THE SPRINGER MORPHISM, POLYNOMIAL REPRESENTATION RINGS, AND THE COHOMOLOGY RING OF GRASSMANNIANS

Sean Rogers

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Approved by:

Shrawan Kumar

Prakash Belkale

Jiuzu Hong

Robert Proctor

Richard Rimanyi

ABSTRACT

Sean Rogers: The Springer Morphism, Polynomial Representation Rings, and the Cohomology Ring of Grassmannians
(Under the direction of Shrawan Kumar)

To any almost faithful representation of a complex, connected, reductive algebraic group G of highest weight λ one can associate a dominant morphism from the group to its Lie algebra $\mathfrak{g}$. This map enjoys many nice properties. In particular, when restricted to a maximal torus it maps to the Cartan subalgebra. This map can be used to give a natural definition of polynomial representations for the classical groups of types B, C, and D. Given a parabolic subgroup $P \subset G$, Kumar showed there is a surjective algebra homomorphism from the polynomial represntations of a Levi subgroup of P to the cohomology of G/P which extends a classical result relating the polynomial representations of GL(r) and the cohomology of the grassmannian of r-planes in n-space $H^*(Gr(r,n))$. In this work we give an explicit determination of the map θ_λ for simple groups and consider Kumar's map for types B, C, and G.

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CHAPTER 1

Introduction

1.1 Historical Context

Schubert calculus as subject emerged from Schubert's work on the calculus of enumerative grometry [Sc1, Sc2]. A typical example of an enumerative problem is as follows (borrowed from [KL]). In three space, how many lines intersect four given lines? The solution for lines in general position turns out to be two. Schubert's approach to such problems was to work in the Grassmannian manifold. Let $V = \mathbb{C}^n$. Then the Grassmannian manifold $Gr(m, V)$ as a set is the set of all m-dimensional subspaces of V. It can be given the structure of a complex manifold (or projective variety) of dimension $n(m-n)$. Let $F_\bullet := \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = V\}$, where $dimF_i = i$, be a complete $flag$. The Grassmannian has a stratification of affine subsets given by geometric intersection conditions with respect to this flag. A partition $\lambda = \{\lambda_1, ..., \lambda_m\}$ is a sequence of weakly decreasing integers. Let $|\lambda| = \sum_{i=1}^{m} \lambda_i$. For a partiton λ such that $\lambda_1 \leq n - m$ we can define a subset of the Grassmannian,

$$\Omega_\lambda(F_\bullet) = \{X \in Gr(m, n) : dim(X \cap V_{n-m+j-\lambda_j} > j \; \forall i \leq j \leq m\}.$$

These subsets are isomorphic to $\mathbb{C}^{m(n-m)-|\lambda|}$ and are called Schubert cells. They form a stratification the Grassmannian. Their closures are the so called Schubert varieties $X_\lambda(F_\bullet)$. Let $[X_\lambda] \in H_{2(m(n-m)-|\lambda|)}(Gr(m, n))$ denote the fundamental class of X_λ in the singular homology of the Grassmannian. Then, take $\sigma_\lambda \in H^{2|\lambda|}(Gr(m, n))$ be their Poincare dual classes. Then the Schubert basis theorem states that these classes, hereafter called Schubert classes, σ_λ form an integral basis of the singular cohomology ring of the Grassmannian $H^*(Gr(m, n))$. These classes are independent of the choice of flag $F_\bullet$. The fundamental insight of classical Schubert calculus is that problems in enumerative geometry and intersection theory can be solved by performing corresponding algebraic calculations in the ring $H^*(Gr(m, n))$. Let σ_i be the Schubert class corresponding to the partition $\lambda = (i, 0, ..., 0)$. These are called the special Schubert classes. Pieri gave a rule for expanding the cup product of a special Schubert class and a general Schubert class in the Schubert basis. Giambelli gave a formula for expressing any Schubert class σ_λ as a polynomial in the special Schubert classes σ_i. Solving the above problem in enumerative geometry amounts to computing $\sigma_1^4 = 2\sigma_{(2,2)}$ in $H^*(Gr(2,4))$. $\sigma_{(2,2)}$ is the class of a point and we arrive at our answer of two. In general we have the structure constants of $H^*(Gr(m,n))$ are

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^{\nu} \sigma_\nu.$$

The constant $c_{\lambda\mu}^{\nu}$ is know to be the number of points in the intersection of general translates of the Schubert varieties $X_\lambda,\ X_\mu$, and $X_{\check{\nu}}$, where $\check{\nu}$ is the dual partition $\check{\nu}_i = m - n - \nu_{m+1-i}$.

A combinatorial rule for computing the coefficients $c_{\lambda\mu}^{\nu}$ in terms of the given partitions was given by Littlewood and Richardson. The context with which the coefficients arose however was not apriori related to intersections of Schubert cycles. Schur polynomials s_λ are a basis for the symmetric polynomials. Then one has $s_\lambda s_\mu = \sum c_{\lambda\mu}^{\nu} s_\nu$. It is well know that polynomial irreducible representations of $GL(m)$ are indexed by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_m)$ where λ represents the highest weight of the representation. Denote this representation by $V(\lambda)$. The character of this representation is the Schur polynomial s_λ. As the character of a tensor product of representations is the product of the characters of the given representations, it holds that

$$V(\lambda) \otimes V(\mu) = \sum c_{\lambda\mu}^{\nu} V(\nu)$$

.

Note that $c_{\lambda\mu}^{\nu}$ counts the dimension of the $GL(m)$ invariant subspace of $V(\lambda) \otimes V(\mu) \otimes V(\check{\nu})$, where $V(\check{\nu})$ is the dual representation of $V(\nu)$. Let $Rep_{poly}(GL(m))$ be the polynomial representation ring. That the structure constants of $H^*(Gr(m,n))$ and $Rep_{poly}(Gl(m))$ coincide as long as the partitions $\lambda,\ \mu,\ \nu$ fit in a $m \times m - n$ rectangle was proved by Lesiur [Les] in 1947. However, the proof is indirect. Essentially, they are both rings are governed by Schur functions. See [F6] for the history of this problem and other contexts in which the Littlewood-Richardson coefficients appear. Thus one wonders if there is a more natural explanation of this fact or a generalization to other classical groups $(Sp(2m),\ SO(2m),\ SO(2m+1))$. Possible explanations have been given by Tamvakis via the Chern-Weil theory of characteristic classes

[T1], by Belkale via tangent spaces to Schubert varieties [Be], and by Mukhin, Tarasov, and Varchenko via represenations of the Bethe algebra [MTV]. One problem of generalization is how to adequately define polynomial representations for other classical groups, or more generally connected reductive groups. Kumar gives one attempt at generalization via the Springer morphism in [Ku2]. This work is the genesis for this thesis.

A modern formulation of Schubert calculus can be given in terms of generalized partial flag varieties. A generalized $flag\ variety$ can be defined for an connected, complex, reductive algebraic group G as the projective variety G/B. Here B is a Borel subgroup, i.e. a maximal, connected, solvable Zariski closed subgroup. For $G = GL(m)$, the standard Borel subgroup is the set of upper triangular matrices and we have

$$G/B = \{F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = \mathbb{C}^m\}$$

is the variety of complete flags $F_\bullet$ with $dim(F_i) = i$. Every reductive goup has a maximal torus $T \subset B \subset G$ and a Weyl group $W = N(T)/T$. Let $\mathfrak{t},\ \mathfrak{b},\ \mathfrak{g}$ be the corresponding Lie algebras. The flag variety G/B has a cell decomposition called the Bruhat decomposition into affine open cells B_w, indexed by elements of the Weyl group $w \in W$. The closures of these cells X_w are also called Schubert varieties and a Schubert calculus can be defined on $H^*(G/B)$. Let ϵ_w be the Kronecker dual to the fundamental homology class of X_w (ϵ_w is then called a Schubert class). As before, $H * (G/B)$ has a basis of Schubert classes. Borel, [Bo], also gave a characterization of the cohomology of $H^*(G/B)$ via the characteristic map

$$\beta : S(\mathfrak{t}^*) \to H^*(G/B).$$

Here $S(\mathfrak{t}^*)$ is the symmetric algebra of the dual of the Cartan subalgebra $\mathfrak{t}$. The map is given by mapping characters to the first Chern class of the related line bundle. A Pieri-type formula is given by Chevalley, and a Giambelli type formula is given by the BGG Scubert polynomials defined by Bernstein, Gelfand, and Gelfand in their seminal work relating the Bruhat and Schubert presentations of the cohomology of G/B [BGG]. They identify a set of polynomials p_w in $S(\mathfrak{t}^*)$ corresponding Schubert classes ϵ_w using divided difference operators $A_w : S(\mathfrak{t}^*) \to S(\mathfrak{t}^*)$ corresponding to elements of the Weyl group. A parabolic group P is any group containing a Borel subgroup $B \subset P$. Then G/P is a generalized partial flag variety and there is an analagous picture of the Schubert calculus on G/P. For maximal parabolics $P \subset GL(m)$, $GL(m)/P$ is a Grassmannian.

Information about reductive groups and their Schubert calculus is discussed in more detail in Chapter 2 (Preliminaries).

1.2 Concerning this work

Here we describe the rest of the thesis. Let G be a connected reductive algebraic group over $\mathbb{C}$ with Borel subgroup B and maximal torus $T \subset B$ of rank n with character group $X^*(T)$. Let P be a standard parabolic subgroup with Levi subgroup L containing T. Let W (resp. W_L) be the Weyl group of G (resp. L). Let V_λ be an irreducible almost faithful representation of G with highest weight λ, i.e. λ is a dominant integral weight and the corresponding map $\rho_\lambda : G \to GL(V_\lambda)$ has finite kernel. Then, Springer defined an adjoint-invariant regular map with Zariski dense image from the group to its Lie algebra, $\theta_\lambda : G \to \mathfrak{g}$, which depends on λ [BR]. Properties of this map are discussed in Chapter 3 Section 1. In particular, when restricted to the maximal torus we have $\theta_{\lambda|T} : T \to \mathfrak{t}$. We note that this map can also be viewed as a generalization of the classical Cayley map. Furthermore, Kumar [Ku2] use this map to define the λ-polynomial representation ring of a group G, $\mathrm{Rep}_\lambda^{\mathbb{C}}(G)$. The ring $\mathrm{Rep}_\lambda^{\mathbb{C}}(G)$ is a subring of representation ring of G which is isomorphic to $S(\mathfrak{t}^*)^W$, the ring of Weyl group invariant polynomials. For any weights λ_1, λ_2 the λ-polynomial representation ring are isomorphic but the isomorphism is different. For $Sp(2n),\ SO(2n)$, and $SO(2n+1)$ we define the polynomial representation ring to be $\mathrm{Rep}_{\omega_1}^{\mathbb{C}}(G)$.

We can restate the classical result relating the polynomial representation ring of $GL(r)$ to the singular cohomology ring of the Grassmannian $H^*(Gr(r,n)$ as follows. There is an explicit surjective ring homomorphism

$$\xi : Rep_{poly}(GL(r)) \to H^*(Gr(r,n)).$$

In recent work by Kumar [Ku2], the Springer morphism is used in a crucial way to extend the above classical result relating the polynomial representation ring of the general linear group GL_r and the singular cohomology ring $H^*(Gr(r,n))$ of the Grassmannian of r-dimensional complex linear subspaces of $\mathbb{C}^n$ to the Levi subgroups of any reductive group G and the cohomology of the corresponding flag varieties G/P. Computing $\theta_\lambda|_T$ is integral to this process. Importantly, θ_λ takes the maximal torus T to its Lie algebra $\mathfrak{t}$, thus inducing an injective $\mathbb{C}$-algebra homomorphism $(\theta_\lambda|_T)^* : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$ between the corresponding affine coordinate rings. Let L be the Levi subgroup of the parabolic P which contains the torus T. The Springer morphism is equivariant under the adjoint action and thus $(\theta_\lambda|_T)^*$ takes $\mathbb{C}[\mathfrak{t}]^{W_L}$ to $\mathbb{C}[T]^{W_L}$. One can then define the $\lambda - polynomial\ subring\ Rep^{\mathbb{C}}_{\lambda-poly}(L)$ to be the image of $\mathbb{C}[\mathfrak{t}]^{W_L}$ under $(\theta_\lambda|_T)^*$ (as $Rep^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$). Here $Rep^{\mathbb{C}}(L)$ is the complex representation ring of L. This leads to a surjective $\mathbb{C}$-algebra homomorphism $\xi^P_\lambda : Rep^{\mathbb{C}}_{\lambda-poly}(L) \to H^*(G/P, \mathbb{C})$, as in [Ku2]. The map θ_λ enjoys many nice properties (see [KM]). In this work we compute $\theta_\lambda|_T$ in a uniform way for all simple algebraic groups G and any dominant integral weight λ.

As $\theta_\lambda|_T$ maps T into $\mathfrak{t}$, we have that for a given simple group G and an irreducible representation V_λ, one may write

$$\theta_\lambda(t) = \sum_{i=1}^{n} c_i(\lambda)\check{\alpha}_i,$$

where we take the simple coroots $\{\check{\alpha}_i\}$ as a basis for $\mathfrak{t}$. We give a complete determination for these coefficients $c_i(t)$ for any simple, simply-connected algebraic group G as a sum over the weights of the torus action on V_λ.

For a given representation V_λ, let Λ_λ be the set of weights appearing in the weight space decomposition of $V_\lambda = \bigoplus V^\mu_\lambda$, listed with multiplicity. Let $\omega_1, ..., \omega_n$ be the fundamental weights in $\mathfrak{t}^*$, and consider the weights $\mu \in \Lambda_\lambda$ written in the fundamental weight basis, i.e. $\mu = (\mu_1, ..., \mu_n) = \mu_1\omega_1 + ... + \mu_n\omega_n$. Let $e^\mu(t) \in X^*(T)$ be the corresponding character of T. Then we find that,

Theorem. *The coefficients $c_i(t)$ are determined by the following set of equations.*

$$\begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^\mu(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^\mu(t) \end{pmatrix} = S(G,\lambda) \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix},$$

where $S(G,\lambda) = \{\sum\limits_{\mu\in\Lambda_\lambda} \mu_i\mu_j\}_{ij}$.

The main result of [R] determines that

Theorem. *The above matrix*

$$S(G,\lambda) := \{\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j\}_{ij} = (\frac{1}{2}\sum_{\mu\in\Lambda_\lambda} \mu_i^2)S\ ,$$

where S is a specific uniform symmetrization of the Cartan matrix A for G, and μ_i is the coordinate of the fundamental weight corresponding to a long root (or any root in the simply-laced case).

In particular, for the simply-laced groups $S(G,\lambda) = (\frac{1}{2}\sum\limits_{\mu\in\Lambda_\lambda} \mu_1^2)A$. The determination of $S(G,\lambda)$ relies on the fact that Λ_λ is invariant under the action of the Weyl group W, and moreover that if $\sigma \in W$ then $dim(V_\mu) = dim(V_{\sigma.\mu})$. The above results are discussed in Chapter 3.

In Chapters 4 and 5 we recall the work *Representation ring of Levi subgroups versus the cohomology ring of flag varieties* by Kumar [Ku2]. In particular, discuss the map

$$\xi^P_{\omega_1}(L) : \text{Rep}^{\mathbb{C}}_{\lambda-poly} \to H^*(G/P,\mathbb{C})$$

in the case of the classical complex algebraic groups $GL(k),\ Sp(2k)$, and $SO(2k+1)$ and their maximal parabolics (V_{ω_1} is the defining representation in each case). Note we do not analyze the type D case, $SO(2n)$. The analysis is more or less uniform. In types $A,\ B$, and C quotienting by a maximal parabolic gives a Grassmannian. Consider the Dynkin diagram in classical type of rank n and take the maximal parabolic P^{n-k} corresponding to the $(n-k)^{th}$ node of the Dynkin diagram. Then, the Grassmannians in question are: For type A,

$$Gr(n-k,n) = \{X \in \mathbb{C}^n : dim(X) = n-k\},$$

For type C (let ϑ be a skew-symmetric bilinear form on $\mathbb{C}^{2n}$),

$$IG(n-k,2n) = \{X \in \mathbb{C}^{2n} : dim(X) = n-k,\ \vartheta(v,w) = 0\ \forall v,w \in X\},$$

For type B (let ϑ be a symmetric bilinear form on $\mathbb{C}^{2n+1}$),

$$OG(n-k, 2n+1) = \{X \in \mathbb{C}^{2n+1} : dim(X) = n-k,\ \vartheta(v,w) = 0\, \forall v, w \in X\}.$$

Then for each of these spaces there is a short exact sequence of vector bundles

$$0 \to S \to V \to Q \to 0.$$

Where V is the trivial rank n (respectively $2n$, $2n+1$) bundle, S is the tautological subbundle (i.e. the fiber over $X \in Gr(n-k, n)$ is $X \subset \mathbb{C}^n$ for the type A case), and Q is the tautological quotient bundle. In the above cases the Chern classes $c_i(Q)$ generate the cohomology ring $H^*(G/P^{n-k})$ [BKT1]. Buch, Kresch, and Tamvakis gave Pieri and Giambelli formulas with the Chern classes $c_i(Q)$ as special classes for both the classical and quantum cohomology of $IG(n-k, 2n)$ and $OG(n-k, 2n+1)$ in terms of $k-$strict partitions (see [BKT1, BKT2, BKT3, BKT4]). We rely heavily on their formalism and presentations of the cohomology rings.

For the parabolic group P^{n-k} above we have that the Levi subgroups of P^{n-k} for types A, B, and C are $L^A_{n-k} = GL(n-k) \times GL(k)$, $L^C_{n-k} = GL(n-k) \times Sp(2k)$, and $L^B_{n-k} = GL(n-k) \times SO(2k+1)$. We then have maps ξ^{n-k}, from the the Levi subgroup L_{n-k} to the corresponding Grassmannian. Factoring through this map allows one to recover the classical map ξ [Ku2, Theorem 8]. We give explicit descriptions of these maps ξ^{n-k}, i.e. we describe the images of the generators of $\text{Rep}^{\mathbb{C}}_{poly}(L_{n-k})$ in terms of the Chern classes of S and Q. If we fix k, and allow n to go to infinity we get the stable cohomology rings. For example, the stable cohomology ring of type A, $\mathbb{H}(Gr_k)$, is the inverse limit in the category of graded rings of the system

$$\cdots \leftarrow H^*(Gr(n-k, n), \mathbb{C}) \leftarrow H^*(Gr(n-k+1, n+1), \mathbb{C}) \leftarrow \dots$$

Then, factoring through ξ^{n-k} gives an isomorphism $\text{Rep}^{\mathbb{C}}_{\omega_1}(GL(k)) \simeq \mathbb{H}(Gr_k)$. We prove that the analogous maps in types B and C are injective.

CHAPTER 2

Preliminaries

2.1 Reductive Groups, Root Data, and Representations

Let G be a linear algebraic group over the complex numbers $\mathbb{C}$, i.e. a group which is also an algebraic variety such that the inverse and multiplication maps are morphisms. The radical of an algebraic group is the identity component of its maximal, closed, solvable subroup and the subgroup of unipotent elements in this group is referred to as its unipotent radical. If the unipotent radiacal is trivial then G is called $reductive$. Further, if the radical of G is trivial then the group is called *semi-simple*. For the rest of this subsection we will assume G is reductive.

A subgroup that is isomophic to $(\mathbb{C}^*)^k$ for some k is called a torus. For a given maximal torus T we can define the Weyl group $W = N(T)/T$, where $N(T)$ is the normalizer of T in G. Since all maximal tori are conjugate, different choices of T will lead to isomorphic Weyl groups. A Borel subgroup is a maximal, solvable, connected, Zariski closed subgroup. All Borel subroups are conjugate and contain a maximal torus. For the rest of this section we will fix $T \subset B \subset G$ and we fix $dim\ T = n$, called the semisimple rank of G (or just rank). Let $X(t)$ denote the character group of T, that is the set of morphisms $T \to \mathbb{C}*$

The tangent space at the identity of an algebraic group has the structure of a Lie algebra and is denoted using the lowercase gothic character $\mathfrak{g}$. Similarly, we let $\mathfrak{t},\ \mathfrak{b}$ denote the Lie algebras of T, and B. There is a natural map $exp:\ \mathfrak{g} \to\ G$. It follows that any representation of G

$$\rho : G \to GL(V)$$

yields a Lie algebra representation by taking the differential

$$d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

If we restrict the representation to $\mathfrak{t}$ and note that all representations of $\mathfrak{t}$ are completely reducible, we can right down a $weight$ space decomposition for the representation $V = \bigoplus V_\lambda$. Here $\lambda \in \mathfrak{t}^* = Hom_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ and $V_\lambda = \{v \in V | \, X \cdot v = \lambda(X)v \, \forall \, X \in \mathfrak{t}\}$

In particular, G naturally acts of $\mathfrak{g}$ by the adjoint action Ad,

$$Ad(g) \cdot X = \frac{d}{dt} g \; exp(tX) \; g^{-1}|_{t=0}$$

Differentiating this action give the adjoint representation of $\mathfrak{g}$

$$ad : \mathfrak{g} \to \; \mathfrak{gl}(\mathfrak{g})$$

$$X \to [X, \cdot]$$

Then under the adjoint action of $\mathfrak{t}$ we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus \mathfrak{g}_\alpha$$

The nonzero weights $\alpha \in \mathfrak{t}^*$ of the adjoint representation are called the roots of $\mathfrak{g}$ (and G respectively). Denote the set of roots by R. R then forms a root system in $\mathfrak{t}^*$. Our choice of Borel B (hence $\mathfrak{b}$) determines a base $\Delta = \alpha_1, ..., \alpha_n$ of simple roots. Every root in R can be written as a linear combination of simple roots with either all non-negative or all non-positive coefficients. Then let R^+ be the set of positive roots and let $R^- = -R^+$ be the set of negative roots. The action of W on T and the adjoint action of G on $\mathfrak{g}$ induce an action of W on $\mathfrak{t}$. For any root α and any $\mu \in \mathfrak{t}^*$ we define the reflection in $\mathfrak{t}^*$

$$s_\alpha(\mu) = \mu - \mu(\check{\alpha})\alpha = \mu - \frac{2\langle \alpha, \mu \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

Then the simple reflections s_i, where $s_i = s_{\alpha_i}$ for any simple root, generate W when W is identified with its action on $\mathfrak{t}$

The Killing form on $\mathfrak{g}$ is a symmetric, adjoint-invariant, W-invariant bilinear form given by $\langle X, Y \rangle = Tr(ad(X)ad(Y))$. This induces an identification of $\mathfrak{t}$ and $\mathfrak{t}^*$. For any root $\alpha \in R$, let $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ be the corresponfing coroot and let $\check{R}$ denote the set of coroots. The simple coroots $\check{\Delta}$ form a basis for $\mathfrak{t}$. Define the *fundameltal weights* ω_i by $\langle \omega_i, \check{\alpha_j} \rangle = \omega_i(\check{\alpha_j}) = \delta_{ij}$. Let $\Lambda = \bigoplus_{i=1}^{n} \omega_i$ be the weight lattice in $\mathfrak{t}^*$, and let $\Gamma = \bigoplus_{\alpha \in \Delta} \alpha$ be the root lattice. Identify $X(T)$ with a lattice in $\mathfrak{t}^*$ by differentiating. Then in general we have

$$\Gamma \subset X(T) \subset \Lambda$$

If G is simply connected we have $X(T) = \Lambda$.

