The Sharp Lifespan for Quasilinear Wave Equations in Exterior Domains with Polynomial Local Energy Decay

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Abstract

JOHN ALBERT HELMS: The Sharp Lifespan for Quasilinear Wave Equations in Exterior Domains with Polynomial Local Energy Decay
(Under the direction of Professor Jason Metcalfe)

We investigate the lifespan of quasilinear Dirichlet-wave equations of the form $(\partial_t^2 - \Delta)u = Q(u, u', u'')$ in $[0, T] \times \mathbb{R}^3 \backslash \mathcal{K}$, where $\mathcal{K}$ is a bounded domain with smooth boundary. Previous results have demonstrated long time existence in the case that $\mathcal{K}$ was assumed to be star-shaped. We show that the same lifespan holds for more general geometries, where we only assume a polynomial local decay of energy with a possible loss of regularity for solutions to the linear homogeneous wave equation.

I dedicate this paper to my parents and brother, David, Cathy and Paul Helms.

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CHAPTER 1

Introduction

This paper shall prove sharp lower bounds of lifespans to solutions to certain quasilinear, multiple-speed Dirichlet-wave equations in exterior domains in three spatial dimensions. In particular we will consider solutions to quasilinear wave equations with small data whose domains are $[0,T]\times\mathbb{R}^3\backslash\mathcal{K}$, where $\mathbb{R}^3\backslash\mathcal{K}$ is the complement of a bounded domain $\mathcal{K}\subset\mathbb{R}^3$ with smooth boundary $\partial\mathcal{K}$.

Before stating the form of the wave equations that we will be considering, we will define all of the common notation that we will be using. We shall take Δ to be the the standard Laplacian on $\mathbb{R}^n$:

$$\Delta=\sum_{j=1}^{n}\partial_j^2.$$

As we are dealing with multiple-speed systems of wave equations, we shall let u be vector-valued in $\mathbb{R}^D$. We will define our wave operator $\Box$ to be the vector-valued, multiple-speed d'Alembertian

$$\Box=\mathrm{diag}(\Box_{c_1},\dots,\Box_{c_D}), \tag{1.1}$$

where

$$\Box_{c_I}u=(\partial_t^2-c_I^2\Delta)u,$$

The constants c_I, which are referred to as the wave speeds, are assumed to be positive but not necessarily distinct. Since we are dealing with systems of wave equations with multiple wave speeds, we will require estimates that apply in this setting. The only estimates where we will have to be mindful of the fact that we are dealing with a system of wave equation with multiple speeds will be those in Section 3.1. The remaining estimates in Chapters 2 and 3 will be proved only for scalar unit-speed wave equations. This shall not present a problem to us since the estimates for the unit-speed wave equation easily extend to equations with multiple speeds. When we are dealing with scalar wave equations, we will slightly abuse the notation in (1.1) and let $\Box = \partial_t^2 - \Delta$, the unit-speed wave operator.

We define the spacetime gradient to be

$$\nabla_{t,x} u = u' = (\partial_t u, \nabla_x u),$$

and u'' to be the collection of all second-order partial derivatives of u. We define the radial derivative ∂_r to be the vector field such that $\partial_r u = \left\langle \frac{x}{r}, \nabla_x \right\rangle u$. We also define the angular derivative of u to be $\not\nabla_i u = \partial_i u - \frac{x_i}{r}\partial_r u$, which is the standard i-th derivative of u minus its radial component in the x_i direction. Similar to the standard gradient, we define the angular gradient to be

$$\not\nabla u = (\not\nabla_1 u, \ldots, \not\nabla_n u).$$

Due to the fact that we will be dealing with a systems of D wave equations, we will write u to represent the vector-valued function u in $\mathbb{R}^D$ whose components are u^I. To see the connection between this definition and the standard gradient, one can use the orthogonality of the angular and radial vector fields to check that the following identity holds:

$$|\nabla_x u|^2 = |\not\nabla u|^2 + |\partial_r u|^2 .$$

One can also decompose the Laplacian into its radial and angular components to obtain the identity:

$$\Delta = \partial_r^2 + \frac{(n-1)}{r}\partial_r + \not\nabla \cdot \not\nabla.$$

In the context of this paper, by requiring the nonlinearity Q to be quasilinear, we shall assume that Q be a smooth function of u, u', u'' vanishing up to second order and linear in u''. We shall also assume that the highest order terms are symmetric. This means that Q has the form:

$$Q^I(u,u',u'') = A^I(u,u') + \sum_{1\le J\le D} \sum_{i,j=0}^{3} B^{ij,IJ}(u,u')\partial_i\partial_j u^J, \quad 1 \le I \le D, \tag{1.2}$$

where A^I and $B^{ij,IJ}$ are smooth functions such that each A^I vanishes to second order and the $B^{ij,IJ}$ vanish to first order and satisfy the following symmetry conditions

$$B^{ij,IJ} = B^{ji,IJ} = B^{ij,JI}. \tag{1.3}$$

Due to the fact that we will be dealing with small initial data, we will only deal with the lowest order terms in the Taylor expansion for Q. By this, we mean that we will truncate Q at the quadratic level so that each A^I is a quadratic form of u and u' and that each $B^{ij,IJ}$ is a linear form in u and u'. The case where Q has higher order terms can be dealt with using the fact that these higher order terms are easier to control in the iteration argument that is presented in Chapter 4. It will be clear that this simplification does not affect the lifespan since the lifespan. We will also write A to represent the vector u in $\mathbb{R}^D$ whose components are A^I.

Throughout the paper, we define a quasilinear function to be a function of u, u', u''. The letter Q shall always denote a quasilinear function. We shall use the notation $Q(u, u', u'')$ to emphasize that Q is a quasilinear function depends on u, u' and u''. In some discussions of previous results where the nonlinearity depends only on u' and u'', we will write Q as $Q(u', u'')$. It should be noted that the dependency of Q on u introduces more complications than are present in cases where Q does not depend on u. This shall be made clearer in the discussion of the past results for nonlinear wave equations.

In many estimates we will use the convention $A \lesssim B$ which will mean there is a constant $C > 0$ that will be independent of important parameters such that $A \leq CB$. Also, unless otherwise specified, if the constant C is explicitly written in an estimate, it is assumed that C is allowed to vary from line to line. From the proofs of the estimates, it will be clear that C is independent of important parameters, such as the time, T, and the size of the initial data, ϵ.

We shall be considering wave equations of the following form:

$$\begin{cases} \Box u(t,x) = Q(u, u', u''), \quad (t,x) \in [0,T] \times \mathbb{R}^3 \backslash \mathcal{K}, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \\ u(t,x) = 0, \quad x \in \partial\mathcal{K}. \end{cases} \tag{1.4}$$

Without loss of generality, one can use scaling and translation to assume that the obstacle $\mathcal{K}$ is contained in $\{|x| < 1\}$ and that $0 \in \mathcal{K}$. We will assume this throughout the paper. Note that $\mathcal{K}$ is not necessarily connected. As indicated in the third line in (1.4), we are also assuming that the solution u vanishes on $\partial\mathcal{K}$.

1.1. Geometric Assumptions

Unless specifically noted, in each estimate of this paper we shall assume that the exterior of $\mathcal{K}$ satisfies a local energy decay condition with a possible loss in regularity. We suppose that there exist a positive integer M and constants $A_0, \sigma > 0$ such that if $u \in C^\infty([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})$ solves

$$\begin{cases} \Box u(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3 \backslash \mathcal{K}, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \\ u(t,x) = 0, \quad x \in \partial \mathcal{K}. \end{cases} \tag{1.5}$$

where (f, g) are supported on $\{|x| < 10\}$, then the following estimate holds:

$$\left(\int_{\{x \in \mathbb{R}^3 \backslash \mathcal{K} : |x| < 10\}} \left| u'(t,x) \right|^2 \, dx \right)^{1/2} \leq A_0 \langle t \rangle^{-2-\sigma} \sum_{|\alpha| \leq M} \left\| \partial_x^\alpha u'(0, \cdot) \right\|_2 . \tag{1.6}$$

Informally, this assumption says that for solutions to the linear wave equation with compactly supported initial data, one has a polynomial local energy decay estimate with a loss in regularity on the right hand side. One can also think of this assumption as embodying the physical notion that after an initial disturbance near a reflecting obstacle, the energy of that disturbance propagates away from the obstacle at a fixed rate. This rate is determined by the geometry of the boundary of the obstacle.

The motivation for our assumption (1.6) comes from a long history of results proving local energy decay for a variety of geometries. One of the main objectives of these studies was to investigate the relationship between the geometry of the boundary of the obstacle and the rate of local energy decay. Two early papers [**50**, **51**] by Morawetz studied the decay of solutions of wave equations in the exterior of star-shaped domains in three spatial dimensions. In [**50**], Morawetz used a local energy decay estimate to prove pointwise decay rates for solutions to the linear wave equation. In [**51**], stronger local energy decay estimates were used to study the asymptotic behavior for wave equations with a harmonic potential. A seminal paper by Lax, Morawetz and Phillips [**36**] used the results in [**50**] to show that the local energy decays exponentially when the obstacle is star-shaped. In the context of our problem, their result can be stated as follows.

THEOREM 1.1. *(Lax, Morawetz, Phillips [**36**]) Suppose that $u \in C^\infty([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})$ solves* (1.5) *and that the initial data f, g have supports that are contained in the set $\{|x| < 10\}$. If $\mathcal{K}$ is star-shaped, then it follows that there are uniform constants $a_1, a_2 > 0$ such that for $0 \leq t \leq T$,*

$$\left\| u'(t, \cdot) \right\|_{L^2(\{x \in \mathbb{R}^3 \backslash \mathcal{K} : |x| < 10\})} \leq a_1 e^{-a_2 t} \left\| u'(0, \cdot) \right\|_2 . \tag{1.7}$$

This result was complemented by a later paper [**52**] which used sharp Huygens' principle to prove that for any exterior domain in $\mathbb{R}^n$, where $n \geq 3$ is odd, if the local energy decays to zero, then it actually does so at an exponential rate.

Lax and Phillips proposed in [**37**] that the decay rate of local energy near an obstacle was closely tied to the behavior of geodesics in the exterior of the obstacle. Specifically, they conjectured that the local energy decays to zero at a uniform rate if and only if the geometry of the exterior of the obstacle is *nontrapping*. A Riemannian manifold M is said to be nontrapping if for every compact set K, all geodesics that start in K escape from K within a fixed period of time. More precisely, this means that for any compact set $K \subsetneq M$, there is a time $T(K) > 0$ such that for any unit-speed geodesic η such that $\eta(0) \in K$, it follows that there a time $0 \leq \tau \leq T(K)$, such that $\eta(\pm\tau) \in M \backslash K$. The conjecture of Lax and Phillips was subsequently proved in one direction by Ralston in [**57**]. He showed that if the exterior geometry has trapped geodesics, then one cannot obtain energy decay estimates such as (1.7). In our context, Ralston's result implies the following theorem.

Theorem 1.2. *(Ralston [**57**]) Suppose that the region $\{x \in \mathbb{R}^3 \backslash \mathcal{K} : |x| \leq 10\}$ fails the nontrapping condition as stated above. Then for any $\mu > 0$ and any time $t_0 > 0$, it follows that one can construct initial data $f, g \in C_c^\infty(\{x \in \mathbb{R}^3 \backslash \mathcal{K} : |x| \leq 10\})$ such that the solution $u \in C^\infty([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})$ that solves* (1.5) *with f, g as initial data satisfies the inequality*

$$\left\| u'(t_0, \cdot) \right\|_{L^2(\{x \in \mathbb{R}^3 \backslash \mathcal{K} : |x| \leq 10\})} > (1 - \mu) \left\| u'(0, \cdot) \right\|_2 .$$

The other direction to the conjecture of Lax and Phillips was resolved for the 3 dimensional case by Morawetz, Ralston and Strauss in [**54**] when they showed that (1.7) holds provided that the geometry of the exterior of the obstacle is nontrapping. They also showed that for all even dimensions, the rate of decay for the local energy is at least $O(t^{-1})$. Melrose [**41**] improved this estimate by showing that for all even dimensions n, the rate of decay is actually $O(t^{-n/2})$. This estimate was further strengthened by Ralston [**56**] who showed that for even dimensions n, one can actually show that the local energy decays like $O(t^{-(n-1)})$. One can also see Vodev [**70**] for results in nontrapping metrics. Thus, it is clear that our hypothesis (1.6) holds when the exterior of the obstacle $\mathcal{K}$ is nontrapping.

The first major step in finding local energy decay estimates in trapping geometries came from Ikawa [**17, 18**]. For exterior domains consisting of a finite number of convex obstacles satisfying certain technical assumptions (see Theorem 1 in [**18**]), there is an exponential decay of local energy with a possible loss in regularity. For the results of this paper, Ikawa's theorem implies the following estimate.

THEOREM 1.3. *(Ikawa [**18**]) Let $\mathcal{K}$ consist of a finite number of convex domains that are sufficiently separated. Also let $u \in C^\infty([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})$ solve* (1.5) *and suppose that the initial data f, g have supports that are contained in the set $\{|x| < 10\}$. Then it follows that there are uniform constants $a_1, a_2 > 0$ such that for $0 \le t \le T$,*

$$\left\|u'(t,\cdot)\right\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<10\})} \le a_1 e^{-a_2 t} \sum_{|\alpha|\le 2} \left\|\partial^\alpha u'(0,\cdot)\right\|_2 .$$

Ikawa's result demonstrates that in certain trapping geometries, it is possible to obtain uniform energy decay estimates provided one allows for a loss of regularity. From this discussion, it is clear that the results of Ralston [**56, 57**], Morawetz-Ralston-Strauss [**54**] and Ikawa [**17, 18**] illustrate a dichotomy between the local energy decay estimates that are available in trapping and those that apply in nontrapping geometries. Thus, our assumption (1.6) holds in some trapping geometries, such as in the examples provided by Ikawa [**17, 18**].

We know from a paper by Burq [**2**] that if one allows for a sufficient loss in regularity, then one can show that local energy decays at a logarithmic rate in *any* domain that is the exterior of a compact obstacle. However, we also know from Ralston [**58**] that if the trapped rays are sufficiently stable in the sense that they cause nearby geodesics to remain near them indefinitely, then one cannot generally expect for any exponential energy decay estimate to hold. Thus, we should only expect our hypothesis to hold provided that all of the trapped geodesics are sufficiently unstable.

It should be noted that the study of local energy decay of solutions to the linear wave equations is closely related to the study of local smoothing for the linear Schrödinger equation, which dates back to the work of Sjölin [**65**], Constantin and Saut [**8**], and Vega [**69**]. There are also more recent results on local energy decay estimates and local smoothing in the presence of hyperbolic trapped rays. For example, the reader should see de Verdière-Parisse [**9**], Burq, Gerard and Tzvetkov [**3**], Burq-Zworski [**4**], Christianson [**5**], Nonnenmacher-Zworski [**55**], Wunsch-Zworski [**71**] and Christianson-Wunsch [**6**].

