ON THE EXISTENCE AND REGULARITY OF SOLUTIONS TO A FAMILY OF PARABOLIC PDE

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A thesis submitted to the faculty at the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics in the College of Arts and Sciences.

Chapel Hill
2022

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ABSTRACT

Benjamin Trigsted: On the Existence and Regularity of Solutions to a Family of Parabolic PDE
(Under the direction of Jeremy Marzuola)

We study mainly a second order parabolic PDE which arises in the modeling of the evolution of crystal surfaces. We prove the existence and uniqueness of analytic solutions to a toy model of the problem when a certain size constraint is imposed on the initial data. We also prove a finite-time existence and uniqueness result while imposing no constraints on the size of the initial data. Lastly, we investigate the possibility of finite-time singularity formation in solutions to the second order and a fourth order problem via self-similar coordinates – a topic still being researched.
To Nanny and Papa T
ACKNOWLEDGEMENTS

I would like to thank my advisor, Jeremy Marzuola, for his time and patience. He has taught me so much, and his guidance over the past year has been invaluable. I would also like to thank David Ambrose and Mark Williams for their time as members on my committee.

Over the past two years, I have learned an incredible amount of mathematics and would like to thank the faculty in the department of mathematics at UNC Chapel Hill for their instruction and for making me feel at home. A special thanks to Jason Metcalfe, Yaiza Canzani, David Rose, and Idris Assani.

Lastly, I would like to thank my parents, Nanny Jean, and Maureen for all of their love and support. I could not have done this without them.
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CHAPTER 1

Introduction

We will begin by making some definitions [LE]. Let $U \subset \mathbb{R}^n$ be an open set. Fix $T > 0$. Define $U_T = U \times (0, T]$. Let $u : U_T \to \mathbb{R}$ be defined by $u = u(x,t)$. For each time $t$, we will let $L$ denote the second-order linear partial differential operator having either divergence form

$$Lu = - \sum_{i,j=1}^{n} (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

or the non divergence form

$$Lu = - \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i,x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

for the given coefficients $a^{ij}, b^i, c$ with $i, j = 1, 2, ..., n$. The operator $\frac{\partial}{\partial t} + L$ is said to be parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta |\xi|^2$$

for all $(x,t) \in U_T$, $\xi \in \mathbb{R}^n$. Letting our coefficients $a^{ij}, b^i, c$ depend on $u$ and $\nabla u$, we define the quasilinear operator $Q$ by its divergence form

$$Qu = - \sum_{i,j=1}^{n} (a^{ij}(x,t,u,\nabla u)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x,t,u,\nabla u)u_{x_i} + c(x,t,u,\nabla u)u$$

or its non divergence form

$$Qu = - \sum_{i,j=1}^{n} a^{ij}(x,t,u,\nabla u)u_{x_i,x_j} + \sum_{i=1}^{n} b^i(x,t,u,\nabla u)u_{x_i} + c(x,t,u,\nabla u)u.$$
The quasilinear operator $\frac{\partial}{\partial t} + Q$ is said to be parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i \xi_j \geq \theta |\xi|^2$$

for all $(x,t) \in U_T, \xi \in \mathbb{R}^n$.

In the second chapter of this paper, we will largely consider the following example:

$$h_t - \Delta h = -\frac{1}{2}(\Delta h)^2$$  \hspace{1cm} (1.1)

with spatial domain $\mathbb{T}^n$, the $n$-dimensional torus, and initial condition

$$h(\cdot, 0) = h_0.$$  

Note that the left hand side of this equation involves the uniformly parabolic operator $\frac{\partial}{\partial t} + L$, where $L = -\Delta$ is the operator arising from the case when $a^{ij} = \delta_{ij}$ and $b^i = c = 0$. We will frequently refer to (1.1) as a “toy problem” as it is an approximation of the second order problem:

$$\frac{\partial}{\partial t} h = 1 - e^{-\Delta h}.$$  \hspace{1cm} (1.2)

While working with spatial domain $\mathbb{T}$, in chapter 3 we will see that under the transformation $z = \partial^2_x h$, equation (1.2) becomes the quasilinear parabolic equation

$$z_t = -\partial^2_x(e^{-z}) = e^{-z} z_{xx} - e^{-z} (z_x)^2.$$  

The first main result we prove in this paper is an existence and uniqueness theorem for (1.1). We will proceed using a contraction mapping argument which is built off of a theory of operators on Analytic Spaces. Using the same notation as in [DA1] we will make some preliminary definitions.

Let $\mathbb{T}^n$ denote the $n$-dimensional torus and define $\mathcal{A}(\mathbb{T}^n)$ to be the space of functions defined on $\mathbb{T}^n$ with Fourier series in $l^1$. Fix $0 < \alpha < 1$ and $j \in \mathbb{N}$. Define $B^j_\alpha$ to be the space of real functions
on $T^n \times [0, \infty)$, continuous in time, such that the following norm is finite:

$$
\| f \|_{B^j_\alpha} = \sum_{k \in \mathbb{Z}^n} |k|^j \sup_{t \in [0, \infty)} e^{\alpha t |k|} |\hat{f}(t, k)|.
$$

The space $B^j_\alpha$ exhibits two key properties. First, it is a Banach Algebra. That is, $B^j_\alpha$ is a Banach Space whose norm satisfies

$$
\| fg \|_{B^j_\alpha} \leq \| f \|_{B^j_\alpha} \| g \|_{B^j_\alpha}.
$$

Second, we have that $\| \Delta f \|_{B^0_\alpha} = \| f \|_{B^2_\alpha}$. One can also define the Free Space, $\hat{F}^{s,p}$, with norm

$$
\| f \|_{\hat{F}^{s,p}}(t) = \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi, t)|^p d\xi, \quad s > -\frac{d}{p}, \quad 1 \leq p \leq 2
$$

and perform analysis there. The similarities between the two spaces $\hat{F}^{s,p}$ and $B^j_\alpha$ are evident when one notes that the Wiener Algebra, $\mathbb{A}(\mathbb{R}^d)$, is $\hat{F}^{0,1}$ and $\Delta f \in \mathbb{A}(\mathbb{R}^d)$ is equivalent to $f \in \hat{F}^{2,1}$, see [LS].