A weight λ is called dominant if $< \lambda, \alpha > \geq 0 \; \forall \; \alpha \in \Delta$. Any dominant weight can be written as a non-negative linear combination of fundamental weights. Denote the set of dominant weights by Λ^+. Given a representation V of G, a vector $v \in V$ is called highest weight if (under the induced representation on $\mathfrak{g}$) v is an eigenvector of the action of $\mathfrak{t}$ and is in the kernel of the action of $\mathfrak{g}_\alpha$ for all roots α. If the highest weight vector v is in the weight space V_λ, we say that λ is a highest weight for the representation V. It is highest in the sense that it will be the highest weight given by the followin partial order on weights. We say that $\lambda > \mu$ if $\mu = \lambda - \sum_{\alpha \in \Delta} k_\alpha \alpha$ with $k_\alpha \geq 0$ and integral. If V is irreducible then there is a unique highest weight. The following classification is a fundamental theorem in Lie theory,

Theorem 2.1. *For any dominant weight $\lambda \in X(T) \cap \Lambda^+$ there is a unique irreducible finite-dimensional representation V_λ of G with highest weight λ.*

This will allow us to concretely describe the representation ring of a complex reductive group (See [FH, §23.2] and [BD, §II.7]). We form the representation ring $Rep(G)$ by taking a free abelian group on the isomorphism classes $[V]$ of finite dimensional representations V, modulo the relations $[V] = [V'] + [V'']$ whenever $V \simeq V' \oplus V''$. Since G is reductive this is indeed a free abelian group on the classes of irreducible representations. The tensor product of representations turns this into a ring $[V] \cdot [W] = [V \otimes W]$. Elements such as $[V] - [W]$ are called virtual representations, or virtual characters if we identify a representation with its character. We note that $Rep(G \times H) \simeq Rep(G) \otimes Rep(H)$. Recall that the k^{th} exterior power $\bigwedge^k V$ of a representation has the following property

$$\bigwedge^k (V \oplus W) = \sum_{i+k=k} \bigwedge^i (V) \otimes \bigwedge^j (W)$$

Then the operators

$$\lambda^i : Rep(G) \to Rep(G)$$

$$[V] \to [\bigwedge^i V]$$

make $Rep(G)$ into a special λ-ring. As we will see this structure will also hold for the subrings of polynomials representations for the classical groups.

Note that for a maximal torus $T \subset G$, the Weyl group W acts on T and thus on $Rep(T)$. The inclusion of the torus $T \hookrightarrow G$ induces in isomorphism $Rep(G) \simeq Rep(T)^W$. If we consider the complexified representation ring $Rep^{\mathbb{C}}(G)$ we have that

$$Rep^{\mathbb{C}}(G) \simeq \mathbb{C}[T]^W$$

. Given the representations V_{ω_i} of highest weight ω_i for the fundamental weights, we can also describe the representation ring and complex representation ring by $Rep(G) = \bigoplus \mathbb{Z}V_{\omega_i}$ and $Rep^{\mathbb{C}}(G) = \bigoplus \mathbb{C}V_{\omega_i}$ respectively.

2.1.1 Weyl Groups

Finally, we will collect some facts about the Weyl group which follow from the fact that is is also a Coxeter group (i.e. (W, S) is a Coxeter system. Again we note that W is generated by the simple reflections $S = s_i$ associated to each simple root. These simple reflections obey several relations dependant on the root system. In particular they all obey

$$s_i^2 = 1$$

$$s_i s_j = s_j s_i \;\; if \;\; |i - j| \geq 2$$

And for say $G = SL(n, \mathbb{C})$, where $W = S_n$ the symmetric group we have the relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

. This allows us to define a length function $l : W \to \mathbb{Z}_{\geq 0}$, where $l(w)$ is the smallest integer n such that w can be written as a product of n elements from S. Geometrically, the length of w is the cardinality of the set $wR^+ \cap R^-$ (note the set of roots is invariant under the action of the Weyl group). A word, or decomposition, $w = s_{i_1}s_{i_2}...s_{i_k}$ is said to be reduced if $l(w) = k$. A Weyl group has a unique longest element w_0 where $w_0R^+ = R^-$. Then $l(ww_0) = l(w_0w) = l(w_0) - l(w)$. Note also that $l(w^{-1}) = l(w)$.

Any Coxeter system (W, S) admits a partial ordering $\geq$ on W called the Bruhat order.

Definition 2.1. (Bruhat Order)

If $w, w' \in W$ and there is a conjugate t of some $s \in S$ such that $w' = tw$ and $l(w') = l(w) + 1$ then we say w' covers w (denoted $w \to w'$). The Bruhat order is the transitive closure of $\to$ (i.e. $u < v$ with $l(v) = l(u) + k$ then there is a sequence

$$u \to u_1 \to ... \to u_{k-1} \to v$$

Note that if the Coxeter system (W, S) comes from a the Weyl group of some root system the the conjugate t of s corresponds to the reflection s_β for some positive root $\beta \in R^+$.

Consider a proper subset $\theta \in S$. The subgroup of W generated by θ is called a parabolic subgroup, which we will denote W_θ. (W_θ, θ) is itself a coxeter system. There is a natural, distinguished set of left coset representatives in W/W_θ given by

$$W^\theta = \{w \in W : \ l(ws) = l(w) + 1, \ for\ all\ s \in S\}$$

This gives the following decomposition [Hi, Chapter I, Section 5]

Theorem 2.2. *If $w \in W$, $\theta \in S$, then there is a unique expression $w = w^\theta w_\theta$ with $w^\theta \in W^\theta$ and $w_\theta \in W_\theta$ with $l(w) = l(w^\theta) + l(w_\theta)$.*

As a corollary we see that the set W^θ is precisely the set of minimal-length representatives in each coset wW_θ.

2.1.2 Dynkin Index

We believe that for the λ-polynomials ring $Rep_{\lambda}^{\mathbb{C}}(G)$ (to be defined in Chapter 4), the most appropriate weight to consider is the fundamental weight of minimum Dynkin index. For the classical groups of types A, B, C, D (that is $GL_n, SL_n, SO_{2n+1}, Sp_{2n}, SO_{2n}$) this is just the defining representation. We define the Dynkin index and describe some of its properties here. This subsection follows [Ku3, §Appendix A]

Definition 2.2. Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism between finite-dimiensional simple Lie algebras over $\mathbb{C}$. Then there exists a unique number $d_f \in \mathbb{C}$ called the Dynkin index of f, satisfying

$$\langle f(x), f(y)\rangle = d_f\langle x, y\rangle,\ for\ all\ x, y \in \mathfrak{g}_1$$

where $\langle\cdot,\cdot\rangle$ is the nondegenerate, invariant, symmetric, bilinear form on $\mathfrak{g}_i$ normalized so that $\langle\theta_i, \theta_i\rangle = 2$ for the highest root θ_i of $\mathfrak{g}_i$.

Note that if $h : \mathfrak{g}_2 \to \mathfrak{g}_3$, for $\mathfrak{g}_3$ simple, then $d_{f\circ g} = d_f \cdot d_g$. Given a finite dimensional representation V of a simple Lie algebra $\mathfrak{g}$, $f_V : \mathfrak{g} \to \mathfrak{gl}(V)$, we set

$$d_V = d_{f_V}$$

. Then for two representations V_1 and V_2, taking their direct sum $V_1 \oplus V_2$ we have

$$d_{V_1\oplus V_2} = d_{V_1} + d_{V_2}$$

For their tensor product $V_1 \otimes V_2$, we have that

$$d_{V_1\oplus V_2} = d_{V_1}dim(V_2) + d_{V_2}dim(V_1)$$

We have the following formula for the Dynkin index of an irreducible representation of highest weigh λ of a simple Lie algebra [Ku3, Lemma A.2, A.3]

Lemma 2.1. *Let $\mathfrak{g}$ be a simple Lie algebra and let $V(\lambda)$ be an irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight λ. Then*

$$d_\lambda = d_{V(\lambda)} = \frac{dim_{\mathbb{C}} V(\lambda)}{dim_{\mathbb{C}} \mathfrak{g}} (\|\lambda + \rho\|^2 - \|\rho\|^2)$$

where $|\mu\|^2$ denotes $\mu,\rangle$ and 2ρ represents the sum of all positive roots. Thus, d is a stictly positive real number for any $\lambda \neq 0$. It is in fact true that d_λ is a positive integer.

Lemma 2.2. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie Algebra as before. Let V be a finite dimensional representation of $\mathfrak{g}$ with its formal character given by*

$$ch\, V = \sum_{\lambda \in \mathfrak{t}^*} n_\lambda e^\lambda, \;\; n_\lambda \in \mathbb{Z}$$

with $\mathfrak{t} \subset \mathfrak{g}$ the Cartan subalgebra. Then,

$$d_V = \frac{1}{2} \sum_\lambda n_\lambda (\lambda(\check{\theta}^2))$$

where $\check{\theta} \in \mathfrak{t}$ is the coroot associated to the highest root θ of $\mathfrak{g}$.

Using Lemma 2.2 and the root data [**?**] one finds that the fundamental representations of minimal Dynkin index are the representations ω_1 of index 1 for the classical groups of types A,B,C, and D. For the exceptional groups we find that for $G2$ it is ω_1 (index 2), for $F4$ it is ω_4 (index 6), for $E6$ it is ω_1 or ω_6 (index 6), for $E7$ it is ω_7 (index 12), and for $E8$ it is ω_8 (index 60).

2.2 Bruhat Decomposition, Schubert varieties, and Parabolic Subgroups

Fix $T \subset B \subset G$. An arbitrary Borel group can be decomposed as $B = T \rtimes U$, where U is the set of unipotent elements contained in B. A choice of Borel B is equivalent to choice of positive roots R^+. The map exp restricted to $\mathfrak{g}_\alpha)$ is an isomorphism of varieties, and hence we write $U_\alpha = exp(\mathfrak{g}_\alpha)$. This is a closed, one-dimensional subroup of G isomorphic to $\mathbb{C}$. In particular, $U \simeq \prod_{\alpha \in R^+} U_\alpha$, with correspnding Lie algebra $\mathfrak{u} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$

For any Borel B there is always an opposite Borel B^- such that $B \cap B^- = T$. Then we can write $B^- = T \rtimes U^-$, where $U^- \simeq \prod_{\alpha \in R^-} U_\alpha$. U^- has Lie algebra $\mathfrak{u}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$. Define $U_w^- = U \cap wU^-w^{-1}$ and its correspoding Lie algrbra $\mathfrak{u}_w^- = (Adw)\mathfrak{u}^- \cap \mathfrak{u} = \bigoplus_{\alpha \in wR^- \cap R^+} g_\alpha$. Note that $\mathfrak{u}_{\mathfrak{w}}^-$ is isomorphic as a variety to $\mathbb{C}^{l(w)}$. The following theorem [Borel, page 147] can be seen as a formal consequence of the fact that the data $G, B, N(T), S$ form what its known as a BN-pair [Tits].

Theorem 2.3. *(Bruhat Decomposition) If G is a complex reductive group and $T \subset B$, then G is the disjoint union of double cosets BwB, i.e.*

$$G = \bigsqcup_{w \in W} BwB$$

. Further, there is an isomorphism of varietes $U_w^- \times B \simeq BwB$.

The homogenous space G/B is a projective variety. We have the following corollary of the Bruhat decomposition. The homogenous space G/B is a disjoint union of double cosets

$$G/B = \bigsqcup_{w \in W} BwB/B$$

Further, we have that BwB/B is a cell of complex dimension $l(w)$ via the sequence of algebraic isomorphisms

$$\mathfrak{u}_{\mathfrak{w}}^- \xrightarrow{exp} U_w^- \to U_w^- wB/B = UwB/B = BwB/B$$

We introduce the notation

$$C_w = BwB/B$$

This open affine variety, known as a Schubert Cell, is an orbit under the left translation action of B on G. We will denote the closure of C_w by X_w. This is know as a Schubert Variety. Note that X_w is also $B - stable$

and can thus be written as the disjoint union

$$X_w = \bigsqcup_{v \le w} C_w$$

, where $v, w \in W$ and $v \le w$ in the Bruhat order from the previous section. Use $[X_w]$ to denote the the image of the fundamental class of X_w in the singular cohomology $H_*(G/B)$, where $[X_w] \in H_{2l(w)}(G/B)$. We will use $PD(X_w)$ to denote the cohomology class of complementary dimension associated to X_w by Poincar$\acute{e}$ duality. The fact that G/B (also known as the flag variety) has a cellular decomposition with cells of only even real dimension has many consequences. In particular G/B is simply connected. Additionally, we have the following well know ([Reiner]):

Theorem 2.4. *The integral singular homology $H_*(G/B)$ and cohomology $H^*(G/P)$ are free $\mathbb{Z}$ modules. They form dual lattices under the Kronecker pairing, having $\mathbb{Z}$-dual basis given by the cellular homology classes $\{[X_w] : \ w \in W\}$ and their Kronecker dual cohomology classes denoted $\{\epsilon_w : \ w \in W\}$.*

Thus, for $v, w \in W$ we have that

$$\langle \epsilon_w, X_v \rangle = \delta_{v,w}$$

where $\langle \cdot, \cdot \rangle$ is the usual Kronecker pairing between homology and cohomology.

Note that $dim(G/B) = l(w_0)$. Thus we have that $\epsilon_w \in H^{2l(w)}(G/B)$ and $PD(X_w) \in H^{2l(w_0)-2l(w)}(G/B)$. The class $PD(X_w)$ can be expressed in terms of the $\{\epsilon_w : w \in W\}$ basis as follows [BGG]

Theorem 2.5. *For $w \in W$ we have,*

$$PD(X_w) = \epsilon_{w_0 w}$$

In other words, for $X_{w_0 w} = Bw_0 wB/B$ we have that $PD(X_{w_0 w}) = \epsilon_w$

Now that we have a preferred basis for the cohomology ring $H^*(G/B)$ we would like to describe the multiplication (i.e. cup product) with respect to this basis. That is, given $v, w \in W$ we want a closed formula for the constants c^u_{vw} appearing in the decomposition of the product

$$\epsilon_v \cdot \epsilon_w = \sum_{w \in W} c^u_{vw} \epsilon_u$$

where $\cdot$ is really the cup product. To this end we have the following Pieri-like formula due to Chevalley [BGG]

Theorem 2.6. *(Chevalley Formula) For any $w \in W$ and any simple root α, with corresponding simple reflection s_α, we have that*

$$\epsilon_w \cdot \epsilon_{s_\alpha} = \sum (\omega_\alpha, \check{\beta}) \epsilon_{ws_\beta}$$

where the sum runs over the positive roots such that $l(ws_\beta) = l(w) + 1$. s_β is the reflection associated to a root given by $s_\beta(\xi) = \xi - (\xi, \check{\beta})\beta$ for $\xi \in \mathfrak{t}^*$.

In the next section we will discuss a Giambelli-like formula due to [BGG]. Now we will discuss parabolic subgroups $B \subset P$ and the generalized partial flag varieties G/P. Aparabolic subgroup is any subgroup such that the quorient G/P can be realized as the orbit of the action of G on $\mathbb{P}(V)$ for some representation V of P. In particular, G/P is a projective variety. Equivalently, parabolic subgroups are those subgroups that contain a conjugate of a Borel subgroup. So Borel subgroups are the minimal parabolic subgroups. Generally, we will fix a Borel and consider parabolics $B \subset P$. Parabolic subgroups are completely characterized by subsets of the simple roots Δ up to conjugacy, and since the simple roots are in one-to-one correspondence to the vertices of the Dynkin diagram for G we have that parabolics are in one to one correspondence with subsets of vertices of the Dynkin diagram.

Consider a subset $\theta \in S = \{\alpha_1, ..., \alpha_n\}$. The we define the parabolic Lie algebra

$$\mathfrak{p}_\theta = \mathfrak{t} \oplus \bigoplus_{\beta \in T(\theta)} \mathfrak{g}_\alpha$$

where $T(\theta)$ is equal to the set of positive roots R^- and all roots generated by the negatives of θ. Let P_θ be the corresponding parabolic group. Note that $P_\emptyset = B$ and $P_S = G$. Parabolic subgroups have a Levi decomposition $P_\theta = L_\theta U_\theta$, where $U_\theta = \prod_{\alpha \in R^+ \setminus T(\theta)}$ is the unipotent radical of P_θ, and $L_\theta = \langle T, U_\alpha | \, \alpha \in R_\theta \rangle$ is a reductive group called the Levi subgroup and R_θ is the root system generated by θ. Let W_θ be the Weyl group of L_θ. If the parabolic P is understood this will sometimes be denoted W_P. Note that W_θ is exactly the parabolic subgroup mentioned at the end of the previous section. We now have the following

Bruhat decompositions for G and G/P

$$G = \bigsqcup_{w \in W^\theta} BwP_\theta$$

$$G/P = \bigsqcup_{w \in W^\theta} BwP_\theta/P_\theta$$

W^θ is the set of minimal length coset representatives in W/W_θ. Geometrically, B-orbits in G/P are obtained by collapsing orbits in G/B if their $w's$ lie in the same W_θ coset. Fix $P_\theta = P$. Again we have schubert cells $C_w^P = BwP/P$ isomorphic to the affine space $\mathbb{C}^{l(w)}$ for $w \in W^P = W^\theta$. Then the schubert variety X_w^P, fundamental class$[X_w^P]$, Poincare and Kronecker duals $PD(X_w^P)$, ϵ_w^P and all defined analogously as for G/B. These cohomology classes are related in the next theorem [Ku1, chapter 11].

Theorem 2.7. *Let $\pi_P : G/B \to G/P$ be the natural projection. Then the induced map $\pi_P^* : H^*(G/P) \to H^*(G/B)$ is injective with image equal to $H^*(G/B)^{W_P}$, the W_P invariants. In particular, for $w \in W^P$, we have*

$$\pi^*(\epsilon_w^P) = \epsilon_w$$

.

Let $K \subset G$ be a maximal compact subgroups of G, with maximal compact torus $T = K \cap H$. Then there is a homoemorphism $K/T \simeq G/B$, and further we can identify $W = N(H)/H = N(T)/T$. W acts on K by conjugation and this action preserves T so there is an action of W on K/T. This induces an action of W on the homology and cohomology of K/T and hence on G/B.

So we can identify $H^*(G/P)$ with a subring of $H^*(G/B)$ and drop the P superscript from ϵ_w^P when it is understood. Note that the action of W on $H^*(G/B)$ is induced from the action of W on $K/T = G/B$. Recall that every element $w \in W$ can be decomposed as $w = w^\theta w_\theta$ where $w_\theta \in W_\theta$, $w^\theta \in W^\theta$, and $l(w) = l(w^\theta) + l(w_\theta)$. Applying this to w_0 we write $w_0 = w^{0,P} w_{0,P}$ for $P_\theta = P$. Then $w_{0,P}$ is the longest element of W_P and $w^{0,P}$ is the longest minimal-length representative in W^P. Then Poincare duality between $H_*(G/P)$ and $H^*(G/P)$ is given by the following theorem (see [KLM, 2.1])

Theorem 2.8. *Let $w \in W^P$. Define the involutive map $\theta^P : W \to W$ by $\theta^P(w) = w_0 w w_{0,p}$. The θ^P carries W^P into itself and we have*

$$PD(X_w^P) = \epsilon_{w_0 w w_{0,P}}$$

The proof relies on the following key lemma. Let $a \cdot b$ denote the intersection pairing on $H_*(G/B)$ with $a \in H_k(G/B)$ and $b \in H_{l(w_{0,P})-k(G/P)}$. Then we have that

$$\langle PD(a), b \rangle = a \cdot v$$

Lemma 2.3. *For $v, w \in W^P$ with $l(w) = l(v)$,*

$$X_w^P \cdot X_{\theta^P(v)}^P = \delta_{v,w}$$

2.3 Borel Characteristic map and BGG-operators

The discussion above may be referred to as the Schubert picture of cohomology [Hi]. In this section we will discuss another point of view called the Borel picture of cohomology and we will discuss the results and fomalism of Bernstein, Gelfand, Gelfand [BGG] and Demazure [D2] to connect the two pictures. Good resources for this material are the origonal papers [BGG], [D2], [Hi, chapter IV], [KLM, sections 2,3], [FP, appendix E], and [P4].

Let X be a variety that B acts freely on from the right such that the qoutient X/B exists and the projection $p : X \to X/B$ is a principal B-bundle. Let $\rho : B \to GL(V)$ be a representation of B. Then consider the complex vector bundle $\mathcal{L}_\rho$=$X \ \times^B \ V$ given by taking the quotient of $X \times \ V$ by the relation

$$(x, v) \sim (xb, \rho(b)^{-1}v)$$

for $x \in X, \ b \in B, \ v \in V$. In particular let $\lambda : B \to \mathbb{C}^*$ be a character of B, and let $\mathcal{L}_\lambda$ be the complex line bundle described above. First define $\beta : X(B) \to H^2(X/B)$ by taking a character of B to the first chern class of the associated line bundle, i.e. $\beta(\lambda) = c_1(\mathcal{L}_\lambda)$. This can be extended symmetrically to a homomorphism of graded rings (doubling degrees)

$$\beta : S(X(B)) \to H^*(X/B, \mathbb{Z})$$

where $S^*(X(B))$ is the symmetric algebra of $X(B)$. This map is called the characteristic map of the fiber bundle $p : X/B$.

For our purposes, we let B act on G on the right and the fiber bundle under consideration is $p : G \to G/B$. Now, consider $\mathfrak{t}^* = Hom_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$. So $\mathfrak{t}^*$ is just the characters of the Cartan subalgebra. Let us assume that G is simply connected. Then any $\chi \in \mathfrak{t}^*$ lifts to a character $\chi : T \to \mathbb{C}^*$ by$(exp(t)) = exp(\chi(t))$ for $t \in \mathfrak{t}$. This character can be further extended to $B = TU$ by setting $\chi|_U = 1$ (and indeed the character group of B and T are euivalent under this identification). Then as above we can define a map $\beta : \mathfrak{t}^* \to H^2(G/B)$ by lifting a character χ to B and taking the first chern class of the associated complex line bundle $\mathcal{L}_\chi$ over G/B. Again, we extend this symmetrically to obtain

$$\beta : S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{Z})$$

This is known as the Borel characteristic map [Bo]. If we consider $\beta \otimes \mathbb{C}$ then the map is surjective and has kernel J generated by the W-invariants with no constant term. So, letting $S = S(\mathfrak{t}^*)$ and taking complex coefficients, we have an isomorphism $\beta : S/J \to H^*(G/B, \mathbb{C})$. Note that β commutes with the action of W on S and on $H^*(G/B)$ [BGG, proposition 1.3(i)]. Also, since W acts as a finite complex reflection group on $\mathfrak{t}^*$, then by Chevalley's theorem the W-invariants $S(\mathfrak{t}^*)^W$ are a polynomial subalgebra $\mathbb{C}[f_1, ...f_n]$ where n is the rank of G [Hi, chapter II, section 3]. Thus, under Borel's presentation we see that $H^*(G/B)$ is a complete intersection ring with n generators and as many relations.

Let $P \subset G$ be a parabolic. Then we also have an isomorphism of graded rings as follows. Let S^{W_P} be the set of W_P invariants under the action of W_P on $\mathfrak{t}^*$). Then if we restrict we have

$$\beta : S^{W_P} \to H^*(G/P) \simeq H^*(G/B)^{W_P}$$

again with $(S/J)^{W_P} \simeq H^*(G/P)$.

In their seminal paper [BGG] Berstein, Gelfand, Gelfand developed a connection between the Schubert and Borel pictures of the cohomology of $H^*(G/B)$. In particular the give polynomials $p_w \in S^{l(w)}(\mathfrak{t}^*)\ mod\ J$ such that

$$\beta(p_w) = \epsilon_w \in H^{2l(w)}(G/B)$$

The key algebraic operator used in this work is the following

Definition 2.3. For each root $\alpha \in R$ define a divided difference operator $A_\alpha : S^k(\mathfrak{t}^*) \to S^{k-1}(\mathfrak{t}^*)$ by

$$A_\alpha(f) = \frac{f - s_i f}{\alpha_i}$$

These operators are also known as BGG or Demazure operators in the literature. We collect some properties of A_{s_i} in the following omnibus lemma [BGG, Lemma 3.3].

