

$$\widehat{R}\left(\widehat{x}^H_\infty, x^L\right) - R_1\left(\widehat{x}^H_\infty, x^L\right)$$
$$= \left(\frac{\mu\widehat{x}^H_\infty}{(1-\delta)(1-\mu)} - \left(\frac{\delta\mu}{(1-\delta)} + \left(1-\widehat{x}^H_\infty\right)\right)\right)\underline{\theta}q_H + \left(\frac{(1-\delta(1-\mu))\left(1-\mu-\widehat{x}^H_\infty\right)}{(1-\delta)(1-\mu)}\right)\underline{\theta}q_L$$
$$= \left(\frac{(1-\rho)\widehat{x}^H_\infty - (1-\rho)(1-\mu)}{(1-\delta)(1-\mu)}\right)\underline{\theta}q_H + \left(\frac{(1-\rho)\left(1-\mu-\widehat{x}^H_\infty\right)}{(1-\delta)(1-\mu)}\right)\underline{\theta}q_L$$
$$= \frac{(1-\rho)}{(1-\delta)(1-\mu)}\quad\left(\widehat{x}^H_\infty - (1-\mu)\right)\underline{\theta}\left(q_H - q_L\right) < 0$$

when $\widehat{x}^H_\infty < 1-\mu$. Moreover, due to the definition of $\underline{\mu}$, $\widehat{x}^* \geq (1-\mu)\widehat{b}$. Additionally, when $x^L < \overline{x}^L$, since $\widehat{R}\left(\widehat{x}^*, x^L\right) < R_1\left(\widehat{x}^*, x^L\right)$, we have $\widehat{x}^* < \widetilde{x}^*$. On the other hand, when $x^L > \overline{x}^L$, since $\widehat{R}\left(\widehat{x}^*, x^L\right) > R_1\left(\widehat{x}^*, x^L\right)$, we have $\widehat{x}^* > \widetilde{x}^*$.

The pentad $\{P^H(\cdot),\, P^L(\cdot),\, t^H(\cdot),\, t^L(\cdot),\, R(\cdot)\}$ is defined as follows.

$$P^H\left(x^H, x^L\right) = \begin{cases} \overline{\theta}q_H & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ \widetilde{p}^H_k & \forall x^H \in \left(y^H_{k-1}, y^H_k\right],\ k \in S \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \widehat{p}^H_k & \forall x^H \in \left(y^H_{k-1}, y^H_k\right],\ k \in S' \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta}q_H & \forall x^H \in \left(y^H_{m+1}, 1\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta}q_H & \forall x^H \in \left(y^H_{m'+1}, 1\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \end{cases}$$

$$P^L\left(x^H, x^L\right) = \begin{cases} \overline{\theta}q_H & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta}q_L & \forall x^H \in \left[0, \widehat{b}\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ \widetilde{p}^L_k & \forall x^H \in \left(y^H_{k-1}, y^H_k\right],\ k \in \{1, \ldots, m+1\} \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta}q_L & \forall x^H \in \left(y^H_{k-1}, y^H_k\right],\ k \in \{1, \ldots, m'+1\} \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta}q_L & \forall x^H \in \left(y^H_{m+1}, 1\right] \text{ and } \forall x^L \in \left[0, \overline{x}^L\left(x^H\right)\right] \\ \underline{\theta}q_L & \forall x^H \in \left(y^H_{m'+1}, 1\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \end{cases}$$

$$t^H\left(x^H, x^L\right) = \begin{cases} \widehat{b} & \forall x^H \in \left[0, x^H_1\right] \text{ and } \forall x^L \\ y^H_{k-1} & \forall x^H \in \left(x^H_{k-1}, x^H_k\right],\ k \in \{2, \ldots, m+1\} \\ 1 & \forall x^H \in \left(y^H_{m'+1}, 1\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \\ 1 & \forall x^H \in \left(y^H_{m'+1}, 1\right] \text{ and } \forall x^L \notin \left[0, \overline{x}^L\left(x^H\right)\right] \end{cases}$$

$$t^L\left(x^H,x^L\right)=\begin{cases}0 & \forall x^H\in\left[0,x_1^H\right] \text{ and } \forall x^L\in\left[0,\overline{x}^L\left(x^H\right)\right]\\ 1-\widehat{b} & \forall x^H\in\left[0,x_1^H\right] \text{ and } \forall x^L\notin\left[0,\overline{x}^L\left(x^H\right)\right]\\ 0 & \forall x^H\in\left(x_{k-1}^H,x_k^H\right],\ k\in S \text{ and } \forall x^L\in\left[0,\overline{x}^L\left(x^H\right)\right]\\ 1-y_{k-1}^H & \forall x^H\in\left(x_{k-1}^H,x_k^H\right],\ k\in S'\text{and } \forall x^L\notin\left[0,\overline{x}^L\left(x^H\right)\right]\\ 0 & \forall x^H\in\left(y_{m+1}^H,1\right] \text{ and } \forall x^L\in\left[0,\overline{x}^L\left(x^H\right)\right]\\ 0 & \forall x^H\in\left(y_{m'+1}^H,1\right] \text{ and } \forall x^L\notin\left[0,\overline{x}^L\left(x^H\right)\right]\end{cases}$$

$$R\left(x^H,x^L\right)=\begin{cases}(\dfrac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\overline{\theta}q_H-\overline{\theta}q_Lx^L\\ (\dfrac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))\left(\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L\right)+C_1\\ (\dfrac{\mu y_{k-1}^H}{1-\delta}-(x^H-x_{k-1}^H))\widetilde{p}_{k-1}^H-x^L\widetilde{p}_{k-1}^L\\ (\dfrac{\mu y_{k-1}^H}{1-\delta}-(x^H-x_{k-1}^H))\widehat{p}_{k-1}^H+C_2\\ (\dfrac{\delta\mu}{(1-\delta)}+(1-x^H))\underline{\theta}q_H-x^L\underline{\theta}q_L\\ (\dfrac{\delta\mu}{1-\delta}+(1-x^H))\underline{\theta}q_H-x^L\underline{\theta}q_L\end{cases}$$

where $S\equiv\{1,\ldots,m+1\}$, $S'\equiv\{2,\ldots,m+1\}$, $C_1\equiv(\dfrac{\delta\mu(1-\widehat{b})}{1-\delta}+(1-\widehat{b}-x^L))\underline{\theta}q_L$, $C_2\equiv(\dfrac{\mu\left(1-y_{k-1}^H\right)}{1-\delta}-\left(x^L-\left(1-\mu-x_{k-1}^H\right)\right))\underline{\theta}q_L$.

Now, we will show that the stationary pentads defined above satisfy the maximization problem of the monopolist. First, we consider the case in which $(1,0)$ coexists with the monopoly steady states, i.e., $x_\infty^H<1-\mu$. Let $\Pi\left(y;x\right)=P^H\left(y^H,y^L\right)\left(y^H-x^H\right)+P^L\left(y^H,y^L\right)\left(y^L-x^L\right)+R\left(\left(1-\mu\right)y\right)$. We show that $T^L\left(x\right)\subset\{0,\ 1-y_{m+1}^H,\ 1-y_m^H,\ \ldots,\ 1-y_0^H\}$.$\Pi\left(y;x\right)$ is strictly increasing in y^H on any of the intervals $[0,y_0]\times[0,1]$, $(y_0,y_1]\times[0,1]$, $\ldots$, $(y_{s+1},1]\times[0,1]$, $s\in\{m,m'\}$. So, $T^H\left(x\right)\subset\{y_0^H,\ y_1^H,\ \ldots,\ y_{m+1}^H,1\}$.

Due to the recursive structure of $\widetilde{x}_k$ and $\widehat{x}_k$ we have $\Pi\left(\widetilde{y}_k;\widetilde{x}_k\right)=\Pi\left(\widetilde{y}_{k-1};\widetilde{x}_k\right)$ and $\Pi\left(\widehat{y}_k;\widehat{x}_k\right)=\Pi\left(\widehat{y}_{k-1};\widehat{x}_k\right)$. Let's define $\widetilde{h},\widehat{h},g$ as follows.[4]

$$\widetilde{h}_{m+1}\left(x^H,x^L\right)=\Pi\left(1;\left(x^H,x^L\right)\right)-\Pi\left(\widetilde{y}_m;\left(x^H,x^L\right)\right),$$

[4] $\widetilde{h}_{m+1}\left(x^H,x^L\right)=\left[\left(\dfrac{\delta\mu}{(1-\delta)}+\left(1-x^H\right)\right)\underline{\theta}q_H-x^L\underline{\theta}q_L\right]-\left[\left(\dfrac{\mu y_m^H}{1-\delta}-\left(x^H-x_{k-1}^H\right)\right)\widetilde{p}_m^H-x^L\widetilde{p}_m^L\right]$
$\widetilde{h}_k\left(x^H,x^L\right)=\left[\left(\dfrac{\mu y_k^H}{1-\delta}-\left(x^H-x_k^H\right)\right)\widetilde{p}_k^H-x^L\widetilde{p}_k^L\right]-\left[\left(\dfrac{\mu y_{k-1}^H}{1-\delta}-\left(x^H-x_{k-1}^H\right)\right)\widetilde{p}_{k-1}^H-x^L\widetilde{p}_{k-1}^L\right]$

$$\widetilde{h}_k\left(x^H,x^L\right)=\Pi\left(\widetilde{y}_k;\left(x^H,x^L\right)\right)-\Pi\left(\widetilde{y}_{k-1};\left(x^H,x^L\right)\right)$$

for $k\leq m$ where $\widetilde{y}_k=\left(\widetilde{y}^H_k,0\right)$, and[5]

$$\widehat{h}_{m'+1}\left(x^H,x^L\right)=\Pi\left(1;\left(x^H,x^L\right)\right)-\Pi\left(\widehat{y}_{m'};\left(x^H,x^L\right)\right),$$

$$\widehat{h}_k\left(x^H,x^L\right)=\Pi\left(\widehat{y}_k;\left(x^H,x^L\right)\right)-\Pi\left(\widehat{y}_{k-1};\left(x^H,x^L\right)\right)$$

for $k\leq m'$ where $\widehat{y}_k=\left(\widehat{y}^H_k,1-\mu-\widehat{y}^H_k\right)$. In this case $\widetilde{h}_k\left(\widetilde{x}^H_k,0\right)=0$ and $\widehat{h}_k(\widehat{x}^H_k,1-\mu-\widehat{x}^H_k)$ $=0$ for all k. The state variable if high quality good affect h's as follows

$$\frac{\partial\widehat{h}_k}{\partial x^H}=\widehat{p}^H_{k-1}-\widehat{p}^H_k>0.$$

Now, the existence of the threshold value of low quality good will be proved. We define

$$g\left(x^H,x^L\right)=\Pi\left(\widetilde{y}_k;\left(x^H,x^L\right)\right)-\Pi\left(\widehat{y}_k;\left(x^H,x^L\right)\right).$$

There exists $\overline{x}^L=x^L\left(x^H\right)$ such that $g\left(x^H,\overline{x}^L\right)=0$. Moreover, $\dfrac{\partial g}{\partial x^L}=-\widehat{p}^L_k+\underline{\theta}q_L<0$.