1.2. Main Theorem

Before stating the main theorem, we must also impose certain well-known "compatibility conditions" on the initial data (f, g) so that they agree with the Dirichlet boundary conditions on $\partial\mathcal{K}$. We first let $J_k u = \{\partial_x^\alpha u : 0 \leq |\alpha| \leq k\}$ denote the collection of spatial derivatives of u up to order k. If we fix m and suppose that u is a formal solution to (1.4) in H^m, then it follows that we can write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, for $0 \leq k \leq m-1$. By using the relationship given by $\Box u = Q(u, u', u'')$, we can see that ψ_k depend on Q as well as $J_k f$ and $J_{k-1} g$. We then say that $(f, g) \in H^m \times H^{m-1}$ satisfy the compatibility conditions up to order m if ψ_k vanishes on $\partial\mathcal{K}$ for $0 \leq k \leq m-1$. We say that f, g satisfy the compability conditions to infinite order if this holds for all m. A simple example of these conditions can be seen by considering the homogeneous wave equation $\Box u = 0$ with initial data $(f, g) \in C_c^\infty$. Due to the fact that we want $\partial_t^2 u(0, x)$ to equal zero on $\partial\mathcal{K}$, we can use the wave equation $\Box u = 0$ to conclude that $\Delta u(0, x) = \Delta f(x) = 0$ on $\partial\mathcal{K}$ is a compatibility condition if we want to solve the linear wave equation in an exterior domain. For more details, the reader should refer to [**27**]. We now state the main theorem:

Theorem 1.4. *Let $\mathcal{K}$ be a fixed bounded domain with smooth boundary that satisfies* (1.6). *Assume also that Q and $\Box$ are as in* (1.2), (1.3) *and* (1.1), *respectively. Suppose that $(f, g) \in C_c^\infty(\mathbb{R}^3 \backslash \mathcal{K})$ satisfy the compatibility conditions to infinite order. Also assume that there is a fixed $R^* > 0$ such that $f(x)$ and $g(x)$ vanish for $|x| > R^*$. Then there are constants $c, \epsilon_0 > 0$ and an integer $N > 0$ such that for all $\epsilon \leq \epsilon_0$, if*

$$\sum_{|\alpha| \leq N} \|\partial_x^\alpha f\|_2 + \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha g\|_2 \leq \epsilon, \tag{1.8}$$

then (1.4) *has a unique solution $u \in C^\infty([0, T_\epsilon] \times \mathbb{R}^3 \backslash \mathcal{K})$, where*

$$T_\epsilon \geq \frac{c}{\epsilon^2}. \tag{1.9}$$

It should be emphasized that in this case, we are assuming that (f, g) are smooth, compactly supported functions that vanish outside of a fixed set $\{|x| < R^*\}$.

Before, we proceed, we shall some explain some vocabulary that is commonly used when discussing lifespans of wave equations with small data. A solution u is said to exist *almost globally* if $u(t, \cdot)$ exists in the classical sense (that is, u solves (1.4) and lies in $C^2([0, T] \times \mathbb{R}^3 \backslash \mathcal{K})$), where the lifespan T grows exponentially as the size of the intial data, ϵ, shrinks to zero. A solution u is said to *exist globally* if $u(t,\cdot)$ exists in the classical sense for all time.

1.3. Past Results on the Wave Equation in Minkowski Space

Early works on the wave equation such as Lax [**35**] and John [**19**] demonstrated that 1-dimensional wave equations in $\mathbb{R}^{1+1}$ that are "genuinely nonlinear" inevitably develop singularities in finite time that is on the order of $1/\epsilon$, where ϵ is the size of the initial data. In a follow-up to [**19**], John [**20**] proved rough lower bounds on the lifespan of wave equations in $\mathbb{R}^{1+n}$, where $n \geq 3$, that demonstrated that one can get better lifespan bounds in dimensions $n \geq 3$. Specifically, John considered scalar quasilinear wave equations of the form

$$\begin{cases} \Box u(t,x) = \sum_{i,j=0}^{n} a_{ij}(u')\partial_i\partial_j u, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \end{cases} \tag{1.10}$$

where (a_{ij}) is a symmetric matrix with coefficients that are smooth for $|u'|$ small. He showed that if ϵ is small, then the lifespan T satisfies the bounds

$$T \geq C(\epsilon \log(1/\epsilon))^{-4}, \quad n = 3,$$
$$T \geq C/\epsilon^{-2}, \quad n > 3.$$

John explained that, in higher dimensions, one should observe that the higher rate of decay of solutions will delay the onset of singularities.

The next major breakthrough came from Klainerman [**29**] who considered scalar nonlinear wave equations of the form

$$\begin{cases} \Box u(t,x) = F(u', u''), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \end{cases} \tag{1.11}$$

where F is a nonlinear function that is smooth near the origin and vanishes up to second order. Note that, unlike the nonlinearity Q in this paper, F can depend on u'' in a nonlinear manner. Klainerman showed that solutions to (1.11) with sufficiently small initial data actually exist globally for all time for dimensions $n \geq 6$. This result was also reproved by Shatah [**59**] and Klainerman-Ponce [**33**] using simpler methods.

A key point of concern in this line of research was the physically significant case $n = 3$ since lifespan bounds for this dimension appeared to be more difficult to attain. A relevant point to our current result is that John was also able to show in [**21**] that solutions to $\Box u = u^2$ in 3 dimensions blow up in finite time, $T = C/\epsilon^2$. For more on the behavior of such solutions, the reader can also see Lindblad [**39**]. Since $\Box u = u^2$ is a special case of (1.4), it follows from finite propagation speed that the best lower bound for the lifespan that we can hope for in our current setting is $T \geq C/\epsilon^2$. John also showed in [**22**] that one cannot generally hope for global existence in 3 dimensions, even when the nonlinearity does not depend on the solution u itself, when he proved that there is a class of solutions to (1.11) which blow up at time $T = c_1 \exp(c_2/\epsilon^2)$. For related results, see John [**23**], Klainerman [**30**] and Sideris [**61**].

John and Klainerman [**25**] later proved that in 3 dimensions, with certain mild conditions on F, solutions to (1.11) exist almost globally with a lower lifespan bound of $T \geq c_1 \exp(c_2/\epsilon)$. The key innovation behind this result was using the translation and rotation-invariance of the wave operator to prove their lifespan in conjunction with weighted estimates for the inhomogeneous wave equation. The collection of vector fields they used consisted of

$$\partial_t = \partial_0 = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \tag{1.12}$$

and Euclidean rotations,

$$\Omega^{jk} = x_j \partial_k - x_k \partial_j, \quad 1 \leq j < k \leq n. \tag{1.13}$$

By translation and rotation-invariant, we refer to the fact that the vector fields in (1.12) and (1.13) commute with the single-speed wave operators $\Box_c = (\partial_t^2 - c^2 \Delta)$. Due to the fact that this collection of vector fields will be essential in the main result of this paper, we will write Z to denote a translation or spatial rotation:

$$Z = \{\partial_i, \Omega^{ij} : 0 \leq i \leq 3, 1 \leq j < k \leq 3\}.$$

Throughout the paper, we will use multi-indices when different kinds of vector fields are being applied to a function, such as with the collection Z of translations and spatial rotations. For example, if $V = \{V_1, \dots, V_N\}$ is a collection of vector fields, then for $u \in C^\infty$,

$$V^\alpha u = V_1^{\alpha_1} \cdots V_N^{\alpha_N} u, \quad \alpha = (\alpha_1, \dots, \alpha_N).$$

Klainerman later employed his method of using the invariant vector fields in [**31**] to prove global existence to (1.11) in dimensions $n \geq 4$ and almost global existence in dimensions $n = 3$ with the same lifespan bound as [**25**]. In this paper, Klainerman used the Lorentz-invariance of the wave operator. That is, in addition to translations and spatial rotations, he also used the fact that Lorentz boosts,

$$\Omega^{0j} = t\partial_j + x_j\partial_t, \quad 1 \leq j \leq n, \tag{1.14}$$

commute with the unit-speed wave operator $\Box$:

$$[\Omega^{0j}, \Box] = 0, \quad 1 \leq j \leq n.$$

He also used the scaling vector field,

$$L = t\partial_t + r\partial_r, \quad \text{where } r = |x|. \tag{1.15}$$

One can check that the scaling vector field L *almost* commutes with the d'Alembertian:

$$[L, \Box] = -2\Box. \tag{1.16}$$

We shall return to discussing these commutator properties in Chapter 2.

While one cannot hope in general to obtain global solutions to (1.11), an advancement was made by placing certain restrictions on the nonlinearity. Using Klainerman's suggestion of imposing a "null condition" on the nonlinearity, Christodoulou [**7**] and Klainerman himself [**32**] independently proved global existence for single-speed systems of nonlinear wave equations with small data in $n = 3$. While we will not go into any detail regarding the null condition, it is worth noting that the null condition requires that the nonlinearity be quasilinear.

While the results of Klainerman and Christodoulou included applications to systems of wave equations, their limitation was that their result only applied to systems of a single wave speed. A major step in understanding multiple-speed systems of nonlinear wave equations came from Klainerman and Sideris [**34**] when they proved almost global existence for multiple-speed systems of quadratic, divergence-form wave equations, which have applications in classical problems such as elasticity. This result not only gave an improved version of an older proof of the same result by John [**24**], but it also used a smaller collection of invariant vector fields than those used in [**31**]. This enabled them to adapt Klainerman's method of invariant vector fields to classical systems that are not relativistic in nature. That is, these systems have multiple, distinct wave speeds associated with them. Specifically, Klainerman and Sideris only used translations, modified Euclidean rotations along with scaling. The absence of Lorentz boosts from this collection comes from the fact that Lorentz boosts have an associated wave speed and do not have suitable commutator properties with the vector-valued d'Alembertian in systems where there are distinct wave speeds. For the remaining vector fields, the same commutator properties hold. It turns out that if we replace $\Box$ with a scalar wave operator with an associated speed $c > 0$, $\Box_c = \partial_t^2 - c^2\Delta$, we get

$$\begin{aligned} &[Z, \Box_c] = 0, \\ &[L, \Box_c] = -2\Box_c. \end{aligned} \tag{1.17}$$

More results concerning multiple speeds and elasticity are presented in Sideris [**62, 63**] and Agemi [**1**]. There are also more recent results on multiple-speed systems of wave equations. For example, the reader can consult Sideris and Tu [**64**], Sogge [**66**], and Yokoyama [**72**].

Lifespan bounds for the boundaryless version of (1.4) for a single wave speed was resolved by Lindblad [**40**] in which he showed that the lifespan of such equations satisfies the bound $T \geq C/\epsilon^2$. Due to the previous work of John [**21**], we know that Lindblad's result is sharp. This also means that the lifespan (1.9) must also be sharp. For more results in this direction, see Hörmander [**15**] and Li-Xin [**38**].

1.4. Past Results on Dirichlet-Wave Equations

The first major results that resolved many of the issues with extending the earlier results for wave equations in Minkowski space to exterior domains were due to Keel, Smith and Sogge [**26–28**]. In [**26**], Keel, Smith and Sogge were able to prove almost global existence in 3 dimensions for semilinear wave equations. Following up on this result in [**27**], Keel, Smith and Sogge were able to prove the global existence theorems of Klainerman [**32**] and Christodoulou [**7**] for certain exterior domains. Specifically, they demonstrated global existence of small data solutions to single-speed systems of quasilinear wave equations

$$\begin{cases} \Box u(t,x) = Q(u', u''), \quad (t,x) \in [0,T] \times \mathbb{R}^3 \backslash \mathcal{K}, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \\ u(t,x) = 0, \quad x \in \partial\mathcal{K}, \end{cases} \tag{1.18}$$

where the obstacle $\mathcal{K}$ is star-shaped and Q is a smooth function that vanishes to second order and satisfies a null condition.

Keel, Smith and Sogge also showed in [**26**] that if one were to eliminate the dependence of Q on the second derivatives of u, then one can show almost global existence to scalar quasilinear wave equations where the exterior of $\mathcal{K}$ is nontrapping. This proved the semilinear analogue of John and Klainerman's [**25**] almost global existence theorem. Under stronger geometric assumptions, Keel, Smith and Sogge [**28**] were able to extend the work of Klainerman and Sideris [**34**] by proving almost global existence to multiple-speed systems of quasilinear wave equations of the form

$$\begin{cases} \Box_{c_I} u^I(t,x) = Q^I(u', u''), \quad (t,x) \in [0,T] \times \mathbb{R}^3 \backslash \mathcal{K}, \quad 1 \le I \le D, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \\ u(t,x) = 0, \quad x \in \partial\mathcal{K}, \end{cases} \tag{1.19}$$

where $\mathcal{K}$ is star-shaped. The main innovation of the last two papers was the adaptation of Klainerman's method of using invariant vector fields to exterior domain problems. Using elliptic regularity estimates, Keel, Smith and Sogge were able to incorporate spatial translations into their estimates. In addition to their difficulties with multiple-speed systems, Lorentz boosts seem to be ill-suited for exterior domain problems, even when the system has only one wave speed. This is due to the fact that the tangential component of the boost becomes unbounded as $t \to \infty$. On the other hand, as demonstrated in [**28**], the scaling vector field can still be useful in exterior domain problems, since its worst component $t\partial_t$ preserves the Dirichlet boundary conditions despite being unbounded as $t \to \infty$.

In [**28**], the scaling vector field was used to prove the necessary L^2 estimates to prove almost global existence. Due to their use of the scaling vector field, Keel, Smith and Sogge modified the proofs to variable coefficient energy estimates so that they could control the energy norms that involved scaling vector fields in exterior domains. They also demonstrated the usefulness of weighted $L^2_t L^2_x$ estimates to handle the lower order terms that arise in dealing with exterior domain problems. Another key ingredient in their proof was establishing an exterior domain version of Hörmander's L^1, L^∞ estimate [**16**] that did not involve Lorentz boosts.

Another tool used by Keel, Smith and Sogge in both [**27**] and [**28**] that played a prominent roll in their proofs were the local energy decay estimates (1.7) of Lax-Morawetz-Phillips [**36**] and Morawetz-Ralston-Strauss [**53**]. However, it should be noted that a later paper by Metcalfe and Sogge [**48**] was able to reprove the result of [**28**] without using the scaling vector field and without using the local energy decay of Lax, Morawetz and Phillips [**36**]. One should also see Metcalfe [**42**], Metcalfe-Sogge [**46**], Shibata and Tsutsumi [**60**], and Hayashi [**13**] for global results for nonlinear equations in higher dimensions $n \geq 4$. For a recent result on certain kinds of semilinear wave equations in $n = 3, 4$ dimensions, see Yu [**73**].