The approach in Chapter 2 is as follows: We will analyze equation (1.1) by considering the Duhamel formulation of solutions to the problem with initial data $h_0$. We will define an operator $T$, based off of the Duhamel formulation of (1.1) and show that it is a well defined operator when one restricts its domain to a particular analytic space. Finally, we will prove multiple lemmas which serve as estimates that ultimately show that $T$ is indeed a contraction under an appropriate size constraint on $h_0$. The results proved in Chapter 2 hold globally in time, differing from the local result discussed in Chapter 3. We also compare our analysis to the analysis of the fourth order problem

$$
\frac{\partial}{\partial t} h = \Delta e^{-\Delta h}
$$

as completed in [DA1].

The second major result of this paper is an existence and uniqueness theorem for the second order problem, (1.2) and is proved in Chapter 3. The main steps are as follows:

We assume a sufficiently regular solution exists, and using mainly integration by parts we perform a series of energy estimates. We then introduce a mollifying operator which we use to approximate the problem. After performing a second sequence of energy estimates on potential
solutions to the regularized problem, we define an energy functional from which a Grönwall argument yields a short-time solution to the regularized problem. We then show that a limiting procedure gives us a solution to the original PDE and that this solution is unique.

Finally, in Chapter 4 we investigate potential singularity formations solutions to (1.2). Under a suitable self-similar coordinate transformation, we are able to transform our problems into ODE which we can solve via separation of variables. The fourth chapter gives a glimpse into the ongoing research in singularity formation of solutions to the problems discussed in the first three chapters. We will state a few preliminary results, qualitative hypotheses, and include numerical data which support our ongoing quantitative analysis.
CHAPTER 2
Analysis of the Toy Problem (Small Data)

We first state the main result we will prove in this section:

**Theorem 2.1.** Given initial data \( h_0 \) satisfying \( \| \Delta h_0 \|_{\mathbb{L}^\infty} < \frac{1}{2} \), there exists \( r_1 > 0 \) and \( 0 < \alpha < 1 \) such that (1.1) has a unique solution in a ball of radius \( r_1 \) in the Analytic Space, \( \mathcal{B}^2_\alpha \).

To prove Theorem 2.1, we will prove a few preliminary lemmas that will ultimately give us the tools needed to apply the Contraction Mapping Principle. But first, let us consider the Duhamel formulation of solutions to (1.1):

\[
h(\cdot, t) = e^{\Delta t} h_0 - \frac{1}{2} \int_0^t e^{\Delta (t-s)} (\Delta h)^2(\cdot, s) \, ds.
\]

(2.1)

By defining \( T h \) to be the right hand side of (2.1), it will be enough to solve the fixed point problem \( h = T h \). Define \( X = \{ f \in \mathcal{B}^2_\alpha : \| f - e^{\Delta t} h_0 \|_{\mathcal{B}^2_\alpha} \leq r_1 \} \) and define \( r_0 = \| e^{\Delta t} h_0 \|_{\mathcal{B}^2_\alpha} \). We will fix \( f \in X \) and calculate the norm

\[
\| T f - e^{\Delta t} h_0 \|_{\mathcal{B}^2_\alpha} = \| - \frac{1}{2} \int_0^t e^{\Delta (t-s)} (\Delta f)^2(\cdot, s) \, ds \|_{\mathcal{B}^2_\alpha}.
\]

(2.2)

As they will be used repeatedly in the proofs of the following lemmas, we recall the definition of the \( \mathcal{B}^j_\alpha \) norm and two properties which immediately follow:

\[
\| f \|_{\mathcal{B}^j_\alpha} = \sum_{k \in \mathbb{Z}^n} |k|^j \sup_{\tilde{t} \in [0, \infty)} e^{\alpha \tilde{t}} | \hat{f}(\tilde{t}, k)|.
\]

(2.3)

\[
\| fg \|_{\mathcal{B}^j_\alpha} \leq \| f \|_{\mathcal{B}^j_\alpha} \| g \|_{\mathcal{B}^j_\alpha}
\]

(2.4)

\[
\| \Delta f \|_{\mathcal{B}^0_\alpha} = \| f \|_{\mathcal{B}^2_\alpha}
\]

(2.5)
Lemma 2.2. Define $J^+ F = -\frac{1}{2} \int_0^t e^{\Delta(t-s)} F(\cdot, s) ds$. Then $J^+ : \mathcal{B}_\alpha^0 \to \mathcal{B}_\alpha^2$ and
$$
\| J^+ f \|_{\mathcal{B}_\alpha^2} \leq \frac{1}{2(1-\alpha)} \| f \|_{\mathcal{B}_\alpha^0}.
$$

