ASYMPTOTIC BOUNDARY OBSERVABILITY FOR THE WAVE EQUATION ON SIMPLICES

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Abstract. In this paper, we consider the wave equation on an n-dimensional simplex with Dirichlet boundary conditions. Our main result is an asymptotic observability identity from any one face of the simplex.

1. Introduction

In this paper, we study the wave equation $(\partial_t^2 - \Delta)u = 0$ on all n-dimensional simplex with Dirichlet boundary conditions. We obtain an asymptotic observability property from any one face of the simplex. This generalizes the result in [CS18] from triangles to to simplices in $\mathbb{R}^n$. The proof is similar to that of [CS18]. It uses commutators and integration by parts arguments, but involves a coordinate transformation and linear algebra as well.

The formal statement of the problem is represented by (1.1):

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 \text{ on } (0,\infty)\times\Omega, \\ u\Big|_{\partial\Omega} = 0, \\ u(0,x_1,x_2,\dots,x_n) = u_0(x_1,x_2,\dots,x_n), \\ u_t(0,x_1,x_2,\dots,x_n) = u_1(x_1,x_2,\dots,x_n) \end{cases} \tag{1.1}$$

where u is real-valued and $u_0 \in H_0^1(\Omega)\cap H^3(\Omega)$ and $u_1 \in H_0^1(\Omega)\cap H^2(\Omega)$. Regarding this problem, the main theorem is the following:

Theorem 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a simplex with faces $F_0, F_1, F_2, \dots, F_n$ and suppose u solves the wave equation on Ω. For any finite time $T > 0$, following we obtain the asymptotic observability identity for any one face of the simplex Ω:

$$\int_0^T \int_{F_0} |\partial_\nu u|^2 dS_0 dt = \frac{T Area(F_0)}{nVol(\Omega)} \tilde{E}(0)\left(1 + \mathcal{O}\left(\frac{1}{T}\right)\right), \tag{1.2}$$

where $\partial_\nu u$ is the normal derivative on $F_0$and dS_0 is the induced surface measure. $\tilde{E(t)}$ is the conserved energy of the solution u to the wave equation, defined by:

$$\tilde{E}(t) = \int_\Omega |\partial_t u|^2 + |\nabla u|^2 dV. \tag{1.3}$$

Remark 1. Observability in this paper means we can observe the initial energy by taking a measurement on one face.

2. History

The study of observability is based on the prerequisite that waves propagate along straight-line paths in a homogeneous medium. Waves reflect off the boundary satisfying the law of reflection, so that the angle of incidence is the same as the reflection angle.

The idea of observability originates from Rauch-Taylor's paper [RT74] , where they studied geometric control for the damped wave equation $u_{tt} - \Delta u + a(x)\partial_t u = 0$. The idea is if every ray passes through the damping region where $a > 0$, energy decays exponentially as $E(t) \leqslant Ce^{-t/c}$. For example, the first picture of **Figure 1** is not geometric control while the second one is.

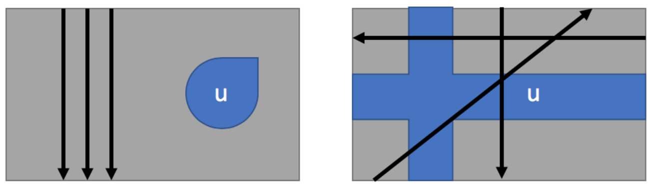

Figure 1. Rays Passing Through Subset Of Domain

The closely related idea of observability asks if you can "see" the initial energy by taking the measurement of a subset of the domain or a subset of the boundary. In the work of Bardos-Lebeau-Rauch [BLR92], the observability from a subset of the boundary was studied in depth. The condition for observability, similar to in Rauch-Taylor is that all rays hit the control region on the boundary transversally, as indicated in **Figure 2**.

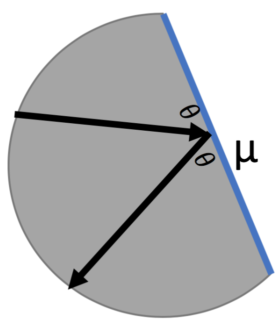

Figure 2. Rays Hitting The Control Region On The Boundary

In the work of Christianson-Stafford [CS18], an asymptotic observability property from any one side of a triangle is proved. The proof was split into the cases of acute triangles and obtuse triangles, shown by **Figure 3**. Waves are assumed to propagate along straight-line paths at an unit speed, traveling from the opposite corner to the interested side. The result was obtained with an argument of the method of this paper, by the use of commutator and integration by parts arguments.

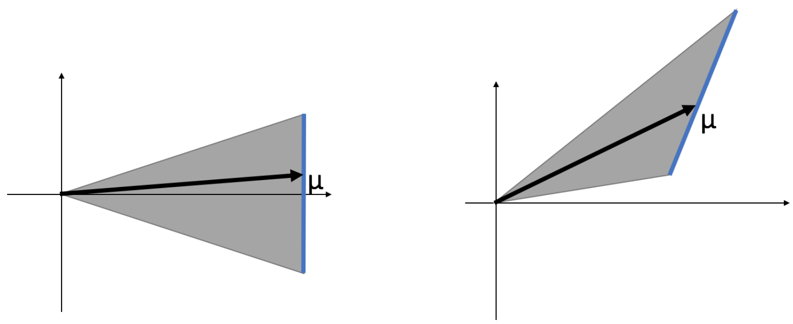

FIGURE 3. Asymptotic Observability On One Side Of Triangles

3. Preliminaries

This section of preliminaries provides lemmas and definitions required in the main proof.

