IMMANENT STRUCTURALISM:
A NEO-ARISTOTELIAN ACCOUNT OF MATHEMATICS

Alfredo Watkins

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Approved by:

Marc Lange

Thomas Hofweber

John Roberts

Ram Neta

Markus Kohl

ABSTRACT

Alfredo Watkins: Immanent Structuralism: A Neo-Aristotelian Account of Mathematics
(Under the direction of Marc Lange)

The aim of this dissertation is to propose and defend a position in the philosophy of mathematics called "immanent structuralism." This can be contrasted with the standard Platonist view in the philosophy of mathematics, which holds that mathematics studies a unique category of non-physical, abstract entities. Platonism immediately leads to the epistemological problem of how we can *know* about these entities if they are not part of the physical world. By contrast, immanent structuralism holds that the things mathematics studies are *structures* or *structural patterns*. These structures or patterns are like other physical universals in that they can be instantiated by physical systems. Therefore, some of them can be known through ordinary perception and, as I argue, the rest can be built out of these ones.

The first half of the dissertation lays out the core of the theory: I discuss what these structural patterns are and how they can constitute the subject-matter of mathematics. I also give an essence-based account of mathematical truth which refers only to these properties. I then argue that this view avoids the epistemological problems with Platonism, since it allows some basic mathematical properties to be literally instantiated in the physical world, making them graspable by perception, while others can be constructed out of these. I claim that this view is better suited to account for the ordinary knowledge of mathematics had by most people.

The second half of the dissertation applies this theory to several special topics in the philosophy of mathematics, including mathematical reduction, mathematical treating-as, mathematics and modality, and mathematical explanation. I also discuss why immanent structuralism presents a unique challenge to indispensability arguments in mathematics and to certain parity arguments in other fields of philosophy. The ultimate hope of the dissertation is to show that a better path forward for realists in the philosophy of mathematics is to move away from object-based accounts like Platonism, and instead move toward more Aristotelian, property-based accounts, like the theory presented here.

For Isabella Fatima

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CHAPTER 1: INTRODUCING IMMANENT STRUCTURALISM

1. Introduction

The aim of this dissertation will be to defend a specific form of structuralism in the philosophy of mathematics which I call "immanent structuralism." Structuralism holds that mathematics is about structures or patterns. *Immanent* structuralism holds that these structures or patterns are universals or properties that can be (but need not always be) literally instantiated by many different kinds of things, particularly physical systems. Immanent structuralism is distinct from the more standard, *ante rem* structuralism, in that it holds structures to be what are called purely structural *universals*, rather than systems consisting of a special sort of intrinsically featureless *object* or *particular*.[1] According to immanent structuralism, a true mathematical statement holds iff – and because – certain facts about the natures of purely structural properties obtain.

2. Platonism and Ante Rem Structuralism

The structuralist approach can be illustrated by contrast with traditional Platonism. According to traditional Platonists, mathematical objects are *sui generis*. They are their own fundamental kind of entity. Moreover, for the Platonist, some mathematical objects are

[1] Cf. *ante rem* theorist Michael Resnik's description in (1997), p. 201: "The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures. And as such they have no identity or distinguishing features outside a structure."

particulars. For instance, the number 2 is to be thought of as an abstract individual, i.e., an object, or a particular thing. Platonism faces several well-known challenges:

> **The Epistemological Challenge**: How are most people able to have reliable beliefs about mathematics if those beliefs are about causally inert abstract objects?[2]
>
> **The Ontological Challenge**: The fewer the fundamental kinds of entities one posits, the better. So a theory that can plausibly amend mathematical ontology to some already-recognized category is preferable.
>
> **The Applicability Challenge**: Why are we able to learn about the physical world by using mathematics (and not just in physics, but in all sorts of natural sciences) if mathematics is about a realm of non-physical entities?

In addition, *reductive* versions of Platonism, which try to reduce other classes of mathematical objects to some subset of them, face a serious problem called the "Multiple Reductions Problem." Consider, for example, a "naïve" set-theoretic Platonism, according to which all mathematical objects *just are* sets.[3] On this view, sets are all we need *ontologically speaking* to make sense of mathematics.

One challenge for this view is that mathematical practice seems to allow for multiple, equally salient reductions of the natural numbers to sets. For instance, take the following proposed reduction from Von Neumann. Let us call the following series of sets the "V-Sets":

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[2] Benacerraf (1973)

[3] Note on this set-theoretic Platonist view sets are the only kind of *sui generis* abstract mathematical object. All others are reducible to them. Set-theoretic Platonism is the most common form of reductive Platonism.

(V-Sets): 0: { }, 1:{ {} }, 2: {{} , { {} }}, …

[where the n+1th set is the power set of the nth set]

Consider also the following series, the "Z-Sets" (due to Zermelo):

(Z-Sets): 0: {}, 1: { {} }, 2: {{ {} }}, …

[where the n+1th set is the set of the nth set]

The problem for naïve set-theoretic Platonism is that the Z-Sets are just as good for a mathematical reduction of arithmetic to set theory as are the V-Sets. Therefore reductive set-theoretic Platonism faces an additional problem:

> **The Multiple-Reductions Problem**: There are multiple, equally good possible reductions of the ontology of numbers to the ontology of sets.

This last problem has inspired structuralist views of mathematics, which treat mathematics as the science of *structures*, or *patterns.* This is based on the insight that what is important for mathematics is not so much which particular series of sets you use – the V-sets or the Z-sets – but rather that the series of sets has the right type of structure to serve as a suitable representation of the natural numbers.

According to the standard version of structuralism advanced by Resnik (1997) and Shapiro (1997) – *ante rem* structuralism – the subject matter of mathematics consists of "an ontology of featureless objects, called 'positions', and … systems of relations or 'patterns' in which these positions figure."[4] Ante rem structuralists view individual mathematical objects as

[4] Resnik (1997) p. 269.

the “nodes” or “positions” within these systems.[5] For example, the *natural number system* according to ante rem structuralists is a system of *intrinsically featureless* objects with the order characteristic of the natural numbers.[6] These mathematical objects are taken to have *no intrinsic nature*, instead being entirely defined and constituted by their relations to other objects in the system or structure.[7]

So, on this view, the number 2 is not intrinsically a set, or a function, or anything else like that. Instead, it is an intrinsically featureless object whose essence simply is “that which comes before 3, and after 1.”[8] Therefore, structuralists will say that the V-sets and the Z-sets both *exemplify* the natural number structure, but that neither is, strictly speaking, *identical* with the series of natural numbers.[9]

However, arguably ante rem structuralism, which conceives of structures and their positions as abstract objects, still suffers from the three problems of traditional Platonism, viz., the epistemological, ontological, and applicability problems. Additionally, ante rem structuralism

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[5] Note that, like the Platonist, ante rem structuralists interpret reference to mathematical objects as straightforwardly singular and referential. See Shapiro (1997) pp. 10-11. See also p. 13: “According to ante rem structuralism, the variables of the theory range over the places of that structure, the singular terms denote places in that structure, and the relation symbols denote the relations of the structure.” And p. 83: “Places in structures are bona fide objects … Bona fide singular terms…like “2” denote bona fide objects.”

[6] I.e., they constitute an omega-series.

[7] Resnik: “In mathematics, I claim, we do not have objects with an ‘internal’ composition arranged in structures, we have only structures. The objects of mathematics … are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure.” (1981) p. 530. Shapiro: “The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other.” (2000) p. 258.

[8] These latter numbers themselves are defined in terms of their relations to other objects in the system in the same way. Thus, we can say that the objects in the system are all defined in terms of each other.

[9] Note that, on the ante rem structuralist view, “The natural-number structure *itself* exemplifies the natural number structure.” (Shapiro 1997, p. 101, emphasis added)

takes on a seemingly more obscure ontology than Platonism, in that it is committed to objects that are not only abstract, but whose natures are entirely exhausted by their relations to other such objects. While I think there is a legitimate insight behind this claim, it would be better if we did not have to expand our ontology to include this seemingly esoteric sort of object with such an unusual nature. I will say a bit more about these issues in Chapter 5.

3. Structural Universals

Faced with these issues, let us look at the version of structuralism I wish to defend: Immanent structuralism. The “immanence” in the phrase “immanent structuralism” refers to the fact that, according to immanent structuralism, mathematics studies structural *universals* or *properties*, some of which are literally had or instantiated by physical objects. In just the way that other properties like *volume*, *mass* and *charge* are “located in” objects or systems of them, mathematical patterns or structures can be as well. Hence, they are “immanent” to the objects that have them.[10]

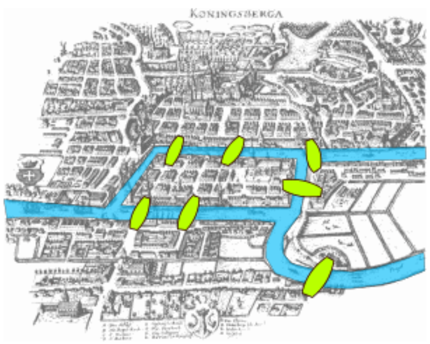
One question for the immanent structuralist is what is meant by a “structural property?” I think the clearest answer to this question comes from philosopher James Franklin:[11]

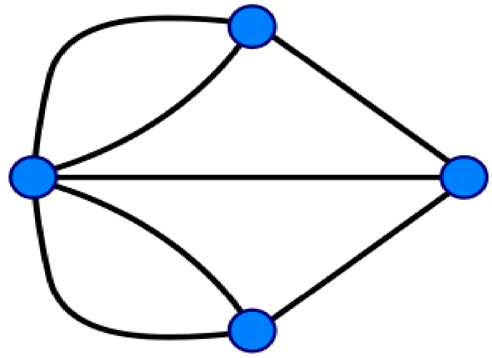
[10] This is in contrast to the ante rem structuralist. Cf., Resnik (1997) p. 261: “Some philosophers … have wanted to take structural properties, construed as metaphysical universals, as primitive entities and interpret mathematics within a theory of universals. … I am a realist about mathematical objects first, without being a realist about properties at all.” See also ibid., p. 269. See also Shapiro (1997) pp. 89-90. For ante rem structuralists, a structure is more like an *exemplar* or *paradigm*, along the lines of Plato’s Ideas or Forms. As such, ante rem theorists do not ultimately understand “exemplification” as straightforward property or universal-instantiation, as I would, but rather as consisting in something analogous to an isomorphism or congruence relation. See Resnik (1997) p. 204 ff. and Shapiro (1997) pp. 90-91.

[11] For this definition, see Franklin (2014) p. 57. My view is deeply indebted to Franklin’s work, although my view takes the account of mathematical truth and ontology in a rather different direction.

(PROP): P is a purely structural property iff P can be defined entirely in terms of ‘part’, ‘whole’, ‘sameness’, ‘difference’, and purely logical vocabulary.

This is best illustrated by an example. Consider Euler’s famous Bridges of Konigsberg problem:





(A): Bridge **(B): K-Graph**

The question Euler set out to answer was whether there was a path through the city that would cross each bridge exactly once. (The bridges are highlighted.) However, the rules are that the islands can only be reached by the bridges (no swimming, flying, or wormhole-ing!) and every bridge, once accessed, must be crossed to the other side (no turning back half-way across the bridge!). One need not end up at the place one started. One only has to cross each bridge once.

Now, as it turns out, the answer to Euler’s question is negative: There is no such path. However, what is most interesting about this case for our purposes is the fact that many of the details mentioned in the question don’t *matter*, at least mathematically speaking: The question can be grasped entirely by looking at the graphical representation in (B).

I will call the type of object (B) represents a K-graph. To state the definition of a K-graph (the type of graph the question is about) all we need to mention are four distinct parts, v1, … , v4 (represented via four nodes), and some relation E between them (holding between the parts in the same way as the seven lines connect the nodes). Thus, the property of being a K-graph would seem to be a purely structural property, since it can be defined as follows:

> **(K-graph):** The property of being a K-graph is the property of being a whole G with some distinct parts $v_1, \dots , v_4$, and some relation E between these parts such that v_1Ev_2, v_1Ev_3, … (etc.).[12]

Contrast this with a paradigmatically non-purely structural property, such as Aristotle's definition of the property of being a human:

> **(Human):** The property of being an animal, and of having a rational nature, (etc.)

Assuming, of course, that being an animal and having a rational nature will have to be defined in irreducibly physical terms (or maybe even irreducibly mental terms), the property of being human, as defined, is not a purely structural property.

As another example, consider the Klein 4-group.[13] It is a group with four elements, e, a, b, c, and an operation * on these elements. Its table is given below:

*****	**e**	**a**	**b**	**c**
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

[12] Note that by specifying the parts of G with variables and the relations with a predicate we are specifying the logical categories of these things, and so we can still say that this property is defined in terms of purely logical vocabulary.

[13] Lewis (1990)

We can define the *Klein 4-group* as a purely structural property:

> **(KLEIN):** The property of being a Klein 4-group is the property of being a whole that is a group[14] with distinct parts e, a, b, c, and a function[15] * on these parts such that: e*e=e, e*a=a , ... (etc.)

The property of being a Klein 4-group is spelled out by specifying the definition of * from the table we saw above. Note that many algebraic structures can be defined by similar tables.

We can also draw examples from topology and analysis. For example, take the property of being a *topological space*:[16]

> **(TOP):** The property of being a topological space is the property of being a whole S with two parts, O and C (called the open parts and closed parts), such that:
>
> 1. There is some part e (called the empty part) that has no parts.
> 2. S is a part of O and e is a part of O.
> 3. Any sum of parts of O is a part of O
> 4. Any finite intersection of parts of O is a part of O.

This important definition from topology allows us to give the definition of *continuity* as well:

[14] The property of being a group is itself purely structural. If one looks at a definition of a group, one will see that it just specifies some whole with some operation obeying closure, associativity, etc., where these properties are themselves definable purely structurally in terms of part, whole and logical vocabulary.

[15] Talk of functions can be reduced to talk of relations, if one finds talk in terms of relations preferable.

[16] See Franklin (2014) p. 61.

> **(CONT):** The property of being a continuous function is the property of being some function f from parts of a topological space S to parts of S, such that the inverse image of any open part is also open.[17]

Immanent structuralism holds that, given the abstract nature of mathematics, all of the structures that mathematics studies can be defined as purely structural properties similar to these.[18] Thus, the subject matter of mathematics consists only in purely structural *properties*, and does not include any Platonist-style mathematical *objects*. I will argue for this more fully in Chapter 4, but for now hopefully these simple examples are illustrative of the idea.

4. Immanent Structuralism: Ontology and Epistemology

One of the primary benefits I claim for immanent structuralism is that we can avoid some of the epistemological and ontological concerns that arise for Platonists and ante rem structuralists.

In the first place, we are able to get rid of *sui generis* Platonic mathematical objects, and only have to deal with properties. Arguably, there will be independent metaphysical pressure from the empirical world to deal with properties or universals anyway. Thus, the categories of being required to account for mathematical truth are reduced.

Secondly, on my view, since mathematical properties are purely structural properties, they can be literally had or instantiated by physical objects, just as other properties like *mass*,

[17] Hopefully it is clear how 'inverse image' would be defined too. And again, if it is easier to think in terms of relations, function talk can be wholly explained in terms of relations.

[18] For further examples, drawn from the higher reaches of mathematics, along with discussion of the role part and whole thinking plays in mathematics, see Bell (2004). I have chosen not to use Bell's examples, not because I think they don't work, but because the concept of a purely structural property is most easily seen via simple cases. Nevertheless, Bell's examples confirm that this is not just a feature of "simple" mathematics. Arguably, it is the defining feature of mathematics.

charge and *color* are.[19] Thus, mathematical patterns, or structures, can be "located in" objects or regions of space in exactly the same way that an object's size, mass, or color can be. Hence, at least in principle, *some* mathematical properties and relations can be accessed directly through perception.[20]

In order for claims about *uninstantiated* mathematical structures to come out true, however, we will need there to be some uninstantiated – and thus, unperceived – properties.[21] Nevertheless, I will argue if immanent structuralism is true then all of the uninstantiated properties of mathematics can be built up *out of* directly perceivable ones – just as the property *golden mountain* can be built out of the perceivable properties *golden* and *mountain*.

For these reasons, immanent structuralism has the potential to provide a more realistic epistemology for higher mathematics, where our initial acquisition of mathematical concepts is similar to our acquisition of concepts of physical properties (viz., through perception). Adult humans can then go on to build and define the more complicated concepts of higher mathematics by means of their general logical concepts.

In these respects, immanent structuralism can hope to provide a more plausible epistemology than standard Platonistic theories. Part of the problem for Platonism has been the explicitly axiomatic model that it takes as its paradigm case: Mathematical theorems are justified by appeal to some fundamental axioms and well-defined rules of inference that are taken to be

[19] Though in virtue of their being "purely structural" properties, they can also be had by not-obviously-physical things too – e.g., I can count ideas, relations, or even angels if there are any.

[20] Often, though by no means exclusively, by visual perception.

[21] Though, in a way, immanent structuralism's ontology is even less committal than this: If one is ultimately a nominalist about properties, then presumably one has a way to effectively translate *all* property-talk – including talk about uninstantiated properties – into language that doesn't require reference to properties. That would be fine with me, so long as this paraphrastic elimination is able to capture all the facts about properties that I will need later.

valid. And so the quest has been to look for the "foundational" axioms which also specify the fundamental "entities" or "objects" of mathematics. This leads to the further problem of how these foundational axioms and their postulation of these basic mathematical objects can be justified. Various proposals have been given for how to do this, including appeals to a quasi-perceptual direct intuition, inference to the best explanation, indispensability arguments, and revised concepts of analyticity.[22] Arguably, none of these solutions is particularly satisfying.

Undoubtedly, advances in the axiomatic method have contributed decisively to the rigor of mathematics. However, it should not be taken as the paradigm case or the starting point for philosophical inquiry. In practice, the axiomatic model is not the primary means by which mathematical understanding is cultivated or how mathematical results are discovered. Formal or quasi-formal proof is very much the last step. Furthermore, proof by fundamental axioms is not the only way that mathematical results can be justified. And it is almost certainly not how mathematical concepts are initially acquired.[23]

Important recent work in cognitive science and philosophy of psychology, such as Carey (2011) and Burge (2010), has argued that mathematical concepts and basic mathematical beliefs are present from an early age, and likely are represented in the perceptual systems even of sufficiently sophisticated animals. And obviously young children learn important mathematical results in grade school, and are taught by non-axiomatic, suggestive methods (with a heavy emphasis on perceptual aids). Presumably these mathematical results are *known*.

[22] See Shapiro (2000) and Linnebo (2017) for overviews of some of these approaches.

[23] I will discuss these claims further in Chapter 3 below.

So, while perhaps rigorous proof provides the *best* justification for mathematical beliefs, rigorous proof is not *necessary* for that. Therefore, even if Platonists were successful in the project of providing adequate "foundations" for mathematics in the form of axioms plus a plausible story of how these axioms can be known (e.g., via Frege's view that they are known qua analytic truths, or via some indispensability argument), this is still an implausible explanation for the vast majority of mathematical knowledge had by most people in most times and places. On immanent structuralism, however, explaining this knowledge is not more difficult than explaining how people can gain knowledge through perception and the concepts built up from perception.

5. A Preview of Things to Come

The rest of the dissertation is roughly in two parts: The first part, consisting of Chapters 2 and 3, constitutes the bulk of the theory. Here I attempt to give a compelling story about the two central issues for any philosophy of mathematics: (a) ontology and truth in mathematics and (b) the epistemology of mathematics. Chapter 2 contains an essence-based account of mathematical truth that assumes the existence of Aristotelian universals. In this chapter I draw on recent work in neo-Aristotelian metaphysics that has been unavailable to or underutilized in previous discussions of mathematics. This chapter also develops and advances the metaphysics of property parthood.[24] In Chapter 3 I take the account of mathematical truth and ontology from Chapter 2 and try to show how a plausible, quasi-empiricist epistemology of mathematics falls out of this account, while clarifying further a few aspects of the ontology.

[24] A notion that has seen a resurgence very recently and is likely to become increasingly important as intensionalizing accounts of semantics gain traction. See especially Craig Warmke (2015), (2016), and (2019), as well as L.A. Paul (2002), (2004) and (2012).

The second part of the dissertation applies the metaphysical and epistemological theory set out in Part I to more specific issues in mathematical practice and in the philosophy of science. Chapter 4 considers the practice of mathematical reduction and what I call “treating-as,” and illustrates how immanent structuralism is ideally situated to explain these phenomena. Having seen the theory put to use a bit, Chapter 5 constitutes a brief interlude, and compares immanent structuralism with some closely related positions – including Shapiro and Resnik’s *ante rem* structuralism, Hellman’s modal structuralism, and Balaguer’s modified “full-blooded” Platonism. I try to show how immanent structuralism avoids some of the significant pitfalls that even these more sophisticated treatments fall into while retaining their advantages.

Chapter 6 develops the notion of “*de re* mathematical necessity,” which has the potential to be confused with other notions of necessity related to mathematics. I argue that immanent structuralism is much better placed to explain cases of *de re* mathematical necessity than traditional Platonism. Indeed, I argue that – perhaps surprisingly – these cases constitute a significant and underappreciated problem for Platonism. Chapter 7 is a concluding chapter, where I attempt to draw some broader lessons for ontology and philosophical theorizing. In particular, I identify a few forms of reasoning that are common among analytic philosophers and ontologists and explain why they are undercut by the theory I’ve presented.

CHAPTER 2: MATHEMATICAL TRUTH AND ONTOLOGY

Introduction

In this chapter I will try to give an account of how all mathematical truths can be seen to be true in virtue of the natures of purely structural properties. In Section 1 I will make clearer Benacerraf's original problem of a uniform semantics. In Sections 2 and 3, I will identify a part of natural language that I call "essentialist pattern statements." I will explain how these sorts of statements can provide us a model for a theory of mathematical truth so that we can resolve Benacerraf's challenge. In Sections 4 and 5 I discuss the notion of "property parthood" that will serve as the basic ontological tool in our theory of mathematical truth. I show that the best account of this notion appeals to an Aristotelian notion of essence. In Section 6 I give the account of mathematical truth in full. Before concluding, Section 7 discusses two ontological interpretations of "essence" that we might take.

1. The Problem of Uniform Semantics

In the previous chapter, I provided examples to illustrate how one might plausibly construe the subject matter of mathematics in terms of purely structural properties. I have also suggested some ways immanent structuralism might provide a more satisfying ontology and epistemology for mathematics than its competitors. Before examining these suggestions more closely, there is a classic technical concern we must address, also due to Benacerraf. This is known as the problem of *uniform semantics*.

The Uniform Semantics Problem: Platonism allows us to give a uniform semantics for mathematical and ordinary discourse. For example, '2<3' and 'Paco is bigger than Lolly' both appear to have a similar logical form; both seem to assert that some relation holds between two objects. But if we reject a Platonist interpretation of mathematical statements, how are we to provide a semantics that is uniform between mathematical and other linguistic discourse?

In Benacerraf's seminal article, "Mathematical Truth" (1973), he notes an apparent tension that arises for any attempt at an adequate theory of mathematical truth, i.e., any candidate for an adequate semantics for mathematical statements.

On the one hand, we want our semantics for mathematical statements to be an account of their *truth* (rather than, say, an account merely of their *theoremhood*, or their *being felicitous utterances on a given occasion*). But on the other hand, the concept of *truth* comes along with some requirements, i.e., there are some conditions that are constitutive of *truth-hood*.

In particular, if we want to give an explication of truth for some area of discourse, that discourse's having a standard, Tarski-inspired semantics for ordinary singular and quantified statements is, *ceteris paribus*, a plausible constraint on that account of truth. Otherwise, it is hard to see how the concept being explicated is a *truth*-concept.

Benacerraf's point can be illustrated by a similar requirement on truth, Tarski's own "Convention T":[25]

[25] Cf. Tarski (1944).

Convention T: Any viable theory of truth for some set of sentences S must satisfy the following condition. For any sentence p in S:

(T) "p" is true iff p.

For instance, if we are trying to give an account of what it is for statements about my dog to be true, that account must entail that "Paco is small" is *true* just in case Paco is small. This is generally taken to be a plausible constraint on any account of truth.

Benacerraf thinks that a further constraint on a theory of truth is that the relevant sentences have a "standard" semantics. What this means is that the semantics explicating our idea of mathematical truth must be the usual semantics for ordinary, non-mathematical statements, i.e., mathematical statements ought to be given the same sorts of truth-conditions as other ordinary indicative sentences. But on the ordinary, Tarski-inspired semantics, names refer to objects, quantifiers range over some domain of entities, and predicates indicate properties or relations that the objects or entities can have. So the standard semantics gives a *Platonist* reading to mathematical statements.

From this follows the second horn of Benacerraf's dilemma: If we *do* accept a "uniform" or "homogeneous" semantics for both mathematical and non-mathematical statements, then we seem to saddle mathematical sentences with commitments to an ontology that would simultaneously make it impossible for us to know their truth.

Although Benacerraf frames his challenge specifically in terms of a causal theory of knowledge, most philosophers have moved beyond this assumption of his argument.[26] Nevertheless embracing this particular machinery is unnecessary for his argument to go through.

[26] See Maddy (1996) and Linnebo (2017).

We can pose the same question this way: How can we plausibly account for most people having reliable knowledge of entities that are *entirely causally disconnected* from the physical world?[27] We will come back to this part of Benacerraf's dilemma later.[28]

To sum things up, Benacerraf's dilemma can be stated succinctly as follows:

> **Benacerraf's Dilemma**: You can either have an adequate theory of mathematical truth – which would require a uniform semantics between mathematical and non-mathematical discourse – or a plausible mathematical epistemology, but not both.

Now, up this point I have not given an explicit and general account of how immanent structuralism would construe the truth-conditions for mathematical statements. But if, as immanent structuralism asserts, all that is needed for some mathematical statement to be true is for some fact about some structural properties to hold, then it must be possible to give the truth-conditions for mathematical statements entirely in terms of facts about these structural properties. Therefore, my concern in the rest of this chapter will be to provide an immanent structuralist account of the truth-conditions for mathematical statements that can meet Benacerraf's challenge. In other words, this chapter will focus on the first horn of Benacerraf's dilemma.

If we want to reject Benacerraf's dilemma, we will have to find some premise of his to deny. So let us look at his argument a bit more closely. First off, Benacerraf begins by noting that a semantics for mathematics must be, in part, an explication of *truth*, and not some other concept (like theoremhood, felicitous-utterance-hood, interesting-metaphor-hood, etc.):

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[27] Cp. Maddy (1990) pp. 36-50.

[28] See Chapter 3 below.

> "For present purposes we can state it [the uniform semantics requirement] as the requirement that there be an over-all theory of truth in terms of which it can be certified that the account of mathematical truth is indeed an account of mathematical *truth*. The account should imply truth conditions for mathematical propositions that are evidently conditions of their truth (and not simply, say, of their theoremhood in some formal system)."[29]

Benacerraf then argues that satisfying this requirement requires a semantics that is uniform between mathematical and non-mathematical discourse:

> "Another way of putting this first requirement is to demand that any theory of mathematical truth be in conformity with a general theory of truth … which certifies that the property of sentences that the account calls 'truth' is indeed truth. This, it seems to me, can be done only on the basis of some general theory for at least the language as a whole ... Perhaps the applicability of this requirement to the present case amounts only to a plea that *the semantical apparatus of mathematics be seen as part and parcel of that of the natural language in which it is done*, and thus that whatever semantical account we are inclined to give of names or, more generally, of singular terms, predicates, and quantifiers in the mother tongue include those parts of the mother tongue which we classify as mathematese."[30]

Now, one crucial assumption of Benacerraf's argument here, which I have highlighted, is that there even *is* a uniform semantics for non-mathematical discourse. That is to say, Benacerraf

[29] Benacerraf (1973) p. 408.

[30] Ibid.

seems to assume that, in general, all superficially singular or quantified sentences will have the same semantics, and that there is such a thing as "the" semantical apparatus of natural language. However, since 1973 when Benacerraf's paper was first published, many cases have been brought up that significantly cast doubt on Benacerraf's original thesis, even leaving aside mathematical discourse.

Consider, for instance, the following cases, which have been widely discussed in the relevant literature:

(1) "Some dog is black." [Ordinary 'count' quantifier]

(2) "Some water is wet." [Mass quantifier]

(3) "Some critics admire only one another." [Plural quantifier]

All three of these sentences superficially have the same grammar, but they appear to have rather different truth conditions. The first sentence can be spelled out as being equivalent to "There is at least one individual thing that is a dog and is black." But arguably this way of cashing out (1) cannot be applied to (2). After all, it seems strange to say (in any normal context) "There is some particular thing that is *a water* and is wet."[31] This suggests that (1) and (2) have a different semantical form. Finally, the third sentence, known as a "Geach-Kaplan sentence," is a famous instance of "non-firstorderizability": its content cannot accurately be captured by a first-order sentence with the same sort of quantifier as (1).[32]

[31] Cf. Lowe (2009).

[32] Cf. Boolos (1984) p. 56. Some argue that there are ways to capture the contents of these sentences using first-order quantification over sets. See, e.g., Quine (1969). But this is not the only way to do so (nor, perhaps, the most plausible). And regardless, the important point for our purposes is that the semantics of (3) will be different from the semantics of (1), despite having syntax superficially similar to a first-order sentence like "Some dogs love some humans."

There is more to be said about these particular cases, of course, and much has been said. To be fair, perhaps there *is* in fact some more general logical form that all three of these quantified statements share. And perhaps even if these sentences do not all share exactly the same semantics, Benacerraf's claim that quantified mathematical statements should have a semantics *similar* to (1) still has *prima facie* plausibility. My point here though is simply to note that Benacerraf's argument importantly depends on an assumption that is at least *challengeable*, namely:

- There is just *one, uniform* semantics for ordinary quantified and singular statements.

