COMING TO UNDERSTAND ANGLE AND ANGLE MEASURE: A DESIGN-BASED RESEARCH CURRICULUM STUDY USING CONTEXT-AWARE UBIQUITOUS LEARNING

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ABSTRACT

HELEN CROMPTON: Coming to understand angle and angle measure: A design-based research curriculum study using context-aware ubiquitous learning
(Under the direction of Susan N. Friel)

This study uses design-based research (DBR) to develop an empirically-substantiated instructional theory about students' development of angle and angle measure, with real-world connections and technological tools through the use of context-aware ubiquitous learning. The research questions guiding this research are:

1. How do students come to understand angle and angle measure through the use of real-world connections and technology enabled learning tasks?
2. What are effective means of support to facilitate students' understanding of angle and angle measure?

A conjectured local instruction theory was developed from a thorough review of the literature in chapter two. This review encompassed research-based developmental trajectories and effective instructional supports for promoting students' understanding of angle and angle measure. It was conjectured that context-aware u-learning was a good support for students coming to understand angle and angle measure. Context-aware u-learning in this study involves the use of real-world connections and a Dynamic Geometry Environment.

The local instruction theory was subject to a cyclical iterative process of anticipation, enactment, evaluation, and revision (Gravemeijer & van Eerde, 2009). This process contributes to the theories of how students come to understand angle and angle measures, while also developing a set of instructional activities which can be utilized and adapted by educators to meet the needs of the students in their classrooms. The instructional sequence was implemented in one classroom-based teaching experiment in the first macro cycle of the DBR process. A second macro cycle was implemented using revised instructional materials in one classroom-based teaching experiment.

Findings indicate that context-aware u-learning is a valuable mathematical context for introducing students to angle and angle measure. Common misconceptions about angle can be avoided as the students study angles in the real world which presents them with angles with rays of different length and in various orientations. Good foundations were built by having the students consider angle by the generalizable properties and over the seven days the students showed good movement across the van Hiele levels of geometric thinking. A revised local instruction theory is presented as a result of the findings from this study.

DEDICATION

This dissertation is dedicated to my mother, who gave me steadfast encouragement and enthusiasm for my studies.

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First and foremost, I would like to thank my husband Stephen. Without his support and sacrifice, I would never have been able to pursue my goals and dreams for furthering my studies. Thank you for believing in me and encouraging me throughout my studies. We did it! I would like to offer my sincere appreciation to my family. I would like to thank my Dad for raising me to have faith and understand that with Him all things are possible. To my sisters, Ruth and Allison, thank you for giving me advice and allowing me to talk for hours about my ideas and projects.

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CHAPTER I: INTRODUCTION

Geometry is the study of the size, shape and position of two-dimensional and three-dimensional shapes and figures. In geometry, one explores spatial sense and geometric reasoning. Geometry is found everywhere: in art, architecture, engineering, robotics, land surveys, astronomy, sculptures, space, nature, sports, machines, cars and much more. In the early years of geometry, the focus tends to be on shapes and solids, then moves to properties and relationships of shapes and solids, and as abstract thinking progresses, geometry becomes much more about analysis and reasoning.

Geometry is linked to many other topics in mathematics, including measurement. Angle is an important concept in geometry and in the study of measurement. Measurement is "the process of assigning a number to a magnitude of some attribute of an object, such as its length, relative to the unit" (Clements & Sarama, 2009, p.163). Students must understand and recognize angles conceptually and also be able to link this knowledge to angle measure. The van Hiele levels (adapted by Scally, 1990), described the way in which children initially identify, characterize and operate on angles according to their appearance. As children become more familiar working with angle, properties of angles are identified, and those properties are used to solve problems. With effective instruction, this progression continues as upper elementary and middle school students use the knowledge gained to formulate and use definitions of angle, as well as provide informal arguments about those angle properties.

Angle measure is formally taught once students have grasped the basic angle concepts. However, Clements and Sarama (2009) described the way that children as young as two begin intuitively using angle measure notions during block play. As children reach the ages of four and five years, they may identify corresponding and congruent angles using physical models. With effective instruction, children at the age of six years can sort angles into groups of smaller or larger angles, although they may become confused with irrelevant features such as length of rays. At seven years, students can recognize right angles and equal angles of other measure. From the age of eight, Clements and Sarama described the way in which children become *angle measurers*, as they understand angle measure in terms of generalizable concepts and procedures.

There are many unique challenges to understanding angle measure that can be difficult for students to grasp. Students may develop many misconceptions and encounter difficulties while learning concepts and skills in angle and angle measure (Clements & Battista, 1989; Clements & Burns, 2000; Kieran, 1986; Magina & Hoyles, 1997). Prototype diagrams can lead students to considering non-relevant attributes (Clements & Battista, 1992; Yerushalmy & Chazan, 1993), such as orientation (Battista, 2009). For example, in Figure 1.1, angle (a) is an example of the prototypical textbook figure of a right angle. Therefore if a right angle, such as (b), is shown to students, they may not consider this figure to be a right angle.

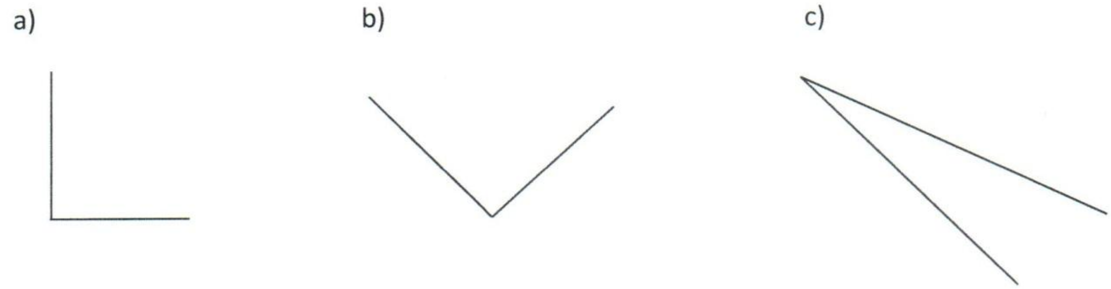

Figure 1.1. Angle Examples.

Nomenclature can also cause misinterpretation; for instance, students can consider a right angle to be an angle that points to the right (Clements & Sarama, 2009), such as the example in Figure 1.1 (a). This is another reason why students may not classify figure (b) as a right angle. As students move on to angle measure, many students believe that the size of the angle is determined by measuring the length of the line segments that are the rays of the angle (Clements, 2004; Clements & Battista, 1989; Berthelot & Salin, 1998; Wilson & Adams, 1992). For example, in Figure 1.1, (c) would be deemed the largest angle as the length of the rays are longer than lengths of the rays on the other two examples.

A complication that adds to student misunderstanding is that the mathematical concept of angle appears to have multiple different definitions. Henderson and Kieran (2005) identified three themes or categories to define angle: (a) a geometric shape, where two rays meet at a common endpoint (Browning, Garza-Kling, & Hill Sundling, 2007), (b) a measure, as the space between the two rays, or (c) a dynamic rotation, as a representation of a turn. Others (Clements & Battista, 1989, 1990; Keiser, 2000; Mitchelmore & White, 2000; Scally,

1990) posited that a definition needs to be developed that is more than just a static explanation (as in (a) above), and the dynamic nature of turn should be considered with angle measure.

Despite the difficulties many children may encounter when learning about angle and angle measure, Clements and Sarama (2009) suggested that these concepts need to be taught within the elementary years. They offer four arguments for early instruction:

- First, children can and do compare angles and turn measures informally.
- Second, use of angle size, at least implicitly, is necessary to work with shapes; for example, children who distinguish a square from a non-square rhombus are recognizing angle size relationships, at least at an intuitive level.
- Third, angle measure plays a pivotal role in geometry throughout school, and laying the groundwork early is a sound curricular goal.
- Fourth, the research indicates that, although only a small percentage of students learn angles well through elementary school, young children can learn these concepts successfully.

(Clements & Sarama, 2009, p. 184)

In addition, evidence indicates that elementary children are developmentally able to learn about angle concepts. For example, in Piaget's studies, he identified children as young as six developing a tacit knowledge of angle and that this develops into extrinsic knowledge around the age of nine (Olson, 1970). Lehrer, Jenkins, and Osana (1998) also found that children's knowledge of angles grows during the elementary years. It makes sense that the learning process involved in students developing understanding of angles should begin in the elementary years (Clements, 2004).

Researchers (e.g., Browning & Garza-Kling, 2009; Clements & Burns, 2000; Fyhn, 2007; Lehrer et al., 1998; Mitchelmore, 1989, 1993, 1997, 1998; Mitchelmore & White, 2000) have explored various pedagogical strategies to provide opportunities for students to develop an understanding of angle and angle measure. Two recurring trends emerged from the research, which are the use of real-world connections and the use of technology as supportive pedagogical components to promote students' understanding of angle concepts. Mitchelmore and White (2004, 2007) postulated that early angle instruction should engender an ever-increasing awareness of angle across real-world contexts. Mitchelmore (1997, 1998) used realistic models such as door knobs, dolls, and roads to successfully support students' understanding of angle as a turn.

Others (e.g., Zbiek, Heid, Blume, & Dick, 2007; Sinclair & Jackiw, 2010) reported the efficacy of using dynamic geometry environments (DGEs) to support learning of angle concepts. For example, Zbiek et al. specifically described how the drag feature in DGEs can be used to change angle measures, leading to conjectures about the way in which the angles of a shape change as one of those angles is dragged into another position. The students' interaction between the mechanical moving of shapes (spatial) and their theoretical (geometrical) understanding supports the development of spatial reasoning (Laborde, Kynigos, Hollebrands, & Strasser, 2006).

Recent technological advancements have led to *context-aware ubiquitous learning* (context-aware u-learning; Hwang, Wu, & Chen, 2007; Yang, 2006), a form of mobile learning (m-learning) that provides a means by which users of mobile devices can study real-world phenomena, while using the mobile devices to provide timely and effective computer support (Lonsdale, Baber, Sharples, & Arvanitis, 2004). For example, it is possible for students to learn about angle concepts by using the portable mobile technologies to take pictures of occurrences of angle in real-world settings, while at the same time, using applications such as the DGEs for further exploration of angles in these contexts.

The Purpose of this Study

Drawing on current research, this study addresses Clements' (2004) call for angle concepts to be taught in the elementary years. The purpose of this dissertation was to develop an instructional theory about students' development of angle and angle measure, making use of real-world connections and technological tools through the use of context-aware u-learning. This study uses Gravemeijer and van Eerde (2009) design-based research (DBR), as it employs methods that enable the research team to develop a local instruction theory and instructional materials to be used to explore the process by which students learn a particular concept in mathematics.

There are two research questions:

1. How do students come to understand angle and angle measure?[1]
2. What are effective means of support to facilitate students' understanding of angle and angle measure?

The research involves studying how children engage and participate in instructional activities, while considering the learning goals. The local instruction theory is subject to a cyclical iterative process of anticipation, enactment, evaluation, and revision (Gravemeijer & van Eerde, 2009). This process contributes to the theories of how students come to understand angle and angle measures, while also developing a set of instructional activities

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[1] The terms measure and measurement are used interchangeably in this dissertation.

which can be utilized and adapted by educators to meet the needs of the students in their classrooms.

This dissertation is comprised of six chapters. In chapter two, the first section of the literature review summarizes research-based developmental trajectories regarding angle and angle measure. The second section of the literature review considers effective instructional supports for promoting students' understanding of angle and angle measure. In chapter three, the DBR approach is discussed and a conjectured local instruction theory about students' learning of angle and angle measure through the use of context aware u-learning tasks is articulated, based on a review of the literature.

The study methodology is developed in greater detail in chapter four, followed by the study findings in chapter five. Finally, in chapter six, a revised local instruction theory is presented with the effective means of support for this learning progression. The final outcomes of this research include an instructional theory about the progression of students' understanding of angle and angle measure through the use of context aware u-learning and the instructional materials to support this learning.

CHAPTER TWO: REVIEW OF THE LITERATURE

The study of geometry and spatial reasoning enable children to understand the "space that the child must learn to know, explore, and conquer, in order to live, breathe and move in it" (NCTM, 1989, p. 48). Geometry is a complex subject incorporating many challenging mathematical concepts. Angle concepts are particularly difficult for students to grasp (Battista, 2007; Clements, 2004; Clements & Battista, 1992; Lindquist & Kouba, 1989; Piaget & Inhelder, 1948/1967). Furthermore, angle measure requires students to consider measure as the relationship between two components (rays) in a dynamic turn, which is different than linear measurement they have typically encountered (Clements & Sarama, 2009).

Empirical findings have highlighted two different methods for supporting students' understanding of angle and angle measures; these are the use of Dynamic Geometry Environments (DGE; e.g., Clements & Battista, 1994; Clements, Wilson, & Sarama, 2004; Laborde et al., 2006) and real-world connections (e.g., Clements & Burns, 2000; Fyhn, 2007; Mitchelmore, 1989, 1993, 1997, 1998). Context-aware u-learning is a type of mobile learning that provides the opportunities for students to utilize the tools in DGEs while learning in a real-world setting.

The purpose of this dissertation is to develop an empirically-substantiated instructional theory about students' development of angle and angle measure, with real-world connections and technological tools through the use of context-aware u-learning. The research questions guiding this research are:

1. How do students come to understand angle and angle measure?
2. What are effective means of support to facilitate understanding of angle and angle measure?

This study utilizes a design-based research (DBR) methodology and the literature review serves to develop a conjectured local instruction theory. DBR is detailed in chapter three; however, it is important to point out that the use of this methodology requires a different format than a typical literature review. This literature review is not intended to point out the gaps in the literature, but to clarify what is known in order to inform the development of the conjectured local instruction theory (Markworth, 2010). This review is comprised of three main sections: how students come to understand angle and angle measure, what students need to know about angle and angle measure, and support for learning about those angle concepts.

Angle and Angle Measure

Understanding angle concepts requires the apperception of the physical properties of angle, including the static (configurational) and dynamic (moving) aspects (Kieran, 1986; Scally, 1986). Two strands of geometry are involved: geometry and measurement, each with its own content, procedures and applications. While there is a dichotomy between the two mathematical strands, angle and angle measure are highly intertwined. Nevertheless, to clearly explicate the empirical and theoretical underpinnings of each of the concepts, angle and angle measure will be discussed separately before making the connections between the two.

How children come to understand angle. Two major theories have dominated the research on angle. The first is the van Hiele levels of geometric thinking (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984), developed by Dutch educators Pierre van Hiele and Dina van Hiele-Geldof. The second is from the work of Piaget and colleagues (viz., Piaget & Inhelder, 1948/1967; Piaget, Inhelder, & Szeminska, 1960), in relation to angle conceptions. More recently, Mitchelmore and colleagues (viz., Mitchelmore, 1989, 1993, 1998; Mitchelmore & White, 2000, 2004, 2007) delineated students' development of angle concepts by progressive abstraction and generalization, and Scally (1990) applied the van Hiele model to develop a specific theory of angle concept development. Each of these theories are detailed in this chapter to develop a rich understanding of students' development of angle.

The van Hiele model (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984) highlights students' development through five levels of geometric thought, from gestalt-like unanalyzed viewing to a highly complex level of thinking. The emphasis of the van Hiele model is placed on the purpose of effective instruction to facilitate progression through the levels. However, even with effective instruction, elementary students typically do not progress beyond the second or third level; therefore, only the first three van Hiele levels are discussed in this study.

The way in which the van Hiele levels are numbered has varied (Clements & Battista, 1992). For the purpose of this paper, the first three levels are listed. The terms visualization, analysis, and informal deduction describe the cognitive levels through which the students progress (De Villiers, 1987; Hoffer, 1981; Teppo, 1991), with the fourth and fifth levels (deduction and rigor) omitted.

Level 1 (Visualization): Students at this initial level identify, name, compare, and operate on shapes and other geometric configurations according to their appearance. Figures are seen as visual gestalts in that individual attributes, such as angle measure, are not explicitly recognized; instead the figures are considered as a collection of a whole. Perception guides the students' reasoning, and visual prototypes are typically used to name a figure. For example, a student may say that a figure is a rectangle because it looks like a door (Clements, 1998).

Level 2 (Analysis): Students at this level have progressed from gestalt perceptions to analyzing figures according to their attributes and are able to identify the relationships among the attributes to discover rules for how figures are named. For instance, a student may think of an equilateral triangle as a figure with three equal angles; therefore, the student has learned that the term "equilateral triangle" refers to a specific collection of properties.

Level 3 (Informal Deduction): Students at this level can provide abstract definitions and informal arguments. They can distinguish between the necessity and sufficiency of a set of properties for a concept, while also ordering those properties logically. It becomes clear, for example, why a square can also be a rectangle. Although the students are showing a method of logical organization, they do not know that it is a method by which geometric truths are established.

The van Hieles theorized that learning is a discontinuous process, with jumps in the learning curve that reveal the five discrete levels. The levels are sequential and hierarchical descriptions of how the student would demonstrate thinking at each level. In order to move through the levels, students need to become proficient in a large portion of the lower level before they can advance to a higher level (Hoffer, 1981). From observations of students' thinking, van Hiele (1957/1984a) noticed that knowledge intrinsic at one level appears in an extrinsic way at the next. For example, while a child may be using particular properties to determine the name of a shape, the actual thinking at that level may not be cognizant of those features.

Similar to the van Hieles (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984), Piaget and colleagues (vis., Piaget & Inhelder, 1948/1967; Piaget et al, 1960) developed a thesis on the way students come to understand geometry and angle. However, Piaget and colleagues also extended this thesis to include spatial reasoning, which Piaget and Inhelder (1948/1967) called *representational space*. Representational space is how children conceptualize and represent physical space. This body of research led to the topological primacy thesis.

The topological primacy thesis refers to Piaget and Inhelder's (1956, 1967) claim that a young child's intrinsic geometry is initially topological; which is where students apperceive relations such as enclosure, connectedness, and continuity. This is followed by the student's ability to learn projective (rectilinearly) and Euclidean (parallelism, angularity, and distance) relationships (Darke, 1982). Congruent with the van Hiele model, Piaget and Inhelder posited that there is a definite order in developmental progression that must be observed. In Piaget's studies, he identified children as young as six developing a tacit knowledge of angle, developing to extrinsic knowledge around the age of nine (Olson, 1970).

As Piaget and Inhelder (1948/1967) study children's perspective taking, they posited that the difference between topological, projective and Euclidean perspectives involves the relationship between the figures and the subject. Topological perspectives consider the figure in isolation, projective perspectives involve perspectives between the figure and the subject, and Euclidean refers to perspectives between figures. Clements (1998), Battista (2007), and Piaget and Inhelder, described perspective taking as a critical developmental step in geometry. As students develop projective and Euclidean perspectives, they are able to move beyond their own perspectives to the perspectives of others. For example, with the development of projective space, students can construct straight lines by putting themselves as one of two points to be linked by a straight line. As students gain the perspective of Euclidean space, concepts such as angularity and parallelism are developed.

Piaget et al. (1977) described students' understanding of angle in terms of the abstraction process. More recently, Mitchelmore and colleagues (viz., Mitchelmore, 1989, 1993, 1998; Mitchelmore & White, 2000, 2004, 2007) conducted studies to focus specifically on angle abstraction as they delineate students' development of angle concepts by progressive abstraction and generalization. The work of Mitchelmore and colleagues was brought about by Skemp (1986), who took Piaget et al.'s notions of abstraction being superficial appearance and extended this to think about the underlying structure.

> Abstracting is an activity by which we become aware of similarities… among our experiences. Classifying, means collecting together our experiences on the basis of these similarities. An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class… To distinguish between abstracting as an activity an abstraction as its end-product, we shall… call the latter a concept. (Skemp, 1986, p. 21)

Mitchelmore and White (2000) postulated that students develop angle concepts through three overlapping stages of abstraction. Each stage represents a progressively more cultured classification of students' experience of angle concepts.

Stage 1: Situated angle concepts. Preschool children learn many everyday concepts such as slide, hill, roof, and bend, and an adult can typically see these as having a connection with angle. Mitchelmore and White (2000) defined these concepts as *situated angle concepts* as they are developed from children's mental classification of situations experienced by the children. A situated angle concept is limited to similar situations which may look alike, have similar actions and are experienced in similar social environments. Empirical findings (e.g., Mitchelmore, 1997; Mitchelmore & White, 1998) led Mitchelmore to declare that children have formed many situated angle concepts as they begin schooling.

Stage 2: Contextual angle concepts. During elementary school, most students learn words such as "slope" and to classify physical angle situations, which Mitchelmore and White (2000) described as *physical angle contexts*. Students are able to develop the meaning of terms (e.g., slope) in that they can provide a number of different examples of slope when asked to do so. The students are first able to use this term in only a few situations, but this understanding is then generalized to other situations. As the term evolves and is generalizable, it has become a mental object in its own right. Mitchelmore and White (2000) called such concepts *contextual angle concepts*. By the age of nine, students have formed an explicit understanding of slope, turn, intersection, and corner; however, the concept of bend is still vague.

Physical angle contexts form from common geometrical configurations and similar physical actions. But they are not formed on similarities between physical or mental operations on those configurations. For example, the concept of turn is abstracted from an observed movement, not an action imposed by the student.

Stage 3: Abstract angle concepts. While students consider angle contexts as distinct, they can also recognize similarities between them. For example, studies (Mitchelmore, 1997; Mitchelmore & White, 1998) indicated that students noted a similarity between intersections and bends, and about half recognized a connection between slopes and corners, and between turns, intersections, and bends. The recognition of similarities is the beginning of an elementary mathematical conception of angle called an *abstract angle concept*.

As Mitchelmore and colleagues developed a more detailed look into the subject of angle, Scally (1990) took the van Hiele model and developed a set of level indicators that focus specifically on angle. The levels correspond with the first three van Hiele levels: visualization, analysis, and informal deduction. Each level has an overall description and then multiple level indicators. The overall descriptions are:

First level: In general, the student identifies, characterizes, and operates on angles according to their appearance.

Second level: In general, the student establishes properties of angles and uses properties to solve problems.

Third level: In general, the student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments.

The detailed list of level indicators can be found in full in Appendix A.

How children come to understand angle measure. Piaget and colleagues (viz., Piaget & Inhelder, 1948/1967; Piaget et al., 1960) provided great insight into students' development of angle conceptions; in addition, they posited that the cognitive development of angle measure can be developed through the use of the Cartesian coordinate system. However, the greatest contribution to the understanding of students' angle measure development has come from the theories and studies conducted by Clements and colleagues (viz., Clements, Battista, Sarama, & Swaminathan, 1996; Clements & Sarama, 2009; Clements & Stephan, 2004). The work of Clements and colleagues is explicated in this section while also drawing on the relevant studies and theories of others. This section also makes the connection between angles as a geometric shape and angle measure.

As with understanding length and area, comprehending angle measure requires students to first understand partitioning, unit iteration (Clements & Stephan, 2004), and equal units. Specifically, Sarama, Clements, Barrett, Van Dine, and McDonel (2011) highlighted a number of essential understandings needed to understand length measure. These include students' ability to: recognize length as an attribute, compare lengths, recognize the need for units of equal length, measure by using multiple identical units, and measure by using a single iterative unit. Students will be expected to have these skills for linear measure before moving onto angle measure (Clements & Stephan, 2004).

There are also other unique challenges to understanding angle measure which have to be considered. This is exacerbated by the multiple definitions given to the concept of angle. For example, angle can be considered (a) a geometric shape, (b) a measure, or (c) a dynamic rotation (Henderson & Kieran, 2005). Freudenthal (1973) proffered that multiple definitions can be appropriate as targets for instruction. However, many consider a multiple definition proposal as problematic. Earlier static definitions may impede students' exploration of angle (Keiser, 2000), and there are those (viz., Clements & Battista, 1989, 1990; Keiser, 2000;

Mitchelmore & White, 2000; Scally, 1990) who think that the definition should be developed more thoughtfully than a static explanation, and the dynamic nature of turn should be considered in order for students to come to understand angle measure.

Students in the early elementary grades often form two separate conceptions of angle: angle as a shape and angle as a movement (Clements et al., 1996; Clements & Sarama, 2009). When students are taught the topic of angle measure, they have to move beyond the conceptions of angle as a static shape. Otherwise they will adopt measurement approaches that involve measuring the rays rather than the measure of angle turn. This can lead to misconceptions that continue well into high school (Lehrer et al., 1998). To understand angle measure, Clements and Sarama posited that students need to overcome misconceptions and difficulties with orientation, discriminate angles as critical parts of geometric figures, and represent the idea of turns and their measure.

Clements and Sarama (2009) developed a trajectory for angle measure for pre-kindergarten and the elementary grades. The developmental progression has five levels organized by age. At each level, a descriptive title has been given to define the abilities of the students.

Ages 2-3 years: Intuitive Angle Builder. The child intuitively uses some angle measure notions in everyday settings, such as building with blocks. (Places blocks parallel to one another and at right angles with the perceptual support of the blocks themselves to build a "road".)

Ages 4-5 years: Implicit Angle User. The child implicitly uses some angles notions, including parallelism and perpendicularity, in physical alignment tasks, construction with blocks, or other everyday contexts (Mitchelmore, 1989, 1992; Seo & Ginsburg, 2004). The child may identify corresponding angles of a pair of congruent triangles using physical models. Child uses the word “angle” or other descriptive vocabulary to describe some of these situations. (Moves a long unit block to be parallel with another set of blocks after adjusting the distance between them, in anticipation of laying several other blocks perpendicular across them.)

Age 6 years: Angle Matcher. Child matches angles concretely and explicitly recognizes parallels from non-parallels in specific contexts (Mitchelmore, 1992). The child sorts angles into “smaller” or “larger” (but may be misled by irrelevant features, such as length of the line segments). Given several non-congruent triangles, the child finds pairs that have one angle that is the same measure, by laying the angles on top of one another.

Age 7 years: Angle Size Comparer. The child differentiates angle and angle size from shapes and contexts and compares angle sizes. The child recognizes right angles, and then equal angles of other measure, in different orientations (Mitchelmore, 1989). Child can compare simple turns.

Ages 8 + years: Angle measurer. Child understands angle and angle measure in both primary aspects and can represent multiple contexts in terms of the standards, generalizable concepts and procedures of angle and angle measure. For example, two rays, the common endpoint, rotation of one ray to the other around that endpoint, and measure of that rotation. (Clements & Sarama, 2009, p. 187)

It must be noted that Clements and Sarama (2009) developed a number of instructional tasks as part of the hypothetical learning trajectory, and without adequate effective instruction, students may not be performing at these levels. However, from the trajectory, it appears that students older than eight can be developmentally ready to understand angle and angle measure.

Curricular expectations for learning about angle and angle measure. In the last section of this chapter, the focus is on describing how students come to understand angle and angle measure. The theoretical and empirical components, such as the van Hiele geometric levels (Clements & Sarama, 2009; van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984) and the developmental process described by Clements and Sarama (2009), highlight the essential understandings that students need to develop. Essential understandings are defined by Karp, Caldwell, Zbiek, and Bay-Williams (2011) as the specific interconnected ideas of a larger mathematical concept. Other similar terms have been used in mathematics; for example, Watt, Clements, and Lehrer (2002) referred to “big ideas” as concepts that underlie understanding and mastering a strand of mathematics, and Wiggins and McTighe (2005) described “enduring understandings” as the important understandings students need to retain to make meaning of the subject. These terms all refer to critical concepts needed in students’ development as they come to understand angle and angle measure.

Although there are many (e.g., Ginsburg, Inoue, & Seo, 1999; Lehrer, Jenkins, & Osana, 1998; Sanberg & Huttenlocher, 1996) who described students’ early development toward understanding angle and angle measure, it appears from the review of the literature that students in the final elementary years are developmentally ready to understand these concepts formally. The National Council of Teachers of Mathematics (NCTM; 2000) expectations are that students in grades 3-5 should identify angles as pertinent properties of shape, understand such attributes as size of angle and select the type of unit for measuring each attribute.

In addition, the NCTM (2006) developed a set of curriculum focal points, described as *indispensable elements* and *core structures* for each grade level. Students in third grade are expected to describe, analyze, compare, and classify two-dimensional shapes through attributes such as angle. As part of understanding two-dimensional shapes, students are asked to measure and classify angles in fourth grade. The authors of the curriculum focal points highlighted geometry and measurement as critical topics for students in mathematics (NCTM, 2006). The Common Core State Standards (CCSS; CCSSO/NGA, 2010) are similar to the NCTM standards, with students being expected to consider defining attributes, such as angle, intrinsically during the early elementary grades and formally identify angle concepts in the fourth grade. Specifically, within the fourth grade geometry strand of the CCSS, students are expected to:

- Draw angles (right, acute, obtuse), and identify these as two-dimensional figures.
- Classify two-dimensional figures based on the presence or absence of angles of a specified size.
- Recognize right angles.

The measurement strand identifies these essential understandings:

- Recognize angles as geometric shapes that are formed whenever two rays share a common endpoint, and understand concepts of angle measure.
- Measure angles with reference to a circle; considering angles as fractions of a circle.

- Measure angles in whole-number degrees using a protractor and sketch angles of a specific size.
- Recognize angles as additive.
- Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematics problems.

It is interesting that the authors of the recent CCSS (CCSSO/NGA, 2010) chose to use only the static definition of angle in the grade that students begin learning about angle measure. This is not congruent with suggested effective practice defined in the literature. The CCSS requirements in relation to angle and angle measure are congruent with the learning progressions framework described by Hess (2011).

Support for Learning about Angle and Angle Measure

In the first two sections of this chapter, a review of the literature has been conducted to determine how students learn about angle and angle measure and to identify curricular expectations. This section provides a further review of the literature to highlight the ways in which educators can support students as they come to understand angle and angle measure. Mitchelmore (1998) proffered that for educators to be effective they need to consider the difficulties and misconceptions students have with these concepts. Therefore, this section begins with a summary of those difficulties and misconceptions that children can have. Through an in-depth study of the literature, five reoccurring problems have been identified. A brief summary will be given of each of the issues.

Problem 1[2]: Angles have an abstract nature. Students struggle with angle conceptualization (Battista, 2007; Clements & Battista, 1992) due to the multiple ways in which angles can be represented (Smart, 2009). Mitchelmore and White (2004) described the way in which students are required to see angle concepts as both abstract-apart and abstract-general. Abstract-general angles embody general properties of the world and can be easier for students to understand than abstract-apart, which are angle representations on diagrams and similar representations (Mitchelmore & White, 2004).

Problem 2: Understanding angle as a turn. Students can have difficulty in understanding angle as a turn (Battista, 2007; Clements & Sarama, 2009), and this has led Mitchelmore and White (2000) to proffer that angle measure should not be taught to beginning learners as the amount of turning (about a point) from one line to another. However, this suggestion is contradicted by some who claim that turns are natural for young children (Hoffer, 1988), and if explicitly and carefully taught, students of elementary age can learn angle as a turn, especially when using supportive technologies which highlight the turn (Clements & Battista, 2001).

Problem 3: Understanding what angle is measuring. This problem is closely tied to problem two. As many students do not perceive angles as turns, students often believe that they need to measure the lengths of the rays, rather than an actual turn or the proximity of two sides (Clements & Battista, 1989; Lehrer et al., 1998). In a study with a group of first,

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[2] Although a numbered list is provided, this does not connote an ordinal position of difficulty. The numbers are included to assist the reader in determining the location and organization of the listed problems.

second, third and fifth grade students, even when asked about angles in dynamic contexts, 95% of the students measured lengths of the rays when asked to measure the angles.

Problem 4: Struggle to see different angles in different contexts. Empirical findings led Mitchelmore and White (2000) to conclude that students had difficulties relating the standard angle concept to various angle contexts. For example, students could not identify the two lines that make up a standard angle in the context of a door.

Problem 5: Salient criteria for judging angles. This problem is connected with the other issues described. As students consider the physical attributes of an angle, they can erroneously include particular attributes as salient. For example, empirical findings indicate that students can often wrongly acknowledge the length of the angle's rays and orientation as salient features of angle (Lehrer et al., 1998). This is a misconception highlighted by many (viz., Battista, 2009; Clements & Battista, 1992; Yerushalmy & Chazan, 1993) who accredited typical angle diagrams as a major cause for this problem. For example, students may not recognize a right angle as it is placed in a nonstandard orientation.

While considering the ways in which students come to understand angle and angle measure, which was delineated in the first section of this chapter, the evidence based curricular recommendations from the second section, and the misconceptions and difficulties highlighted in this section, the literature was once again reviewed to determine theories and empirically based instructional practice to support students' understanding of angle and angle measure. Two trends emerged from this review, which are that real-world connections and the use of technological supports have a positive effect on student learning.

Real-world connections. Early mathematicians noted the importance of connecting mathematical concepts to the real world. Clairaut (1741/2006) described how he learned all that he could about the principles of geometry with real-world connections. He posited that in order to teach geometry effectively, it was necessary to start by applying those principles to measuring land. It is worth noting here that the Greek etymology of geometry is the measure of earth or land. Comenius (1657/1986) described the importance of mathematics being presented to the senses as much as possible as concrete relevant items. This sentiment is still echoed by many in the mathematics community today, with many mathematicians and governments advocating for a connection to be made between mathematics and the real world (e.g., Bartolini-Bussi, Taimina, & Isoda, 2010; Gainsburg, 2008; Hiebert & Carpenter, 1992; NCTM, 2000; National Research Council, 1990).

Using real-world connections in mathematics has many recorded benefits, such as enhancing students' understanding of the mathematical concepts (De Lange, 1996; Steen & Forman, 1995), amplifying students' ability to think mathematically outside the classroom (Lehrer & Chazan, 1998; National Research Council, 1998), and motivating students to learn about mathematics (National Academy of Sciences, 2003). There have been a number of studies to determine the affordance of teaching angle concepts with real-world connections. There are those who have used real-world objects; for example Piaget and Inhelder (1948/1967) used tongs, and Mitchelmore and White (2000) used adjustable models of wheels, doors, scissors, and fans. Others used real-life physical situations; for instance, Munier, Devichi, and Merle (2008) had students determine angles in a playground experience, Fyhn (2007) used a climbing project for the students to study angles made by body formations during climbing activities, and Clements et al. (1996) began their study by having students use their experience of body movements to consider angle and help them mathematize their physical experiences.

Mathematizing real-world contexts has been a repeating theme during the literature review. The term *mathematizing* is described as "…the organizing and structuring activity in which acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structures" (Treffers, 1987, p. 247). The very surroundings and actions which the students have been involved with since birth have developed the students' intrinsic knowledge of geometry. This intrinsic knowledge can be especially efficacious in developing students' conceptualizations of angle concepts (Clements et al., 1996). Furthermore, both measurement (Sarama et al., 2011) and geometry are principal real-world applications of mathematics.

Battista (2009) lamented that "geometry instruction and curricula generally neglect the process of forming concepts from physical objects and instead focus on using diagrams and objects to represent formal shape concepts" (p. 97). Consequently, students connect irrelevant attributes of the diagram or object to the geometric concept (Clements & Battista, 1992), for example the orientation or the length of angle rays. Understanding salient criteria needed for judging angles is a common difficulty or misconception students have. In the study conducted by Munier et al. (2008), the researchers conclude that real-world situations enable students to invalidate the idea that length is an appropriate way to compare angles. Mitchelmore (1997) added that studying several angle contexts ensures that length of rays and angle orientation would not become a part of the students' developing angle concept.

Students find angles a difficult, abstract topic, and it is essential to have students link different angle contexts for that very reason (Wilson & Adams, 1992). "It is only by recognizing the similarity between angle situations with and without both arms visible that the standard angle concept can be generalized" (Mitchelmore & White, 2000, p. 234).

Nevertheless, finding angles within real-world situations can be a difficult task for students, considering the vast amount of visual information that the students have to sift through to find the angles. However, Gutiérrez (1996) pointed out that these are skills that need to be developed in relation to students' spatial reasoning. *Figure-ground perception* is the ability to identify a specific figure by isolating it out of a complex background, and *perception of spatial positions* is the ability to relate an object, picture, or mental image to oneself (Gutiérrez, 1996)[3]. The latter is especially important, for the angle a student sees in the edge of a window can be a different size, or kind of angle, when viewed from a different position.

From the review of the literature, it is evident that real-world connections are crucial in the development of students' understanding of angle and angle measure. Real-world contexts provide opportunities for students to explore, make conjectures, display, and clarify their understanding of angle concepts in motivating and meaningful ways (Munier et al., 2008). Specifically, through the use of real-world connections students can mathematize intrinsic environmental and physical experiences (Clements et al., 1996), determine relevant attributes of angles from those that are irrelevant (Clements & Battista, 1992; Munier et al., 2008), make abstract angles comprehensible, and generalize the standard angle (Mitchelmore & White, 2000).

Technology: Dynamic Geometry Environments. A considerable connection was made between students' developing understanding of angle concepts and technology. However, to understand the role technology has in students' understanding of angle requires an acknowledgement of two different types of activities: the technical and the conceptual

[3] The full list is available in Gutiérrez, 1996, p. 10.

(Hoyles & Noss, 2003; Zbiek et al., 2007). The technical component describes mechanical or procedural performance. It is the way in which students interact with the technologies to construct, manipulate, and measure angles. While performing these tasks, students are also developing sequences of mathematical actions. Conceptual activities involve students' understanding, communicating, and developing the mathematical connections, relationships, and structures (Zbiek et al., 2007). Although a dichotomy between the two activities has been described, students need to be involved in both tasks for technology to positively influence student learning (Borwein, 2005; Borwein & Bailey, 2003; Zbiek et al., 2007).

Two technological environments have dominated the research on angle concepts: Dynamic Geometry Environments (DGEs) and Logo-based environments. DGEs provide the students with figures (e.g., lines, points, circles) and basic tools (e.g., the ability to draw a perpendicular line from one point to another) to create composite figures. Various dynamic transformations can also be performed, with the ability to trace the path of the movements for later visual inspection. Logo is a computer programming language used for programs such as Logo-based Turtle Geometry (TG) and the related Microworlds. TG typically involves a robotic turtle that is directed to move around the screen by typing commands into the computer; as the turtle moves, it draws lines creating various figures. Microworlds are computational environments in which students can engage in exploration and construction activities (Noss & Hoyles, 1996; Sarama & Clements, 2002).

In the late 1980's, Logo became recognized as angle-making computer software (Smart, 2009). Logo environments are action-based as they have the students using perceptual (viz., watching the movements of the turtle) and physical (viz., as the students interpret the movement of the turtle as an actual physical motion of walking) demands.

Technological environments such as Logo are more beneficial than static diagrams in helping students understand that angles are dynamic turns (Clements & Battista, 1992). It would appear that Logo was one computer program that brought attention to the way in which technology can support the teaching and learning of angle and angle measure. However, it has been reported that students could not link their own body movements to those of the turtle (Clements et al., 1996), and there is a lack of transfer of angle concepts to physical angle concepts in general (Mitchelmore & White, 2000).

DGEs are a more recent type of computer program credited with supporting students' developing understanding of angle concepts. There are a number of ways in which DGEs can extend and enhance students' understanding while avoiding the common difficulties and misconceptions students have. As the name suggests, it is also a program that provides dynamic images that may assist students in recognizing that angle measure is based on a turn. Having the ability to create and manipulate objects assists students in perceiving the angles as geometric entities, rather than just visual objects (Zbiek et al., 2007). Therefore, students are more likely to reflect on the appropriate properties to determine the categorization of the angles, as they are able to simultaneously take into account the specific and grounded with the abstract and generalized (Clements & Battista, 1994). In other words, DGEs support students in understanding the abstract nature of angles while understanding salient criteria for judging angles. DGEs expand the repertoire of representations available, beyond the prototypical angles often displayed in textbooks (Clements & Battista, 1992; Zbiek et al., 2007).

DGEs provide cognitive tools to support students as they come to understand angle and angle measure. Cognitive tools are defined as technologies that act as external aids to amplify students' cognitive capacities during thinking, learning, and problem solving (Lajoie, 2000; Lajoie & Azevedo, 2006). Other terms have been used to name these tools. Pea (1987) described them as *cognitive technologies*; Zbiek et al. (2007) as *cognitive technological tools*; and Hoyles (1995) as *computational scaffolding*. Hoyles and Noss (2003) used the term *expressive tools* to specifically refer to the tools available in DGEs.

The tools within the DGEs provide students with a way to access the mathematical characteristics underlying geometry and spatial reasoning (Laborde et al., 2006). The software tools become an extension of the students' thinking once students begin to use the programs (Mason, 1992). Hoyles (1995) described this extension as computational scaffolding, a support process to aid in constructing situated abstractions. The tools affect the very way in which the students think and solve tasks (Vérillon & Rabardel, 1995). As students create or access the visual representations within the software, the cognitive tools act as *user agents* (Kaput, 1992) to perform geometric actions or procedures under the direction of the student (Zbiek et al., 2007).

Cognitive tools have the potential to enhance and extend students' learning of geometry in a number of ways. Pea (1987) describes the way in which the cognitive tools can amplify intellectual activity. That is, DGEs can increase the speed in which mathematical tasks are accomplished, with higher accuracy (Pea, 1987). In addition, students can work with the tools within the geometry software to assist in discerning regularities that may have otherwise remained hidden (Heid, 1997; Pea, 1987). Meagher (2006) extended Pea's theories to a two-way amplification perspective as he reiterated how students can be amplified by the computer, but also described the way in which students can amplify the technology as they refine educational goals and make the technology provide the best fit for those goals.

Ben-Zvi (2000) proposed a number of different ways mathematical software can also reorganize students' activities. She described the way

(a) tools may shift the students' activity to a higher level, as they have to integrate tasks and focus attention on detailed planning;

(b) tools change the objects and form of the activity;

(c) tools focus the activity on transforming and analyzing representations;

(d) tools support the situated cognitive mode of thinking and problem solving; and

(e) tools enable students in constructing meaning of conceptions by the use of the representative ambiguity.

Clements and Burns (2000) described the way in which students used computer tools to manipulated angles in a computer program:

> The mental image-based version of movements [on screen]; that is the new mental scheme. Eventually, these mental schemes become operational; that is, they can be created, maintained, and transformed internally. Students then have a conceptual protractor that they can mentally project onto objects and situations to measure turns or angles (p. 42).

Physical protractors may be a typical tool of choice for many mathematicians; however, as students initially learn about angle concepts, the design of the typical static protractor has been found to be problematic. For example, to use a protractor, the student has to identify two lines on the protractor and match those to the rays of the angles on the paper. This is difficult as students have several lines to choose for the base, then the second line has to be imagined (Mitchelmore & White, 2000), and the absence of structural angle components often leads to failure to establish effective structural mappings (Battista, 2007). Furthermore, the activity of finding corresponding lines on the protractor does not assist students in visualizing the concept of turn while using a static procedure (Clements & Burns, 2000).

Mitchelmore and White (2000) posited that “it would seem that the ability to interpret a turn as a relation between two lines, and hence to recognize the angular similarity between a turn and a corner, is an essential prerequisite to angle measure using a protractor” (p. 234). This can be helped by using the protractor with a moving arm, but the typical protractors used in schools are the static, semi-circle protractors. At this point the next problem arises. As educators intend to develop students’ understanding of angle as a turn of up to 360°, the use of a protractor of up to 180° can cause confusion and fuel the misconceptions and difficulties that students have (Close, 1982). Clements and Burns (2000) posited that dynamic computer programs can overcome these problems as the dynamic nature of the programs aid students in internalizing angle benchmarks (e.g., 90°, 180°) and in cognitively comprehending the notion of unit iterations within the image of an angle turn.

DGEs provide a window on the students’ conceptions and understandings. Physical tools do not automatically react to students’ actions with feedback, and often mistakes can go unnoticed or be misinterpreted by students (Zbiek et al., 2007). Researchers have reported that the design of DGEs does not allow students to hide what they do not know (Clements & Battista, 1994). Therefore, mistakes and student misconceptions can be clearly identified, allowing the opportunity for educators to plan appropriate tasks and activities to fill those gaps in the students’ geometric understanding.

The feedback provided from the DGEs can act as a catalyst for large or small group discussions (Mariotti, 2000). For students to develop a rich understanding, it is crucial that they have the opportunity to interact with others to share mathematical ideas and findings

(Chaplin, O'Connor, & Canavan-Anderson, 2009; Richardson, 1999, 2002). “Reflective thought and, hence, learning is enhanced when the learner is engaged with others working on the same ideas” (Van de Walle & Lovin, 2006, p. 4). Theoretical and empirical evidence has indicated that discussion is particularly essential in overcoming many of the misconceptions students develop in relation to angle and angle measure (Browning et al., 2007; Mitchelmore & White, 2000; Munier et al., 2008).

Through the use of DGEs, teachers are fostering mathematical discourse, augmenting communication from teacher-to-student, or computer-to-student, to a richer student-to-student communication (Roblyer & Doering, 2010). In addition, interactive geometry software allows discussion of geometric objects in a manner that was once impossible with traditional paper and pencil representations (Yu, Barrett, & Presmeg, 2009). DGEs enable students to produce detailed external representations of their internal mental representations. Once externalized, there are visible phenomena that can be shared and discussed with others. Although the representations are idiosyncratic, the visuals and computer activities provide a common context for students to effectively share their ideas (Yu et al., 2009), and the mediating function of the computer can create a channel of communication based on shared language (Hoyles & Noss, 1996).

From the review of the literature, it is evident that DGEs are efficacious in developing students’ knowledge of angle and angle measure. To summarize, this section has delineated the way in which DGEs specifically aids students in learning angle concepts while avoiding the highlighted difficulties and misconceptions. For example, the dynamic attributes of computer programs allow students to see angle measures as turns (Clements & Burns, 2000) and enable students to uncover the salient geometric attributes of angle to take into account the abstract and the generalized (Clements & Battista, 1994; Zbiek et al., 2007). In addition, this section has revealed the potential of DGEs to act as cognitive tools and to promote discussion as a means for students to extend and enhance their understanding of angle and angle measure.

Mobile learning: Context-aware ubiquitous learning. The theories and empirical findings surrounding the teaching and learning of angle and angle measure, clearly advocate for the use of real-world connections and DGEs to support learning. There are many (e.g., Balacheff & Kaput, 1996; Sarama & Clements, 2009; Sinclair & Jackiw, 2010) who have made the connection between the two supports as they describe how mathematical computer programs have sought to mathematize the world by adding real-world referents.