Uniqueness:

Let $\left\{P^H_0(\cdot),P^L_0(\cdot),t^H_0(\cdot),t^L_0(\cdot),R_0(\cdot)\right\}$ be the pentad with some stationary equilibrium having $\left(\widehat{b},0\right)$ as steady state. We will show that the pentad is unique.

Let's define $\ddot{p}^H_1=\sup\limits_{y^H>\widehat{b}}P^H_0\left(y^H,y^L\right)$, and $\ddot{p}^L_1=\sup\limits_{y^H+y^L>\widehat{b}}P^L_0\left(y^H,y^L\right)$. *First, suppose that* $\ddot{p}^H_1=\underline{\theta}q_H$. In this case, $P^H_0\left(y^H,y^L\right)=\begin{cases}\overline{\theta}q_H & \text{for } \; y^H\in\left(0,\widehat{b}\right] \text{ and all } y^L\\ \underline{\theta}q_H & \text{for } \; y^H\in\left(\widehat{b},1\right] \text{ and all } y^L\end{cases}$. For $y^H>\widehat{b}$, buyer's arbitrage equation for high quality good implies that $t^H_0\left((1-\mu)\,y^H,(1-\mu)\,y^L\right)>\widehat{b}$ for all

[5] $\widehat{h}_{m'+1}\left(x^H,x^L\right)=\left[\left(\frac{\delta\mu}{(1-\delta)}+\left(1-x^H\right)\right)\underline{\theta}q_H-x^L\underline{\theta}q_L\right]$
$-\left[\left(\frac{\mu y^H_{m'}}{1-\delta}-\left(x^H-x^H_{m'}\right)\right)\widehat{p}^H_{m'}+\left(\frac{\mu\left(1-y^H_{m'}\right)}{1-\delta}-\left(x^L-\left(1-\mu-x^H_{m'}\right)\right)\right)\underline{\theta}q_L\right]$
$\widehat{h}_k\left(x^H,x^L\right)=\left[\left(\frac{\mu y^H_k}{1-\delta}-\left(x^H-x^H_k\right)\right)\widehat{p}^H_k+\left(\frac{\mu\left(1-y^H_k\right)}{1-\delta}-\left(x^L-\left(1-\mu-x^H_k\right)\right)\right)\underline{\theta}q_L\right]$
$-\left[\left(\frac{\mu y^H_{k-1}}{1-\delta}-\left(x^H-x^H_{k-1}\right)\right)\widehat{p}^H_{k-1}+\left(\frac{\mu\left(1-y^H_{k-1}\right)}{1-\delta}-\left(x^L-\left(1-\mu-x^H_{k-1}\right)\right)\right)\underline{\theta}q_L\right]$

$(1-\rho)\underline{\theta}q_H + \rho P_0^H\left(t_0^H\left((1-\mu)y^H,(1-\mu)y^L\right)\right) = P_0^H\left(y^H\right) = \underline{\theta}q_H$. Therefore, since $\underline{\theta}q_H$ for all $y^H \in \left(\widehat{b},1\right]$ and all y^L, we have $t_0^H\left(x^H,x^L\right) = 1$ for all $x^H > (1-\mu)\widehat{b}$. We now show that $\ddot{p}_1^L = \underline{\theta}q_L$. Suppose not. Since $\ddot{p}_1^L > \underline{\theta}q_L$ for some $y^H + y^L > \widehat{b}$, buyer's indifference equation for low quality good implies that $t_0^H\left(x^H,x^L\right)+t_0^L\left(x^H,x^L\right) < \dfrac{x^H+x^L}{1-\mu}$ for $x^H+x^L > (1-\mu)\widehat{b}$ since $\underline{\theta}q_L - P_0^L\left(y^H+y^L\right) = \rho(\underline{\theta}q_L - P_0^L(t_0^H((1-\mu)y^H,(1-\mu)y^L) + t_0^L((1-\mu)y^H,(1-\mu)y^L)))$. This leads to a contradiction because $t_0^H\left(x^H,x^L\right) = 1$ for all $x^H > (1-\mu)\widehat{b}$. Therefore, $P_0^H\left(y^H,y^L\right) = \begin{cases} \overline{\theta}q_L & \text{for} \quad y^H \in \left(0,\widehat{b}\right] \text{ and all } y^L \\ \underline{\theta}q_L & \text{for} \quad y^H \in \left(\widehat{b},1\right] \text{ and all } y^L \end{cases}$. It follows that, since $P_0^H\left(y^H,y^L\right) > P_0^H\left(y^H,y^L\right)$ for $y^H > \widehat{b}$, we have $t_0^L\left(x^H,x^L\right) = 0$ for all $x^H \in \left(\widehat{b},1\right]$. Now, define $\dot{p}_1^L = \inf\limits_{y^H+y^L\leq\widehat{b}} P_0^L\left(y^H,y^L\right)$. Suppose first that $\dot{p}_1^L = \overline{\theta}q_L$. Now, suppose that $\dot{p}_1^L < \overline{\theta}q_L$. Define. Let $\dot{b}_1^L = \sup\left\{x^H+x^L < (1-\mu)\widehat{b} : t_0^L\left(x^H,x^L\right) = 0\right\}$. Then, since $\left(\widehat{b},0\right)$ is a steady state, we have

$$P_0^H\left(y^H,y^L\right) = \begin{cases} \overline{\theta}q_H & \text{for} \quad y^H \in \left(0,\widehat{b}\right] \text{ and all } y^L \\ \underline{\theta}q_H & \text{for} \quad y^H \in \left(\widehat{b},1\right] \text{ and all } y^L \end{cases},$$

$$P_0^H\left(y^H,y^L\right) = \begin{cases} \overline{\theta}q_L & \text{for} \quad y^H \in \left(0,\widehat{b}\right] \text{ and all } y^L \\ \underline{\theta}q_L & \text{for} \quad y^H \in \left(\widehat{b},1\right] \text{ and all } y^L \end{cases},$$

$$t_0^H\left(x^H,x^L\right) = \begin{cases} \widehat{b} & \text{for} \quad y^H \in \left(0,\widehat{b}\right] \text{ and all } y^L \\ 1 & \text{for} \quad y^H \in \left(\widehat{b},1\right] \text{ and all } y^L \end{cases},$$

$$t_0^L\left(x^H,x^L\right) = \begin{cases} 0 & \text{for} \quad \text{all } y^H \text{ and all } y^L \end{cases},$$

$$R\left(x^H,x^L\right) = \begin{cases} \widehat{b} & \text{for} \quad y^H \in \left(0,\widehat{b}\right] \text{ and all } y^L \\ 1 & \text{for} \quad y^H \in \left(\widehat{b},1\right] \text{ and all } y^L \end{cases}.$$

Therefore, $\left\{P_0^H(\cdot), P_0^L(\cdot), t_0^H(\cdot), t_0^L(\cdot), R_0(\cdot)\right\}$ is uniquely defined.

Secondly, suppose that $\ddot{p}_1^H > \underline{\theta}q_H$ and $\ddot{p}_1^L = \underline{\theta}q_L$. Now we show that $\ddot{p}_1^H = \widehat{p}_1^H$. First one should observe that $t_0^H\left((1-\mu)y^H,(1-\mu)y^L\right) < y^H$ for all $y^H \in \left(\widehat{b},1\right]$ and y^L when $P_0^H\left(y^H,y^L\right) > \underline{\theta}q_H$. Suppose not. Let $t_0^H\left((1-\mu)y^H,(1-\mu)y^L\right) \geq y^H$.This implies that $P_0^H\left(t_0^H\left((1-\mu)y^H,(1-\mu)y^L\right)\right) < P_0^H\left(y^H\right)$. Due to the indifference equation we have $P_0^H(y^H) = (1-\rho)\underline{\theta}q_H + \rho P_0^H\left(t_0^H\left((1-\mu)y^H,(1-\mu)y^L\right)\right) \leq (1-\rho)\underline{\theta}q_H + \rho P_0^H\left(y^H\right)$. Therefore, $P_0^H\left(y^H\right) \leq \underline{\theta}q_H$ which contradicts with the initial assumption. We now claim that there exists

$\varepsilon > 0$ such that $t_0^H\left((1-\mu)\,y^H,(1-\mu)\,y^L\right) = \widehat{b}$ for all $y^H \in \left(\widehat{b},\widehat{b}+\epsilon\right]$. Let's choose $y_0^H, y_0^L \in \left(\widehat{b},1\right]$ with $P_0^H\left(y_0^H\right) > \underline{\theta}q_H$ and iteratively define y_K^H as $y_K^H = t_0^H\left((1-\mu)\,y_{k-1}^H,(1-\mu)\,y_{k-1}^L\right)$. The reasoning is, if there is no such ϵ then $\left\{y_K^H\right\}$ would be decreasing sequence bounded below by $\widehat{b}$. In this case, the indifference equation implies that $P_0^H\left(y_0^H\right) = \left(1-\rho^k\right)\underline{\theta}q_H + \rho^k P_0^H\left(y_k^H\right)$. As k goes to infinity, this would imply that $\lim_{k\to\infty} P_0^H\left(y_0^H, 1-y_0^H\right) = \underline{\theta}q_H$ which yields to a contradiction. This implies that $\ddot{p}_1^H = \widehat{p}_1^H$. Let's define $\check{x}_1^H = \max\{x \le 1-\mu : t_0^H\left(x_H, 1-\mu-x_H\right) = \widehat{b}\}$ and $\ddot{p}_2^H = \sup_{y^H > \widehat{b}}\left\{P_0^H\left(y^H,y^L\right) \mid y^H > \dfrac{\check{x}_1^H}{1-\mu}\right\}$. Now, we show that $\check{x}_1^H = \widehat{x}_1^H$. There are three cases.

Suppose $\check{x}_1^H = 1-\mu$. Then we have $P_0^H\left(y^H,y^L\right) = \widehat{p}_1^H$ for all $y^H \in \left(\widehat{b},1\right]$ and all y^L, $R_0\left(x^H,x^L\right) = \widehat{R}\left(x^H,x^L\right)$, $t_0^H\left(x^H,x^L\right) = \widehat{b}$ and $t_0^L\left(x^H,x^L\right) = 1-\widehat{b}$ for all $x^H \in [0,1-\mu]$ and x^L. Therefore, the pentad is uniquely defined and $\widehat{x}_1^H = 1-\mu$ and $m' = 0$.