Proof. Note that $\mathcal{J}^+ f = -\frac{1}{2} \int_0^t e^{-|k|^2(t-s)} \hat{f}(k, s) ds$ and the following inequalities hold:

$$
\| J^+ f \|_{\mathcal{B}_\alpha^2} = \sum_{k \in \mathbb{Z}^n} |k|^2 \sup_{t \in [0, \infty)} e^{\alpha |k|} |\mathcal{J}^+ f| \leq \sum_{k \in \mathbb{Z}^n} |k|^2 \sup_{t \in [0, \infty)} e^{\alpha |k|} \cdot \frac{1}{2} \int_0^t e^{-|k|^2(t-s)} |\hat{f}(k, s)| ds
$$

$$
\leq \sum_{k \in \mathbb{Z}^n} |k|^2 \sup_{t \in [0, \infty)} e^{\alpha |k|} \cdot \frac{1}{2} \int_0^t e^{-|k|^2(t-s)-\alpha |s|} \cdot \left( \sup_{\tau \in [0, \infty)} e^{\alpha \tau} |\hat{f}(k, \tau)| \right) ds
$$

$$
\leq \left( \sum_{k \in \mathbb{Z}^n} \sup_{\tau \in [0, \infty)} e^{\alpha \tau |k|} |\hat{f}(k, \tau)| \right) \left( \sup_{k \in \mathbb{Z}^n} \sup_{t \in [0, \infty)} \frac{1}{2} |k|^2 e^{\alpha |k|} \int_0^t e^{-|k|^2(t-s)-\alpha |s|} ds \right).
$$

Observe also that $\sum_{k \in \mathbb{Z}^n} \sup_{\tau \in [0, \infty)} e^{\alpha \tau |k|} |\hat{f}(k, \tau)| = \| f \|_{\mathcal{B}_\alpha^0}$. Using elementary calculus we have that

$$
\int_0^t e^{-|k|^2(t-s)-\alpha |s|} ds = \frac{e^{-\alpha |k| t} - e^{-|k|^2 t}}{|k|^2 - \alpha |k|}.
$$

Therefore,

$$
\sup_{t \in [0, \infty)} \frac{1}{2} |k|^2 e^{\alpha |k|} \int_0^t e^{-|k|^2(t-s)-\alpha |s|} ds \leq \frac{1}{2(1-\alpha)}
$$

which implies that

$$
\| J^+ f \|_{\mathcal{B}_\alpha^2} \leq \frac{1}{2(1-\alpha)} \| f \|_{\mathcal{B}_\alpha^0}
$$

as desired.

Lemma 2.3. Provided that $\frac{(r_0 + r_1)^2}{2(1-\alpha)} \leq r_1$, $\mathcal{T} : X \to X$ is well defined.

Proof. Recall that $X = \{ f \in \mathcal{B}_\alpha^2 : \| f - e^{\Delta t} h_0 \|_{\mathcal{B}_\alpha^2} \leq r_1 \}$. Fix $f \in X$. By Lemma 2.2 and (2.4), the Banach Algebra property of $\mathcal{B}_\alpha^2$, we have:

$$
\| \mathcal{T} f - e^{\Delta t} h_0 \|_{\mathcal{B}_\alpha^2} = \| J^+ (\Delta f)^2 \|_{\mathcal{B}_\alpha^2} \leq \frac{1}{2(1-\alpha)} \| (\Delta f)^2 \|_{\mathcal{B}_\alpha^2} \leq \frac{1}{2(1-\alpha)} (\| \Delta f \|_{\mathcal{B}_\alpha^2})^2.
$$

Then by (2.5) and the triangle inequality we have that

$$
\frac{1}{2(1-\alpha)} (\| \Delta f \|_{\mathcal{B}_\alpha^2})^2 = \frac{(\| f \|_{\mathcal{B}_\alpha^2})^2}{2(1-\alpha)} \leq \frac{(r_0 + r_1)^2}{2(1-\alpha)}.
$$
Now we will derive a condition for $T$ to be a contraction.

**Lemma 2.4.** $T : X \to X$ is a contraction if $\frac{r_0 + r_1}{1 - \alpha} < 1$.

**Proof.** Fix $f, \tilde{f} \in X$. By applying Lemma 2.2, properties (2.4) and (2.5), and the triangle inequality, the following chain of inequalities holds:

$$
\|Tf - \tilde{Tf}\|_{B^2_{\alpha}} = \|J^+((\Delta f)^2) - (\Delta \tilde{f})^2)\|_{B^0_{\alpha}} \leq \frac{1}{2(1 - \alpha)} \|((\Delta f)^2) - (\Delta \tilde{f})^2)\|_{B^0_{\alpha}}
$$

$$
\leq \frac{1}{2(1 - \alpha)} \|\Delta f - \Delta \tilde{f}\|_{B^0_{\alpha}} \|\Delta f + \Delta \tilde{f}\|_{B^0_{\alpha}} = \frac{1}{2(1 - \alpha)} \|f - \tilde{f}\|_{B^0_{\alpha}} \|\Delta f + \Delta \tilde{f}\|_{B^0_{\alpha}}
$$

$$
\leq \frac{1}{2(1 - \alpha)} \|f - \tilde{f}\|_{B^0_{\alpha}} \left(\|\Delta f\|_{B^0_{\alpha}} + \|\Delta \tilde{f}\|_{B^0_{\alpha}}\right) \leq \left(\frac{r_0 + r_1}{1 - \alpha}\right) \|f - \tilde{f}\|_{B^0_{\alpha}}.
$$

We now prove a Lemma which estimates the size of the initial data in the Wiener Algebra.

**Lemma 2.5.** $\|e^{\Delta t}h_0\|_{B^2_{\alpha}} \leq \|\Delta h_0\|_{A(\mathbb{T}^n)}$.

**Proof.** Indeed,

$$
\|e^{\Delta t}h_0\|_{B^2_{\alpha}} = \sum_{k \in \mathbb{Z}^n} |k|^2 \sup_{t \in [0, \infty)} e^{\alpha t |k|} \left|\widehat{e^{\Delta t}h_0}\right| = \sum_{k \in \mathbb{Z}^n} |k|^2 \sup_{t \in [0, \infty)} e^{\alpha t |k|} \left|\widehat{e^{\Delta t}h_0}\right|
$$

$$
\leq \sum_{k \in \mathbb{Z}^n} |k|^2 \left|\hat{h}_0\right| \left(\sup_{k \in \mathbb{Z}^n} \sup_{t \in [0, \infty)} e^{t \alpha (|k| - |k|)}\right) = \|\Delta h_0\|_{A(\mathbb{T}^n)}.
$$

By the Contraction Mapping Principal, Theorem 2.1 will be proven if we can find $r_0, r_1 > 0$ such that Lemmas 2.3 and 2.4 are satisfied. To that end, we state and prove another Lemma:

**Lemma 2.6.** Fix $0 < r_0 < \frac{1}{2}$, set $r_1 = r_0$, and choose $0 < \alpha < 1 - 2r_0$. Then the hypotheses of Lemma 2.3 and Lemma 2.4 hold. Moreover, if the hypothesis of Lemma 2.3 holds, then $r_0 < \frac{1}{2}$.