Mobile learning (m-learning) has provided a new phase in the evolution of technology enhanced learning (Looi, Seow, Zhang, So, Chen, & Wong, 2010). M-learning is defined as "learning across multiple contexts, through social and content interactions, using personal electronic devices" (Crompton, 2013, p. 4). As m-learning developed, the multiple affordances the device offered to extend traditional pedagogies became evident. Traxler (2011) described five ways in which m-learning offers new learning opportunities: 1) contingent learning, allowing learners to respond and react to the environment and changing experiences, 2) situated learning, in which learning takes place in the surroundings applicable to the learning, 3) authentic learning, with the tasks directly related to the immediate learning goals, 4) context aware learning, in which learning is informed by the history and the environment, and 5) personalized learning, customized for each unique learner in terms of abilities, interests, and preferences.

Mobile devices have been used in a number of mathematical studies and the findings indicate that m-learning can develop students' understanding of estimation (Lan, Sung, Tan, Lin, & Chang, 2010), addition, subtraction (Zurita & Nussbaum, 2007), and multiplication (Wei, Hung, Lee, & Chen, 2011). In addition, research on m-learning has shown that both the mobility and the connectivity of the devices allow students to become active in the learning process and make those real-world mathematical connections. Rather than sitting in front of a conventional tethered computer, the students can use smaller portable devices to learn by physically exploring the real world (Colella, 2000; Squire & Klopfer, 2007).

This real-world connection has developed into a sub category of m-learning and is referred to as context-aware ubiquitous learning (context-aware u-learning; Lonsdale et al., 2004). Hwang, Tsai, and Yang (2008) described context-aware u-learning as:

> The learner's situation or the situation of the real-world environment in which the learner in location can be sensed, implying that the system is able to conduct the learning activities in the real world… context-aware u-learning can actively provide supports and hints to the learners in the right way, in the right place, and at the right time, based on the environmental contexts in the real world. (p. 84)

To develop an idea of what this looks like in practice, Hwang et al. (2008) provided a table of context-aware u-learning models and examples of each. Table 1 provides a few of those models and examples[4].

Table 1.1

Models and examples of context-aware u-learning activities

[4] The full table can be accessed at Hwang et al., 2008, p. 86.

Model	Context- Aware Ubiquitous Learning Examples
Learning in the real world with online guidance	The students learning in the real world and are guided by the system, based on the real-world data collected by the sensors. For example, for the students who takes a chemistry course, hints are provided automatically based on his or her real-world actions during the chemistry procedures.
Learning in the real-world with online support	The students learn in the real world, and support is automatically provided by the system based on the real-world data collected by the sensors. For example, for the student who is learning to identify the types of plants on campus, relevant information concerning the features of each type of plant is provided automatically based on his or her location and the plants around him or her.
Collect data in the real world via observations	The students are asked to collect data by observing objects in the real world and to transfer the data to the server via wireless communications. For example, observe the plants in this area and transfer the data (including the photos you take and your own descriptions of the features of each plant) to the server.
Identification of a real-world object	Students are asked to answer the questions concerning the identification of the real-world objects. For example, what is the name of the insects shown by the teacher?
Observations of the learning environment	Students are asked to answer the questions concerning the observation of the learning environment around them. For example, observe the school garden, and upload the names of all the insects you find.
Cooperative data collecting	A group of students are asked to cooperatively collect data in the real world and discuss their findings with others via mobile devices. For example, Cooperatively draw a map of the school by measuring each area and integrate the collected data.
Cooperative problem solving	The students are asked to cooperatively solve problems in the real world by discussing through mobile devices. For example, search each corner of the school and find the evidence that can be used to determine the degree of air pollution.

However, not all learning need take place in the real world: Mobile devices may be used to complement decontextualized learning of mathematics within the classroom with contextualized learning outside the classroom (Tangney, O'Hanlon, Munnelly, Watson, & Jennings, 2010)

In a recent study, students were using context-aware u-learning to study elementary geometry concepts within the real-world setting. Elisson and Ramberg (2012) used Design-based research to conduct a study where students were asked to complete two activities. In the first activity, students worked with the concept of volume as they were asked to play the role of architects planning for new buildings. In the second activity, the students studied area as they relocated imaginary species from the local zoo to a field close to the school. Both activities required the use of a mobile software application which measures the distance between two mobile devices via Global Positioning System (GPS). For example, in the second activity, students were placed into groups of three and taken to a nearby field. They were asked to estimate the area of two small rectangles marked by plastic cones, then using pre-made cardboard squares the students measured the area and typed this answer into the mobile device.

Once the measure was inputted into the device correctly, a new task was given. This task asked students to estimate the area of the rectangle ($4000m^2$) in a larger field. Next, to measure the rectangle, students stood at either end of the sides and used the GPS measuring application on the mobile devices. Multiplying two sides, the students typed the area of the rectangle into the application. The final part of this task required the students to use the mobile device to construct a coned area of $4000\ m^2$ using the measurement application.

The purpose of the study was to consider guidelines for designing contextual mobile learning activities to ensure that mobile devices enhance and support learning, rather than distract students from the mathematical content to be studied. Results of this study identified the following guidelines.

The design of the mobile devices should:

- Let students assume roles
- Be used by students as contextual tools for measuring or probing the physical environment.

The location-based and contextual mobile learning activities should:

- Be designed for physical interaction with the environment.
- Let teachers assume roles
- Encourage face-to-face collaboration

Learning activities should

- Introduce unfamiliar aspects of the location-based and contextual mobile learning activities before going into the field.

This study used one of many measurement applications for mobile devices which can extend and enhance students understanding of elementary geometry and measurement concepts. Sketchpad Explorer (2012) is a type of DGE which is now available on mobile devices. With this application, specific add-ons allow the students to interact with the real world to take photographs of physical objects in the environment environments. The many tools within the DGEs can be used while the student is still in the same location. However, while these tools are available, as Elisson and Ramberg (2012) reported, there are also considerations that need to be made to ensure the activities are well designed in order to utilize these applications for learning to take place.

Task design. It is clear from the literature review thus far that mobile devices, DGEs, and real-world environments can provide a cocktail of supports for students to effectively learn about angle and angle measure. This section highlights the importance of task design in considering appropriate activities to challenge and extend students' thinking. Next, this section describes ways in which activities can be constructed to successfully incorporate mathematical discussion and to use the guidance provided by van Hiele-Geldof's (1957/ 1984) instructional phases, to think about the way students develop geometrical understandings.

Doyle (1983) described students' work in terms of academic tasks. The nature of the tasks contributes not only to what students learn, but also how they think about, develop, use, and make sense of mathematics (Stein, Grover, & Henningsen, 1996). Doyle defined four categories of academic tasks: memory tasks, procedural or routine tasks, comprehension or understanding tasks, and opinion tasks. He claims that each of these categories varies in terms of the cognitive operations required during each different task. Using this idea of cognitive load, Stein, Smith, Henningsen, and Silver (2000) developed a model which delineates four levels of cognitive tasks: low cognitive demand tasks - memorization, low cognitive demand tasks –procedures without connections, high cognitive demand tasks – procedures with connections, high cognitive demand tasks – doing mathematics.

For example, the least taxing of these, memorization tasks, involve reproducing facts, formulas or definitions from memory without understanding, and doing mathematics, a high cognitive demand task requires effort, exploration, understanding, knowledge and non-algorithmic thinking. To provide students with tasks to deepen and extend their mathematical knowledge, tasks should have three features (Hiebert & Wearne, 1996). First, the tasks should be a challenge to students; leading to higher cognitive demand as the students are required to think and problem solve. Second, the tasks must connect with where the students are at in terms of learning. In other words, they should have prior skills and learning to draw from. Third, the tasks must engage students in thinking about important mathematical ideas and have the students to reflect on these ideas.

Student interaction is an essential component of tasks (Hiebert et al., 1997). Using tasks designed within a context-aware u-learning approach, students can take advantage of the portability, size, and sensory features (e.g., camera and scanners) of mobile devices, to interact easily with peers and the environment. Connectivity is a key feature in learning with mobile devices (Laurillard, 2007; Sharples, Sánchez, Milrad, & Vavoula, 2009), and cooperative, discussion based approaches to learning are well documented as being advantageous in students developing a deep understanding of the mathematical concepts (Richardson, 1999; Van de Walle & Lovin, 2006). DGEs were also highlighted earlier in the literature review for the way in which the programs fostered mathematical discourse. Nonetheless, discussions need to be well planned and purposeful in order for the mathematical ideas to be heard (Nyikos & Hashimoto, 1997; van Hiele, 1957/1984b).

These practices, often described as *academic talk* or *accountable talk*, which highlights discourse as being accountable to the learning community in which participants listen to and build their contributions in response to those of others (Michaels, O'Connor, & Resnick, 2008). Van de Walle and Lovin (2006) proffered that effective discussions include: active-participation, reflective responses, and turn taking. Richardson (1999) and Hiebert et al. (1997) suggested that students should be given time during discussions to reflect on the ideas of others. Although talking is a simple activity for many people, to engage students in effective mathematical discussions can take a lot more practice. To support students in conducting these conversations, Chaplin et al. (2009) devised a set of talk moves which can be used by the teacher to support mathematical thinking. There are five talk moves in total which are listed in Table 2.1

Table 2.1

Talk Moves

Talk Moves	
Move 1	Revoicing. (“So you’re saying that it’s an odd number?”)
Move 2	Repeating: Asking students to restate someone else’s reasoning. (“Can you repeat what he just said in your own words?”)
Move 3	Reasoning: Asking students to apply their own reasoning to someone else’s reasoning. (“Do you agree or disagree and why?”)
Move 4	Adding on: Prompting students for further participation. (“Would someone like to add something more to this?”)
Move 5	Waiting: Using wait time. (“Take your time…. We’ll wait…”)

Table 2.1 Adapted from “Classroom Discussions: Using Math Talk to Help Students Learn” (2nd ed.), by S. H. Chaplin, C. O’Connor, N. Canavan-Anderson, 2009. Sausalito, CA: Math Solutions.

Strategies, such as the talk moves, assist students in participating in academically productive conversations.

Mathematical discussions play an important role in van Hiele-Geldof’s (1957/1984) instructional phases. There are five phases in total, designed to promote learning through each of the van Hiele levels of geometrical thinking (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984). The phases are sequential, although students may go back and forth through the phases as the students encounter new concepts.

Phase 1: *(Inquiry/Information)* During this initial stage, students get acquainted with the geometric concepts as the students engage in conversations and activities about the objects of study. For example, students examine examples and non-examples of angles. Students make observations and questions are raised.

Phase 2: *(Guided orientation)* Students explore the concept through a carefully designed sequence of activities. The activities are designed to slowly reveal particular characteristics of the concept.

Phase 3: *(Explication)* Students have now gained some understanding of the geometric concept from the earlier activities. Technical language will be introduced, and during this phase in the activities, students will be encouraged to express and exchange views about the geometrical phenomena while using the technical language.

Phase 4: *(Free orientation)* Students work on more difficult activities to use the knowledge they have gained in the other phases. They will be asked to select parts of this newly gained knowledge to solve problems, or develop further relationships.

Phase 5: *(Integration)* Activities would involve students summarizing all that they have learned about the subject. Students will be asked to develop a newly organized network of what they understand about the geometric concept.

Context-aware u-learning connects students to real-world phenomena and technological tools, such as DGE, to support learning of angle and angle concepts. However, the design of activities needs to be intentionally developed in a way that will allow students to take advantage of these supports, while also ensuring that they progress in their understanding of angle and angle measure. The van Hiele-Geldof's (1957/1984) instructional phases provide a way in which activities can be structured to extend and enhance students' understanding, while building on prior knowledge.

CHAPTER THREE: DESIGN-BASED RESEARCH

Design-based research (DBR) is a methodology that supports the development of and research concerning a local instruction theory to be used to support students learning concepts in mathematics. DBR is used in this study to address the following research questions:

1. How do students come to understand angle and angle measure?
2. What are effective means of support to facilitate understanding of angle and angle measure?

The DBR methodology is discussed as the theoretical framework to undergird this study. This chapter has two main sections. In the first section, the tenets of DBR are explained. In the second section, the methodology is applied and reflects literature reviewed in chapter two. A conjectured local instruction theory is proposed and a brief summary of instructional activities reflecting the application of this local instruction theory is provided. The testing and revision of this conjectured local instruction theory through the use of the instructional activities is the focus of this dissertation.

Design-Based Research

The terms "design-research" (Oha & Reeves, 2010), "development research" (Conceicao, Sherry, & Gibson, 2004) and "design experiments" (Brown, 1992) have been used interchangeably to describe the DBR methodology. The more current term in the literature is “design-based research” (DBR; Anderson & Shattuck, 2012) and is the term that has been selected for use within this study.

DBR emerged as the practical research methodology to bridge the gap between theories and practice (Anderson & Shattuck, 2012). The methodology permits the researcher to focus on learners’ understanding. The goals of DBR are to develop local instruction theories and to extend theoretical frameworks related to learning mathematics concepts (Gravemeijer & Cobb, 2006).

DBR is “a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings” (Barab & Squire, 2004, p. 2). Anderson and Shattuck (2012) highlighted seven characteristics of this methodology. The research is:

1. *Situated in a real educational contexts:* As the research is conducted in the real educational context, this provides validity to the research, and the results can effectively be used to inform, assess, and improve practice in one (and often other) contexts.
2. *Focuses on the design and testing of a significant intervention:* The intervention is one that can be used in other classrooms, by teachers with students, and is not simply an intervention to be used for experimental purposes. The design of the intervention is a key feature in DBR.
3. *Uses mixed methods:* DBR typically involves a mixed methods approach. As Maxcy notes, it is logical for researchers to select and use different methods, chosen as needed (Maxcy, 2003, p. 59).

4. *Involves multiple iterations:* The implementation of design-based interventions involves the testing of prototypes, through iterative refinement, and evolution of the design tested in authentic practice.
5. *Involves a collaborative partnership between researchers and practitioners:* In DBR the teacher and the researcher work collaboratively. The partnership recognizes that the teachers are often may be ill equipped to conduct rigorous research and also have limited time is to do so. The researcher may not understand the complexities of the classroom culture and the politics of the specific educational system to effectively create and measure the intervention. A collaborative partnership supports joint understanding of the instructional implications.
6. *Involves the evolution of design principles:* The methodology leads to the development of practical design principles, patterns, and grounded theorizing. The design principles reflect the conditions in which they operate and provide tools and conceptual models to help understand and adjust the intervention to maximize learning.
7. *Provides practical impact on practice:* Anderson and Shattuck (2012) noted that research is often disconnected from practice. Often research that seeks to advance theory but does not demonstrate the value of the design by creating an impact on learning in the local context of study does not adequately justify the value of the proposed theory (Barab & Squire, 2004, p. 6). Effective DBR has a direct impact on the theory and the practice.

Local instruction theory. One of the primary aims of DBR is the development of a local instruction theory (Gravemeijer & Cobb, 2006). Gravemeijer's (1999, 2004) construct of a *local instruction theory* is developed within the context of DBR, and describes a frame of reference for designing and engaging students in a set of exemplary instructional activities which support students' learning of a particular mathematical concept (Nickerson & Whitacre, 2010). In the DBR process, initially, a *conjectured local instructional theory* is developed from empirical evidence (i.e., literature review) and proposed theories of learning and pedagogy addressing a particular mathematical domain.

Through the process of DBR, a conjectured local instruction theory is modified and strengthened. Analysis is ongoing and the implementation of instructional interventions provides information about how students are, or are not, learning and the methods by which learning is made possible (Gravemeijer & van Eerde, 2009). The information collected from an instructional experiment contributes to the revision of the conjectured local instructional theory (through a thought experiment), and results in potential revision of the instructional sequence and the subsequent instructional experiment (Markworth, 2010).

In DBR, the identification of a local instruction theory occurs in the first phase of the research and is then revised throughout the research process and provides a framework of analysis (Markworth, 2010). This revision begins during the micro cycle process. Figure 3.1 provides a graphical representation of the micro cycle process. For example, during the course of a two-week instructional cycle, mini cycles occur approximately ten times during an instructional sequence, which is referred to as a teaching experiment.

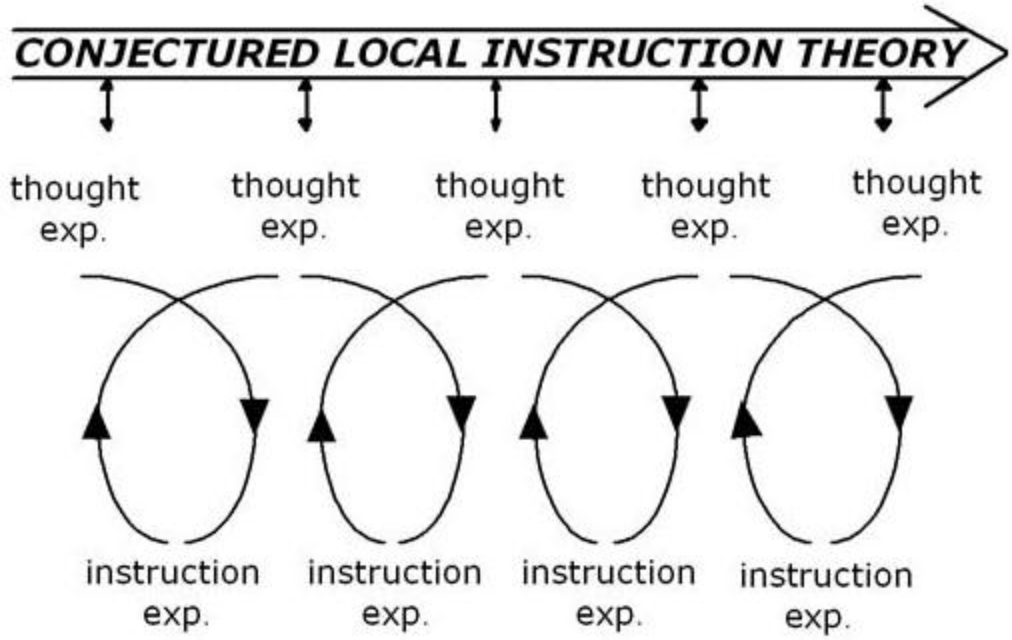

Figure 3.1. Reflexive Relation between Theory and Experiments

Figure 3.1. Adapted from "Design Research from a learning Design Perspective" by Gravemeijer and Cobb, 2006. In K. Gravemeijer. J. van den Akker, S. McKenney, & N. Nieveen (Eds.), *Educational design research*, London: Routledge, pp. 17-51.

The micro cycles comprise the long term macro cycle. For example, a ten day instructional sequence, when completed, is a macro cycle, which is followed by a second macro cycle as shown in Figure 3.2. The second macro cycle consists of the implementation of the revised instructional sequences based on the revisions to the conjectured local instruction theory.

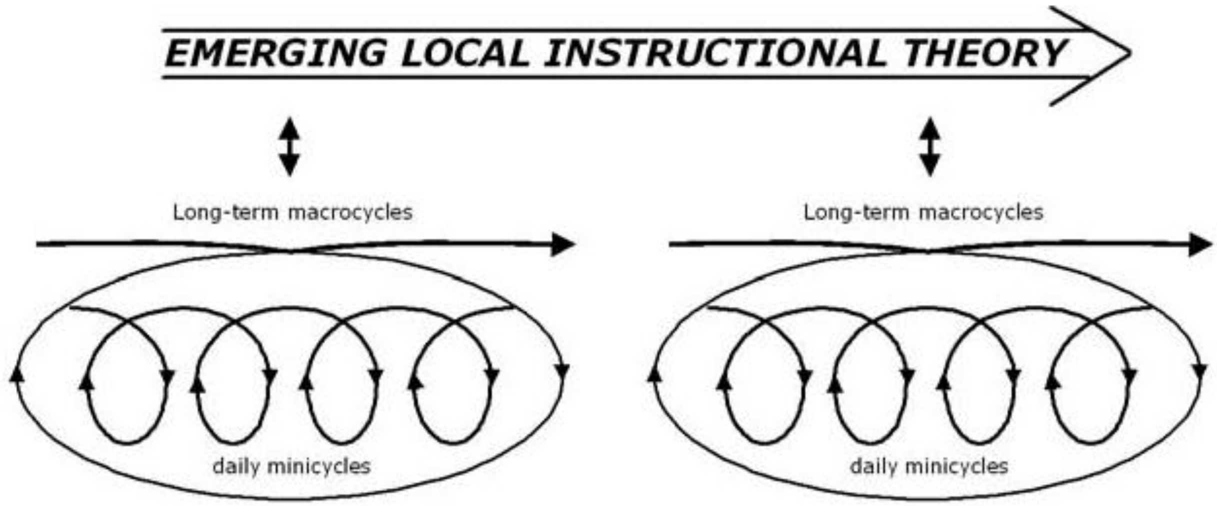

Figure 3.2. The Micro and Macro Cycles

Figure 3.2.. Adapted from "Design Research from a learning Design Perspective" by Gravemeijer and Cobb, 2006. In K. Gravemeijer. J. van den Akker, S. McKenney, & N. Nieveen (Eds.), *Educational design research*, London: Routledge, pp. 17-51.

There are three phases conducted within DBR (Gravemeijer & van Eerde, 2009; Shavelson, Phillips, Towne, & Feuer, 2003; Simon, 1995). Gravemeijer and van Eerde (2009) described DBR as a cyclical iterative process of anticipation, enactment, and evaluation. The anticipation is the development of the conjectured local instruction theory and the design of the instructional activities, the second component is the teaching experiment with the daily mini cycle analysis, and the third component involves the retrospective analysis (the reflection on the macro cycle). Iterations of these three components make up the macro cycle, which underpins the emerging conjectured local instruction theory.

Theoretical Framework: Conjectured Local Instruction Theory

The purpose of this section is to articulate a conjectured local instruction theory about students' development of angle and angle measure through the use of context-aware ubiquitous learning tasks. The conjectured local instruction theory was framed as a result of the literature review. This framework is the initial conjecture of a local instruction theory about a how students develop their knowledge of the concept of angle and angle measure. Based on this theory, a proposed set of tasks and anticipated students' responses are developed. Throughout the study, the conjectures can be refuted and alternative conjectures developed and tested (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

Gravemeijer and Cobb (2006) described the conjectured local instruction theory consisting of a learning process and a means for supporting that process. The literature review identified a number of different frameworks to use as lenses for the way in which students develop an understanding of angle and angle measure. In particular, the van Hiele levels utilized by Scally (1990) provided a set of level indicators that encompass both angle and angle measure. Mitchelmore and colleagues (viz., Mitchelmore, 1989, 1993, 1998; Mitchelmore & White, 2000, 2004, 2007) provided a focus on angle abstraction and generalization, and Piaget and Inhelder (1948/1967) offered a thesis on spatial reasoning in relation to angle concepts.

What also emerges from the review of the literature is the importance of context-aware u-learning tasks using real-world connections and applied technology learning tasks to support students' understanding of angle concepts. Through the use of real-world connections students can mathematize intrinsic environmental and physical experiences (Clements et al., 1996), determine relevant attributes of angles from those that are irrelevant (Clements & Battista, 1992; Munier et al., 2008), make abstract angles comprehensible, and the standard angle generalizable (Mitchelmore & White, 2000). Dynamic Geometry Environments provide effective supports to aid students in learning angle concepts while avoiding the potential difficulties and misconceptions. The dynamic attributes of computer programs also allow students to see angle measures as turns (Clements & Burns, 2000), enabling students to uncover the salient geometric attributes of angle to take into account the abstract and the generalized concept of angle (Clements & Battista, 1994; Zbiek et al., 2007).

In preparation for the classroom design experiment, Gravemeijer and Cobb (2006) described how goals must first be selected based on history, tradition, and assessment practices, then those goals must be problematized to consider the essential understandings for the mathematical topic. During the review of the literature, it appears that students are developmentally ready to learn about angle concepts by fourth grade. Curricular expectations were reviewed (CCSSO/NGA, 2010; NCTM, 2006); angle instruction typically begins in fourth grade and trajectories for this instruction appear to be well aligned to the research.

However, the CCSSO/NGA (2010) suggests students be taught the static definition of angle in the fourth grade while also introducing angle measures. This is problematic as theoretical and empirical evidence indicates that static definitions inhibit students thinking in regard to understanding angle measure and other angle concepts. Therefore, aligned to the research, the students in this study are supported in develop their own definition of angle based on angle as a turn. The goals determined for this study will be based on the essential understandings highlighted in the literature review.

The goal of the instructional intervention was to develop an empirically-substantiated instructional theory about students' development of angle and angle measure, with real-world connections and technological tools through the use of context-aware u-learning. The essential understandings identified in the literature review are[5]:

1. Recognize angles as geometric shapes that are formed whenever two rays share a common endpoint.
2. Understand that angles can be identified in a real world setting.
3. Recognize that there are an infinite number of angles.
4. Recognize and compare angles based on size using non-standard and standard language (acute, obtuse and right angles).
5. Recognize acute, obtuse, and right angles in different contexts (real-world and paper and pencil).
6. Recognize acute, obtuse, and right angles in different orientations and with rays of different lengths.
7. Recognize salient attributes of angles, such as two rays with a common endpoint.
8. Understand that angles can be measured with reference to a circle and that angles are fractions of a circle.
9. Understand that angles are measured by units called degrees.

[5] Although numbered, this is not to connote a hierarchy or developmental progression. It is conjectured that students may develop some understandings before others, which may be different than the progression of another student.

10. Understand that benchmarks can be used to understand angle measures. For example, a full circle turn is 360°, straight angle is 180°, and right angle is 90°.
11. Recognize that the same angle can appear to be a different size depending on different visual perspectives.
12. Understand that angles are defined by particular attributes which involve angle as a turn (e.g., "two rays, the common endpoint, rotation of one ray to the other around that endpoint, and measure of that rotation"; (Clements & Sarama, 2009, p. 186).

Instructional materials. A sequence of six lessons was designed for use in fourth grade classrooms. The lessons involve seven class periods; five lasting approximately 60 minutes long, and lesson three taking 120 minutes. An overview of the instructional sequence is provided in Table 3.1. The table includes the learning progression and the instructional activity. This is followed by a more detailed description of each of the lessons. However, the full lesson plans can be found as Appendix B.

This instructional sequence is comprised of seven lessons that utilize van Hiele-Geldof's (1957/1984) five phases of geometric instruction: 1) inquiry/information, 2) guided orientation, 3) explication, 4) free orientation, and 5) integration. The phases are described in chapter two. The progression of these phases is tied to the mathematical concepts. Therefore, the phases follow a somewhat linear path beginning at the initial inquiry phase as the students begin to explore the angle concept, but the activities move back and forth between the stages during the lessons. These lessons have been influenced by the format of Van de Walle and Lovin's (2006) three part format for problem-based lessons, and discussion has a critical role in the lessons which also include Chaplin, O'Connor, and Canavan-Anderson’s (2009) Talk Moves described in the literature review.

Table 3.1

Overview of the Instructional Sequence

Lesson	Learning Progression	Instructional Phases (van Hiele-Geldof, 1957/1984)	Instructional Activity
1	Recognize angles as geometric shapes that are formed whenever two rays share a common endpoint. Identify angles in a real-world setting.	• Initial Inquiry • Direct Orientation	Students are introduced to the concept of angle via projected images of different examples of angles in different orientations with sides of different lengths. The term angle is introduced. Students look for angles in the real-world.
2	Identify angles in a real-world setting.	• Explication	Students are introduced to the application Sketchpad Explorer and taught how to use the DGEs to take

	Recognize that there are an infinite number of angles.		photographs and how to use the dynamic protractor. Students take photographs of angles in a real-world setting disregarding orientation and length of rays. Students will use the tools in the DGEs to highlight the angles found.
3 & 4	Recognize and compare angles based on size using non-standard and standard language (right, obtuse, acute, and straight angles).	• Guided orientation • Explication	Students will work in groups making angles with straws and compare size of those angles using non-standard language. Introduced to the terms: right, obtuse, acute, and straight angles. Using the benchmark of 90° on the dynamic protractor, students find examples of right, obtuse, acute, and straight angles in a real-world environment. An angle gallery will be created from the screenshots. Students will work in pairs to discuss the categorization of an angle in the real-world and check their accuracy

			using QR codes.
5	Understand that angles can be measured with reference to a circle and that angles are fractions of a circle. Understand that angles are measured by units called degrees. Understand that benchmarks can be made for angle measures. For example, a full circle turn is 360°, therefore a straight angle is 180° and a right angle is 90°.	• Explication • Free orientation • Integration	Wedge activity to create benchmarks. Using the wedges to measure a set of materials such as a coat hanger, books, scissors, and a car ramp, noting that the latter two can be changed to vary angle size.
6	Recognize acute, obtuse, right, and straight angles in different	• Guided orientation • Explication	Students work in groups to identify and categorize right, acute and obtuse angles in paper and pencil and real-

	contexts (real-world and paper and pencil. Recognize salient attributes of angle.	• Free orientation • Integration	world contexts. Angle walk to identify angles in different settings. Class discussion to determine salient attributes of angles.
7	Recognize that the same angle can appear to be a different size depending on different visual perspectives (positions). Understand that angles are defined by particular attributes which involve angle as a turn (e.g., “two rays, the common endpoint, the rotation of one ray to the other around that endpoint, and measure of that rotation”; Clements and	• Free orientation • Integration	Students work in pairs to photograph and measure angles from different perspectives. Work in groups to create a poster to define angle to students who have not yet studied angle.

	Sarama, 2009, p.186).

The content and structure of the lessons.

A design-based researcher resembles a *bricoleur*, a French term to denote an experienced tinker/handy person who uses the materials that happen to be available (Gravemeijer & Cobb, 2006). Therefore, resources such as mathematical curricula and texts are adapted to construe an instructional sequence, with the selections and adaptations guided by the conjectured domain specific instruction theory (Gravemeijer, 1994; Gravemeijer & Cobb, 2006). This instructional sequence employed this *theory-guided bricolage* (Gravemeijer, 1994) approach with curricula adapted where possible. However, as context-aware u-learning is a relatively new field of learning, many of the activities were designed for this study.

Lesson One. This is the initial inquiry phase where student become acquainted with angle. The goal of this lesson was for students to recognize angles as geometric shapes that are formed whenever two rays share a common endpoint, and to begin to identify angles in a real-world setting. It was conjectured that the students were working within the van Hiele-Geldof (1957/1984) initial inquiry phase where student become acquainted with angle, then move into the direct orientation phase as the students explore the topic of angle though finding and discussing angles in a real-world setting. Furthermore, the activities were gradually revealing the geometric concepts of angle as they were designed to have the students begin considering the salient attributes of angle.

Students were introduced to the concept of angle via projected images of different examples of angles. The angles were intentionally portrayed in different orientations with sides of different lengths, to avoid the misconception that orientation and length of sides are salient attributes of angle. Students will initially work in pairs to describe what they can visually observe from the figures (e.g., lines, a point, and two lines in different directions). This was followed by whole group discussion to determine the similarities of the figures. The students' language was recorded and used to determine what an angle is. The term angle was formally introduced at this time.

Students went out into the area surrounding the school to identify angles in the real-world setting. Some difficulties were to be expected as students could have struggled to see angles in a different context. The teacher supported the students in pointing out some examples and non-examples to discuss with the class; the angles were chosen of various sizes and orientations. In addition, students were given cardboard tubes to use as a viewer to minimize the amount of visual information being processed while the students were searching for angles. For the final phase of the lesson, the students returned to the classroom for a discussion on what they found out about angle. The objective of the discussion was to determine if students can identify what an angle looks like using non-formal language, and if students could identify angles in a real-world setting connecting that angle attributes identified earlier in the lesson to the angles identified.

Lesson Two. The goal of this lesson was for students to find angles in a real-world setting, and recognize that there are an infinite number of angles. It was conjectured that the students would be involved in the van Hiele-Geldof (1957/1984) explication phase as students began to become conscious of the relationships of angles as geometrical shapes and began to express those ideas as words. Each student was given an iPad2, with Sketchpad Explorer loaded onto the device with the add-on sketch titled Measure a Picture (Steketee & Crompton, 2012). At the beginning of Lesson Two, the students were introduced to Sketchpad Explorer, and taught how to use the DGEs to take photographs and how to use the dynamic protractor. Students were also taught how to take screenshots to save their work to the device. The viewfinder of the camera minimized the viewing area, similar to the effect of cardboard tube from the day before. Students practiced using Sketchpad Explorer to take photographs of angles in class. The teacher demonstrated getting into position to take photographs of the angles from a direct front view. Later in the sequence, students took photographs from different angles.

Students went back outside, to the area surrounding the school, and were asked to work in pairs to take photographs of angles using Sketchpad Explorer. As the students found angles, they were asked to use the protractor to place against the angle to identify the different angles found in the one picture. Students focused on one angle or multiple, and they worked with a partner to initially confirm with each other that they have found an angle based on the discussion from the day before. Then the students continue to work in pairs to study the differences or similarities between the angles they have found. For the final part of the lesson, students came back to the classroom to share screenshots with the rest of the class via a projected screen. Probing questions started leading to the conclusion that there are an infinite number of angles.

Lesson Three. The goals of this double lesson were that students recognize and compare angles based on size using non-standard and standard language (right, obtuse, acute, and straight angles). During this lesson, it was conjectured that the students were involved in the van Hiele-Geldof (1957/1984) guided orientation phase involved in looking for relationships, and the explication phase as new terminology was introduced. This lesson started by having the students recap on what they have learned over the last couple of lessons. The teacher facilitated a discussion to cover the essential points to ensure understanding. Next, students will worked in groups of about four to make angles from different lengths of straws. The different straw lengths worked toward avoiding students' misconceptions of the length of the rays being salient angle attributes. To create the angles the students placed the straws with one end of each straw touching.

Students compared the angles they made with straws, to the angles the other students made in the group. To avoid having students consider orientation, they were specifically asked to think about the dynamic protractor and the movement of the turning sides and think about the difference in angle size. The movement of the dynamic protractor was displayed on a projected screen for the class to observe. The teacher refrained at this time from explaining any further details about angle size, or using any further measurement terms, beyond the description of the turning sides and the words angle size.

As the students worked in groups to categorize the angles, they were required to share some of the findings with the class. Diagrams and notes on poster paper supported students in explaining what they had found. The teacher guided the discussion to finally introduce the concepts right, straight, acute, and obtuse angle. The words were posted on the classroom wall with various examples. Students were told that a right angle is 90° and this was displayed on the dynamic protractor. At this time, the full meaning of measure was not described in any further detail. Working in pairs, students used the iPads to take photographs of angles in the real-world using Sketchpad Explorer. Students worked in pairs to find examples of right, straight, acute, and obtuse angles. These angles were identified with the dynamic protractor. Students used the screenshots on the iPads to create a gallery walk for students to look at other examples. Students then asked questions to other students during this time.

The final activity had students consider their understanding of right, straight, acute, and obtuse angles. Various angles in a real-world environment were indicated by using colored tape. Students worked in pairs to discuss each angle to determine the categorization. Next to each of the angles was a QR code and the students scanned the codes to see if they were correct. The codes also took the students to a website to find further examples and learn more about the categorization.

Lesson Four. The goals of this lesson were for students to recognize acute, obtuse, right, and straight angles in different contexts and determine the salient attributes of angle. It was conjectured that the activities in Lesson Six would involve a number of different van Hiele-Geldof (1957/1984) instructional phases, including guided orientation, explication, free orientation and integration. The lesson began with a brief recap on the prior lessons, this was conducted by using a photograph of a house and the teacher used talk moves to determine what the students do or do not understand.

For the main activity, students worked in groups of 4 or 5, and each group of students were given a set of cards with a selection of pencil drawn angles. Angles had various orientation and ray lengths. Students had to sort the angle cards into categories of acute, obtuse, right, and straight angles. Some cards were non-examples which were placed in the non-angle category. Students were encouraged to use mathematical discussions to determine which group each angle should be place in. The final closing activity involved a class discussion on salient and non-salient attributes of angles. From this discussion a chart was developed and posted on the classroom wall.

Lesson Five. The goals of this lesson were for students to understand that angles can be measured with reference to a circle, and that angles are fractions of a circle. Students were told that angles are measure by units called degrees and that benchmarks can be used to assist in recognizing approximate angle measures. As the students completed the activities in this lesson, it was conjectured that the students would be involved in the van Hiele-Geldof (1957/ 1984) explication phase as they learn new terminology, but also move into the free orientation and the integration phase.

To begin, students were asked various questions to think about angle measure. The dynamic protractor was used to demonstrate the angle enlarging to 360° with the angle creating a full circle. The main component of this lesson used an adapted version of Browning et al.'s (2007) and Millsaps' (2012) wedge activity. Students worked with paper circle of different sizes to create benchmarks. For example, a full circle turn is 360°, therefore a straight angle is 180° and a right angle is 90°. Students used the wedges to measure angles on a worksheet, then moved on to a set of materials such as a coat hanger, books, scissors, and a car ramp.

The measures were determined using the benchmarks to decide an approximate measure and if it is an acute, right, obtuse, or straight angle. Students used reasoning skills as they considered an approximate measure in degrees. Finally, the class had a discussion on the measurement activities. During this discussion, students demonstrated to the class the various strategies they used and their thinking behind those strategies.

Lesson Six. This was the final lesson in the instructional sequence. The two main goals of this lesson were for students to recognize that the same angle can appear to be a different size depending on different visual perspectives (positions), and to understand that angles are defined by particular attributes which involve angle as a turn (e.g., "two rays, the common endpoint, the rotation of one ray to the other around that endpoint, and measure of that rotation"; Clements and Sarama, 2009, p.186). It was conjectured that the first objective would involve the free orientation van Hiele-Geldof (1957/1984) phase and the final activity would involve the integration phase. The initial component of this lesson required the students to consider angle measurement and following a class question and recap session the students were given the opportunity to look at the dynamic protractor and consider how it moved and the size of angles.

Spatial perception plays an important role in geometry, and photography provides an excellent example of how angles can appear different depending on where the photographer stands. It was made very clear to the students that the actual angle does not change; however, the angle can appear to be a different size depending on the spatial perspective the photographer has of that angle. For the main activity, students worked in pairs to create two different screenshots of the same angle, but from different perspectives. Students used the dynamic protractor to measure the two different angle perspectives, and the students were challenged to find the greatest difference in angle size. Students had to determine the difference in degrees by using simple calculations.

For the final part of this series of lesson, students worked together in groups of four or five to create a poster to explain angle and angle measure. The students were informed that they were creating the poster to explain angle to other fourth grade students who have not yet studied angle. The students were first directed to create a list of what should be included on the poster, then once the lists had been checked by a teacher/researcher they were to begin the poster. The teacher moved around the room posing questions to extend students 'thinking and provide support where necessary.

In this chapter, the tenets of DBR were explained. DBR was then applied to the literature reviewed in chapter two. A conjectured local instruction theory was proposed with a description of the instructional activities reflecting the application of the local instruction theory. The full detailed lesson plans can be found in Appendix B. In summary, context-aware u-learning was identified as a means through which students could learn about angle and angle measure supported by real-world connections and technological tools. Seven lessons were developed to connect the conjectured local instructional theory to activities based on van Hiele-Geldof's (1957/1984) five phases of geometric instruction. The next chapter describes the participants involved in the study, the DBR protocol and the methods used for data collection and analysis.

CHAPTER FOUR: METHODOLOGY

Design-based research (DBR) is a systematic yet flexible methodology utilizing an iterative cyclical process of design, implementation, analysis, and revision. The purpose of this particular DBR methodology is to develop a local instruction theory that details the process by which students learn a particular concept in mathematics (Gravemeijer & van Eerde, 2009). The central tenets of DBR are delineated in chapter three. DBR is a methodology designed for use in real-world settings and involves a collaborative partnership between researchers and practitioners (Anderson & Shattuck, 2012). This methodology is used in this study to address the following research questions:

1. How do students come to understand angle and angle measure?
2. What are effective means of support to facilitate understanding of angle and angle measure?

This chapter is composed of three sections. First, those involved in the study are described, including the participants and the research team. Next, the DBR protocol is detailed. Finally, the methods used in the data collection and analysis are described in full.

Participants

The protocol for this research study involved two macro cycles with two teaching experiments. The two teaching experiments were carried out, one each with a class of fourth grade students. There were 30 students in each class, for a total of 60 student participants in the study. Eight of the 60 students completed the pre and post instruction clinical interviews.

The eight students were made up of four randomly selected students from each class. This particular grade was chosen as the Common Core Standards require teachers to formally begin teaching angle concepts at fourth grade. In addition, empirical evidence indicates that fourth grade students are developmentally ready to learn about angle concepts (Lehrer et al., 1998; Olson, 1970), and studies of this concept should begin during the elementary years (Clements, 2004). The study was conducted at the beginning of the school year, when it was anticipated that the fourth grade students would have little prior experience with angle or angle measure.

Two teachers were selected to participate in the study. There were three teachers in total for that grade level. Two of the teachers each had over six years' experience and they were selected for the study. The third teacher was a first year teacher who chose not to be included in the study. Once the two teacher participants were determined, the fourth grade students taught by those teachers were recruited for participation in the data collection procedures. Recruitment scripts and the consent/assent forms were preapproved by the University of North Carolina at Chapel Hill Institutional Review Board (IRB) can be found in Appendix C. The class teachers and the fourth grade students were recruited from Phillips School in Walker County[6]. This district was chosen for three reasons: (a) it does not follow a restrictive pacing guide, (b) it is more flexible in allowing the incorporation of alternative instructional sequences, and (c) the district staff were willing to have the researcher carry out instruction in fourth grade classrooms.

[6] All names have been changed to pseudonyms to protect participants' identities.

The school was chosen because they have access to a full set of iPad 2s, enough to equip a class of up to 30 students. Both students and teachers were familiar with the basic operation of the iPad 2 and did not need any further instruction beyond the use of the new applications, which was utilized in the design experiment. The Technology Coach at the school provided lessons on how to operate the iPads, such as searching for apps and taking screenshots.

Research Team

The researcher acted as the teacher in both of the teaching experiments. In the DBR process it is not uncommon for one researcher to serve as the teacher implementing the instructional intervention (e.g., Cummings-Smith, 2010; Markworth, 2010). For both teaching experiments, the class teacher served as a witness to the teaching episodes, and another mathematics PhD student and prior educator acted as co-researcher.

Design-Based Research Protocol for this Study

The specific DBR selected for this study was developed by Gravemeijer and colleagues (Gravemeijer, 1994; Gravemeijer & Cobb, 2006; Gravemeijer & van Eerde, 2009) to connect directly with mathematics education. This form of DBR has been used in mathematical research methodologies within the K-12 environment (e.g., Markworth, 2010). This specific DBR methodology was delineated in chapter three of this study. The study involves two macro cycles with one teaching experiment occurring in each macro cycle. The teaching experiments consisted of seven days of mini cycles of thought and instruction experiments to serve the development of the local instruction theory. The macro cycles for this study are illustrated in Figure 4.1. Note the occurrence of the three phases within each macro cycle: the design of instructional materials, classroom-based teaching experiments and mini cycle analysis, and the retrospective analysis of the teaching experiments which informed the next macro cycle.

One day prior to the commencement of the teaching experiment, the clinical interview was administered to the four students from the first class. Next, using the instructional materials described in chapter three, the first teaching experiment was conducted in early fall, for seven consecutive school days. During the teaching experiments, the co-researcher and witness observed and took notes on all classroom instruction, and the instruction was videotaped. Students' work was collected at the end of each day. Also, at the end of the day's instruction, the researcher, co-researcher, and witness met to discuss the lesson. The conversations were audio recorded. Following this meeting, the researcher completed a daily reflection journal, recording impressions, feelings and thoughts for each of the teaching episodes during each mini cycle.

During each daily mini cycle during a teaching experiment, the researcher utilized the collected data to modify the next day's instruction when necessary. The second teaching experiment took place two weeks after the conclusion of the first teaching experiment. There were two retrospective analyses conducted, one at the conclusion of each macro cycle. The local instruction theory came from the final retrospective analysis.

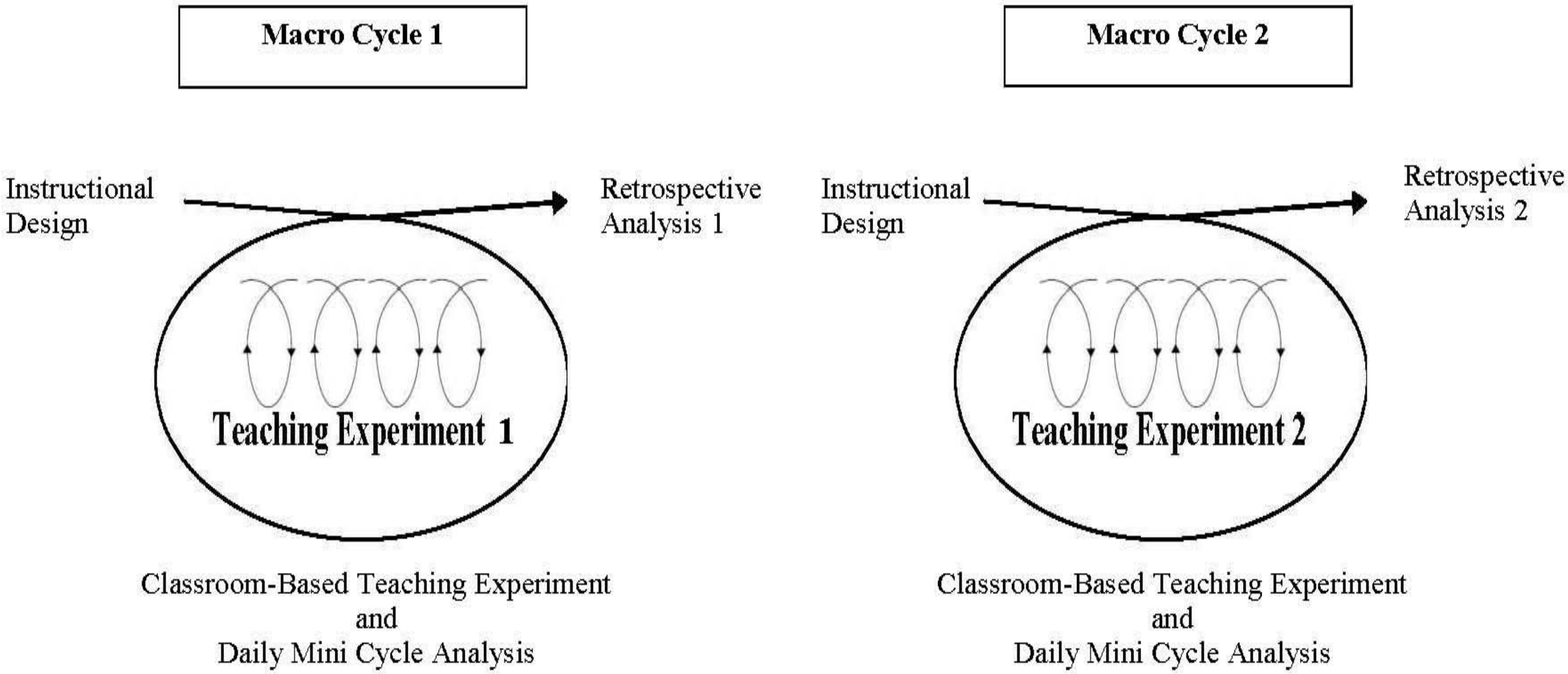

Figure 4.1. A Diagrammatic Representation of the Study.