The fact that examples like those above cast doubt on Benacerraf's assumption makes his original formulation of the dilemma unworkable. If Benacerraf's criterion for an adequate semantics for mathematics is that it be uniform between mathematics and all non-mathematical discourse, then arguably this is impossible for anyone, Platonist or otherwise, if only because there is *no such thing* as "the" semantics for all non-mathematical singular and quantified statements.[33]

I think other cases I will discuss below show further why Benacerraf's assumption that there is a single uniform semantics is mistaken. With that said, I still agree that it would be nice if we could provide a mathematical semantics that is uniform with at least *a certain part* of natural language, while also providing a plausible mathematical epistemology. I believe this is possible, and will try to do so in the sections that follow.

[33] According to John Burgess, regarding Boolos' 1984 and 1985 articles, "plural quantification had prior to these papers received little or no attention." (See Burgess in Boolos (1998) p. 10) It is understandable therefore that at the time of Benacerraf's original (1973) discussion he did not take them into consideration.

2. Pattern Statements

My strategy is to consider mathematical statements along the lines of what I call "pattern statements." The basic idea is this: There are many true statements, in grammatical singular form, that are about properties or universals, and that can be true even when the relevant universal is not instantiated.[34] Arguably, some of these "pattern statements" are true because of facts about the *natures* of the relevant properties or universals. Let's look at some examples.

Consider a case in biology where one might say "The cell membrane is an organelle" or "The cell membrane is semipermeable." Or, in physics, one might say of some hypothetical particle, "The neutralino is a hypothetical particle" and "The neutralino is a fermion with spin ½," or "The graviton is a boson." Or, in the case of baseball, if one were explaining to a friend the different positions, one might say "The pitcher is the player who throws the ball"[35] or "The umpire enforces the rules" or "The coach picks the batting order."

Arguably, none of these statements commits one to the existence of some strange, *particular individual* called *the cell membrane*, or *the neutralino*, or *the graviton*, or *the coach*. Arguably, these are true just because of some facts about the natures of the relevant properties, e.g., *being a cell membrane*, *being a pitcher*, etc.

The strategy, then, will be to assimilate mathematical statements to singular-form pattern-statements of this sort. To say "0 is even" is to say "the number with no successor is even," and this can be given a reading similar to "the neutralino has spin 1/2." To say "0 is less than 1" is to

[34] And therefore, as I will argue, the definite descriptions occurring in them cannot be interpreted as disguised, first-order existentially-quantified statements.

[35] From Wikipedia's entry on "Pitcher": "In baseball, the pitcher is the player who throws the baseball from the pitcher's mound toward the catcher to begin each play, with the goal of retiring a batter, who attempts to either make contact with the pitched ball or draw a walk."

say “the number with no successor is less than the number that succeeds 1,” and this can be given a semantics similar to “the nucleus is positioned behind the cell membrane.” In addition, these mathematical statements can themselves be parsed further, so that they are phrased in purely structural vocabulary. If this strategy is successful, then we will be able to meet Benacerraf’s challenge and show how mathematical statements are uniform with at least a *part* of ordinary natural language.

3. Properties, Property-Parthood, and Pattern Statements

Though this gives us a general strategy for making the semantics of mathematical statements uniform with (a part of) ordinary discourse, that still leaves open the question of how, *precisely*, to spell out the semantics of the relevant pattern statements entirely in terms of universals, thereby allowing us to show that facts about universals are all that is necessary for the right mathematical statements to come out true.

The method I propose is to think in terms of constitutive relations between properties. As a simple case, think of the superficially singular-form pattern statement “The dog is an animal.” One way to cash this out is to say that this is true if the property of *being an animal* is “part of” the property of *being a dog*.[36] In other words, *being an animal* is *part of the very nature* of the property *being a dog*. It is part of what *constitutes* that property. The strategy will be to assimilate mathematical statements to statements of this sort.

But first, we should make a qualification: In the case of *being a mammal* and *being a dog*, it seems plausible that the relation between these two properties is, in fact, a necessary and

[36] Cf. Warmke (2015).

essential one. But it is not clear that *all* pattern statements have this feature. Consider, for instance, the following piece of discourse:

> "The bonobo, also historically called the pygmy chimpanzee and less often, the dwarf or gracile chimpanzee, is an endangered great ape and one of the two species making up the genus Pan; the other being the common chimpanzee. … The bonobo is distinguished by relatively long legs, pink lips, dark face and tail-tuft through adulthood, and parted long hair on its head. The bonobo is found in a 500,000 km2 (190,000 sq mi) area of the Congo Basin in the Democratic Republic of the Congo, Central Africa."[37]

At least *some* of the pattern statements here appear to be contingent. For instance, the last sentence would be false if the bonobo were to leave its natural habitat and migrate to Tanzania (which could happen). So, evidently, these particular pattern statements cannot be explained in terms of what properties are part of the nature of *being a bonobo*.

It is an interesting question whether contingent pattern statements like "The bonobo is found in the Congo Basin" might be understood to have the same underlying structure as metaphysically necessary "essentialist" pattern statements like "The dog is an animal." However, I will not try to resolve the issue here. Doing so is not necessary here because all distinctively mathematical truths express necessary and essential truths about structural properties.[38]

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[37] See https://en.wikipedia.org/wiki/Bonobo. Thanks to Ram Neta for mentioning this point and the example.

[38] I can propose one hypothesis though: Perhaps singular-form pattern-statements can be thought of as expressing a type of modal relation between properties, but with a stronger or weaker modal force depending on the context. The relation between *being a mammal* and *being a bonobo* would be the (very) strong modal relation of *is part of the nature of*. The relation between *having pink lips* and *being a bonobo* will be a contingent, but still relatively strong and stable (presumably nomic-like) relationship. (Cf. Armstrong (2010) p. 37.) The relation between *being a bonobo* and *being in the Congo Basin* will apparently be still weaker – perhaps just the degenerate one of "having an extension that is (mostly) a sub-extension of." But this is only a suggestion, and I will not attempt to spell this out into a worked-out theory for all pattern-statements. As I have mentioned, this is happily unnecessary for my purposes.

For our purposes, then, it is only important that there be a sub-class of pattern statements – the *essentialist pattern statements*, you might call them – that are plausibly made true in virtue of essential-parthood relations holding between properties, just like the case of "The dog is a mammal."[39] Examples like these show that the truth-conditions for at least *some* pattern-statements can be given entirely in terms of facts about properties.

My goal is to give a similar account for more complicated statements, such as those found in mathematics, and expand a semantics resting on property-mereological relations of this sort to the rest of mathematical discourse. If this is successful, then by showing that the truth-conditions of mathematical claims can be given entirely in terms of relations between purely structural properties we can show that the truths of mathematics do not commit us to anything more than certain properties or universals – i.e., that no abstract mathematical *objects* or *particulars* are necessary.[40]

While our approach will have to be generalized to more complex cases later, let us for now consider how this might go just for some simple mathematical statement. Consider, for instance, the subject-predicate singular-form claim, "0 has a successor." The *Platonist* semantics for this claim is obvious, and is the same as many ordinary singular subject-predicate statements:

[39] This sort of example is used by Warmke (2015), Roy (1993) and Jubien (2009) as a clear illustration of statements expressing constitutive property-parthood relations. Warmke, Roy and Jubien have already attempted to give an intensional semantics for *modal* statements in terms of mereological relations among properties/intensions. My hope is if I can give a purely intensional account of the truth-conditions for mathematical statements then that will provide an interesting contribution to the larger Warmke-Jubien project of "intensionalizing" semantics more generally.

[40] If this part of the theory goes through, then that reduces the number of problems to be solved. The problems of *mathematical* ontology and epistemology become the problem of the ontology and epistemology of *universals* more generally.

- '0 has a successor' is true iff the object zero has a certain property (viz., having a successor)[41]

My semantics necessarily will be somewhat less straightforward than this, since I want to give a semantics that makes reference only to constitutive parthood relations between purely structural properties.

First, let me reiterate the definition from Section 1.3 of the structural property of *being a Peano Arithmetic system*. To make this easier, we will use some abbreviations. Note that these are only abbreviations of natural language, not parts of a formal language:

- Σx: - "For some x such that"
- Σ!x - "For exactly one x"
- ix: - "The x such that"
- Πx: - "For all x such that"
- x∘S -"x is a part of S"
- U≪S - "U is a whole which is a part of S or is identical to S"

The clauses of the following definition then correspond to the traditional Peano axioms:

(PA): The property of instantiating Peano Arithmetic is the property of being a whole S and function ′ from the parts of S to the parts of S such that:

(1) $\Sigma!x{\circ}S{:}\ {\sim}\Sigma y{\circ}S{:}\ x=y'$; $0 \overset{\text{def}}{=} \iota x{\circ}S{:}\ {\sim}\Sigma y{\circ}S{:}\ x=y'$

(2) $\Pi x{:}\ (x{\circ}S \Rightarrow x'{\circ}S)$

[41] On *ante rem* structuralism, too, this sentence will have the same referential, first-order semantics (cf. Chapter 1 above). The primary difference from a *sui generis* Platonist is that the ante rem structuralist understands the nature of the object zero to be different. See Shapiro (1997) p. 72.

(3) $\Pi x: \Pi y: (x'=y' \Rightarrow x=y)$

(4) $\Pi U: U \ll S \Rightarrow [[0 \circ U \ \& \ \Pi x \circ S: (x \circ U \Rightarrow x' \circ U)] \Rightarrow \Pi x \circ S: x \circ U]$

What this definition essentially does is to take the traditional Peano axioms and to summarize them in purely structural terms, i.e., in terms of part, whole and purely logical vocabulary. For example, the second clause states that if x is a part of the system instantiating Peano arithmetic, then so is the successor of x.

So, "PA(S,')" is true for some whole S and relation ' iff S and ' together instantiate this complex structural property. This is what it means to say some physical "system" instantiates Peano arithmetic.

For the immanent structuralist, in the statement "0 has a successor" the phrase "0" connotes some structural property. So for our semantics, the content of the phrase "0" in the sentence "0 has a successor" will be some structural property defined in terms of the natural number pattern as a whole, namely: "the natural number that is not a successor":

[0**] = [**"The natural number that is not a successor"**]**

$= \lambda z{:}\exists S \forall x\big(G(x) \rightarrow F(x)\big)'[z \circ S \ \& \ PA(S,') \ \& \ z = \text{i}x \circ S{:} \sim\Sigma y \circ S{:}\ x'=y]$[42]

= the property of being a z such that (z is in a PA system S with a function ' satisfying the Peano axioms) and z is the unique x in that system which is not the successor of anything in the system

Spelled out explicitly in this way, this can be seen to be a purely structural property. Similarly for the content of "has a successor":

[42] Note: The semantics for the lambda operator here will not be in terms of functions from individuals to truth values, as in the traditional way. Rather, lambda expressions will denote actual *properties*.

[“has a successor”**]**

$= \lambda z{:}\exists S \exists '[z \circ S \ \&\ PA(S,') \ \&\ \Sigma y \circ S{:}\ y = z']$

= the property of being a z such that (z is in a PA system)[43] & some y in the system is the successor of z

So, fully spelling out the structural properties involved, the immanent structuralist will translate “0 has a successor” as “the natural number that is not a successor has a successor.”

On the semantics I am proposing, this is to be read as a pattern statement, like “the pitcher throws the ball.” So our strategy will be to apply the insight that the above two properties stand in constitutive, property parthood relationships to each other. Thus, the semantics for “0 has a successor” will be in terms of a constitutive property-parthood relation, which we can label $\Subset$:

(CONS): [“has a successor”**]** $\Subset$ **[“**The natural number that is not a successor”**]**

I.e.:

$\lambda z{:}\exists S \exists '[z \circ S \ \&\ PA(S,') \ \&\ \Sigma y \circ S{:}\ y = z'] \Subset \lambda z{:}\exists S \exists '[z \circ S \ \&\ PA(S,') \ \&\ z = \imath x \circ S{:}\ {\sim}\Sigma y \circ S{:}\ x = y']$

The way to read **(CONS)** is that it is true just in case the former property *is part of what it is to be* the latter property.[44] The property *having a successor* is part of the nature of the property

[43] In the rest of the discussion, for ease of readability I will summarize “z is in a PA system S with a function ‘ satisfying the Peano axioms” by simply saying “z is in a PA system.”

[44] Similar to how the property *being a mammal* is part of the nature of the property *being a dog*, which makes the pattern statement “the dog is a mammal” come out true.

being the member of a PA-series that is not any other member's successor.[45] In that sense, the property **[“**has a successor**”]** is a *constituent* or a *part* of the property **[“**the natural number with no successor**”]**.[46]

In sum: The basic idea for a simple singular mathematical sentence “a is F” would be that “a is F” is true if the property *being F* “follows from” or is “part of” the essence of the property *being a*. In the next section, I propose a more general semantics for the rest of mathematical discourse based entirely on property-parthood relations of this sort.

4. Language and Denotation: A First Stab

Before we begin, we should briefly specify the language we will be working with. The semantics I give here will be for a standard sort of language:

Language:

Let a1, a2, ... be individual constants.

Let F^n, G^n, H^n, … be n-place predicates (for each n).

Let x1, x2, … xn, … be countably many variables.

We also include the usual first-order logical connectives and symbols, including identity. The usual rules for generating formulas and sentences apply.[47]

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[45] Note that we do *not* want to say that “0 has a successor” is true iff the property **[**“has a successor”**]** is true of *the property* **[**“the natural number with no successor”**]** – that doesn't really make much sense, and at best seems false.

[46] *Important*: Note that this is a different sense of “part” than the sense in which it occurs in the definitions of the mathematical structural properties themselves. There is “property-parthood,” and then there is the use of “part” in the ordinary sense. Mathematical *definitions* make use of the *latter* sense, while our semantics makes use of the former sense.

[47] Since the aim is to provide a semantics to rival Platonism, we will want to analyze the same sorts of formulas and sentences that a standard semantics does.

Let us think about denotation for our language. From our discussion in the previous section we gleaned some idea of how an immanent structuralist should spell out a simple mathematical statement, as well as what denotation will have to look like. So we begin by drawing this out a bit more.

First, note again that on immanent structuralism, the ontology of mathematics consists of just the purely structural properties. All that is available, for mathematical purposes, are purely structural properties. Therefore, individual constants must connote purely structural properties.[48]

Which ones? Well, naturally, an individual constant will connote the property of being a certain *individual* fulfilling a unique role within some kind of purely structural system.[49] We saw this in the case of the term "0" for example. This stood for the *property* of being *the* unique thing in a Peano-arithmetic successor series that is not the successor of anything else in the series.

So, where [] is a denotation function, and where a is an individual constant:

(D1): [a] denotes some purely structural *individual*-property,

- i.e., properties of the form: [being an x s.t. x is *the* thing of such and such sort in a purely structural system]

[48] Note that occasionally I will use the phrase "connotes" instead of "denotes." While these both express the same thing, "connotes" is sometimes used to emphasize the fact that, in our theory, the only thing referred to is *properties*, including when we are using individual names, and furthermore that predicating something of a name in our language amounts to saying one property is part of another rather than that an individual entity instantiates a property.

[49] Note: An individual constant connotes the *property* of being the individual with a certain role in a kind of structural system; individual constants do *not* necessarily denote *individuals* that *actually are* the role-holders in some *instantiation* of the relevant structure.

- For example, [0] denotes λz:∃S∃'[z∘S & PA(S,') & z = ix∘S: ~Σy∘S: x'=y] = the property of being a z such that (z is part of a system instantiates PA) and z is the unique x in that system which is not the successor of anything in the system.

From this, we might take a first stab at a simple atomic, subject-predicate sentence. For example, in the last section we saw that "0 has a successor" will have the following truth-condition:

(Cons): [“has a successor”**] ⋐ [**0**]**

So we might conclude that in general, where a is an individual constant and [F] a purely structural property:

(T0): [a is F] is true when [F] ⋐ [a]

- i.e., the purely structural property *being F* is part of the nature of the purely structural individual property [a][50]

Now, up to this point, I have left the "property-parthood" notion that is being used in our semantics, and which we have denoted by **⋐,** to be understood at an intuitive level. This is a good point at which to briefly diverge and elaborate on it a bit more. As the discussion below will show, we ultimately will have reasons slightly to modify **(T0)**.

5. What is Property Parthood?

What does it mean when we assert an essentialist pattern-statement? Take our example:

(Dog) 'The dog is a mammal' is true because *being a mammal* is part of *being a dog*.

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[50] Note that this truth-condition is ultimately the same in form as other, non-mathematical essentialist pattern statements. For example: **[**The dog is a mammal**]** is true when **[**mammal**] ⋐ [**the dog**].**

Specifically, what does it mean to say that one property is "part of" another, as in this sentence?

One attractive account takes this sort of talk literally. Call this *the literal account*. On this view, "part" here just means *part* in the ordinary sense of the word. So properties like *being a dog* and *being a horse* will both literally have *being a mammal* as a part.[51] That is why "The dog is a mammal" comes out true.

Craig Warmke, who also uses the example of "dogs are mammals" to illustrate the property-parthood relation, takes this view. He develops a formal property mereology with definitions and axioms similar to those found in the traditional mereology of physical objects.[52] Warmke includes classic axioms for proper parthood, such as Asymmetry, Transitivity, and Weak Supplementation, and for sums he endorses Universal Composition.[53] The axioms are formally similar to traditional mereological axioms, but for Warmke the variables are intended to range over properties:

- **Asymmetry**: If x is a proper part of y, y isn't a proper part of x.
- **Transitivity**: If x is a proper part of y, and y is a proper part of z, then x is a proper part of z.
- **Weak Supplementation**: If x is a proper part of y, then y has another proper part disjoint from x.

[51] Thus, these properties (*being a dog* and *being a horse*) will also, literally, *overlap*.

[52] Cf. Warmke (2016). See also Michael Jubien (2009). L.A. Paul also explicitly develops a property mereology, where the notion of "part" in property parthood is understood in an entirely literal manner, and axioms similar to the traditional mereological axioms for ordinary physical composites are given. The main difference from Warmke is in Paul's conception of properties. Paul is a trope theorist, taking properties to be tropes, i.e., particular, non-repeatable, individual metaphysical constituents of ordinary objects that are local to the objects that have them. Traditional Platonists and Aristotelians take properties to be *universals* – meaning at least that they are repeatable – with Aristotelians taking them to be literally located in all their instances, and Platonists taking them not to be literally located anywhere. See Paul (2002) pp. 582-3.

[53] For a helpful overview of the various axiomatizations of mereology, cf. Varzi (2016).

- **Unrestricted Composition**: For any specifiable set of properties whatever, there is a sum of those properties which is itself a property.[54]

So, for instance, on Warmke's ontology the property *being a golden mountain* will be a literal mereological sum of *being golden* and *being a mountain*. Given the latter two properties, the existence of the former follows from Unrestricted Composition. Also, "Golden mountains are physical" will be true because of transitivity, together with the fact that *being physical* is, presumably, part of *being a mountain*.

One virtue of this account is that it is relatively straightforward. It takes the phrase "being a part of" seriously, allowing us straightforwardly to account for our ways of speaking. Another virtue is that it is amenable to clear formal treatment via a familiar system of definitions and axioms.

Furthermore, the literal account nicely explains some of our intuitive judgments about property parthood. For example, the transitivity axiom straightforwardly explains the "golden mountains are physical" case just mentioned. And asymmetry explains why we think that "cats are mammals" is true, but not "mammals are cats": We think there is more to *being a cat* than *being mammalian* – i.e., that *being mammalian* is a *proper* part of *being a cat* – and so we can infer from this fact together with Asymmetry that "mammals are cats" is false.[55]

Nevertheless, I have a few objections to the literal mereological account.

[54] From this it quickly follows that some impossible properties exist, i.e., ones that cannot be instantiated. For example, *being round and square*. Arguably this is a *benefit* of Warmke's account, at least if non-trivial reasoning about impossibilities is possible.

[55] Since "mammals are cats" is only true on the account if [cat] is a part of [mammal], i.e., if [cat] is a proper part of or identical to [mammal]. And they clearly are not identical. So by Asymmetry and the fact that [mammal] is a proper part of [cat] it follows that [cat] is not a part of [mammal].

For one, it is somewhat difficult to assess whether the ordinary property parthood talk we use ought to be taken literally enough to merit a formal system of mereology, especially considering that properties are a rather different kind of thing from, say, physical atoms. We have a relatively clear understanding of "parthood" and "composition" as applied to physical objects. We can even use "parthood" clearly in an extended sense to talk about more abstract subjects, like "parts" of theories. When it comes to properties, on the other hand, talk of one property's being *part* of another is perhaps better thought of as a sort of analogy or *facon de parler* to express a different sort of relation.

This worry is not decisive, however. A more important concern has to do with whether the literal mereological account can actually do the theoretical work it is needed to do – in particular, whether it can adequately capture the modal force of property-parthood statements. This is my more central objection. First, recall our **(Dog)** example:

(Dog) "Dogs are mammals" is true because *being a mammal* is part of *being a dog*.

On the literal account, the right-hand side is supposed to be read as a mereological statement, with "part" being understood entirely univocally with our ordinary notion.

The problem is that it is hard to see how, on the literal reading of property "parthood," the right-hand side is equivalent to or even *entails* the left-hand side. This is because the pattern statement "Dogs are mammals" is like "Water is H2O" or "Hydrogen atoms are physical:" It is an assertion of a *necessary, essential* truth. But it is not the case that, *in general*, the *parts* of a thing are necessary or essential to that thing: From "A is part of B" we cannot, in general, infer that "A is *necessarily* part of B." And so "*being a mammal* is part of *being a dog*" does not, *by itself*, entail the necessary pattern statement "Dogs are mammals." Therefore, so long as "part" is understood in the usual sense, the right-hand side of **(Dog)** does not entail the left-hand side, and so the two propositions are not necessarily equivalent.

Tony Roy makes a similar point in a different context using the case of the properties *being red* and *being colored*:

> "Now, r is the *actual* BEING RED and c is the *actual* BEING COLORED … And, in reaching the conclusion that it follows from the nature of c that if something instantiates r, it instantiates c, have we reached the conclusion that necessarily whatever is red is colored? I think not. For if there is a possible world … where BEING RED exists without being a disjunctive constituent of BEING COLORED then (plausibly) there is a world where it is not the case that everything red is colored. … [G]iven that the structure of BEING COLORED is as in actuality, it follows that whatever is red is colored. But to move from this to the conclusion that *necessarily* whatever is red is colored, we need that BEING COLORED has its structure necessarily …"[56]

In other words (modifying Roy's example slightly), even if the mereological claim "part of *being red* is *being colored*" is *actually* true, if this parthood statement is *contingent* then from the literal account it will follow that "Red is a color" is contingent – which it should not be. In short: The relation between a property and its "property-parts" is stronger than that between, e.g., my body and any particular atom that happens to be part of it. It is not clear how the *literal* reading of property parthood accounts for this.

[56] Roy (1993) p. 345.

Of course, the problem might be solved by saying that abstract objects like properties *do* have their parts necessarily. But it is difficult to see how the literal mereological account, by itself, has the resources to justify this claim. Consider Roy's discussion:

> "It might be argued that intrinsic properties of abstract objects and, in particular, structures of properties are essential to objects that have them. However, in general, the structural properties of a thing may be distinguished from those that are essential to it. Let's say (vaguely) that the structure of a thing has to do with its parts and their interrelations. Then it is at least arguable that a lectern's HAVING THIS (PARTICULAR) BIT OF WOOD AS A PART is a nonessential but structural property of it."[57]

If the "structures" of other objects are not essential to them, and if the notion of "part" in property-parthood statements is not different in kind from the ordinary mereological notion, then why is the structure of a *property* any different? What *grounds* the fact that abstract objects like properties have their intrinsic structures essentially?

For these reasons, I fear that the literal mereological account is at least incomplete.[58]

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[57] Roy (1993) p. 343.

[58] Incidentally, this also suggests a worry for the "modal intensionalist" theory of modality advocated by Warmke. See Warmke (2015) and Warmke (2016). Warmke's system takes the literal interpretation of property-parthood, and defines "p is necessary" as true when the property *being such that p* is part of the property *being a world*: On Warmke's view, $\Box p$ is true iff *being such that p* is part of *being a world*. My worry, based on our discussion above, would be that the definiens cannot constitute an analysis of the definiendum if *being a world* only has *being such that p* as a part *contingently*. Of course, the two sides might be contingently, materially equivalent, and so Warmke's analysis might still be formally useful. But Warmke explicitly intends his analysis of modal notions to be more than another mere formalism: He wishes for it to be *explanatory* of *why* the particular truths that are necessary are the ones that are necessary. Cf. (2015) pp. 309 and 331. Perhaps Warmke's analysis can be combined with the discussion below that tries to resolve some of these worries for the literal account of property parthood.

To help us see what the right account is, consider the following sorts of property-parthood statements, all of which are apparently ways we might put the same point –and, therefore, all of which are potential paraphrases of some essentialist pattern-statements. Where F and G are properties:

- (Being) F is part of (being) G.
- F is part of what it is to be G.
- F is part of the essence of G.
- F is part of the nature of G.
- F is part of the definition of G.
- Being F is constitutive of being G.

These all seem to be equivalent. But some of these ways of speaking suggest, or even explicitly appeal to, essentialist notions.[59] What I propose then is that if we are to get the right modality for essentialist pattern statements we should interpret the $\Subset$ directly as an essentialist notion.

The preferred conception of essence that I will work with is an Aristotelian conception, inspired by Kit Fine's early discussion in "Essence and Modality."[60] Kit Fine's view is largely in response to a *modal* conception of essence, where the notion of an *essential* property is understood as a *metaphysically necessary* property.

In the case of property-parthood statements, the modal analysis would likely go something like this (with the necessity operator □ read as a metaphysical necessity):

[59] Others have noted the essentialist nature of these sorts of statements, cf. Correia (2006).

[60] See Fine (1993) and (1995).

- (**Modal**): F is part of the essence of G iff $\Box\forall x(Gx \rightarrow Fx)$

Though I think this analysis is in some ways closer to the truth than the literal mereological account – note, for instance, that in this analysis, the notion of "part" plays no ultimate role – I prefer an interpretation of essence closer to Fine's notion, which I will explain.

Fine begins by noting that the modal account of essence fails to capture certain asymmetries between necessarily co-existent objects. For example, one might think it is part of the essence of Jack – Tarzan's son – that if Jack exists then Tarzan is his father; but it does not seem to be part of *Tarzan's* essence that if Jack exists then he is Jack's father. Although it might be a *de re* necessary property of both that *if Jack exists then Tarzan is his father* (say, because of the essentiality of origins), it does not seem to be *definitive* of Tarzan in the way that it is of Jack; this property is not part of Tarzan's very definition or nature in the way that it is of Jack's.[61]

In short, there seems to be an asymmetry in essence between Jack and Tarzan that is not captured by the modal account. Although it may be true that an essentialist statement *entails* the relevant metaphysical necessity, it does not seem that every metaphysical necessity entails a corresponding essentialist statement.

Other examples of essentialist asymmetries might be: It is part of the very definition of salt to contain chlorine, but not part of the very definition of chlorine to be a potential constituent of salt (even though it might be a *de re* necessary feature of both, since necessarily salt is NaCl). Or to take Fine's famous example: It seems to be part of the nature of the set singleton-Socrates that it contain Socrates, but not part of Socrates' own nature or definition that singleton-Socrates

[61] The example comes from Gideon Rosen's helpful discussion in Rosen (2015).

contain him. This latter property, although it is necessarily true of him, is not really part of the intrinsic nature of him; it does not seem like an intrinsic part of *who he is*.

To remedy this, Fine proposes that we introduce a new, hyperintensional modal operator: "$\Box_G$": "$\Box_G$ p" which is to be read as "It is true in virtue of the nature of G that p."

G can be any predicate, including the property of *being identical to x,* for some specific individual x.[62] Going back to the chlorine/salt example, the modal definition of essence will fail to capture the definitional asymmetry, since both of the following are true:

- **(M1)** $\Box\forall x\big(\lambda zP(z)(x) \rightarrow F(x)\big)$
- **(M2)** $\Box\forall x\big(Chlorine(x) \rightarrow PossiblyContainedinSalt(x)\big)$

By **(Modal)** it follows that *containing-chlorine* is part of *being salt*, which is good. But it also follows that *being possibly contained in salt* is part of *being chlorine*, which seems false.[63] This motivates Fine's hyperintensional operator, which can be used to provide an alternative definition to **(Modal)**:

- **(Fine)** Being F is part of being G iff $\Box_G\forall x\big(\lambda zP(z)(x) \rightarrow F(x)\big)$

What this says is that *being F* is part of *being* G iff it is true in virtue of the nature of G that all Gs are Fs. This interpretation of property-parthood statements will preserve the essentialist asymmetry between chlorine and salt, because the right-hand side of **(Fine)** does not

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[62] Fine (1993) and (1995) seems to waver a bit on whether to make the operator relative to an individual, to some set of individuals, or to a predicate, but the formulation I've given is most suitable to our purposes here.

[63] Just to be clear: What Fineans question is not whether it is, in fact, a necessary truth that all chlorine is possibly contained in salt. What they would insist is that this feature about *salt* does not pertain intrinsically to *the definition of what it is to be chlorine*.

follow from the mere metaphysical necessity **(M2)**. Thus, the property *containing-chlorine* will be part of *being salt*, but *being possibly contained in salt* will not be part of *being chlorine*.

So Fine's essentialist operator seems a more promising notion for the purpose of analyzing property-parthood statements. However, there is a further relevant distinction Fine makes between two possible interpretations of "$\Box_G$". "$\Box_G$ p" can be read as expressing either a *constitutive*-essential fact or a *consequentialist*-essential fact. Let us briefly draw out this distinction.

Very briefly, the consequentialist essence of something will include all those features that are essential to it *plus those that follow from the essence together with the natures of the logical connectives*. In other words, the consequentialist essence of something is the logical closure of the essence. The *constitutive* essence, on the other hand, is restricted to the "core" essential features of the thing, from which the rest of the consequentialist essence follows.