Suppose $\check{x}_1^H < 1-\mu$ and $\ddot{p}_2^H = \underline{\theta}q_H$ Then as above, $P_0^H\left(y^H,y^L\right) = \underline{\theta}q_H$ for all $y^H \in \left(\dfrac{\check{x}_1^H}{1-\mu},1\right]$ and y^L, $R_0\left(x^H,x^L\right) = R_1\left(x^H,x^L\right)$, $t_0^H\left(x^H,x^L\right) = 1$ and $t_0^L\left(x^H,x^L\right) = 0$ for all $x^H \in \left[\check{x}_1^H,1-\mu\right]$ and x^L. Moreover, $R_0\left(x^H,x^L\right) = \widehat{R}\left(x^H,x^L\right)$ for all $x^H < \check{x}_1^H$. The continuity of R_0 at $x^H = \check{x}_1^H$ implies that $\widehat{R}\left(\check{x}_1^H,x^L\right) = R_1\left(\check{x}_1^H,x^L\right)$. Therefore, $\check{x}_1^H = \widehat{x}^* = \widehat{x}_1^H$.

Suppose $\check{x}_1^H < 1-\mu$ and $\ddot{p}_2^H > \underline{\theta}q_H$. Then we claim that $\ddot{p}_2^H = \widehat{p}_2^H$. To establish the claim we need to show that there exists $\epsilon > 0$ such that $t_0^H\left(x_H,x_L\right) = \dfrac{\check{x}_1^H}{1-\mu}$ for all $x^H \in \left(\check{x}_1^H,\check{x}_1^H+\epsilon\right]$. Then we show that $\check{x}_1^H = \widehat{x}_1^H$. Let's define a function $\widehat{V}\left(x^H,x^L\right)$ as

$$\widehat{V}\left(x^H,x^L\right) = \widehat{\Pi}\left(\left(\widehat{b},1-\widehat{b}\right);\left(x^H,x^L\right)\right) - \left(\frac{\mu\widehat{p}_1^H x^H}{(1-\mu)(1-\delta)} + \frac{\mu\underline{\theta}q_L x^L}{(1-\mu)(1-\delta)}\right)$$

where $\widehat{\Pi}((\widehat{b},1-\widehat{b}),x) = (\dfrac{\delta\mu\widehat{b}}{1-\delta}+(\widehat{b}-x^H))(\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L) + (\dfrac{\delta\mu(1-\widehat{b})}{1-\delta}+(1-\widehat{b}-x^L))\underline{\theta}q_L$. By definition of $\widehat{x}_1^H$, we have $\widehat{V}\left(\widehat{x}_1^H,1-\mu-\widehat{x}_1^H\right) = 0$. Since $t_0^H\left(x^H,x^L\right) = \widehat{b}$ and $t_0^L\left(x^H,x^L\right) = 1-\widehat{b}$ for $x^H \le \check{x}_1^H$ and all x^L, and $t_0^H\left(x^H,x^L\right) = \dfrac{\check{x}_1^H}{1-\mu}$ and $t_0^L\left(x^H,x^L\right) = 1-\dfrac{\check{x}_1^H}{1-\mu}$ for $x^H \in \left(\check{x}_1^H,\check{x}_1^H+\epsilon\right]$ and all x^L. Continuity of R_0 implies that $\widehat{V}\left(\check{x}_1^H,1-\mu-\check{x}_1^H\right) = 0$ as well. Moreover, since $\dfrac{d\widehat{V}\left(x,1-\mu-x\right)}{dx} < 0$ we must have $\check{x}_1^H = \widehat{x}_1^H$.

The same argument can be applied inductively. Given $\check{x}_k^H = \max\left\{x : t_0^H\left(x^H,x^L\right) = \check{x}_{k-1}^H\right\}$

then it must be that $\check{x}_k^H = \widehat{x}_k^H$. If $\check{x}_k^H = 1-\mu$ then $\widehat{x}_k^H = 1-\mu$ and $m = k-1$. If $\check{x}_k^H < 1-\mu$ then $\ddot{p}_{k+1}^H = \underline{\theta} q_H$ then $\check{x}_k^H = \widehat{x}^* = \widehat{x}_k^H$ and $m = k$, $\ddot{p}_{k+1}^H > \underline{\theta} q_H$ then $m > k-1$ and define $\check{x}_{k+1}^H = \max\left\{x : t_0^H\left(x^H, x^L\right) = \check{x}_k^H\right\}$.

Thirdly, suppose that $\ddot{p}_1^H > \underline{\theta} q_H$ and $\ddot{p}_1^L > \underline{\theta} q_L$. We claim that $\ddot{p}_1^H = \widetilde{p}_1^H$ and $\ddot{p}_1^L = \widetilde{p}_1^L$. To prove the claim we need to show that there exist ϵ and δ such that $t_0^H\left(\left(1-\mu\right)y^H, \left(1-\mu\right)y^L\right) = \widehat{b}$ and $t_0^L\left(\left(1-\mu\right)y^H, \left(1-\mu\right)y^L\right) = 0$ for all $y^H \in \left(\widehat{b}, \widehat{b}+\epsilon\right]$ and $y^L \in [0,\delta]$. We know from the previous discussion that there exists $\epsilon > 0$ such that $t_0^H\left(\left(1-\mu\right)y^H, \left(1-\mu\right)y^L\right) = \widehat{b}$ for $y^H \in \left(\widehat{b}, \widehat{b}+\epsilon\right]$ and all y^L. Now, we show that there exists δ such that $t_0^L((1-\mu)y^H, (1-\mu)y^L) = 0$ for all $y^H \in (\widehat{b}, \widehat{b}+\epsilon]$ and $y^L \in [0,\delta]$ where $\ddot{p}_H > \underline{\theta} q_H$ and $\ddot{p}_L > \underline{\theta} q_L$. Since $\ddot{p}_L > \underline{\theta} q_L$. stock of low quality good should be decreasing. That is, $\widehat{b} + t_0^L\left(\left(1-\mu\right)y^H, \left(1-\mu\right)y^L\right) < y^H + y^L$. As y^H approaches $\widehat{b}$, we would have $\lim_{y^H \to \widehat{b}^+} t_0^L\left(\left(1-\mu\right)y^H, \left(1-\mu\right)y^L\right) < y^L$. Now let's choose y_0^H and y_0^H with $P_0^L\left(y_0^L\right) > \underline{\theta} q_L$ and $y_0^H + y_0^L \in \left(\widehat{b}, 1\right]$. If there was no such δ then $\left\{y_0^H + y_k^L\right\}$ would be a decreasing sequence bounded below by y_0^H. Then $P_0^L\left(y_0^L\right) = \left(1-\rho^k\right)\underline{\theta} q_L + \rho^k$ $P_0^L\left(y_k^L\right)$ where $y_k^L = t_0^H\left(\left(1-\mu\right)y_0^H, \left(1-\mu\right)y_{k-1}^L\right)$ which would imply that $\lim\limits_{k\to\infty} P_0^L\left(y_0^L\right) = \underline{\theta} q_L$. This yields a contradiction.

Now define $\check{x}_1^H = \sup\{x^H \le 1-\mu : t_0^H\left(x^H, 0\right) = \widehat{b}\}$ and $\check{x}_1^L = \sup\{x^L \le 1-\mu-\widehat{b} : t_0^H\left(\check{x}_1^H, x^L\right) = 0\}$. *Suppose* $\check{x}_1^H = 1-\mu$. Then $P_0^H\left(y^H, y^L\right) = \widetilde{p}_1^H$ and $P_0^H\left(y^H, y^L\right) = \widetilde{p}_1^L$ for all $y^H \in \left(\widehat{b}, 1\right]$ and all y^L, and $R_0\left(x^H, x^L\right) = \widetilde{R}\left(x^H, x^L\right)$, $t_0^H\left(x^H, x^L\right) = \widehat{b}$, $t_0^L\left(x^H, x^L\right) = 0$ and $x^H \in [0, 1-\mu]$ and all x^L. $P_0^H\left(y^H, y^L\right) = \widetilde{p}_1^H$, $R_0\left(x^H, x^L\right) = \widetilde{R}\left(x^H, x^L\right)$, $t_0^H\left(x^H, x^L\right) = \widehat{b}$, $t_0^L\left(x^H, x^L\right) = 0$ for all $y^H \in \left(\widehat{b}, 1\right]$ and $x^H \in [0, 1-\mu]$ and all x^L. *Suppose* $\check{x}_1^H < 1-\mu$ and $\check{x}_1^L = 1-\mu-\check{x}_1^H$ and $\ddot{p}_2^H = \underline{\theta} q_H$. Then as above, $P_0^H\left(y^H, y^L\right) = \underline{\theta} q_H$, $P_0^L\left(y^H, y^L\right) = \underline{\theta} q_L$, $R_0\left(x^H, x^L\right) = R_1\left(x^H, x^L\right)$, $t_0^H\left(x^H, x^L\right) = 1$, $t_0^L\left(x^H, x^L\right) = 0$ for $y^H \in \left(\dfrac{\check{x}_1^H}{1-\mu}, 1\right]$ and $x^H \in \left[\check{x}_1^H, 1-\mu\right]$ and all $x^L \ge 0$ and $P_0^H\left(y^H, y^L\right) = \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L$, $P_0^L\left(y^H, y^L\right) = \underline{\theta} q_L$ and $t_0^H\left(x^H, x^L\right) = \widehat{b}$, $t_0^L\left(x^H, x^L\right) = 1-\widehat{b}$, $R_0\left(x^H, x^L\right) = \widehat{R}\left(x^H, x^L\right)$ for all $y^H \in \left[0, \dfrac{\check{x}_1^H}{1-\mu}\right]$ and $x^H \in \left[0, \check{x}_1^H\right]$ and all y^L. $R_0\left(x^H, x^L\right) = \widehat{R}\left(x^H, x^L\right)$ for all $x^H \le \check{x}_1^H$. Continuity of R_0 at $x^H = \check{x}_1^H$ implies that $\widehat{R}\left(\check{x}_1^H, x^L\right) = R_1\left(\check{x}_1^H, x^L\right)$. Therefore, $\check{x}_1^H = \widehat{x}^* = \widehat{x}_1^H$.Define $\ddot{p}_2^L = \sup\limits_{y^H > \widehat{b}}\left\{P_0^L\left(y^H, y^L\right) \mid y^L > \dfrac{\check{x}_1^L}{1-\mu}\right\}$. Then the same argument can be applied inductively.