Data Collection and Analysis

One of the distinct characteristics of DBR methodology is that the researchers develop a deeper understanding of the phenomenon while the research is in progress. Therefore, it is crucial that the research team generated a comprehensive record of the entire process (Cobb et al., 2003). There were several sources of data that were used in this DBR process. This section includes details of the purpose, design, and collection procedures for each of these data. These data sources are:

- a pre and post instruction clinical interview
- co-researcher and witness classroom observations
- whole class video recording
- daily mini cycle reflection audio-recording with research team
- artifact collection of student classwork
- researcher's daily reflection journal
- retrospective analysis at the end of a macro-cycle

These data sources served various purposes and are utilized at various points during both the daily mini cycle analysis and the retrospective analysis phases at the end of each macro cycle. Table 4.1 illustrates the points at which the information from these data was used.

Table 4.1

Data Sources and when these Data were Analyzed

	Select Students for Interviews	Daily Mini Cycle Analysis	Retrospective Analysis 1 Macro Cycle 1	Retrospective Analysis 2 Macro Cycle 2
Pre instruction Clinical Interview	✓		✓	✓
Post instruction Clinical Interview			✓	✓
Co-Researcher and Witness Classroom Observations		✓	✓	✓
Whole-class and Small Group Video			✓	✓
Daily Mini Cycle Reflection		✓	✓	✓
Artifact Collection			✓	✓
Researcher Reflection Journal		✓	✓	✓

Pre and post instruction clinical interviews. Van Hiele (1957/1984a) believed that students' levels of geometric thought are achieved largely as a result of effective geometry instruction. In this study, the pre and post instruction assessments are clinical interviews based on the van Hiele levels of geometrical thinking. The interviews determined students' initial understanding of angles and overall growth following instruction. Clinical interviews were chosen for this particular study as this method of data collection allowed the researcher flexibility in pursuing comments made by the student (Ginsburg, 1981), and can be used as a method for eliciting and recording naturalistic forms of thinking in mathematics (Clement, 2000).

Scally’s (1990) clinical interview allowed the investigator to react responsively to data, asking new questions in order to clarify and extend student thinking. In addition, the interviews permitted the researcher to gain insight into the depth of student understanding with a collection of both oral and graphical explanations. The credibility of Scally’s clinical interview has been determined with 83% reliability and the content validity of the instrument established. Furthermore, Scally’s (1990) study provided evidence for her to claim that the instruments and scoring procedures could be used effectively by other researchers and in other settings. The design underpinning Scally’s interviews is threefold: the discovery of cognitive activities (structures, processes, and thought patterns), the identification of cognitive activities, and the evaluation of levels of competence (Ginsburg, 1981), which is similar to the framework adopted by Piaget.

Adopting the first three levels of the van Hiele’s model of geometric thinking (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984), Scally (1990) developed a set of level indicators that focused specifically on angle concepts. The level indicators are visualization, analysis, and informal deduction.

- *Level 1 (visualization):* In general, the student identifies, characterizes, and operates on angles according to their appearance.

- *Level 2 (analysis):* In general, the student establishes properties of angles and uses properties to solve problems.
- *Level 3 (informal deduction):* In general, the students formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments.

Each of these levels came with a list of level indicators that were used to assess the angle understanding of the fourth grade students in the study. These level indicators can be found as Appendix A.

The clinical interview is made up of six angle activities: drawing angles, identifying and defining angles, sorting angles, measuring angles, determining the relationship between angles, and deducing angles. The clinical interview has a scoring guide which correlates to these three levels. For each activity there can be multiple parts and a full script is provided. However, not all the activities and questions needed to be used in the interview (Scally, 1990). For example, if students were struggling with many of the early tasks, activity six could be omitted as it is considerably more difficult than the other tasks. For the purpose of this study, activities 5c and 6 were omitted as they cover concepts such as parallelism that would not have been formally taught prior to the study. The activity descriptions and the scripts can be found in Appendix D.

The same clinical interview was used for both the pre instruction and post instruction interview. The pre instruction interview was administered to the four selected participants one day before the teaching experiment began, and the post instruction interview administered one day following the conclusion of the teaching experiment. The interviews were administered and scored by the primary researcher who conducted a number of pilot interviews with fourth grade students in preparation for this study. The interviews lasted for approximately 30 minutes, although there were no temporal restraints on this procedure.

Following the pre and post instruction interviews, Scally's (1990) scoring criteria was used to determine the van Hiele level at which each student was working. Interviewer notes and scoring criteria can be found in Appendix E. The audio transcripts of the interviews were analyzed to determine whether the student exhibited behaviors characteristic of the van Hiele level descriptors assigned to the interview tasks. Each student's performance was then compared across the two interviews. The results of this analysis are reported in narratives for each student and summarized in the tables of van Hiele level performance. There were two tables completed for each student. Table F.1 recorded the students' van Hiele level behavior indicators during the six activities. The van Hiele indicators were labeled with numbers which correspond to the leveled scoring criteria. There are 19 van Hiele behavior indicators in total and a copy of the levels and this table can be found in Appendix F. Table G.1. was used to record the van Hiele levels at each interview, so a comparison can be made between the pre and post instruction levels. This table can be found in Appendix G. Aligned with Scally's interview protocol, Table F.1, Table G.1., and the narrative were used together to interpret students' progress during the study.

All interview paperwork was coded to identify the participants and names were avoided during the interview. If a participant's name was used during the interview, the name was swapped for the participant code on the transcripts. The interviews were audio recorded and transcribed. The transcripts were used during the reflective analysis at the end of each macro cycle.

Co-researcher and witness classroom observations. While the researcher was conducting the teaching experiment, the respective classroom teacher and the mathematics education colleague acted as witnesses to the process. They observed the class and took notes during each of the teaching experiments. Furthermore, they were participant observers, interacting with students and assisting in whole group and small group instruction. This participation involved answering questions and posing questions to further students' thinking. The observation notes were collected at the end of each day by the researcher.

Whole class and small group video. Each teaching episode was video recorded to capture both the instruction and student participation. The camera was situated on a tripod to obtain a good overall view of the teaching. As the students worked in small groups, a video camera was positioned to focus on the group of four students who conducted the pre and post instructional interviews. Activities performed outside the classroom were recorded with the video camera. For example, during a particularly interesting small group debate, the video camera was situated in a position to capture this discussion. The video recordings were downloaded at the end of each day and transcribed. The transcripts were coded using Scally's (1990) van Hiele level indicators**.**

Daily mini cycle reflection. Following each of the seven teaching episodes, the researcher, co-researcher and teacher meet to discuss the instructional activities of that day and student progress in understanding the angle concepts taught. The sessions were audio recorded and transcribed. Cobb et al. (2003) recommended having these conversations and making audio recordings as a method of documenting the evolving conjectures, and to reflect on these data together with the observations of the teaching episodes that may support or question the conjectures made.

Artifact collection. Hard copies of students' work were collected at the end of each teaching episode. Students' work had the identifiers removed and participant codes were attached to each piece of work for identification. Photocopies were then made for further analysis. In addition, screen captures were taken of students' work on the iPads and downloaded at the end of each day. Screen captures are images taken by the iPad to record the visible items displayed on the device. For example, if a student took a photograph and used tools in the DGEs to highlight or measure angles on the photograph, a snapshot of this image was recorded and saved for later analysis. Participant identification codes were included in the file names of the screen captures. The students work was coded using Scally's (1990) van Hiele level indicators.

Researcher reflection journal. The primary researcher completed a personal reflection journal for each of the teaching episodes during each mini cycle. The journal is an instrument that allows the researcher to step back from the action to record impressions, feelings, and thoughts (Holly, 2002), and within the context of DBR, future plans can also be recorded. This form of data collection provides a medium for thinking aloud and is a reflective tool for "trying out ideas for action and assessing their implication, and evaluating the effectiveness of attempts to introduce changes" Holly, 2002, p. v). The researcher reflection journal completed during each mini cycle was a catalyst for change during the teaching experiment and the retrospective analysis.

Retrospective analysis. During this study, there were two retrospective analyses, one after each teaching experiment. Although this particular phase considers all the data collected to that point in time (e.g., video, discussions, interviews), this phase generates a new synthesized set of data. In other words, the entire data during the macro cycle was studied collectively, to look for “patterns in the data, framing assumed patterns as conjectures about the data, testing those conjectures on the complete data set, and using the findings as data for a subsequent round of analysis” (Gravemeijer & van Eerde, 2009, p. 517). The data from the first retrospective analysis was used for the next macro cycle, and the data from the final retrospective analysis was used to create a more robust local instructional theory. Figure 4.2 indicates when each of these data were collected using the diagrammatic representation of the study.

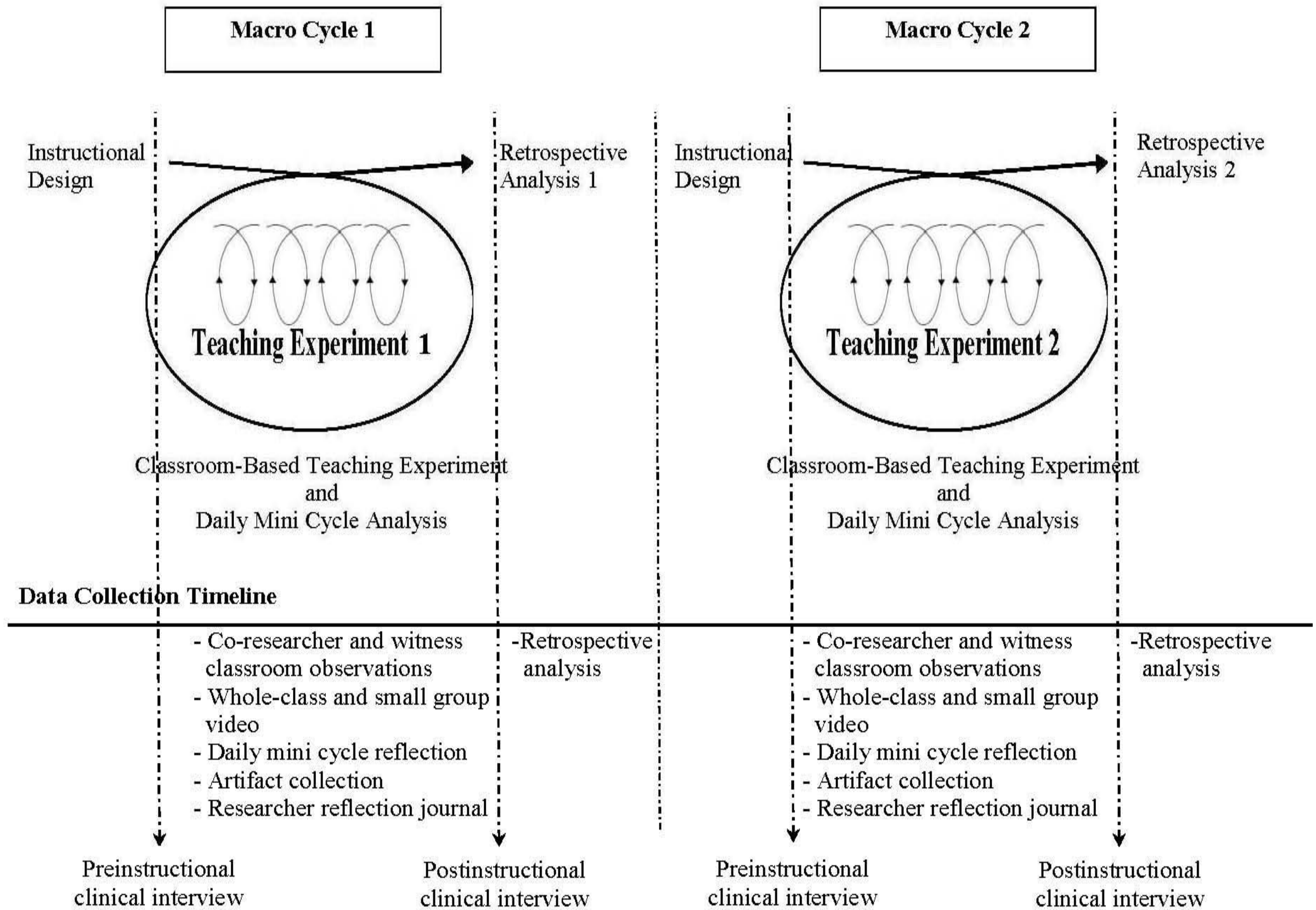

Figure 4.2. A Diagrammatic Representation of the Study with Points of Data Collection.

CHAPTER FIVE: FINDINGS

In Chapter three, a conjectured local instruction theory about students' development of angle and angle measure through the use of context-aware ubiquitous learning tasks was presented as a proposed theoretical framework for this study. In addition, an instructional sequence of tasks were designed and summarized. This sequence included six lessons designed for use in fourth grade classrooms. The lessons were implemented in two macro cycles (Figure 4.1). In the previous chapter, the design-based research (DBR) protocol is detailed and the methods used in the data collection and analysis are explicated. Data from multiple sources were collected from macro cycle one and two to answer the following questions:

1. How do students come to understand angle and angle measure?
2. What are effective means of support to facilitate understanding of angle and angle measure?

In this chapter, the findings from the retrospective analysis are presented. Each teaching experiment consisted of six lessons over seven teaching episodes. The lessons utilize van Hiele-Geldof's (1957/1984) five phases of geometric instruction: 1) inquiry/information, 2) guided orientation, 3) explication, 4) free orientation, and 5) integration. The lesson format has been based loosely on the format of Van de Walle and Lovin's (2006) three part format of Before phase, During Phase and After Phase for problem-based lessons.

The Before Phase typically involved activities that had the students exploring their own knowledge about a mathematical concept. This was often based on the concepts from the prior lesson/s. The During Phase had the students completing tasks that involved the students actively finding, measuring, and/or categorizing angles. These tasks were completed inside or outside the classroom. The students often used Sketch Explorer on the iPad during this time. For the After Phase, this was generally the time that students came together as a class to hold mathematical discussions and synthesize the information they had gained from the lesson. Students' classwork and screenshots from the iPads were collected and saved at the end of each day to be considered in the daily mini cycle analyses.

Various changes were made to the instructional materials following the retrospective analysis at the conclusion of macro cycle one. These changes were implemented during macro cycle two as part of the teaching experiment. Changes were made to reflect concerns about the activities and student learning in regard to those instructional activities. Those changes are discussed in this chapter. The findings from retrospective analysis one, two and the entire DR process affected the final changes to the instructional sequence. The changes are discussed in chapter six and they are also reflected in the instructional materials provided in Appendix B.

The framework for this chapter is based on the two research questions for this study. The first section of this chapter presents findings around how students come to understand angle and angle measure. The second part of this chapter discusses the effective means of support to facilitate understanding of angle and angle measure. In consideration of the way in which the students learn and the supports to be provided, these created changes to the local instruction theory. This revised theory is presented in chapter six.

How Students Come to Understand Angle and Angle Measure

The learning goal for this instructional intervention was the development of students' understanding of angle and angle measure. Understanding angle concepts requires the apperception of the physical attributes of angle; these include the static (configurational) and dynamic (moving) aspects (Kieran, 1986; Scally, 1986), and the relationship to angle measure. Furthermore, students should understand that angles can be represented in multiple contexts in regards to standards, generalizable concepts and procedures for measuring angle (Clements & Sarama, 2009).

Context- aware u-learning was one type of task that was proposed to lead to the support and development of students' understanding of angle and angle measure. Context-aware u-learning connects students to real-world phenomena and technological tools, such as DGE, to support learning of angle and angle concepts. In addition, mathematical discourse was also included as a support which is enhanced by the use of mobile technologies. Within the instructional sequence, context-aware u-learning was intertwined with traditional instruction, as the mobile devices were used to complement decontextualized learning of mathematics taking place within the classroom with the contextualized learning outside the classroom (Tangney B., O'Hanlon P., Munnelly J., Watson R., & Jennings K, 2010).

This section is organized into three parts to represent the first three van Hiele levels of geometric thinking. These three levels encompass the 12 essential understandings identified in the literature review which constituted the lesson objectives for the instructional experiment. Findings about students understanding of angle and angle measure in relation to these three levels of thinking are presented along with a discussion on angle and angle measure as applicable. These three levels are followed by the findings of the pre and post instruction interviews for macro cycle one and two.

Level one: Visual level of geometric thinking.

Explanation and conjectures. In Scally's (1990) adapted van Hiele levels for angle, level one is the first of five levels. Students working within this level identify, characterize, and operate on angles according to their appearance[7]. In the sequence of six lessons, it was conjectured that the students would be working at level one during the first two lessons. The objectives for Lessons One and Two were developed to have the students move to working at level two; they were asked to focus on angle properties rather than attending to the visual appearance. Many of the students were expected to be novice learners with regard to angle concepts' and it was anticipated that many may be working more at the visualization level of geometric thinking than level two.

Summary of Lessons One and Two and student responses. In Lesson One, students were introduced to a set of angles and are required to determine whether the angles are alike or different. Students then went out into the area surrounding the school to identify angles in the real-world setting. This initial lesson was summarized with a discussion and students' journal entries focusing on the properties of angles. In Lesson Two, students explored the use of a Dynamic Geometry Environment (DGE) and then used this program to identify angles in the real world using screenshots from Lesson One. Possible angles were discussed with a partner. The lesson was summarized with the students' screenshots shared in class and a discussion about how the students identify angles.

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[7] A detailed list of level one indicators can be found in Appendix A.

Although it was conjectured that students would begin working at level one and move to level two during Lessons One and Two, students may have reverted to level one thinking as new concepts were introduced. Therefore, it is likely that evidence of level one thinking may continue to appear throughout the sequence as the teaching experiment was only seven consecutive school days. Some students may have worked partially within level one for longer than others as they processed, internalized and grew in understanding to move onto subsequent levels.

The first activity in Lesson One required students to recognize that angles have a number of salient attributes, such as two rays and a common end point. At the beginning of the first lesson, students were given a sheet of angles and asked to work in pairs to study the figures and are asked to answer two questions stated verbally:

What can you tell me about these figures from what you have noticed?

What do all these figures have in common?

Data was triangulated from the video and observer comments from teaching experiment one (TE1), these data suggest that approximately two thirds of the students in the class described the important attributes of angles to their partners. However, other students in the class appeared not to be able to decompose the figures into the individual attributes.

The video and observation data show that students also made visual comparisons, such as one pair who based their observations on the gestalt angle appearance. The following excerpt[8] is taken from a discussion during this initial activity:

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[8] Some of the excerpts of the transcripts were edited for readability. In these cases, the content of the discussion did not change, but unrelated segments were removed.

Teacher: What do all of these figures have in common?

Samantha: They look sort of like a corner.

Teacher: What else do you notice about the figures?

Cara: They all look like some sort of triangle.

These comments are indicative of students working at the visual thinking level. They did not notice that each figure had two straight lines that were connected at one end point. They see the figures as a collection of a whole rather than the individual attributes. Clements (1998) described how students at this level are guided by perception and that visual prototypes are used to name a figure. In this case, Cara connects the figures as being similar to triangles. It is interesting how various orientations and sizes are used, yet still she wants to connect the figure to a gestalt shape with which she is familiar.

Early in macro cycle one, one teacher explained that all the fourth grade students had been taught about angle in third grade. From the observation notes, it appeared that the students had rote learned a number of angle categories/names and had little understanding of what an angle was. For example, during this initial activity on day one, this was an extract from another pair discussion:

Teacher: So, what do all of these figures have in common?

Jeremy: They are angles.

Teacher: What is an angle?

Jeremy: These (pointing to the figures).

Teacher: How do you know that these are angles? What makes you believe that these are angles?

Carl: Because we learned about angles a bit last year.

Teacher:	So are all figures angles? If I drew more figures on another sheet would they be angles?
Jeremy:	That depends.
Teacher:	Depends on what?
Jeremy:	If they look like that (pointing to the sheet of figures). They all look like angles.
Teacher:	Would this be an angle (drawing a circular open shape).
Carl:	No…because it does not look like an angle.

The two students were working at level one as they had the idea that angles look a certain way to fit with particular angle categories. They are again not able to identify specific attributes of the angles.

From these sorts of discussions, it was evident that students were unable to reduce their observations and their language to focus on the attributes of angles. The researcher used these data to make some adjustments in instructional plans for the second round of instruction to have students imagine that their partners were kindergarten students and that they had to describe the figures carefully using simple understandable language for their partners to understand. Their partners were instructed to say that they did not understand and seek further clarification if students reverted to technical mathematical language. The intent was to help students move past any rote material from early grade levels. Based on the video and observation evidence, this change appears to have been effective in the second teaching experiment. In the second iteration of this set of lessons, there appear to be only a few instances of students using visual or technical mathematical language noted in the observer notes.

In Lesson Two, the students used the Dynamic Geometry Environment (DGE) Measure a Picture (Steketee, & Crompton, 2012), the add-on program of Sketchpad Explorer (2012). They used this program with iPad mobile devices to photograph angles they identified in their playground environment. In TE1, as students went out to find angles in the playground, video evidence, observation notes and students' work show that many of the students gravitated towards natural artifacts to find angles in places such as trees. The students would often find an artifact visually resembling an angle, but if students considered the attributes of angle, such as two straight lines, they would determine that it was not always an angle. For example, in Figure 5.1 Claire found angle like shapes on a tree stump and marked those as angles with the dynamic protractor. Under the protractor, the lines are distinctly bent and distorted on the natural curves of the wood.

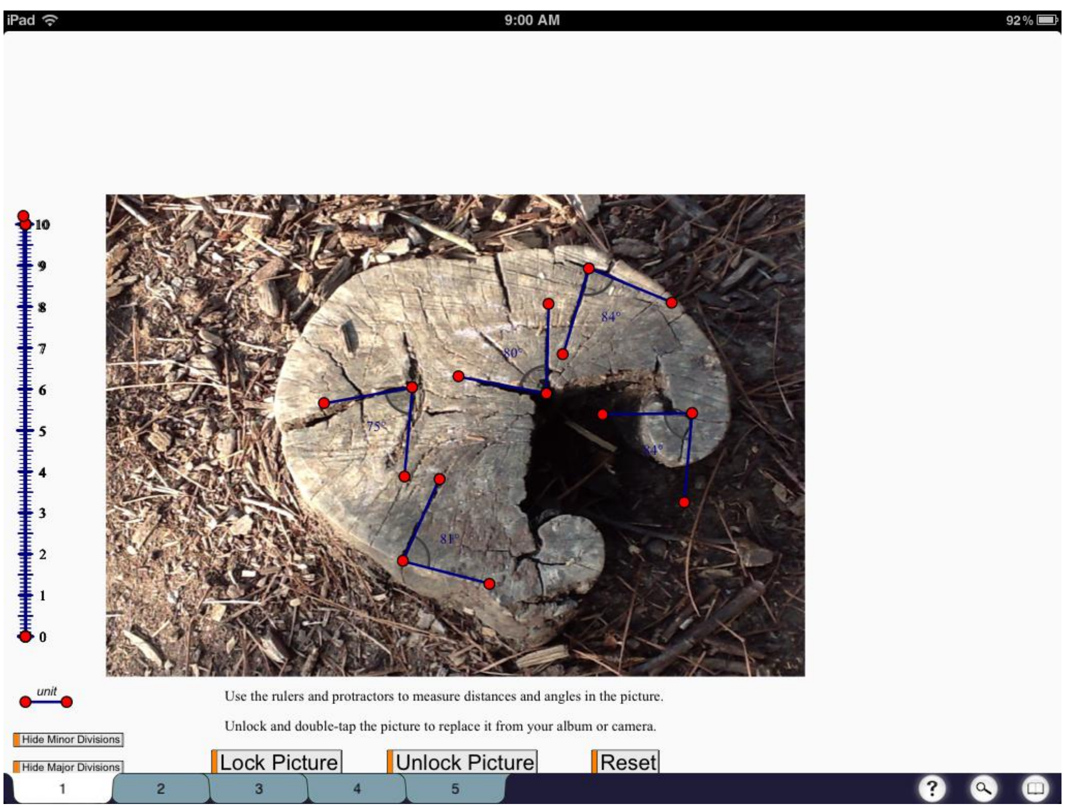

Figure 5.1. Student Found Angle Like Shapes in the Tree Stump.

Claire was identifying angles based on the visual appearance, searching for shapes that look like angles and was not identifying angles by the properties of angles. While she is actively looking for angles in the real-world, Claire is working within the visualization level of geometric thinking. Other students did similar work.

In light of this issue and before the second teaching experiment, the instructional materials were altered to include the instructor's conducting a brief class discussion about the best places to look for angles based on salient angle properties. This discussion focused primarily on the point that straight lines are more likely to be found on manufactured artifacts than those found in nature. This discussion was included to encourage students to work towards the analysis level of geometric thinking as they had to consider the properties rather than the gestalt appearance.

During this activity, students were required to take screenshots of the angles they found in both TE1 and TE2. The screenshots were coded for those pictures that were (actually) angles or were (actually) non-angles. Students often identified more than one angle in the screenshot, although there were no more than five potential angles identified on a screenshot. For each angle identified a code was given (i.e., example of angle or not an angle). This was completed for both teaching experiments and the results are presented in Table 5.1.

Table 5.1

Real-World Angle Identification

	Teaching Experiment 1 (n = 30)	Teaching Experiment 2 (n = 30)
Angle	26 (28%)	55 (87%)
Non-Angle	68 (72%)	8 (13%)

Note. There were 30 students in each class; however, each student may have identified between one and five angles on each screenshot.

In TE1, 30 students took screenshots of angles and identified them using the dynamic protractor. Of the 94 potential angles found by the students, 28% were examples of angles with 72% not being examples of angles, i.e., non-angles, as they did not have the relevant attributes required to be an angle. In experiment two (TE2), 30 students took screenshots of angles and identified them using the dynamic protractor. Of the 63 potential angles identified by the students, 87% were examples of angles and 13% were not examples of angles, i.e., non-angles. This was evidence that there was a change between the two teaching experiments in students' ability to identify angles in real-world contexts.

It would appear from the findings summarized on Table 5.1 that this added discussion implemented in TE2 was helpful as fewer non-angles were identified in TE2. However, even in the TE2 some students were still working at level one at the end of Lesson Two. For example, Matthew believed that he had found an angle in Figure 5.2.

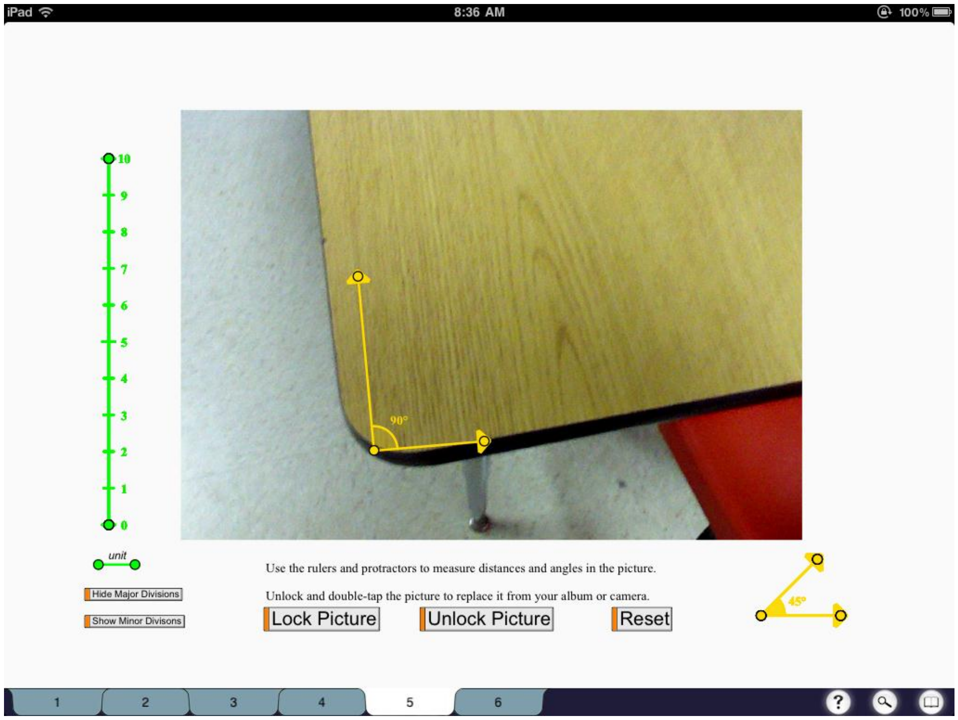

Figure 5.2. Searching for Real-World Angles.

This is an extract from a conversation following Matthew's potential angle find.

Teacher: In your screenshot where is the angle Matthew?

Matthew: There (Pointing to the angle indicated on the screenshot).

Teacher: How do you know that is an angle?

Matthew: This is the corner of the table and …angles are corners.

In the van Hiele level indicators for the visualization level, one of those indicators describes the way that a student can exclude relevant angle properties. As Matthew chose this potential angle, he has failed to consider relevant angle attributes, i.e., that the two lines need to be straight lines and that the two lines should meet at one end point. To triangulate the screenshot data I asked Matthew why he thought it was an angle and he said that it was a corner so it was an angle. Matthew may need supplementary activities to support his development towards level two thinking. For future iterations of the instructional sequence, it would be useful for students to have an assessment at the end of Lesson Two to determine how many students, like Matthew, need supplementary instruction to move from level one to level two thinking.

Level two: Analysis level of geometric thinking.

Explanation and conjectures. In Scally's (1990) adapted van Hiele levels for angle, level two is the second of five levels. Students working within this level establish properties of angles and uses properties to solve problems[9]. In the sequence of six lessons, it was conjectured that the students would be working at level two during Lessons Three and Four and begin moving into level three during Lesson Five.

Summary of Lessons Three and Four and student responses. The objective of Lesson Three was for students to recognize acute, obtuse, right and straight angles in different contexts (viz., real-world and paper and pencil). Students had to sort angles they had made with wooden coffee stirrers into similar groups.

Level one thinking beyond the first two lessons. In TE1 Lesson Three, students were still showing some evidence of working within van Hiele level one. On day three, the objective was to have students consider angle attributes to move towards the analysis level of geometric thinking. The objective of Lesson Three was to recognize and compare angles based on size using non-standard and standard language (acute, obtuse and right angle). The students made triangles using wooden coffee stirrers cut to different lengths. Then, working in groups, the students sorted those angles into similar groups. The students had to determine their own groups using what they had learned about salient and non-salient angle attributes.

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[9] A detailed list of level two indicators can be found in Appendix A.

Triangulating the data by using the video and the video transcripts coded using Scally's van Hiele level indicators, as well as observer notes, these data show that four fifths of the students in TE1 class were moving into level two. However, the other one fifth, represented as two groups of three students were working at the visualization level. One of those groups of students sorted the angles by their rays, a non-salient angle attribute. This inclusion of irrelevant properties is listed in the van Hiele levels as an indicator of a student working at the initial visualization level. One of the groups recognized some of the salient attributes, such as two lines and an end point, but the sort was based on the length of the rays classed as small angles for the short rays and big angles for the long rays, see Figure 5.3.

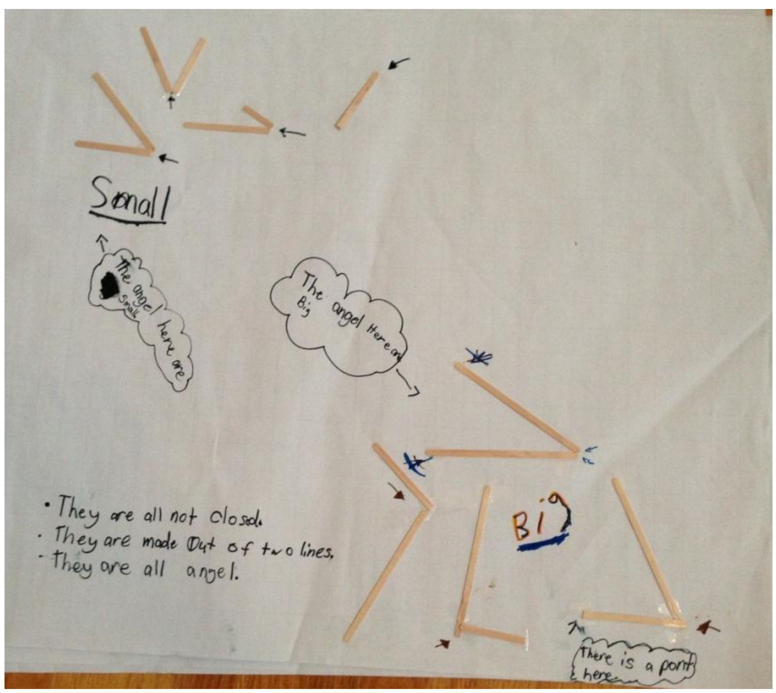


Figure 5.3. Sorting Angles by the Length of the Rays.

This finding led to a modification to the add-on program Measure a Picture. In the initial program, the dynamic protractor did not have adjustable ray lengths. The rays appeared more like line segments with another end point. Modifications were made for the ray to have an arrow and for the length to be adjustable, see Figure 5.4. In addition, the color of the rays was changed to make the protractor more visible on photographs.

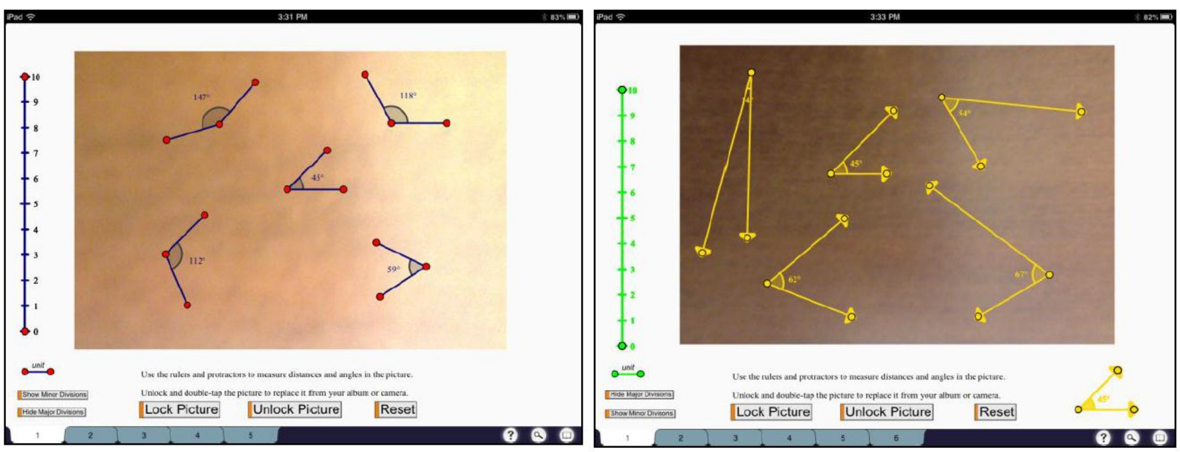


Figure 5.4. Modifications to Measure a Picture.

There were a total of ten posters with three students working on each poster in TE1 and TE2. In TE1, two groups provided evidence of working within the visualization level of geometric thinking. One group included non-salient attributes (see Figure 5.3) and the other group based their sort on those that look like corners (right angles) and those that do not look like corners. In TE2, all ten groups did not provide any evidence of geometric thinking below van Hiele level two.

Level two thinking in Lessons Three and Four. From the angle sorting activity, using data from the student work artifacts, video evidence, and observation notes it appears that students in TE2 were analyzing and comparing angles in terms of their properties and were able to formulate and use generalizations about properties of angles in problem solving situations. This is congruent with the van Hiele level two indicators for thinking about angles. For example in Figure 5.5, the three students created a set of angles and they were able to categorize the angles into the four groups (acute, obtuse, right and straight angles). The angles were in different orientations with rays of different lengths. This indicates that the students understand which were the salient angle attributes and those that were non-salient.

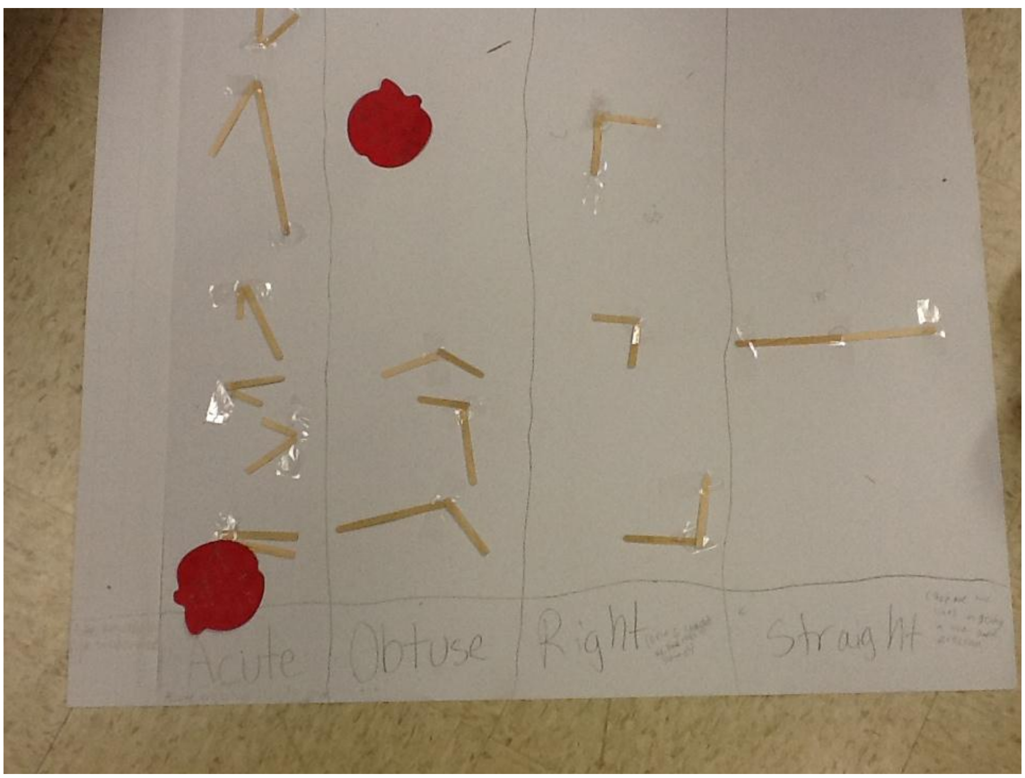


Figure 5.5. Angle Sorting Activity in Teaching Experiment Two.

The changes to the DGE program appear to have also supported students earlier in the instructional sequence. During Lesson Two, as the students in TE2 found angles using the modified program, from the video evidence and observational notes it appears that students were focused on salient angle attributes with 87% of the angles found by students in TE2 correctly identified in comparison to the 28% correctly found by the students in TE1, see Table 5.1. In addition, students often made the rays of different lengths to point out that the length of the rays were non-salient attributes. For example, Catrin took this screenshot of angles, see Figure 5.6, and the following discussion ensued.

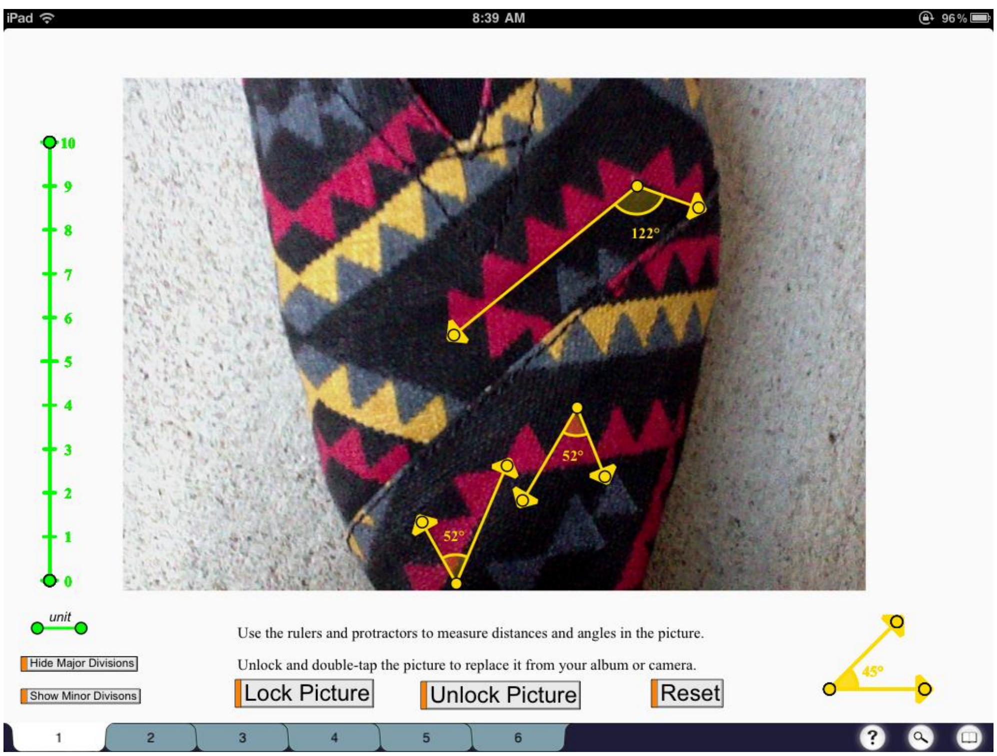


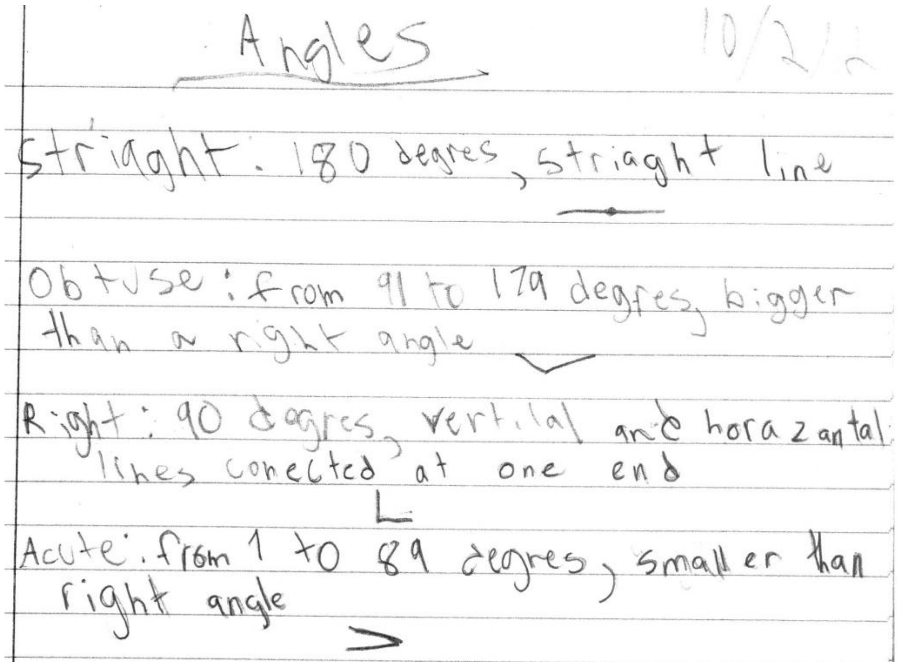
Figure 5.6. Rays are a Non-salient Angle Attribute.

Teacher: I notice that the rays are of different lengths.

Catrin: Because, that does not matter. I have put the rays against where I see the angles, like there (pointing to the top angle), that is only short and that is long, but it does not make a difference to the angle size as it is not measuring the length of the lines.

Catrin's screenshot and response is indicative of a student working within the second level of geometric thinking as she has analyzed the angles based on their properties rather than the gestalt appearance.

During the mini cycle reflection at the end of day three TE1, it was concluded that students needed time to reflect upon and synthesize the information they had gathered on angles. Therefore, another change that was made to the instructional experiment in TE2 was to have students write a journal entry at the end of Lessons One and Three. Journaling provided students with a time to reflect on their own understanding of a particular concept. Figure 5.7 is a good example of a student working at level two identifying angles by their individual properties, see Figure 5.7.



Angles 10/2/

striaght. 180 degres, striaght line

Obtuse: from 91 to 179 degres, bigger than a right angle

Right: 90 dagres, vertital and horazantal lines conected at one end

Acute: from 1 to 89 degres, smaller than right angle

Figure 5.7. Journaling to Describe the Categories of Angle.

Each of the student journals from day one and day three of TE2 were coded using Scally's van Hiele level indicators for geometric thinking. All of the journals were coded at a van Hiele level two.

Level three: Informal deduction level of geometric thinking.

Explanation and conjectures. In Scally's (1990) adapted van Hiele levels for angle, level three is the third of five levels. Students working within this level formulate and use definitions, provide informal arguments that order previously discovered properties, and follow and give deductive arguments[10]. In the sequence of six lessons, it was conjectured that the students would begin working at level three during Lesson Five and Six.

[10] A detailed list of level three indicators can be found in Appendix A.

