RELATIONS BETWEEN 6D $\mathcal{N} = (2,0)$ CONFORMAL FIELD THEORY AND 5D, 4D GAUGE THEORIES

Yang Sun

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics and Astronomy.

Chapel Hill
2015

Approved by:

Louise Dolan

Charles Evans

Jonathan Heckman

Jack Ng

Frank Tsui

ABSTRACT

YANG SUN: RELATIONS BETWEEN 6D $\mathcal{N} = (\mathbf{2}, \mathbf{0})$ CONFORMAL FIELD THEORY AND 5D, 4D GAUGE THEORIES.
(Under the direction of Louise Dolan.)

The six-dimensional (6D), $\mathcal{N} = (2,0)$ super conformal field theory (SCFT), which contains a tensor multiplet, is considered to govern some of the lower dimensional supersymmetric gauge theories. After a general introduction to the 6D, $\mathcal{N} = (2,0)$ theory with sixteen supercharges and supersymmetric Yang-Mills theory in 4D and 5D, there follows a description of the partition function approach for a single M5-brane of which the world volume theory is the abelian 6D, $\mathcal{N} = (2,0)$ SCFT. We introduce the conjecture by Michael Douglas and Neil Lambert that the $(2,0)$ SCFT on S^1 is equivalent to the 5D maximally supersymmetric Yang-Mills theory. S-duality is an important property first found in Maxwell theory and later generalized to different supersymmetric gauge theories, such as 4D, $\mathcal{N} = 4$ super Yang-Mills and 4D supersymmetric QCD. We briefly discuss the origin of the S-duality of the 4D abelian gauge theory with an theta angle from the 6D tensor theory. By computing and comparing the explicit formulas for the partition functions, we will show that the 4D and 5D abelian gauge theories share fundamental properties with the 6D tensor theory.

In Chapter 2, we give our preliminary test of the conjecture of Douglas and Lambert by using the partition functions computation. We give an explicit computation of the partition function of a five-dimensional abelian gauge theory on a five-torus T^5 with a general flat metric using the Dirac method of quantizing with constraints. We compare this with the partition function of a single fivebrane compactified on S^1 times T^5, which is obtained from the six-torus calculation of Dolan and Nappi [`arXiv:hep-th/9806016`]. The radius R_1 of the circle S^1 is set to the dimensionful gauge coupling constant $g^2_{5YM} = 4\pi^2 R_1$. We find the two partition functions are equal only in the limit where R_1 is small relative to T^5, a limit which removes the

Kaluza-Klein modes from the 6D sum. This suggests the 6D, $\mathcal{N} = (2,0)$ tensor theory on a circle is an ultraviolet completion of the 5D gauge theory, rather than an exact quantum equivalence.

In Chapter 2, we compute the partition function of four-dimensional abelian gauge theory on a general four-torus T^4 with flat metric using Dirac quantization. In addition to an $SL(4,\mathcal{Z})$ symmetry, it possesses $SL(2,\mathcal{Z})$ symmetry that is electromagnetic S-duality. We show explicitly how this $SL(2,\mathcal{Z})$ S-duality of the 4D abelian gauge theory has its origin in symmetries of the 6D $(2,0)$ tensor theory, by computing the partition function of a single fivebrane compactified on $T^2 \times T^4$, which has $SL(2,\mathcal{Z}) \times SL(4,\mathcal{Z})$ symmetry. If we identify the couplings of the abelian gauge theory $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ with the complex modulus of the T^2 torus, $\tau = \beta^2 + i\frac{R_1}{R_2}$, then in the small T^2 limit, the partition function of the fivebrane tensor field can be factorized, and contains the partition function of the 4D gauge theory. In this way the $SL(2,\mathcal{Z})$ symmetry of the 6D tensor partition function is identified with the S-duality symmetry of the 4D gauge partition function. Each partition function is the product of zero mode and oscillator contributions, where the $SL(2,\mathcal{Z})$ acts suitably. For the 4D gauge theory, which has a Lagrangian, this product redistributes when using path integral quantization.

ACKNOWLEDGEMENTS

I am grateful to my advisor, Louise Dolan, for all her dedication, guidance, and encouragement throughout graduate school. I am thankful for all the members of my committee and the work they have done.

I am also grateful to Marcus Berg for his collaboration in research and getting me interested with the scattering amplitudes.

I appreciate all the help from Justin Sawon, Robert Lipshitz, and the rest of the mathematics department have done for me.

Many thanks to Mom, Dad for all their love, support, and good advice. I appreciate the long-time friendship of Kevin Ludwick, Bill Shively, Adam Kehller,Xin Lin, Jian Chen and all my college mates and my friends outside physics, Renyuan Zhang, David Lax, Mike Schuster and to others I did not mention here, thank you.

TABLE OF CONTENTS

Chapter 1

Introduction

1.1 Fundamentals of the 6D Theory

6D Supersymmetric Field Theory with 16 Super Charges

In six dimensions, massless particles are specified by representations of the little group, which is spin(4) or $SU(2) \times SU(2)$. Using representations by $SU(2) \times SU(2)$, one can have two different super charges $Q_{\alpha,i}$ and $Q_{\dot{\beta},j}$, where $i, j = 1, 2$ which transform as $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ of $SU(2) \times SU(2)$. These can be seen as annihilation operators acting on the spin state of the particles and their adjoint operators can be seen as the creation operators.

$(1, 1)$ *Supersymmetry*

By taking different combinations of the two raising operators $Q^{\dagger}_{\alpha,i}$ and $Q^{\dagger}_{\dot{\beta},j}$, one obtains the representations

$$r = (\mathbf{2}, \mathbf{2}) + \mathbf{2}\,(\mathbf{2}, \mathbf{1}) + \mathbf{2}\,(\mathbf{1}, \mathbf{2}) + \mathbf{4}\,(\mathbf{1}, \mathbf{1}), \tag{1.1}$$

acting on the state $|\mathbf{1}, \mathbf{1}\rangle$ to give

$$|\mathbf{2}, \mathbf{2}\rangle + \mathbf{2}\,|\mathbf{2}, \mathbf{1}\rangle + \mathbf{2}\,|\mathbf{1}, \mathbf{2}\rangle + \mathbf{4}\,|\mathbf{1}, \mathbf{1}\rangle, \tag{1.2}$$

which is the vector multiplet. Acting on the state $|\mathbf{2}, \mathbf{2}\rangle$ gives the supergravity multiplet,

$$\begin{aligned} &|\mathbf{3}, \mathbf{3}\rangle \oplus |\mathbf{3}, \mathbf{1}\rangle \oplus |\mathbf{1}, \mathbf{3}\rangle \oplus |\mathbf{1}, \mathbf{1}\rangle \oplus \mathbf{4}\,|\mathbf{2}, \mathbf{2}\rangle \\ &\quad \oplus 2\,|\mathbf{3}, \mathbf{2}\rangle \oplus \mathbf{2}\,|\mathbf{2}, \mathbf{3}\rangle \oplus \mathbf{2}\,|\mathbf{1}, \mathbf{2}\rangle \oplus \mathbf{2}\,|\mathbf{2}, \mathbf{1}\rangle. \end{aligned} \tag{1.3}$$

The bosonic field content contains a graviton, an antisymmetric tensor, a scalar and four vectors.

$(2,0)$ *Supersymmetry*

Another possible 6D supersymmetry has a $(2,0)$ whose supercharges have a complex $2(\mathbf{2},\mathbf{1})$ representation. The raising operator form

$$r' = (\mathbf{3},\mathbf{1}) \oplus \mathbf{4}\,(\mathbf{2},\mathbf{1}) \oplus \mathbf{5}\,(\mathbf{1},\mathbf{1}). \tag{1.4}$$

Acting on $|\mathbf{1},\mathbf{3}\rangle$, these produce the supergravity multiplet

$$|\mathbf{3},\mathbf{3}\rangle \oplus \mathbf{4}\,|\mathbf{2},\mathbf{3}\rangle \oplus \mathbf{5}\,|\mathbf{1},\mathbf{1}\rangle, \tag{1.5}$$

which contains one graviton, four gravitinos, and five anti-self-dual antisymmetric tensors. Acting on $|\mathbf{1},\mathbf{1}\rangle$ we obtain the tensor supermultiplet,

$$|\mathbf{3},\mathbf{1}\rangle \oplus \mathbf{4}\,|\mathbf{2},\mathbf{1}\rangle \oplus \mathbf{5}\,|\mathbf{1},\mathbf{1}\rangle \tag{1.6}$$

with one chiral two-form B_{MN} with self-dual anti-symmetric tensor H_{MNL}, five scalars and four fermions. If we only consider the tensor supermultiplet of the $(2,0)$ supersymmetry, it is the 6D $N = (2,0)$ superconformal field theory (SCFT) in six-dimension that enjoys an $\mathrm{OSp}(2,6|2)$ super conformal symmetry that we will study later. Interesting features in lower-dimensional gauge theory are found to have their origin in this theory.

1.2 Fundamentals

5D Maximally Supersymmetric Yang-Mills

The 5D maximally supersymmetric Yang-Mills theory has 16 supercharges with the little group $SO(3) \simeq SU(2)$. It has the field content of a vector A_m with $m = 0,1,2,3,4$, five scalars X_I with $I = 6,7,8,9,10$ and fermions Ψ, which transform under the little group as $\big(1(\mathbf{3}), 2(\mathbf{2}), 5(\mathbf{1})\big)$ and all taking values in the adjoint representation of a Lie-algebra. The supersymmetry transformations are

$$
\begin{aligned}
&\delta_\epsilon X_I = i\bar{\epsilon}\Gamma_I\Psi, \\
&\delta_\epsilon A_m = i\bar{\epsilon}\Gamma_m\Gamma_5\Psi, \\
&\delta_\epsilon\Psi = \big(\frac{1}{2}F_{mn}\Gamma^{mn} + D_m X_I\Gamma^{m\,I} + \frac{i}{2}[X_I, X_J]\Gamma^{IJ}\big)\epsilon,
\end{aligned}
\tag{1.7}
$$

where the spinor ϵ satisfies $\Gamma_{012345}\epsilon = \epsilon$. The covariant derivative acts as $D_m X_I = \partial_m X_I - i[A_m, X_I]$, and $F_{mn} = \partial_m A_n - \partial_n A_m - i[A_m, A_n]$. The supersymmetric invariant action is given by

$$
\begin{aligned}
S = -\frac{1}{g_{YM}^2}\int d^5x\, tr\big(&F_{mn}F^{mn} + \int D^m X_I D_m X_I \\
&- i\bar{\Psi}\Gamma^m D_m\Psi + \frac{1}{2}\bar{\Psi}\Gamma^5\Gamma^I[X^I, \Psi] - \frac{1}{4}[X^I, X^J]^2\big),
\end{aligned}
\tag{1.8}
$$

where the spinors are eleven-dimensional spinors.

4D, $\mathcal{N} = 4$ Supersymmetric Yang-Mills

The 4D, $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory has 16 supercharges and the little group $SO(2)$. It has the field content of one gauge field A^i, $i = 0, 1, 2, 3$, six massless real scalar fields X_I , $I = 1...6$ and four chiral fermions $\Psi_{\alpha,a}$ with $a = 1...4$ and the indices $\alpha = 1, 2$, and helicities $\big((\pm1), 4(\pm\frac{1}{2}), 6(0)\big)$. All fields transform in the adjoint representation of the gauge group. Similarly, the supersymmetric transformation for the four-dimensional super Yang-Mills are

$$
\begin{aligned}
&\delta_\epsilon X_I = -i\bar{\epsilon}\Gamma^I\Psi, \\
&\delta_\epsilon A_i = -i\bar{\epsilon}\Gamma_i\Psi, \\
&\delta_\epsilon\Psi = \big(\frac{1}{2}F_{ij}\Gamma^{ij} + D_i X_I\Gamma^{iI} + \frac{i}{2}[X_I, X_J]\Gamma^{IJ}\big)\epsilon.
\end{aligned}
\tag{1.9}
$$

The supersymmetric invariant Euclidean action is given by

$$
\begin{aligned}
S = & -\int d^4x\, tr\big(\frac{1}{2g_{YM}^2}F_{ij}F^{ij} + \frac{i\theta}{16\pi^2}F_{ij}\widetilde{F^{ij}}\big) \\
& -\frac{1}{2g_{YM}^2}\big(2D_iX_ID^iX_I - [X_I, X_J]^2 - 2i\bar{\Psi}\Gamma^iD_i\Psi - 2\bar{\Psi}\Gamma^I[\Phi_I, \Psi]\big). \qquad (1.10)
\end{aligned}
$$

The $\mathcal{N} = 4$ theory is conformal and enjoys $PSU(2, 2|4)$ symmetry, even at the quantum level. In particular, the β-function vanishes to all orders in perturbation theory. A salient feature of $\mathcal{N} = 4$ is that it is conjectured to be invariant under an $SL(2, \mathcal{Z})$ transformation acting on the Yang-Mills coupling constant, known as the S-duality.

1.3 Free Abelian Version of the Actions

Restricted to the non-supersymmetric and abelian case, the 5D and 4D action is given by (1.8) and (1.10),

$$
\begin{aligned}
S_{5D} &= -\frac{1}{4g_{YM}^2}\int d^5x\, F_{mn}F^{mn}, \\
S_{4D} &= -\int d^4x\, \big(\frac{1}{2g_{YM}^2}F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{16\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\big). \qquad (1.11)
\end{aligned}
$$

1.4 Partition Function of the M5-brane and Gauge Theory

M5-branes describes an SCFT with $(2, 0)$ supersymmetry. The world-volume of a single M5-brane propagates an abelian chiral two-form potential with self-dual field strength as discussed in the previous section. Therefore, the 6D $(2, 0)$ theory does not have a covariant Lagrangian description. For a single M5-brane, one can write down its partition function [1]. The first partition function for the tensor field is computed explicitly for a T^6 manifold [2] with a flat metric. To circumvent the difficulty of lacking a covariant Lagrangian for a self-dual three-form tensor field, one can first write down an $SL(5, \mathcal{Z})$ covariant Hamiltonian and momentum for a T^6 and compute the partition function by separating the field strength H_{MNL} into a zero mode part and an oscillator mode part,

$$H = H^0 \oplus H'. \tag{1.12}$$

where the zero modes are the harmonic representatives of the self-dual three-form H^0, and the oscillator modes are $H' = dB$ for the chiral two-form. One obtains a fully $SL(6, \mathcal{Z})$ invariant partition function by tracing over the zero modes and the oscillator modes. Later, a single M5-brane partition function is computed on an arbitrary six-dimensional manifold by the holomorphic factorization approach [3].

On the gauge theory side, for convenience to compare to the chiral two-form, we adopt the Hamiltonian formulation, and separated the partition function again into zero and oscillator modes. After choosing the appropriate holonomy condition, the zero mode partition function is computed by summing over the electric and magnetic fields as integers from integrals on homology cycles due to the Dirac charge quantization condition. We treat the oscillators of the abelian gauge theory on the general torus by the method of Dirac quantization. In this dissertation, we will present explicit formulas for the partition functions for both gauge theory and the tensor theory.

1.5 Motivation and Plan of this Work

Part I

It has been known that the M5-brane can be dimensionally reduced to a D4-brane by compactifying the M5-brane on a circle. But recently it had been conjectured that these two quantum theories are completely equivalent [4] [5]. When we compactify the x^1 direction of the six-torus on a circle of circumference $2\pi R_1$, using dimensional reduction of the self-dual three-form H_{MNL}, one obtain the 5D gauge field strength $F_{mn} = H_{1mn}$, where $1 \leq M, N, L \leq 6$ and $2 \leq m, n \leq 6$. The other component, H_{mnl} are obtained by the self-duality condition. As a preliminary test of this conjecture, we consider the partition function of five-dimensional Maxwell theory on a five-torus and compare it with that obtained in [2] for a single M5-brane restricted to $T^5 \times S^1$.

Part II

It has been suggested that S-duality of the 4D, $\mathcal{N} = 4$ Yang-Mills has its origin in the 6D, non-abelian $\mathcal{N} = (2,0)$ SCFT and this can be tested at the abelian and non-supersymmetric level by comparing the partition functions. We consider two different quantum field theories in four and six dimensions. One is a 4D free abelian gauge theory with coupling constant $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ on T^4. The other is an abelian chiral two-form potential B_{MN} with the self-dual field strength tensor H_{MNL} compactified on $T^2 \times T^4$. The two-torus T^2 is described by the complex modulus $\tau' = i\frac{R_1}{R_2} + \beta^2$ and identified with the 4D coupling constant τ. We will show that in the limit that T^2 is small, the 6D partition function is equivalent to the 4D partition function times the contribution of an additional scalar. The 6D partition function on $T^2 \times T^4$ is invariant under the modular group $SL(2,\mathcal{Z})$, which is the mapping class group of the two-torus T^2. Therefore, the $SL(2,\mathcal{Z})$ of T^2 symmetry implies the S-duality in the 4D abelian gauge theory.

Chapter 2

Partition Functions for T^5 and $S^1 \times T^5$

2.1 Introduction

A quantum equivalence between the six-dimensional $\mathcal{N} = (2,0)$ theory of multiple fivebranes compactfied on a circle S^1, and five-dimensional maximally supersymmetric Yang Mills has been conjectured by Douglas and Lambert *et al.* in [4, 5]. in this chapter we will study an abelian version of the conjecture where the common five-manifold is a five-torus T^5 with a general flat metric, and find an equivalence only in the weak coupling limit.

The physical degrees of freedom of a single fivebrane are described by an $\mathcal{N} = (2,0)$ tensor supermultiplet which includes a chiral two-form field potential, so even a single fivebrane has no fully covariant action. In order to investigate its quantum theory [2] computes the partition function instead, which we carried out on the six-torus T^6. We will use this calculation to investigate the partition function of the self-dual three-form field strength restricted to $S^1 \times T^5$ and compare it with the partition function of the five-dimensional Maxwell theory on a twisted five-torus quantized via Dirac constraints in radiation gauge.

Because both the theory and the manifold are so simple, we do not use localization techniques fruitful for non-abelian theories and their partition functions on spheres [7]-[12].

The five-dimensional Maxwell partition function on T^5 is defined as in string theory [13],

$$
\begin{aligned}
Z^{5D,Maxwell} &\equiv tre^{-2\pi H^{5D}+i2\pi\gamma^i P_i^{5D}} = Z^{5D}_{zero\ modes}\cdot Z^{5D}_{osc},\\
H^{5D} &= \frac{R_6}{g^2_{5YM}}\int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\,\big(\frac{1}{2R_6^2} g^{ii'}\,F_{6i}F_{6i'} + \frac{1}{4}g^{ii'}g^{jj'}F_{ij}F_{i'j'}\big),\\
P_i^{5D} &= \frac{1}{g^2_{5YM}R_6}\int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\; g^{jj'}\,F_{6j'}F_{ij},
\end{aligned}
\tag{2.1}
$$

in terms of the gauge field strength $F_{\tilde m\tilde n}(\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$, and constant metric g^{ij}, R_6, γ^i. The partition function of the abelian chiral two-form on a space circle times the five-torus is

$$
\begin{aligned}
Z^{6D,chiral} &= tr\, e^{-2\pi R_6\mathcal{H}+i2\pi\gamma^i\mathcal{P}_i} = Z^{6D}_{zero\ modes}\cdot Z^{6D}_{osc},\\
\mathcal{H} &= \frac{1}{12}\int_0^{2\pi} d\theta^1\ldots d\theta^5\sqrt{G_5}{G_5}^{ll'}{G_5}^{mm'}{G_5}^{nn'}H_{lmn}(\vec\theta,\theta^6)\,H_{l'm'n'}(\vec\theta,\theta^6),\\
\mathcal{P}_i &= -\frac{1}{24}\int_0^{2\pi} d\theta^1...d\theta^5\epsilon^{rsumn}H_{umn}(\vec\theta,\theta^6)\,H_{irs}(\vec\theta,\theta^6)
\end{aligned}
\tag{2.2}
$$

where θ^1 is the direction of the circle S^1. The time direction θ^6 we will use for quantization is common to both theories, and the angles between the circle and the five-torus denoted by α, β^i in [2] have been set to zero. The final results are given in (3.73), (3.74).

We use (3.2,2.2) to compute both the zero mode and oscillator contributions, and find an exact equivalence between the zero mode contributions,

$$
Z^{6D}_{zero\ modes} = Z^{5D}_{zero\ modes}. \tag{2.3}
$$

Not surprisingly, we find the oscillator traces differ by the absence in Z^{5D}_{osc} of the Kaluza-Klein modes generated in Z^{6D}_{osc} from compactification on the circle S^1.

The Kaluza-Klein modes have been associated with instantons in the five-dimensional non-abelian gauge theory in [4, 5, 17, 18], with additional comments given for the abelian limit. It would be interesting to find a systematic way to incorporate these modes in a generalized five-dimensional partition function along the lines of a character, in order to match the partition functions exactly, but we have not done that here. Rather our explicit expressions show an equivalence between the oscillator traces of the two theories only in the limit where the compactification radius R_1 of the circle is small compared to the five-torus T^5.

Other approaches to $\mathcal{N} = (2, 0)$ theories formulate fields for non-abelian chiral two-forms [19]-[24] which would be useful if the non-abelian six-dimensional theory has a classical description and if the quantum theory can be described in terms of fields. On the other hand the partition functions on various manifolds [20]-[29] can demonstrate aspects of the six-dimensional finite quantum conformal theory presumed responsible for features of four-dimensional gauge theory [30].

In section 2, the contribution of the zero modes to the partition function for the chiral theory on a circle times a five-torus is computed as a sum over the ten integer eigenvalues, and its relation to that of the gauge theory is shown via a fiber bundle approach. In section 3, the abelian gauge theory is quantized on a five-torus using Dirac constraints, and the Hamiltonian and momenta are computed in terms of the oscillator modes. In section 4, we construct the oscillator trace contribution to the partition function for the gauge theory and compare it with that of the chiral two-form. Section 5 contains discussion and conclusions. We presents details of the Dirac quantization and Appendix B verifies the Hamilton equations of motion. Appendix C regularizes the vacuum energy. Appendix D proves the $SL(5, \mathcal{Z})$ invariance of both partition functions.

2.2 Zero Modes

The $\mathcal{N} = (2, 0)$ 6D world volume theory of the fivebrane contains five scalars, two four-spinors and a chiral two-form B_{MN}, which has a self-dual three-form field strength

$H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$ with $1 \leq L, M, N \leq 6$,

$$H_{LMN}(\vec{\theta}, \theta^6) = \frac{1}{6\sqrt{-G}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6). \tag{2.4}$$

(2.4) gives $H_{LMN}(\vec{\theta}, \theta^6) = \frac{i}{6\sqrt{|G|}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6)$ for a Euclidean signature metric. In the absence of a covariant Lagrangian, the partition function of the chiral field is defined via a trace over the Hamiltonian [2] as is familiar from string calculations. We display this expression in (2.2) where the metric has been restricted to describe the line element for $S^1 \times T^5$,

$$ds^2 = {R_1}^2 (d\theta^1)^2 + {R_6}^2 (d\theta^6)^2 + \sum_{i,j=2\ldots5} g_{ij} (d\theta^i - \gamma^i d\theta^6)(d\theta^j - \gamma^j d\theta^6) \tag{2.5}$$

with $0 \leq \theta^I \leq 2\pi$, $1 \leq I \leq 6$. The parameters R_1 and R_6 are the radii for directions 1 and 6, g_{ij} is a 4D metric, and γ^j are the angles between between 6 and j. So from (3.5),

$$G_{ij} = g_{ij}\,;\; G_{11} = {R_1}^2; \quad G_{i1} = 0\,; \quad G_{66} = {R_6}^2 + g_{ij}\gamma^i\gamma^j; \quad G_{i6} = -g_{ij}\gamma^j\,; \quad G_{16} = 0; \tag{2.6}$$

and the inverse metric is

$$G^{ij} = g^{ij} + \frac{\gamma^i\gamma^j}{R_6^2}; \quad G^{11} = \frac{1}{R_1^2}; \quad G^{1i} = 0; \quad G^{66} = \frac{1}{R_6^2}; \quad G^{i6} = \frac{\gamma^i}{R_6^2}; \quad G^{16} = 0. \tag{2.7}$$

We want to keep the time direction θ^6 common to both theories, so in the 5D expressions (3.2) the indices are on $2 \leq \tilde{m}, \tilde{n} \leq 6$; and the Hamiltonian and momenta in (2.2) sum on $1 \leq m, n \leq 5$. The common space index is labeled $2 \leq i, j \leq 5$. To this end, for the metric G_{MN} in (2.6) we introduce the 5-dimensional inverse (in directions 1,2,3,4,5)

$$G_5{}^{ij} = g^{ij}; \qquad G_5{}^{i1} = 0; \qquad G_5{}^{11} = \frac{1}{R_1^2}; \tag{2.8}$$

and the 5-dimensional inverse (in directions 2,3,4,5,6) for the five-torus T^5,

$$\widetilde{G}_5^{ij} = g^{ij} + \frac{\gamma^i\gamma^j}{R_6^2}\,; \qquad \widetilde{G}_5^{i6} = \frac{\gamma^i}{R_6^2}\,; \qquad \widetilde{G}_5^{66} = \frac{1}{R_6^2}. \tag{2.9}$$

The determinants of the metrics are related simply by $\sqrt{G} = R_6\sqrt{G_5} = R_1\sqrt{\widetilde{G}_5} = R_6 R_1 \sqrt{g}$, and $\epsilon_{23456} \equiv \widetilde{G}_5\, \epsilon^{23456} = \widetilde{G}_5$, with corresponding epsilon tensors related by G, G_5, g.

To compute $Z^{6D}_{zero\ modes}$ we neglect the integrations in (2.2) and get

$$\begin{aligned} -2\pi R_6 \mathcal{H} &= -\frac{\pi}{6} R_6 R_1 \sqrt{g} g^{ii'} g^{jj'} g^{kk'} H_{ijk} H_{i'j'k'} - \frac{\pi}{4}\frac{R_6}{R_1}\sqrt{g}(g^{jj'}g^{kk'} - g^{jk'}g^{kj'}) H_{1jk} H_{1j'k'}, \\ i2\pi\gamma^i P_i &= -\frac{i\pi}{2}\gamma^i \epsilon^{jkj'k'} H_{1jk} H_{ij'k'} = \frac{i\pi}{3}\gamma^i \epsilon^{jj'kk'} H_{j'kk'} H_{1ij}, \end{aligned} \tag{2.10}$$

where the zero modes of the four fields H_{ijk} are labeled by the integers $n_7, \ldots, n_{10}$. The six fields H_{1jk} have zero mode eigenvalues $H_{123} = n_1$, $H_{124} = n_2$, $H_{125} = n_3$, $H_{134} = n_4$, $H_{135} = n_5$, $H_{145} = n_6$, and the trace on the zero mode operators in (2.2) is

$$\begin{aligned} Z^{6D}_{zero\ modes} = &\sum_{n_1,\ldots,n_6} \exp\{-\frac{\pi}{4}\frac{R_6}{R_1}\sqrt{g}(g^{jj'}g^{kk'} - g^{jk'}g^{kj'}) H_{1jk} H_{1j'k'}\} \\ &\cdot \sum_{n_7,\ldots,n_{10}} \exp\{-\frac{\pi}{6} R_6 R_1 \sqrt{g} g^{ii'} g^{jj'} g^{kk'} H_{ijk} H_{i'j'k'} - \frac{i\pi}{2}\gamma^i \epsilon^{jkj'k'} H_{1jk} H_{ij'k'}\}. \end{aligned} \tag{2.11}$$

The same sum is obtained from the 5D Maxwell theory (3.2) where the gauge coupling is identified with the radius of the circle $g^2_{5YM} = 4\pi^2 R_1$, as follows. The zero modes of the gauge theory are eigenvalues of operator-valued fields that satisfy Maxwell equations with no sources. Even classically these solutions have constant F_{ij} which lead to non-zero flux through closed two-surfaces that are not a boundary of a three-dimensional submanifold in T^5. Working in $A_6 = 0$ gauge, if we consider the $U(1)$ gauge field A_i at any time θ^6 as a connection on a principal U(1) bundle with base manifold T^4, then the curvature $F_{ij} = \partial_i A_j - \partial_j A_i$

for $2 \leq i, j \leq 5$ must have integer flux [31, 32], in the sense that

$$n_I = \frac{1}{2\pi} \int_{\Sigma_2^I} F \equiv \frac{1}{2\pi} \int_{\Sigma_2^I} \frac{1}{2} F_{ij}\, d\theta^i \wedge d\theta^j, \qquad n_I \in \mathcal{Z}, \text{ for each } 1 \leq I \leq 6. \tag{2.12}$$

In T^4, the six representative two-cycles Σ_2^I are each a 2-torus constructed by the six ways of combining the four S^1 of T^4 two at a time, given by the cohomology class, $\dim H_2(T^4) = 6$. Relabeling n_I as $n_{i,j}$ and Σ_2^I as $\Sigma_2^{i,j}$, $2 \leq i < j \leq 5$, we have $\int_{\Sigma_2^{g,h}} d\theta^i \wedge d\theta^j = (2\pi)^2 (\delta_g^i \delta_h^j - \delta_h^i \delta_g^j)$. So (3.25) is

$$F_{ij} = \frac{n_{i,j}}{2\pi}, \qquad n_{i,j} \in \mathcal{Z} \text{ for } i < j. \tag{2.13}$$

Furthermore we show how the zero mode eigenvalues of F_{6i} are found from those of the conjugate momentum Π^i. In section 3 we derive the form of H^{5D} and P_i^{5D} given in (3.2) from a canonical quantization using a Lorentzian signature metric. In (3.44) the conjugate momentum is defined as

$$\Pi^i = \frac{\sqrt{g}}{4\pi^2 R_1 R_6} g^{ii'} F_{6i'}. \tag{2.14}$$

From the commutation relations (3.52) we can compute its commutator with the holonomy $\int_{\Sigma_1^k} A \equiv \int_{\Sigma_1^k} A_i(\vec{\theta}, \theta^6) d\theta^i$ where Σ_1^k are the four representative one-cycle circles in T^4,

$$\left[\int_{\Sigma_1^k} A_i(\vec{\theta}, \theta^6) d\theta^i, \int \frac{d^4\theta'}{2\pi} \Pi^j(\vec{\theta}', \theta^6) \right] = \frac{i}{2\pi} \int_{\Sigma_1^k} d\theta^j = i\, \delta_k^j. \tag{2.15}$$

Hence an eigenstate ψ of the the zero mode operator $\frac{1}{2\pi} \int d^4\theta' \Pi^k(\vec{\theta}', \theta^6)$ with eigenvalue λ is

$$\psi = e^{i\lambda \int_{\Sigma_1^k} A} |0\rangle, \qquad \left(\frac{1}{2\pi} \int d^4\theta' \Pi^k(\vec{\theta}', \theta^6) \right) e^{i\lambda \int_{\Sigma_1^k} A} |0\rangle = \lambda\ e^{i\lambda \int_{\Sigma_1^k} A} |0\rangle.$$

Since the holonomy is defined mod 2π, thus allowing A to vary by gauges when crossing neighborhoods, but ensuring $e^{i\int_{\Sigma_1^k} A}$ to be a single valued element of the structure group $U(1)$, then the states

$$e^{i\lambda\int_{\Sigma_1^k} A}\,|0\rangle \qquad \text{and} \qquad e^{i\lambda\left(2\pi+\int_{\Sigma_1^k} A\right)}\,|0\rangle \tag{2.16}$$

must be equivalent, so the eigenvalue λ of the operator $\frac{1}{2\pi}\int d^4\theta'\Pi^k(\vec{\theta}', \theta^6)$ must have integer values $n^{(k)}$,

$$\Pi^k(\vec{\theta}', \theta^6) = \frac{n^{(k)}}{(2\pi)^3}, \qquad n^{(k)} \in \mathcal{Z}^4. \tag{2.17}$$

In this normalization of the zero mode eigenvalues for the gauge theory, we are taking the $d\theta^i$ space integrations into account. So (3.2) gives

$$\begin{aligned} &-2\pi H^{5D} + i2\pi\gamma^i P_i^{5D} \\ &= \Big(-\frac{\pi\sqrt{g}}{R_1R_6}g^{ii'}F_{6i}F_{6i'} - \frac{\pi R_6}{2R_1}\sqrt{g}g^{ii'}g^{jj'}F_{ij}F_{i'j'} + 2\pi i\gamma^i\frac{\sqrt{g}}{R_1R_6}g^{jj'}F_{6j'}F_{ij}\Big)\,(2\pi)^2. \end{aligned} \tag{2.18}$$

We can use the identity

$$-\frac{1}{4}\epsilon^{jkj'k'}H_{1jk}H_{ij'k'} = \frac{1}{6}\epsilon^{jj'kk'}H_{j'kk'}H_{1ij},$$

to rewrite the last term in (2.11) as

$$-\frac{i\pi}{2}\gamma^i\epsilon^{jkj'k'}H_{1jk}H_{ij'k'} = \frac{i\pi}{3}\gamma^i\epsilon^{jj'kk'}H_{j'kk'}H_{1ij},$$

which is equal to the last term in (2.18) if we identify

$$\frac{1}{6}\epsilon^{jj'kk'}H_{j'kk'} = \frac{2\pi\sqrt{g}}{R_1R_6}g^{jj'}F_{6j'}, \qquad H_{1ij} = 2\pi F_{ij}. \tag{2.19}$$

Then, from (2.19) we have that the first term in (2.18) becomes

$$-\frac{4\pi^3\sqrt{g}}{R_1R_6}g^{ii'}F_{6i}F_{6i'}=-\frac{\pi}{6}\sqrt{g}R_1R_6\;g^{j'g'}g^{gh}g^{g'h'}H_{j'kk'}H_{g'hh'}.$$

Thus with the identifications in (2.19), the 5D Maxwell expression in (2.18) is equal to the 6D chiral exponent in (2.11),

$$\begin{aligned}-2\pi H^{5D}+i2\pi\gamma^iP_i^{5D}&=\Big(-\frac{\pi\sqrt{g}}{R_1R_6}g^{ii'}\,F_{6i}F_{6i'}-\frac{\pi R_6\sqrt{g}}{2R_1}g^{ii'}g^{jj'}F_{ij}F_{i'j'}+\frac{i2\pi\sqrt{g}}{R_1R_6}\gamma^ig^{jj'}\,F_{6j'}F_{ij}\Big)(2\pi)^2\\ =-t\mathcal{H}+i2\pi\gamma^iP_i&=-\frac{\pi}{6}R_6R_1\sqrt{g}g^{ii'}g^{jj'}g^{kk'}H_{ijk}H_{i'j'k'}-\frac{\pi}{4}\frac{R_6}{R_1}\sqrt{g}(g^{jj'}g^{kk'}-g^{jk'}g^{j'k})H_{1jk}H_{1j'k'}\\ &\quad-\frac{i\pi}{2}\gamma^i\epsilon^{jkj'k'}H_{1jk}H_{ij'k'}.\end{aligned}$$

We now discuss the sum over integers in (2.11). From (2.19), if H_{1jk} are integers, then $2\pi\,F_{ij}$ are integers. If H_{ijk} are integers, then $\frac{1}{6}\epsilon^{jj'kk'}H_{j'kk'}$ are also integers. This implies, again from (2.19), that $\frac{2\pi\sqrt{g}}{R_1R_6}g^{jj'}F_{6j'}$ should be integers, which we justify in (3.27) and (2.17) with (2.14). Thus the Maxwell zero mode trace can be written as

$$\begin{aligned}Z^{5D}_{zero\ modes}=&\sum_{n_1\ldots n_6}\exp\{-2\pi^3\frac{R_6\sqrt{g}}{R_1}g^{ii'}g^{jj'}F_{ij}F_{i'j'}\}\\ &\cdot\sum_{n^7\ldots n^{10}}\exp\{-\frac{4\pi^3\sqrt{g}}{R_1R_6}g^{ii'}F_{6i}F_{6i'}+\frac{i(2\pi)^3\sqrt{g}}{R_1R_6}\gamma^ig^{jj'}\,F_{6j'}F_{ij}\}\end{aligned}\tag{2.20}$$

where the integer eigenvalues are $n_1=2\pi F_{23},n_2=2\pi F_{24},n_3=2\pi F_{25},n_4=2\pi F_{34}$, $n_5=2\pi F_{35},n_6=2\pi F_{45}$; $(n^7,n^8,n^9,n^{10})\equiv(n^{(2)},n^{(3)},n^{(4)},n^{(5)})$, for $n^{(k)}\equiv\frac{2\pi\sqrt{g}}{R_1R_6}g^{ki'}F_{6i'}\in\mathcal{Z}^4$. So we have proved the relation (2.3)

$$Z^{6D}_{zero\ modes}=Z^{5D}_{zero\ modes}\tag{2.21}$$

and the explicit expression is given by (2.11) or (2.20).