**Proof.** Given $0 < r_0 < \frac{1}{2}$, set $r_1 = r_0$ and choose $\alpha < 1 - 2r_0$. Then

$$
\frac{(2r_0)^2}{2r_0} \leq 1 - \alpha \iff \frac{(2r_0)^2}{2(1 - \alpha)} \leq r_0 \iff \frac{(r_0 + r_1)^2}{2(1 - \alpha)} \leq r_1.
$$
and
\[ \frac{r_0 + r_1}{1 - \alpha} < 1 \]
so that Lemmas 2.3 and 2.4 are satisfied. Now suppose that Lemma 2.3 is satisfied. Since we are only considering \(0 < \alpha < 1\), we have
\[ \frac{(r_0 + r_1)^2}{2} < \frac{(r_0 + r_1)^2}{2(1 - \alpha)} \leq r_1^2 + 2r_0 r_1 + r_1^2 - 2r_1 < 0. \]

If we treat the expression above as a quadratic inequality in the variable \(r_1\) with \(r_0\) fixed, then some algebra yields
\[ 2r_0 - 1 \leq r_0^2 + 2r_0 r_1 + r_1^2 - 2r_1 < 0. \]
Therefore, Lemma 2.3 implies that \(r_0 < \frac{1}{2}\).

An application of the Contraction Mapping Principal proves Theorem 2.1. Thus, we have shown that when the initial data, \(h_0\), satisfies \(\|\Delta h_0\|_{L^\infty(T^a)} < \frac{1}{2}\), (1.1) has a unique solution in a ball of positive radius in Analytic Space. Moreover, our bound of \(\frac{1}{2}\) is sharp – in the sense that if our initial data were any larger, the argument outlined in the section would fail.

This is a key difference between the toy problem and the fourth order problem, (1.3). Sufficient bounds for \(r_0\) that guarantee solutions to (1.3) have been found [DA1], but they are not sharp. Specifically, the following theorem has been proven:

**Theorem 2.7.** If \(r_0, r_1 > 0\) and \(\alpha \in (0, 1)\) satisfy
\[ \frac{1}{1 - \alpha} \sum_{j=2}^{\infty} \frac{(r_0 + r_1)^j}{j!} \leq r_1 \]
and
\[ e^{r_0 + r_1} - \frac{1}{1 - \alpha} < 1 \]
and if
\[ \|h_0\|_{L^\infty(T^a)} \leq r_0, \]
then (1.3) has a unique solution in the ball of radius $r_1$ centered at $e^{-\Delta^2 t}h_0$ in $B^2_\alpha$.

It has been checked numerically that $r_0 = .38$, $r_1 = .3029$, and $\alpha = .02$ satisfy the hypotheses of Theorem 2.7. However, for $r_0 = .39$ no choice of $r_1 > 0$ and $\alpha \in (0, 1)$ appears to satisfy the regularity condition stated in the hypothesis of Theorem 2.7.
CHAPTER 3
Local Existence for a Second Order Problem

We will now examine the equation

$$h_t = 1 - e^{-\partial^2_x h}$$  \hspace{1cm} (3.1)

with spatial domain $T$ and initial condition

$$h(x, 0) = h_0.$$  

Under the transformation $z = \partial^2_x h$, we arrive at the equation

$$z_t = -\partial^2_x (e^{-z})$$  \hspace{1cm} (3.2)

with initial condition

$$z(x, 0) = z_0.$$  

We will perform our analysis by way of energy estimates. We will begin with a few preliminary lemmas where we assume sufficient differentiability conditions so that we may take derivatives and repeatedly use integration by parts.

**Lemma 3.1.** Let $z$ be a sufficiently regular solution to (3.2). Then $\frac{d}{dt} \|z(\cdot, t)\|^2_{L^2} \leq 0$.

**Proof.** The right hand side of (3.2) is equal to

$$e^{-z} z_{xx} - (z_x)^2 e^{-z}.$$  \hspace{1cm} (3.3)
Thus, by multiplying both sides of (3.2) by \( z \) and integrating over \( T \) we have

\[
\frac{1}{2} \frac{d}{dt} \int_T z^2 dx = \int_T z \cdot z_{xx} e^{-z} dx - \int_T z \cdot (z_x)^2 e^{-z} dx.
\]

Integrating by parts on the second integral on the right hand side yields

\[
\frac{1}{2} \frac{d}{dt} \int_T z^2 dx = \int_T z \cdot z_{xx} e^{-z} dx - \int_T (e^{-z} \cdot z \cdot z_{xx} + e^{-z} \cdot (z_x)^2) dx
\]

\[
= - \int_T e^{-z} \cdot (z_x)^2 dx \leq 0.
\]

Therefore, we have \( \frac{d}{dt} \|z(\cdot, t)\|_{L^2} \leq 0 \).

\[
\square
\]

**Lemma 3.2.** \( \frac{d}{dt} \|z(\cdot, t)\|_{L^2}^2 = - \int_T e^{-z} (z_{xx})^2 dx + \int_T e^{-z} (z_x)^2 \cdot z_{xx} dx. \)