Summary of lessons five and six and student responses. The objectives for Lesson Five required students to understand that angles can be measured with reference to a circle and that angles are fractions of a circle. Students also studied the use of benchmarks, understand the meaning of degrees of measure and gained experience using a nonstandard unit of measure. In addition, two other objectives were included in this lesson, which required level three thinking to complete. The objective was that students are to recognize that there are an infinite number of angles and to consider angle as a turn. These two objectives are specifically referenced in the van Hiele level three indicators.

To meet these objectives, the lesson used an adapted version of Browning, Garza-Kling, and Hill Sundling's (2007) and Millsaps' (2012) wedge activity. The students used a folded paper circle to create a wedge to measure various angles on paper and real-world objects. The objectives for Lesson Six required the students to recognize that angle size can appear different based on different visual perspectives. The activity for this objective was to have the students taking photographs of angles from various positions. The photographs were taken within the DGE and students then use the tools to measure the angles and discuss their findings. The second objective is for the students to understand that angles are defined by particular attributes which involve angle as a turn. This was the culminating activity of the sequence as students discussed in groups what they knew about angle and the students then create a poster for other fourth grade students to explain what angles are.

Level two thinking during Lessons Five and Six. During Lesson Five students had to complete a worksheet during which they had to estimate the size of nine angles and categorized the angles as acute, obtuse, right and straight angles. The results of the categorized angles can be found in Figure 5.8. The chart provides evidence that students in TE2 gained higher scores on the categorizing angle worksheet assessment than those in TE1. All 12 students from TE2 got all nine answers correct which is double the amount in TE1. The minimum score of TE2 is six correct and for TE1 the minimum score is five. TE1 n = 29, TE2 n = 28.

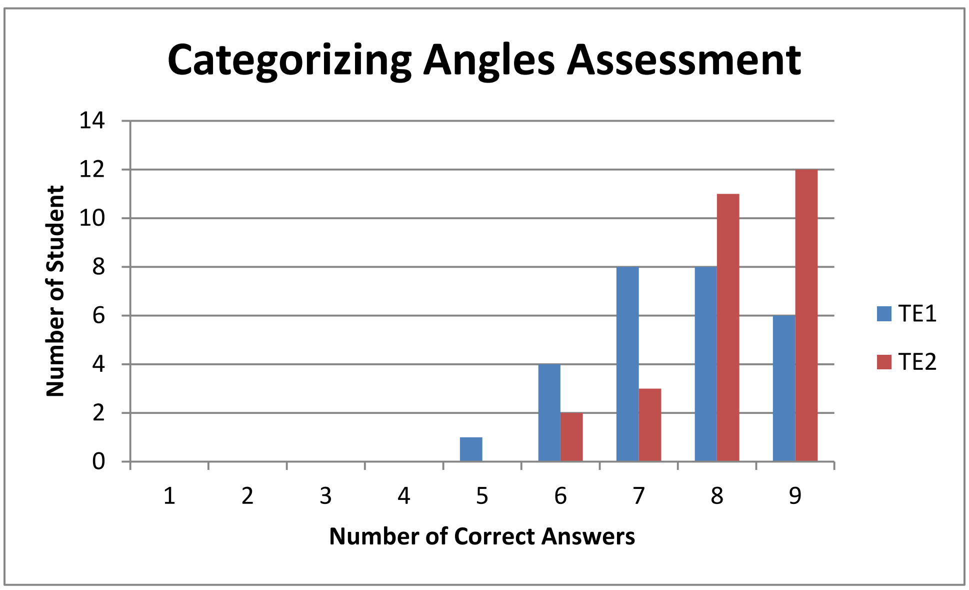


Figure 5.8. Categorizing Angles Assessment.

In addition to determining if the students could provide the correct nomenclature, the worksheet also assessed if students were using properties rather than visual appearance to establish the category. For example, many students who gained an incorrect answer for question number six did so because the angle looked like a right angle and they marked it as such. However, those who did not estimate the size of the angle on visual appearance alone instead studied the angle properties and found that it was an obtuse angle of 100°. Those who gained a score of nine on the test had to be working at the minimum at van Hiele level two for each question.

The worksheet assessment also required the students to provide an estimation of the angle measure using the paper wedge they had created using a circle of paper. This was the first time the students had studied angle measure and the observation notes and artifact collection identified that students needed further support beyond that provided in the instructional plans. Therefore, a number of changes were made to this activity following the completion of macro cycle one.

Clements and Burns (2000) advocated for supports to help students to internalize angle benchmarks (e.g., 90°, 180°). The wedge activity helped students develop these benchmarks, but observation notes pointed out that students often still forgot the 90°, and 180° benchmarks. The researcher determined from the evidence provided in the mini cycle reflection that until students had practice at using these physical benchmarks they would often forget the actual measure of the benchmarks.

To assist students in remembering these benchmarks, in TE2 students were asked to write on the degrees of measure onto the paper wedges for 90° and 180°. The ability to use benchmark measures was one of the objectives for Lesson Five and this skill can also be found in the van Hiele level two indicators. Another change to the measurement activity was the added discussion about the important of beginning the measure at zero. From the student work artifacts and observational notes, it appears that students would not always place the protractor to begin the measure at zero. As students are taught to conduct linear measurement with the zero mark on a ruler, students were asked to remember this as they place the side of the wedge to match one side of the angle. These changes appear to have slightly helped the students as they provided estimations of angle measure on the worksheets in TE2, the results from TE1 and TE2 can be found in Figure 5.9. The chart shows a slight increase from TE1 to TE2 with a mean score of 5.5 for TE1 and 5.93 for TE2.

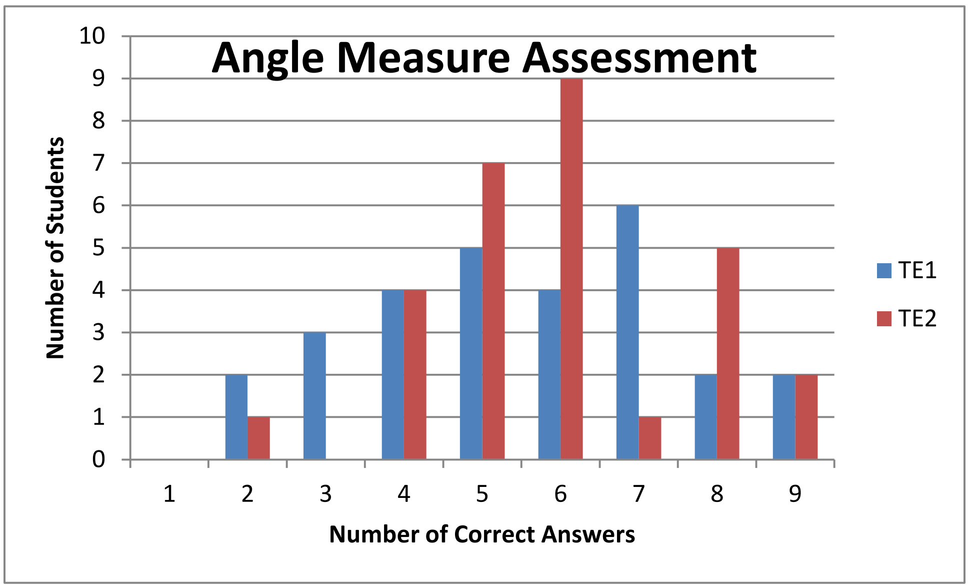


Figure 5.9. Angle Measure Assessment.

One final change made to the measurement activity was to provide the name *reflex* angle to students when asked. Observational notes show that during TE1 and TE2 students asked what the name of this category was as they began to consider a full turn as 360°. Students understood 1-89° was an acute angle, 90° a right angle, 91-179° an obtuse angle and 180° a straight angle. As the dynamic protractor continued beyond 180° students asked the name of this other category. This change was not based on student's achievement, but on the basis of just-in-time learning, that the students had identified that a category was missing from their understanding and they wanted to know the answer to fill this gap in their learning.

Evidence of level three thinking in Lessons Five and Six. Triangulated data, gathered from the video recording, classroom observations and collectively the daily mini cycle reflections did not highlight any issues with Lessons Five and Six. However, at the end of macro cycle one, it was concluded that one of the lesson objectives had not been fully met, namely, that students did not recognize that there could be an infinite number of angles. The instructional plans for TE1 led the students to understand that there are 360° in a circle and this is where the teaching stopped. An additional component was added in TE2 to have students connect with fractions of a degree. As the concept of fractions was also relatively new to fourth grade students an addition was made that allowed students to connect to linear measure and consider ½ a whole unit and a ¼ of a unit. The intention was to have students understand that there are more than 360° as each individual unit could be split into many smaller parts of a degree.

This addition to the instructional plans appeared effective as data from the TE2 clinical interviews and also in the estimating angles assessment, in Lesson Five. Figure 5.10 displays Christine's work from this assessment as some of her estimations included half a degree. For problem *h* she used the 90° benchmark wedge to indicate where the 90° would be. Then split 90° in half for two 45° angles and determined that the angle was about half of 45° for 22½°. Although on problem *g* she marked out 90° with the wedge benchmark and estimated the angle size to be 112½°. When questioned she reported that she chose to estimate based on the benchmarks that she had and she did not think it was exactly 113 but just a little less than that.

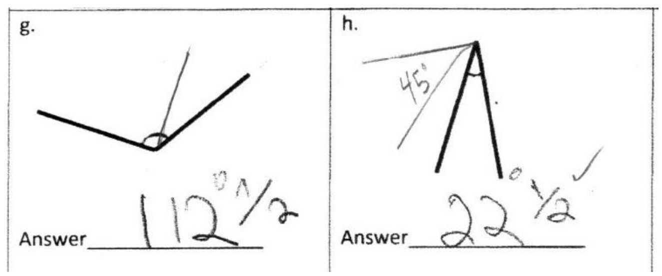


Figure 5.10. Estimating Measures Using Fractions of a Degree.

In TE2, the video and observation data show that students were typically working within van Hiele level two as the students often demonstrated the ability to list the salient properties of angle. From these data, it would appear that the added journal entries in TE2 were helpful as students had time to consider what they understood angle to be. The final activity in the instructional experiment required students to work in groups to explain on poster paper what they knew about angle. The posters were coded for comments that reflected the three van Hiele levels. In the final teaching experiment, there were no comments matching van Hiele level one. These would have been comments referring to what angles look like. The indicators for level two are that the lists describe angles with a litany of properties or insufficient properties rather than necessary and sufficient properties. For the eight posters, there was no evidence of insufficient properties. The students that described other properties talked about the irrelevance of those properties. For example, the students in one group wrote:

1. They have two connected lines.
2. It has straight lines.
3. There are different groups of angles like acute, right and obtuse.
4. The lines on the angle donst hase to be the same length.
5. Angles can go in different directions.
6. The corner of the angle dose not move.
7. There is space inside of each angle.

They have listed the salient attributes of angle using standard and non-standard language and four and five point out that some features are not important. The group was questioned further about the last two points. The students justified their answers:

Allison: The corner is like the pivot foot in baseball. It does not move.

Ruth: One of the lines moves, the other stays where it is. As the line moves away from the other line the space gets bigger (the student is demonstrating with her arms).

Allison: And it does not matter if the lines are long and there is a lot of space in the middle. It is only measuring the space right where the two lines connect.

These two girls from that group are explaining a very difficult concept to understand. With the properties listed by the group and justifications provided by the two girls, this is a simple example of students starting to work within level three. However, individual questioning of the rest of the students would be needed to determine if they can also provide the justifications to also be working at a level three. To help the students understand angle as a turn, during TE2 students were asked to think about the pivot foot in baseball as they considered the end point. That the end point did not move but the ray (the leg) moved while the other stayed still. This provided a real-world connection to a confusing concept for many students. Allison, in this justification is making that connection.

Pre and Post Instruction Interviews

The purpose of the pre and post instruction clinical interviews (see Appendix D for protocol) in this research study was to gain an accurate understanding of the student's level of geometric thinking before and after instruction in order to inform the conjectured local instruction theory and the development of learning activities in the instructional unit. The interviews were conducted and scored using Scally's (1990) coding instructions. Narratives were written for each student the same day of the interview. These narratives formed the basis for reporting the findings in this section.

In this section, a summary of each student's response pre and post instruction for each of Macro cycles one and two is presented. The summaries of interviews are intended to address the following questions:

1. How do student responses to the interview questions confirm the conjectured local instruction theory and/or the effectiveness of the learning activities in the instructional unit?
2. What questions/issues do student responses to the interview questions raise with respect to the conjectured local instruction theory and/or the learning activities in the instructional unit?

3. What actions/adjustments/modifications are made in the conjectured local instruction theory and/or the learning activities in the instructional unit based on the results of the interview responses?

Summary by Student – Pre and Post Instruction Interviews

Interviews: Macro cycle one.

Mia. Mia's van Hiele levels indicate gains from pre to post instruction interviews. This can be seen in Table 5.3; she is working at the visual level for much of the pre instruction interview. In the pre instruction interview, Mia appears to hold the common misconception that an angle is measured by using a linear measure of the distance between the ends of the two rays. For example, in Figure 5.11 she described the straight angle as being about two inches long. She considered the straight angle to be measured based on the lengths of the rays. Therefore, if both of the rays were drawn two inches long, she would describe the measure of the angle as four inches. When questioned about angles of other sizes Mia always determined the measure on the distance between the where the drawn rays appeared to finish.

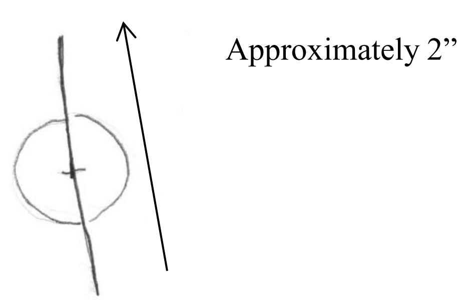


Figure 5.11. Mia's Pre Instruction Interview Angle Measuring.

Mia did not demonstrate this misconception during the post instruction interview.

In the pre instruction interview, Mia often appears to exclude relevant attributes and include irrelevant attributes to define an angle. For example, in Figure 5.12 Mia stated that the figure with curved lines was an angle. When questioned where the angles were on the drawing, she said that the angle was at the end of each ray. She ignored the salient attributes that angles have two straight lines and a common end point, also evidenced earlier in Figure 5.11. Both these examples characterize responses that appeared throughout the pre instruction interview.

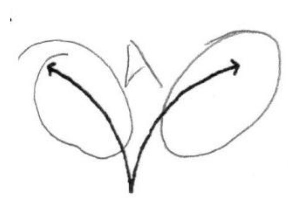


Figure 5.12. Identified in the Pre Instruction Interview as Two Angles.

In the post instruction interview, Mia excludes irrelevant attributes and includes the relevant attributes of angle. Mia provided evidence that she was very clear on what angles could be found in a picture of a building with a clock tower. She found a number of angles as requested and also voluntarily explained what criteria she was using to determine if they were angles. For example, "This is an angle because they are two straight lines and they are not parallel as they are touching at one point and they are not bent in any way; it is an obtuse angle." When she was probed to respond if other figures in the picture were angles, Mia responded either yes, as it has these salient attributes (straight lines connected), or no, that it was not an angle as it had curved lines.

In the pre instruction interview, Mia demonstrated little understanding of angle measure. She did have the idea that angles turn in a full circle, but in organizing the degrees she described half of the circle as 50 degrees, then split the other half into two quarters, one of those quarters she describes as being 50 degrees and the other as being 25 degrees, see Figure 5.13. In the post instruction interview, Mia provided reasonable angle estimations and used 90° quadrants and the 180° straight angle benchmark to estimate measures.

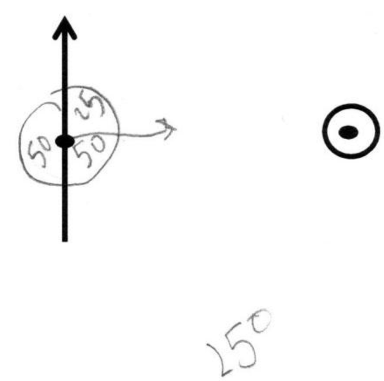


Figure 5.13. Mia's Pre Instruction Interview Turn Estimations.

Table 5.2 provides a summary of Mia's levels of thinking during the pre and post instruction interview. A clear distinction can be seen as Mia moves from working at the visual thinking level in the pre instruction interview, to the analysis level in the post instruction interview. In the pre instruction interview she compared and sorted angles by looks and in the analysis thinking level Mia studies the angles by their properties. In the pre instruction interview, for the drawing, identifying and sorting tasks Mia is working primarily at the visual level with some evidence of thinking in the analysis level. For the angle measure and relations tasks she is working in the visual level of thinking. In the post instruction interview, she is able to generalize angle attributes across tasks and is working well within the analysis level for drawing, identifying, sorting, measuring angles, and angle relations. There is no evidence of Mia working within the visual level during any of the tasks.

Table 5.2

van Hiele Levels by Interview and Task - Mia

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	Va	A
Identifies Angle	Va	A
Sorts Angle	Va	A
Angle Measure	V	A
Angle Relations	V	A

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Claire. Claire's van Hiele levels indicate gains from pre to post instruction interviews. In the pre instruction interview, it appeared from her responses that she had rote learned some angle facts, such as the names of angle categories, but there was no understanding behind these facts. For example, during the interview, Claire described acute angles as being the same size and that there were two different sizes of obtuse angles. In Table 5.4 it shows that in the pre instruction interview Claire often included irrelevant attributes of angle and excluded relevant attributes. For example, when she was shown a drawing of a building with a clock tower she chose two curved lines with a common endpoint as an angle. In this particular case she is excluding straight lines as a relevant angle attribute. In the post instruction interview, this was corrected, although it appears that she is still including some irrelevant attributes in the angle measurement activities.

In the pre instruction interview, Claire was given credit for generalizing property as she used the terms acute, obtuse, and right angles. However, she was not given credit for analyzing by property as she was unable to do this as the categories were used without understanding. In the post instruction interview, Claire was able to analyze by properties and also generalize these properties. In Table 5.4 it appears that Claire was working mainly at the visual thinking level in the pre instruction interview and was starting to move into the analysis level for drawing, identifying and sorting angle, moving to the analysis level for the post instruction interview.

For the pre instruction interview of angle measure and angle relations she was working within the visual thinking level. In the post interview Claire was primarily working at the analysis level of thinking, although during the angle measure activity she occasionally transition from the analysis level back to the visual thinking level. For example, in the post instruction interview, Claire described a right angle as “sort of like a square, like if you take a right angle and put it next to a piece of paper then it should be the same. The corner should be the same as a right angle and it is 90°”. Claire seemed to be grappling with what she understands about a right angle. While these are all correct, the most salient attributes are not forefront in her mind, she seems to be working between the visual and the analysis level of thinking.

Table 5.3 provides a summary of Claire’s levels of thinking during the pre and post instruction interview. In the pre instruction interview, Claire was working at the visual level of thinking for angle measure and angle relations. She did show some indication of working in the analysis level for drawing and sorting angles, but the visual level was still the dominant method of thinking. For identifying angles she was working in both the visual and analysis level. Data gathered from the post instruction interview shows a large increase in thinking about angle. Claire was working well within the analysis level for drawing and sorting angles as well as angle relations. For angle measure and identifying angles, she was working primarily in the analysis level, but did show some indication of still working in the visual level.

Table 5.3

van Hiele Levels by Interview and Task - Claire

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	Va	A

Identifies Angle	VA	vA
Sorts Angle	Va	A
Angle Measure	V	vA
Angle Relations	V	A

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from “The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment,” by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Chloe. Chloe’s van Hiele levels indicate gains from pre to post instruction interviews. In the pre instruction interview, when Chloe was asked how many angles could be drawn she responded that there were about 70, when asked how she came to that number she could not answer. When asked the same question in the post instruction interview she said that if they just have a 1° difference there would be 360 different angles. When questioned if it could be smaller than 1° she said no.

In 50% of the pre instruction interview activities, she includes irrelevant attributes of angles and excludes relevant attributes. For example, when asked about the curved joining lines at the top of the tower on the picture she erroneously said that it was an angle. In addition, Chloe seemed a bit unsure if the arches over the windows would be angles, but said that as it did not have a triangle in it she thought it may not be. In the post instruction interview, Chloe found a number of angles quickly. When asked initially to find three angles she chose a point on the picture where three angles met, see Figure 5.14, and was keen to say how a line on one angle can be used as a line on another angle. Chloe described the criteria for choosing angles as two lines that meet at one point. When other possible angles were suggested, she used the criteria to determine if it was an angle or not. She found the clock hands as another angle, but pointed out that the archway over the top of the window and the doors were not angles as the lines are curved.

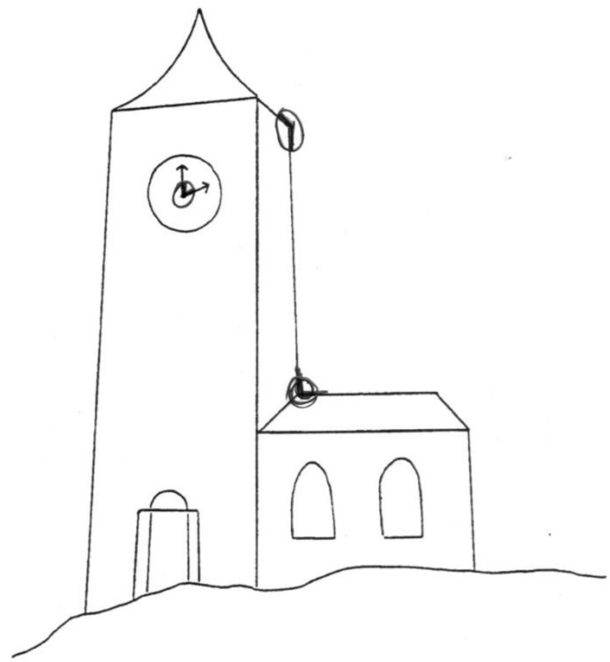


Figure 5.14. Chloe's Angle Identification in the Post Instruction Interview.

In the pre instruction interview, Chloe held the misconception that the length of the rays equated to the size of the angle. For example, in the sorting angles activity, she explained that the rays were an important factor in determining how the angles were alike. She spoke about the length of these rays being similar and that this was connected with the measure of the angles. In the post instruction interview, she does not hold this misconception and she only compares and sorts angles by their properties.

Table 5.4 provides a summary of Chloe's levels of thinking during the pre and post instruction interview. In the pre instruction interview, for angle relations and angle measure Chloe is working at the visual level of thinking. In the sorting angles activity she was showing evidence of some analysis thinking and for the drawing angles activity she is working equally in the visual and the analysis levels. For identifying angles, she is working primarily in the analysis level but still uses some visual thinking. For the post instruction interview, Chloe is working fully within the analysis thinking level for drawing, identifying, sorting, and measuring angles and angle relations, with no evidence of thinking below this level.

Table 5.4

van Hiele Levels by Interview and Task - Chloe

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	VA	A
Identifies Angle	vA	A
Sorts Angle	Va	A
Angle Measure	V	A
Angle Relations	V	A

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Nick. Nick’s van Hiele levels indicate gains from pre to post instruction interviews. In the pre instruction interview, Nick struggled to find angles in pictures and figures. On the majority of the activities he included irrelevant angle attributes and excluded relevant angle attributes in his actions and descriptions. He held the misconception that orientation was a salient factor in identifying angles. For example, during the angle relations activity, Nick stated that a right angle is when the angle is on the right side of the shape and a left angle is on the left side. When asked if he could see any of these angles he was describing from a sheet of figures he pointed out the angle in Figure 5.15 and said that this was a left angle as the angle is on the left side of the figure.

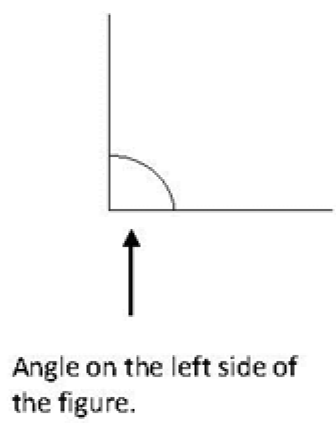


Figure 5.15. Nick's Pictorial Representation of a Left Angle.

In addition to the misconceptions about salient attributes, Nick also used language indicative of level one thinking, as he often used the words “looks like”.

In the post instruction interview, Nick only included salient angle attributes and did not include any irrelevant attributes. Furthermore, during the drawing angles activity (see Figure 5.16), Nick pointed out that he had drawn four angles in various orientations and with rays of different lengths to show that these were irrelevant angle attributes. Throughout the post instruction interview Nick analyzed angle attributes by their properties and did not sort or compare by the visual appearance of the angles.

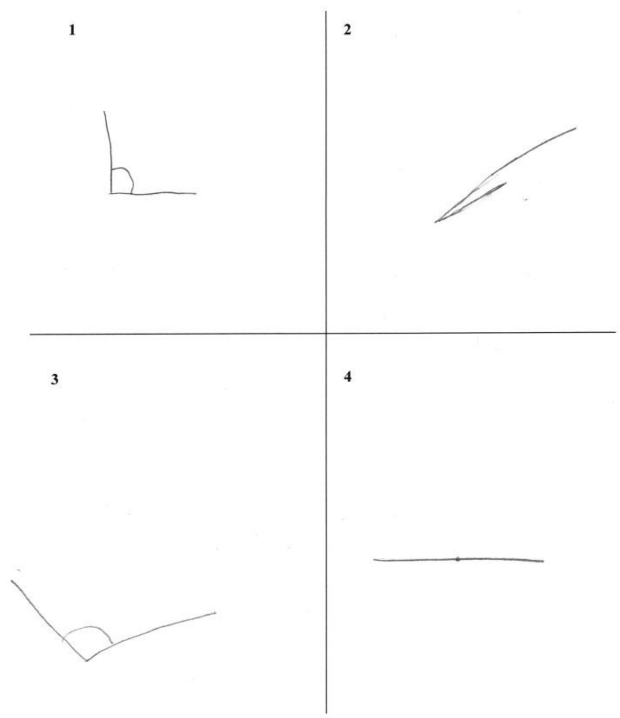


Figure 5.16. Nick’s Angle Drawings for the Post Instruction Interview.

In the pre instruction interview, Nick was not able to organize the cut-out shapes to make a right angle or a straight angle. It appeared from the way he was putting the shapes together that he was struggling with his spatial reasoning as he could not determine which way to turn the shapes to make the straight angle. In the post instruction interview, he was nervous and reluctant about putting the cut-out shapes to form a right angle, but when requested to try he was able to do so. However, he was unable to organize the cut-out shapes to form a straight angle. During the pre and post instruction interview he was unable to use spatial visualization to place two static shapes together to form right and straight angles.

Table 5.5 provides a summary of Nick's levels of thinking during the pre and post instruction interview. Nick transitioned from predominantly working within the visual thinking level in the pre instruction interview, to the analysis level of thinking in the post interview. In the pre instruction interview, during the angle measure and relations activities, Nick was working at the visual thinking level. He was showing some analysis thinking as he drew and sorted angles, but visual thinking was the predominant method. For identifying angles, he was mainly working in the analysis level, but there were indications of working within the visual level. In the post instruction interview, he was working fully within the analysis level of thinking with no indications of visual thinking.

Table 5.5

van Hiele Levels by Interview and Task - Nick

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	Va	A
Identifies Angle	vA	A
Sorts Angle	Va	A

Angle Measure	V	A
Angle Relations	V	A

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Summary of macro cycle one. The students in TE1 began working between the visual and the analysis level for drawing, identifying, and sorting angles. This can be seen in Table 5.6. For angle measure and relations the students were working within the visual level. For the post instruction interviews, see Table 5.7, the four students in TE1 improved and moved from the visual to the analysis level. The majority of the students were working fully within the analysis level (level two) at the end of the macro cycle.

Table 5.6

Teaching Experiment One: Pre Instruction Interview Summary

	V	VA	A	AI	I
Draws Angles		4			
Identifies Angle		4			
Sorts Angle		4			
Angle Measure	4				
Angle Relations	4				

Note. V indicates that those students are working at the visual level; A indicates that those students are working at the analysis level, and I indicates that those students are working at the informal deduction level. Two letters indicate that those students are working between two levels. Dominance in one level is not denoted on this table. The numbers represent the students working at that level. Table adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Table 5.7

Teaching Experiment One: Post Instruction Interview Summary

	V	VA	A	AI	I
Draws Angles			4		
Identifies Angle		1	3		
Sorts Angle			4		
Angle Measure		1	3		
Angle Relations			4		

Note. V indicates that those students are working at the visual level; A indicates that those students are working at the analysis level, and I indicates that those students are working at the informal deduction level. Two letters indicate that those students are working between two levels. Dominance in one level is not denoted on this table. The numbers represent the students working at that level. Table adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

In the pre instruction interview, these data show that the four students interviewed had misconceptions and a lack of understanding in various areas of angle and angle measure. Specifically:

- Students included irrelevant angle attributes and excluded relevant angle attributes when identifying, drawing, sorting and measuring angles.
- One student considered angle measure as the distance between the ends of the two rays.

From the literature review, these findings were expected and the instructional plans were developed to support students in these areas.

In the post instruction interview, these data show that students were still lacking in certain understandings, specifically that:

- Students lacked in spatial reasoning as students were unable to mentally manipulate angles to piece them together to create right or straight angles. Students also struggled to do this physically.
- Students had not begun to consider an infinite number of angles. Some of the students understood that there were 360° in a full circle but had not considered a fraction of a degree.

Changes were made to the instructional plans in TE2 to have students further consider what they understood about angles in order to avoid any misconceptions about orientation and the length of the rays. To work towards being able to formulate complete definitions of angle and be able to justify those conclusions. Other changes were made to support students in understanding that there are an infinite number of angles. These changes to the instructional plans are intended to provide the students with a deeper understanding that would move them towards level three thinking.

Interviews: Macro cycle two.

Ava. Ava's van Hiele levels indicate gains from pre to post instruction interviews. In the pre instruction interviews she included irrelevant attributes of angles in all of the various activities and in three of the five tasks she excluded relevant attributes of angles. For example, during the drawing activity when she described the difference between the angles she had drawn, she described some as being wider than others. When Ava was asked how many angles she would possibly draw, she guessed at five angles, although she drew three and was struggling to think of another angle. For the fourth angle she drew the angle in a different orientation than the others. When questioned, she said it was a different angle as it looked different as it was drawn a different way (orientation). Here she is including a non-salient angle attribute and her language “looked different” is indicative of a student using level one thinking.

In the post interviews she excluded the irrelevant attributes and included only the relevant attributes on all the tasks. For example, As Ava moved onto the task where she had to find angles from various figures she quickly found simple angles as well as external and internal angles in complex open and closed figures (see Figure 5.17). These angles were found without the need to prompt the Ava. There were a couple of figures where she had not circled any angles, she accurately responded that they could not be angles as they had curved lines or the lines did not connect. When asked how she would help others find the angles on the sheet she said that they have to look for two straight lines that were connected.

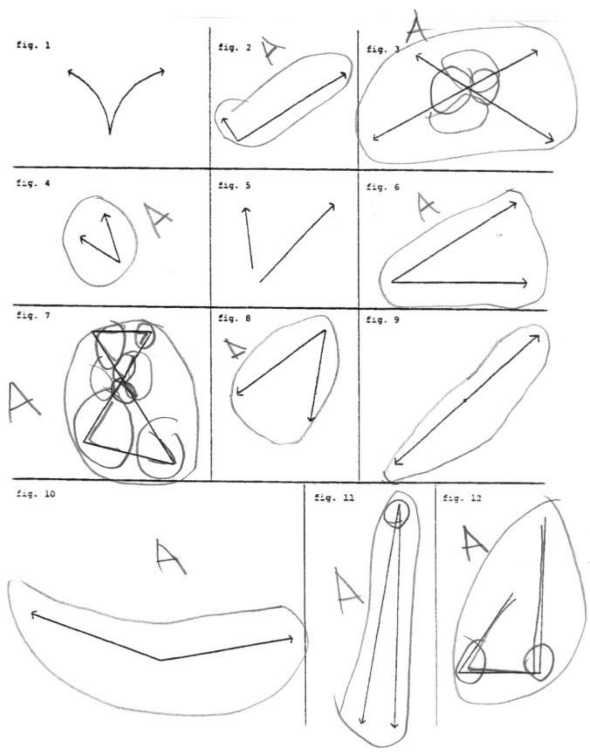


Figure 5.17. Ava's Post Instruction Interview Angle Identification.

In the pre instruction interview, Ava was only able to generalize angle properties during the sorting task; however, on the post instruction interview she was able to generalize these properties across all the tasks.

In the pre-interview, Ava showed little to no understanding of angle measurement. For a 90° angle, she guessed that it was 43° and her ideas of measurement seemed to change for each question. For example, the 90° angle she reported as 43° but when it came to the 110° angle, she reported this as being 40° and a 165° angle as 50°. Ava's ability to understand and estimate angle measure greatly improved on the post instruction interview. She was able to use the benchmark measure of 90° to draw quadrants during the angle measurement activity, to provide accurate or reasonable angle estimations, see Figure 5.18.

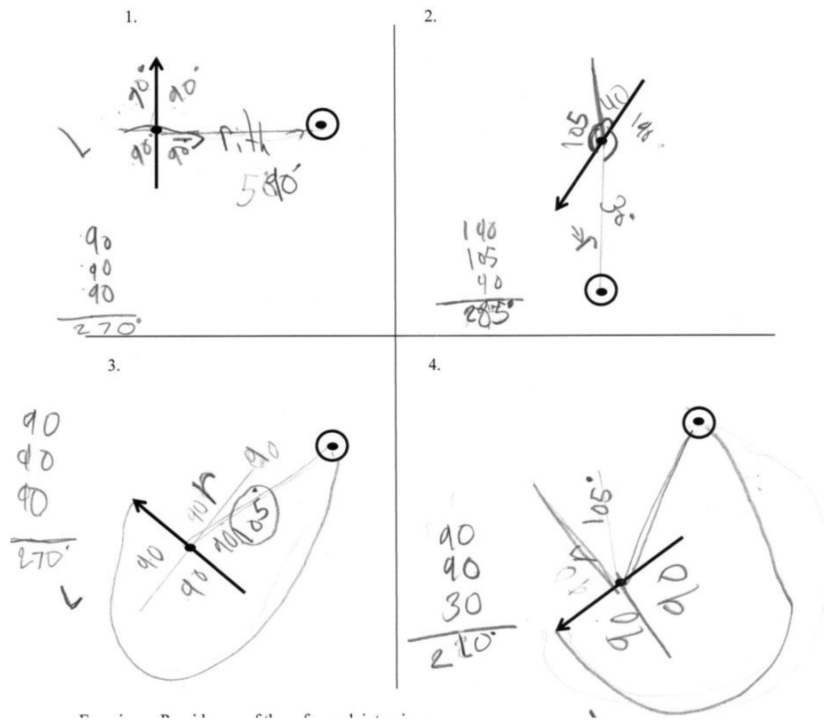


Figure 5.18. Ava’s Post Instruction Interview Angle Measure Estimations.

Table 5.8 provides a summary of Ava’s levels of thinking during the pre and post instruction interview. There is a clear progression from Ava working at the visual thinking level in the pre instruction interview and moving to the analysis level of thinking on the post instruction interview. In the pre instruction interview, she is working within the visual level for drawing angles, angle relations and angle measure. There is some evidence of her working in the analysis level as she sorts and identifies angles, but her thinking is primarily within the visual level of thinking. For the post instruction interview, Ava is working well within the analysis level for drawing, identifying and sorting angles, but for angle measure there is some movement to the analysis level but she is still working in the visual level. In the angle relations activity, Ava’s thinking is primarily in the analysis level of thinking, but her definitions of right and straight angles are indicative of a move towards the informal deduction thinking level, which is the third van Hiele level of thinking.

Table 5.8

van Hiele Levels by Interview and Task -Ava

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	V	A
Identifies Angle	Va	A
Sorts Angle	Va	A
Angle Measure	V	Va
Angle Relations	V	Ai

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Carl. Carl's van Hiele levels indicate considerable gains from pre to post instruction interviews. In the pre instruction interview, he believed that angles were straight lines pointing different directions. An example of this is when he was asked to draw four different angles he provided the drawings in Figure 5.19. He indicated that the difference between the angles (line) were that they pointed in different directions. Throughout the pre instruction interview, Carl had major misconceptions about angle and described orientation and length of the rays as the only attributes of angle, and that angles were lines that could be straight or curved. He stated that a right angle is a line pointing to the right (north-east) and a straight angle is a vertical or horizontal line.

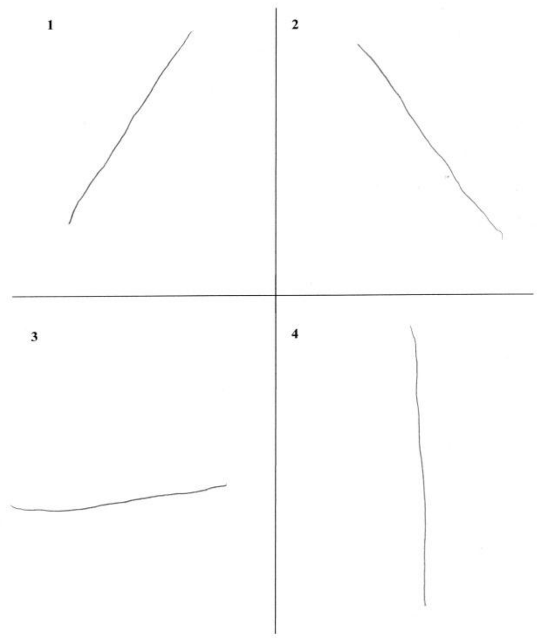


Figure 5.19. Carl's Angle Drawings in the Pre Instruction Interview.

The pre instruction interview results show that he includes irrelevant, excludes relevant, compares and sorts by looks and has little to no knowledge of angle measure.

In the post instruction interview, Carl made considerable gains in the understanding of angle. In the post instruction interview, Carl had removed the misconceptions that he had about angle attributes. He included only the relevant attributes of angle in his actions and descriptions, he analyzed by property and was able to generalize those properties in all the activities. For example, when asked to identify the angles on a sheet of figures, Carl correctly identified all the angles in various orientations and with different ray lengths. He also identified the internal angles on complex closed and open figures. On one particular complex figure he noticed straight angles, which had not been noticed by children in any of the pre and post instruction interviews, or on pilot interviews; see Figure 5.20, the straight angles are marked in red on the complex figure. Carl correctly described how he had not included figures if they had curved lines, or lines that did not meet at a common point.

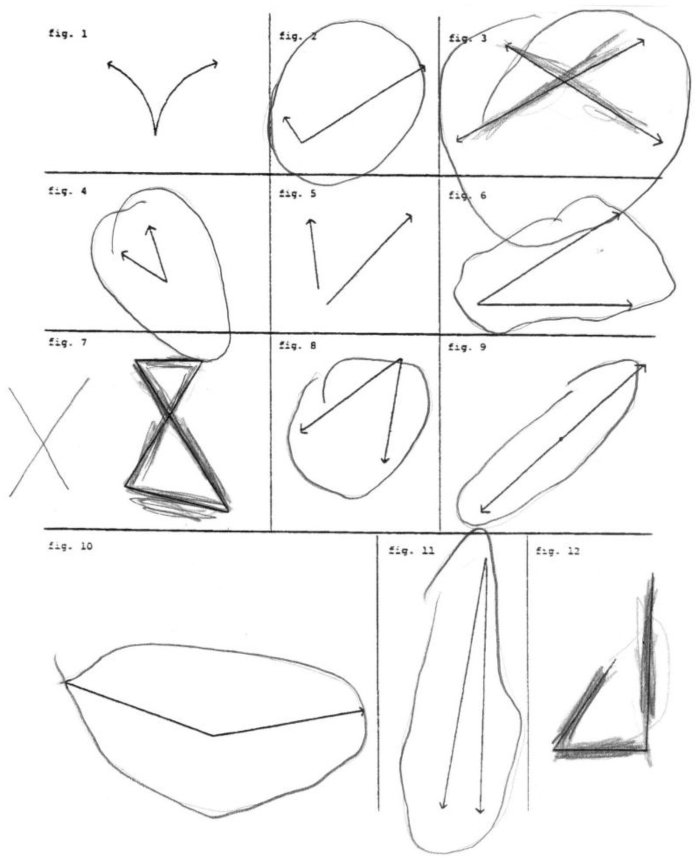


Figure 5.20. Carl’s Post Instruction Interview Angle Identification.

Table 5.9 provides a summary of Carl’s levels of thinking during the pre and post instruction interview. During the pre instruction interview, Carl was working in the visual thinking level in the pre instruction interview with some movement to the analysis level in the identifying, sorting, and relations activities. In the post instruction interview he was working well within the analysis level of thinking for the identification, sorting and measure of angles. During the drawing angles activity, Carl moved partially into the informal deduction level as he was able to indicate that there could be an infinite number of angles.

Table 5.9

van Hiele Levels by Interview and Task - Carl

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	V	Ai
Identifies Angle	Va	A
Sorts Angle	Va	A
Angle Measure	V	A
Angle Relations	Va	vA

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Grace. Grace's van Hiele levels indicate considerable gains from pre to post instruction interviews. On all five of the pre instruction interview tasks Grace included irrelevant attributes in her actions and descriptions of angle. For example, in the pre-interview, Grace was able to identify two angles correctly from the picture. Both angles were right angles, but in different orientations. However, she also identified the two curved lines on the top of the tower as an angle and when questioned, she said that the archways over the windows were both angles, see Figure 5.21.

Figure 5.21. Grace's Pre Instruction Interview Angle Identification.

In the post instruction interview, Grace was able to talk in detail about the angles she chose correctly in the picture. She described how angles have to have two straight lines and that they also had to be connected. She then said that straight lines would really be rays as the lines could go on forever and it did not matter if the lines on an angle were different lengths. When questioned about other possibilities for angles, she correctly accepted some as angles, and she said that the top of the tower and the window arches could not be angles as they were curved lines. During the pre instruction interview Grace excluded relevant attributes on four of the five tasks and sorted and compared by the visual appearance of the figures. In the post instruction interview, Grace only included relevant attributes and she only used properties and not visual appearance for each of the five tasks. In addition she was able to generalize those properties.

In both the pre and the post instruction interview, Grace did struggle with spatial visualization and in an activity which required her to physically place two cut-out shapes together she was not able to do so. Figure 5.22 is an example of when Grace made a right angle by just using the straight sides of two cut-out figures.

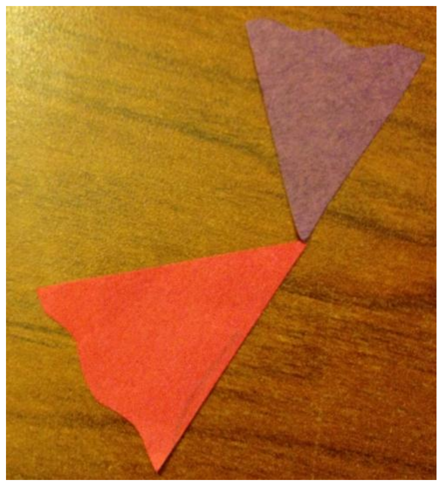


Figure 5.22. Using the Cut-Out Shapes Grace Made a Right Angle.

Table 5.10 provides a summary of Grace's levels of thinking during the pre and post instruction interview. For the pre instruction interview Grade was working within the visual thinking level for drawing, sorting, measuring, and relations categories. For the identification of angles, Grace did show some movement into the analysis level of thinking. For the post interview she was working well within the analysis thinking level with indications of working within the informal deduction thinking level for aspects of the drawing and relations activities.

Table 5.10

van Hiele Levels by Interview and Task - Grace

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	V	Ai
Identifies Angle	V	A
Sorts Angle	Va	A
Angle Measure	V	A
Angle Relations	V	Ai

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Isabel. Isabel's van Hiele levels indicate gains from pre to post instruction interviews. In the pre instruction interviews she included irrelevant attributes of angles in four of the five tasks and excluded relevant attributes in three of the five tasks. For example, When Isabel was shown a when shown a set of figures that were angles and non-angles, she pointed to a figure with two curved lines and said that the figure was an angle. Isabel compared by gestalt visual appearance and only in two of the five tasks did she analyze angles by their properties. In the post instruction interview Isabel only focused on the relevant angle attributes; she always analyzed angles by their properties and was able to generalize these properties to all angles. For example, when asked what people should specifically look for when identifying angles on a page Isabel said that they should look for two straight lines connected, and it does not matter how long the rays are or what direction it is facing.

In the pre instruction interview, Isabel showed no understanding of angle measure. This was interesting as she mentioned 90° during the angle identification activity. But this knowledge appears to be rote learned with no understanding as she was unable to give any estimate of degrees when a 90° angle was pointed out. When asked to estimate the angle size of a 20° angle she responded that it was zero degrees. In the post instruction interview, Isabel provided reasonable angle estimations. She also drew 90° quadrants to estimate the measure of the angle. For example, on the first measurement problem the 90° quadrants were oriented similar to the edges of the paper which was correct for the way the figure was positioned, see Figure 5. 23. However on the each of the following questions Isabel oriented the quadrants to the spinner and ball. This provided evidence that she was clear on where the 90° quadrants should appear and where appropriate she split a 90° quadrant into two 45° parts. For these larger estimations she used the 90° and 180° benchmark or she subtracted the initial number from the 360° full turn.