Lemma 3.1 (Conserved Energy)**.** For the solution u to the wave equation (1.1). The energy is conserved:

$$\tilde{E}(t) = \tilde{E}(0). \tag{3.1}$$

Proof. Start with the wave equation $(\partial_t^2 - \Delta)u = 0$. By multiplying u_t and by integrating the wave equation on the domain, we have the following computations:

$$\begin{aligned}
&\int_\Omega (\partial_t^2 - \Delta) u u_t dV = 0 \\
\Rightarrow &\int_\Omega \partial_t^2 u u_t dV - \int_\Omega \Delta u u_t dV = 0 \\
\Rightarrow &\int_\Omega \partial_t^2 u u_t dV + \int_\Omega \nabla^T u \nabla u_t dV - \int_{\partial\Omega} \partial_\nu u u_t dS = 0,
\end{aligned} \tag{3.2}$$

where ν is the outward unit normal vector and dS is the induced surface measure.

Because u vanishes on the boundary, $u_t\Big|_{\partial\Omega} = 0$ holds. We could therefore cancel out the last term in the previous computation and prove that the energy is conserved due to result of (3.2):

$$\begin{aligned}
\tilde{E}'(t) &= \int_\Omega u_{tt} u_t + u_t u_{tt} + \nabla^T u_t \nabla u + \nabla^T u \nabla u_t dV \\
&= 2 \int_\Omega u_{tt} u_t + \nabla^T u \nabla u_t dV \\
&= 0.
\end{aligned} \tag{3.3}$$

$\square$

Definition 3.1 (Elliptic Operator)**.** A constant coefficient elliptic operator P on $\Omega \subseteq \mathbb{R}^n$ is defined by

$$P = -\sum_{i,j=0}^{n} K_{ij}\partial_{x_i}\partial_{x_j}, \tag{3.4}$$

where K is an $n \times n$ symmetric, positive definite matrix.

Lemma 3.2 (Ellipticity)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a simplex. If $u \in H_0^1(\Omega) \cap H^3(\Omega)$, then there exists an constant C such that

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C\langle Pu, u\rangle_{L^2}. \tag{3.5}$$

The following lemma is a modified version of Theorem 4 published in the paper [Chr17] by Christianson. The modified version can be directly used in the main proof of this paper.

Lemma 3.3 (Green's formula for (3.4))**.** Let $\Omega' \in \mathbb{R}^n$ be the standard simplex and $g\Big|_{\partial\Omega'} = 0$. Let P be an elliptic operator. Then for functions $f, g \in C^\infty(\Omega)$, we have,

$$\int_{\Omega'} (Pf)gdV = \int_{\Omega'} f(Pg)dV + \int_{\partial\Omega'} f(\nu^T K)\partial g dSdt \tag{3.6}$$

where ν is the outward unit vector on every face of the simplex Ω.

This paper inherits the notation that Christianson used in the paper [Chr17] to define higher dimension simplicies.

Definition 3.2 (Simplex)**.** Let independent vectors $\vec{p_1}, \vec{p_2}, \ldots, \vec{p_n} \in \mathbb{R}^n$ span from the origin, then a simplex Ω in $\mathbb{R}^n$ is defined as:

$$\Omega = \{\sum_{i=1}^{n} c_i\vec{p_i} : \sum_{i=0}^{n} c_i \leqslant 1 \ and \ c_i \geqslant 0, \vec{p_i} \in \mathbb{R}^n\} \tag{3.7}$$

We denote the face where $c_i = 0, i = 1, \ldots, n$ as F_i and the remaining face F_0[1]. Let matrix A be $\begin{bmatrix} | & | & & | \\ \vec{p_1} & \vec{p_2} & \ldots & \vec{p_n} \\ | & | & & | \end{bmatrix}$. Because the column vectors $\vec{p_1}, \vec{p_2}, \ldots, \vec{p_n}$ are linearly independent, there exists an inverse matrix of A. Denote this inverse matrix by B.

In particular, we have standard simplex $\Omega' \in \mathbb{R}^n$.

Definition 3.3 (Standard Simplex)**.** Let unit vectors $\vec{e_1} = [1, 0, 0, \ldots, 0]^T$, $\vec{e_2} = [0, 1, 0, \ldots, 0]^T, \ldots, \vec{e_n} = [0, 0, 0, \ldots, 1]^T \in \mathbb{R}^n$ be n linear independent vectors. The standard simplex, denoted by Ω', is defined by all convex combinations of these linearly independent unit vectors:

$$\Omega' = \{\sum_{i=1}^{n} d_i\vec{e_i} : \sum_{i=0}^{n} d_i \leqslant 1 \ and \ d_i \geqslant 0, \vec{e_i} \in \mathbb{R}^n\} \tag{3.8}$$

[1]By choosing a different corner as the origin, we can get the result for any one of the faces.