Consider, for example, the classic Aristotelian definition of *human*:

- **(Human)** To be a human is to be a rational animal:

  $\Box_{\mathrm{Human}} \forall x \big(Human(x) \leftrightarrow Rational(x) \land Animal(x) \big)$

This definition, which is meant to express the essence of *being human*, is supposed to provide a *real definition* of human. A real definition, on the traditional Aristotelian understanding, will include only those properties that are part of the very core of whatever is being defined.[64] You might say that the *constitutive* essence is *sparse*. So the constitutive essence of *being human* will include the properties *being rational* and *being animal*. On the other hand, it

[64] Really, in Aristotle's original understanding, a real definition must be by genus and species. In this case, "rational" is the species and "animal" the genus. See David Oderberg, *Real Essentialism* (2007).

will not include conjunctive features like *being rational or irrational* or *being such that if animals are material then humans are material*. These features *would* be part of the *consequentialist* essence, however, since they follow from the constitutive essence via logical closure.

Although the idea of constitutive essence is *relatively* clear – it consists of those features or facts that belong to the consequential essence but not in virtue of anything *else* in the consequential essence[65] – Fine tends to work more with the consequentialist concept, as it can sometimes be difficult in practice to separate out the features that are the "core" of a thing's essence from those that are not. For our purposes, the consequentialist essence is the more important notion. This is because mathematical truths include not *just* claims about the real definitions of various mathematical properties, but also claims about what *follows from* those definitions.

With this in mind, we are now in a position to give an interpretation of property-parthood statements:

- (**Part**) Being F is part of being G (i.e., F $\Subset$ G) iff $\Box_G \forall x\big(G(x) \rightarrow F(x)\big)$
  - (Where "$\Box_G$" is to be read as "It is true in virtue of the *consequentialist-essence* of G")

This resolves the problem the literal account of property parthood faced, which was that it could not acccount for the essentialist modal force of property parthood statements. On our interpretation, this essentialist force is built into property parthood statements from the beginning.

[65] Cf. Rosen (2015) pp. 195-6.

6. A Property-Constituent Theory of Mathematical Truth

Now that the notion of property-parthood has been given a definite ontological interpretation, we can go on to give the rest of our semantics for a mathematical language entirely in terms of essentialist relations between properties.

As we noted earlier in section 2.3, singular phrases should be given the denotation of a purely structural individual property.

Denotation:

(**D1**) [a] will denote some purely structural *individual*-property

- i.e., properties of the form: [being an x s.t. x is *the* thing of such and such sort in a purely structural system]
    - Again, a purely structural property or system is one that can be defined entirely in terms of part, whole, sameness, difference, and purely logical vocabulary.
    - e.g., [0] denotes λz:∃S∃'[z∘S & PA(S,') & z = ix∘S: ~Σy∘S: x'=y] = the property of being a z such that (z is part of a system instantiates PA) and z is the unique x in that system which is not the successor of anything in the system.

Going back to our earlier attempt at an analysis of singular predications, as a first stab we said:

(T0): [a is F] is true when [F] ⋐ [a]

Here, [a] is some purely structural individual property. Let '$\lambda zP(z)$' denote this property. Now that we have an understanding of what the ⋐ operator means, we know '[F] ⋐ [a]' should be interpreted as follows:

- (**T0***) [F] ⋐ [a] iff $\Box_{\lambda zP(z)}\forall x\big(\lambda zP(z)(x) \rightarrow F(x)\big)$

Since we are going with a consequentialist interpretation of the essentialist operator, it is helpful, formally speaking, to have a convenient way of referring to all the properties that are part of a property's consequentialist essence.

(**D2**) [a]* denotes the set of properties that are in the *consequential essence* of [a]

- The consequential essence of [a] is the set of all properties that are part of the real definition of [a] or that logically follow from it – in other words, all the properties that 'logically follow from the essence' of [a].
    - Where [a] is some purely structural individual property and where $\lambda zP(z)$ denotes [a]: [F] is in [a]* iff $\Box_{\lambda zP(z)}\forall x\big(\lambda zP(z)(x) \rightarrow F(x)\big)$

With all of this laid out, we can now give the full truth-condition for a singular essentialist pattern-statement:

(**T1**) [a is F] is true if [F] is in [a]*; otherwise not.

- That is, if *being F* is part of the consequential-essence of *being a*.

For example, [0 is even] will be true if *being even* is in the consequential essence of *being the thing in a PA-system with no predecessor* or else follows logically *from* something in the consequential essence.[66]

There is one other technical issue to be resolved before we can give all the other truth-conditions: We've seen what to do in the central case of a singular subject-predicate sentence. But how should we handle relational statements that assert a relation between *several* terms?

Here I will borrow an idea from Craig Warmke.[67] First, let us consider the n-place relation F^n:

(**D3**) [F^n_, _, … _,] denotes an n-ary purely structural relation between individuals.

- For example, the arithmetical relation [<_,_] denotes one thing's being less than another in a PA-system (which, given that this is a purely structural property, would be what is denoted by the phrase "one thing's coming before another in a successor series").

The way to solve the issue is essentially to turn this into a one-place property. We can then give the same truth-condition as in (**T1**).

Let a1, a2, … , an be some names. [a1], … , [an] are the purely structural individual-properties that these names denote. Suppose "$\lambda zP^1(z)$," … "$\lambda zP^n(z)$" are specifications of these properties in purely structural terms. Then [$\lambda z(z{=}\iota xP^1(x))$], … [$\lambda z(z{=}\iota xP^n(x))$] are properties of

[66] Given what the consequentialist essence is, this is just a redundant way of saying "is in the consequential essence."

[67] See Warmke (2019).

being identical to the unique thing that has P^1, … the property of being identical to the unique thing that has P^n , etc.[68]

We can now turn the n-place relation F^n into a one-place property:

(**D4**) [$F^n a_1, a_2, a_3, \_\_, a_5, \ldots a_n$] denotes the property of being F-related to a1, a2, a3, and a5, a6, … an

- Or, a bit more formally: Let [$F^n\_, \_, \ldots \_$,] = $\lambda x_1 \lambda x_2 \ldots \lambda x_n R(x_1, x_2, \ldots, x_n)$
- Then [$F^n a_1, a_2, a_3, \_\_, a_5, \ldots a_n$] = $\lambda x_4 R(\iota x P^1(x), \iota x P^2(x), \iota x P^3(x), x_4, \ldots \iota x P^n(x))$

With this definition in place we can then state the truth-conditions for n-place relational statements with names. Where F^2 is a two-place predicate:

(**T2**) [F^2ab] is true if [$F^2a\_$] is in [b]* and [F^2_b] is in [a]*; o/w not

- More informally: F^2ab is true iff *being F-related to a* is in the essence of b, and *being F-related to b* is in the essence of a.[69]
- For example, informally, [2<3] iff
  - The property [_< 3], i.e., the property of *coming before the S(S(S(0))),* is part of [2], i.e., *being the S(S(0))* ***and***
  - property [2 < _], i.e., *coming after the S(S(0)),* is part of [3], i.e., *being the S(S(S(0))).*

All other n-place predicates will have a semantics similar to the last clause.

[68] This divergence from Warmke's account in definition (D4) is necessary because for Warmke constants essentially denote haecceities that are unique to specific, individual entities, whereas in our case constants denote general properties that can be had by more than one individual.

[69] Keep in mind that in our language, which is purely structural "a" and "b" will actually have the form of "definite descriptions" that ultimately denote some purely structural property. For example, "a" might be "the successor of 0."

The rest of the semantics for basic statements is then relatively straightforward:

- (**T3**) [a = b] is true iff [a]* = [b]* [70]
- (**T4**) [~p] iff [p] is not true
- (**T5**) [p & q] iff [p] and [q] are true

Furthermore, all the other propositional connectives, such as material conditionals and disjunctions, can be defined in the usual way in terms of these ones. Also, with the account of singular and relational statements being laid down, we can give a straightforward definition for quantification as well that parallels the usual one.

First, we add a countable infinity of variables x1, x2, … xn, and so on to the language just as we normally might, and we define formulas with variables in the usual way. We then add the idea of a variable assignment f. However, again, since we are working with a purely structural ontology, the function f assigns variables to purely structural *individual-properties*. For example, f might assign x to the property "being the successor of 0." Other than this qualification about the variable assignment, the definition of quantification will be rather similar to the usual one:

- **(T6)** $\forall x Fx$ is true relative to variable assignment f if for any variable assignment f' just like f except perhaps for the variable assigned to x: [F] is part of f'([x])*
    - Here f'([x])* is, of course, the consequential essence of whatever individual structural property f' assigns to x. See **(D2)** above.

[70] Note that this clause gets at the idea – which *ante rem* structuralists say – that the natures of "mathematical objects" are entirely exhausted by their relations to other objects within the structure. Being 2 *just is* being an object in relation to objects in a certain sort of system, because ultimately that's all there is in the *essence* of *being 2*; or at least, so structuralists would contend. I just do not think it is necessary to conclude that for any structural mathematical property there actually *are* intrinsically featureless mathematical *objects* that *instantiate* that property.

With this definition, we can define the existential quantifier in the usual way in terms of negation and the universal quantifier, and so we have a language adequate for mathematics together with truth-conditions that only make reference to purely structural properties.

7. Ontology and Ideology

Given that the truth-conditions of mathematical claims can be given entirely in terms of essentialist, property-parthood relations between purely structural properties, it follows that the truths of mathematics do not commit us to anything more than certain properties or universals – i.e., that no "mathematical objects" are necessary for mathematical truth. This then reduces the problems of *mathematical* ontology and epistemology to the problems of the ontology and epistemology of properties or universals more generally.

Even by itself this is a step forward, insofar as we've made the problem of mathematical ontology and epistemology part of a more general problem. Furthermore, the latter problem seems more soluble *prima facie*.

Nevertheless, we should look more closely at the ontological commitments of the theory. Before considering the epistemology of mathematical properties in the next chapter, I'd like to propose a "pure essentialist" or "primitivist" reading of the theory presented in sections 2.5 and 2.6 that allows us to have mathematical truths without even the need for *properties*.

So far, I have freely referred to properties, and I have couched my theory in terms of them. And indeed, this is the most natural way to formulate our theory. However, is it *essential* that we do so?

From our discussion in 2.5, we settled on an interpretation of property parthood in terms of essence. In particular, we gave the following definition, in terms of Fine's consequentialist essentialist operator (Where λzP(z) denotes the individual-property of being [a]):

(**T0***) [F] ⋐ [a] iff $\Box_{\lambda zP(z)}\forall x\big(\lambda zP(z)(x) \rightarrow F(x)\big)$

What I will call the *referential reading* of the right-hand side is this:

- (**Referential**) "It is true in virtue of the nature of the property λzP(z) that whatever is λzP(z) is also F."

I call this the "referential" reading because it seems to commit us to a real ontology of properties in two ways: First, it seems to refer straightforwardly to a property. Secondly, it seems to posit a ground or truthmaker for the truth of the relevant conditional, viz., the *nature* of the relevant property. But presumably:

- (**Grounds**) Grounds for truths must *exist*.

So, **Referential** seems to commit us to *properties* and *natures*.

In general, I will work with this reading of (**T0***) and assume that it is true. But I believe it is worth flagging an alternative way of thinking about the essentialist operator, namely, to think of it as a sort of *primitive modal operator*, the way some philosophers have thought about the necessity operator □.

To make the comparison clearer I will first say a little bit about the more general position known as *modal primitivism*[71] which primarily shows up in discussions of metaphysical

[71] Cf. Bueno and Shalkowski (2009), (2013) and (2014). See also Vaidya (2015).

modality. This position is also known as "modalism." In Anand Vaidya's characterization it consists of the following theses:

1. Alethic metaphysical modality is primitive.
2. Modality does not reduce to quantification over possible worlds. That is, "it is possible that P" does not mean that "P is true in some possible world." And "it is necessary that P" does not mean "P is true in all possible worlds".
3. The theory does not appeal to or use possible worlds.
4. An empiricist-friendly approach to the epistemology of modality does not depend on conceivability or the postulation of possible worlds.

One *prima facie* argument for this view is that any attempt to reduce alethic modality to quantification over worlds will seemingly have to involve a specification of those worlds which are *possible*. For example, if we take worlds to be sets of propositions, we will want to analyze "possibly p" as being true when p is true in all those sets of propositions that are *possibly realized*.[72] But this of course appeals to the notion of possibility again. Thus, the "possibility" remains unreduced, and it is difficult to see how we are going to get rid of it. Therefore we should just take the possibility operator as primitive.[73]

[72] One might think that we can define the sets of propositions that are "possibly realized" as those that are consistent. But that leaves the question of what is meant by consistency. It is widely accepted that a syntactic criterion will be inadequate for a number of reasons, with one being that the propositions that are metaphysically possible extend beyond those that are describable in any canonical language. But without a syntactic criterion we are then still left with the problem of what *really* constitues a "broadly logically consistent" set of propositions, and the modality remains ultimately unanalyzed. Moreover, we cannot simply handwaive away this problem as not being amenable to formal analysis, since what we are concerned with is not just formal semantics, but the ontological question of what "broadly logical" or "metaphysical" modality *really is*.

[73] Obviously this argument is strong only if one does not already have a worked-out strategy for reducing the modal to the entirely non-modal. Lewisians, of course, do, and will therefore not find the above argument for primitivism compelling. However, when it comes to Lewis' proposal – requiring, as it does, acceptance *a priori* of dogs with dragon heads, or of baseball bats that compose sonatas – few are likely to be willing to accept this strategy. While

I propose that we can say something similar about the language of "essence" that we have been using. Here is the *primitivist* or *pure essentialist reading* of (**T0***):

- (**Primitivist**) "$\Box_G$" is a primitive operator, for any G. Thus, there really are very *many* essentialist operators. You might read it aloud as: "G-essentially: p." And for another predicate H, it might be the case that "H-essentially: q." And so on.

In other words, this reading purports to describe an essentialist fact without quantifying over or referring to any properties. As Quine might put it, it is an attempted trade in ontology for ideology.[74]

The downside of this trade, of course, is that we can no longer refer to a specific class of entities to *ground* the truth of the essentialist claims. The referentialist reading, on the other hand, appeals to *properties* (and, what seems to come along with them automatically, *their natures*[75]) as those things in virtue of which the essentialist facts come out true. However, it is not clear that this is a decisive advantage for the referentialist, for the following reason.

Suppose that some property G grounds the truth of the essentialist fact that $\Box_G$ P. Now, it seems plausible that essential truths entail metaphysically necessary truths (albeit not vice-versa). After all, if it is true because of the very *definition* or *essence* of something that P, then surely it *cannot not* be the case that P. For example, if it is true in virtue the nature of water that all water contains hydrogen, then surely it couldn't be the case that some water *not* contain hydrogen. To

Lewis's view of the world certainly is fascinating to think about, one can only stare at it for so long before primitive modality begins to attract one's gaze.

[74] Cf. Quine (1951) and Sider (2011) pp. 12-14.

[75] Actually, long ago when Aristotelian metaphysics dominated philosophical discussions, it was once considered a substantive and highly controverted question whether a thing and its essence were merely one thing or two. Cf. Aquinas *On Being and Essence* for the classic case that a thing and its essence are distinct. I side with others, like Duns Scotus, in assuming that a property and its nature are not really two separate things.

put the point another way: The essentialist operator seems to express a *stronger* modality than the metaphysical-necessity operator, in the sense that an essentialist truth always entails a metaphysically necessary truth, but perhaps not all metaphysically necessary truths are essentialist ones.

Given this connection between essential truths and metaphysical necessities then, we have the following *essentialist-necessity principle*:

- (**ENP**) $\Box_G$ P entails $\Box$ P

Returning to the referential reading, essentialist statements like '$\Box_G$ P' are supposed to be true in virtue of the existence of some properties G and their natures.

Now let us ask a question: Assume some essentialist truth, like $\Box_G$ P. Does the existence of G *entail* that $\Box_G$ P? Here is why it must: If the existence of G does not *entail* $\Box_G$ P, then $\Box_G$ P *could fail to be true.*[76] In that case it seems that P could fail to be true (since P is, in fact, true *in virtue of the nature of G,* so that it seems to depend on the nature of G for its truth). In that case, $\Box$ P would be false. But by (ENP) it cannot be the case that $\Box_G$ P is true and $\Box$ P is false. So, the existence of G must entail that $\Box_G$ P.

But that raises a new problem: *Why* does the existence of G entail $\Box_G$ P?

You might say it is because the existence of G is metaphysically necessary. Of course, one might ask "Why?" But suppose we do not press for an explanation of this. Even so, it still does not seem to be enough to ground the entailment.

[76] Because p entails q when it is not possible for p to be true and q false. So if the existence of G does not entail $\Box_G$ P then it is possible for G to exist but $\Box_G$ P to be false.

Suppose, for example, that G is the property of *being a brown chair*. Let our proposition P be "Brown chairs are colored." Let "$\Box_G$ P" be the corresponding essentialist claim. Now, suppose that G is a *necessary* being, existing in all worlds. However, what if in some other possible world, G is *instead* the property *being a ghost*.[77] By itself, this supposition does not contradict the claim that G is a necessary being. I only said that G *exists* necessarily, i.e., that G exists in all possible worlds. It does not follow that G has all the same features in every world, any more than I have the same features I do in every possible world.

But in this world where G is the property of *being a ghost* it does not seem true in virtue of G that brown chairs are colored. So, it is not the case that in every world where G exists "$\Box_G$ P" is true. Which is to say, the existence of G does not entail $\Box_G$ P.

The upshot of this discussion is that even if our property G is a necessary being this does not by itself explain the *entailment* relation between the existence of property G and $\Box_G$ P.

You might say: "Fine, it is the existence of G *together with the claim that being a brown chair is its nature* that entails $\Box_G$ P." But then we might ask: What if G has a different *nature* in some other possible world?[78] Now perhaps one might say that it *has its nature* necessarily. If *that* is true, then the existence of G *does* seem to entail $\Box_G$ P, and perhaps we are able to explain EMP in this way.

[77] By itself, this supposition does not contradict the claim that G is a necessary being. I only said that G *exists* necessarily, i.e., that G *exists* in all possible worlds. It does not follow that G *has all the same features* in every world. For example, you might say that God is a necessary being, but it doesn't follow that he is *the same in all respects* in every world. In one world God might choose to create the sun, and in another world he might choose not to. That is consistent with God's being a necessary being.

[78] This echoes an objection made in Section 2.5 above against the literal reading of property parthood.

However, we may ask yet further: W*hy* does G have its nature necessarily? You might just say that this is a primitive, ungrounded metaphysical necessity. Admittedly, that is not an obviously bad position to take. But we might wonder: Wouldn't it have just been better all along simply to say that the ultimate, ungrounded "modal" truth was the bare essentialist claim that $\Box_G$ P? It is not clear what advantage in explanatory depth is gained by appealing to a primitive metaphysically necessary law that properties have their natures in all possible worlds. Why not simply say "$\Box_G$ P" is the primitive modal fact?

For these reasons, it is not clear that the ability of the referential reading to provide grounds for essentialist truths is completely decisive, since this dialectic seems to show that it can provide adequate grounds only by appealing to another primitive metaphysically necessary law. But if that law is indeed primitive, then it is itself a modal fact without grounds, and so it is not clear that the referentialist's desire for "grounds" was ultimately satisfiable in the first place.

Still, there is one last potential response the referentialist might give. Going back to what we said a bit earlier, some Aristotelians have taken a thing and its nature not to be really distinct.[79] In particular then, if G is a property and N(G) is the nature of G, then G = N(G). We could then say that the necessity of identity explains why a property has its nature necessarily. Of course, this relies on the slightly controversial claim that a thing and its nature are really identical. Nevertheless, it may provide a reasonable stopping point for referentialists about essence.

If the disadvantage of the primitive essentialist reading is that it does not provide ontological grounds or truthmakers, it nevertheless has the advantage that it is more

[79] See, e.g., footnote 75 above.

parsimonious.[80] On the primitivist or "pure" essentialist reading, there are primitive essentialist facts that are intrinsic to reality, but they need not be localized in some particular set of *things* called "properties" or "natures."[81] This strikes me as a rather elegant picture.

Nevertheless, going forward, I will officially speak in terms of the standard referentialist reading of properties and essences. But the reader should keep in mind that immanent structuralists have the primitivist option in their back pockets as well.

8. Conclusion

In this chapter, I have provided a story of how all mathematical truths can be grounded in purely structural universals, with no reference to specifically "mathematical objects" whatsoever. In the process, I have given a theory of property parthood in terms of essentialist facts, and have put forward two possible readings of what we are saying when we speak of "the essence" of something: the *referentialist* reading and the *primitivist* reading. I also have attempted to meet Benacerraf's "uniform semantics" challenge. I have done so by identifying "pattern statements" in natural language and then showing how mathematical statements can be taken to have the same underlying semantics as a subclass of these, namely, the essentialist pattern statements. Now, armed with our account of *what it is* for a mathematical proposition to be true, we can turn to considering how we such a thing might come to be known.

[80] And as I have argued, the lack of entities to ground the primitive essentialist facts is not obviously a disadvantage, at least if referentialists themselves also have to posit a further, ungrounded and metaphysically primitive modal law which says that properties have their natures necessarily.

[81] A further potential advantage I will discuss later is that the pure essentialist view might allow Aristotelians better to deal with alien properties. See Chapter 3 below.

CHAPTER 3: MATHEMATICAL EPISTEMOLOGY (AND SOME METAPHYSICS)

1. Benacerraf's Second Horn

Let us summarize a bit. In Chapter 1 I explained the notion of a purely structural property and illustrated how several areas of mathematics can be characterized as dealing with purely structural properties of one sort or another. In Chapter 2 I laid out Benacerraf's dilemma, which claimed that it is desirable, but impossible, both to have a semantics for mathematical discourse that is uniform with the rest of ordinary language while simultaneously possessing a plausible mathematical epistemology.

However, as we saw, Benacerraf's assumption that even *ordinary* singular and quantified statements have *just one* semantics is questionable. At best, then, mathematics can only be uniform with a *part* of ordinary non-mathematical language.

I then argued that the part of ordinary language that mathematics is uniform with is a class of what I've called "pattern statements." If I am correct, at least a certain subclass of these statements – the essentialist pattern statements – includes the mathematical ones. I also argued these statements should be cashed out in terms of constitutive, property-parthood relations holding between various properties.

In this chapter I would like to address the second part of Benacerraf's dilemma and spell out a bit the epistemology of mathematics on immanent structuralism. To help guide our thinking it will be helpful to return to Benacerraf's classic discussion.

In the original formulation of his epistemological argument against Platonism, Benacerraf assumed a causal theory of knowledge:

> "I favor a causal account of knowledge on which for X to know that S is true requires some causal relation to obtain between X and the referents of the names, predicates, and quantifiers of S. I believe in addition in a causal theory of *reference*, thus making the link to my saying knowingly that S *doubly* causal."[82]

Benacerraf argues this causal theory of knowledge (and reference) rules out the possibility of mathematical knowledge given a Platonist semantics and ontology:

> "[C]ombining *this* view of knowledge with the 'standard' view of mathematical truth makes it difficult to see how mathematical knowledge is possible. If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have mathematical knowledge cannot be made out. It will be impossible to account for how anyone knows any properly number-theoretical propositions."[83]

Benacerraf's argument runs into some problems, however. For one thing, a natural response to all of this is simply to deny the rather strong causal theory of knowledge on which

[82] Bencerraf (1973) p. 413.

[83] Ibid. p. 414.

the argument is based, as many now do.[84] This has been the strategy, for example, of those who attempt to provide an IBE or "indispensability" argument for the existence of mathematical objects: Perhaps mathematical objects can be known by virtue of their having *explanatory* power without having *causal* power.

2. Platonism and Seabiscuit Epistemology

This response can be challenged however. For one thing, we might ask whether IBE or indispensability arguments are successful at establishing the existence of all the mathematical objects (and the facts about them) that mathematicians take themselves to know.[85] One might doubt this. On the face of it, indispensability arguments leave unjustified much of higher mathematics that has no adequate justification in terms of playing a role in well-confirmed physical theories. The postulation of certain infinities of sets, for instance, might not receive adequate justification in virtue of the minimal role they play in physics.[86] But arguably all normal day-to-day mathematics (including the well worked out theory of higher infinities) has at least prima facie justification, in the same way that ordinary physical or biological science does.

However, suppose we leave aside this objection, and grant that inference to the best explanation (IBE) arguments can justify all of the parts of mathematics we want to be justified. We are still left with another version of the challenge:

- (**Challenge**) Given that most people (including mathematicians) do not (and cannot) justify their mathematical beliefs via IBE or indispensability arguments,

[84] See Linnebo (2017) for an up to date discussion on Benacerraf's dilemma.

[85] See, e.g., Maddy (1990) Ch. 1. A bit more on indispensability arguments later, however. In particular, see Chapters 6 and 7.

[86] This appears to have been Quine's view, for example.

> how can *their* mathematical knowledge be reliable, given that we can have no causal connections to abstract mathematical objects?[87]

To illustrate the problem for Platonists, consider what I will call ***the Seabiscuit Example***. Imagine the following scenario:

> Suppose I were simply to *assert*, that in a galaxy many light years away, which we have never even remotely observed or interacted with, there is a constellation of thirteen planets with 80 billion intelligent inhabitants. I say that all of these extraterrestrials are less than four feet tall, except on one planet (which I call Planet Seabiscuit), where everyone is nine feet tall. I also assert that everyone on Planet Seabiscuit has only one hand, and everyone on the other planets has only one foot. Finally, I claim that the twelve other planets' denizens, having two hands and greater numbers, were successfully able to enslave those on Planet Seabiscuit, and that they force Seabiscuit's two-footed inhabitants to carry them around like horses.

The question is this: For most people who have not run any sophisticated philosophical argument, how is speculation about mathematical objects any different from wildly speculating *a priori*, from the armchair, about the existence of people in some distant galaxy we have never had any experience of or interaction with? Given that the latter *a priori* speculations cannot be knowledge, how could the former be?

In short, on Platonism, it is hard to see how most of us are not just shooting in the dark.

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[87] This challenge is slightly different from both Benacerraf's original version and Maddy's updated version in terms of "reliable connections." First, it is different in that it is restricted to only a part of the mathematical knowledge that exists. Second, it is open to the possibility that IBE or indispensability arguments could, in principle, work for some parts of mathematics. Really, the version of the challenge I am proposing is a challenge to adequately account for mathematical practice, both that of the average mathematician and that of the ordinary layman.

Now, one response is that most average people are justified in their mathematical beliefs because *mathematicians* are justified, and most people learned their mathematics from a mathematician.[88] But for one thing, it's not clear that most people with justified mathematical beliefs did this or had to. More importantly, this merely pushes back the question as to why all the courses mathematicians took and methods they learned are not merely methods of vibrant imagining. Why are *mathematicians'* beliefs not also on a par with wild speculation about causally distant planets?

You might say it is because the mathematicians, or at least enough of them, learned about foundations from a philosopher.[89] But this response seems to make mathematical knowledge dependent on the arguments of the philosophers. Here I will merely reproduce the wisdom of David Lewis in response to the notion that philosophical objections might undermine mathematics:

> "I'm moved to laughter at the thought of how *presumptuous* it would be to reject mathematics for philosophical reasons. How would *you* like the job of telling the mathematicians that they must change their ways and abjure their countless errors[?] Can you tell them, with a straight face, to follow philosophical argument wherever it may lead? If they challenge your credentials, will you boast of philosophy's other great discoveries? … Not me!"[90]

[88] Or, they learned from someone who learned from [someone who learned from … someone who learned from] a mathematician. Thanks to Alex Campbell for making this point.

[89] Or learned from [someone who learned from … someone who learned from] a philosopher.

[90] Lewis *Parts of Classes* (1991), p. 59.

The same sentiment seems applicable to the idea that mathematicians' knowledge rests on the condition that some philosopher has previously justified the methods of mathematics. In short, we cannot solve the access problem by appealing to the authority of philosophers.

3. A More Adequate Epistemology

Whatever the issues with Benacerraf's original formulation of the argument, I think he is right that the access problem is most pressing if we take mathematical statements to be uniform with statements about the existence of persons on other planets. To know the proposition "some prime numbers are even" would then have to be like knowing *a priori* about the wretched inhabitants of Seabiscuit.

On my picture, however, mathematical statements are not known this way, because they do not make assertions similar to our celestial conjectures. On the contrary, to know that a mathematical statement is true is to know that something is part of the nature of some purely structural property.[91] Just to take a simple singular statement, like "a is F", what I need to know is that *being F* is part of the nature of *being a*.

In the mathematical case these will both be purely structural properties. That means they are ultimately definable entirely in terms of part, whole, sameness, difference and purely logical vocabulary. But presumably it is unproblematic how we come to understand these notions from experience, or at least not more problematic than how we grasp other empirical concepts.

Arguably we can learn about concepts like part and whole from our perception of more or less complex objects. Vision obviously plays an important role, but we can also discern different parts of objects by feeling them, and we can even distinguish different "parts" of sounds, tunes,

[91] See Chapter 2 above.

etc. Perhaps part of the story also involves some practical experience with putting things together and taking them apart. At the very least, the problem of how we have an adequate understanding of part and whole does not seem to be a uniquely *mathematical* problem, but part of the more general question of how we gain concepts like these through perception. Similarly for "sameness" and "difference" and logical concepts like conjunction, conditionality, and negation.

These, then, are the building blocks of purely structural properties. But of course, most of the time we do not learn mathematics by explicitly constructing formal definitions. So we still need a story of how most people form their basic mathematical knowledge.