2.3 Dirac Quantization of Maxwell Theory on a Five-torus

To evaluate the oscillator contribution to the partition function in (3.2), we will first quantize the abelian gauge theory on the five-torus with a general metric. The equation of motion is $\partial^{\tilde{m}} F_{\tilde{m}\tilde{n}} = 0$. For $F_{\tilde{m}\tilde{n}} = \partial_{\tilde{m}} A_{\tilde{n}} - \partial_{\tilde{n}} A_{\tilde{m}}$, a solution is given by a solution to

$$\partial^{\tilde{n}} \partial_{\tilde{n}} A_{\tilde{m}} = 0, \qquad \partial^{\tilde{m}} A_{\tilde{m}} = 0. \tag{2.22}$$

These have a plane wave solution $A_{\tilde{m}}(\vec{\theta}, \theta^6) = f_{\tilde{m}}(k) e^{ik\cdot\theta} + (f_{\tilde{m}}(k) e^{ik\cdot\theta})^*$ when

$$\widetilde{G}_L^{\tilde{m}\tilde{n}} k_{\tilde{m}} k_{\tilde{n}} = 0, \qquad k^{\tilde{m}} f_{\tilde{m}} = 0. \tag{2.23}$$

In order for the operator formalism (3.2) to reproduce a path integral quantization with spacetime metric (2.9), we must canonically quantize H^{5D} and P_i^{5D} via a metric that has zero angles with the time direction, *i.e.* $\gamma^i = 0$, and insert γ^i in the partition function merely as the coefficient of P_i^{5D} [13]. Furthermore a Lorentzian signature metric is needed for quantum mechanics, so we modify the metric on the five-torus (2.6), (2.9) to be

$$\widetilde{G}_{L\,ij} = g_{ij}\,;\ \widetilde{G}_{L\,66} = -{R_6}^2;\ \widetilde{G}_{L\,i6} = 0\,; \quad \widetilde{G}_L^{ij} = g^{ij};\ \widetilde{G}_L^{66} = -\frac{1}{R_6^2};\ \widetilde{G}_L^{i6} = 0, \quad \widetilde{G}_L = \det \widetilde{G}_{L\,\tilde{m}\tilde{n}}. \tag{2.24}$$

Solving for k_6 from (2.23) we find

$$k_6 = \pm \frac{\sqrt{-\widetilde{G}_L^{66}}}{\widetilde{G}_L^{66}}\,|k|, \tag{2.25}$$

where $2 \le i, j \le 5$, and $|k| \equiv \sqrt{g^{ij} k_i k_j}$. Use the gauge invariance $f_{\tilde{m}} \to f'_{\tilde{m}} = f_{\tilde{m}} + k_{\tilde{m}}\lambda$ to fix $f'_6 = 0$, which is the gauge choice $A_6 = 0$. This reduces the number of components of $A_{\tilde{m}}$ from 5 to 4. To satisfy (2.23), we can use the $\partial^{\tilde{m}} F_{\tilde{m}6} = -\partial_6 \partial^i A_i = 0$

component of the equation of motion to eliminate f_5 in terms of the three f_2, f_3, f_4,

$$f_5 = -\frac{1}{p^5}(p^2 f_2 + p^3 f_3 + p^4 f_4),$$

leaving just three independent polarization vectors corresponding to the physical degrees of freedom of the 5D one-form with Spin(3) content 3. From the Lorentzian Lagrangian

$$\mathcal{L} = -\frac{1}{4}\frac{\sqrt{-\widetilde{G}_L}}{g_{5YM}^2}\widetilde{G}_L^{\tilde{m}\tilde{m}'}\widetilde{G}_L^{\tilde{n}\tilde{n}'}F_{\tilde{m}\tilde{n}}F_{\tilde{m}'\tilde{n}'} = \frac{R_6\sqrt{g}}{4\pi^2 R_1}\Big(-\frac{1}{4}g^{ii'}g^{jj'}F_{ij}F_{i'j'} - \frac{1}{2}\widetilde{G}_L^{66}g^{jj'}F_{6j}F_{6j'}\Big), \tag{2.26}$$

the energy-momentum tensor

$$\mathcal{T}^{\,m}_{\ \ n} = \frac{\delta\mathcal{L}}{\delta\partial_m A_p}\partial_n A_p - \delta^{\,m}_{\ \ n}\,\mathcal{L} \tag{2.27}$$

leads to the Hamiltonian and momenta operators

$$H_c \equiv \int d^4\theta\, \mathcal{T}^{\,6}_{\ \ 6} = \int d^4\theta\,\Big(\frac{R_6\sqrt{g}}{4\pi^2 R_1}\big(-\frac{1}{2}\widetilde{G}_L^{66}g^{ii'}\,F_{6i}F_{6i'} + \frac{1}{4}g^{ii'}g^{jj'}F_{ij}F_{i'j'} - F^{6i}\partial_i A_6\big) + \Pi^6\partial_6 A_6\Big), \tag{2.28}$$

$$P_i \equiv \int d^4\theta\, \mathcal{T}^{6}_{\ \ i} = \int d^4\theta\Big(\frac{R_6\sqrt{g}}{4\pi^2 R_1}\big(-\widetilde{G}_L^{66}g^{jj'}\,F_{6j'}F_{ij} - F^{6j}\partial_j A_i\big) + \Pi^6\partial_i A_6\Big), \tag{2.29}$$

where the conjugate momentum is

$$\Pi^i = \frac{\delta\mathcal{L}}{\delta\partial_6 A_i} = -\frac{R_6\sqrt{g}}{4\pi^2 R_1}F^{6i} = -\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66}g^{ii'}F_{6i'}, \qquad \Pi^6 = \frac{\delta\mathcal{L}}{\delta\partial_6 A_6} = 0. \tag{2.30}$$

We quantize the Maxwell field on the five-torus with the metric (2.24) in radiation gauge using Dirac constraints [50, 51]. The theory has a primary constraint $\Pi^6(\vec{\theta},\theta^6) \approx$

0. We can express the Hamiltonian (3.42) in terms of the conjugate momentum as

$$H_{can} = \int d^4\theta \Big(-\frac{2\pi^2 R_1}{R_6\sqrt{g}\widetilde{G}_L^{66}}\, g_{ii'}\Pi^i\,\Pi^{i'} + \frac{R_6\sqrt{g}}{16\pi^2 R_1} g^{ii'}g^{jj'}F_{ij}F_{i'j'} - \partial_i\Pi^i\, A_6\Big), \tag{2.31}$$

where the last term has been integrated by parts. The primary Hamiltonian is defined by

$$H_p = \int d^4\theta \Big(-\frac{2\pi^2 R_1}{R_6\sqrt{g}\widetilde{G}_L^{66}}\, g_{ii'}\Pi^i\,\Pi^{i'} + \frac{R_6\sqrt{g}}{16\pi^2 R_1} g^{ii'}g^{jj'}F_{ij}F_{i'j'} - \partial_i\Pi^i\, A_6 + \lambda_1\Pi^6\Big), \tag{2.32}$$

with λ_1 as a Lagrange multiplier. In Appendix A, we use the Dirac method of quantizing with constraints for the radiation gauge conditions $A_6 \approx 0$, $\partial^i A_i \approx 0$, and find the equal time commutation relations (A.13), (A.14):

$$\begin{aligned} &[\Pi^j(\vec{\theta},\theta^6), A_i(\vec{\theta}',\theta^6)] = -i\Big(\delta_i^j - g^{jj'}(\partial_i \frac{1}{g^{kk'}\partial_k\partial_{k'}}\partial_{j'})\Big)\ \delta^4(\theta-\theta'),\\ &[A_i(\vec{\theta},\theta^6), A_j(\vec{\theta}',\theta^6)] = 0, \qquad [\Pi^i(\vec{\theta},\theta^6), \Pi^j(\vec{\theta}',\theta^6)] = 0. \end{aligned} \tag{2.33}$$

Appendix B shows the Hamilitonian (3.51) to give the correct equations of motion.

In $A_6 = 0$ gauge, the free quantum vector field on the torus is expanded as

$$A_i(\vec{\theta},\theta^6) = \ \textit{zero modes}\ + \sum_{\vec{k}\neq 0, \vec{k}\in\mathcal{Z}_4} (f_i^\kappa a_{\vec{k}}^\kappa e^{ik\cdot\theta} + f_i^{\kappa *} a_{\vec{k}}^{\kappa\dagger} e^{-ik\cdot\theta}),$$

where $1 \le \kappa \le 3$, $2 \le i \le 5$ and k_6 defined in (3.40). The sum is on the dual lattice $\vec{k} = k_i \in \mathcal{Z}_4 \neq \vec{0}$. Having computed the zero mode contribution in (2.20), here we consider

$$A_i(\vec{\theta},\theta^6) = \sum_{\vec{k}\neq 0} (a_{\vec{k}\,i} e^{ik\cdot\theta} + a^\dagger_{\vec{k}\,i} e^{-ik\cdot\theta}), \tag{2.34}$$

with polarizations absorbed in

$$a_{\vec{k}\,i} = f_i^\kappa a_{\vec{k}}^\kappa. \tag{2.35}$$

From (3.52) the commutator in terms of the oscillators is

$$\int \frac{d^4\theta d^4\theta'}{(2\pi)^8} e^{-ik_i\theta^i} e^{-ik'_i\theta'^i}[A_i(\vec{\theta},0), A_j(\vec{\theta}',0)] = [(a_{\vec{k}\,i} + a^\dagger_{-\vec{k}\,i}), (a_{\vec{k}'\,j} + a^\dagger_{-\vec{k}'\,j})] = 0. \quad (2.36)$$

The conjugate momentum $\Pi^j(\vec{\theta},\theta^6)$ in (3.44) is expressed in terms of $a_{\vec{k}\,i}, a^\dagger_{-\vec{k}\,i}$ by

$$\Pi^j(\vec{\theta},\theta^6) = -i\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66} g^{jj'} \sum_{\vec{k}} k_6\,(a_{\vec{k}\,j'} e^{ik\cdot\theta} - a^\dagger_{\vec{k}\,j'} e^{-ik\cdot\theta}). \quad (2.37)$$

Then taking the Fourier transform of $\Pi^j(\vec{\theta},\theta^6)$ at $\theta^6 = 0$, we have

$$\int \frac{d^4\theta}{(2\pi)^4} e^{-ik_i\theta^i}\Pi^j(\vec{\theta},0) = -i\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66} g^{jj'} k_6\,(a_{\vec{k}\,j'} - a^\dagger_{-\vec{k}\,j'}). \quad (2.38)$$

From (2.38) and the commutators (3.52) and (3.55), we find

$$\begin{aligned} &\int \frac{d^4\theta d^4\theta'}{(2\pi)^8} e^{-ik_i\theta^i} e^{-ik'_i\theta'^i}[\Pi^j(\vec{\theta},0), A_i(\vec{\theta}',0)] \\ &= -i(\delta_i^j - \frac{g^{jj'}k_i k_{j'}}{g^{kk'}k_k k_{k'}})\delta_{\vec{k},-\vec{k}'}\ \frac{1}{(2\pi)^4} = -i\frac{R_6\sqrt{g}}{4\pi^2 R_1}\widetilde{G}_L^{66} g^{jj'} k_6\,[(a_{\vec{k}\,j'} - a^\dagger_{-\vec{k}\,j'}), (a_{\vec{k}'\,i} + a^\dagger_{-\vec{k}'\,i})]. \end{aligned} \quad (2.39)$$

To reach the oscillator commutator (2.45), we define

$$A_{\vec{k}\,i} \equiv a_{\vec{k}\,i} + a^\dagger_{-\vec{k}\,i} = A^\dagger_{-\vec{k}\,i}, \qquad E_{\vec{k}\,i} \equiv a_{\vec{k}\,i} - a^\dagger_{-\vec{k}\,i} = -E^\dagger_{-\vec{k}\,i}, \quad (2.40)$$

$$a_{\vec{k}\,i} = \frac{1}{2}(A_{\vec{k}\,i} + E_{\vec{k}\,i}), \qquad a^\dagger_{\vec{k}\,i} = \frac{1}{2}(A^\dagger_{\vec{k}\,i} + E^\dagger_{\vec{k}\,i}) = \frac{1}{2}(A_{-\vec{k}\,i} - E_{-\vec{k}\,i}). \quad (2.41)$$

Now inverting (2.39) we have

$$[E_{\vec{k}\,j}, A_{\vec{k}'\,i}] = \frac{R_1}{R_6\sqrt{g}\widetilde{G}_L^{66} k_6}\frac{1}{(2\pi)^2}\big(g_{ji} - \frac{k_j k_i}{g^{kk'}k_k k_{k'}}\big)\delta_{\vec{k},-\vec{k}'}, \quad (2.42)$$

and from (2.38) and the relations (3.52) and (3.55),

$$[A_{\vec{k}\,i}, A_{\vec{k}'\,j}] = 0, \qquad [E_{\vec{k}\,i}, E_{\vec{k}'\,j}] = 0. \quad (2.43)$$

Using (2.41),

$$[a_{\vec{k}\,i}, a^{\dagger}_{\vec{k}'\,j}] = \frac{1}{4}\Big([A_{\vec{k}\,i}, A_{-\vec{k}'\,j}] - [E_{\vec{k}\,i}, E_{-\vec{k}'\,j}] - [A_{\vec{k}\,i}, E_{-\vec{k}'\,j}] + [E_{\vec{k}\,i}, A_{-\vec{k}'\,j}]\Big), \tag{2.44}$$

together with (2.42), (2.43) we find the oscillator commutation relations

$$\begin{aligned}
&[a_{\vec{k}\,i}, a^{\dagger}_{\vec{k}'\,j}] = \frac{R_1}{R_6\sqrt{g}\widetilde{G}^{66}_L k_6}\frac{1}{2(2\pi)^2}\big(g_{ij} - \frac{k_i k_j}{g^{kk'}k_k k_{k'}}\big)\delta_{\vec{k},\vec{k}'},\\
&[a_{\vec{k}\,i}, a_{\vec{k}'\,j}] = 0, \qquad [a^{\dagger}_{\vec{k}\,i}, a^{\dagger}_{\vec{k}'\,j}] = 0.
\end{aligned} \tag{2.45}$$

In the gauge $\partial^i A_i(\vec{\theta}, \theta^6) = 0$, then $k^i a_{\vec{k}\,i} = g^{ij}k_j a_{\vec{k}\,i} = 0,\ k^i a^{\dagger}_{\vec{k}\,i} = g^{ij}k_j a^{\dagger}_{\vec{k}\,i} = 0$ as in (2.23), and these are consistent with the commutator (3.56). We will use this commutator to proceed with the evaluation of the Hamiltonian and momenta in (3.42,2.29). In $A_6 = 0$ gauge,

$$H_c = \int d^4\theta\,\frac{R_6\sqrt{g}}{4\pi^2 R_1}\Big(-\frac{1}{2}\widetilde{G}^{66}_L g^{ii'}\,\partial_6 A_i \partial_6 A_{i'} + \frac{1}{4}g^{ii'}g^{jj'}F_{ij}F_{i'j'}\Big), \tag{2.46}$$

which is the Hamiltonian H^{5D} in (3.2). In (2.29) after integrating by parts, we also set the second constraint described in Appendix A $\partial_i \Pi^i = 0$, to find

$$P_i = \frac{1}{4\pi^2 R_1 R_6}\int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\; g^{jj'}\,F_{6j'}F_{ij}, \tag{2.47}$$

which is the momenta P^{5D}_i in (3.2).

From (2.46), in terms of the normal mode expansion (3.53),

$$\begin{aligned}
H_c = {} & (2\pi)^2\frac{R_6\sqrt{g}}{R_1}\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}\big(\frac{1}{2}\widetilde{G}^{66}_L g^{ii'}k_6k_6 + \frac{1}{2}(g^{ii'}g^{jj'} - g^{ij'}g^{ji'})k_j k_{j'}\big)\big(a_{\vec{k}\,i}a_{-\vec{k}\,i'}e^{2ik_6\theta^6} + a^{\dagger}_{\vec{k}\,i}a^{\dagger}_{-\vec{k}\,i'}e^{-2ik_6\theta^6}\big)\\
& + (2\pi)^2\frac{R_6\sqrt{g}}{R_1}\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}\big(-\frac{1}{2}\widetilde{G}^{66}_L g^{ii'}k_6k_6 + \frac{1}{2}(g^{ii'}g^{jj'} - g^{ij'}g^{ji'})k_j k_{j'}\big)\big(a_{\vec{k}\,i}a^{\dagger}_{\vec{k}\,i'} + a^{\dagger}_{\vec{k}\,i}a_{\vec{k}\,i'}\big),
\end{aligned} \tag{2.48}$$

with the delta function

$$\int \frac{d^4\theta}{(2\pi)^4} e^{i(k_i - k'_i)\theta^i} = \delta_{\vec{k},\vec{k}'}, \tag{2.49}$$

and where k_6 is given in (3.40). From the on-shell and transverse conditions (2.23), $\widetilde{G}_L^{66} k_6 k_6 + |k|^2 = 0$, and $k^i a_{\vec{k}\,i} = k^i a^\dagger_{\vec{k}\,i} = 0$, so the time-dependence of H_c on θ^6 cancels and

$$H_c = (2\pi)^2 \frac{R_6\sqrt{g}}{R_1} \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} g^{ii'}|k|^2 \left(a_{\vec{k}\,i} a^\dagger_{\vec{k}\,i'} + a^\dagger_{\vec{k}\,i} a_{\vec{k}\,i'}\right). \tag{2.50}$$

Similarly the momenta from (2.47) become

$$P_i = -\frac{R_6\sqrt{g}}{R_1} \widetilde{G}_L^{66} g^{jj'} (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} k_6 k_i \left(a_{\vec{k}\,j'} a^\dagger_{\vec{k}\,j} + a^\dagger_{\vec{k}\,j'} a_{\vec{k}\,j}\right). \tag{2.51}$$

Then

$$\begin{aligned} -H_c + i\gamma^i P_i &= \mp\sqrt{-\widetilde{G}_L^{66}} \frac{R_6\sqrt{g}}{R_1} (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k| \Big(\pm \frac{|k|}{\sqrt{-\widetilde{G}_L^{66}}} + i\gamma^i k_i\Big) g^{jj'} \left(a_{\vec{k}\,j} a^\dagger_{\vec{k}\,j'} + a^\dagger_{\vec{k}\,j} a_{\vec{k}\,j'}\right) \\ &= \mp i\sqrt{-\widetilde{G}_L^{66}} \frac{R_6\sqrt{g}}{R_1} (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k| \Big(\pm i \frac{\sqrt{-\widetilde{G}_L^{66}}}{\widetilde{G}_L^{66}} |k| + \gamma^i k_i\Big) g^{jj'} \left(a_{\vec{k}\,j} a^\dagger_{\vec{k}\,j'} + a^\dagger_{\vec{k}\,j} a_{\vec{k}\,j'}\right). \end{aligned} \tag{2.52}$$

Since we are using a Lorentzian signature metric at this point, $-\widetilde{G}_L^{66} > 0$. Then rewriting in terms of a real Euclidean radius R_6, and making the upper sign choice in (3.40), we have

$$-H_c + i\gamma^i P_i = -i\frac{1}{R_6} \frac{R_6\sqrt{g}}{R_1} (2\pi)^2 \sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}} |k| \big(-iR_6|k| + \gamma^i k_i\big) g^{jj'} \left(a_{\vec{k}\,j} a^\dagger_{\vec{k}\,j'} + a^\dagger_{\vec{k}\,j} a_{\vec{k}\,j'}\right). \tag{2.53}$$

Inserting the polarizations as $a_{\vec{k}\,i} = f_i^\kappa a_{\vec{k}}^\kappa$ and $a_{\vec{k}\,i}^\dagger = f_i^{\lambda *} a_{\vec{k}}^{\lambda\dagger}$ from (3.54) in the commutator (3.56) gives

$$[a_{\vec{k}\,i}, a_{\vec{k}'\,j}^\dagger] = \frac{R_1}{R_6\sqrt{g}}\frac{R_6}{|k|}\frac{1}{2(2\pi)^2}\Big(g_{ij} - \frac{k_i k_j}{|k|^2}\Big)\delta_{\vec{k},\vec{k}'} = f_i^\kappa f_j^{\lambda *}[a_{\vec{k}}^\kappa, a_{\vec{k}}^{\lambda\dagger}], \tag{2.54}$$

where we choose the normalization

$$[a_{\vec{k}}^\kappa, a_{\vec{k}'}^{\lambda\dagger}] = \delta^{\kappa\lambda}\delta_{\vec{k},\vec{k}'}. \tag{2.55}$$

Then the polarization vectors satisfy

$$f_i^\kappa f_j^{\lambda *}\delta^{\kappa\lambda} = \frac{R_1}{\sqrt{g}\,|k|}\frac{1}{2(2\pi)^2}\Big(g_{ij} - \frac{k_i k_j}{|k|^2}\Big), \qquad g^{jj'} f_j^\kappa f_{j'}^{\lambda *}\delta^{\kappa\lambda} = \frac{R_1}{\sqrt{g}\,|k|}\frac{1}{2(2\pi)^2}\cdot 3,$$
$$g^{jj'} f_j^\kappa f_{j'}^{\lambda *} = \delta^{\kappa\lambda}\frac{R_1}{\sqrt{g}\,|k|}\frac{1}{2(2\pi)^2}.$$

So the exponent in (3.2) is given by

$$\begin{aligned}-H_c + i\gamma^i P_i &= -i\frac{1}{R_6}\,\frac{R_6\sqrt{g}}{R_1}(2\pi)^2\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}|k|\big(-iR_6|k| + \gamma^i k_i\big)g^{jj'}\big(2a_{\vec{k}\,j}^\dagger a_{\vec{k}\,j'} + [a_{\vec{k}\,j}, a_{\vec{k}\,j'}^\dagger]\big)\\ &= -i\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}\big(\gamma^i k_i - iR_6|k|\big)\,a_{\vec{k}}^{\kappa\dagger}a_{\vec{k}}^\kappa \quad - \frac{i}{2}\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}\big(-iR_6|k|\big)\,\delta^{\kappa\kappa}.\end{aligned} \tag{2.56}$$

Then the partition function is

$$Z^{5D,Maxwell} \equiv tr\ \exp\{2\pi(-H_c + i\gamma^i P_i)\} = Z^{5D}_{zero\ modes}\ Z^{5D}_{osc}, \tag{2.57}$$

where from (3.62),

$$Z^{5D}_{osc} = tr\ e^{-2\pi i\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}\big(\gamma^i k_i - iR_6|k|\big)\,a_{\vec{k}}^{\kappa\dagger}a_{\vec{k}}^\kappa\ -\pi R_6\sum_{\vec{k}\in\mathcal{Z}^4\neq\vec{0}}|k|\,\delta^{\kappa\kappa}}. \tag{2.58}$$

2.4 Comparison of Oscillator Traces Z^{5D}_{osc} and Z^{6D}_{osc}

In order to compare the partition functions of the two theories, we first review the calculation for the 6D chiral field from [2] setting the angles between the circle and five-torus $\alpha, \beta^i = 0$. The oscillator trace is evaluated by rewriting (2.2) as

$$
\begin{aligned}
-2\pi R_6 \mathcal{H} + i2\pi\gamma^i \mathcal{P}_i = \frac{i\pi}{12}\int_0^{2\pi} d^5\theta H_{lrs}\epsilon^{lrsmn} H_{6mn} = \frac{i\pi}{2}\int_0^{2\pi} d^5\theta \sqrt{-G} H^{6mn} H_{6mn} \\
= -i\pi \int_0^{2\pi} d^5\theta (\Pi^{mn} H_{6mn} + H_{6mn}\Pi^{mn})
\end{aligned}
\tag{2.59}
$$

where the definitions $H^{6mn} = \frac{1}{6\sqrt{-G}}\epsilon^{mnlrs}H_{lrs}$ and $H_{6mn} = \frac{1}{6\sqrt{-G}G^{66}}\epsilon_{mnlrs}H^{lrs}$ follow from the self-dual equation of motion (2.4). $\Pi^{mn}(\vec{\theta}, \theta^6)$, the field conjugate to $B_{mn}(\vec{\theta}, \theta^6)$ is defined from the Lagrangian for a general (non-self-dual) two-form $I_6 = \int d^6\theta (-\frac{\sqrt{-G}}{24}) H_{LMN}H^{LMN}$, so $\Pi^{mn} = \frac{\delta I_6}{\delta \partial_6 B_{mn}} = -\frac{\sqrt{-G}}{4} H^{6mn}$. The commutation relations of the two-form and its conjugate field $\Pi^{mn}(\vec{\theta}, \theta^6)$ are

$$
\begin{aligned}
[\Pi^{rs}(\vec{\theta}, \theta^6), B_{mn}(\vec{\theta}', \theta^6)] =& - i\delta^5(\vec{\theta} - \vec{\theta}')(\delta^r_m \delta^s_n - \delta^r_n \delta^s_m), \\
[\Pi^{rs}(\vec{\theta}, \theta^6), \Pi^{mn}(\vec{\theta}', \theta^6)] =& [B_{rs}(\vec{\theta}, \theta^6), B_{mn}(\vec{\theta}', \theta^6)] = 0.
\end{aligned}
$$

From the Bianchi identity $\partial_{[L} H_{MNP]} = 0$ and the fact that (2.4) implies $\partial^L H_{LMN} = 0$, then a solution to (2.1) is given by a solution to the homogeneous equations $\partial^L \partial_L B_{MN} = 0$, $\partial^L B_{LN} = 0$. These have a plane wave solution

$$
B_{MN}(\vec{\theta}, \theta^6) = f_{MN}(p) e^{ip\cdot\theta} + (f_{MN}(p)e^{ip\cdot\theta})^*; \qquad G^{LN} p_L p_N = 0\,; \qquad p^L f_{LN} = 0; \tag{2.60}
$$

and quantum tensor field expansion

$$
B_{mn}(\vec{\theta}, \theta^6) = zero\ modes \quad + \sum_{\vec{p} = p_l \in \mathcal{Z}^5 \neq \vec{0}} (f^\kappa_{mn} b^\kappa_{\vec{p}} e^{ip\cdot\theta} + f^{\kappa *}_{mn} b^{\kappa\dagger}_{\vec{p}} e^{-ip^*\cdot\theta}) \tag{2.61}
$$

for the three physical polarizations of the 6D chiral two-form [2], $1 \leq \kappa \leq 3$. Because oscillators with different polarizations commute, each polarization can be treated separately and the result then cubed. Without the zero mode term,

$$B_{mn}(\vec{\theta},\theta^6) = \sum_{\vec{p}\neq 0}(b_{\vec{p}mn}\, e^{ip\cdot\theta} + b^{\dagger}_{\vec{p}mn} e^{-ip^*\cdot\theta})\,, \tag{2.62}$$

for $b_{\vec{p}mn} = f^1_{mn} b^1_{\vec{p}}$ for example, with a similar expansion for $\Pi^{mn}(\vec{\theta},\theta^6)$ in terms of $c^{6mn\dagger}_{\vec{p}}$. From (2.60) the momentum p_6 is

$$p_6 = -\gamma^i p_i - iR_6\sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}. \tag{2.63}$$

For the gauge choice $B_{6n} = 0$, the exponent (3.68) becomes

$$\begin{aligned}
&- i\pi(2\pi)^5 \sum_{\vec{p}=p_l\in\mathcal{Z}^5\neq 0} ip_6(\mathcal{C}^{6mn\dagger}_{\vec{p}} B_{\vec{p}mn} + B_{\vec{p}mn}\mathcal{C}^{6mn\dagger}_{\vec{p}}) \\
&= -2i\pi\sum_{\vec{p}\neq 0} p_6 \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\lambda}_{\vec{p}} f^{\kappa mn}(p) f^{\lambda}_{mn}(p) - i\pi\sum_{\vec{p}\neq 0} p_6 f^{\kappa mn}(p) f^{\kappa}_{mn}(p) \\
&= -2i\pi\sum_{\vec{p}\neq 0} p_6 \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - i\pi\sum_{\vec{p}\neq 0} p_6 \delta^{\kappa\kappa},
\end{aligned} \tag{2.64}$$

with $B_{\vec{p}mn} \equiv b_{\vec{p}mn} + b^{\dagger}_{-\vec{p}mn}$, $\mathcal{C}^{6mn\dagger}_{\vec{p}} \equiv c^{6mn}_{-\vec{p}} + c^{6mn\dagger}_{\vec{p}}$. The polarization tensors have been restored where $1 \leq \kappa, \lambda \leq 3$ and the oscillators $B^{\kappa}_{\vec{p}}, \mathcal{C}^{\lambda\dagger}_{\vec{p}}$ satisfy the commutation relation

$$[B^{\kappa}_{\vec{p}}, \mathcal{C}^{\lambda\dagger}_{\vec{p}}] = \delta^{\kappa\lambda}\ \delta_{\vec{p},\vec{p}'}. \tag{2.65}$$

So restricting the manifold to a circle times a five-torus in [2] we have

$$\begin{aligned}
&- 2\pi R_6\mathcal{H} + i2\pi\gamma^i P_i \\
&= -2i\pi \sum_{\vec{p}\in\mathcal{Z}^5\neq 0}\Big(- \gamma^i p_i - iR_6\sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}\Big)\mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - \pi R_6 \sum_{\vec{p}\in\mathcal{Z}^5}\sqrt{g^{ij}p_ip_j + \frac{p_1^2}{R_1^2}}\ \delta^{\kappa\kappa}
\end{aligned} \tag{2.66}$$

The oscillator trace (2.2) is

$$Z^{6D}_{osc} = tr\, e^{-t\mathcal{H}+i2\pi\gamma^i P_i} = tr\, e^{-2i\pi\sum_{\vec{p}\neq 0} p_6 C^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - \pi R_6 \sum_{\vec{p}} \sqrt{g^{ij}p_i p_j + \frac{p_1^2}{R_1^2}}\ \delta^{\kappa\kappa}},$$

$$Z^{6D,chiral} = Z^{6D}_{zero\, modes} \cdot \big(e^{-\pi R_6 \sum_{\vec{n}\in\mathcal{Z}^5} \sqrt{g^{ij}n_i n_j + \frac{n_1^2}{R_1^2}}} \prod_{\vec{n}\in\mathcal{Z}^5\neq 0} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_i n_j + \frac{n_1^2}{R_1^2}}\ + i2\pi\gamma^i n_i}}\big)^3. \tag{2.67}$$

Regularizing the vacuum energy as in [2], the chiral field partition function (2.2) becomes

$$Z^{6D,chiral} = Z^{6D}_{zero\, modes} \cdot \Big(e^{R_6\pi^{-3}\sum_{\vec{n}\neq\vec{0}} \frac{\sqrt{G_5}}{(g_{ij}n^i n^j + R_1^2 (n^1)^2)^3}} \prod_{\vec{n}\in\mathcal{Z}^5\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_i n_j + \frac{(n_1)^2}{R_1^2}} + i2\pi\gamma^i n_i}}\Big)^3, \tag{2.68}$$

where $Z^{6D}_{zero\, modes}$ is given in (2.11). Lastly we compute the 5D Maxwell partition function (3.2) from (3.64),

$$Z^{5D,Maxwell} = Z^{5D}_{zero\ modes} \cdot\ tr\ e^{-2i\pi\sum_{\vec{k}\neq\vec{0}}(\gamma^i k_i - iR_6\sqrt{g^{ij}k_i k_j})\, a^{\kappa\dagger}_{\vec{k}}\ a^{\kappa}_{\vec{k}} - \pi\sum_{\vec{k}\neq\vec{0}}(R_6\sqrt{g^{ij}k_i k_j})\,\delta^{\kappa\kappa}}, \tag{2.69}$$

where $\vec{k} = k_i = n_i \in \mathcal{Z}^4$ on the torus. From the standard Fock space argument

$$tr\ \omega^{\sum_p p a_p^\dagger a_p} = \prod_p \sum_{k=0}^{\infty} \langle k|\omega^{p a_p^\dagger a_p}|k\rangle = \prod_p \frac{1}{1-\omega^p},$$

we perform the trace on the oscillators,

$$Z^{5D}_{osc} = \Big(e^{-\pi R_6 \sum_{\vec{n}\in\mathcal{Z}^4}\sqrt{g^{ij}n_i n_j}} \prod_{\vec{n}\in\mathcal{Z}^4\neq\vec{0}} \frac{1}{1-e^{-i2\pi(\gamma^i n_i - iR_6\sqrt{g^{ij}n_i n_j})}}\Big)^3, \tag{2.70}$$

$$Z^{5D,Maxwell} = Z^{5D}_{zero\ modes} \cdot\ \Big(e^{-\pi R_6 \sum_{\vec{n}\in\mathcal{Z}^4}\sqrt{g^{ij}n_i n_j}} \prod_{\vec{n}\in\mathcal{Z}^4\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_i n_j} - 2\pi i\gamma^i n_i}}\Big)^3, \tag{2.71}$$

where $Z^{5D}_{zero\ modes}$ is given in (2.20). (3.66) and (2.67) are each manifestly $SL(4,\mathcal{Z})$

invariant due to the underlying $SO(4)$ invariance we have labeled as $i = 2,3,4,5$. We use the $SL(4,\mathcal{Z})$ invariant regularization of the vacuum energy reviewed in Appendix C to obtain

$$Z^{5D,Maxwell} = Z^{5D}_{zero\ modes} \cdot \Big(e^{\frac{3}{8}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{ij}n^in^j)^{\frac{5}{2}}}} \prod_{\vec{n}\in\mathcal{Z}^4\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}-2\pi i\gamma^i n_i}}\Big)^3, \tag{2.72}$$

where the sum is on the original lattice $\vec{n} = n^i \in \mathcal{Z}^4 \neq \vec{0}$, and the product is on the dual lattice $\vec{n} = n_i \in \mathcal{Z}^4 \neq \vec{0}$. In Appendix D we prove that the product of the zero mode contribution and the oscillator contribution in (3.67) is $SL(5,\mathcal{Z})$ invariant. In (F.48) we give an equivalent expression,

$$\begin{aligned} Z^{5D,Maxwell} = Z^{5D}_{zero\ modes} &\cdot \Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_2}|n|+2\pi i\gamma^2 n}}\Big)^3 \\ &\cdot \Big(\prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}+2\pi i\gamma^i n_i}}\Big)^3, \end{aligned} \tag{2.73}$$

with $< H >_{p_\perp}$ defined in (B.13). In Appendix D we also prove the $SL(5,\mathcal{Z})$ invariance of the 6D chiral partition function (2.68), using the equivalent form (C.44),

$$\begin{aligned} Z^{6D,chiral} = Z^{6D}_{zero\ modes} &\cdot \big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\in\mathcal{Z}\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_2}|n|+2\pi i\,\gamma^2\,n\,)}}\big)^3 \\ &\cdot \big(\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 <H>^{6D}_{p_\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j+\frac{n_1^2}{R_1^2}}\ +i2\pi\gamma^i n_i}}\big)^3 \end{aligned} \tag{2.74}$$

with $< H >^{6D}_{p_\perp}$ in (F.64). Thus the partition functions of the two theories are both $SL(5,\mathcal{Z})$ invariant, but they are not equal.

The comparison of the 6D chiral theory on $S^1 \times T^5$ and the abelian gauge theory on T^5 shows the exponent of the oscillator contribution to the partition function for the 6D theory (2.66),

$$
\begin{aligned}
& -2\pi R_6 \mathcal{H} + i2\pi\gamma^i P_i \\
& = -2\pi \sum_{\vec{p}\in\mathcal{Z}^5\neq 0} \Big(-i\gamma^i p_i + R_6\sqrt{g^{ij}p_i p_j + \frac{p_1^2}{R_1^2}}\,\Big)\, \mathcal{C}_{\vec{p}}^{\kappa\dagger} B_{\vec{p}}^{\kappa} - \pi R_6 \sum_{\vec{p}\in\mathcal{Z}^5} \sqrt{g^{ij}p_i p_j + \frac{p_1^2}{R_1^2}}\ \delta^{\kappa\kappa},
\end{aligned}
\tag{2.75}
$$

and for the gauge theory (3.62),

$$
-2\pi H^{5D} + 2\pi i\gamma^i P_i^{5D} = -2\pi \sum_{\vec{k}\in\mathcal{Z}^4\neq 0} \Big(i\gamma^i k_i + R_6\sqrt{g^{ij}k_i k_j}\Big)\ a_{\vec{k}}^{\kappa\dagger}\, a_{\vec{k}}^{\kappa} - \pi R_6 \sum_{\vec{k}\in\mathcal{Z}^4} \sqrt{g^{ij}k_i k_j}\, \delta^{\kappa\kappa},
\tag{2.76}
$$

differ only by the sum on the Kaluza-Klein modes p_1 of S^1 since for the chiral case $\vec{p}\in\mathcal{Z}^5$, and for the Maxwell case $\vec{k}\in\mathcal{Z}^4$. Both theories have three polarizations, $1\leq\kappa\leq 3$, and from (3.69), (3.61) the oscillators have the same commutation relations,

$$
[B_{\vec{p}}^{\lambda}, \mathcal{C}_{\vec{p}}^{\lambda\dagger}] = \delta^{\kappa\lambda}\ \delta_{\vec{p},\vec{p}'}, \qquad [a_{\vec{k}}^{\kappa}, a_{\vec{k}'}^{\lambda\dagger}] = \delta^{\kappa\lambda}\ \delta_{\vec{k},\vec{k}'}.
\tag{2.77}
$$

If we discard the Kaluza-Klein modes p_1^2 in the usual limit [30] as the radius of the circle R_1 is very small with respect to the radii and angles g_{ij}, R_6, of the five-torus, then the oscillator products in (3.74) and (3.73) are equivalent. This holds as a precise limit since we can separate the product on $n_\perp = (n_1, n_\alpha) \neq 0_\perp$ in (3.74), into $(n_1 = 0, n_\alpha \neq (0,0,0))$ and $(n_1\neq 0, \text{all } n_\alpha)$, to find at fixed n_2,

$$
\begin{aligned}
& \prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{ij}n_i n_j + \frac{(n_1)^2}{R_1^2}} + 2\pi i\gamma^i n_i}} \\
& \quad = \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{ij}n_i n_j} + 2\pi i\gamma^i n_i}} \cdot \prod_{n_1\neq 0, n_\alpha\in\mathcal{Z}^3} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{ij}n_i n_j + \frac{(n_1)^2}{R_1^2}} + 2\pi i\gamma^i n_i}}.
\end{aligned}
\tag{2.78}
$$

In the limit of small R_1 the last product reduces to unity, thus for S^1 smaller than T^5

$$\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j+\frac{(n_1)^2}{R_1^2}}+2\pi i\gamma^i n_i}} \to \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}+2\pi i\gamma^i n_i}}. \tag{2.79}$$

Inspecting the regularized vacuum energies $< H >_{p_\perp}$ and $< H >^{6D}_{p_\perp}$ in (B.13),(F.64),

$$\begin{aligned}
< H >_{p_\perp\neq 0} &= -\pi^{-1}\,|p_\perp|\sum_{n=1}^{\infty}\cos(p_\alpha\kappa^\alpha 2\pi n)\frac{K_1(2\pi nR_2|p_\perp|)}{n}, \quad \text{for} \quad |p_\perp|\equiv\sqrt{\widetilde{g}^{\alpha\beta}n_\alpha n_\beta},\\
< H >^{6D}_{p_\perp\neq 0} &= -\pi^{-1}\,|p_\perp|\sum_{n=1}^{\infty}\cos(p_\alpha\kappa^\alpha 2\pi n)\frac{K_1(2\pi nR_2|p_\perp|)}{n}, \quad \text{for} \quad |p_\perp|\equiv\sqrt{\frac{(n_1)^2}{R_1^2}+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta},
\end{aligned} \tag{2.80}$$

we see they have the same form of spherical Bessel functions, but the argument differs by Kaluza-Klein modes. Again separating the product on $n_\perp = (n_1, n_\alpha)$ in (3.74), into

$(n_1 = 0, n_\alpha \neq (0,0,0))$ and $(n_1 \neq 0, \text{all } n_\alpha)$ we have

$$\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6<H>^{6D}_{p_\perp}} = \Big(\prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6<H>_{p_\perp}}\Big)\cdot\Big(\prod_{n_1\neq 0, n_\alpha\in\mathcal{Z}^3} e^{-2\pi R_6<H>^{6D}_{p_\perp}}\Big). \tag{2.81}$$

In the limit $R_1 \to 0$, the last product is unity because for $n_1 \neq 0$,

$$\begin{aligned}
&\lim_{R_1\to 0}\sqrt{\frac{(n_1)^2}{R_1^2}+\widetilde{g}^{\alpha\beta}n_\alpha n_\beta} \sim \frac{|n_1|}{R_1},\\
&\lim_{R_1\to 0}|p_\perp|\;K_1(2\pi nR_2|p_\perp|) = \lim_{R_1\to 0}\frac{|n_1|}{R_1}\;K_1\big(2\pi nR_2\frac{|n_1|}{R_1}\big) = 0,
\end{aligned} \tag{2.82}$$

since $\lim_{x\to\infty} xK_1(x) \sim \sqrt{x}\,e^{-x} \to 0$. [52]. So (3.78) leads to

$$\lim_{R_1\to 0}\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6<H>^{6D}_{p_\perp}} = \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6<H>_{p_\perp}}. \tag{2.83}$$

Thus in the limit where the radius of the circle S^1 is small with respect to T^5, which is the limit of weak coupling g^2_{5YM}, we have proved

$$
\begin{aligned}
&\lim_{R_1\to 0} \prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6 <H>^{6D}_{p_\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j+\frac{n_1^2}{R_1^2}}+i2\pi\gamma^i n_i}} \\
&= \prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}+2\pi i\gamma^i n_i}}.
\end{aligned} \tag{2.84}
$$

So together with (2.3), we have shown the partition functions of the chiral theory on $S^1\times T^5$ and of Maxwell theory on T^5, which we computed in (3.74) and (3.73), are equal only in the weak coupling limit,

$$
\lim_{R_1\to 0} Z^{6D,chiral} = Z^{5D,Maxwell}. \tag{2.85}
$$

2.5 Discussion and Conclusions

We have addressed a conjecture of the quantum equivalence between the six-dimensional conformally invariant $N=(2,0)$ theory compactified on a circle and the five-dimensional maximally supersymmetric Yang-Mills theory. in this chapter we consider an abelian case without supersymmetry when the five-dimensional manifold is a twisted torus. We compute the partition functions for the chiral tensor field B_{LN} on $S^1\times T^5$, and for the Maxwell field $A_{\tilde{m}}$ on T^5. We prove the two partition functions are each $SL(5,\mathcal{Z})$ invariant, but are equal only in the limit of weak coupling g^2_{5YM}, a parameter which is proportional to R_1, the radius of the circle S^1.