**Proof.** By differentiating (3.3) in \( x \) and multiplying by \( z_x \), we have that

\[
\frac{1}{2} \partial_t (z_x)^2 = z_x e^{-z} \cdot z_{xxx} - z_{xx} e^{-z} \cdot (z_x)^2 + (z_x)^4 e^{-z} - 2e^{-z} \cdot (z_x)^2 \cdot z_{xx}.
\]

Integrating, we see then that

\[
\frac{1}{2} \frac{d}{dt} \int_T (z_x)^2 dx = \int_T z_x e^{-z} \cdot z_{xxx} dx - \int_T z_{xx} e^{-z} \cdot (z_x)^2 dx + \int_T (z_x)^4 e^{-z} dx - \int_T 2e^{-z} \cdot (z_x)^2 \cdot z_{xx} dx
\]

\[
= (A) - (B) + (C) - (D).
\]

Integrating by parts, we see that

\[
(A) - (B) = - \int_T (z_{xx})^2 e^{-z} dx.
\]

Similarly, integration by parts yields that

\[
(C) = 3 \int_T e^{-z} (z_x)^2 \cdot z_{xx} dx
\]
so that

\[(C) - (D) = \int_{\mathbb{T}} e^{-z}(z_x)^2 \cdot z_{xx} dx.\]

Therefore our conclusion holds.

**Lemma 3.3.** \[\frac{1}{2} \frac{d}{dt} \|z_{xx}\|_{L^2}^2 = \frac{9}{2} \int_{\mathbb{T}} e^{-z}(z_x)^2 \cdot z_{xx} dx - \frac{3}{2} \int_{\mathbb{T}} e^{-z}(z_{xx})^2 dx - \int_{\mathbb{T}} e^{-z}(z_{xxx})^2 dx - \frac{1}{5} \int_{\mathbb{T}} e^{-z}(z_x)^6 dx.\]

**Proof.** The approach is the same as in the previous two lemmas. By differentiating the right hand side of (3.4) in \(x\) and multiplying by \(z_{xx}\), we have that

\[
\frac{1}{2} \partial_t (z_{xx})^2 = (3.5)
\]

\[
z_{xx} \cdot z_{xxxx} e^{-z} - 4 e^{-z} \cdot z_x \cdot z_{xx} \cdot z_{xxx} + 6 e^{-z} \cdot (z_x)^2 \cdot (z_{xx})^2 - 3 e^{-z}(z_{xx})^3 - e^{-z}(z_x)^4 \cdot z_{xx}.
\]

Integration by parts on the fourth and fifth terms yields that

\[
\frac{1}{2} \frac{d}{dt} \|z_{xx}\|_{L^2}^2 = \int_{\mathbb{T}} e^{-z} \cdot z_{xx} \cdot z_{xxxx} dx + 2 \int_{\mathbb{T}} e^{-z} \cdot z_x \cdot z_{xx} \cdot z_{xxx} dx + 3 \int_{\mathbb{T}} e^{-z}(z_x)^2 (z_{xx})^2 dx - \frac{1}{5} \int_{\mathbb{T}} e^{-z}(z_x)^6 dx.
\]

Integrating by parts on the first term, we see

\[
\frac{1}{2} \frac{d}{dt} \|z_{xx}\|_{L^2}^2 = 3 \int_{\mathbb{T}} e^{-z} \cdot z_x \cdot z_{xx} \cdot z_{xxx} dx + 3 \int_{\mathbb{T}} e^{-z}(z_x)^2 (z_{xx})^2 dx - \int_{\mathbb{T}} e^{-z}(z_{xxx})^2 dx - \frac{1}{5} \int_{\mathbb{T}} e^{-z}(z_x)^6 dx.
\]

One last application of integration by parts to the first term yields our result.

To define our energy, we are going to need a bound on one more derivative, so we state a final estimation lemma.

**Lemma 3.4.**

\[
\frac{1}{2} \frac{d}{dt} \|z_{xxx}\|_{L^2}^2 = 2 \int_{\mathbb{T}} e^{-z} \cdot z_{xx} \cdot (z_{xxx})^2 dx + 8 \int_{\mathbb{T}} e^{-z}(z_x)^2 \cdot (z_{xxx})^2 dx + 15 \int_{\mathbb{T}} e^{-z} \cdot z_{xxx} \cdot (z_{xx})^2 \cdot z_x dx - 10 \int_{\mathbb{T}} e^{-z} \cdot z_{xxx} \cdot (z_x)^3 dx - 10 \int_{\mathbb{T}} e^{-z}(z_{xxx})^2 \cdot z_{xx} dx + \int_{\mathbb{T}} e^{-z} \cdot (z_{xx})^5 dx - \int_{\mathbb{T}} e^{-z} \cdot (z_{xxx})^2 dx.
\]
Proof. We differentiate the right hand side of (3.5) and multiply by $z_{xxx}$ to arrive at

$$\frac{1}{2} \frac{d}{dt} \|z_{xxx}(\cdot, t)\|_{L^2}^2 =$$

$$\int \left( e^{-z} \cdot z_{xxxx} \cdot z_{xxx} - 5e^{-z} \cdot z_{xxxx} \cdot z_{xxx} \cdot z_x + 10e^{-z}(z_x)^2 \cdot (z_{xxx})^2 + 15e^{-z} \cdot z_{xxx} \cdot (z_{xx})^2 \cdot z_x - 10e^{-z} \cdot z_{xxx} \cdot (z_x)^3 - 10e^{-z} \cdot (z_{xxx})^2 \cdot z_{xx} + e^{-z} \cdot (z_x)^5 \right) dx.$$  

Integrating by parts, we have that

$$\int \frac{e^{-z}}{T} \cdot z_{xxxx} \cdot z_{xxx} dx = - \int \frac{e^{-z}}{T} \cdot (z_{xxxx})^2 dx + \int \frac{e^{-z}}{T} \cdot z_{xxxx} \cdot z_{xxx} \cdot z_x dx.$$

After combining like terms, a final integration by parts yields our result.

