Fermion Mixings in SU(9) Family Unification

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Abstract

In an SU(9) model of gauged family unification, we propose an explanation for why angles observed in the lepton flavor (PMNS) mixing matrix are significantly larger than those measured for any analogous quark flavor (KM) mixing angle. It is directly related to a see-saw mechanism that we assume to be responsible for the generation of neutrino masses. Our model is more constrained and therefore even more predictive than a model previously proposed by Barr.

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Introduction.

When grand unified theories (GUTs) of strong and electroweak interactions were at their most popular (from the 1970s through the 1980s) non-zero neutrino masses had been predicted but not experimentally established. Thus any effort to address the neutrino mixing issue was not then motivated for want of empirical data.

Nevertheless, by resurrecting ideas about family unification which were suggested at that time, in the present note we shall show that the unexpected nature of neutrino mixing, especially that two of the neutrino mixing angles are substantially larger than any of the quark mixing angles, can be explained.

The idea of family unification is to embed a one-family GUT based on gauge group $G'$ in a three-family GUT based on a bigger gauge group $G$, usually with $G' \subset G$. Sequential family replication in $G'$ arises from the complex representation $\rho$ of $G$ and its decomposition $\rho \rightarrow \rho'$ under $G \rightarrow G'$.

Minimal SU(9) model of family unification

By far the simplest family unification model \#3 is the SU(9) model suggested in 1980 by one of the present authors [1].

Since the low energy fermions are chiral, the representation $\rho$ of $G$ must be arranged to be anomaly free. This is straightforward especially if one restricts attention to the totally antisymmetric irreps $[k]$ of rank $k$ for $SU(N)$ as is necessary to avoid color irreps bigger than triplet and antitriplet. Normalizing the anomaly $A(N, k)$ such that $A(N, 1) = 1$ the values of $A(N, k)$ satisfy a generalized Pascal triangle with rows labeled by $N$ (the first row is $N = 3$) and columns by $k$ values $1 \leq k \leq N$:

\[
\begin{array}{cccccccc}
1 & -1 \\
1 & 0 & -1 \\
1 & 1 & -1 & -1 \\
1 & 2 & 0 & -2 & -1 \\
1 & 3 & 2 & -2 & -3 & -1 \\
1 & 4 & 5 & 0 & -5 & -4 & -1 \\
1 & 5 & 9 & 5 & -5 & -9 & -5 & -1 \\
\end{array}
\]

and so on ad infinitum.

\#3 A more general family unification scheme is in [2]. Other $SU(N)$ models are in [3–14]
Family unification for three families requires that when the irreps \((N, k)\) contained in \(\rho\) are decomposed with respect to the standard model gauge group \(G_{SM} \equiv SU(3) \times SU(2) \times U(1)\) then \(\rho \rightarrow \rho'\) under \(G \rightarrow G_{SM}\) leads to a \(\rho'\) with three chiral families up to possible \(G_{SM}\)-singlets and real representations of \(G_{SM}\). These real representations can all pair up into Dirac mass terms and generally acquire masses inaccessible to present colliders. We shall refer to all such states as superheavy.

The appropriate decomposition is facilitated by employment of \(SU(5)\) which is equivalent for the group theory to the standard model \(^4\) and the decomposition then involves binomial coefficients \(C_q^M \equiv N![q!(N - q)!]^{-1}\) according to

\[
\rho \equiv (N, k) \rightarrow \rho' \equiv \Sigma_p C_p^{N-5}(5, k - p)
\]

There is one more issue, cancellation of anomalies in \(\rho\), which automatically implies anomaly-freedom and complete families in the standard model. Writing \(A(N, k)\) to represent the chiral anomaly of representation \((N, k)\) normalization such that the defining representation \((N, 1)\) has anomaly \(A(N, k) = +1\) one employs the well-known result

\[
A(N, k) = \left[ \frac{(N - 3)!(N - 2k)}{(N - k - 1)!(k - 1)!} \right]
\]

which yields the generalized Pascal triangle mentioned above. Anomaly freedom for \(\rho \equiv \Sigma_p B_p(N, p)\) is assured if and only if \(\Sigma B_p A(N, p) = 0\).