This standard simplex has $n+1$ faces $F'_0, F'_1, \ldots, F'_n$, where F'_i is the face with $d_i = 0, i = 1, 2, \ldots, n$ while the remaining face is F'_0.

The standard rectangular coordinates of the standard simplex Ω' are denoted as $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ while the rectangular coordinates of the original simplex Ω are denoted as $(x_1, x_2, \ldots, x_n)$ in this paper.

The following transformation takes the arbitrary simplex Ω in $\mathbb{R}^n$ to the standard simplex Ω'. Let $\vec{x} = [x_1, x_2, \ldots, x_n]^T$ denote one vector in the simplex Ω and $\vec{y} = [y_1, y_2, \ldots, y_n]^T$ denote the corresponding vector of $\vec{x}$ in the standard simplex Ω', then we could obtain the following equation by considering the relation between the sets of basis of the simplex Ω and that of the standard simplex Ω':

$$\vec{x} = A\vec{y} \tag{3.9}$$

For example, when $\vec{y} = \vec{e_j} = [0, \ldots, 1, 0, \ldots, 0]^T$, where 1 is at the jth position of the vector, we have this relation:

$$\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{p_1} & \vec{p_2} & \vdots & \vec{p_n} \\ | & | & & | \end{bmatrix} \vec{e_j} = \vec{p_j} \tag{3.10}$$

By observing this relation, we claim that $\nabla_x = B^T \nabla_y$ after the transformation. The proof of this claim is in **Appendix A**.

Since the Laplacian operator is $-\Delta_x = -\nabla_x^T \nabla_x$ on the simplex Ω, the above claim implies this equation is equivalent to $P = -(B^T\nabla_y)^T(B^T\nabla_y) = -\nabla_y^T BB^T \nabla_y$ on the standard simplex Ω'. Denote it by P.

According to (1.3), the energy of the solution to the wave equation in y-coordinate can be defined as :

$$E(t) = \int_{\Omega'} |u_t|^2 + |B^T \nabla u|^2 dV. \tag{3.11}$$

We claim that this energy is also conserved:

$$E(t) = E(0). \tag{3.12}$$

Proof of (3.12).

$$\begin{aligned} E(t)' &= \int_{\Omega'} u_{tt}u_t + u_t u_{tt} + (B^T\nabla u_t)(B^T\nabla u) + (B^T\nabla u)(B^T\nabla u_t) \\ &= 2\int_{\Omega'} u_{tt}u_t + (B^T\nabla u)(B^T\nabla u_t) \\ &= 0. \end{aligned} \tag{3.13}$$

□

The method used in this proof is similar to that of **Lemma 3.1**.

Lemma 3.4. Consider the vector field $X = \sum_{i=0}^n x_i \partial_{x_i}$ and the second order constant coefficient symmetric operator $T = -\sum_{i,j=1}^n a_{ij}\partial_{x_i}\partial_{x_j}$. Then:

$$[T, X] = 2T \tag{3.14}$$

Proof. When $i = j$, $i = 1, \ldots, n$,

$$
\begin{aligned}
&[-a_{ii}\partial_{x_i}^2, x_1\partial_{x_1} + x_2\partial_{x_2} + \cdots + x_n\partial_{x_n}] \\
&= \sum_{k=1}^{n}[-a_{ii}\partial_{x_i}^2, x_k\partial_{x_k}] \\
&= [-a_{ii}\partial_{x_i}^2, x_i\partial_{x_i}] + \sum_{k=1,k\neq i}^{n}[-a_{ii}\partial_{x_i}^2, x_k\partial_{x_k}] \\
&= -a_{ii}\partial_{x_i}^2(x_i\partial_{x_i}) + x_i\partial_{x_i}(a_{ii}\partial_{x_i}^2) + \sum_{k=1,k\neq i}^{n}(-a_{ii}\partial_{x_i}^2(x_k\partial_{x_k}) + x_k\partial_{x_k}(a_{ii}\partial_{x_i}^2)) \\
&= -a_{ii}\partial_{x_i}(\partial_{x_i} + x_i\partial_{x_i}^2) + x_i a_{ii}\partial_{x_i}^3 + 0 \\
&= -a_{ii}\partial_{x_i}^2 - a_{ii}\partial_{x_i}^2 - x_i a_{ii}\partial_{x_i}^3 + x_i a_{ii}\partial_{x_i}^3 \\
&= -2a_{ii}\partial_{x_i}^2
\end{aligned} \tag{3.15}
$$