The explicit use of the star-shaped hypothesis in papers such as [**28**,**48**] begged the question if it were possible to prove analogous results in exterior domains where one were to weaken the geometric assumptions to allow for some degree of trapping. Metcalfe-Sogge [**45**] gave an affirmative answer to this question by proving global existence to (1.19) under the assumption that Q satisfies a null condition and that one assumes exponential decay of local energy with a possible loss of regularity for solutions to the linear wave equation whose initial data are supported in a bounded neighborhood containing $\mathcal{K}$. Using interpolation, one can deduce that the example of Ikawa [**17**, **18**] satisfies this condition. Not only did their methods prove the theorem of [**27**] under weaker geometric assumptions but they were also able to handle multiple-speed systems. To do this, they also had to devise estimates that used their local energy decay assumption to control L^2 energy norms where scaling vector fields were being applied to the solution. For our purposes, the estimates of Metcalfe-Sogge [**45**] that utilize local energy decay to deal with the scaling vector fields shall be vital in this paper. The result of Metcalfe-Sogge [**45**] was further generalized by Metcalfe, Nakamura and Sogge [**44**] who showed that under a weaker null condition, one can still prove global existence. Metcalfe, Nakamura and Sogge [**43**] later strengthened this result to include a larger class of quasilinear wave equations, including some such that the nonlinearity depends on u at the cubic level. Another paper by Metcalfe and Sogge [**47**] was able to prove global existence for certain null form wave equations that were addressed in [**44**] using techniques that did not require one to distinguish the scaling vector fields from translations and rotations.

The result of Lindblad [**40**] was later extended by Du and Zhou [**11**] to include domains that are the exterior of a star-shaped obstacle in $n = 3$. Their chief innovation with this paper involved an application of a weighted Sobolev inequality to prove suitable L^2 bounds for the solution itself. A subsequent paper by Du, Metcalfe, Sogge and Zhou [**10**] refined these methods in order to prove almost global existence when $n = 4$. As we mentioned earlier, Theorem 1.4 shows that we can relax the geometric assumptions of Du and Zhou by only requiring that the exterior of $\mathcal{K}$ only satisfy a sufficiently rapid polynomial decay estimate (1.6).

1.5. Methods of This Paper

In this paper, we will be using the modified version of Klainerman's invariant vector field method, which was developed by Keel, Smith and Sogge [**26**, **28**] which was further illustrated in Metcalfe-Sogge [**45**]. It will be convenient for us to utilize translations as these are easy to control using elliptic regularity estimates coupled with variable coefficient energy estimates. We will utilize the weighted L^2 estimates of Keel, Smith and Sogge [**28**] to handle the error terms that arise from handling rotations. However, we will also need to use scaling vector fields in order to control the terms that arise from the applying the L^1, L^∞ estimatesof Hörmander [**16**]. The issue of controlling energy norms that involve the scaling vector field will be dealt with using Theorem 3.8, which was originally proved in Metcalfe-Sogge.

An intuitive way to understand the logic behind the estimates and their role in this paper is to view the vector fields we are using in the context of a "hierarchy." The vector fields that are the highest on the hierarchy are heuristically the ones that are the easiest to bound. Elements in lower positions on the hierarchy will almost always be bounded by elements that are higher on the hierarchy. This will require elements that are higher on the hierarchy to occur in higher quantities than those that are lower on the hierarchy. Our hierarchy is as follows:

$$\partial_t^j u'$$

$$\partial^\alpha u',$$

$$Z^\alpha \partial^\beta u,$$

$$L^\mu \partial^\alpha u',$$

$$L^\mu Z^\alpha \partial^\beta u.$$

Thus, the order of vector fields in the hierarchy, from easiest to bound to the most difficult to bound, is: $\partial_t, \partial_x, \Omega^{ij}, L$.

1.6. Background Information

Before proceeding any further, we review a couple of standard results that play an important role in many of the estimates in this paper. We first prove the version of Gronwall's inequality that we shall be using in this paper. The version presented here is the same as proved in Sogge [**67**].

THEOREM 1.5. *(Gronwall's Inequality) Suppose A, β, E are bounded, nonnegative functions on $[0,T]$ and suppose E is also increasing on $[0,T]$. Then it follows that if $0 \le t \le T$, and*

$$A(t) \le E(t) + \int_0^t \beta(s)A(s)\,ds, \tag{1.20}$$

then it follows that

$$A(t) \le E(t)\exp\left(\int_0^t \beta(s)\,ds\right).$$

PROOF. Without loss of generality, it suffices to consider the case when $t = T$. Because of this, we can replace $E(t)$ with a constant $E_1 := E(T)$ since $E(t) \le E(T)$. Define $B(t) = E_1 + \int_0^t \beta(s)A(s)\,ds$. We see that

$$\begin{aligned} B'(t) &= \beta(t)A(t) \\ &\le \beta(t)B(t). \end{aligned}$$

It follows that $\partial_t\left(B(t)\exp\left(-\int_0^t \beta(s)\,ds\right)\right) \le 0$. Integrating both sides with respect to t, we see that

$$B(t) \le B(0)\exp\left(\int_0^t \beta(s)\,ds\right) = E_1\ \exp\left(\int_0^t \beta(s)\,ds\right),$$

for $0 \le t \le T$. Since $A(T) \le B(T)$, this proves the theorem. □

Another estimate we shall need is a standard L^2 regularity estimate for an elliptic operator P with smooth coefficients. We write

$$P = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha$$

as a differential operator of order k with C^∞ coefficients. An operator P is *elliptic* at a point $x_0 \in \mathbb{R}^n$ if $\sum_{|\alpha|=k} a_\alpha(x_0)\xi^\alpha \neq 0$ for all nonzero $\xi \in \mathbb{R}^n$. Using this definition, one can see that there are constants $A, R > 0$ such that

$$\left|\sum_{|\alpha|\le k} a_\alpha(x_0)\xi^\alpha\right| \ge A|\xi|^k, \quad \text{for } |\xi| \ge R.$$

Let Ω be an open subset of $\mathbb{R}^n$. We also define $H_0^s(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$. We are now ready to state the estimate.

THEOREM 1.6. *Suppose Ω is a bounded open set of $\mathbb{R}^n$ and $P = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha$ is elliptic on an open neighborhood Ω_0 of the closure of Ω. Then for any $s \in \mathbb{R}$, there is a constant $C > 0$ such that for all $u \in H_0^s(\Omega)$,*

$$\|u\|_{H^s(\mathbb{R}^n)} \le C\left(\|Pu\|_{H^{s-k}(\mathbb{R}^n)} + \|u\|_{H^{s-1}(\mathbb{R}^n)}\right).$$

We now state the corollary that we shall use in our estimates.

COROLLARY 1.7. *Let $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ vanish on $\partial\mathcal{K}$. Then it follows that for $R \ge 2$, $\delta > 0$ and $0 \le t \le T$,*

$$\begin{aligned}\sum_{|\alpha|\le N} \|\partial_x^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}\cap\{|x|<R\})} \le C &\sum_{|\alpha|\le N-2} \|\Delta\partial_x^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}\cap\{|x|<R+\delta\})} \\ &+ C\sum_{|\alpha|\le N-1} \|\partial_x^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}\cap\{|x|<R+\delta\})},\end{aligned}$$

where C depends on δ.

This paper is organized as follows. Chapter 2 proves the free space estimates for the wave equation that we will be using. Section 2.1 is devoted to the L^2 estimates that we shall need. Section 2.2 covers the weighted L^2 estimates for u'. Section 2.3 covers the Du-Zhou [**11**] estimates bounding the weighted L^2 and L^2 norms of u without the spacetime gradient. Section 2.4 is devoted to a couple of well-known Sobolev lemmas. Section 2.5 covers the Keel, Smith and Sogge [**28**] variants of the L^1, L^∞ estimates. We will also need divergence-form estimates which are covered in Section 2.6. Chapter 3 covers the Dirichlet-wave equation analogues of the estimates in Chapter 2. Section 3.1 covers the necessary L^2 estimates that we will be using in the exterior domain setting. Section 3.2 deals with the weighted L^2 estimates. Section 3.3

proves the exterior domain variant of Hörmander's estimate that was proved by Keel, Smith and Sogge [**28**]. Chapter 4 covers the proof of Theorem 1.4.

CHAPTER 2

Estimates for Wave Equations in Free Space

2.1. The Energy Inequality

In this section, we will motivate the general method that we will use for proving useful estimates in Minkowski space $[0,T]\times\mathbb{R}^3$. In the cases where proving the estimate on $[0,T]\times\mathbb{R}^n$ does not provide any more difficulty, the more general case shall be considered. The norm in which it is most natural to control solutions to the wave equation in free space is the L^2 norm of the spacetime gradient,

$$\sup_{0\le t\le T}\left\|u'(t,\cdot)\right\|_2,$$

which corresponds to the conservation of energy law for the homogeneous wave equation (see Proposition 2.1 below).

Proposition 2.1. *(Conservation of Energy) Suppose $u\in C^\infty([0,T]\times\mathbb{R}^n)$ solves $\Box u(t,x)=0$ and that for any fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then it follows that for $0\le t\le T$,*

$$\left\|u'(t,\cdot)\right\|_2=\left\|u'(0,\cdot)\right\|_2. \tag{2.1}$$

Proof. Differentiating with respect to t and integrating by parts, we see that

$$\begin{aligned}\partial_t\left(\left\|u'(t,\cdot)\right\|_2^2\right)&=2\int\left(\partial_t^2u\partial_tu+\sum_{j=1}^n\partial_t\partial_ju\partial_ju\right)\,dx\\&=2\int\left(\partial_t^2u\partial_tu-\partial_tu\Delta u\right)\,dx\\&=2\int\partial_tu\Box u\,dx=0.\end{aligned}$$

$\Box$

By Duhamel's Principle and after an application of the Minkowski integral inequality, we get the following corollary.

Corollary 2.2. *(Energy Inequality) Suppose that $u \in C^\infty([0,T]\times\mathbb{R}^n)$ solves*

$$\begin{cases} \Box u(t,x) = G(t,x), \\ u(0,x) = f(x), \partial_t u(0,x) = g(x). \end{cases} \tag{2.2}$$

Also suppose that for any fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then it follows that for $0 \le t \le T$,

$$\left\|u'(t,\cdot)\right\|_2 \lesssim \|(\nabla_x f, g)\|_2 + \int_0^t \|G(s,\cdot)\|_2 \; ds. \tag{2.3}$$

To illustrate the true power of this method, we will recall the commutator relations for translations, spatial rotations and the scaling vector field with $\Box$ that were discussed in the introduction to this paper:

$$[Z, \Box] = 0,$$

$$[L, \Box] = -2\Box.$$

We also see that a few simple calculations also yield

$$\begin{aligned}
[\Omega^{ij}, \Omega^{kl}] &= \delta_{jk}\Omega^{il} + \delta_{ik}\Omega^{lj} + \delta_{il}\Omega^{kj} + \delta_{jl}\Omega^{ki}, && 1 \le i,j,k,l \le 3, \\
[\Omega^{ij}, L] &= 0, && 1 \le i,j \le 3, \\
[\partial_i, \Omega^{jk}] &= \delta_{ij}\partial_k - \delta_{ik}\partial_j, && 0 \le i \le 3, 1 \le j,k \le 3, \\
[\partial_i, L] &= \partial_i, && 0 \le i \le 3.
\end{aligned}$$

where δ_{ij} is the Kronecker delta. We will often implicitly make use of the above facts in our calculations. Because of this, it follows that for any fixed positive integers ν, N, we also have the estimate

$$\sum_{\substack{|\alpha|\le N \\ \mu\le\nu}} \left\|(L^\mu Z^\alpha u)'(t,\cdot)\right\|_2 \lesssim \sum_{\substack{|\alpha|\le N \\ \mu\le\nu}} \left\|(L^\mu Z^\alpha u)'(0,\cdot)\right\|_2 + \int_0^t \sum_{\substack{|\alpha|\le N \\ \mu\le\nu}} \|L^\mu Z^\alpha \Box u(s,\cdot)\|_2 \; ds.$$

Thus, it is clear that using Lorentz-invariant vector fields in conjunction with estimates for u' such as the energy equality yields new estimates for $L^\mu Z^\alpha u'$.

2.2. Weighted L^2 Estimates Involving the Spacetime Gradient

We shall need estimates in $[0,T]\times\mathbb{R}^3$ that bound certain weighted L^2 norms. Using the local energy decay assumption (1.6), we shall be able to extend these estimates to the exterior domain setting $[0,T]\times\mathbb{R}^3\backslash\mathcal{K}$. We shall prove an estimate similar to Theorem 3.6 in [**11**] and Proposition 3.1 in [**28**]. The following estimates are proved in the more general case that the spatial dimension $n\geq 3$.

Theorem 2.3. *Suppose $\phi\in C^\infty([0,T]\times\mathbb{R}^n)$ vanishes for large $|x|$ for every fixed t. Then it follows that if $n\geq 3$, then*

$$\langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4}\phi'\right\|_{L^2([0,T]\times\mathbb{R}^n)}\lesssim\left\|\phi'(0,\cdot)\right\|_2+\int_0^T\|\Box\phi(s,\cdot)\|_2\,ds, \tag{2.4}$$

and

$$\begin{aligned}&(\log(2+T))^{-1/2}\left(\left\|\langle x\rangle^{-1/2}\phi'\right\|_{L^2([0,T]\times\mathbb{R}^n)}+\left\|\langle x\rangle^{-3/2}\phi\right\|_{L^2([0,T]\times\mathbb{R}^n)}\right)\\&\qquad\lesssim\left\|\phi'(0,\cdot)\right\|_2+\int_0^T\|\Box\phi(s,\cdot)\|_2\,ds.\end{aligned} \tag{2.5}$$

Also for any $n\geq 3$ and any fixed $\delta>0$, we have

$$\left\|\langle x\rangle^{-1/2-\delta}\phi'\right\|_{L^2([0,T]\times\mathbb{R}^n)}+\left\|\langle x\rangle^{-3/2-\delta}\phi\right\|_{L^2([0,T]\times\mathbb{R}^n)}\lesssim\left\|\phi'(0,\cdot)\right\|_2+\int_0^T\|\Box\phi(s,\cdot)\|_2\,ds, \tag{2.6}$$

where the implicit constant in (2.6) *depends on δ.*

The original proof of the first inequality (2.4) in 3 spatial dimensions used sharp Huygens' principle (see Lemma 3.2 in Du-Zhou [**11**] and Proposition 2.1 in Keel-Smith-Sogge [**26**]). This method of proof, however, is not robust as sharp Huygens' principle holds only for the flat wave equations in $[0,T]\times\mathbb{R}^n$ for odd n. For more generality, we shall use the energy methods employed in the proof of Lemma 4.1 in Metcalfe-Sogge [**48**], which do not involve Huygens' principle. Earlier examples of proofs that use the same method can be found in the works of Morawetz [**54**] and the appendix to [**68**] by Rodnianski.

Since the exposition is eased by using tensor calculus, we shall introduce some new notation for this proof. We shall use the Einstein convention where repeated indices are summed. We will use Greek indices $\alpha,\beta,\gamma,\delta$ and so on, when the summations run from $0,\dots,n$. We will use Roman indices a, b, c, d and so on, when the summations run from $1, \ldots, n$. We define the Minkowski metric $g_{\alpha\beta} = \mathrm{diag}(-1, 1, \ldots, 1)$. We define ∇ to be the Levi-Civita connection associated with the metric g. From this, one can define the covariant derivative D^α associated with the connection ∇. Due to the fact that Minkowski space is flat, we have the correspondence $D^\alpha = \partial^\alpha$. ∂^α is associated with the standard derivative ∂_α via the equivalence

$$\partial^\alpha = g^{\alpha\beta}\partial_\beta,$$

where $g^{\alpha\beta}$ is the inverse of the matrix $g_{\alpha\beta}$. Note that in this new notation, the d'Alembertian operator becomes

$$\Box\phi = -\partial^\gamma\partial_\gamma\phi. \tag{2.7}$$

We define $Q_{\alpha\beta}$ to be the *energy-momentum tensor* by

$$Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi. \tag{2.8}$$

We shall now prove a well-known fact.