Now, if mathematical properties are purely structural properties, then they are just like other properties that can be instantiated by concrete systems of physical objects. In that case, there is no reason our perceptual systems could not be adapted to pick up on some simple mathematical properties directly, just as I come to understand the property of being red by seeing, abstracting, and grasping it from its many instances. If so, then there is some stock of simple mathematical properties that we can come to know about via perception. A property like *being circular* might be an example. Small numerical quantities might be another case, such as *being two* or *being three*.

Putting all of this together, we can give a plausible account of the mathematical knowledge ordinary people have. I know that squares have four sides, for example, because I've *perceived* them, and have come to know enough about what squares *are* to know that part of it is being four-sided. Simple mathematical knowledge of this sort typically comes at a young age, and it usually involves children generalizing by being shown visual examples.

Once I gain some concepts of mathematical properties via perception, I can then combine my concepts of these properties together via logical operations and thereby understand some things about the more complex properties I “build up.”

For example, I can build the property of being red *and* circular, and know from my knowledge of redness that *red circles are crimson or ruby or scarlet, but not teal or beige*. Note that this is knowledge of a truth about red *circles*, and it follows from my knowledge of redness.

In a similar way, I can conceptually “build” new mathematical properties out of the ones I already know, and infer various things about the natures of these properties from my knowledge of the natures of part, whole, sameness, difference, and my logical concepts.

Suppose, for instance, I define the property of *being a three-tiered-system* as the property of being an object with three parts, x, y, z, where z is part of y and y is part of x. An example might be a system of three concentric circles. Then I can know the relation *‘part z’ is part of ‘part x’* is part of the nature of *being a three-tiered system*. That is, I can infer that in a three-tiered system, the smaller part z is part of the bigger part x. I know this in virtue of my knowledge of the natures of part and whole.

Arguably, this sort of reasoning about properties captures the type of reasoning that occurs in mathematics, except that the structural properties mathematicians are working with generally are far more complicated.

Note that the reasoning need not be formalized. Often, long before any proof has been discovered, a seasoned mathematician might simply have good enough “insight” to allows him or her reliably to “see” the relevant result rather quickly. This is sometimes referred to as “intuition.” However, this “insight” is not analogous to the perception of an *object*, pace

Platonists.[92] Rather, it is the understanding of what follows from a *nature*. The final, formal proof (if there is one) is then the explicit drawing out – via steps that are ultimately justifiable by the natures of part, whole, and logical relations – of what truly does follow from that nature.

4. The Emptiness Objection

On the whole then, the epistemology of purely structural properties seems more tractable than the epistemology of abstract mathematical objects. However, there is a worry lurking in the background. It starts from what we might call the ***empty definition principle***:[93]

- (**Empty**) It is not in general true that any entity that can be *defined*[94] actually *exists*.

One obvious problem is that we can define inconsistent objects, such as square circles. But even if we restrict ourselves to consistent definitions, it does not follow that what is defined actually exists. For example, if I define *Jack* as *the goblin that stole my bag from the gym*, it does not follow that Jack exists or that there are any goblins. Although this principle is not entirely uncontroversial,[95] something close to it is correct, so I will go with the relatively straightforward formulation just given.

Interestingly, the tension between **Empty** and our epistemological access to mathematical entities partly motivates the position in philosophy of mathematics known as *full-blooded*

[92] Compare Brown (2014) *Platonism, Naturalism and Mathematical Knowledge*, especially Chapter 5. See also Godel (1944) and Maddy (1990) for discussion of Platonist theories based on "intuition."

[93] Thanks to Thomas Hofweber for making this problem clearer to me.

[94] Or "consistently defined"

[95] Fictional object theorists and some theorists about transworld identity might take issue with it.

Platonism.[96] Full-blooded Platonists simply deny **Empty**, and boldly claim that *any consistently definable mathematical entity exists*, so that there will always be enough objects to correspond to all the new sorts of systems and structures that mathematicians might come up with. I will talk about full-blooded Platonism more in Chapter 5, but for now I will just note it is worth trying to avoid these troubles without embracing this somewhat shocking position. I will therefore try to give a response based on immanent structuralism instead.

Prima facie, **Empty** raises an issue for our view too. Insofar as we wish to account for all the truths of higher mathematics, we will need there to be uninstantiated mathematical properties. Since these properties are not instantiated, they obviously cannot be directly perceived. Therefore, as was explained above, they will have to be built up conceptually, i.e., defined in terms of previously grasped properties.

But given (**Empty**), the mere fact that uninstantiated mathematical properties can be *defined* does not guarantee that they *exist*. Let us call this problem the *emptiness objection*. Two conclusions seem to follow from this objection, one general and one about mathematics specifically.

First, just because I know about the properties which could constitute a further complex property, it does not follow that I know about that complex property. For example, just because I know about *being a quadrilateral* and *having four right angles*, it does not follow that I know about the property *being a rectangle*: For given (**Empty**), the existence of *being a rectangle* does not follow from the mere fact that it is definable in terms of the other two properties.

[96] Also known as "plenitudinous Platonism." The most thorough exposition of the view is Balaguer's *Platonism and Anti-Platonism in Mathematics* (1998).

The second conclusion, which seems to follow from the first, is that we do not have knowledge of uninstantiated mathematics, including much of higher mathematics. For if we are not justified in assuming the existence of a mathematical property based on the existence of the properties it is defined in terms of, then there is no way to know about uninstantiated mathematical properties – or, perhaps a bit more carefully, at least *most* people's purported knowledge of those properties is unjustified.[97]

Furthermore, it seems to follow that we lack knowledge even of much *instantiated* mathematics. While I have assumed that *some* mathematical properties can be physically instantiated and therefore are discoverable through perception,[98] it is clearly false that *all* physically instantiated mathematical properties are directly perceivable. For example, the large number-property *971342580146982676025* is probably physically instantiated, but it is clearly impossible for this property to be visually perceived and then abstracted from that perception. It is far too large.

Therefore, for most people, even many *instantiated* mathematical properties will have to be known by being constructed conceptually based on previously known mathematical properties. So the emptiness objection seems to impugn the ability of immanent structuralism to explain quite a lot of mathematical knowledge, both of uninstantiated *and instantiated* structures.

[97] As I discussed earlier, I admit that *perhaps* we could establish the existence of some uninstantiated mathematical properties by some argument from best explanation, indispensability, etc. But most of the time mathematicians and ordinary people don't think about that. Rather, they *define* new mathematical structures (or learn the relevant definitions) and proceed to reason on that basis without any second thoughts. They certainly do not worry in that moment about whether an independent philosophical or scientific argument could be given for the existence of the new structural property.

[98] No small conclusion I might note! Even if immanent structuralism faced problems about uninstantiated mathematics similar to Platonism, that would not make it a philosophical tie. For Platonists have trouble even accounting for – perhaps *especially* accounting for – the most basic mathematical knowledge that ordinary folk including young children can have. See Franklin (2014) on this point.

Now perhaps there is a simple reply here: Perhaps we should simply *posit* that there is an independent world of properties, and that any logical combination of these properties is a property. And so in particular, any logical combination of *instantiated* properties will be a property. Perhaps we might motivate this by holding an *abundant theory* of properties, according to which any predicate corresponds to a property.[99]

While this might work, the trouble is that it is not enough simply to *posit* that this view is true. In fact, to do so seemingly begs the question against the emptiness objection, since the empty definition principle (**Empty**), as applied to properties, straightforwardly entails that one cannot simply posit a property without justification.

To see this, recall the example of Planet Seabiscuit. Imagine we define Planet Seabiscuit by describing a group of planets and their denizens, and then asserting their existence without any special justification. Clearly this is running afoul of the empty definition principle.

We say that the same problem applies to the average person doing mathematics on Platonism. But then it is difficult to see how it is not falling into the *same* fallacy simply to *posit* that there are a bunch of properties – many of which we do not have any causal connection with because they are uninstantiated. If we simply *assume* there are abundant properties in order to salvage higher mathematics, we will have moved the philosophy of mathematics no further than a hundred years ago, when Russell objected to the defining of classes into existence:

[99] Or *almost* any predicate: See Van Inwagen, "A Theory of Properties" (2004) for a defense of the "abundant," property-for-every-predicate view. Van Inwagen qualifies the unrestricted nature of his view when it comes to predicates subject to Russellian-type paradoxes (for example, Van Inwagen argues the property of *being non-self-exemplifying* cannot exist).

> "The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil."[100]

So, what if instead of "postulating" we were to *argue* for the existence of abundant properties via a Quinean argument? This is the route Van Inwagen takes,[101] as do many others. They argue that we are committed to properties because for any predicate F that might be instantiated we can say "something could have the feature of being F." A Russellian analysis of descriptions quickly shows that talking about "*the* feature of being F" involves quantification over features or properties. Thus, we are committed to properties for every predicate (or at least every logically coherent predicate).

This is an interesting suggestion, but as the reader might have guessed, I am uncomfortable with neo-Quinean arguments of this sort. The same sort of argument could be given for Platonic numbers, after all, and it would be ironic to say the least if my attempts to provide an alternative to Platonism rested on this sort of argumentation. More fundamentally, I doubt whether we can truly draw heavyweight ontological conclusions from reasoning of this sort, based as it is on mere paraphrastic constructions in ordinary language. These arguments seem "cheap."[102] Furthermore, given the discussion in chapter 2 above, it should come as no surprise that I do not take surface-level quantification or definite descriptions to be a simple guide to ontological commitment in the way neo-Quineans do.[103] Rather than open myself to the

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[100] Russell, B. (1919) *Introduction to Mathematical Philosophy*. Dover Publications.

[101] See Van Inwagen, ibid.

[102] If "postulating" all the properties we need is analogous to *theft* over honest toil, then perhaps cheap Quinean arguments are analogous to fraud.

charge of metaphysical hypocrisy, then, we will find another way to justify belief in uninstantiated mathematical properties.

5. Uninstantiated Structural Universals

So, let us return to our honest toil: To explain how it is that one can reliably assume the existence of a newly *defined* mathematical property, given that one has knowledge of the properties it is defined *in terms of*.[104] We must explain how the world guarantees that all definable mathematical properties exist given the existence of the perceivable ones.

What the immanent structuralist should say here, I think, is that all the defined, higher-level mathematical properties supervene on some basic mathematical properties that are instantiated and perceivable. In other words, given that the instantiated and perceivable mathematical properties exist, the existence of the ones definable in terms of them comes for free.

So how do we show that the uninstantiated mathematical properties supervene on the instantiated ones?

The first step is to appeal to the core thesis of immanent structuralism: The subject matter of mathematics is *structure*, where structures are understood as *purely structural properties*:

[103] For discussion of similar issues, see some of the contributions in Chalmers, Manley, and Wasserman (2009), especially Hofweber (2009) pp. 282-283, and also Hofweber (2005). Hofweber's discussion brings out a further difference in the way that Aristotelians like myself think of properties and the way neo-Quineans do. The latter take properties to be the semantic correlates of certain paraphrastic linguistic constructions, whereas I and other Aristotelians take them primarily to be metaphysical constituents that are present in concrete entities. Properties ground concrete facts about these entities, including their modal attributes and causal powers. This results in a different picture of the nature of properties and, importantly, of the exemplification relation. As some metaphysicians put it, Aristotelians tend to favor a *constituent* ontology when it comes to properties and their exemplification, while Platonists favor a *relational* one. See Van Inwagen, "Constituent vs. Relational Ontologies" (2011).

[104] Where this latter knowledge may come, for example, from perception or previous definition.

Properties that are definable wholly in terms of part, whole, sameness, difference, and logical vocabulary. Note that these basic properties, which are the mathematical "building blocks," actually exist in the world and are instantiated.

The second step is to affirm a "reducibility thesis": The existence of properties that are *logical constructs* is reducible to the existence of their basic *components*. Fellow-Aristotelian David Armstrong explains it this way – he contrasts what he calls "second-rate" properties with the "sparse" properties or universals that he takes to be ontologically fundamental:

> "At the same time, though, even when doing philosophy, we often need to refer to properties that are not universals, for instance *being a game* or *being a householder*. I call these 'second-rate' properties. … My idea … for dealing with these properties is to deploy a supervenience thesis. Suppose you had a God-like complete account of the world organized as the instantiations of all the universals, both properties and relations. Then, I suggest, you would at the same time have, with no ontological addition to the world, all the instantiation of the second-rate properties and relations. … It is a case of 'nothing over and above' – always an interesting claim because it gives us the more ontologically economical theory, a virtue if one can get it."[105]

This reducibility thesis seems especially plausible in the case of properties that are logical constructs, like "being F and G," "being H or not-F or not-G," and so on. Just as it seems no more is needed for the conjunctive fact (Fa & Fb) to obtain than that Fa obtains and Fb obtains,

[105] D.M. Armstrong, (2010) *Sketch for a Systematic Metaphysics*, pp. 19-20. See also Armstrong's (1997) *A World of States of Affairs*.

in a similar way, it seems there is no more needed for the existence of the property of *being F and G* than for both *being F* and *being G* to exist.[106]

Now, I do not necessarily agree with Armstrong's characterization of these properties as "second-rate," because I do not take reducibility to impose any sort of inferior ontological status on a thing. For example, I do not think dogs are second-rate beings simply because they can be reduced to their components.[107] Indeed, in more than one respect, dogs are superior to their components.[108] So I would prefer to say the "constructed" properties are just "non-basic," "non-foundational," or perhaps even "grounded" as opposed to "ungrounded," not that they are "second-rate."

Apart from this quibble, however, Armstrong's description illustrates how constructed properties can be taken to exist so long as their components do. Therefore, if the mathematical properties are precisely the purely structural ones, then the existence of the properties "part," "whole," "sameness," "difference," together with the basic logical relations is sufficient for the existence of all mathematical properties, instantiated or uninstantiated. Therefore, all mathematical properties exist. This appears to resolve the emptiness objection.

However, one might have a lingering worry. Perhaps the reducibility thesis cannot get us *all* uninstantiated properties, thereby leaving some true propositions about possible but actually uninstantiated properties ungrounded. For example, Armstrong's view notoriously faces a

[106] One suggestive way you might put this is that it seems like the existence of the natures of F and G is enough to determine *what it would be like* for something to be both F and G. In other words, the actually existent natures of F and G metaphysically fix or determine what an F-and-G thing *would have to be.* And that seems to be all that is necessary to say the property of being F-and-G exists. However, if this way of putting the point is less perspicuous to the reader, one can simply go with the original reducibility or supervenience claim mentioned above.

[107] Assuming they can be, anyway – an admittedly questionable assumption.

[108] I certainly prefer their company.

challenge in accounting for modal facts about *alien* properties, i.e. properties that are neither instantiated in the actual world at any time nor definable in terms of ones that are. How can we account for the possibility that there might be aliens properties, in this sense?

I have two responses to this objection. The first response is somewhat speculative. Recall the discussion at the end of Chapter 2 explaining two different interpretations of essence one might give: The *"referential"* conception and the *"primitivist"* conception. Recall that on the primitivist conception, to say that "the essence of X is to be F" is to say something like "it is essential to X's that: X's are Fs," where "it is essential to X" is a primitive modal operator. This is in contrast to the "referential" conception, where an essence is taken to be some kind of irreducible *object* or *thing*.

Arguably, if the primitivist conception is correct, we get a picture of the world as intrinsically and irreducibly modal and essentialist. In that case, one can take all essentialist facts as basic.[109] One will then have the required grounds for modal facts about alien properties.

Admittedly, it is less easy to see how the *referential* conception will account for alien properties. The problem is similar to one that David Armstrong faced in his account of metaphysical modality.[110] In essence, according to Armstrong, we can think of possible worlds as combinations of "basic" states of affairs, where a "basic" state of affairs is when some simple individual(s) instantiate some simple universal(s). So, for example, we can say there is a possible

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[109] Obviously, on the primitivist view, the essentialist facts will be left ungrounded by anything further. It is not clear, however, that this is a bad thing. After all, should "the essence of water is H2O" have some non-modal, non-essentialist grounds? Arguably, this is a reasonable stopping point for explanation. If someone asks "*Why* is the essence of water to be H2O?" it seems reasonable to say "Well, it just is. That's just what the essence is."

[110] See Armstrong (2010) pp 86-7.

world where there are green humans because we can logically construct the state of affairs of some individual human having the simple universal "green."

The problem for Armstrong's account is that it does not seem to explain how there can be possible worlds with "alien" properties, i.e., properties that are not instantiated in the actual world nor definable in terms of ones that are. It does not seem that we can get possible worlds with alien properties simply by combining actual properties together.

Now, there are a few things Armstrong might say. For one, Armstrong could simply deny that there are possible worlds where totally alien properties are instantiated.[111] Perhaps this could be justified by an appeal to the overall elegance of Armstrong's combinatorial theory of modality. If the theory is strongly justified on other grounds, perhaps it can override the initial presumption in favor of alien properties. However, this at least seems like a cost that it would at least be preferable to avoid, if possible.

In later work, Armstrong changes his mind, and allows for the introduction of alien properties.[112] However, it is obscure how they can be admitted given his overall commitment to grounding modality in actual entities, together with his claim that only instantiated universals exist. Perhaps alien properties can be thought of as logical fictions or idealizations of some sort.[113]

I will not pursue this line of thought further, however, and instead will turn to my second and more central response to the problem. And that is that, whether we go primitivist or

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[111] Armstrong takes this view in (1989).

[112] See Armstrong (1997).

[113] Cf. Sider (2002) and (2005) for further discussion of this proposal.

referentialist about essences, I can bracket the question of alien properties, as I am not concerned here to ground all facts about uninstantiated properties, but rather all the facts of *mathematics*. I have not set myself on the task of solving all problems about the metaphysics of uninstantiated properties, but only the case of uninstantiated mathematical ones

6. Conclusion

To sum things up: If we can grasp the essences of the *basic* components of more complex uninstantiated mathematical properties, then we can use this grasp together with our reliable logical abilities to work out what else is part of the mathematical property's essence. A particularly simple example of this was given in Section 3 above, with the case of the property *being a three-tiered system*. In its idealized form, this reasoning takes place through *proof*. But some people have the ability to reliably grasp logical relations among mathematical properties *without* formal proof. We call this "grasping" mathematical insight, and we call people who have this insight mathematicians.

Overall then, we have a sketch of a story about how mathematical knowledge can be had by a combination of knowledge of simple mathematical properties grasped through perception plus logical construction based on that knowledge. Admittedly, there are some broader epistemological issues that have not been resolved here. For example, there are more general epistemological issues about how we come to have knowledge of essences based on perception at all, as well as about the reliability of our logical knowledge. These are much broader questions with large literatures behind them, and I will not try to address them here.[114] Suffice it to say

[114] However, for an account of essentialist knowledge that I find plausible, see Lowe (2013) "What is the Source of Our Knowledge of Modal Truths," especially pp. 934 ff. Incidentally, Lowe finds mathematical propositions to be the *clearest* case of essentialist knowledge, and regularly appeals to them throughout most of his work on essence; Lowe especially likes geometrical cases. For further discussion, see also Vaidya (2010) and (2017), and Horvath (2014).

that, if my account has shown we can have mathematical knowledge so long as our logical knowledge is reliable and we can gain some basic essentialist knowledge from perception, I will consider my account to have made some progress.

CHAPTER 4: MATHEMATICAL REDUCTION, TREATING-AS, AND INSTANTIATION

1. The Problem of Multiple Reductions

It is standard fare that Platonic numbers have a reduction problem. And I am not talking about their weight.[115] I am referring to the well-known fact that mathematical reduction is a multifarious matter: There are equally salient ways of reducing one part of the mathematical world to another.

The classic example is the V-sets and the Z-sets. Here is the reduction of the natural numbers in terms of von Neumann's "V-sets":

- **(V-Sets):** 0: {}, 1:{{}}, 2: {{},{{}}}, …
  - [where the n+1th set is the power set of the nth set]

And here it is in terms of the "Z-Sets" as I'm dubbing them, after Zermelo:

- **(Z-Sets):** 0: {}, 1: {{}}, 2: {{{}}}, …
  - [where the n+1th set is the set of the nth set]

[115] That would be impossible.

The problem for set-theoretic Platonism is that the Z-Sets are just as good for a mathematical reduction of arithmetic to set theory as are the V-Sets. Thus, reductive, set-theoretic Platonism faces a problem:

> **The Multiple-Reductions Problem**: There are multiple, equally good possible reductions of the ontology of numbers to the ontology of sets.

This problem is not new. In its original formulation, it was the inspiration for the idea that we should treat mathematics as the science of *structures*, or *patterns*. We discussed the problem a bit before in Chapter 1 above.

What is less often discussed, however, is what exactly mathematical reduction *is*.[116] If it is not an ontological relation – as the non-identity of the V-sets and Z-sets seems to show[117] – then how should we understand it? And what sort of ontology best accounts for it?

In this chapter I will argue that mathematical reduction should be understood as a particular instance of a more general phenomenon called mathematical "treating-as," and that our version of structuralism can nicely account for this phenomenon. In Section 2, I will describe "treating-as," and how mathematical reduction should be understood as a specific instance of it. Here we will draw out some of the formal properties of treating-as. Section 3 will discuss some of the useful epistemic and explanatory functions treating-as serves in mathematical practice. We will see that treating-as can allow us both to see *why* certain results hold, as well as discover *new* results. Finally, Section 4 will discuss how mathematical treating-as can be understood in terms

[116] One underappreciated account is Daniel A. Bonevac, *Reduction in the Abstract Sciences* (1982). However, Bonevac's discussion is centered around the issue of abstract objects generally, and is not primarily interested in mathematical reduction for its own sake as a phenomenon in mathematical practice.

[117] They are non-identical because of the axiom of extensionality. Assuming A = B and A = C entails B = C, it follows from multiple reducibility that mathematical reduction cannot be identity.

of the ontology of immanent structuralism. In doing so, I will discuss two important types of instantiation relevant to the account. The properties of treating-as will be seen to fall elegantly out of this analysis.

2. Phenomena: Mathematical Treating-As and Reduction

I claim that mathematical reduction is an instance of a more general phenomenon that we can call "mathematical treating-as." To illustrate this phenomenon, recall the following sort of locution from one's undergraduate mathematics textbooks or lectures:

- "You can think of vectors as directed line segments on a plane."
- "You can think of the set S as a function, which assigns 1 to any object in S, 0 otherwise."
- "You can think of a function as a set of ordered n-tuples."
- "You can think of an integral as the area under a curve."
- "You can think of complex numbers as rotations around the origin of a plane."
- "You can think of complex numbers as ordered pairs of real numbers."

And, of course, we have the set-theoretical case that we've seen already:

- "You can treat the natural numbers as the V-sets (or the Z-sets)."

As even these rather straightforward examples show, mathematical "treating-as" is not confined to set theory, but occurs over a wide variety of subject matters. These elementary cases deal with vectors, lines, planes, sets, functions, integrals, and complex numbers.

What's clear about all of these examples of mathematical "treating-as" is that they do not, on the face of it, carry any heavy ontological implications. At least in the examples where sets are not involved, when you treat X as Y it does not seem plausible to literally *identify* the X's with the Y's in any metaphysically significant sense. After all, it seems unlikely that a "rotation" is what a complex number *really is*.

We can further argue based on its formal properties that treating-as is not straightforward ontological identification. For one thing, treating-as can be symmetric. In the set/function case above, for instance, functions are treated as primitive, and sets are defined in terms of them. However, mathematicians will also treat functions as a type of set, viz., as sets of ordered n-tuples. So sets can be treated as functions, but functions also can be treated as sets. Therefore, the "treating-as" relation sometimes is symmetric. However, ontological reduction is *never* symmetric.[118]

Furthermore, if treating-as were an ontic relation, what we would normally take to be category errors would in fact be mathematical truths. For example, you can treat numbers as arrays of dots, and treat addition as the concatenation of such arrays.[119] On an ontic interpretation of treating-as, therefore, the physical and the abstract can become one.

Moreover, in some cases where X is treated as Y, it is not totally clear we should be committed to the *existence* of the Y's at all: Should we really admit the existence of such "things" as rotations, or lines *in abstracto*? It is not immediately obvious we should. However, if treating-as is not an ontic relation then it is not necessarily problematic to treat an accepted entity as a dubious one: For it seems that even if the ontological status of these items Y is less certain

[118] For example, if heat reduces to molecular motion, molecular motion does *not* reduce to heat.

[119] See the discussion in Bigelow (1990).

and clear than the X we are *treating* as Y, it is fine within mathematical practice for you to treat X as Y.

The takeaway from all of this is that the mathematical treatment of X as Y does not have any substantive ontological implications for the relationship between X and Y.

Now, returning to the example of the set-theoretic reduction of the natural numbers, it seems plausible that mathematical reduction is also a case of mathematical “treating-as.” After all, the reduction of the natural numbers to set theory is often done using the same locution: You can “think of” the natural numbers as certain sets.[120] Similarly, you can treat the complex numbers as ordered pairs. There certainly is no prima facie objection to thinking of mathematical reduction as an example of treating-as. So what distinguishes reductions from any other case of treating-as?

The main difference between mathematical *reductions* and less specific instances of treating-as seems to be that in reductions, we usually treat *an entire system* that we are concerned with as a part of another system, one whose “credentials” are already firmly established. For instance, we treat the *entire system* of the natural numbers as the Z-sets or the V-sets, and thereby “reduce” them. Similarly for the case of the complex numbers and ordered pairs of reals.

Now, if mathematical reductions are indeed merely a special case of mathematical treating-as, and if the latter relation is not ontically significant, it makes sense why we find mathematical reductions not to be either. In other words: We know that mathematical reductions cannot be ontological reductions because of the multiple reductions problem. The theory that

[120] For this, consult a textbook. See, for example, the classic set theory texts of Stoll (1963) and Quine (1969).

we've proposed – that "reductions" are just instances of "treating-as" – explains this, since we've already seen independent reasons that "treating-as" is not an ontologically reductive relation.

3. Functions: The Uses of Mathematical Treating-As

While at this point we have seen instances of "treating-as" and learned a bit about its formal features, that still leaves the question of what mathematical treating-as *is*, exactly, and what it is for. Let us begin with the latter question. The former question will be addressed in the next section.

Arguably, mathematical treating-as serves several functions. For one, treating-as can be *heuristic*. It can give students an easier way of conceptualizing or visualizing some piece of mathematics. For instance, most students find it easier to think of an integral as an area under a curve instead of thinking directly in terms of Riemann sums. Physical or pictorial examples of treating-as generally serve these sorts of heuristic functions well. Using simpler examples from other areas of mathematics that may be more familiar to the student can serve this heuristic function as well.

Looking at the particular case of mathematical *reduction*, this type of mathematical treating-as can serve various "*house-cleaning*" functions: It can allow us to show how one type of mathematical entity X (maybe one that has been considered dubious for some time) is no more problematic than another type of mathematical entity Y, either in virtue of the well-definedness of Y or in virtue of the assumed consistency of axioms about the Ys.

This seems to be one reason why mathematicians began to think of complex numbers as ordered pairs, for example. By showing how the whole collection of complex numbers can be *treated* as a collection ordered pairs, we can conclude that complex numbers are no more mysterious than ordered pairs. So, mathematical treating-as can help us prove *well-definedness* and *relative consistency* results.

Furthermore, treating-as has explanatory functions. This has been well-illustrated in a number of cases by William D'Alessandro.[121] I will focus on a simple case.

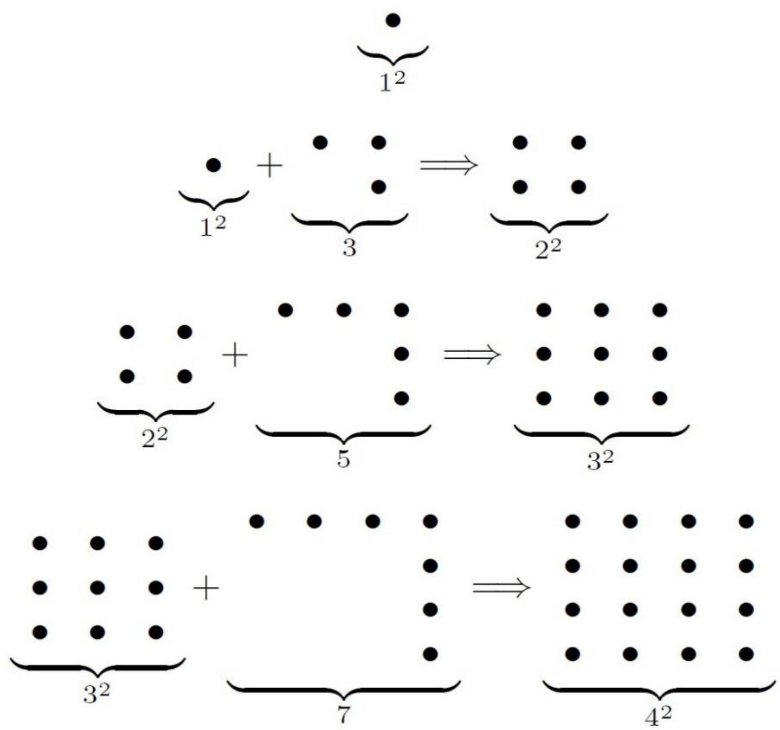
Suppose we want to understand why the sum of the first n odd natural numbers is n-squared. In other words, why is it that $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$, $1 + 3 + 5 + 7 = 4^2$, and so on?

While there are various proofs that can be given for this result, not all of them will help you to see *why* this result holds. The following picture proof, however, *does* provide explanatory help. D'Alessandro explains the proof as follows:

> Start with 1, the first odd number, which can be regarded as a square array of side 1 (and hence of area 1^2). The second odd number is 3, and we can think of adding 3 to 1 as augmenting the original square array so as to make a new one of side 2 (and hence of area 2^2). Similarly, adding 5 to 1 + 3 gives a square array of side 3, and so on, as in the diagram.