Lemma 2.4. *Let $\alpha \in S$ and $w \in W$. Let $f, g \in S(\mathfrak{t}^*)$*

(i) $A_\alpha^2 = 0$

(ii) $A_{-\alpha} = -A_\alpha$

(iii) $wA_\alpha w^{-1} = A_{w\alpha}$

(iv) $s_\alpha A_\alpha = A_\alpha$

(v) $s_\alpha = 1 - \alpha A_\alpha$

(vi) $A_\alpha(f) = 0 \leftrightarrow s_\alpha f = f$

(vii) $A_\alpha(fg) = A_\alpha(f)g + (s_\alpha f)A_\alpha(g)$

(viii) $A_\alpha J \subset J$

By (viii) above we see that A_α induces an operator on S/J. For any $w \in W$ we further define

$$A_w := A_{s_{\alpha_1}} \circ \cdots \circ A_{\alpha_{s_k}}$$

where $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced decomposition for w. The we have that [Hi, Chapter IV, Proposition 1.7]

Proposition 2.1. *The operators A_w are well defined, i.e. they do not depend on the choice of reduced decomposition for w. Further, we have that $A_w \circ A_v = A_{wv}$ if $l(wv) = l(w) + l(v)$ and $A_w \circ A_v = 0$ otherwise.*

The Borel characteristic map, the Schubert classes and the BGG operators are all related by the following equation [BGG, section 4].

Proposition 2.2. *Let $\beta : S(\mathfrak{t}^*) \to H^*(G/B)$. For $f \in S^k$*

$$\beta(f) = \sum_{l(w)=k} A_w(f)\epsilon_w$$

The above equation is valid for partial flag varieties as well if we restrict the summation to the set $\{w \in W^P : \ l(w) = k\}$.

There is an analogue of the BGG operator D_{s_i} on $H^*(G/B)$ which commutes with the Borel characteristic map, i.e. for $f \in S$ we have that $\beta(A_{s_i} f) = D_{s_i}\beta(f)$. Hence we will just use A_w to refer to the BGG operator on both S and $H^*(G/B)$. For an explicit description of the geometric operator D_{s_i} see [KLM, section 3.3]. Then we have the following description of the action of A_w on the Schubert classes ϵ_v [BGG, Theorem 3.14]

Theorem 2.9. *For $v, w \in W$ such that $l(vw^{-1}) = l(v) - l(w)$, we have that*

$$A_w\epsilon_v = \epsilon_{vw^{-1}}$$

and equals 0 otherwise.

We also give the following formula for the Weyl group action on a Schubert class. For a simple root α and $w \in W$

$$s_\alpha\epsilon_w = \epsilon_w \quad if\ l(ws_\alpha) = l(w) + 1$$

$$s_\alpha\epsilon_w = -\epsilon_w - \sum(\alpha, \check{\gamma})\epsilon_{ws_\alpha s_\gamma} \quad if\ l(ws_\alpha) = l(w) - 1$$

where the sum is over all positive roots $\gamma \neq \alpha$ such that $l(ws_\alpha s_\gamma) = l(w)$.

Now define elements $p_w \in S$ as follows. Starting with the longest element w_0 we let $p_{w_0} = \frac{1}{|W|}\prod_{\gamma\in R^+}\gamma$. Then for arbitrary $w \in W$ recursively define p_w by $p_w = A_{w^{-1}w_0}p_{w_0}$. Then the main result of [BGG] is

Theorem 2.10.

$$\beta(p_w) = \epsilon_w$$

where really we are taking $p_w \; mod \; J$ in S/J.

These are polynomial representatives in S of the Schubert classes. Lascoux and Schurtzenburger [**?**] introduced another set of polynomial representatives in type A, called Schubert polynomials, which enjoy many nice combinatorial properties. These are obtained by applying divided difference operators to the monomial $x_1^{n-1}x_2^{n-2}...x_{n-1}$ which represents the top class (here $x_1, ..., x_n$ are the coordinates of the standard representation of A_{n-1}. There are natural analogues for the other classical types such as the Schubert polynomials of Billey and Haiman [BH], the theta and eta polynomials of Buch, Kresch, and Tamvakis [BKT1, BKT2].

We also note that the the Chevalley formula (Theorem 2.6) is a partial solution to the general Littlewood-Richardson problem of describing the coefficients on the expansion

$$\epsilon_w\epsilon_v = \sum_{u\in W} c^u_{wv}\epsilon^u_{wv}$$

These represent geometric intersections of the varities $X_{w_0u},\ X_{w_0v}$ and X_u and thus must be positive. We now give a brief description of these coeffiecients due to Pragacz [P3, P4]. Then by combining the above expansion with Theorem 2.9 we see that

$$c^u_{wv} = A_u(\epsilon_w \cdot \epsilon_v)$$

Now suppose that $l(w) = k$ and that $l(v) = l$. Let $u = s_{\alpha_1}...s_{\alpha_{k+l}}$ be a reduced decomposition. Then by iterating 2.2 (vii) we have

$$c^u_{wv} = A_{\alpha_1} \circ ... \circ A_{\alpha_{k+l}}(\epsilon_w \cdot \epsilon_v) = \sum A_I(\epsilon_w) \cdot A^I_\alpha(\epsilon_v)$$

where the sum is over all subsequences $I = (i_1, ..., i_k) \subset [1, ...k + l]$ and $A_I = A_{\alpha_{i_1}} \circ ... \circ A_{\alpha_{i_k}}$ and A^I_α is obtained by taking $A_{\alpha_1} \circ ... \circ A_{\alpha_{k+l}}$ and replacing each A_{α_i} by s_{α_i} for all $i \in I$. Then by Theorem 2.9 we can deduce that

$$c^u_{wv} = \sum A^I_\alpha(\epsilon_v)$$

where the sum is over all subsequences I such that $s_{\alpha_{i_1}} ... s_{\alpha_{i_k}}$ is a reduced decomposition for w. The Chevalley formula can then be derived form this rule.

CHAPTER 3

The Springer Morphism

We will first briefly set the notation for this chapter, primarily in §3.2

Let G be a simply-connected semi-simple algebraic group over $\mathbb{C}$ (though the constructions of this §3.1 are valid in the more general case of a connected reductive complex group). Denote its Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$ of rank n, and fixed base of simple roots $\Delta = \{\alpha_j\}$. Take the set of simple co-roots $\check{\Delta} = \{\check{\alpha}_j\}$ as a basis for the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Then $\mathfrak{t}_{\mathbb{Z}} = \bigoplus_{j=1}^{n} \mathbb{Z}\check{\alpha}_j$ is the co-root lattice. Further, the weight lattice is $\mathfrak{t}^*_{\mathbb{Z}} = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i$, where $\omega_i \in \mathfrak{t}^*$ is the i^{th} fundamental weight of $\mathfrak{g}$ defined by $\omega_i(\check{\alpha}_j) = \delta_{ij}$. Then the maximal torus $T \subset G$ (with Lie algebra $\mathfrak{t}$) can be identified with $T = Hom_{\mathbb{Z}}(\mathfrak{t}^*_{\mathbb{Z}}, \mathbb{C}^*)$ as in [Sp]. Finally, let W be the Weyl group of G, generated by the simple reflections s_i. So for $\mu \in \mathfrak{t}^*$, $s_i(\mu) = \mu - \mu(\check{\alpha}_i)\alpha_i$.

Let V_λ be the irreducible representation of G with highest weight λ. Then V_λ has weight space decomposition

$$V_\lambda = \bigoplus V_\lambda^\mu$$

where $V_\lambda^\mu = \{v \in V_\lambda |\ t.v = ((\mu_1\omega_1 + ... + \mu_n\omega_n)(t))v\ \forall v \in V_\lambda\}$ is the weight space with weight $\mu = \mu_1\omega_1 + ... + \mu_n\omega_n$.

So for $t \in T$ and $v \in V_{\mu_1,\mu_2,...,\mu_n}$ we have that the action of t on v is given by

$$t.v = t(\mu_1, ..., \mu_n)v = e^\mu(t)v$$

where $(\mu_1, ...\mu_n) = \mu_1\omega_1 + ... + \mu_n\omega_n$. Additionally $\check{\alpha}_j \in \mathfrak{t}$ acts on v by

$$\check{\alpha}_j.v = (\mu_1\omega_1 + ... + \mu_n\omega_n)(\check{\alpha}_j)v = \mu_j v.$$

A representation $\rho : G \to GL(V)$ is called almost faithful if it has finite kernel, i.e. the induced representation $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is injective.

3.1 Definition and Properties

In the literature this construction is also known as the Generalized Cayley Transform. Some references for this material are [BR, KM]. The following construction is in fact a special case of what [LPR1, §10] call a generalized Cayley map which is any dominant algebraic morphism $G \to \mathfrak{g}$.

Given a connected reductive group G, its Lie algebra $\mathfrak{g}$, and an almost faithful representation V_λ, the Springer morphism is a map

$$\theta_\lambda : G \to \mathfrak{g}$$

given by

$$\begin{tikzcd} G \arrow[r] \arrow[drr, "\theta_\lambda"] & Aut(V(\lambda)) \subset & End(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \arrow[d, "\pi_\lambda"] \\ & & \mathfrak{g} \end{tikzcd}$$

where $\mathfrak{g}$ sits canonically inside $End(V_\lambda)$ via the derivative $d\rho_\lambda$, the orthogonal complement $\mathfrak{g}^\perp$ is taken via the adjoint invariant form $< A, B >= tr(AB)$ on $End(V_\lambda)$, and π is the projection onto the $\mathfrak{g}$ component. So $\theta_\lambda = \pi_\lambda \circ \rho_\lambda$. By considering a compact form $K \subset G$, it is easy to see that the restiction of trace form to $d\rho_\lambda(\mathfrak{g})$ is non degenerate and thus $\mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$. Note, that since $\pi \circ d\rho_\lambda$ is the the identity map, θ_λ is a local diffeomorphism at 1, and hence has Zariski dense image. By construction, θ_λ is an algebraic morphism.

Let $d\theta_\lambda = \pi_\lambda \circ T\theta_\lambda : TG \to T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denote the differential of θ_λ, so that $d\theta_\lambda(g) = \pi_\lambda \circ T_g G \to \mathfrak{g}$. We let $X_1, ..., X_n$ be a linear basis of $\mathfrak{g}$ and let $L_{X_1}, ..., L_{X_n}$ denote the corresponding left-invariant vector fields on G. Let

$$\Psi_\lambda(g) = det(d\theta_\lambda(g))$$

be the Springer determinant (or Cayley Determinant) for the representation V_λ. Note that Ψ_λ does not depend on choice of basis for $\mathfrak{g}$. We list the following basic properties of θ_λ and $\Psi_\lambda(g)$.

Theorem 3.1. *Let G be a connected, reductive, complex algebraic group and let θ_λ be the Springer morphism, where V_λ is an almost faithful representation. Let $T \subset G$ be a maximal torus and let B_λ be the restiction of the inner product $\langle A, B\rangle = tr(AB)$ on $d\theta_\lambda(\mathfrak{g}) \subset End(V)$ Then,*

1. $\theta_\lambda \circ Conj_b = Ad_b \circ \theta_\lambda$
2. $\theta_{\lambda|T} : T \to \mathfrak{t}$
3. Ψ_λ *is invariant under conjugation.*
4. $d\theta_\lambda(e) : \mathfrak{g} \to \mathfrak{g}$ *is the identity mapping. So* $d\theta_\lambda(g)$ *is invertible for* g *in the non-empty Zariski open dense subset* $\{h \in G : \ \theta_\lambda(h) \neq 0\}$ *and is not invertible on the hypersurface* $\{h \in G : \ \theta_\lambda(h) = 0\}$
5. *Let* χ_λ *be the character of* ρ_λ*, i.e.* $\chi_\lambda(g) = tr(\rho_\lambda(g))$. *Then* $d\chi_\lambda(g)(T_e(\mu_g)X) = tr(d\rho(\theta_\lambda(g))d\rho_\lambda(X)) = B_\lambda(\theta_\lambda(g), X)$
6. *The differential* $d\theta_\lambda(g).T_e(\mu_g).X \in \mathfrak{g}$ *is given by the implicit equation* $tr(d\rho_\lambda(d\theta_\lambda(g)T_e(\mu_g)X)d\rho_\lambda(Y)) = tr(\theta_\lambda(g)d\rho_\lambda(X)d\rho_\lambda(Y))$ *for* $Y \in \mathfrak{g}$
7. *If* $\theta_\lambda(e) = 0$ *and* $a \in G$ *is such that* $\rho_\lambda(a) \in d\rho_\lambda(\mathfrak{g})$ *then* $d\theta_\lambda(a^{-1})$ *is not invertible.*

Proof. We give a proof of (2) because of its importance to the rest of the paper. Let $t \in T$. We then write

$$\theta_\lambda(t) = h + \sum_{\alpha \in R} x_\alpha, \ \ for\ h \in \mathfrak{t},\ and\ x_\alpha \in \mathfrak{g}_\alpha$$

Then by conjugation invariance (see (1) which follows from the invaraiance of trace) we have

$$\theta_\lambda(t) = \theta_\lambda(sts^{-1}) = h + \sum_{\alpha \in R} Ad_s(x_\alpha)\ for\ any\ s \in T$$

Thus we see that $x_\alpha = 0$, i.e., $\theta_\lambda(t) \in \mathfrak{t}$ □

Example 3.1. The Springer morphism $\theta_\lambda : G \to \mathfrak{g}$, in general, indeed depends upon the choice of λ. For example, the Springer morphism $\theta_{\omega_1} : Sl_2 \to \mathfrak{sl}_2$ restricted to the diagonal torus can easily seen to be

$$\theta_{\omega_1}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{z-z^{-1}}{2} & 0 \\ 0 & -\frac{z-z^{-1}}{2} \end{pmatrix}.$$

On the other hand, the Springer morphism $\theta_{2\omega_1} : \mathbf{SL}_2 \to \mathfrak{sl}_2$ restricted to the diagonal torus is given by

$$\theta_{2\omega_1}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{z^2-z^{-2}}{4} & 0 \\ 0 & -\frac{z^2-z^{-2}}{4} \end{pmatrix}.$$

$\Diamond$

We also record the following theorems from Kostant and Michor [KM, 2.7,2.8]

Theorem 3.2. *Let G be semisimple and let $\rho : G \to GL(V)$ be an almost faithful representation. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}$ into simple ideals $\mathfrak{g}_i$. Let $G_i, ..., G_k$ be the corresponding connected subgroups of G. Then we have that*

$$\theta_\rho|_{G_i} = \theta_{\rho|_{G_i}},\ for\ i = 1, ...k$$

Theorem 3.3. *Let G be a simple algebraic group and let $\rho_i : G \to GL(V_i)$ be non-trivial representations for $i = 1, 2$. The inner product B_{ρ_i} on $\mathfrak{g}$ is a multiple of the Cartan Killing form B on $\mathfrak{g}$, so we write $B_{\rho_i} = j_{\rho_i}B$. Then we have*

1. *For the direct sum representation $\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$ we have*

$$\theta_{\rho_1\oplus\rho_2}(g) = \frac{j_{\rho_1}}{j_{\rho_1\oplus\rho_2}}\theta_{\rho_1}(g) + \frac{j_{\rho_2}}{j_{\rho_1\oplus\rho_2}}\theta_{\rho_2}(g) \in \mathfrak{g}$$

2. *For the tensor representation* $\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ *we have*

$$\theta_{\rho_1\otimes\rho_2}(g) = \frac{j_{\rho_1}\chi_{\rho_2}}{j_{\rho_1\otimes\rho_2}}\theta_{\rho_1}(g) + \frac{j_{\rho_2}\chi_{\rho_1}}{j_{\rho_1\otimes\rho_2}}\theta_{\rho_2}(g) \in \mathfrak{g}$$

3. *For the n-fold tensor product representation* $\otimes^n\rho : G \to GL(\otimes^n V)$ *we have*

$$\theta_{\otimes^n\rho}(g) = (\frac{\chi_\rho(g)}{dim(V)})^{n-1}\theta_\rho(g)$$

4. *For the contragradient representation* $\rho^T : G \to GL(V^*)$ *given by* $\rho^T(g) = \rho(g^{-1})^T$ *we have*

$$\theta_{\rho^T}(g) = -\theta_\rho(g^{-1})$$

Where, for a complex simple algebraic group, j_{ρ_i} *is amultiple of the Dynkin Index of the representation* ρ_i*. It is non-negative and satisfies*

$$j_{\rho_1\oplus\rho_2} = j_{\rho_1} + j_{\rho_2},$$

$$j_{\rho_1\otimes\rho_2} = dim(V_2)j_{\rho_1} + dim(V_1)j_{\rho_2},$$

$$j_{\rho_\lambda} = \frac{dim(V_\lambda)}{dim(\mathfrak{g})}B(\lambda, \lambda+\rho)$$

where in the last line ρ *is the half sum of all positive roots.*

One motivation for studying such maps comes from a result of Springer which states that the Unipotent varity $U \subset G$ of unipotent elements is isomorphic as an algebraic variety to the nilcone $N \subset \mathfrak{g}$ of nilpotent elements in the lie algebra. Bardsley and Richardson [BR] used Springer morphisms, even in finite characteristic for good primes, to give examples of such isomorphisms. Kostant and Michor [KM, 4.5] then consider the complex case and generalize this to reductive algebraic groups to show

Theorem 3.4. *Let* $a \in G$ *be regular and assume that* $d\theta_\lambda(s)$ *is invertible. Then* θ_lambda *restricts to an isomorphism of affine varieties*

$$\theta_\lambda : \overline{Conj_G(a)} \to \overline{Ad_G(\theta_\lambda(a)}$$

Additionally, the Springer morphisms preserve the Jordan decompostion. Recall that any element $a \in G$ has a multiplicitave Jordan decomposition $a = a_s a_u$, where a_s and a_u are semisimple and unipotent elements. Similiarly, for any $X \in fg$ we have that $X = X_s + X_n$, where X_s and X_n are semisimple and nilpotent elements respectively. Then we have [KM, 4.11] that $\theta_\lambda(a_s) = \theta_\lambda(a)_s$ and $\theta_\lambda(a_u) = \theta_\lambda(a)_u$.

Finally we also want to consider the degree of the map θ_λ. To that end we have the following theorems from [KM, 2.9,3.3] and [LPR2, Corrolary 2]

Theorem 3.5. *For the Springer morphism θ_λ the induced mapping $\theta_\lambda^* : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[G]$ between the algebra of regular functions is injective, equivatiant, and maps the subalgebras of invariant regular functions to each other, $\theta_\lambda^* : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[G]^G$. Thus, $\theta_\lambda : G \to \mathfrak{g}$ is a dominant algebraic morphism. By the algebraic Peter-Weyl theorem we have that $\mathbb{C}[G] = \oplus_{\mu \in D} \mathbb{C}[G]_\mu$ where D is the set of dominant integral weights, and where*

$$\mathbb{C}[G]_\mu = \{f \in \mathbb{C}[G] : f(g) = tr(\rho_\mu(g)B) \; for \; some \; B \in End(V_\mu)\}$$

For an irreducible representation ρ_λ we thus have $\theta_\lambda^*(\mathfrak{g}^*) \in \mathbb{C}[G]_\lambda$.

Finally, we have the following result about the degrees of Springer morphisms.

Theorem 3.6. *For a Springer morphism θ_λ of a connected reductive group G. Then,*

$$deg\ \theta_\lambda = [Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G] = [Q(T)^W : Q(\mathfrak{t})^W]$$

We hope then that the results of the next section could help determine the degree of a Springer map for any semi-simple group.

3.2 An Explicit Determination of the Springer Morphism

Let V_λ be a d dimensional almost faithful irreducible representation of G of highest weight λ. Let $\Lambda_\lambda = \{(\mu_1^i, ..., \mu_n^i)\}_{i=1}^d$ be an enumeration of the set of weights considered with their multiplicity that appear in the weight space decomposition of V_λ (so μ_j^i is the coordinate of the j^{th} fundamental weight for the i^{th} weight in the decomposition) Then we can take a basis of weight vectors $\{v_{\mu_1^i,...,\mu_n^i}\}_{i=1}^d$ on which the torus T and hence each simple co-root acts diagonally. Thus,

$$\rho_\lambda(t) = diag\{e^{\mu^1}(t), ..., e^{\mu^d}(t)\} \in Aut(V_\lambda)$$

and for a simple co-root $\check{\alpha}_j$ we have that

$$d\rho_\lambda(\check{\alpha}_j) = diag\{\mu_j^1, ..., \mu_j^d\} \in End(V_\lambda).$$

In order to compute the projection to $\mathfrak{g} \in End(V_\lambda)) \simeq \mathfrak{g} \oplus \mathfrak{g}^\perp$ we calculate $d\rho_\lambda(\mathfrak{g})^\perp \in End(V_\lambda)$ with respect to the symmetric bilinear form $tr(AB)$. Recall that $d\rho_\lambda$ is faithful so we identify $\mathfrak{g}$ with its image under $d\rho_\lambda$. Let $X = (x_{ij}) \in End(V_\lambda)$. Then for X to be contained in $d\rho_\lambda(\mathfrak{g})^\perp$ it follows that

$$tr(d\rho_\lambda(\check{\alpha}_j) \cdot X) = 0 \implies \sum_{i=1}^{d} \mu_j^i x_{ii} = 0$$

for all co-roots, $\check{\alpha}_j \in \mathfrak{t}$.

So $\sum_{\mu \in \Lambda_\lambda} \mu_1^i x_{ii} = \sum_{\mu \in \Lambda_\lambda} \mu_2^i x_{ii} = ... = \sum_{\mu \in \Lambda_\lambda} \mu_n^i x_{ii} = 0$. Now to project $\rho_\lambda(t)$ onto $d\rho_\lambda(\mathfrak{t})$ we write ρ_λ as a sum

$$\rho_\lambda(t) = \sum_{j=1}^{n} c_j(t) d\rho_\lambda(\check{\alpha}_j) + X(t).$$

where $c_j : T \mapsto \mathbb{C}$ is a function that depends on λ, and $X(t) \in d\rho_\lambda(\mathfrak{g})^\perp$. It follows then that

$$\theta_\lambda(t) = \sum c_j(t)\check{\alpha}_j$$

We aim to solve for the coefficients $c_j(t)$. Note that for the root space $\mathfrak{g}_\alpha$, we have that $\mathfrak{g}_\alpha.V_\mu \subset V_{\mu+\alpha}$. Thus, $d\rho_\lambda(e_\alpha)$ for $e_\alpha \in \mathfrak{g}_\alpha$ will only have off diagonal entries, and as such the condition $tr(d\rho_\lambda(e_\alpha) \cdot X) = 0$ will only add constraints to the off diagonal entries of $X \in d\rho_\lambda(\mathfrak{g})^\perp$. As the action of t and $\check{\alpha}_j$ are both diagonal, by comparing coordinates we have the following set of d equations

$$e^{\mu^1}(t) = c_1(t)\mu_1^1 + ... + c_n(t)\mu_n^1 + x_{11}(t)$$

$$e^{\mu^2}(t) = c_1(t)\mu_1^2 + ... + c_n(t)\mu_n^2 + x_{22}(t)$$

$$\vdots$$

$$e^{\mu^d}(t) = c_1(t)\mu_1^d + ... + c_n(t)\mu_n^d + x_{dd}(t).$$

This can be reduced to n equations by utilizing the fact that $\sum\limits_{i=1}^{d} \mu_j^i x_{ii} = 0$, as follows. Multiply each equation above by μ_1^i and sum (then repeat with $\mu_2^i, ..., \mu_n^i$)

$$\sum_{i=1}^{d} \mu_1^i e^{(\mu_1^i,...,\mu_n^i)}(t) = \sum_{i=1}^{d} (\mu_1^i)^2 c_1(t) + \sum_{i=1}^{d} \mu_1^i \mu_2^i c_2(t) + ... + \sum_{i=1}^{d} \mu_1^i \mu_n^i c_n(t)$$

$$\vdots$$

$$\sum_{i=1}^{d} \mu_n^i e^{(\mu_1^i,...,\mu_n^i)} = \sum_{i=1}^{d} \mu_1^i \mu_n^i c_1(t) + \sum_{i=1}^{d} \mu_2^i \mu_n^i c_2(t) + ... + \sum_{i=1}^{d} (\mu_n^i)^2 c_n(t)$$

More concisely this can be written as

$$\begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^{\mu}(t) \end{pmatrix} = S(G,\lambda) \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

where

$$S(G,\lambda) := \begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1^2 & \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_2 & ... & \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_n \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_2 & \sum\limits_{\mu\in\Lambda_\lambda} \mu_2^2 & ... & \sum\limits_{\mu\in\Lambda_\lambda} \mu_2\mu_n \\ \vdots & \ddots & & \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_n & ... & \sum\limits_{\mu\in\Lambda_\lambda} \mu_{n-1}\mu_n & \sum\limits_{\mu\in\Lambda_\lambda} \mu_n^2 \end{pmatrix}$$

In the next section we will show that $S(G,\lambda)$ is a multiple of a symmetrization of the Cartan matrix for G, and is thus invertible. So, we have that

$$\begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix} = S^{-1}(G,\lambda) \begin{pmatrix} \sum\limits_{\mu\in\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \vdots \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_n e^{\mu}(t) \end{pmatrix}$$

We calculate the matrix $S(G,\lambda)$ for the classical and exceptional simple algebraic groups. In the following sections, we continue the notation

$$\Lambda_\lambda = \{(\mu_1, ...\mu_n) |\ \mu_1\omega_1 + ... + \mu_n\omega_n\ is\ a\ weight\ of V_\lambda\}$$

counted with multiplicity.