LEMMA 2.4. *Let $Q_{\alpha\beta}$ be defined as above and let $\phi \in C^\infty([0,T]\times\mathbb{R}^n)$. Then $Q_{\alpha\beta}[\phi]$ satisfies the equation*

$$D^\alpha Q_{\alpha\beta}[\phi] = -\Box\phi\partial_\beta\phi.$$

PROOF. This follows from a simple calculation. We see that

$$\begin{aligned} D^\alpha Q_{\alpha\beta}[\phi] &= \partial^\alpha\left(\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi\right). \\ &= \partial^\alpha\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha\phi\partial^\alpha\partial_\beta\phi \\ &- \frac{1}{2}g_{\alpha\beta}\partial^\alpha\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial^\alpha\partial_\gamma\phi \\ &= -\Box\phi\partial_\beta\phi + \partial_\alpha\phi\partial^\alpha\partial_\beta\phi \\ &- \frac{1}{2}g_{\alpha\beta}\partial^\alpha\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial^\alpha\partial_\gamma\phi. \end{aligned} \tag{2.9}$$

We see that the sum of last 2 terms in the right hand side of (2.9) are equal to $-\partial_\gamma\phi\partial^\gamma\partial_\beta\phi$, which shows that the sum of the last three terms in right hand side of (2.9) is zero. $\Box$

We now define the *momentum density* that is obtained by contracting $Q_{\alpha\beta}[\phi]$ with a radial vector field X^β,

$$P_\alpha[\phi, X] = Q_{\alpha\beta}[\phi]X^\beta, \tag{2.10}$$

where

$$X = f(r)\partial_r. \tag{2.11}$$

This means that $X^a = \dfrac{f(r)}{r}x^a$ for $a = 1, \dots, n$ and $X^0 = 0$. If one also defines the *deformation tensor of* X,

$$\pi_{ab} = \frac{1}{2}\left(D_a X_b + D_b X_a\right), \tag{2.12}$$

using Lemma 2.4, one can prove the following.

Lemma 2.5. *Let* $Q_{\alpha\beta}, P_\alpha, X$, *and* π *be defined as in* (2.8), (2.10), (2.11) *and* (2.12), *respectively. Then it follows that*

$$D^\alpha P_\alpha[\phi, X] = -\Box\phi f(r)\partial_r\phi + f'(r)\left|\partial_r\phi\right|^2 + \frac{f(r)}{r}\left|\nabla\!\!\!/\,\phi\right|^2 - \frac{1}{2}\mathrm{tr}\ \pi\ \partial^\gamma\phi\partial_\gamma\phi, \tag{2.13}$$

where

$$\mathrm{tr}\ \pi = f'(r) + (n-1)\frac{f(r)}{r}. \tag{2.14}$$

Proof. We shall first establish the following:

$$D^\alpha P_\alpha[\phi, X] = -\Box\phi f(r)\partial_r\phi + Q_{ab}[\phi]\pi^{ab}. \tag{2.15}$$

We can establish this using Lemma 2.4, and via the following calculation:

$$\begin{aligned}
D^\alpha P_\alpha[\phi, X] &= D^\alpha Q_{\alpha\beta}[\phi]X^\beta + Q_{ab}[\phi]D^a X^b \\
&= -\Box\phi\partial_\beta\phi X^\beta + Q_{ab}[\phi]D^a X^b \\
&= -\Box\phi f(r)\frac{x^\beta}{r}\partial_\beta\phi + \frac{1}{2}\left(Q_{ab}[\phi]D^a X^b + Q_{ba}[\phi]D^b X^a\right) \\
&= -\Box\phi f(r)\partial_r\phi + \frac{1}{2}Q_{ab}[\phi]\left(D^a X^b + D^b X^a\right).
\end{aligned}$$

Using (2.15), we see that

$$\begin{aligned} D^\alpha P_\alpha[\phi, X] &= -\Box\phi f(r)\partial_r\phi + \left(\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial^\gamma\phi\partial_\gamma\phi\right)\pi^{ab} \\ &= -\Box\phi f(r)\partial_r\phi + \partial_a\phi\partial_b\phi\pi^{ab} - \frac{1}{2}\mathrm{tr}\,\pi\;\partial^\gamma\phi\partial_\gamma\phi. \end{aligned}$$

Thus, to establish (2.13), it suffices to prove that $\partial_a\phi\partial_b\phi\pi^{ab}$ equals the sum of the second and third terms in the right hand side of (2.13). We see that

$$\begin{aligned} \partial_a\phi\partial_b\phi\pi^{ab} &= \frac{1}{2}\partial_a\phi\partial_b\phi\left(\partial^a\left(\frac{f(r)}{r}x^b\right) + \partial^b\left(\frac{f(r)}{r}x^a\right)\right) \\ &= \partial_a\phi\partial_b\phi\left(\frac{x^a x^b}{r^2}f'(r) + \left(\delta^{ab} - \frac{x^a x^b}{r^2}\right)\frac{f(r)}{r}\right) \\ &= |\partial_r\phi|^2 f'(r) + |\nabla\!\!\!/\phi|^2\frac{f(r)}{r}. \end{aligned}$$

We now establish (2.14). This follows from the calculation:

$$\begin{aligned} \delta_a{}^b\pi^a{}_b &= \frac{1}{2}\delta_a{}^b\partial^a\left(\frac{f(r)}{r}x_b\right) + \frac{1}{2}\delta_a{}^b\partial_b\left(\frac{f(r)}{r}x^a\right) \\ &= \delta_a{}^b\frac{x^a x_b}{r^2}\left(f'(r) - \frac{f(r)}{r}\right) + \delta_a{}^b\delta_b{}^a\frac{f(r)}{r} \\ &= f'(r) + (n-1)\frac{f(r)}{r}. \end{aligned}$$

□

We wish to introduce a modified momentum density that will enable us to control $\partial_t\phi$. We observe that

$$\begin{aligned} \frac{f(r)}{r}\partial^\gamma\phi\partial_\gamma\phi &= D^\gamma\left(\frac{f(r)}{r}\phi\partial_\gamma\phi - \frac{1}{2}\partial_\gamma\left(\frac{f(r)}{r}\right)|\phi|^2\right) \\ &+ \frac{1}{2}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2 + \frac{f(r)}{r}\phi\Box\phi. \end{aligned}$$

Using Lemma 2.5, we see that

$$\begin{aligned} D^\alpha P_\alpha[\phi, X] &= -\Box\phi f(r)\partial_r\phi + f'(r)|\partial_r\phi|^2 + \frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 \\ &- \frac{1}{2}\left(f'(r) + (n-1)\frac{f(r)}{r}\right)\partial^\gamma\phi\partial_\gamma\phi \\ &= -\Box\phi f(r)\partial_r\phi + f'(r)|\partial_r\phi|^2 + \frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 - \frac{1}{2}f'(r)\partial^\gamma\phi\partial_\gamma\phi \end{aligned}$$

$$
-\frac{(n-1)}{2} D^{\gamma}\left(\frac{f(r)}{r}\phi\partial_{\gamma}\phi - \frac{1}{2}\partial_{\gamma}\left(\frac{f(r)}{r}\right)|\phi|^2\right) - \frac{(n-1)}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2
- \frac{(n-1)}{2}\frac{f(r)}{r}\phi\Box\phi.
$$

If we define our modified momentum density to be

$$
\overline{P}_{\alpha}[\phi, X] = P_{\alpha}[\phi, X] + \frac{(n-1)}{2}\frac{f(r)}{r}\phi\partial_{\alpha}\phi - \frac{(n-1)}{4}\partial_{\alpha}\left(\frac{f(r)}{r}\right)|\phi|^2,
$$

then it follows that

$$
\begin{aligned}
D^{\alpha}\overline{P}_{\alpha}[\phi, X] &= -\Box\phi f(r)\partial_r\phi + f'(r)\,|\partial_r\phi|^2 + \frac{f(r)}{r}\,|\not\nabla\phi|^2 \\
&\quad - \frac{1}{2}f'(r)\partial^{\alpha}\phi\partial_{\alpha}\phi - \frac{(n-1)}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2 \\
&\quad - \frac{(n-1)}{2}\frac{f(r)}{r}\phi\Box\phi.
\end{aligned} \tag{2.16}
$$

Our radial function f shall be conveniently chosen so that $-\Delta\left(\frac{f(r)}{r}\right)$ shall be positive for $n \geq 3$. We are now ready to prove Theorem 2.3.

Proof. (Theorem 2.3) By Duhamel's principle, it suffices to prove this theorem in the case $\Box\phi = 0$. From (2.16) and the divergence theorem, we see that

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} \left(f'(r)\,|\partial_r\phi|^2 + \frac{f(r)}{r}\,|\not\nabla\phi|^2 - \frac{1}{2}f'(r)\partial^{\gamma}\phi\partial_{\gamma}\phi - \frac{(n-1)}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2\right)\,dx\,dt \\
&\quad = \int_0^T \int_{\mathbb{R}^n} D^{\alpha}\overline{P}_{\alpha}[\phi, X](t,x)\,dx\,dt \\
&\quad = \int_{\mathbb{R}^n} \overline{P}_{\alpha}[\phi, X](T,x)\,dx - \int_{\mathbb{R}^n} \overline{P}_{\alpha}[\phi, X](0,x)\,dx.
\end{aligned} \tag{2.17}
$$

If we choose f so that $|f(r)| \lesssim 1$ and $|f'(r)| \lesssim \frac{1}{r}$, it follows that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n} \overline{P}_0[\phi, X](0,x)\,dx\right| &\lesssim \left(\left\|r^{-1}\phi(0,\cdot)\right\|_2 + \left\|\phi'(0,\cdot)\right\|_2\right)\left\|\phi'(0,\cdot)\right\|_2 \\
&\lesssim \left\|\phi'(0,\cdot)\right\|_2^2,
\end{aligned}
$$

by applying a Hardy inequality in the last step. Applying the same argument and energy conservation for the linear homogeneous wave equation, we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \overline{P}_0[\phi, X](T,x)\,dx\right| &\lesssim \left\|\phi'(T,\cdot)\right\|_2^2 \\ &= \left\|\phi'(0,\cdot)\right\|_2^2 .\end{aligned}$$

From the previous two inequalities and (2.17), we get the inequality

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} \left(f'(r)\left|\partial_r\phi\right|^2 + \frac{f(r)}{r}\left|\nabla\!\!\!/\phi\right|^2 - \frac{1}{2}f'(r)\partial^\gamma\phi\partial_\gamma\phi - \frac{(n-1)}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2\right)\,dx\,dt \\ &\quad\lesssim \left\|\phi'(0,\cdot)\right\|_2^2 .\end{aligned} \tag{2.18}$$

We now choose our weight function f:

$$f(r) = \frac{r}{\rho + r}, \tag{2.19}$$

where $\rho > 0$. Observing that $f(r)/r > f'(r)$, $\partial^\gamma\phi\partial_\gamma\phi = -\left|\partial_t\phi\right|^2 + \left|\partial_r\phi\right|^2 + \left|\nabla\!\!\!/\phi\right|^2$, and that

$$-\Delta\left(\frac{f(r)}{r}\right) = \frac{(n-3)r + (n-1)\rho}{r(\rho+r)^3},$$

we get the following estimate for $\rho > 0$,

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} \left(\frac{\rho}{(r+\rho)^2}\left|\partial_r\phi\right|^2 + \frac{1}{r+\rho}\left|\nabla\!\!\!/\phi\right|^2 + \frac{\rho}{(r+\rho)^2}\left|\partial_t\phi\right|^2 + \frac{\rho}{(r+\rho)^3}|\phi|^2\right)\,dx\,dt \\ &\quad\lesssim \left\|\phi'(0,\cdot)\right\|_2^2 .\end{aligned} \tag{2.20}$$

When we set $\rho = 1$ and restrict x such that $|x| < 1$, we get

$$\begin{aligned}&\int_0^T \int_{|x|<1} \left(\left|\phi'\right|^2 + |\phi|^2\right)\,dx\,dt \\ &\qquad\lesssim \left\|\phi'(0,\cdot)\right\|_2^2 .\end{aligned} \tag{2.21}$$

Setting $\rho = 2^k$, for $k \geq 0$, we get

$$\begin{aligned}&\int_0^T \int_{2^k<|x|<2^{k+1}} \left(\langle x\rangle^{-1}\left|\phi'\right|^2 + \langle x\rangle^{-3}|\phi|^2\right)\,dx\,dt \\ &\qquad\lesssim \left\|\phi'(0,\cdot)\right\|_2^2 .\end{aligned} \tag{2.22}$$

We shall first prove (2.6) and show that (2.5) follows from similar arguments. Summing over dyadic regions $\{2^k < |x| < 2^{k+1}\}$ and $\{|x| < 1\}$ and applying the previous two inequalities, we see that

$$
\begin{aligned}
&\left\|\phi'\right\|^2_{L^2([0,T]\times\{|x|<1\})} + \|\phi\|^2_{L^2([0,T]\times\{|x|<1\})} \\
&+ \sum_{k=0}^{\infty} 2^{-2k\delta} \left(\left\| \langle x\rangle^{-1/2} \phi' \right\|^2_{L^2([0,T]\times\{2^k<|x|<2^{k+1}\})} + \left\| \langle x\rangle^{-3/2} \phi \right\|^2_{L^2([0,T]\times\{2^k<|x|<2^{k+1}\})} \right) \\
&\lesssim \sum_{k=0}^{\infty} 2^{-2k\delta} \left\|\phi'(0,\cdot)\right\|_2^2 \\
&\lesssim \left\|\phi'(0,\cdot)\right\|_2^2 ,
\end{aligned}
\tag{2.23}
$$

where the implicit constant in the last inequality depends on δ. This proves (2.6). To prove (2.5), we divide the proof into two cases: $|x| < T$ and $|x| > T$. The latter case is handled by a Hardy inequality and energy conservation:

$$
\begin{aligned}
\left\| \langle x\rangle^{-1/2} \phi' \right\|^2_{L^2([0,T]\times\{|x|>T\})} &+ \left\| \langle x\rangle^{-3/2} \phi \right\|^2_{L^2([0,T]\times\{|x|>T\})} \\
&\lesssim \sup_{0\le t\le T} \left\|\phi'(t,\cdot)\right\|_2^2 \\
&\lesssim \left\|\phi'(0,\cdot)\right\|_2^2 .
\end{aligned}
\tag{2.24}
$$

The former case is handled in manner similar to (2.23). By summing over $\{|x| < 1\}$ and dyadic regions $\{2^k < |x| < 2^{k+1}\}$, where $0 \le k \le \log(2 + T)$, we see that

$$
\begin{aligned}
\left\| \langle x\rangle^{-1/2} \phi' \right\|^2_{L^2([0,T]\times\{|x|<T\})} &+ \left\| \langle x\rangle^{-3/2} \phi \right\|^2_{L^2([0,T]\times\{|x|<T\})} \\
&\lesssim \log(2+T) \left\|\phi'(0,\cdot)\right\|_2^2 .
\end{aligned}
$$

To prove (2.4), we shall again split this in two cases: $|x| < T$ and $|x| > T$. The latter case is dealt with by applying the same argument used in (2.24). To deal with the former case, we shall apply (2.21) and (2.22) to see that

(2.25)

$$\begin{aligned}
&\left\|\phi'\right\|^2_{L^2([0,T]\times\{|x|<1\})} + \sum_{k=0}^{\log(2+T)} \left\|\langle x\rangle^{-1/4}\phi'\right\|^2_{L^2([0,T]\times\{2^k<|x|<2^{k+1}\})} \\
&\qquad\lesssim \left\|\phi'\right\|^2_{L^2([0,T]\times\{|x|<1\})} + \sum_{k=0}^{\log(2+T)} 2^{k/2}\left\|\langle x\rangle^{-1/2}\phi'\right\|^2_{L^2([0,T]\times\{2^k<|x|<2^{k+1}\})} \\
&\qquad\lesssim \sum_{k=0}^{\log(2+T)} 2^{k/2}\left\|\phi'(0,\cdot)\right\|_2^2 \\
&\qquad\lesssim \langle T\rangle^{1/2}\left\|\phi'(0,\cdot)\right\|_2^2.
\end{aligned}$$

This completes the proof. □

From this proof, we also obtain a useful corollary.