[121] See D'Alessandro (2018) and (2020). D'Alessandro instead refers to the phenomenon as mathematical "viewing-as," and does so in a somewhat different context, but the phenomena appear to me to be the same. Incidentally, D'Alessandro is interested in treating-as precisely *because* it is not an ontic relation. D'Alessandro tries to use this fact to show that explanations in mathematics do not have to be a species of grounding, ontological dependence, or causation. After all, if treating-as can be explanatory without being an ontic relation, then it seems some explanations are not instances of grounding or ontological dependence.

While I agree with D'Alessandro that treating-as explanations cannot be identified with grounding or dependence relations in a naïve way, I will briefly raise a few questions for D'Alessandro's argument below. See footnote 138.



What is interesting about the picture proof is, if you study it closely, it helps you to see *why* the result holds: One way to think about it is if I take the next odd number, I'm going to have a "dot" in the upper right corner, because the number is odd, and getting the "next" odd number is a matter of adding two dots to the previous odd number.

That extra dot is always going to allow me to form a new square by combining it with the previous square. And so I will always be able to produce a new square. As this point is clearly perfectly general, I can now better explain *why* the result holds by means of this picture proof. Treating-as, then, can have very useful explanatory functions.

Mathematical treating-as can also serve useful *epistemic* functions. If we reduce one part of mathematical discourse X to another part Y, this can give us clues as to how we should axiomatize X, since we may be able to catch onto a pattern that is more clear or evident in Y. Or, we may able to see that certain proposed axioms are redundant, unnecessary, or perhaps don't even accurately describe the relevant parts of Y. We might even discover results about X that we wouldn't have seen if we'd not thought of X as Y.

To illustrate this, consider another example brought up by D'Alessandro.[122] The example comes from Marc Konvisser's linear algebra textbook. Suppose we want to know what the solutions are to the polynomial $x^n - 1 = 0$. Finding the desired results when n = 1, 2, 3, 4 is rather straightforward. We simply use standard algebraic methods of factoring or applying the quadratic formula.

What is interesting though, is that we can discover a much more general result by treating complex numbers as rotations on the plane. In particular, we are to think of multiplying a complex number z by *i* as rotating the vector associated with z counterclockwise by an angle of $\pi/2$ i.e., 90 degrees. Konvisser then describes how this is relevant to the original problem of finding the solutions of $x^n - 1 = 0$:

> Now let us see how this interpretation of multiplication by i as rotation by $\pi/2$ can help us solve our original problem of finding the roots of equations of the form $0 = x^n - 1$ ... In order to find complex numbers that satisfy the equation $z^n = 1$, let us see if we can find complex numbers z so that multiplication by z represents a rotation of 1/n way around, that is, a rotation of $2\pi/n$.[123]

In other words, we want complex numbers z which, when applied n times, bring you back to the same place – which is equivalent to just multiplying by "1." You might say we are treating the formula $z^n = 1$ as a statement about rotations.

[122] D'Alessandro (2020)

[123]In ibid.

Konvisser then notes a basic result of simple trigonometry:

> Let $z = \cos \theta + i\sin \theta$. Multiplication by z represents a counterclockwise rotation by the angle θ. (31)

This can then be used to give us the desired general solution. As D'Alessandro notes:[124]

> This essentially solves the original problem. We wanted to find the solutions to $x^n - 1 = 0$, i.e. the complex numbers z corresponding to rotations of $2\pi/n$. By the previous proposition, one such solution is $z = \cos(2\pi/n) + i\sin(2\pi/n)$. In fact, z is a "primitive nth root of unity", and the remaining roots are the powers $z^2, z^3, \ldots, z^{n-1}$.

What this example shows is that a very simple case of treating-as – treating complex numbers as rotation on the plane – can be immensely useful in discovering a new, more general result about polynomials. Thus, treating-as serves useful *epistemic* functions.

In summary, mathematical treating-as serves important *heuristic, house-cleaning, explanatory,* and *epistemic* functions. I will now attempt an account of what treating-as *is*.

4. Treating-As: An Immanent Structuralist Analysis

Let's return to structuralism for a moment. Recall that on the standard version of structuralism advanced by Resnik (1997) and Shapiro (1997) – *ante rem* structuralism – the subject matter of mathematics consists of "an ontology of featureless objects, called 'positions', and … systems of relations or 'patterns' in which these positions figure."[125] Ante rem

[124] Ibid.

[125] Resnik (1997) p. 269.

structuralists view individual mathematical objects as the “nodes” or “positions” within these systems.

For example, the *natural number system* is a structure of intrinsically featureless objects with the order characteristic of the natural numbers. Thus, structuralists will say that the V-sets and the Z-sets both *exemplify* the natural number pattern, but that neither is, strictly speaking, *identical* with the series of natural numbers.

On this view, the natures of mathematical objects (such as the number two, say) are wholly constituted by the relations they bear to other positions within the structure. In other words, the intrinsic natures of mathematical objects are exhausted by their relations to the other objects in the larger structure they are a part of.

While ante rem structuralism is motivated by a desire to account for how there can be multiple salient reductions, unfortunately, as I briefly argued in Chapter 1 and will explain further in Chapter 5, *ante rem* structuralism suffers from some of the same epistemological problems as standard Platonism. This seems plausible just based on the fact that it *is* a version of Platonism, albeit one with a particular view about the nature of the relevant Platonica. Furthermore, the objectual character of ante rem structuralism that we have just described opens it up to objections based on the nature of identity. This will be discussed more in Chapter 5.

For these reasons, I will instead propose an explanation of treating-as and reduction in terms of *immanent* structuralism. I believe our version of structuralism can explain why treating-as has the heuristic, house-cleaning, explanatory, and epistemic functions that it does, and why the “treating-as” relation is successful in serving them.

So, on to our analysis: I propose that we understand mathematical treating-as in terms of instantiation of purely structural properties. More specifically, to say that you can treat a mathematical structure X as Y is to say X is instantiated by Y:

(TREAT): We can treat a mathematical structure X as Y if Y instantiates X.

So, for example, on this account, you can treat vectors as line-segments on a plane, because line-segments on a plane *instantiate* the structure of a vector space. Or you can treat complex numbers as rotations, because rotations *instantiate* the structure of the complex numbers. And in general, if we can treat X as Y, that means Y *instantiates* the pattern X. Because of that, we can use information we gain from thinking about Y to tell us about the original structure or pattern, X. Of course, different "treatments" might serve some of the various functions better than others; this analysis does not mean to say that all "treatments" will be useful. But, in principle, one can mathematically treat the X's as Y's just in case Y is an instance of the X's.

Before showing how the properties of treating-as fall out from this analysis, I need to clarify a point about universals. The analysis in **TREAT** implies that when X is treated as Y, Y instantiates X. Now, it might seem to follow from this that Y must be physical or concrete, or at the very least an *object* or *individual*, since it is something that instantiates a universal. However, on immanent structuralism, the subject-matter of mathematics consists only in universals. This might seem to pose a problem. After all, we can treat one mathematical system as another, as we saw in the case of the natural numbers in and the V-sets and Z-sets.

To resolve this worry, I will need to say that universals can instantiate other universals.[126] In doing so, I will appeal to a distinction between two sorts of instantiation, corresponding to two sorts of predication. This has been known since Aristotle wrote the *Categories*,[127] but the thought has arguably been developed most fully by the late metaphysician and fellow neo-Aristotelian, E. J. Lowe.[128]

Consider the two following sorts of things we might say:

- To be a dog is to be a certain sort of individual animal.
    - [Or: "*The dog* is a certain sort of animal."]
- To be a dog is a property.
    - [Or: "*The dog* is a property."]

Clearly, these two utterances can both be consistently affirmed, even though in the first case being an "animal" is predicated of the subject, while in the second being a "property" is. The reason these two statements can be consistent is that the first sort of predication draws out the content of *doghood*, while the latter is a metaphysical claim about the ontological correlate of the kind-term "dog."[129] You might say that the first statement talks about what is *contained* in the property *being a dog*, while the second talks about the property of *being a dog* qua property. You might also say that the latter is an "external" predication, while the former is "internal," i.e. specifying the internal nature of the property's contents.

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[126] Thanks to Daniel Bonevac for helping me see this problem clearly.

[127] See Aristotle *Categories*.

[128] See, especially Lowe's *Four-Category Ontology* (2007) and *More Kinds of Being* (2009).

[129] And, moreover, one that is unlikely to be uttered outside the context of a metaphysical discussion.

So we can say that the property of being a dog instantiates *being an individual* in what I have called the "internal" sense, even though the property itself is *not* an individual (rather, it is a property). This is ordinarily the sort of instantiation that is relevant in **TREAT** when we treat one mathematical system as part of another.[130]

Note that if **TREAT** is right, then immanent structuralism is uniquely suited to account for treating-as. A Platonist cannot agree with **TREAT** because Platonic objects, being individuals, do not instantiate each other. Only universals can be instantiated. If the number 3 or the function y = 2x are objects like Alfredo and Bob, then like Alfredo and Bob, they cannot be "instantiated" by each other. They just exist and have the features they do.

On the immanent structuralist view, you can treat vectors and complex numbers as lines and rotations because these latter things *instantiate* the former mathematical patterns.[131] In the set-theoretic case in particular, we can think of numbers as V-sets or Z-sets because these series of sets are paradigmatic *instances* of the natural number pattern.

This analysis predicts that treating-as will have the various formal properties that we've seen it to have.[132] First off, clearly, on this account, treating-as need not have any substantive ontological implications, at least not in the sense that if A can be treated as B, A *is* B. For the mere fact that something *instantiates* a pattern does not in any way tend to make us think that it and the pattern are *identical*.

[130] For further discussion, see Chapter 2 above, particularly the discussion of singular pattern statements.

[131] Again, given what we've just said about instantiation, it is fine if "lines" and "rotations" are themselves universals that might be exemplified in various concrete systems.

[132] Recall the discussion toward the beginning of Section 2 above.

Moreover, treating-as will sometimes be symmetric, since if we have two patterns X and Y, then it may sometimes be that *both* X instantiates Y *and* Y instantiates X. That is why sets can be treated as functions and functions treated as sets.

Furthermore, there is no reason why treating-as should imply *uniqueness*: There is no reason why we can't treat the number 2 as both a certain V-set *and* as a certain Z-set, since both the V-sets and the Z-sets might instantiate the natural number pattern. In this way we can easily end up with multiple reductions. And that is fine.

Finally, it can at least sometimes be useful to treat a pattern X as something else Y, even when the existential status of Y is dubious. This is because the existential status of Y is *irrelevant*, at least for mathematical purposes, so long as the treatment of X as Y accomplishes the relevant heuristic, house-cleaning, explanatory, or epistemic goals. In fact, for mathematical purposes, when treating X in terms of Y, it doesn't matter *what* Y is intrinsically, so long as it instantiates the relevant pattern.

Allow me briefly to digress and emphasize the last point that was just made. "It doesn't matter what Y is." If Y instantiates the pattern X, then we can treat X as Y.

This fact about mathematics is worth highlighting explicitly: For the purposes of mathematical *truth*, it seems that the *intrinsic natures* of the objects referred to does not matter. In other words, it doesn't matter *what* you take mathematical statements to be *about*, so long as what you take them to be about instantiates the relevant pattern. I will call this phenomenon *mathematical abstraction*: Mathematics abstracts from the intrinsic natures of objects. Arguably, the phenomenon of mathematical abstraction favors immanent structuralism too. Let's consider a few examples.

First, the one from number theory that we've been discussing repeatedly:

> Consider the student who asks his or her teacher, "Is the number 2 the V-set or is it the Z-set?" The teacher will probably say, "Think of it whichever way you like! It doesn't matter."

Indeed, from this hypothetical discourse we can draw the more general conclusion that, so far as the truth of the number statement goes, it doesn't matter *what* you take number statements to be about. This makes sense on our view, because according to immanent structuralism statements asserting features of the number 2 are true so long as certain statements about *the natural number pattern in general* are true. It therefore *does not matter* which particular *instance* of the pattern you find it helpful to think in terms of, so long as it *is an instance*.

Another example comes to mind from David Lewis. Consider the Klein 4-group:

*	**E**	**A**	**B**	**C**
E	E	A	B	C
A	A	E	C	B
B	C	B	E	A
C	C	B	A	E

Lewis nicely explains how this example illustrates mathematical abstraction:

> "Compare an algebraist's answer to a protesting student who says he hasn't been told what the Klein 4-group is just by being shown the table for it. What are these four things e, a, b, c? The Prof may answer: 'They're anything you like. No one thing is *the* Klein 4-

group; rather, any function (or, equivalently, any four-things-and-a-function) that obeys the table is *a* Klein 4-group. Anything I tell you about 'the' Klein 4-group is tacitly general.'"[133]

This abstraction from subject matter seems to be characteristic of mathematics: Geometry can be applied to color spaces, physical spaces, and probability spaces. Linear algebra can be applied to rotations, lines, and utility functions. Analysis can be applied to physical curves, demand curves, and n-dimensional-manifold "curves."

This feature of mathematics – its "abstractness" – is both predicted by and lends support to immanent structuralism. Recall that we defined a *purely structural property* as one that is definable entirely in terms of part, whole, sameness, difference, and purely logical vocabulary. Immanent structuralism asserts that these are *precisely* the structures studied by mathematics. It is because of the extreme generality of these notions that mathematics *can* be so abstract. At the same time, the fact that mathematics *is* so abstract suggests that all of its structures can be defined in these terms.

So much for mathematical abstraction. Let us return to treating-as. We have seen how **TREAT** easily explains the formal properties of treating-as, viz., that it is not ontological, that it non-unique, that it can be symmetric, and that it is abstract. Let us see whether **TREAT** can also shed light on the heuristic, house-cleaning, explanatory, and epistemic functions that we discussed earlier.

[133] See Lewis (1993) p. 15. In fact, Lewis in some places describes his view as "structuralist." Though Lewis' view is not the same as our view insofar as he does not rest his theory on an ontology of universals, his view is indeed similar to the immanent structuralist's "in spirit."

First, it is clear on our analysis how treating-as serves the various logical “house-cleaning” functions. Consider the following two highly plausible principles about universals:

1. If concrete system Y instantiates X, then X is a coherent universal.

That is to say, if the universal X has an instance, then X is not a logically contradictory universal.[134] This principle is obviously true, and it is the ontological correlate of the principle in model theory that consistency can be shown by witnessing a concrete model. This principle clearly is most applicable when we are treating one mathematical system as some array of concrete physical objects.

We also have a more general principle, covering both types of “instantiation” discussed earlier:

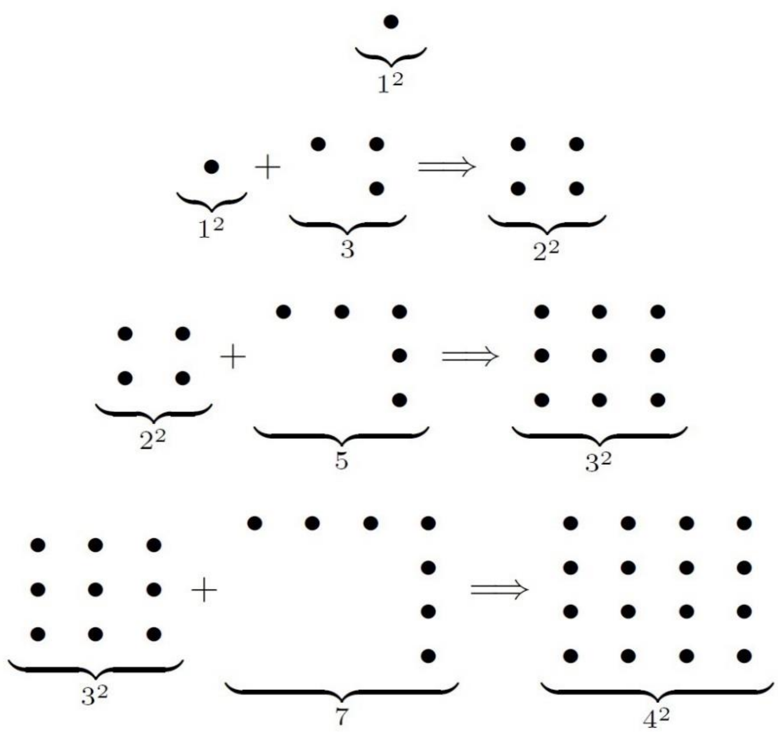
2. If Y is a concrete object or a coherent universal, and Y instantiates X, then X is a coherent universal.

This principle grounds arguments for and proofs of relative consistency. In cases where Y is some mathematical system, this clearly is useful only when Y is *itself* believed to be coherent. But of course that is in fact what is assumed when one tries to give a model of some mathematical theory in terms of, say, set-theoretic objects or arithmetic, and it is no secret that the consistency of much of mathematics is hostage to the consistency of these more elementary parts.

[134] An example of a “logically contradictory” universal might be “being green and not-green” or “being a round square.”

As for heuristics, explanations, and mathematical insight, our analysis in **TREAT** can help us see why treating one pattern as another might help us be more attentive to some aspect of the pattern in question.

For instance, consider the example given earlier, where we treat the natural numbers as arrays of dots, and we use this treatment to explain why the first n odd numbers add up to n^2. I would claim that in this picture proof we are literally able to *perceive* some necessary facts about the pattern in question.[135]



By our looking closely at a few examples, we can *perceive* the property of *being an (n x n) array of dots*,[136] and we can grasp that the two sides of an (n x n) array of dots *must* have an

[135] On literally perceiving mathematically necessary relations in physical instances, see Legg and Franklin (2017) "Perceiving Necessity". Legg and Franklin give several other instances where it seems plausible that the viewer of the picture proof is perceiving a necessity. See also Chapter 6 below.

[136] Note that this property is *literally instantiated* in each of the square arrays of dots.

odd number of dots – namely, (2n - 1) dots – because the two sides must "share" a dot. We also thereby grasp that (2n – 1) must be the nth odd number of dots. And we can visually *see* that (2n - 1) and $(n - 1)^2$ dots must always fit together to produce n^2. In short, what we are perceiving are necessary relations between certain numerical properties.

However, we also can clearly grasp that these necessities are not generated by anything about the dots in particular, but rather by the relations between the cardinality properties they instantiate, viz., being n^2, being (2n - 1) and being $(n - 1)^2$. Thus, by perceiving several concrete instances of these properties we are able to grasp a fact about the nature of cardinality in general, and we can now see *why* the general theorem is true, since now we can see *what it is about the nature of cardinality* that *makes it true*. In short, we can now grasp why something is an essential fact about a mathematical pattern (the natural numbers).[137] A mathematical induction proof, on the other hand, may show us *that* the theorem is true of the pattern, but does not allow us so clearly to see *why*.[138]

[137]Arguably, a similar story can be told for cases where we treat a mathematical system X as another *mathematical system* Y. In this case, however, what does the work is not the physical perception of certain necessities but rather our clearer grasp of some essential feature of Y.

[138] If the account of treating-as explanations given here works, it may raise questions for D'Alessandro's argument that viewing-as explanations need not be parallel to ontological dependence or grounding relations. (See note 121 above.) Recall that, treating-as is not an ontological relation: When X is treated as Y, this does not mean Y ontologically "grounds" facts about X. D'Alessandro concludes from this that not all explanations involve ontological grounding, since viewing X as Y can help explain some fact about X.

However, I am proposing that in a viewing-as explanation, the explanans (whatever it is that a pattern is being treated-as) allows us to see why the explanandum (a pattern, i.e. a certain universal) has some feature as part of its essence. But arguably the relation between the essential features of a property and that property is one of ontological determination or grounding: The essential features of a property do, in some sense, ground the existence of the property. If that is right, then perhaps what D'Alessandro calls the "dependency thesis" needn't be rejected, at least not wholesale: Perhaps explanatory relations really do parallel ontological dependence or grounding relations.

I should also note that this may nevertheless be consistent with D'Alessandro's hypothesis that explanatoriness should be understood as a type of cognitive relation, for Aristotelians have long thought there to be a tight connection between essence and intelligibility. See Koslicki (2011) as well as Reeve (2000).

5. Conclusion

The multiple-reductions problem provides a compelling reason for doubting that "mathematical reductions" are, in fact, *ontological* reductions. This makes it difficult for any "reductive" form of Platonism to get off the ground. Instead, we should think of mathematical reduction as a specific instance of treating-as.

As we have seen, treating-as serves important heuristic and epistemic functions in the practice of mathematics. Treating-as also possesses certain formal properties. Understanding these functions and features of treating-as helps us to see *why* mathematical reduction is not an ontological relation.

Furthermore, thinking of treating-as in terms of the ontology of immanent structuralism allows us nicely to explain how treating-as can have precisely these features and functions. In particular, by identifying two sorts of property-instantiation, we can explain why it is possible *both* to treat mathematical systems as *concrete* systems as well as other *mathematical* systems. In short, the phenomenon of mathematical treating-as is helpfully understood in immanent structuralist terms. This provides some confirmation of the immanent structuralist's framework.

CHAPTER 5: COMPARISONS: PLATONISMS AND OTHER STRUCTURALISMS

Introduction

Up to this point we have discussed the ontology, semantics, and epistemology of immanent structuralism. Before moving on to an important application of the view, I would like to spend some time discussing where it differs from other related theories in the philosophy of mathematics. While I obviously will not be able to discuss every picture of mathematics that has been given, I will spend some time comparing immanent structuralism with other versions of structuralism, as well as other views in the vicinity of structuralism.

In Section 1 I will briefly discuss further the *ante rem* view, which is what most people mean when they refer simply to “structuralism.” In Section 2 I will discuss a specific problem about identity that *ante rem* structuralists have long struggled with, but which is easily resolved on the immanent version. In Section 3 I first briefly recount what is known as “eliminative structuralism.” The difficulties with this theory will help us to see why Hellman develops his “modal” version of structuralism. Here I will demonstrate how immanent structuralism avoids some important difficulties for Hellman’s view. Finally, in Section 4, I discuss “full-blooded Platonism.” Perhaps surprisingly, this particularly thorough version of Platonism has a number of features in common with immanent structuralism. But as we will see, immanent structuralism and full-blooded Platonism differ on the category of entities they ascribe “fullness” to, and the immanent view is far more plausible precisely because of this difference.

1. Ante Rem Epistemology and Ontology: Still Too Platonist

In previous chapters we have touched a bit on the more traditional version of structuralism called *ante rem* structuralism. On ante rem structuralism, mathematics is committed to an infinity of *structures*. These structures are composed of mathematical objects that we can call *places*. They are unique objects in that they are mere "places" in a system, having no intrinsic properties apart from their abstract relations to other such "featureless" objects. Nevertheless, they still are individual entities that can be named and quantified over just like any other entity. As Hellman and Shapiro (2019) explain:

> The *ante rem* SGS [*sui generis* structuralism] view, for example, has a straightforward account of reference and of the semantics of the languages of mathematics: the variables of a branch of mathematics, such as arithmetic, real analysis, or complex analysis, range over the places in an *ante rem* structure. Singular terms denote individual places, so the language is understood at face value.
>
> In other words, an advocate of SGS has it that the straightforward grammatical structure of a mathematical language reflects the underlying logical form of the propositions. For example, in the simple arithmetic equation 3 x 8 = 24, the numerals '3', '8', and '24' at least seem to be singular terms – proper names. In the SGS view, they *are* singular terms. The role of a singular term is to denote an individual object and, in the SGS view, each of these numerals denotes a place in the natural number structure. … In this respect, then, SGS is a variation on traditional Platonism. For this perspective to make sense, however, one has to think of a place in a structure as a *bona fide* object, the sort of thing that can be

denoted by a singular term, and the sort of thing that can be in the range of first-order variables.[139]

From Chapter 2, it should be clear how different the *ante rem* account of mathematical truth and ontology is from the immanent structuralist's: In the end, a*nte rem* structuralism is still a version of *Platonism*.

We can bring out the differences between the views in a few ways. For one, consider how the two views would parse a simple predication, like "2 is even." For the ante rem theorist, "2" refers to some specific object that has no intrinsic features, and "is even" predicates of it a certain relation between this thing and other such featureless objects. By contrast, for the immanent theorist, "2" refers to a *property* that can be instantiated by concrete physical systems, and the predicate "is even" amounts to the claim that a certain structural feature is part of this property's essence. Arguably, this difference gives the immanent structuralist a leg up on at least two fronts: epistemology and ontology.

As we have already discussed in Chapter 3, the specific ontology and account of the truth-conditions for mathematics that we have given allows us to say that mathematical knowledge starts from perception and "builds up" from there by logical construction. Insofar as *ante rem* theorists still posit causally isolated Platonic objects as the subject matter of mathematics, its advantages over traditional sui generis Platonism do not seem quite as substantial as the immanent theorist's.[140]

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[139] Hellman and Shapiro (2019) *Mathematical Structuralism*, p. 3.

[140] Resnik (1997) in particular, relies heavily throughout the book on indispensability arguments in order to justify mathematical knowledge. Shapiro on the other hand does not place so much weight on indispensability arguments. However, partly for that reason his picture of mathematical knowledge is a bit obscure. While Shapiro suggests that mathematical knowledge occurs in some way by abstraction, it is not clear how, precisely, the process he describes

On the ontological front, ante rem structuralism has the disadvantage of all Platonist theories, in that it increases the number of categories of being we must countenance. By positing abstract Platonic individuals, we are no longer able to have simple, clean world of concrete objects and their properties. This disadvantage of the ante rem view is in fact part of the motivation for Hellman's modal structuralism, which purports to be a version of "structuralism without structures." Before considering Hellman's view, however, I would like to address an important problem about identity that sometimes comes up for structuralists. This problem is another drawback of the ante rem ontology that I think can be avoided by the immanent theorist.

2. Solving the "Identity Problem" for Structuralism

To reiterate, on the Shapiro-Resnik *ante rem* view, Platonism is still true insofar as numbers are conceived as abstract individuals with mathematical properties, and mathematical claims are still thought of as being true in virtue of these individuals and their properties. The difference is that *ante rem* structuralists consider these mathematical objects not to have any *intrinsic* natures. Instead, their identities are exhausted by their relations to other (also intrinsically featureless) objects within the structure.[141]

This leads to a classic problem for structuralists, which was raised by John Burgess in a review of Shapiro's (1997) treatise.[142] Fraser MacBride elegantly summarizes the issue:

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would reliably track the causally disconnected realm that he takes to constitute the subject matter of mathematics. See Shapiro (1997) pp 112 ff.

[141] Traditional Platonists, by contrast, consider mathematical objects to be *sui generis*, and to have non-relational intrinsic natures. See the beginning of Chapter 4, Section 3 above.

[142] John Burgess (1999) "Book Review: Stewart Shapiro, *Philosophy of Mathematics*"

> Far more worrisome is the concern that Shapiro's notion of a structural position cannot be made to cohere with the existence of structures admitting nontrivial automorphisms. The concern arises from structuralist slogans like “The essence of a natural number is its relation to other natural numbers … there is no more to the individual numbers ‘in themselves’ than the relations they bear to each other” (PMSO, pp. 72–73). This makes it appear as if Shapiro is committed to a version of the Identity of Indiscernibles for mathematical objects: if x and y share just the same intrastructural relations to other items, then x = y. There are, however, systems of mathematical objects that contain structurally indiscernible elements (systems that admit nontrivial automorphisms): the complex numbers (i and −i), the additive integers (+1 and −1), points in the Euclidean plane, geometric figures with reflectional symmetry, and so on. This means that Shapiro is committed to identifying these indiscernible elements: i with −i, +1 with −1, and so on. But we know mathematically that these elements are distinct. It follows that ante rem structuralism must be rejected (see Burgess [1999], pp. 287–288; and Keränen [2001]).[143]

This is a pressing issue for ante rem structuralism, and among the main hesitations one might have with the standard *ante rem* view are the strange nature and identity conditions of the objects it posits.

Luckily, our view does not suffer from this problem. For simplicity’s sake, let’s take the case of +1 and -1. Shapiro and Resnik’s view has the difficulty that it is unable to distinguish between these two objects, because they are both intrinsically featureless and are in all extrinsic respects symmetric. Since Shapiro claims that there is *nothing more* to mathematical objects than

[143] Fraser MacBride (2005) “Structuralism Reconsidered” pp. 581-582, in Shapiro (2005) *The Oxford Handbook of Philosophy of Mathematics and Logic*.

their mutual relations (that's the structuralism part), he seems to be committed to the identity of indiscernibles for mathematical objects. We can put the problem as follows.

- (1) Suppose ante rem structuralism is true.
- (2) +1 and -1 in the structure of the integers are in all extrinsic respects symmetric.
- (3) But if ante rem structuralism is true, then for mathematical objects there is nothing more to the identity of an object than its extrinsic relations to other objects in the structure.
- (4) Therefore, if two mathematical objects cannot be distinguished by their relational properties, then they are identical.
- (5) So, by (2) and (4): +1 and -1 are identical.
- (6) However, +1 and -1 are not identical.
- (7) Therefore, ante rem structuralism is not true.

This seems like a rather difficult problem for structuralism. However, I will argue below that on immanent structuralism, it appears completely to dissolve away.

To see this, recall the discussion in Chapter 2. For simplicity's sake, let's consider the case of the positive and negative integers.

On immanent structuralism, what does it mean to say that +1 and -1 are identical? Well, "+1" and "-1" are individual constants like "0." They are therefore abbreviations for purely structural individual properties.[144] Without going through the process of formalizing their descriptions, they refer to the following properties:

- [+1] denotes the property *being in an integer system with an ordering relation < and being the thing between 0 and 2.*[145]
- [-1] denotes the property *being in an integer system with an ordering relation < and being the thing between -2 and 0.*

There is no doubt that one and the same concrete system can be ordered by different ordering relations <, and that therefore one thing occupying the "+1" role under *one ordering* can also occupy the "-1" role under a *different* ordering. However, that is not at all the same as saying that the +1 and -1 role are the same *under a particular ordering.*

As an analogy, consider the property *being the first in a line*. This property is defined relative to some orientation or direction D. Of course, from some *specific* orientation D1 I could be considered *last,* while from another orientation D2 I could be considered *first*. But that does not at all make the property of *being the first in a line* the same *property* as *being the last in a line*.