To carry out the computations we first restricted an earlier calculation [2] of the chiral partition function on T^6 to $S^1\times T^5$. Then we used an operator quantization to compute the Maxwell partition on T^5 as defined in (3.2) which inserts non-zero γ^i as the coefficient of P^{5D}_i, but otherwise quantizes the theory in a 5D Lorentzian signature metric that has zero angles with its time direction, *i.e.* $\gamma^i=0,\ 2\leq i\leq 5$, [13]. We used this metric and form (3.2) to derive both the zero mode and oscillator contributions. The Maxwell field theory was thus quantized on T^5, with the Dirac method of constraints resulting in the commutation relations in (3.56).

Comparing the partition function of the Maxwell field on a twisted five-torus T^5 with that of a two-form potential with a self-dual three-form field strength on $S^1 \times T^5$, where the radius of the circle is $R_1 \equiv g_{5YM}^2/4\pi^2$, we find the two theories are not equivalent as quantum theories, but are equal only in the limit where R_1 is small relative to the metric parameters of the five-torus, a limit which effectively removes the Kaluza-Klein modes from the 6d partition sum. How to incorporate these modes rigorously in the 5D theory, possibly interpreted as instantons in the non-abelian version of the gauge theory with appropriate dynamics remains difficult [36]-[39], suggesting that the 6d finite conformal $N = (2,0)$ theory on a circle is an ultraviolet completion of the 5D maximally supersymmetric gauge theory rather than an exact quantum equivalence.

Furthermore, it would be compelling to find how expressions for the partition function of the 6d $N = (2,0)$ conformal quantum theory computed on various manifolds using localization should reduce to the expression in [2] in an appropriate limit, providing a check that localization is equivalent to canonical quantization.

Chapter 3

EM Duality on T^4 from the Fivebrane on $\mathbf{T^2} \times \mathbf{T^4}$

3.1 Introduction

Four-dimensional $N = 4$ Yang-Mills theory is conjectured to possess S-duality, which implies the theory with gauge coupling g, gauge group G, and theta parameter θ is equivalent to one with $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ transformed by modular transformations $SL(2, \mathcal{Z})$, and the group to $G^\vee$ [41]-[43], with the weight lattice of $G^\vee$ dual to that of G . The conjecture has been tested by the Vafa-Witten partition function on various four-manifolds [44]. More recently, a computation of the $N = 4$ Yang-Mills partition function on the four-sphere using the localization method for quantization, enables checking S-duality directly [8].

This duality is believed to have its origin in a certain superconformal field theory in six dimensions, the M5 brane $(2, 0)$ theory. When the $6d$, $N = (2, 0)$ theory is compactified on T^2, one obtains the $4d$, $N = 4$ Yang-Mills theory, and the $SL(2, \mathcal{Z})$ group of the torus should imply the S-duality of the four-dimensional gauge theory [45]-[32].

in this chapter, we compare the partition function of the $6d$ chiral tensor boson of one fivebrane compactified on $T^2 \times T^4$, with that of $U(1)$ gauge theory with a θ parameter, compactified on T^4. We use these to show explicitly how the $6d$ theory is the origin of S-duality in the gauge theory. Since the $6d$ chiral two-form has a self-dual three-form field strength and thus lacks a Lagrangian [1], we will use the Hamiltonian formulation to compute the partition functions for both theories.

As motivated by [13], the four-dimensional $U(1)$ gauge partition function on T^4

is

$$Z^{4d,Maxwell} \equiv tre^{-2\pi H^{4d}+i2\pi\gamma^\alpha P^{4d}_\alpha} = Z^{4d}_{\text{zero modes}} \cdot Z^{4d}_{\text{osc}}, \tag{3.1}$$

where the Hamiltonian and momentum are

$$H^{4d} = \int_0^{2\pi} d^3\theta \Big(\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\Pi^\alpha\Pi^\beta + \frac{e^2}{32\pi^2}\sqrt{g}\left[\frac{\theta^2}{4\pi^2}+\frac{16\pi^2}{e^4}\right] g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta} + \frac{\theta e^2}{16\pi^2}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}F_{\gamma\delta}\Pi^\beta\Big),$$
$$P^{4d}_\alpha = \int_0^{2\pi} d^3\theta \; \Pi^\beta F_{\alpha\beta} \tag{3.2}$$

in terms of the gauge field strength tensor $F_{ij}(\theta^3,\theta^4,\theta^5,\theta^6)$, the conjugate momentum Π^α, and the constant parameters $g_{\alpha\beta}, R_6$ and γ^α in the metric G_{ij} of T^4. They will be derived from the abelian gauge theory Lagrangian, given here for Euclidean signature

$$I = \frac{1}{8\pi}\int_{T^4} d\theta_3 d\theta_4 d\theta_5 d\theta_6 \left(\frac{4\pi}{e^2}\sqrt{g}F^{ij}F_{ij} - \frac{i\theta}{4\pi}\epsilon^{ijkl}F_{ij}F_{kl}\right), \tag{3.3}$$

with $\epsilon^{3456}=1$, $\epsilon_{ijkl} = g\epsilon^{ijkl}$, and $g = \det(G_{ij})$.

In contrast, the partition function of the abelian chiral two-form on $T^2\times T^4$ is [2]

$$Z^{6d,chiral} = tr\, e^{-2\pi R_6\mathcal{H}+i2\pi\gamma^\alpha\mathcal{P}_\alpha} = Z^{6d}_{\text{zero modes}} \cdot Z^{6d}_{\text{osc}},$$
$$\mathcal{H} = \frac{1}{12}\int_0^{2\pi} d\theta^1\dots d\theta^5\sqrt{G}_5 G_5{}^{mm'}G_5{}^{nn'}G_5{}^{pp'}H_{mnp}(\vec\theta,\theta^6)\,H_{m'n'p'}(\vec\theta,\theta^6),$$
$$\mathcal{P}_\alpha = -\frac{1}{24}\int_0^{2\pi} d\theta^1...d\theta^5\epsilon^{mnprs}H_{mnp}(\vec\theta,\theta^6)\,H_{\alpha rs}(\vec\theta,\theta^6) \tag{3.4}$$

where θ^1 and θ^2 are the coordinates of the two one-cycles of T^2. The time direction θ^6 is common to both theories, the angle between θ^1 and θ^2 is β^2, and $G_5{}^{mn}$ is the inverse metric of G_{5mn}, where $1\le m,n\le 5$. The eight angles between the two-torus and the four-torus are set to zero.

Section 2 is a list of our results; their derivations are presented in the succeeding sections. In section 3, the contribution of the zero modes to the partition function for the chiral theory on the manifold $M = T^2 \times T^4$ is computed as a sum over ten integer eigenvalues using the Hamiltonian formulation. The zero mode sum for the gauge theory on the same $T^4 \subset M$ is calculated with six integer eigenvalues. We find that once we identify the modulus of the T^2 contained in M, $\tau = \beta^2 + i\frac{R_1}{R_2}$, with the gauge couplings $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, then the two theories are related by $Z^{6d}_{zero\,modes} = \epsilon Z^{4d}_{zero\,modes}$, where ϵ is due to the zero modes of the scalar field that arises in addition to F_{ij} from the compactification of the $6d$ self-dual three-form. In section 4, the abelian gauge theory is quantized on a four-torus using Dirac constraints, and the Hamiltonian and momentum are computed in terms of oscillator modes. For small T^2, the Kaluza-Klein modes are removed from the partition function of the chiral two-form, and in this limit it agrees with the gauge theory result, up to the scalar field contribution. In Appendix E, we show the path integral quantization gives the same result for the 4d gauge theory partition function as canonical quantization. However, the zero and oscillator mode contributions differ in the two quantizations. In Appendix F, we show how the zero and oscillator mode contributions transform under $SL(2,\mathcal{Z})$ for the 6d theory, as well as for both quantizations of the 4d theory. We prove the partition functions in 4d and 6d are both $SL(2,\mathcal{Z})$ invariant. In Appendix G, the vacuum energy is regularized. In Appendix H, we introduce a complete set of $SL(4,\mathcal{Z})$ generators, and then prove the 4d and 6d partition functions are invariant under $SL(4,\mathcal{Z})$ transformations.

We compute partition functions for a chiral two-form on $T^2 \times T^4$ and for a $U(1)$ gauge boson on the same T^4. The geometry of the manifold $T^2 \times T^4$ will be described by the line element,

$$\begin{aligned} ds^2 = & R_2^2(d\theta^2 - \beta^2 d\theta^1)^2 + R_1^2(d\theta^1)^2 \\ & + \sum_{\alpha,\beta} g_{\alpha\beta}(d\theta^\alpha - \gamma^\alpha d\theta^6)(d\theta^\beta - \gamma^\beta d\theta^6) + R_6^2(d\theta^6)^2, \end{aligned} \tag{3.5}$$

with $0 \leq \theta^I \leq 2\pi$, $1 \leq I \leq 6$, and $3 \leq \alpha \leq 5$. R_1, R_2 are the radii for directions $I = 1, 2$ on T^2, and β^2 is the angle between them. $g_{\alpha\beta}$ fixes the metric for a T^3 submanifold of T^4, R_6 is the remaining radius, and γ^α is the angle between those. So, from (3.5) the metric is

$$\begin{aligned} T^2: \quad & G_{11} = {R_1}^2 + R_2^2\beta^2\beta^2, \qquad G_{12} = -R_2^2\beta^2, \qquad G_{22} = R_2^2; \\ T^4: \quad & G_{\alpha\beta} = g_{\alpha\beta}, \qquad G_{\alpha 6} = -g_{\alpha\beta}\gamma^\beta, \qquad G_{66} = {R_6}^2 + g_{\alpha\beta}\gamma^\alpha\gamma^\beta; \\ & G_{\alpha 1} = G_{\alpha 2} = 0, \qquad G_{16} = G_{26} = 0; \end{aligned} \tag{3.6}$$

and the inverse metric is

$$\begin{aligned} T^2: \quad & G^{11} = \frac{1}{{R_1}^2}, \qquad G^{12} = \frac{\beta^2}{{R_1}^2}, \qquad G^{22} = \frac{1}{R_2^2} + \frac{\beta^2\beta^2}{{R_1}^2} \equiv g^{22} + \frac{\beta^2\beta^2}{{R_1}^2}; \\ T^4: \quad & G^{\alpha\beta} = g^{\alpha\beta} + \frac{\gamma^\alpha\gamma^\beta}{{R_6}^2}, \qquad G^{\alpha 6} = \frac{\gamma^\alpha}{{R_6}^2}, \qquad G^{66} = \frac{1}{{R_6}^2} \\ & G^{1\alpha} = G^{2\alpha} = 0, \qquad G^{16} = G^{26} = 0. \end{aligned} \tag{3.7}$$

θ^6 is chosen to be the time direction for both theories. In the $4d$ expression (3.3) the indices of the field strength tensor have $3 \leq i, j, k, l \leq 6$, whereas in (3.4), the Hamiltonian and momentum are written in terms of fields with indices $1 \leq m, n, p, r, s \leq 5$.

The 5-dimensional inverse in directions 1, 2, 3, 4, 5 is ${G_5}^{mn}$,

$$
\begin{aligned}
&G_5^{11} = \frac{1}{{R_1}^2}, \qquad G_5^{12} = \frac{\beta^2}{{R_1}^2}, \qquad G_5^{22} = g^{22} + \frac{\beta^2\beta^2}{{R_1}^2} \\
&G_5^{\alpha\beta} = g^{\alpha\beta}, \qquad G_5^{1\alpha} = 0, \qquad G_5^{2\alpha} = 0.
\end{aligned} \tag{3.8}
$$

$g^{\alpha\beta}$ is the $3d$ inverse of $g_{\alpha\beta}$. The determinants are related by

$$
\sqrt{G} = \sqrt{\det G_{IJ}} = R_1 R_2 \sqrt{g} = R_1 R_2 R_6 \sqrt{\tilde{g}} = R_6 \sqrt{G_5}, \tag{3.9}
$$

where G is the determinant for $6d$ metric G_{IJ}. G_5, g and $\tilde{g}$ are the determinants for the $5D$ metric G_{mn}, $4d$ metric G_{ij}, and $3d$ metric $g_{\alpha\beta}$ respectively.

The zero mode partition function of the $6d$ chiral two-form on $T^2 \times T^4$ with the metric (3.7) is

$$
\begin{aligned}
Z^{6d}_{\text{zero modes}} = &\sum_{n_8,n_9,n_{10}} \exp\{-\frac{\pi R_6}{R_1 R_2}\sqrt{\tilde{g}} g^{\alpha\alpha'} H_{12\alpha} H_{12\alpha'}\} \\
&\cdot \sum_{n_7} \exp\{-\frac{\pi}{6} R_6 R_1 R_2 \sqrt{\tilde{g}} g^{\alpha\alpha'} g^{\beta\beta'} g^{\delta\delta'} H_{\alpha\beta\delta} H_{\alpha'\beta'\delta'} - i\pi\gamma^\alpha \epsilon^{\gamma\beta\delta} H_{12\gamma} H_{\alpha\beta\delta}\} \\
&\cdot \sum_{n_4,n_5,n_6} \exp\{-\frac{\pi}{2} R_6 R_1 R_2 \sqrt{\tilde{g}} (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{{R_1}^2}) g^{\alpha\alpha'} g^{\beta\beta'} H_{2\alpha\beta} H_{2\alpha'\beta'}\} \\
&\cdot \sum_{n_1,n_2,n_3} \exp\{-\pi \frac{R_6 R_2}{R_1} \sqrt{\tilde{g}} \beta^2 g^{\alpha\alpha'} g^{\beta\beta'} H_{1\alpha\beta} H_{2\alpha'\beta'} + i\pi\gamma^\alpha \epsilon^{\gamma\beta\delta} H_{1\gamma\beta} H_{2\alpha\delta} \\
&\qquad - \frac{\pi}{4}\frac{R_6 R_2}{R_1} \sqrt{\tilde{g}} (g^{\alpha\alpha'} g^{\beta\beta'} - g^{\alpha\beta'} g^{\beta\alpha'}) H_{1\alpha\beta} H_{1\alpha'\beta'}\}
\end{aligned} \tag{3.10}
$$

where the zero mode eigenvalues of the field strength tensor are integers, and (3.10) factors into a sum on $H_{\alpha\beta\gamma}$ as $H_{345} = n_7$, $H_{12\alpha}$ as $H_{123} = n_8$, $H_{124} = n_9$, $H_{125} = n_{10}$; and a sum over $H_{1\alpha\beta}$ defined as $H_{134} = n_1$, $H_{145} = n_2$, $H_{135} = n_3$ and $H_{2\alpha\beta}$ as $H_{234} = n_4$, $H_{245} = n_5$, $H_{235} = n_6$, as we will show in section 3.

The zero mode partition function of the $4d$ gauge boson on T^4 with the metric

(3.6) is

$$Z^{4d}_{\text{zero modes}} = \sum_{n_4,n_5,n_6} \exp\{-\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta\} \cdot \sum_{n_1,n_2,n_3} \exp\{-\frac{\theta e^2}{8\pi}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta\}$$
$$\cdot \exp\{-\frac{e^2\sqrt{g}}{8}(\frac{\theta^2}{4\pi^2}+\frac{16\pi^2}{e^4}) g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}\}, \tag{3.11}$$

where $\widetilde{\Pi}^\alpha$ take integer values $\widetilde{\Pi}^3 = n_4$, $\widetilde{\Pi}^4 = n_5$, $\widetilde{\Pi}^5 = n_6$, and $\widetilde{F}_{34} = n_1$, $\widetilde{F}_{35} = n_2$, $\widetilde{F}_{45} = n_3$, from section 3. We identify the integers

$$H_{2\alpha\beta} = \widetilde{F}_{\alpha\beta} \qquad \text{and} \qquad H_{1\alpha\beta} = \frac{1}{\tilde{g}}\epsilon_{\alpha\beta\gamma}\widetilde{\Pi}^\gamma, \tag{3.12}$$

where $\tilde{g} = g\,R_6^{-2}$ from (3.9), and the modulus

$$\tau = \beta^2 + i\frac{R_1}{R_2} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2},$$

so that as shown in section 3, we have the factorization

$$Z^{6d}_{\text{zero modes}} = \epsilon\, Z^{4d}_{\text{zero modes}}, \tag{3.13}$$

where ϵ comes from the remaining four zero modes $H_{\alpha\beta\gamma}$ and $H_{12\alpha}$ due to the additional scalar that occurs in the compactification of the $6d$ self-dual three-form field strength,

$$\epsilon = \sum_{n_8,n_9,n_{10}} \exp\{-\frac{\pi R_6}{R_1R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'}\}$$
$$\cdot \sum_{n_7} \exp\{-\frac{\pi}{6}R_6R_1R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'} - i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}\}. \tag{3.14}$$

From section 4, there is a similar relation between the oscillator partition functions

$$\lim_{R_1,R_2\to 0} Z^{6d}_{\text{osc}} = \epsilon' Z^{4d}_{\text{osc}}, \tag{3.15}$$

where

$$Z^{6d}_{\rm osc} = \Big(e^{R_6\pi^{-3}\sum_{\vec{n}\neq\vec{0}}\frac{\sqrt{G_5}}{(G_{mp}n^m n^p)^3}}\prod_{\vec{p}\in\mathcal{Z}^5\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta+\tilde{p}^2}+2\pi i\gamma^\alpha p_\alpha}}\Big)^3, \tag{3.16}$$

$$Z^{4d}_{\rm osc} = \Big(e^{\frac{1}{2}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}}\cdot\prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2, \tag{3.17}$$

where $\tilde{p}^2 \equiv \frac{p_1^2}{R_1^2} + (\frac{1}{R_1^2} + \frac{\beta^2\beta^2}{R_2^2})p_2^2 + \frac{2\beta^2}{R_1^2}p_1p_2$, and ϵ' is the oscillator contribution from the additional scalar,

$$\epsilon' = e^{\frac{1}{2}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}}\cdot\prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}. \tag{3.18}$$

Therefore, in the limit of small T^2, we have

$$\lim_{R_1,R_2\to 0} Z^{6d,\,chiral} = \epsilon\,\epsilon'\,Z^{4d,\,Maxwell}. \tag{3.19}$$

We use this relation between the $6d$ and $4d$ partition functions to extract the S-duality of the latter from a geometric symmetry of the former. For $\tau = \beta^2 + i\frac{R_1}{R_2} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, under the $SL(2,\mathcal{Z})$ transformations

$$\tau \to -\frac{1}{\tau}; \quad \tau \to \tau - 1, \tag{3.20}$$

$Z^{6d}_{\rm zero\,modes}$ and $Z^{6d}_{\rm osc}$ are separately invariant, as are $Z^{4d}_{\rm zero\,modes}$ and $Z^{4d}_{\rm osc}$, which we will prove in Appendix F. In particular, $Z^{4d}_{\rm osc}$ is independent of e^2 and θ. A path integral computation agrees with our $U(1)$ partition function, as we review in Appendix E [47]. Nevertheless, in the path integral quantization the zero and non-zero mode contributions are rearranged, and although each is invariant under $\tau \to \tau - 1$, they transform differently under $\tau \to -\frac{1}{\tau}$, with $Z^{PI}_{\rm zero\,modes} \to |\tau|^3 Z^{PI}_{\rm zero\,modes}$ and $Z^{PI}_{\rm non-zero\,modes} \to$

$|\tau|^{-3} Z^{PI}_{\text{non-zero modes}}$. For a general spin manifold, the $U(1)$ partition function transforms as a modular form under S-duality [48], but in the case of T^4 which we consider in this chapter the weight is zero.

3.2 Zero Modes

In this section, we show details for the computation of the zero mode partition functions. The $N = (2,0)$, $6d$ world volume theory of the fivebrane contains a chiral two-form B_{MN}, which has a self-dual three-form field strength $H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$ with $1 \leq L, M, N \leq 6$,

$$H_{LMN}(\vec{\theta}, \theta^6) = \frac{1}{6\sqrt{-G}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6). \tag{3.21}$$

Since there is no covariant Lagrangian description for the chiral two-form, we compute its partition function from (3.4). As in [2],[6] the zero mode partition function of the $6d$ chiral theory is calculated in the Hamiltonian formulation similarly to string theory,

$$Z^{6d}_{\text{zero modes}} = tr\left(e^{-t\mathcal{H} + iy^l \mathcal{P}_l}\right) \tag{3.22}$$

where $t = 2\pi R_6$ and $y^l = 2\pi \frac{G^{l6}}{G^{66}}$, with $l = 1, ..5$. However, y^1 and y^2 are zero due to the metric (3.7). Neglecting the integrations and using the metric (3.8) in (3.4), we find

$$\begin{aligned}
-t\mathcal{H} = &-\frac{\pi}{6} R_6 R_1 R_2 \sqrt{\tilde{g}} g^{\alpha\alpha'} g^{\beta\beta'} g^{\lambda\lambda'} H_{\alpha\beta\lambda} H_{\alpha'\beta'\lambda'} - \frac{\pi}{2} R_6 \frac{R_1}{R_2} \sqrt{\tilde{g}} g^{\alpha\alpha'} g^{\beta\beta'} H_{2\alpha\beta} H_{2\alpha'\beta'} \\
&- \frac{\pi}{2} \frac{R_6}{R_1} R_2 {\beta^2}^2 g^{\alpha\alpha'} g^{\beta\beta'} \sqrt{\tilde{g}} H_{2\alpha\beta} H_{2\alpha'\beta'} - \pi \frac{R_6}{R_1} R_2 \beta^2 \sqrt{\tilde{g}} g^{\alpha\alpha'} g^{\beta\beta'} H_{1\alpha\beta} H_{2\alpha'\beta'} \\
&- \frac{\pi R_6}{R_1 R_2} \sqrt{\tilde{g}} g^{\alpha\alpha'} H_{12\alpha} H_{12\alpha'} - \frac{\pi}{4} R_2 \frac{R_6}{R_1} \sqrt{\tilde{g}} (g^{\alpha\alpha'} g^{\beta\beta'} - g^{\alpha\beta'} g^{\alpha'\beta}) H_{1\alpha\beta} H_{1\alpha'\beta'},
\end{aligned} \tag{3.23}$$

and the momentum components $3 \leq \alpha \leq 5$ are

$$\mathcal{P}_\alpha = -\frac{1}{2}\epsilon^{\gamma\beta\delta} H_{12\gamma} H_{\alpha\beta\delta} + \frac{1}{2}\epsilon^{\gamma\beta\delta} H_{1\gamma\beta} H_{2\alpha\delta}, \tag{3.24}$$

where the zero modes of the ten fields H_{lmp} are labeled by integers $n_1, \ldots n_{10}$ [2]. Then (3.22) is given by (3.10).

Similarly, we compute the zero mode partition function for the 4d $U(1)$ theory from (3.1). We consider the charge quantization condition

$$n_I = \frac{1}{2\pi}\int_{\Sigma_2^I} F \equiv \frac{1}{2\pi}\int_{\Sigma_2^I} \frac{1}{2} F_{\alpha\beta}\, d\theta^\alpha \wedge d\theta^\beta, \qquad n_I \in \mathcal{Z}, \text{ for each } 1 \leq I \leq 3. \tag{3.25}$$

as well as the commutation relation obtained from (3.52)

$$\left[\int_{\Sigma_1^\gamma} A_\alpha(\vec{\theta}, \theta^6) d\theta^\alpha, \int \frac{d^3\theta'}{2\pi} \Pi^\beta(\vec{\theta'}, \theta^6)\right] = \frac{i}{2\pi}\int_{\Sigma_1^\gamma} d\theta^\beta = i\,\delta_\gamma^\beta, \tag{3.26}$$

and use the standard argument [6],[3] to show that the field strength $F_{\alpha\beta}$ and momentum Π^α zero modes have eigenvalues

$$F_{\alpha\beta} = \frac{n_{\alpha,\beta}}{2\pi}, \quad n_{\alpha,\beta} \in \mathcal{Z} \text{ for } \alpha < \beta, \qquad \text{and} \qquad \Pi^\alpha(\vec{\theta'}, \theta^6) = \frac{n^{(\alpha)}}{(2\pi)^2}, \quad n^{(\alpha)} \in \mathcal{Z}^3. \tag{3.27}$$

Thus we define integer valued modes $\widetilde{F}_{\alpha\beta} \equiv 2\pi F_{\alpha\beta}$ and $\widetilde{\Pi}^\alpha \equiv (2\pi)^2 \Pi^\alpha$. Taking into account the spatial integrations $d\theta^\alpha$, (3.2) gives

$$\begin{aligned}
&- 2\pi H^{4d} + i2\pi\gamma^\alpha P_\alpha^{4d} \\
&= -\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta - \frac{e^2\sqrt{g}}{8}\left[\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4}\right] g^{\alpha\beta} g^{\gamma\delta} \widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} - \frac{\theta e^2}{8\pi}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta \\
&\quad + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta},
\end{aligned} \tag{3.28}$$

where (3.2) itself is derived in section 4. So from (3.28) and (3.1),

$$
\begin{aligned}
Z^{4d}_{\text{zero modes}} = \sum_{n_4,n_5,n_6} \exp\{-\frac{e^2}{4}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta\} \cdot \sum_{n_1,n_2,n_3} \exp\{-\frac{\theta e^2}{8\pi}\frac{R_6^2}{\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta\} \\
\cdot \exp\{-\frac{e^2\sqrt{g}}{8}(\frac{\theta^2}{4\pi^2}+\frac{16\pi^2}{e^4}) g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}\},
\end{aligned}
\tag{3.29}
$$

where n_I are integers, with $\widetilde{F}_{34} = n_1,\ \widetilde{F}_{35} = n_2,\ \widetilde{F}_{45} = n_3$, and $\widetilde{\Pi}^3 = n_4,\ \widetilde{\Pi}^4 = n_5,\ \widetilde{\Pi}^6 = n_6$. (3.29) is the zero mode contribution to the $4d$ $U(1)$ partition function (3.1), and is (3.11).

If we identify the gauge couplings $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ with the modulus of T^2, $\tau = \beta^2 + i\frac{R_1}{R_2}$, then

$$
\frac{e^2}{4\pi} = \frac{R_2}{R_1}, \qquad \frac{\theta}{2\pi} = \beta^2,
\tag{3.30}
$$

and (3.29) becomes

$$
\begin{aligned}
Z^{4d}_{\text{zero modes}} = \sum_{n_4,n_5,n_6} \exp\{-\pi\frac{R_2R_6^2}{R_1\sqrt{g}} g_{\alpha\beta}\widetilde{\Pi}^\alpha\widetilde{\Pi}^\beta\} \cdot \sum_{n_1,n_2,n_3} \exp\{-\pi\beta^2\frac{R_2R_6^2}{R_1\sqrt{g}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}\widetilde{\Pi}^\beta\} \\
\cdot \exp\{-\frac{\pi}{2}\frac{R_2}{R_1}\sqrt{g}(\beta^{2^2}+\frac{R_1^2}{R_2^2}) g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta} + 2\pi i\gamma^\alpha\widetilde{\Pi}^\beta\widetilde{F}_{\alpha\beta}\}.
\end{aligned}
\tag{3.31}
$$

Then the last four terms in the chiral two-form zero mode sum (3.10) are equal to (3.11) since

$$
\begin{aligned}
-\frac{\pi}{2}\frac{R_2R_6\sqrt{\tilde{g}}}{R_1}(\frac{R_1^2}{R_2^2}+\beta^{2^2}) g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'} &= -\frac{\pi}{2}\frac{R_2R_6}{R_1}\sqrt{\tilde{g}}\,(\frac{R_1^2}{R_2^2}+\beta^{2^2}) g^{\alpha\beta}g^{\gamma\delta}\widetilde{F}_{\alpha\gamma}\widetilde{F}_{\beta\delta}, \\
-\pi\frac{R_6R_2}{R_1}\sqrt{\tilde{g}}\,\beta^2 g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'} &= -\pi\beta^2\frac{R_6R_2}{R_1\sqrt{\tilde{g}}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\tilde{F}_{\gamma\delta}\tilde{\Pi}^\beta, \\
i\pi\gamma^\alpha\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta} &= 2\pi i\gamma^\alpha\,\tilde{\Pi}^\beta\tilde{F}_{\alpha\beta}, \\
-\frac{\pi}{2}\frac{R_6R_2}{R_1}\sqrt{\tilde{g}} g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{1\alpha'\beta'} &= -\pi\frac{R_6R_2}{R_1\sqrt{\tilde{g}}} g_{\alpha\beta}\tilde{\Pi}^\alpha\tilde{\Pi}^\beta,
\end{aligned}
\tag{3.32}
$$

when we identify the integers

$$H_{2\alpha\beta} = \widetilde{F}_{\alpha\beta} \qquad \text{and} \qquad H_{1\alpha\beta} = \frac{1}{\tilde{g}}\epsilon_{\alpha\beta\gamma}\widetilde{\Pi}^{\gamma}, \tag{3.33}$$

with $\tilde{g} = g\,R_6^{-2}$ from (3.9). Thus the $6d$ and $4d$ zero mode sums from (3.10) and (3.11) are related by

$$Z^{6d}_{\text{zero modes}} = \epsilon\, Z^{4d}_{\text{zero modes}}, \tag{3.34}$$

where

$$\begin{aligned}\epsilon = &\sum_{n_8,n_9,n_{10}} \exp\{-\frac{\pi R_6}{R_1 R_2}\sqrt{\tilde{g}}g^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'}\} \\ &\cdot \sum_{n_7} \exp\{-\frac{\pi}{6}R_6 R_1 R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}g^{\delta\delta'}H_{\alpha\beta\delta}H_{\alpha'\beta'\delta'} - i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta}\}.\end{aligned} \tag{3.35}$$

3.3 Oscillator modes

To compute the oscillator contribution to the partition function (3.1), we quantize the $U(1)$ gauge theory with a theta term on the T^4 manifold using Dirac brackets. From (3.3), the equations of motion are $\partial^i F_{ij} = 0$, since the theta term is a total divergence and does not contribute to them. So in Lorenz gauge, the gauge potential A_i with field strength tensor $F_{ij} = \partial_i A_j - \partial_j A_i$ is obtained by solving the equation

$$\partial^i \partial_i A_j = 0, \qquad \text{with} \qquad \partial^i A_i = 0. \tag{3.36}$$

The potential has a plane wave solution

$$A_i(\vec{\theta}, \theta^6) = \text{zero modes } + \sum_{\vec{k}\neq 0}(f_i(k)e^{ik\cdot\theta} + (f_i(k)e^{ik\cdot\theta})^*) \tag{3.37}$$

with momenta satisfying the on shell condition and gauge condition

$$\widetilde{G}_L^{ij} k_i k_j = 0, \qquad k^i f_i = 0. \tag{3.38}$$

As in [13],[6] the Hamiltonian H^{4d} and momentum P_α^{4d} are quantized with a Lorentzian signature metric that has zero angles with the time direction, $\gamma^\alpha = 0$. So we modify the metric on the four-torus (3.6), (3.7) to be

$$\begin{aligned}
&\widetilde{G}_{L\,\alpha\beta} = g_{\alpha\beta}\,, \quad \widetilde{G}_{L\,66} = -{R_6}^2, \quad \widetilde{G}_{L\,\alpha 6} = 0 \\
&\widetilde{G}_L^{\alpha\beta} = g^{\alpha\beta}, \quad \widetilde{G}_L^{66} = -\frac{1}{R_6^2}, \quad \widetilde{G}_L^{\alpha 6} = 0, \quad \widetilde{G}_L = \det \widetilde{G}_{L\,ij} = -g.
\end{aligned} \tag{3.39}$$

Solving for k_6 from (3.38) we find

$$k_6 = \frac{\sqrt{-\widetilde{G}_L^{66}}}{\widetilde{G}_L^{66}}\,|k|, \tag{3.40}$$

where $3 \leq \alpha, \beta \leq 5$, and $|k| \equiv \sqrt{g^{\alpha\beta} k_\alpha k_\beta}$. Employ the remaining gauge invariance $f_i \to f_i' = f_i + k_i \lambda$ to fix $f_6' = 0$, which is the gauge choice

$$A_6 = 0.$$

This reduces the number of components of A_i from 4 to 3. To satisfy (3.38), we can use the $\partial^i F_{i6} = -\partial_6 \partial^\alpha A_\alpha = 0$ component of the equation of motion to eliminate f_5 in terms of f_3, f_4,

$$f_5 = -\frac{1}{p^5}(p^3 f_3 + p^4 f_4),$$

leaving just two independent polarization vectors corresponding to the physical degrees of freedom of a four-dimensional gauge theory.

From the Lorentzian Lagrangian and energy-momentum tensor given by

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2e^2}\sqrt{-\widetilde{G}_L}\widetilde{G}_L^{ik}\widetilde{G}_L^{jl}F_{ij}F_{kl} + \frac{\theta}{32\pi^2}\epsilon^{ijkl}F_{ij}F_{kl}, \\
\mathcal{T}^i_{\ j} &= \frac{\delta\mathcal{L}}{\delta\partial_i A_k}\partial_j A_k - \delta^i_j\mathcal{L},
\end{aligned} \tag{3.41}
$$

we obtain the Hamiltonian and momentum operators

$$
H_c \equiv \int d^3\theta\, \mathcal{T}^6_{\ 6} = \int d^3\theta \Big(-\frac{\sqrt{g}}{e^2}\widetilde{G}_L^{66} g^{\alpha\beta}\, F_{6\alpha}F_{6\beta} + \frac{\sqrt{g}}{2e^2} g^{\alpha\alpha'}g^{\beta\beta'}F_{\alpha\beta}F_{\alpha'\beta'} - \partial_\alpha \Pi^\alpha\, A_6\Big), \tag{3.42}
$$

$$
P_\alpha \equiv \int d^3\theta\, \mathcal{T}^6_{\ \alpha} = \int d^3\theta \Big(-\frac{2}{e^2}\sqrt{g}\,\widetilde{G}_L^{66} g^{\beta\gamma}F_{6\gamma}F_{\alpha\beta} - \partial_\beta\Pi^\beta\, A_\alpha + \Pi^6\partial_\alpha A_6\Big), \tag{3.43}
$$

where we have integrated by parts; and the conjugate momentum is

$$
\Pi^\alpha = \frac{\delta\mathcal{L}}{\delta\partial_6 A_\alpha} = -\frac{2}{e^2}\sqrt{g}\,\widetilde{G}_L^{66} g^{\alpha\beta}F_{6\beta} - \frac{\theta}{8\pi^2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}, \qquad \Pi^6 = \frac{\delta\mathcal{L}}{\delta\partial_6 A_6} = 0. \tag{3.44}
$$

Then we have

$$
\begin{aligned}
H_c - i\gamma^\alpha P_\alpha = \int d\theta^3 \Big(& \frac{R_6^2}{4}\frac{e^2}{\sqrt{g}} g_{\alpha\beta}\big(\Pi^\alpha + \frac{\theta}{8\pi^2}\epsilon^{\alpha\gamma\delta}F_{\gamma\delta}\big)\big(\Pi^\beta + \frac{\theta}{8\pi^2}\epsilon^{\beta\rho\sigma}F_{\rho\sigma}\big) \\
&+ \frac{\sqrt{g}}{2e^2} g^{\alpha\tilde{\alpha}}g^{\beta\tilde{\beta}}F_{\alpha\beta}F_{\tilde{\alpha}\tilde{\beta}} - i\gamma^\alpha\big(\Pi^\beta + \frac{\theta}{8\pi^2}\epsilon^{\beta\gamma\delta}F_{\gamma\delta}\big)F_{\alpha\beta}\Big),
\end{aligned} \tag{3.45}
$$

up to terms proportional to A_6 and $\partial_\alpha\Pi^\alpha$ which vanish in Lorenz gauge. Note the term proportional to $\epsilon^{\beta\gamma\delta}F_{\gamma\delta}F_{\alpha\beta}$ vanishes identically. (3.45) is equal to $H^{4d} - i\gamma^\alpha P^{4d}_\alpha$ given in (3.2), and is used to compute the zero mode partition function in (3.11) via (3.28).