When $i \neq j$, $i, j = 1, \ldots, n$,

$$
\begin{aligned}
&[-a_{ij}\partial_{x_i}\partial_{x_j}, x_1\partial_{x_1} + x_2\partial_{x_2} + \cdots + x_n\partial_{x_n}] \\
&= \sum_{k=1}^{n}[-a_{ij}\partial_{x_i}\partial_{x_j}, x_k\partial_{x_k}] + \sum_{k=1,k\neq i,j}^{n}[-a_{ij}\partial_{x_i}\partial_{x_j}, x_k\partial_{x_k}] \\
&= [-a_{ij}\partial_{x_i}\partial_{x_j}, x_i\partial_{x_i}] + [-a_{ij}\partial_{x_i}\partial_{x_j}, x_j\partial_{x_j}] + 0 \\
&= -a_{ij}\partial_{x_i}\partial_{x_j}(x_i\partial_{x_i}) + x_i\partial_{x_i}(a_{ij}\partial_{x_i}\partial_{x_j}) \\
&\quad - a_{ij}\partial_{x_i}\partial_{x_j}(x_j\partial_{x_j}) + x_j\partial_{x_j}(a_{ij}\partial_{x_i}\partial_{x_j}) \\
&= -a_{ij}\partial_{x_j}(\partial_{x_i} + x_i\partial_{x_i}^2) + x_i a_{ij}\partial_{x_i}^2\partial_{x_j} \\
&\quad - a_{ij}\partial_{x_i}(\partial_{x_j} + x_j\partial_{x_j}^2) + x_j a_{ij}\partial_{x_j}^2\partial_{x_i} \\
&= -2a_{ij}\partial_{x_j}\partial_{x_i}
\end{aligned} \tag{3.16}
$$

□

4. Proof of the Theorem

Let the vector field be $Y = y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n}$ on the standard simplex Ω'. For $P = -\nabla^T BB^T\nabla$, we will compute:

$$
\int_0^T \int_{\Omega'} [\partial_t^2 + P, Y]uu dV, \tag{4.1}
$$

in two different ways. The two approaches are based on different ways of dealing with the commutator. One approach starts with using **Lemma 3.4** while in the other approach we evaluate the commutator explicitly.

To start our first approach, we could apply **Lemma 3.4** to compute the commutator, because P is an elliptic operator. Then, since u the satisfies wave equation, integration by parts gives:

$$
\begin{aligned}
\int_0^T \int_{\Omega'} [\partial_t^2 + P, Y] uudVdt &= \int_0^T \int_{\Omega'} 2PuudVdt \\
&= \int_0^T \int_{\Omega'} (P - \partial_t^2) uudVdt \\
&= \int_0^T \int_{\Omega'} -\nabla^T BB^T \nabla uudVdt - \int_0^T \int_{\Omega'} \partial_t^2 uudVdt \\
&= \int_0^T \int_{\Omega'} -(B^T\nabla)^T B^T \nabla uudVdt - \int_0^T \int_{\Omega'} \partial_t^2 uudVdt \\
&= \int_0^T \int_{\Omega'} (B^T\nabla)u(B^T\nabla)udVdt + \int_0^T \int_{\Omega'} \partial_t u \partial_t udVdt \\
&\quad - \int_{\Omega'} \partial_t uudV\Big|_0^T \\
&= \int_0^T \int_{\Omega'} |B^T\nabla u|^2 dVdt + \int_0^T \int_{\Omega'} |\partial_t u|^2 dVdt \\
&\quad - \int_{\Omega'} \partial_t uudV\Big|_0^T \\
&= \int_0^T \int_{\Omega'} |B^T\nabla u|^2 + |\partial_t u|^2 dVdt - \int_{\Omega'} \partial_t uudV\Big|_0^T \\
&= TE(0) - \int_{\Omega'} \partial_t uudV\Big|_0^T
\end{aligned}
\tag{4.2}
$$

Notice that at the last step of the previous computation, we used result of (3.12) .

We next compute (4.1) by a different approach. We first evaluate the commutator explicitly and then use integration by parts. The second term generated by the commutator cancels out because of the homogeneous wave equation. Indeed,

$$
\begin{aligned}
\int_0^T \int_{\Omega'} [\partial_t^2 + P, Y] uudVdt &= \int_0^T \int_{\Omega'} (\partial_t^2 + P)Yuu - Y(\partial_t^2 + P)uudVdt \\
&= \int_0^T \int_{\Omega'} \partial_t^2 Yuu + PYuudVdt
\end{aligned}
\tag{4.3}
$$

After we apply **Lemma 3.3** to the second term and use integration by parts twice on the first term, we have:

$$\begin{aligned}
&\int_0^T \int_{\Omega'} \partial_t^2 Yuu + PYuudVdt \\
&= \int_0^T \int_{\Omega'} Yu\partial_t^2 udVdt + \int_{\Omega'} \partial_t YuudV\Big|_0^T - \int_{\Omega'} Yu\partial_t udV\Big|_0^T \\
&\quad + \int_0^T \int_{\Omega'} YuPudVdt + \int_0^T \int_{\partial\Omega'} Yu(\nu^T BB^T)\nabla udSdt \\
&= \int_0^T \int_{\Omega'} Yu(\partial_t^2 + P)udVdt + \int_{\Omega'} \partial_t YuudV\Big|_0^T \\
&\quad - \int_{\Omega'} Yu\partial_t udV\Big|_0^T + \int_0^T \int_{\partial\Omega'} Yu(\nu^T BB^T)\nabla udSdt \\
&= \int_{\Omega'} \partial_t YuudV\Big|_0^T - \int_{\Omega'} Yu\partial tudV\Big|_0^T + \int_0^T \int_{\partial\Omega'} Yu(\nu^T BB^T)\nabla udSdt
\end{aligned} \tag{4.4}$$

where ν is the outward normal vector to every face, and dS is the reduced differential displacement.