The result of searching the possible \(\rho\) for each \(N\) leads to the conclusion reached three decades ago that the simplest family unification occurs for \(N = 9\) and the model is \([1]\)

\[
(9, 3) + 9(9, 1)^* \tag{3}
\]

which may be rewritten with \(N = 9\) suppressed as \([5]\)

\[
[3] + 9[1]^* \tag{4}
\]

with dimensions \(84 + 9(\bar{9})\)'s. One may perhaps best mnemonicize it by

\[
9^3 + 9(\bar{9}) \tag{5}
\]

\(^{#4}\)This does not imply that \(SU(5)\) is a good symmetry at any energy.

\(^{#5}\)Recall that \([k]\) denotes a totally antisymmetric \(k^{th}\)-rank irrep of \(SU(9)\).
When this $SU(9)$-model was built in 1979, there was little experimental evidence for neutrino mass and absolutely nothing was known about PMNS mixing. At present comparably as much is known about PMNS as about the much longer-studied KM quark mixing matrix.

In what follows we shall use the $SU(9)$-model to attempt to explain the significant difference between observed lepton and quark mixings.

**Group Theoretic Representations**

Let us examine more carefully the fermion fields extant in the $SU(9)$-model. The group-theoretic bookkeeping is facilitated by the use of a $SU(5)$ subgroup of $SU(9)$, yet at any point we may rewrite in the standard model $SU(3)_C \times SU(2)_L \times U(1)_Y$ subgroup and there is no implication whatsoever that $SU(5)$ is a symmetry or that the additional twelve cofactor gauge bosons exist physically. I.e., we could reduce the $SU(9)$-model to a $SU(3)_C \times SU(2)_L \times U(1)_Y \times SU(4)_{family}$ model. However, keeping the full $SU(9)$ imposes constraints among coupling constants and other parameters.

We introduce the notation $A, B, C, .. = 1 - 9$ for $SU(9)$; $I, J, K, .. = 1 - 5$ or $SU(5)$; and $k, l, m, .. = 1 - 4$ for the $SU(4) \subset SU(9)$ which commutes with the $SU(5) \subset SU(9)$.

The $(9, 3)$ fermions in Eq.(3) can thereby be rewritten

$$\Psi^{ABC} \equiv \Psi^{IJK} + \Psi^{IKk} + \Psi^{Ikl} + \Psi^{klm}$$

while the $9(9, 1)^*$ fermions of Eq.(3) become

$$9\Psi_A \equiv 9(\Psi_I + \Psi_k)$$

The quarks and leptons of the three-family standard model and their respective mixings are not difficult to read off from Eqs. (6) and (7).

To keep track of quarks and leptons, we first recall the locations of the flavor eigenstates in the first fermion family. The second and third families are *mutatis mutandis*.

We denote the three QCD colors as Red (R), Green (G) and Blue (B).

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\#6The smallest appropriate irreducible representation of $SU(N)$ occurs for $N = 6$ with dimension [15–17] greater than $3 \times 10^5$ so the reducibility in Eqs. (3–15) is inevitable.
Of these fifteen chiral fermions, ten are located in

$$\Psi^{IJ} \equiv \begin{pmatrix}
0 & \bar{u}^B & -\bar{u}^G & -u^R & -d^R \\
-\bar{u}^B & 0 & \bar{u}^R & -u^G & -d^G \\
\bar{u}^G & -\bar{u}^R & 0 & -u^B & -d^B \\
u^R & u^G & u^B & 0 & -e^+ \\
d^R & d^G & d^B & e^+ & 0
\end{pmatrix}$$  \hspace{1cm} (8)

and the remaining five are in

$$\Psi_I \equiv (\bar{d}^R, \bar{d}^G, \bar{d}^B | \nu_e, e^-)$$  \hspace{1cm} (9)

In Eqs. (8) and (9) the $SU(3)_C$ and $SU(2)_L$ factors of the standard model gauge group are indicated.