To compute the oscillator modes, the appearance of θ solely in the combination $\Pi^\alpha + \frac{\theta}{8\pi^2}\epsilon^{\alpha\gamma\delta}F_{\gamma\delta}$ in (3.45) suggests we make a canonical transformation on the oscillator fields $\Pi^\alpha(\vec{\theta},\theta^6), A_\beta(\vec{\theta},\theta^6)$ [49]. Consider the equal time quantum bracket, suppressing the θ^6 dependence,

$$\left[\int d^3\theta' \epsilon^{\alpha\beta\delta} F_{\alpha\beta} A_\delta,\ \Pi^\gamma(\vec{\theta}) \right] = 2i\epsilon^{\gamma\alpha\beta} F_{\alpha\beta}(\vec{\theta}), \tag{3.46}$$

and the canonical transformation

$$U(\theta) = \exp\{i\frac{\theta}{32\pi^2} \int d^3\theta' \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma\}, \tag{3.47}$$

under which $\Pi^\alpha(\vec{\theta}, \theta^6), A_\beta(\vec{\theta}, \theta^6)$ transform to $\widehat{\Pi}^\alpha(\vec{\theta}, \theta^6), \widehat{A}_\beta(\vec{\theta}, \theta^6)$,

$$\begin{aligned} \widehat{\Pi}^\alpha(\vec{\theta}) = U^{-1}(\theta)\, \Pi^\alpha(\vec{\theta})\, U(\theta) &= \Pi^\alpha(\vec{\theta}) + \frac{\theta}{8\pi^2} \epsilon^{\alpha\gamma\delta} F_{\gamma\delta}(\vec{\theta}) \\ \widehat{A}_\beta(\vec{\theta}) = U^{-1}(\theta)\, A_\beta(\vec{\theta})\, U(\theta) &= A_\beta(\vec{\theta}). \end{aligned} \tag{3.48}$$

Therefore the exponent (3.45) contains no theta dependence when written in terms of $\widehat{\Pi}^\alpha$, which now reads

$$\big(H_c - i\gamma^\alpha P_\alpha\big) = \int d\theta^3 \big(-\frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \widehat{\Pi}^\alpha \widehat{\Pi}^\beta + \frac{\sqrt{g}}{2e^2} g^{\alpha\tilde{\alpha}} g^{\beta\tilde{\beta}} F_{\alpha\beta} F_{\tilde{\alpha}\tilde{\beta}} - i\gamma^\alpha \widehat{\Pi}^\beta F_{\alpha\beta} \big). \tag{3.49}$$

Thus, for the computation of the oscillator partition function we will quantize with $\theta = 0$. Note that had we done this for the zero modes, it would not be possible to pick the zero mode integer charges consistently. Since the zero and oscillator modes commute, we are free to canonically transform the latter and not the former.

In the discussion that follows we assume $\theta = 0$ and drop the hats. We directly quantize the Maxwell theory on the four-torus with the metric (3.39) in Lorenz gauge using Dirac constraints [50, 51]. The theory has a primary constraint $\Pi^6(\vec{\theta}, \theta^6) \approx 0$. We can express the Hamiltonian (3.42) in terms of the conjugate momentum as

$$H_c = \int d\theta^3 \frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{\sqrt{g}}{2e^2} g^{\alpha\tilde{\alpha}} g^{\beta\tilde{\beta}} F_{\alpha\beta} F_{\tilde{\alpha}\tilde{\beta}}. \tag{3.50}$$

The primary Hamiltonian is defined by

$$H_c = \int d\theta^3 \Big(\frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{\sqrt{g}}{2e^2} g^{\alpha\tilde{\alpha}} g^{\beta\tilde{\beta}} F_{\alpha\beta} F_{\tilde{\alpha}\tilde{\beta}} - \partial_\alpha \Pi^\alpha A_6 + \lambda_1 \Pi^6 \Big), \tag{3.51}$$

with λ_1 as a Lagrange multiplier. As in [6], we use the Dirac method of quantizing with constraints for the radiation gauge conditions $A_6 \approx 0$, $\partial^\alpha A_\alpha \approx 0$, and find the equal time commutation relations:

$$\begin{aligned} &[\Pi^\beta(\vec{\theta}, \theta^6), A_\alpha(\vec{\theta}', \theta^6)] = -i \Big(\delta_\alpha^\beta - g^{\beta\beta'} (\partial_\alpha \frac{1}{g^{\gamma\gamma'} \partial_\gamma \partial_{\gamma'}} \partial_{\beta'}) \Big)\, \delta^3(\theta - \theta'), \\ &[A_\alpha(\vec{\theta}, \theta^6), A_\beta(\vec{\theta}', \theta^6)] = 0, \qquad [\Pi^\alpha(\vec{\theta}, \theta^6), \Pi^\beta(\vec{\theta}', \theta^6)] = 0. \end{aligned} \tag{3.52}$$

In $A_6 = 0$ gauge, the vector potential on the torus is expanded as

$$A_\alpha(\vec{\theta}, \theta^6) = \text{ zero modes } + \sum_{\vec{k} \neq 0, \vec{k} \in \mathcal{Z}_3} (f_\alpha^\kappa a_{\vec{k}}^\kappa e^{ik\cdot\theta} + f_\alpha^{\kappa *} a_{\vec{k}}^{\kappa\dagger} e^{-ik\cdot\theta}),$$

where $1 \leq \kappa \leq 2$, $3 \leq \alpha \leq 5$ and k_6 defined in (3.40). The sum is on the dual lattice $\vec{k} = k_\alpha \in \mathcal{Z}_3 \neq \vec{0}$. Here we only consider the oscillator modes expansion of the potential and the conjugate momentum in (3.44) with vanishing θ angle

$$\begin{aligned} A_\alpha(\vec{\theta}, \theta^6) &= \sum_{\vec{k} \neq 0} (a_{\vec{k}\,\alpha} e^{ik\cdot\theta} + a_{\vec{k}\,\alpha}^\dagger e^{-ik\cdot\theta}), \\ \Pi^\beta(\vec{\theta}, \theta^6) &= -i \frac{2\sqrt{g}}{e^2} \widetilde{G}_L^{66} g^{\beta\beta'} \sum_{\vec{k}} k_6\, (a_{\vec{k}\,\beta'} e^{ik\cdot\theta} - a_{\vec{k}\,\beta'}^\dagger e^{-ik\cdot\theta}). \end{aligned} \tag{3.53}$$

and the polarizations absorbed in

$$a_{\vec{k}\,\alpha} = f_\alpha^\kappa a_{\vec{k}}^\kappa. \tag{3.54}$$

From (3.52), the commutator in terms of the oscillators is

$$\int \frac{d^3\theta d^3\theta'}{(2\pi)^6} e^{-ik_\alpha\theta^\alpha} e^{-ik'_\alpha\theta'^\alpha} [A_\alpha(\vec{\theta},0), A_\beta(\vec{\theta}',0)] = [(a_{\vec{k}\,\alpha} + a^\dagger_{-\vec{k}\,\alpha}), (a_{\vec{k}'\,\beta} + a^\dagger_{-\vec{k}'\,\beta})] = 0. \tag{3.55}$$

We consider the Fourier transform (3.55) of all the commutators (3.52), so the commutator of the oscillators is found to be:

$$\begin{aligned} &[a_{\vec{k}\,\alpha}, a^\dagger_{\vec{k}'\,\beta}] = \frac{e^2}{2\sqrt{g}\widetilde{G}^{66}_L k_6} \frac{1}{2(2\pi)^3} \big(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{g^{\gamma\gamma'} k_\gamma k_{\gamma'}}\big) \delta_{\vec{k},\vec{k}'}, \\ &[a_{\vec{k}\,\alpha}, a_{\vec{k}'\,\beta}] = 0, \qquad [a^\dagger_{\vec{k}\,\alpha}, a^\dagger_{\vec{k}'\,\beta}] = 0. \end{aligned} \tag{3.56}$$

In $A_6 = 0$ gauge, we use (3.53) and (3.56) to evaluate the Hamiltonian and momentum in (3.42) and (3.43)

$$\begin{aligned} H_c &= \int d^3\theta\, \frac{2\sqrt{g}}{e^2} \Big(-\frac{1}{2} \widetilde{G}^{66}_L g^{\alpha\alpha'}\, \partial_6 A_\alpha \partial_6 A_{\alpha'} + \frac{1}{4} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} \Big), \\ P_\alpha &= \frac{2}{R_6^2 e^2} \int_0^{2\pi} d\theta^3 d\theta^4 d\theta^5 \sqrt{g}\; g^{\beta\beta'}\; F_{6\beta'} F_{\alpha\beta}. \end{aligned} \tag{3.57}$$

With (3.53), (3.57) can be expressed in terms of the oscillator modes where time-dependent terms cancel,

$$\begin{aligned} H_c &= (2\pi)^3 \frac{2\sqrt{g}}{e^2} \sum_{\vec{k}\in\mathcal{Z}^3\neq\vec{0}} g^{\alpha\alpha'} |k|^2\ (a_{\vec{k}\,\alpha} a^\dagger_{\vec{k}\,\alpha'} + a^\dagger_{\vec{k}\,\alpha} a_{\vec{k}\,\alpha'}), \\ P_\alpha &= -\frac{2\sqrt{g}}{e^2} \widetilde{G}^{66}_L g^{\beta\beta'} (2\pi)^3 \sum_{\vec{k}\in\mathcal{Z}^3\neq\vec{0}} k_6 k_\alpha \big(a_{\vec{k}\,\beta'} a^\dagger_{\vec{k}\,\beta} + a^\dagger_{\vec{k}\,\beta'} a_{\vec{k}\,\beta}\big). \end{aligned} \tag{3.58}$$

and we have used the on-shell condition $\widetilde{G}^{66}_L k_6 k_6 + |k|^2 = 0$, and the transverse condition $k^\alpha a_{\vec{k}\,\alpha} = k^\alpha a^\dagger_{\vec{k}\,\alpha} = 0$. Then,

$$-H_c + i\gamma^\alpha P_\alpha = -i\frac{1}{R_6} \frac{2\sqrt{g}}{e^2} (2\pi)^3 \sum_{\vec{k}\in\mathcal{Z}^3\neq\vec{0}} |k| \big(-iR_6|k| + \gamma^\alpha k_\alpha \big) g^{\beta\beta'} \big(a_{\vec{k}\,\beta} a^\dagger_{\vec{k}\,\beta'} + a^\dagger_{\vec{k}\,\beta} a_{\vec{k}\,\beta'}\big). \tag{3.59}$$

Inserting the polarizations as $a_{\vec{k}\,\alpha} = f^{\kappa}_{\alpha} a^{\kappa}_{\vec{k}}$ and $a^{\dagger}_{\vec{k}\,\alpha} = f^{\lambda *}_{\alpha} a^{\lambda\dagger}_{\vec{k}}$ from (3.54) in the commutator (3.56) gives

$$[a_{\vec{k}\,\alpha}, a^{\dagger}_{\vec{k}'\,\beta}] = \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3} \Big(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} \Big) \delta_{\vec{k},\vec{k}'} = f^{\kappa}_{\alpha} f^{\lambda *}_{\beta} [a^{\kappa}_{\vec{k}}, a^{\lambda\dagger}_{\vec{k}}], \tag{3.60}$$

where we choose the normalization

$$[a^{\kappa}_{\vec{k}}, a^{\lambda\dagger}_{\vec{k}'}] = \delta^{\kappa\lambda} \delta_{\vec{k},\vec{k}'}, \tag{3.61}$$

with $1 \leq \kappa, \lambda \leq 2$. Then the polarization vectors satisfy

$$f^{\kappa}_{\alpha} f^{\lambda *}_{\beta} \delta^{\kappa\lambda} = \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3} \Big(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} \Big), \qquad g^{\beta\beta'} f^{\kappa}_{\beta} f^{\lambda *}_{\beta'} \delta^{\kappa\lambda} = \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3} \cdot 2,$$
$$g^{\beta\beta'} f^{\kappa}_{\beta} f^{\lambda *}_{\beta'} = \delta^{\kappa\lambda} \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \frac{1}{(2\pi)^3}.$$

So the exponent in (3.1) is given by

$$\begin{aligned} -H_c + i\gamma^\alpha P_\alpha &= -iR_6 \, \frac{2\sqrt{g}}{e^2} (2\pi)^3 \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} |k| \big(-iR_6|k| + \gamma^\alpha k_\alpha \big) g^{\beta\beta'} \big(2a^{\dagger}_{\vec{k}\,\beta} a_{\vec{k}\,\beta'} + [a_{\vec{k}\,\beta}, a^{\dagger}_{\vec{k}\,\beta'}] \big) \\ &= -i \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} \big(\gamma^\alpha k_\alpha - iR_6|k| \big)\, a^{\kappa\dagger}_{\vec{k}} a^{\kappa}_{\vec{k}} \quad - \frac{i}{2} \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} \big(-iR_6|k| \big)\, \delta^{\kappa\kappa}. \end{aligned} \tag{3.62}$$

The $U(1)$ partition function is

$$Z^{4d,Maxwell} \equiv tr \; \exp\{2\pi(-H_c + i\gamma^i P_i)\} = Z^{4d}_{\text{zero modes}} \, Z^{4d}_{\text{osc}}, \tag{3.63}$$

so from (3.62),

$$Z^{4d}_{\text{osc}} = tr \; e^{-2\pi i \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} \big(\gamma^\alpha k_\alpha - iR_6|k| \big)\, a^{\kappa\dagger}_{\vec{k}} a^{\kappa}_{\vec{k}} \; - \pi R_6 \sum_{\vec{k} \in \mathcal{Z}^3 \neq \vec{0}} |k| \, \delta^{\kappa\kappa}}. \tag{3.64}$$

From the usual Fock space argument

$$tr\ \omega^{\sum_p p a_p^\dagger a_p} = \prod_p \sum_{k=0}^{\infty} \langle k|\omega^{p a_p^\dagger a_p}|k\rangle = \prod_p \frac{1}{1-\omega^p},$$

we perform the trace on the oscillators,

$$Z^{4d}_{\rm osc} = \Big(e^{-\pi R_6 \sum_{\vec n\in \mathcal{Z}^3}\sqrt{g^{\alpha\beta}n_\alpha n_\beta}} \prod_{\vec n\in\mathcal{Z}^3\neq\vec 0} \frac{1}{1-e^{-i2\pi(\gamma^\alpha n_\alpha - iR_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta})}}\Big)^2, \tag{3.65}$$

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}}\cdot\Big(e^{-\pi R_6 \sum_{\vec n\in \mathcal{Z}^3}\sqrt{g^{\alpha\beta}n_\alpha n_\beta}} \prod_{\vec n\in\mathcal{Z}^3\neq\vec 0} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2, \tag{3.66}$$

where $Z^{4d}_{\text{zero modes}}$ is given in (3.11). (3.66) and (3.72) are each manifestly $SL(3,\mathcal{Z})$ invariant due to the underlying $SO(3)$ invariance we have labeled as $\alpha=3,4,5$. We use the $SL(3,\mathcal{Z})$ invariant regularization of the vacuum energy reviewed in Appendix G to obtain

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}}\cdot\Big(e^{\frac{1}{2}R_6\pi^{-2}\sum_{\vec n\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}} \prod_{\vec n\in\mathcal{Z}^3\neq\vec 0} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2, \tag{3.67}$$

which leads to (3.17).

On the other hand, one can evaluate the oscillator trace for the $6d$ chiral two-form from (3.4) as in[2],[6]. The exponent in the trace is

$$\begin{aligned}
-2\pi R_6\mathcal{H} + i2\pi\gamma^i P_i &= \frac{i\pi}{12}\int_0^{2\pi} d^5\theta H_{lrs}\epsilon^{lrsmn}H_{6mn} = \frac{i\pi}{2}\int_0^{2\pi} d^5\theta\sqrt{-G}H^{6mn}H_{6mn}\\
&= -i\pi\int_0^{2\pi} d^5\theta(\Pi^{mn}H_{6mn} + H_{6mn}\Pi^{mn})\\
&= -2i\pi\sum_{\vec p\neq 0} p_6\mathcal{C}^{\kappa\dagger}_{\vec p}B^\kappa_{\vec p} - i\pi\sum_{\vec p\neq 0} p_6\delta^{\kappa\kappa},
\end{aligned} \tag{3.68}$$

where $\Pi^{mn} = -\frac{\sqrt{-G}}{4}\Pi^{6mn}$, and Π^{6mn} is the momentum conjugate to B_{MN}. In the gauge $B_{6n}=0$, the normal mode expansion for the free quantum fields B_{mn} and Π^{mn} on a torus is given in terms of oscillators $B^{\kappa}_{\vec{p}}$ and $\mathcal{C}^{\kappa\dagger}_{\vec{p}}$ defined in [2], with the commutation relations

$$[B^{\kappa}_{\vec{p}}, \mathcal{C}^{\lambda\dagger}_{\vec{p}'}] = \delta^{\kappa\lambda}\ \delta_{\vec{p},\vec{p}'} \tag{3.69}$$

where $1 \leq \kappa, \lambda \leq 3$ labels the three physical degrees of freedom of the chiral two-form, and $\vec{p} = (p_1, p_2, p_\alpha)$ lies on the integer lattice $\mathcal{Z}^5$. From the on-shell condition $G^{LM}p_L p_M = 0$,

$$p_6 = -\gamma^\alpha p_\alpha - iR_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta + \frac{p_1^2}{R_1^2} + (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{R_1^2})p_2^2 + 2\frac{\beta^2}{R_1^2}p_1 p_2}. \tag{3.70}$$

Thus the oscillator partition function of the chiral two-form on $T^2 \times T^4$ is obtained by tracing over the oscillators

$$\begin{aligned} Z^{6d}_{\text{osc}} &= tr\ e^{-2i\pi\sum_{\vec{p}\neq 0} p_6 \mathcal{C}^{\kappa\dagger}_{\vec{p}} B^{\kappa}_{\vec{p}} - i\pi\sum_{\vec{p}\neq 0} p_6 \delta^{\kappa\kappa}} \\ &= \Big(e^{-\pi R_6 \sum_{\vec{p}}\sqrt{g^{\alpha\beta}p_\alpha p_\beta + \tilde{p}^2}} \prod_{\vec{p}\neq 0} \frac{1}{1 - e^{-2\pi i p_6}}\Big)^3 \\ &= \Big(e^{-\pi R_6 \sum_{\vec{p}\in\mathcal{Z}^5}\sqrt{g^{\alpha\beta}p_\alpha p_\beta + \tilde{p}^2}} \prod_{\vec{p}\in\mathcal{Z}^5\neq\vec{0}} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta + \tilde{p}^2} + 2\pi i\gamma^\alpha p_\alpha}}\Big)^3, \end{aligned} \tag{3.71}$$

where $\tilde{p}^2 \equiv \frac{p_1^2}{R_1^2} + (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{R_1^2})p_2^2 + 2\frac{\beta^2}{R_1^2}p_1 p_2$. Regularizing the vacuum energy in the oscillator sum [2] yields

$$Z^{6d,chiral} = Z^{6d}_{\text{zero modes}} \cdot \Big(e^{R_6\pi^{-3}\sum_{\vec{n}\neq\vec{0}}\frac{\sqrt{G_5}}{(G_{mp}n^m n^p)^3}} \prod_{\vec{p}\in\mathcal{Z}^5\neq\vec{0}} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{\alpha\beta}p_\alpha p_\beta + \tilde{p}^2} + 2\pi i\gamma^\alpha p_\alpha}}\Big)^3, \tag{3.72}$$

where $\vec{n} \in \mathcal{Z}^5$ is on the dual lattice, G_{mp} is defined in (3.6), and $Z^{6d}_{\text{zero modes}}$ is given in (3.10).

Comparing the $4d$ and $6d$ oscillator traces (3.66) and (3.71), the $6d$ chiral two-form sum has a cube rather than a square, corresponding to one additional polarization, and it contains Kaluza-Klein modes. In Appendix H, we prove that the product of the zero mode and the oscillator mode partition function for the $4d$ theory in (3.67) is $SL(4,\mathcal{Z})$ invariant. In (F.48) we give an equivalent expression,

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}} \cdot \Big(e^{\frac{\pi R_6}{6R_3}} \prod_{n_3\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_3}|n_3|+2\pi i\gamma^3 n_3}}\Big)^2$$

$$\cdot \Big(\prod_{(n_a)\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6<H>_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}\Big)^2, \tag{3.73}$$

where $4\leq a\leq 5$, with $< H >_{p_\perp}$ defined in (E.3).

In Appendix H, we also prove the $SL(4,\mathcal{Z})$ invariance of the $6d$ chiral partition function (3.72), using the equivalent form (F.65),

$$Z^{6d,chiral} = Z^{6d}_{\text{zero modes}} \cdot \big(e^{\frac{\pi R_6}{6R_3}} \prod_{n_3\in\mathcal{Z}\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_3}|n_3|+2\pi i\,\gamma^3\,n_3\,)}}\big)^3$$

$$\cdot \big(\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)} e^{-2\pi R_6<H>^{6d}_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\tilde{n}^2}\;+i2\pi\gamma^\alpha n_\alpha}}\big)^3, \tag{3.74}$$

with $< H >^{6d}_{p_\perp}$ in (F.64), and $\tilde{n}^2 = \frac{(n_1)^2}{R_1^2} + (\frac{1}{R_2^2}+\frac{\beta^2}{R_1^2})n_2^2 + 2\frac{\beta^2}{R_1^2}n_2 n_1$. In the limit when R_1 and R_2 are small with respect to the metric parameters $g_{\alpha\beta}, R_6$ of the four-torus, the contribution from each polarization in (3.73) and (3.74) is equivalent. To see this limit, we can separate the product on $n_\perp = (n_1, n_2, n_a) \neq 0_\perp$ in (3.74), into $(n_1 = 0, n_2 = 0, n_a \neq (0,0))$, $(n_1 \neq 0, n_2 \neq 0,$ all $n_a)$, $(n_1 = 0, n_2 \neq 0,$ all $n_a)$, $(n_1 \neq 0, n_2 = 0,$ all $n_a))$

to find, at fixed n_3,

$$\prod_{n_\perp\in\mathcal{Z}^4\neq(0,0,0,0)}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\frac{(n_1)^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{\beta^2}{R_1^2})n_2^2+2\frac{\beta^2}{R_1^2}n_2n_1}+2\pi i\gamma^\alpha n_\alpha}}$$
$$=\prod_{n_a\in\mathcal{Z}^2\neq(0,0)}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}$$
$$\cdot\prod_{n_1\neq0,n_2\neq0,(n_a\in\mathcal{Z}^2)}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\frac{(n_1)^2}{R_1^2}+(\frac{1}{R_2^2}+\frac{\beta^2}{R_1^2})n_2^2+\frac{\beta^2}{R_1^2}n_2n_1}+2\pi i\gamma^\alpha n_\alpha}}$$
$$\cdot\prod_{n_1=0,n_2\neq0,(n_a\in\mathcal{Z}^2)}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+(\frac{1}{R_2^2}+\frac{\beta^2}{R_1^2})n_2^2}+2\pi i\gamma^\alpha n_\alpha}}$$
$$\cdot\prod_{n_2=0,n_1\neq0,(n_a\in\mathcal{Z}^2)}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\frac{(n_1)^2}{R_1^2}}+2\pi i\gamma^\alpha n_\alpha}}\tag{3.75}$$

Thus for T^2 smaller than T^4, the last three products reduce to unity, so

$$\prod_{n_\perp\in\mathcal{Z}^4\neq\vec{0}}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta+\tilde{n}^2}+2\pi i\gamma^\alpha n_\beta}}\xrightarrow{R_1,R_2\to0}\prod_{n_a\in\mathcal{Z}^2\neq(0,0)}\frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}}.\tag{3.76}$$

The regularized vacuum energies in (E.3) and (F.64),

$$< H >_{p_\perp\neq0}=-\pi^{-1}\,|p_\perp|\sum_{n=1}^\infty\cos(p_a\kappa^a2\pi n)\frac{K_1(2\pi nR_3|p_\perp|)}{n},\quad\text{for}\quad|p_\perp|\equiv\sqrt{\widetilde{g}^{ab}n_an_b},$$
$$< H >^{6d}_{p_\perp\neq0}=-\pi^{-1}\,|p_\perp|\sum_{n=1}^\infty\cos(p_a\kappa^a2\pi n)\frac{K_1(2\pi nR_3|p_\perp|)}{n},\quad\text{for}\quad|p_\perp|\equiv\sqrt{\tilde{n}^2+\widetilde{g}^{ab}n_an_b},\tag{3.77}$$

have the same form of spherical Bessel function, but the argument differs by modes (p_1,p_2). Again separating the product on $n_\perp=(n_1,n_2,n_a)$ in (3.74), into $(n_1=0,n_2=0,n_a\neq(0,0))$, $(n_1\neq0,n_2\neq0\,\text{all}\;n_a)$, $(n_1=0,n_2\neq0,\;\text{all}\;n_a)$, $(n_1\neq$

$0, n_2 = 0,$ all n_a)) we have

$$\prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} = \Big(\prod_{n_a \in \mathcal{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>^{4d}_{p_\perp}} \Big) \cdot \Big(\prod_{n_1 \neq 0, n_2 \neq 0, n_a \in \mathcal{Z}^2} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \Big)$$
$$\cdot \Big(\prod_{n_1 \neq 0, n_2 = 0, n_a \in \mathcal{Z}^2} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \Big) \cdot \Big(\prod_{n_1 = 0, n_2 \neq 0, n_a \in \mathcal{Z}^2} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \Big) \tag{3.78}$$

In the limit $R_1, R_2 \to 0$, the last three products are unity. For example, the second is unity because for $n_1, n_2 \neq 0$,

$$\lim_{R_1, R_2 \to 0} \sqrt{\tilde{n}^2 + \tilde{g}^{\alpha\beta} n_\alpha n_\beta} \sim \sqrt{\tilde{n}^2},$$
$$\lim_{R_1, R_2 \to 0} (|p_\perp| \; K_1(2\pi n R_3 |p_\perp|) = \lim_{R_1, R_2 \to 0} \sqrt{\tilde{n}^2} \; K_1\Big(2\pi n R_3 \big(\sqrt{\tilde{n}^2}\big)\Big) = 0, \tag{3.79}$$

since $\lim_{x \to \infty} x K_1(x) \sim \sqrt{x}\; e^{-x} \to 0$ [52]. So (3.78) leads to

$$\lim_{R_1, R_2 \to 0} \prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} = \prod_{n_a \in \mathcal{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>_{p_\perp}}. \tag{3.80}$$

Thus in the limit when T^2 is small with respect to T^4,

$$\lim_{R_1, R_2 \to 0} \prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \prod_{n_3 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{(g^{\alpha\beta} n_\alpha n_\beta + \frac{n_1^2}{R_1^2} + \frac{\beta^2}{R_1^2}) n_2^2 + \frac{\beta^2}{R_1^2} n_2 n_1} + i2\pi\gamma^\alpha n_\alpha}}$$
$$= \prod_{n_a \in \mathcal{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n_3 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta} + 2\pi i \gamma^\alpha n_\alpha}}. \tag{3.81}$$

So we have shown the partition functions of the chiral theory on $T^2 \times T^4$ and of gauge theory on T^4, agree in the small T^2 limit upon neglecting the less interesting contribution ϵ',

$$\lim_{R_1, R_2 \to 0} Z^{6d}_{osc} = \epsilon' \cdot Z^{4d}_{osc}, \tag{3.82}$$

which is (3.15). Again, ϵ' is equivalently the oscillator contribution from one polarization, that is

$$\epsilon' = \Big(e^{\frac{1}{8}R_6\pi^{-2}\sum_{\vec{n}\neq 0}\frac{\sqrt{g}}{(g_{\alpha\beta}n^\alpha n^\beta)^2}} \cdot \prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}} \Big) \tag{3.83}$$

The relation between the $4d$ gauge and $6d$ tensor partition function is shown in the small T^2 limit,

$$\lim_{R_1,R_2\to 0} Z^{6d,chiral} = \epsilon\epsilon' \cdot Z^{4d,Maxwell}, \tag{3.84}$$

which is (3.19). $\epsilon\epsilon'$ is the partition function of a real scalar field in 4d, and is independent of the gauge coupling τ.

3.4 S-duality of $Z^{4d,Maxwell}$ from $Z^{6d,chiral}$

In Appendices B and D we show explicitly how the $SL(2,\mathcal{Z})\times SL(4,\mathcal{Z})$ symmetry of the partition function of the $6d$ tensor field of the M-fivebrane of $N=(2,0)$ theory compactified on $T^2\times T^4$ implies the $SL(2,\mathcal{Z})$ S-duality of the $4d$ $U(1)$ gauge field partition function. These computations use the Hamiltonian formulation. In Appendix E we review the path integral formalism for the $4d$ zero and non-zero mode partition functions, and give their relations to the quantities computed in the Hamiltonian formulation. The results are summarized here.

$$Z^{4d}_{\text{zero modes}} = (\operatorname{Im}\tau)^{\frac{3}{2}}\frac{g^{\frac{1}{4}}}{R_6^2}\, Z^{PI}_{\text{zero modes}}. \tag{3.85}$$

$$Z^{4d}_{osc} = (\operatorname{Im}\tau)^{-\frac{3}{2}} g^{-\frac{1}{4}} R_6^2 Z^{PI}_{osc}. \tag{3.86}$$

$$Z^{4d}_{\text{zero modes}} \longrightarrow Z^{4d}_{\text{zero modes}}, \qquad Z^{PI}_{\text{zero modes}} \longrightarrow |\tau|^{3} Z^{PI}_{\text{zero modes}} \qquad \text{under } S$$
$$Z^{4d}_{\text{zero modes}} \longrightarrow Z^{4d}_{\text{zero modes}}, \qquad Z^{PI}_{\text{zero modes}} \longrightarrow Z^{PI}_{\text{zero modes}} \qquad \text{under } T \tag{3.87}$$

and

$$Z^{4d}_{\text{osc}} \longrightarrow Z^{4d}_{\text{osc}}, \qquad Z^{PI}_{\text{non-zero modes}} \longrightarrow |\tau|^{-3} Z^{PI}_{\text{non-zero modes}} \qquad \text{under } S$$
$$Z^{4d}_{\text{osc}} \longrightarrow Z^{4d}_{\text{osc}}, \qquad Z^{PI}_{\text{non-zero modes}} \longrightarrow Z^{PI}_{\text{non-zero modes}} \qquad \text{under } T. \tag{3.88}$$

S and T are the generators of the duality symmetry $SL(2,\mathcal{Z})$, $S : \tau \to -\frac{1}{\tau}$, $T : \tau \to \tau - 1$, where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ is also given by the modulus of the two-torus, $\tau = \beta^2 + i\frac{R_1}{R_2}$.

3.5 Discussion and Conclusions

We computed the partition function of the abelian gauge theory on a general four-dimensional torus T^4 and the partition function of a chiral two-form compactified on $T^2 \times T^4$. The coupling for the 4D gauge theory, $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, is identified with the complex modulus $\tau = \beta^2 + i\frac{R_1}{R_2}$ of T^2. Assuming the metric of T^2 is much smaller than T^4, the 6D partition function factorizes to a partition function for gauge theory on T^4 and a contribution from the extra scalar arising from compactification. The 6D partition function has a manifest $SL(2,\mathcal{Z}) \times SL(4,\mathcal{Z})$ symmetry. Therefore the $SL(2,\mathcal{Z})$ symmetry with the group action on the coupling, $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, known as S-duality becomes manifest in the 4D Maxwell theory.

The 6D chiral two-form has no Lagrangian, so we use the Hamiltonian approach to compute both the 4D and 6D partition functions. For gauge theory, the integration of the electric and magnetic fields as observables around one- and two-cycles respectively take integer values due to charge quantization. We sum over all possible integers to get the zero mode partition function. For the oscillator mode calculation, we quantize the gauge theory using the Dirac method with constraints. In 6D, the partition function follows from [2],[6].

We have also given the result of the 4D partition function, computed by the path integral formalism. It agrees with the partition function obtained with the Hamiltonian formulation. However, the path integral form factors into zero modes and oscillator modes differently, which leads to different $SL(2, \mathcal{Z})$ transformation properties for the components. The 6D and 4D partition functions share the same $SL(2, \mathcal{Z}) \times SL(4, \mathcal{Z})$ symmetry.

If we consider supersymmetry, compactification of the 6D theory on T^2 leads to $N = 4$ gauge theory in the limit of small T^2. On the other hand, an $N = 2$ theory of class S [53],[54] arises when the 6D, $(2, 0)$ theory is compactified on a punctured Riemann surface with genus g. Here the mapping class group of the Riemann surfaces acts as a generalized S-duality on 4D super-Yang-Mills theory [55]-[57]. In Additional data about the gauge theory such as the discrete θ angle where the S-duality group acts can also specified [57]. another direction, we can study the 2D conformal field theory present when 6D theory is compactified on a four-dimensional manifold. The 2D-4D relation can also be studied from a topological point of view [58],[59]. Finding explicit results, such as we have derived for $T^2 \times T^4$, for these more general investigations would be advantageous.

Chapter 4

Conclusions and Future Directions

We computed the partition function of the five-dimensional abelian gauge theory on a five-torus T^5 with a general flat metric by separating into zero modes and oscillator modes. The computation is familiar from the one-loop modular invariant partition function computation in string theory [13]. Although neither the zero mode nor the oscillator mode partition functions are $SL(5, \mathcal{Z})$ invariant, the product of them is an $SL(5, \mathcal{Z})$ invariant function of the metric parameters. This arises from the T^5 compactification. We compared this with the partition function of a single fivebrane compactified on a circle S^1 times T^5, which is computed by reducing the six-torus calculation of Dolan and Nappi [2]. The two partition functions agree for the zero modes, but the Kaluza-Klein modes (KK modes) associated with the compactification on the circle are missing from the $5D$ oscillator expression. Hence, these two theories only agree in the weak coupling limit, thus disproving the conjecture by Douglas et al. at the abelian level [6]. However, it is still interesting to understand the relation between these two theories at the non-abelian level. When the 6D $(2, 0)$ is compactified on a circle S^1, the associated KK modes could be identified with the instantons in the $5D$ Yang-Mills theory. One might try to include instantons in the computation of the $5D$ partition function. The full 6D spectrum might thus be obtained from the $5D$ theory. One of my future research projects will be to give a systematic way to account for instantons in the partition function which is itself a very challenging topic [20]. When one computes the partition function for the $5D$ supersymmetric gauge theory on a more general manifold, one can use the supersymmetric localization technique to quantize the theory, which is under active investigation [27].

S-duality has its origin in a supersymmetric conformal field theory in six dimensions, the 6D, $\mathcal{N} = (2,0)$ theory. When the 6D, $\mathcal{N} = (2,0)$ theory is compactified on T^2, we obtain the 4D, $\mathcal{N} = 4$ super Yang-Mills theory, and the $SL(2,\mathcal{Z})$ symmetry of the torus implies the electromagnetic duality of the four-dimensional gauge theory. To test this, we compute the partition function for the 6D self-dual two-form potential on $T^2 \times T^4$, which posesses $SL(2,\mathcal{Z}) \times SL(4,\mathcal{Z})$ symmetry. Also, we compute the 4D gauge theory on a general T^4 torus with the gauge coupling, $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ set to the complex modulus of the torus T^2, $\tau = \beta^2 + i\frac{R_1}{R_2}$. In the small T^2 limit, the 6D chiral two-form partition function contains the partition function for the 4D abelian gauge theory combined with a factor that represents the extra modes that transforms trivially under the $SL(2,\mathcal{Z})$. Therefore, the $SL(2,\mathcal{Z})$ symmetry of the gauge theory on T^4 follows from the 6D theory. For the 4D gauge theory, since there exists a Lagrangian description, we also compute the partition function using the path integral formalism which turns out to be consistent with the result obtained from the canonical quantization. However, it factorizes differently into the zero modes and the oscillator modes [6].

Our partition function computation shows explicitly that the S-duality of four-dimensional gauge theory has a six-dimensional origin. More generally, one can consider an $\mathcal{N} = 2$ theory of class $\mathcal{S}$ arises when the 6D, $(2,0)$ is compactified on a punctured Riemann surface with genus g [53]. In such a way, the mapping class group of the Riemann surfaces acts as a generalized S-duality on the 4D super Yang-Mills theory. Viewed differently, we obtain a $2d$ Toda conformal field theory by compactification on a four-dimensional manifold. The equivalence of the $2d$ and 4D theory, known as the AGT correspondence [55], has been studied via their superconformal index computation [11]. However, since the AGT correspondence has its root in the 6D theory, it will be even more interesting to understand this duality from the 6D partition function.