Proof of Corollary 10. First, we derive the threshold for x^L. Define $g\left(x\right) = \Pi\left(\widetilde{y}_k, \left(x^H, x\right)\right) -$

$\Pi\left(\widehat{y}_k, \left(x^H, x\right)\right)$ where

$$\Pi\left(\widetilde{y}_k, \left(x^H, x^L\right)\right) = \left(\frac{\mu y_k^H}{1-\delta} - \left(x^H - x_k^H\right)\right)\widetilde{p}_k^H - x^L \widetilde{p}_k^L$$

and

$$\Pi\left(\widehat{y}_k, \left(x^H, x^L\right)\right) = \left(\frac{\mu y_k^H}{1-\delta} - \left(x^H - x_k^H\right)\right)\widehat{p}_k^H + \left(\frac{\mu\left(1-y_k^H\right)}{1-\delta} - \left(x^L - \left(1-\mu - x_k^H\right)\right)\right)\underline{\theta} q_L.$$

The threshold x^L, $\overline{x}^L\left(x^H\right)$, solves $g\left(x\right) = 0$ Since $\dfrac{dg\left(x\right)}{d\left(x\right)} < 0$. Therefore, for all $x^L \leq \overline{x}^L\left(x^H\right)$ the seller prefers $\widetilde{y}_k$ to $\widehat{y}_k$; otherwise, the seller prefers $\widehat{y}_k$ to $\widetilde{y}_k$. When the time horizon between two successive offers is z,

$$\overline{x}^L\left(x^H\right) = y_k^H\left(\frac{\underline{\theta}}{\rho^k \Delta\theta} + 1\right)\frac{1-\rho}{1-\delta} - \frac{\left(1-\rho\right)\underline{\theta}}{\left(1-\delta\right)\rho^k \Delta\theta} - x^H.$$

In the limit, as the length of time period z approaches zero, we have

$$\overline{x}^L\left(x^H\right) = y_k^H \frac{\overline{\theta}}{\Delta\theta}\frac{\lambda + r}{r} - \frac{\underline{\theta}}{\Delta\theta}\frac{\lambda + r}{r} - x^H.$$

Second, we derive the market prices of the goods.

i. Derivation of prices on the standard monopoly path, $P^H\left(\widetilde{y}_k^H\right)$: Since $P^H\left(\widetilde{y}_k^H\right) = \widetilde{p}_k = \underline{\theta} q_H + \rho^k \Delta\theta q_H$ we have $\widetilde{p}_k - \widetilde{p}_{k-1} = -\rho^{k-1}\left(1-\rho\right)\Delta\theta q_H$. This implies that

$$\frac{P^H\left(\widetilde{y}_k^H\right) - P^H\left(\widetilde{y}_{k-1}^H\right)}{\widetilde{y}_k^H - \widetilde{y}_k^H} = \frac{-\rho^{k-1}\left(1-\rho\right)\Delta\theta q_H\left(\mu\underline{\theta} q_H + \left(1-\mu\right)\widetilde{p}_{k-1}^H\right)}{\widetilde{y}_{k-1}^H\left(\widetilde{p}_{k-1}^H - \left(\mu\underline{\theta} q_H + \left(1-\mu\right)\widetilde{p}_{k-1}^H\right)\right)}$$

Since $\widetilde{p}_k^H - \underline{\theta} q_H = \rho^k \Delta\theta q_H$,

$$\frac{P^H\left(\widetilde{y}_k^H\right) - P^H\left(\widetilde{y}_{k-1}^H\right)}{\widetilde{y}_k^H - \widetilde{y}_{k-1}^H} = \frac{-\left(1-\rho\right)\left(\widetilde{p}_{k-1}^H - \mu\left(\widetilde{p}_{k-1}^H - \underline{\theta} q_H\right)\right)}{\mu \widetilde{y}_{k-1}^H}.$$

As z approaches 0, $\widetilde{y}_k^H - \widetilde{y}_{k-1}^H$ converges to zero. Hence, the above equation converges to[6]

$$P_1^H(y,0)\,y = -\frac{\lambda + r}{\lambda} P^H(y,0)\,.$$

The solution of this differential equation gives us $P^H(y,0) = \widetilde{k}_0 y^{-\dfrac{\lambda+r}{\lambda}}$. Since $P^H\left(\widehat{b},0\right) = \overline{\theta} q_H$, $\widetilde{k}_0$ is defined by $\widetilde{k}_0 = \overline{\theta} q_H \left(\widehat{b}\right)^{\dfrac{\lambda+r}{\lambda}}$. Therefore,

$$P^H(y,0) = \overline{\theta} q_H \left(\frac{\widehat{b}}{y}\right)^{\dfrac{\lambda+r}{\lambda}}.$$

ii. Derivation of prices on the segmented monopoly path, $P^H\left(\widetilde{y}_k^H\right)$*:* Since $P^H\left(\widehat{y}_k^H\right) = \widehat{p}_k = \underline{\theta} q_H + \rho^k \Delta\theta\Delta q$ we have $\widehat{p}_k - \widehat{p}_{k-1} = -\rho^{k-1}(1-\rho)\Delta\theta\Delta q$. This implies that

$$\frac{P^H\left(\widehat{y}_k^H\right) - P^H\left(\widehat{y}_{k-1}^H\right)}{\widehat{y}_k^H - \widehat{y}_k^H} = \frac{-\rho^{k-1}(1-\rho)\Delta\theta\Delta q\left(\mu\underline{\theta} q_H - \underline{\theta} q_L + (1-\mu)\widehat{p}_{k-1}^H\right)}{\widehat{y}_{k-1}^H\left(-\mu\underline{\theta} q_H + \mu\widehat{p}_{k-1}^H\right)}.$$

Since $\widehat{p}_k^H - \underline{\theta} q_H = \rho^k \Delta\theta\Delta q$,

$$\frac{P^H\left(\widehat{y}_k^H\right) - P^H\left(\widehat{y}_{k-1}^H\right)}{\widehat{y}_k^H - \widehat{y}_{k-1}^H} = \frac{-(1-\rho)\left(\mu\underline{\theta} q_H - \underline{\theta} q_L + (1-\mu)\widehat{p}_{k-1}^H\right)}{\mu\widehat{y}_{k-1}^H}.$$

As z approaches 0, $\widehat{y}_k^H - \widehat{y}_{k-1}^H$ converges to zero. Hence, the above equation converges to[7]

$$P_1^H(y,1-y)\,y = -\frac{\lambda + r}{\lambda}\left(P^H(y,1-y) - \underline{\theta} q_L\right).$$

---

[6]Since $\lim_{z\to 0^+} \dfrac{P^H\left(\widetilde{y}_k^H\right) - P^H\left(\widetilde{y}_{k-1}^H\right)}{\widetilde{y}_k^H - \widetilde{y}_{k-1}^H} = P_1^H(y,0)$ and $\lim_{z\to 0^+} \dfrac{-(1-\rho)\left(\widetilde{p}_{k-1}^H - \mu\left(\widetilde{p}_{k-1}^H - \underline{\theta} q_H\right)\right)}{\mu\widetilde{y}_{k-1}^H} = -\dfrac{\lambda+r}{\lambda}\dfrac{P^H(y,0)}{y}$.

[7]Since
$\lim_{z\to 0^+} \dfrac{P^H\left(\widehat{y}_k^H\right) - P^H\left(\widehat{y}_{k-1}^H\right)}{\widehat{y}_k^H - \widehat{y}_{k-1}^H} = P_1^H(y,1-y)$
$\lim_{z\to 0^+} \dfrac{-(1-\rho)\left(\mu\underline{\theta} q_H - \underline{\theta} q_L + (1-\mu)\widehat{p}_{k-1}^H\right)}{\mu\widehat{y}_{k-1}^H} = -\dfrac{\lambda+r}{\lambda}\dfrac{\left(P^H(y,0) - \underline{\theta} q_L\right)}{y}$.

The solution of this differential equation gives us $P^H(y,1-y) = \widehat{k}_0 y^{-\dfrac{\lambda+r}{\lambda}} + \underline{\theta} q_L$. Since $P^H\left(\widehat{b},0\right) = \overline{\theta} q_H - \overline{\theta} q_L + \underline{\theta} q_L$, $\widehat{k}_0$ is defined by $\widehat{k}_0 = \overline{\theta}\Delta q\left(\widehat{b}\right)^{\dfrac{\lambda+r}{\lambda}}$. Therefore,

$$P^H(y,1-y) = \overline{\theta}\Delta q\left(\frac{\widehat{b}}{y}\right)^{\dfrac{\lambda+r}{\lambda}} + \underline{\theta} q_L.$$

Third, we derive $\dot{y}$.

i. Derivation of $\dot{y}$ on the standard monopoly path, $\dfrac{d\widetilde{y}}{dz}$: Upon dividing both sides by z, the equation for $\widetilde{y}_k^H - \widetilde{y}_{k-1}^H$ becomes

$$\frac{\widetilde{y}_k^H - \widetilde{y}_{k-1}^H}{z} = \frac{\mu\widetilde{y}_{k-1}^H}{z}\left(\frac{\widetilde{p}_{k-1}^H - \underline{\theta} q_H}{\mu\underline{\theta} q_H + (1-\mu)\,\widetilde{p}_{k-1}^H}\right).$$

Since $\lim_{z\to 0^+}\dfrac{\widetilde{y}_k^H - \widetilde{y}_{k-1}^H}{z} = \dfrac{d\widetilde{y}}{dz}$ and $\lim_{z\to 0^+}\dfrac{\mu\widetilde{y}_{k-1}^H}{z}\left(\dfrac{\widetilde{p}_{k-1}^H - \underline{\theta} q_H}{\mu\underline{\theta} q_H + (1-\mu)\,\widetilde{p}_{k-1}^H}\right) = \lambda\widetilde{y}\dfrac{P^H(y,0) - \underline{\theta} q_H}{P^H(y,0)}$, we have

$$\dot{\widetilde{y}} = \lambda\widetilde{y}\frac{P^H(y,0) - \underline{\theta} q_H}{P^H(y,0)} = \lambda\widetilde{y}\left(1 - \frac{\underline{\theta}}{\overline{\overline{\theta}}}\left(\frac{\widetilde{y}}{\widehat{b}}\right)^{\dfrac{\lambda+r}{\lambda}}\right).$$

ii. Derivation of $\dot{y}$ on the segmented monopoly path, $\dfrac{d\widehat{y}}{dz}$: Upon dividing both sides by z, the equation for $\widehat{y}_k^H - \widehat{y}_{k-1}^H$ becomes

$$\frac{\widehat{y}_k^H - \widehat{y}_{k-1}^H}{z} = \frac{\mu\widehat{y}_{k-1}^H}{z}\left(\frac{\widehat{p}_{k-1}^H - \underline{\theta} q_H}{\mu\underline{\theta} q_H - \underline{\theta} q_L + (1-\mu)\,\widehat{p}_{k-1}^H}\right).$$

By the same reasoning we have, $\dfrac{d\widehat{y}}{dz} = \lambda\widehat{y}\left(1 - \dfrac{\underline{\theta}\,(q_H - q_L)}{P^H(\widehat{y},1-\widehat{y}) - \underline{\theta} q_L}\right) = \lambda\widehat{y}\left(1 - \dfrac{\underline{\theta}}{\overline{\overline{\theta}}}\left(\dfrac{\widehat{y}}{\widehat{b}}\right)^{\dfrac{\lambda+r}{\lambda}}\right)$.

Proof of Corollary 11. See the proof of Theorem 5.