To simplify the term integrate on the boundary in (4.4), we study every face of the simplex by writing out the vector field $Y = y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n}$. On face F_1', we have that:

$$\begin{aligned}
Yu\Big|_{F_1'} &= (y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n})u \\
&= (0 \cdot \partial_{y_1})u + y_2 \cdot 0 + y_3 \cdot 0 + \cdots + y_n \cdot 0 \\
&= 0
\end{aligned} \tag{4.5}$$

by observing that $y_1 = 0$ on F_1' and that the tangential derivatives $\partial_{y_2}u, \partial_{y_3}u, \ldots, \partial_{y_n}u$ of F_1' are all equal to 0 since $u\Big|_{\partial\Omega'} = 0$. Therefore, we could conclude that $Yu\Big|_{F_1'} = 0$. Similarly, the same result applies on the other $n-1$ faces of the standard simplex:

$$\begin{cases}
Yu\Big|_{F_1'} = (y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n})u = 0 \\
Yu\Big|_{F_2'} = (y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n})u = 0 \\
\vdots \\
Yu\Big|_{F_n'} = (y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n})u = 0.
\end{cases} \tag{4.6}$$

However, on face F_0', the condition is different because none of the spatial variables is 0 or none of $\partial_{y_2}u, \partial_{y_3}u \ldots, \partial_{y_n}u$ are tangential derivatives. We need to find its tangential vectors.

Notice that the unit normal derivative on this face is $\partial_\nu u = \frac{1}{\sqrt{n}}[1, 1, \ldots, 1]\nabla u$. Because the tangential derivatives are orthogonal to the normal derivative, we could choose different tangential vectors and take advantage of the fact that they all equal to 0. The first tangential derivative we choose is $\frac{1}{\sqrt{n}}[1,-1,0,\dots,0]\nabla u$ and it satisfies:

$$\frac{1}{\sqrt{n}}[1,-1,0,\dots,0]\nabla u = 0 \quad \Rightarrow \quad \partial_{y_1}u = \partial_{y_2}u \tag{4.7}$$

Similarly, by choosing other tangential derivatives for the face F_0 and by setting them equal to 0, we conclude that:

$$\partial_{y_1}u = \partial_{y_2}u = \cdots = \partial_{y_n}u \tag{4.8}$$

Therefore, using the conclusion above, the normal vector can be represented as:

$$\begin{aligned}\partial_\nu u &= \frac{1}{\sqrt{n}}(\partial_{y_1} + \partial_{y_2} + \cdots + \partial_{y_n})u \\ &= \frac{1}{\sqrt{n}}(n\partial_{y_1}u) \\ &= \sqrt{n}\partial_{y_1}u \\ &= \sqrt{n}\partial_{y_2}u \\ &= \dots \\ &= \sqrt{n}\partial_{y_n}u\end{aligned} \tag{4.9}$$

which implies that:

$$\begin{cases}\partial_{y_1}u = \frac{1}{\sqrt{n}}\partial_\nu u \\ \partial_{y_2}u = \frac{1}{\sqrt{n}}\partial_\nu u \\ \vdots \\ \partial_{y_n}u = \frac{1}{\sqrt{n}}\partial_\nu u\end{cases} \tag{4.10}$$

Since $y_1 + y_2 + \cdots + y_n = 1$,

$$\begin{aligned}&Yu\Big|_{F_0'} \\ &= (y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_n\partial_{y_n})u \\ &= (y_1 + y_2 + \cdots + y_n)\frac{1}{\sqrt{n}}\partial_\nu u \\ &= 1 \times \frac{1}{\sqrt{n}}\partial_\nu u \\ &= \frac{1}{\sqrt{n}}\partial_\nu u.\end{aligned} \tag{4.11}$$

Now the integration on the boundary can be simplified into the form: $\int_0^T \int_{F_0'} \frac{1}{\sqrt{n}}\partial_\nu u(\nu^T BB^T)\nabla u dS_0' dt$, which only involves face F_0'. As a result, the second approach (4.4) is simplified to:

$$\begin{aligned}\int_0^T \int_{\Omega'} [\partial_t^2 + P, Y]uudVdt = &\int_{\Omega'} \partial_t YuudV\Big|_0^T - \int_{\Omega'} Yu\partial tudV\Big|_0^T \\ &+ \int_0^T \int_{F_0'} \frac{1}{\sqrt{n}}\partial_\nu u(\nu^T BB^T)\nabla udS_0'dt\end{aligned} \tag{4.12}$$