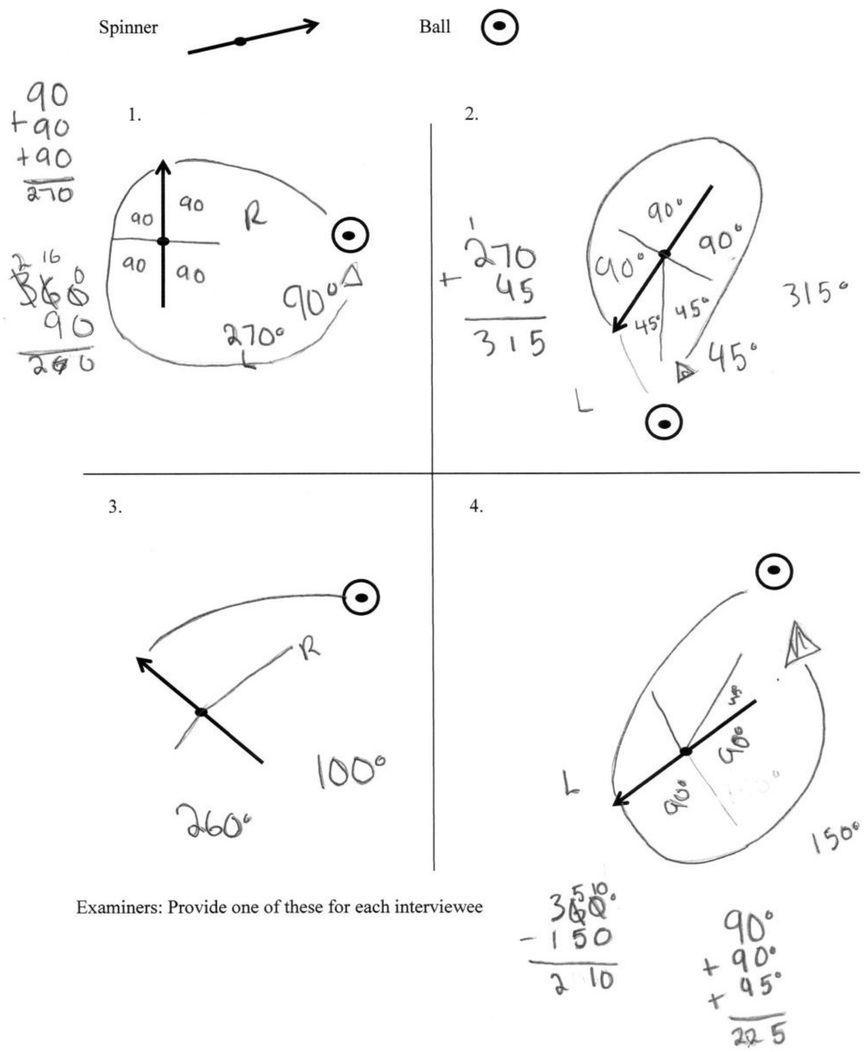


Figure 5.23. Isabel's Post Instruction Interview Angle Measure Estimations.

Table 5.11 provides a summary of Isabel's levels of thinking during the pre and post instruction interview. Isabel moved from the visual thinking level to the analysis level of thinking from the pre to the post instruction interview. In the pre instruction interviews, Isabel was at thinking at the visual level in angle measure and angle relations. For drawing and identifying angles she was working primarily in the visual level of thinking, but there is some evidence of analysis thinking. In the post instruction interviews, Isabel was working well within the analysis level for all activities. In addition, during the angle drawing and angle relations activities, Isabel showed signs of working within the informal deduction thinking level as he was able to define straight and right angles and described how that there were an infinite number of angles as the degrees could be broken into fractions of a degree.

Table 5.11

van Hiele Levels by Interview and Task - Isabel

van Hiele Tasks	Pre Instruction Interview	Post Instruction Interview
Draws Angles	Va	Ai
Identifies Angle	Va	A
Sorts Angle	A	A
Angle Measure	V	A
Angle Relations	V	Ai

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from “The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment,” by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Summary of macro cycle two. Students in TE2 predominantly scored within the visual level in the pre instruction interview with some students working partially between the visual and analysis level, see Table 5.12. One student was working in the analysis level for sorting angle during the pre instruction interview. For the post instruction interview, the majority of the students moved into the analysis level of geometric thinking, however, for drawing angles and angle relations three of the four students were working between the analysis level of thinking and the informal deduction level, see Table 5.13. This could be attributed to the addition of the infinite angle discussion in Lesson Five as students moved into level three thinking for drawing angles. For the angle relations this could be due to the addition of journal writing, requiring students to consider the language they used as they imagined their partner as a kindergartner student and with the changes to the DGE program to include extendable rays.

Table 5.12

Teaching Experiment Two: Pre Instruction Interview Summary

	V	VA	A	AI	I
Draws Angles	3	1			
Identifies Angle	1	3			
Sorts Angle		3	1		
Angle Measure	4				
Angle Relations	3	1			

Note. V indicates that those students are working at the visual level; A indicates that those students are working at the analysis level, and I indicates that those students are working at the informal deduction level. Two letters indicate that those students are working between two levels. Dominance in one level is not denoted on this table. The numbers represent the students working at that level. Table adapted from “The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment,” by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

Table 5.13

Teaching Experiment Two: Post Instruction Interview Summary

	V	VA	A	AI	I

Draws Angles		1	3
Identifies Angle		4	
Sorts Angle		4	
Angle Measure	1	3	
Angle Relations	1		3

Note. V indicates that those students are working at the visual level; A indicates that those students are working at the analysis level, and I indicates that those students are working at the informal deduction level. Two letters indicate that those students are working between two levels. Dominance in one level is not denoted on this table. The numbers represent the students working at that level. Table adapted from “The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment,” by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

In the post instruction interview, these data show that students were still lacking in certain understandings, specifically that:

- Angle is developed by a turn and angles are measured by the degree of that turn.
- Benchmark measures can assist students in estimating the measure of an angle.
- Practice in spatial reasoning is needed to gain these skills.

Changes were made to the instructional plans to have students label the benchmark to support students in internalizing these benchmark measures. Further discussion on angle as a turn were included using the dynamic protractor to support this understanding. For the spatial reasoning difficulties, students will need ongoing practice and this will need to be considered a skill to be practiced by students on a regular basis. As spatial reasoning is not a mathematical skill pertinent to angle and angle measure, changes were not addressed in the instructional sequence.

Effective Means of Support to Facilitate Understanding of Angle and Angle Measure

The theories and empirical findings surrounding the teaching and learning of angle and angle measure, clearly advocate for the use of real-world connections and Dynamic Geometry Environments (DGEs) to support learning. Context-aware u-learning is a term to describe a method of teaching and learning that brings together the DGEs and real-world referents that have been used in this study. In the local instructional theory it was conjectured that context-aware u-learning would be an effective support in helping students understand angle and angle measure. Task design was another identified means of supporting students' understanding of angle and angle measure. The task design used in this study encompassed the use of mathematical discussion, van Hiele-Geldof's (1957/1984) instructional phases, and consideration of cognitive load and the type of academic task.

Analysis to determine the efficacy of these supports was conducted through coding and in-depth clinical interviews to determine students' progression within the van Hiele levels of geometric thinking. The findings of this analysis were provided in the prior section of this chapter. However, this section provides an additional review of the data from both macro cycles including mini cycle reflective conversations, researcher's journal notes, and video transcripts to provide confirmatory and contradictory evidence. This evidence was then triangulated with students' artifacts from class and the clinical interviews.

Context-aware ubiquitous learning. An important progression in students' understanding of angle concepts is the movement into the analysis level (level one) from the visualization level (level two). This requires students to become aware of the physical attributes of angle. On the first day of the instructional experiments students went out into the real-world to find angles. As the students came back to class it was sometimes difficult for the students to recall the details about their angles. Some of the students were able to provide accounts of the attributes they remembered, but when quizzed about an additional property they could not always answer these questions.

As the students moved onto using the DGE to take photographs of the angles this was described as being very helpful by the observers. Once the students found an angle the photographs provided a visual record of the angle chosen. The teacher was also able to use those static visuals to determine if the students did actually understand. The screenshot provided a concrete artifact to evaluate and use to direct future instruction. For example, Figure 5.24 shows a screenshot taken by Stephen.

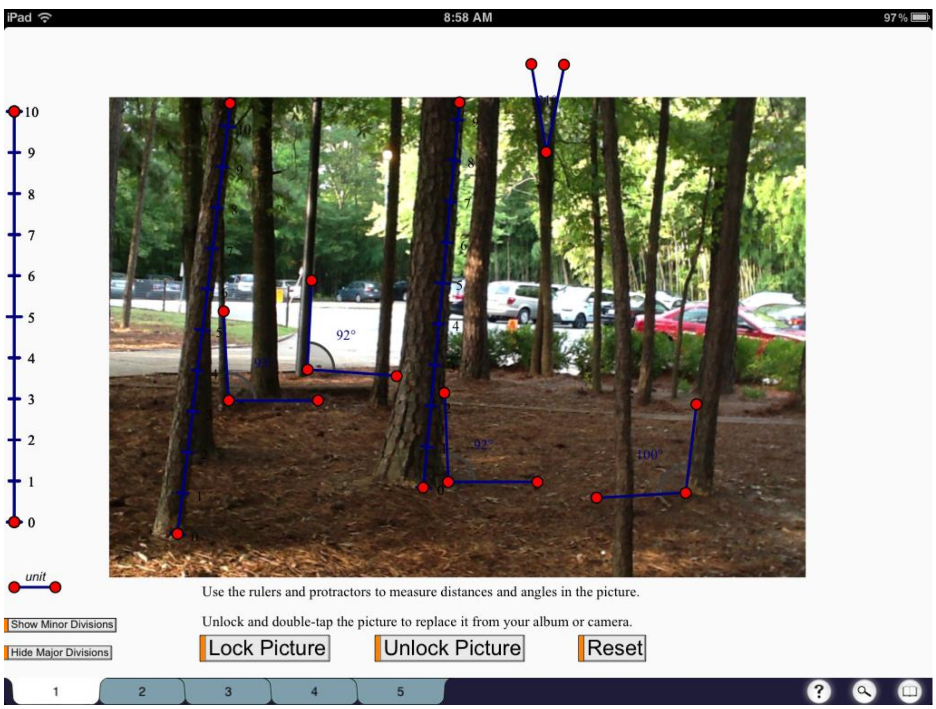


Figure 5.24. Angles in the Real-World.

From this photograph it appears that Stephen is not searching for angles based on knowledge of angle properties but on visual components that look like angles. Using this screenshot, the teacher was able to direct Stephen to focus on angle properties, such as looking for two straight lines that meet. In this particular case, it was also easier to provide feedback as the student was able to look at the screenshot to see if the angles met these criteria. The students were able to zoom in on an area to look more carefully at the attributes in the photographs.

In addition, there were a number of recorded cases during TE1 and TE2 when the DGE appeared to support the students thinking. This is an extract of one of the conversations from TE2:

> Isabel: When I measured the edge of that window it was 90°, then I measured the other window and it was 90°, (pointing to the adjacent window), but underneath I also have a straight angle and I just noticed that they are the same.

This was a point when Isabel was starting to notice supplementary angles. This has been supported by the ability the dynamic protractor gave her to measure the two windows and to also place another protractor underneath to see the three protractors together. However, in some cases, students needed to be reminded how to use the tools properly for the program to be helpful. For example, the screenshot in Figure 5.25 shows how one student placed four different protractors on the photograph but has not considered (a) measuring the artifact with the measure beginning at zero, (b) which angle they intend on measuring, or (c) if a measure seems appropriate (e.g., 266°).

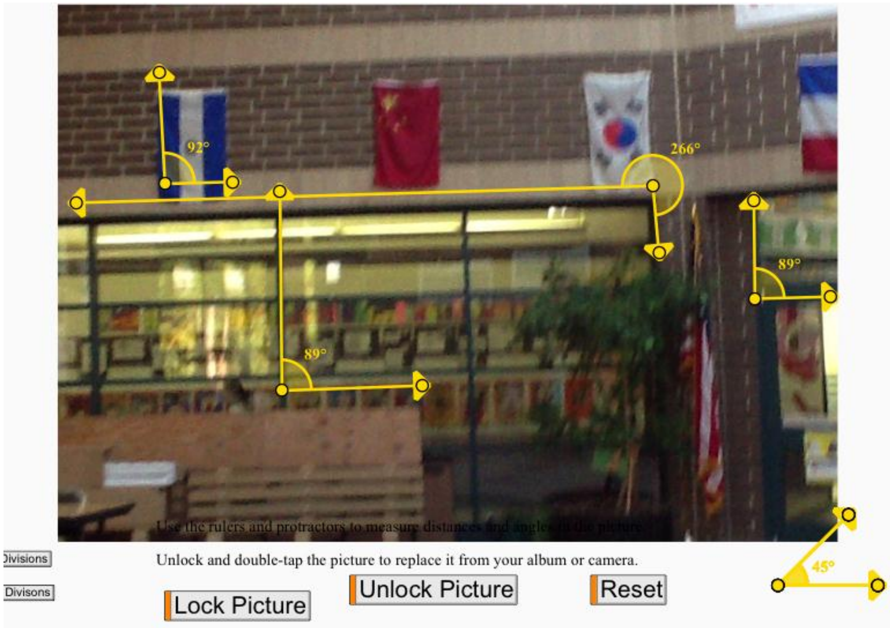


Figure 5.25. It is Essential that the Students Know How to Use the Tools Correctly.

In the majority of the screenshots taken by the students this could be evidence of the students mathematizing their everyday world. Mathematizing was described by Treffers (1987) as "…the organizing and structuring activity in which acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structures" (p. 247). Each student mathematized artifacts from the real-world across the three van Hiele levels as they use what they know about angle concepts and see how this fits with their environment and used the tools to explore further.

As the students moved from studying angles on paper to studying angles in a real-world context this may have supported students in generalizing angle properties. As students applied the properties to the different situations their accuracy increased in finding angles.

This was noticed from the first screenshots the students took on day two to the screenshots on day seven. See Table 5.14.

Table 5.14.

Accuracy in Identifying Angles on Day Two and Day Seven

	Teaching Experiment 1 Day 2 (n= 30)	Teaching Experiment 1 Day 7 (n= 30)	Teaching Experiment 2 Day 2 (n= 30)	Teaching Experiment 2 Day 7 (n= 29)
Angle	26 (28%)	35 (67%)	55 (87%)	43 (96%)
Non-Angle	68 (72%)	17 (33%)	8 (13%)	2 (4%)

Note. There were 29-30 students in each class; however, each student may have identified between one and five angles on each screenshot.

Other factors may have attributed to this increase in students finding angles; however, this positive increase was also noted in the researcher, co-researcher, and teacher’s notes and comments.

In the literature review Clements and Batista, (1992), Lehrer et al, (1998) and Yerushalmy and Chazan (1993) reported on the difficulty students have with determining salient criteria for judging angles. There were numerous occasions during the teaching experiment when it was observed how students were supported by the real-world connections and the DGE to determine which attributes were salient and those that were not. For example, Figure 5.25 and Figure 5.26 highlight two of many screenshots taken by students who have emphasized how the length of the rays does not matter when measuring angles.

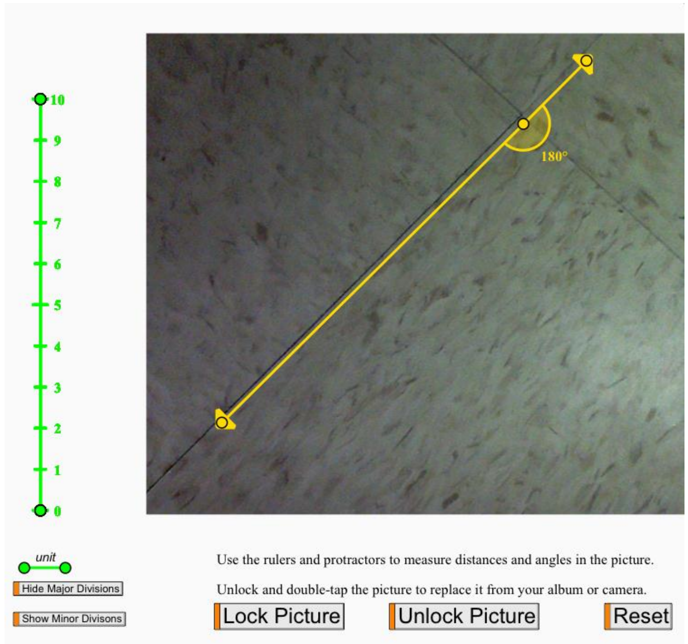


Figure 5.26. A Student Recognizing Salient and Non-salient Attributes of Angle Measure.

Many examples can also be found where students find angles in various orientations and during the screenshot activities angle orientation was not addressed as an issue for the students. For example, students did not look for right angles in the typical orientation found in text books as students were faced with right angles facing many different directions. This also corresponds with the findings during the post instruction interview as students often pointed out the non-salient attributes of angles, even without prompting. The change in the *Measure a Picture* program to have the students change the length of the rays was very helpful with students often seen demonstrating to teachers and other peers how changing the length of the ray did not make a difference to the angle size.

During both macro cycle one and two there were issues with students' lack of spatial reasoning. Congruent with the call by Gutiérrez (1996) to have students master finding angles in complex backgrounds, the real-world environment allowed students the opportunity to do this. However, the picture viewer in the program helped reduce this backdrop to a manageable size. However, in the final retrospective analysis it was noted from a number of different sources that students had difficulty with spatial orientation tasks and this was also evident in the post instruction interviews from TE1 and 2. In the final lesson, students were given the task of finding angles from different perspectives and to look at how the same angle appeared to be of a different measure. For example, Figure 5.27 shows two different pictures taken by a student of the video tripod that was in the classroom. The same angle measures 153° from a photograph taken from one angle and 184° taken from another direction.

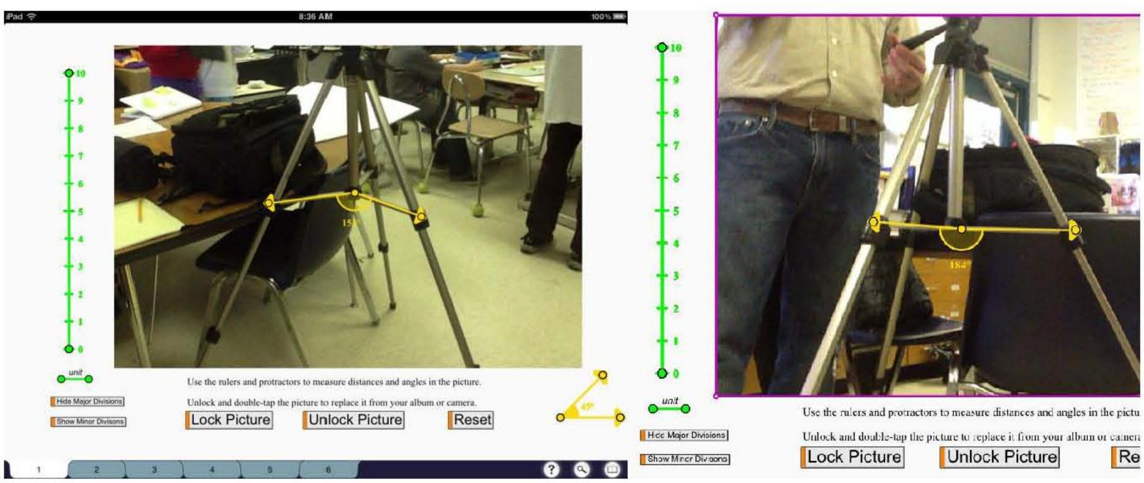


Figure 5.27. Angles Measures Taken from Different Perspectives.

Some of the students found this activity a little confusing. This is an extract from one such conversation:

Chris: So, I have measured this angle (the edge of the calendar on the wall), and it is 112°, but when I measure it from here (straight on) it is 90°. So how can you tell what the angle is if is like different from where you are?

Teacher: When you look at it straight on, that is the true measure of the angle.

Chris: So how is it different when you are somewhere like else? I don’t get it.

This was similar to questions and comments other students had. Following further examples and discussion, a few students still struggled to understand how it could appear to be a different angle as you stand in different positions. During the interview the students had to use spatial orientation to compose right and straight angles from cut-out shapes. Many of the students appeared to have difficulties with this task. The findings are displayed in Table 5.15.

Table 5.15

Results of Spatial Visualization Tasks

	Teaching Experiment 1 Pre interview		Teaching Experiment 1 Post interview		Teaching Experiment 2 Pre interview		Teaching Experiment 2 Post interview		Total
	Right Angle	Straight Angle	Right Angle	Straight Angle	Right Angle	Straight Angle	Right Angle	Straight Angle	
Correctly from the Cut-Out Shapes	2		4	2			2	2	12
Incorrectly from the Cut-Out Shapes	2	4		2	4	4	2	2	20

Correctly from the Angles on Paper			1				1		2
Incorrectly from the Angles on Paper	4	4	3	4	4	4	3	4	30

Note. n = 4 for TE1 and n = 4 for TE2.

A slight improvement was evident from the pre instruction interview to the post instruction interview. However, on the post instruction interview only two of the eight children could piece two angles visually together to make a right angle and none of the students could make a straight angle by putting two of the angles together.

Task design. In looking at the task design as a means of support for students' coming to understand angle and angle measure, two broad areas were considered. First, the academic tasks were considered in regard to Doyle's (1983) four categories of academic tasks: procedural or routine tasks, comprehension or understanding tasks, and opinion tasks. Stein, Smith, Henningsen, and Silver's (2000) cognitive load model was also addressed to complement Doyle's model. This model delineates four levels of cognitive tasks: low cognitive demand tasks - memorization, low cognitive demand tasks –procedures without connections, high cognitive demand tasks – procedures with connections, high cognitive demand tasks – doing mathematics.

Furthermore, van Hiele-Geldof's (1957/1984) instructional phases were utilized in the lessons. These phases were designed to promote learning through each of the van Hiele levels of geometrical thinking (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/ 1984). The second broad area addressed in this section is the practice of academic talk or accountable talk supported by Chaplin et al.'s (2009) Talk Moves. During the teaching experiments Talk Moves were included to assist students in participating in academically productive conversations.

Academic tasks. The lesson plans provide the conjectured van Hiele-Geldof instructional phase the students are expected to be working within. The lessons were carefully constructed guided by the instructional phases. Table 5.16 provides a summary of these levels and following is a brief description of each of the phases.

Phase 1: *(Inquiry/Information)* During this initial stage, students get acquainted with the geometric concepts as the students engage in conversations and activities about the objects of study. For example, students examine examples and non-examples of angles. Students make observations and questions are raised.

Phase 2: *(Guided orientation)* Students explore the concept through a carefully designed sequence of activities. The activities are designed to slowly reveal particular characteristics of the concept.

Phase 3: *(Explication)* Students have now gained some understanding of the geometric concept from the earlier activities. Technical language will be introduced, and during this phase in the activities, students will be encouraged to express and exchange views about the geometrical phenomena while using the technical language.

Phase 4: *(Free orientation)* Students work on more difficult activities to use the knowledge they have gained in the other phases. They will be asked to select parts of this newly gained knowledge to solve problems, or develop further relationships.

Phase 5: *(Integration)* Activities would involve students summarizing all that they have learned about the subject. Students will be asked to develop a newly organized network of what they understand about the geometric concept.

Table 5.16

Van Hiele-Geldof Instructional Phases for each Lesson

Lesson	Van Hiele-Geldof Instructional Phases
1	• Initial Inquiry • Direct Orientation
2	• Explication
3	• Guided orientation • Explication
4	• Guided orientation • Explication • Free orientation • Integration
5	• Explication • Free orientation • Integration
6	• Free Orientation • Integration

In Lesson Two, students were working in the Explication phase as part of the van Hiele-Geldof instructional phases. Students in this stage have gained some understanding of the geometric concepts from the earlier activities in day one as they have learned the term angle and studied what attributes constitute an angle. During Lesson Two the students went out into the playground or the atrium to identify, photograph and demarcate angles using the tools within the DGE. Students were then asked to check with their partner to see if they also agreed that it was an angle that they had found. These data gathered from this activity in TE1 and TE2 was coded using the instructional phases to determine if students appear to be working within the Explication phases described by the van Hiele-Geldof�given phases of instruction.

The coding showed that the majority of the students appeared to be working within the guided orientation phase rather than the explication phase. The majority of the students spent their time looking for angles, taking the photographs and marking the angles by using the dynamic protractor. As students found the angles in various orientations with rays of different lengths, the students were developing the idea of what angles were salient and those that were not. This matches the description of phase two as the activity was slowly revealing particular characteristics of the concept.

The few that were engaged in discussion with a partner may have moved into phase three as they began to use the technical language. Data gathered from the mini cycle discussion and the researcher's initial reflection notes have the students working within phase three describing how these conversations were happening. However, the classroom video camera and small group camera recorded very little technical language used by the students. The majority of the student interactions recorded involved students pointing to the iPad screen to show their partner with pointing often replacing the majority of the conversation, especially in TE1. This was slightly improved upon in TE2 as students were specifically asked to work with a partner to have these conversations, although the discussion was still minimal with screen showing the dominant form of communication.

One of the main activities in Lesson Four was the angle card sorting activity. This lesson was listed as covering multiple instructional phases including guided orientation, explication, free orientation and integration. For the card sorting activity students would be expected to be working within instructional phase three and four (explication and free orientation). These data from this activity were again coded using the instructional phases. TE1 was considerably different than TE2. The activity was intended as a group activity to have the students working together; however, in TE1, once the cards were handed out to the group the students worked individually. In most of the groups the students took a handful of cards each and started placing the cards in individual piles. This was evident from watching the video and was reported by the observers.

For TE2, a change was made to the instructional plans to have the students deal the cards so each person had a set number of cards. Next, the students had to take it in turns to place a card down in a pile. The rest of the group had to see if they agreed if it was correct or not and the student had to justify their answer. This task was a high cognitive task as the students had to use what they had learned about the salient and the non-salient attributes of angle and the different angle categories to choose where to place the cards. In addition, the students have to justify their answers. Below is an extract from the video transcripts of a group of three children, Carl, Grace, and David conducting the sorting task. This extract also shows the coding for the van Hiele level of geometric thinking.

David: Okay, okay, these two are acute angles. Do you agree?

Isabel: Yes

Carl: Yes, because they look like acute angles (*level 1*).

David: And then it is your turn.

Grace: Well…I think…these (sorting through the cards she has) are all acute.

David: Yes, agree.

Teacher: What do you think Carl? Did you see what happened over here? Do you think these are all acute angles?

Carl: I'm not sure.

Teacher: Grace, tell us why you think these are all acute angles.

Grace: Well, they all have two lines and meet together at one point so they are angles (*level 2*). But... they have a narrow opening and they are smaller than 90° like a right angle, like that (pointing to a right angle; *level 2*).

Carl: Erm. well I think these are are straight angles

David: Erm, okay, I agree.

Carl: They are straight angles because they look like straight lines with a dot on them (*level 1*).

Although this is a high cognitively demanding task, students are working at different levels. Carl is working at Van Hiele level one and Grace shows evidence of working at a level two. There is evidence of students working at the explication phase of instruction as they use the names of the various angle categories. In addition, there are also times during the activity when students are working at the free orientation phase of instruction as they use the knowledge they have gained from earlier instruction to sort the cards. Carl may also add to his understanding as he listens to Grace justify her answers by recapping on the salient attributes of the angles.

Mathematical discussion. The researcher made a number of comments about the mathematical discussions as these were easier to conduct that expected in TE1 and TE2. In both classes, the classroom teachers often held many discussion activities and students were accustomed to this practice. The teachers reported that they did not use any specific model but what they did use was similar to the Talk Moves model. To move into fully working within van Hiele level three, students need to be able to create definitions and modify these following discussions with others. Students also need to justify their answers and practice doing this. Table 5.17 provides a list of the talk moves used in Lessons One, Three, Four, and Five.

Table 5.17

Talk Moves

Talk Moves	
Move 1	Revoicing. ("So you're saying that it's an odd number?")
Move 2	Repeating: Asking students to restate someone else's reasoning. ("Can you repeat what he just said in your own words?")
Move 3	Reasoning: Asking students to apply their own reasoning to someone else's reasoning. ("Do you agree or disagree and why?")
Move 4	Adding on: Prompting students for further participation. ("Would someone like to add something more to this?")
Move 5	Waiting: Using wait time. ("Take your time…. We'll wait…")

Table 5.25. Adapted from "Classroom Discussions: Using Math Talk to Help Students Learn" (2nd ed.), by S. H. Chaplin, C. O'Connor, N. Canavan-Anderson, 2009. Sausalito, CA: Math Solutions.

The observers' comments were very positive towards the discussions with talk moves incorporated. Recorded comments include:

- It really made the students think?
- They knew that they may be called upon to say what someone else had said in their own words so it makes them really pay attention.
- I was amazed at the connections the students were making.
- I loved how they were clearly making connections from what they have learned from prior lessons and integrating this into the comments to add to what someone else had said.
- They knew that you were going to wait for an answer. Some just keep quiet so you quickly go on to someone else (Talking about what often happens in classrooms).

The video transcripts also provided some evidence that students were really thinking mathematically as students were actively involved making conjectures and using justifications for their responses. It was also recorded how students were often forward thinking and considering topics that were to be addressed later in the lesson or the following day. For example, this extract was taken from Lesson Four as the students in the class make a chart to determine salient and non-salient attributes of angle.

Teacher:	What would go on this side? What are the things that are not important about angle?
David:	How long the lines are.
Teacher:	Would someone like to add to Scott's comment?

Scott: But, how can that be right. I know what we have talked about but, but the space inside the angle will change if the lines are longer.

Teacher: What do you think to that comment?

Scott: The space inside will change in the lines are longer so when you measure the angle it will be different.

Claire: You don't measure the space but the little thing in the corner.

As students had this debate, they wanted to know how angles were going to be measured and this led into the measurement lesson the following day.

Changes to instructional materials

A number of changes were made to the instructional materials throughout TE1 and TE2 from the findings during the teaching experiments and the interviews. A summary of these changes can be found in the lists below. The first list has the main changes that were made to the instructional materials in TE1. Although many of the changes were made throughout the mini cycle analysis, the students in TE1 did not get those changes. Only the TE2 students got those changes. Similarly, the second list has the changes that were made throughout TE2 and were changed within the instructional materials found in Appendix B. Students in TE2 did not get these changes.

Summary of changes made to instructional materials in TE1

- Mathematical language reduced in Lesson One.
- Mathematical journaling was added.
- Revisions made to Measure a Picture.
- One discussion added about angles found in manufactured or natural settings.

- Students work in teams for angle sorting activity.
- An infinite angle discussion was included.
- Discussion included about the importance of beginning at zero measure.
- Describe the pivot foot to help students understand angle as a turn.
- Students write the degrees of measure on the benchmark wedges.

Summary of changes made to instructional materials in TE2

- After Lesson Two, the instructional plans have the teacher telling the students to just focus on one angle in their photograph, not multiple angles.
- An additional emphasis on having students discussing the mathematical concepts when working in pairs.

In summary, the findings in this chapter have been presented organized on the two research questions: (1) How do students come to understand angle and angle measure through the use of real-world connections and technology enabled learning tasks? (2) What are effective means of support to facilitate understanding of angle and angle measure? The findings of the initial question are provided by examples taken from the teaching experiments relating to the van Hiele levels of geometric thinking. At the end of this section the findings of the clinical interviews are reported. As the findings of the initial question are provided, there is substantial overlap with the findings for the second question; what are effective means of support. However, the final section of this chapter uncovers further evidence from the various data sources triangulated with students' artifacts from class and the clinical interviews to supply additional information that was not already covered. An interpretation of these findings is offered in chapter six. This includes a revised instruction theory about students’ development of angle and angle measurement concepts, making use of real-world connections and technological tools through the use of context-aware u-learning.

CHAPTER SIX: CONCLUSION

This study began with the design of a conjectured local instruction theory about students' development of angle and angle measure, making use of real-world connections and technological tools through the use of context-aware ubiquitous learning (context-aware u-learning). A conjectured local instruction theory consists of a learning process and a means for supporting that process (Gravemeijer & Cobb, 2006). The means for support of the learning process involved the development of a set of instructional materials that reflected the conjectured local instruction theory.

The purpose of this research was to develop an empirically-based instruction theory for students' learning in this mathematical context along with a set of instructional materials that reflected this theory. Therefore, this dissertation aims to address the following research questions:

1. How do students come to understand angle and angle measure?
2. What are effective means of support to facilitate students' understanding of angle and angle measure?

The initial conjectured local instruction theory was developed from a thorough review of the literature that is presented in Chapter Two. This review encompassed research-based developmental trajectories and effective instructional supports for promoting students' understanding of angle and angle measure. It was conjectured that context-aware u-learning was a useful instructional support for students coming to understand angle and angle measure. Context-aware u-learning in this study involves the use of real-world connections, iPads, with the applications *SketchPad Explorer, Measure a Picture*, and *Inigma*. In addition, learning activities were constructed to incorporate mathematical discussion and used van Hiele-Geldof's (1957/1984) instructional phases.

An instructional sequence of six lessons was designed for use in fourth grade classrooms. These lessons involve seven class periods. Using a cyclical iterative process of anticipation, enactment, evaluation, and revision (Gravemeijer & van Eerde, 2009), two macro cycles were conducted. In the previous chapter, the findings from the two macro cycles are presented. The purpose of this chapter is to present a revised theoretical framework and set of instructional materials based on the results from the design-based research (DBR) presented in this dissertation.

A local instruction theory developed within the context of DBR describes a frame of reference for designing and engaging students in a set of exemplary instructional activities which support students' learning of a particular mathematical concept (Gravemeijer's, 1999, 2004; Nickerson & Whitacre, 2010). These two main components of the local instruction theory serve as the foundation of the two research questions addressed through this research. Therefore, this revised local instruction theory presented in this section is a discussion of how students come to understand angle and angle measure and of effective means of support to facilitate understanding of angle and angle measure.

The first half of this chapter addresses revisions of the conjectured local instruction theory concerning how students come to understand angle and angle measure through the use of real-world connections and technology enabled learning tasks. It builds from the original theory found in Chapter Three (pages 48-52). This is followed by the revised sequence of tasks which reflect the changes and additions made to the six lesson, seven-day instructional design that was initially presented in Chapter three as part of the conjectured local instruction theory. Finally, the implications for this research are provided, as well as the limitations and areas for future research are discussed.

Revised Local Instruction Theory

The initial conjectured local instruction theory is explained in Chapter Three (pages 48-52). Gravemeijer and Cobb (2006) described the conjectured local instruction theory consisting of a learning process and a means for supporting that process. The literature review identified a number of different frameworks to use as lenses for the way in which students develop an understanding of angle and angle measure. In particular, the van Hiele levels utilized by Scally (1990) provided a set of level indicators that encompass both angle and angle measure. Mitchelmore and colleagues (viz., Mitchelmore, 1989, 1993, 1998; Mitchelmore & White, 2000, 2004, 2007) provided a focus on angle abstraction and generalization, and Piaget and Inhelder (1948/1967) offered a thesis on spatial reasoning in relation to angle concepts. A list of 12 essential understandings were developed from the review of the literature, these are:

1. Recognize angles as geometric shapes that are formed whenever two rays share a common endpoint.
2. Understand that angles can be identified in a real world setting.
3. Recognize that there are an infinite number of angles.
4. Recognize and compare angles based on size using non-standard and standard language (acute, obtuse and right angles).

5. Recognize acute, obtuse, and right angles in different contexts (real-world and paper and pencil).
6. Recognize acute, obtuse, and right angles in different orientations and with rays of different lengths.
7. Recognize salient attributes of angles, such as two rays with a common endpoint.
8. Understand that angles can be measured with reference to a circle and that angles are fractions of a circle.
9. Understand that angles are measured by units called degrees.
10. Understand that benchmarks can be used to understand angle measures. For example, a full circle turn is 360°, straight angle is 180°, and right angle is 90°.
11. Recognize that the same angle can appear to be a different size depending on different visual perspectives.
12. Understand that angles are defined by particular attributes which involve angle as a turn (e.g., "two rays, the common endpoint, rotation of one ray to the other around that endpoint, and measure of that rotation"; (Clements & Sarama, 2009, p. 186).

The local instruction theory did not change from the beginning of the first macro cycle to the end of the retrospective analysis at the end of the second macro cycle. What emerged from the review of the literature is the importance of context-aware u-learning tasks using real-world connections and applied technology learning tasks to support students' understanding of angle concepts.

The revised instructional sequence is presented in Table 6.1. This table includes the lesson objectives organized in sequence as the learning progression across the six lessons. The van Hiele-Geldof (1957/1984) instructional phases covered within each lesson are listed. Next, a brief overview of the instructional activity is provided. The complete lesson plans can be found in Appendix B. In the following sections, the changes in the instructional sequence are discussed.

The instructional sequence in Table 6.1 is developed from the local conjectured instruction theory to include the way in which students learn about angle and means of supporting that process. The way in which students learn about angle is included in the sequence of lesson objectives developed from the essential understandings highlighted in the literature review. The means of supporting that process are the activities listed for each lesson which include real-world connections and technological tools as part of context-aware ubiquitous learning. During the instructional sequence and the retrospective analysis, changes were made to the initial instructional sequence. These changes are reflected in Table 6.1 and described in full in the following section.

Table 6.1

Overview of the Instructional Sequence

Lesson	Day	Learning Progression	Instructional Phases (van Hiele-Geldof, 1957/1984)	Instructional Activity
1	1	Recognize angles as geometric shapes that are formed whenever two rays share a common endpoint. Identify angles in a real-world setting.	• Initial Inquiry • Direct Orientation	Students are introduced to the concept of angle via projected images of different examples of angles in different orientations with sides of different lengths. The term angle is introduced. Students look for angles in the real-world.
2	2	Identify angles in a real-world setting. Begin to recognize that there are an	• Explication	Students are introduced to the application Sketchpad Explorer and taught how to use the DGEs to take photographs and how to use the

		infinite number of angles.		dynamic protractor. Students take photographs of angles in a real-world setting disregarding orientation and length of rays. Students will use the tools in the DGEs to highlight the angles found.
3	3 & 4	Recognize and compare angles based on size using non-standard and standard language (right, obtuse, acute, and straight angles).	• Guided orientation • Explication	Students will work in groups making angles with straws and compare size of those angles using non-standard language. Introduced to the terms: right, obtuse, acute, and straight angles. Using the benchmark of 90° on the dynamic protractor, students find examples of right, obtuse, acute, and straight angles in a real-world environment. An angle gallery will be created from the screenshots. Students will work in pairs to discuss the

				categorization of an angle in the real-world and check their accuracy using QR codes.
4	5	Recognize acute, obtuse, right, and straight angles in different contexts (real-world and paper and pencil. Recognize salient attributes of angle.	• Guided orientation • Explication • Free orientation • Integration	Students work in groups to categorize acute, obtuse, right, and straight angles. Class discussion to create a table of important and non-important attributes of angles.
5	6	Understand that angles can be measured with reference to a circle and that angles are fractions of a circle. Experience using a nonstandard unit of measure (a wedge). Recognize that the attribute being measured is the space between the two line segments caused by the turn	• Explication • Free orientation • Integration	Wedge activity to create benchmarks. Using the wedges to measure a set of materials such as a coat hanger, books, scissors, and a car ramp, noting that the latter two can be changed to vary angle size.

		of the line segments. Understand that angles are measured by units called degrees. Understand that benchmarks can be made for angle measures. For example, a full circle turn is 360°, therefore a straight angle is 180° and a right angle is 90°. Recognize that there an infinite number of angles.		
6	7	Recognize that the same angle can appear to be a different size depending on different visual perspectives (positions). Understand that angles are defined by particular attributes which involve	• Free orientation • Integration	Students work in pairs to photograph and measure angles from different perspectives. Work in groups to create a poster to define angle to students who have not yet studied angle.

	angle as a turn (e.g., “two rays, the common endpoint, the rotation of one ray to the other around that endpoint, and measure of that rotation”; Clements and Sarama, 2009, p.186).	

Changes to the instructional sequence.

Lesson One. The first lesson in this instructional sequence requires the students to focus on the properties of angle. Students are introduced to angles, and they are required to decompose the figure into the salient components. Various angles are displayed with rays of different lengths and in various orientations to avoid the students considering these non-salient attributes.

There were two changes made to Lesson One following macro cycle one. First, as students were initially asked to study the figures (angles), some of the students were trying to focus on mathematical words, such as angle. To avoid this happening, students were asked to describe the figures as though they were speaking to a kindergarten student. This pushed the students to describe the basic properties, such as two lines.

Second, during this lesson, the students are exposed to a number of different new ideas and concepts. To help the students reflect on this new information and consolidate what they had learned. There was an addition made the lesson plan to have students write in their math journals to respond to the following question “What is an angle?”

Lesson Two. The objectives for Lesson Two were to have students identifying angles in a real-world setting and begin to recognize that there are an infinite number of angles. A small yet substantial change was made to the second lesson objective. The plans initially said that the students will recognize that there are an infinite number of angles. The activities in this lesson are developed to have the students notice that there are many different types of angles; however this objective was moved to Lesson Five as students learn in greater detail angle measure and fractions of an angle.

A considerable change was made to the computer program *Measure a Picture* which students begin using in Lesson Two. To avoid the students considering the length of the rays to be salient angle attributes, the rays in the program were altered to allow students to change the lengths. The dynamic protractor was also changed to a brighter color so it could be easily seen on the real-world photographs that the students took. An addition was added to Lesson Two to include a discussion about the best places to find angles from manufactured or natural settings. This focused on the salient properties of angle and that straight lines were more easily found in man-made constructs rather than nature.

A portion of Lesson Two involved teaching the students how to use the program *Measure a Picture*. After the students were given this tutorial, they are asked to conduct a sequence of tasks, which involve remembering how to use the program. To assist the students in remembering the tasks, a small addition was added to the lesson plans to include a list of directions for the students to follow.

At the end of Lesson Two, the findings indicated that a short whole class assessment would be useful in identifying those students who have not yet moved onto level two thinking. This way, the teacher can then provide additional materials to support that transition. This was not added to the lesson during the research cycles; its inclusion would be part of future research plans.

Lesson Three. The lesson objective for Lesson Three was for students to recognize and compare angles based on size using non-standard and standard language. The modification to the computer program was especially useful with this objective. At the beginning of the lesson, as the students summarized what they have learned over the past couple of days, this discussion was facilitated by the teacher. An additional point was added to the instructional plans that large classes should initially allow students to hold the initial discussion in pairs before feeding back those discussions to the whole class. This would allow the students to reflect on the question and provide their answers to their partners who could work with them to refine their answers.

At the end of the final retrospective analysis, a change was made to the iPad activities. Often students would take a photograph of one angle, and then, as they looked at the photograph, the students would find a number of other angles and also highlight them with the dynamic protractors available. As the students did this they would not always spend the time focusing on the initial angle. The instructions were changed to have the students focus on just one angle for the screenshots. The final change to this lesson was to have students finish Lesson Three by writing descriptions of the angle categories in their math journals. Students would be provided with the names of the categories, but have to describe each category in their own words.

Lesson Four. This lesson has two objectives: recognize acute, obtuse, right, and straight angles in different contexts (real-world and paper and pencil) and recognize salient attributes of angle. One of the main activities is to have students sort angle cards into different categories. During teaching experiment one (TE1), the students did not work in teams, but instead, they each took the cards and worked on this activity alone. They did check some of the answers with each other at the end as the students had various piles of cards, but this was not the purpose of placing students in teams. The activity was to have students discussing the placement of the cards throughout the activity.

This was changed to have the cards dealt to each student in the team. Then they would take turns to place a card into the correct category pile. The rest of the team would then look to see if they agreed with the category the students had chosen. The student would have to verbally justify why they had placed the card in that particular pile. This addition to the instructional plans appeared to be effective during teaching experiment two (TE2) as the students were seen working in groups having discussions about why the cards had been placed in particular groups.

At the end of Lesson Four, the findings indicated that a short whole class assessment would be useful in identifying those students who do not yet indicate any movement onto level three thinking. This way, the teacher can then provide additional materials to support that transition. This was not added to the lesson during the research cycles; its inclusion would be part of future research plans.

Lesson Five. This lesson was the first in the instructional sequence to formally focus on angle measure. The lesson had five objectives which focused on various aspects of angle measure as students: understand that angles can be measured with reference to a circle and that angles are fractions of a circle; experience using a nonstandard unit of measurement (a "wedge"); recognize that the attribute being measured is the space between the two line segments caused by the turn of the line segments.; understand that angles are measured by units called degrees; and understand that benchmarks can be made for angle measures. For example, a full circle turn is 360°, therefore a straight angle is 180° and a right angle is 90°, and how to use those benchmarks to think about approximate angle measures.

There were three major changes or additions made to the instructional plans. The first change was for students to write onto the benchmark paper wedges the degrees of measure, specifically, on the 90° and the 180° benchmark. This was to provide an extra visual reminder to support the students in internalizing the benchmarks. As the students use the paper wedges to measure angles, an addition was added to the instructional plans to include a brief discussion about the importance of measuring by always beginning at zero. In other words, to have students measure angles by lining up the wedge measure with one side of the measure on one of the angle rays. During this discussion the students are asked to make the connection to their experience of linear measure during which they have to place the ruler next to the object and line up the start of the object with zero.

A change to Lesson Five was to include the lesson objective, *recognize that there are an infinite number of angles.* This was moved from Lesson Two as the students would not have the experience with angle or the knowledge foundations to understand fractions of a degree at that time. This objective was moved to Lesson Five and the connection to fractions was made with linear angle measure that the students are already familiar with. An additional discussion was added to the lesson to have students understand that there can be fractions of a degree not just the 360 different measures. This is a difficult concept for the students in this grade as they are only just learning about fractions. It can be hard for students to transfer this new understanding to a different context. Therefore, this discussion is basic in that it connects to linear measurement that the students are familiar with and asks them to consider parts of a unit not a whole unit but ½ or ¼ of a unit.

The final change to this lesson was to help the students understand angle as a turn by making another real-world connection by having the students think about the pivot foot in baseball as similar to the end point where the two lines meet in an angle. This would help the students to understand that the end point does not move, but the ray (the leg) moves while the other stays still. With the dynamic protractor this provides another real-world connection to a concept which can be confusing for many students.

Lesson Six. The objectives for Lesson Six require students to recognize that the same angle can appear to be a different size depending on different visual perspectives (locations) and to understand that angles are defined by particular attributes which involve angle as a turn. A few of the earlier changes supported students during this lesson in TE2. For example, the changes that were made to measure a picture were helpful as students considered the salient attributes of angle. The connection to the movement of angles as a rotation around the endpoint as similar to the pivot foot in baseball was an addition that helped students consider and describe angles in Lesson Six.

In summary, the results from this study affected the local instruction theory as the embodiment of this theoretical framework. The resulting instructional plan appears to have been an effective sequence of tasks that enable students to understand angle and angle measure while avoiding many common misconceptions that can arise during instruction. In the final section of this dissertation, limitations of the study and areas for future research are discussed.