Our main result will be calculating the matrix $S(G,\lambda)$ as defined in section 3, for the simple algebraic groups. We use the convention that the Cartan matrix associated to the root system of $\mathfrak{g}$ is $A = (A_{ij})$, where $A_{ij} = \alpha_i(\check{\alpha_j})$. Then A is a change-of-basis matrix for $\mathfrak{t}^*$ between the fundamental weights and the simple roots. Furthermore, A satisfies the following properties

- For diagonal entries $A_{ii} = 2$
- For non-diagonal entries $A_{ij} \leq 0$
- $A_{ij} = 0$ iff $A_{ji} = 0$
- A can be written as DS, where D is a diagonal matrix, and S is a symmetric matrix.

Let D be the diagonal matrix defined by $D_{ij} = \frac{\delta_{ij}}{2}(\alpha_i, \alpha_j)$, where if we realize the root system R associated to $\mathfrak{g}$ as a set of vectors in a Euclidean space E, then $(\cdot,\cdot)$ is the standard inner product. In this framework we can write $A_{ij} = \alpha_i(\check{\alpha_j}) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ Then, writing $A = DS$, we find that the matrix S has coordinate entries given by

$$S_{ij} = \frac{4(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

and is clearly symmetric.

$(\cdot, \cdot)$ is an invariant bilinear form on $\mathfrak{t}^*$, normalized so that so that $(\alpha_i, \alpha_i) = 2$ where α_i is the highest root. Note that under this formulation, if G is of simply-laced type then D is the identity matrix and S is the Cartan matrix. We find that in general for a given simple group G that $S(G, \lambda)$ is a multiple of S. Before stating our result precisely we fix the following notation. If α_j is any long simple root (for the simply laced case α_j can be any simple root), consider the corresponding fundamental weight ω_j. Let $x_j(\lambda) := \sum_{\mu \in \Lambda_\lambda} \mu_j^2$, where μ_j is the j^{th} coordinate of the weight $\mu \in \Lambda_\lambda$ in the fundamental weight basis.

Proposition 3.1. *Let G be a simple algebraic group. Let $S(G, \lambda)$ be defined as in section 3. Set $x_j(\lambda) := \sum_{\mu \in \Lambda_\lambda} \mu_j^2$ for a long root α_j. Let S be a symmetrization of the Cartan matrix as above. Then $S(G, \lambda)$ is a multiple of S. More precisely,*

$$S(G, \lambda) = \frac{1}{2} x_j(\lambda) \cdot S$$

and this is independent of the choice of long root α_j.

Proof. The proof will rely on the fact that the set of weights Λ_λ of V_λ is invariant under the action of the Weyl group on $\mathfrak{t}^*$, i.e. for $w \in W$, $w.\Lambda_\lambda = \Lambda_\lambda$. The following Lemma is true for all simple groups. The following two lemmas are sufficient to prove the simply-laced case but also hold for the non-simply laced cases.

Lemma 3.1. *For a given simple group G, if the Cartan matrix entry $A_{ij} = 0$, i.e the nodes representing the simple roots α_i and α_j are not connected on the associated Dynkin diagram, then*

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = 0,$$

where $\mu = (\mu_1, ..., \mu_n)$.

Proof. Consider the simple reflection s_i acting on a weight $\mu = (\mu_1, ...\mu_n) \in \Lambda_\lambda$. Then

$$s_i(\mu) = (\mu_1, ...\mu_n) - ((\mu_1, ...\mu_n)(\check{\alpha_i}))(\alpha_i)$$

where $(\mu_1, ...\mu_n)(\check{\alpha_i}) = (\mu_1\omega_1 + ...\mu_n\omega_n)(\check{\alpha_i}) = \mu_i$. Using the Cartan matrix to write the simple roots α_i in the fundamental weight basis gives $\alpha_i = (A_{i,1}, ..., A_{i,n})$. Then the above reflection yields

$$s_i(\mu) = (\mu_1, ...\mu_n) - \mu_i(A_{i,1}, ..., A_{i,n}) = (\mu_1 - \mu_i A_{i1}, ..., \mu_n - \mu_i A_{in})$$

Now note that $A_{ii} = 2$ and $A_{ij} = 0$. So the i^{th} coordinate of $s_i(\mu)$ is $[s_i(\mu)]_i = \mu_i - \mu_i A_{ii} = -\mu_i$ and the j^{th} coordinate of $s_i(\mu)$ is $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij} = \mu_j$. Thus we find that

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = \sum_{s_i(\mu)\in\Lambda_\lambda} \mu_i\mu_j = \sum_{\mu\in\Lambda_\lambda} [s_i(\mu)]_i \cdot [s_i(\mu)]_j = \sum_{\mu\in\Lambda_\lambda} -\mu_i\mu_j,$$

by invariance of Λ_λ under s_i. Thus, the result follows. □

Lemma 3.2. *If simple roots α_i and α_j of G are connected via the Dynkin diagram and have the same length then*

$$\sum_{\mu\in\Lambda_\lambda} \mu_i^2 = \sum_{\mu\in\Lambda_\lambda} \mu_j^2.$$

Furthermore,

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -\frac{1}{2}\sum_{\mu\in\Lambda_\lambda} \mu_i^2$$

Proof. We have that $A_{ij} = A_{ji} = -1$. Then as above with $\mu = (\mu_1, ...\mu_n) \in \Lambda_\lambda$, we have that $s_i(\mu) = (\mu_1 - \mu_i A_{i1}, ..., \mu_n - \mu_i A_{in})$. Now consider

$$s_j s_i(\mu) = ((\mu_1 - \mu_i A_{i1}) - (\mu_j - \mu_i A_{ij})A_{j1}, ..., (\mu_n - \mu_i A_{in}) - (\mu_j - \mu_i A_{ij})A_{jn})$$

Thus, $[s_j s_i(\mu)]_i = (\mu_i - \mu_i A_{ii}) - (\mu_j - \mu_i A_{ij})A_{ji} = -\mu_i - (\mu_j + \mu_i)(-1) = \mu_j$. Thus,

$$\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_i = \sum_{\mu \in \Lambda_\lambda} [s_j s_i(\mu)]_i \cdot [s_j s_i(\mu)]_i = \sum_{\mu \in \Lambda_\lambda} \mu_j \mu_j$$

The second part of the lemma follows from the fact that $[s_i(\mu)]_j = \mu_j - \mu_i A_{ij}$ with $A_{ij} = -1$. It follows that

$$\sum_{\mu \in \Lambda_\lambda} \mu_j^2 = \sum_{\mu \in \Lambda_\lambda} [s_i(\mu)]_j^2 = \sum_{\mu \in \Lambda_\lambda} (\mu_j + \mu_i)^2$$

Thus, $\sum\limits_{\mu \in \Lambda_\lambda} \mu_i \mu_i = -2 \sum\limits_{\mu \in \Lambda_\lambda} \mu_i \mu_j$ □

Recall that the root systems of simple groups of type B_n, C_n, G_2, F_4 contain long and short simple roots. Our convention will be the same as in [Bou]. That is, for B_n that $\alpha_1, ..., \alpha_{n-1}$ are the long roots and α_n is short, for C_n that $\alpha_1, ...\alpha_{n-1}$ are short and α_n is long, for G_2 that α_1 is short and α_2 is long, and for F_4 that the first and second are long and that the third and fourth are short.

Proposition 3.2. *Let G be a rank n simple group of types B_n, C_n, or F_4. For any long root α_j, set $x_j(\lambda) = \sum_{\mu \in \Lambda_\lambda} \mu_j^2$. If α_i is a short root, then $\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$. If either or both of α_i and α_j are short, then $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -x_j(\lambda)$*

Proof. Note that if α_i and α_j are both long roots, connected via the Dynkin diagram, then $A_{ij} = A_{ji} = -1$ So Lemma 4.3 shows that

$$\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = \sum_{\mu \in \Lambda_\lambda} \mu_j^2,$$

and that $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -\frac{1}{2} \sum_{\mu \in \Lambda_\lambda} \mu_i^2$. The same is true for the short roots as $A_{ij} = A_{ji} = -1$ for connected short roots. So we need to show that if α_i and α_j are short and long roots respectively and connected via the Dynkin diagram, then $\sum_{\mu \in \Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$, and that $\sum_{\mu \in \Lambda_\lambda} \mu_i \mu_j = -x_j(\lambda)$. To show this we first note that $A_{ij} = -1$ and $A_{ji} = -2$ and then compare $[s_i(\mu)]_i, [s_j(\mu)]_j, [s_j(\mu)]_i$ and $[s_i(\mu)]_j$. Note that $[s_i(\mu)]_i = -\mu_i$ and $s_j(\mu_j) = -\mu_j$ as before. Also, $[s_i(\mu)]_j = \mu_j - \mu_i A_{i,j} = \mu_j + \mu_i$ and $[s_j(\mu)]_i = \mu_i - \mu_j A_{ji} = \mu_i + 2\mu_j$. Thus, we have that

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = \sum_{\mu\in\Lambda_\lambda} [s_j(\mu)]_i \cdot [s_j(\mu)]_j = \sum_{\mu\in\Lambda_\lambda} (\mu_i + 2\mu_j)(-\mu_j) = \sum_{\mu\in\Lambda_\lambda} -\mu_i\mu_j - 2\mu_j^2$$

Thus $\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = -\sum_{\mu\in\Lambda_\lambda} \mu_j^2 = -x_j(\lambda)$. Applying, s_i to μ gives

$$\sum_{\mu\in\Lambda_\lambda} \mu_i\mu_j = \sum_{\mu\in\Lambda_\lambda} [s_i(\mu)]_i \cdot [s_i(\mu)]_j = \sum_{\mu\in\Lambda_\lambda} -\mu_i\mu_j - \mu_i^2$$

Thus, $\sum\limits_{\mu\in\Lambda_\lambda} \mu_i^2 = 2x_j(\lambda)$ □

So it follows that with $x_j(\lambda) = \sum\limits_{\mu\in\Lambda_\lambda} \mu_j^2$, where α_j is a long root, then

$$S(B_n,\lambda) = \frac{x_j(\lambda)}{2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -2 & 4 \end{pmatrix}, S(C_n,\lambda) = \frac{x_j(\lambda)}{2}\begin{pmatrix} 4 & -2 & & & & \\ -2 & 4 & -2 & & & \\ & -2 & \ddots & & & \\ & & & 4 & -2 & \\ & & & -2 & 4 & -2 \\ & & & & -2 & 2 \end{pmatrix}$$

$$S(F_4,\lambda) = \frac{x_j(\lambda)}{2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

We give inverses of these matrices in the next section.

Let α_1 be the short root, and α_2 the long root of G_2.

Proposition 3.3. $\sum\limits_{\mu\in\Lambda_\lambda} \mu_1^2 = -2 \sum\limits_{\mu\in\Lambda_\lambda} \mu_1\mu_2 = 3 \sum\limits_{\mu\in\Lambda_\lambda} \mu_2^2$

Proof. Let $\mu = (\mu_1, \mu_2) \in \Lambda_\lambda$. Then since $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, we find that $s_1(\mu) = (-\mu_1, \mu_1 + \mu_2)$ and that $s_2(\mu) = (\mu_1 + 3\mu_2, -\mu_2)$. So,

$$\sum_{\mu\in\Lambda_\lambda} \mu_1^2 = \sum_{\mu\in\Lambda_\lambda} (\mu_1 + 3\mu_2)^2$$

from which it follows that $\sum_{\mu\in\Lambda_\lambda} \mu_1\mu_2 = -\frac{3}{2}\sum_{\mu\in\Lambda_\lambda} \mu_2^2$. Additionally, we have that

$$\sum_{\mu\in\Lambda_\lambda} \mu_2^2 = \sum_{\mu\in\Lambda_\lambda} (\mu_1 + \mu_2)^2$$

from which we can see that $\sum_{\mu\in\Lambda_\lambda} \mu_1^2 = -2\sum_{\mu\in\Lambda_\lambda} \mu_1\mu_2 = 3\sum_{\mu\in\Lambda_\lambda} \mu_2^2$. Thus,

$$S(G_2, \lambda) = \frac{1}{2} \sum_{\mu\in\Lambda_\lambda} \mu_2^2 \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$$

$\square$

In particular, we can solve for $c_1(t)$ and $c_2(t)$ as

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = S(G_2, \lambda)^{-1} \begin{pmatrix} \sum\limits_{\Lambda_\lambda} \mu_1 e^{\mu}(t) \\ \sum\limits_{\mu\in\Lambda_\lambda} \mu_2 e^{\mu}(t) \end{pmatrix}$$

then, letting $x = \sum\limits_{\mu \in \Lambda_\lambda} \mu_2^2$ we have that $S^{-1}(G, \lambda) = \frac{2}{3x} \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$. Thus,

$$c_1(t, \lambda) = \frac{2}{3x} \sum_{\mu \in \Lambda_\lambda} (2\mu_1 + 3\mu_2) e^{\mu}(t)$$

$$c_2(t, \lambda) = \frac{2}{3x} \sum_{\mu \in \Lambda_\lambda} (3\mu_1 + 6\mu_2) e^{\mu}(t)$$

.

3.2.1 Examples

Consider $G = Sp(2n, \mathbb{C})$=$\{A \in GL(2n) | M = A^t M A\}$ where $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ where I_n is the $n \times n$ identity matrix, and $\mathfrak{sp}(2n, \mathbb{C})$=$\{X \in \mathfrak{gl}(2n) | X^t M + M X = 0\}$.

Let $\lambda = \omega_1$, the defining representation. Then we have that Λ_λ=$\{\pm\omega_1$ and $\pm(\omega_i - \omega_{i+1})$ for $1 \le i \le n-1\}$. So, $x = \sum\limits_{\Lambda_\lambda} \mu_n^2 = 2$. Let $T = diag\{t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}\}$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = 2\epsilon_n$. The simple coroots in $\mathfrak{t}$ are then $\check{\alpha}_i = E_i - E_{i+1} - E_{n+i} + E_{n+i+1}$ for $1 \le 1 \le n-1$ and $\check{\alpha}_n$=$E_n - E_{2n}$ where E_i is the diagonal matrix with a 1 in the i^{th} slot and 0's elsewhere [FH]. In the orthogonal basis for $\mathfrak{t}$, $\omega_i = \epsilon_1 + ... + \epsilon_i$. Thus, the character $e^{\mu}(t)$ is given by $e^{\mu}(t) = t_1^{\mu_1 + ... \mu_n} \cdot t_2^{\mu_2 + ... + \mu_n} \cdot ... \cdot t_n^{\mu_n}$. Then, we have that

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 2 & 2 & ... & 2 \\ 1 & 2 & 3 & ... & 3 \\ ... & ... & ... & ... & ... \\ 1 & 2 & 3 & ... & n \end{pmatrix} \begin{pmatrix} t_1 - t_1^{-1} - t_2 + t_2^{-1} \\ t_2 - t_2^{-1} - t_3 + t_3^{-1} \\ \vdots \\ t_{n-1} - t_{n-1}^{-1} - t_n + t_n^{-1} \\ t_n - t_n^{-1} \end{pmatrix}$$

which gives

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t_1 - t_1^{-1} \\ \vdots \\ t_{n-1} - t_{n-1}^{-1} \\ t_1 - t_1^{-1} + ... + t_n - t_n^{-}1 \end{pmatrix}$$

Thus,

$$\theta_\lambda(t) = c_1(t)\check{\alpha}_1 + ... + c_n(t)\check{\alpha}_n = diag(\frac{t_1 - t_1^{-1}}{2}, ..., \frac{t_n - t_n^{-1}}{2}, -\frac{t_1 - t_1^{-1}}{2}, ..., -\frac{t_n - t_n^{-1}}{2}).$$

Note that this is equivalent to the Cayley transform as in §6 of [Ku2]. Similar results hold for $\theta_{\omega_1}(t)$ for the standard maximal tori of $SO(2n+1,\mathbb{C})$ and $SO(2n,\mathbb{C})$. □

The inverses of the Cartan matrices for A_n, D_n, E_6, E_7, E_8 respectively have the form (as in [Ro])

$$\frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & ... & 3 & 2 & 1 \\ n-1 & 2(n-1) & 2(n-3) & ... & 6 & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & ... & 9 & 6 & 3 \\ ... & ... & ... & ... & ... & ... & ... \\ 2 & 4 & 6 & ... & (2n-2) & 2(n-1) & n-1 \\ 1 & 2 & 3 & ... & n-2 & n-1 & n \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & ... & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & ... & 2 & 1 & 1 \\ 1 & 2 & 3 & ... & 3 & \frac{3}{2} & \frac{3}{2} \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & 2 & 3 & ... & n-2 & \frac{n-2}{2} & \frac{n-2}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & ... & \frac{n-2}{2} & \frac{n}{4} & \frac{n-2}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & ... & \frac{n-2}{2} & \frac{n-2}{4} & \frac{n}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 2 & 3 & 2 & 1 \\ \frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & 3 & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix}, \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{2}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$$

The inverse of the matrix S for types C_n, B_n, G_2, F_4 have the form

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 2 & 2 & ... & 2 \\ 1 & 2 & 3 & ... & 3 \\ ... & ... & ... & ... & ... \\ 1 & 2 & 3 & ... & n \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & ... & 2 & 1 \\ 2 & 4 & 4 & ... & 4 & 2 \\ 2 & 4 & 6 & ... & 6 & 3 \\ ... & ... & ... & ... & ... & ... \\ 2 & 4 & 6 & ... & 2(n-1) & n-1 \\ 1 & 2 & 3 & ... & n-1 & 2 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} & 1 \end{pmatrix}$$

CHAPTER 4

Representation Ring of Levi Subgroups vs Cohomology Ring of Flag Varieties

4.1 Classical Result and Polynomials Invariants

Let $Gr(r,n)$ be the Grassmanian of r-planes in $\mathbb{C}^n$. Then a classical result states that the tensor product of irreducible polynomial representations of the general linear group $GL(r)$ over $\mathbb{C}$ corresponds in a certain sense to the cup product in the cohomology of the flag manifold, $H^*(Gr(r,n),\mathbb{Z})$.

Note that the Lie group $GL(r)$ is contained in its Lie algebra $\mathfrak{gl}(r) = M_{r\times r}$.

Definition 4.1. An irrep $V(\lambda)$ of $GL(r)$ is called a polynomial rep if its character lifts to a character on the Lie algebra $\mathfrak{gl}(r)$

$$\begin{tikzcd} GL(r) \arrow[r, "\chi_\lambda"] \arrow[d, "i"] & \mathbb{C} \\ \mathfrak{gl}(r) \arrow[ur, dashed] & \end{tikzcd}$$

Alternately, a finite dimensional representation $\rho : GL(r) \to GL(V)$ is said to be polynomial if there exists a basis of V such that entries of $\rho(g)$ are polynomials in the matrix entries of g. Every irreducible polynomial representation of $GL(r)$ is indexed by a partition (its highest weight)

$$\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0\}$$

such that the action of the torus is given by

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \to t_1^{\lambda_1}...t_r^{\lambda_r}$$

Note that $Gr(r,n) = GL(r)/P_r$ where P_r is a maxiamal parabolic subgroups containing the the standard upper triangular Borel subgroup $B \subset P_r \subset GL(r)$. P_r is taken by deleting the r^{th} node of the Dynkin diagram for $GL(r)$ (or in the language of Chapter 2, $P_r = P_\theta$ with $\theta = \Delta - \{\alpha_r\}$. Then we have the following Bruhat decomposition

$$\bigsqcup_{w \in W_G/W_{P_r}} BwP_r/P_r$$

where $W_G = S_n$ and $W_{P_r} = S_r \times S_{n-r-1}$ and $W_G/W_{P_r} = W^\theta$ is the following set of length r subsequences of $[n]$, $S(r,n) = \{A : 1 \leq a_1 \leq a_2 < ... < a_r \leq n\}$. Any such tuple represents the permutation

$$\nu_A = (a_1, ..., a_r, a_{r+1}, ..., a_n),\ i \mapsto a_i$$

Then we have that

$$H^*(Gr(r,n), \mathbb{Z}) = \bigoplus_{A \in S(r,n)} \mathbb{Z}_{\epsilon^{P_r}_{\nu(A)}}$$

where $\epsilon^{P_r}_{\nu(A)} \in H^{2l(\nu(A))}(Gr(r,n))$. This leads to the classical result that

Theorem 4.1. *The following map ξ is a surejective ring homomorphism*

$$\xi : Rep_{poly}(Gl(r)) \to H^*(Gr(r,n), \mathbb{Z})$$

where

$$[V(\lambda)] \to \epsilon^{P_r}_{\nu(A)}\ if\ \lambda_1 \leq n - r$$

$$\to 0\ \ otherwise$$

and $A(\lambda) = \{1 + \lambda_r < 2 + \lambda_{r-1} < ... < r + \lambda_1\}$ is a surjective homomorphism.

The work [Ku2] (on which this thesis is largely based) aimed at generalizing the above classical result to the larger context of the Levi subgroups of reductive groups and the cohomology of the corresponding partial flag varieties. For another attempt at generalization, see [**?**]. We will use the Borel characteristic map of §2.3 and the Springer morphism of the previous chapter to do so in the next section. A historical difficulty for extending the above result to other classical types is that it is not clear how to define polynomial representations for other groups. The polynomial ring of invariants $S(\mathfrak{t}^*)^W$ for a Weyl group will serve as the model for the polynomial representations of a group with said Weyl group. We now give some basic facts about the ring $S(\mathfrak{t}^*)^W$ and examples for the Weyl groups of simple groups.

4.1.1 Weyl Group Invariants

More generally, let G be a group acting linearly on a vector space V. If $\mathbb{C}[V]$ is the space of polynomial functions on V, then there is an induced action of G on $\mathbb{C}[V]$ given by $(g \cdot f)(x) = f(g^{-1}(x))$. Classical invariant theory was concerned itself with the structure of the space of invariant polynomials $\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] |\ g \cdot f = f\ \forall g \in G\}$, particularly finiteness results [Hu]. For example, Hilbert and Noether showed that the ring of invariants is a finitely generated $\mathbb{C}-$algebra. A theorem of Chevalley-Shepard-Todd showed that the ring of invariants is a polynomial ring if and only if G is a complex reflection group. Furthermore the degrees of the generators are unique. As Weyl groups are complex reflection groups, their ring of invariants $S(\mathfrak{t}^*)^W$ is a polynomial ring on rank($\mathfrak{t}$) generators. The degrees d_i of these generators are listed below.

Type	Degrees
A_n	2,3,...,n+1
B_n	2,4,6,...,2n
C_n	2,4,6,...,2n
D_n	2,4,6,...,2n-2,n
G_2	2,6
F_4	2,6,8,12
E_6	2,5,6,8,9,12
E_7	2,6,8,10,12,14,18
E_8	2,8,12,14,18,20,24,30

Table 4.1: Degrees of Basic Invariants

In particular, we also have that $\prod_{i=1}^n d_i = |W|$ and $\sum_{i=1}^n (d_i - 1)$ is the number of reflections. We can now describe the well-known polynomial invariants for the classical groups. For examples for the exceptional groups see [Lee, Me, Ts].