APPENDIX A

EQUATION OF MOTION

The 5D Maxwell theory on a five-torus with metric (2.24) has the Hamiltonian (3.51),

$$H_p = \int d^4\theta \Big(- \frac{2\pi^2 R_1}{R_6 \sqrt{g} \widetilde{G}_L^{66}} g_{ii'} \Pi^i \Pi^{i'} + \frac{R_6 \sqrt{g}}{16\pi^2 R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \partial_i \Pi^i A_6 + \lambda_1 \Pi^6 \Big), \tag{A.1}$$

with λ_1 as a Lagrange multiplier. To quantize and derive the commutation relations, we start with the equal-time *canonical Poisson brackets*

$$\begin{aligned} &\{\Pi^{\tilde{m}}(\vec{\theta}, \theta^6), A_{\tilde{n}}(\vec{\theta}', \theta^6)\} = -\{A_{\tilde{n}}(\vec{\theta}', \theta^6), \Pi^{\tilde{m}}(\vec{\theta}, \theta^6)\} = -\delta^4(\vec{\theta} - \vec{\theta}')\, \delta^{\tilde{m}}_{\tilde{n}}, \\ &\{\Pi^{\tilde{m}}(\vec{\theta}, \theta^6), \Pi^{\tilde{n}}(\vec{\theta}', \theta^6)\} = \{A_{\tilde{m}}(\vec{\theta}, \theta^6), A_{\tilde{n}}(\vec{\theta}', \theta^6)\} = 0. \end{aligned} \tag{A.2}$$

The constraints are required to be time-independent, so for $\phi^1(\theta) \equiv \Pi^6(\vec{\theta}, \theta^6)$,

$$\partial_6 \phi^1(\vec{\theta}, \theta^6) = \{\phi^1(\vec{\theta}, \theta^6), H_p\} = -\int d^4\theta' \{\Pi^6(\theta), A_6(\theta')\}\, \partial_i \Pi^i(\theta') = \partial_i \Pi^i(\theta) \approx 0. \tag{A.3}$$

Thus the secondary constraint is

$$\phi^2(\theta) \equiv \partial_i \Pi^i(\vec{\theta}, \theta^6) \approx 0, \tag{A.4}$$

which is time-independent from the contribution

$$\partial_6 \phi^2(\vec{\theta}, \theta^6) = \{\phi^2(\vec{\theta}, \theta^6), H_p\} = \frac{R_6 \sqrt{g}}{16\pi^2 R_1} g^{ii'} g^{jj'} \int d^4\theta' \{\partial_k \Pi^k(\theta), F_{ij}(\theta') F_{i'j'}(\theta')\} = 0. \tag{A.5}$$

The two constraints ϕ^1, ϕ^2 are first class constraints since they have vanishing Poisson bracket,

$$\{\Pi^6(\theta), \partial_i \Pi^i(\theta')\} = 0. \tag{A.6}$$

We introduce the gauge conditions

$$\phi^3(\theta) \equiv A_6(\theta) \approx 0, \qquad \phi^4(\theta) \equiv \partial^i A_i(\theta) = g^{ij}\partial_j A_i \approx 0. \tag{A.7}$$

These convert all four constraints to second class, *i.e.* all now have at least one non-vanishing Poisson bracket with each other, where the non-vanishing brackets are

$$\begin{aligned}
&\{\phi^1(\theta), \phi^3(\theta')\} = \{\Pi^6(\theta), A_6(\theta')\} = -\delta^4(\theta - \theta') = -\{A_6(\theta), \Pi^6(\theta')\}, \\
&\{\phi^2(\theta), \phi^4(\theta')\} = \{\partial_i \Pi^i(\theta), g^{jj'}\partial_{j'} A_j(\theta')\} = g^{ij}\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j}\delta^4(\theta - \theta') = -\{g^{jj'}\partial_{j'} A_j(\theta), \partial_i \Pi^i(\theta')\}.
\end{aligned} \tag{A.8}$$

Furthermore, there are no new constraints since $\partial_6 \phi^A(\vec{\theta}, \theta^6) = \{\phi^A(\vec{\theta}, \theta^6), H\} \approx 0$, when all $\phi^A \approx 0,\ 1 \leq A \leq 4$, and $\lambda_1 = \partial_6 A_6$. We can write (A.8) as a matrix $C^{AB}(\theta, \theta') \equiv \{\phi^A(\theta), \phi^B(\theta')\}$,

$$C^{AB} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & g^{ij}\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j} \\ 1 & 0 & 0 & 0 \\ 0 & -g^{ij}\frac{\partial}{\partial\theta^i}\frac{\partial}{\partial\theta^j} & 0 & 0 \end{pmatrix} \delta^4(\theta - \theta'). \tag{A.9}$$

The inverse matrix is

$$(C_{AB})^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{g^{kk'}\frac{\partial}{\partial\theta^k}\frac{\partial}{\partial\theta^{k'}}} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{g^{kk'}\frac{\partial}{\partial\theta^k}\frac{\partial}{\partial\theta^{k'}}} & 0 & 0 \end{pmatrix} \delta^4(\theta - \theta'). \tag{A.10}$$

The Dirac bracket is defined to vanish with any constraint,

$$\begin{aligned}\{A_{\tilde m}(\theta),\Pi^{\tilde n}(\theta')\}_D = \{A_{\tilde m}(\theta),\Pi^{\tilde n}(\theta')\} - \int d^4\rho d^4\rho' \Big(&\{A_{\tilde m}(\theta),\Pi^6(\rho)\}C_{13}^{-1}\{A_6(\rho'),\pi^{\tilde n}(\theta')\} \\ &+ \{A_{\tilde m}(\theta),\partial_i\Pi^i(\rho)\}C_{24}^{-1}\{\partial^j A_j(\rho'),\Pi^{\tilde n}(\theta')\} \\ &+ \{A_{\tilde m}(\theta),A_6(\rho)\}C_{31}^{-1}\{\Pi^6(\rho'),\pi^{\tilde n}(\theta')\} \\ &+ \{A_{\tilde m}(\theta),\partial^j A_j(\rho)\}C_{42}^{-1}\{\partial_i\Pi^i(\rho'),\Pi^{\tilde n}(\theta')\}.\Big)\end{aligned} \tag{A.11}$$

So

$$\begin{aligned}\{A_i(\theta),\Pi^j(\theta')\}_D &= \{A_i(\theta),\pi^j(\theta')\} - \int d^4\rho d^4\rho' \Big(\{A_i(\theta),\partial_k\Pi^k(\rho)\}\; C_{24}^{-1}\; \{\partial^{k'}A_{k'}(\rho'),\pi^j(\theta')\}\Big) \\ &= \Big(\delta_i^j - g^{jj'}\partial_i \frac{1}{g^{kk'}\partial_k\partial_{k'}}\partial_{j'}\Big)\; \delta^4(\theta-\theta'),\end{aligned} \tag{A.12}$$

where here all ∂_j are with respect to θ^j. So promoting the Dirac Poisson bracket to a quantum commutator, we derive the equal time commutation relations

$$[\Pi^j(\vec\theta,\theta^6),A_i(\vec\theta',\theta^6)] = -i\Big(\delta_i^j - g^{jj'}(\partial_i\frac{1}{g^{kk'}\partial_k\partial_{k'}}\partial_{j'})\Big)\;\delta^4(\theta-\theta'), \tag{A.13}$$

and similarly,

$$[A_i(\vec\theta,\theta^6),A_j(\vec\theta',\theta^6)] = 0, \qquad [\Pi^i(\vec\theta,\theta^6),\Pi^j(\vec\theta',\theta^6)] = 0. \tag{A.14}$$

Furthermore we can check explicitly that Dirac brackets with a constraint vanish, for example

$$\begin{aligned}&\{\Pi^j(\theta),\partial^i A_i(\theta')\}_D = \{\Pi^j(\theta), g^{ik}\partial_k A_i(\theta') - g_{ik}\gamma^k\Pi^i(\theta') \\ &= \widetilde G_L^{jk}\frac{\partial}{\partial\theta^k}\delta^4(\theta-\theta') - \widetilde G_L^{jl}\frac{\partial}{\partial\theta^l}\delta^4(\theta-\theta') = 0 = [\Pi^j(\theta),\partial^i A_i(\theta')],\end{aligned} \tag{A.15}$$

and

$$[\partial_j \Pi^j(\theta), A_i(\theta')] = \partial_j \Big(\delta_i^j - g^{jj'} (\partial_i \frac{1}{g^{kk'} \partial_k \partial_{k'}} \partial_{j'}) \Big) \; \delta^4(\theta - \theta') = 0. \tag{A.16}$$

APPENDIX B

REGULARIZATION FOR 5D MAXWELL THEORY

The Fourier transform of powers of a radial function is

$$|\vec{p}|^{\alpha-n} = \frac{c_\alpha}{(2\pi)^n} \int d^n y \sqrt{G_n}\, e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^\alpha}, \qquad where \qquad c_\alpha \equiv \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. \tag{B.1}$$

This formula holds by analytic continuation, since for general n, α, where the area of the unit sphere S_{n-2} is

$$\omega_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \equiv \int_0^\pi d\theta_1 d\theta_2 \ldots d\theta_{n-3}\ \sin\theta_1\ \sin^2\theta_2 \ldots \sin^{n-3}\theta_{n-3} \int_0^{2\pi} d\phi, \tag{B.2}$$

the Fourier integral is

$$\begin{aligned}
\int d^n y \sqrt{G_n}\, e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^\alpha} &= \int_0^\infty dy\, y^{n-1-\alpha} \int_0^\pi d\theta \sin^{n-2}\theta\ e^{-i|\vec{p}|y\cos\theta}\ \omega_{n-2} \\
&= \int_0^\infty dy\, y^{n-1-\alpha}\ \frac{(2\pi)^{\frac{n}{2}}}{(|\vec{p}|y)^{\frac{n-2}{2}}}\ J_{\frac{n-2}{2}}(|\vec{p}|y) \\
&= |\vec{p}|^{\alpha-n}\ (2\pi)^{\frac{n}{2}}\ \frac{2^{\frac{n}{2}-\alpha}\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})},
\end{aligned} \tag{B.3}$$

where the last expression is valid for the integral when $-\frac{n}{2} < \frac{n}{2} - \alpha < \frac{1}{2}$, but can be analytically continued for all $\alpha \neq -n, -n-1, \ldots$

So expressing $|\vec{p}|$ in terms of its 4D Fourier transform,

$$\begin{aligned}
|\vec{p}| &= -\frac{3}{4\pi^2} \int d^4 y \sqrt{g}\, e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^5}, \\
< H > &= \frac{1}{2} \sum_{\vec{p}\in\mathcal{Z}^4} |\vec{p}|\ e^{i\vec{p}\cdot\vec{x}}|_{\vec{x}=0} = \frac{1}{2} \sum_{\vec{p}\in\mathcal{Z}^4} \sqrt{g^{ij} p_i p_j},
\end{aligned} \tag{B.4}$$

we have for the sum on the dual lattice, $p_i \in \mathcal{Z}^4$,

$$\sum_{\vec{p}\in\mathcal{Z}^4} |\vec{p}| e^{i\vec{p}\cdot\vec{x}} = -\frac{3}{4\pi^2}\sqrt{g}\int d^4y \frac{1}{|\vec{y}|^5}\sum_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$
$$= -\frac{3}{4\pi^2}\sqrt{g}\int d^4y \frac{1}{|\vec{y}|^5}(2\pi)^4\sum_{\vec{n}\neq 0}\delta^4(\vec{x}-\vec{y}+2\pi\vec{n}) = -12\pi^2\sqrt{g}\sum_{\vec{n}\in\mathcal{Z}^4\neq 0}\frac{1}{|\vec{x}+2\pi\vec{n}|^5} \tag{B.5}$$

where the regularization consists of removing the $\vec{n}=0$ term from the equality,

$$\sum_{\vec{p}\in\mathcal{Z}^4} e^{i\vec{p}\cdot\vec{x}} = (2\pi)^4\sum_{\vec{n}\in\mathcal{Z}^4}\delta^4(\vec{x}+2\pi\vec{n}) \tag{B.6}$$

and the sum on $\vec{n}$ is on the original lattice $\vec{n} = n^i \in \mathcal{Z}^4$. The regularized vacuum energy is

$$< H >= -\frac{3}{16\pi^3}\sqrt{g}\sum_{\vec{n}\in\mathcal{Z}^4\neq 0}\frac{1}{(g_{ij}n^in^j)^{\frac{5}{2}}} = -6\pi^2\sqrt{g}\sum_{\vec{n}\in\mathcal{Z}^4\neq 0}\frac{1}{|2\pi\vec{n}|^5}\,. \tag{B.7}$$

For the discussion of $SL(5,\mathcal{Z})$ invariance in Appendix G, it is also useful to write the regularized sum (E.1), as

$$< H > = \sum_{p_\perp\in\mathcal{Z}^3} < H >_{p_\perp} = < H >_{p_\perp=0} + \sum_{p_\perp\in\mathcal{Z}^3\neq 0} < H >_{p_\perp}, \tag{B.8}$$

where $p_\perp = p_\alpha \in \mathcal{Z}^3,\ \alpha = 3,4,5$, and

$$< H >_{p_\perp=0} = \frac{1}{2}\sum_{p_2\in\mathcal{Z}}\sqrt{g^{22}p_2p_2} = \frac{1}{R_2}\sum_{n=1}^{\infty} n = \frac{1}{R_2}\zeta(-1) = -\frac{1}{12R_2} \tag{B.9}$$

by zeta function regularization. For general $p_\perp$, we express (E.1) as a sum of terms at fixed transverse momentum [2],

$$< H >_{p_\perp} = -6\pi^2\sqrt{g}\frac{1}{(2\pi)^3}\int d^3z_\perp e^{-ip_\perp\cdot z_\perp}\sum_{\vec{n}\in\mathcal{Z}^4\neq 0}\frac{1}{|2\pi\vec{n}+z_\perp|^5}, \tag{B.10}$$

using the equality for the periodic delta function, $\sum_{p_\alpha\in\mathcal{Z}^3} e^{ip\cdot z} = (2\pi)^3\sum_{n^\alpha\in\mathcal{Z}^3}\delta^3(\vec{z}+2\pi\vec{n})$. Changing variables $z^\alpha \to y^\alpha + 2\pi n^\alpha$, (B.10) becomes

$$< H >_{p_\perp} = -6\pi^2\sqrt{g}\frac{1}{(2\pi)^3}\int d^3y_\perp e^{-ip_\perp\cdot y_\perp}\sum_{n\in\mathcal{Z}\neq 0}\frac{1}{|2\pi n+y_\perp|^5} \tag{B.11}$$

where n is the n^2 component on the original lattice, and the denominator is $|2\pi n + y_\perp|^2 \equiv [(2\pi n)^2G_{22}+2(2\pi n)G_{2\alpha}y_\perp^\alpha+y_\perp^\alpha y_\perp^\beta G_{\alpha\beta}] = [(2\pi n)^2(R_2^2+g_{\alpha\beta}\kappa^\alpha\kappa^\beta)-2(2\pi n)g_{\alpha\beta}\kappa^\beta y_\perp^\alpha+y_\perp^\alpha y_\perp^\beta g_{\alpha\beta}]$. We can extract the $p_\perp = 0$ part of (B.11) to verify (B.9),

$$\begin{aligned}< H >_{p_\perp=0} &= -6\pi^2\sqrt{g}\frac{1}{(2\pi)^3}\sum_{n\in\mathcal{Z}\neq 0}\int d^3y_\perp\frac{1}{|2\pi n+y_\perp|^5}\\ &= -6\pi^2\sqrt{g}\frac{1}{(2\pi)^3}\sum_{n\in\mathcal{Z}\neq 0}\frac{4\pi}{3}\frac{1}{(2\pi)^2R_2^2}\frac{1}{n^2}\frac{1}{\sqrt{\tilde{g}}} = -\frac{\zeta(2)}{2\pi^2R_2} = -\frac{1}{12R_2},\end{aligned} \tag{B.12}$$

by performing the y integrations. For general $p_\perp \in \mathcal{Z}^3 \neq 0$, (B.11) integrates to give the spherical Bessel functions,

$$\begin{aligned}< H >_{p_\perp\neq 0} &= |p_\perp|^2R_2\sum_{n=1}^{\infty}\cos(p_\alpha\kappa^\alpha 2\pi n)\big[K_2(2\pi nR_2|p_\perp|)-K_0(2\pi nR_2|p_\perp|)\big]\\ &= -\pi^{-1}\,|p_\perp|R_2\sum_{n=1}^{\infty}\cos(p_\alpha\kappa^\alpha 2\pi n)\frac{K_1(2\pi nR_2|p_\perp|)}{n},\end{aligned} \tag{B.13}$$

where $|p_\perp| = \sqrt{\tilde{g}^{\alpha\beta}n_\alpha n_\beta}$ can be viewed as the mass of three scalar bosons [2].

For a d-dimensional lattice sum, the general formula used in (B.4) for regulating the divergent sum is [2],

$$\begin{aligned}|\vec{p}| &= 2\pi^{-\frac{d}{2}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(-\frac{1}{2})}\int d^dy\sqrt{G_d}\,e^{-i\vec{p}\cdot\vec{y}}\frac{1}{|\vec{y}|^{d+1}},\\ < H > &= \frac{1}{2}\sum_{\vec{p}\in\mathcal{Z}^d}|\vec{p}|\,e^{i\vec{p}\cdot\vec{x}}|_{\vec{x}=0} = \frac{1}{2}\sum_{\vec{p}\in\mathcal{Z}^d}\sqrt{g^{\alpha\beta}p_\alpha p_\beta}\\ &= 2^d\pi^{\frac{d}{2}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(-\frac{1}{2})}\sqrt{G_d}\sum_{\vec{n}\in Z^d\neq\vec{0}}\frac{1}{|2\pi\vec{n}|^{d+1}}.\end{aligned} \tag{B.14}$$

APPENDIX C

$SL(5,\mathcal{Z})$ INVARIANCE

Rewriting the 5D metric (2,3,4,5,6)
From (2.6) the metric on the five-torus, for $i,j=2,3,4,5$, is

$$G_{ij}=g_{ij},\qquad G_{i6}=-g_{ij}\gamma^j,\qquad G_{66}=R_6^2+g_{ij}\gamma^i\gamma^j,$$
$$\widetilde{G}_5\equiv\det G_{\tilde{m}\tilde{n}}=R_6^2\,\det g_{ij}\equiv R_6^2\,g. \tag{C.1}$$

We can rewrite this metric using $\alpha,\beta=3,4,5$,

$$g_{22}\equiv R_2^2+\widetilde{g}_{\alpha\beta}\kappa^\alpha\kappa^\beta,\qquad g_{\alpha 2}\equiv-\widetilde{g}_{\alpha\beta}\kappa^\beta,\qquad g_{\alpha\beta}\equiv\widetilde{g}_{\alpha\beta},\qquad (\gamma^2)\kappa^\alpha-\gamma^\alpha\equiv-\widetilde{\gamma}^\alpha,$$
$$G_{22}=R_2^2+\widetilde{g}_{\alpha\beta}\kappa^\alpha\kappa^\beta,\qquad G_{26}=-(\gamma^2)R_2^2+\widetilde{g}_{\alpha\beta}\kappa^\beta\widetilde{\gamma}^\alpha,\qquad G_{2\alpha}=-\widetilde{g}_{\alpha\beta}\kappa^\beta,$$
$$G_{\alpha\beta}=\widetilde{g}_{\alpha\beta},\qquad G_{\alpha 6}=-\widetilde{g}_{\alpha\beta}\widetilde{\gamma}^\beta,\qquad G_{66}=R_6^2+(\gamma^2)^2R_2^2+\widetilde{g}_{\alpha\beta}\,\widetilde{\gamma}^\alpha\widetilde{\gamma}^\beta. \tag{C.2}$$

The 4D inverse of g_{ij} is

$$g^{\alpha\beta}=\widetilde{g}^{\alpha\beta}+\frac{\kappa^\alpha\kappa^\beta}{R_2^2},\qquad g^{\alpha 2}=\frac{\kappa^\alpha}{R_2^2},\qquad g^{22}=\frac{1}{R_2^2}, \tag{C.3}$$

where $\widetilde{g}^{\alpha\beta}$ is the $3d$ inverse of $\widetilde{g}_{\alpha\beta}$.

$$g\equiv\det g_{ij}=R_2^2\,\det\widetilde{g}_{\alpha\beta}\equiv R_2^2\,\widetilde{g}.$$

The line element can be written as

$$
\begin{aligned}
ds^2 &= R_6^2(d\theta^6)^2 + \sum_{i,j=2,\ldots,5} g_{ij}(d\theta^i - \gamma^i d\theta^6)(d\theta^j - \gamma^j d\theta^6) \\
&= R_2^2(d\theta^2 - (\gamma^2)d\theta^6)^2 + R_6^2(d\theta^6)^2 \\
&\quad + \sum_{\alpha,\beta=3,4,5} \widetilde{g}_{\alpha\beta}(d\theta^\alpha - \widetilde{\gamma}^\alpha d\theta^6 - \kappa^\alpha d\theta^2)\,(d\theta^\beta - \widetilde{\gamma}^\beta d\theta^6 - \kappa^\beta d\theta^2). \qquad \text{(C.4)}
\end{aligned}
$$

We define

$$
\widetilde{\tau} \equiv \gamma^2 + i\frac{R_6}{R_2}. \qquad \text{(C.5)}
$$

The 5D inverse is

$$
\begin{aligned}
&\widetilde{G}_5^{22} = \frac{|\widetilde{\tau}|^2}{R_6^2} = \widetilde{G}_5^{66}|\widetilde{\tau}|^2, \qquad \widetilde{G}_5^{66} = \frac{1}{R_6^2}, \qquad \widetilde{G}_5^{26} = \frac{\gamma^2}{R_6^2}, \qquad \widetilde{G}_5^{2\alpha} = \frac{\kappa^\alpha|\widetilde{\tau}|^2}{R_6^2} + \frac{\gamma^2\widetilde{\gamma}^\alpha}{R_6^2}, \\
&\widetilde{G}_5^{\alpha\beta} = \widetilde{g}^{\alpha\beta} + \frac{\kappa^\alpha\kappa^\beta}{R_6^2}|\widetilde{\tau}|^2 + \frac{\widetilde{\gamma}^\alpha\widetilde{\gamma}^\beta}{R_6^2} + \frac{\gamma^2(\widetilde{\gamma}^\alpha\kappa^\beta + \kappa^\alpha\widetilde{\gamma}^\beta)}{R_6^2}, \qquad \widetilde{G}_5^{6\alpha} = \frac{\gamma^\alpha}{R_6^2} = \frac{\gamma^2\kappa^\alpha + \widetilde{\gamma}^\alpha}{R_6^2}. \qquad \text{(C.6)}
\end{aligned}
$$

Generators of $SL(n,\mathcal{Z})$

The $SL(n,\mathcal{Z})$ unimodular groups can be generated by two matrices For $SL(5,\mathcal{Z})$ these can be taken to be U_1, U_2,

$$
U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{(C.7)}
$$

so that every matrix M in $SL(5,\mathcal{Z})$ can be written as a product $U_1^{n_1}U_2^{n_2}U_1^{n_3}\ldots$. Therefore to prove the $SL(5,\mathcal{Z})$ invariance of (3.67), we will show it is invariant under U_1 and U_2. Matrices U_1 and U_2 act on the basis vectors of the five-torus $\vec{\alpha}_{\tilde{m}}$ where

$\vec{\alpha}_{\tilde{m}} \cdot \vec{\alpha}_{\tilde{n}} \equiv \alpha^{\tilde{p}}_{\tilde{m}} \alpha^{\tilde{q}}_{\tilde{n}} G_{\tilde{p}\tilde{q}} = G_{\tilde{m}\tilde{n}}$,

$$\begin{aligned}
\vec{\alpha}_2 &= (1,0,0,0,0) \\
\vec{\alpha}_6 &= (0,1,0,0,0) \\
\vec{\alpha}_3 &= (0,0,1,0,0) \\
\vec{\alpha}_4 &= (0,0,0,1,0) \\
\vec{\alpha}_5 &= (0,0,0,0,1).
\end{aligned} \tag{C.8}$$

For our metric (F.3), the U_2 transformation

$$\begin{pmatrix} \vec{\alpha}'_2 \\ \vec{\alpha}'_6 \\ \vec{\alpha}'_3 \\ \vec{\alpha}'_4 \\ \vec{\alpha}'_5 \end{pmatrix} = U_2 \begin{pmatrix} \vec{\alpha}_2 \\ \vec{\alpha}_6 \\ \vec{\alpha}_3 \\ \vec{\alpha}_4 \\ \vec{\alpha}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.9}$$

results in $\vec{\alpha}'_2 \cdot \vec{\alpha}'_2 \equiv \alpha'^{\tilde{p}}_2 \alpha'^{\tilde{q}}_2 G_{\tilde{p}\tilde{q}} = G_{22} = G'_{22},\ \vec{\alpha}'_2 \cdot \vec{\alpha}'_6 \equiv \alpha'^{\tilde{p}}_2 \alpha'^{\tilde{q}}_6 G_{\tilde{p}\tilde{q}} = G_{22} + G_{26} = G'_{26}$, etc. So U_2 corresponds to

$$R_2 \to R_2,\ R_6 \to R_6,\ \gamma^2 \to \gamma^2 - 1,\ \kappa^\alpha \to \kappa^\alpha,\ \widetilde{\gamma}^\alpha \to \widetilde{\gamma}^\alpha + \kappa^\alpha,\ \widetilde{g}_{\alpha\beta} \to \widetilde{g}_{\alpha\beta}, \tag{C.10}$$

or equivalently

$$R_6 \to R_6,\ \gamma^2 \to \gamma^2 - 1,\ g_{ij} \to g_{ij},\ \gamma^\alpha \to \gamma^\alpha, \tag{C.11}$$

which leaves invariant the line element (F.5) if $d\theta^2 \to d\theta^2 - d\theta^6,\ d\theta^6 \to d\theta^6,\ d\theta^\alpha \to d\theta^\alpha$. U_2 is the generalization of the usual $\widetilde{\tau} \to \widetilde{\tau} - 1$ modular transformation. The 4D inverse metric $g^{ij} \equiv \{g^{\alpha\beta}, g^{\alpha 2}, g^{22}\}$ does not change under U_2. It is easily checked that U_2 is an invariance of the 5D Maxwell partition function (3.66) as well as the chiral partition function (2.68). It leaves the zero mode and oscillator contributions invariant separately.

The other generator, U_1 is related to the $SL(2, \mathcal{Z})$ transformation $\widetilde{\tau} \to -(\widetilde{\tau})^{-1}$ that we discuss as follows:

$$U_1 = U' M_4 \tag{C.12}$$

where M_4 is an $SL(4, \mathcal{Z})$ transformation given by

$$M_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.13}$$

and U' is the matrix corresponding to the transformation on the metric parameters (F.16),

$$U' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C.14}$$

Under U', the metric parameters transform as

$$R_2 \to R_2|\widetilde{\tau}|, \quad R_6 \to R_6|\widetilde{\tau}|^{-1}, \quad \gamma^2 \to -\gamma^2|\widetilde{\tau}|^{-2}, \quad \kappa^\alpha \to \widetilde{\gamma}^\alpha, \quad \widetilde{\gamma}^\alpha \to -\kappa^\alpha, \quad \widetilde{g}_{\alpha\beta} \to \widetilde{g}_{\alpha\beta}.$$

$\widetilde{\tau} \to -\frac{1}{\widetilde{\tau}}.$ Or equivalently,

$$G_{\alpha\beta} \to G_{\alpha\beta}, \quad G_{\alpha 2} \to G_{\alpha 6}, \quad G_{\alpha 6} \to -G_{\alpha 2}, \quad G_{22} \to G_{66}, \quad G_{66} \to G_{22}, \quad G_{26} \to -G_{26},$$

$$\widetilde{G}_5^{\alpha\beta} \to \widetilde{G}_5^{\alpha\beta}, \quad \widetilde{G}_5^{\alpha 2} \to \widetilde{G}_5^{\alpha 6}, \quad \widetilde{G}_5^{\alpha 6} \to -\widetilde{G}_5^{\alpha 2}, \quad \widetilde{G}_5^{22} \to \frac{\widetilde{G}_5^{22}}{|\widetilde{\tau}|^2}, \quad \widetilde{G}_5^{66} \to |\widetilde{\tau}|^2 \widetilde{G}_5^{66}, \quad \widetilde{G}_5^{26} \to -\widetilde{G}_5^{26}, \tag{C.15}$$

where $3 \leq \alpha, \beta \leq 5$, and

$$\widetilde{\tau} \equiv \gamma^2 + i\frac{R_6}{R_2}, \qquad |\widetilde{\tau}|^2 = (\gamma^2)^2 + \frac{R_6^2}{R_2^2}. \tag{C.16}$$

The transformation (F.16) leaves invariant the line element (F.5) when $d\theta^2 \to d\theta^6$, $d\theta^6 \to -d\theta^2,\ d\theta^1 \to d\theta^1,\ d\theta^\alpha \to d\theta^\alpha$. The generators have the property $\det U_1 = 1$, $\det U_2 = 1,\ \det U' = 1,\ \det M_4 = 1$.

Under M_4, the metric parameters transform as

$$R_6 \to R_6, \qquad \gamma^2 \to -\gamma^3, \quad \gamma^\alpha \to \gamma^{\alpha+1}, \quad g_{\alpha\beta} \to g_{\alpha+1,\beta+1}, \quad g_{\alpha 2} \to -g_{\alpha+1,3}, \quad g_{22} \to g_{33},$$

$$g^{\alpha\beta} \to g^{\alpha+1,\beta+1}, \quad g^{\alpha 2} \to -g^{\alpha+1,3}, \quad g^{22} \to g^{33}, \quad \det g_{ij} = g, \quad g \to g.$$ Or equivalently,

$$\begin{aligned}
&G_{\alpha\beta} \to G_{\alpha+1,\beta+1}, \quad G_{\alpha 2} \to -G_{\alpha+1,3}, \quad G_{\alpha 6} \to G_{\alpha+1,6}, \quad G_{22} \to G_{33}, \quad G_{66} \to G_{66}, \quad G_{26} \to -G_{36},\\
&\widetilde{G}_5^{\alpha\beta} \to \widetilde{G}_5^{\alpha+1,\beta+1}, \quad \widetilde{G}_5^{\alpha 2} \to -\widetilde{G}_5^{\alpha+1,3}, \quad \widetilde{G}_5^{\alpha 6} \to \widetilde{G}_5^{\alpha+1,6}, \quad \widetilde{G}_5^{22} \to \widetilde{G}_5^{33}, \quad \widetilde{G}_5^{26} \to -\widetilde{G}_5^{36}, \quad \widetilde{G}_5^{66} \to \widetilde{G}_5^{66},\\
&\det \widetilde{G}_5 = R_6\ g, \qquad \det \widetilde{G}_5 \to \det \widetilde{G}_5,
\end{aligned} \tag{C.17}$$

where $3 \le \alpha, \beta \le 5$, and $\alpha + 1 \equiv 2$ for $\alpha = 5$.

We can check that $Z^{5D}_{zero\ modes}$ is invariant under M_4 given in (C.13) as follows. Letting the M_4 transformation (C.17) act on (2.20), we find that the three subterms in the exponent

$$\begin{aligned}
&-2\pi^3 \frac{R_6\sqrt{g}}{R_1}\Big(g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta}F_{\alpha'\beta'} + 4g^{\alpha\alpha'} g^{\beta 2} F_{\alpha\beta}F_{\alpha' 2} + 2g^{\alpha\alpha'} g^{22} F_{\alpha 2}F_{\alpha' 2} - 2g^{\alpha 2} g^{\alpha' 2} F_{\alpha 2}F_{\alpha' 2}\Big),\\
&-\pi \frac{R_1 R_6}{\sqrt{g}} m^i g_{ij} m^j,\\
&\ i4\pi^2 \gamma^i m^j F_{ij}
\end{aligned} \tag{C.18}$$

are separately invariant under (C.17), if we replace the the integers $2\pi F_{ij} \in \mathcal{Z}^6, m^i \in \mathcal{Z}^4$ by

$$2\pi F_{\alpha\beta} \to 2\pi F_{\alpha+1,\beta+1}, \quad 2\pi F_{\alpha 2} \to -2\pi F_{\alpha+1,3}, \quad m^2 \to -m^3, \qquad m^\alpha \to m^{\alpha+1}, \tag{C.19}$$

where $m^i \equiv \frac{2\pi\sqrt{g}}{R_1 R_6} g^{ii'} F_{6i'}$ relabels $(n^7, n^8, n^9, n^{10}) = (m^2, m^3, m^4, m^5)$.
Therefore under M_4, for the zero mode contribution,

$$\sum_{n_1,\ldots,n_6,n^7,\ldots n^{10}} e^{-2\pi H^{5D} + i2\pi\gamma^i P_i^{5D}} \to \sum_{n_1,\ldots,n_6,n^7,\ldots n^{10}} e^{-2\pi H^{5D} + i2\pi\gamma^i P_i^{5D}}. \tag{C.20}$$

So $Z^{5D}_{zero\,modes}$ is invariant under M_4. The origin of this is the $SO(4)$ invariance in the coordinate space labeled by $i = 2, 3, 4, 5$.

Next we show under U' that $Z^{5D}_{zero\,modes}$ transforms to $|\widetilde{\tau}|^3\, Z^{5D}_{zero\,modes}$. From (2.20),

$$
\begin{aligned}
Z^{5D}_{zero\;modes} &= \sum_{n_1\dots n_6} \exp\{-2\pi^3 \frac{R_6\sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\} \sum_{m^2\dots m^5} \exp\{-\pi \frac{R_1 R_6}{\sqrt{g}} m^i g_{ij} m^j + i4\pi^2 \gamma^i m^j\, F_{ij}\} \\
&= \sum_{n_1\dots n_6} \exp\{-2\pi^3 \frac{R_6\sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\} \sum_{m^2\dots m^5} \exp\{-\pi m\cdot A^{-1}\cdot m + 2\pi i\, m\cdot x\},
\end{aligned}
\tag{C.21}
$$

where $A^{-1}_{ij} = \frac{R_1 R_6}{\sqrt{g}} g_{ij}$ and $x_j = 2\pi\gamma^i F_{ij}$. Using a generalization of the Poisson summation formula

$$
\sum_{m\in\mathcal{Z}^p} e^{-\pi m\cdot A^{-1}\cdot m} e^{2\pi i m\cdot x} = (\det A)^{\frac{1}{2}} \sum_{m\in\mathcal{Z}^p} e^{-\pi(m+x)\cdot A\cdot(m+x))}
$$

we obtain from (C.21),

$$
\begin{aligned}
Z^{5D}_{zero\;modes} = (\det A)^{\frac{1}{2}} &\sum_{n_1\dots n_6\in\mathcal{Z}^6} \exp\{-2\pi^3 \frac{R_6\sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\} \\
&\cdot \sum_{m_2\dots m_5\in\mathcal{Z}^4} \exp\{-\pi \frac{\sqrt{g}}{R_1 R_6} g^{jj'} (m_j + \gamma^i 2\pi F_{ij})(m_{j'} + \gamma^{i'} 2\pi F_{i'j'})\},
\end{aligned}
\tag{C.22}
$$

where

$$
A^{jj'} = \frac{\sqrt{g}}{R_1 R_6} g^{jj'}, \qquad \det A = (\det A^{-1})^{-1} = \frac{g}{(R_1 R_6)^4}.
\tag{C.23}
$$

To check how this transforms under U' as given in (F.16), it is convenient to express (C.22) in terms of the metric $\widetilde{G}_5^{\tilde{l}\tilde{m}}$ found in (2.9),

$$
\begin{aligned}
Z^{5D}_{zero\;modes} = \frac{\sqrt{g}}{(R_1 R_6)^2} &\sum_{n_1\dots n_6\in\mathcal{Z}^6} \exp\{-\frac{\pi}{2} \frac{R_6\sqrt{g}}{R_1} \widetilde{G}_5^{ii'} \widetilde{G}_5^{jj'} (2\pi F_{ij})(2\pi F_{i'j'})\} \\
&\cdot \sum_{m_2\dots m_5\in\mathcal{Z}^4} \exp\Big\{-2\pi \frac{\sqrt{g} R_6}{R_1} \widetilde{G}_5^{6i'} \widetilde{G}_5^{jj'} m_{j'} (2\pi F_{ij}) - \pi \frac{R_6\sqrt{g}}{R_1 1} g^{jj'} m_j m_{j'}\Big\}.
\end{aligned}
\tag{C.24}
$$

Curiously we can identify the exponent in (C.24) as the Euclidean action, if we relabel the integers m_i by f_{6i}, and the $2\pi F_{ij}$ by f_{ij}; and neglect the integrations. In this form it will be easy to study its U' transformation, where (C.24) and (2.20) can also be written as

$$Z^{5D}_{zero\ modes} = \frac{\sqrt{g}}{(R_1R_6)^2} \sum_{f_{\tilde{m}\tilde{n}} \in \mathcal{Z}^{10}} \exp\Big\{ -2\pi \frac{\sqrt{\widetilde{G}_5}}{4R_1} \widetilde{G}_5^{\tilde{m}\tilde{m}'} \widetilde{G}_5^{\tilde{n}\tilde{n}'} f_{\tilde{m}\tilde{n}} f_{\tilde{m}'\tilde{n}'} \Big\}. \tag{C.25}$$

Under U' from (F.16), the coefficient transforms as

$$U' : \qquad \frac{\sqrt{g}}{(R_1R_6)^2} \to \frac{\sqrt{g}}{(R_1R_6)^2}\, |\widetilde{\tau}|^3, \tag{C.26}$$

since $\frac{\sqrt{g}}{(R_1R_6)^2} = \frac{R_2\sqrt{\widetilde{g}}}{(R_1R_6)^2}$. The Euclidean action for the zero mode computation is invariant under U', as we show next by first summing $\tilde{m} = \{2, \alpha, 6\}$, with $3 \leq \alpha \leq 5$.

$$\begin{aligned}
& -2\pi \frac{\sqrt{\widetilde{G}_5}}{4R_1} \widetilde{G}^{\tilde{m}\tilde{m}'} \widetilde{G}^{\tilde{n}\tilde{n}'} f_{\tilde{m}\tilde{n}} f_{\tilde{m}'\tilde{n}'} \\
&= -\frac{\pi R_2 R_6 \sqrt{\widetilde{g}}}{2R_1} \Big(\widetilde{G}_5^{\alpha\alpha'} \widetilde{G}_5^{\beta\beta'} f_{\alpha\beta} f_{\alpha'\beta'} + 4\widetilde{G}_5^{\alpha\alpha'} \widetilde{G}_5^{\beta 2} f_{\alpha\beta} f_{\alpha' 2} + 4\widetilde{G}_5^{\alpha\alpha'} \widetilde{G}_5^{\beta 6} f_{\alpha\beta} f_{\alpha' 6} + 2\widetilde{G}_5^{\alpha\alpha'} \widetilde{G}_5^{22} f_{\alpha 2} f_{\alpha' 2} \\
&\qquad - 2\widetilde{G}_5^{\alpha 2} \widetilde{G}_5^{\alpha' 2} f_{\alpha 2} f_{\alpha' 2} + 4\widetilde{G}_5^{\alpha\alpha'} \widetilde{G}_5^{26} f_{\alpha 2} f_{\alpha' 6} - 4\widetilde{G}_5^{\alpha 6} \widetilde{G}_5^{\alpha' 2} f_{\alpha 2} f_{\alpha' 6} + 4\widetilde{G}_5^{\alpha 2} \widetilde{G}_5^{\alpha' 6} f_{\alpha\alpha'} f_{26} \\
&\qquad + 2\widetilde{G}_5^{\alpha\alpha'} \widetilde{G}_5^{66} f_{\alpha 6} f_{\alpha' 6} - 2\widetilde{G}_5^{\alpha 6} \widetilde{G}_5^{\alpha' 6} f_{\alpha 6} f_{\alpha' 6} + 4\widetilde{G}_5^{\alpha 2} \widetilde{G}_5^{26} f_{\alpha 2} f_{26} - 4\widetilde{G}_5^{\alpha 6} \widetilde{G}_5^{22} f_{\alpha 2} f_{26} \\
&\qquad + 4\widetilde{G}_5^{\alpha 2} \widetilde{G}_5^{66} f_{\alpha 6} f_{26} - 4\widetilde{G}_5^{\alpha 6} \widetilde{G}_5^{26} f_{\alpha 6} f_{26} - 2\widetilde{G}_5^{26} \widetilde{G}_5^{26} f_{26} f_{26} + 2\widetilde{G}_5^{22} \widetilde{G}_5^{66} f_{26} f_{26} \Big).
\end{aligned} \tag{C.27}$$

Letting the U' transformation (F.16) act on (F.25), we see (F.25) changes to

$$
\begin{aligned}
&\Big(-2\pi\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\Big)' \\
&= -\frac{\pi R_2 R_6\sqrt{\widetilde{g}}}{2R_1}\Big(\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta\beta'}f_{\alpha\beta}f_{\alpha'\beta'} + 4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta 6}f_{\alpha\beta}f_{\alpha' 2} - 4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{\beta 2}f_{\alpha\beta}f_{\alpha' 6} + \frac{2}{|\widetilde{\tau}|^2}\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{22}f_{\alpha 2}f_{\alpha' 2} \\
&\qquad - 2\widetilde{G}_5^{\alpha 6}\widetilde{G}_5^{\alpha' 6}f_{\alpha 2}f_{\alpha' 2} - 4\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{26}f_{\alpha 2}f_{\alpha' 6} + 4\widetilde{G}_5^{\alpha 2}\widetilde{G}_5^{\alpha' 6}f_{\alpha 2}f_{\alpha' 6} - 4\widetilde{G}_5^{\alpha 6}\widetilde{G}_5^{\alpha' 2}f_{\alpha\alpha'}f_{26} \\
&\qquad + 2|\widetilde{\tau}|^2\widetilde{G}_5^{\alpha\alpha'}\widetilde{G}_5^{66}F_{\alpha 6}F_{\alpha' 6} - 2\widetilde{G}_5^{\alpha 6}\widetilde{G}_5^{\alpha' 2}f_{\alpha 6}f_{\alpha' 6} - 4\widetilde{G}_5^{\alpha 6}\widetilde{G}_5^{26}f_{\alpha 2}f_{26} + \frac{4}{|\widetilde{\tau}|^2}\widetilde{G}_5^{\alpha 2}\widetilde{G}_5^{22}f_{\alpha 2}f_{26} \\
&\qquad + 4|\widetilde{\tau}|^2\widetilde{G}_5^{\alpha 6}\widetilde{G}_5^{66}f_{\alpha 6}f_{26} - 4\widetilde{G}_5^{\alpha 2}\widetilde{G}_5^{26}f_{\alpha 6}f_{26} - 2\widetilde{G}_5^{26}\widetilde{G}_5^{26}F_{26}F_{26} + 2\widetilde{G}_5^{22}\widetilde{G}_5^{66}f_{26}f_{26}\Big).
\end{aligned}
\tag{C.28}
$$

In the partition sum $\sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} e^{-2\pi\left(\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\right)'}$, we can replace the integers as follows: $f_{\alpha 2}\to f_{\alpha 6},\ f_{\alpha 6}\to -f_{\alpha 2}$. Then using (F.7), we have

$$
\sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} e^{-2\pi\left(\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\right)'} = \sum_{f_{\tilde{m}\tilde{n}}\in\mathcal{Z}^{10}} e^{-2\pi\left(\frac{\sqrt{\widetilde{G}_5}}{4R_1}\widetilde{G}^{\tilde{m}\tilde{m}'}\widetilde{G}^{\tilde{n}\tilde{n}'}f_{\tilde{m}\tilde{n}}f_{\tilde{m}'\tilde{n}'}\right)}. \tag{C.29}
$$

So we have proved that under the U' transformation (F.16),

$$
Z^{5D}_{zero\ modes}(R_2|\widetilde{\tau}|, R_6|\widetilde{\tau}|^{-1}, \widetilde{g}_{\alpha\beta}, -\gamma^2|\widetilde{\tau}|^2, \widetilde{\gamma}^\alpha, -\kappa^\alpha) = |\widetilde{\tau}|^3\ Z^{5D}_{zero\ modes}(R_2, R_6, \widetilde{g}_{\alpha\beta}, \gamma^2, \kappa^\alpha, \widetilde{\gamma}^\alpha);
\tag{C.30}
$$

and thus under the $SL(5,\mathcal{Z})$ generator U_1, $Z^{5D}_{zero\ modes}$ transforms to $|\widetilde{\tau}|^3\ Z^{5D}_{zero\ modes}$. (F.28) also holds for $Z^{6D}_{zero\ modes}$, from (2.21). This is sometimes referred to as an $SL(2,\mathcal{Z})$ anomaly of the zero mode partition function, because U' includes the $\widetilde{\tau}\to-\frac{1}{\widetilde{\tau}}$ transformation. Finally we will show how this anomaly is canceled by the oscillator contribution. The 5D and 6D oscillator contributions are not equal, as given in (3.65) and (2.67). By inspection each is invariant under M_4, (C.17).