Corollary 2.6. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^n)$ solves $\Box u = G$ with vanishing initial data. Also suppose that $u(t,x)$ vanishes for large $|x|$ for every fixed t. Then it follows that if $n \geq 3$, then*

$$\left\|u'\right\|_{L^2([0,T]\times\{|x|<4\})} + \|u\|_{L^2([0,T]\times\{|x|<4\})} \lesssim \int_0^T \|G(s,\cdot)\|_2\,ds. \tag{2.26}$$

We will also need a lemma to deal with the spatial cutoffs that occur in the proofs of the weighted estimates in Chapter 3. Estimates of this type were originally proved by Keel, Smith and Sogge [**26**, **28**] using sharp Huygens' principle. We will use techniques such as those presented in Metcalfe-Sogge [**47**, **48**] that instead rely on the energy methods that were just discussed in the proof of Theorem 2.3.

Lemma 2.7. *Suppose that $R > 1$. Let $G \in C_c^\infty([0,T]\times\mathbb{R}^n)$, where $n \geq 3$ and the support of G is contained in $\{1 < |x| \leq R\}$. Suppose that $\phi \in C^\infty([0,T]\times\mathbb{R}^n)$ solves the boundaryless wave equation $\Box\phi = G$ with vanishing initial data. Then it follows that*

$$\sup_{0\leq t\leq T}\left\|\phi'(t,\cdot)\right\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4}\phi'\right\|_{L^2([0,T]\times\mathbb{R}^n)} \lesssim \|G\|_{L^2([0,T]\times\{x\in\mathbb{R}^n:1<|x|<R\})}, \tag{2.27}$$

where the implicit constant depends on R.

Proof. Applying the fundamental theorem of calculus and integrating by parts in each x_i variable, we get

$$\frac{1}{2}\int |\phi'(t,x)|^2\,dx = \int_0^t \int_{1<|x|<R} G(s,x)\partial_s\phi(s,x)\,dx\,ds. \tag{2.28}$$

From (2.16) and the divergence theorem, we see that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} &\left(f'(r)\,|\partial_r\phi|^2 + \frac{f(r)}{r}\,|\nabla\!\!\!/\phi|^2 - \frac{1}{2}f'(r)\partial^\gamma\phi\partial_\gamma\phi - \frac{(n-1)}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2\right)\,dx\,ds\\ &= \int_{\mathbb{R}^n}\overline{P}_\alpha[\phi,X](T,x)\,dx - \int_{\mathbb{R}^n}\overline{P}_\alpha[\phi,X](0,x)\,dx\\ &\quad+\int_0^T\int_{\mathbb{R}^n}\Box\phi(s,x)f(r)\partial_r\phi(t,x) + \frac{(n-1)}{2}\frac{f(r)}{r}\phi(s,x)\Box\phi(s,x)\,dx\,ds.\end{aligned}\tag{2.29}$$

By a Hardy inequality, it follows that

$$\left|\int_{\mathbb{R}^n}\overline{P}_\alpha[\phi,X](0,x)\,dx\right| \lesssim \left\|\phi'(0,\cdot)\right\|_2^2 = 0.$$

By a similar argument and (2.28), we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\overline{P}_\alpha[\phi,X](T,x)\,dx\right| &\lesssim \left\|\phi'(T,\cdot)\right\|_2^2\\ &\lesssim \int_0^T\int_{1<|x|<R}|G(s,x)\partial_s\phi(s,x)|\,dx\,ds.\end{aligned}$$

From these calculations, we obtain the inequality

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} &\left(f'(r)\,|\partial_r\phi|^2 + \frac{f(r)}{r}\,|\nabla\!\!\!/\phi|^2 - \frac{1}{2}f'(r)\partial^\gamma\phi\partial_\gamma\phi - \frac{(n-1)}{4}\Delta\left(\frac{f(r)}{r}\right)|\phi|^2\right)dxds\\ &\lesssim \int_0^T\int_{1<|x|<R}|G(s,x)|\left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r}\right)\,dxds.\end{aligned}\tag{2.30}$$

If we let f be as in (2.19), then, as in (2.20), we get the inequality for $\rho > 0$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} &\left(\frac{\rho}{(r+\rho)^2}\,|\partial_r\phi|^2 + \frac{1}{r+\rho}\,|\nabla\!\!\!/\phi|^2 + \frac{\rho}{(r+\rho)^2}\,|\partial_t\phi|^2 + \frac{\rho}{(r+\rho)^3}\,|\phi|^2\right)\,dx\,ds\\ &\lesssim \int_0^T\int_{1<|x|<R}|G(s,x)|\left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r}\right)\,dx\,ds.\end{aligned}\tag{2.31}$$

From (2.31) in the case $\rho = 1$, if we restrict x such that $1 < |x| < R$, then we get

$$
\begin{aligned}
&\int_0^T \int_{1<|x|<R} \left(|\phi'|^2 + \frac{|\phi|^2}{r^2} \right) \, dx \, ds \\
&\qquad \lesssim \int_0^T \int_{1<|x|<R} |G(s,x)| \left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r} \right) \, dx \, ds \\
&\qquad \lesssim \|G\|_{L^2([0,T]\times\{x\in\mathbb{R}^n : 1<|x|<R\})} \times \\
&\qquad\quad \left(\|\phi'\|_{L^2([0,T]\times\{x\in\mathbb{R}^n : 1<|x|<R\})} + \|r^{-1}\phi\|_{L^2([0,T]\times\{x\in\mathbb{R}^n : 1<|x|<R\})} \right).
\end{aligned} \tag{2.32}
$$

This implies that

$$
\begin{aligned}
&\|\phi'\|_{L^2([0,T]\times\{x\in\mathbb{R}^n : 1<|x|<R\})} + \|r^{-1}\phi\|_{L^2([0,T]\times\{x\in\mathbb{R}^n : 1<|x|<R\})} \\
&\qquad \lesssim \|G\|_{L^2([0,T]\times\{x\in\mathbb{R}^n : 1<|x|<R\})}.
\end{aligned} \tag{2.33}
$$

Setting $\rho = 2^k$, for $k \geq 0$, we get

$$
\begin{aligned}
&\int_0^T \int_{2^k<|x|<2^{k+1}} \langle x \rangle^{-1} |\phi'|^2 \, dx \, ds \\
&\qquad \lesssim \int_0^T \int_{1<|x|<R} |G(s,x)| \left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r} \right) \, dx \, ds.
\end{aligned} \tag{2.34}
$$

For $|x| > T$, we apply (2.28) see that

$$
\begin{aligned}
\langle T \rangle^{-1/2} \left\| \langle x \rangle^{-1/4} \phi' \right\|^2_{L^2([0,T]\times\{|x|>T\})} &\lesssim \sup_{0\leq t\leq T} \|\phi'(t,\cdot)\|_2^2 \\
&\lesssim \int_0^T \int_{1<|x|<R} |G(s,x)||\phi'(s,x)| \, dx \, ds.
\end{aligned}
$$

For $|x| < T$, we sum over dyadic intervals just as in (2.25) to see that

$$
\begin{aligned}
&\langle T \rangle^{-1/2} \left\| \langle x \rangle^{-1/4} \phi' \right\|^2_{L^2([0,T]\times\{|x|<t\})} \\
&\qquad \lesssim \int_0^T \int_{1<|x|<R} |G(s,x)| \left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r} \right) \, dx \, ds.
\end{aligned}
$$

The previous two inequalities and (2.28) show that

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \|\phi'(t,\cdot)\|_2^2 + \langle T \rangle^{-1/2} \left\| \langle x \rangle^{-1/4} \phi' \right\|^2_{L^2([0,T]\times\mathbb{R}^n)} \\
&\qquad \lesssim \int_0^T \int_{1<|x|<R} |G(s,x)| \left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r} \right) \, dx \, ds.
\end{aligned} \tag{2.35}
$$

Applying Cauchy-Schwarz, we see that

$$\begin{aligned}
&\int_0^T \int_{1<|x|<R} |G(s,x)| \left(|\phi'(s,x)| + \frac{|\phi(s,x)|}{r} \right) \, dx \, ds \\
&\qquad \leq \|G\|_{L^2([0,T]\times\{x\in\mathbb{R}^n:1<|x|<R\})} \times \\
&\qquad\quad \left(\|\phi'\|_{L^2([0,T]\times\{x\in\mathbb{R}^n:1<|x|<R\})} + \|r^{-1}\phi\|_{L^2([0,T]\times\{x\in\mathbb{R}^n:1<|x|<R\})} \right).
\end{aligned} \tag{2.36}$$

Combining this inequality with (2.35) and (2.33), we get

$$\sup_{0\leq t\leq T} \left\|\phi'(t,\cdot)\right\|_2^2 + \langle T\rangle^{-1/2} \left\| \langle x\rangle^{-1/4} \phi' \right\|^2_{L^2([0,T]\times\mathbb{R}^n)} \lesssim \|G\|^2_{L^2([0,T]\times\{x\in\mathbb{R}^n:1<|x|<R\})},$$

which proves the lemma. □

2.3. L^2 and Weighted L^2 Estimates without the Spacetime Gradient

We shall now prove the estimates that first appeared in Du-Zhou [**11**]. These estimates will be necessary to control norms that arise in the Picard iteration where there is no spacetime gradient being applied to u. Specifically, the estimate of Du-Zhou allows us to apply the energy inequality when no spacetime gradient is present. While these estimates were originally proved in [**11**], we shall present the proof from a subsequent paper by Du, Metcalfe, Sogge and Zhou [**10**]. We will start by defining a mixed L^pL^q-norm:

$$\|h\|_{L^p_r L^q_\omega(\mathbb{R}^n)} := \left\| \|h(r\cdot)\|_{L^q(S^{n-1},d\omega)} \right\|_{L^p([0,\infty),r^{n-1}\,dr)},$$

where $d\omega$ is the induced surface measure on S^{n-1}. We also define $C_0^\infty(\mathbb{R}^n)$ to be the space of smooth functions that vanish at infinity. That is, $f \in C_0^\infty(\mathbb{R}^n)$ if, for every $\eta > 0$, the set $\{x \in \mathbb{R}^n : |f(x)| \geq \eta\}$ is compact. The key ingredient to proving estimates for the L^2 norm of $u(t,\cdot)$ without the spacetime gradient shall be the following proposition, which was proved by Du and Zhou [**11**].

Proposition 2.8. *Suppose $h \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$. Then it follows that*

$$\|h\|_{\dot{H}^{-1}(\mathbb{R}^n)} \lesssim \|h\|_{L^{2n/(n+2)}(|x|<3)} + \left\| |x|^{-(n-2)/2} h \right\|_{L^1_r L^2_\omega(|x|>2)}. \tag{2.37}$$

To prove this proposition, we shall need the following estimate (See Lemma 4.1 in [**34**], Lemma 2.1 in [**10**]).

LEMMA 2.9. *Let $v \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$, and $R > 0$. Then it follows that*

$$R^{1/2} \|v\|_{L_r^\infty L_\omega^2(|x|>R)} \lesssim \left\| |x|^{-(n-3)/2} \nabla_x v \right\|_{L^2(|x|>R)},$$

where the implicit constants are independent of R.

PROOF. Applying the Fundamental Theorem of Calculus and Cauchy-Schwarz, we see that if $r > R$,

$$\begin{aligned}
&R^{1/2} \left(\int_{S^{n-1}} |v(r\omega)|^2 \, d\omega \right)^{1/2} \lesssim R^{1/2} \left(\int_{S^{n-1}} \int_R^\infty |\partial_\rho v(\rho\omega)||v(\rho\omega)| \, d\rho d\omega \right)^{1/2} \\
&\lesssim \left(\int_{S^{n-1}} \int_R^\infty |\partial_\rho v|^2 \rho^2 \, d\rho d\omega \right)^{1/4} R^{1/2} \left(\int_{S^{n-1}} \int_R^\infty |v|^2 \, \frac{d\rho}{\rho^2} d\omega \right)^{1/4} \\
&\lesssim \left\| |x|^{-(n-3)/2} \nabla_x v \right\|_{L^2(|x|>R)}^{1/2} R^{1/2} \|v\|_{L_r^\infty L_\omega^2(|x|>R)}^{1/4} \left(\int_R^\infty \rho^{-2} \, d\rho \right)^{1/4} \\
&\lesssim \left\| |x|^{-(n-3)/2} \nabla_x v \right\|_{L^2(|x|>R)}^{1/2} R^{1/4} \|v\|_{L_r^\infty L_\omega^2(|x|>R)}^{1/4}.
\end{aligned}$$

□

We are now ready to prove Proposition 2.8.