$Type\ A_n$: It is convenient to work in $\mathbb{C}^{n+1}$ restricted to the hyperplane $x_1 + ... + x_{n+1} = 0$. Then $W_{A_n} = S_{n+1}$ acts on $\mathbb{C}[x_1, ..., x_n + 1]$ by permuting the variables. Recall [Hu] that the simple roots are given by $\Delta = \{e_i - e_{i+1} |\ i = 1, ..., n\}$. Then the simple reflections s_i act by permuting x_i with x_{i+1}. Then we have the following set of basic invariants

$$f_i = e_i(x_1, ..., x_{n+1})$$

for $i = 2, 3, .., n+1$, where e_i is the i^{th} elementary symmetric polynomial (Note that $e_1(x_1, ..., x_{n+1}) = x_1 + ...x_{n+1} = 0$.

$Type\ B_n\ and\ C_n$: Note that C_n and B_n have the same Weyl Group. The simple roots of type B_n are $\Delta = \{e_i - e_{i+1} |\ i = 1, .., n\} \cup \{e_n\}$. So the simple reflections s_i act by permuting x_i and x_{i+1} and s_n acts by taking x_n to $-x_n$. In particular, the Weyl group $W_{B_b} \simeq S_n \rtimes \mathbb{Z}_{\not\vDash}$ is the hyperoctahedral group. We have the following set of basic invariants

$$f_i = e_i(x_1^2, ...x_n^2)$$

for $i = 1, ..., n$.

$TypeD_n$ The simple roots of type D_n are given by $\Delta = \{e_i - e_{i+1} |\ i = 1, .., n-1 \cup \{e_{n-1} + e_n\}$. The first $n-1$ simple reflections act as before and s_n acts by permuting x_{n-1} and x_n and changing their sign. The Weyl group W_{D_n} is the subgroup of W_{B_n} of elements with an even number of sign changes. We have the following set of basic invariants.

$$f_i = e_i(x_1^2, ..., x_n^2)$$

for $i = 1, ..., n-1$ and

$$f_n = e_n(x_1, ..., x_n) = x_1...x_n$$

4.2 Main Result

We are now ready to state the main result of [Ku2]. Let G be a connected reductive algebraic group over $\mathbb{C}$ and P a standard parabolic subgroup with Levi subgroup L containing the chosen maximal torus T. Let W_L be the Weyl group of L.

Recall the surjective Borel morphism from §2.3,

$$S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{C})$$

which takes a chacter $\mu \in X(T)$ to the first chern class of the line bundle $\mathcal{L}(\mu)$. We can realize $X(T)$ as a lattice in $\mathfrak{t}^*$ via taking derivative. W_L acts on both $S(\mathfrak{t}^*)$ and $H^*(G/B, \mathbb{C})$, and restricting we get a surjective graded algebra homomorphism:

$$\beta^P : S(\mathfrak{t}^*)^{W_L} \to H^*(G/B, \mathbb{C})^{W_L} \simeq H^*(G/P, \mathbb{C}),$$

. where the last isomorphism is induced from the projection $G/B \to G/P$.

Take an almost faithful G-module V_λ. Let $\theta_\lambda : G \to \mathfrak{g}$ be the associated Springer morphism from §3. Restricting $\theta_{\lambda|T} : T \to \mathfrak{t}$ induces the corresponding $W-$equivariant injective algebra homomorphism on the affine coordinate rings:

$$\theta^*_{\lambda|T} : \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \to \mathbb{C}[T]$$

So, resticting to W_L invariants we get the following injective algebra homomorphism:

$$\theta_{\lambda|T}(P)^* : \mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L} \to \mathbb{C}[T]^{W_L}$$

Now we let $Rep(L)$ be the representation ring of L and let $Rep^{\mathbb{C}}(L) = Rep(L) \otimes \mathbb{C}$ be its complexified representation ring. Then, recall from §2.1 that $Rep^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$ obtained by taking the character of an $L-$module restricted to T. Note again that a representation V of L is denoted by $[V]$ as an element of $Rep(L)$.

Then we make the following definition inspired by the definition for a polynomial representation of $GL(r)$ given earlier

Definition 4.2. A virtual character $\chi \in Rep^{\mathbb{C}}(G)$ is called $\lambda-$poly if the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\chi"] \arrow[d, "\theta_\lambda"'] & \mathbb{C} \\ \mathfrak{g} \arrow[ur] & \end{tikzcd}$$

I.e. $\chi \in Rep^{\mathbb{C}}(L)$ is $\lambda - poly$ iff the corresponding function in $\mathbb{C}[T]^{W_L}$ is in the image of $\theta_{\lambda|T}(P)^*$. The set $Rep^{\mathbb{C}}_{\lambda-poly}(L)$ of all $\lambda-$polynomial characters is a subalgebra of $Rep^{\mathbb{C}}(L)$ isomorphic to the algebra

$S(\mathfrak{t}^*)^{W_L}$ of Weyl polynomial invaraints. Thus, $\theta_{\lambda|T}(P)^*$ induces an isomporphism

$$\theta_{\lambda|T}(P)^* : S(\mathfrak{t}^*)^{W_L} \to Rep^{\mathbb{C}}_{\lambda-poly}(L)$$

Now, the main result is as follows by composing the above maps

Theorem 4.2. *Let V_λ be an almost faithful irreducible G-module and let P be any standard parabolic subgroup. Then, the above maps (specifically $\beta^P \circ (\theta_{\lambda|T}(P)^*)^{-1}$) give rise to a surjective $\mathbb{C}$-algebra homomorphism*

$$\xi^P_\lambda : Rep^{\mathbb{C}}_{\lambda-poly}(L) \to H^*(G/P, \mathbb{C}).$$

Moreover, let Q be another standard parabolic subgroup with Levi subgroup R containing T such that $P \subset Q$ (and hence $L \subset R$). Then, we have the following commutative diagram:

$$\begin{array}{ccc} Rep^{\mathbb{C}}_{\lambda-poly}(R) & \xrightarrow{\xi^Q_\lambda} & H^*(G/Q, \mathbb{C}) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \pi^*} \\ Rep^{\mathbb{C}}_{\lambda-poly}(L) & \xrightarrow{\xi^P_\lambda} & H^*(G/P, \mathbb{C}), \end{array}$$

where π^* is induced from the standard projection $\pi : G/P \to G/Q$ and γ is induced from the restriction of representations.

Example 4.1. The subalgebra $Rep^{\mathbb{C}}_{\lambda-poly}(G) \subset Rep^{\mathbb{C}}(G)$, in general, indeed depends upon the choice of λ. For example, for $G = \mathbf{SL}_2$, following Example 3.1,

$$Rep^{\mathbb{C}}_{\omega_1-poly}(\mathbf{SL}_2) = \mathbb{C}[(z - z^{-1})^2],$$

whereas

$$Rep^{\mathbb{C}}_{2\omega_1-poly}(\mathbf{SL}_2) = \mathbb{C}[(z^2 - z^{-2})^2],$$

for the maximal torus in $\mathbf{SL}_2$ given by

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^* \right\}.$$

$\diamondsuit$

4.3 Type A

In this section we will show how we can recover the classical result stated at the beginning of this section from Theorem 4.2 and give more details on the Cohomology of the Grassmannian.

4.3.1 Recovering the Classical Result

We adopt the same notation as from the beginning of the chapter and follow [Ku2, §5]. The torus $T \subset Gl(n)$ is the set of diagonal matrices in $GL(n)$ and the cartan subalgebra is then given by set of diagonal matrices $\mathfrak{t} = \{diag(t_1, ..., t_n) : \ t_i \in \mathbb{C}\}$. Then the simple roots and coroots are given by

$$\alpha_i = t_i - t_{i+1} \text{ and } \breve{\alpha}_i = diag(0, , , 0, 1, -1, 0, ...0), \text{ for any } 1 \leq i \leq n-1$$

where 1 is in the i^{th} place. And the fundamental weights are given by

$$\omega_i = t_1 + ... + t_i$$

The Weyl group of type A_n is the symmetric group S_n generated by the reflections S_{α_i} associated to the simple roots. Here, $S_{\alpha_i} = s_i = (i, i+1)$. Now let P_r be the standard maximal parabolic associated to the subset $\theta = \Delta - \{\alpha_r\}$ of simple roots. Then L_r is the unique Levi subgroup containing T such that its simple roots are θ. Then as mentioned before $GL(n)/P_r \simeq Gr(r, n)$, the Grassmannian of n-planes in $\mathbb{C}^n$. Furthermore the set of minimal length cosets which index the Schubert classes in the cohomology of the grassmannian can be parametrized by the following set of strictly increasing sequences,

$$S(r, n) = \{I := 1 \leq a_1 < ... < a_r \leq n\}.$$

These sequences represent the permutation $\nu_A \in S_n$ given by

$$_I = (a_1, ..., a_r, a_{r+1}, ..., a_n),\ i \mapsto a_i$$

where the above permutation is written in one-line notation and the $\{a_{r+1}, ..., a_n\}$=$[n]\backslash\{a_1, ..., a_r\}$ are put in increasing order. Such permutations are said to have a descent at r, i.e. $w(r+1) < w(r)$ or equivalently $l(ws_r) < l(w)$. There is also a paremtrization by partitions $\lambda = \{n - r \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_1 \geq 0\}$. The partition λ_I corresponding to the sequence I is given by $\lambda_i = I_{r+1-i} - (r + 1 - i)$. So For example, if n=5 and r=3, the sequence $I = (1, 3, 5)$ corresponds to the one-line permutation $w_I = (1, 3, 5, 2, 4)$ and to the partition $\lambda_I = (2, 1, 0)$. The corresponding Schubert variety will then be denoted by either $X_I,\ X_{w_I}$, or X_{λ_I}.

Now as mentioned above, irreducible polynomial representations of $GL(r)$ are also parametrized by permutations. The map ξ from Theorem 4.1 can then be stated as mapping $[V(\lambda)] \mapsto \epsilon_\lambda$.

Let $G = GL(n)$ and let $\lambda = \omega_1$ so V_λ is the defining representation. Then we have that,

$$\theta_{\omega_1} : GL(n) \to \mathfrak{g}(n)$$

. Furthermore, $Rep_{\omega_1-poly}(G)$ coincides with the usual notion of polynomial representation (where $Rep_{\omega_1-poly}(G) = Rep^{\mathbb{C}}_{\omega_1-poly}(G) \cap Rep(G)$. For $1 \leq r \leq n - 1$, the Levi subgroup L_r is the subgroup

$$L_r \simeq GL(r) \times GL(n - r) \subset GL(n).$$

Then, from Theorem 4.2 we have a $\mathbb{C}$-algebra homomorphism:

$$\xi^{P_r}_{\omega_1} : Rep^{\mathbb{C}}_{\omega_1-poly}(L_r) \to H^*(Gr(r, n), \mathbb{C})$$

where,

$$\text{Rep}^{\mathbb{C}}_{\omega_1-poly}(L_r) \simeq \left[Rep_{poly}(GL(r)) \otimes Rep_{poly}(GL(n - r))\right] \otimes_{\mathbb{Z}} \mathbb{C}$$

In order to get a map from $\text{Rep}^{\mathbb{C}}_{\omega_1}(GL(r))$ we factor through the ring homorphism

$$i : Rep_{poly}(GL(r)) \to Rep_{poly}(GL(r)) \otimes Rep_{poly}(GL(n-r))$$

where we tensor a $GL(r)$ representation with the trivial one-dimensional $GL(n-r)$ representation.

Theorem 4.3. *$\xi^{P_r}_{\omega_1} \circ i$ coincides with ξ from Theorem 4.1.*

Proof. Note that since these are $\mathbb{C}$ algebra homomorphisms, we need only check that they correspond on the fundamental representations $[V_{\omega_1}]$ since they generate $Rep_{poly}(GL_r)$. Note that for $\lambda = (1, .., 1, 0, .., 0)$ with i one's we have $[V(\omega_i)] = [V(\lambda)]$. Furthermore, $w_\lambda = s_{r-i+1}...s_r$. Thus by definition,

$$\xi([V(\omega_i)]) = \epsilon^{P_r}_{s_{r-i+1}...s_r}.$$

Moreover, the character of $[V(\omega_i)]$ is the i$^{\text{th}}$ elementary symmetric polynomial $e_i(x_1, ..., x_r)$ where x_i is the i^{th} coordinate map on $\mathfrak{t}$. Thus,

$$\xi^{P_r}_{\omega_1}([V(\omega_1)]) = \beta(e_i(x_1, ..., x_r))$$

where β is the Borel characteristic map. Then, by [Hi, Chapter 4 Lemma 5.4] we have

$$\beta(e_i(x_1, ..x_r)) = \epsilon^{P_r}_{s_{r-i+1}...s_r}$$

completing the proof.

$\square$

4.3.2 Cohomology of the Grassmannian

As a model for what we will discuss in Chapter 5 for the other classical groups we will briefly discuss in more detail the structure of the cohomology of the Grassmannian. References for this material are [T2, T7]

From the previous section we know that $X = Gr(r, n)$ can be realized as the homogenous space $GL(n)/P_r$. Then from chapter 2, sections 2 and 3, we saw that $H^*(Gr(r, n), \mathbb{C})$ has an additive schubert basis indexed by the minimal length elements of W/W_{P_r}

Fix the complete flag of vector subspaces ($F_i = \langle e_1, ..., e_i \rangle \subset \mathbb{C}^n$)

$$F_\bullet \;:\; 0 = F_0 \subset F_1 \subset ... \subset F_n = \mathbb{C}^n$$

. Consider the set of index sets $S(r,n) = \{I : 1 \leq i_1 < ... < i_r \leq n\}$. Now let $\Omega \in X$, then we can define a corresponding index set $I(\Omega)$ by

$$I(\Omega) = \{(i_1, .., i_r) | \; \Omega \cap F_{i_j} \supsetneq \Omega \cap F_{i_j - 1}\}$$

Then we can define the following subvariety of X for a given index set I,

$$X_I^\circ(F_\bullet) := \{\Omega \in X | I(\Omega) = I\}$$

$X_I^\circ(F_\bullet)$ is then isomorphic to an affine space of dimension $\sum_{j=1}^r (i_j - j)$ and these give a familiar cell decomposition of the grassmannian

$$Gr(r,n) = \coprod_I X_I^\circ(F_\bullet)$$

These are exactly the open Bruhat cells(up to choice of full flag or Borel subgroup) and an index set corresponds to an element W/W_{P_r} as in the previous subsection. Let $X_I(F_\bullet)$ be the closure. We can also parametrize subvarieties by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r)$ where we require $n - r \geq \lambda_1$. The subvarieties are

$$X_\lambda(F_\bullet) = \{\Omega \in X | \; dim(\Omega \cap F_{n-r+j-\lambda_j}) \geq j, \; 1 \leq j \leq r\}$$

. We can associate a partition making the codimension of X_I apparent to an index set I by letting

$$\lambda_j = n - r + j - i_j.$$

Then $X_\lambda(F_\bullet)$ is a subvariety of co-dimension $|\lambda| = \sum_{j=1}^r \lambda_j$. We can also associate the dual partition $\tilde{\lambda}$ given by

$$\tilde{\lambda}_j = i_{r+1-j} - (r + 1 - j).$$

Then, $X_{\tilde{\lambda}(F_\bullet)}$ is a variety of dimension $|\lambda|$. Associate a permutation w_λ to a partition λ by

$$w_\lambda(i) = \lambda_{r+1-i} + i$$

. Now, let $[X_\lambda]$ be the fundamental homology class of the subvariety X_λ (the class is independent of choice of flag, see [Br]). Associate ϵ_λ and $\epsilon_{\tilde{\lambda}}$ be cohomology classes associated to an index set I. Then $\epsilon_{\tilde{\lambda}}$ and ϵ_λ are related to each other as ϵ_w and $\epsilon_{w_0 w w_0^P}$ are from §2.2 ($w_0 w_{0,P}$ is the longest element of W_P).

For example, let $w = (1, 2, 3, 6, 4, 6)$ be an element of $S(4, 6)$, i.e. the associated index set is $I = (1, 2, 3, 6)$. Then, $\lambda = (2, 2, 2, 0)$ and $\tilde{\lambda} = (2, 0, 0, 0)$ based of I as above. Then we have that

$$\epsilon_w = \epsilon_{\tilde{\lambda},}$$

and $\epsilon_w = PD[X_\lambda]$ or equivalently that $\epsilon_{w_0 w w_{0_P}} = \epsilon_\lambda$.

The varieties indexed by partitions of a single part $X_p := X_{(p,0,..,0)}$ for $1 \leq p \leq n - r$ play a special role in determining the cohomology ring. They depend only on a single Schubert condition,

$$X_p(F_\bullet) = \left\{\Omega \in X |\, \Omega \cap F_{n+1-p} \neq 0\right\}.$$

Note, $\epsilon_p \in H^p(X, \mathbb{C})$. These are called the special Schubert classes.

We now want to give a presentation for the colomology ring $H^*(Gr(r, n), \mathbb{C})$. The idea here is the same for all grassmannians of classical type [BKT1, §0]. Over the grassmannian there is a universal short exact sequence of vector bundles

$$0 \to S \to V \to Q \to 0.$$

V is the trivial rank n bundle. S is the tautological rank r subbundle where the fiber over a point $\Omega \in X$ is Ω, and Q is the tautological rank $d = n - r$ quotient bundle. Then we have that $\epsilon_p = c_p(Q)$, i.e. ϵ_p is the p^{th} Chern class of the quotient bundle. Now, multiplication in the cohomology ring is determined by the classical Pieri rule, which is a type of Chevalley rule from §2.2. It states that the product of a Schubert class

$\epsilon_\lambda \in H^{|\lambda|}$ with a special Schubert class is given by,

$$\epsilon_\lambda \cdot \epsilon_p = \sum \epsilon_\mu,$$

where the sum is over all partitions μ obtained from λ by adding p blocks to λ while adding no two in the same column. Additionally, any Schubert class ϵ_λ can be expressed as a polynomial in the special Schubert classes. This is the classical Giambelli formula,

$$\epsilon_\lambda = det(\epsilon_{\lambda_i+j-i})_{1\leq i<j\leq n}$$

The Pieri formula implies that the special Schubert classes ϵ_p generate the cohomology of the grassmannian. We can present the cohomology as a qutient of the polynomial ring $\mathbb{C}[\epsilon_1, ...\epsilon_d]/I_{r,d}$ where $I_{r,d}$ is generated by the determinantal relations

$$det(\epsilon_{1+j-i})_{1,j\leq m} = 0, \; r+1 \leq m \leq n$$

The Whitney sum folmula applied to S and Q, $c(S)c(Q) = 1$ where $c(Q)$ is the total Chern class of the bundle Q, can be used to show that these relations hold in $H^*(X, \mathbb{C})$ and dimensional considerations show that they are sufficient. We will be able to give presentations for the cohomology of grassmannians of type B and C in terms of the Chern classes of a universal quotient bundle as well.

4.3.3 Result in the Inverse Limit

We will look at the situation from §4.3.1 again this time focusing on the second factor. Fix an integer d. We want to compare the ring $Rep_{poly}(GL(d))$ to the cohomology ring $H^*(Gr(n-d, n), \mathbb{C})$ of the grassmannian of codimension d subspaces of $\mathbb{C}$. Again we consider the maximal parabolic P_{n-d} associated to the subset of simple roots $\Delta - \{\alpha_{n-d}\}$. Then as before $L_r = GL(n-d) \times GL(d)$ and theorem 4.2, and we have

$$\mathrm{Rep}^{\mathbb{C}}_{\omega_1}(L_r) = \mathrm{Rep}^{\mathbb{C}}_{\omega_1}(GL(n-d)) \otimes_{\mathbb{C}} \mathrm{Rep}^{\mathbb{C}}_{\omega_1}(GL(d)) = \mathbb{C}_{sym}[t_1, ..., t_{n-d}] \otimes \mathbb{C}_{sym}[t_{d+1}, ...t_n]$$

Here again $\mathbb{C}_{sym}[x_1, ..., x_k]$ is the subring of of polynomials invariant under permutation of variable. The fundamental theorem of symmetric polynomials says that any symmetric polynomial can be written as a polynomial in the elementary symmetric polynomials $e_i(x_1, ..., x_k)$ for $1 \leq i \leq k$. The elementary symmetric polynomial is defined as

$$e_i(x_1, ...x_k) = \sum_{1 \leq j_1 < .. < j_i \leq k} x_{j_1}...x_{j_i}$$

We saw in 4.3.1 that

$$\xi^{P_{n-d}}_{\omega_1}(e_i(t_1, .., t_{n-d})) = \epsilon_{s_{n-d-i+1}...s_{n-d}}$$

Consider $e_1(t_{n-d+1}, ..., t_n) = t_{n-d+1} + ... + t_n$. Let $T = \{t = (t_1, ..., t_n)|t_i \neq 0\}$ be the maximal torus in $GL(n)$. Then with $\theta_{\omega_1}(t) = t \in \mathfrak{gl}_n$. Let $x_i : \mathfrak{t} \to \mathbb{C}$ be the linear map taking

$$diag(x_1, ..., x_n) \in \mathfrak{t} \text{ to } x_i$$

Then since $\theta_{\omega_1|T}$ is just the inclusion $T \subset \mathfrak{t}$, $(\theta^*_{\omega_1|T})^{-1}(x_i) \mapsto t_i$, so $e_i(t_1, ..., t_n) \mapsto e_i(x_1, ..., x_n)$. Then we just need to compute $\beta(e_i(x_{n-d+1}, ..., x_n))$. Recall that $\beta(\omega_i) = \epsilon_{s_i}$. In the fundamental weight basis $x_i = \omega_i - \omega_{i-1}$, so

$$\beta(x_i) = \beta(\omega_i) - \beta(\omega_{i-1}) = \epsilon_{s_i} - \epsilon_{s_{i-1}},$$

except for $\beta(x_n) = \beta(\omega_n - \omega_{n-1}) = -\epsilon_{s_{n-1}}$. Then, finally

$$\beta(e_1(x_{n-d+1}, ..., x_n)) = \sum_{i=1}^{d} \beta(\omega_{n-d+i} - \omega_{n-d+i-1}) = -\epsilon_{s_{n-d}}$$

In general, we claim

Lemma 4.1. *For any $k = 1, ..., d$, the following eqution holds in $H^*(Gr(n-d, n), \mathbb{C})$*

$$\beta(e_k(x_{n-d+1,...,x_n}) = (-1)^k(\epsilon_{s_{n-d+k-1}...s_{n-d}})$$

Proof. Recall that we can relate the Borel morphism to Schubert classes by

$$\beta(f) = \sum_{w \in W^P} A_w(f)\epsilon_w.$$

So we just need to show that $A_w(e_k(x_{n-d+1}, ..., x_n)) = (-1)^k$ if $w = s_{n-d+k}...s_{n-d}$ and equals 0 otherwise. This follows by inducting on the number of variables and noting the following properties of A_{s_i}

$$A_{s_i}(e_k(x_j, ..., x_n)) \neq 0 \text{ only if } i = j - 1$$

$$A_{s_j}((e_k(x_j, ..., x_n)) = -e_{k-1}(x_{j+1}, ..., x_n)$$

□

If we also consider the bijections between elements of W^{P_r} and $S(n-d, d)$ we can associate $\beta(e_k(x_{n-d+1}, ..., x_n))$ to ϵ_λ for some partition λ. The Weyl group element $s_{n-d+k}s_{n-d+k-1}...s_{n-d}$ corresponds to the one-line permutation $[1, 2, .., n-d-1, n-d+k, ...]$ where $w(n-d+1)$ through $w(n)$ are taken from the remaining number and put in increasing order. This element then corresponds to the partition $(k, 0, .., 0)$. Thus,

$$\epsilon_{s_{n-d+k-1}...s_{n-d}} = \epsilon_k = c_k(Q).$$

We also note that $\epsilon_{s_{n-d-k+1}...s_{n-d}} = \epsilon_(1, .., 1, 0, ..0) = c_k(S)$, where $(1, ..1, 0, .., 0)$ is the partition with k leading ones and $c_k(S)$ is the k^{th} chern class of the tautological subbundle. We collect the above results and discussions into the following proposition