U' acts on Z^{5D}_{osc}

To derive how U' acts on Z^{5D}_{osc}, we first separate the product on $\vec{n}=(n,n_\alpha)\neq\vec{0}$ into a product on (all n, but $n_\alpha\neq(0,0,0)$) and on ($n\neq 0$, $n_\alpha=(0,0,0)$). Then using the regularized vacuum energy (E.1) expressed as sum over zero and non-zero transverse momenta $p_\perp = n_\alpha$ in (E.2),(B.9),(B.13), we find that (3.65) becomes

$$\begin{aligned} Z^{5D,Maxwell} &= Z^{5D}_{zero\ modes} \cdot \Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_2}|n|-2\pi i\gamma^2 n}} \Big)^3 \\ &\cdot \Big(\prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6<H>_{p_\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}-2\pi i\gamma^i n_i}} \Big)^3. \end{aligned} \tag{C.31}$$

As in [2] we observe the middle expression above can be written in terms of the Dedekind eta function $\eta(\widetilde{\tau}) \equiv e^{\frac{\pi i\widetilde{\tau}}{12}} \prod_{n\in Z\neq 0}(1-e^{2\pi i n\widetilde{\tau}})$, with $\widetilde{\tau} = \gamma^2 + i\frac{R_6}{R_2}$,

$$\Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n\neq 0} \frac{1}{1-e^{-2\pi\frac{R_6}{R_2}|n|-2\pi i\gamma^2 n}} \Big)^3 = (\eta(\widetilde{\tau})\bar{\eta}(\bar{\widetilde{\tau}}))^{-3}. \tag{C.32}$$

This transforms under U' in (F.16) as

$$(\eta(-\widetilde{\tau}^{-1})\bar{\eta}(-\bar{\widetilde{\tau}}^{-1}))^{-3} = |\widetilde{\tau}|^{-3}\,(\eta(\widetilde{\tau})\bar{\eta}(\bar{\widetilde{\tau}}))^{-3}, \tag{C.33}$$

where $\eta(-\widetilde{\tau}^{-1}) = (i\widetilde{\tau})^{\frac{1}{2}}\eta(\widetilde{\tau})$. In this way the anomaly of the zero modes in (F.28) is canceled by the massless part of the oscillator partition function (F.50). Lastly we demonstrate the third expression in (F.48) is invariant under U',

$$\Big(\prod_{n_\alpha\in\mathcal{Z}^3\neq(0,0,0)} e^{-2\pi R_6<H>_\perp} \prod_{n_2\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}-2\pi i\gamma^i n_i}} \Big)^3 = (PI)^{\frac{3}{2}} \tag{C.34}$$

where $(PI)^{\frac{3}{2}}$ is the modular invariant 2d partition function of three massive scalar bosons of mass $\sqrt{\widetilde{g}^{\alpha\beta}n_\alpha n_\beta}$, coupled to a worldsheet gauge field following [2]. From (3.66),

$$Z^{5D}_{osc} = (e^{-\pi R_6\sum_{\vec{n}\in\mathcal{Z}^4}\sqrt{g^{ij}n_in_j}} \prod_{\vec{n}\in\mathcal{Z}^4\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{ij}n_in_j}}}\Big)^3 \tag{C.35}$$

we can extract for fixed $n_\alpha \neq 0$,

$$
\begin{aligned}
(PI)^{\frac{1}{2}} &\equiv e^{-\pi R_6 \sum_{n_2\in \mathcal{Z}} \sqrt{g^{ij} n_i n_j}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1-e^{-2\pi R_6 \sqrt{g^{ij}n_i n_j}+2\pi i \gamma^i n_i}} \\
&= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta' E}{2}}}{1-e^{-\beta' E + 2\pi i(\gamma^2 s+\gamma^\alpha n_\alpha)}} \qquad \text{where } s\equiv n_2, \quad E \equiv \sqrt{g^{ij}n_i n_j}, \quad \beta' \equiv 2\pi R_6 \\
&= \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\,\sqrt{\cosh \beta' E - \cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)}} \qquad \text{for } n_\alpha \to -n_\alpha \\
&= e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}} \big(\ln[\cosh\beta' E - \cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)]+\ln 2\big)} \equiv e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\, \nu(E)},
\end{aligned}
\tag{C.36}
$$

where

$$
\begin{aligned}
\sum_{s\in\mathcal{Z}} \nu(E) &\equiv \sum_{s\in\mathcal{Z}} \big(\ln[\cosh\beta' E - \cos 2\pi(\gamma^2 s+\gamma^\alpha n_\alpha)]+\ln 2\big) \\
&= \sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}} \ln\,[\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2+E^2].
\end{aligned}
\tag{C.37}
$$

(F.54) follows in a similar way to steps (B.3)-(B.3) in [2], thus confirming its U' invariance due to the modular invariance of the massive 2d partition function, which we discuss further in the next section. We can also show directly that (F.54) is invariant under U', since

$$
\begin{aligned}
&E^2 = g^{ij}n_i n_j = g^{22}s^2 + 2g^{2\alpha} s n_\alpha + g^{\alpha\beta} n_\alpha n_\beta = \frac{1}{R_2^2}(s+\kappa^\alpha)^2 + \widetilde{g}^{\alpha\beta} n_\alpha n_\beta, \\
&\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2 = \frac{1}{R_6^2}(r+\widetilde{\gamma}^\alpha n_\alpha + \gamma^2(s+\kappa^\alpha n_\alpha))^2,
\end{aligned}
\tag{C.38}
$$

then

$$
\begin{aligned}
&\frac{4\pi^2}{\beta'^2}(r+\gamma^2 s+\gamma^\alpha n_\alpha)^2 + E^2 \\
&= \frac{1}{R_6^2}(s+\kappa^\alpha n_\alpha)^2\,|\widetilde{\tau}|^2 + \frac{1}{R_6^2}(r+\widetilde{\gamma}^\alpha n_\alpha)^2 + \frac{2\gamma^2}{R_6^2}(r+\widetilde{\gamma}^\alpha n_\alpha)(s+\kappa^\alpha n_\alpha) + \widetilde{g}^{\alpha\beta} n_\alpha n_\beta.
\end{aligned}
\tag{C.39}
$$

So we see the transformation U' given in (F.16) leaves (F.56) invariant if $s \to r$ and $r \to -s$. Therefore (F.54) is invariant under U', so that $(PI)^{\frac{1}{2}}$ given in (F.53) is invariant under U'.

In this way, we have established invariance under U_1 and U_2, and thus proved the partition function for the 5D Maxwell theory on T^5, given alternatively by (3.67) or (F.48), is invariant under $SL(5, \mathcal{Z})$, the mapping class group of T^5.

U' acts on Z^{6D}_{osc}

For the 6D chiral theory on $S^1 \times T^5$, the regularized vacuum energy from (B.14),

$$< H >^{6D} = -32\pi^2 \sqrt{G_5} \sum_{\vec{n} \neq \vec{0}} \frac{1}{(2\pi)^6 (g_{ij} n^i n^j + R_1^2 (n^1)^2)^3} \tag{C.40}$$

can be decomposed similarly to (E.2),

$$< H >^{6D} = \sum_{p_\perp \in \mathcal{Z}^3} < H >^{6D}_{p_\perp} = < H >^{6D}_{p_\perp = 0} + \sum_{p_\perp \in \mathcal{Z}^3 \neq 0} < H >^{6D}_{p_\perp}, \tag{C.41}$$

where

$$< H >^{6D}_{p_\perp} = -32\pi^2 \sqrt{G_5} \frac{1}{(2\pi)^4} \int d^4 y_\perp e^{-ip_\perp \cdot y_\perp} \sum_{n^2 \in \mathcal{Z} \neq 0} \frac{1}{|2\pi n^2 + y_\perp|^6}, \tag{C.42}$$

with denominator $|2\pi n^2 + y_\perp|^2 = G_{22}(2\pi n^2)^2 + 2(2\pi n^2) G_{2k} y^k_\perp + G_{kk'} y^k_\perp y^{k'}_\perp$,

$$\begin{aligned}
< H >^{6D}_{p_\perp = 0} &= -\frac{1}{12 R_2}, \\
< H >^{6D}_{p_\perp \neq 0} &= |p_\perp|^2 R_2 \sum_{n=1}^{\infty} cos(p_\alpha \kappa^\alpha 2\pi n) \big[K_2(2\pi n R_2 |p_\perp|) - K_0(2\pi n R_2 |p_\perp|) \big] \\
&= -\pi^{-1} \, |p_\perp| R_2 \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^\alpha 2\pi n) \frac{K_1(2\pi n R_2 |p_\perp|)}{n},
\end{aligned} \tag{C.43}$$

where $p_\perp = (p_1, p_\alpha) = n_\perp = (n_1, n_\alpha) = (n_1, n_3, n_4, n_5) \in \mathcal{Z}^4$, $|p_\perp| = \sqrt{\frac{(n_1)^2}{R_1^2} + \widetilde{g}^{\alpha\beta} n_\alpha n_\beta}$.

The U' invariance of (2.68) follows when we separate the product on $\vec{n} \in \mathcal{Z}^5 \neq \vec{0}$ into a product on ($n_2 \neq 0, \; n_\perp \equiv (n_1, n_3, n_4, n_5) = (0, 0, 0, 0)$,) and on (all n_2, but

$n_\perp = (n_1, n_3, n_4, n_5) \neq (0,0,0,0))$. Then

$$\begin{aligned}
Z^{6D,chiral} &= Z^{6D}_{zero\,modes} \cdot \Big(e^{\frac{\pi R_6}{6R_2}} \prod_{n_2 \in \mathcal{Z} \neq 0} \frac{1}{1 - e^{2\pi i\,(\gamma^2\, n_2 + i\frac{R_6}{R_2}|n_2|\,)}}\Big)^3 \\
&\quad \cdot \Big(\prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6D}_{p_\perp}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i2\pi\gamma^i n_i}}\Big)^3 \\
&= Z^{6D}_{zero\,modes} \cdot \big(\eta(\widetilde{\tau})\ \bar{\eta}(\bar{\widetilde{\tau}})\big)^{-3} \\
&\quad \cdot \Big(\prod_{(n_1,n_3,n_4,n_5) \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6D}_{p_\perp}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i2\pi\gamma^i n_i}}\Big)^3,
\end{aligned} \tag{C.44}$$

where $\widetilde{\tau} = \gamma^2 + i\frac{R_6}{R_2}$. So from the previous section together with (2.3), U' leaves invariant

$$Z^{6D}_{zero\,modes} \cdot \big(\eta(\widetilde{\tau})\ \bar{\eta}(\bar{\widetilde{\tau}})\big)^{-3}. \tag{C.45}$$

The part of the 6D partition function (C.44) at fixed $n_\perp \neq 0$,

$$e^{-2\pi R_6 <H>_{n_\perp \neq 0}} \prod_{n_2 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i2\pi\gamma^i n_i}} \tag{C.46}$$

corresponds to massive bosons on a two-torus and is invariant under the $SL(2, \mathcal{Z})$ transformation U' given in (F.16), as follows [2]. Each term with fixed $n_\perp \neq 0$ given in (F.67) is the square root of the partition function on T^2 (in the directions 2,6) of a massive complex scalar with $m^2 \equiv G^{11} n_1^2 + \widetilde{g}^{\alpha\beta} n_\alpha n_\beta$, $3 \leq \alpha, \beta \leq 5$, that couples to a constant gauge field $A^\mu \equiv iG^{\mu i} n_i$ with $\mu, \nu = 2, 6$; $i, j = 1, 3, 4, 5$. The metric on T^2 is $h_{22} = R_2^2\,, h_{66} = R_6^2 + (\gamma^2)^2 R_2^2\,,\ h_{26} = -\gamma^2 R_2^2$. Its inverse is $h^{22} = \frac{1}{R_2^2} + \frac{(\gamma^2)^2}{R_6^2}$, $h^{66} = \frac{1}{R_6^2}$

and $h^{26} = \frac{\gamma^2}{R_6^2}$. The manifestly $SL(2,\mathcal{Z})$ invariant *path integral* on the two-torus is

$$
\begin{aligned}
P.I. &= \int d\phi\, d\bar{\phi}\ e^{-\int_0^{2\pi} d\theta^2 \int_0^{2\pi} d\theta^6\ h^{\mu\nu}(\partial_\mu + A_\mu)\bar{\phi}(\partial_\nu - A_\nu)\phi + m^2\bar{\phi}\phi} \\
&= \int d\bar{\phi}\, d\phi e^{-\int_0^{2\pi} d\theta^2 \int_0^{2\pi} d\theta^6 \bar{\phi}(-(\frac{1}{R_2^2} + \frac{(\gamma^2)^2}{R_6^2})\partial_2^2 - (\frac{1}{R_6})^2\partial_6^2 - 2\frac{\gamma^2}{R_6^2}\partial_2\partial_6 + 2A^2\partial_2 + 2A^6\partial_6 + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta)\phi} \\
&= \det\Big([-(\tfrac{1}{R_2^2} + (\tfrac{\gamma^2}{R_6})^2)\partial_2^2 - (\tfrac{1}{R_6})^2\partial_6^2 - 2\gamma^2(\tfrac{1}{R_6})^2\partial_2\partial_6 + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta + 2iG^{2\alpha}n_\alpha\partial_2 + 2iG^{6\alpha}n_\alpha\partial_6 \\
&= e^{-tr\,ln\left[-(\frac{1}{R_2^2} + (\frac{\gamma^2}{R_6})^2)\partial_2^2 - (\frac{1}{R_6})^2\partial_6^2 - 2\gamma^2(\frac{1}{R_6})^2\partial_2\partial_6 + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta + 2iG^{2\alpha}n_\alpha\partial_2 + 2iG^{6\alpha}n_\alpha\partial_6\right]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}}\left[\ln(\frac{4\pi^2}{\beta'^2}r^2 + (\frac{1}{R_2^2} + (\frac{\gamma^2}{R_6})^2)s^2 + 2\gamma^2(\frac{1}{R_6})^2 rs + G^{11}n_1n_1 + G^{\alpha\beta}n_\alpha n_\beta + 2G^{1\alpha}n_\alpha\, s + 2G^{6\alpha}n_\alpha\, r)\right]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\nu(E)} \qquad\qquad (C.47)
\end{aligned}
$$

where from (2.7), $G^{11} = \frac{1}{R_1^2}, G^{\alpha\beta} = g^{\alpha\beta} + \frac{\gamma^\alpha\gamma^\beta}{R_6^2}, G^{2\alpha} = g^{2\alpha} + \frac{\gamma^2\gamma^\alpha}{R_6^2}, G^{6\alpha} = \frac{\gamma^\alpha}{R_6^2}$, and $\beta' \equiv 2\pi R_6$, and $\partial_2\phi = -is\phi;\ \partial_6\phi = -ir\phi$, and $n_2 \equiv s$. The sum on r is

$$
\nu(E) = \sum_{r\in\mathcal{Z}} \ln\Big[\frac{4\pi^2}{\beta'^2}(r + \gamma^2 s + \gamma^\alpha n_\alpha)^2 + E^2\Big], \qquad (C.48)
$$

with $E^2 \equiv G_5^{lm}n_l n_m = G_5^{11}n_1n_1 + G_5^{\alpha\beta}n_\alpha n_\beta + 2G_5^{\alpha 2}n_\alpha n_2 + G_5^{22}n_2n_2$, and $G_5^{11} = \frac{1}{R_1^2}$, $G_5^{12} = 0, G_5^{1\alpha} = 0, G_5^{2\alpha} = g^{2\alpha} = \frac{\kappa^\alpha}{R_2^2}, G_5^{22} = g^{22} = \frac{1}{R_2^2}, G_5^{\alpha\beta} = g^{\alpha\beta} = \widetilde{g}^{\alpha\beta} + \frac{\kappa^\alpha\kappa^\beta}{R_2^2}$. We evaluate the divergent sum $\nu(E)$ on r by

$$
\begin{aligned}
\frac{\partial\nu(E)}{\partial E} &= \sum_r \frac{2E}{\frac{4\pi^2}{\beta'^2}(r + \gamma^2 s + \gamma^\alpha n_\alpha)^2 + E^2} \\
&= \partial_E \ln\Big[\cosh\beta' E - \cos 2\pi\big(\gamma^2 s + \gamma^\alpha n_\alpha\big)\Big], \qquad (C.49)
\end{aligned}
$$

using the sum $\sum_{n\in\mathcal{Z}} \frac{2y}{(2\pi n + z)^2 + y^2} = \frac{\sinh y}{\cosh y - \cos z}$. Then integrating (F.70), we choose the integration constant to maintain modular invariance of (F.68),

$$
\nu(E) = \ln\big[\cosh\beta' E - \cos 2\pi\big(\gamma^2 s + \gamma^\alpha n_\alpha\big)\big] + \ln 2. \qquad (C.50)
$$

It follows for $n_2 \equiv s$ we have that (F.68) is

$$
\begin{aligned}
(P.I.)^{\frac{1}{2}} &= \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\sqrt{\cosh\beta E - \cos 2\pi(\gamma^2 s + \gamma^\alpha n_\alpha)}} \\
&= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta E}{2}}}{1 - e^{-\beta E + 2\pi i(\gamma^2 s + \gamma^\alpha n_\alpha)}} \\
&= e^{-\pi R_6 \sum_{s\in\mathcal{Z}} \sqrt{G_5^{lm} n_l n_m}} \prod_{s\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6\sqrt{G_5^{lm} n_l n_m} + 2\pi i \gamma^2 s + 2\pi i \gamma^\alpha n_\alpha}} \\
&= e^{-2\pi R_6 <H>_{n\perp}} \prod_{n_2\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6\sqrt{G_5^{lm} n_l n_m} + 2\pi i \gamma^2 n_2 + 2\pi i \gamma^\alpha n_\alpha}}\,, \qquad \text{(C.51)}
\end{aligned}
$$

which is (F.67). Its invariance under U' follows since (F.16) is an $SL(2,\mathcal{Z})$ transformation on T^2 combined with a gauge transformation on the 2d gauge field, $A_\mu \equiv h_{\mu\nu} i n_i G^{\nu i}$ where $\mu,\nu = 2,6$, $A_\mu \to A_\mu + \partial_\mu \lambda$, and $\phi \to e^{i\lambda}$, $\bar{\phi} \to e^{-i\lambda}$,

$$
\lambda(\theta^1, \theta^6) = \theta^2\, i(\widetilde{\gamma}^\alpha - \kappa^\alpha) - \theta^6\, i(\widetilde{\gamma}^\alpha + \kappa^\alpha) \qquad \text{(C.52)}
$$

since $A_2 = i\kappa^\alpha n_\alpha,\ A_6 = i\,\widetilde{\gamma}^\alpha\, n_\alpha$. Hence (F.72) and thus (F.67) are invariant under U'. So we have proved the 6D partition function for the chiral field on $S^1 \times T^5$, given by (2.68) or equivalently (C.44), is invariant under U_1 and U_2 and is hence $SL(5,\mathcal{Z})$ invariant.

APPENDIX D

CANONICAL AND PATH INTEGRAL APPROACHES

For convenience in comparing the $4d$ gauge theory with the $6d$ chiral theory in sections 2 and 3, we quantized both using canonical quantization. Since a Lagrangian exists for the $4d$ gauge theory, it is useful to verify that its path integral quantization agrees with canonical quantization. We find the two quantizations distribute zero and oscillator mode contributions differently, and thus these factors transform differently under the action of $SL(2, Z)$. We summarize the path integral quantization results from [32], [47], [48], [60]. Following [32], [48], the two-form zero mode part, $\frac{F}{2\pi}$ is the harmonic representative and can be expanded in terms of the basis $\alpha_I = \frac{1}{(2\pi)^2} d\theta^1 \wedge d\theta^2$, etc., $I = 1, 2, .., 6$ namely

$$\frac{F}{2\pi} \equiv m = \sum_I m_I \alpha_I, \tag{D.1}$$

where m_I are integers. Define (m, n) to be the intersection form such that $(m, n) = \int m \wedge n$, and thus

$$\begin{aligned}(m, m) &= \frac{1}{16\pi^2} \int d^4\theta \epsilon^{ijkl} F_{ij} F_{kl} \\ (m, *m) &= \frac{1}{8\pi^2} \int d^4\theta \sqrt{g} F^{ij} F_{ij}.\end{aligned} \tag{D.2}$$

So the action (3.3) is given as

$$I = \frac{4\pi^2}{e^2}(m, *m) - \frac{i\theta}{2}(m, m) = \frac{1}{2e^2} \int d^4\theta \sqrt{g} F^{ij} F_{ij} - \frac{i\theta}{32\pi^2} \int d^4\theta \epsilon^{ijkl} F_{ij} F_{kl}. \tag{D.3}$$

The zero mode partition function from the path integral formalism can be expressed as a lattice sum over the integral basis of m_I [32], [48],

$$
\begin{aligned}
Z^{PI}_{\text{zero modes}} &= \sum_{m_I\in\mathcal{Z}^6} \exp\Big[-\frac{4\pi^2}{e^2}\big(m,*m\big)+\frac{i\theta}{2}\big(m,m\big)\Big] \\
&= \sum_{m_I\in\mathcal{Z}^6} \exp\Big[\frac{i\pi}{2}\tau\Big(\big(m,m\big)+\big(m,*m\big)\Big)-\frac{i\pi}{2}\bar\tau\Big(-\big(m,m\big)+\big(m,*m\big)\Big)\Big],
\end{aligned} \tag{D.4}
$$

where $\tau=\frac{\theta}{2\pi}+i\frac{4\pi}{e^2}$, and we have chosen the θ dependence of the action as in [32]. Alternatively the zero mode sum be can written in terms of the metric using (D.3)

$$
\begin{aligned}
Z^{PI}_{\text{zero modes}} = \sum_{\widetilde F_{ij}\in\mathcal{Z}^6} \exp\{ [&-\frac{\pi}{2}R_6\sqrt{\tilde g}g^{\alpha\beta}g^{\gamma\delta}\widetilde F_{\alpha\gamma}\tilde F_{\beta\delta}-\pi\frac{\sqrt{\tilde g}}{R_6}g^{\delta\delta'}\widetilde F_{\delta\beta}\gamma^\beta\widetilde F_{\delta'\beta'}\gamma^{\beta'}-\pi\frac{\sqrt{\tilde g}}{R_6}g^{\alpha\beta}\widetilde F_{6\alpha}\widetilde F_{6\beta} \\
&+2\pi\frac{\sqrt{\tilde g}}{R_6}g^{\alpha\delta}\widetilde F_{6\alpha}\widetilde F_{\delta\beta}\gamma^\beta-i\frac{\theta e^2}{8\pi}\epsilon^{\alpha\beta\gamma}\widetilde F_{6\alpha}\tilde F_{\beta\gamma}]\ \frac{4\pi}{e^2}\}
\end{aligned} \tag{D.5}
$$

where $\widetilde F_{ij}=2\pi F_{ij}=m_I$ are integers due to the charge quantization (D.1), and where we have taken into account the integrations $\int d^4\theta=(2\pi)^4$ in (D.5). To compare the zero mode partition functions from the Hamiltonian and path integral formalisms, we rewrite the Hamiltonian formulation result (3.11) as

$$
\begin{aligned}
&Z^{4d}_{\text{zero modes}} \\
&=\sum_{\widetilde\Pi^\alpha,\widetilde F_{\alpha\beta}} \exp\Big[-\frac{e^2R_6}{4\sqrt{\tilde g}}g_{\alpha\beta}\big(\widetilde\Pi^\alpha+i\frac{4\pi\sqrt{\tilde g}}{e^2R_6}g^{\alpha\delta}\widetilde F_{\delta\lambda}\gamma^\lambda+\frac{\theta\epsilon^{\alpha\gamma\delta}}{4\pi}\widetilde F_{\gamma\delta}\big)\cdot\big(\widetilde\Pi^\beta+i\frac{4\pi\sqrt{\tilde g}}{e^2R_6}g^{\beta\delta'}\widetilde F_{\delta'\lambda'}\gamma^{\lambda'}+\frac{\theta\epsilon^{\beta\gamma'\delta'}}{4\pi}\widetilde F_{\gamma'\delta'}\big) \\
&\qquad -\frac{4\pi^2}{e^2}\frac{\sqrt{\tilde g}}{R_6}g^{\delta\delta'}\widetilde F_{\delta\beta}\gamma^\beta\widetilde F_{\delta'\beta'}\gamma^{\beta'}-\frac{2\pi^2}{e^2}\sqrt{g}g^{\alpha\beta}g^{\gamma\delta}\widetilde F_{\alpha\gamma}\widetilde F_{\beta\delta}\Big].
\end{aligned} \tag{D.6}
$$

After Poisson resummation,

$$
\sum_{n\in\mathcal{Z}^3}\exp[-\pi(n+x)\cdot\mathcal{A}\cdot(n+x)]=(\det\mathcal{A})^{-\frac12}\sum_{\mathrm{n}\in\mathcal{Z}^3}\mathrm{e}^{-\pi\mathrm{n}\cdot\mathcal{A}^{-1}\cdot\mathrm{n}}\mathrm{e}^{2\pi\mathrm{i}\mathrm{n}\cdot\mathrm{x}}, \tag{D.7}
$$

where $\mathcal{A}_{\alpha\beta} \equiv \frac{e^2 R_6}{4\pi\sqrt{\tilde{g}}} g_{\alpha\beta}$ and $x^\alpha \equiv i\frac{4\pi\sqrt{\tilde{g}}}{e^2 R_6} g^{\alpha\delta}\widetilde{F}_{\delta\lambda}\gamma^\lambda + \frac{\theta}{4\pi}\epsilon^{\alpha\gamma\delta}\widetilde{F}_{\gamma\delta}$, we get the Hamiltonian expression as

$$
\begin{aligned}
Z^{4d}_{\text{zero modes}} &= (\frac{e^2}{4\pi})^{-\frac{3}{2}} \frac{\widetilde{g}^{\frac{1}{4}}}{R_6{}^{\frac{3}{2}}} \sum_{\hat{\Pi}_\alpha, \tilde{F}_{\alpha\beta}} \exp\{-\frac{4\pi^2\sqrt{\tilde{g}}}{e^2 R_6} g^{\alpha\beta}\hat{\Pi}_\alpha\hat{\Pi}_\beta - i\frac{\theta}{2}\hat{\Pi}_\alpha\epsilon^{\alpha\gamma\delta}\tilde{F}_{\gamma\delta} + \frac{8\pi^2\sqrt{\tilde{g}}}{e^2 R_6} g^{\alpha\beta}\hat{\Pi}_\alpha\tilde{F}_{\beta\delta}\gamma^\delta \\
&\qquad - \frac{4\pi^2}{e^2}\frac{\sqrt{\tilde{g}}}{R_6} g^{\delta\delta'}\tilde{F}_{\delta\beta}\gamma^\beta\tilde{F}_{\delta'\beta'}\gamma^{\beta'} - \frac{2\pi^2 R_6}{e^2}\sqrt{\tilde{g}} g^{\alpha\beta}g^{\gamma\delta}\tilde{F}_{\alpha\gamma}\tilde{F}_{\beta\delta}\} \\
&= (\operatorname{Im}\tau)^{\frac{3}{2}} \frac{\widetilde{g}^{\frac{1}{4}}}{R_6{}^{\frac{3}{2}}} Z^{PI}_{\text{zero modes}},
\end{aligned}
\tag{D.8}
$$

where $\hat{\Pi}_\alpha$ is the integer value of $\widetilde{\Pi}^\alpha$, and we identify $\hat{\Pi}_\alpha$ with $\widetilde{F}_{6\alpha}$ in (D.5). Then

$$
Z^{PI}_{\text{zero modes}} = (\operatorname{Im}\tau)^{-\frac{3}{2}} \frac{R_6^2}{g^{\frac{1}{4}}} Z^{4d}_{\text{zero modes}}, \tag{D.9}
$$

which is (3.85).

We review from [47] how the non-zero mode partition function is defined by a path integral,

$$
Z^{PI}_{\text{non-zero modes}} = \int_A DA^\mu e^{-I}. \tag{D.10}
$$

Performing the functional integration with the Fadeev-Popov approach, [47] regularizes the path integral by

$$
Z^{PI}_{\text{non-zero modes}} = \frac{1}{(2\pi)^{\frac{b_1-1}{2}}} \left(\frac{g}{\text{vol}T^4}\right)^{\frac{1}{2}} \left[\det(\Delta_0)\frac{\det(2\pi\text{Im}\tau\Delta_0)}{\det(2\pi\text{Im}\tau\Delta_1)}\right]^{\frac{1}{2}} = (\frac{g}{(2\pi)^4\sqrt{g}})^{\frac{1}{2}} (2\pi\text{Im}\tau)^{\frac{b_1-1}{2}} \frac{\det\Delta_0}{\det\Delta_1^{\frac{1}{2}}}, \tag{D.11}
$$

where $b_1 = 4$ is the dimension of the group $H^1(T^4)$. $\Delta_p = (d^\dagger d + dd^\dagger)_p$ is the kinetic energy operator acting on the p-form. $g = \text{det}G_{ij}$. So $\Delta_0 = -G^{ij}\partial_i\partial_j$, and $\det(\Delta_1) = \det(\Delta_0)^4$. Thus

$$
Z^{PI}_{\text{non-zero modes}} = \frac{g^{\frac{1}{4}}}{\sqrt{2\pi}}(\text{Im}\tau)^{\frac{3}{2}} \ \det\Delta_0^{-1}. \tag{D.12}
$$

The determinant can be computed

$$\det\Delta_0^{-\frac{1}{2}} = \exp\{-\frac{1}{2}\mathrm{trln}A\}, \tag{D.13}$$

$$\begin{aligned}\exp\{-\frac{1}{2}\mathrm{trln}\Delta_0\} &= \exp\Big(-\frac{1}{2}\mathrm{trln}\big(-\mathrm{G}^{66}\partial_6^2 - 2\mathrm{G}_{6\alpha}\partial_6\partial_\alpha - \mathrm{G}^{\alpha\beta}\partial_\alpha\partial_\beta\big)\Big) \\ &= \exp\Big(-\frac{1}{2}\sum_{\mathrm{n}_\alpha\neq\tilde{0}}\sum_{\mathrm{n}_6}\ln\big(\frac{1}{\mathrm{R}_6^2}\mathrm{n}_6^2 + 2\frac{\gamma^\alpha}{\mathrm{R}_6^2}\mathrm{n}_\alpha\mathrm{n}_6 + \mathrm{G}^{\alpha\beta}\mathrm{n}_\alpha\mathrm{n}_\beta\big)\Big) \\ &= \exp\Big(-\frac{1}{2}\sum_{\mathrm{n}_\alpha\neq\tilde{0}}\sum_{\mathrm{n}_6}\ln\big(\frac{1}{\mathrm{R}_6^2}(\mathrm{n}_6+\gamma^\alpha\mathrm{n}_\alpha)^2 + \mathrm{g}^{\alpha\beta}\mathrm{n}_\alpha\mathrm{n}_\beta\big)\Big).\end{aligned} \tag{D.14}$$

Let $\mu(E) \equiv \sum_{n_6}\ln\big(\frac{1}{R_6^2}(n_6+\gamma^\alpha n_\alpha)^2 + E^2\big)$, where $E^2 \equiv g^{\alpha\beta}n_\alpha n_\beta,\ \rho = 2\pi R_6$,

$$\begin{aligned}\frac{\partial\mu(E)}{\partial E} &= \sum_{n_6}\frac{2E}{\frac{1}{R_6^2}(n_6+\gamma^\alpha n_\alpha)^2 + E^2} = \frac{\rho\sinh(\rho E)}{\cosh(\rho E) - \cos(2\pi\gamma^\alpha n_\alpha)} \\ &= \partial_E\ln\big[\cosh(\rho E) - \cos(2\pi\gamma^\alpha n_\alpha)\big].\end{aligned} \tag{D.15}$$

After integration, we have

$$\mu(E) = \ln\Big[\cosh(\rho E) - \cos(2\pi\gamma^\alpha n_\alpha)\Big] + \ln\left(R_6^2\sqrt{\frac{2}{\pi}}\right). \tag{D.16}$$

where the constant $\ln\left(R_6^2\sqrt{\frac{2}{\pi}}\right)$ maintains $SL(4,Z)$ invariance of the partition function. So,

$$\begin{aligned}\det\Delta_0^{-\frac{1}{2}} = \exp\Big(-\frac{1}{2}\mathrm{trln}\Delta_0\Big) &= e^{-\frac{1}{2}\sum_{n_\alpha\neq\vec{0}}\mu(E)} \\ &= \frac{(2\pi)^{\frac{1}{4}}}{R_6}\prod_{n_\alpha\in\mathcal{Z}^3\neq\vec{0}}\frac{1}{\sqrt{2}\sqrt{\cosh(\rho E) - \cos(2\pi\gamma^\alpha n_\alpha)}} \\ &= \frac{(2\pi)^{\frac{1}{4}}}{R_6}\prod_{n_\alpha\in\mathcal{Z}^3\neq\vec{0}}\frac{e^{-\frac{\rho E}{2}}}{1-e^{-\rho E+2\pi i\gamma^\alpha n_\alpha}}.\end{aligned} \tag{D.17}$$

Therefore, using (D.12), we have

$$Z^{PI}_{\text{non-zero modes}} = (\operatorname{Im}\tau)^{\frac{3}{2}} \frac{g^{\frac{1}{4}}}{R_6^2} Z^{4d}_{osc}, \tag{D.18}$$

which is (3.86).

Together with (D.9), the partition functions from the two quantizations agree but they factor differently into zero and oscillator modes of the $Z^{6d,chiral}$ and $Z^{4d,Maxwell}$ partition functions The S-duality group $SL(2,\mathcal{Z})$ group has two generators S and T which act on the parameter τ to give

$$S : \tau \to -\frac{1}{\tau}, \qquad T : \tau \to \tau - 1. \tag{D.19}$$

Since $\tau = \beta^2 + i\frac{R_1}{R_2} = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, the transformation S corresponds to

$$R_1 \to R_1|\tau|^{-1}, \qquad R_2 \to R_2|\tau|, \qquad \beta^2 \to -|\tau|^{-2}\beta^2, \tag{D.20}$$

and T corresponds to

$$\beta^2 \to \beta^2 - 1. \tag{D.21}$$

Or equivalently

$$\begin{aligned} S: &\qquad \frac{4\pi}{e^2} \to \frac{4\pi}{e^2}|\tau|^{-2}, \quad \theta \to -\theta|\tau|^{-2} \\ T: &\qquad \theta \to \theta - 2\pi, \end{aligned} \tag{D.22}$$

which for $\theta = 0$ is the familiar electromagnetic duality transformation $\frac{e^2}{4\pi} \to \frac{4\pi}{e^2}$.