Limitations and Future Research

The purpose of DBR is a methodology that supports the development of and research concerning a local instruction theory to be used to support students learning concepts in mathematics. DBR is used in this study to address the following research questions: (1) How do students come to understand angle and angle measure through the use of real-world connections and technology enabled learning tasks? (2) What are effective means of support to facilitate understanding of angle and angle measure?

This research process began with a comprehensive review of the literature leading to a conjectured local instruction theory. This conjectured theory described the process by which students come to understand angle and angle measure and a means for supporting this process was theorized. The outcomes of this research include the revised local instruction theory which was presented in the previous section and the revised instructional materials which can be found in full in Appendix B. In this final section, general conclusions, limitations of this study, and areas for future research are discussed.

Overall, context-aware u-learning is a valuable mathematical context for introducing students to angle and angle measure. Common misconceptions about angle can be avoided as the students study angles in the real world which presents them with angles with rays of different length and in various orientations. Good foundations were built by having the students consider angle by the generalizable properties and over the seven days the students who were interviewed showed great movement across the van Hiele levels of geometric thinking. The students interviewed also showed considerable gains of the basic understanding of angle measure. Nonetheless, the content and the length of the instructional sequence are not adequate to fully understand angle and angle measure and further work is needed especially in understanding angle measure.

It was essential that students gained crucial foundational understandings of angle before moving on to angle measure. This moved the formal measurement component to Lesson Five of six. Students need to now build on this knowledge to provide practice to enable the students to internalize benchmarks more effectively and to better understand how angle is developed by a turn. This will provide students with a more sophisticated understanding of what angle is measuring.

A limitation to this study is the method by which the data was collected. The clinical interviews appeared highly effective in allowing the researcher an opportunity to fully probe to determine at what levels of thinking the students she interviewed were working. However, each interview took approximately 30 minutes to conduct. This is impractical in a class of thirty students and to determine the whole class other methods must be sought. Pierre van Hiele (1957/1984) pointed out that students may appear knowledgeable, but that knowledge may be rote learning. In addition, he noticed that knowledge intrinsic at one level appears in an extrinsic way at the next. For example, while a child may be using particular properties to determine the name of a shape, the actual thinking at that level may not be cognizant of those features.

To fully determine students thinking, van Hiele posits that students must be observed. For future research of this nature, researchers should make every effort to determine the level of the other students in the class. This could be conducted by a modified version of the clinical interview to include just a few questions to determine students' thinking in the core angle understandings. This will provide data of the whole class to determine how much learning has taken place. In addition, a pre and posttest could be administered to the entire class to again provide evidence of learning. As well as answering various questions on angle and angle measure, the students would also be asked to explain their thinking in words.

As there are no formative or summative assessments of all students included in the instructional materials this limits the amount of data I can collect and therefore cannot generalize that data beyond the small group of students who were interviewed. The limited number of student participants in this study does not effectively allow for comparison of students' learning across demographic variables, such as class, race, or gender. These could be important variables to consider, especially with context-aware u-learning requiring students to be highly active participants in the learning process.

From the data that were collected from the class, the researcher noted some limitations with the screenshot data. The screenshot data gathered from each student often had multiple pieces of data on each screenshot. The instructional plans were changed to purposefully have the students only identify one angle per screenshot. This would enable students to focus on the properties of just one angle instead of multiple. However, for the data collection for this study some students found up to five angles in one screenshot while others identified only one. In the tables, each angle was identified separately and this may have skewed the data either positively or negatively. If this research was to be conducted again, it would be crucial to have students only identify one angle per screenshot.

To build from this study, future research could develop the continuing sequence of lessons to have students fully understand both angle and angle measure. In particular, that study would need to incorporate activities to building students' spatial reasoning skills and the transfer of basic measurement skills to angle measure. In doing so, students gain further practice to internalize angle benchmarks. To have students move into the third van Hiele level, these activities would involve the Talk Moves to have students making justifications to explain their reasoning. Future studies with a larger participant pool may be able to determine how this instructional sequence meets the needs of learners from various demographic variables. In addition, future studies could focus on the design of formative and summative assessments which can be used in the instructional materials and to interview larger participant pools. Using these assessments there are indications from this study to say that there may be some slight changes to the local instruction theory.

This study is significant as it appears at a time when elementary mathematics teachers are being required to rethink their mathematical practices with the implementation of the

Common Core State Standards. These standards describe the authentic, real-world problems that students should be challenged with. In addition, the promise and potential of using mobile devices is now rapidly becoming apparent and there is widespread interest amongst parents, students, principals and teachers. One significant challenge to this implementation is the lack of teacher training and knowledge on how to successfully implement such technological tools. This study provides a set of instructional materials that can be adapted for use in other fourth grade classrooms and could provide ideas to other educators.

APPENDIX A

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990, pp.346-348)

First Level: Visualization

In general, the student identifies, characterizes, and operates on angles according to their appearance. Specifically the student:

1. Draws angles.
2. Identifies, names or labels angles in a simple drawing or more complex figure by relying primarily on visual cues rather than properties of the angle. The students may use standards or non-standard language (such as referring to angles as corners).
3. Includes irrelevant properties or relationships when describing angles, such as length of ray.
4. Excludes relevant properties or relationships when characterizing angles, such as straightness.
5. Sorts angles on the basis of their appearance as a whole. Specifically not having the 90 degree referent, making inconsistent sorting, or sorting by an irrelevant attribute.
6. Analyzes or compares angles (in tasks including, but not limited to: turning angles, congruent angles, complementary angles, or supplementary angles) on a looks-like basis.

Second Level: Analysis

In general, the student establishes properties of angles and uses properties to solve problems. Specifically, the student:

1. Analyzes and compares angles in terms of their properties.
2. Identifies relationships among angles within figures.
3. Recalls and uses appropriate vocabulary for relationships, such as corresponding angles are congruent.
4. Describes angles with a litany of properties or insufficient properties rather than necessary and sufficient properties.
5. Is able to decentrate in a task to determine angle measure, as indicated by accurate estimates of amount of turn and by deciding which way to turn. (Whether a student can orient turning relative to a spinner's position rather than to his or her own body's position is called the ability to decentrate.)
6. Accurately estimates angle measure by using known properties (such as right angles measure 90 degrees) or by insightful approaches.

Formulates and uses generalizations about properties of angles in problem solving situations and may use related language (all, every, none) but (a) does not explain how properties are interrelated, (b) does not use formal textbook definitions[11], (c) does not explain

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[11] The literature review highlighted the multiple ways in which definitions can differ and while this will not be omitted from the scoring criteria, the researchers will remain cognizant of the type of definition provided and if the definition and other explanations focus on the physical attributes, or if the dynamic properties are also included.

subclass relations, (d) does not see a need for logical explanations of such generalizations and does not use language related to explanations (if-then).

Third Level: Informal Deduction

In general the student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments. Specifically the student:

1. Identifies necessary and sufficient properties in the context of a justification.
2. Formulates and uses complete definitions, (a) explicitly referring to them, (b) accepting equivalent forms of definitions, and (c) accepting new definitions of previously learned concepts.
3. Is able to conceive of an infinite number of angles.
4. Explicitly describes relationships between properties, including sub-class relations.
5. Presents an informal argument/informal proof. Justifying the conclusion using logical relationships of properties: orders properties, interrelates several properties, and/or discovers new properties by deduction.
6. Present an informal argument/informal proof deductively (implicitly using such logical forms as the chain rule, or explicitly using “if-then” for example).

APPENDIX B

Lesson One
Day 1

Objectives

1. Recognize angles as geometric shapes that are formed whenever two line segments share a common endpoint.
2. Identify angles in a real-world setting.

Van Hiele-Geldof Instructional Phases

- Initial Inquiry
- Direct Orientation

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990)

I expect that students will be working at level one as they identify angles based on their appearance as a whole using standard and non-standard language. Students will often use irrelevant properties or relationships when describing angles.

Materials and Equipment

- Computer and projector (angles can also be drawn onto poster paper).
- Image of various angles
- Cardboard tubes
- Small students whiteboards and pens

Prior Knowledge

As the first lesson in this sequence, students are expected to already have some knowledge about geometry and measurement. In geometry, students should be able to identify vertices in shapes and be able to understand the terms line segment and endpoint. However, if students do not use this language early in this lesson they should not be asked to do so. It is important that the students initially use their own language to describe the figures. Geometrical terms will be introduced at the end of the introductory activity.

In measurement, students should already be able to: recognize length as an attribute, compare lengths, recognize the need for units of equal length, measure by using multiple identical units, and measure by using a single iterative unit. Students will be expected to have these skills for linear measure before moving onto angle measure.

Learning Environment

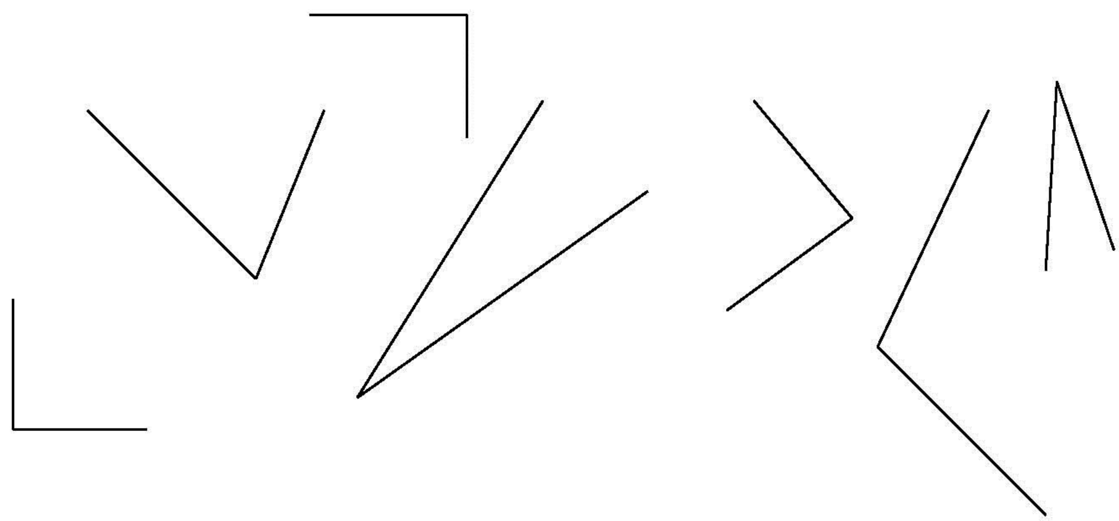
The activities in this lesson will take place both inside and outside the classroom. When selecting a place outside the classroom, steer the students towards man-made artifacts that have straight lines, e.g., climbing frames. During the lesson, students are asked to work in pairs. Where possible, students should be able to select their partners. This will allow students to work with someone they feel comfortable sharing with. During the discussion, students will be asked to use the talk moves of adding on, revoicing, and wait time. Adding on is where the teacher prompts the students for further participation in asking them to add on to what one student has said. In the revoicing move, the teacher repeats what the student has said and asks the student if this is correct. This move is useful if the students' initial response is not fully clear. Wait time is used to allow students time to think: the teacher will tell the student to take his/her time in answering the question.

Teaching Notes

One of the main objectives of this lesson is for children to recognize angles as geometric shapes that are formed whenever two line segments share a common endpoint. However, this lesson follows a constructivist approach in that children must come to this understanding themselves through the carefully planned activities. The teacher is a facilitator throughout this lesson and only teaches directly at certain points which have been identified in the plan. For example, in the initial activity, various angles are displayed for students to see, it is imperative that the teacher does not teach at this point in the lesson, students should describe what they observe and do so using their own language. One the students have identified what an angle is, it is then appropriate to use the formal mathematical terminology.

Before Phase

- Students will be introduced to the concept of angle via a computer projected image of a collection of different examples of angles. The teacher will introduce the shapes as figures at this time and <u>not use the term angle/s</u>. Write the word *figure* on the board so students are clear of the word.



The angles are intentionally portrayed in different orientations with sides of different lengths, to avoid the misconception that orientation and length of sides are salient attributes of angle.

- Students will initially work in pairs to answer the questions:
    - What can you tell me about these figures from what you have noticed?
    - What do all these figures have in common?

Have the students imagine that they have to explain to a kindergarten student what they see in simple terms. At this point visual observations such as *lines, a point, and two lines in different directions* are expected. If students are able to use the terms line segment and endpoint these should be accepted, but other language such as lines should also be included if this is the language used by the students. If students use the word angle, this will not be ignored, but it will not be discussed further at this point in the lesson. It is important for the students to recognize what constitutes an angle before giving the figure a name.

- The whole class will come back together to answer the question:
    - What do all these figures have in common?

Pairs will share their ideas and the students' descriptions will be written onto chart paper, using the language the students used. During the discussion, students will be asked to use the talk moves of revoicing, adding on, and wait time.

- Following the discussion, the term angle will be formally introduced by the teacher. It will be written on the board and the teacher will model using the term and connecting this back to the attributes identified at this time from the language the students used.
- The teacher will then recap on prior teaching of the terms ray, line segment and endpoint. The words will be posted on the classroom wall for students to see. These terms will then be matched to the angles from this activity.

During Phase

- Students go out into the area surrounding the school to identify angles in the real-world setting. They will be specifically asked:
    - "As you look around our environment (clarify meaning of environment), can you see any angles?"

They will be asked to look and not to answer at this time. Some difficulties can be expected as students may struggle to see angles in a different context than the paper and pencil angles observed earlier.

- The teacher will support the

VOCABULARY CHECK

Figure

Some students may not have been exposed the term figure. Ensure that all students understand that a *figure* can be a 2D or 3D shape that is open or closed. Therefore it may be a shape such as a square, a circle, or an open shape such as the angle examples.

End Point

A point is an exact location in space.

Line Segment

A line segment is a part of a line that has two end points.

Ray

A ray is part of a line that has one endpoint and continues on in one direction.

students in pointing out some examples and non-examples to discuss with the class; the angles will be chosen of various sizes and orientations. Remind the students about the salient properties of angle, such as two straight lines and ask them to consider if you would typically find straight lines on natural objects or man-made. Then guide the students towards man-made objects to find examples.

- Following the teacher examples, three or four students can be chosen to point out angles they have seen. For each answer, the teacher will work with the class to go through the attributes described earlier to see if they fit the angles pointed out by the students.
- Students will then work in partners to look for angles. Students will all be given cardboard tubes (such as kitchen tissue tubes) to use as a viewer to minimize the amount of visual information being processed while the students are searching for angles. Teacher should model the use of the tube by putting the cardboard tube up to one eye and closing the other eye.

- As the students find an angle, have them draw this angle on the white boards.
- If students talk about the length of the line segments with the angles they are finding in the real-world setting, address this by asking the students if it mattered how long the "lines" (use the language the students used in class) were in the angle figures during the first part of the lesson. If students do not bring up this point, do not address it. When

students find angles in the real-world the line segments will be of different lengths and it is helpful for the students to implicitly understand that this does not matter, before addressing this formally.

After Phase

- Student return to the class for the class discussion. The discussion will be based on the following the questions:
    - "What did you notice about the angles you found?"
    - "When you found an angle, how did you know that it was an angle?"
    - "Can you describe one of the angles you partner found?"(Students will be warned to expect this question during the last pair activity). This final question uses the talk move of repeating.

  Tell the student that the objective of the lesson was to determine if students can identify what an angle looks like using non-formal language, and if students could identify angles in a real-world setting connecting that angle attributes identified earlier in the lesson to the angles identified.
- Have students complete a quick write in their math journals using the question *What properties does an angle have?* This will help the students synthesize what they have learned from the lesson and for the teacher to assess what the students have understood.

Lesson Two
Day 2

Objectives

1. Identify angles in a real-world setting.
2. Begin to recognize that there are an infinite number of angles. (This will be continued in Lesson Five as students look at actual degrees of measure.)

Van Hiele-Geldof Instructional Phases

- Guided Orientation
- Explication

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990)

It is expected that students will still be working at level one as sorting will primarily be based on the appearance of the angles as a whole. Angles will be analyzed on a looks-like basis.

Materials and Equipment

- iPad 2 for each student.
- SketchPad Explorer app downloaded to each iPad 2. http://www.dynamicgeometry.com/General_Resources/Sketchpad_Explorer_for_iPad.html
- SketchPad add-on program downloaded to each iPad 2. http://sketchexchange.keypress.com/browse/topic/all-topics/by-recent/1/448/measure-a-picture For instructions on how to include the add on program please use the first link in this list and scroll to near the bottom of the screen for simple instructions.
- iPad display program (e.g., AirServer) to display the iPad screen on the projected computer screen.

Prior Knowledge

This lesson begins with a tutorial on how to use the add-on program *Measure a Picture* within the application (app) *SketchPad Explorer*. Before this lesson, students are expected to have a basic knowledge of how to use an iPad. For example, students should be able to turn the device off and on, open and close apps, and being familiar with the way they have to touch the screen rather than using a mouse. This may only take a 20 minute lesson with students to gain this familiarity.

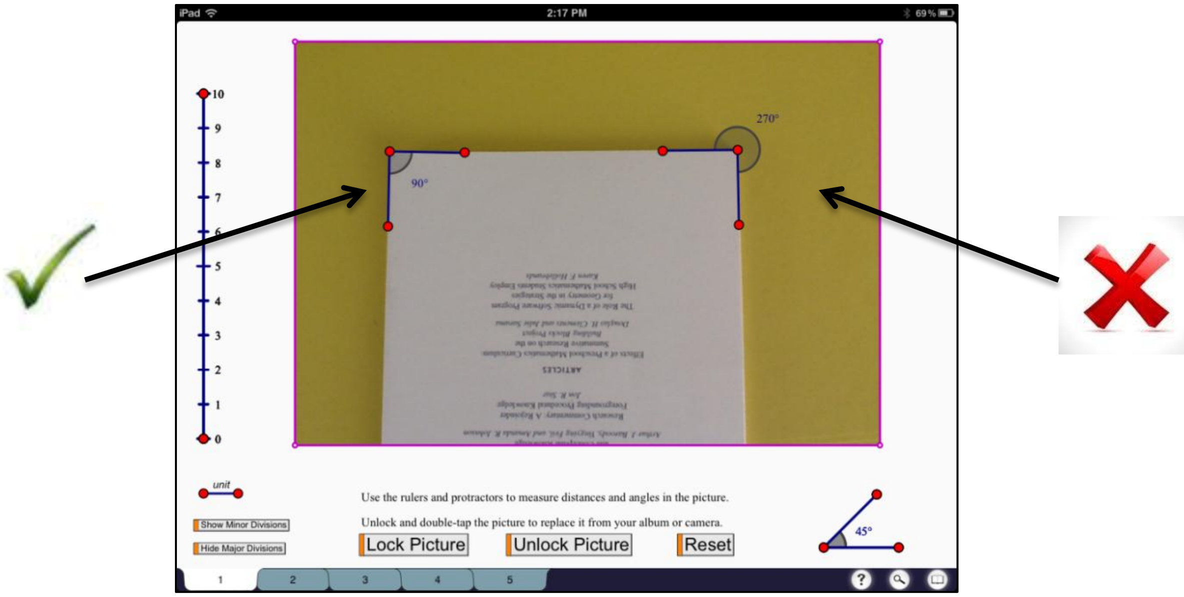
For the main component of the lesson, students will be expected to draw from the prior lesson to be able to identify and describe angles. Students should be able to recognize angles draw as paper and pencil images, and begin to identify angles in a real-world setting. Students should be able to use the term angle, line segment and point/end point.

Learning Environment

Students will be working inside the outside the classroom for this lesson. There are two main goals to this lesson: identify angles in a real-world setting, and recognize that there an infinite number of angles. The first part of this lesson involves students learning how to use the program.

Students will hopefully become accustomed to using the app. quickly, although working in pairs will help support this process. Students will use the program to photograph angles in a real-world setting and then place the dynamic protractor onto the angle they have identified. Students have learned how to identify angles in paper and pencil form and also had a short lesson in identifying angles in a real-world setting. However, the teacher needs to be aware that there may be some difficulty in students identifying angles in the photographs as they have changed from three-dimensional images to two dimensional images. The teacher should work with those students who are having difficulties to help them see the real-world example and how it is represented on screen.

In this lesson, the dynamic protractor is used by the students to identify the angle/s that they have found. Although the dynamic protractor also identifies the degrees of those angles, this should not be pointed out by the teacher. It is most likely that the students notice the numbers, but the teacher should say that the students can look at these numbers as they are going to be discussed in the following lessons. It is important that the students observe the turn of the dynamic protractor as it moves and that students ensure that the measure is the correct way. For example:

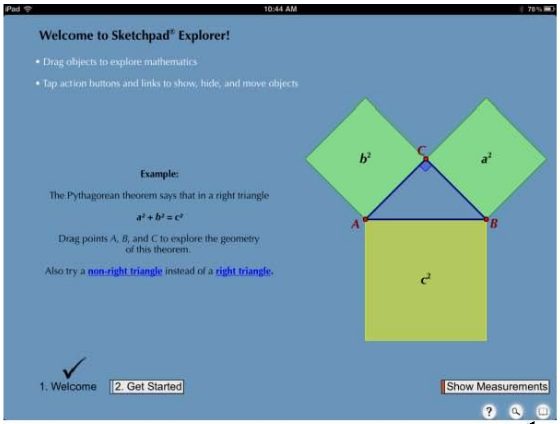


Before Phase

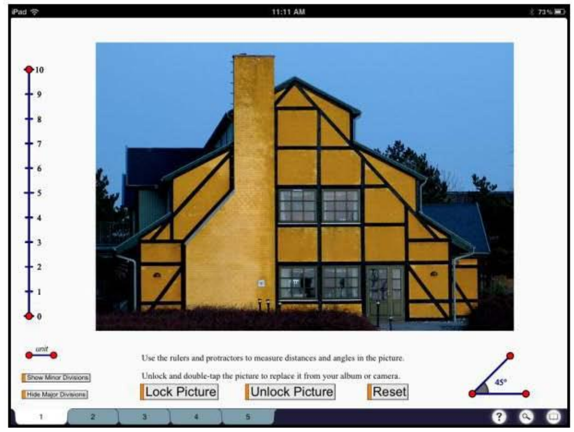
- During the before phase for this lesson, the main aim is having the students explore how to use the application Sketchpad Explorer on the iPad. Each student will be given an iPad2, with Sketchpad Explorer loaded onto the device with the add-on sketch titled Measure a Picture. At the beginning of lesson two, the students will be introduced to Sketchpad Explorer, and taught how to use the DGEs to take photographs and use the dynamic protractor.

  The SketchPad Explorer icon looks like this and will be found on the main iPad screen. Have students tap on this to start the program.
  Using a program (e.g., AirServer) to share the iPad screen the teacher can demonstrate these steps to the students.

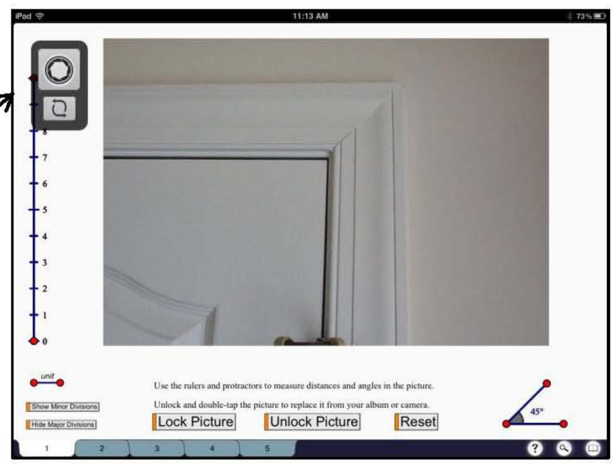


1. This is the screen you will arrive at. Click on the very corner button to get to the **Measure a Picture** add on program.

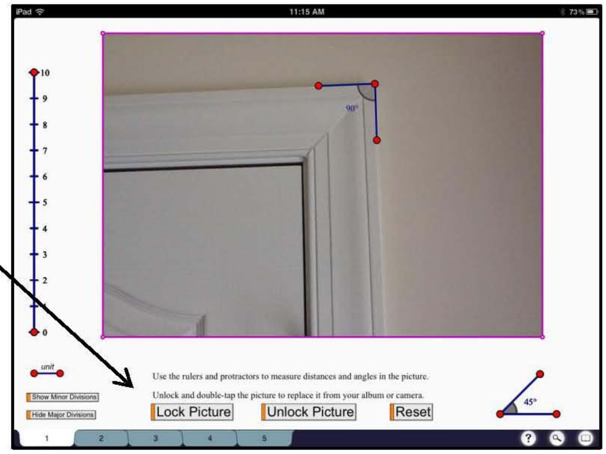


2. This is the Measure a Picture add on program. Students can practice using the Dynamic protractor tool (bottom right). The red circles allow the user to move the protractor.

3. To change the picture, the student must double tap on the picture and choose *Replace from Camera*. This will bring up this button on the top left of the screen to take photographs. Note, this button can be moved anywhere on screen to where is comfortable to the student. With the help of the viewer, the student will move the iPad until they obtain the angle they wish to photograph.



4. Once the student has taken the photograph, tell the student to select lock picture from the bottom middle of the screen.
This will stop the picture moving, then the student can practice using the dynamic protractor on the photograph they have taken.



- The dynamic protractor will not be named as such; the tool will just be demonstrated and described as a tool to be used in the DGE. However, through questioning the students will focus on the angle attributes this shape has which were described in the last lesson. The demonstration will be done with the iPad connected to the projected screen so the class can see what the screen will look like.

- Students will also be taught how to take screenshots to save their work to the device.

Screen Shots

To take a screen shot, press the on/off switch and the center home button at the same time. A camera click sound will indicate that the photograph has been taken.

- Students will practice using Sketchpad Explorer to take photographs of angles in class. The viewfinder of the camera will minimize the viewing area, similar to the effect of cardboard tube from the day before.
- The teacher will demonstrate getting into position to take photographs of the angles from a direct front view. Later in the sequence, students will take photographs from different angles.

During Phase

- Students will go back outside, to the area surrounding the school, and will be asked to work in pairs to take photographs of angles using Sketchpad Explorer. Conduct a brief recap on the main points from yesterday's lesson. This can be conducted through questions such as:
  - What did we learn about angles yesterday?
  - How would you identify an angle?
- As the students find angles, they will be asked to use the protractor to place against the angle to identify the different angles found in the one picture. Students may focus on one angle or multiple within one photograph, and they will work with a partner to initially confirm with each other that they have found an angle based on the discussion of angle attributes. The teacher will move among the pairs checking for understanding, or any difficulties and misconceptions
- Once the students have had a practice at finding the angles, each student should choose one angle to photograph. They will then place the dynamic protractor on top of the angle and take a screen shot. Students may need to be reminded how to take screen shots at this time. Once the students have their angle photographs, they should continue to work in pairs to answer the questions:
  - How could you prove that your angle is actually an angle?
  - How is your angle different than your partner's angle? (Are they still both angles?)

After Phase

- Go over the lesson objectives and point out that these are not told at the beginning of the lesson as they have to come to these understandings through the activities rather than being told.
- For the final part of the lesson, students will come back to the classroom to share screenshots with the rest of the class via a projected screen. It is useful at this time to create good examples that can be used for future classes, or have teacher created examples on hand to use. Probing questions will lead to the students in the class coming to the conclusion that there are an infinite number of angles. Probing questions include:
    - Are any of these angles the same (and if so, how)?
    - How are the angles different? Students talk in partners and then as a class to answer the questions
    - Could we find other different angles?
    - Could you tell me how many different angles you could find?

Lesson Three
Days 3 & 4

Objectives

3. Recognize and compare angles based on size using non-standard and standard language (right, obtuse, acute, and straight angles).

Van Hiele-Geldof Instructional Phases

- Guided orientation
- Explication

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990)

During this double lesson, it is expected that students will move on to level two. Students will be analyzing and comparing angles in terms of their properties and appropriate vocabulary will be used. Students will estimate angle measure by using known properties or benchmarks (such as right angles measure 90°. Angles less than 90° are acute, and angles that are greater than 90°, but less than a straight angle are obtuse angles).

Materials and Equipment

- Straws
- Document camera
- Poster Paper
- Sticky tape or stick glue
- iPad 2 for each student
- Computer Projector
- Cable to connect the iPad to the computer projector
- QR Reader app. (e.g., i-nigma) loaded onto the iPads
- QR Codes (printed in this lesson plan)
- Sticky Putty

- iPad display program (e.g., AirServer) to display the iPad screen on the projected computer screen.

Prior Knowledge

Before moving onto this lesson, students should have a met the objectives from the first two lessons. Specifically, be able to recognize angles as geometric shapes that are formed whenever two line segments share a common endpoint, and to identify angles in a real-world setting. The final objective of lesson two was to recognize that there are an infinite number of angles and if students do not fully have this understanding, the gallery activity will help in developing that understanding.

Learning Environment

This lesson lasts for approximately 90 minutes, two 45 minute lessons. The goals of this lesson are that students will recognize and compare angles based on size using non-standard and standard language (right, obtuse, acute, and straight angles). During this lesson, students will be involved in the guided orientation phase involved in looking for relationships, and the explication phase as new terminology is introduced. Students will be working inside and outside the classroom for this double lesson. A lot of the work will be students working in pairs. It is helpful where possible for students to work with their choice of partner, to ensure the students are comfortable talking with that student.

Before Phase

- The teacher will have students recap on what they have learned over the past couple of days. This will be done through a question and answer session with the teacher using the talk moves *repeating, adding on, and wait time*. The repeating will ensure students are both listening to other students in the class, but also understanding their reasoning. Adding on will have students delve in deeper to what they have learned as they have to add on to the answers of others. Wait time will enable students to have time to think about what they have learned.

  The questions could include:

- How would you describe an angle to a new student who had not learned about angle? (Students would be prompted to describe the angle in detail using geometric terms: angle, line segment, and end point.)
- (Drawing non examples and examples onto chart paper, or a white board) Is this an angle? Why/why not?

  These questions could be done with pair share if the class is large, then they can feed back to the group. When working with a partner or the class, have the students be clear in their descriptions, using non-standard or standard language.

During Phase

- Students will be placed into groups of four/five. They will be given a set of straws and asked to make angles using the straws. The teacher will demonstrate by placing two straws together on the document camera, making a 45°angle, then another as a 150° angle (approximately). Straws will be of different lengths to help avoid students' misconceptions of the length of the line segments being salient angle attributes. Tell the students to make two or three angles with the straws.
- Students will compare the angles they have made with straws, to the angles the other students have made in the group. To avoid having students consider orientation, they will be specifically asked to think about the dynamic protractor and the movement of the turning sides and think about the difference in angle size.
- Using a program e.g., AirServer, the iPad screen can be displayed for all the class to see the movement of the dynamic protractor. Starting with the dynamic protractor closed, or at an angle size of one or two, demonstrate the turn of the protractor as it opens to a larger angle. The teacher will refrain at this time from explaining any further details about angle size (in degrees).
- Working in the same groups, the teacher will then ask:
  - "Look at the angles you have made in your group. Can you think of a way to group the angles you have, based on the size of the angles? Work in your group to consider how you can do this. In 15 minutes you will be asked to share back with the class on what you have done. You have been given poster paper so you can

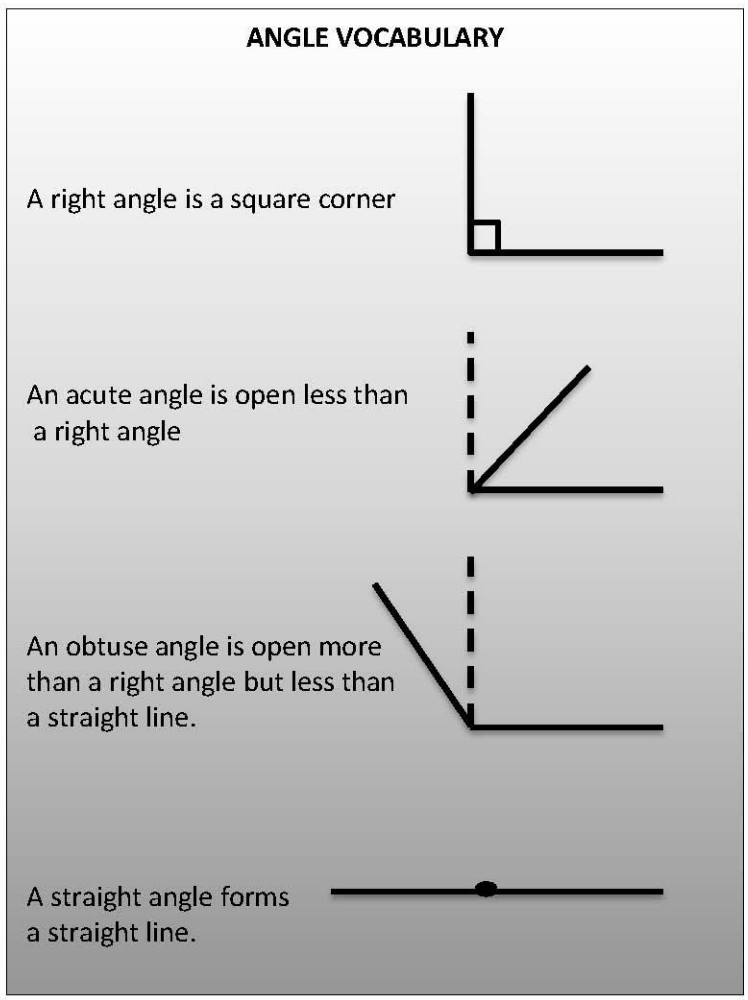
draw, write, and perhaps stick your angles onto the paper so you can share back with the class".

Students then work in the same groups to categorize the angles. Diagrams and notes on poster paper will support students in explaining how they have chosen to do this. If a group finishes early, they can be asked to clearly define how they have grouped the angles. This can be done on the posters or on paper.

- Students will be given time to share their observations with the class. The teacher will direct the attention of the class to point out group size especially those that are acute and obtuse (without using this language). As the class share their posters they can be photographed with the iPad and displayed on the projected screen. The weight of the materials on the posters can make the poster difficult for students to hold.
- The teacher will guide the discussion to finally introduce the concepts right, straight, acute, and obtuse angle.

  The words will be posted on the classroom wall with various examples (including various line segment lengths and orientations).

  Students will be told that a right angle is 90° and this will be displayed on the dynamic protractor. The protractor will then be increased to a straight angle and to zero degrees.



- Using SketchPad Explorer, the teacher will model taking a photograph of an angle found in the classroom. With the iPad connected to the computer projector, the teacher will model placing the dynamic protractor on the angle to determine the size of the angle. The teacher will model what a right, acute, obtuse, and straight angle would look like on the iPad.

- Each student in the class will be given an iPad and the class will move to outside the school. It is important that you are in a location where there are some good examples of different angles. This can be the playground with climbing frames and also near the school building.
- Working in pairs, students will use the iPads to take photographs of angles in the real-world in Sketchpad Explorer. Students will work in pairs to find examples of right, straight, acute, and obtuse angles. These are to be identified with the dynamic protractor. The students need to be able to recognize the relationship of the angle they have found to 90° to determine the name of the angle. Students may begin by using the terms “smaller than” or “larger than”. This is acceptable at this stage and teachers should respond by repeating the language the student used and following it with the formal mathematical name. For example, “smaller than 90° would be an acute angle”.
- Students will take screenshots of two or three of what they would describe as good examples of the angles.

- The class will go back to the classroom where each student will choose their favorite angle example from the two or three screenshots. Each student will place the iPad upright on the desks creating a *Gallery of Angles*. Acute angles will be in one area, obtuse in another, etc. Students will write their names on a piece of paper and place this sticking out from under the iPad. This will help students find their iPad afterward and if other students have questions, they will know who to find. The students will then walk around the gallery to view the different angle photographs. Students can ask questions to other students during this time.

GALLERY WALK

For the gallery walk, the students will clasp their hands behind their backs and walk around the room looking at the photographs on the iPads. Students will be asked to think about the size of the angle and how it fits with that size category (acute, obtuse etc.). For the first three/four minutes, students will walk around in silence. Next, students will discuss the photographs and ask questions to other students about the angle they have identified in the photograph.

- At the end of the gallery walk, the teacher will recap on yesterday's lesson that there are an infinite number of angles. This will be supported by the many different angles displayed in the gallery.
- If the students ask what category is larger than a straight angle, students can be told that the next category of angles from 181° to 360° are reflex angles.

After Phase

- The final activity has students check their understanding of right, straight, acute, and obtuse angles. The students will go back outside the school with the iPads and get into pairs.
- The teacher will use tape to identify various acute, obtuse, right, and straight angles in the environment. Students work in pairs to discuss each angle to determine the categorization. Next to each of the angles will be a QR code and the students will scan the codes to see if they are correct. Once the students scan the codes, this will take the students to a website to find further examples.

- Go over the lesson objectives and point out that these are not told at the beginning of the lesson as they have to come to these understandings through the activities rather than being told initially.
- Students finish with a journal entry to describe the various categories of angles.

HOW TO USE THE QR CODES

1. Make copies of the codes in the charts below and cut out the codes (the square with the border). Remember to keep them separate so you know which they are.
2. With sticky putty, stick the codes next to the angles that students have to identify. (These can be backed onto card to help the codes last longer.)
3. When the students are ready to scan the codes, the will open up the QR Code reader (e.g., i-nigma), the students will point the iPad camera at the codes.
4. The app. will read the codes and a website will appear to let the students know if they were correct, and also provide more examples of that type of angle.

QR Codes

Acute angle	**Obtuse angle**
URL: http://www.mathsisfun.com/acute.html QR Code	URL: http://www.mathsisfun.com/obtuse.html QR Code:

Right angle	Straight angle
URL: http://www.mathsisfun.com/rightangle.html	URL: http://www.mathsisfun.com/geometry/straight-angle.html

Lesson Four
Day 5

Objectives

4. Recognize acute, obtuse, right, and straight angles in different contexts (real-world and paper and pencil).
5. Recognize salient attributes of angle.

Van Hiele-Geldof Instructional Phases

- Guided orientation
- Explication
- Free orientation
- Integration

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990)

At the end of this lesson, students will be moving onto level three as the students identify necessary and sufficient properties in the context of a justification. However, many students may still be advancing in level two as they continue to work on appropriate vocabulary and may describe angles with a litany of properties, rather than necessary and sufficient properties.

Materials and Equipment

- Angle Cards
- Computer projector
- iPad
- iPad display program (e.g., AirServer) to display the iPad screen on the projected computer screen.
- SketchPad Explorer
- Chart paper
- Tape

Prior Knowledge

Students will be expected to draw on the knowledge gained from yesterday's lesson where they learned about the categorization of angles dependent on angle size into the four groups: acute, obtuse, right and straight.

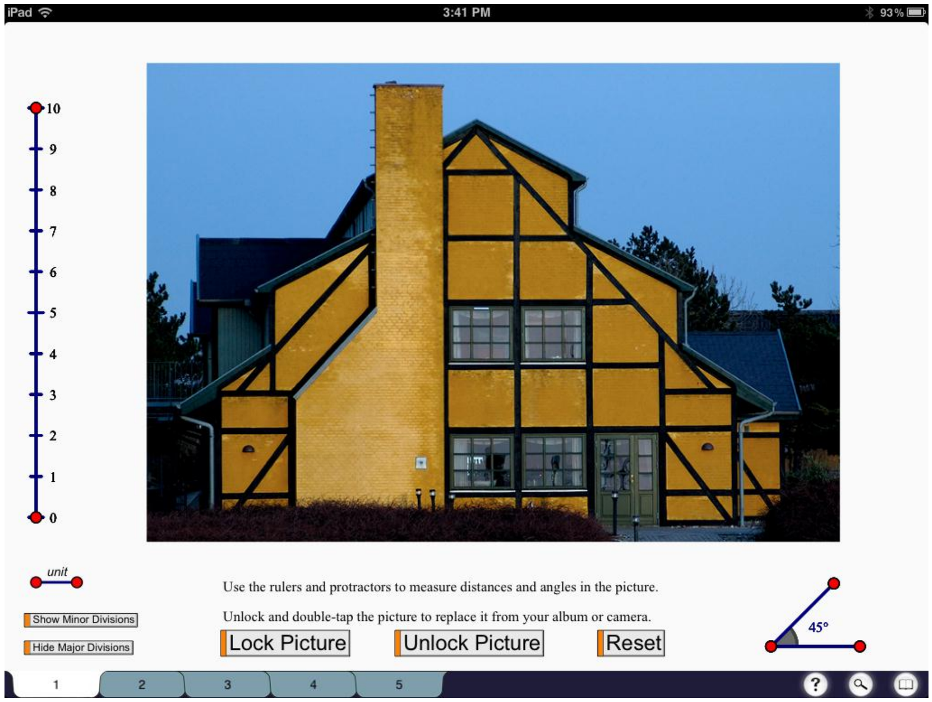
Learning Environment

The goals of this lesson are for students to recognize acute, obtuse, right, and straight angles in different contexts and determine the salient attributes of angle. The activities in lesson six involve a number of different van Hiele instructional phases, including guided orientation, explication, free orientation and integration. The card sorting activity is completed in groups of four heterogeneous groups. The social dynamics also need to be considered in choosing groups to ensure mathematical talk does not convert to social talk.

This format of this lesson is designed to have the students recognize categories of angle in different contexts. The activity begins with a real life photograph, and changes to paper and pencil angles. This transition is important in that it requires students to be able to transfer the knowledge from the three-dimensional angles from the photographs taken outside the school in lesson three, to two-dimensional photographs and paper and pencil angles. However, students may need support during this process and it should not be expected that what they could do in one context they will be able to do in another.

Before Phase

- For this lesson, go over the lesson objectives at the beginning of the lesson with the students.
- Attach the iPad to the computer projector and show the students the photograph that is pictured on the first page of Measure a Picture app.



As a recap on yesterday's lesson, ask the students:

- Can you see any right angles, acute, obtuse, and straight angles in the design of this house?

Have students come up and to point out one angle they can see and have them describe their reasoning for the name of that angle (acute, obtuse etc.). During this time the teacher will use talk moves of adding on, revoicing, and wait time. Adding on is where the teacher prompts the students for further participation in asking them to add on to what one student has said. In the revoicing move, the teacher repeats what the student has said and asks the student if this is correct. This move is useful if the students' initial response is not fully clear. Wait time is used to allow students time to think: the teacher will tell the student to take his/her time in answering the question.

During Phase

- Students will get into groups of four. Each group of students will be given a set of 24 cards with a selection of pencil drawn angles. Angles will have various orientation and line segment lengths. Each group will also get a piece of poster paper. The students deal out the cards so they have six each. Students have to sort the angle cards into categories of acute, obtuse, right, straight angles and non-angles. Each student takes a turn in placing a card on the chart paper under the correct heading. Some cards will be non-examples which will be placed in the non-angle category. Students will be encouraged to use mathematical discussions to determine which group each angle should be place in. Teacher will move around the groups listening to the mathematical discussion.

Teacher should use open probing questions to ensure students are thinking in depth about angle size (e.g., Why did you choose to place that angle in xxx group rather than the xxx group? What is your reasoning?). This will also be an opportunity for the teacher to check for misconceptions, such as students considering incorrect attributes such as orientation or line segment length. Teacher will check the organized cards to make sure students have chosen correctly (answer sheet can be found following the sort cards at the end of this lesson plan).

Once the first groups begin to finish sorting the cards ask the groups to stop. Students should then stick the cards on the poster paper with tape. Ask the students to walk around the room looking at the other posters. The students will place sticky notes with any comments or questions they may have about the posters. Have the groups finally go back to their posters and look the sticky notes they may have.

Bring the students back together and ask:

- Which cards did you have trouble sorting and why?

Students may describe the different line segment lengths as being confusing e.g., P, Q and U. The teacher will at this point raise the question

- Does it matters that the line segments are of different lengths? For students who are struggling ask them to think about a window they may have photographed – Are the line segments the same length? (No).

Students may describe the cards that do not have a common end point e.g., H and W. Again lead the students back to the first lesson of what an angle looks like.

Ask the student if they could determine which angles were right angles. Question how they were sure if the angles were actually right angles. Ask the students to recap on what a right angle is (90° as shown on the dynamic protractor). Ask students:

- Is (L) a right angle? How can you be sure? Note: students can try to answer this final question, but the teacher should ask students to think about this question and they will come back to it in the next lesson.

After Phase

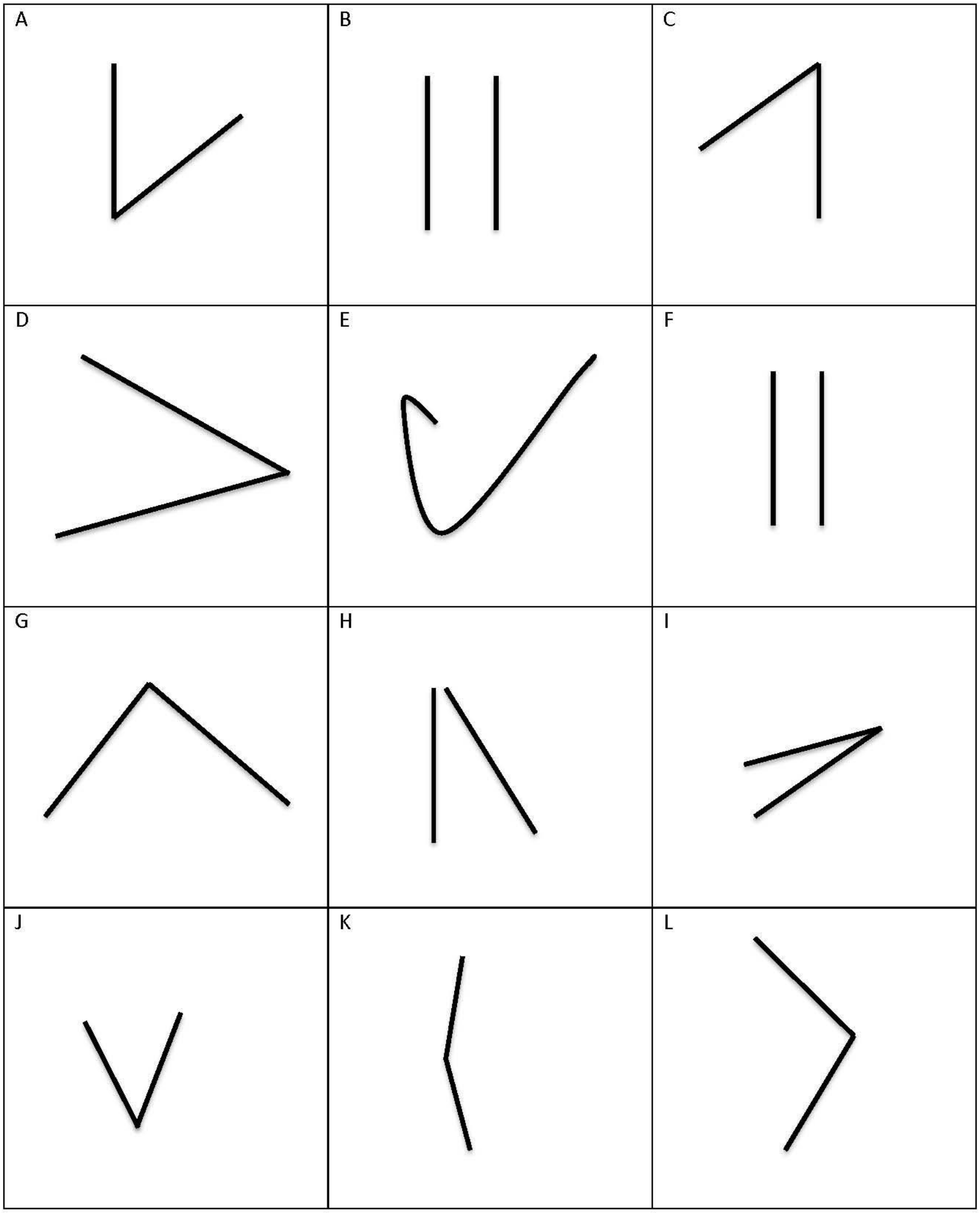
- The lesson will close with a class discussion on salient (important) and non-salient (not important) attributes of angles. Students will begin with a pair share to answer the questions:
  - What are the important parts of an angle? What does an angle have?
  - When measuring an angle, what is not important?