Proposition 4.1. *Let $L_{n-d} = GL(n-d) \times GL(d) \subset GL(n) = G$ be the Levi subgroup of the maximal parabolic P_{n-d} associated to subset of simple roots $\Delta - \{\alpha_{n-d}\}$. Then the map*

$$\xi_{\omega_1}^{P_{n-d}} : Rep_{\omega_1}^{\mathbb{C}}(L_{n-d}) \to H^*(GL(n-d, n), \mathbb{C})$$

takes,

$$e_k(t_1, ..., t_{n-d}) \mapsto \epsilon_{1^k} = c_k(S^*)$$

and

$$e_k(t_{n-d+1}, ..., t_n) \mapsto \epsilon_k = (-1)^k c_k(Q)$$

Now, consider the map

$$\iota_2^n : Rep_{poly}(Gl(d)) \to Rep_{poly}(GL(n-d)) \otimes Rep_{poly}(GL(d))$$

given by tensoring a $GL(d)$ polynomials representation with the trivial $GL(n-d)$ representation. Then this gives a map from $\xi_{\omega_1}^{n,d} : Rep_{poly}(GL(d)) \mapsto H^*(Gr(n-d,n))$ by composing $\xi_{\omega_1}^{P_{n-d}} \circ i_2$. Now consider the following inclusion of varieties

$$... \to Gr(n-d,n) \to Gr(n+1-d,n+1) \to ...$$

This yields a corresponding sequence

$$... \leftarrow H^*(Gr(n-d,n),\mathbb{C}) \leftarrow H^*(Gr(n-d+1,n+1),\mathbb{C}) \leftarrow ...$$

Note the Chern classes of the universal quotient bundles are stable in this system, i.e. in the map $H^*(Gr(n-d,n)\mathbb{C}) \leftarrow H^*(Gr(n-d+1,n+1),\mathbb{C})$ one has $c_p(Q_n) \leftarrow c_p(Q_{n+1})$. Then let

$$\mathbb{H}(Gr_d) = \varprojlim H^*(Gr(n-d,n),\mathbb{C}),$$

that is the inverse limit in the categrory of graded rings in the above system. Consider the diagram

$$\begin{tikzcd}
 & {} \\
\mathrm{Rep}_{\omega_1}^{\mathbb{C}}(Gl(d)) \arrow[r, "\xi_{\omega_1}^{n,d}\circ\iota_2^n"] \arrow[dr, "\xi_{\omega_1}^{n+1,d}\circ\iota_2^{n+1}"] & H^*(Gr(n-d,n),\mathbb{C}) \arrow[u, "i_n^*"] \\
 & H^*(Gr(n+1-d,n),\mathbb{C}) \arrow[u, "i_{n+1}^*"] \\
 & {} \arrow[u, "i_{n+2}^*"]
\end{tikzcd}$$

Then due to the stability of the chern classes, we have

$$i_{n+1}^* \circ \xi_{\omega_1}^{n+1,d}(e_k(x)) = i_{n+1}^*((-1)^k c_k(Q_{n+1})) = (-1)^k c_k(Q_n) = \xi_{\omega_1}^{n,d}(e_k(x))$$

. Thus, we have a map from

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(GL(d)) \to \mathbb{H}(Gr_d)$$

. Looking at the presentation for the coholomology of $H^*(Gr(n-d,n),\mathbb{C})$, none of the relations hold in the inverse limit. This yields the following theorem

Theorem 4.4. *There is an graded algebra isomorphism between $Rep_{poly}(GL(d))$ and $\mathbb{H}(Gr_d) = \varprojlim H^*(Gr(n-d,n),\mathbb{C}) = \mathbb{C}[\epsilon_1, ..., \epsilon_d]$, given by mapping the k^{th} elementary symmetric polynomial to the class $\epsilon_k = c_k(Q)$.*

We will attempt to derive similar results fro other Lie types in the next chapter.

CHAPTER 5

Types B,C, and G

5.1 Representation Ring of the Classical groups

In this section all results are due to and we follow closely [Ku2, §6]. In accordance with our expectation that the fundamental weight of minimal dynkin index is the most appropriate dominant weight for which to consider the λ-polynomial representation ring, we let $\lambda = \omega_1$ (which is the defining representation for the classical groups $Sp(2n, \mathbb{C}), SO(2n+1, \mathbb{C})$, and $SO(2n, \mathbb{C})$).

Take symmetric forms on $\mathbb{C}^{2n}, \mathbb{C}^{2n+1}$ (resp. an alternating form on $\mathbb{C}^{2n}$) so that $SO(2n)$, $SO(2n+1)$ (resp. $Sp(2n)$) are given respectively by

$$SO(2n) = \{g \in SL_{2n} : (g^t)^{-1} = E_D g E_D^{-1}\}$$

$$SO(2n+1) = \{g \in SL_{2n+1} : (g^t)^{-1} = E_B g E_B^{-1}\}$$

$$Sp(2n) = \{g \in SL_{2n} : (g^t)^{-1} = E_C g E_C^{-1}\},$$

where E_D is the antidiagonal matrix with all its antidiagonal entries 1; E_B is the antidiagonal matrix with all its antidiagonal entries 1 except the $(n+1, n+1)$-th entry which is 2; E_C is the block matrix

$$E_C = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix},$$

where J_n is the antidiagonal $n \times n$ matrix with all its antidiagonal entries 1. (The suffix D, B, C refers to the types of the corresponding groups.)

Depending upon the case, denote E_D, E_B or E_C by the common symbol E. Consider the Springer morphism for these groups with $\lambda = \omega_1$, which is their defining representation. Then, Springer morphism in this case is just the Cayley transform [Ku2, Lemma 10].

Lemma 5.1. *The Springer morphism $\theta : G \to \mathfrak{g}$ for $G = So_{2n}, So_{2n+1}$ or Sp_{2n} is given by*

$$g \mapsto \frac{g - E^{-1} g^t E}{2}, \quad \text{for } g \in G.$$

Proof. The lemma follows immediately since under the decomposition

$$End(V(\omega_1)) = \mathfrak{g} \oplus \mathfrak{g}^{\perp},$$

any $A \in End(V(\omega_1))$ decomposes as

$$A = \frac{(A - E^{-1} A^t E)}{2} + \frac{(A + E^{-1} A^t E)}{2}.$$

□

We choose the maximal tori in $Sp(2n)$, $SO(2n)$ and $SO(2n+1)$ respectively as follows:

$$T_C = T_D = \left\{ \mathbf{t} = \left(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1} \right) : t_i \in \mathbb{C}^* \right\} \tag{5.1}$$

$$T_B = \left\{ \mathbf{t} = \left(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1} \right) : \ t_i \in \mathbb{C}^* \right\}. \tag{5.2}$$

Their associated Cartan subalgebras are then given by

$$\mathfrak{t}_C = \mathfrak{t}_D = \left\{ \bar{\mathbf{t}} = \left(x_1, \ldots, x_n, -x_n, \ldots, -x_1 \right) : x_i \in \mathbb{C} \right\} \tag{5.3}$$

$$\mathfrak{t}_B = \left\{ \bar{\mathbf{t}} = \left(x_1, \ldots, x_n, 0, -x_n, \ldots, -x_1 \right) : \ x_i \in \mathbb{C} \right\}. \tag{5.4}$$

From the description of the Springer morphism given above, we immediately get the following:

Lemma 5.2. *Restricted to the maximal torus as above, we get the following description of the Springer morphism θ_{ω_1} (which can also easily be derived from Proposition 3.1 as the example in §3.2.1 was for type C) :*

(a) $G = SO2n) : \theta(\mathbf{t}) = \left(\frac{t_1-t_1^{-1}}{2}, \ldots, \frac{t_n-t_n^{-1}}{2}, -(\frac{t_n-t_n^{-1}}{2}), \ldots, -(\frac{t_1-t_1^{-1}}{2})\right)$

(b) $G = Sp(2n)$: *Same as in the above case of* $G = So(2n)$.

(c) $G = So(2n) : \theta(\mathbf{t}) = \left(\frac{t_1-t_1^{-1}}{2}, \ldots, \frac{t_n-t_n^{-1}}{2}, 0, -(\frac{t_n-t_n^{-1}}{2}), \ldots, -(\frac{t_1-t_1^{-1}}{2})\right)$.

Recall that $\mathrm{Rep}_{\omega_1}^{\mathbb{C}}(G)$ is isomophic to the ring of Weyl group invariants $S(\mathfrak{t}^*)^W$. Then the above Lemma together with the description of $S(\mathfrak{t}^*)^W$ given in §4.1 yields the following result

Lemma 5.3.
- *Let $G = SO(2n+1)$ or $Sp(2n)$. Then the polynomial representation ring is given by*

$$Rep_{\omega_1}^{\mathbb{C}}(G) \simeq \mathbb{C}_{sym}\Big[\Big(\frac{t_1 - t_1^{-1}}{2}\Big)^2, ..., \Big(\frac{t_n - t_n^{-1}}{2}\Big)^2\Big]$$

- *Let $G = SO(2n)$. Then the polynomial representation ring is given by*

$$Rep_{\omega_1}^{\mathbb{C}}(SO(n)) = \mathbb{C}_{sym}\Big[\Big(\frac{t_1 - t_1^{-1}}{2}\Big)^2, ..., \Big(\frac{t_{n-1} - t_{n-1}^{-1}}{2}\Big)^2, \Big(\frac{t_1 - t_1^{-1}}{2}\Big)...\Big(\frac{t_n - t_n^{-1}}{2}\Big)\Big]$$

Furthermore, like the standard representation rings $Rep(G)$ the ω_1-polynomial rings of type B and C also carry the structure of a special λ-ring as mentioned in §2.1. We give the following more complete definition, from [AT, §1].

Definition 5.1. A *special λ-ring* is, by definition, a commutative ring R with identity with a map

$$\lambda : R \to R[[q]], \;\; x \mapsto \sum_{i \geq 0} \lambda^i(x) q^i,$$

which satisfies the following:

(1) $\lambda^0(x) = 1$

(2) $\lambda^1(x) = x$, for all $x \in R$

(3) $\lambda(x + y) = \lambda(x)\lambda(y)$, for all $x, y \in R$

(4) $\lambda(1) = 1 + q$, and

(5) There are universal (independent of R) polynomials P_k and $P_{k,l}$ over $\mathbb{Z}$ such that

$$\lambda^k(xy) = P_k\left(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y)\right)$$

$$\text{and} \quad \lambda^k(\lambda^l x) = P_{k,l}\left(\lambda^1(x), \ldots, \lambda^{kl}(x)\right), \quad \text{for all} \quad k, l \geq 1.$$

Then as mentioned in Chapter 2, the operation

$$\lambda^i([V]) = [\wedge^i(V)]$$

turns $Rep(G)$ into a special $\lambda - ring$ and extend it to virtual representations by property (3). Then we have the following Theorem [Ku2, Lemma 15, Lemma 16, Lemma 18] for the classical groups.

Theorem 5.1. *a) Let $G = SO(2n+1)$ or $Sp(2n)$. Then, the subring $Rep(G) \subset Rep(G)$ of ω_1-polynomial characters is a special λ-subring, where*

$$Rep(G) := \textbf{Rep}^{\mathbb{C}}(G) \cap Rep(G).$$

b) Moreover, the character

$$\chi(\mathbf{t}) = \sum_{i=1}^{n} \left(t_i^2 + t_i^{-2}\right) \in (G), \quad \textit{for } \mathbf{t} \in T_C \ \textit{ given by } (5.1) \textit{ or } \mathbf{t} \in T_B \ \textit{ given by } (5.2)$$

generates (G) as a λ-ring, i.e., $\chi(\mathbf{t})$, $\lambda^2(\chi(\mathbf{t})), \ldots, \lambda^n(\chi(\mathbf{t}))$ generate the ring (G).

In the case $G = Sp(2n)$, $\chi(\boldsymbol{t})$ is the character of the virtual representation $[S^2V] - [\bigwedge V]$, where $[V]$ is the standard representation of $Sp(2n)$.

In the case $G = SO(2n+1)$, $\chi(\boldsymbol{t})$ is the character of the virtual representation $[S^2V] - [\bigwedge V] - [\epsilon]$, where $[V]$ is the standard representation of $So(2n+1)$ and $[\epsilon]$ is the trivial one-dimensional representation.

c) The ring $Rep_{poly}(SO(2n))$ is not a λ-subring of Rep(G). (Consider the function $\prod_{i=1}^{n}(t_i - t_i^{-1}) \in Rep_{poly}(SO(2n))$

d) For $SO(2n)$ $(n \geq 3)$, $\prod_{i=1}^{n}(t_i - t_i^{-1})$ is the character of the virtual represnentation $[V(2\omega_n)] - [V(2\omega_{n-1})]$ where $\omega_i - poly$ is the i^th fundamental representation of $Spin(2n)$.

e) For $G = Sp(2n)[n \geq 2],\ SO(2n+1)[n \geq 3],\ SO(2n)[n \geq 4]$, no non-trivial irreducible representation $[V(\lambda)]$ belong to Rep$^{\mathbb{C}}_{\omega_1}(G)$.

Note the contrast of Theorem 5.1(e) with the type A case in which $V[\omega_i] \in \text{Rep}^{\mathbb{C}}_{\omega_1}(Gl(n))$.

5.2 Type C

We try to generalize Theorem 4.4 to type C. We will describe the cohomology of isotropic grassmannians, how theorem 4.2 specializes to said case, and the extension to the inverse limit.

5.2.1 Cohomology of IG(n-k,2n)

In this section we will describe the additive and multiplicative structure of the cohomology ring of isotropic grassmannians $X = IG(n-k, 2n)$. Again we fix an integer k, the reason for this will become apparent when we want to derive a partial analogue to Theorem 4.4 in type C. As for type A we have parametrizations of Schubert varieties, classes, and Poincare dual classes via index sets, partitions, and minmal length coset representatives of $W/W_{P_{n-k}}$. There are special Schubert classes and Pieri and Giambelli formulas as well. The ring structure can also be described by Chern classes of certain tautological bundles.

References for the following parametrizations can be found in [BK2, BKT1, LL, PR1, T2, T7].

Equip $V = \mathbb{C}^{2n}$ with a non-degenerate skew-symmetric bilinear form ϑ. Fix a complete *isotropic* flag $F_\bullet$,

$$0 = F_0 \subset F_1 \subset ... \subset F_{2n} = V$$

where $F_i = F_{2n-i}^{\perp}$ with respect to ϑ. Note, F_n is a maximal isotropic subspace. Then, we define the isotropic grassmannian $IG(n-k, 2n)$ as,

$$IG(n-k, 2n) := \left\{\Omega \in Gr(n-k, 2n) :\ \vartheta(v, v') = 0,\ \forall v, v' \in \Omega\right\}$$

There exists numerous parametrizations of the Schubert varieties in the isotropic grassmannian. For one, they are parametrized by index sets $\{I : 1 \leq p_{i_1} < ... < p_{i_{n-k}} \leq 2n\}$ such that $p_i + p_j \neq 2n + 1$. The corresponding Schubert cell is given by

$$X_I^\circ = \{\Omega \in X |\ \Omega \cap F_p \supsetneq \Omega \cap F_{p-1}\},$$

and the Schubert variety is given by

$$X_I(F_\bullet) = \{\Omega \in X | dim(\Omega \cap F_{p_j}) \geq j,\ \forall 1 \leq j \leq n-k\}.$$

We also note that the dual index set is then given by $\check{I}$ given by setting $\check{p_j} = 2n + 1 - p_{n-k+1-j}$. These index sets of course correspond to minimal length coset representatives of course.

Note that $Sp(2n)$ can be realized as the fixed point subgroup G^σ of $G = SL(2n)$ under the involution $\sigma(A) = E(A^t)^{-1}E^{-1}$ where $E = E_C$ as in §5.1. Here, we follow [BK2]. If $T^A \subset B^A$ are the maximal torus and Borel subgroup of $SL(2n)$, then $T^\sigma = T$, and $B^\sigma = B$ as in the previous section. Let$\Delta_C = \{\beta_1, ..., \beta_n\}$ be the simple roots of $Sp(2n)$. Then, $\beta_i = \alpha_i|\mathfrak{t}$ where $\{\alpha_1, ..., \alpha_{2n-1}\}$ are the simple roots of $SL(2n)$. The corresponding simple coroots are given by

$$\check{\beta}_i = \check{\alpha}_i + \check{\alpha}_{2n-1},\ \text{for } 1 \leq i \leq n,$$

and

$$\check{\beta}_n = \check{\alpha}_n$$

Under the inclusion $W_C \subset S_{2n}$ we have that the simple reflections of $Sp(2n)$ are given by

$$s_i = r_i r_{2n-i}\ \ if\ \ 1 \leq i \leq n-1$$

$$= r_n\ \ if\ \ i = n$$

where r_i is the i^{th} simple reflection for $Sl(2n)$. The Weyl group W_{C_n} can be identified with the subset of $W_{A_{2n}}$ invariant under σ:

$$\{(a_1, ..., a_{2n}) \in S_{2n} : a_{2n+1-i} = 2n + 1 - a_i \; \forall 1 \leq i \leq 2n\}$$

. Consider the parabolic weyl subgoup generated by $\Delta_C - \{\beta_{n-k}\}$. Then the minimal length coset representatives of $W_C/W_{C,P_{n-k}}$ are can be identified with the set

$$\mathrm{I}(n-k, 2n) = \{I := 1 \leq p_1 < \cdots < p_n \leq 2n \;\; and \; I \cap \overline{I}) = \emptyset\},$$

where $\overline{I} = \{2n+1-p_1, ..., 2n+1-p_{n-k}\}$. But this is just an index set. It represents the permutation in S_{2n} given by taking $p_{n-k}+1, ..., p_n = [n]\backslash(I \sqcup \overline{I})$ and setting $p_{2n+1-i} = 2n+1-p_i$

Finally we can also associate a k-strict permutation to an index set or Weyl group element. First, a partition λ is said to be $k-$strict if no part greater than k is repeated (i.e. $\lambda_j > k \Rightarrow \lambda_{j+1} < \lambda_j$. This is the combinatorial object with which Buch, Kresch, and Tamvakis derive their Pieri and Giambelli rules in both the classical and quantum cohomology of the isotropic grassmannian. The bijection between index sets and k-strict partitions (contained in an $(n-k) \times (n+k)$) rectangle is defined as follows [BKT1, 4.1]. Let $I = \{1 \leq i_1 < ... < i_{n-k} \leq 2n\}$ be an index set. Then

$$\lambda_j(I) = n + k + 1 - I_j + \{i < j : I_i + I_j > 2n + 1\}.$$

In the reverse, given a k-strict partition $\lambda = (\lambda_1, ..., \lambda_{n-k})$ associates to an index set $I(\lambda)$ by

$$I_j(\lambda) := n + k + 1 - \lambda_j + \{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}.$$

The Schubert class of codimension $|\lambda|$ is then simply $X_\lambda = X_{I(\lambda)}$ where the Schubert variety with index set I is defined as above. The dual $\check{\lambda}$ is the $k-$ strict partition associated to the dual index set. The set of barred permuations in $W/W_{P_{n-k}}$ can also be bijectively associated to $k-$strict partitions [T7, 4.2].

For an example, let $n = 5, \; k = 2$. Then $s_2 s_3 \in W/W_{P_3}$ is associated to the index set $(1, 3, 4)$. The associated $k-$strict partition is then $(7, 5, 4)$. The dual index set is $(7, 8, 10)$ and the dual partition is $(1,1,0)$. For our purposes, we actually wish to associate to a Weyl group element w a partition λ_w such that $l(w) = |\lambda|$. So in the above case, we will write $\epsilon_{s_3s_2} = \epsilon_{(1,1,0)}$. In general, when we write $\epsilon_w = \epsilon_\lambda \in H^{l(w)=|\lambda|}(IG(n-k,n))$ the partition we refer to can be arrived at as follows. The index set is $I_w = \{w(1), ..., w(n-k)\}$. Then take the dual index set $\check{I}$ and take $\lambda(\check{I})$. We write $\lambda(w)$ for this partition, and will interchange between writing ϵ_w and ϵ_λ. The classes ϵ_i for $1 \leq i \leq n+k$ are referred to as the special Schubert classes.

We will also need the Giambelli formula of [BKT2] and we follow closely some of the exposition given their. A fundamental insight of theirs is that classical Giambelli formulas can be restated in terms of Young's raising operators [**?**](see also [T3]). For any integer sequence $(a_1, a_2, ...)$ with finite support and $i < j$, they define $R_{ij} = (\alpha_1, ..., \alpha_i + 1, ..., \alpha_j - 1, ..)$. Then a raising operator is any monomial in the $R'_{ij}s$. Setting $m_\alpha = \prod_i \epsilon_i$, then $Rm_\alpha = m_{R_\alpha}$ for any raising operator. They show that the classical Giambelli formula for $H^*(Gr(n-k,n))$ can be restated as

$$\epsilon_\lambda = \prod_{i<j}(1 - R_{ij})m_\lambda.$$

For example, in $H^*(Gr(3,5))$ (with the convention $\epsilon_0 = 1$ and $\epsilon_i = 0$ for $i < 0$ and $i > 5$) one has

$$\epsilon_{(3,2,1)} = (1-R_{12})(1-R_{13})(1-R_{23})m_{321}$$

$$= (1 - R_{12} - R_{23} - R_{13} + R_{12}R_{23} + R_{12}R_{13} + R_{13}R_{23} - R_{12}R_{13}R_{23})m_{321}$$

$$= m_{321} - m_{411} - m_{4,2,0} + m_{4,2,0} + m_{5,1,-1} + m_{4,3,-1} - m_{5,2,-1} = \epsilon_3\epsilon_2\epsilon_1 - \epsilon_4\epsilon_1^2$$

To any $k-$strict partition λ the associated raising operator is

$$R^\lambda = \prod_{i<j}(1-R_{ij}) \prod_{\lambda_i + \lambda_k > 2k+j-i} \frac{1}{1+R_{ij}}$$

. The Giambell formula of Buch, Kresch, and Tamvakis can then be simply stated as

Theorem 5.2. *[BKT2, Theorem 1] For any k-strict partition ϵ_λ, we have $\epsilon_\lambda = R^\lambda m_\lambda$ in the cohomology ring of $IG(n-k, 2n)$.*

Indeed, in the computations we need it for the partitions will only have 2 non-zero parts with $\lambda_1+\lambda_2 < 2k$ so the Giambelli formula reduces to $\epsilon_{(\lambda_1,\lambda_2)} = \epsilon_{\lambda_1}\epsilon_{\lambda_2} - \epsilon_{\lambda_1+1}\epsilon_{\lambda_2-1}$

As in type A we have the following short exact sequence of vector bundles,

$$0 \to S \to V_C \to Q \to 0,$$

where V_C is the trivial bundle of rank $2n$, S is the tautogogical subbundle of rank $n-k$, and Q is the tautological quotient bundle of rank $n+k$. Then the Schubert classes ϵ_i equal to the i^{th} Chern class of the qoutient bundle $c_i(Q)$, and these classes generate the cohomology ring. We give reduced decompositions so that ϵ_i in the next section such that $\epsilon_i = \epsilon_w \in H^*(IG(n-k,2n),\mathbb{C})$. Also, like the type A case there is a Pieri formula for the product of any Schubert class with that of a special Schubert class. Then we have the following presentation of the cohomology ring due to [BKT2, Theorem 1.2]. By convention we set $\epsilon_0 = 1$ and $\epsilon_p = 0$ if $p < 0$ or $p > n+k$.

Theorem 5.3. *The cohomology ring $H^*(IG(n-k,2m),\mathbb{C})$ is presented as the quotient of the polynomial ring $\mathbb{C}[\epsilon_1, ..., \epsilon_{n+k}]$ by the relations*

$$det(\epsilon_{1+j-i})_{1\le i<j\le r}, \quad n-k+1 \le r \le n+k,$$

and

$$\epsilon_r^2 + \sum_{i=1}^{n+k-r} (-1)^i \epsilon_{r+i}\epsilon_{r-i} = 0, \quad k+1 \le r \le n$$

As in type A, the determinantal relations come from the Whitney sum formula $c(S)c(Q) = 1$. The quadratic relations come from considering that the symplectic form gives a pairing $SQ \to \mathcal{O}$ which yields an injection $S \to Q^*$. Chern classes $c_j(Q^*/S)$ vanish for $j > k$ and one can deduce that $c(Q)c(Q^*)$ vanishes in degree $> 2k$.