We now study the first term of this simplified version (4.12), using integration by parts and the chain rule:

$$\begin{aligned}\int_{\Omega'} \partial_t Y u u dV\Big|_0^T &= -\int_{\Omega'} \partial_t u \sum_{j=1}^n \partial_{y_j}(y_j u) dV\Big|_0^T \\ &= -\int_{\Omega'} \partial_t u (nu + \sum_{j=1}^n y_j \partial_{y_j} u) dV\Big|_0^T \\ &= -n\int_{\Omega'} \partial_t u u dV\Big|_0^T - \int_{\Omega'} \partial_t u Y u dV\Big|_0^T .\end{aligned} \tag{4.13}$$

Then we have the second approach summarized as:

$$\begin{aligned}\int_0^T \int_{\Omega'} [\partial_t^2 + P, Y] u u dV dt = -n\int_{\Omega'} \partial_t u u dV\Big|_0^T &- 2\int_{\Omega'} \partial_t u Y u dV\Big|_0^T . \\ + \int_0^T \int_{F_0'} &\frac{1}{\sqrt{n}} \partial_\nu u (\nu^T B B^T) \nabla u dS_0' dt\end{aligned} \tag{4.14}$$

Combining this and (4.2), and re-organizing terms, we have:

$$\int_0^T \int_{F_0'} \frac{1}{\sqrt{n}} \partial_\nu u (\nu^T B B^T) \nabla u dS_0' dt = TE(0) + (n-1)\int_{\Omega'} \partial_t u u dV\Big|_0^T + 2\int_{\Omega'} \partial_t u Y u dV\Big|_0^T \tag{4.15}$$

Now to obtain the observability from face F_0', we are going to analyze the last two terms of (4.15) to determine whether we could absorb them into initial energy through estimation.

Firstly, to estimate the third term on the right side of (4.15), for some fixed time t_0, we use Cauchy's inequality and triangle equality to obtain:

$$\begin{aligned}|\int_{\Omega'} \partial_t u Y u dV\Big|_{t_0}| &\leqslant C\int_{\Omega'} |\partial_t u|^2 dV\Big|_{t_0} + C\int_{\Omega'} (\sum_{j=1}^n |y_j \partial_{y_j} u|)^2 dV\Big|_{t_0} \\ &\leqslant C\int_{\Omega'} |\partial_t u|^2 dV\Big|_{t_0} + C\int_{\Omega'} (\sum_{j=1}^n |\partial_{y_j} u|)^2 dV\Big|_{t_0} \\ &\leqslant C\int_{\Omega'} |\partial_t u|^2 dV\Big|_{t_0} + C\int_{\Omega'} (\sum_{i=1}^n |\partial_{y_i} u|)(\sum_{k=1}^n |\partial_{y_k} u|) dV\Big|_{t_0} \\ &= C\int_{\Omega'} |\partial_t u|^2 dV\Big|_{t_0} + C\int_{\Omega'} \sum_{i=1}^n \sum_{k=1}^n (|\partial_{y_i} u||\partial_{y_k} u|) dV\Big|_{t_0} \\ &\leqslant C\int_{\Omega'} |\partial_t u|^2 dV\Big|_{t_0} + C\int_{\Omega'} (|\partial_{y_1} u|^2 + |\partial_{y_2} u|^2 + \cdots + |\partial_{y_n} u|^2) dV\Big|_{t_0} \\ &\leqslant C\int_{\Omega'} |\partial_t u|^2 + |\nabla u|^2 dV\Big|_{t_0}\end{aligned} \tag{4.16}$$

Note that every coefficient C changes from line to line but they are independent from time variable t_0. By applying the **Lemma 3.2**, which can be done since $B^T B$ is positive definite, we have:

$$\begin{aligned} C\int_{\Omega'}|\partial_t u|^2+|\nabla u|^2dV\Big|_{t_0} &\leqslant C\int_{\Omega'}|\partial_t u|^2+\langle BB^T\nabla u,\nabla u\rangle dV\Big|_{t_0} \\ &\leqslant C\int_{\Omega'}|\partial_t u|^2+|B^T\nabla u|^2dV\Big|_{t_0} \\ &= CE(0) \end{aligned} \tag{4.17}$$

Thus, combining (4.16) and (4.17) gives :

$$\begin{aligned} |\int_{\Omega'}\partial_t uYudV\Big|_0^T| &\leqslant |\int_{\Omega'}\partial_t uYudV\Big|_{t=T}|+|\int_{\Omega'}\partial_t uYudV\Big|_{t=0}| \\ &\leqslant CE(0). \end{aligned} \tag{4.18}$$

Similarly, we perform another estimation for the second term of (4.15) by using the Cauchy inequality and the Poincaré inequality. Again, the coefficient C changes but are not depend on t.