Proof of Theorem 6. Consider a reputational equilibria in which $\left(\dot{y}^H, \dot{y}^L\right)$ where $\dot{y}^H < \widehat{b}$ and $\dot{y}^H + \dot{y}^L \leq \widehat{b}$ is a steady state. By the proof of Proposition 3, we have $P^H\left(b^H\right) = \overline{\theta} q_H$ for $b \leq \dot{y}^H$, $P^H\left(b^H\right) < \overline{\theta} q_H$ for $b \in \left(\dot{y}^H, \widehat{b}\right]$, and $P^H\left(b^H\right) = \underline{\theta} q_H$ for $b \in \left(\widehat{b}, 1\right]$. Moreover, the proof of Proposition 3 also implies that $P^L(b) = \overline{\theta} q_L$ for $b \leq \dot{y}^H + \dot{y}^L$, $P^L(b) < \overline{\theta} q_L$ for

$b \in \left(\grave{y}^H + \grave{y}^L, \widehat{b}\right]$, and $P^L(b) = \underline{\theta} q_L$ for $b \in \left(\widehat{b}, 1\right]$. Therefore, for $b \in \left(\grave{y}^H, 1\right]$, the stock of high quality good is strictly increasing and for $b \in \left(\grave{y}^H + \grave{y}^L, 1\right]$, total stock of durable goods (high quality and low quality) is strictly increasing. It follows that when the stock of high quality good is greater than $\grave{y}^H$, then the stationary pentad must coincide with the Coase Conjecture equilibrium pentad.

The proof has three main parts. First, we show that the existence of a reputational equilibrium implies $\mu \in \left(\underline{\mu}, \overline{\mu}\right]$. Then, we show that $\underline{\mu} < \overline{\mu}$. Finally, we prove that for each $\mu \in \left(\underline{\mu}, \overline{\mu}\right]$ there exists a reputational equilibrium.

First, we prove that $\mu > \underline{\mu}$ if a reputational equilibrium exists. Since $P^H(b) = \overline{\theta} q_H$ for all $b \leq \grave{y}^H$ and $P^L(b) = \overline{\theta} q_L$ for all $b \leq \grave{y}^H + \grave{y}^L$, we must have $t^H\left((1-\mu) y^H, (1-\mu) y^L\right) = \grave{y}^H$ and $t^L\left((1-\mu) y^H, (1-\mu) y^L\right) = \grave{y}^L$ for all $y^H \leq \grave{y}^H$ and $y^L \leq \grave{y}^H + \grave{y}^L$. This implies that $R\left((1-\mu)\grave{y}^H, (1-\mu)\grave{y}^L\right) = \dfrac{\mu \grave{y}^H \overline{\theta} q_H}{1-\delta} + \dfrac{\mu \grave{y}^L \overline{\theta} q_L}{1-\delta}$. By the continuity of function R, $\grave{y}^H$ and $\grave{y}^L$ solve $\zeta\left(y^H, y^L\right) = 0$ where $\zeta(\cdot)$ is defined as

$$\zeta\left(y^H, y^L\right) = \frac{\mu y^H \overline{\theta} q_H}{1-\delta} + \frac{\mu y^L \overline{\theta} q_L}{1-\delta} - R^C\left((1-\mu) y^H, (1-\mu) y^L\right)$$

with $y^H \leq \grave{y}^H$ and $y^L \leq \grave{y}^H + \grave{y}^L$. Since $\zeta\left(\widehat{b}, 0\right) = 0$ at $\underline{\mu}$, we have $\zeta\left(\widehat{b}, 0\right) < 0$ for all $\mu < \underline{\mu}$. Moreover, since ζ is strictly increasing, the existence of $0 < \grave{y}^H < \widehat{b}$ and $0 \leq \grave{y}^H + \grave{y}^L \leq \widehat{b}$ requires $\mu > \underline{\mu}$.

Second, we prove that the existence of a reputational equilibrium implies $\mu \leq \overline{\mu}$. The proof follows a contradiction. Suppose that for some $\mu > \overline{\mu}$, a reputational equilibrium with $\left(\grave{y}^H, \grave{y}^L\right)$ exists. Let's define

$$\overline{K} = \min\left\{k : \overline{x}^H_j < \overline{x}^H_{j-1} \text{ for all } j \leq k\right\}$$

and

$$\widetilde{K} = \min\left\{k : \widetilde{x}^H_j < \widetilde{x}^H_{j-1} \text{ for all } j \leq k\right\}.$$

Since the Coase Conjecture equilibrium does not exist for $\mu > \overline{\mu}$, we have $\overline{x}^H_{\overline{K}} > 0$ and $\widetilde{x}^H_{\widetilde{K}} > 0$. Since, $\overline{x}^H_{\overline{K}+1} > \overline{x}^H_{\overline{K}}$, when the initial state is $\left(\overline{x}^H_{\overline{K}}, x^L\right)$, for all $x^L \leq (1-\mu)\widehat{b} - \overline{x}^H_{\overline{K}}$, the monopolist prefers selling to $\left(\overline{y}^H_{\overline{K}}, y^L\right)$ to selling to $\left(\overline{y}^H_{\overline{K}-1}, \widehat{b} - \overline{y}^H_{\overline{K}-1}\right)$. That is,

$$\mu p^H \overline{y}^H_{\overline{K}} + \mu p^L y^L + \delta R^C \left(\overline{x}^H_{\overline{K}}, x^L\right) > R^C \left(\overline{x}^H_{\overline{K}}, x^L\right)$$

which can be rewritten as

$$\frac{\mu p^H \overline{y}^H_{\overline{K}}}{1-\delta} + \frac{\mu p^L \overline{y}^H_{\overline{K}}}{1-\delta} > R^C \left(\overline{x}^H_{\overline{K}}, x^L\right).$$

Therefore, $\zeta\left(\overline{y}^H_{\overline{K}}, y^L\right) > 0$. Since $\zeta(\cdot)$ is increasing we have $\grave{y}^H < \overline{y}^H_{\overline{K}}$. This is not possible because the sequence of the Coase Conjecture equilibrium cannot be extended to initial states below $\overline{x}^H_{\overline{K}}$.

Now, we will show that $\underline{\mu} < \overline{\mu}$. To do that we need to prove that if $\mu \leq \overline{\mu}$ then the sequence $\left\{\overline{x}^H_k\right\}$ and $\left\{\widetilde{x}^H_k\right\}$ are strictly decreasing and m and m' are finite. Because the continuity of $\overline{x}^H_k$ and $\widetilde{x}^H_k$ in μ implies that the same property holds in a right neighborhood of μ which implies that $\underline{\mu} \geq \overline{\mu}$. The proof of Theorem 2 establishes that $\mu \leq \overline{\mu}$ iff $\frac{\mu \widehat{b\theta} q_H}{1-\delta} \leq \Pi\left((1,0), \left((1-\mu)\widehat{b}, 0\right)\right)$ which is equivalent to

$$\frac{\mu\widehat{b}\left(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L\right)}{1-\delta} + \frac{\mu\left(1-\widehat{b}\right)\underline{\theta}q_L}{1-\delta} \leq \Pi((1,0),((1-\mu)\widehat{b},(1-\mu)(1-\widehat{b}))).$$

To prove that the sequence $\left\{\overline{x}^H_k\right\}$ and $\left\{\widetilde{x}^H_k\right\}$ are strictly decreasing and m and m' are finite, we use induction. First, we show that $x^H_2 < x^H_1$ hold for any $\mu \leq \underline{\mu}$. To establish that $\overline{x}^H_2 < \overline{x}^H_1$, we need to show that $\overline{h}_2\left(\overline{x}^H_1\right)$ which implies that $\overline{x}^H_2 < \overline{x}^H_1$, since $\overline{h}_2(\cdot)$ is decreasing and $\overline{h}_2\left(\overline{x}^H_2\right) = 0$. We define $\overline{h}_2\left(\overline{x}^H_1\right)$ as

$$\begin{aligned}
\overline{h}_2\left(\overline{x}^H_1\right) &= \Pi\left(\left(\overline{y}^H_1, 0\right); \left(\overline{x}^H_1, 0\right)\right) - \Pi\left(\left(\overline{y}^H_0, 0\right); \left(\overline{x}^H_1, 0\right)\right) \\
&= P^H\left(\overline{y}^H_1, 0\right)\mu\widehat{b} + \delta\Pi\left(\left(\overline{y}^H_0, 0\right); \left(\overline{x}^H_1, 0\right)\right) - \Pi\left(\left(\overline{y}^H_0, 0\right); \left(\overline{x}^H_1, 0\right)\right) \\
&= P^H\left(\overline{y}^H_1, 0\right)\mu\widehat{b} - (1-\delta)\Pi\left(\left(\overline{y}^H_0, 0\right); \left(\overline{x}^H_1, 0\right)\right)
\end{aligned}$$

Since $\Pi\left(\left(\overline{y}_0^H,0\right);\left(\overline{x}_1^H,0\right)\right) \geq \mu\widehat{b}\overline{\theta}q_H$, we have

$$\overline{h}_2\left(\overline{x}_1^H\right) \leq P^H\left(\overline{y}_1^H,0\right)\mu\widehat{b} - \mu\widehat{b}\overline{\theta}q_H = \mu\widehat{b}\left(P^H\left(\overline{y}_1^H,0\right) - \overline{\theta}q_H\right) < 0.$$

Similarly to establish that $\widetilde{x}_2^H < \widetilde{x}_1^H$, we need to show that $\widetilde{h}_2\left(\widetilde{x}_1^H\right)$ which implies that $\widetilde{x}_2^H < \widetilde{x}_1^H$, since $\widetilde{h}_2(\cdot)$ is decreasing and $\widetilde{h}_2\left(\widetilde{x}_2^H\right) = 0$. We define $\widetilde{h}_2\left(\widetilde{x}_1^H\right)$ as

$$\begin{aligned}
\widetilde{h}_2\left(\widetilde{x}_1^H\right) &= \Pi\left(\left(\widetilde{y}_1^H,1-\widetilde{y}_1^H\right);\left(\widetilde{x}_1^H,1-\mu-\widetilde{x}_1^H\right)\right) \\
&\qquad - \Pi\left(\left(\widetilde{y}_0^H,1-\widetilde{y}_0^H\right);\left(\widetilde{x}_1^H,1-\mu-\widetilde{x}_1^H\right)\right) \\
&= P^H\left(\widetilde{y}_1^H,1-\widetilde{y}_1^H\right)\mu\widehat{b} + P^L\left(\widetilde{y}_1^H,1-\widetilde{y}_1^H\right)\mu\left(1-\widehat{b}\right) \\
&\qquad - (1-\delta)\,\Pi\left(\left(\widetilde{y}_0^H,1-\widetilde{y}_0^H\right);\left(\widetilde{x}_1^H,1-\mu-\widetilde{x}_1^H\right)\right).
\end{aligned}$$