5.2.2 Theorem 4.2

As for type A, we aim to explicitly determine the map in Theorem 4.2 for $G = Sp(2n)$ and any maximal parabolic P_{n-k}. Take $V = \mathbb{C}^{2n}$ and $IG(n-k, 2n)$ as in the previous section. Here we follow and expand on [Ku2, §7].

We take $B_C := B \cap Sp_{2n}$ as the Borel subgroup of Sp_{2n}, where B is the standard Borel subgroup of SL_{2n} consisting of upper triangular matrices of determinant 1. Then, $IG(n-k, 2n)$ is the quotient Sp_{2n}/P_{n-k} of Sp_{2n} by the standard maximal parabolic subgroup P_{n-k} with $\Delta \setminus \{\alpha_r\{n-k\}$ as the set of simple roots of its Levi component L_{n-k}. (We take L_{n-k} to be the unique Levi subgroup of P_r containing T_C.)

$$L_{n-k} \simeq GL(n-k) \times Sp(2k).$$

From Lemma 5.2, we have

$$\theta_{\omega_1}(\mathbf{t}_C) = \left(\frac{t_1 - t_1^{-1}}{2}, ..., \frac{t_n - t_n^{-1}}{2}, -(\frac{t_n - t_n^{-1}}{2}), ..., -(\frac{t_1 - t_1^{-1}}{2})\right)$$

Also, recall that $Rep_{\omega_1}^{\mathbb{C}}(L_{n-k}) \simeq S(\mathfrak{t}_C^*)^{W_P}$. Using the fundamental invariants from §4.1.1 we find that the representation ring is given by,

$$\text{Rep}_{\omega_1}^{\mathbb{C}}(L_{n-k}) \simeq \mathbb{C}_{sym}\Big[\Big(\frac{t_1 - t_1^{-1}}{2}\Big), ..., \Big(\frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2}\Big)\Big]$$

$$\otimes_{\mathbb{C}} \mathbb{C}_{sym}\Big[\Big(\frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2}\Big)^2, ..., \Big(\frac{t_n - t_n^{-1}}{2}\Big)^2\Big]$$

where $\mathbb{C}_{sym}$ denotes the subalgebra of the polynomial ring consisting of symmetric polynomials. Further, by Theorem 5.1,

$$\mathbb{C}_{sym}[(\frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2})^2, \ldots, (\frac{t_n - t_n^{-1}}{2})^2]$$

is generated (as a $\mathbb{C}$-algebra) by the virtual representations:

$$\{\lambda^d\left([S^2(V_{2(n-r)})] - [\Lambda^2(V_{2(n-r)})]\right)\}_{1\leq d\leq k},$$

where $V_{2k} = \mathbb{C}^{2k}$ is the standard representation of $Sp(2k)$ and λ is the λ-ring structure on $Rep(G)$.

The following theorem [Ku2, $Proposition 19$] partially determined the homomorphism of Theorem 4.2

Theorem 5.4. *The map $\xi^{P_{n-k}} : Rep^{\mathbb{C}}_{poly}(L^C_{n-k}) \to H^*(IG(n-k, 2n), \mathbb{C})$ of Theorem 4.2 takes*

$$\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \to \epsilon_{s_{n-k}},$$

and

$$\frac{1}{4}\left[(t_{n-k+1} - t_{n-k+1}^{-1})^2 + \cdots + (t_n + t_n^{-1})^2\right] \to \epsilon^2_{s_{n-k}} + 2\sum_{j=n-k+1}^{n-1} \epsilon^2_{s_j} + \epsilon^2_{s_n} - 2\sum_{j=n-k}^{n-1} \epsilon_{s_j}\epsilon_{s_{j+1}}$$

Proof. For $1 \leq n$, let $x_i : \mathfrak{t} \to \mathbb{C}$ be the linear map which takes

$$diag(x_1, ..., x_n, -x_n, ..., -x_1) \to x_i$$

. Then by Lemma 5.2, the homomorphism $\theta^*_{\omega_1|T} : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$ induced from the Springer morphism θ_{ω_1} takes

$$x_1 + \cdots + x_{n-k} \to \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}).$$

However, note that the weight $x_1 + ... + x_{n-k}$ is the first fundamental weight ω_{n-k}. Thus,

$$\beta \circ (\theta^*_{\omega_1|T})^{-1}(\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) = \beta(x_1 + \cdots + x_{n-k}) = \beta(\omega_{n-k}) = \epsilon_{s_{n-k}},$$

where the last inequality comes from the fact that by 2.2

$$\beta(\omega_{n-k}) = \sum_{\alpha \in \Delta} A_\alpha(\omega_{n-k})\epsilon_{s_\alpha}$$

. Note that $s_{n-k}(\omega_{n-k}) = \omega_{n-k} - \alpha_{n-k}$ and $s_i(\omega_{n-k}) = 0$ otherwise. Then $\beta(\omega_{n-k}) = A_{\alpha_{n-k}}(\omega_{n-k})\epsilon_{s_{n-k}} = \epsilon_{s_{n-k}}$. Indeed, in general for $\beta : S(\mathfrak{t}^*) \to H^*(G/B)$, one has that $\beta(\omega_j) = \epsilon_{s_j}$

Similiarly, under $(\theta^*_{\omega_1|T})^{-1}$,

$$\frac{1}{4}\bigg((t^2_{n-k+1} - t_{n-k+1})^{-1} + \cdots + (t_n + t_n^{-1})^2\bigg) \to x^2_{n-k} + \cdots + x^2_n$$

Since in type C we have $\omega_i = \sum_{j=1}^{i} x_i$, then we can right the coordinate functions in the fundamental weight basis $x_i = \omega_i - \omega_{i-1}$. So, $x^2_{n-k+1} + \cdots + x^2_n = (\omega_{n-k+1} - \omega_{n-k})^2 + \cdots + (\omega_n - \omega_{n-1})^2$. Then from the remark above it is clear that $\xi^{P_{n-k}}$ takes

$$\frac{1}{4}\bigg((t^2_{n-k+1} - t_{n-k+1})^{-1} + \cdots + (t_n + t_n^{-1})^2\bigg) \to (\epsilon_{s_{n-k+1}} - \epsilon_{s_{n-k}})^2 + \cdots + (\epsilon_{sn} - \epsilon_{s_{n-1}})^2,$$

which expands to give the stated result. □

Note that the term

$$\epsilon^2_{s_{n-k}} + 2\sum_{j=n-k+1}^{n-1} \epsilon^2_{s_j} + \epsilon^2_{s_n} - 2\sum_{j=n-k}^{n-1} \epsilon_{s_j}\epsilon_{s_{j+1}}$$

is not written in the basis $\{\epsilon_w : \ w \in P^{n-k}\}$ of $H^*(Sp(2n)P_{n-k}, \mathbb{C}) = H^*(IG(n-k, 2n), \mathbb{C})$. Indeed, s_{n-k} is the only simple reflection in W^P_{n-k}. We can in theory use Chevalley's formula (§2.6 to expand the quadratic terms into the additive Schubert basis and all nonvanishing terms should be elements of W^P_{n-k}. Since the terms are all low degree this is feasible. We have the following lemma on products $\epsilon_{s_i}\epsilon_{s_j}$ in $H^*(Sp(2n), \mathbb{C})$ which is just a corollary to Chevalley's theorem.

Lemma 5.4. *1. If $|i-j| \geq 2$, then*

$$\epsilon_i \epsilon_j = \epsilon_{s_i s_j}$$

2. If $i, i+1 \neq n$,

$$\epsilon_{s_i} \epsilon_{s_{i+1}} = \epsilon_{s_i s_{i+1}} + \epsilon_{s_{i+1} s_i}$$

3. If $i \neq 1, n$,

$$\epsilon_i^2 = \epsilon_{s_{i-1} s_i} + \epsilon_{s_i s_{i+1}}$$

4. If $i = 1$,

$$\epsilon_{s_1}^2 = \epsilon_{s_2 s_1}$$

5. If $i = n$,

$$\epsilon_{s_n}^2 = 2\epsilon_{s_{n-1} s_n}$$

Using the Lemma 5.4, the quadratic term simplifies to

$$\epsilon_{s_{n-k}}^2 + 2 \sum_{j=n-k+1}^{n-1} \epsilon_{s_j}^2 + \epsilon_{s_n}^2 - 2 \sum_{j=n-k}^{n-1} \epsilon_{s_j} \epsilon_{s_{j+1}} = \epsilon_{s_{n-k-1} s_{n-k}} - \epsilon_{s_{n-k+1} s_{n-k}},$$

with $\{s_{n-k+1} s_{n-k}, s_{n-k-1} s_{n-k}\} \in W_{n-k}^P$. In order to fully realize the map $\xi_{n-k\omega_1}^P$, the images of the elementary symmetric plynomials e_k in the above torus variables must be determined. The above strategy using the Chevalley formula would be difficult as the degrees get large. Rather, as in type A we want to determine the image of $\xi_{\omega_1}^{P_{n-k}}$in terms of the Chern classes $c_k(Q)$ of the tautological quotient bundle. Indeed, we have the following expansion of the previous theorem.

Theorem 5.5. *The map $\xi^{P_{n-k}} : Rep_{poly}^{\mathbb{C}}(L_{n-k}^C) \to H^*(IG(n-k, 2n), \mathbb{C})$ of Theorem 4.2 takes*

$$e_i \left[(\frac{t_1 - t_1^{-1}}{2}), ..., (\frac{t_{n-k} - t_{n-k}}{2}) \right] \to c_i(S) = \epsilon_{(1)^i}(S) = \epsilon_{s_{n-k+i-1} \cdots s_{n-k}}$$

For $1 \leq i \leq n-k$. Now, let $\epsilon_i = c_i(Q)$ and let $\epsilon_0 = 1$ and $\epsilon_p = 0$ for $p < 0$ or $p > n+k$. Then

$$e_i\Big[(\frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2})^2, ..., (\frac{t_n - t_n}{2})^2\Big] \to \epsilon_i^2 + 2\sum_{j=1}^{n+k-r}(-1)^i \epsilon_{i+j}\epsilon_{i-j}$$

for $1 \leq i \leq k$

Proof. The $'Type\ A'$ part follows exactly as before. We see that

$$(\theta^*_{\omega_1|T})^{-1}\Big(e_i\Big[(\frac{t_1 - t_1^{-1}}{2}), ..., (\frac{t_{n-k} - t_{n-k}}{2})\Big]\Big) = e_i(x_1, ..., x_{n-k})$$

Assume $k > 1$, then

$$\beta(e_i(x_1, ..., x_{n-k})) = \sum_{l(w)=i} A_w(e_i(x_1, ..., x_{n-k})\epsilon_w.$$

Note that we are taking $\Delta = \{e_1^* - e_2^*, ..., e_{n-2}^* - e_{n-1}^*, 2e_n^*\}$ to be the simple roots.

Then, just as in §4.3.1, we have $A_w(e_i(x_1, ..., x_{n-k})) = \delta_{w,w_{1^i}}$, where $w_{1^i} = s_{n-k-1+i} \dots s_{n-k-1}s_{n-k}$. Furthermore, $\epsilon_{w_{1^i}}$ is the i^{th} Chern class of the tautological subbundle [LL, 4.1].

Now let $k = 0$. So $IG(n, 2n)$ is the variety of maximal isotropic planes in $\mathbb{C}^{2n}$. Then its clear $A_{s_m}(e_1(x_1, ..., x_n)) = 0$ if $s_m \neq s_n$. Otherwise,

$$A_{s_n}(e_i(x_1, ..., x_n)) = \frac{e_i(x_1, ..., x_n) - e_i(x_1, ..., -x_n)}{2x_n}$$

Recall the following useful identity for elementary symmetric polynomials,

$$e_i(x_1, ..., x_m) = e_i(x_1, ..., x_{n-1}) + x_n(e_{i-1}(x_1, ..., x_{n-1}))$$

. Then,

$$A_{s_n}(e_i(x_1, ..., x_n)) = \frac{e_i(x_1, ..., x_{n-1}) + x_n e_{i-1}(x_1, ..., x_{n-1}) - e_i(x_1, ..., x_n) + x_n e_{i-1}(x_1, ..., x_{n-1})}{2x_n}$$

$$= \frac{2x_n e_{i-1}(x_1, ..., x_n)}{2x_n} = e_{i-1}(x_1, ..., x_{n-1})$$

From here the result follows as before.

The $'typeC'$ part of the above theorem is trickier. First, we examine the polynomials associated to the Chern classes of the tautological quotient bundle. We note [LL, Remark 4.3] that for $1 \leq i \leq k$, $c_i(S) = \epsilon_{s_{n-k+i-1} \cdots s_{n-k}}$ as before. For $i \geq k+1$, then $c_i(S) = \epsilon_{s_{n-i+k+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{n-k}}$. Let $1 \leq i \leq k$, then $\beta(e_i(x_{n-k+1}, ... x_n)) = \epsilon_{s_{n-k+1-1} \cdots s_{n-k}} = c_i(Q)$. This is a type A element and so the proof is the same as for the quotient bundle over the grassmannian. The polynomials mapping to elements $c_i(Q)$ for $i \geq k$ are certain interpolations of Schur-Q and elementary symmetric polynomials called theta functions. These were developed in [BKT2, §5] (see also [W, TW]). Nevertheless, we observe how to compute $\beta(e_i(x_{n-k+1}^2, ..., x_n^2))$. Under the association $\epsilon_w = \epsilon_\lambda$ described in §5.2.1. The idea is to show that $A_w(e_i(x_{n-k+1}^2, ..., x_n^2)) = \pm 1$ if $\lambda(w) = (\lambda(w)_1, \lambda(w)_2) = (2i - j + 1, j - 1)$ for $1 \leq j \leq i + 1.$, and $A_w(e_i(x_{n-k}^2, ..., x_n^2)) = 0$ otherwise. For example, if $i = 3$ then the partitions which show up are $(6, 0),\ (5, 1),\ (4, 2),\ (3, 3)$. Then in general we have

$$\beta(e_i(x_{n-k+1}^2, ..., x_n^2)) = \sum_{j=1}^{i+1} (-1)^{i+j-1} \epsilon_{(2i-j+1, j-1)}$$

Furthermore, applying to the Giambelli formula,

$$\epsilon_{(2i+1-j, j-1)} = \epsilon_{2i+1-j}\epsilon_{j-1} - \epsilon_{2i+2-j}\epsilon_{j-2}$$

and simplifying then yields

$$(e_i(x_{n-k}^2, ..., x_n^2)) = \epsilon_i^2 + 2\sum_{j=1}^{j=i} \epsilon_{i-j}\epsilon_{i+j}.$$

To simplify notation, we let $l = n - k$. Then we want to evaluate $A_w(e_i(x_{l+1}^2, ..., x_n^2))$ for words $w \in W^P$ with $l(w) = i^2$. We have

$$A_{s_l} e_i(x_{l+1}^2, ..., x_n^2) = (-x_l - x_{l+1}) e_{i-1}(x_{l+2}^2, ..., x_n^2)$$

and $A_{s_j}e_i(x_{l+1}^2, ..., x_n^2) = 0$ if $j \neq l$. From here the options are $A_{s_{l+1}}$ or $A_{s_{l-1}}$. If we apply $A_{s_{l-1}}$, using the Leibniz formula we get.

$$A_{s_{l-1}s_l}(e_i(X^2)) = A_{s_{l-1}}(-x_l - x_{l+1})e_{i-1}(x_{l+2}^2, ..., x_n^2) + s_{l-1}(-x_{l-1} - x_l)A_{s_{l-1}}e_{i-1}(x_{l+2}^2, ..., x_n^2)$$

But $A_{l-1}e_{i-1}(x_{l+2}^2, ..., x_n^2) = 0$ and we have

$$A_{s_{l-1}s_l}e_i(x_{l+1}^2, ..., x_n^2) = e_{i-1}(x_{l+2}^2, ..., x_n^2).$$

From here one must apply $A_{s_{l+1}}$ and we are essentially back where we started. We remind the reader that if a word $\tilde{w}$ is not reduced, then $A_{\tilde{w}=0}$. We also adopt a preferred reduced decomposition in which 'lower' reflections are moved to the right if possible. I.e., if $i < j$ and s_i and s_j commute we will move s_i to the right of s_j if possible via the commutation or braid relations.

Now, given the above, we will prove the theorem for $k = 2$ and then proceed by induction. So, we want to consider find $w \in W$ such that $A_w e_2(x_{n-1}^2, x_n^2) \neq 0$. From above we have

$$\beta(e_1(x_{n-1}, x_n) = \epsilon_{s_{n-3}s_{n-2}} - \epsilon_{s_{n-1}s_{n-2}}$$

The first operator applied to $e_2(x_{n-1}^2, x_n^2)$ must be $A_{s_{n-2}}$ as above which gives

$$A_{s_{n-2}}e_2(x_{n-1}^2, x_n^2) = (-x_{n-2} - x_{n-1})(x_n^2).$$

Choosing the lowest reduced decomposition, apply $A_{s_{n-3}}$,

$$A_{s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = x_n^2.$$

From here only $A_{s_{n-1}}$ can be applied to give

$$A_{s_{n-1}s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = -x_{n-1} - x_n.$$

From here apply $A_{s_{n-2}}$ to get

$$A_{s_{n-2}s_{n-1}s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = 1,$$

or apply A_{s_n} to get

$$A_{s_n s_{n-1}s_{n-3}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = \frac{-x_{n-1} - x_n + x_{n-1} - x_n}{2x_n} = -1.$$

If instead after $A_{s_{n-2}}$ we were to apply $A_{s_{n-1}}$ we would get

$$A_{s_{n-1}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = -x_n^2 + (x_{n-2} + x_n)(x_{n-1} + x_n).$$

Then the only choice (which we have not seen before under preferred reduced decomposition) is A_n,

$$A_{s_n s_{n-1}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = x_{n-1} + x_{n-2}.$$

Finally again our only choice to produce a new word is $A_{s_{n-1}}$,

$$A_{s_{n-1}s_n s_{n-1}s_{n-2}}e_2(x_{n-1}^2, x_n^2) = 1$$

. Thus, collecting the above gives

$$\beta(e_2(x_{n-1}^2, x_n^2)) = \epsilon_{s_{n-2}s_{n-1}s_{n-3}s_{n-2}} - \epsilon_{s_n s_{n-1}s_{n-3}s_{n-2}} + \epsilon_{s_{n-1}s_n s_{n-1}s_{n-2}}.$$

In terms of $k-$strict partitions this give

$$\beta(e_2(x_{n-1}^2, x_n^2)) = \epsilon_{\ydiagram{2,2}} - \epsilon_{\ydiagram{3,1}} + \epsilon_{\ydiagram{4}}$$

Then, under the Giambelli formula of [BKT2] one has in this case $\epsilon_\lambda = (1 - R_{12})m_\lambda$, or

$$\beta(e_2(x_{n-1}^2, x_n^2)) = (\epsilon_2^2 - \epsilon_3\epsilon_1) - (\epsilon_3\epsilon_1 - \epsilon_4) + \epsilon_4 = \epsilon_2^2 - 2\epsilon_3\epsilon_1 + 2\epsilon_4$$

$\square$

Now we are ready to state our (weaker) analog to Theorem 4.4.

5.2.3 Inverse Limit

As in type A, define the stable cohomology ring [BKT3, §1.3] as

$$\mathbb{H}(IG_k) = \varprojlim H^*(IG(n-k, 2n), \mathbb{C})$$

as the inverse limit in the category of graded rings of the inverse system

$$\cdots \leftarrow H^*(IG(n-k, 2n), \mathbb{C}) \leftarrow H^*(IG(n-k+1, 2n+2), \mathbb{C}) \leftarrow \dots$$

This ring has an additive basis of Schubert classes ϵ_λ for each $k-$ strict partition λ. There is a natural surjective ring homorphism $\mathbb{H}(IG_k) \to H^*(IG(n-k, 2n), \mathbb{C})$ given by mapping ϵ_λ to ϵ_λ whenever λ fits in a $(n-k) \times (n+k)$ rectangle and to zero otherwise. Furthermore, from the presentation of the ring $H^*(IG(n-k, n), \mathbb{C})$ (Theorem 5.3), none of the determinantal relations hold in the inverse limit. So, $\mathbb{H}(IG_k)$ is isomorphic to the polynomial ring $\mathbb{C}[\epsilon_1, \epsilon_2, ...]$ modulo the relations

$$\epsilon_m^2 + 2\sum_{i=1}^{m}(-1)^i \epsilon_{m+i}\epsilon_{m-i}$$

for $m > k$. To get a map from $\text{Rep}^{\mathbb{C}}_{\omega_1}(SP(2k))$ to $\mathbb{H}(IG_k)$ we map a polynomial $f(h) \in \text{Rep}^{\mathbb{C}}_{\omega_1}(Sp(2k)) = \mathbb{C}[(\frac{h_1-h_1^{-1}}{2})^2, ..., (\frac{h_k-h_k^{-1}}{2})^2]$ to $1 \otimes f(t) \in \text{Rep}^{\mathbb{C}}_{\omega_1}(L^C_{n-k})$, where $f(t)$ is the same polynomial written in the variables $(\frac{t_i-t_i^{-1}}{2})^2$ for $n-k+1 < i < n$. Then we have the map $\xi_{n,k} := \xi^{P_{n-k}} \circ \iota_2 : \text{Rep}^{\mathbb{C}}_{\omega_1}(SP(2k)) \to$

$H^*(IG(n-k,2n),\mathbb{C})$. Consider the following diagram

$$\begin{tikzcd}
& {} \\
\mathrm{Rep}^{\mathbb{C}}_{\omega_1}(Sp(2k)) \arrow[r, "\xi_{n,k}"] \arrow[dr, "\xi_{n+1,k}"] & H^*(IG(n-k+1,2n+2),\mathbb{C}) \arrow[u, "\pi_n^*"] \\
& H^*(IG(n-k,2n),\mathbb{C}) \arrow[u, "\pi_{n+1}^*"] \\
& {} \arrow[u, "\pi_{n+2}^*"]
\end{tikzcd}$$

This map is compatible with the system since Chern classes are stable.

Then we have the following analog to 4.2,

Theorem 5.6. *Define the map $\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\omega_1}(Sp(2k)) \to \mathbb{H}(IG_k)$ by mapping generators*

$$e_i(h_1^2, ..., h_k^2) \to \epsilon_m^2 + 2\sum_{i=1}^{m}(-1)^i \epsilon_{m+j}\epsilon_{m-j}.$$

Then this map is injective.

Proof. The above map is equivalent to the map $\mathbb{C}[e_1, ..., e_k] \to \mathbb{C}[\epsilon_1, ...]/I$ where I is given by the relations

$$\epsilon_m^2 + 2\sum_{i=1}^{m}(-1)^i \epsilon_{m+i}\epsilon_{m-i} \ \ m > k.$$

Reduce coeffiecients to $\mathbb{Z}$. Note that $\mathbb{H}(IG_k)$ over $\mathbb{Z}$ is a free, torsion-free, $\mathbb{Z}$ module. Reduce coefficients to $\mathbb{Z}_2$. Then the map becomes,

$$\mathbb{Z}_2[e_1, , ..., e_k] \to \mathbb{Z}_2[\epsilon_1, ..., \epsilon_k] \otimes \mathbb{Z}_2[\epsilon_{k+1}, ...]/\langle \epsilon_m^2 = 1\rangle_{m>k}$$

, with $e_i \to \epsilon_i$. Then, clearly this map is injective. This suffices. $\square$

5.3 Type B

The results here are nearly identical to those in type C.