Therefore, on immanent structuralism [+1 = -1] is simply false, because these two properties do not share the same consequential essence. After all, there is no way you could prove that the thing between -2 and 0 in a system is the same as the thing between 0 and 2 in *the same* system. A similar story could be told about -i and +i.

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[144] See Chapter 2 above, sections 2.3 and especially 2.4.

[145] In principle, the denotation of these latter constants can be spelled out in purely structural terms as well. I am simply not doing so, for the sake of brevity. For an example of a more complete abbreviation, see section 2.3 above.

I wish to emphasize how simple a matter this is for *immanent* structuralism, in contrast to the serious challenge it has been taken to be for *ante rem*. Rather than positing a structural *universal* that *can* be instantiated by concrete things, ante rem theorists posit strange featureless objects that *themselves exemplify* the relevant pattern (and, therefore, exemplify *themselves*![146]). From the immanent point of view, ante rem theorists have gotten themselves into an unnecessary quagmire of objections about the nature of identity within structures.[147]

Perhaps ante rem theorists were driven to posit this strange class of featureless objects to secure a subject-matter for mathematics, in order that much of mathematics will not be empty (and therefore false). This is sometimes called the "Empty Model" worry. We will discuss this worry further in a moment. But assume for the moment that it is a valid one. If we reject the contention that mathematics aims to describe some distinct class of objects, and instead construe it as *explicating the natures of certain universals*, this problem doesn't arise. You don't need any actual *instances* of the structural universals in order to ground mathematical truth if you've already got the universals themselves![148] And after all, why *shouldn't* these alone be enough to "determine" or "ground" all the facts that mathematics aims to account for?

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[146] "The natural-number structure *itself* exemplifies the natural number structure." (Shapiro 1997, p. 101, emphasis added.) Note that for ante rem theorists, a structure is more like an *exemplar* or *paradigm*, along the lines of Plato's Ideas or Forms. As such, ante rem theorists do not ultimately understand "exemplification" as straightforward property or universal-instantiation, as I would, but rather as consisting in something analogous to an isomorphism or congruence relation. See Resnik (1997) p. 204 ff. and Shapiro (1997) pp. 90-91.

[147] I should note that some structuralist responses have been given to the problem of identity. One response is to "rigidify" the structures by adding a symbols like +i and -i and making a decision to designate determinate places. Cf. Halimi 2019. One worry about this however is that we may have uncountably many indiscernibles, and there will not be enough names. Another option is to take identity as primitive, rather than a notion defined in terms of how an object relates to other objects. But one might wonder whether this undercuts the structuralism of structuralism. Cf. Reck and Schiemer (2019) for discussion. Regardless of whether the ante rem theorist can get out of the problem or not, the immanent theorist would claim these issues can be easily avoided by abandoning the ante rem approach.

[148] When put this way, does this not simply seem intuitive?

3. Eliminative Structuralism and Hellman's Modal Structuralism

We have seen that immanent and *ante rem* structuralism differ in their being property-based and object-based theories, respectively. However, it is worth asking whether we can come up with a structuralist theory that appeals *neither* to properties *nor* objects. In other words: Can we construct a structuralist theory without any special *ontology* at all?

In this section I will consider two attempts to do just that. Specifically, we will look at what is known as "eliminative" structuralism, as well as Hellman's modal structuralism. As we will see, a problem that is central to both of these views is the *model problem*. I will try to show how immanent structuralism can retain some of the benefits of these views while side-stepping other problems these theories face.

Eliminative Structuralism

According to eliminative structuralism, mathematics is essentially universal quantification, and we can "eliminate" mathematical *objects* from our ontology entirely – hence the name. Marc Gasser helpfully summarizes the distinction between the eliminative and non-eliminative approaches as follows:

> The first approach, *eliminative structuralism,* is motivated by the thought that objects with a purely structural nature are deeply problematic, and that they really shouldn't count as objects at all. Eliminative structuralists seek to paraphrase statements referring to mathematical objects into general statements about what holds in any collection of objects satisfying certain conditions. Such paraphrase saves the eliminative structuralist from referring to mathematical objects—and indeed from referring to any particular

> structure to which these objects might belong—and thereby frees her from ontological commitments she deems problematic.
>
> *Noneliminative structuralism*, by contrast, doesn't shy away from an essentially relational conception of mathematical objects. Ordinary mathematical statements are taken at face-value, as statements with singular terms denoting objects like triangles, numbers, sets, and so on—it's just that the nature of these objects is constituted by the structural relations they bear to one another, and nothing more. If these objects seem metaphysically queer, so be it.[149]

So, for example, take a statement like "2 + 3 = 5" An eliminative structuralist would paraphrase this as saying something along the lines of the following

- "In any system of objects S that exemplifies the natural number structure:
  - $2_S + 3_S = 5_S$ "

Here, 2_S , 3_S and 5_S are objects playing the "2, 3, and 5" roles, respectively, and + is a function defined according to the relevant Peano axioms. The important point is that the eliminative structuralist's reading does not imply there are any strange "relational" or "structural" entities. Arithmetical statements, for instance, are just universal quantification. Therefore, whenever we are talking about two systems sharing a "common structure," this is really just a useful way of expressing the fact that two systems are instances of the quantified statement above. It does not involve commitment to some independently subsisting abstract

[149] Cf. Gasser (2015). Arguably an example of eliminative structuralism would be Putnam (1967) and perhaps also Benacerraf (1965).

"structure." Obviously, then, the quintessentially *noneliminative* version of structuralism would be *ante rem* structuralism.[150]

One major problem for eliminative structuralists, however, is that it seems at least possible that there not be enough objects to satisfy all the axioms of the various mathematical systems or "structures" we want to study. For example, if there are finitely many objects in the world – even if the number is astronomically large – then there is no system of objects satisfying the axioms of Peano arithmetic, and so every arithmetical statement will be vacuously true. In particular, then, it will be true both that 2 + 3 = 5 and 2 + 3 = 92, clearly an undesirable result. Moreover, even if there are enough objects to satisfy the relevant axioms, it does not seem like the truths of mathematics should be hostage to the state of the world in this way. The truths of mathematics are not accidentally true. Call this the problem of *model existence*.

Modal Structuralism vs. Immanent Structuralism

It is in response to this worry that Geoffrey Hellman developed his "modal" variety of structuralism.[151] In essence, modal structuralism holds that mathematical statements are, indeed, disguised universal quantifications. However, they are not *just* that: They are also inherently *modal*. They are claims about the *necessity* of some universally quantified statement.

[150] It is interesting to consider how we should categorize immanent structuralism in terms of the eliminative vs. non-eliminative distinction. On the one hand, if non-eliminative structuralism just is understood as the *negation* of eliminative structuralism, immanent structuralism would count as a non-eliminative view, since it does not take mathematics-talk merely to be disguised universal quantification. However, Gasser positively describes non-eliminative structuralism as endorsing "an essentially relational conception of mathematical objects," and as not "shying away" from the metaphysically strange nature of such objects. By describing it this way, Gasser seems simply to identify non-eliminative structuralism with Resnik and Shapiro's *ante rem* structuralism. In that case, it seems immanent structuralism would fall into neither category, and so Gasser's categorization is not exhaustive of the various possible views.

[151] Putnam in (1967) is the inspiration for the use of modality to resolve some of the problems associated with model existence, but Hellman provides the first fully-developed account. See Hellman (1989).

Take again, for instance, the claim that "2 + 3 = 5". For Hellman, this will get a reading that is similar to the one the eliminativist gives, but it will also include a necessity operator at the beginning:

- "*Necessarily*: In any system of objects S that exemplifies the natural number structure: $2_S + 3_S = 5_S$"

Furthermore, Hellman specifies that in a mathemtical assertion the antecedent of this claim is taken to be metaphysically possible:

- "Possibly, there is some system of objects S that exemplifies the natural number structure."

These two conditions allow Hellman to avoid the problem of model existence, since even if there is no *actual* model of Peano arithmetic, arguably it is at least *possible* there is such a model. The arithmetical claim, then, would be saying in effect that in any such *possible* model, 2 + 3 = 5 will be satisfied. Therefore, even if there is no actual model of Peano arithmetic, the claim that "2 + 3 = 5" is not just trivially true because we still have to consider whether the relevant universal claim holds in other possible models.

Hellman is able to claim a few important benefits for his view. Most importantly, in addition to the fact that it seems to avoid the problem of model existence, the theory is intended to be *entirely nominalistic*. Indeed, Hellman sometimes refers to his view as a "structuralism without structures." On Hellman's picture mathematics is apparently not committed to any abstract objects at all, let alone the esoteric, essentially-relational ones found in *ante rem* structuralism. Mathematics only asserts various claims about what is necessary *if* certain structural conditions hold, together with the claim that those structural conditions are at least logically *possible*.

Objection 1: Epistemology

Despite the attractions of Hellman's approach, however, his view faces a few potential worries. The first problem is one of modal epistemology: How do we know the logical possibility claim that Hellman takes to be part of the mathematical assertion?

Note that this part of Hellman's view is not a mere appendage. It is essential to avoiding the problem of model existence. Take again the case of "2 + 3 = 5." For Hellman this reads as follows:

- "Necessarily: For any system of objects satisfying the natural number structure: $2_S + 3_S = 5_S$"

Now suppose it is *not* possible that some system of objects is an instance of Peano arithmetic. In that case, the above claim will be trivially true. To see this, we can think in terms of possible worlds: Since there *are* no possible worlds where the natural number structure is satisfied, then the claim *is* true in all the possible worlds where the natural number structure is satisfied; for there *are* no such worlds. In that case, we will end up again with the undesirable result that 2 + 3 = 5, but also that 2 + 3 = 92.

This raises the question of how we can know which mathematical structures are logically possible and which are not. Note that, for Hellman, the "possibility" involved is not a mere *epistemic* possibility, in the sense that it "might" turn out true, but a *logical* one. Otherwise, mathematical systems whose structure we are not yet aware of are not "possible," but presumably the truths of mathematics do not turn on the limitations of our knowledge in this way. So, we must be able to give a story of how we know Peano arithemtic is *possibly* instantiated, in a broadly logical sense.[152] Otherwise the epistemological issues with modal structuralism will be no more tractable than the ones Platonists face.

One might press this objection further by noting the implications of possible worlds semantics for Hellman's view. One might argue that Hellman's commitment to a broadly logical notion of modality implies he cannot avoid the problem of abstract objects. This is because Hellman still is committed to the model theory and semantics of modality.[153] But possible worlds semantics has shown that we should cash out claims of logical "necessity" and "possibility" in terms of possible worlds:

- "Necessarily p" is true iff p holds in all logically possible worlds.
- "Possibly p" is true iff p holds in some logically possible world.

If Hellman's possibility and necessity claims ultimately must be cashed out in terms of possible worlds, however, then we do not actually avoid abstract objects at all, in which case it is not clear what epistemological benefits there are for Hellman's "structuralism without structures" over Platonism.

This is admittedly a powerful challenge. Nevertheless, I think there are a few good responses Hellman can give. First of all, it is not clear that Hellman would (or should) agree that cashing out modal claims in terms of possible worlds gives a reductive or explanatory analysis. Hellman could adopt any number of fictionalist or non-reductivist views about possible worlds, and this would certainly fit better with the spirit of Hellman's views.

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[152] For Hellman, the type of modality in play is the sort that is captured precisely by the laws of modal system S5.

[153] See Shapiro (1997) pp. 229-30 for a similar objection.

Furthermore, I actually suspect Hellman is at least *somewhat* better off than the Platonist, *even if* we were to grant that logical possibility claims ought to be semantically cashed out in terms of possible worlds. For Hellman can appeal to a de re/de dicto ambiguity in the argument: While it might be true that claims about logical possibility *de re* refer to and are made true by facts about possible worlds, this doesn't entail they simply are assertions *about* possible worlds *de dicto*.

Consider an analogy: "Paco (my dog) exists" might be true because various atoms, which are themselves composed of subatomic particles, are arranged a certain way. But this in no way implies someone must *know* this *latter* fact in order to know that my dog exists. After all, people knew about dogs long before they knew about quarks. Similarly, one might know a proposition is possible without having any conception of "possible worlds" at all.

For these reasons, I am not sure that the epistemological challenge to Hellman, at least as found in these versions, is very compelling. With that said, I do suspect that immanent structuralism has something of a leg up on Hellman's view, although ultimately I donot place too much weight on this advantage. But let me first explain what I think the advantage is, as it will help to bring out some of the differences between Hellman's view and the immanent view.

According to immanent structuralism, there is indeed an element of "modal knowledge" present in mathematics. However, it is a specific *kind* of modal knowledge: *essentialist* knowledge. Moreover, we can give at least a sketch of a story of how we *get* this essentialist knowledge: It is through a posteriori knowledge of basic mathematical essences and of logical properties, together with the "building up" of more complicated mathematical properties from these ones. Hellman's modal structuralism is not specific in this way about the relevant *kind* of modal knowledge in play. For that reason, Hellman appears stuck with the bigger problem of how it is that we can know about “broadly logical possibility” in general.

I suspect that this general problem about modal epistemology is indeed more difficult to solve than the one of how we grasp a few basic essences and then understand properties that are logically defined in terms of them. A full epistemology of logical possibility likely will have to resolve long-standing issues about a priori knowledge and the reliability of imagination, conceivability, etc. The immanent theorist’s epistemological problems by contrast seem, at least on their face, a bit more tractable.

In addition to this, the epistemology of immanent structuralism overall seems better suited to account for the mathematical knowledge had by most ordinary people, which starts from perception. By making mathematical assertions equivalent to rather abstract claims about broadly logical necessity and possibility, however, it is not clear how Hellman’s picture connects with the ordinary sort of mathematical knowledge that the average person gains, particularly from an early point in his or her knowing.

Nevertheless, I certainly would not claim that a broader story about our knowledge of logical possibility *cannot* be given. For this reason, I consider the epistemological advantages of immanent structuralism over Hellman’s view to be a smaller argument in my favor.

Objection 2: The Possibility Assumption

I would like now to turn to a second challenge I wish to put forth for Hellman’s view. This challenge questions the possibility assumption that Hellman claims is built into mathematical assertions.

Perhaps this challenge will strike the reader as surprising: After all, don't we want our mathematical systems at least to be *possible*? Isn't the coherence of a mathematical theory a virtue, if not a *requirement*?

I will try to argue that, in fact, this requirement is not a requirement at all, and it is a problem with Hellman's account that it claims it must be. For just as *eliminative* structuralism ties the fate of mathematics too closely to the accidents of the actual world, *modal* structuralism ties the fate of mathematics too closely to what happens to be metaphysically possible. Consider the following case.

Case: The Metaphysics of Infinity

It is conceivable that some philosopher might construct an argument for why the world *must* have a finite number of objects. These arguments may or may not succeed, but arguments of this sort can be and have been given. Consider, for instance, Immanuel Kant's First Antinomy on Space and Time, which argues for the finitude of the past:

> If we assume that the world has no beginning in time, then up to every given moment an eternity has elapsed, and there has passed away in that world an infinite series of successive states of things. Now the infinity of a series consists in the fact that it can never be completed through successive synthesis. It thus follows that it is impossible for an infinite world-series to have passed away, and that a beginning of the world is therefore a necessary condition of the world's existence.[154]

Now, I have nothing here to say directly about the merits or demerits of this or any other such arguments. The point is simply that the coherence of an actual, physically instantiated

[154] Kant (1929), p. 396.

infinity with no beginning has been challenged philosophically, whether successfully or not. And yet we can freely discuss such the hypothesis of such infinities in the mathematical sphere.

So, what if it turned out that certain infinities were not necessarily mathematically *inconsistent,* but simply could not be instantiated in a metaphysically possible, concrete world? Would this really cast doubt on the mathematics? Arguably not.

In fact, it seems the mathematician should not have a care in the world about whether the infinities he studies are metaphysically possible according to philosophers, any more than whether the theories he constructs are physically possible according to physicists. By building a possibility assumption into mathematical statements, however, Hellman seemingly has made the facts of mathematics hostage to the terrain of metaphysical possibility. This seems to imply mathematicians should be worried about what philosophers have to say about the possibility or not of the mathematical systems they develop. But that would be like tying the legitimacy of certain areas of mathematical study to whether physicists consider the systems under consideration to be physically possible. Except tying it to philosophy seems much worse.

It is interesting to see why this problem is not pressing for the immanent structuralist: The reason is that immanent structuralism goes beyond standard theories of metaphysical modality and provides a *hyperintensional* account of the truth-conditions of mathematics.[155] Recalling our discussion from Chapter 2, we say that an arithmetical statement like "2 + 3 = 5" is true just in case certain essential parthood relations hold between the properties *being 2, being 3,* and *being 5*. Or, to take an even simpler example, "2 is even" is true just in case *being even* is

[155] This is in line with the manner in which hyperintensional approaches to metaphysics have been steadily gaining in popularity in the last decade. See Nolan (2014) on how discussions in metaphysics have tended to move beyond the mere "modal" approach which leaned heavily on the notion of broadly logical possibility. Arguably, Fine with his work on essence has been a key figure in this reorientation.

part of the essence of the property *being 2*. These properties themselves are to be spelled out in terms of the larger structural property of *being a system of Peano arithmetic*. What is interesting is that these essentialist relations can hold *even if the relevant structure is not metaphysically possible*. This is a consequence of the fact that we decided to go with the Finean interpretation of essence rather than the "modal" interpretation.[156] On this interpretation, *being even* can still be part of the property *being two* even if, hypothetically, it were shown by philosophers that an actual infinity is not concretely realizable in any metaphysically possible world. Therefore, immanent structuralism explains the fact that mathematics is not hostage to metaphysics.

More broadly, the issue between the modal and immanent structuralist is whether mathematics should be tied to a specific form of possibility: "broadly logical" possibility. While I have illustrated the problem by means of a hypothetical philosophical argument, this is not really the central concern. I am not particularly worried that philosophers *actually will* decisively overturn the metaphysical possibility of infinity, for example. The point is just that modal structuralism seems to place a limit on mathematical structures that the practice of mathematics itself does not, viz., that mathematical structures be broadly logically possible. I would claim that we should not tie mathematics to this rather philosophical notion of possibility. It is a benefit of immanent structuralism that it does not do so.

4. Revised Platonism: Full-Blooded Platonism

Before concluding this chapter, I would briefly like to compare immanent structuralism to a very special version of Platonism. This is the position known as *plenitudinous* or *full-blood Platonism*. First, let me explain what that is.

[156]See the discussion in Section 2.5 above.

The name is meant to be descriptive. According to this rare and somewhat remarkable version of Platonism, there is a *plenitude* of mathematical objects. This plenum is specified by a *plenitude principle*, which in essence says that any mathematical objects that *could* exist *actually do* exist. The idea is supposed to be that any (coherent) mathematical system we might develop will have corresponding Platonic entities, no matter how arcane or unique the system might be. This position has been most prominently defended by Mark Balaguer, especially in his (1998).[157]

To illustrate the basic idea behind the plenitude principle, consider the case of the Continuum Hypothesis (CH). CH states that there is no set whose cardinality is strictly between that of the integers and the real numbers. In other words, if we are considering sizes of sets, after the size of the integers, the size of the reals is the immediately next "largest" size.

Now, according to a famous result, CH is independent of the standard axiomatization of set theory, the Zermelo-Frankel axioms plus Choice (ZFC). That is, CH cannot be proven or disproven frosm ZFC.

What the full-blooded Platonist would say, then, is that there are *multiple* universes of sets. There is one universe of sets which obeys ZFC plus CH, and another universe of sets which obeys ZFC plus not-CH. Moreover, neither universe is more or less real than the other. More traditional versions of Platonism, by contrast, typically assert that there is just one universe of sets, and that the Continuum Hypothesis is either determinately true or determinately false of these sets. Whether we can know it or not is a separate question.[158]

[157] See Balaguer (1998), *Platonism and Anti-Platonism in Mathematics.*

[158] See Linnebo (2018) sec. 4.2.

Balaguer's view is, of course, open to some objections. For one thing, your first reaction on hearing about full-blooded Platonism might be like the reaction you had on your first exposure to David Lewis – that is, a combination of curiosity and staring. After all, how could someone be so bold as to propose not only a world of abstract Platonic objects, but a *plenitude* of them? How can anyone do such a thing? More importantly, how can he be allowed to get away with it? If people are scandalized by traditional Platonism, wait until they hear about *full-blooded* Platonism.

On consideration though, it may not be as bad as it sounds. While there is a sense in which full-blooded Platonism posits "more" objects for us to consider, in a very literal sense, it doesn't. If the more traditional Platonist hierarchy of cardinals, ordinals, and all sorts of unimaginable infinities is not a scandal, then actually it is not so clear why the plenitude should be.

Besides, full-blooded Platonism has some benefits. The main one is that it purports to answer the million-dollar-question: How can we know reliably about causally dislocated Platonic objects? On full-blooded Platonism the answer is simple: We know how to come up with coherent systems, and all of those systems exist. When it comes to positing abstract mathematical objects, you really can't go wrong!

While there are other objections to full-blooded Platonism, I will not consider them here.[159] What I would instead like to do is explain how, perhaps surprisingly, immanent structuralism and full-blooded Platonism are in some ways rather similar.

[159] Though see, e.g. Restall (2003) for worries about whether it is actually possible to specify the "plenitude principle" in a precise way.

The core of the similarity is that both full-blooded Platonism and immanent structuralism at least in principle accept a plenum of mathematical properties. Recall that, for the immanent structuralist, a purely structural property is any one that can be defined in terms of part, whole, and purely logical vocabulary. Now, some of these mathematical properties might certainly be more natural than others, or more useful, or more elegant, so on. But they're all there: They come for free because of the reducibility thesis mentioned in Chapter 3.

The difference is that the full-blooded Platonist goes on to posit an additional plenum: The s*ui generis* mathematical *objects*. You could think of full-blooded Platonism as being sort of like immanent structuralism, but where every structural property is (for some reason) instantiated. Or, conversely, you could think of immanent structuralism as full-blooded Platonism minus the entitative bloat. However, given the story about mathematical truth told in Chapter 2, why not just go with the properties and ditch the objects? They are ontologically cheaper, being reducible to a few simple properties and abstract logical relations, and the story of how we know about them is straightforward.

5. Conclusion

In this chapter I have tried to show how immanent structuralism compares to other versions of structuralism and to a unique version of Platonism. In particular, I have tried to show that immanent structuralism can handily avoid some of the problems the *ante rem* view faces in its ontology, particularly when it comes to the thorny issue of "identity" between abstract, intrinsically featureless "positions." Immanent structuralism side-steps the issue by instead speaking about the identity or non-identity of certain relational *properties*.

Furthermore, immanent structuralism avoids the problem of model existence, insofar as it defines mathematical truth in terms of various necessary essentialist truths rather than models of objects that instantiate mathematical systems. In other words, we do not have to worry about having a "model" for our mathematical theories; all we need is for the relevant essentialist truths to hold. Moreover, because of this, we do not even need to worry about *possible* models. If, for some strange reason, philosophy or some other discipline showed a certain infinity studied in mathematics to be metaphysically impossible, the mathematician does not have to worry, since that still need not affect what is an essential truth about the relevant mathematical property being considered.

Finally, we looked at full-blooded or "plenitudinous" Platonism and compared it to immanent structuralism. We saw that the two stories are, in one sense, remarkably similar, while in another sense they are drastically different. They are similar in that they both are happy with a mathematical "plenum." They are different in that the plenum of immanent structuralism, being one merely of reducible mathematical *properties*, seems far less scandalous and shocking.

CHAPTER 6: EXPLAINING DE RE MATHEMATICAL NECESSITIES

Introduction

Toward the beginning of his 1921 address, “Geometry and Experience,” Albert Einstein remarks on mathematics’ subject-matter and epistemic status:

> ”At this point an enigma presents itself which in all ages has agitated inquiring minds. How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?
>
> In my opinion the answer to this question is, briefly, this: **As far as the propositions of mathematics refer to *reality*, they are not *certain*; and as far as they are *certain*, they do not refer to *reality*.**”[160]

If we replace the word “certain” in this passage with “necessary,” one will have grasped a common attitude toward mathematical necessity central to many standard philosophies of mathematics.[161]

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[160] Albert Einstein, “Geometry and Experience, Address of 27 January 1921 to the Prussian Academy of Sciences in Berlin. Emphasis added.

[161] This thesis seems to accurately characterize most versions of formalism, verificationism, constructivism, nominalism, and fictionalism. Indeed, as I will argue below, on this picture of mathematical necessity one will not have a wholly inaccurate account of Platonism.

I will argue, contrary to this thesis, that there are mathematical truths about physical, concrete objects that carry a particularly strong sort of necessity – mathematical necessity. In other words, there are *physical facts* which are *mathematically necessary*. I call these mathematically necessary facts "*de re* mathematical necessities." I argue that these physical facts carry a type of necessity that is stronger than mere natural or metaphysical necessity, and that this sort of necessity requires explanation. I will attempt to give such an explanation below.

In Section 1, I will clarify what I mean by "mathematical truths about physical, concrete objects" – what I call "*de re* mathematical necessities" – illustrating with examples. In Section 2, I will explain what I mean in saying mathematical necessity is a "particularly strong sort of necessity," and will try to make clear what is so "strong" about it. In Section 3, I argue against Platonism as a strategy for explaining *de re* mathematical necessities. I will try to show that the standard Platonist ontology of mathematics faces insuperable difficulties with these sorts of cases. Section 4 will show how immanent structuralism can better explain these necessities than Platonism. The conclusion, Section 5, will briefly relate the story in Section 4 to some recent discussions about mathematical explanation in science.

1. Mathematically Necessary Facts about the World

I wish to discuss certain necessary facts about concrete, physical objects. However, I claim these facts do not carry only nomic or metaphysical necessity. They carry *mathematical necessity*, the same strong sort of necessity characteristic of mathematical theorems.

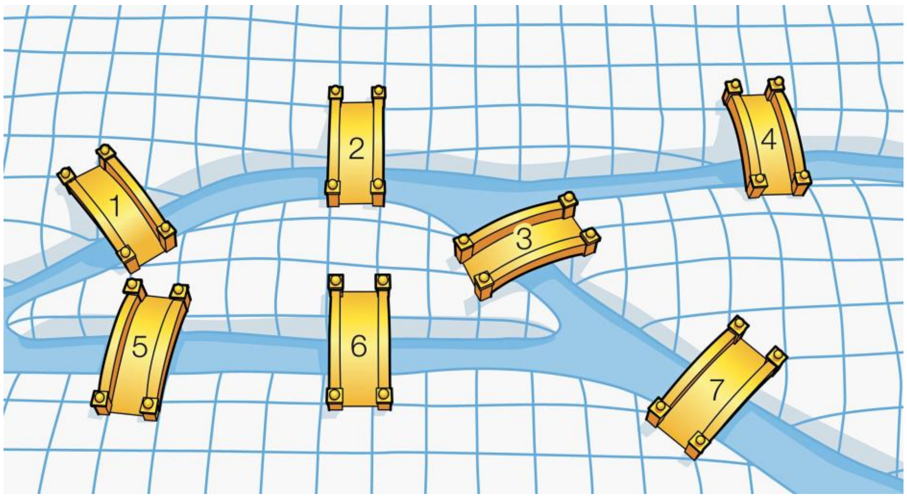
Let us first examine some of these putative necessities. Consider the following five cases:

1. It is impossible to walk a path where one crosses over each of the bridges of Konigsberg exactly once. (Pincock 2007)

2. My bathroom floor cannot be tiled with regular equal pentagons. (Franklin 2014)
3. Necessarily, any physical body that is symmetric about both axes is symmetrical about the point of intersection of the axes. (Franklin 2014)
4. Five apples cannot be equally divided among three children. (Braine 1972)
5. This trefoil knot cannot be unknotted without cutting. (Lange 2017)

I will discuss each case in turn and ask the reader to notice the *necessity* involved in each case.

- **The Konigsberg Bridges:** It is impossible to walk a path where one crosses over each of the bridges of Konigsberg exactly once.



Here is a graph of the Konigsberg bridges as they were in 1736, when Leonard Euler proved a famous result. Euler asked whether there is a path through the city that crosses each bridge *exactly once*. The rules are that the islands can only be reached by walking across the bridges (e.g., no swimming) and every bridge, once accessed, must be crossed to the other side (no turning back half-way across the bridge). One need not end up at the place one started. One need only cross each bridge once.

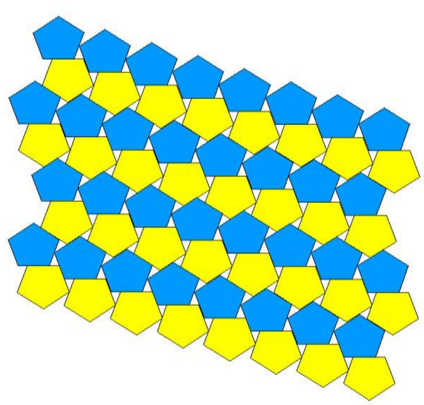
Euler ultimately proved that there is no such path, but you can probably convince yourself of the result right now. To see the impossibility of crossing each bridge exactly once, just try a few times to do it. (Go ahead!)

Suppose, for instance, that you start in the top-left corner and attempt 1-5-6-2-4. You will either have to choose 3 or 7. Say you choose 3. Then you cannot get to 7 without crossing a bridge twice. However, if you had instead chosen 7, you'd have then been unable to get to *3* without crossing a bridge twice. Sorry!

After trying out even just a one or two more paths, you may become convinced of the *necessity* of your never walking a path through the city by crossing each bridge exactly once. No matter how many times you try to do it, you find that you just *can't*.