$$\begin{aligned} (n-1)\int_{\Omega'}\partial_t uudV\Big|_0^T &\leqslant (n-1)(C\int_{\Omega'}|\partial_t u|^2dV\Big|_0^T+C\int_{\Omega'}|u|^2dV\Big|_0^T) \\ &\leqslant (n-1)(C\int_{\Omega'}|\partial_t u|^2dV\Big|_0^T+C\int_{\Omega'}|\nabla u|^2dV\Big|_0^T) \\ &\leqslant C\int_{\Omega'}|\partial_t u|^2+\langle BB^T\nabla u,\nabla u\rangle dV\Big|_{t_0} \\ &\leqslant C\int_{\Omega'}|\partial_t u|^2+|B^T\nabla u|^2dV\Big|_0^T \\ &= CE(0) \end{aligned} \tag{4.19}$$

Therefore combining (4.19) and (4.18) into (4.15) yields:

$$\int_0^T\int_{F_0'}\frac{1}{\sqrt{n}}\partial_\nu u(\nu^TBB^T)\nabla udS_0'dt=TE(0)+\mathcal{O}(1)E(0) \tag{4.20}$$

To obtain the observability on face of the original simplex Ω, we make the following transformation from the standard simplex Ω' back to the original simplex Ω.

Starting from the right side of (4.20), we first transform the energy back to the original simplex. By using the results introduced in C.2 and C.3 in **Appendix C**,

$dy = \frac{1}{\det(A)}dx$ and $\det(A) = n!Vol(\Omega)$. Therefore,

$$
\begin{aligned}
TE(0) + \mathcal{O}(1)E(0) &= (T + \mathcal{O}(1)) \int_{\Omega'} (|\partial_t u_1|^2 + |B^T \nabla u_0|^2) dV \\
&= (T + \mathcal{O}(1)) \int_{\Omega'} (|\partial_t u_1|^2 + |B^T \nabla u_0|^2) dy_1 dy_2 \dots dy_n \\
&= \frac{T + \mathcal{O}(1)}{det(A)} \int_{\Omega} |\partial_t u_1|^2 + |\nabla u_0|^2 dx_1 dx_2 \dots dx_n \\
&= \frac{T + \mathcal{O}(1)}{n!Vol(\Omega)} \int_{\Omega} |\partial_t u_1|^2 + |\nabla u_0|^2 dx_1 dx_2 \dots dx_n \\
&= \frac{T + \mathcal{O}(1)}{n!Vol(\Omega)} \tilde{E}(0)
\end{aligned}
\tag{4.21}
$$

On the left side of (4.20), to transform from face F_0' of standard simplex Ω' back to the face F_0 of original simplex Ω, we first change the graph coordinate dS_0' back to the rectangular coordinate:

$$
\begin{aligned}
F_0' &= \{y_n = 1 - y_1 - y_2 - \dots - y_{n-1}\} \\
\Rightarrow dS_0' &= (1^2 + (-1)^2 + \dots + (-1)^2)^{\frac{1}{2}} dy_1 dy_2 \dots dy_{n-1} \\
&= \sqrt{n} dy_1 dy_2 \dots dy_{n-1}.
\end{aligned}
\tag{4.22}
$$

Then, the left side of (4.20) can be written as:

$$
\begin{aligned}
\int_0^T \int_{F_0'} \frac{1}{\sqrt{n}} \partial_\nu u (\nu^T B B^T) \nabla u dS_0' dt &= \int_0^T \int_{\Omega'_{n-1}} \frac{\sqrt{n}}{\sqrt{n}} \partial_\nu u (\nu^T B B^T) \nabla u dy_1 dy_2 \dots dy_{n-1} dt \\
&= \int_0^T \int_{\Omega'_{n-1}} \partial_\nu u (\nu^T B B^T) \nabla u dy_1 dy_2 \dots dy_{n-1} dt \\
&= \frac{1}{(n-1)!Area(F_0)} \int_0^T \int_{F_0} \partial_\nu u (\nu^T) \nabla u dS_0 dt \\
&= \frac{1}{(n-1)!Area(F_0)} \int_0^T \int_{F_0} (\partial_\nu u)(\partial_\nu \bar{u}) dS_0 dt \\
&= \frac{1}{(n-1)!Area(F_0)} \int_0^T |\partial_\nu u|^2 dS_0 dt.
\end{aligned}
\tag{4.23}
$$

By equating (4.21) and (4.23) through (4.20), we could get our final conclusion of the observability from one face of the original simplex $\Omega \subseteq \mathbb{R}^n$:

$$
\begin{aligned}
\int_0^T \int_{F_0} |\partial_\nu u|^2 dS_0 dt &= \frac{(n-1)!Area(F_0)}{n!Vol(\Omega)} E(0)(T + \mathcal{O}(1)) \\
&= \frac{TArea(F_0)}{nVol(\Omega)} E(0) \left(1 + \mathcal{O}\left(\frac{1}{T}\right)\right).
\end{aligned}
\tag{4.24}
$$

This finishes the proof of **Theorem 1.1**.

Appendices

A. CHANGE VARIABLES

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$. Based on the relation, we have $v(x) = v(Ay)$ where v is a function.