Since $\Pi\left(\left(\widetilde{y}_0^H,1-\widetilde{y}_0^H\right);\left(\widetilde{x}_1^H,1-\mu-\widetilde{x}_1^H\right)\right) \geq \mu\left(\left(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L\right)\widehat{b} + \underline{\theta}q_L\left(1-\widehat{b}\right)\right)$, we have

$$\begin{aligned}
\widetilde{h}_2\left(\widetilde{x}_1^H\right) &\leq \left(P^H\left(\widetilde{y}_1^H,1-\widetilde{y}_1^H\right) - \left(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L\right)\right)\mu\widehat{b} \\
&\qquad + \left(P^L\left(\widetilde{y}_1^H,1-\widetilde{y}_1^H\right) - \underline{\theta}q_L\right)\mu\left(1-\widehat{b}\right) < 0.
\end{aligned}$$

To establish the second step of induction, we need to show that there exist Δ such that $x_3^H < x_2^H - \Delta$. The idea is the same. We prove that $\overline{h}_3\left(\overline{x}_2^H\right) < 0$ which directly implies that $\overline{x}_3^H < \overline{x}_2^H$ since $\overline{h}_3(\cdot)$ is decreasing and $\overline{h}_3\left(\overline{x}_3^H\right) = 0$. We define $\overline{h}_3\left(\overline{x}_2^H\right)$ as

$$\begin{aligned}
\overline{h}_3\left(\overline{x}_2^H\right) &= \Pi\left(\overline{y}_2;\overline{x}_2\right) - \Pi\left(\overline{y}_1;\overline{x}_2\right) \\
&= P^H\left(\overline{y}_2\right)\mu\overline{y}_2^H + P^L\left(\overline{y}_2\right)\mu\left(\widehat{b} - \overline{y}_2^H\right) - (1-\delta)\,\Pi\left(\overline{y}_1;\overline{x}_2\right) \\
&= P^H\left(\overline{y}_2\right)\mu\overline{y}_2^H + P^L\left(\overline{y}_2\right)\frac{\mu}{1-\mu}\left((1-\mu)\,\widehat{b} - \overline{x}_2^H\right) - (1-\delta)\,\Pi\left(\overline{y}_1;\overline{x}_2\right).
\end{aligned}$$

We know that $\Pi\left(\overline{y}_1;\overline{x}_2\right) = \Pi\left(\overline{y}_0;\overline{x}_2\right)$. This implies that $\Pi\left(\overline{y}_1;\overline{x}_2\right) = \left(\overline{x}_1^H - \overline{x}_2^H\right)\underline{\theta}q_H -$

$\left((1-\mu)\widehat{b}-\overline{x}_2^H\right)\underline{\theta}q_L+\Pi\left(\overline{y}_0;\overline{x}_1\right)$. Therefore,

$$
\begin{aligned}
\overline{h}_3\left(\overline{x}_2^H\right) &< P^H\left(\overline{y}_2\right)\frac{\mu\overline{x}_2^H}{1-\mu}+P^L\left(\overline{y}_2\right)\left(\mu\widehat{b}-\frac{\mu\overline{x}_2^H}{1-\mu}\right)-(1-\delta)\left(\overline{x}_1^H-\overline{x}_2^H\right)\underline{\theta}q_H \\
&\quad+(1-\delta)\left((1-\mu)\widehat{b}-\overline{x}_2^H\right)\underline{\theta}q_L-\mu\widehat{b}\overline{\theta}q_H \\
&=\left(P^H\left(\overline{y}_2\right)-P^L\left(\overline{y}_2\right)\right)\frac{\mu\overline{x}_2^H}{1-\mu}+P^L\left(\overline{y}_2\right)\mu\widehat{b}-(1-\delta)\left(\overline{x}_1^H-\overline{x}_2^H\right)\underline{\theta}\Delta q-\mu\widehat{b}\overline{\theta}q_H \\
&=\left(P^H\left(\overline{y}_2\right)-P^L\left(\overline{y}_2\right)\right)\frac{\mu\overline{x}_2^H}{1-\mu}-\left(\overline{\theta}q_H-P^L\left(\overline{y}_2\right)\right)\mu\widehat{b}-(1-\delta)\left(\overline{x}_1^H-\overline{x}_2^H\right)\underline{\theta}\Delta q \\
&<\left(\left(P^H\left(\overline{y}_2\right)-P^L\left(\overline{y}_2\right)\right)-\left(\overline{\theta}q_H-P^L\left(\overline{y}_2\right)\right)\right)\frac{\mu\overline{x}_2^H}{1-\mu}-(1-\delta)\left(\overline{x}_1^H-\overline{x}_2^H\right)\underline{\theta}\Delta q \\
&=\left(P^H\left(\overline{y}_2\right)-\overline{\theta}q_H\right)\frac{\mu\overline{x}_2^H}{1-\mu}-(1-\delta)\left(\overline{x}_1^H-\overline{x}_2^H\right)\underline{\theta}\Delta q<0.
\end{aligned}
$$

Similarly we now prove that $\widetilde{h}_3\left(\widetilde{x}_2^H\right)<0$ which directly implies that $\widetilde{x}_3^H<\widetilde{x}_2^H$ since $\widetilde{h}_3(\cdot)$ is decreasing and $\widetilde{h}_3\left(\widetilde{x}_3^H\right)=0$. We define $\widetilde{h}_3\left(\widetilde{x}_2^H\right)$ as

$$
\begin{aligned}
\widetilde{h}_3\left(\widetilde{x}_2^H\right) &=\Pi\left(\widetilde{y}_2;\widetilde{x}_2\right)-\Pi\left(\widetilde{y}_1;\widetilde{x}_2\right) \\
&=P^H\left(\widetilde{y}_2\right)\mu\widetilde{y}_2^H+P^L\left(\widetilde{y}_2\right)\mu\left(1-\widetilde{y}_2^H\right)-(1-\delta)\Pi\left(\widetilde{y}_1;\widetilde{x}_2\right) \\
&=P^H\left(\widetilde{y}_2\right)\mu\widetilde{y}_2^H+P^L\left(\widetilde{y}_2\right)\frac{\mu}{1-\mu}\left((1-\mu)-\widetilde{x}_2^H\right)-(1-\delta)\Pi\left(\widetilde{y}_1;\widetilde{x}_2\right).
\end{aligned}
$$

We know that $\Pi\left(\widetilde{y}_1;\widetilde{x}_2\right)=\Pi\left(\widetilde{y}_0;\widetilde{x}_2\right)$. This implies that $\Pi\left(\widetilde{y}_1;\widetilde{x}_2\right)=\left(\overline{x}_1^H-\overline{x}_2^H\right)\underline{\theta}q_H-\left((1-\mu)\widehat{b}-\overline{x}_2^H\right)\underline{\theta}q_L+\Pi\left(\overline{y}_0;\overline{x}_1\right)$. Therefore,

$$
\begin{aligned}
\widetilde{h}_3\left(\widetilde{x}_2^H\right) &< P^H\left(\widetilde{y}_2\right)\frac{\mu\widetilde{x}_2^H}{1-\delta}+P^L\left(\widetilde{y}_2\right)(\mu-\frac{\mu\widetilde{x}_2^H}{1-\delta})-(1-\delta)\left(1-\mu-\widetilde{x}_2^H\right)\underline{\theta}\Delta q-(1-\delta)\Pi\left(\widetilde{y}_0;\widetilde{x}_0\right) \\
&=\left(P^H\left(\widetilde{y}_2\right)-P^L\left(\widetilde{y}_2\right)\right)\frac{\mu\widetilde{x}_2^H}{1-\mu}+P^L\left(\widetilde{y}_2\right)\mu-(1-\delta)\left(1-\mu-\overline{x}_2^H\right)\underline{\theta}\Delta q \\
&\quad-\mu\widehat{b}\left(\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L\right)-\mu\left(1-\widehat{b}\right)\underline{\theta}q_L \\
&=\left(P^H\left(\widetilde{y}_2\right)-P^L\left(\widetilde{y}_2\right)\right)\frac{\mu\widetilde{x}_2^H}{1-\delta}-(1-\delta)\left(1-\mu-\widetilde{x}_2^H\right)\underline{\theta}\Delta q-\mu\widehat{b}\left(\overline{\theta}q_H-\overline{\theta}q_L\right) \\
&<\left(P^H\left(\widetilde{y}_2\right)-P^L\left(\widetilde{y}_2\right)-\overline{\theta}q_H+\overline{\theta}q_L\right)\frac{\mu\widetilde{x}_2^H}{1-\delta} \\
&\quad-(1-\delta)\left(1-\mu-\widetilde{x}_2^H\right)\underline{\theta}\Delta q<0.
\end{aligned}
$$

Now, suppose that there exist Δ such that $x_k^H < x_{k-1}^H - \Delta$ for $k \geq 3$. We will show that it also holds for $k+1$. Since the sequence is decreasing until $k-1$, we have

$$
\begin{aligned}
\Pi\left(\overline{y}_{k-1}, \overline{x}_k\right) \geq \Pi\left(\overline{y}_0, \overline{x}_k\right) & \\
&= \left(1 - \overline{x}_k\right) \underline{\theta} q_H - \left(\left(1-\mu\right)\widehat{b} - \overline{x}_k\right) \underline{\theta} q_L + \delta R\left(\overline{x}_0\right) \\
&= \Pi\left(\overline{y}_0, \overline{x}_1\right) + \left(\overline{x}_1^H - \overline{x}_k^H\right) \underline{\theta} q_H - \left(\left(1-\mu\right)\widehat{b} - \overline{x}_k\right) \underline{\theta} q_L \\
&\geq \frac{\mu \widehat{b\overline{\theta}} q_H}{1-\delta} + \left(\overline{x}_1^H - \overline{x}_k^H\right) \underline{\theta} q_H - \left(\left(1-\mu\right)\widehat{b} - \overline{x}_k\right) \underline{\theta} q_L.
\end{aligned}
$$

Therefore, we have $\Pi\left(\overline{y}_{k-1}, \overline{x}_k\right) \geq \dfrac{\mu \widehat{b\overline{\theta}} q_H}{1-\delta} + \left(\overline{x}_1^H - \overline{x}_k^H\right) \underline{\theta} \Delta q$. The last inequality follows the first inequality stated above.