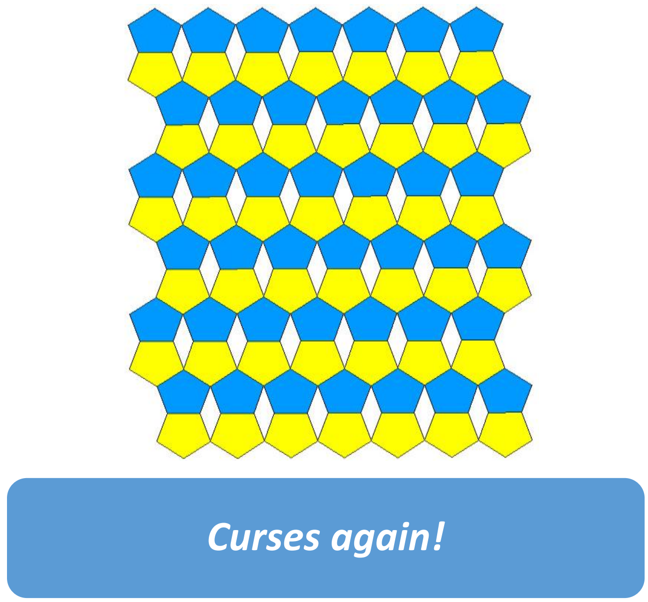
- **Tiling the Bathroom Floor:** My bathroom floor cannot be tiled with regular, equal pentagons.

Suppose my wife asks me to tile the bathroom floor. Not being much of a handyman, I buy pentagon tiles. When I first attempt the tiling, I am met with difficulties:



... *Curses*!

Despite these setbacks, I take solace in the knowledge that what I lack in native cerebral talent I make up for in persistence. Undeterred, I give it a second go, trying a different arrangement this time:

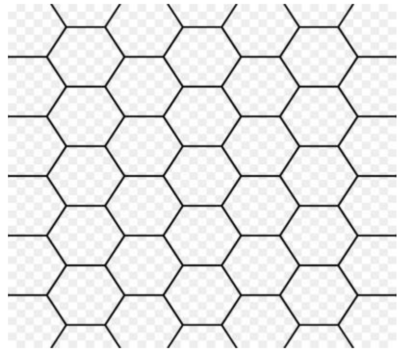


After a few more tries, I recall another famous (though likely apocryphal) Einstein dictum, and realize it is insane for me to continue. After ruling out the possibility that I've been sold defective pentagons, I eventually come to grasp that *one simply cannot tile the bathroom floor with equal, regular pentagon tiles*, and that I will be hauling my van and future self back to Home Depot.[162]

[162] The facts about my bathroom floor as well as the trajectory of my future self are both confirmed later that day, when I open my geometry textbook and learn that a Euclidean plane cannot be tiled by regular pentagons, because the internal angle of a regular pentagon is 108°, which does not divide 360° (the angle measure of a full, circular turn). Thus, in the diagram above, when I put the sides of my pentagons together and look at the corners where three pentagons touch, there must (!) always be a "gap," which is to say that the pentagons do not tile the plane.

Perhaps I will try hexagon tiles instead:

(Much better…)



- **Symmetry about Axes**: Necessarily, any physical body that is symmetric about both axes is symmetrical about the point of intersection of the axes.

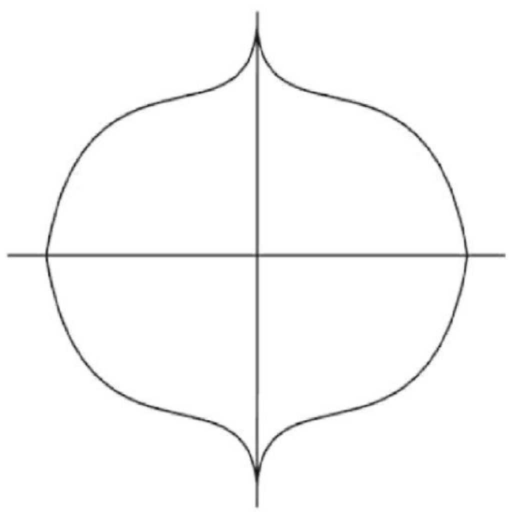
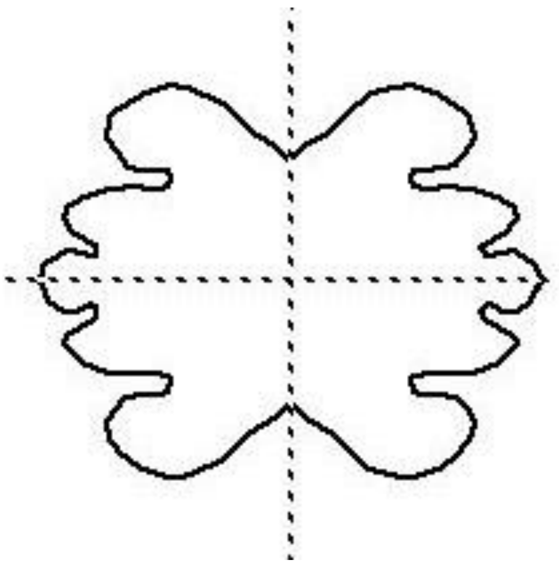


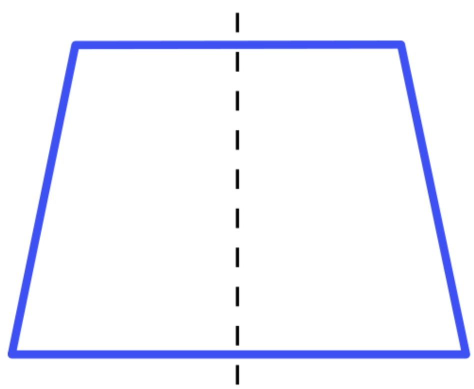
Figure 5.5 A figure vertically, horizontally, and centrally symmetric

To illustrate this case, take the drawing in front of you, which is symmetric about both axes. Now put your finger on some point A inside the figure. You will find that, for any point you choose, you will be able to find another point B inside of the figure that is the same distance from the center, and where the line from A to B runs through the center.

Furthermore, if you draw other diagrams that are symmetric about both axes, you will find that the same thing happens. You cannot draw a diagram symmetric about both axes without it being symmetric about the point of intersection as well. This can be seen by drawing a circle, square, or even other, stranger figures, such as this one:



You will also find that you *can* draw diagrams that are symmetric about one axis without being symmetric about the center – for example, an isosceles trapezoid, such as this:

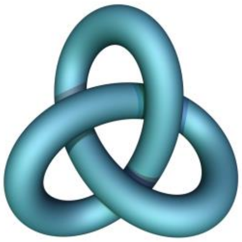


These necessities and possibilites apply not only to the scribblings on paper you might draw, but also to rocks you might find, pottery you might build, or even animals you might breed.

- **Dad's Apples**: Five apples cannot be equally divided among three children.

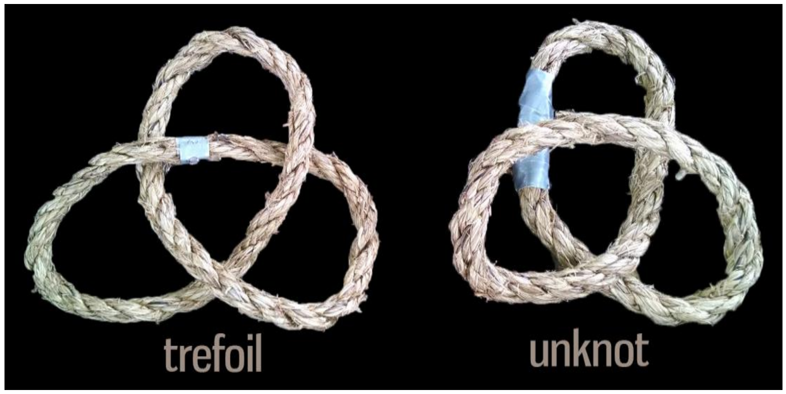
As we are out on a picnic, my three children start to become hungry. Embracing egalitarian ideals, but not having been particularly quick at math, I try to divide my five apples equally among my children. No matter how many times I try, I fail. Of course, that is because I have set myself upon an impossible task. Eventually accepting that the laws of mathematics prevent me from achieving an equal outcome any other way, I do the fair thing, and toss two of the apples so that everyone receives an equal share.

- **Trefoil Knot**: This trefoil knot cannot be unknotted without cutting.



(Abstract Trefoil Knot)

Suppose I have a rope in the form of a trefoil knot. No matter how I contort it, I am unable to unknot it. (See the picture on the left, below.)



As you will see, if you have one yourself, you can pull one part of the rope through one loop, push it back through another, twist it, turn it, pull it, squeeze it. But the only way you will be able to *undo* it is by *breaking* it. One way, of course, would be to rip the rope apart with your bare hands. But I don't make it to the gym quite that often. So I get a pair of scissors, some duct tape, cut the rope at a point, and after undoing the knot I tape the ends back together. (See the image on the right, above.) That is what I *must* do if I am to undo my trefoil knotted rope.

I contend that in all of these cases, we are dealing with *physical facts* that are *necessary*. That is, in each case, the fact or proposition that is necessary is a *physical* fact, involving the properties of *real*, *concrete physical* objects or systems of them. What are necessary are facts about *bridges*, bathroom *floors* and *tiles*, works of *pottery*, pieces of *fabric*, *drawings*, *apples*, and *ropes*.

Furthermore, I claim that these physical facts carry a *stronger-than-usual* sort of necessity, *mathematical* necessity.[163] This necessity is the same type of necessity characteristic

[163] And *a fortiori*, they carry the more common "metaphysical" and "physical" flavors of necessity too.

of mathematical theorems. I will denote this necessity with a special box symbol, together with a special diamond for the corresponding sort of possibility:

⧈p for “It is mathematically necessary that p.”

♢p for “It is mathematically possible that p.”

I will also use the following abbreviations for the metaphysical modalities:

□p for “It is metaphysically necessary that p.”

◇p for “It is metaphysically possible that p.”

And for the physical modalities:

■p for “It is physically necessary that p.”

◆p for “It is physically possible that p.”

Like the other forms of modality, we can define the one mathematical modality in terms of the other. For example, we can define mathematical possibility in terms of mathematical necessity:

Def: ♢ $p \leftrightarrow \neg$ ⧈ $\neg p$

My claim, then, is that if f is one of our relevant necessary physical facts, and q is a mathematical theorem, then when we say that ⧈f and ⧈q, it is the same sort of necessity that is expressed in each case. Thus, there are physical facts that are mathematically necessary. I will call these “*de re* mathematical necessities.”

These are necessities attaching to physical realities. They are not about our ideas, or concepts, or any such mental item. They prevent you, and me, and the world from doing certain things. In fact, even if there had been no humans, and so no human ideas or concepts, the *de re* mathematical necessities would still be there. To see this, consider a further example.

- **Ball and Cloth**: Necessarily, any sphere of radius 1 meter is not covered by 12 square meters of material.

I claim that it is *mathematically necessary* that a physical sphere of radius 1 meter is not covered by 12 square meters of material (such as fabric). Moreover, even if nobody were around to contemplate it or to try it, and even if there had been no concepts or ideas by which someone might entertain it, the possibility of such a material covering such a sphere would be foreclosed, and the world would still be limited in its arrangements by this necessity. For instance, no matter how the wind blows around a $12m^2$ cloth – even given infinite time – it will never cover any ball of radius 1m. And *it cannot be otherwise*.

2. Strong Mathematical Modality

I have claimed that the type of necessity possessed by these physical facts about what does and does not happen in the world – mathematical necessity – is the same type of necessity characteristic of mathematical theorems.

Note that this is plausible, because in the examples we have seen the necessity that applies to the relevant physical facts seems to be a very *strong* type of modality. To tile a floor with pentagons, or to walk each bridge of Konigsberg exactly once, do not seem just to be very difficult things to do. They seem, in some sense, positively *incoherent*.[164]

[164] Nevertheless, I would hold that it is not a strictly logical necessity that is in play, *pace* logicists, but a synthetic, mathematical one. The necessity is strong, but not quite that strong.

Violating these mathematical laws seems more impossible than a violation of mere physical law, e.g., a violation of the conservation of energy. Indeed, such feats as we have imagined seem more impossible than violations of certain *metaphysical* laws, e.g., a contingent being (say, a pink elephant) popping into being out of nothing.[165]

Another way to bring out the point is in terms of counterfactuals. Even if the conservation of energy had been false, I would not have been able to take a path crossing each bridge of Konigsberg only once. Or even *if* objects were able to pop into being without a cause, I still would not be able to cover a sphere of radius 1 meter with 12 square meters of material.

So the modality that attaches to our cases is as strong as, but also stronger than, mere physical and metaphysical necessity. In other words, if ⧈p holds, then so does □p and ■p, but not necessarily vice versa.

Objections: These Aren't *Real* Necessities

I will now consider a few objections to the idea of "*de re* mathematical necessities." There are a few ways one might argue these examples are somehow not "real" necessities.

Objection 1: Perhaps one might object to the *negativity* of the examples I've given so far. After all, aren't these all just cases of something *failing* to hold? (For instance, the failure of anyone to take a certain path across the Konigsberg bridges.) These are not "positive" necessities, only "negative" impossibilities.

[165] Assuming it is a metaphysical law that "large-scale physical objects do not pop into existence for no reason." If one doesn't consider this a metaphysical law, then one can choose a different example, e.g., the truth (or falsity) of nominalism, the truth of three- or four-dimensionalism about time, etc.

Reply: I have two responses: (A) First, the necessity of something *not* happening is still a type of necessity. Nobody would contend that the physical necessity of nothing's traveling faster than the speed of light, for example, is not a real necessity merely because it is a "negative" necessity. (B) Second, it is not true that all the cases of *de re* mathematical necessity that I've given are negative examples. For instance, Example 3 shows how certain things positively *must* happen: Necessarily, all physical bodies that you find that are symmetrical about both axes will be symmetric about the point of intersection.

Objection 2: These cases are not *absolutely* necessary. They are only necessary *given certain assumptions*. For example, you cannot take a certain path across the Konigsberg bridges *if* you do not swim across the river, *and* the laws of physics hold, *and* the bridges remain in the same pattern, etc.

Reply: Again, I have two responses. First, even if the necessities described are conditional, still, the relevant *conditionals* are mathematically necessary – and they are not trivially true, since in some cases their physically contingent antecedents are satisfied. Secondly, some of the cases I've described are *unconditional*, such as the ball and cloth example (Case 6) mentioned toward the end of section 1.

Objection 3: One might argue the notion of a "strength of modality," which is necessary for the idea of a strong mathematical necessity to get off the ground, is only subjective, and is perhaps tied to one's credences. For example, the apparent "absurdity" or "incoherence" involved in someone's tiling a flat surface with regular equal pentagons is really just a consequence of one's near certainty in *belief* that this won't happen.

Reply: It does not seem like the idea of a "strength of modality" can be reduced to credences in general. For example, take the physically necessary fact that nothing can move faster than the speed of light. Assuming the lawhood of this fact, I would argue that our rational confidence is not the issue. After all, we may not think the speed of light any more *likely* to be surpassed than that some mathematical proposition will be found to be false. Even so, imagining a violation of this law *even though we know that it is an actual law* does not seem incoherent or absurd in the same way as a violation of the relevant mathematical proposition, *even though we know the truth of the mathematical proposition*. So our sense of metaphysical "incoherence" does not necessarily track the strength of our credences.

We can see this further by looking at certain indicative conditionals. Consider the following assertions:

- "*Even if* it is a true law of physics that the speed of light cannot be surpassed, the world could have been set up differently so that something could surpass it."
    - [or: "...one can *imagine* it being set up differently so that something could surpass it."]

Compare that to these two indicative conditionals:

- "*Even if* there is a proof that P ≠ NP, the world could have been set up differently, so that for any problem verifiable in polynomial time, a computer program could be made that solves it in polynomial time."
- "Let us call a "prime group" of objects any collection of objects which is numbered by a prime number. Then *even if* Goldbach's Conjecture is true, the world could have been set

up differently, so that there was some even number of apples (greater than 2) that *could not* be split up into two prime groups."

These conditionals help bring out the "incoherence" involved in the second and third scenarios (and absent from the first) irrespective of confidence levels, since they ask us to consider what would be involved if we let our confidence in the antecedents be 1 (certainty). But since only the *second and third* conditionals strike us as incoherent, the counter-intuitiveness of the second and third statements is due to the *modal strength* of mathematical theoremhood and of the associated *de re* mathematical necessities, not the level of credence involved.

3. Platonic Explanations of *De Re* Mathematical Necessities

We have seen that there are *de re* mathematical necessities that are about the physical world. It would be nice if we could explain the strong sort of necessity that attaches to these physical facts. The aim of this and the following section will be to try and do so. I will first consider, and reject, Platonist attempts to explain *de re* mathematical necessity. Then, in the following section, I will give an immanent structuralist take on the matter.

I should first clarify what needs explaining. Later in Section 4 I will appeal to my own account of mathematical truth which claims that the truths of mathematics are *essentialist* truths, akin to the claim that the essence of water is H2O. What I will *not* aim to explain is why the various essentialist truths hold. I am not sure that there *is* a deeper set of facts or entities that can explain why, for example, the essence of being a dog includes being a mammal, other than the fact that that is the essence of the thing. Since I claim mathematical truths are essentialist truths of this sort, this may be rock bottom for us.

What I *will* try to explain are the *de re* mathematical necessities, i.e. those physical facts that carry mathematical necessity. I will try to explain *why* they are necessary, and why with *such strength*. For instance, take some physical generality – say, the fact that any sphere of radius 1 meter is not covered by 12 square meters of material – and call it f.[166] This claim does not just happen to hold. It is necessary in a particularly strong way. $\Box$f. Why is that?[167]

The first theory we might turn to is the standard view of mathematical truth: Platonism. *Prima facie*, Platonists seem naturally poised to resolve our question. After all, according to Platonism, mathematical objects are abstract entities, and exist timelessly, spacelessly, *and necessarily*. Perhaps these necessary beings and their necessary features can help explain the *de re* mathematical necessities, because the *de re* mathematical necessities are in some way derivative from the Platonic objects and *their* necessity.

Consider for instance the following facts about the Platonic realm:

- 1'. There is no path through a K-graph [the abstract graph corresponding to the Konigsberg bridges] that touches each edge only once.
- 2'. The Euclidean plane cannot be tiled by regular, equal pentagons.
- 3'. Any figure that is symmetric about both axes is symmetrical about the point of intersection of the axes.
- 4'. The number 5 is not divisible by the number 3.

[166] This fact f is what Marc Lange calls a "sub-nomic fact." See Lange (2007). It is a true proposition describing the physical world without any modal operators. (In this case, it is a universally quantified one.)

[167] The problem can be thought of via analogy with the laws of nature. While some generalities merely *happen* to hold – like the fact that there are no gold cubes exceeding a cubic mile – others hold non-accidentally, such as the fact that no body accelerates from rest to beyond the speed of light. Well, it is the same thing here, except that our generality f seems to hold with an even *greater* non-accidentality than the one about the speed of light. If this latter non-accident's non-accidentality calls out for explanation, then so does f's.

- 5’. The trefoil knot in three-dimensional space is not isotopic to the unknot.

These facts refer to abstract, non-physical mathematical objects and their relations to each other. According to Platonists, these facts are necessarily true. They correspond to the five examples of *de re* mathematical necessities we discussed earlier:

1. It is impossible to walk a path where one crosses over each of the bridges of Konigsberg exactly once.
2. My bathroom floor cannot be tiled with regular equal pentagons.
3. Necessarily, any physical body that is symmetric about both axes is symmetrical about the point of intersection of the axes.
4. Five apples cannot be equally divided among three children.
5. This trefoil knot cannot be unknotted without cutting.

The Platonist explanation for *de re* mathematical necessities, then, would be that facts 1’ – 5’ are necessarily true in virtue of the relevant *necessary beings* (the Platonica), and 1 – 5 are necessarily true because they are grounded in or explained by the truth of 1’ – 5’.

This heavily metaphysical picture seems initially promising, insofar as it gives us objective, necessary beings in order to ground the relevant *de re* mathematical necessities. However, there are several objections to the Platonist strategy. I will present three worries.

Objection 1: Pushing the Question Back

One concern is that the Platonist’s solution seems merely to push back the problem we began with. There are two parts to this objection.

First off, we can grant that there are mathematical objects, but still ask why they exist with *necessity*? In other words, why are they *necessary* beings? (Note that they must be, if they are to explain necessary truths like 1 – 5, since in order to do so 1' – 5' must be necessary, and that requires that the Platonic objects be necessary.)

Second, even if we grant that they *are* necessary beings, in order to explain the relevant *de re* mathematical necessities, the claims in 1' – 5' must be necessarily true, and so the mathematical objects must have their features and relations necessarily. But we can then ask, "Why must they have their features and relations necessarily?"

Perhaps the Platonist can appeal to the *natures* or *essences* of these entities to explain the necessity of 1' – 5'. The features and relations predicated of the Platonic horde are essential to them, and so 1' – 5' are necessarily true.

One might worry that this move seems slightly *ad hoc*. It would be better, for example, if we could see *why* the relevant predications are essential to these objects, other than a bare assertion that this is so. Nevertheless, I grant that if the features mentioned in 1' – 5' *are* parts of the essences of the Platonic objects then 1' – 5' will be necessary. Thus, I will place less weight on the second part of Objection 1.[168]

Objection 2: Getting the Right Grade

Still, that leaves the first part of Objection 1 unanswered, and raises a further issue. For even granting that the Platonic beings *are* necessary and have their features essentially, it still is not clear how the Platonist can adequately explain the *strength* of the necessity involved in 1 – 5.

[168] With that said, in Section 4 I will try to show how we *can* explain *why* the predications in 1' – 5' are necessarily, and indeed *essentially* true. I take it as counting in favor of my theory that it does not merely have to *assert* that 1' – 5' are essentially true, but purports to *explain* this fact.

For the necessity in play is *mathematical* necessity, the sort of necessity characteristic of mathematical theorems, and the sort that makes walking a certain kind of path along the Konigsberg bridges seem *incoherent*.

It is not enough to say that the Platonic objects are necessary beings, since they could be merely *metaphysically* necessary being. If they were *merely metaphysically* necessary beings, then the strength of the necessity attaching to facts 1 – 5 would remain unexplained. Here is why.

Take one of the sub-nomic physical facts f that is mathematically necessary according to 1 – 5.[169] For example, we can let f be the fact (corresponding to Example 4) that I never divide my five apples equally among my three children. Then (4) says: ⧈f.

Now take the purported Platonic explanans:

4'. The number 5 is not divisible by the number 3.

Note that (4'), as stated, is also a sub-nomic fact. Although it is about abstracta, it does not contain any modal operators. However, in order to fully explain (4) (i.e., the fact that ⧈f) it must be the case not only that the number 5 *happens* to not be divisible by 3, but further:

4*. ⧈(the number 5 is not divisible by the number 3)

[169] Again, a sub-nomic fact is essentially one that does not contain any modal operators. It can include, e.g., basic predications, conjunctions and disjunctions, universally quantified statements, etc. "All objects travel no faster than the speed of light," for example, is a sub-nomic fact. "*It is a law that* all objects travel no faster than the speed of light" however is not.

The reason for this is simple: A weaker necessity, taken just by itself, cannot explain a stronger necessity.[170] "You cannot give what you don't have."[171]

It would be bizarre, for instance, to try to explain the *metaphysical* law (if it is one) that every event with a beginning has a cause *merely* by appealing to the fact that the *physical* laws we have happen to verify this law (if they do). That would be like trying to explain *the lawhood* of the fact no body accelerates from rest to beyond the speed of light by appealing to the bare sub-nomic generality that *at no time* does any material body in fact do so. Similarly, if one believed that it is *metaphysically* necessary that all beings are physical, it would not be sufficient to explain this merely if one thought that somehow *the actual laws of physics* showed all *actual* entities to be physical.[172]

So 4* must be true in order for the Platonist to explain 4. However, for 4* to be true, it must be the case that in all mathematically-possible worlds there are the numbers 3 and 5. So it also must be the case that:

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[170] I would also claim that in order for one necessary fact to explain another, the first must entail the second. But again, since mathematical necessity apparently is stronger than metaphysical necessity, in general $\Box p \nVdash \boxdot f$. (Where $\Box p$ is the claim that p is metaphysically necessary.)

[171] Compare Descartes's discussion in Meditation III: "[H]ow could the cause give reality to the effect unless it first had that reality itself? Two things follow from this: that something cannot arise from nothing, *and that what is more perfect – that is, contains in itself more reality – cannot arise from what is less perfect*." It seems likely that Descartes is here using "cause" in the more general Aristotelian/scholastic sense of "explanation."

[172] To be clear, I am not saying that the laws of nature are irrelevant to discovering the metaphysical laws, or that an assertion about the laws of nature could not function in an argument for some putative metaphysical law. I am only claiming that the connection will not be as straightforward as explaining $\Box q$ by the mere fact that $\blacksquare q$. For example, a proper metaphysical argument from the *natural* necessity of there only being physical beings to the *metaphysical* necessity of this might go as follows: Suppose one held the theory – as some do (e.g., Vetter *Potentiality* (2015), Jacobs (2010), and Pruss *Actuality, Possibility and Worlds* (2011)) – that in order for something to be metaphysically possible, there must be some entity with the *power* of bringing it about. Then, if the laws of nature established there to be only physical entities, and one held that physical entities only have the power to create other physical entities, then it follows that it is metaphysically necessary that all beings are physical. This explanation of the metaphysical necessity of there being only physical entities appeals to a putative natural law, but only works in conjunction with other *metaphysical* premises. Another strategy might be if one were a scientific essentialist, in the manner of Brian Ellis (2001), and held for independent reasons that all physical laws are essential truths.

⧈(the number 5 exists) and ⧈(the number 3 exists)

That is, the relevant mathematical objects must exist with the mathematical necessity characteristic of theorems like 4'. Otherwise, 4* would be false and the Platonist explanation of 4 would fail.

So, one must posit not only that the Platonic objects are necessary, but also that they exist with a very strong sort of necessity. It is not clear, however, that the Platonist can give a satisfying explanation for why this should be so.

In fact, there is a more positive worry for the mathematical Platonist here. For abstract objects are posited in all sorts of philosophical contexts, often by arguments similar to the standard ones given by mathematical Platonists.[173] For instance, philosophers have posited abstract properties, propositions, and worlds, using arguments quite similar to ones given for the existence of mathematical objects.

But it is not clear that the necessity that is posited for *those* objects is the strong kind that is expressed by "⧈f". Or, wording the point slightly differently: To deny the existence of some abstract objects, like properties or propositions, and to instead be a nominalist, does not seem incoherent in the same way that tiling a floor with regular equal pentagons does, *even if* nominalism is necessarily false.

With this in mind, we can put the argument in the form of a dilemma. Either mathematical objects exist with the same sort of necessity as other abstract objects, or they exist with a stronger sort of necessity than other abstract objects. If the former, then they cannot

[173] Peter van Inwagen in "A Theory of Properties," (2004) offers a Quinean argument for the existence of properties as abstract objects, and explicitly attempts to demonstrate that they must be necessary beings.

explain *de re* mathematical modality. If the latter, then their necessary existence *can* explain *de re* mathematical modality, but this stronger necessity seems ad hoc and unmotivated. For there does not seem to be anything in one abstract object that would make it a more necessary being than another.

But perhaps all of this is simply putting the original point differently: There does not seem to be any Platonist explanation for why mathematical abstracta have a stronger sort of necessary existence than other abstracta. However, as I will argue below, immanent structuralism *does* have an explanation for the type of necessity attaching to de re mathematical modalities. Thus, we should favor immanent structuralism.

Before turning to that argument, however, there is a final objection for the Platonist we should consider.

Objection 3: The Kripkean Objection (i.e., the "Humphrey" Worry)

Suppose we leave aside the previous objections and grant that the Platonist facts in 1' – 5' are necessary, and furthermore that they and the beings they refer to are necessary with the right *kind* of necessity. Even so, I claim there is a further problem. The final and most important worry for the Platonist is that there is not a sufficiently close *connection* between the Platonic and physical realms to explain the *de re* mathematical necessities.

The issue is simply that it is unclear what the necessary properties of some distant, abstract, causally isolated Platonicum, like the number 5, has to do with the physical world. Perhaps this non-physical thing we call the number 5 is related to this other non-physical thing we call the number 3 in a certain way – but what does that have to do with my *apples*?

The problem is similar to one raised by Saul Kripke in reference to David Lewis's modal realism. Lewis proposed to make modal statements come out true in virtue of the happenings in various causally isolated, concrete possible worlds. In particular, modal statements about me and my abilities come out true or false in virtue of what my concrete counterparts in these other worlds do.

Kripke, however, laid out an important worry for Lewisianism about possibility, namely, that it is unclear how other concrete worlds have anything to do with what it is possible for those of us *in this world* to do. It is not obvious, for example, how the fact that Humphrey could have won is related in any way to occurrences involving someone quite somewhere else who reminds us of him:

> "[According to modal realism] if we say 'Humphrey might have won the election (if only he had done such-and-such),' we are not talking about something that might have happened to Humphrey, but to someone else, a 'counterpart'. Probably, however, Humphrey could not care less whether someone else, no matter how much resembling him, would have been victorious in another possible world."[174]

Similarly, there is no obvious reason why the bridges of Konigsberg should care enough about the abstract Platonic graphs to never fail to emulate them. More generally, it is unclear how the truths about Plato's non-physical heaven manage to regulate the physical realm in order to produce *any de re* mathematical necessities.

That is at least how things seem to stand. But perhaps the Platonist might appeal to the existence of a *correspondence* or *structural similarity* between the relevant abstracta and

[174] Cf. Kripke (1980), p. 45. Probably Nixon did not care either.

concreta, e.g., between the physical Konigsberg bridges and the relevant Platonic graph. Maybe it is this *correspondence* or structural congruence that transfers the necessity from the Platonic realm to the physical.