According to chain rule, we know that,

$$\begin{aligned} \partial_{y_j} v(Ay) &= \partial_{y_j} v \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n \end{bmatrix} \\ &= v_{x_1}a_{1j} + v_{x_2}a_{2j} + \dots + v_{x_n}a_{nj} \\ &= \sum_{k=1}^{n} v_{x_k}(Ay)a_{kj} \\ &= \nabla_x^T v\Big|_{x=Ay} \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}, j = 1, 2, \dots, n. \end{aligned} \tag{A.1}$$

Then we have

$$\begin{aligned} \nabla_y(v(Ay)) = \begin{bmatrix} \partial_{y_1} v(Ay) \\ \partial_{y_2} v(Ay) \\ \vdots \\ \partial_{y_n} v(Ay) \end{bmatrix} &= \begin{bmatrix} \nabla_x^T v\Big|_{x=Ay} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \\ \nabla_x^T v\Big|_{x=Ay} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \\ \vdots \\ \nabla_x^T v\Big|_{x=Ay} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \end{bmatrix} \\ &= A^T \nabla_x v(Ay) \end{aligned} \tag{A.2}$$

Therefore, $\nabla_x = (A^{-1})^T \nabla_y = B^T \nabla_y$.

B. Simplex Volume

We used the fact that the volume of a n-dimensional standard simplex is $\frac{1}{n!}$ in the main proof. The proof by induction is presented as following:

Proof. When $n = 2$, the standard simplex is spanned by two vectors $\vec{v_1} = [0,1]$ and $\vec{v_2} = [1,0]$. We have its area equals to $\frac{1}{2} = \frac{1}{2!}$.

Given a standard simplex $S \in \mathbb{R}^{n-1}$, assume $Vol(S) = \frac{1}{(n-1)!}$. Then for the standard simplex $T \in \mathbb{R}^n$ with rectangular coordinates $[t_1, t_2, \ldots, t_n]$, we have this relation satisfied:

$$t_1 + t_2 + \cdots + t_n \leqslant 1 \tag{B.1}$$

Assume $t_n = k$ for some constant $0 \leqslant k \leqslant 1$, then: $t_1 + t_2 + \cdots + t_{n-1} \leqslant 1 - k$. The volume of this simplex $S' \in \mathbb{R}^{n-1}$ is:

$$\int_{t_1=0}^{1-k} \int_{t_2=0}^{(1-k)-t_1} \cdots \int_{t_{n-1}=0}^{(1-k)-t_1-t_2-\cdots-t_{n-2}} d_{t_{n-1}} \ldots dt_2 dt_1 \tag{B.2}$$

To use the method of integration by substitution, for each j from 1 to $n-1$, let $s_j = \frac{t_j}{1-k}$ and therefore $(1-k)ds_j = dt_j$. Regarding the bounds of the integral, all t_js can be substituted by $(1-k)s_j$s. In particular, we have every s_j bounded by:

$$\begin{aligned} &\Rightarrow 0 \leqslant (1-k)s_j \leqslant (1-k) - (1-k)s_1 - (1-k)s_2 - \cdots - (1-k)s_{j-1} \\ &\Rightarrow 0 \leqslant s_j \leqslant 1 - s_1 - s_2 - \cdots - s_{j-1} \end{aligned} \tag{B.3}$$

Because we assumed that the volume of the standard simplex S is $\frac{1}{(n-1)!}$, the volume of this (n-1)-dimensional simplex S' can be simplified to the form of:

$$(1-k)^{n-1} \int_{s_1=0}^{1} \int_{s_2=0}^{1-s_1} \cdots \int_{s_{n-1}=0}^{1-s_1-s_2-\cdots-s_{n-2}} ds_{n-1} \ldots ds_2 ds_1 = \frac{(1-k)^{n-1}}{(n-1)!}. \tag{B.4}$$

After integrating by variable k from 0 to 1, we get the volume of the standard simplex $T \in \mathbb{R}^n$:

$$\int_0^1 \frac{(1-k)^{n-1}}{(n-1)!} dk = \frac{1}{n!}. \tag{B.5}$$

□

C. Determinant and Volume

Using the same notation indicated in the main proof. The rectangular coordinate of the original simplex x and that of the standard simplex Ω' is y is related by:

$$\vec{x} = A\vec{y} \tag{C.1}$$

where A is the matrix with its columns equal to vectors $\vec{p_1}, \vec{p_2}, \ldots, \vec{p_n}$. Their derivatives satisfies:

$$dx_1 dx_2 \ldots dx_n = det(A) dy_1 dy_2 \ldots dy_n \tag{C.2}$$

where $det(A)$ denotes the determinant of the Jacobian matrix of the first partial derivatives. Furthermore, by the conclusion of **Appendix B**, we know that the volume of the standard simplex Ω' is $\frac{1}{n!}$ and thus we have:

$$\begin{aligned}
&\frac{1}{det(A)}\int_{\Omega} dx_1 dx_1 \ldots dx_n = \int_{\Omega'} dy_1 dy_2 \ldots dy_n \\
&\Rightarrow \frac{1}{n!} = \frac{1}{det(A)} Vol(\Omega) \\
&\Rightarrow det(A) = n! Vol(\Omega)
\end{aligned} \tag{C.3}$$

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