Weak hypercharge $Y$ defined by $Q = (T_{L3} + \frac{1}{2}Y)$ corresponds to the historically normalized $SU(5)$ generator

$$Y = \text{diag} \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, 1, 1\right)$$  \hspace{1cm} (10)

**Quark Mixings**

Keeping renormalizability, hence avoiding irrelevant operators, quark mixings will arise from two Yukawa couplings

$$\lambda \Psi^{ABC} \Psi^{DEF} H^{GHI} \epsilon_{ABCDEFGHI}$$  \hspace{1cm} (11)

and

$$\lambda'_{(b)} \Psi^{ABC} \Psi^{(b)}_A H_{BC}$$  \hspace{1cm} (12)

where indices in parentheses run over the nine $\bar{9}$s. At the $SU(5) \times SU(4)$ level, these terms reduce to

$$\lambda \Psi^{IJ} \Psi^{KLh} H^{Mcd} \epsilon_{IJKLM} \epsilon_{abcd}$$  \hspace{1cm} (13)
and

$$\lambda'_{(b)} \Psi^{IJa} \bar{\Psi}^{(b)} H_{Ja}$$

(14)

where $\hat{b} = 1, 2, 3$ is a family label for those $\bar{\Psi}$s of $SU(5)$ that stay light.

When the Higgs doublet acquires a VEV in the neutral component $H_4$, there result mass matrices $U$ and $D$ for up- and down- quarks respectively. One way to proceed is to identify the flavor and mass eigenstates for the up-quarks then diagonalize

$$D \equiv DD^\dagger$$

(15)

by

$$V_{KM}^\dagger D V_{KM} = \text{diag}(M_b^2, M_s^2, M_d^2)$$

(16)

To zeroth order in the Cabibbo angle $\beta = \sin \Theta_{12} \sim 0.22$, the $KM$ matrix $V_{KM}$ is a unit matrix. The three mixing angles in $V_{KM}$ are all small, not more than 0.22 radians.

Thus, the Yukawa couplings in Eq.(13) and Eq.(14) are approximately diagonal in family space. Although it could have been otherwise, Nature chooses Yukawa couplings with this property.

The big question is: once we accept that neutrinos are massive and there exists a lepton counterpart $V_{P_{MNS}}$ to the quark $V_{KM}$ mixing, why is it not similar to $V_{KM}$?

When the neutrino angles $\theta_{ij}$ were measured it seemed a surprise to many in the theory community that the atmospheric and solar angles, $\theta_{23}$ and $\theta_{12}$, are larger than $\Theta_{12}$. In retrospect, could this have been a anticipated?

**Lepton Mixings**

The Yukawa coupling in Eq.(14) gives a mass matrix for the charged leptons

$$L_{(\hat{b})} = \lambda'_{(\hat{b})} < H_4 >$$

(17)

For the neutrinos we adopt the idea that they have Majorana (not Dirac) masses. Then there must be right-handed neutrino fields $N^R_n$ where the label $n$ is usually taken to be $n = 1, 2, 3$ one for each family. It is also possible that there are only two right-handed
neutrinos $n = 1, 2$ that contribute to the sea saw with the consequence that one mass eigenvalue vanish.