PROOF. (Proposition 2.8) We shall split $h = h_1 + h_2$, where h_1 is a smooth function that equals $h(x)$ when $|x| < 2$ and zero when $|x| > 3$. Let $H_0^s(|x| < 3)$ denote the completion of $C_c^\infty(|x| < 3)$ in $H^s(|x| < 3)$. It follows from Proposition 6.15 in Folland [**12**] that in $H_0^s(|x| < 3)$, the norms for H^s and $\dot{H}^s$ are equivalent. By Sobolev Embedding $L^{2n/(n+2)}(\mathbb{R}^n) \hookrightarrow \dot{H}^{-1}(\mathbb{R}^n)$, we see that $\|h_1\|_{\dot{H}^{-1}(\mathbb{R}^n)}$ is controlled by the first term in the right hand side of (2.37). If we let $v \in C_c^\infty(\mathbb{R}^n) \cap \dot{H}^1(\mathbb{R}^n)$, we can see that

$$\int_{|x|>2} h_2(x) v(x) \, dx \leq \left\| |x|^{-(n-2)/2} h_2 \right\|_{L_r^1 L_\omega^2(|x|>2)} \left\| |x|^{(n-2)/2} v \right\|_{L_r^\infty L_\omega^2(|x|>2)}.$$

To control the second term in the right hand side, fix $\rho > 2$. Applying the previous lemma, we see that

$$\begin{aligned}
\rho^{(n-3)/2} \cdot \rho^{1/2} \|v(\rho \, \cdot)\|_{L^2(S^{n-1})} &\lesssim \rho^{(n-3)/2} \left\| |x|^{-(n-3)/2} \nabla_x v \right\|_{L^2(|x|>\rho)} \\
&\lesssim \|\nabla_x v\|_{L^2(|x|>2)}.
\end{aligned}$$

By the definition of the $\dot{H}^{-1}$-norm, this shows that $\|h_2\|_{\dot{H}^{-1}(\mathbb{R}^n)}$ is controlled by the second term in (2.37). $\square$

We are now ready to prove an estimate that shall enable us to control the L^2 and weighted L^2 norms of u without the spacetime gradient.

THEOREM 2.10. *Suppose $u \in C^\infty([0,T] \times \mathbb{R}^n)$ is a solution to*

$$\begin{cases} \Box u(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = 0, \quad \partial_t u(0,x) = g(x), \end{cases}$$

Also suppose that for each fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then it follows that

$$\begin{aligned} \sup_{0<t<T} \|u(t,\cdot)\|_2 + \langle T \rangle^{-1/4} \left\| \langle x \rangle^{-1/4} u \right\|_{L^2([0,T]\times\mathbb{R}^n)} \\ \lesssim \|g\|_{L^{2n/(n+2)}(|x|<3)} + \left\| |x|^{-(n-2)/2} g \right\|_{L^1_r L^2_\omega(|x|>2)}. \end{aligned} \tag{2.38}$$

PROOF. Define $D_j = \partial_j / i$. Let

$$v_j(t,x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \widehat{u}(t,\xi) \frac{\xi_j}{|\xi|^2}\, d\xi,$$

where $\widehat{u}(t,\cdot)$ is the the Fourier transform of $u(t,\cdot)$. It follows that $u = \sum_{j=1}^n D_j v_j$. Observe that each v_j solves $\Box v_j = 0$ with initial data $v_j(0,x) = 0, \partial_t v_j(0,x) = g_j(x)$. Observe that

$$g_j(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \widehat{g}(\xi) \frac{\xi_j}{|\xi|^2}\, d\xi.$$

By Theorem 2.3 and the energy inequality, we see that

$$\begin{aligned} &\sup_{0<t<T} \|u(t,\cdot)\|_2 + \langle T \rangle^{-1/4} \left\| \langle x \rangle^{-1/4} u \right\|_{L^2([0,T]\times\mathbb{R}^n)} \\ &\quad \lesssim \sum_{j=1}^n \|D_j v_j(t,\cdot)\|_2 + \langle T \rangle^{-1/4} \left\| \langle x \rangle^{-1/4} D_j v_j \right\|_{L^2([0,T]\times\mathbb{R}^n)} \\ &\quad \lesssim \sum_{j=1}^n \|g_j\|_2 \\ &\quad \lesssim \|g\|_{\dot{H}^{-1}}. \end{aligned}$$

Applying Proposition 2.8 completes the proof. $\square$

By Duhamel's principle, we also have the following corollary,.

Corollary 2.11. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^n)$ is a solution to*

$$\begin{cases} \Box u(t,x) = G(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^3, \\ u(0,x) = \partial_t u(0,x) = 0. \end{cases}$$

Also suppose that for each fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then it follows that

$$\begin{aligned} &\sup_{0<t<T} \|u(t,\cdot)\|_2 + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} u \right\|_{L^2([0,T]\times\mathbb{R}^n)} \\ &\qquad \lesssim \int_0^T \|G(s,\cdot)\|_{L^{2n/(n+2)}(|x|<3)} + \left\| |x|^{-(n-2)/2} G(s,\cdot) \right\|_{L^1_r L^2_\omega(|x|>2)} \, ds. \end{aligned} \tag{2.39}$$

To control the second term on the right hand side of (2.39), we will need to pass to the weighted L^2 norms discussed in Section 2.2. Due to the amount of x decay present in this term, we will only be able to use weighted norms with a $\langle T\rangle^{-1/4}$ weight instead of a $\log(2+T)^{-1/2}$ weight in our iteration argument. It shall be clear in the proof of Theorem 1.4 that using these weighted norms that involve $\langle T\rangle^{-1/4}$ necessitates our lifespan bound (1.9). In dimensions 4 and higher, however, one does get sufficient x decay in the right hand side of (2.39) to use weighted L^2 norms that involve only a $\log(2+T)^{-1/2}$ weight, which tend to allow for longer lifespans.

We will also need an estimate analogous to Lemma 2.7 in case where no spacetime gradient is being applied to the solution. This estimate was adapted from the proofs of Keel, Smith and Sogge [**26**, **28**] by Du and Zhou [**11**].

Lemma 2.12. *Let $G \in C^\infty_c([0,T]\times\mathbb{R}^3)$ and the support of G is contained in $\{1 < |x| \le 3\}$. Suppose that $u \in C^\infty([0,T]\times\mathbb{R}^3)$ solves $\Box u = G$ with vanishing initial data. Then it follows that*

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_2 + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} u \right\|_{L^2([0,T]\times\mathbb{R}^3)} \lesssim \|G\|_{L^2([0,T]\times\{x\in\mathbb{R}^3 : |x|<3\})} . \tag{2.40}$$

Proof. Fix a smooth function χ such that $\chi(s) = 0$ for $|s| > 2$ and $\displaystyle\sum_{j=-\infty}^{\infty} \chi(s-j) = 1$. We will let $G_j(t,x) = \chi(t-j)G(t,x)$. Thus, $G = \displaystyle\sum_{j=-\infty}^{\infty} G_j$. Let u_j solve the boundaryless wave equation $\Box u_j = G_j$ with vanishing initial data. Since

$$u_j(t,x) = \int_0^t \int E(t-s, x-y) G_j(s,y) \, dy \, ds,$$

where E is the fundamental solution to the linear wave equation in 3 dimensions, it follows from sharp Huygens' principle that u_j is supported on $\{(t,x) : j-5 \leq |t-|x|| \leq j+5\}$. Since $u = \sum_j u_j$ and the supports of the functions $\{u_j\}$ have finite overlap, it follows that $|u|^2 \lesssim \sum_j |u_j|^2$. By the above argument and Corollary 2.11, it follows that

$$\begin{aligned}
&\|u(t,\cdot)\|_2^2 + \langle T\rangle^{-1/2} \sum_{\substack{|\alpha|+\mu\leq N+\nu \\ \mu\leq\nu}} \left\| \langle x\rangle^{-1/4} u \right\|_{L^2([0,T]\times\mathbb{R}^3)}^2 \\
&\lesssim \sum_j \left(\|u_j(t,\cdot)\|_2^2 + \langle T\rangle^{-1/2} \left\| \langle x\rangle^{-1/4} u_j \right\|_{L^2([0,T]\times\mathbb{R}^3)}^2 \right) \\
&\lesssim \sum_j \left(\int_0^T \|G_j(s,\cdot)\|_2 \, ds \right)^2 \\
&\lesssim \int_0^T \sum_j \|G_j(s,\cdot)\|_2^2 \, ds \\
&= \|G\|_{L^2([0,T]\times\{|x|<3\})}^2 .
\end{aligned}$$

□

2.4. Sobolev Estimates

We will need to prove an analogue of the Sobolev embedding theorem for the sphere S^{n-1} (see Klainerman [**32**]).

PROPOSITION 2.13. *Suppose $h \in C^\infty(S^{n-1})$. Then it follows that*

$$\|h\|_{L^\infty(S^{n-1})} \lesssim \sum_{|\alpha|\leq n-1} \|\Omega^\alpha h\|_{L^1(S^{n-1})}$$

and

$$\|h\|_{L^\infty(S^{n-1})} \lesssim \sum_{|\alpha|\leq (n+1)/2} \|\Omega^\alpha h\|_{L^2(S^{n-1})} .$$

PROOF. Define a partition of unity $\{\chi_k\}$ subordinate to an atlas for S^{n-1} that consists of finitely many coordinate charts $\{\varphi_k : U_k \to V_k\}$. For the first inequality, we apply the fundamental theorem of calculus in each coordinate direction on each V_k. This gives us

$$\|h\|_{L^\infty(S^{n-1})} \lesssim \sum_k \int_{V_k} \left| \partial_1 \cdots \partial_{n-1}((\chi_k h)\circ\varphi_k^{-1})(x) \right| \, dx.$$

Since the collection $\{\Omega^{ij}\}$ span the tangent space at each point on S^{n-1}, this proves the first inequality. To obtain the second inequality, we apply Sobolev embedding on each V_k to get

$$\begin{aligned}\|h\|_{L^\infty(S^{n-1})} &\lesssim \sum_k \left\|((\chi_k h)\circ\varphi^{-1})\right\|_{L^\infty(V_k)}\\ &\lesssim \sum_{|\alpha|\le(n+1)/2} \|\Omega^\alpha h\|_{L^2(S^{n-1})}.\end{aligned}$$

□

We also need the following local version of the Sobolev embedding theorem where x is taken over an annulus.

Proposition 2.14. *Suppose $h \in C^\infty(\mathbb{R}^n)$. Then for $R \ge 1$,*

$$\|h\|_{L^\infty(R<|x|<2R)} \lesssim R^{-(n-1)/2} \sum_{|\alpha|+j\le(n+2)/2} \left\|\Omega^\alpha \partial_r^j h\right\|_{L^2(R/2<|x|<4R)}, \tag{2.41}$$

and

$$\|h\|_{L^\infty(R<|x|<R+1)} \lesssim R^{-(n-1)/2} \sum_{|\alpha|+j\le(n+2)/2} \left\|\Omega^\alpha \partial_r^j h\right\|_{L^2(R-1<|x|<R+2)}. \tag{2.42}$$

Proof. Fix a cutoff $\rho \in C^\infty(\mathbb{R})$ such that $\rho(s) = 1$ when $1 < s < 2$ and zero when $s < 1/2$ or $s > 4$. Thus, $\rho(s/R) = 1$ when $R < s < 2R$ and equals zero when $r < R/2$ or $r > 4R$. Just as in the previous proposition, if we define a finite partition of unity $\{\varphi_k : U_k \to V_k\}$ on S^{n-1} and apply Sobolev embedding on $\mathbb{R} \times V_k$, it follows that

$$\sup_{(r,\omega)\in(R,2R)\times S^{n-1}} \chi(r/R)|h(r\omega)| \lesssim \sum_{|\alpha|+j\le(n+2)/2} \left(\int_{S^{n-1}} \int_{R/2}^{4R} |\Omega^\alpha \partial_r^j h(r\omega)|^2 \, dr d\omega\right)^{1/2}. \tag{2.43}$$

Since the volume element for $\mathbb{R}^n$ in polar coordinates is $r^{n-1} dr d\omega$, the last quantity in (2.43) is controlled by the right hand side of (2.41). We obtain (2.42) via a similar argument. Define $\xi \in C^\infty(\mathbb{R})$ such that $\xi(s)$ equals 1 when $0 < s < 1$ and equals zero when $s < -1$ or $s > 2$. By Sobolev embedding, it follows that

$$\sup_{(r,\omega)\in(R,R+1)\times S^{n-1}} \xi(r-R)|h(r\omega)| \lesssim \sum_{|\alpha|+j\le(n+2)/2} \left(\int_{S^{n-1}} \int_{R-1}^{R+2} |\Omega^\alpha \partial_r^j h(r\omega)|^2 \, dr d\omega\right)^{1/2}, \tag{2.44}$$

which is bounded by the right hand side of (2.42). □

2.5. Pointwise Estimates

We shall need the L^1, L^∞ estimates of Hörmander [**14**,**16**] and Klainerman [**32**] who proved them for wave equations in $[0,T]\times\mathbb{R}^3$. We shall give the proof of Keel, Smith and Sogge [**28**] which uses the positivity of the fundamental solution to the 3-dimensional wave equation. Their proof was an improvement to previous versions of this estimate in that it only relied on scaling, spatial rotations and spatial translations. The absence of Lorentz boosts in the right hand side of the inequality will allow us to apply this estimate in the exterior domain setting.

Proposition 2.15. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3)$ solves*

$$\begin{cases} \Box u(t,x) = G(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^3, \\ u(0,x) = \partial_t u(0,x) = 0. \end{cases} \tag{2.45}$$

Then it follows that for $0 \le t \le T$,

$$(1+t+|x|)|u(t,x)| \lesssim \int_0^t \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} |L^\mu Z^\alpha G(s,y)| \; \frac{dy\,ds}{|y|}. \tag{2.46}$$

Before we prove this proposition, we need to prove a couple of lemmas.

Lemma 2.16. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3)$ is a solution to (2.45). Fix $x\in\mathbb{R}^3$, and let $|x| = r$. Then it follows that for $0\le t\le T$,*

$$|x|\,|u(t,x)| \le \frac{1}{2}\int_0^t \int_{|r-(t-s)|}^{r+t-s} \sup_{|\theta|=1} |G(s,\rho\theta)| \; \rho\, d\rho\, ds. \tag{2.47}$$

Proof. Let U solve

$$\begin{cases} \Box U(t,x) = F(t,|x|), \quad (t,x)\in[0,T]\times\mathbb{R}^3, \\ U(0,x) = \partial_t U(0,x) = 0, \end{cases} \tag{2.48}$$

where $F(t,|x|) = \sup_{\theta\in S^2} |G(t,|x|\theta)|$. Note that with a slight abuse of notation, we will write $U(t,x) = U(t,r)$ since U is a radial function. Since $|G(t,x)| \le F(t,|x|)$, it follows that $|u(t,x)| \le U(t,x)$. We will now show that rU solves a 1-dimensional wave equation. If we define $\Box_r = \partial_t^2 - \partial_r^2$, we see that $\Box_r(rU) = r\Box U$. From the solution to the 1-dimensional wave equation, we see that

$$rU(t,r) = \frac{1}{2}\int_0^t \int_{|r-(t-s)|}^{r+(t-s)} F(s,\rho)\rho\; d\rho ds.$$

From this solution and the fact that $|u(t,x)| \le U(t,x)$, one sees that (2.47) holds. $\Box$

For the next lemma, we will be using the proof that appears in Keel, Smith and Sogge [**28**].