I have two replies to this. First, it is obscure on this picture why it is reference specifically to the *mathematical facts* that explains the relevant *de re* mathematical necessity. If it is the *correspondence* that confers the necessity on the relevant *de re* mathematical fact, then why is it correspondence specifically with the *Platonic* entities that explains the *de re* necessity?

For example, take the second case we mentioned earlier, and the relevant mathematical fact that explains it.

2. My bathroom floor cannot be tiled with regular, equal pentagons.

Because:

2’. The Euclidean plane is not tiled by regular, equal pentagons.

The proposal then would seem to be that (2’) explains (2) in virtue of the mathematical necessity of (2’) together with the correspondence or structural similarity between the objects mentioned in (2’) and those mentioned in (2). However, if (2’) explains (2) in virtue of the *correspondence* between the objects mentioned in (2) and those in (2’), then it is unclear why any other mathematically necessary fact that mentions structurally similar objects couldn’t do the job. For instance, why isn’t it the correspondence between the objects in (2) and some *other* Platonic objects – e.g., certain sets? For that matter, why couldn’t it be the correspondence between the objects in (2) and those in the following *physical* fact (which is also a *de re* mathematical necessity)?

2''. I cannot tile my sheet of paper with drawn pentagons.

Note that (2'') has the right kind of necessity – the same kind as (2'). However, it is not (2'') that explains (2), nor (2) that explains (2''), but rather (2') that explains *both*.[175] It is not clear on the current proposal, however, why (2') better explains (2) than does (2''). In both cases you have objects with the right sort of correspondence and the right sort of necessity.

However, the second and more important response is that the "immanent structuralist" solution that I will propose renders the Platonic objects superfluous, while also being able better to deal with Objections 1 and 2. For if there is a structural similarity between the Platonic objects and the physical ones, then why not just appeal to that structural relation *directly* to explain 1 - 5 and avoid positing special abstract objects that necessarily instantiate that relation? This is exactly what I will do. It is to this solution that we will now turn.

4. A Structuralist Explanation of *De Re* Mathematical Necessities

Before we can give an adequate explanation of the *de re* mathematical necessities, let us briefly recall the theory of mathematical truth from Chapter 2, which does not involve appeal to abstract Platonic objects. Instead, *immanent structuralism* appeals to Aristotelian essences which can be literally instantiated in physical systems.

On this view, mathematics does not study some abstract mathematical objects.[176] Rather, mathematics studies a certain class of *universals* or *properties* – what I call, following James

[175] An argument for this is that if (2'') explained (2), it seems that (2) could explain (2'') just as easily. There is no reason to think that the fact about sheets of paper is prior to the one about plaster tiles. So if (2'') explains (2), then (2) explains (2''). But then we would have a rather short circle of explanation. But (2'') does not explain itself, for it *is* explained (by 2'). So (2'') does not explain (2). *Mutatis mutandis* for (2) explaining (2'').

[176] In contrast to the Platonist, who, as W.L. Craig puts it, conceives of mathematical objects as "just like automobiles, only more numerous, abstract and eternal." See Craig (2016), p. 9. Craig gets this description from Resnik (1980), p. 162.

Franklin, the *purely structural properties*[177] – and these can be literally instantiated by physical systems. Thus, mathematical patterns, or structures, can be "located in" objects in exactly the same way that an object's size, mass, or color can.

So, the ontology of immanent structuralism, unlike that of Platonism, does not consist in mathematical *objects*. It instead consists of physically instantiable universals or properties, namely, the purely structural ones: those that can be "built up" out of the properties of *part*, *whole*, *sameness*, *difference,* and purely logical vocabulary.

In addition to its special ontology, immanent structuralism gives a different account of the *truth-conditions* of mathematical statements. Platonists understand a singular mathematical statement like "2 is even" as being true when there exists some abstract object, the number 2, and it has a certain property, *evenness*.

As we saw above, I understand mathematical truth differently.[178] I will briefly reiterate that account. On immanent structuralism, what actually makes a mathematical statement like "a is F" true is when the property *being F* is part of the essence of the structural property *being a*.

In the case of "2 is even," this statement is true when the property of "being 2" (which I claim can be understood as a purely structural property)[179] has "being even" (also a purely structural property) as *part of its essence*.

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[177] See Franklin (2014) *A Neo-Aristotelian Realist Philosophy of Mathematics.*

[178] The full account was given in Chapter 2 above.

[179] Namely, the property of *occupying a certain place in a successor series*. As I argue in Chapter 2 above, this is indeed a purely structural property, and can ultimately be reduced to one definable entirely in terms of part, whole, sameness, difference, and purely logical vocabulary.

To refresh the reader on what I mean by one property's being "part of the essence" of another, consider a non-mathematical example. Take the statement "Dogs are mammals." Arguably, this is an *essentialist* truth about dogs. This statement is true because *being a mammal* is "part of" the property of *being a dog*. Being a mammal is part of *what it is to be* a dog. We can use a special symbol, "⋐" to denote this "property-parthood" relation:

being a mammal ⋐ *being a dog*

In a similar way, a statement like "0 has a successor," which is shorthand for the purely structural formulation, "The thing that is not a successor (the number zero) has a successor," asserts:

having a successor ⋐ *being the object that is not a successor*

I understand "property parthood" statements like these to just be equivalent to statements about the essences of the relevant properties:

"*Being a mammal* ⋐ *Being a dog*" is true iff *being a mammal* is essential to (is "part" of the essence of) *being a dog*.

Thus, if my theory of mathematical truth is right, then *mathematical theorems express essential truths about the natures of mathematical structural properties*. I think this is the key to providing a structuralist explanation of *de re* mathematical necessities.[180]

Considering that true mathematical propositions express essential truths about mathematical *properties*, and that these properties, being purely structural, can be literally

[180] As a side-note: We can now make the distinction between *de re* mathematical necessities and the theorems of mathematics. *De re* mathematical necessities are ordinary physical generalizations with a certain necessity operator attached, viz., mathematical necessity. Mathematical *theorems*, on the other hand, are ultimately essentialist statements about universals.

instantiated by physical systems, the necessity attaching to the essences of these properties comes to be had by their physical instances. This produces a strong *de re* mathematical necessity. Allow me to illustrate.

To see how this works, let's consider a non-mathematical case where the necessity of the essentialist facts transfers to the physical facts. Take my chihuahua, Paco, in all his mammality. Why is it the case that he *must* be an animal, given that he is a mammal? How do we explain the *de re* necessity that *Paco is an animal if he is a mammal*?

Well, we can explain the necessity of Paco's being an animal given that he is a mammal by virtue of the fact that it is necessary that *all mammals are animals*. But then how do we explain the necessity of *this* general fact, that all mammals are animals?

We can explain the necessity of this by the fact that *part of being a mammal* is *being an animal*. That is, being an animal is *essential to* being a mammal. And since whatever is essentially true is also metaphysically necessary,[181] it is *metaphysically necessary* that all mammals are animals. In short, we have the following route from essential truths to explaining necessary generalizations, as well as general and particular sub-nomic facts:

[Essentially: (x)p] --> [metaphysically necessarily: (x)p] --> (x)p and p(a)

So, I can explain the necessity of Paco's being an animal given that he is a mammal in this way. Similarly, going back to mathematical *de re* necessities, we can explain why the Konigsberg bridges *cannot* be crossed once each given their arrangement: It is because the

[181] Cf. Fine (1994).

Konigsberg bridges instantiate the mathematical property of being a K-graph,[182] and, as Euler's theorem has shown us, *having no Eulerian path* is part of the essence of *being a K-Graph*.[183]

Or, to take another example, suppose I am sorting through a bunch of drawings. Why is it that I *must* find one that is symmetrical about the point of intersection whenever I find one that is symmetric about both axes? Or why is it that, no matter what way the wind blows, it *cannot* cover a sphere of radius 1 meter by 12 square meters of material?

On the immanent structuralist explanation, it is because *being symmetric about the point of intersection* is part of the essence of *being something symmetrical about both axes*. And *not being covered by 12 square meters* is essential to *being a sphere of radius 1 meter*. The necessity that derives from the essences of these universals – *being symmetrical about both axes* and *being a sphere of radius 1 meter* – is "transferred" to whatever objects have them.

To put it somewhat imagistically, we can think of the mathematical universals as creating necessities from the "inside out" rather than the "top down." The universals are metaphysical "components" of various physical objects, and by being there they generate necessities in them. Whenever these mathematical universals enter the world, making their habitations in various systems of physical objects, they, like other universals, bring their essences, and their attendant necessities with them.

In sum: If I am correct that mathematical theorems express essential truths about mathematical properties, and that these mathematical properties are purely structural and therefore directly instantiable by physical systems, then we can explain *de re* mathematical

[182] Just as Paco instantiates *being a mammal*.

[183] Just as *being an animal* is part of the essence of *being a mammal*.

necessities in the same way we sometimes explain other physical necessities, viz., by appeal to the essences of various properties and kinds.[184]

To bring out our solution out more fully, let us consider how the immanent structuralist explanation of *de re* mathematical necessities fares when confronted with the problems Platonism faced.

Objection 1: Pushing the Question Back

For any Platonist explanation of *de re* mathematical necessities to get off the ground, Platonic objects must be necessary beings, and their mathematical properties must be had necessarily. But we saw that Platonists have some difficulty explaining *why* mathematical objects must be necessary, and why the mathematical properties they have are necessary.

The immanent structuralist on the other hand does not need to posit any necessary objects. Instead, she can appeal to the essential truths and take these as basic.[185] And it is straightforward for the immanent structuralist to explain why it is metaphysically necessary that, say, the number 2 is even – unlike the Platonist, who seems to posit it as a brute fact. This truth can be *explained* and *made intelligible* in terms of the essence of the relevant mathematical structural property, viz., *being the successor of 1 in a Peano series*.

Objection 2: Getting the Right Grade

[184] It should be noted that Platonists cannot take advantage of the essentialist strategy I use here. On Platonism, you might argue that the relevant theorem of graph theory is necessary because having no Eulerian path is of the essence of *the abstract graph-theoretic objects* mentioned in the theorem. However, there is no direct route from this to explaining why *those physical bridges over there* cannot have a direct path that crosses each bridge once. See Objection 3 of the previous section.

[185] Which, presumably, the Platonist will *eventually* have to do as well, since the best Platonist explanation appeals to the essences of the *mathematical* objects. See the discussion of Objection 1 in Section 6.3 above.

How about the issue of the grade of necessity? Take again our theorem (2'):

2'. The Euclidean plane has no tiling by regular equal pentagons.

Again, Platonists have difficulty explaining why mathematical theorems like (2) are necessary with *mathematical* necessity – which is stronger than mere *metaphysical* necessity – since they must explain why Platonic abstracta exist with a stronger sort of necessity than other abstracta.

Immanent structuralists get a stronger grade of necessity for free. From the structuralist perspective, the prospects of tiling my floor with pentagons are about as bad as trying to make hydrogen-less water, or of trying to breed the first mammal that is not an animal.[186] Assuming these things really are inconsistent with the essences of *water* and *mammal*, these tasks should not only strike one as bold, hubristic, or even just outside the bounds of what is physically possible, but also as *incoherent* and *absurd*. Even if, like some of the saints, you had God's ear, so that He would be willing to change the laws of physics for you, He would have to disappoint you in your request for hydrogen-less water.[187]

I would suggest it is the same sort of incoherence at play in these non-mathematical cases as in the impossible situations posited in our discussion of the *de re* mathematical necessities.

Objection 3: The Kripkean Objection (the "Humphrey" Worry)

[186] That is, assuming you agree that hydrogen is part of the essence of water, and that being an animal is part of being a mammal. If not, the reader may pick his or her preferred example of an essential truth.

[187] Even Descartes – perhaps the most thoroughgoing theological voluntarist of all the scholastic philosophers – did not think God could *change* the essences of things, but only that God could make certain essential truths to *have no truth value*. God could make it fail to be the case that water is essentially H2O, but not that there could actually *be* water with a composition *other* than H2O.

It is clear how the structuralist view avoids the Kripkean objection that the Platonist faced. If one grants that the essential truths about gold can explain the necessity of the bare physical *generality* “All gold we might find in the world has a proton number of 79, and none will be found without it,” then the structuralist explanation for *de re* mathematical necessities will be exactly the same. Thus, no worry similar to the Humphrey objection seems to arise, because the structuralist explanation does not appeal to some special class of causally inefficacious, spatially disconnected, and existentially apart abstract objects. The *de re* mathematical necessities are instead grounded in the properties that are *actually in* the physical systems that instantiate them.

Furthermore, we can see more clearly than the Platonist the connection between the *de re* mathematical necessities and the relevant mathematical theorems. Consider the following *de re* mathematical necessities 1 – 5, and the corresponding *mathematical* facts, 1’ – 5’:

1. It is impossible for someone to walk over the Konigsberg bridges exactly once each.
    - 1’. Because there is no such path through a K-graph.
2. This room’s floor cannot be tiled with regular, equal pentagons.
    - 2’. Because the Euclidean plane is not tiled by regular, equal pentagons.
3. Necessarily, any physical body that is symmetric about both axes is symmetrical about the point of intersection of the axes. (Franklin 2014)
    - 3’.Because a similar theorem is true of any symmetric *figure*.
4. Five apples cannot be equally divided among three children. (Braine 1972)
    - 4’. Because the number 5 is not divisible by the number 3.
5. This trefoil knot cannot be unknotted without cutting. (Lange 2017)

- 5’. Because a trefoil knot in three-dimensional space is not isotopic to the unknot.

The relevant *mathematical* statements 1’ – 5’ should be interpreted as telling us something about the essences of the relevant mathematical *universals*. They tell us about the “clothes” that the various physical objects and systems might put on. The *de re* mathematical necessities in 1 – 5 tell us about what happens (and, in virtue of 1’-5’, what *must* happen) to the actual physical objects *when they wear them*. Here, the connection between the mathematical fact and the de re necessity in the physical world is made perspicuous.

5. Conclusion

Let us summarize our discussion. In Section 1, I argued that there are *physical* facts that carry *mathematical* necessity. These are called *de re* mathematical necessities. In Section 2, I argued that this “mathematical necessity” is indeed a genuine, strong grade of necessity, stronger than both physical and metaphysical necessity. In Section 3, I explained and critiqued possible Platonist attempts to explain *de re* mathematical necessities. Finally, in Section 4, I illustrated how our alternative account of mathematical ontology and truth – immanent structuralism – handily explains cases of *de re* mathematical necessity. I will close with a few brief remarks on some of the implications of this discussion.

CHAPTER 7: CONCLUSION: LESSONS FOR ANALYTIC PHILOSOPHY AND METAPHYSICS

Introduction

At this point, I have provided an account of the ontology, semantics, and epistemology of immanent structuralism. I have also illustrated some of the applications of the view, specifically to mathematical reduction, mathematical modality, and mathematical explanation. In this concluding chapter I would like to reflect briefly on a few implications of our theory for broader debates about the methodology of philosophy of mathematics, as well as metaphysics and ethics.

Sections 1 and 2 will discuss indispensability arguments in mathematics and analytic ontology, in both their classic as well as more recent forms. My hope is that immanent structuralism can shed light on why these arguments do not work. However, more recent versions of the indispensability argument will be seen to fail for different reasons than older versions.

In Section 3 I will discuss some parity arguments that ethicists have given to defend non-naturalistic accounts of the epistemology and ontology of ethical properties. These parity arguments appeal to the alleged need in mathematics for a rationalistic faculty of pure *a priori* intuition. I argue immanent structuralism demonstrates that these arguments fail.

Finally, in Section 4, I summarize what I take to be some of the key contributions of the dissertation. In particular, I try to highlight a few places where I have tried to move the ontology and ideology of structuralism forward. I then briefly note some of the ideas we've discussed that might provide avenues for further research among philosophers of mathematics and metaphysicians.

1. Classic Quinean Arguments in Mathematics and Metaphysics

In this section I will briefly discuss how immanent structuralists should think about indispensability arguments. Although immanent structuralism can be considered a "realist" theory in the sense that it takes mathematics to be true, objective, and to have some sort of ontological explanation, its realism still is rather different from the realism of Quine, Putnam, and other Platonists.

In the classic form proposed by Quine and Putnam, the indispensability argument for mathematical objects is rather simple, and can be summarized as follows:[188]

- (1) We ought to be ontologically committed to whatever entities are indispensable to our best scientific theories. (Premise)
- (2) Mathematical entities are indispensable to our best scientific theories. (Premise)
- (3) Therefore, we ought to be ontologically committed to mathematical entities. (Conclusion)

[188]See Quine (1949/1980) and Putnam (1979) for the classic discussion. See also Colyvan (2019) for an overview.

The justification for the first premise is partly based on the theory of ontological commitment put forward by Quine. For Quine, "to be is to be the value of a variable." Therefore, when the scientific theories we hold to quantify over something, that is *ipso facto* to be committed to those entities.

The route from this view of ontological commitment to Premise 1 is most straightforward if you are a realist about science. In that case, our "best scientific theories" are just true. And if they are true, then whatever is quantified over in those theories exists – since "to exist" is to be the value of a variable.[189]

It is generally argued in favor of the second premise that there is no way to rid our successful scientific theories of mathematical discourse without unacceptably compromising the theory in regards to its simplicity, explanatory power, or other theoretical virtues.

Of course, some nominalists will get off board at this point, claiming we can do the relevant science perfectly well without the mathematical discourse.[190] I will not assess these attempts here. At this point I would only like to flag a couple of ways immanent structuralists might interact with the argument.

First, it is interesting to note that we can accept the first premise without issue. An immanent structuralist could, at least in principle, agree that the entities our best scientific theories commit us to should be accepted. However, immanent structuralists want to reject the

[189]Things get trickier when we recognize the fact that our best theories probably are *not* true – at least not quite. While I am not primarily concerned to resolve this problem here, others have recognized and provided thorough discussion of the issue. See, e.g., Lewis (1970). Suffice it to say we probably can get an indispensability argument going even when we are dealing with approximately true theories.

[190]Field (1980) is the classic nominalist attempt.

conclusion, since we want to avoid the commitment to abstract mathematical entities. So the trouble seems to be in the second premise.

Ultimately, this trouble goes back to Quine's criterion of ontological commitment. As we saw in Chapter 2, even though mathematical discourse contains statements whose superficial syntax is quantificational and includes definite descriptions, immanent structuralists interpret this language in a certain way. We claim that these statements can be given sufficient truth-conditions in terms of purely structural *properties* without the need to commit ourselves to specifically mathematical *objects*. If the theory proposed in Chapter 2 is successful then, Premise 2 of the indispensability argument ought to be rejected. Mathematical *entities* are not indispensable to our best scientific theories after all. All we need are mathematical *properties*. Indeed, as the next section will try to show, the case for the indispensability of mathematical *properties* is far stronger than the case for mathematical *objects*.

2. Modified Indispensability Arguments: Explanatory Arguments

While the classic indispensability argument continues to exert an influence, other more sophisticated forms of argument have been developed in recent years, partly in response to worries about what "indispensability" amounts to. In particular, some have taken note of the fact that mathematics seems to figure in a special way in certain *scientific explanations*. From these cases of mathematical explanations of scientific phenomena some have constructed modified "indispensability" or "best explanation" arguments for the existence of mathematical entities.

One famous case of alleged mathematical explanations in science is the example of the cicadas popularized by Baker (2005).[191] Mark Colyvan summarizes the case nicely:

> One example of how mathematics might be thought to be explanatory is found in the periodic cicada case (Yoshimura 1997 and Baker 2005). North American Magicicadas are found to have life cycles of 13 or 17 years. It is proposed by some biologists that there is an evolutionary advantage in having such prime-numbered life cycles. Prime-numbered life cycles mean that the Magicicadas avoid competition, potential predators, and hybridisation. The idea is quite simple: because prime numbers have no non-trivial factors, there are very few other life cycles that can be synchronised with a prime-numbered life cycle. The Magicicadas thus have an effective avoidance strategy that, under certain conditions, will be selected for. While the explanation being advanced involves biology (e.g. evolutionary theory, theories of competition and predation), a crucial part of the explanation comes from number theory, namely, the fundamental fact about prime numbers. Baker (2005) argues that this is a genuinely mathematical explanation of a biological fact. There are other examples of alleged mathematical explanations in the literature but this remains the most widely discussed and is something of a poster child for mathematical explanation.[192]

Of course, there are questions that might be raised about the philosophical implications of the example. Indeed, it is possible to question whether the mathematics really plays an essential explanatory role in the case at all. But rather than adjudicate those questions, let us suppose the

[191]Indeed, it would seem that no discussion of mathematical explanation would be complete without it. Perhaps we can even say it was necessary *de re* of this dissertation that it mention it. (At least, according to a certain sense of "necessity.")

[192]Colyvan (2019), Sec. 5.

mathematics *does* play a role in the explanation here. What can we infer from this about the status of mathematical entities?

Some have taken this as an occasion to present a new twist on the indispensability argument. Rather than getting entangled in the Quinean aporia of “best scientific theories” and the relationship between existence and quantification, the modified argument simply appeals to the presence of mathematical entities as theoretical posits in the relevant mathematical explanations. Paolo Mancosu summarizes how these sorts of arguments go as follows:[193]

- (1) There are genuinely mathematical explanations of empirical phenomena.
- (2) We ought to be committed to the theoretical posits postulated by such explanations.
- (3) We ought to be committed to the entities postulated by the mathematics in question.

As we have seen in Chapter 6, the immanent structuralist has something to say about this sort of argument. First off, the idea in Premise 1 of mathematical explanations of empirical phenomena fits nicely with the idea that there are *de re* mathematical necessities.

Take the case of the cicadas for instance: Arguably, what is happening here is that the life cycles – which are concrete cycles of time – are instantiating various number properties. Some instantiate the number 13, others instantiate the number 17. Moreover, it is part of the essence of the numerical properties *being 13* and *being 17* that numbers other than them cannot be multiplied so as to get these numbers as a result. That is to say, they are prime. When the life cycles instantiate these properties, then, it is *not possible* for life cycles of a different number to

[193]Mancosu (2008), p. 137.

be synchronized with them. In other words, the cicadas are taking advantage of a *de re* mathematical necessity! The fact about prime numbers at least *partly* explains the situation because it *makes necessary* the fact that life cycles of different numbers do not synchronize with the life cycles of the cicadas.

However, as we saw in Chapter 6, not only do mathematical *entities* not seem to play a role in this “making necessary,” it is not at all clear how they *could* do so. Abstract, causally inert numbers do not seem to make anything necessary at all. On the other hand, the essences of the relevant mathematical properties *do* help make certain things necessary, and so they are ideally suited to play a role in the explanation.

The lesson to draw from this is that even modified indispensability arguments based on mathematical explanations do not have sufficient justification for the second premise. What the argument from Chapter 6 seems to imply is that mathematical entities not only are not necessary for mathematical explanations of scientific phenomena, but it is difficult to even see how they *could* play a role given their isolated and abstract nature.

3. Parity Arguments in Non-Naturalist Ethics

A final broader debate that I would like to touch on occurs in ethics. This is the debate about ethical non-naturalism and ethical intuitionism. According to ethical non-naturalists, ethical properties like goodness, duty, obligation, and so on, are *non-natural* properties: They cannot be reduced to anything physical or natural. This raises the question of how we are able to gain ethical knowledge. If ethical properties are non-natural, then presumably they cannot be accessed in the way any other ordinary natural property can, such as through perception. So how are they accessed?

To answer this question, ethical intuitionists propose that we have a faculty of *moral intuition* that allows us to grasp the ethical properties.[194] This is understood to be a non-perceptual rational faculty that gives us direct (albeit fallible) access to the non-natural ethical facts.

Naturally, one worries that it is difficult to see how such a faculty could be reliable or naturalistically explicable. On the face of it, positing an *a priori* faculty that gives us a "direct grasp" on non-natural features of the world seems somewhat mystical. To put the worry a little more precisely, insofar as there is no obvious causal or other necessary connection between non-natural properties and any faculty we might have, a faculty of this sort is not likely to be reliable.

Now, a number of responses can be given on behalf of the ethical non-naturalist here. I certainly do not intend here to adjudicate the overall question. However, I would like to flag one particular response that ethical intuitionists have given to this objection against their non-naturalism. And that is to make a *parity* argument.

The strategy of intuitionist parity arguments is not necessarily to give a complete story of how our *a priori* faculty of moral intuition can be reliable. Instead, the response is to point out the parity between our *a priori* moral faculty and *other a priori* faculties we might have. If there is nothing particularly troubling about the latter faculties, then there cannot be anything wrong *in principle* with the former. For there is nothing that makes the one kind of *a priori* knowledge special but not the other.

[194] Among the most prominent treatments of ethical intuitionism are Michael Huemer's *Ethical Intuitionism* (2005) and Russ Shafer-Landau's *Moral Realism* (2003). Both of these theorists are ethical non-naturalists.

Michael Huemer in his popular book *Moral Intuitionism* brings out the point by comparing moral knowledge with our knowledge of mathematics. In both cases, Huemer contends, we must rely on our rational perception of at least certain basic facts. The only difference, he claims, is in the objects of our rational perception:

> It is important to remember that intuitionists do not say that we have a 'special moral sense', that is, a separate faculty dedicated solely to cognizing moral truths. Intuition is a function of *reason*. Moral intuition differs from mathematical intuition in the way that perceptions of cars differ from perceptions of trees – that is, merely in having different objects. Thus there is no special explanation required for … 'the faculty of moral intuition'[195]

In a similar way, Russ Shafer-Landau makes an argument based on the parity between ethics and metaphysics. He counters critiques of the notion of "intuition" as follows:

> Yet this criticism, if successful, is sufficient to eliminate the justification we might have for *any* of our philosophical beliefs. … In philosophy, as in ethics, we rely very heavily on intuitions and considered judgments to adjudicate between conflicting claims. Try doing modal metaphysics or analytic epistemology without such convictions. It just doesn't seem possible.[196]

Yet despite Huemer's assertions of a parity between ethical reasoning and mathematics, what I have tried to show in the previous pages is that the case of mathematics is *not* like Huemer's moral intuitionism at all. On the contrary, mathematical knowledge is far more similar

[195]Huemer (2005) pp. 215-16.

[196]Shafer-Landau (2003) p. 220.

to the type of knowledge we gain about ordinary natural properties through perception. Admittedly, mathematics goes somewhat beyond this, in that it allows for us to define new mathematical properties and infer things from those definitions. But to that extent it appears *even more* to be *unlike* ethical reasoning, which does not purport merely to deduce from what has been explicitly defined by logical operations, but rather to discover substantive ethical truths.

Similarly, Shafer-Landau's argument for parity between non-naturalistic ethics and discussions about modal metaphysics may hold in some cases. But I hope to have also illustrated some cases where it might not, viz., when we are considering essential truths about properties that we have gained through knowledge by perception.

In short, I believe the discussion in the previous pages strongly shows that parity arguments between ethical intuitionism and mathematics fail. Also, parity arguments between ethical intuitionism and modal metaphysics may fail in some cases too. If anything, if there *were* a strong parity between mathematical and ethical reasoning, this would support a more naturalistic and broadly *Aristotelian* picture of ethics, where we gain knowledge of ethical properties through experience and by abstraction from concrete examples.

4. Immanent Structuralist Metaphysics: Future Hopes

Before concluding, I would like to highlight a few of the goals I have tried to accomplish in the previous pages.

First, I have tried to separate mathematical *truth*, *objectivity* and *necessity* from mathematical *objects* or *entities*. In my opinion, there is no reason why in order to have *mathematical realism* we must also accept *object realism*. It is hard to see why, if we have all the mathematical properties we need together with their natures, we would then also need specifically mathematical *objects*. Why shouldn't a plenum of mathematical properties be enough to ground the truths of mathematics? What *more* could be gained by positing, in addition, mathematical entities?[197] What is lost without them?

At this point in our discussion, discarding the mathematical objects as an unnecessary appendage may seem obvious. Who needs them? I am glad if by this point it *does* seem obvious. However, for the majority of the history of philosophy of mathematics in the last hundred and fifty years or so, this point has not been considered obvious at all. Mathematical entities seem constantly to draw mathematical realists to themselves. Even structuralists have tended for the most part to develop their theories in a Platonist direction, taking talk of "places" in "structures" quite literally, and it is frequently difficult when considering structuralism to think beyond this mindset. If I have made this paradigm seem somewhat less inevitable in the reader's mind, I will take that as an accomplishment. I hope that the ontological, semantic and epistemological apparatus developed in previous chapters has succeeded to some degree in this goal.

I also believe the discussion in the previous pages opens up a number of avenues for further research. Throughout this work, I have at various places tried to tease out metaphysical distinctions that perhaps have not received enough attention or else are only beginning to do so. In particular, I believe the following topics which have been touched on here might be worth investigating further in the future:

- The ontology and semantics of specifically *essentialist* pattern statements

[197]Note that mathematical entities are, indeed, an *addition*, because mathematical entities *themselves* will have to instantiate mathematical properties.

- The metaphysics of property parthood, and the problems that face the literal account
- Essentialist accounts of property parthood and intensionalist semantics based on these accounts
- *Referentialist* metaphysics of essences vs. *primitivist* metaphysics of essences
- How the epistemology of simple mathematical essences might engage with the cognitive science literature on mathematical education
- The functions of mathematical reductions when these reductions are considered instances of "treating-as"
- The notion of *de re* mathematical necessity
- The connection between essences, *de re* mathematical necessities, and scientific explanationsHow the metaphysics of mathematical properties interacts with mathematical explanations in science

I have tried to make progress on a number of these topics, but certainly there is more to be investigated. I hope the discussion here will help open up that research.

Ultimately, I hope that the work I have done on immanent structuralism shows it to be a plausible and powerful alternative to the more dominant Platonist and structuralist theories that have been proposed. And even if one does not buy it in the end, I hope the words of Giordano Bruno may at least ring true of it: "Se non è vero, è ben trovato."[198]

[198]"Even if it is not true, it is well conceived."

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