The neutrino mass matrix $M_{\nu}$ is assumed to arise [18] from a see-saw mechanism

$$M_{\nu} = M_D M_N^T M_D^T$$ (18)

The underlying couplings are

$$[(M_D)_{an} \nu^a N_R^n + h.c.] + (M_N)_{nn'} N_R^n N_R^{n'}$$ (19)

It is general to adopt a basis where the charged leptons have degenerate flavor and mass eigenstates whereupon the lepton mixings $V_{PMNS}$ are contained in

$$V_{PMNS}^\dagger V_{PMNS} = \text{diag}(m_3^2, m_2^2, m_1^2)$$ (20)

where we have introduced the matrix $\mathcal{N}$ by

$$\mathcal{N} \equiv M_{\nu} M_{\nu}^\dagger$$ (21)

Of course, we know the answer for $V_{PMNS}$ but let us objectively scrutinize the see-saw in Eq.(18) with the repeated index summation convention

$$(M_{\nu})_{ab} = (M_D)_{an} (M_N^T)_{nn'} (M_D^T)_{n'b}$$ (22)

and for general $(M_N^T)_{nn'}$ which involves arbitrary mixing between the $N_R^n$, there is no reason for $(M_{\nu})_{ab}$ to be approximately diagonal. Consequently there is every reason for the lepton mixings in $V_{PMNS}$ to be of order one, not small like the quark mixings in $V_{KM}$.

In the $SU(9)$-model, according to Eqs.(6,7) there are as many as fourty chiral fermions without 3-2-1 charge. The right-handed neutrinos $N_R^n$ with $n = 1, 2$ are among these. All are superheavy but only two participate in the see-saw mechanism with the three light left-handed neutrinos.

We note that without breaking $SU(4)$ eight states can be paired up according to

$$\Psi_{klm}^k \Psi_n \epsilon_{klm}^n$$ (23)

Once $SU(4)$ is broken to $SU(2)$ all of the remaining thirty-two singlet chiral fermions can successfully acquire superheavy masses.
Larger Mixing Angles for Leptons than for Quarks

In $SU(9)$ the qualitative difference between neutrino and quark mixings arises from the different $SU(4)$ dependences appearing in Eqs. (13) and (14), for quarks, which are all $SU(4)$ non-singlets and in Eq. (23), for right-handed neutrinos, which are all $SU(4)$ singlets.

As already pointed out after Eq. (16), the matrix $V_{KM}$ has small off-diagonal elements corresponding to the fact that Nature chooses simple Yukawa couplings for quarks which are almost flavor diagonal. For leptons, on the other hand, the see-saw mechanism Eq. (18) leads to the neutrino mass matrix Eq. (22) which is not simply related to Yukawa couplings, but instead is more closely related to the Dirac and Majorana mass terms in Eqs. (19) and (21) respectively. This underlies why the lepton mixing angles are larger than the quark angles using the difference between $SU(4)$ transformation properties evident in our Eqs. (6) and (7) from reference [1].

Discussion

We have seen that one could have anticipated before the measurements that some neutrino mixing angles would be significantly larger than any quark mixing.

It could be of some interest to investigate fermion mixings in extensions of the standard model such as the chiral color model [19] or in the presence of a fourth family [20].

An $SU(8)$ model has been proposed by Barr [21] with the similar aim. Our model improves on it because we use a simpler family unification.

Furthermore it is not necessary to limit ourselves to non-supersymmetric models. In [22] it was shown that models exist with supersymmetric family unification. $SU(8)$ and $SU(9)$ models were explored and gauge symmetry breaking was carried out that preserved supersymmetry [23] to a low scale. While the particle content is somewhat different in these models, the basic conclusion stays the same. We expect a low energy theory, here the MSSM, extended and constrained by a gauged family symmetry broken at an intermediate scale.

As our final open question we ask: Can a discrete flavor symmetry be successfully embedded in a gauged family unification model? For example, the $SU(4)$ gauged symmetry is contained within $SU(9)$ and commutes with the standard model gauge group. This $SU(4)$ contains $SU(2)$ subgroups which have the binary tetrahedral group as a subgroup. Can such a binary tetrahedral subgroup in the present model be identified with the flavor symmetry used in e.g. [24–26]? Numerous other models with continuous non-abelian family symmetries exist [27], and the same question can be posed for all such models.
Acknowledgements

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