Now, we will show that $\overline{h}_{k+1}\left(\overline{x}_k^H\right) = \Pi\left(\overline{y}_k, \overline{x}_k\right) - \Pi\left(\overline{y}_{k-1}, \overline{x}_k\right) < 0$. Since,

$$
\Pi\left(\overline{y}_k, \overline{x}_k\right) = \mu \overline{y}_k^H P^H\left(\overline{y}_k\right) + \mu\left(\widehat{b} - \overline{y}_k^H\right) P^L\left(\overline{y}_k\right) + \delta \Pi\left(\overline{y}_{k-1}, \overline{x}_k\right)
$$

we have

$$
\overline{h}_{k+1}\left(\overline{x}_k^H\right) = \mu \overline{y}_k^H\left(P^H\left(\overline{y}_k\right) - P^L\left(\overline{y}_k\right)\right) + \mu \widehat{b} P^L\left(\overline{y}_k\right) - \left(1-\delta\right) \Pi\left(\overline{y}_{k-1}, \overline{x}_k\right).
$$

By using the above inequality

$$
\begin{aligned}
\overline{h}_{k+1}\left(\overline{x}_k^H\right) \geq \mu \overline{y}_k^H\left(P^H\left(\overline{y}_k\right) - P^L\left(\overline{y}_k\right)\right) + \mu \widehat{b} P^L\left(\overline{y}_k\right) & \\
& - \mu \widehat{b\overline{\theta}} q_H - \left(1-\delta\right)\left(\overline{x}_1^H - \overline{x}_k^H\right) \underline{\theta} \Delta q \\
= \mu \overline{y}_k^H\left(P^H\left(\overline{y}_k\right) - P^L\left(\overline{y}_k\right)\right) - \mu \widehat{b}\left(\overline{\theta} q_H - P^L\left(\overline{y}_k\right)\right) & - \left(1-\delta\right)\left(\overline{x}_1^H - \overline{x}_k^H\right) \underline{\theta} \Delta q \\
& < -\mu \overline{y}_k^H\left(\overline{\theta} q_H - P^H\left(\overline{y}_k\right)\right) - \left(1-\delta\right)\left(\overline{x}_1^H - \overline{x}_k^H\right) \underline{\theta} \Delta q < 0.
\end{aligned}
$$

Therefore, since $\overline{h}_{k+1}\left(\overline{x}_{k+1}^H\right) = 0$ and $\overline{h}_{k+1}(\cdot)$ is decreasing, we must have $\overline{x}_{k+1}^H < \overline{x}_k^H$.

Similarly,

$$\Pi\left(\widetilde{y}_{k-1},\widetilde{x}_k\right)\geq\Pi\left(\widetilde{y}_0,\widetilde{x}_k\right)=\left(1-\widetilde{x}_k^H\right)\underline{\theta}q_H-\left((1-\mu)-\widetilde{x}_k^H\right)\underline{\theta}q_L+\delta R\left(\widetilde{x}_0\right)$$
$$=\Pi\left(\widetilde{y}_0,\overline{x}_1\right)+\left(\widetilde{x}_1^H-\widetilde{x}_k^H\right)\underline{\theta}q_H-\left((1-\mu)-\widetilde{x}_1^H\right)\underline{\theta}q_L-\left((1-\mu)-\widetilde{x}_k^H\right)\underline{\theta}q_L$$
$$=\Pi\left(\widetilde{y}_0,\overline{x}_1\right)+\left(\widetilde{x}_1^H-\widetilde{x}_k^H\right)\Delta q$$
$$\geq\frac{\mu\widehat{b}\left(\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L\right)}{1-\delta}+\frac{\mu\left(1-\widehat{b}\right)\underline{\theta}q_L}{1-\delta}+\left((1-\mu)\,\widehat{b}-\overline{x}_k\right)\underline{\theta}\Delta q.$$

The last inequality follows the first inequality stated above.

Now, we will show that $\widetilde{h}_{k+1}\left(\widetilde{x}_k^H\right)=\Pi\left(\widetilde{y}_k,\widetilde{x}_k\right)-\Pi\left(\widetilde{y}_{k-1},\widetilde{x}_k\right)<0$. Since,

$$\Pi\left(\widetilde{y}_k,\widetilde{x}_k\right)=\mu\overline{y}_k^HP^H\left(\widetilde{y}_k\right)+\mu\left(1-\widetilde{y}_k^H\right)P^L\left(\widetilde{y}_k\right)+\delta\Pi\left(\widetilde{y}_{k-1},\widetilde{x}_k\right)$$

we have

$$\widetilde{h}_{k+1}\left(\widetilde{x}_k^H\right)=\mu\widetilde{y}_k^H\left(P^H\left(\widetilde{y}_k\right)-P^L\left(\widetilde{y}_k\right)\right)+\mu P^L\left(\widetilde{y}_k\right)-(1-\delta)\,\Pi\left(\widetilde{y}_{k-1},\widetilde{x}_k\right).$$

By using the above inequality, we will have $\widetilde{h}_{k+1}\left(\widetilde{x}_k^H\right)<\mu\widetilde{y}_k^HP^H\left(\widetilde{y}_k\right)+\mu(1-\widetilde{y}_k^H)P^L\left(\widetilde{y}_k\right)$ $-\mu\widehat{b}(\overline{\theta}q_H-\overline{\theta}q_L+\underline{\theta}q_L)$ $-\mu(1-\widehat{b})\underline{\theta}q_L-(1-\delta)\left((1-\mu)\widehat{b}-\widetilde{x}_k^H\right)\underline{\theta}\Delta q=\mu\widetilde{y}_k^H(P^H\left(\widetilde{y}_k\right)-P^L\left(\widetilde{y}_k\right))+$ $\mu P^L\left(\widetilde{y}_k\right)$ $-\mu((\widehat{b\overline{\theta}}q_H-\overline{\theta}q_L)+(1-\widehat{b})\underline{\theta}q_L)$ $-(1-\delta)((1-\mu)\widehat{b}-\widetilde{x}_k^H)\underline{\theta}\Delta q=\mu\widetilde{y}_k^H(P^H\left(\widetilde{y}_k\right)-P^L\left(\widetilde{y}_k\right))$ $-\mu\widehat{b}(\overline{\theta}q_H-\overline{\theta}q_L-\underline{\theta}q_L)-(1-\delta)((1-\mu)\widehat{b}-\widetilde{x}_k^H)\underline{\theta}\Delta q<0$. Therefore, since $\widetilde{h}_{k+1}\left(\widetilde{x}_{k+1}^H\right)=0$ and $\widetilde{h}_{k+1}(\cdot)$ is decreasing, we must have $\widetilde{x}_{k+1}^H<\widetilde{x}_k^H$. This implies that $m\leq 2+\dfrac{\widehat{b}-\overline{x}_2^H}{\epsilon}$ and $m'\leq 2+\dfrac{\widehat{b}-\widetilde{x}_2^H}{\epsilon}$.

Finally, we show that a reputational equilibrium exists for any $\mu\in\left(\underline{\mu},\overline{\mu}\right]$. Let's define $\zeta(\cdot)$ with $y^H<\widehat{b}$ and $y^H+y^L\leq\widehat{b}$ as

$$\zeta\left(y^H,y^L\right)=\frac{\mu y^H\overline{\theta}q_H}{1-\delta}+\frac{\mu y^L\overline{\theta}q_L}{1-\delta}-R^C\left((1-\mu)\,y^H,(1-\mu)\,y^L\right).$$

First, we show that there exists $\grave{y}^H$ such that $\zeta\left(\grave{y}^H,0\right)=0$. Since $\zeta\left(0,0\right)=-R^C\left(0,0\right)<0$ and $\zeta\left(\widehat{b},0\right)>0$ and $\zeta(\cdot)$ is increasing, there must exist $\grave{y}^H\in\left(0,\widehat{b}\right)$ such that $\zeta\left(\grave{y}^H,0\right)=0$.

Second, we show that there exists $\check{y}^H \in \left(0, \grave{y}^H\right)$ such that $\zeta\left(\check{y}^H, \widehat{b} - \check{y}^H\right) = 0$. Since ζ is increasing with respect to both arguments, $\zeta\left(\grave{y}^H, \widehat{b} - \grave{y}^H\right) > 0$ and $\zeta_1 > \zeta_2$, there exists $\check{y}^H < \grave{y}^H$ such that $\zeta\left(\check{y}^H, \widehat{b} - \check{y}^H\right) = 0$.

To establish that $0 < \check{y}^H$, we will show that $\zeta\left(0, \widehat{b}\right) < 0$ which directly implies that there exists $\underline{\grave{y}}^H \in \left(0, \grave{y}^H\right)$ such that $\zeta\left(\underline{\grave{y}}^H, \underline{\grave{y}}^H - \widehat{b}\right) = 0$. By definition of $\underline{\mu}$, $\zeta\left(\widehat{b}, 0\right) > 0$ and as y^H decreases $\zeta\left(y^H, \widehat{b} - y^H\right)$ decreases as well. Since

$$R^C\left(0, (1-\mu)\widehat{b}\right) > R^C\left((1-\mu)\widehat{b}, (1-\mu)\left(1-\widehat{b}\right)\right)$$
$$> \frac{\mu\widehat{b}\left(\overline{\theta}q_H - \rho\Delta\theta\Delta q\right)}{1-\delta} + \frac{\mu\left(1-\widehat{b}\right)\underline{\theta}q_L}{1-\delta},$$

we have

$$\zeta\left(0, \widehat{b}\right) = \frac{\mu\widehat{b}\overline{\theta}q_L}{1-\delta} - R^C\left(0, (1-\mu)\widehat{b}\right)$$
$$< \frac{\mu\widehat{b}\overline{\theta}q_L}{1-\delta} - \frac{\mu\widehat{b}\left(\overline{\theta}q_H - \rho\Delta\theta\Delta q\right)}{1-\delta} - \frac{\mu\left(1-\widehat{b}\right)\underline{\theta}q_L}{1-\delta} < 0.$$

This finishes the proof that $\check{y}^H \in \left(0, \grave{y}^H\right)$.

Additionally, if $\overline{\theta}\left(\grave{y}^H + \grave{y}^L\right) > \underline{\theta}$ then the monopolist moves to $\left(\grave{y}^H, \grave{y}^L\right)$ otherwise the monopolist would move to $\left(\grave{y}^H, 1 - \grave{y}^H\right)$ immediately. Therefore, there exists $\left(\grave{y}^H, \grave{y}^L\right)$ such

that $P^H\left(y^H, y^L\right) = \overline{\theta}q_H$, $P^L\left(y^H, y^L\right) = \overline{\theta}q_L$for $y^H \in \left[0, \grave{y}^H\right]$ and for $y^L \in \left[0, \grave{y}^L\right]$ and $P^H\left(y^H, y^L\right) = \left(\overline{\theta}q_H - \overline{\theta}q_L + \underline{\theta}q_L\right)$, $P^L\left(y^H, y^L\right) = \underline{\theta}q_L$for $y^H \in \left[0, \grave{y}^H\right]$ and for $y^L \in \left(\grave{y}^L, 1\right]$. Moreover, if $\overline{\theta}\left(\grave{y}^H + \grave{y}^L\right) \geq \underline{\theta}$ then we have $t^H\left(y^H, y^L\right) = \grave{y}^H$ and $t^L\left(y^H, y^L\right) = \grave{y}^L$ for $y^H \in \left[0, \grave{y}^H\right]$ and for $y^L \in \left[0, \grave{y}^L\right]$. If $\overline{\theta}\left(\grave{y}^H + \grave{y}^L\right) < \underline{\theta}$ then we have $t^H\left(y^H, y^L\right) = \grave{y}^H$ and $t^L\left(y^H, y^L\right) = 1 - \grave{y}^H$ for $y^H \in \left[0, \grave{y}^H\right]$ and for all y^L. Thus, $R\left(x^H, x^L\right) = \left(\grave{y}^H - x^H\right)\overline{\theta}q_H + \left(\grave{y}^L - x^L\right)\overline{\theta}q_L + \dfrac{\delta\mu\grave{y}^H\overline{\theta}q_H}{1-\delta} + \dfrac{\delta\mu\grave{y}^L\overline{\theta}q_L}{1-\delta}$ for $\overline{\theta}\left(\grave{y}^H + \grave{y}^L\right) \geq \underline{\theta}$ and for $y^H \in \left[0, \grave{y}^H\right]$, $y^L \in \left[0, \grave{y}^L\right]$, whereas $R\left(x^H, x^L\right) = \left(\grave{y}^H - x^H\right)\left(\overline{\theta}\Delta q + \underline{\theta}q_L\right) + \left(1 - \grave{y}^H - x^L\right)\underline{\theta}q_L + \dfrac{\delta\mu\grave{y}^H\left(\overline{\theta}\Delta q + \underline{\theta}q_L\right)}{1-\delta} + \dfrac{\delta\mu\left(1-\grave{y}^H\right)\underline{\theta}q_L}{1-\delta}$ for $\overline{\theta}\left(\grave{y}^H + \grave{y}^L\right) < \underline{\theta}$ and for $y^H \in \left[0, \grave{y}^H\right]$, all y^L. The pentad is defined by the Coase Conjecture pentad otherwise.