5.3.1 Cohomology of OG(n-k,2n+1)

In this section we will describe the additive and multiplicative structure of the cohomology ring of orthogonal grassmannians $X = OG(n-k, 2n)$. Again we fix an integer k, the reason for this will become apparent when we want to derive a partial analogue to Theorem 4.4 in type B. As for type A we have parametrizations of Schubert varieties, classes, and Poincare dual classes via index sets, partitions, and minmal length coset representatives of $W/W_{P_{n-k}}$. There are special Schubert classes and Pieri and Giambelli formulas as well. The ring structure can also be described by Chern classes of certain tautological bundles.

References for the following parametrizations can be found in [BK2, BKT1, LL, PR1, T2, T7].

Equip $V = \mathbb{C}^{2n+1}$ with a non-degenerate symmetric bilinear form ϑ. Fix a complete *orthogonal* flag $F_\bullet$,

$$0 = F_0 \subset F_1 \subset ... \subset F_{2n+1} = V$$

where $F_i = F^{\perp}_{2n+1-i}$ with respect to ϑ. Note, F_n is a maximal isotropic subspace. Then, we define the orthogonal grassmannian $OG(n-k, 2n+1)$ as,

$$OG(n-k, 2n+) := \left\{ \Omega \in Gr(n-k, 2n+1) : \ \vartheta(v, v') = 0, \ \forall v, v' \in \Omega \right\}$$

Schubert varieties of $OG(n-k, 2n+1)$ are also paremtrized by index sets $\{I : 1 \leq p_{i_1} \leq \cdots \leq p_{i_{n-k}} \leq 2n+1\}$ such that $p_i + p_j \neq 2n+2$. The open Schubert cell X°_I and the closed Schubert variety $X_I(F_\bullet)$ are defined in the same way as for type C.

Following [BK2], we can realize $SO(2n+1)$ as the fixed point subgroup G^θ of $G = SL(2n+1)$ under the involution $\sigma(A) = E^{-1}(A^t)^{-1}E$ where $E = E_B$ as in §5.1. If $T^A \subset B^A$ are the maximal torus and Borel subgroup of $SL(2n+1)$, then $T^\sigma = T$, and $B^\sigma = B$ as in the §5.1. Let$\Delta_B = \{\beta_1, ..., \beta_n\}$ be the simple roots of $SO(2n+1)$. Then, $\beta_i = \alpha_i|\mathfrak{t}$ where $\{\alpha_1, ..., \alpha_{2n}\}$ are the simple roots of $SL(2n+1)$. The corresponding simple coroots are given by

$$\check{\beta}_i = \check{\alpha}_i + \check{\alpha}_{2n+1-1}, \text{ for } 1 \leq i \leq n,$$

and

$$\check{\beta}_n = 2\check{\alpha}_n + 2\check{\alpha}_{n+1}$$

Under the inclusion $W_B \subset S_{2n}$ we have that the simple reflections of $SO(2n)$ are given by

$$s_i = r_i r_{2n+1-i} \;\; if \;\; 1 \leq i \leq n-1$$

$$= r_n r_{n+1} r_n \;\; if \;\; i = n$$

where r_i is the i^{th} simple reflection for $Sl(2n+1)$. The Weyl group W_{B_n} can be identified with the subset of $W_{A_{2n+1}}$ invariant under σ:

$$\{(a_1, ..., a_{2n+1}) \in S_{2n+1} : a_{2n+2-i} = 2n+2-a_i \; \forall 1 \leq i \leq 2n\}$$

. Consider the parabolic Weyl subgoup generated by $\Delta_B - \{\beta_{n-k}\}$. Then the minimal length coset representatives of $W_B/W_{B,P_{n-k}}$ can be identified with the set

$$\mathrm{I}(n-k, 2n+1) = \{I := 1 \leq p_1 < \cdots < p_n \leq 2n+1 \; p_j \neq n+1 \; for \; any \; j \; and \; I \cap \overline{I}) = \emptyset\},$$

where $\overline{I} = \{2n+2-p_1, ..., 2n+2-p_{n-k}\}$. But this is just an index set. It represents the permutation in S_{2n+} given by taking $p_{n-k}+1, ..., p_n = [n] \backslash (I \sqcup \overline{I})$ and setting $p_{2n+1-i} = 2n+1-p_i$.

The Schubert varieties are also parametrized by the same set of k-stict partitions which fit in a $(n-k) \times (n+k)$ rectangle. Following [BKT1, §4.2], to any k-strict partition, let $I_j(\lambda) = n+k+1-\lambda_j + \{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}$. Then the appropriate index set for the type B Schubert variety is given by $\bar{I}$ where,

$$\bar{I}_j(\lambda) = \begin{cases} I_j(\lambda) + 1 & if \; \lambda_j \leq k, \\ I_j(\lambda) & it \; \lambda_j > k \end{cases}$$

Let ϵ_λ be the Schubert class associated to a $k-$ strict partition as before. Then, again, ϵ_i are the special Schubert classes.

As in types A and C, there is a short exact sequence of vector bundles

$$0 \to S_B \to V_B \to Q_B \to 0$$

where V_B is the trivial bundle and S_B and Q_B are the tautological sub and quotient bundles. For a given $k-$strict partition λ, let $l_k(\lambda)$ be the number of parts of λ which are strictly greater than k. Then a well known result of [BS], when translated into the language of $k-$strict partitions in [BKT4], says that the map taking $c_p(Q_C)$ to $c_p(Q_B)$ extends to a ring isomorphism $\phi : H^*(IG(n-k, 2n), \mathbb{C}) \to H^*(OG(n-k, 2n+1), \mathbb{C})$ such that $\phi(\epsilon_\lambda) = 2^{l_k(\lambda)}\epsilon_\lambda$.

Buch, Kresch, and Tamkvakis have also shown that

$$c_i(Q_B) = \begin{cases} \epsilon_i & if\ i \leq k \\ 2\epsilon_i & if\ i > k \end{cases}$$

The Giambelli formula is then given by

$$\epsilon_\lambda = 2^{-l_k(\lambda)} R_\lambda m_\lambda$$

Note that $m_\lambda = \prod_i c_{\lambda_1}$ is given in terms of the Chern classes (which unlike type C do not exactly match up the special Schubert classes ϵ_i). In terms of the variables $c_i = c_i(Q_B)$ for $1 \leq i \leq n-k$ (with $c_0 = 1$ and $c_i = 0$ if $i < 0$ or $n-k > 0$, $H^*(OG(n-k, 2n+1), \mathbb{C})$ has the same presentation $H^*(IG(n-k, 2n), \mathbb{C})$ from Theorem 5.3.

5.3.2 Theorem 4.2

As for type A, we aim to explicitly determine the map in Theorem 4.2 for $G = So(2n+1)$ and any maximal parabolic P_{n-k}. Take $V = \mathbb{C}^{2n+1}$ and $OG(n-k, 2n+1)$ as in the previous section. Here we follow and expand on [Ku2, §8].

We take $B_B := B \cap SO_{2n+1}$ as the Borel subgroup of SO_{2n+1}, where B is the standard Borel subgroup of SL_{2n+1} consisting of upper triangular matrices of determinant 1. Then, $OG(n-k, 2n+1)$ is the quotient SO_{2n+1}/P_{n-k} of SO_{2n+1} by the standard maximal parabolic subgroup P_{n-k} with $\Delta \setminus \{\alpha_{n-k}\}$ as the set of simple roots of its Levi component L_{n-k}. (We take L_{n-k} to be the unique Levi subgroup of P_r containing T_B). Then,

$$L_{n-k} \simeq GL(n-k) \times SO(2k+1).$$

From Lemma 5.2, we have

$$\theta_{\omega_1}(\mathbf{t}_C) = \left(\frac{t_1 - t_1^{-1}}{2}, ..., \frac{t_n - t_n^{-1}}{2}, 0, -(\frac{t_n - t_n^{-1}}{2}), ..., -(\frac{t_1 - t_1^{-1}}{2})\right)$$

Also, recall that $Rep^{\mathbb{C}}_{\omega_1}(L_{n-k}) \simeq S(\mathfrak{t}_B^*)^{W_{P_{n-k}}}$. Using the fundamental invariants from §4.1.1 we find that the representation ring is given by,

$$\text{Rep}^{\mathbb{C}}_{\omega_1}(L_{n-k}) \simeq \mathbb{C}_{sym}\left[\left(\frac{t_1 - t_1^{-1}}{2}\right), ..., \left(\frac{t_{n-k} - t_{n-k}^{-1}}{2}\right)\right]$$

$$\otimes_{\mathbb{C}} \mathbb{C}_{sym}\left[\left(\frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2}\right)^2, ..., \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right]$$

where $\mathbb{C}_{sym}$ denotes the subalgebra of the polynomial ring consisting of symmetric polynomials. Further, by Theorem 5.1,

$$\mathbb{C}_{sym}[(\frac{t_{r+1} - t_{r+1}^{-1}}{2})^2, \ldots, (\frac{t_n - t_n^{-1}}{2})^2]$$

is generated (as a $\mathbb{C}$-algebra) by the virtual representations:

$$\{\lambda^d\left([S^2(V'_{2k+1})] - [\Lambda^2(V'_{2k+1})]\right) - [\epsilon]\big)\}_{1 \le d \le k},$$

where $V'_{2k+1} = \mathbb{C}^{2k}$ is the standard representation of $SO(2k+1)$, $[\epsilon]$ is the trivial one-dimensional representation, and λ is the λ-ring structure on $Rep(G)$.

The following theorem [Ku2, $Proposition 20$] partially determined the homomorphism of Theorem 4.2

Theorem 5.7. *The map $\xi^{P_{n-k}} : Rep^{\mathbb{C}}_{poly}(L^C_{n-k}) \to H^*(OG(n-k, 2n+1), \mathbb{C})$ of Theorem 4.2 takes*

$$\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \to \epsilon_{s_{n-k}},\ if\ k > 0,$$

$$\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) \to 2\epsilon_{s_{n-k}},\ if\ k = 0,$$

and

$$\frac{1}{4}\left[(t_{n-k+1} - t_{n-k+1}^{-1})^2 + \cdots + (t_n + t_n^{-1})^2\right] \to \epsilon^2_{s_{n-k}} + 2\sum_{j=n-k+1}^{n-1} \epsilon^2_{s_j} + 4\epsilon^2_{s_n} - 2\sum_{j=n-k}^{n-1} \epsilon_{s_j}\epsilon_{s_{j+1}} - 4\epsilon_{s_{n-1}}\epsilon_{s_n}$$

Proof. For $1 \leq n$, let $x_i : \mathfrak{t} \to \mathbb{C}$ be the linear map which takes

$$diag(x_1, ..., x_n, 0, -x_n, ..., -x_1) \to x_i$$

. Then by Lemma 5.2, the homomorphism $\theta^*_{\omega_1|T} : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$ induced from the Springer morphism θ_{ω_1} takes

$$x_1 + \cdots + x_{n-k} \to \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}).$$

However, note that the weight $x_1 + ... + x_{n-k}$ is the first fundamental weight ω_{n-k} if $k > 0$. Thus,

$$\beta \circ (\theta^*_{\omega_1|T})^{-1}(\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_{n-k} + t_{n-k}^{-1}) = \beta(x_1 + \cdots + x_{n-k}) = \beta(\omega_{n-k}) = \epsilon_{s_{n-k}}.$$

If $k = 0$, one has

$$x_1 + \cdots + x_n \to \frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_n + t_n^{-1}).$$

Note that for type B that $\omega_n = \frac{x_1+...+x_n}{2}$. Thus,

$$\beta \circ (\theta^*_{\omega_1|T})^{-1}(\frac{1}{2}(t_1 - t_1^{-1} + \cdots + t_n + t_n^{-1}) = \beta(x_1 + \cdots + x_n) = \beta(2\omega_n) = 2\epsilon_{s_n}.$$

Similiarly, under $(\theta^*_{\omega_1|T})^{-1}$,

$$\frac{1}{4}\left((t^2_{n-k+1} - t_{n-k+1})^{-1} + \cdots + (t_n + t_n^{-1})^2\right) \to x^2_{n-k} + \cdots + x_n^2$$

In type B we have $\omega_i = \sum_{j=1}^{i} x_i$ for $1 \leq i \leq n-1$, and $\omega_n = \frac{x_1+...+x_n}{2}$. Then we can right the coordinate functions in the fundamental weight basis $x_i = \omega_i - \omega_{i-1}$ for $1 \leq i \leq n-1$, and $x_n = 2\omega_n - \omega_{n-1}$. So, $x^2_{n-k+1} + \cdots + x_n^2 = (\omega_{n-k+1} - \omega_{n-k})^2 + \ldots (\omega_{n-1} - \omega_{n-2}) + (2\omega_n - \omega_{n-1})^2$. Then from the remark above it is clear that $\xi^{P_{n-k}}$ takes

$$\frac{1}{4}\left((t^2_{n-k+1} - t_{n-k+1})^{-1} + \cdots + (t_n + t_n^{-1})^2\right) \to (\epsilon_{s_{n-k+1}} - \epsilon_{s_{n-k}})^2 + \cdots + (\epsilon_{s_{n-1}} - \epsilon_{s_{n-2}}) + (2\epsilon_{s_n} - \epsilon_{s_{n-1}})^2,$$

which expands to give the stated result. To write it in the $W_B^{P_{n-k}}$ basis one can use the Chevalley formula or we note that

$$A_{s_j}(x^2_{n-k+1} + ... + x_n^2) = -x_{n-k} - x_{n+1} \; if \; j = n-k \; and \; 0 \; otherwise.$$

Then we have $A_{s_{n-k-1}}(-x_{n-k} - x_{n-k+1}) = 1$ and $A_{s_n-k+1}(-x_{n-k} - x_{n-k+1}) = -1$. Thus, we have exactly as in type C,

$$\xi^{P_{n-k}}\left[(\frac{1}{4}\left[(t_{n-k+1} - t^{-1}_{n-k+1})^2 + \cdots + (t_n + t_n^{-1})^2\right] \to \epsilon_{s_{n-k-1}s_{n-k}} - \epsilon_{s_{n-k+1}s_{n-k}}\right.$$

$\square$

In terms of the Chern classes c_i we get essentially the same result as in Theorem 5.5

Theorem 5.8. *The map $\xi^{P_{n-k}} : Rep^{\mathbb{C}}_{poly}(L^B_{n-k}) \to H^*(OG(n-k, 2n+1), \mathbb{C})$ of Theorem 4.2 takes*

$$e_i\left[(\frac{t_1 - t_1^{-1}}{2}), ..., (\frac{t_{n-k} - t_{n-k}}{2})\right] \to c_i(S) = \epsilon_{(1)^i}(S) = \epsilon_{s_{n-k+i-1} \dots s_{n-k}} \text{ if } k>0$$

$$e_i\left[(\frac{t_1 - t_1^{-1}}{2}), ..., (\frac{t_{n-k} - t_{n-k}}{2})\right] \to c_i(S) = \epsilon_{(1)^i}(S) = 2\epsilon_{s_{n-k+i-1} \dots s_{n-k}} \text{ if } k=0$$

For $1 \le i \le n-k$, let $c_i = c_i(Q_B)$. Define $c_0 = 1$ and $c_p = 0$ for $p < 0$ or $p > n+k$. Then,

$$e_i\left[(\frac{t_{n-k+1} - t_{n-k+1}^{-1}}{2})^2, ..., (\frac{t_n - t_n}{2})^2\right] \to c_i^2 + 2\sum_{j=1}^{n+k-r} (-1)^i c_{i+j} c_{i-j}$$

for $1 \le i \le k$.

Proof. For the type A part with $k > 0$ the proof is the same as for Theorem 5.5. For $k = 0$, it is essentially the same except in this case

$$A_{s_n}(e_i(x_2, ..., x_n)) = \frac{e_i(x_1, ..., x_n) - e_i(x_1, ..., -x_n)}{x_n},$$

and we see that

$$A_{s_n}(e_i(x_1, ..., x_n)) = \frac{2x_n e_i - 1(x_1, ..., x_{n-1}}{x_n} = 2e_{i-1}(x_1, ..., x_{n-1}).$$

The result follows from here.

The type B part is analogous to the proof of Theorem 5.5 up to accounting for factors of two when applying A_{s_n} and using the type B Giambelli formula. □

5.3.3 Inverse Limit

Analogously, we define the stable cohomology ring [BKT3, §3.2] as

$$\mathbb{H}(OG_k) = \varprojlim H^*(OG(n-k, 2n+1), \mathbb{C})$$

as the inverse limit in the category of graded rings of the inverse system

$$\cdots \leftarrow H^*(OG(n-k,2n+1),\mathbb{C}) \leftarrow H^*(OG(n-k+1,2n+3),\mathbb{C}) \leftarrow \dots$$

This ring has an additive basis of Schubert classes ϵ_λ for each $k-$ strict partition λ. There is a natural surjective ring homorphism $\mathbb{H}(OG_k) \to H^*(OG(n-k,2n+1),\mathbb{C})$ given by mapping ϵ_λ to ϵ_λ whenever λ fits in a $(n-k)\times(n+k)$ rectangle and to zero otherwise. In terms of the variables c_i we have that $\mathbb{H}(OG_k)$ has the exact same presentation as $\mathbb{H}(IG_k)$ from §5.2.3. Factoring through $\xi^{P_{n-k}}$: $\mathrm{Rep}^{\mathbb{C}}_{omega_1}(L_{n-k}) \to H^*(SO(n-k,2n+1)$ as in §5.2.3, we get a map from $\mathrm{Rep}^{\mathbb{C}}SO(2k+1)$ to $\mathbb{H}(OG_k)$. Then we have the analogous theorem,

Theorem 5.9. *The map ξ_k : $Rep^{\mathbb{C}}_{\omega_1}(SO(2k+1)) \to \mathbb{H}(OG_k)$ given by mapping generators*

$$e_i(h_1^2,...,h_k^2) \to c_i^2 + 2\sum_{j=1}^{i}(-1)^i c_{i+j}c_{i-j}$$

is injective.

5.4 G2

5.4.1 Representation Ring of G2

Here we compute the ω_1-polynomial representation ring of $G2$. Note that ω_1 is the fundamental representation of minimal Dynkin index. We first need to compute the Springer morphism. We write $\theta_{\omega_1}(t) = c_1(t)\check{\alpha}_1 + c_2(t)\check{\alpha}_2$. From §3.2 we saw that

$$c_1(t,\lambda) = \frac{2}{3x}\sum_{\mu\in\Lambda_\lambda}(2\mu_1+3\mu_2)e^{\mu}(t)$$

$$c_2(t,\lambda) = \frac{2}{3x}\sum_{\mu\in\Lambda_\lambda}(3\mu_1+6\mu_2)e^{\mu}(t)$$

, for θ_λ. In this case $x = \sum_{\Lambda_\lambda} \mu_2^2$ where μ_2 is the coordinate of the second fundamental weight for a given weight $\mu \in \Lambda_\lambda$. For $\lambda = \omega_1$ the weights are

$$\Lambda_{\omega_1} = \{(1,0), (-1,0), (1,-1), (-1,1), (2,-1), (-2,1)\}.$$

Let s be the simple reflection asscoiated to the first simple root α_1 and t be the simple reflection for α_2. To state the result more clearly we note the following from [A, Appendix A]. There is an embedding $W_{G_2} \to W_{A_6}$ given by $s \to r_{12}r_{35}r_{67}$ and $t \to r_{23}r_{56}$ where r_{ij} is the transposition (i,j) in S_7. This inclusion corresponds to an inclusion $G2 \to GL(7)$. The inclusion of tori is given by

$$(t_1, t_2) \to (t_1, t_2, t_1 t_2^{-1}, 1, t_1^{-1} t_2, t_2^{-1}, t_1^{-1}).$$

The inclusion on Cartan subaglebras is then given by

$$(h_1, h_2) \to (h_1, h_2, h_1 - h_2, 0, h_2 - h_1, -h_2, -h_1).$$

Let x_i be the i^{th} coordinate function on the cartan subalgebra. We then have the following root data for $G2$. The simple roots are given by

$$\alpha_1 = x_1 - x_2, \quad \alpha_2 = -x_1 + 2x_2.$$

The simple coroots are given by

$$\check{\alpha}_1 = 3h_1 - 3h_2, \quad \check{\alpha}_2 = -h_1 + 2h_2.$$

The fundamental weights are given by

$$\omega_1 = x_1, \quad \omega_2 = x_1 + x_2.$$

We also note that $\alpha_3 = \alpha_1 + \alpha_2 = x_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2 = x_1$. We can also now write $e^\mu(t) = e^{(\mu_1,\mu_2)}(t_1, t_2) = t_1^{\mu_1+\mu_2} t_2^{\mu_2}$. From the fundamental weights given above and the formulas from §3.2, we have

$$c_1(t) = \frac{1}{6}(2e^{(1,0)}(t) + e^{(-1,1)}(t) + e^{(2,-1)}(t) - e^{(-2,1)}(t) - e^{(1,-1)}(t) - 2e^{(-1,0)}(t))$$

$$c_2(t) = \frac{1}{6}(3e^{(1,0)}(t) + 3e^{(-1,1)}(t) - 3e^{(1,-1)}(t) - 3e^{(-1,0)}(t)).$$

More explicity using $e^{\mu}(t) = t_1^{\mu_1+\mu_2} t_2^{\mu_2}$ we have,

$$c_1(t) = \frac{1}{6}(2t_1 + t_2 + t_1 t_2^{-1} - t_1^{-1} t_2 - t_2^{-1} - 2t_1^{-1})$$

$$c_2(t) = \frac{1}{2}(t_1 + t_2 - t_1^{-1} - t_2^{-1})$$

Then, we have $\theta_{\omega_1}(t_1, t_2) = c_1(t)\check{\alpha_1} + c_2(t)\check{\alpha_2} = (3c_1(t) - c_2(t), -3c_1(t) + 2c_2(t))$ which gives

$$\theta_{\omega_1}(t_1, t_2) = \left(\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2} \right)$$

Now, recall that by definition $\mathrm{Rep}^{\mathbb{C}}_{\omega_1}(G) = S(\mathfrak{t}^*)^W$. The Weyl group of $G2$ is the dihedral group $D_6 = \{s, t | s^2 = t^2 = (st)^6 = 1\}$ and has fundamental invariants 2,6 (see Table 4.1). The following set of polynomial invariants is given by [Ts, 3.3](other sets of invariants can be found in [Lee, Me]). in terms of the simple roots α_1 and α_2:

$$f_{2k} = \alpha_1^{2k} + (\alpha_1 + \alpha_2)^{2k} + (2\alpha_1 + \alpha_2)^{2k}.$$

Then we have $S(\mathfrak{t}^*) = \mathbb{C}[f_2, f_6]$. Using the coordinates from above we can re-write these as

$$f_{2k} = (x_1 - x_2)^{2k} + x_1^{2k} + x_2^{2k}.$$

So,

$$f_2 = 2x_1^2 + 2x_2^2 - 2x_1 x_2$$

$$f_6 = 2x_1^6 - 6x_1^5 x_2 + 15x_1^4 x_2^2 - 20x_1^3 x_2^3 + 15x_1^2 x_2^4 - 6x_1 x_2^5 + 2x_2^6.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\omega_1}(G_2) = \mathbb{C}[f_2(\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2}),$$

$$f_6(\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2})].$$

We have the following analogue of [Ku2, Proposition 24]. Under the coordinates of $\theta_{\omega_1}(t_1, t_2)$ on the maximal torus $T \subset G2$.

$$\mathrm{Rep}^{\mathbb{C}}_{\omega_1}(T) = \mathbb{C}[\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2}, \frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2}]$$

where we think if T as the Levi subgroup of B.

Theorem 5.10. *Under the homomorphism $\xi^B : Rep^{\mathbb{C}}_{\omega_1}(T) \to H^*(G2/B, \mathbb{C})$, we have*

$$\frac{t_1 - t_1^{-1} + t_1 t_2^{-1} - t_1^{-1} t_2}{2} \to \epsilon_{s_1}$$

and,

$$\frac{t_2 - t_2^{-1} + t_1^{-1} t_2 - t_1 t_2^{-1}}{2} \to \epsilon_{s_2} - \epsilon_{s_1}$$

Proof. Observe that $x_1 = \omega_1$ and $x_2 = \omega_2 - \omega_1$, and $\beta(\omega_i) = \epsilon_{s_i}$. □

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