$6d$ partition function

The $6d$ chiral two-form zero mode partition function (3.10),

$$
\begin{aligned}
Z^{6d}_{\text{zero modes}} = \sum_{n_8,n_9,n_{10}} & \exp\{-\frac{\pi R_6}{R_1 R_2}\sqrt{\tilde{g}} g^{\alpha\alpha'} H_{12\alpha} H_{12\alpha'}\} \\
& \cdot \sum_{n_7} \exp\{-\frac{\pi}{6} R_6 R_1 R_2 \sqrt{\tilde{g}} g^{\alpha\alpha'} g^{\beta\beta'} g^{\delta\delta'} H_{\alpha\beta\delta} H_{\alpha'\beta'\delta'} - i\pi\gamma^\alpha \epsilon^{\gamma\beta\delta} H_{12\gamma} H_{\alpha\beta\delta}\} \\
& \cdot \sum_{n_4,n_5,n_6} \exp\{-\frac{\pi}{2} R_6 R_1 R_2 \sqrt{\tilde{g}} (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{{R_1}^2}) g^{\alpha\alpha'} g^{\beta\beta'} H_{2\alpha\beta} H_{2\alpha'\beta'}\} \\
& \cdot \sum_{n_1,n_2,n_3} \exp\{-\pi \frac{R_6 R_2}{R_1} \sqrt{\tilde{g}} \beta^2 g^{\alpha\alpha'} g^{\beta\beta'} H_{1\alpha\beta} H_{2\alpha'\beta'} + i\pi\gamma^\alpha \epsilon^{\gamma\beta\delta} H_{1\gamma\beta} H_{2\alpha\delta} \\
& \qquad - \frac{\pi}{4} \frac{R_6 R_2}{R_1} \sqrt{\tilde{g}} (g^{\alpha\alpha'} g^{\beta\beta'} - g^{\alpha\beta'} g^{\beta\alpha'}) H_{1\alpha\beta} H_{1\alpha'\beta'}\}
\end{aligned}
\tag{D.23}
$$

where $H_{134} = n_1$,$H_{145} = n_2$, $H_{135} = n_3$, $H_{234} = n_4$, $H_{245} = n_5$, $H_{235} = n_6$, $H_{345} = n_7$, $H_{123} = n_8$, $H_{124} = n_9$, $H_{125} = n_{10}$, is invariant under both S and T. To show the invariance using (D.20,D.21) we group the exponents in (D.23) into two sets,

$$
-\frac{\pi R_6}{R_1 R_2}\sqrt{\tilde{g}} g^{\alpha\alpha'} H_{12\alpha} H_{12\alpha'} - \frac{\pi}{6} R_6 R_1 R_2 \sqrt{\tilde{g}} g^{\alpha\alpha'} g^{\beta\beta'} g^{\delta\delta'} H_{\alpha\beta\delta} H_{\alpha'\beta'\delta'} - i\pi\gamma^\alpha \epsilon^{\gamma\beta\delta} H_{12\gamma} H_{\alpha\beta\delta},
\tag{D.24}
$$

and

$$
\begin{aligned}
& -\frac{\pi}{2} R_6 R_1 R_2 \sqrt{\tilde{g}} (\frac{1}{R_2^2} + \frac{{\beta^2}^2}{{R_1}^2}) g^{\alpha\alpha'} g^{\beta\beta'} H_{2\alpha\beta} H_{2\alpha'\beta'} - \pi \frac{R_6}{R_1} R_2 \sqrt{\tilde{g}} \beta^2 g^{\alpha\alpha'} g^{\beta\beta'} H_{1\alpha\beta} H_{2\alpha'\beta'} \\
& + i\pi\gamma^\alpha \epsilon^{\gamma\beta\delta} H_{1\gamma\beta} H_{2\alpha\delta} - \frac{\pi}{2} \frac{R_6 R_2}{R_1} \sqrt{\tilde{g}}\, g^{\alpha\alpha'} g^{\beta\beta'}\, H_{1\alpha\beta} H_{1\alpha'\beta'}.
\end{aligned}
\tag{D.25}
$$

(D.24) has no β^2 dependence and therefore is invariant under T. (D.25) transforms under under T to become

$$
\begin{aligned}
&-\frac{\pi}{2}R_6R_1R_2\sqrt{\tilde{g}}(\frac{1}{R_2^2}+\frac{{\beta^2}^2}{{R_1}^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}-\pi\frac{R_6}{R_1}\,R_2\sqrt{\tilde{g}}\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'}\\
&+i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta}-\frac{\pi}{2}\frac{R_6}{R_1R_2}\sqrt{\tilde{g}}\,g^{\alpha\alpha'}g^{\beta\beta'}\,H_{1\alpha\beta}H_{1\alpha'\beta'}\\
&+\pi\frac{R_6}{R_1}\sqrt{\tilde{g}}R_2\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}-\frac{\pi}{2}\frac{R_6}{R_1}\sqrt{\tilde{g}}R_2g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}+\pi\frac{R_6}{R_1}R_2\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'}
\end{aligned}
\tag{D.26}
$$

which is equivalent to (D.25) in the sum where we shift the integer zero mode field strength $H_{1\alpha\beta}$ to $H_{1\alpha\beta}-H_{2\alpha\beta}$.

Under S, we see (D.24) as a function of R_1R_2 is invariant, and find (D.25) transforms to

$$
\begin{aligned}
&-\frac{\pi}{2}\frac{R_6R_2}{R_1}\sqrt{\tilde{g}}g^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}+\pi\frac{R_6}{R_1}R_2\sqrt{\tilde{g}}\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{2\alpha'\beta'}\\
&+i\pi\gamma^{\alpha}\epsilon^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta}-\frac{\pi}{2}R_1R_6R_2\sqrt{\tilde{g}}(g^{22}+\frac{{\beta^2}^2}{R_1^2})g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{1\alpha'\beta'}.
\end{aligned}
\tag{D.27}
$$

So by shifting the integer field strength tensors $H_{1\alpha\beta}\to H_{2\alpha\beta}$ and $H_{2\alpha\beta}\to -H_{1\alpha\beta}$, the sum on (D.25) is left invariant by S. Thus we have proved $SL(2,\mathcal{Z})$ invariance of the $6d$ zero mode partition function (3.10), and that its factors ϵ and $Z^{4d}_{\text{zeromodes}}$ in (3.13) are separately $SL(2,\mathcal{Z})$ invariant.

For the oscillator modes (3.71), the only term that transforms in the sum and product is

$$
\tilde{p}^2\equiv\frac{{p_1}^2}{{R_1}^2}+(g^{22}+\frac{{\beta^2}^2}{{R_1}^2})p_2^2+\frac{2\beta^2}{{R_1}^2}p_1p_2,
\tag{D.28}
$$

which is invariant under T by shifting the momentum $p_1\to p_1+p_2$. With the S transformation, $\tilde{p}^2$ becomes

$$
{p_1}^2(g^{22}+\frac{{\beta^2}^2}{{R_1}^2})+\frac{1}{{R_1}^2}p_2^2-\frac{2\beta^2}{{R_1}^2}p_1p_2,
\tag{D.29}
$$

and by also exchanging the momentum $p_1\to p_2$ and $p_2\to -p_1$, the term remains the same. So the 6d oscillator partition function (3.71) is $SL(2\mathcal{Z})$ invariant, which holds also for regularized vacuum energy as given in (3.72).

$4d$ $U(1)$ partition function

In the Hamiltonian formulation, $SL(2, \mathcal{Z})$ leaves invariant the $U(1)$ oscillator partition function (3.65), since it is independent of e^2 and θ. We have also checked above, starting from $6d$, that the zero mode $4d$ partition function (3.11) is invariant. Thus the $U(1)$ partition function from the Hamiltonian formalism is S-duality invariant.

The S-duality transformations on the corresponding quantities in the path integral quantization can be derived from (D.9) and (D.18). Since $\mathrm{Im}\,\tau \to \frac{1}{|\tau|^2}\mathrm{Im}\tau$ under S, and is invariant under T, we have

$$
\begin{aligned}
&Z^{4d}_{\text{zero modes}} \longrightarrow Z^{4d}_{\text{zero modes}}, \qquad Z^{PI}_{\text{zero modes}} \longrightarrow |\tau|^3 Z^{PI}_{\text{zero modes}} \qquad &&\text{under } S \\
&Z^{4d}_{\text{zero modes}} \longrightarrow Z^{4d}_{\text{zero modes}}, \qquad Z^{PI}_{\text{zero modes}} \longrightarrow Z^{PI}_{\text{zero modes}} \qquad &&\text{under } T
\end{aligned}
\tag{D.30}
$$

and

$$
\begin{aligned}
&Z^{4d}_{\text{osc}} \longrightarrow Z^{4d}_{\text{osc}}, \qquad Z^{PI}_{\text{osc}} \longrightarrow |\tau|^{-3} Z^{PI}_{\text{osc}} \qquad &&\text{under } S \\
&Z^{4d}_{\text{osc}} \longrightarrow Z^{4d}_{\text{osc}}, \qquad Z^{PI}_{\text{osc}} \longrightarrow Z^{PI}_{\text{osc}} \qquad &&\text{under } T,
\end{aligned}
\tag{D.31}
$$

which is (3.87) and (3.88).

APPENDIX E

REGULARIZATION OF $4d$ MAXWELL THEORY

The sum in (3.65) is divergent. We regularize the vacuum energy following [2],[6]. For $< H >= \frac{1}{2}\sum_{p_\alpha\in\mathcal{Z}^3}\sqrt{g^{\alpha\beta}p_\alpha p_\beta}$, the $SL(3,\mathcal{Z})$ invariant regularized vacuum energy becomes

$$< H >= -\frac{1}{4\pi^3}\sqrt{\tilde{g}}\sum_{n^\alpha\in\mathcal{Z}^3\neq 0}\frac{1}{(g_{\alpha\beta}n^\alpha n^\beta)^2} = -4\pi\sqrt{\tilde{g}}\sum_{\vec{n}\in\mathcal{Z}^3\neq 0}\frac{1}{|2\pi\vec{n}|^4}. \tag{E.1}$$

For the proof of $SL(4,\mathcal{Z})$ invariance in Appendix H, it is also useful to write the regularized sum (E.1), as

$$< H > = \sum_{p_\perp\in\mathcal{Z}^2} < H >_{p_\perp} = < H >_{p_\perp=0} + \sum_{p_\perp\in\mathcal{Z}^2\neq 0} < H >_{p_\perp}, \tag{E.2}$$

where $p_\perp = p_a \in \mathcal{Z}^2,\ a = 4,5$, and

$$\begin{aligned} < H >_{p_\perp=0} &= \frac{1}{2}\sum_{p_3\in\mathcal{Z}}\sqrt{g^{33}p_3p_3} = \frac{1}{R_3}\sum_{n=1}^{\infty} n = \frac{1}{R_3}\zeta(-1) = -\frac{1}{12R_3}; \\ < H >_{p_\perp\neq 0} &= |p_\perp|^2 R_3 \sum_{n=1}^{\infty}\cos(p_a\kappa^a 2\pi n)\big[K_2(2\pi n R_3|p_\perp|) - K_0(2\pi n R_3|p_\perp|)\big] \end{aligned} \tag{E.3}$$

$|p_\perp| = \sqrt{p_a p_b \tilde{g}^{ab}}$, using the 2d inverse metric as defined in Appendix H.

APPENDIX F

$SL(4,\mathcal{Z})$ INVARIANCE OF $Z^{4d,\text{Maxwell}}$ AND $Z^{6d,\text{chiral}}$

Rewriting the $4d$ metric (3,4,5,6) From (3.6) the metric on the four-torus, for $\alpha, \beta = 3, 4, 5$, is

$$G_{\alpha\beta} = g_{\alpha\beta}, \qquad G_{\alpha 6} = -g_{\alpha\beta}\gamma^{\beta}, \qquad G_{66} = R_6^2 + g_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}. \tag{F.1}$$

We can rewrite this metric using $a, b = 4, 5$,

$$g_{33} \equiv R_3^2 + g_{ab}\kappa^a\kappa^b, \qquad g_{a3} \equiv -g_{ab}\kappa^b, \qquad g_{ab} \equiv g_{ab}, \qquad (\gamma^3)\kappa^a - \gamma^a \equiv -\widetilde{\gamma}^a, \tag{F.2}$$

$$\begin{aligned} &G_{33} = R_3^2 + g_{ab}\kappa^a\kappa^b, \qquad &&G_{36} = -(\gamma^3)R_3^2 + g_{ab}\kappa^b\widetilde{\gamma}^a, \qquad &&G_{3a} = -g_{ab}\kappa^b, \\ &G_{ab} = g_{ab}, &&G_{a6} = -g_{ab}\widetilde{\gamma}^b, &&G_{66} = R_6^2 + (\gamma^3)^2 R_3^2 + g_{ab}\,\widetilde{\gamma}^a\widetilde{\gamma}^b \end{aligned} \tag{F.3}$$

The $3d$ inverse of $g_{\alpha\beta}$ is

$$g^{ab} = \widetilde{g}^{ab} + \frac{\kappa^a\kappa^b}{R_3^2}, \qquad g^{a3} = \frac{\kappa^a}{R_3^2}, \qquad g^{33} = \frac{1}{R_3^2}, \tag{F.4}$$

where $\widetilde{g}^{ab}$ is the $2d$ inverse of g_{ab}.

$$g \equiv \det G_{ij} = R_6^2\ \det g_{\alpha\beta} \equiv R_6^2\,\widetilde{g} = R_6^2 R_3^2\ \det g_{ab} \equiv R_6^2 R_3^2\,\bar{g}.$$

The line element can be written as

$$
\begin{aligned}
ds^2 &= R_6^2(d\theta^6)^2 + \sum_{\alpha,\beta=3,4,5} g_{\alpha\beta}(d\theta^\alpha - \gamma^\alpha d\theta^6)(d\theta^\beta - \gamma^\beta d\theta^6) \\
&= R_3^2(d\theta^3 - (\gamma^3)d\theta^6)^2 + R_6^2(d\theta^6)^2 \\
&\quad + \sum_{a,b=4,5} g_{ab}(d\theta^a - \widetilde{\gamma}^a d\theta^6 - \kappa^a d\theta^3)\,(d\theta^b - \widetilde{\gamma}^b d\theta^6 - \kappa^b d\theta^3).
\end{aligned} \tag{F.5}
$$

We define

$$
\widetilde{\tau} \equiv \gamma^3 + i\frac{R_6}{R_3}. \tag{F.6}
$$

The 4d inverse is

$$
\begin{aligned}
&\widetilde{G}_4^{33} = \frac{|\widetilde{\tau}|^2}{R_6^2} = \widetilde{G}_4^{66}|\widetilde{\tau}|^2, \qquad \widetilde{G}_4^{66} = \frac{1}{R_6^2}, \qquad \widetilde{G}_4^{36} = \frac{\gamma^3}{R_6^2}, \qquad \widetilde{G}_4^{3a} = \frac{\kappa^a|\widetilde{\tau}|^2}{R_6^2} + \frac{\gamma^3\widetilde{\gamma}^a}{R_6^2}, \\
&\widetilde{G}_4^{ab} = \widetilde{g}^{ab} + \frac{\kappa^a\kappa^b}{R_6^2}|\widetilde{\tau}|^2 + \frac{\widetilde{\gamma}^a\widetilde{\gamma}^b}{R_6^2} + \frac{\gamma^3(\widetilde{\gamma}^a\kappa^b + \kappa^a\widetilde{\gamma}^b)}{R_6^2}, \qquad \widetilde{G}_4^{6a} = \frac{\gamma^a}{R_6^2} = \frac{\gamma^3\kappa^a + \widetilde{\gamma}^a}{R_6^2}.
\end{aligned} \tag{F.7}
$$

Generators of $GL(n, \mathcal{Z})$

The $GL(n, \mathcal{Z})$ unimodular group can be generated by three matrices For $GL(4, \mathcal{Z})$ these can be taken to be U_1, U_2 and U_3,

$$
U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad U_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{F.8}
$$

so that every matrix M in $GL(4, \mathcal{Z})$ can be written as a product $U_1^{n_1}U_2^{n_2}U_3^{n_3}U_1^{n_4}U_2^{n_5}U_3^{n_6}\ldots$, for integers n_i. Matrices U_1, U_2 and U_3 act on the basis vectors of the four-torus $\vec{\alpha}_i$

where $\vec{\alpha}_i \cdot \vec{\alpha}_j \equiv \alpha_i^k \alpha_j^l G_{kl} = G_{ij}$,

$$\begin{aligned}
\vec{\alpha}_3 &= (1,0,0,0) \\
\vec{\alpha}_6 &= (0,1,0,0) \\
\vec{\alpha}_4 &= (0,0,1,0) \\
\vec{\alpha}_5 &= (0,0,0,1).
\end{aligned} \tag{F.9}$$

For our metric (F.3), the U_2 transformation

$$\begin{pmatrix} \vec{\alpha}'_3 \\ \vec{\alpha}'_6 \\ \vec{\alpha}'_4 \\ \vec{\alpha}'_5 \end{pmatrix} = U_2 \begin{pmatrix} \vec{\alpha}_3 \\ \vec{\alpha}_6 \\ \vec{\alpha}_4 \\ \vec{\alpha}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{F.10}$$

results in $\vec{\alpha}'_3 \cdot \vec{\alpha}'_3 \equiv \alpha'^i_3 \alpha'^j_3 G_{ij} = G_{33} = G'_{33},\ \vec{\alpha}'_3 \cdot \vec{\alpha}'_6 \equiv \alpha'^i_3 \alpha'^j_6 G_{ij} = G_{33} + G_{36} = G'_{36}$, etc. So U_2 corresponds to

$$R_3 \to R_3,\ R_6 \to R_6,\ \gamma^3 \to \gamma^3 - 1,\ \kappa^a \to \kappa^a,\ \widetilde{\gamma}^a \to \widetilde{\gamma}^a + \kappa^a,\ g_{ab} \to g_{ab}, \tag{F.11}$$

or equivalently

$$R_6 \to R_6,\ \gamma^3 \to \gamma^3 - 1,\ g_{\alpha\beta} \to g_{\alpha\beta},\ \gamma^a \to \gamma^a, \tag{F.12}$$

which leaves invariant the line element (F.5) if $d\theta^3 \to d\theta^3 - d\theta^6,\ d\theta^6 \to d\theta^6,\ d\theta^a \to d\theta^a$. U_2 is the generalization of the usual $\widetilde{\tau} \to \widetilde{\tau} - 1$ modular transformation. The 3d inverse metric $g^{\alpha\beta} \equiv \{g^{ab}, g^{a3}, g^{33}\}$ does not change under U_2. It is easily checked that U_2 is an invariance of the 4d Maxwell partition function (3.67) as well as the $6d$ chiral two-form partition function (3.72). It leaves the zero mode and oscillator contributions invariant separately.

The other generator, U_1 is related to the $SL(2, \mathcal{Z})$ transformation $\widetilde{\tau} \to -(\widetilde{\tau})^{-1}$ that we discuss as follows:

$$U_1 = U' M_3 \tag{F.13}$$

where M_3 is a $GL(3, \mathcal{Z})$ transformation given by

$$M_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{F.14}$$

and U' is the matrix corresponding to the transformation on the metric parameters (F.16),

$$U' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{F.15}$$

Under U', the metric parameters transform as

$$\begin{gathered} R_3 \to R_3|\widetilde{\tau}|, \quad R_6 \to R_6|\widetilde{\tau}|^{-1}, \quad \gamma^3 \to -\gamma^3|\widetilde{\tau}|^{-2}, \quad \kappa^a \to \widetilde{\gamma}^a, \quad \widetilde{\gamma}^a \to -\kappa^a, \quad g_{ab} \to g_{ab}. \\ \widetilde{\tau} \to -\frac{1}{\widetilde{\tau}}. \qquad \text{Or equivalently,} \\ G_{ab} \to G_{ab}, \quad G_{a3} \to G_{a6}, \quad G_{a6} \to -G_{a3}, \quad G_{33} \to G_{66}, \quad G_{66} \to G_{33}, \quad G_{36} \to -G_{36}, \\ \widetilde{G}_4^{ab} \to \widetilde{G}_4^{ab}, \quad \widetilde{G}_4^{a3} \to \widetilde{G}_4^{a6}, \quad \widetilde{G}_4^{a6} \to -\widetilde{G}_4^{a3}, \quad \widetilde{G}_4^{33} \to \frac{\widetilde{G}_4^{33}}{|\widetilde{\tau}|^2}, \quad \widetilde{G}_4^{66} \to |\widetilde{\tau}|^2 \widetilde{G}_4^{66}, \quad \widetilde{G}_4^{36} \to -\widetilde{G}_4^{36}, \end{gathered} \tag{F.16}$$

where $4 \le a, b \le 5$, and

$$\widetilde{\tau} \equiv \gamma^3 + i\frac{R_6}{R_3}, \qquad |\widetilde{\tau}|^2 = (\gamma^3)^2 + \frac{R_6^2}{R_3^2}. \tag{F.17}$$

The transformation (F.16) leaves invariant the line element (F.5) when $d\theta^3 \to d\theta^6$, $d\theta^6 \to -d\theta^3,\ d\theta^a \to d\theta^a$. The generators have the property $\det U_1 = -1, \det U_2 = 1$, $\det U_3 = -1,\ \det U' = 1,\ \det M_3 = -1$.

Under M_3, the metric parameters transform as

$$R_6 \to R_6, \qquad \gamma^3 \to -\gamma^4, \quad \gamma^a \to \gamma^{a+1}, \quad g_{ab} \to g_{a+1,b+1}, \quad g_{a3} \to -g_{a+1,4}, \quad g_{33} \to g_{44},$$

$$g^{ab} \to g^{a+1,b+1}, \quad g^{a3} \to -g^{a+1,4}, \quad g^{33} \to g^{44}, \quad \det g_{\alpha\beta} = \tilde{g}, \quad \tilde{g} \to \tilde{g}.$$ Or equivalently,

$$\begin{aligned}
&G_{ab} \to G_{a+1,b+1}, \quad G_{a3} \to -G_{a+1,4}, \quad G_{a6} \to G_{a+1,6}, \quad G_{33} \to G_{44}, \quad G_{66} \to G_{66}, \quad G_{36} \to -G_{46}, \\
&\widetilde{G}_4^{ab} \to \widetilde{G}_5^{a+1,b+1}, \quad \widetilde{G}_4^{a3} \to -\widetilde{G}_5^{a+1,4}, \quad \widetilde{G}_4^{a6} \to \widetilde{G}_4^{a+1,6}, \quad \widetilde{G}_4^{33} \to \widetilde{G}_4^{44}, \quad \widetilde{G}_4^{36} \to -\widetilde{G}_4^{46}, \quad \widetilde{G}_4^{66} \to \widetilde{G}_4^{66}, \\
&\det \widetilde{G}_4 = R_6\, \tilde{g}, \qquad \det \widetilde{G}_4 \to \det \widetilde{G}_4,
\end{aligned} \tag{F.18}$$

where $4 \leq a, b \leq 5$, and $a+1 \equiv 3$ for $a = 5$. We see that M_3 takes $Z^{4d}_{\text{zero modes}}$ to its complex conjugate as follows. Letting the M_3 transformation (F.18) act on (3.11), we find that the three subterms in the exponent

$$\begin{aligned}
&-\frac{e^2}{8} R_6 \sqrt{\tilde{g}} \left(\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4}\right) \Big(g^{aa'} g^{bb'} \tilde{F}_{ab} \tilde{F}_{a'b'} + 4 g^{aa'} g^{b3} \tilde{F}_{ab} \tilde{F}_{a'3} + 2 g^{aa'} g^{33} \tilde{F}_{a3} \tilde{F}_{a'3} - 2 g^{a3} g^{a'3} \tilde{F}_{a3} \tilde{F}_{a'3} \Big), \\
&-\frac{e^2 R_6}{4\sqrt{\tilde{g}}}\, \tilde{\Pi}^\alpha g_{\alpha\beta} \tilde{\Pi}^\beta, \\
&-\frac{\theta e^2 R_6}{8\pi^2 \sqrt{\tilde{g}}} g_{\alpha\beta} \epsilon^{\alpha\gamma\delta} \tilde{F}_{\gamma\delta} \tilde{\Pi}^\beta,
\end{aligned} \tag{F.19}$$

are separately invariant under (F.18) if we replace the integers $\widetilde{F}_{\alpha\beta} \in \mathcal{Z}^3, \tilde{\Pi}^\alpha \in \mathcal{Z}^3$ by

$$\tilde{F}_{ab} \to \tilde{F}_{a+1,b+1}, \quad \tilde{F}_{a3} \to -\tilde{F}_{a+1,4}, \quad \tilde{\Pi}^3 \to \tilde{\Pi}^4, \qquad \tilde{\Pi}^a \to -\tilde{\Pi}^{a+1}. \tag{F.20}$$

However, acted on by M_3 with the field shift (F.20), the term

$$2\pi i \gamma^\alpha \tilde{\Pi}^\beta \widetilde{F}_{\alpha\beta} \to -2\pi i \gamma^\alpha \tilde{\Pi}^\beta \widetilde{F}_{\alpha\beta} \tag{F.21}$$

changes sign. Thus we have

$$M_3 : \qquad Z^{4d}_{\text{zero modes}} \to Z^{4d\ *}_{\text{zero modes}} \tag{F.22}$$

The action of U' on $Z^{4d}_{\text{zero modes}}$

Next we show that under U', $Z^{4d}_{\text{zero modes}}$ transforms to $|\widetilde{\tau}|^2\, Z^{4d}_{\text{zero modes}}$. From (D.5) and (D.9), we have

$$Z^{4d}_{\text{zero modes}} = (\frac{4\pi}{e^2})^{-\frac{3}{2}} \frac{\tilde{g}^{\frac{1}{4}}}{R_6^{\frac{3}{2}}} \sum_{\tilde{F}_{ij}\in\mathcal{Z}^6} \exp\{-\frac{2\pi^2}{e^2} R_6\sqrt{\tilde{g}} g^{ij} g^{i'j'} \tilde{F}_{ii'}\tilde{F}_{jj'} - \frac{i}{2}\theta\epsilon^{\alpha\beta\gamma}\tilde{F}_{6\alpha}\tilde{F}_{\beta\gamma}\}, \quad \text{(F.23)}$$

from which it will be easy to see how it transforms under the U' transformation. Under U' from (F.16), the coefficient transforms as

$$U': \qquad (\frac{4\pi}{e^2})^{-\frac{3}{2}} \frac{\tilde{g}^{\frac{1}{4}}}{R_6^{\frac{3}{2}}} \to (\frac{4\pi}{e^2})^{-\frac{3}{2}} \frac{\tilde{g}^{\frac{1}{4}}}{R_6^{\frac{3}{2}}}\ |\widetilde{\tau}|^2. \quad \text{(F.24)}$$

The Euclidean action for the zero mode computation is invariant under U', as we show next by first summing $i = \{3, a, 6\}$, with $4 \leq a \leq 5$.

$$\begin{aligned}
&-\frac{2\pi^2 R_6\sqrt{\tilde{g}}}{e^2}\frac{R_1}{R_2} g^{ij} g^{i'j'}\tilde{F}_{ii'}\tilde{F}_{jj'}\\
&= -\frac{2\pi^2 R_6\sqrt{\tilde{g}}}{e^2}\Big(\widetilde{G}_4^{aa'}\widetilde{G}_4^{bb'}\tilde{F}_{ab}\tilde{F}_{a'b'} + 4\widetilde{G}_4^{aa'}\widetilde{G}_4^{b3}\tilde{F}_{ab}\tilde{F}_{a'3} + 4\widetilde{G}_4^{aa'}\widetilde{G}_4^{b6}\tilde{F}_{ab}\tilde{F}_{a'6} + 2\widetilde{G}_4^{aa'}\widetilde{G}_4^{33}\tilde{F}_{a3}\tilde{F}_{a'3}\\
&\qquad - 2\widetilde{G}_4^{a3}\widetilde{G}_4^{a'3}\tilde{F}_{a3}\tilde{F}_{a'3} + 4\widetilde{G}_4^{aa'}\widetilde{G}_4^{36}\tilde{F}_{a3}\tilde{F}_{a'6} - 4\widetilde{G}_4^{a6}\widetilde{G}_4^{a'3}\tilde{F}_{a3}\tilde{F}_{a'6} + 4\widetilde{G}_4^{a3}\widetilde{G}_4^{b6}\tilde{F}_{ab}\tilde{F}_{36}\\
&\qquad + 2\widetilde{G}_4^{aa'}\widetilde{G}_4^{66}\tilde{F}_{a6}\tilde{F}_{a'6} - 2\widetilde{G}_4^{a6}\widetilde{G}_4^{a'6}\tilde{F}_{a6}\tilde{F}_{a'6} + 4\widetilde{G}_4^{a3}\widetilde{G}_4^{36}\tilde{F}_{a3}\tilde{F}_{36} - 4\widetilde{G}_4^{a6}\widetilde{G}_4^{33}\tilde{F}_{a3}\tilde{F}_{36}\\
&\qquad + 4\widetilde{G}_4^{a3}\widetilde{G}_4^{66}\tilde{F}_{a6}\tilde{F}_{36} - 4\widetilde{G}_4^{a6}\widetilde{G}_4^{36}\tilde{F}_{a6}\tilde{F}_{36} - 2\widetilde{G}_4^{36}\widetilde{G}_4^{36}\tilde{F}_{36}\tilde{F}_{36} + 2\widetilde{G}_4^{33}\widetilde{G}_4^{66}\tilde{F}_{36}\tilde{F}_{36}\Big).
\end{aligned} \quad \text{(F.25)}$$

Letting the U' transformation (F.16) act on (F.25), we see the first term in the exponent of (F.23) changes to

$$
\begin{aligned}
-\frac{2\pi^2 R_6\sqrt{\widetilde{g}}}{e^2}\Big(&\widetilde{G}_4^{aa'}\widetilde{G}_4^{bb'}\tilde{F}_{ab}\tilde{F}_{a'b'}+4\widetilde{G}_4^{aa'}\widetilde{G}_4^{b6}\tilde{F}_{ab}\tilde{F}_{a3}-4\widetilde{G}_4^{aa'}\widetilde{G}_4^{b3}\tilde{F}_{ab}\tilde{F}_{a'6}+\frac{2}{|\widetilde{\tau}|^2}\widetilde{G}_4^{aa'}\widetilde{G}_4^{33}\tilde{F}_{a3}\tilde{F}_{a'3}\\
&-2\widetilde{G}_4^{a6}\widetilde{G}_4^{a'6}\tilde{F}_{a3}\tilde{F}_{a'3}-4\widetilde{G}_4^{aa'}\widetilde{G}_4^{36}\tilde{F}_{a3}\tilde{F}_{a'6}+4\widetilde{G}_4^{a3}\widetilde{G}_4^{a'6}\tilde{F}_{a3}\tilde{F}_{a'6}-4\widetilde{G}_5^{\alpha 6}\widetilde{G}_5^{\alpha' 3}\tilde{F}_{aa'}\tilde{F}_{36}\\
&+2|\widetilde{\tau}|^2\widetilde{G}_4^{aa'}\widetilde{G}_4^{66}F_{a6}F_{a'6}-2\widetilde{G}_4^{a6}\widetilde{G}_4^{a'3}\tilde{F}_{a6}\tilde{F}_{a'6}-4\widetilde{G}_4^{a6}\widetilde{G}_4^{36}\tilde{F}_{a3}\tilde{F}_{36}+\frac{4}{|\widetilde{\tau}|^2}\widetilde{G}_4^{a3}\widetilde{G}_4^{33}\tilde{F}_{a3}\tilde{F}_{36}\\
&+4|\widetilde{\tau}|^2\widetilde{G}_4^{a6}\widetilde{G}_4^{66}\tilde{F}_{a6}\tilde{F}_{36}-4\widetilde{G}_4^{a3}\widetilde{G}_4^{36}\tilde{F}_{a6}\tilde{F}_{36}-2\widetilde{G}_4^{36}\widetilde{G}_4^{36}\tilde{F}_{36}\tilde{F}_{36}+2\widetilde{G}_4^{33}\widetilde{G}_4^{66}\tilde{F}_{36}\tilde{F}_{36}\Big).
\end{aligned}
\tag{F.26}
$$

The second term in the exponential of (F.23) is a topological term, and is left invariant under the action of U' by inspection. If we replace the integers $\widetilde{F}_{3a}\to\widetilde{F}_{6a}$ and $\widetilde{F}_{a6}\to -\widetilde{F}_{a3}$, the two terms are left invariant, so the sum

$$
\sum_{\tilde{F}_{ij}\in\mathcal{Z}^6} e^{-\frac{2\pi^2\sqrt{g}}{e^2}g^{ij}g^{i'j'}\tilde{F}_{ii'}\tilde{F}_{jj'}+i\frac{\theta}{2}\epsilon^{\alpha\beta\gamma}\tilde{F}_{6\alpha}\tilde{F}_{\beta\gamma}} \tag{F.27}
$$

is invariant. Thus we have shown that under the U' transformation (F.16),

$$
Z^{4d}_{\text{zero modes}}(R_3|\widetilde{\tau}|,R_6|\widetilde{\tau}|^{-1},g_{ab},-\gamma^3|\widetilde{\tau}|^{-2},\widetilde{\gamma}^a,-\kappa^a)=|\widetilde{\tau}|^2\;Z^{4d}_{\text{zero modes}}(R_3,R_6,g_{ab},\gamma^3,\kappa^a,\widetilde{\gamma}^a). \tag{F.28}
$$

Also from (F.23), we can write (F.22) as

$$
M_3:\qquad Z^{4d}_{\text{zero modes}}(e^2,\theta,G_{ij})\to Z^{4d}_{\text{zero modes}}(e^2,-\theta,G_{ij}). \tag{F.29}
$$

and thus under the $GL(4,\mathcal{Z})$ generator U_1,

$$
Z^{4d}_{\text{zero modes}}\to|\widetilde{\tau}|^2\left(Z^{4d}_{\text{zero modes}}\right)^*. \tag{F.30}
$$

The residual factor $|\widetilde{\tau}|^2$ is sometimes referred to as an $SL(2,\mathcal{Z})$ anomaly of the zero mode partition function, because U' includes the $\widetilde{\tau}\to-\frac{1}{\widetilde{\tau}}$ transformation. Finally we will show how this anomaly is canceled by the oscillator contribution.

Under U_3, the metric parameters transform as

$$R_6 \to R_6, \qquad \gamma^3 \to -\gamma^3, \quad \gamma^a \to \gamma^a, \quad g_{ab} \to g_{ab}, \quad g_{a3} \to -g_{a3}, \quad g_{33} \to g_{33},$$
$$g^{ab} \to g^{ab}, \quad g^{a3} \to -g^{a3}, \quad g^{33} \to g^{33}, \quad \det g_{\alpha\beta} = \tilde{g}, \quad \tilde{g} \to \tilde{g}.$$ Or equivalently,

$$\begin{aligned}
&G_{ab} \to G_{ab}, \quad G_{a3} \to -G_{a3}, \quad G_{a6} \to G_{a6}, \quad G_{33} \to G_{33}, \quad G_{66} \to G_{66}, \quad G_{36} \to -G_{36},\\
&\widetilde{G}_4^{ab} \to \widetilde{G}_4^{ab}, \quad \widetilde{G}_4^{a3} \to -\widetilde{G}_4^{a3}, \quad \widetilde{G}_4^{a6} \to \widetilde{G}_4^{a6}, \quad \widetilde{G}_4^{33} \to \widetilde{G}_4^{33}, \quad \widetilde{G}_4^{36} \to -\widetilde{G}_4^{36}, \quad \widetilde{G}_4^{66} \to \widetilde{G}_4^{66},\\
&\det \widetilde{G}_4 = R_6\, \tilde{g}, \qquad \det \widetilde{G}_4 \to \det \widetilde{G}_4,
\end{aligned} \tag{F.31}$$

where $4 \leq a, b \leq 5$ and $\widetilde{G}^{\alpha\beta}$ is the 3d inverse. We can check that $Z^{4d}_{\text{zero modes}}$ becomes its complex conjugate under U_3 given in (F.31) as follows. Letting the U_3 transformation (F.31) act on (3.11), we find that three of the terms in the exponent

$$\begin{aligned}
&-\frac{e^2 R_6 \sqrt{\tilde{g}}}{8}\left(\frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4}\right)\Big(g^{aa'}g^{bb'}F_{ab}F_{bb'} + 4g^{aa'}g^{b3}F_{ab}F_{a'3} + 2g^{aa'}g^{33}F_{a3}F_{a'3} - 2g^{a3}g^{a'3}F_{a3}F_{a'3}\Big),\\
&-\frac{e^2 R_6}{4\sqrt{\tilde{g}}}\tilde{\Pi}^\alpha g_{\alpha\beta}\tilde{\Pi}^\beta,\\
&-\frac{\theta e^2 R_6}{8\pi\sqrt{\tilde{g}}} g_{\alpha\beta}\epsilon^{\alpha\gamma\delta}\tilde{F}_{\gamma\delta}\tilde{\Pi}^\beta,
\end{aligned} \tag{F.32}$$

are separately invariant under (F.18), if we replace the the integers $\widetilde{F}_{\alpha\beta} \in \mathcal{Z}^3, \tilde{\Pi}^\alpha \in \mathcal{Z}^3$ by

$$\widetilde{F}_{ab} \to \widetilde{F}_{ab}, \quad \tilde{F}_{a3} \to -\tilde{F}_{a3}, \quad \tilde{\Pi}^3 \to \tilde{\Pi}^3, \qquad \tilde{\Pi}^a \to -\tilde{\Pi}^a, \tag{F.33}$$

However the subterm

$$2\pi i\gamma^\alpha\tilde{\Pi}^\beta\tilde{F}_{\alpha\beta} \to -2\pi i\gamma^\alpha\tilde{\Pi}^\beta\tilde{F}_{\alpha\beta} \tag{F.34}$$

acted by U_3 with the field shift in (F.33). Therefore the zero mode partition function goes to its complex conjugate under U_3.