Proof of Corollary 12. Consider $(\grave{y}^H, \grave{y}^L)$. We set $\grave{y}^H$and $\grave{y}^L$ such that $\zeta(\grave{y}^H, \grave{y}^L) = 0$. $\lim_{z\to 0^+} \zeta(y^H, y^L) = \lim_{z\to 0^+}((\frac{\mu\overline{\theta}q_H y^H}{1-\delta} + \frac{\mu\overline{\theta}q_L y^L}{1-\delta}) - ((1-(1-\mu)y^H)\underline{\theta}q_H - (1-\mu)y^L\underline{\theta}q_L + \frac{\delta\mu\underline{\theta}q_H}{1-\delta})) = \lim_{z\to 0^+} \frac{1}{1-\delta}((y^H q_H + y^L q_L)(\mu\overline{\theta} + (1-\delta)(1-\mu)\underline{\theta}) - ((1-\delta)\underline{\theta}q_H + \delta\mu\underline{\theta}q_H))$. Therefore, at $\left(\grave{y}^H, \grave{y}^L\right)$ we have $(\grave{y}^H q_H + \grave{y}^L q_L) = \lim_{z\to 0^+} \frac{(1-\delta)\,\underline{\theta}q_H + \delta\mu\underline{\theta}q_H}{\mu\overline{\theta} + (1-\delta)\,(1-\mu)\,\underline{\theta}} = \frac{(\lambda+r)\,\underline{\theta}q_H}{\lambda\overline{\theta} + r\underline{\theta}}$. Since $(\grave{y}^H + \grave{y}^L)\overline{\theta} > (\grave{y}^H + \grave{y}^L \frac{q_L}{q_H})\overline{\theta} = \frac{(\lambda+r)\,\underline{\theta}\overline{\theta}}{\lambda\overline{\theta} + r\underline{\theta}} > \underline{\theta}$, the seller moves to $(\grave{y}^H, \grave{y}^L)$ from the initial state $(0,0)$.

Proof of Proposition 9. The functions $\overline{\mu}\,(\delta)$ and $\underline{\mu}\,(\delta)$ are decreasing, with $\overline{\mu}\,(0) = \underline{\mu}\,(0) = \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, $\lim_{\delta\to 1^-} \underline{\mu}\,(\delta) = 0$, and $\lim_{\delta\to 0^+} \overline{\mu}\,(\delta) > 0$. Consider $\overline{\mu}$. We know that $\overline{\mu}$ is derived from the existence of the Coase Conjecture equilibrium. By definition we have $\overline{x}_0^H = 1-\mu$, $\overline{x}_1^H = (1-\mu)\widehat{b}$, $\overline{x}_2^H = \frac{\widehat{b\theta}-\underline{\theta}}{\overline{\theta}-\underline{\theta}} \frac{q_H}{(q_H - q_L)} - (1-\mu)\,\widehat{b}\frac{q_L}{(q_H-q_L)}$, and $\overline{x}_k^H = (1-\mu)^{-1}(\overline{x}_{k-1}^H - \left(\overline{x}_{k-2}^H - \overline{x}_{k-1}^H\right)\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)}) + \mu\widehat{b}\frac{q_L}{(q_H-q_L)}$. Define $\Delta_k = \overline{x}_{k-1}^H - \overline{x}_k^H$. Then it follows that $\Delta_1 = (1-\mu)(1-\widehat{b})$, $\Delta_2 = (\frac{\left(1-\widehat{b}\right)\underline{\theta} - \mu\widehat{b}\left(\overline{\theta}-\underline{\theta}\right)}{\overline{\theta}-\underline{\theta}})\frac{q_H}{q_H-q_L}$, $\Delta_k = (1-(1-\mu)^{-1})\overline{x}_{k-1}^H + (1-\mu)^{-1}\Delta_{k-1}\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)} - \mu\widehat{b}\frac{q_L}{(q_H-q_L)}$. First, we show that $\lim_{\delta\to 0^+} \overline{\mu}\,(\delta) = \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta}-\underline{\theta}}$. We set $\overline{\mu}$ so that for all $\mu \le \overline{\mu}$, we have $\Delta_k > 0$ for $k = 1,\ldots,m$ and there exists m such that $\overline{x}_{m+1}^H < 0 \le \overline{x}_m^H$ holds. As $\delta \to 0$, Δ_3 increases drastically and so does Δ_k for all $k \ge 3$. In this case, we need to make sure that Δ_2 is positive. For all $\mu \le \frac{\left(1-\widehat{b}\right)\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, we have $x_2^H \le \overline{x}_3^H$. Therefore, we guarantee that for $k = 1,\ldots,m$ we have $\Delta_k > 0$ and m is set such that $\overline{x}_{m+1}^H < 0 \le \overline{x}_m^H$.

We now claim that $\lim_{\delta\to 1^-} \overline{\mu}\,(\delta) > 0$. When $\delta = 1$ and $\mu = 0$, for $k = 3,\ldots,m+1$, Δ_k is pinned down to $\Delta_k = \Delta_{k-1}\frac{\overline{\theta}}{\left(\overline{\theta}-\underline{\theta}\right)}$. Hence $\Delta_k = \Delta_2(\frac{\overline{\theta}}{\left(\overline{\theta}-\underline{\theta}\right)})^{k-1} > 0$. Let m be such that $\overline{x}_{m+1}^H < 0 \le \overline{x}_m^H$. Since $\overline{x}_k^H$ is a continuous function of μ, it follows that there exists $\mu' > 0$ such that for all $\mu \in [0,\mu')$ we have $\Delta_k > 0$ and $\overline{x}_{m+1}^H < 0 \le \overline{x}_m^H$.

Finally, we show that $\overline{\mu}$ is a decreasing function of δ. First, we choose $\mu < \overline{\mu}\,(\delta)$ so that $\Delta_k > 0$ for all $k = 1,\ldots,m+1$. Then we show that Δ_k is decreasing in δ which gives us that $\Delta_k\,(\delta') > \Delta_k\,(\delta) > 0$ where $\delta' > \delta$. This implies that $\overline{\mu}\,(\delta') > \overline{\mu}\,(\delta)$ for

$\delta' > \delta$. Observe that Δ_2 and $\overline{x}_2^H$ are independent of δ and $\overline{x}_3^H$ is increasing in δ. Moreover, $\frac{d\Delta_3}{d\delta} = -\frac{\Delta_2}{\delta\rho}\frac{\overline{\theta}}{(1-\mu)\left(\overline{\theta}-\underline{\theta}\right)} < 0$. The proof comes from induction. If $\frac{d\overline{x}_{k-1}^H}{d\delta} > 0$ and $\frac{d\Delta_{k-1}}{d\delta} < -(\frac{\overline{\theta}}{(1-\mu)\left(\overline{\theta}-\underline{\theta}\right)})^{k-1}$ then we prove that $\frac{d\overline{x}_k^H}{d\delta} > 0$ and $\frac{d\Delta_k}{d\delta} < -(\frac{\overline{\theta}}{(1-\mu)\left(\overline{\theta}-\underline{\theta}\right)})^k$. Since $\Delta_k = (1-(1-\mu)^{-1})\overline{x}_{k-1}^H + (1-\mu)^{-1}\Delta_{k-1}\frac{\overline{\theta}}{\rho^{k-2}(\overline{\theta}-\underline{\theta})} - \mu\widehat{b}\frac{q_L}{(q_H-q_L)}$, $\frac{d\Delta_k}{d\delta} < (1-(1-\mu)^{-1})\frac{d\overline{x}_{k-1}^H}{d\delta} + (1-\mu)^{-1}\frac{d\Delta_{k-1}}{d\delta}\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)} - (k-2)(1-\mu)^{-1}\Delta_{k-1}\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)\delta}$. Moreover, since $(1-(1-\mu)^{-1})\frac{d\overline{x}_{k-1}^H}{d\delta} < 0$ and $(k-2)(1-\mu)^{-1}\Delta_{k-1}\frac{\overline{\theta}}{\rho^{k-2}(\overline{\theta}-\underline{\theta})\delta} > 0$, we have $\frac{d\Delta_k}{d\delta} < (1-\mu)^{-1}\frac{d\Delta_{k-1}}{d\delta}\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)}$. Due to the assumption on $\frac{d\Delta_{k-1}}{d\delta}$, $\frac{d\Delta_k}{d\delta} < -(\frac{\overline{\theta}}{(1-\mu)\left(\overline{\theta}-\underline{\theta}\right)})^k\frac{1}{\rho^{k-2}} < -(\frac{\overline{\theta}}{(1-\mu)\left(\overline{\theta}-\underline{\theta}\right)})^k$ since $\frac{1}{\rho^{k-2}} > 1$. Now, to finish the induction we need to show that $\frac{d\overline{x}_k^H}{d\delta} > 0$. By definition, we have

$$\frac{d\overline{x}_k^H}{d\delta} = (1-\mu)^{-1}(\frac{d\overline{x}_{k-1}^H}{d\delta} - \frac{d\Delta_{k-1}}{d\delta}\frac{\overline{\theta}}{\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)} + (k-2)\,\Delta_{k-1}\frac{\overline{\theta}}{\delta\rho^{k-2}\left(\overline{\theta}-\underline{\theta}\right)}).$$

Since $\frac{d\Delta_{k-1}}{d\delta} < 0$ it follows that $\frac{d\overline{x}_k^H}{d\delta} > 0$. Therefore, we conclude that if $\mu \leq \overline{\mu}(\delta)$ then $\mu \leq \overline{\mu}(\delta')$ for all $\delta' < \delta$, $\overline{\mu}$ is a decreasing function of δ. The same results hold for $\underline{\mu}(\delta)$.

Proof of Corollary 13. Trivial.

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