Lemma 2.17. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3)$ solves (2.45). Then it follows that for $0 \le t \le T$,*

$$t|u(t,x)| \lesssim \int_0^t \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} |L^\mu \Omega^\alpha G(s,y)| \ \frac{dy\ ds}{|y|}. \tag{2.49}$$

Proof. Let $\tilde{u}(s,x) = u(ts,tx)$ and $\tilde{G}(s,x) = t^2 G(ts,tx)$. We see that $\tilde{u}$ solves $\Box \tilde{u}(s,x) = \tilde{G}(s,x)$. Suppose we know that

$$|\tilde{u}(1,x/t)| \lesssim \int_0^1 \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} \left|L^\mu \Omega^\alpha \tilde{G}(s,y)\right| \ \frac{dy\ ds}{|y|}. \tag{2.50}$$

Introducing a change of coordinates, we set $y' = ty$ and $s' = ts$. By utilizing the fact that homogeneous vector fields, such as those present in (2.50), are invariant under scaling, we see that the right hand side of (2.50) is equal to

$$\frac{1}{t}\int_0^t \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|+\mu\le 3 \\ \mu\le 1}} \left|L^\mu \Omega^\alpha G(s',y')\right| \ \frac{dy'\ ds'}{|y'|}.$$

This would imply that

$$\begin{aligned} t|u(t,x)| &= t|\tilde{u}(1,x/t)| \\ &\lesssim \int_0^t \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|\le 2 \\ \mu\le 1}} \left|L^\mu \Omega^\alpha G(s',y')\right| \ \frac{dy'\ ds'}{|y'|}. \end{aligned} \tag{2.51}$$

Since the right hand side of (2.50) is independent of x, (2.51) shows that it suffices to prove (2.49) in the case $t = 1$.

When $|x| > 1/10$, we shall apply (2.47) to see that

$$|x||u(1,x)| \lesssim \int_0^1 \int_{|r-(1-s)|}^{r+1-s} \sup_{\theta\in S^2} |G(s,\rho\theta)| \ \rho\, d\rho\, ds. \tag{2.52}$$

Applying Sobolev embedding on S^2, we see that the left hand side of (2.52) is controlled by

$$\sum_{|\alpha|\le 2} \int_0^1 \int_{\mathbb{R}^3} |\Omega^\alpha G(s,y)| \ \frac{dy\ ds}{|y|}.$$

In the case that $|x| > 1/10$, this proves (2.49).

To deal with the case where $|x| \leq 1/10$, we will want to restrict the support of G. Define $\psi \in C^\infty(\mathbb{R})$ such that $\psi(y) = 0$ when $|y| > 4$ and $\psi(y) = 1$ when $|y| < 2$. For our purposes, we will need the fact that $\psi(y/|x|) = 1$ when $|y| < 2|x|$ and zero when $|y| > 4|x|$. We shall observe that

$$\Omega^{ij}\left(\psi(y/|x|)G(s,y)\right) = \frac{y_i}{|x|}\partial_j\psi(y/|x|)G(s,y) - \frac{y_j}{|x|}\partial_i\psi(y/|x|)G(s,y) + \psi(y/|x|)\,\Omega^{ij}G(s,y),$$

and that

$$L\left(\psi(y/|x|)G(s,y)\right) = \left(\sum_{i=1}^{3}\frac{y_i}{|x|}\partial_i\psi(y/|x|)\right)G(s,y) + \psi(y/|x|)LG(s,y).$$

Due to the fact that $|y|/|x| < 4$ when $\psi(y/|x|) \neq 0$, it follows from the two previous equalities that

$$|\Omega^{ij}\left(\psi(y/|x|)G(s,y)\right)| \lesssim |G(s,y)| + |\Omega^{ij}G(s,y)|,$$

and

$$|L\left(\psi(y/|x|)G(s,y)\right)| \lesssim |G(s,y)| + |LG(s,y)|,$$

where the implicit constants depend only on the choice of the cutoff function ψ. So if we use the cutoff function to split G,

$$G(s,y) = \psi(y/|x|)G(s,y) + (1-\psi(y/|x|))G(s,y),$$

then it follows that we can reduce matters to considering two different cases:

- Case 1: supp $G \subseteq \{(s,y) : |y| \geq 2|x|\}$.
- Case 2: supp $G \subseteq \{(s,y) : |y| \leq 4|x|\}$.

Case 1: From the fundamental solution to the linear wave equation, it follows that

$$u(1,x) = \frac{1}{4\pi}\int_{|y|<1} G(1-|y|, x-y)\,\frac{dy}{|y|}. \tag{2.53}$$

Following the proof of Keel, Smith and Sogge [**28**], we will first bound the integral by using our assumptions on x and the support of G. We will then introduce a change of coordinates to simplify the integrand. This will enable us to bound $u(1,x)$ with relative ease.

To simplify the integral, we observe that when $|x| \geq |y|$, it follows that $|x-y| \leq |x|+|y| \leq 2|x|$. This implies that $G(1-|y|, x-y) = 0$, when $|x| \geq |y|$. Hence, on supp $G(1-|\cdot|, x-\cdot)$, $\dfrac{1}{|y|} \leq \dfrac{2}{|x-y|}$. Thus, we obtain the inequality

$$|u(1,x)| \lesssim \int_{|y|<1} |G(1-|y|, x-y)| \ \frac{dy}{|x-y|}.$$

To introduce a suitable change of coordinates, we will first show that $|(1-|y|, x-y)| \geq 2/5$ on the support of $G(1-|\cdot|, x-\cdot)$. Suppose that $|1-|y|| < 1/2$. Then it follows that

$$\begin{aligned}|x-y| &\geq |y| - |x| \\ &= -1 + |y| + 1 - |x| \\ &\geq -1/2 + 1 - 1/10 = 2/5.\end{aligned}$$

Thus, it follows that $|(1-|y|, x-y)| \geq 2/5$ on the support of $G(1-|\cdot|, x-\cdot)$. We take a cut-off function $\rho \in C^\infty(\mathbb{R})$ such that $\rho(s) = 1$ if $s > 2/5$ and $\rho(s) = 0$ if $s < 1/5$. Using this cutoff, we define the function

$$H(s,y) = \rho(|(s,y)|)|G(s,y)|/|y|.$$

Thus, it follows that

$$|u(1,x)| \lesssim \int_{|y|<1} |H(1-|y|, x-y)| \ dy.$$

We are now ready to define our change of coordinates. Let the map $\varphi(s,y) = s(1-|y|, x-y)$. Noting that the Jacobian of φ is equal to $s^3 \left(\dfrac{\langle x, y\rangle}{|y|} - 1 \right)$ it follows from the fact that $|x| \leq 1/10$ and $H(s,y) = 0$ for $|(s,y)| < 1/5$ that the Jacobian is always bounded away from 0 on the support of $H(\varphi(\cdot,\cdot))$. Using this observation and applying the fundamental theorem of calculus, we see that

$$\begin{aligned}\int_{|y|<1} |H(1-|y|, x-y)| \ dy &= \int_{|y|<1} |(H\circ\varphi)(1,y))| \ dy \\ &\lesssim \int_0^1 \int_{|y|<1} |\partial_s (H\circ\varphi)(s,y))| \ dy\, ds + \int_{|y|<1} |(H\circ\varphi)(0,y))| \ dy.\end{aligned}$$

Observe that $(H \circ \varphi)(0, y) = 0$ and $|\partial_s (H \circ \varphi)(s, y)| = |(LH)(\varphi(s, y))|/s$. Since the Jacobian of φ is bounded below when $H(\varphi(s, y)) \neq 0$, it also follows that s is bounded below when $H(\varphi(s, y)) \neq 0$. Thus, we get the inequality

$$\begin{aligned}
|u(1,x)| &\lesssim \int_0^1 \int_{|y|<1} |(LH)(\varphi(s,y))| \ dy \ ds \\
&\lesssim \int_0^1 \int_{\mathbb{R}^3} |LH(s,y)| \left|J_{\varphi^{-1}}\right| \ dy \ ds \\
&\lesssim \int_0^1 \int_{\mathbb{R}^3} \sum_{\mu \leq 1} |L^\mu G(s,y)| \ \frac{dy \ ds}{|y|},
\end{aligned}$$

where $J_{\varphi^{-1}}$ is the determinant the Jacobian of φ^{-1}. This deals with Case 1.

Case 2: If one rewrites (2.53) slightly differently (see Sogge [**67**]), one can see that

$$u(1,x) = \frac{1}{4\pi} \int_0^1 \int_{S^2} (1-s) G(s, x + (1-s)z) \ d\omega(z) \ ds. \tag{2.54}$$

Since we are assuming that $G(s, y) = 0$ if $|y| > 4|x|$, then we see that the integrand of (2.54) is nonzero only when $|x + (1 - s)z| \leq 4|x|$. Hence, it is nonzero only when $s \geq 1 - 5|x|$. It follows that $u(1, x) = u_0(1, x)$, where u_0 solves the inhomogeneous wave equation $\Box u_0(s, y) = G_0(s, y)$, where $G_0(s, y) = G(s, y)$ if $s > 1 - 5|x|$ and zero otherwise with vanishing initial data. By Lemma 2.16 and Sobolev embedding, it follows that

$$\begin{aligned}
|u(1,x)| = |u_0(1,x)| &\lesssim \frac{1}{|x|} \int_{1-5|x|}^1 \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 2} |\Omega^\alpha G_0(s,y)| \ \frac{dy \ ds}{|y|} \\
&\lesssim \sup_{1-5|x|<s<1} \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 2} |\Omega^\alpha G_0(s,y)| \ \frac{dy}{|y|}.
\end{aligned}$$

Again we apply the fundamental theorem of calculus and the chain rule to see that

$$\begin{aligned}
G_0(s,y) &= \int_0^1 s(\partial_s G_0)(\tau s, \tau y) + \langle y, (\nabla_y G_0) \rangle (\tau s, \tau y) \ d\tau \\
&= \int_0^1 \frac{1}{\tau} (L G_0)(\tau s, \tau y) \ d\tau.
\end{aligned}$$

From this observation, we see that

$$\sup_{1-5|x|<s<1} \int_{\mathbb{R}^3} \sum_{|\alpha|\le 2} |\Omega^\alpha G_0(s,y)| \, \frac{dy}{|y|} \lesssim \sup_{1-5|x|<s<1} \int_0^1 \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|\le 2\\ \mu\le 1}} |L^\mu \Omega^\alpha G_0(\tau s, \tau y)| \, \frac{dy\, d\tau}{\tau|y|}. \tag{2.55}$$

Notice that the determinant of the Jacobian of the map $(\tau, y) \mapsto (\tau s, \tau y)$ is $s\tau^3$. Due to the fact that $G_0(\tau s, \tau y)$ is supported on $1-5|x| < \tau s < 1$ and that s in the right hand side of (2.55) is taken over $1-5|x| < s < 1$, it follows that $(1-5|x|)^3 < s\tau^3 < (1-5|x|)^{-2}$. Recalling that $|x| < 1/10$, we see that the right hand side of (2.55) is controlled by

$$\int_0^1 \int_{\mathbb{R}^3} \sum_{\substack{|\alpha|\le 2\\ \mu\le 1}} |L^\mu \Omega^\alpha G(s,y)| \, \frac{dy\, ds}{|y|},$$

where the implicit constant is independent of x. This completes the proof. □

We are now ready to prove Proposition 2.15.

PROOF. (Proposition 2.15) By applying (2.47) and Sobolev embedding, we only need to consider the case when the weight $(1+t+|x|)$ in the left hand side of (2.46) is replaced with $(1+t)$. Using cut-offs, it suffices to consider the cases in which supp $G \subseteq \{(t,x) : t \ge 1\}$ and supp $G \subseteq \{(t,x) : 0 \le t \le 2\}$. The first case follows from the previous lemma and the observation that our assumption about the support of G implies that the support of u is also contained in $\{(t,x) : t \ge 1\}$. When G is supported on $\{0 \le t \le 2\}$, we define $\tilde{u}$ such that $\tilde{u}$ solves the inhomogeneous wave equation $\Box \tilde{u}(t,x) = G(t-2,x)$ with vanishing initial data. By the first case, we see that

$$\begin{aligned}(1+t)|\tilde{u}(t,x)| &\lesssim \sum_{\substack{|\alpha|\le 2\\ \mu\le 1}} \int_0^t \int_{\mathbb{R}^3} |L^\mu \Omega^\alpha G(s-2,x)| \, \frac{dy\, ds}{|y|} \\ &\lesssim \sum_{\substack{|\alpha|+\mu\le 3\\ \mu\le 1}} \int_0^{t-2} \int_{\mathbb{R}^3} |L^\mu Z^\alpha G(s,x)| \, \frac{dy\, ds}{|y|}.\end{aligned} \tag{2.56}$$

Note that to get from the first inequality to the second in (2.56), we introduce time translations from the substitution in the s variable. Because $u(t,x) = \tilde{u}(t+2,x)$ and $t > 1$ on the support of $\tilde{u}$, it follows that

$$(1+t)|u(t,x)| \lesssim (3+t)|\tilde{u}(t+2,x)|$$

$$\lesssim \sum_{\substack{|\alpha|+\mu\leq 3\\ \mu\leq 1}} \int_0^t \int_{\mathbb{R}^3} |L^\mu Z^\alpha G(s,x)| \, \frac{dy\, ds}{|y|}.$$

The general case follows from using cutoffs in the t variable. $\square$

2.6. Divergence-Form Estimates

We will need a variant of Proposition 4 from Metcalfe and Sogge [**49**]. This will enable us to control some of the terms where no spacetime gradient is being applied to the solution u in the special case that $\Box u = \sum_j a_j\partial_j G$ for some smooth function G. Earlier estimates that were obtained using these techniques can also be found in [**15**] and [**40**].