Appropriate generators for $SL(4,\mathcal{Z})$

We claim that U_1^2, U_2 and U_1U_3 generate the group $SL(4,\mathcal{Z})$ since $GL(n,\mathcal{Z})$ is generated by U_1, U_2 and U_3 or alternatively $R_1 = U_1$, $R_2 = U_3^{-1}U_2$ and $R_3 = U_3$, i.e., any element in $GL(n, \mathcal{Z})$ U can be written as

$$U = R_1{}^{n_1} R_2{}^{n_2} R_3{}^{n_3} R_1{}^{n_4} R_2{}^{n_5} R_3{}^{n_6} .... \tag{F.35}$$

It is understood that $SL(n, \mathcal{Z})$ is generated by even numbers of R_1, R_2 and R_3. Thus, the possible set of group generators for $SL(n, \mathcal{Z})$ is

$$R_1^2, R_2^2, R_3^2, \quad R_1R_2, R_2R_3, R_3R_1, \quad R_2R_1, R_3R_2, R_1R_3 \tag{F.36}$$

with the properties that $R_2^2 = 1$ and $R_3^2 = 1$. A smaller set of the $SL(4, \mathcal{Z})$ generators is

$$R_1^2, R_1R_3, R_2R_3, \tag{F.37}$$

since other generators in (F.36) can be expressed with the generators in (F.39) through the following relations

$$\begin{aligned} R_1R_2 &= R_1R_3(R_2R_3)^{-1}, \qquad R_2R_1 = (R_1R_2)^{-1}R_1^2 \\ R_3R_2 &= (R_2R_3)^{-1}, \qquad R_3R_1 = (R_1R_3)^{-1}R_1^2. \end{aligned} \tag{F.38}$$

Notice that

$$\{R_1^2, R_1R_3, R_2R_3\} = \{U_1^2, U_1U_3, U_2^{-1}\}. \tag{F.39}$$

These three matrices generate $SL(4, \mathcal{Z})$*.* They can be shown to generate Trott's twelve generators B_{ij}

Since we have tested the invariance of the zero mode partition function under U_2, we only need to check invariance under U_1U_3 and U_1^2. For U_1U_3, as previously we separate U_1 into U' and M_3,

$$U_1U_3 = U'M_3U_3 = U'(M_3U_3). \tag{F.40}$$

Since both M_3 and U_3 take $Z^{4d}_{\text{zero modes}}$ to its complex conjugate, M_3U_3 is an invariance of the zero mode partition function. Thus from (F.28),

$$U_1U_3 : \qquad Z^{4d}_{zero\ modes} \to |\widetilde{\tau}|^2 Z^{4d}_{zero\ modes}. \tag{F.41}$$

$U_1{}^2$ acts on $Z^{4d}_{\text{zero modes}}$

Since we have shown before

$$U_1 : \qquad Z^{4d}_{zero\ modes} \to |\widetilde{\tau}|^2 Z^{4d\,*}_{zero\ modes}, \tag{F.42}$$

then

$$U_1{}^2 : \qquad Z^{4d}_{zero\ modes} \to Z^{4d}_{zero\ modes}. \tag{F.43}$$

To summarize, we have

$$\begin{aligned} U_2 &: Z^{4d}_{\text{zero modes}} \to Z^{4d}_{\text{zero modes}}, \\ U_1U_3 &: Z^{4d}_{\text{zero modes}} \to |\widetilde{\tau}|^2 Z^{4d}_{\text{zero modes}}, \\ U_1{}^2 &: Z^{4d}_{\text{zero modes}} \to Z^{4d}_{\text{zero modes}}. \end{aligned} \tag{F.44}$$

One can derive a similar transformation property for $Z^{6d}_{\text{zero modes}}$ using (3.13),

$$\begin{aligned} U_2 &: Z^{6d}_{\text{zero modes}} \to Z^{6d}_{\text{zero modes}}, \\ U_1U_3 &: Z^{6d}_{\text{zero modes}} \to |\widetilde{\tau}|^3 Z^{6d}_{\text{zero modes}}, \\ U_1{}^2 &: Z^{6d}_{\text{zero modes}} \to Z^{6d}_{\text{zero modes}}, \end{aligned} \tag{F.45}$$

which follows from transformations on the factor ϵ, given in (3.14). By inspection ϵ is invariant under U_2 and M_3, and transforms as

$$U' : \epsilon \to |\widetilde{\tau}|\epsilon. \tag{F.46}$$

This can be seen by Poisson resummation since ϵ can be written as

$$\epsilon = \sum_{n_a} \exp\{-\frac{\pi R_6\sqrt{\tilde{g}}}{R_1 R_2} g^{ab} n_a n_b - \frac{\pi R_6\sqrt{\tilde{g}}}{R_3 R_1 R_2 |\widetilde{\tau}|^2} \widetilde{\gamma}^a \widetilde{\gamma}^b n_a n_b\} \sum_{m,n_3} \exp\{-\pi(N+x)\cdot A \cdot (N+x)\},$$
$$= |\widetilde{\tau}|^{-1}\, U'\, \epsilon, \tag{F.47}$$

where

$$H_{12\alpha} = n_\alpha, \qquad H_{\alpha\beta\delta} = \frac{\epsilon_{\alpha\beta\delta}}{\tilde{g}}\, m, \qquad m, n_\alpha \in \mathcal{Z}^4,$$
$$A = \begin{pmatrix} \frac{R_6\sqrt{\tilde{g}}}{R_3 R_1 R_2} & i\gamma^3 \\ i\gamma^3 & \frac{R_6 R_1 R_2}{R_3\sqrt{\tilde{g}}} \end{pmatrix}, \qquad \det A = |\widetilde{\tau}|^2, \qquad N = \begin{pmatrix} n_3 \\ m \end{pmatrix}, \qquad x = \begin{pmatrix} \kappa^a n_a + \frac{\gamma^3 \widetilde{\gamma}^a n_a}{|\widetilde{\tau}|^2} \\ i\frac{R_6\sqrt{\tilde{g}}\,\widetilde{\gamma}^3 n_a}{R_3 R_1 R_2 |\widetilde{\tau}|^2}, \end{pmatrix}.$$

U' acts on Z^{4d}_{osc}

To derive how U' acts on Z^{4d}_{osc}, we first separate the product on $\vec{n} = (n, n_a) \neq \vec{0}$ into a product on (all n, but $n_\alpha \neq (0,0)$) and on ($n \neq 0$, $n_a = (0,0)$). Then using the regularized vacuum energy (E.1) expressed as sum over zero and non-zero transverse momenta $p_\perp = n_a$ in (E.2), we find that (3.67) becomes

$$Z^{4d,Maxwell} = Z^{4d}_{\text{zero modes}} \cdot \Big(e^{\frac{\pi R_6}{6R_3}} \prod_{n\neq 0} \frac{1}{1 - e^{-2\pi\frac{R_6}{R_3}|n| - 2\pi i\gamma^3 n}}\Big)^2$$
$$\cdot \Big(\prod_{n_a \in \mathcal{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>_{p_\perp}} \prod_{n^3\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6\sqrt{g^{\alpha\beta} n_\alpha n_\beta} - 2\pi i \gamma^\alpha n_\alpha}}\Big)^2. \tag{F.48}$$

As in [2] we observe the middle expression above can be written in terms of the Dedekind eta function $\eta(\widetilde{\tau}) \equiv e^{\frac{\pi i\widetilde{\tau}}{12}} \prod_{n\in\mathcal{Z}\neq 0}(1 - e^{2\pi i n\widetilde{\tau}})$, with $\widetilde{\tau} = \gamma^3 + i\frac{R_6}{R_3}$,

$$\Big(e^{\frac{\pi R_6}{6R_3}} \prod_{n\neq 0} \frac{1}{1 - e^{-2\pi\frac{R_6}{R_3}|n| - 2\pi i\gamma^3 n}}\Big)^2 = (\eta(\widetilde{\tau})\bar{\eta}(\bar{\widetilde{\tau}}))^{-2}. \tag{F.49}$$

This transforms under U' in (F.16) as

$$(\eta(-\widetilde{\tau}^{-1})\bar{\eta}(-\bar{\widetilde{\tau}}^{-1}))^{-2} = |\widetilde{\tau}|^{-2}\ (\eta(\widetilde{\tau})\bar{\eta}(\bar{\widetilde{\tau}}))^{-2}, \tag{F.50}$$

where $\eta(-\widetilde{\tau}^{-1}) = (i\widetilde{\tau})^{\frac{1}{2}}\eta(\widetilde{\tau})$. In this way the anomaly of the zero modes in (F.28) is canceled by (F.50). Lastly we demonstrate the third expression in (F.48) is invariant under U',

$$\Big(\prod_{n_a\in\mathcal{Z}^2\neq(0,0)} e^{-2\pi R_6 <H>_\perp} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2 = PI, \tag{F.51}$$

where PI is the modular invariant $2d$ path integral of two massive scalar bosons of mass $\sqrt{\widetilde{g}^{ab}n_a n_b}$, coupled to a worldsheet gauge field, on a two-torus in directions 3,6. Following [2], with more detail in (F.68), we extract from (3.65)

$$Z^{4d}_{\rm osc} = (e^{-\pi R_6\sum_{\vec{n}\in\mathcal{Z}^3}\sqrt{g^{\alpha\beta}n_\alpha n_\beta}} \prod_{\vec{n}\in\mathcal{Z}^3\neq\vec{0}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}-2\pi i\gamma^\alpha n_\alpha}}\Big)^2 \tag{F.52}$$

the $2d$ path integral of free massive bosons coupling to the gauge field, where n_a is fixed and non-zero,

$$\begin{aligned}
(PI)^{\frac{1}{2}} &\equiv e^{-\pi R_6\sum_{n_3\in\mathcal{Z}}\sqrt{g^{\alpha\beta}n_\alpha n_\beta}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1-e^{-2\pi R_6\sqrt{g^{\alpha\beta}n_\alpha n_\beta}+2\pi i\gamma^\alpha n_\alpha}} \\
&= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta' E}{2}}}{1-e^{-\beta' E+2\pi i(\gamma^3 s+\gamma^a n_a)}} \qquad \text{where } s\equiv n_3,\quad E\equiv\sqrt{g^{\alpha\beta}n_\alpha n_\beta},\quad \beta'\equiv 2\pi R_6 \\
&= \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\ \sqrt{\cosh\beta' E-\cos 2\pi(\gamma^3 s+\gamma^a n_a)}} \qquad \text{for } n_a\to -n_a \\
&= e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\big(\ln[\cosh\beta' E-\cos 2\pi(\gamma^3 s+\gamma^a n_a)]+\ln 2\big)} \equiv e^{-\frac{1}{2}\sum_{s\in\mathcal{Z}}\ \nu(E)},
\end{aligned} \tag{F.53}$$

where

$$\begin{aligned}
\sum_{s\in\mathcal{Z}}\nu(E) &\equiv \sum_{s\in\mathcal{Z}}\big(\ln[\cosh\beta' E-\cos 2\pi(\gamma^3 s+\gamma^a n_a)]+\ln 2\big) \\
&= \sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}}\ln\,[\frac{4\pi^2}{\beta'^2}(r+\gamma^3 s+\gamma^a n_a)^2+E^2].
\end{aligned} \tag{F.54}$$

We can show directly that (F.54) is invariant under U', since

$$E^2 = g^{\alpha\beta}n_\alpha n_\beta = g^{33}s^2 + 2g^{3a}sn_a + g^{ab}n_a n_b = \frac{1}{R_3^2}(s + \kappa^a n_a)^2 + \widetilde{g}^{ab}n_a n_b,$$
$$\frac{4\pi^2}{\beta'^2}(r + \gamma^3 s + \gamma^a n_a)^2 = \frac{1}{R_6^2}(r + \widetilde{\gamma}^a n_a + \gamma^3(s + \kappa^a n_a))^2, \tag{F.55}$$

then

$$\frac{4\pi^2}{\beta'^2}(r + \gamma^3 s + \gamma^a n_a)^2 + E^2$$
$$= \frac{1}{R_6^2}(s + \kappa^a n_a)^2 |\widetilde{\tau}|^2 + \frac{1}{R_6^2}(r + \widetilde{\gamma}^a n_a)^2 + \frac{2\gamma^3}{R_6^2}(r + \widetilde{\gamma}^a n_a)(s + \kappa^a n_a) + \widetilde{g}^{ab}n_a n_b. \tag{F.56}$$

So we see the transformation U' given in (F.16) leaves (F.56) invariant if $s \to r$ and $r \to -s$. Therefore (F.54) is invariant under U', so that $(PI)^{\frac{1}{2}}$ given in (F.53) is invariant under U'.

M_3 acts on Z^{4d}_{osc}

M_3 leaves the Z^{4d}_{osc} invariant as can be seen from (F.48) by shifting the integer n_α as

$$n_3 \to -n_4, \qquad n_a \to n_{a+1}. \tag{F.57}$$

So, under $U_1 = U'M_3$,

$$Z^{4d}_{\text{osc}} \to |\widetilde{\tau}|^{-2} Z^{4d}_{\text{osc}}. \tag{F.58}$$

U_2 is an invariance of the oscillator partition function by inspection.

U_3 acts on Z^{4d}_{osc}

U_3 leaves the Z^{4d}_{osc} invariant as can be seen from (F.48) by shifting the integers n_α as

$$n_3 \to -n_3, \qquad n_a \to n_a. \tag{F.59}$$

Thus, the oscillator partition function transforms under the $SL(4, \mathcal{Z})$ generators $\{U_1^2, U_1U_3, U_2\}$

as

$$
\begin{aligned}
U_2 : Z^{4d}_{\text{osc}} &\to Z^{4d}_{\text{osc}},\\
U_1 U_3 : Z^{4d}_{\text{osc}} &\to |\widetilde{\tau}|^{-2} Z^{4d}_{\text{osc}},\\
{U_1}^2 : Z^{4d}_{\text{osc}} &\to Z^{4d}_{\text{osc}}.
\end{aligned}
\tag{F.60}
$$

So together with (F.44) we have established invariance under (F.39), and thus proved the partition function for the $4d$ Maxwell theory on T^4, given alternatively by (3.67) or (F.48), is invariant under $SL(4,\mathcal{Z})$, the mapping class group of T^4.

U' acts on Z^{6d}_{osc}

For the $6d$ chiral theory on $T^2 \times T^4$, where $< H >^{6d} \equiv \frac{1}{2}\sum_{\vec{p}\in\mathcal{Z}^5}\sqrt{G_5^{lm}p_l p_m}$ appears in (3.71), the $SL(3,\mathcal{Z})$ invariant regularized vacuum energy [2] becomes,

$$
\begin{aligned}
< H >^{6d} &= -\frac{1}{2\pi^4}\sqrt{G_5}\sum_{\vec{n}\neq\vec{0}}\frac{1}{(G_{lm}n^l n^m)^3}\\
&= -32\pi^2\sqrt{G_5}\sum_{\vec{n}\neq\vec{0}}\frac{1}{(2\pi)^6\Big(g_{\alpha\beta}n^\alpha n^\beta + (R_1^2 + R_2^2\beta^2\beta^2)(n^1)^2 - 2\beta^2 R_2^2 n^1 n^2 + R_2^2(n^2)^2\Big)^3}
\end{aligned}
\tag{F.61}
$$

and can be decomposed similarly to (E.2),

$$
< H >^{6d} = \sum_{p_\perp\in\mathcal{Z}^4} < H >^{6d}_{p_\perp} = < H >^{6d}_{p_\perp=0} + \sum_{p_\perp\in\mathcal{Z}^4\neq 0} < H >^{6d}_{p_\perp}, \tag{F.62}
$$

where

$$
< H >^{6d}_{p_\perp} = -32\pi^2\sqrt{G_5}\frac{1}{(2\pi)^4}\int d^4y_\perp e^{-ip_\perp\cdot y_\perp}\sum_{n^3\in\mathcal{Z}\neq 0}\frac{1}{|2\pi n^3 + y_\perp|^6}, \tag{F.63}
$$

with denominator $|2\pi n^3 + y_\perp|^2 = G_{33}(2\pi n^3)^2 + 2(2\pi n^3)G_{3k}y_\perp^k + G_{kk'}y_\perp^k y_\perp^{k'}$, with $k =$

$1,2,4,5,$

$$
\begin{aligned}
< H >^{6d}_{p_\perp=0} &= -\frac{1}{12R_3}, \\
< H >^{6d}_{p_\perp\neq 0} &= |p_\perp|^2 R_3 \sum_{n=1}^{\infty} \cos(p_a\kappa^a 2\pi n)\big[K_2(2\pi n R_3|p_\perp|) - K_0(2\pi n R_3|p_\perp|)\big] \\
&= -\pi^{-1}|p_\perp| R_3 \sum_{n=1}^{\infty} \cos(p_a\kappa^a 2\pi n)\frac{K_1(2\pi n R_3|p_\perp|)}{n},
\end{aligned}
\tag{F.64}
$$

with $p_\perp = (p_1, p_2, p_a) = n_\perp = (n_1, n_2, n_a) = (n_1, n_2, n_4, n_5) \in \mathcal{Z}^4$,
$|p_\perp| = \sqrt{\frac{(n_1)^2}{R_1^2} + 2\frac{\beta^2}{R_1^2} + (\frac{1}{R_2^2} + \frac{\beta^{2\,2}}{R_1^2})n_2^2 + \widetilde{g}^{ab} n_a n_b}$.

The U' invariance (3.72) follows when we separate the product on $\vec{n} \in \mathcal{Z}^5 \neq \vec{0}$ into a product on ($n_3 \neq 0,\ n_\perp \equiv (n_1, n_2, n_4, n_5) = (0,0,0,0)$), and on (all n_3, but $n_\perp = (n_1, n_2, n_4, n_5) \neq (0,0,0,0)$). Then

$$
\begin{aligned}
Z^{6d}_{\text{osc}} &= \Big(e^{\frac{\pi R_6}{6R_3}} \prod_{n_3 \in \mathcal{Z}\neq 0} \frac{1}{1 - e^{2\pi i\,(\gamma^3\, n_3 + i\frac{R_6}{R_3}|n_3|\,)}}\Big)^3 \\
&\quad \cdot \Big(\prod_{n_\perp \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{\widetilde{n}^2 + g^{\alpha\beta} n_\alpha n_\beta}\ + i2\pi\gamma^\alpha n_\alpha}}\Big)^3 \\
&= \big(\eta(\widetilde{\tau})\ \bar{\eta}(\bar{\widetilde{\tau}})\big)^{-3} \\
&\quad \cdot \Big(\prod_{(n_1,n_2,n_4,n_5) \in \mathcal{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 <H>^{6d}_{p_\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta + \widetilde{n}^2}\ + i2\pi\gamma^\alpha n_\alpha}}\Big)^3,
\end{aligned}
\tag{F.65}
$$

where $\widetilde{\tau} = \gamma^3 + i\frac{R_6}{R_3}$, and $\widetilde{n}^2 \equiv \frac{n_1^2}{R_1^2} + 2\frac{\beta^2}{R_1^2} n_1 n_2 + (\frac{1}{R_2^2} + \frac{\beta^{2\,2}}{R_1^2}) n_2^2$. Under U',

$$
\eta(\widetilde{\tau})\ \bar{\eta}(\bar{\widetilde{\tau}}) \to |\widetilde{\tau}|\ \eta(\widetilde{\tau})\ \bar{\eta}(\bar{\widetilde{\tau}}). \tag{F.66}
$$

U' leaves invariant the part of the $6d$ oscillator partition function (F.65) at fixed $n_\perp \neq 0$, since

$$
e^{-2\pi R_6 <H>^{6d}_{n_\perp \neq 0}} \prod_{n_3 \in \mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta + \frac{n_1^2}{R_1^2} + 2\frac{\beta^2}{R_1^2} n_1 n_2 + (\frac{1}{R_2^2} + \frac{\beta^{2\,2}}{R_1^2}) n_2^2}\ + i2\pi\gamma^\alpha n_\alpha}}
\tag{F.67}
$$

is the square root of the partition function on T^2 (now in the directions 3,6) of a massive complex scalar with $m^2 \equiv G^{11}n_1^2 + G^{22}n_2^2 + 2G^{12}n_1 n_2 + \widetilde{g}^{ab}n_a n_b$, $4 \le a, b \le 5$, that couples to a constant gauge field $A^\mu \equiv iG^{\mu i}n_i$ with $\mu, \nu = 3, 6$; $i, j = 1, 2, 4, 5$. The metric on this T^2 is $h_{33} = R_3^2\,, h_{66} = R_6^2 + (\gamma^3)^2 R_3^2\,, h_{36} = -\gamma^3 R_3^2$. Its inverse is $h^{33} = \frac{1}{R_3^2} + \frac{(\gamma^3)^2}{R_6^2}$, $h^{66} = \frac{1}{R_6^2}$ and $h^{36} = \frac{\gamma^3}{R_6^2}$. The manifestly $SL(2, \mathcal{Z})$ invariant path integral is

$$
\begin{aligned}
\text{P.I.} &= \int d\phi\, d\bar{\phi}\; e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6\; h^{\mu\nu}(\partial_\mu + A_\mu)\bar{\phi}(\partial_\nu - A_\nu)\phi + m^2\bar{\phi}\phi} \\
&= \int d\bar{\phi}\, d\phi e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6 \bar{\phi}(-(\frac{1}{R_3^2} + \frac{(\gamma^3)^2}{R_6^2})\partial_3^2 - (\frac{1}{R_6})^2\partial_6^2 - 2\frac{\gamma^3}{R_6^2}\partial_3\partial_6 + 2A^3\partial_3 + 2A^6\partial_6 + G^{11}n_1n_1 + G^{22}n_2n_2 + 2G^{12}n_1n_2 + G^{ab}n_an_b)\phi} \\
&= \det\Big([-(\tfrac{1}{R_3^2} + (\tfrac{\gamma^3}{R_6})^2)\partial_3^2 - (\tfrac{1}{R_6})^2\partial_6^2 - 2\gamma^3(\tfrac{1}{R_6})^2\partial_3\partial_6 + G^{11}n_1n_1 + G^{22}n_2n_2 \\
&\qquad + 2G^{12}n_1n_2 + G^{ab}n_an_b + 2iG^{3a}n_a\partial_3 + 2iG^{6a}n_a\partial_6]\Big)^{-1} \\
&= e^{-\mathrm{trln}\Big[-(\frac{1}{R_3^2} + (\frac{\gamma^3}{R_6})^2)\partial_3^2 - (\frac{1}{R_6})^2\partial_6^2 - 2\gamma^3(\frac{1}{R_6})^2\partial_3\partial_6 + G^{11}n_1n_1 + G^{22}n_2n_2 + 2G^{12}n_1n_2 + G^{ab}n_an_b + 2iG^{3a}n_a\partial_3 + 2iG^{6a}n_a\partial_6\Big]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\sum_{r\in\mathcal{Z}}\Big[\ln(\frac{4\pi^2}{\beta'^2}r^2 + (\frac{1}{R_3^2} + (\frac{\gamma^3}{R_6})^2)s^2 + 2\gamma^3(\frac{1}{R_6})^2 rs + G^{11}n_1n_1 + G^{22}n_2n_2 + 2G^{12}n_1n_2 + G^{ab}n_an_b + 2G^{3a}n_a\, s + 2G^{6a}n_a\, r)\Big]} \\
&= e^{-\sum_{s\in\mathcal{Z}}\nu(E)},
\end{aligned}
\tag{F.68}
$$

where from (3.7), $G^{11} = \frac{1}{R_1^2}$, $G^{22} = \frac{1}{R_2^2} + \frac{\beta^{2\,2}}{R_1^2}$, $G^{12} = \frac{\beta^2}{R_1^2}$, $G^{ab} = g^{ab} + \frac{\gamma^a\gamma^b}{{R_6}^2}$, $G^{3a} = g^{3a} + \frac{\gamma^3\gamma^a}{R_6^2}$, $G^{6a} = \frac{\gamma^a}{R_6^2}$, $G^{63} = \frac{\gamma^3}{R_6^2}$, and $\partial_3\phi = -is\phi,\ \partial_6\phi = -ir\phi$, $s = n_3$, and $\beta' = 2\pi R_6$. The sum on r is

$$
\nu(E) = \sum_{r\in\mathcal{Z}} \ln\Big[\frac{4\pi^2}{\beta'^2}(r + \gamma^3 s + \gamma^a n_a)^2 + E^2\Big]\,,
\tag{F.69}
$$

with $E^2 \equiv G_5^{lm}n_l n_m = G_5^{11}n_1n_1 + G_5^{22}n_2n_2 + G_5^{21}n_2n_1 + G_5^{ab}n_an_b + 2G_5^{a3}n_an_3 + G_5^{33}n_3n_3$, and $G_5^{11} = \frac{1}{R_1^2}$, $G_5^{12} = \frac{\beta^2}{R_1^2}$, $G_5^{22} = \frac{1}{R^2} + \frac{\beta^2\beta^2}{R_1^2}$, $G_5^{1\alpha} = G_5^{2\alpha} = 0$, $G_5^{3a} = g^{3a} = \frac{\kappa^a}{R_3^2}$, $G_5^{33} = g^{33} = \frac{1}{R_3^2}$, $G_5^{ab} = g^{ab} = \widetilde{g}^{ab} + \frac{\kappa^a\kappa^b}{R_3^2}$. We evaluate the divergent sum $\nu(E)$ on r by

$$
\begin{aligned}
\frac{\partial\nu(E)}{\partial E} &= \sum_r \frac{2E}{\frac{4\pi^2}{\beta'^2}(r + \gamma^3 s + \gamma^a n_a)^2 + E^2} \\
&= \partial_E \ln\Big[\cosh\beta' E - \cos 2\pi\big(\gamma^3 s + \gamma^a n_a\big)\Big],
\end{aligned}
\tag{F.70}
$$

using the sum $\sum_{n\in\mathcal{Z}} \frac{2y}{(2\pi n + z)^2 + y^2} = \frac{\sinh y}{\cosh y - \cos z}$. Then integrating (F.70), we choose the integration constant to maintain modular invariance of (F.68),

$$\nu(E) = \ln\big[\cosh\beta' E - \cos 2\pi\big(\gamma^3 s + \gamma^a n_a\big)\big] + \ln 2. \tag{F.71}$$

It follows for $s = n_3$ that (F.68) gives

$$\begin{aligned}
(\text{P.I.})^{\frac{1}{2}} &= \prod_{s\in\mathcal{Z}} \frac{1}{\sqrt{2}\sqrt{\cosh\beta' E - \cos 2\pi(\gamma^3 s + \gamma^a n_a)}} \\
&= \prod_{s\in\mathcal{Z}} \frac{e^{-\frac{\beta' E}{2}}}{1 - e^{-\beta' E + 2\pi i(\gamma^3 s + \gamma^a n_a)}} \\
&= e^{-\pi R_6 \sum_{s\in\mathcal{Z}} \sqrt{G_5^{lm} n_l n_m}} \prod_{s\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{G_5^{lm} n_l n_m} + 2\pi i\gamma^3 s + 2\pi i \gamma^a n_a}} \\
&= e^{-2\pi R_6 \lt H \gt_{n\perp}} \prod_{n_3\in\mathcal{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{G_5^{lm} n_l n_m} + 2\pi i\gamma^3 n_3 + 2\pi i \gamma^a n_a}},
\end{aligned} \tag{F.72}$$

which is (F.67). Its invariance under U' follows from the U' invariance of (F.54), which differs from (F.69) only by an additional contribution of $\widetilde{n}^2$ to the mass m^2.

Hence (F.72) and thus (F.67) are invariant under U'.

Furthermore Z^{6d}_{osc} is invariant under M_3, U_2, U_3 by inspection.

Using the same approach for proving $SL(4,\mathcal{Z})$ symmetry of the $4d$ partition function, we have shown the $6d$ oscillator partition function for the chiral two-form given by (3.71), or equivalently (F.65), transforms as

$$\begin{aligned}
U_2 &: Z^{6d}_{\text{osc}} \to Z^{6d}_{\text{osc}}, \\
U_1 U_3 &: Z^{6d}_{\text{osc}} \to |\widetilde{\tau}|^{-3} Z^{6d}_{\text{osc}}, \\
{U_1}^2 &: Z^{6d}_{\text{osc}} \to Z^{6d}_{\text{osc}}.
\end{aligned} \tag{F.73}$$

Together with (F.45), the $6d$ partition function $Z^{6d,chiral} \equiv Z^{6d}_{\text{zero modes}} Z^{6d}_{\text{osc}}$ is $SL(4,\mathcal{Z})$ invariant.

BIBLIOGRAPHY

[1] E. Witten, *Five-brane Effective Action in M theory*, J.Geom.Phys. **22**, 103 (1997), [`arxiv:hep-th/9610234`].

[2] L. Dolan and C. R. Nappi, *A modular invariant partition function for the five-brane*, Nucl. Phys. B **530**, 683 (1998) [`arXiv:hep-th/9806016`]; *The Ramond-Ramond selfdual five form's partition function on T**10*, Mod. Phys. Lett. A **15**, 1261 (2000) [`arXiv:hep-th/0005074`].

[3] M. Henningson, *The quantum Hilbert space of a chiral two-form in d = (5+1)-dimensions*, JHEP **0203**, 021 (2002) [`arXiv:hep-th/0111150`].

[4] M. R. Douglas, *On D=5 super Yang-Mills theory and (2,0) theory*, JHEP **1102**, 011 (2011) [`arXiv:1012.2880 [hep-th]`].

[5] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *M5-Branes, D4-Branes and quantum 5D super-Yang-Mills*, JHEP **1101**, 083 (2011) [`arXiv:1012.2882 [hep-th]`]; *M-Theory and maximally supersymmetric gauge theories*, [`arXiv:1203.4244 [hep-th]`]; N. Lambert, H. Nastase and C. Papageorgakis, *5D Yang-Mills instantons from ABJM Monopoles*, Phys. Rev. D **85**, 066002 (2012) [`arXiv:1111.5619 [hep-th]`].

[6] L. Dolan and Y. Sun, *Partition functions for Maxwell theory on the five-torus and for the fivebrane on* $S^1 \times T^5$ JHEP **1309**, 011 (2013) [`arXiv:hep-th/012085971`].

[7] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, J. Diff. Geom. **70**, 183 (2005) [`arXiv:hep-th/0503126]`].

[8] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313**, 71 (2012) [`arXiv:0712.2824 [hep-th]`].

[9] A. Kapustin, B. Willett and I. Yaakov, *Tests of Seiberg-like duality in three dimensions*, [`arXiv:1012.4021 [hep-th]`].

[10] T. Dimofte, D. Gaiotto and S. Gukov, *3-Manifolds and 3d Indices*, [`arXiv:1112.5179 [hep-th]`].

[11] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, *The 4D superconformal index from q-deformed 2d Yang-Mills*, Phys. Rev. Lett. **106**, 241602 (2011) [`arXiv:1104.3850 [hep-th]`].

[12] Y. Tachikawa, *4D partition function on* $S^1 \times S^3$ *and 2d Yang-Mills with nonzero area*, [`arXiv:1207.3497 [hep-th]`].

[13] M.B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press: Cambridge, U.K. 1987. See vol.II, p40.

[14] D. Bak and A. Gustavsson, *M5/D4 brane partition function on a circle bundle,* JHEP **1212**, 099 (2012) [`arXiv:1209.4391`].

[15] A. Gustavsson, *A preliminary test of abelian D4-M5 duality,* Phys. Lett. B **706**, 225 (2011) [`arXiv:1111.6339 [hep-th]`]; *On the holomorphically factorized partition function for abelian gauge theory in six-dimensions,* Int. J. Mod. Phys. A **17**, 383 (2002) [`arXiv:hep-th/0008161]`].

[16] R. Dijkgraaf, E. P. Verlinde and M. Vonk, *On the partition sum of the NS five-brane,* [`arXiv:hep-th/0205281]`].

[17] H. -C. Kim, S. Kim, E. Koh, K. Lee and S. Lee, *On instantons as Kaluza-Klein modes of M5-branes,* JHEP **1112**, 031 (2011) [`arXiv:1110.2175 [hep-th]`]; S. Bolognesi and K. Lee, *Instanton partons in 5-dim SU(N) gauge theory,* Phys. Rev. D **84**, 106001 (2011) [`arXiv:1106.3664 [hep-th]`].

[18] B. Collie and D. Tong, *The partonic nature of instantons,* JHEP **0908**, 006 (2009) [`arXiv:0905.2267 [hep-th]`].

[19] H. Singh, *Super-Yang-Mills and M5-branes,* JHEP **1108**, 136 (2011) [`arXiv:1107.3408 [hep-th]`].

[20] N. Nekrasov, *Instanton partition functions and M-theory*, Japanese Journal of Mathematics, **4**, 63 (2009) [`springerlink`]; N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting,* Adv. Theor. Math. Phys. **7**, 831 (2004) [`arXiv:hep-th/0206161`].

[21] P. -M. Ho, K. -W. Huang and Y. Matsuo, *A non-abelian self-dual gauge theory in 5+1 dimensions,* JHEP **1107**, 021 (2011) [`arXiv:1104.4040 [hep-th]`].

[22] H. Samtleben, *Actions for non-abelian twisted self-duality,* Nucl. Phys. B **851**, 298 (2011) [`arXiv:1105.3216 [hep-th]`].

[23] C. -S. Chu and S. -L. Ko, *Non-abelian action for multiple five-branes with self-dual tensors,* JHEP **1205**, 028 (2012) [`arXiv:1203.4224 [hep-th]`].

[24] K. -M. Lee and J. -H. Park, *5-D actions for 6-D selfdual tensor field theory,* Phys. Rev. D **64**, 105006 (2001) [`arXiv:hep-th/0008103]`].

[25] J. Källén, J. Qiu and M. Zabzine, *The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere,* [`arXiv:1206.6008 [hep-th]`].

[26] J. Källén, J. A. Minahan, A. Nedelin and M. Zabzine, N^3*-behavior from 5D Yang-Mills theory,* [`arXiv:1207.3763 [hep-th]`].

[27] H. -C. Kim and S. Kim, *M5-branes from gauge theories on the 5-sphere*, [arXiv:1206.6339 [hep-th]].

[28] H. Linander and F. Ohlsson, *(2,0) theory on circle fibrations*, JHEP **1201**, 159 (2012) [arXiv:1111.6045 [hep-th]].

[29] B. Haghighat and S. Vandoren, *Five-dimensional gauge theory and compactification on a torus*, JHEP **1109**, 060 (2011) [arXiv:1107.2847 [hep-th]].

[30] E. Witten, *Conformal field theory in four and six dimensions*, [arXiv:0712.0157 [math.RT]]; *Some comments on string dynamics*, In *Los Angeles 1995, Future perspectives in string theory* 501-523 [arXiv:hep-th/9507121].

[31] E. Witten, *On S-duality in abelian gauge theory*, Selecta Math. **1**, 383 (1995) [arXiv:hep-th/9505186].

[32] E. P. Verlinde, *Global aspects of electric - magnetic duality*, Nucl. Phys. B **455**, 211 (1995) [arXiv:hep-th/9506011].

[33] P .A. M. Dirac, Lectures on quantum mechanics, New York: Belfer Graduate School of Science, Yeshiva University (1964) 87p.

[34] A. Das, Lectures on quantum field theory, Hackensack, USA: World Scientific (2008) 775 p.

[35] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, New York: Dover Publications (1972), p378.

[36] M. Berkooz, M. Rozali and N. Seiberg, *Matrix description of M theory on T^4 and T^5*, Phys. Lett. B **408**, 105 (1997) [arXiv:hep-th/9704089].

[37] N. Seiberg, *Notes on theories with 16 supercharges*, Nucl. Phys. Proc. Suppl. **67**, 158 (1998) [arXiv:hep-th/9705117].

[38] C. M. Hull, *BPS supermultiplets in five-dimensions*, JHEP **0006**, 019 (2000) [arXiv:hep-th/0004086].

[39] Z. Bern, J. J. Carrasco, L. J. Dixon, M. R. Douglas, M. von Hippel and H. Johansson, *D = 5 maximally supersymmetric Yang-Mills theory diverges at six loops*, Phys. Rev. D **87**, 025018 (2013) [arXiv:hep-th/1210.7709].

[40] L. Dolan and Y. Sun, *Electric-magnetic duality of abelian gauge theory on the four-torus, from the fivebrane on $T^2 \times T^4$, via their partition functions*, [arxiv:1411.2563].

[41] C. Montonen and D. I. Olive, *Magnetic Monopoles as Gauge Particles*, Phys. Lett. B**72**, 117 (1977).

[42] P. Goddard, J. Nuyts and D. I. Olive, *Gauge Theories and Magnetic Charge*, Nucl. Phys. B **125**, 1 (1977).

[43] E. Witten and D. I. Olive, *Supersymmetry Algebras That Include Topological Charges* Phys. Lett. B**78**, 97 (1978).

[44] C.Vafa and E.Witten, *A Strong Coupling Test of S Duality,* Nucl. Phys. B **431**, 3 (1994), [`arxiv:hep-th/9408074`].

[45] E. Witten, *Geometric Langlands from Six Dimensions*, [`arxiv:0905.2720 [hep-th]`].

[46] E. Witten, *Conformal Field Theory in Four and Six Dimensions,* [`arxiv:0712.0157 [math.RT]`].

[47] R. Zucchini, *Abelian Duality And Abelian Wilson Loops,* Commun. Math. Phys. **242**, 473 (2003) [`arXiv:hep-th/0210244`].

[48] E. Witten, *On S-duality In Abelian Gauge Theory* Selecta Math. **1**, 383 (1995) [`arXiv:hep-th/9505186`].

[49] H. Kikuchi *Poincare Invariance in Temporal Gauge Canonical Quantization and Theta-Vacua,* Int. J. Mod. Phys. A **9**, 2741 (1994), [`arXiv:hep-th/9302045`].

[50] P .A. M. Dirac, Lectures on Quantum Mechanics, New York: Belfer Graduate School of Science, Yeshiva University (1964) 87p.

[51] A. Das, Lectures on Quantum Field Theory, Hackensack, USA: World Scientific (2008) 775 p.

[52] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, New York: Dover Publications (1972), p378.

[53] D. Gaiotto, *$N = 2$ dualities,* JHEP **1208**, 034 (2012), [`arXiv:hep-th/0904.2715 [hep-th]`].

[54] D. Gaiotto, G. W. Moore, and A. Neitzke, *Four-dimensional Wall-Crossing via Three-dimensional Field Theory,* Commun. Math. Phys. **299** 163 (2010), [`arXiv:1306.4320 [hep-th]`].

[55] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories,* Lett. Math. Phys. **91** 167 (2010), [`arXiv:0906.3219[hep-th]`].

[56] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading Between the Lines of Four-dimensional Gauge Theories,* JHEP **1308**, 115 (2013) [`arXiv:1305.0318 [hep-th]`].

[57] Y. Tachikawa, *On the 6D Origin of Discrete Additional Data of 4D Gauge Theories*, JHEP **1405**, 020 (2014), [`arXiv:1309.0697 [hep-th]`].

[58] A. Gadde, S. Gukov, and P. Putrov, *Fivebranes and 4-Manifolds,* [`arXiv:1306.4320 [hep-th]`].

[59] A. Gadde, S. Gukov and P. Putrov, *(0,2) Trialities*, JHEP **1403**, 076 (2014), [arXiv:1310.0818 [hep-th]].

[60] G. Etesi and A. Nagy, *S-duality in Abelian Gauge Theory Revisited*, J. Geom. Phys. **61**, 693 (2011), [arXiv:1005.5639 [math.DG]]