We now have enough data to define our Energy. Define

$$E(t) = \|z(\cdot, t)\|_{L^2}^2 + \|z_x(\cdot, t)\|_{L^2}^2 + \|z_{xx}(\cdot, t)\|_{L^2}^2 + \|z_{xxx}(\cdot, t)\|_{L^2}^2 = \|z(\cdot, t)\|_{H^3}^2. \quad (3.7)$$

**Lemma 3.5.** For some constant, $C > 0$, $\frac{d}{dt} E(t) \leq Ce^{CE(t)}$.

Proof. As our spatial dimension is a subset of $\mathbb{R}$, we employ the following Sobolev estimate:

$$\|z\|_{\infty} \leq C \|z\|_{H^1} \quad (3.8)$$

Thus, by Lemmas 3.1 through 3.4, we have the following estimate:

$$\frac{d}{dt} E \leq C e^{CE^\frac{1}{2}} (E^3 + E^2 + E^\frac{3}{2}). \quad (3.9)$$

Since the expression in parenthesis in the right hand side of (3.9) is bounded by the power series expansion of $\tilde{C} e^{E^\frac{1}{2}}$, take $\tilde{C} = 5! = 120$ for example, thus we have

$$\frac{d}{dt} E \leq C e^{CE^\frac{1}{2}} \leq Ce^{CE}. \quad \square$$
Lemmas 3.1 through 3.4, and in turn our energy estimate, innately rely on the ability to take derivatives as we please. Since we are not guaranteed that solutions to (3.2) will be sufficiently regular, we will state a reformulation of the problem using a mollifier \([AS]\). To that end, for \(\epsilon > 0\) we let \(J\) denote the mollifier which projects – i.e. \(J = J^2\) – onto Fourier modes of wavenumber at most \(\frac{1}{\epsilon}\). If we let \(F\) denote the Fourier Transform, we have

\[
F(J\epsilon g - g)(k) = \begin{cases} 
0 & |k| \leq \left\lceil \frac{1}{\epsilon} \right\rceil \\
-F(g)(k) & |k| > \left\lceil \frac{1}{\epsilon} \right\rceil 
\end{cases}
\]  

(3.10)

We also list the following mollifier inequalities as in \([AS]\) which hold for all \(\epsilon > 0\) as a consequence of Plancherel’s Theorem. Fix \(f \in L^2\), then

\[
\|J\epsilon f\|_{L^2} \leq \|f\|_{L^2}.
\]  

(3.11)

Next, fix \(f \in L^2\) and \(s \geq 0\), then

\[
\|J\epsilon f\|_{H^s} \leq \frac{c}{\epsilon^s} \|f\|_{L^2}.
\]  

(3.12)

Lastly, for \(g \in H^m\) with \(m \geq 0\):

\[
\lim_{\epsilon \to 0^+} \|J\epsilon g - g\|_{H^m} = 0.
\]  

(3.13)

With this in mind, we reformulate our problem and state a mollified version of (3.2):

\[
\frac{\partial z\epsilon}{\partial \epsilon} = -J\epsilon (\partial^2_x (e^{-J\epsilon z\epsilon})),
\]  

(3.14)

where our initial condition

\[
z\epsilon(x, 0) = z_0
\]

is unchanged.

We remark that due to (3.12), we have for each \(\epsilon > 0\) that \(J\epsilon\) is a bounded operator. Therefore, the right hand side of (3.14) is Lipschitz and an application of Picard’s Theorem for ODE on a
Banach space guarantees that (3.14) has a solution for each fixed \( \epsilon \).

Now we will try to make sense of Lemmas 3.1 through 3.4 without making extra assumptions about the smoothness of our solutions. We begin with stating a result analogous to Lemma 3.1 but for (3.14) instead of (3.2).

**Lemma 3.6.** Let \( z_\epsilon \) solve (3.14). Then 
\[
\frac{1}{2} \frac{d}{dt} \| z_\epsilon(\cdot, t) \|_{L^2}^2 \leq 0.
\]

**Proof.** After differentiating, the right hand side of (3.14) is equal to 
\[
\mathcal{J}_\epsilon [ e^{-\mathcal{J}_\epsilon z_\epsilon} \cdot \mathcal{J}_\epsilon (z_\epsilon, xx) - (\mathcal{J}_\epsilon (z_\epsilon, x))^2 e^{-\mathcal{J}_\epsilon z_\epsilon} ].
\]

Then after multiplying both sides of (3.14) by \( z_\epsilon \) and using the fact that \( \mathcal{J}_\epsilon \) is self-adjoint, we have that
\[
\frac{1}{2} \frac{d}{dt} \| z_\epsilon(\cdot, t) \|_{L^2}^2 = \int_\Omega \mathcal{J}_\epsilon z_\epsilon \cdot e^{-\mathcal{J}_\epsilon z_\epsilon} \cdot \mathcal{J}_\epsilon (z_\epsilon, xx) dx - \int_\Omega \mathcal{J}_\epsilon z_\epsilon \cdot (\mathcal{J}_\epsilon z_\epsilon, x)^2 \cdot e^{-\mathcal{J}_\epsilon z_\epsilon} dx.
\]

Using integration by parts on the second integral on the right hand side, we see that
\[
\frac{1}{2} \frac{d}{dt} \| z_\epsilon(\cdot, t) \|_{L^2}^2 = -\int_\Omega e^{-\mathcal{J}_\epsilon z_\epsilon} (\mathcal{J}_\epsilon z_\epsilon, x)^2 \cdot e^{-\mathcal{J}_\epsilon z_\epsilon} dx \leq 0.
\]

Using the same strategy as in the proof of Lemma 3.6, one can derive similar estimates for (3.14) analogous to Lemmas 3.2, 3.3, and 3.4. Define
\[
E_\epsilon(t) = \| z_\epsilon(\cdot, t) \|_{L^2}^2 + \| z_\epsilon, x(\cdot, t) \|_{L^2}^2 + \| z_\epsilon, xx(\cdot, t) \|_{L^2}^2 + \| z_\epsilon, xxx(\cdot, t) \|_{L^2}^2 = \| z_\epsilon(\cdot, t) \|_{H^3}^2.
\]

We state the following result, a reformulation of Lemma 3.5, as a Corollary. Its proof follows from energy estimates on solutions to (3.14) which follow from Lemmas 3.1 through 3.4 and the techniques employed in the proof of Lemma 3.6.