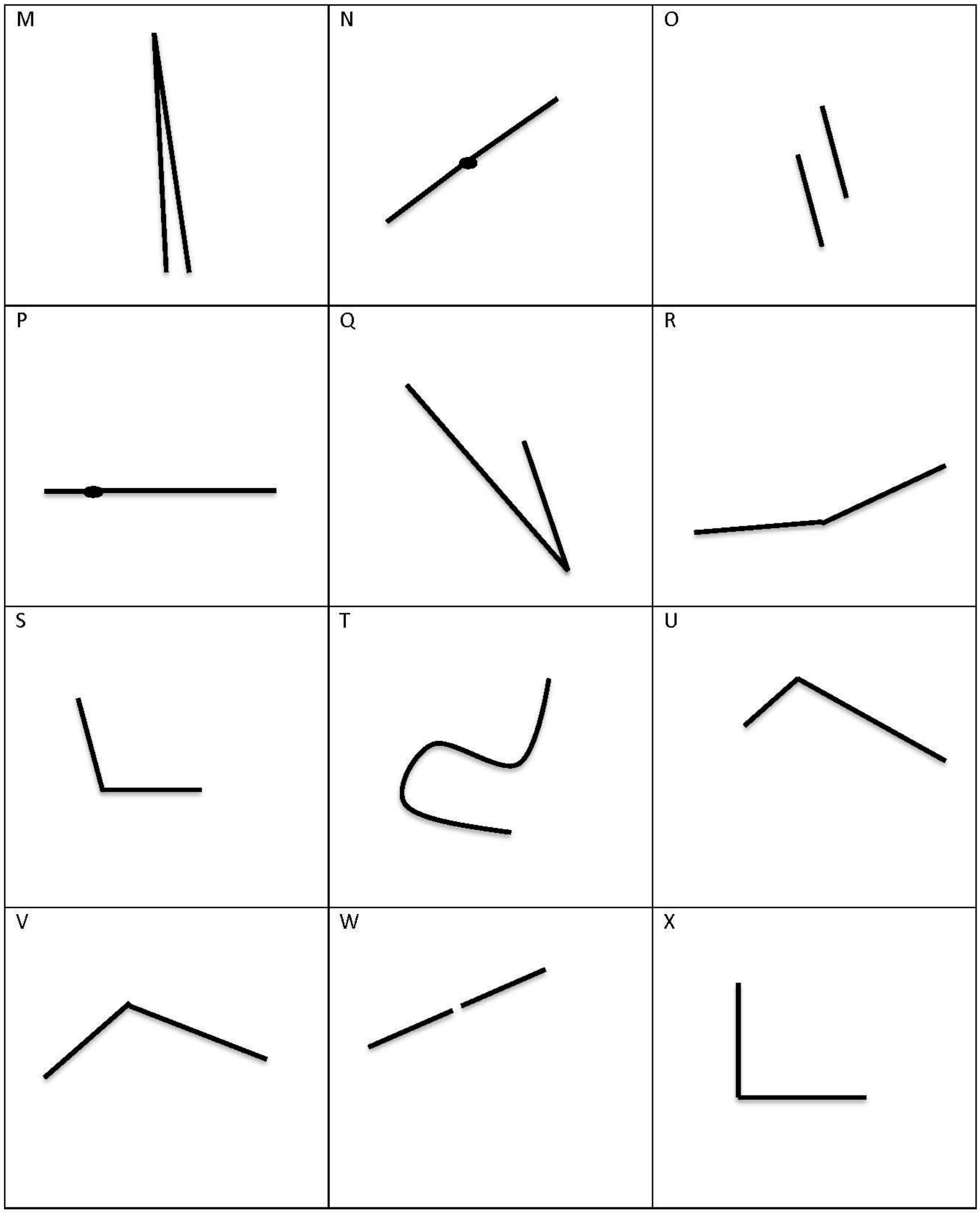
As a class, list the important and unimportant things about angle. For example:

IMPORTANT	**NOT IMPORTANT**
Two line segments/lines	The direction of the angle (orientation)
Two line segments with a common end point	Length of the line segments
The distance of the wedge or the turn	

- The teacher will help guide the discussion with talk moved used in the initial activity (adding on, revoicing, and wait time). Create the list on chart paper so this can be posted on the classroom wall.

Lesson 4 Angle Sorting Cards




M
N
O
P
Q
R
S
T
U
V
W
X

Sort Cards Answer Sheet

Acute angles	A, C, D, I, J, M, Q,
Obtuse angles	K, L, R, U, V, S
Right angles	G, X
Straight Angles	N, P,
Non-angles	B, E, F, H, O, T, W.

Lesson Five
Day 6

Objectives

1. Understand that angles can be measured with reference to a circle and that angles are fractions of a circle.
2. Experience using a nonstandard unit of measurement (a “wedge”)
3. Recognize that the attribute being measured is the space between the two line segments caused by the turn of the line segments.
4. Understand that angles are measured by units called degrees.
5. Understand that benchmarks can be made for angle measures. For example, a full circle turn is 360°, therefore a straight angle is 180° and a right angle is 90°, and how to use those benchmarks to think about approximate angle measures.
6. Recognize that there an infinite number of angles.

Van Hiele-Geldof Instructional Phases

- Explication
- Free orientation
- Integration

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990)

Many students will still be working within level two, but students are expected to begin working in level three as they should start to explicitly describe relationships between properties and identify sufficient properties in the context of a justification. Students also begin to present informal arguments/informal proof deductively (implicitly using such logical forms as the chain rule, or explicitly using “if-then”).

Materials and Equipment

- Paper
- Computer projector
- One iPad

- SketchPad Explorer app.
- Paper circles
- Copies of worksheet 5.1 and 5.2
- Scissors/ books/ blocks or other real world materials to create angles
- Document Camera
- Sorting cards from Lesson Four.

Prior Knowledge

For this lesson, students will need to draw from their understanding of lesson three and four to be able to recognize and compare angles based on size using non-standard and standard language (right, obtuse, acute, and straight angles). For this lesson, students should also have a basic knowledge of fractions including ½, ¼, and 1/8. Students will also be called upon to use strategies such as halving numbers and comparative reasoning to determine the size of the angles.

Learning Environment

The goals of this lesson are for students to understand that angles can be measured with reference to a circle, and that angles are fractions of a circle. Students will understand that angles are measure by units called degrees and that benchmarks can be used to assist in recognizing approximate angle measures. Continuing from lesson two, the students will also understand that there are an infinite number of angles as they consider fractions of an angle as well as the 360° as whole units. As the students complete the activities in this lesson, students will be involved in the explication phases as they learn new terminology, but also moves into the free orientation and the integration phase categorized by the van Hiele's. This lesson uses an adapted version of Browning, Garza-Kling, and Hill Sundling's (2007) and Millsaps' (2012) wedge activity.

Note: When using the dynamic protractor, demonstrate by having the angle at zero, line the ray up with the first side of the angle and then open the angle to the correct measure. This will help the students in understanding angle as a turn.

Before Phase

- Begin the lesson with a recap on what the students know about angle. First ask the question:
  - *What can you tell me about angles?*
- Allow for wait time and let the students know that you want any information they can tell you about angles. Remind the students of the poster made at the end of the last lesson which described things that are important about angles and those that are not important.
- Try to focus initially on the visual aspects of angles, then ask the students:
  - *What do you know about angle measurement?*

  Probe for the categories acute, obtuse, straight and right. Draw an acute angle on the board and ask what type of angle it is. Then an obtuse angle, again asking what angle it is. Ask the students:
  - *What is the difference between the two angles?*

  The students may focus on the lines, but have students focus on the space between the line segments and explain the difference.
  - *How did the SketchPad Explorer helped you decide the categorization of an angle?*
- Have students remember from lesson three that 90° is a right angle and they could use that to check if it was acute and obtuse. Connect the iPad to the projector and show the students the 90° angle and point to the space between the line segments. Take a photograph of a plain piece of paper so the picture does not distract the students from focusing on the dynamic protractor. Tell the students that the ° symbol is read *degrees*. Say *"If you measured the length of a book (running your finger along one side of a book), you may measure in inches or centimeters. When we are measuring the turn of the line segments and the space in between, we measure in degrees".*
  - *What else do you notice about this angle?*

  Pointing to the dynamic protractor. Have the students notice the part of the circle in the angle. Tell the students that this shows that this is the angle that is being measured – the space being measured.
- Demonstrate the dynamic protractor turning and getting larger and larger. Stop at points so the students can think about the numbers and the way in which the part of the circle is

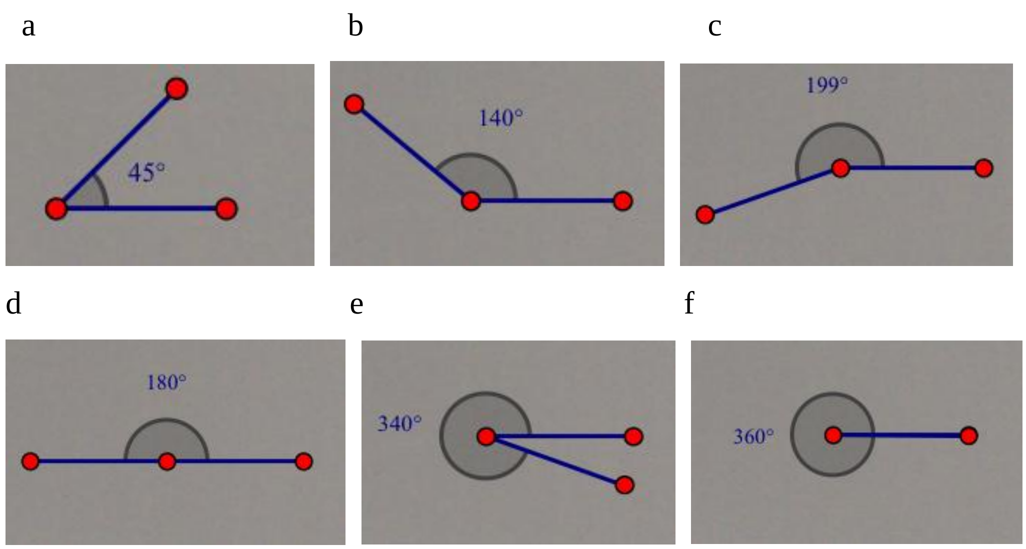
getting towards a whole circle. However, do not mention the circle at this point, just point to what you want the students to look at (the formation of the circle).

- *What do you notice about the numbers?*

Probe students to answer that the numbers are getting larger.

- Complete the angle to reach 360° Ask:
    - *What do you notice?*

Probe students to answer that it has reached 360° and it is a full circle.



During Phase

- Recap that a full turn of one of the line segments is 360°, that the space between the two line segments is 360° and makes a full circle. Be clear to physically point out that the space is around the outside as students may become confused as the two line segments are together and therefore there is no space between them. If students look confused go back to around 340° to point out where the space is.
- Show the students a paper circle. Ask:
    - *How much of a turn of an angle is this circle?* (360°)
- Fold that paper in half
    - *If a full circle turn is 360°, what would this be?* (180°)

Hold the semi-circle against the 180° shown on the dynamic protractor (figure (d) above).

- *How did you know that it would be 180°?* (180 is half of 360)
- *So if I folded the semi-circle in half again, which would give me ¾ of a circle, what would be the angle size of this wedge shaped paper?* (90°, and probe again for how this was determined, showing the calculation on the board).
- *What would happen if I used a smaller square, would this make a difference to the size of the angle?* (Ensure that the students are not mistaking the circle wedge for measuring area. Clarify that the size of the circle does not matter.)
- *Does it matter where I put the wedge to begin measuring?* (Have the students ensuring that they line up the edge of the wedge at zero measure with one side of the angle when measuring.)

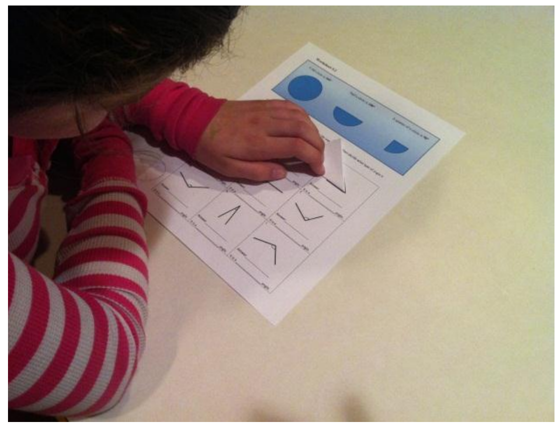
- Put the students in groups of four and give each student worksheet 5.1 and 5.2, and ask the students to cut out the circle on worksheet 5.1. Given the students some time to fold the circle in half and in quarters, reminding students not to rush and that they need to make sure the paper lines up for accuracy. Then, tell the groups:
  - *On worksheet 5.2, there are some angles. Think about how you can measure those angles using the circle. You know that the full circle is 360°, half a circle is 180° and a quarter is 90° and there is a reminder of this at the top of your worksheet. Your task is to use the folded circle as a tool to help you estimate the size of each of the angles on your worksheet. You will need to think of different ways in which you can do this. You may want to fold the circle into different parts, or use other methods.*

  As the students begin the activity the teacher is to use probing questions to support students in thinking about how to use the wedge of the circle to estimate the measure of the angles. The aim is for students to come up with strategies for using the circle to estimate the angle size; to fold the circle multiple times to make different benchmarks, or use the 180° and 90° benchmarks to estimate how much greater or smaller the angle is than the benchmark. Students could write on the folded paper to remind themselves of the benchmarks.

As students are finishing, ask them to check their answers with their partners and if they have a different number of degrees work out why and if they are still correct. Also have the pairs checking that the number of degrees match the angle category. As students are checking their answers with their partners, ask the students to use a different colored pencil so the teacher can see what answers they have revised.

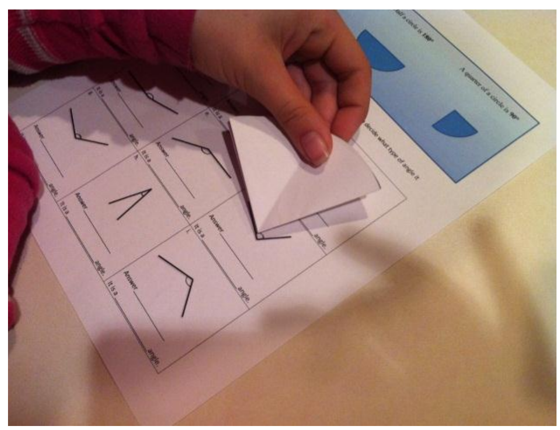
STRATEGIES

1. Making additional folds to the paper to estimate the angle size. For example, in this photograph the student has folded the circle into quarters to determine that angle (c) on worksheet 5.2 is 45°.



2. Making estimations based on comparative reasoning.

For example, the student uses the 90° angle to determine that angle (f) on worksheet 5.2 is a little over 90°. Considering the size of 90° the student estimates that it is just a little more at 95°.



3. Using both the above strategies.

For example, to estimate angle (h) on worksheet 5.2, the student first folds the circle into eighths – 45°. Then the student notices that the angle is slightly smaller than the benchmark of 45° and using comparative reasoning decides that it is 1/9 smaller than the 45° benchmark so it is approximately 40°.

- As groups finish estimating the angles on paper, the groups can be set further tasks using real world objects such as child's scissors, books, ramps made with books leaning against blocks. Place the items in front of each group and point out the angle created by the objects. Ask the students to again estimate the measure of the angles using the circle as a tool to help with the estimation.
- Have the students think about other linear measurement tasks and how there can be half a unit or a quarter of a unit. At this age the students may not be very familiar with fractions so keep it simple to have the students thinking about ½ a degree and that you can make that smaller by cutting it in half again (1/4) and so on. Therefore, there are 360° in a circle, but there are an infinite number of different angle sizes between those degrees.

After Phase

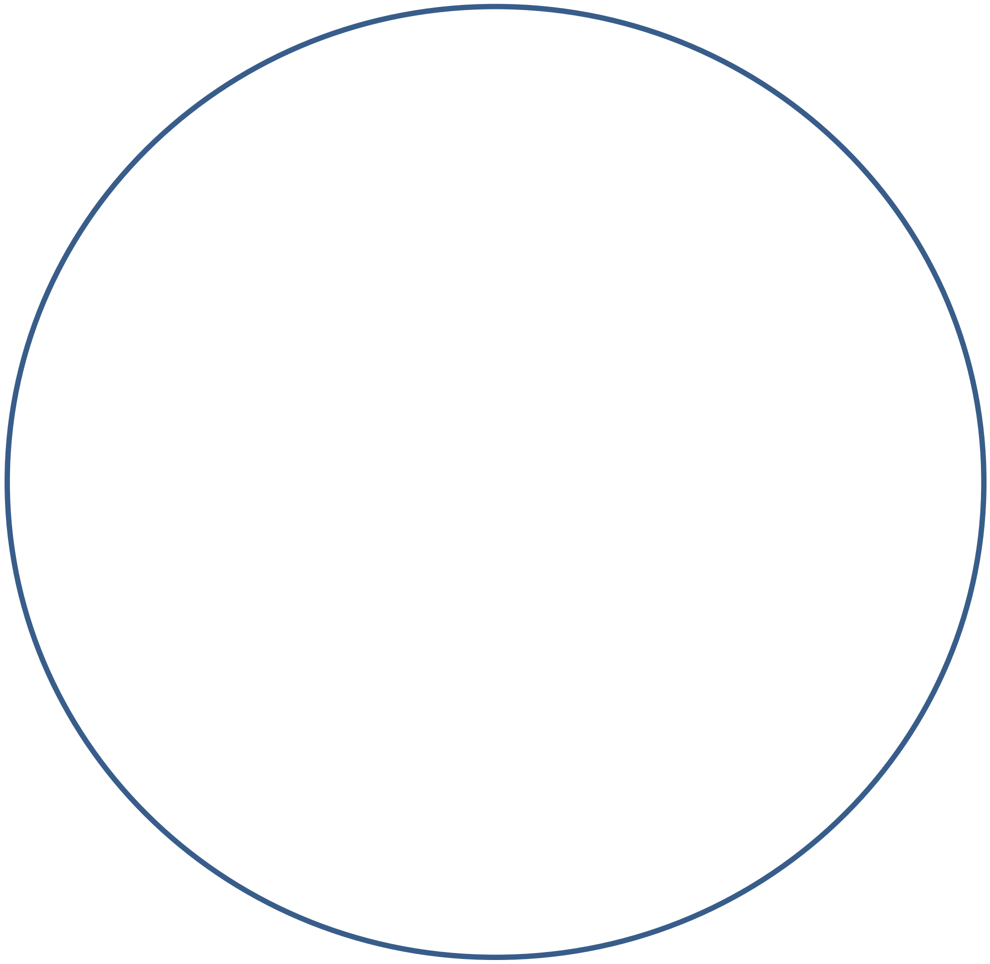
- Bring the class back together for a whole class discussion on the strategies used to estimate the angles on paper and the real-world objects. With worksheet 5.2 placed under the document camera, ask the students how they measured angle (a). Students should all be able to say that they used the 90° benchmark to find out it actually was 90°. This should be the same for angle (b) as students should have used the 180° benchmark. For angles (c-i), ask students from the groups to come and share how they estimated the angle. Be sure to have multiple groups sharing if they used a different strategy.
  As the students share, they can demonstrate on the document camera what they did to the rest of the class and students should be encouraged to show any calculations on the board. Then ask the students to explain how they estimated the angles on the real-world objects. Where possible, place the scissors or book from the group on the document camera and have students demonstrate again.
- Have the students recap on yesterday's lesson where the students had to sort angles into different categories and the difficulty as some angles were close to 90°, but they could have been a little more, or a little lesson. Have the students connecting with what they learned from this lesson as a way to solve that problem. As the teacher, model using the 90° wedge to check the cards.
- During the discussion, to ensure that they students speaking or listening are fully understanding what is being described in the strategies. To do this, the talk moves of

adding on, revoicing, and wait time should be used. Adding on is where the teacher prompts the students for further participation in asking them to add on to what one student has said. In the revoicing move, the teacher repeats what the student has said and asks the student if this is correct. This move is useful if the students' initial response is not fully clear. Wait time is used to allow students time to think: the teacher will tell the student to take his/her time in answering the question.

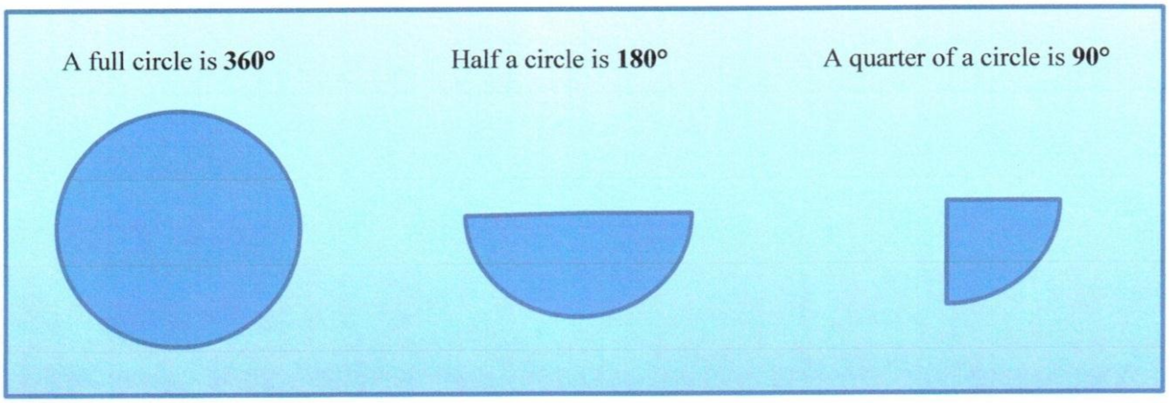
- Go over the lesson objectives and point out that these are not told at the beginning of the lesson as they have to come to these understandings through the activities rather than being told early on.

Worksheet 5.1

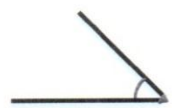
Cut out the circle.



Worksheet 5.2



Use the folded circle as a tool to help you measure the angles below. Then decide what type of angle it is (acute, obtuse, straight, and right angle).

a.  Answer________________ It is a ________________angle.	b.  Answer________________ It is a ________________angle.	c.  Answer________________ It is a ________________angle.
d.  Answer________________ It is a ________________angle.	e.  Answer________________ It is a ________________angle.	f.  Answer________________ It is a ________________angle.
g.  Answer________________ It is a ________________angle.	h.  Answer________________ It is a ________________angle.	i.  Answer________________ It is a ________________angle.

Worksheet 5.2 Answers

Problem	Degrees	Type of Angle
a	90°	Right
b	180°	Straight
c	45°	Acute
d	(160) Grade correct within a range of 145°-175°	Obtuse
e	(145°) Grade correct within a range of 130°-160°	Obtuse
f	(100) Grade correct within a range of 92°-107°	Obtuse
g	(125°) Grade correct within a range of 113°-140°	Obtuse
h	(30°) Grade correct within a range of 20°-40°	Acute
i	(115°) Grade correct within a range of 100°-130°	Obtuse

Lesson Six
Day 7

Objectives

1. Recognize that the same angle can appear to be a different size depending on different visual perspectives (positions).
2. Understand that angles are defined by particular attributes which involve angle as a turn (e.g., "two line segments, the common endpoint, the rotation of one line segment to the other around that endpoint, and measure of that rotation"); Clements and Sarama, 2009, p.186).

Van Hiele-Geldof Instructional Phases

- Free Orientation
- Integration

Van Hiele Level Indicators for the Topic of Angle (Scally, 1990)

Students will be working further in level three as they are able to explicitly describe relationships between properties and identify sufficient properties in the context of a justification. Students also begin to present informal arguments/informal proof deductively (implicitly using such logical forms as the chain rule, or explicitly using "if-then").

Materials and Equipment

- An iPad for each student in the class
- Poster paper
- Poster pens

Prior Knowledge

This lesson will require the students to draw from all the lessons in this angle unit to create a final poster of all that they have learned. Students will be required to consider the angle attributes, as well as the measure of angles. In addition, the students will be asked to consider their real-world experiences to think about perspective taking.

Learning Environment

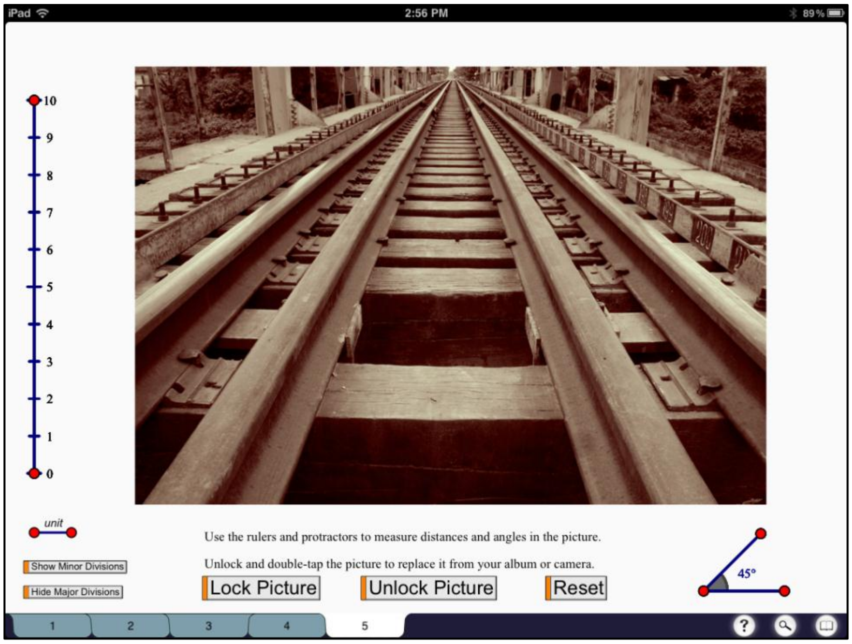
Spatial perception plays a considerable role in geometry, and photography provides an excellent example of how angles can appear different depending on where the photographer stands. It will be made very clear to the students that the actual angle does not change; however, the angle can appear to be a different size depending on the spatial perspective the photographer has of that angle.

Before Phase

- Go over the lesson objectives.
- The teacher will question the students to recap on the activities and the findings of lesson 5. For example, questions could include:
  - *What units are used to measure angles?* (Degrees)
  - *What is the symbol for degrees?* (small raised circle)
  - *What are we measuring when we measure angle?* (Space between the two line segments caused by the turn of the line segments).
  - *How large is a right angle?* (90°)
  - *If an angle measures 100° will it be a right, obtuse, acute, or straight angle?* Have the students get into pairs to discuss this in a pair share for two minutes before responding to this question. Continue with other similar questions to this including 80°, 120°, 40°, 1°, 180° again allowing students to pair share before responding.
- Give each child an iPad. Ask the students to spend a minute looking at the dynamic protractor; looking at how it moves, and the size of the angles. Connections can be made to the end point being like a pivot foot in basketball that the end point stays still while the ray moves around that pivot point.

During Phase

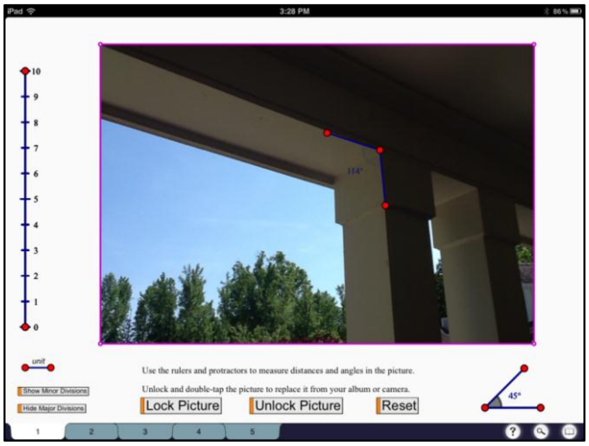
- Take the class outside into the school grounds. Students will be asked to turn to tab five on in SketchPad Explorer and to look at the photograph. Ask students:

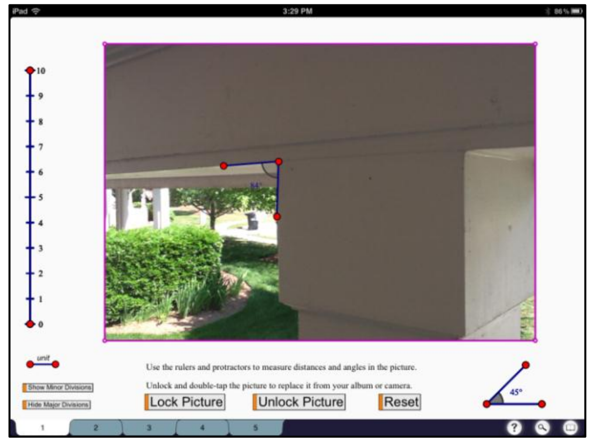


- *What is pictured in the photograph?* (Railway lines/tracks)
- *Are you sure?* (Yes) But railway tracks are all the same distance apart all the way along the tracks, these tracks seem to be getting narrower. How is that? (It just looks that way as things get further into the distance.) *That is correct; it depends on the position of the person as to what they see. That can also happen when you look at something close up. Thinks can look different, when you are stood in different places. This can also happen when you look at angles from different positions.*

- Students will work in pairs with one iPad each to create two different screenshots of the same angle, but from different perspectives. Students are to use the dynamic protractor to measure the two different angle perspectives, and the students are challenged to find the greatest difference in angle size. Students will therefore have to determine the difference in degrees by using simple calculations. Students can try multiple times to get different angle sizes, but the photograph must be of the same angle. Remind the students to take screen shots once they have measured the angle.

  For example,





In the photograph to the left, the angle measures 114° and the same angle photographed from the other side is 84°. Therefore, 114°- 84° = 30° difference. Students can take notepads outside to conduct the calculations, or use the calculator on the iPads to find the difference. Bring the students back together and find out what differences they found. Ask the students where would be the best place to stand to get an accurate angle measurement as possible. (Straight on).

- For the second part of the lesson, the students go back to the classroom. Students are placed into groups of four or five to create a poster to explain angle and angle measure. The students will be informed that they are creating the poster to explain angle to other fourth grade students who have not yet studied angle. The directions will be to first create a list of what should be included on the poster. Ask students to think back to what they first learned about angle. If students are struggling, the teacher can use probing questions to have students think about attributes that are important and those that are not important, angle size categorizations (acute, obtuse etc.), and how angles are measured. Once the lists are finished, the lists are to be checked by the teacher before beginning the poster.

After Phase

- Once the posters are finished, have the students display the posters around the classroom for all the students to see. Students are to look at the posters and think about any questions they may have for the other groups. Students can then share those questions and have the groups respond to the questions. For example, a student may see a comment on another poster and want to know more about that comment. The group that designed the poster will have a chance to respond. This discussion will be facilitated by the teacher and any misconceptions will be addressed.

APPENDIX C

Student Recruitment Script – Face to Face

Good morning/afternoon. My name is Helen Crompton. I am a graduate student at UNC in Chapel Hill, but I used to be a classroom teacher. You are being asked to participate in a research study. I am doing this study to learn more about how fourth grade students learn about angle and angle measure in math. In particular, I am interested in the way you could learn about angle both inside and outside the classroom while also using apps. on the iPad. The reason for doing this research is to develop activities that can be done with students like you to help them learn about angle and angle measure.

Your time commitment for this research will be small. During your regular class time, I will teach your math class seven times. Each lesson will be video-taped with the camera focusing on me. When you work in small groups, I may want to video-record your group work. If you are willing, you may be on those videos, if you do not wish to be on the videos you will be seated so you are off camera. I will collect your work from the lessons, which I will copy and return the original work to your teacher. I would also like to interview some students from this class over the course of the research. Not everyone who agrees to be interviewed will be asked. If you are selected, I would interview you twice, once before, and once after the set of lessons. Each interview will last about 25 minutes.

Everyone in the class will be taught the same lessons. Your participation in this study is completely voluntary. You may choose not to be in the study, or stop being in the study before it is over. If you choose not to be in the study it will not affect your math grade in any way. When I come in to collect the consent forms, I will ask your teacher to leave the room so he/she will not know who has said yes or no. Your participation and any data collected will be kept confidential. Fake names for participants, school, and the school district will be used in publication and presentations. Additionally, other information that might identify you will be removed or changed. If you are not in the study, I will not collect copies of your work and you will not be on any video or audio recordings.

There is a form that you will have to sign if you would like to be a part of the study. There is also a parent permission form for your paren. Both your parent and you need to agree to be in the study; but even if your parent says you may, you can still choose not to be in the study, or stop being in the study at any time. Additional information about the study is provided on the parent permission form and on your assent form. There are two copies of both of these forms, one for your parent and you to sign and return, and one copy of each for your family to keep for you records.

Place take them home and talk about this with your parents, and them bring the signed copies back to school as soon as possible, the forms have a place where it says if you want to be in the study or not. I do hope that many of you will be interested in being in my study.

I will return on ____________ to collect the parent permission and assent forms, whether you say yes or no. I am happy to answer any questions you may have. Thank you.

Student Assent

University of North Carolina at Chapel Hill
Assent to Participate in a Research Study
Minor Subjects (7-14 yrs)

Consent Form Version Date: April 18, 2012
IRB Study # [IRBNO WILL BE INSERTED]
Title of Study: Coming to Understand Angle and Angle Measure: A Design-Based Research Curriculum Study Using Context-Aware Ubiquitous Learning
Person in charge of study: Helen Crompton
Where they work at UNC-Chapel Hill: School of Education
Other people working on this study: May need to list co-researcher here

The people named above are doing a research study.

<u>These are some things we want you to know about research studies:</u>
Your parent needs to give permission for you to be in this study. You do not have to be in this study if you don't want to, even if your parent has already given permission. You may stop being in the study at any time. If you decide to stop, no one will be angry or upset with you. Sometimes good things happen to people who take part in studies, and sometimes things happen that they may not like. We will tell you more about these things below.

<u>Why are they doing this research study?</u>
The reason for doing this research is to find out how the programs on the iPad, and learning both inside and outside the classroom can help us learn about angle and angle measure.

The reason for doing this research is to make a set of activities that can be used with other fourth grade students to help them learn about angle and angle measure.

<u>Why are you being asked to be in this research study?</u>
The purpose of this research study is to find out if a particular set of lessons will help students learn about angle and angle measure. The lessons involve students using programs on the iPad, and learning both inside and outside the classroom to look at real-world angles.

The reason for doing this research is to make a set of activities that can be used with other fourth grade students to help them learn about angle and angle measure.

<u>How many people will take part in this study?</u>
If you decide to be in this study, you will be one of about 52 people in this research study.

<u>What will happen during this study?</u>
During this study, your participation will involve:

- A total of seven lessons will be taught for this study. You will be observed during this time and there will be a video camera pointed at the person teaching the lesson. It is possible that you might be on the video-recording, if you are willing. As you work in groups, the video recorder may be moved to focus on small group work.
- Your class work will be collected from each lesson. The researcher will make copies of it so they can study it later, and the original work will be returned to your teacher.
- Four students in the class will be asked to be interviewed before and after the set of lessons. This may take a little extra time as each interview will take approximately 25 minutes. You may agree to be in the study, but not everyone who agrees to be interviewed will actually get interviewed. Interviews will be audio-recorded.

Please note that all students in this class, whether they participate in the study or not, will receive all the same lessons.
These studies will take place at your school and will last for three class periods, but almost all of this time (except for interviewing) is what would normally happen in your class.

Check the line that best matches your choice:

_____ OK to record me during the study

_____ Not OK to record me during the study

This study will take place at XXXXX school, and will last approximately six weeks.

Who will be told the things we learn about you in this study?
The researcher and class teacher will be the only people who have access to your information in the study.

What are the good things that might happen?
Research is designed to benefit society by gaining new knowledge. The benefits to you from being in this study may be that you learn about angle and angle measure.

What are the bad things that might happen?
Sometimes things happen to people in research studies that may make them feel bad. These are called “risks.” There are no known risks for your participation in this study. However, if you are concerned, you should report any problems to the researcher.

Will you get any money or gifts for being in this research study?
You will not receive any money or gifts for being in this research study.

Who should you ask if you have any questions?
If you have questions you should ask the people listed on the first page of this form. If you have other questions, complaints or concerns about your rights while you are in this research study you may contact the Institutional Review Board at 919-966-3113 or by email to IRB_subjects@unc.edu.

If you sign your name below, it means that you agree to take part in this research study.

____ ____________________
Sign your name here if you want to be in the study Date

____
Print your name here if you want to be in the study

____ ____________________
Signature of Research Team Member Obtaining Assent Date

____
Printed Name of Research Team Member Obtaining Assent

Parental Consent

University of North Carolina at Chapel Hill
Parental Permission for a Minor Child to Participate in a Research Study

Consent Form Version Date: April 19, 2012

IRB Study # [IRBNO WILL BE INSERTED]

Title of Study: Coming to Understand Angle and Angle Measure: A Design-Based Research

Curriculum Study Using Context-Aware Ubiquitous Learning

Principal Investigator: Helen Crompton
Principal Investigator Department: School of Education
Principal Investigator Phone number: (919) 962-6605
Principal Investigator Email Address: hcromp@live.unc.edu
Faculty Advisor: Dr. Susan N. Friel
Faculty Advisor Contact Information: sfriel@email.unc.edu; (919) 962-6605

What are some general things you and your child should know about research studies?

You are being asked to allow your child to take part in a research study. To join the study is voluntary.

You may refuse to give permission, or you may withdraw your permission for your child to be in the study, for any reason, without penalty. Even if you give your permission, your child can decide not to be in the study or to leave the study early.

Research studies are designed to obtain new knowledge. This new information may help people in the future. Your child may not receive any direct benefit from being in the research study. There also may be risks to being in research studies.

Details about this study are discussed below. It is important that you and your child understand this information so that you and your child can make an informed choice about being in this research study.

You will be given a copy of this consent form. You and your child should ask the researchers named above, or staff members who may assist them, any questions you have about this study at any time.

What is the purpose of this study?

The purpose of this research study is to find out if a particular set of lessons will help students learn about angle and angle measure. The lessons involve students using programs on the iPad, and learning both inside and outside the classroom to look at real-world angles.

Before entering the doctoral program at UNC Chapel Hill, Helen Crompton successfully taught in schools for 16 years. Her prior experience as an elementary teacher, as well as her graduate work at UNC Chapel Hill, helped her to realize the importance and possibility of developing mathematical understandings through the use of real-world connections. For example, students may look for angles on buildings and climbing frames which the students are already familiar with. Helen Crompton, the lead researcher is confident that real-world connections and the use of technologies such as programs on the iPad can help fourth grade students successfully learn about angles and angle measure. It is her aim to develop an effective series of activities and a way to teach them by implementing them in the classroom herself and studying its effects.

Your child is being asked to be in the study because his/her teacher is allowing us to do research in the fourth grade math class. Your child is a member of that particular class.

How many people will take part in this study?

A total of approximately 52 people at one school will take part in this study.

How long will your child's part in this study last?

The researcher will come into the class to teach for seven days. Each lesson will last approximately 45 minutes. All of this time will be spent on activities which would normally happen in your child's class. The only real additional time commitment would involve two brief interviews about the lessons, and only four children will be involved in those. The interviews will last 25 minutes and will take place before and after the set of lessons. In total, your child's participation will last for approximately seven to ten days.

What will happen if your child takes part in the study?

If you give permission for your child to participate in this study the following things will happen:

- He/she will be observed in the mathematics classroom for seven class periods. During this lesson, one of the researchers will take notes while the lead researcher, Helen Crompton, teaches the lessons.
- The class lesson will be recorded on video, if you and your child are willing. If not, then the camera will be positioned and seating arranged so that your child will not be on camera. Students work during the lesson will be video-recorded.
- You child's class work will be collected from each observed lesson. After the researcher copies the work, the originals will be returned to the regular classroom teacher.
- The researcher will be asking four students from this class to be interviewed about what they know about angle and angle measure. Two interviews will take place, one before the set of lessons, and one after the lessons. Therefore your child may be interviewed twice if you give permission and your child agrees. However, not everyone who agrees will be interviewed will actually get interviewed. These

interviews will be audio recorded. As noted in the section above, the interviews are the ONLY additional time commitment involved in being in the study.

What are the possible benefits from being in this study?

Research is designed to benefit society by gaining new knowledge. The benefits to your child from being in this study may be that they learn about angle and angle measure, which is a requirement in the fourth grade Common Core Standards.

What are the possible risks or discomforts involved from being in this study?

There are no known or anticipated risks for participation in this research study. There may be uncommon or previously unknown risks. You should report any problems to the researcher.

What if we learn about new findings or information during the study?

You and your child will be given any new information gained during the course of the study that might affect your willingness to continue your child's participation in the study.

How will your child's privacy be protected?

The only written documentation indicating the identities of the participants will be the parent permission, student assent forms and a pre-assigned participant number identification sheet. These will be kept in a locked cabinet in the lead investigator's home. Care will be taken to ensure that all identifying information is removed upon document collection or during data transcription. The lead researcher will be the transcriber of all interviews and videos.

If you child is mentioned in the transcription, your child will be referred to only by his/her pre-assigned participant number. All names of people or places stated in conversation will be replaced with fake names or participant numbers during transcription. Prior to transcription, all notes, artifacts, documents, video, and audio-recordings will be stored in a locked cabinet in the lead researcher's home. All data collected, when transcribed, will be stored on a laptop and an external hard drive in the lead investigator's home. This data will be password protected. Original observation notes and other written documentation will be shredded after transcription. Video recordings will be stored for possible analysis beyond what can be recorded through transcription (i.e. body language). After transcription, the videos will be stored in a locked cabinet in the lead investigators home.

Once the study has ended, the videos and transcripts will be destroyed. In addition, participants will not be identified in any report or publication about this study. Fake names for participants, the school, the school district, will be used in publications or presentations. All other possible identifiers will be removed or changed.

Check the line that best matches your choice:

_____ OK to record me during the study

_____ Not OK to record me during the study

What if you or your child wants to stop before your child's part in the study is complete?
You can withdraw your child from this study at any time, without penalty. The investigators also have the right to stop your child's participation at any time.

Will your child receive anything for being in this study?
Neither you nor your child will receive anything for being in this study.

Will it cost you anything for your child to be in this study?
It will not cost anything to be in this study.

What if you or your child has questions about this study?
You and your child have the right to ask, and have answered, any questions you may have about this research. If there are questions about the study (including payments), complaints, concerns, or if a research-related injury occurs, contact the researchers listed on the first page of this form**.**

What if there are questions about your child's rights as a research participant?
All research on human volunteers is reviewed by a committee that works to protect your child's rights and welfare. If there are questions or concerns about your child's rights as a research subject, or if you would like to obtain information or offer input, you may contact the Institutional Review Board at 919-966-3113 or by email to IRB_subjects@unc.edu.

Parent's Agreement:

I have read the information provided above. I have asked all the questions I have at this time. I voluntarily give permission to allow my child to participate in this research study.

Printed Name of Research Participant (child)

Signature of Parent | Date

Printed Name of Parent

Signature of Research Team Member Obtaining Permission | Date

Printed Name of Research Team Member Obtaining Permission

Teacher Recruitment Script – Email

Good morning! My name is Helen Crompton, and I am a graduate student in the School of Education at UNC. I am interested in the way students come to understand angle and angle measure through the use of real-world and applied technology learning tasks, and effective ways of supporting students understanding. In other words, understanding ways in which a set of lessons which use applications on the iPads in real-world environments to help students understand angle and angle measure. I am excited about the lesson activities that have been developed and I am looking for fourth grade classrooms where I can study the effects of these activities. I would like to invite you to be a participant in my dissertation research.

If you decide to take part in this research study, I will teach one of your mathematics classes for seven consecutive class periods. The day before lessons begin, I will administer a pre-instruction interview to four of the students in your class. The day following the interviews I will administer the same interview. Each interview will require approximately 25 minutes. While I teacher your class, I request that you share your expertise by observing the lesson. You may take notes during your observation if you choose to do so. At a later point in the day we would meet together to discuss the lesson, how the mathematical understandings were developed, and how the lesson for the next day should proceed. With your permission, these discussions would be audio-recorded, and with your permission, I would like to keep a copy of your notes.