**Corollary 3.7.** For some constant, \( C > 0 \), 
\[
\frac{d}{dt} E_\epsilon(t) \leq C e^{CE_\epsilon(t)}.
\]

We use this bound to show the existence of a solution to (3.2) by following the techniques for general quasilinear parabolic PDE as described in [MT]. By [DA2], the differential inequality stated
in Corollary 3.7 has a solution on the time interval \( I = [0, T^*) \), where

\[
T^* = \frac{e^{-CE_\epsilon(0)}}{C^2}, \tag{3.15}
\]

\( T^* \) is independent of \( \epsilon \), and until that time

\[
E_\epsilon(t) \leq -\ln \left( \frac{e^{-CE_\epsilon(0)} - C^2t}{C} \right). \tag{3.16}
\]

Therefore if we fix \( 0 < R < T^* \), we have that for all \( \epsilon > 0 \),

\[
z_\epsilon(x, t) \in C([0, R]; H^3(\mathbb{T})) \cap C^1([0, R]; H^1(\mathbb{T})). \tag{3.17}
\]

Thus there exists a weak limit point \( z \) such that

\[
z \in L^\infty([0, R]; H^3(\mathbb{T})) \cap Lip([0, R]; H^1(\mathbb{T})).
\]

Furthermore, there is a subsequence

\[
z_{\epsilon, \nu} \rightharpoonup z \text{ in } C([0, R]; H^1(\mathbb{T})) \tag{3.18}
\]

as the inclusion \( H^3 \hookrightarrow H^1 \) is compact. We also have, by interpolation inequalities, that \( \{ z_\epsilon : 0 < \epsilon \leq 1 \} \) is bounded in \( C^\sigma([0, R]; H^{3-2\sigma}(\mathbb{T})) \) for each \( \sigma \in (0, 1) \). Moreover, the inclusion \( H^{3-2\sigma}(\mathbb{T}) \hookrightarrow C^2(\mathbb{T}) \) is compact for small \( \sigma > 0 \), as \( 3 > \frac{1}{2} + 2 \), so we have that

\[
z_{\epsilon, \nu} \rightharpoonup z \text{ in } C([0, R]; C^2(\mathbb{T})). \tag{3.19}
\]

It follows that by taking \( \epsilon = \epsilon_\nu \),

\[
-J_\epsilon(\partial_\epsilon^2(e^{-J_\epsilon z_\epsilon})) \rightharpoonup -\partial_\epsilon^2(e^{-z}) \tag{3.20}
\]
in $C(\mathbb{T} \times H^2[0, R])$ for $\frac{1}{2} < \beta < 1$ so that $-\partial_x^2(e^{-z}) \in C(\mathbb{T} \times [0, R])$ by Sobolev embedding. As

$$\frac{\partial z_\epsilon}{\partial t} \rightarrow \frac{\partial z}{\partial t} \quad (3.21)$$

weakly, equating the limits in (3.20) and (3.21) gives us our result. Therefore $z$ satisfies (3.2), and the following theorem has been proven:

**Theorem 3.8.** Suppose $z_0 \in H^3(\mathbb{T})$. Then there exists $R > 0$ and

$$z \in L^\infty([0, R]; H^3(\mathbb{T})) \cap Lip([0, R]; H^1(\mathbb{T}))$$

such that $z$ is a solution to (3.2).

Thus far, we have not addressed the topic of uniqueness. We do so now.

**Theorem 3.9.** Equation (3.2) has a unique solution on $\mathbb{T} \times [0, R]$.

**Proof.** Let $z_1$ and $z_2$ be solutions to (3.2) as in Theorem 3.8. Let $w = z_1 - z_2$. Then we have

$$w_t = \partial_x^2(e^{-z_2} - e^{-z_1}). \quad (3.22)$$

Multiplying by $w$ and integrating, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} w^2 dx = \int_{\mathbb{T}} w \cdot \partial_x^2(e^{-z_2} - e^{-z_1}) dx$$

$$= - \int_{\mathbb{T}} w \cdot \partial_x(e^{-z_2} - e^{-z_1}) dx. \quad (3.23)$$

Taylor expanding $e^{-s}$ about $z_1$ and evaluating at $z_2$, we have that

$$e^{-z_2} = e^{-z_1} - \int_{z_1}^{z_2} e^{-s} ds = e^{-z_1} - \int_{0}^{1} e^{-z_1 - \tilde{s}(z_2 - z_1)}(z_2 - z_1) d\tilde{s}. $$
Thus, we have that
\[ e^{-z_2} - e^{-z_1} = G(x, t)w(x, t) \]
where
\[ G(x, t) = \int_0^1 e^{-z_1 - \tilde{s}(z_2 - z_1)} d\tilde{s}. \]

Plugging this into (3.23), differentiating in \( x \), and then integrating by parts, we see that
\[
\frac{1}{2} \frac{d}{dt} \int_T w^2 \, dx = -\int_T G_x \cdot w \cdot w_x \, dx - \int T G \cdot (w_x)^2 \, dx
\]
\[
= \int_T G_x \cdot w \cdot w_x \, dx + \int_T G_{xx} \cdot w^2 \, dx - \int T G \cdot (w_x)^2 \, dx.
\]

The second equality gives us that
\[-\int_T G_x \cdot w \cdot w_x \, dx = \frac{1}{2} \int_T G_{xx} \cdot w^2 \, dx.\]

Thus,
\[
\frac{d}{dt} \| w(\cdot, t) \|_{L^2}^2 \leq \int_T G_{xx} \cdot w^2 \, dx \leq C \| w(\cdot, t) \|_{L^2}^2
\]
where
\[
C = \max_{(x, t) \in T \times [0, R]} G_{xx}(x, t). \tag{3.24}
\]

Grönwall's Inequality then yields
\[
\| w(\cdot, t) \|_{L^2}^2 \leq e^{Ct} \| w(\cdot, 0) \|_{L^2}^2. \tag{3.25}
\]
But as $w(x,0) = 0$, we must have that

$$\|w\|_{L^2}^2 = 0.$$ 

Therefore, $z_1 = z_2$ and the proof is complete. \qed

Following from Theorems 3.8 and 3.9, we can state a result about the original problem, (3.1)

**Theorem 3.10.** Suppose $h_0 \in H^5(\mathbb{T})$. Then there exists $R > 0$ such that on $\mathbb{T} \times [0, R]$, (3.1) has a unique solution

$$h \in L^\infty([0, R]; H^5(\mathbb{T})) \cap Lip([0, R]; H^3(\mathbb{T})).$$
CHAPTER 4
Self-Similar Coordinates and Singularities

We can continue to analyze equations (1.2) and (1.3) using a self-similar coordinate change. Indeed, with spatial domain $\mathbb{R}$, under the transformation

$$u = e^{-\Delta h},$$

equation (1.2) becomes

$$\frac{\partial u}{\partial t} = u\Delta u$$

and equation (1.3) becomes

$$\frac{\partial u}{\partial t} = -u\Delta^2 u.$$  

We can attempt to solve these equations via separation of variables, and will begin by focusing our attention on (4.2). To that effect, we set

$$u(x, t) = X(x)T(t)$$

and by (4.2) we obtain

$$X(x)T'(t) = X(x)X''(x)T^2(t).$$

Equivalently, we have the equation

$$\frac{T'(t)}{T^2(t)} = X''(x).$$
Fix $\lambda \in \mathbb{R}$ and set

$$\frac{T'}{T^2} = \lambda$$

and $X'' = \lambda$.

Integrating, we arrive at the following:

$$T(t) = -\frac{1}{\lambda(t - t_0)}$$

and $X(x) = \frac{\lambda}{2} x^2 + a_1 x + a_0$.

Choosing our constants, we have the family of solutions to (4.2):

$$u(x, t) = \tilde{U}(x, t; \lambda, a_0, a_1, t_0) = \frac{\lambda}{2} x^2 + a_1 x + a_0 - \lambda(t - t_0).$$

(4.4)

The change of variables in (4.1) gives us that $u$ is non-negative while $\tilde{U}$ isn’t necessarily. Therefore we consider

$$U(x, t; \lambda, a_0, a_1, t_0) = \left(\frac{\frac{\lambda}{2} x^2 + a_1 x + a_0}{-\lambda(t - t_0)}\right).$$

(4.5)

By changing the parameters, we can arrive at a much nicer family of possible solutions. We can set $a_1 = 0$ and define $\alpha_0 = -\frac{2a_0}{\lambda}$ (we will take $\alpha_0 > 0$) to obtain the spatially even system

$$u(x, t) = S(x, t; \alpha_0, t_0) = \left(\frac{x^2 + \alpha_0}{2(t_0 - t)}\right).$$

(4.6)

which solves (4.2). Similarly, one can use separation of variables and a similar analysis to obtain the spatially even family of solutions

$$u(x, t) = S(x, t; \alpha_0, \alpha_2, t_0) = \left(\frac{x^4 + \alpha_2 x^2 + \alpha_0}{24(t - t_0)}\right).$$

(4.7)

which solves (4.3).

It is appropriate to ask how this analysis can be used to construct singular solutions to (1.2) and (1.3) – here we will focus primarily on (1.2) as analysis on (1.3) is still ongoing. Numerical data leads us to believe that we can construct approximate solutions that are smooth up to some finite time by choosing an appropriate way to "glue together" self-similar solutions. Using a backwards Euler finite difference scheme, we have obtained an evolutionary model of numerical solutions to
(4.2). Figure 4.1 showcases a fixed time slice of such a model. The vertical axis – which represents $u(x)$ for a fixed time – is labeled Crystal Height as the second order problem frequently arises in the mathematical modeling of the evolution of crystal surfaces.

![Figure 4.1: A fixed time-slice of a self-similar solution](image)

Zooming in, we see that $u(x)$ is smooth and appears to be parabolic near its maximum and minimum.

![Figure 4.2: Zooming in near the max](image)
If we zoom in near the area where the upward facing parabola and the two downward facing parabolas meet, we see that this time-slice of the self-similar solution appears to be smooth.

In fact, it appears that each time-slice of the self-similar solution is smooth up until the two downward facing parabolas collide. This happens in finite time, and we showcase an image of the phenomenon in Figure 4.5.
Based off of our numerical data, it appears that as the self-similar solution evolves in time the width of the upward facing parabola shrinks to zero. This happens at some finite time, $T$. At time $T$, there will be a jump in the derivative of $u(x)$ as evident by the cusp in Figure 4.5.

It is at the point $T$ that we believe singularity formation will occur in solutions to (1.2). Though constructing such a solution is still an ongoing research topic, we believe that by using methods of analysis similar to those used in finding solutions to the Viscous Berger’s Equation we will be able to appropriately "glue" together self-similar solutions in a way such that they will limit to a solution of (1.2).
REFERENCES