Your students will also be invited to be in this research. They will be asked for separate assent and parental permission. With their parents' permission and their own assent, students' class and group work will be video-recorded for later transcription and analysis. Student written work will also be collected to examine their understanding of angle and angle measure. Participation in this study is completely voluntary. You may choose not to be in the study or to stop being in the study at any time. Your participation and any data collected will be kept confidential. Pseudonyms for participants, school, and school district will be used. Additionally, other identifiers will be removed, masked, or changed. You will have access to the data collected at any time during the study. You will also have the opportunity to review the final publication and make requests for changes to any potential identifying information.

Additional information about the study is provided on the consent form. I am also happy to answer any questions you may have. Thank you! If you are interested in participating, please reply to this email. I would be so happy to work with you!

Kind regards

Helen Crompton

Teacher Consent

University of North Carolina at Chapel Hill
Consent to Participate in a Research Study
Adult Participants

Consent Form Version Date: April 18, 2012
IRB Study # [IRBNO WILL BE INSERTED]
Title of Study: Coming to Understand Angle and Angle Measure: A Design-Based Research Curriculum Study Using Context-Aware Ubiquitous Learning
Principal Investigator: Helen Crompton
Principal Investigator Department: School of Education
Principal Investigator Phone number: (919) 962-6605
Principal Investigator Email Address: hcromp@live.unc.edu
Faculty Advisor: Dr. Susan N. Friel
Faculty Advisor Contact Information: sfriel@email.unc.edu; (919) 962-6605

What are some general things you should know about research studies?
You are being asked to take part in a research study. To join the study is voluntary.
You may refuse to join, or you may withdraw your consent to be in the study, for any reason, without penalty.

Research studies are designed to obtain new knowledge. This new information may help people in the future. You may not receive any direct benefit from being in the research study. There also may be risks to being in research studies.

Details about this study are discussed below. It is important that you understand this information so that you can make an informed choice about being in this research study.

You will be given a copy of this consent form. You should ask the researchers named above, or staff members who may assist them, any questions you have about this study at any time.

What is the purpose of this study?
The purpose of this research study is to explore how the use of real-world connections and technological tools supports learning about angle and angle measure. In this study I will conduct a teaching experiment in your fourth grade class using a self-designed instructional sequence of tasks.

You are being asked to be in the study because you are a fourth grade teacher.

How many people will take part in this study?
If you decide to be in this study, you will be one of two elementary school teachers in this research study.

How long will your part in this study last?
If you decide to be in this study, your time commitment will be approximately seven hours 35 minutes; this includes the time I will be teaching your class.

What will happen if you take part in the study?
If you decide to be in the study, you will allow me, the principal investigator, to teach seven consecutive mathematics lessons to your fourth grade students. Four randomly selected students from your class will be interviewed prior to the instructional sequence of tasks. The interviews will take approximately 25 minutes. The same four students will be interviewed after the sequence of tasks. You will observe each lesson and discuss the lesson with me and a co-researcher later the same day. These discussions will be audio-recorded, with your permission. You may choose to take notes during the lessons that I will be teaching. If so, I would like to make copies of your notes to help me in my analysis.

What are the possible benefits from being in this study?
Research is designed to benefit society by gaining new knowledge. You may also expect to benefit by becoming more aware of the ways in which iPads can be used to teach angle and angle measure. You will be able to watch your students in the classroom for several lessons. This may give you insights into your students that you may not otherwise gain. At the end of the study, you will be given a copy of the instructional sequence which can be used in your future teaching.

What are the possible risks or discomforts involved from being in this study?
There are no known or anticipated risks for participation in this research study. There may be uncommon or previously unknown risks. You should report any problems to the researcher.

What if we learn about new findings or information during the study?
You will be given any new information gained during the course of the study that might affect your willingness to continue your participation.

How will your privacy be protected?
Consent forms and the pre-assigned participant number identification sheet that links study identification codes to names will be kept in a locked cabinet in the principal investigator's home. Care will be taken to ensure that all identifying information is removed and placed with the assigned participant number upon artifact collection or during data transcription. Pseudonyms for participants, school, and school district will be used in publications or presentations. Additionally, other identifiers will be removed masked, or changed.

All documents will be shredded following transcription. All audio-recordings will be password protected on a computer laptop until transcription, after which they will be destroyed. The video recordings will be either password protected on a computer laptop and external hard drive or stored in a locked cabinet in the principal investigator's home. Once the study has come to an end, the videos will be destroyed.

You will be referred to in transcriptions of the audio-recordings of our discussions after I

teach each lesson (if you give permission for audio-recording) by your pre-assigned participant number. All names of people or places stated in conversation will be replaced with pseudonyms or participant numbers during transcription. Prior to transcription, all notes, artifacts, documents, video-recordings and audio-recordings will be stored in a locked cabinet in the co-investigator's home. All data collected, when transcribed, will be stored on a laptop and an external hard drive in the principal investigator's home. This data will be password protected. Field notes and other written documentation will be shredded after transcription into Word documents.
Participants will not be identified in any report or publication about this study.

There is a slight possibility for deductive disclosure, which means that other people such as the staff at the school might be able to figure out which class of teacher is being discussed in a research report. Therefore, to avoid this, we are using pseudonyms for participants, school, and school district in all reports, publications or presentations. Additionally, other identifiers will be removed, masked, or changed. Teacher participants will have access to transcripts, recordings, and reports at any point. Teacher participants will also have the opportunity to review the final study and make requests for changes to any potential identifying information.

Check the line that best matches your choice:

_____ OK to record me during the study

_____ Not OK to record me during the study

What if you want to stop before your part in the study is complete?

You can withdraw from this study at any time, without penalty. The investigators also have the right to stop your participation at any time. This could be because you have had an unexpected reaction, or have failed to follow instructions, or because the entire study has been stopped.

Will you receive anything for being in this study?

You will not receive anything for taking part in this study.

What if you have questions about this study?

You have the right to ask, and have answered, any questions you may have about this research. If you have questions about the study including complaints or concerns, you should contact the researcher listed on the first page of this form.

What if you have questions about your rights as a research participant?

All research on human volunteers is reviewed by a committee that works to protect your rights and welfare. If you have questions or concerns about your rights as a research subject, or if you would like to obtain information or offer input, you may contact the Institutional Review Board at 919-966-3113 or by email to IRB_subjects@unc.edu.

Participant's Agreement:

I have read the information provided above. I have asked all the questions I have at this time. I voluntarily agree to participate in this research study.

____ ________________
Signature of Research Participant   Date

____
Printed Name of Research Participant

Signature of Research Team Member Obtaining Consent Date

Printed Name of Research Team Member Obtaining Consent

APPENDIX D

Angle Interview Protocol[12]

Angle Activities

Activity 1: Drawing Angles

Purpose:	To discover what attributes (size, proportion, orientation, etc.) the student attends to when drawing distinct angles (student-generated angles).
Materials:	Grid paper and pencil

Script:

1. Draw an angle in the box marked #1.
2. Draw another angle in box #2 that is different in some way from the first angle.
3. Draw other angles in box #3 that is different from the first two angles.
4. Can you draw another angle different from #1, #2, and #3? If so, draw it in box #4.
5. How many different angles could you draw?
6. How is #2 different from #1?
7. How is #3 different from #1 and #2?
8. How is #4 different from #1, #2, and #3?

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[12] Clinical interview from "The impact of experience in a Logo environment on adolescents' understanding of angle: A van Hiele-based clinical assessment, by Scally, 1990. Unpublished doctoral dissertation, Emory University, Atlanta, Georgia. Reprinted with permission.

9. Do you agree or disagree with your answer to the pervious question? How many different angles do you think you could draw? How would they all be different from each other?

   If the student focuses only on inappropriate attributes such as orientation or length of rays, ask.

   Can you find some other way to make angles different, other than just turning them (or making the sides longer/shorter, etc.)?

   Ask about any marks the student uses to indicate interior angle, vertex, or continuation of rays.

Activity 1

1

2

3

4

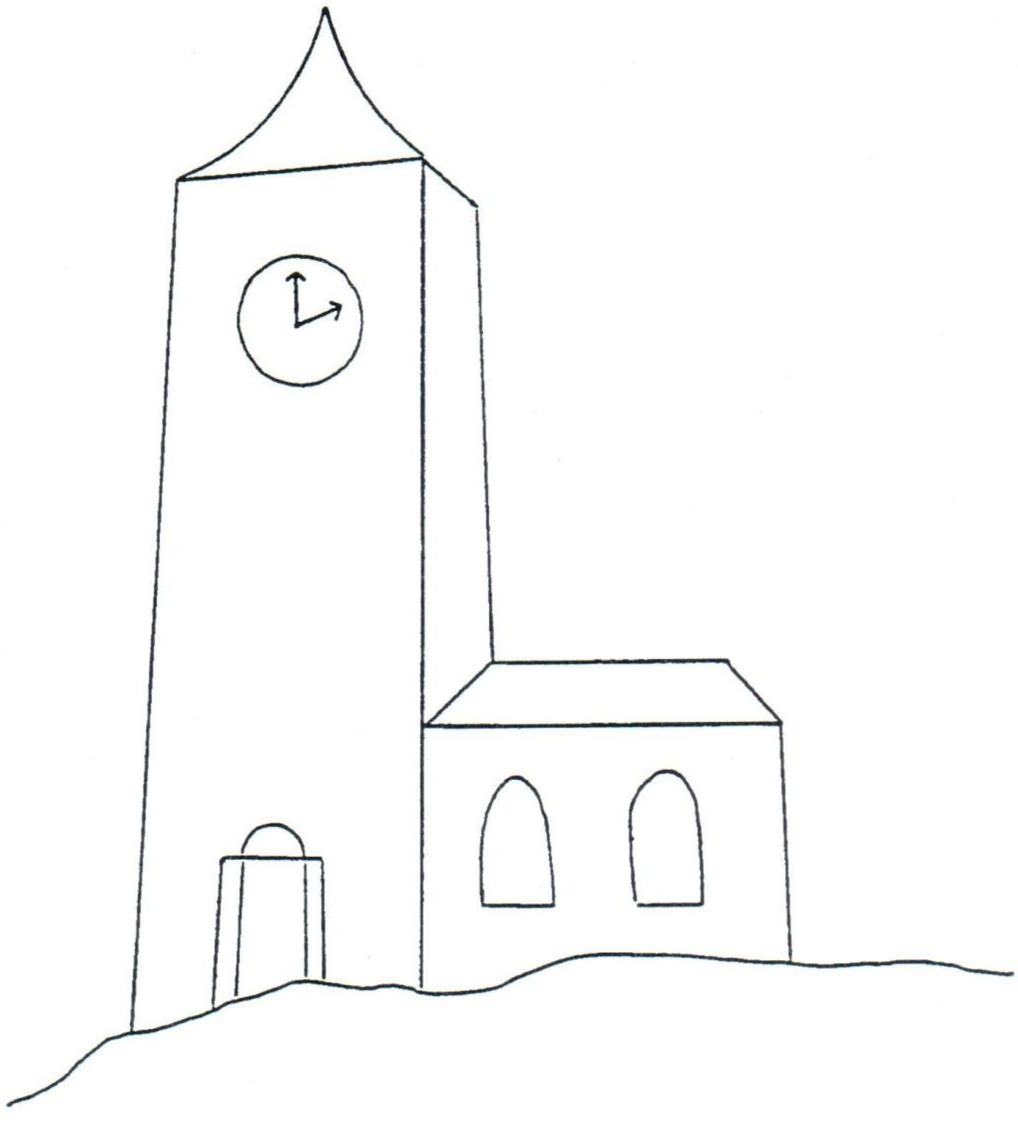
Activity 2a: Identifying and Defining Angles

Purpose:	To determine whether the student can identify angles in a real life picture.
Materials:	Ink drawing of a building and colored pencils.

Script:

1. Find an angle in this drawing and mark it with a colored pencil.
2. Find another angle and mark it with the (colored) pencil.
3. Find a third angle and mark it with the (colored) pencil.
4. For one or two angles marked, ask “Why is that [point to figure xxx] an angle?”
5. Is this an angle (1) clock hands, (2) arch over window, (3) roof over tower, (4) arch over door? (Select one or two not marked previously by the student.)
6. For each ask “Why or why not?”

Activity 2a



Activity 2b: Identifying and Defining Angles

Part A

Purpose:	To determine whether the student can identify models of angles and define them.
Materials:	Angle worksheet and colored pencils.

Script: Put an A inside each angle on this sheet.

Part B.

Purpose:	To determine the properties that the student focuses on when identifying angles.

Script:

1. Why did you put an A in __________?

   (Pick out several of those marked. Include #9 if appropriate, but do not include #3, #7, or #12.)

   Be sure to include all “unusual” responses.

2. If student had marked A on any part of #3, do not ask the first part of the question: Are there any angles in #3? (If so:) How many do you see? (If the student’s response is ambiguous ask him/her to mark the angles with colored pencils.)

3. If student had marked A on any part of #7 do not ask the first part of the question: Are there any angles in #7? (If so:) How many do you see? (If the student’s response is ambiguous ask him/her to mark the angles in colored pencils.)

4. If student had marked A on any part of #12 do not ask the first part of the question: Are there any angles in #12 (If so :) How many do you see?

5. Pick out at least four (if Possible) not marked as angles. Ask. Why did you not put an A in _____? (For each one, including #9 if appropriate.)

Activity 2b: Identifying and Defining Angles

Part C

Purpose:	To elicit properties the student perceives as necessary for defining an angle.

Script: What would you tell someone to look for to pick out all the angles on a sheet of figures?

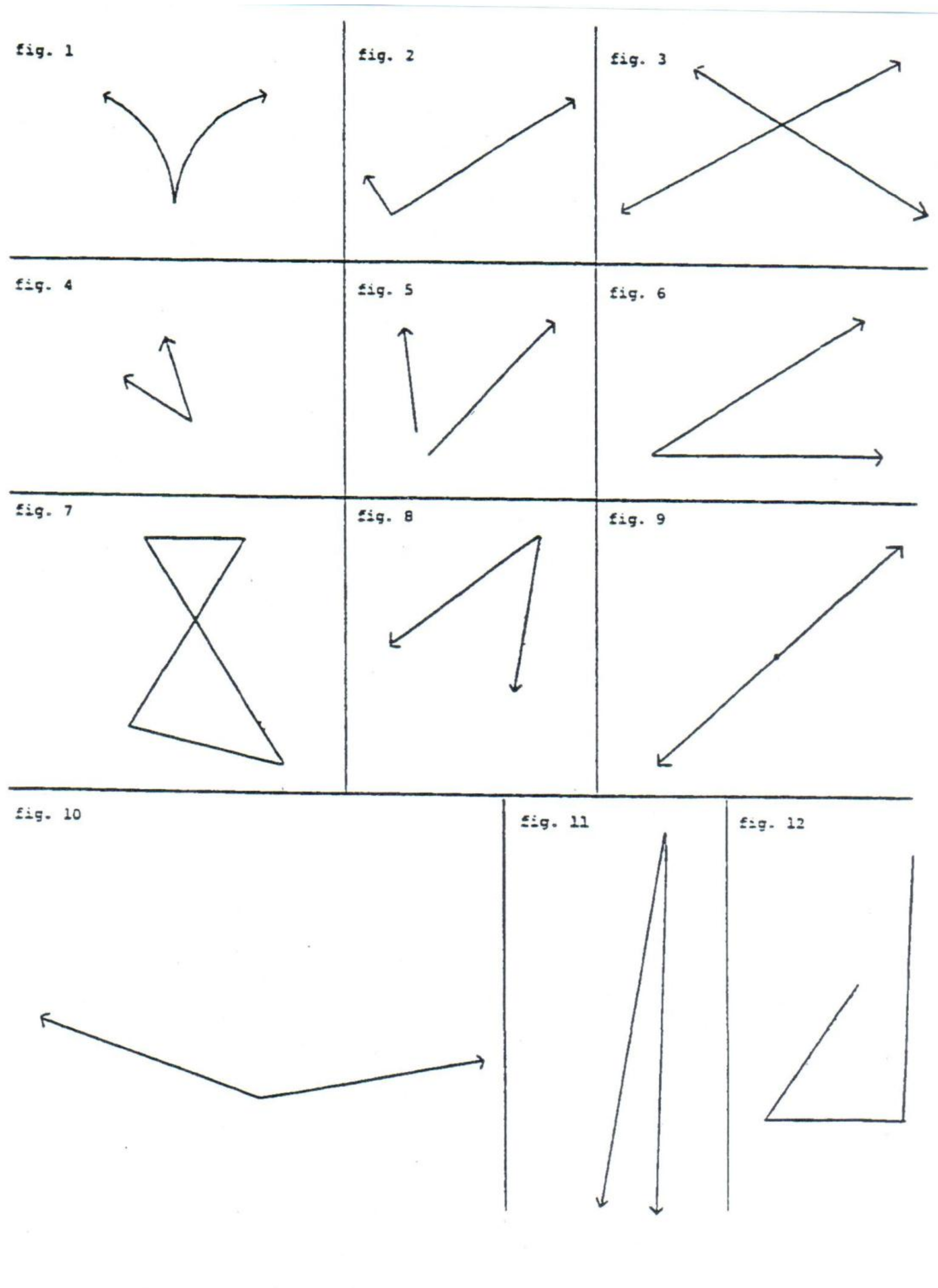
Part D

Purpose:	To elicit properties the student perceives as necessary and sufficient for defining an angle.

Script: (If the students lists more conditions that are necessary and sufficient for defining and angles, ask)

Is that the shortest list of things you could tell someone to look for to pick out all the angles on a sheet of figures?

Activity 2b, 2c, and 2d



Activity 3: Sorting Angles

Part A

Purpose:	To determine extent to which the student focuses on properties when comparing angles.
Materials:	Colored paper mat and angle cutouts. (Place cutouts and colored mat on the table.)

Script:

1. Look at these angles. Select some that are alike in some way and put them on the colored paper.
2. How are they alike? (Put them all together again.)
3. Look at the angles again. Select some that are alike in another way and put them on the colored paper. (Record the grouping.)
4. How are they alike? (Repeat as long as sorting appear useful. Remind students, if necessary, that they can reuse figure.)

Part B (Do not use this activity if the student sorted successfully acute, obtuse, and right angles above.)

Purpose:	To determine the student's ability to distinguish common properties of preselected angles.

Script:

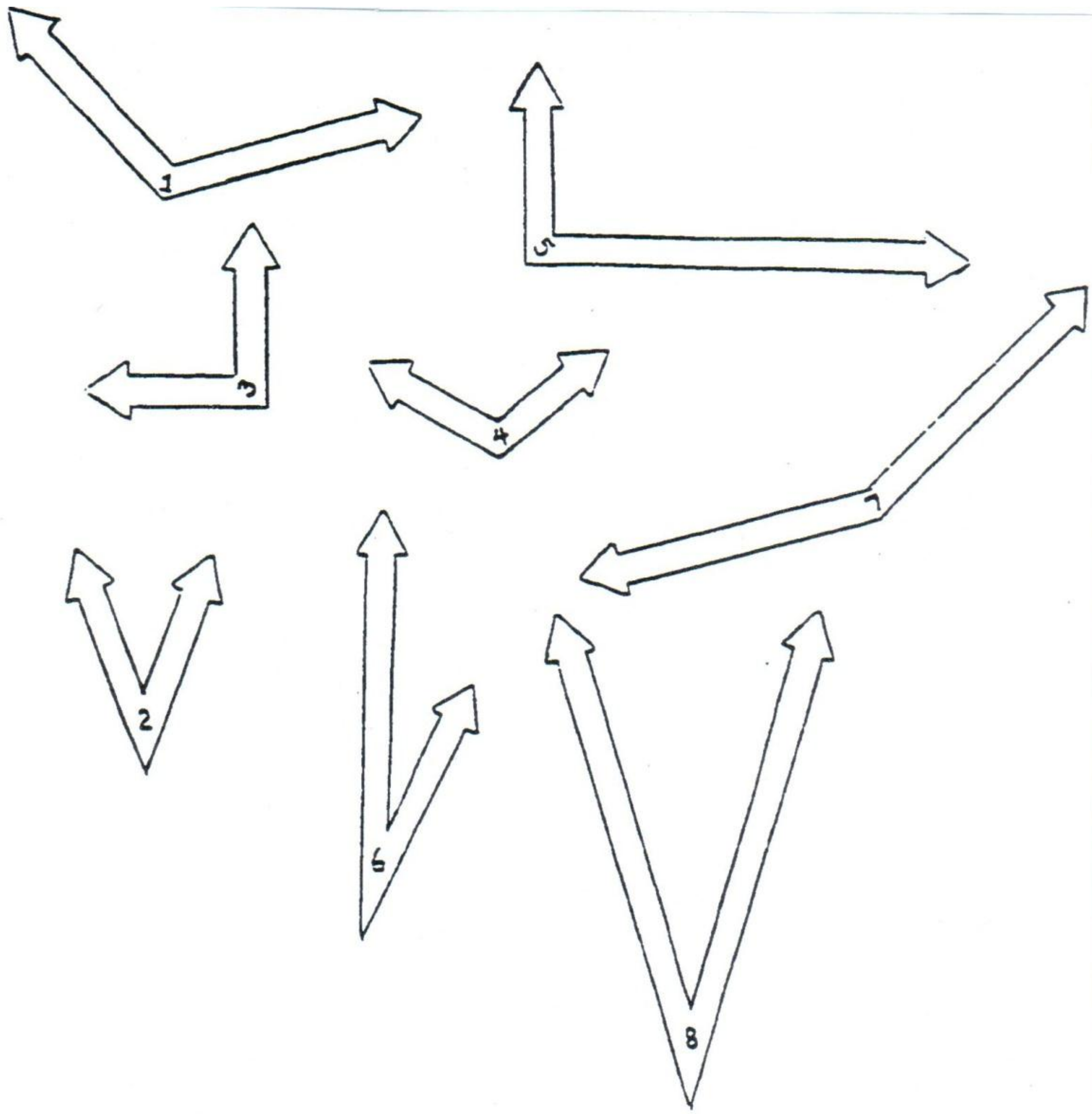
1. (Interviewer selects a set of angles that have some common property: all acute, all obtuse, all right.)

   All of these shapes are alike in some way. How are they alike?

   (The student may find property that the shapes share, but which does not distinguish them from the others. If this happens, praise can be given, and the student can be told. “There is another way… can you find it?”)

2. Repeat part 1 for each of the three possible sorts.

Activity 3 Cutouts



Answer Sheet for Activity 3: Sorting Angles
Part A. Student sorting notes

1.

2.

3.

Part B. Interviewer sorting notes

1.

2.

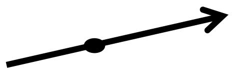
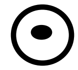
Activity 4: Angle Measure

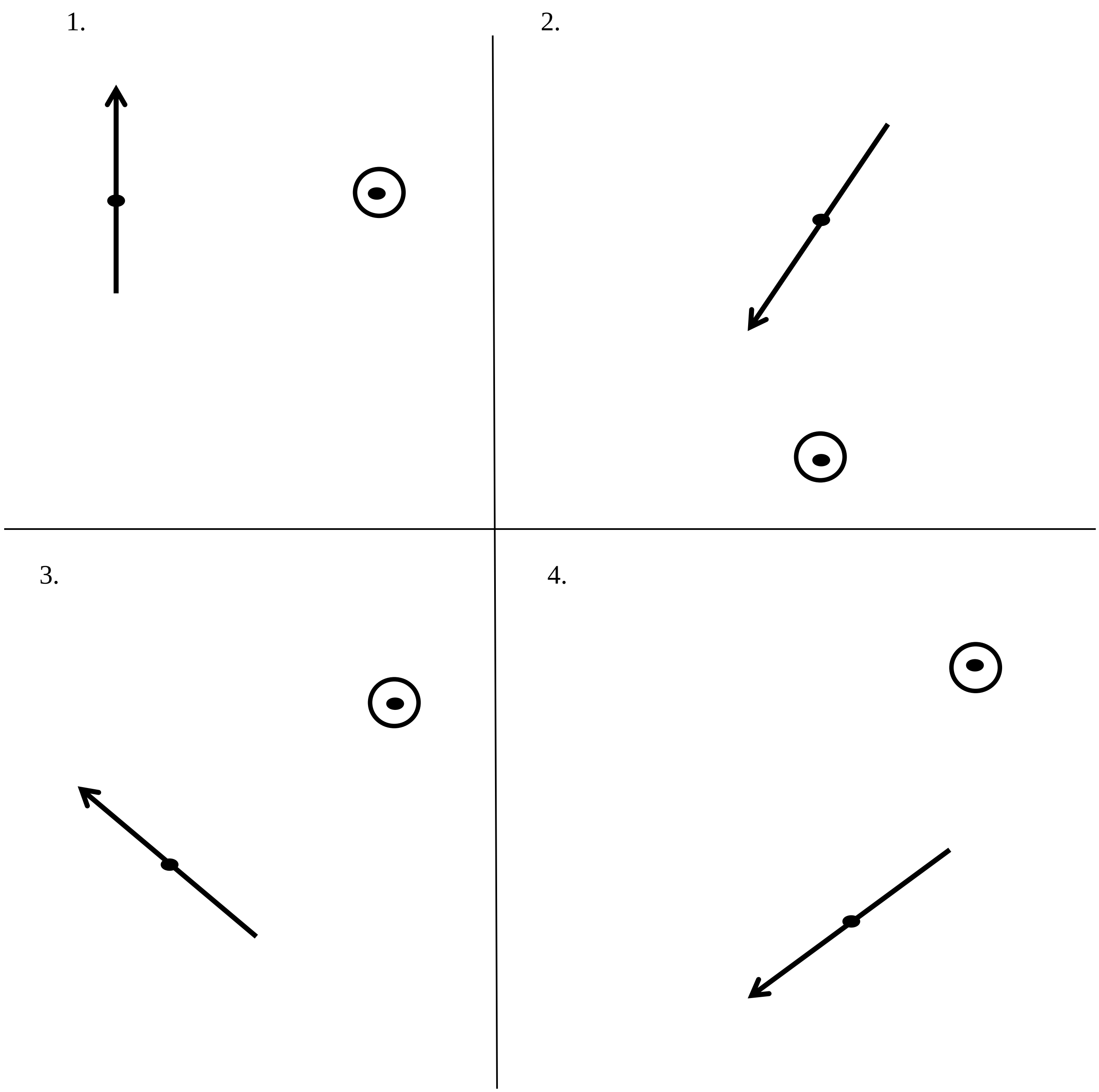
Purpose:	To determine whether students understand angle turning and to discover whether they approximate angle size correctly.
Materials:	One spinner model and spinner worksheets.

Script: (Before you begin this activity demonstrate turning the spinner provided. Orient the spinner so the arrow points to the heading 225°. Ask the student to turn the arrow to its right. Then ask the student to turn the arrow to its left. Note if student reverses these, or has any difficulty, do not correct.)

1. For the picture on this page, estimate how many degrees and in which direction (right or left) you should turn the arrow to aim it directly at the center of the ball. (Explain that students may use a range if they are concerned about estimation.)
2. Feel free to draw on the sheet if you would like to (ask students to write their answers on the sheet.)
3. Think out loud and tell me what you are doing as you work on these.
4. Now look again at picture 1. If you turned the arrow in the opposite direction to aim it at the ball, how many degrees would you turn it? How did you figure that out?
5. Repeat question #4 for each of the other pictures as long as it is productive.

Activity 4: Angle Measure

Spinner  Ball 



Examiners: Provide one of these for each interviewee

Activity 5: Angle Relationships

Purpose:	To determine whether the student can perceive relationships between and among angles such as congruence, angle complements, and angle supplements.
Materials:	Four angle model cutouts and two angle worksheets.

Part 5a Script:

1. Would you tell me what a right angle is? (Record response.)
2. Could you show me a right angle on this sheet of figures? (Activity 5, students' worksheet 1: record any angles(s) the students indicate.)

   (Remove the worksheet from view. Show the student the four angle cutouts.)
3. Would you put two of these angles together to form a right angle? (If the student demonstrated some understanding of the concept "right angle" then continue. Remove the cutouts from view. Show the angle worksheet again.)
4. Find as many pairs of angles as you can on this page that when put together would form a right angle. (After the student responds with the pair (s), ask: why would these form a right angle?

Part 5b Script:

1. Would you tell me what a straight angle is? (Record response.)
2. Could you show me a straight angle on this sheet of figures? (Activity 5, student worksheet 1: record any angle(s) the student indicates.)

   (Remove the worksheet from view. Show the student the four angle cutouts.)

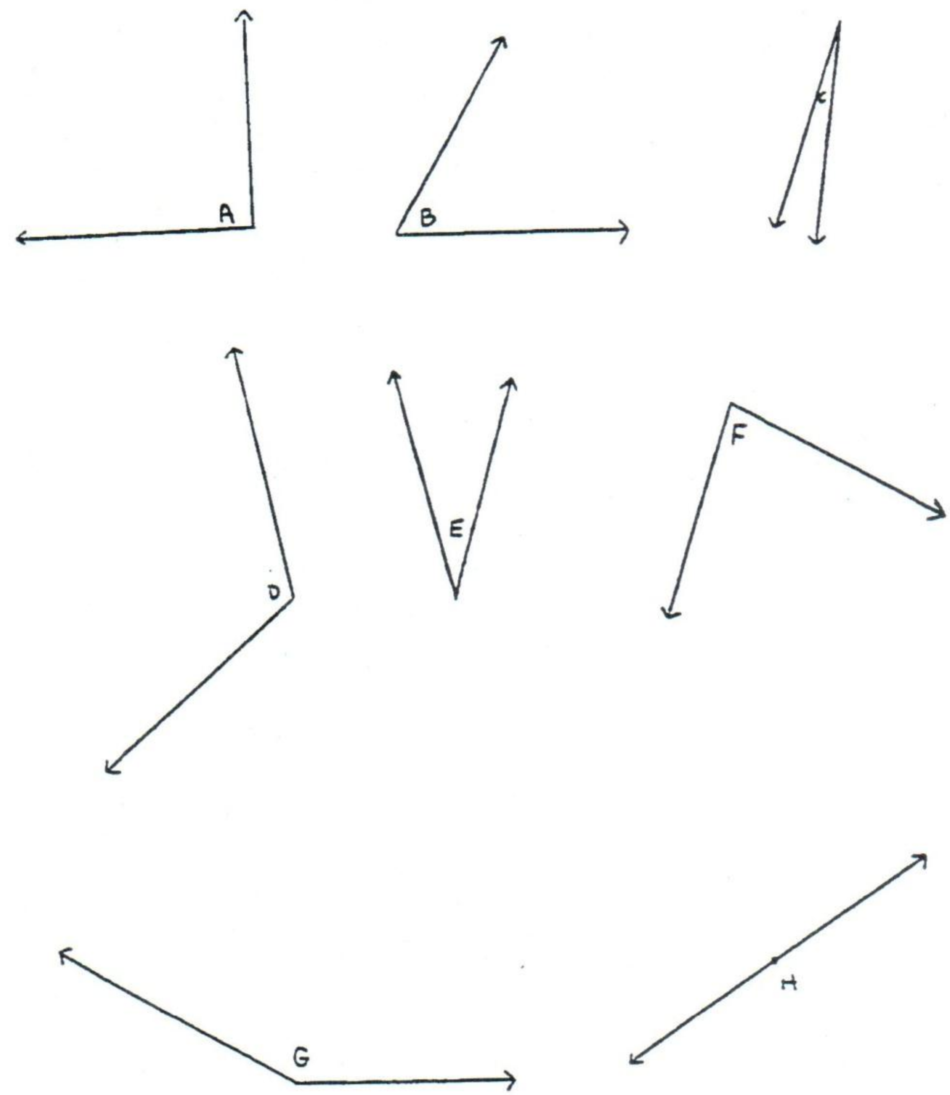
3. Would you put two of these angles together to form a straight angle?

   (If the student demonstrated some understanding of the concept "straight angle" then continue. Remove the cutouts from view. Show the angel worksheet again.)

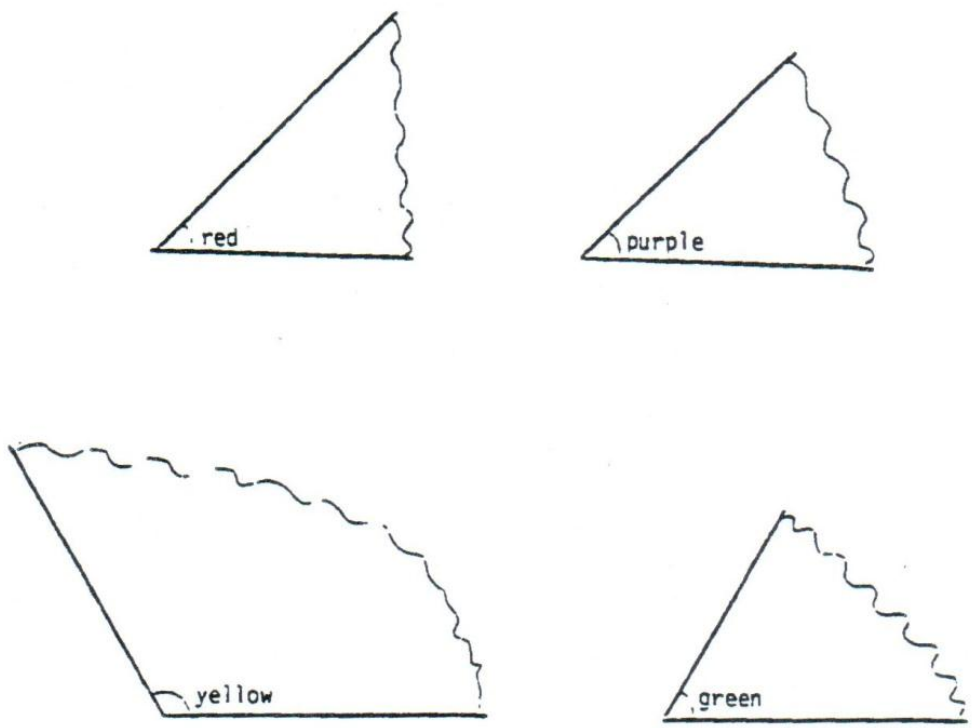
4. Find as many pairs of angles as you can on this page that when put together would form a straight angle.

   (After the student responds with the pair(s), ask: Why would these form a straight angle?

Angle Activity 5. Worksheet 1.



Activity 5. Angle Cutouts



APPENDIX E

Angle Instrument Scoring Criteria[13]

Interviewer Notes on Angle

Task 1: Drawing Angles

The student draws angles independently. Yes_____ No_____

Student's initial response to how many you can draw _____

How does the student indicate angles may differ?

Size____ (Is response explicit / can student justify?)

Degrees____

Length of Ray____

Orientation____

Other____

Does the student agree or disagree with his/her previous response to how many you can draw? ____

Scoring Criteria

1.1 If the student is unable to draw angles in one or more instances this would be indicative of pre-first level thinking. Drawing angles does not preclude 2nd level.

1.3 The student may include orientation, for example, or may refer to the "shape" of the figure and be attending to the rays. Relevance can best be determined when probing of what the student meant by "shape" occurred.

1.4 This would be scored when the student excluded such attributes as straightness or connectedness (perhaps drew some other figure).

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[13] From "The impact of experience in a Logo environment on adolescents' understanding of angle: A van Hiele-based clinical assessment, by Scally, 1990, 350-359. Unpublished doctoral dissertation, Emory University, Atlanta, Georgia. Reprinted with permission.

1.6 The student refers to visual appearance rather than relevant properties to compare some or all of the different angles drawn. The student may use a standard name (such as right angle), but indicate that his or her justification is “looks like.”

2.1 The student describes angles in some or all cases in terms of their properties (including, but not limited to degrees or descriptions such as acute), but is confused about necessary and sufficient conditions for describing angles. Credit should be given for any property identification. The student may compare the angles using non-mutually exclusive properties yet receive credit for this indicator.

2.7 If the student explicitly generalizes that a property of one type of angle drawn on the page is also a property of all other angle of that type one may give credit for this indicator. The emphasis in this indicator is on application. If the student indicates by word or deed that all angles fall into one of three categories (right, acute or obtuse) one may give credit.

3.3 The student indicates the possibility of drawing an infinite number of angles.

Tasks 2a and 2b: Identifying and Defining Angles

a) The student identifies angles in a drawing (tower)

# of initial 3 appropriately identified ____ (list): ____________

# of initial 3 inappropriately identified ____ (list): ___________

The student identifies after probing

appropriately ____ (list):_______________________

inappropriately ____ (list): ________________________

Reasons for inclusion (responses to “Why?”)

Two lines meet _______

Vertex ________

Point ________

Other ________

Reasons for exclusion (responses to “Why not?”)

Curved _______

Disjoint ________

Other ________

b) The student identifies angles

appropriately – independent mode is 2 ___ 4 ___ 6 ___ 8 ___ 10 ___ 11 ___

straight angle 9 ____

angles in multiple figures 3 ___ 7 ___ 12 ___

Angles recognized in complex closed curve? Interior _____ exterior _____

Inappropriately – curved 1 _____ disjoint 5 _____

Reasons for inclusion (responses to “Why?”)

Two lines meet ______

Vertex ______

Point ______

Other ______

Reasons for exclusion (responses to “Why not?”)

Curved ______

Not meeting (disjoint) ______

Other ______

How does the student say he/she would describe angles?

Scoring Criteria

1.2 The student relies on visual clues and does not attend to relevant properties when identifying and labeling some or all angles. (The student's performance may be inconsistent.) If the student is unable to identify angles this would be indicative of pre-first level thinking.

1.3 The student may include orientation, or exclude angles within closed figures believing openness to be a property for example, or may refer to the "shape" of the figure and be attending to the rays. Relevance can best be determined when probing what the student meant by "shape" occurred. A student may indicate that an angle cannot measure 180°, and include some idea of bend as characteristic of angle. Do not check this indicator on that behavior alone, for many texts teach that straight angles are impossible.

1.4 This would be scored when the student excluded such attributes as straightness or connectedness (Fig. 1 and Fig. 5 of Task 2b for example).

2.1 The student describes angles in terms of their properties (including, but not limited to, degrees) but is confused about necessary and sufficient conditions for describing angles. Credit should be given for the identification of any or all properties.

2.4 When responding to "What would you tell someone to look for to pick out all the angles on a sheet of figures?" the student lists insufficient or excessive properties. The student's description is not complete in terms of necessary and sufficient conditions.

2.7 If the student explicitly indicates by word or deed that a property is generalizable to all angles (that all angles are unions of two rays, for example) one may give credit for this indicator. The emphasis of this indicator is on application. The generalization must be one that would lead the student to produce correct rather than incorrect angle identifications.

Task 3: Sorting Angles

Set includes: 1-115°, 3-90°, 4-110°, 5-90°, 6-28°, 7-150°, 8-35°

List which angles student sorted together:

First grouping   Second

Third   Fourth (if any)   etc.

Student's method and rational for the sorts (check those that apply):

Implicit right, obtuse, acute ____________

Explicit right, obtuse, acute w/standard______ or non-standard language ________

States 90° delimiter____________ Degrees___________

Length of ray __________   Looks like basis______________

Left angle ______________   Clock face analogy_______________

Other ______________

Scoring criteria

(Although these cutout representations are in themselves "shapes, the task will not be scored in regard to properties of angles, not of shapes.)

1.3 The student may include orientation, for example, or may refer to the "shape" or "size" of the figure and be attending to the rays. Relevance can best be determined when probing of what the student meant by "shape" or "size" occurred.

1.4 If the student does not seem to be sorting by the 90° referent for sorting, he or she has excluded a relevant property.

1.5 The student is able to sort at least some angles, but the sorting is inconsistent, or the student sorts by non-distinguishing properties. The student sorts on the basis or appearance.

2.1 The student is able to sort at least some angles, but may or may not be confused about the need for justification of the sorting. The student who makes incomplete but consistent sorting may receive credit for this indicator.

2.7 Give credit if the student approaches the sorting task in a generalized manner, implicitly or explicitly sorting by right, acute, and obtuse. The sorts do not have to be complete.

Task 4: Angle Measure

For each check is student was accurate in direction + or – 10°

1. Right _______________ 90°_______________ Other_____________
2. Left ________________ 20°_______________ Other ____________
3. Right_______________ 110°______________ Other_____________
4. Right_______________ 165°______________ Other_____________

Do the same of each reversal

1. Left_______________ 270°______________ Other_____________
2. Right______________ 340°______________ Other_____________
3. Left_______________ 250°______________ Other_____________
4. Left_______________ 195°______________ Other_____________

Strategy used (check all that apply)

Looks like ___________

90° quadrants oriented to edges of paper _________

90° quadrants oriented to spinner and ball _________

180° turn plus estimate of more needed turn ___________

360° subtraction____________

Nonstandard unit of measure (i.e. clock) ____________

Other ____________

Scoring Criteria

1.3 An example might include imposing a quadrant on top of the spinner and ball, with axes oriented horizontally and vertically in relation to the page rather than in relation to the spinner, and assigning values for the turn in a fixed manner (say left is 90, down is 180, right is 270) in relation to where the spinner would point in this fixed system. Another might be trying to interpret turn like the hands of a clock, and assigning values similarly.

1.4 If the students excludes relevant properties (such as 180° opposite turn or 360° total turn) when determining angle turn or opposite turn then this indicator should be checked.

1.6 The student refers to visual appearance to estimate turn in one or more cases. Accuracy of angle estimate (with or without verbalization) does indicate the use of properties rather than just appearance.

2.2 The student identifies relationships in one or more cases in the task (the turning angle and opposite turning angle sum to 360°; turning the spinner in the opposite direction requires a turn of 180° for example) but may or may not apply them consistently, and need not use the relationships in a deductive explanation. The level of response, however, should be beyond rote.

2.3 The student uses appropriate vocabulary for these relationships. (As a counter-example, consider the student who uses imprecise language and describes both 180° turns and 90° turns as "complete" turns.)

2.5 The student is able to orient his or her perspective to that of the spinner in giving both measure and direction of angle turns in at least three of the four original turns.

2.6 The student accurately estimates one or more angle measure by half turn is 180° to estimate angle measure between these) or by using a property (such as "quarter turn" or right angle measure). One may give credit for an intuitive use of degree measure.

2.7 The students explicitly generalizes that a property is true of turning angles in the course of estimating measure (that a complete turn is always 360° no matter the orientation of where the turn begins, or that a half turn equals 180° for example). Generalizing means that the property should be recognized by the student applying in all cases.

Tasks 5a and 5b: Angle Relations

a) Defines right angle as:

____

____

Identifies right angle: A___________ other(s) ______________

Forms right angle: red and purple __________ others___________

(A=90° B=60° C=10° D=120° E=30° F=80° G=150° and H=180°)

Pairs angles to form right angle: B and E ____ C and F ____ others ______

b) Defines straight angles: _________

Identifies straight angles: H ____ other(s) _____

Forms straight angle: green and yellow _______ others ________

Pairs of angles form straight angles: B and D _____ E and G ______ others ______

Scoring Criteria[14]

1.3 Orientation, such as stating that right angles point right, might be an irrelevant attribute. This response may be best scored when probed). Do not check this indicator in the case of a student who defines straight angle as including some "bend."

1.4 One may describe a straight angle as a line (ignoring that the angle is the union of two half-lines and must have a point), but do not check this indicator unless other relevant attributes are ignored in the task.

2.1 The student analyzes/compares angles in terms of their properties (including, but not limited to, degrees), but may not see the need for justification of the comparisons. Credit should be given for the identification of any or all properties. The student who applies these properties to comparisons inconsistently may receive credit for this indicator.

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[14] This study is not utilizing interview questions 5d and 6 as they cover concepts such as parallelism which has not been taught to the participants. Therefore, parts of the scoring criteria have been omitted to reflect this change.

2.7 The student generalizes by word or deed that a property true of one angles is also true of other angles of that type (that right angles measure 90°, that straight angles measure 180°), but the student need not order properties, or describe relationships between properties. This scoring should apply only to generalizations which are accurate and consistently applied.

3.2 The student formulates and explicitly states complete definitions for right and straight angles.

3.4 The student explicitly describes relationships between properties.

3.5 The student presents an informal argument/proof to justify his or her estimates using a) logical, correct relationships between properties or…

APPENDIX F

Table F1.

Occurrence of Indicative van Hiele Level Behaviors by Task

Van Hiele Indicators	Task 1 Draws Angle	Task 2 ID Angle	Task 3 Sorts Angle	Task 4 Measure Angle	Task 5 Relates Angle
1.1 Draws angles		XXXX	XXXX	XXXX	XXXX
1.2 ID/Names < Visually	XXXX		XXXX	XXXX	XXXX
1.3 Includes irrelevant					
1.4 Excludes relevant					
1.5 Sorts by looks	XXXX	XXXX		XXXX	XXXX
1.6 Compares by looks		XXXX	XXXX		
2.1 Analyze by property				XXXX	
2.2 IDs relations	XXXX	XXXX	XXXX		
2.3 Relation vocabulary	XXXX	XXXX	XXXX		
2.4 Excess/Insuff prop	XXXX		XXXX	XXXX	XXXX
2.5 Decentrates	XXXX	XXXX	XXXX		XXXX
2.6 Measure by property	XXXX	XXXX	XXXX		XXXX
2.7 Generalize property					
3.1 Necessary Sufficient	XXXX	XXXX	XXXX	XXXX	
3.2 Forms definition	XXXX	XXXX	XXXX	XXXX	
3.3 Infinite angles		XXXX	XXXX	XXXX	XXXX

Note. XXXX indicates that the task did not elicit that indicator. 1 indicates that the student demonstrated that indicator during the prior interview. 2 indicates that the student demonstrated that indicator during the post instruction interview. - indicates that the student did not demonstrate that indicator during the interview. Adapted from “The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment,” by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

APPENDIX G

Table G1.

Van Hiele Levels by Interview and Task

Van Hiele Tasks	Pre instruction interview	Post instruction interview
Draws Angles		
Identifies Angle		
Sorts Angle		
Angle Measure		
Angle Relations		

Note. V indicates that the student is working at the visual level; A indicates that the student is working at the analysis level, and I indicates that the student is working at the informal deduction level. The predominant level is indicated by upper case letters. Adapted from "The impact of experience in a Logo learning environment on adolescents' understanding of angle: a van Hiele-based clinical assessment," by S. P. Scally, 1990, Unpublished doctoral dissertation, Emory University, Atlanta, Georgia.

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