Theorem 2.18. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3)$ is a solution to*

$$\begin{aligned} \Box u(t,x) &= \sum_{j=0}^{3} a_j\partial_j G(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^3 \\ u(0,x) &= \partial_t u(0,x) = 0, \end{aligned} \tag{2.57}$$

where $a_j\in\mathbb{R}$ and $G\in C^\infty([0,T]\times\mathbb{R}^3)$. Also suppose that $G(0,x)=0$ and that for each fixed t, $G(t,x)$ vanishes for sufficiently large $|x|$. Then it follows that

$$\sup_{0<t<T} \|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \langle T\rangle^{-1/4} \left\|\langle x\rangle^{-1/4} u\right\|_{L^2([0,T]\times\mathbb{R}^3)} \lesssim \int_0^T \|G(s,\cdot)\|_2 \, ds. \tag{2.58}$$

Proof. To prove this, we observe that

$$u = \sum_{j=0}^{3} a_j\partial_j w,$$

where w solves $\Box w(t,x) = G(t,x)$ with vanishing initial data. To bound the first term on the left hand side of (2.58), we apply Corollary 2.2 to w. It follows that

$$\begin{aligned} \sup_{0\leq t\leq T} \|u(t,\cdot)\|_2 &= \sup_{0<t<T} \left\|w'(t,\cdot)\right\|_2 \\ &\lesssim \int_0^T \|G(s,\cdot)\|_2 \, ds. \end{aligned}$$

Bounds for the second term on the left hand side of (2.58) follow from the same argument except that one applies Theorem 2.3 to w. $\square$

CHAPTER 3

Estimates for Dirichlet-Wave Equations in Exterior Domains

3.1. L^2 Estimates

We shall need to prove some estimates for solutions $u \in C^\infty([0,T]\times\mathbb{R}^n\backslash\mathcal{K})$ to a perturbed Dirichlet wave equation

$$\begin{cases} \Box_\gamma u(t,x) = G(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^n\backslash\mathcal{K}, \\ u(t,x)=0, \quad x\in\partial\mathcal{K}, \\ u(0,x)=f(x), \quad \partial_t u(0,x) = g(x). \end{cases} \tag{3.1}$$

Let

$$(\Box_\gamma u)^I = (\partial_t^2 - c_I^2\Delta)u^I + \sum_{J=1}^{D}\sum_{j,k=0}^{n}\gamma^{jk,IJ}(t,x)\partial_j\partial_k u^J, \tag{3.2}$$

where the $\gamma^{jk,IJ}$ are our perturbation terms. We assume that each $\gamma^{jk,IJ}\in C^\infty$, that $\gamma^{jk,IJ}$ satisfy the following symmetry conditions

$$\gamma^{jk,IJ} = \gamma^{kj,IJ} = \gamma^{jk,JI}. \tag{3.3}$$

Slightly abusing notation, we set

$$\sum_{I,J=1}^{D}\sum_{j,k=0}^{n}\left\|\gamma^{jk,IJ}(t,\cdot)\right\|_{L^\infty(\mathbb{R}^n\backslash\mathcal{K})} := \|\gamma(t,\cdot)\|_\infty .$$

We also assume that

$$\|\gamma(t,\cdot)\|_\infty \le \delta, \tag{3.4}$$

where δ is taken to be sufficiently small. We will also be concerned with norms that involve the gradient of γ. With another abuse of notation, we shall write

$$\left\|\gamma'(t,\cdot)\right\|_\infty := \sum_{I,J=1}^{D}\sum_{j,k,l=0}^{n}\left\|\partial_l\gamma^{jk,IJ}(t,\cdot)\right\|_\infty .$$

We shall define the energy form $e_0(u) := \sum_{I=1}^D e_0^I(u)$ that is associated with $\Box_\gamma$, where for $I = 1, \dots, D$, we define

$$
\begin{aligned}
e_0^I = e_0^I(u) = (\partial_0 u^I)^2 + \sum_{k=1}^n c_I^2 (\partial_k u^I)^2 \\
+ 2 \sum_{J=1}^D \sum_{k=0}^n \gamma^{0k,IJ} \partial_0 u^I \partial_k u^J - \sum_{J=1}^D \sum_{j,k=0}^n \gamma^{jk,IJ} \partial_j u^I \partial_k u^J.
\end{aligned} \tag{3.5}
$$

We also define the following quantity, which will be the primary ingredient for the estimates in this section:

$$
E_N(t) = E_N(u)(t) = \int \sum_{j=0}^N e_0(\partial_t^j u)(t,x)\, dx. \tag{3.6}
$$

This particular quantity is important since $\partial_t^k u$ satisfies the Dirichlet boundary conditions. We now state our most basic estimate which shall enable us to control energy norms that involve time translations ∂_t, which will be essential in the proof of Theorem 1.4. The proof of this theorem shall also serve as a model for the proofs of many subsequent estimates in this paper. The first estimate is a standard energy estimate that was employed in earlier works, such as Keel, Smith and Sogge [**28**] and Metcalfe and Sogge [**45**].

THEOREM 3.1. *Fix $N = 0, 1, 2, \dots$ and assume the peturbation terms γ^{ij} are as in* (3.3) *and* (3.4). *Also assume that δ in* (3.4) *is small. Assume that $u \in C^\infty([0,T] \times \mathbb{R}^n \backslash \mathcal{K})$ solves* (3.1) *and that for every fixed t, $u(t,x) = 0$ for $|x|$ sufficiently large. Then it follows that there is a constant $C > 0$ such that*

$$
\partial_t \left[E_N^{1/2}(t) \right] \le C \sum_{j=0}^N \left\| \Box_\gamma \partial_t^j u(t,\cdot) \right\|_2 + C \left\| \gamma'(t,\cdot) \right\|_\infty E_N^{1/2}(t). \tag{3.7}
$$

PROOF. Due to the fact that $\partial_t^k u$ satisfies the Dirichlet boundary conditions, it follows that we need only prove (3.7) in the case that $N = 0$. To do this, we shall need to define the remaining components for the energy-momentum vector. For $k = 1, \dots, n$, and $I = 1, \dots, D$, we define the remaining components of the energy-momentum vector:

$$
e_k^I = e_k^I(u) = -2c_I^2 \partial_0 u^I \partial_k u^I + 2 \sum_{J=1}^D \sum_{j=0}^n \gamma^{jk,IJ} \partial_0 u^I \partial_j u^J. \tag{3.8}
$$

From (3.5), we see that

$$\begin{aligned}
\partial_0 e_0^I &= 2\partial_0 u^I \partial_0^2 u^I + 2c_I^2 \sum_{k=1}^{n} \partial_k u^I \partial_0 \partial_k u^I \\
&\quad + 2\sum_{J=1}^{D}\sum_{k=0}^{n} \gamma^{0k,IJ} \partial_0 u^I \partial_0 \partial_k u^J + 2\sum_{J=1}^{D}\sum_{k=0}^{n} \gamma^{0k,IJ} \partial_0^2 u^I \partial_k u^J \\
&\quad - \sum_{J=1}^{D}\sum_{j,k=0}^{n} \gamma^{jk,IJ} \left[\partial_0 \partial_j u^I \partial_k u^J + \partial_j u^I \partial_0 \partial_k u^J\right] + R_0^I,
\end{aligned} \tag{3.9}$$

where

$$R_0^I = 2\sum_{J=1}^{D}\sum_{k=0}^{n} \partial_0(\gamma^{0k,IJ}) \partial_0 u^I \partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{n} \partial_0(\gamma^{jk,IJ}) \partial_j u^I \partial_k u^J. \tag{3.10}$$

Using the symmetry conditions (3.3), upon summing over I, we see that

$$\begin{aligned}
&2\sum_{I,J=1}^{D}\sum_{k=0}^{n} \gamma^{0k,IJ} \partial_0^2 u^I \partial_k u^J - \sum_{I,J=1}^{D}\sum_{j,k=0}^{n} \gamma^{jk,IJ} \left[\partial_0 \partial_j u^I \partial_k u^J + \partial_j u^I \partial_0 \partial_k u^J\right] \\
&\qquad = -2\sum_{I,J=1}^{D}\sum_{j=0}^{n}\sum_{k=1}^{n} \gamma^{jk,IJ} \partial_0 \partial_k u^I \partial_j u^J.
\end{aligned} \tag{3.11}$$

Thus, it follows that

$$\begin{aligned}
\sum_{I=1}^{D} \partial_0 e_0^I &= 2\sum_{I=1}^{D} \partial_0 u^I \partial_0^2 u^I + 2\sum_{I,J=1}^{D}\sum_{k=1}^{n} \partial_k u^I \partial_0 \partial_k u^J \\
&\quad + 2\sum_{I,J=1}^{D}\sum_{k=0}^{n} \gamma^{0k,IJ} \partial_0 u^I \partial_0 \partial_k u^J - 2\sum_{I,J=1}^{D}\sum_{j=0}^{n}\sum_{k=1}^{n} \gamma^{jk,IJ} \partial_0 \partial_k u^I \partial_j u^J + \sum_{I=1}^{D} R_0^I.
\end{aligned} \tag{3.12}$$

We also see that

$$\begin{aligned}
\sum_{k=1}^{n} \partial_k e_k^I &= -2c_I^2 \partial_0 u^I \Delta u^I - 2\sum_{k=1}^{n} c_I^2 \partial_0 \partial_k u^I \partial_k u^I \\
&\quad + 2\sum_{J=1}^{D}\sum_{j=0}^{n}\sum_{k=1}^{n} \gamma^{jk,IJ} \partial_0 \partial_k u^I \partial_j u^J + 2\sum_{J=1}^{D}\sum_{j=0}^{n}\sum_{k=1}^{n} \gamma^{jk,IJ} \partial_0 u^I \partial_j \partial_k u^J + \sum_{k=1}^{n} R_k^I,
\end{aligned} \tag{3.13}$$

where

$$R_k^I = 2\sum_{J=1}^{D}\sum_{j=0}^{n} \partial_k(\gamma^{jk,IJ}) \partial_0 u^I \partial_j u^J. \tag{3.14}$$

We set

$$e_j := e_j(u) = \sum_{I=1}^{D} e_j^I(u), \quad j = 0, 1, \ldots, n, \tag{3.15}$$

and

$$R(u', u') := \sum_{I=1}^{D} \sum_{k=0}^{n} R_k^I.$$

Note that when we sum over I, the third term in right hand side of (3.13) results in a quantity that is equal to -1 times the term appearing in the right hand side of (3.11). Thus, we see that

$$\begin{aligned}\sum_{j=0}^{n} \partial_j e_j &= 2\,\langle \partial_0 u, \Box u\rangle + 2 \sum_{I,J=1}^{D} \sum_{j,k=0}^{n} \gamma^{jk,IJ} \partial_0 u^I \partial_j \partial_k u^J + R(u', u') \\ &= 2\,\langle \partial_0 u, \Box_\gamma u\rangle + R(u', u'),\end{aligned} \tag{3.16}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^D$. Thus, we have the equation

$$\partial_0 e_0 + \sum_{j=1}^{n} \partial_j e_j = 2\,\langle \partial_0 u, \Box_\gamma u\rangle + R(u', u'). \tag{3.17}$$

Integrating with respect to x and applying the divergence theorem, we see that

$$\partial_0 \int_{\mathbb{R}^n \setminus \mathcal{K}} e_0(t,x)\, dx - \sum_{j=1}^{n} \int_{\partial \mathcal{K}} e_j n_j \, d\omega = 2 \int_{\mathbb{R}^n \setminus \mathcal{K}} \langle \partial_0 u, \Box_\gamma u\rangle \; dx + \int_{\mathbb{R}^n \setminus \mathcal{K}} R(u', u')\, dx. \tag{3.18}$$

In the previous equation, $\vec{n} = (n_1, \ldots, n_n)$ is the outward unit vector normal to $\mathcal{K}$. However, because $\partial_0 u$ vanishes on $\partial\mathcal{K}$, we see that

$$\partial_0 \int_{\mathbb{R}^n \setminus \mathcal{K}} e_0(t,x)\, dx = 2 \int_{\mathbb{R}^n \setminus \mathcal{K}} \langle \partial_0 u, \Box_\gamma u\rangle \; dx + \int_{\mathbb{R}^n \setminus \mathcal{K}} R(u', u')\, dx. \tag{3.19}$$

Noting that when δ is small, then

$$\left(5 \max_I \{c_I^2, c_I^{-2}\}\right)^{-1} \left|u'(t,x)\right|^2 \le e_0(u)(t,x) \le 5 \max_I \{c_I^2, c_I^{-2}\} \left|u'(t,x)\right|^2. \tag{3.20}$$

Applying Cauchy-Schwarz to the first term in the right hand side of (3.19) and (3.20) to both terms in the right hand side of (3.19), we see that

$$\begin{aligned}\partial_0 \int_{\mathbb{R}^n \setminus \mathcal{K}} e_0(t,x))\, dx \le C \left(\int_{\mathbb{R}^n \setminus \mathcal{K}} e_0(t,x)\, dx \right)^{1/2} \left\| \Box_\gamma u(t,\cdot) \right\|_2 \\ + C \left\| \gamma'(t,\cdot) \right\|_\infty \int_{\mathbb{R}^n \setminus \mathcal{K}} e_0(t,x)\, dx.\end{aligned}$$

Dividing both sides by $\left(\int_{\mathbb{R}^n\backslash\mathcal{K}} e_0(t,x)\,dx\right)^{1/2}$, we have established (3.7) in the case $N = 0$. This proves the theorem. $\square$

Now that we can control basic energy norms that involve only time translations ∂_t, it follows that we now would like to control energy norms that involve a larger collection of admissible vector fields. Specifically, we would like to be able to control L^2 norms where spatial and time translations are being applied to u. We also want to control energy norms where we allow at most one scaling vector field to be applied to u. We shall first prove the following estimate to establish local control of norms where no spacetime gradient is applied u. In this lemma and in the estimates to follow, we will often shorten notation by writing

$$\|u(t,\cdot)\|_{L^2(|x|<R)} := \|u(t,\cdot)\|_{L^2(\{x\in\mathbb{R}^n\backslash\mathcal{K}:|x|<R\})}\,.$$

Lemma 3.2. *Let $u \in C^\infty([0,T]\times\mathbb{R}^n\backslash\mathcal{K})$ vanish on $\partial\mathcal{K}$. Then it follows that for $0 \le t \le T$, and $1 \le p \le \infty$,*

$$\|u(t,\cdot)\|_{L^p(|x|<2)} \lesssim \|\nabla_x u(t,\cdot)\|_{L^p(|x|<2)} \tag{3.21}$$

Proof. Let us write for $\omega \in S^{n-1}$,

$$S(\omega) = \{0 < r < 2 : r\omega \in \mathbb{R}^n\backslash\mathcal{K}\}.$$

By the Fundamental Theorem of Calculus and the Dirichlet boundary conditions, we see that

$$\begin{aligned}
\|u(t,\cdot)\|^p_{L^p(|x|<2)} &\lesssim \int_{S^{n-1}}\int_{S(\omega)}\int_{S(\omega)} |\partial_\rho u(t,\rho\omega)|\,|u(t,\rho\omega)|^{p-1}\;d\rho\;dr\;d\omega \\
&\lesssim \int_{S^{n-1}}\int_{S(\omega)} |\partial_\rho u(t,\rho\omega)|\,|u(t,\rho\omega)|^{p-1}\;d\rho\;d\omega \\
&\lesssim \|\nabla_x u(t,\cdot)\|_{L^p(|x|<2)}\,\|u(t,\cdot)\|^{p-1}_{L^p(|x|<2)}\,,
\end{aligned}$$

where we are applying Hölder's inequality in the last step. Dividing both sides by

$$\|u(t,\cdot)\|^{p-1}_{L^p(|x|<2)}$$

proves (3.21) for $1 \le p < \infty$. The case when $p = \infty$ follows from the fact that if $r \le 2$, then

$$u(t,r\omega) \le \int_{S(\omega)} \partial_\rho u(t,\rho\omega)\;d\rho \le 2\sup_{|x|<2}|\nabla_x u(t,x)|\,.$$

□

We shall next prove a very useful elliptic regularity estimate (see Keel-Smith-Sogge [**28**]). This will enable us to use the local energy decay assumption (1.6) and the previous lemma to control L^2-norms that involve spatial translations ∂_i.

PROPOSITION 3.3. *Suppose $u \in C^\infty([0,T] \times \mathbb{R}^n \backslash \mathcal{K})$ solves (3.1) and suppose that for every fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then it follows that for fixed N, ν and for $0 \le t \le T$,*

$$\begin{aligned}\sum_{|\alpha| \le N} \left\| L^\nu \partial^\alpha u'(t,\cdot) \right\|_2 &\lesssim \sum_{\substack{j+\mu \le N+\nu \\ \mu \le \nu}} \left\| L^\mu \partial_t^j u'(t,\cdot) \right\|_2 \\ &\quad + \sum_{\substack{|\alpha|+\mu \le N+\nu-1 \\ \mu \le \nu}} \left\| L^\mu \partial^\alpha \Box u(t,\cdot) \right\|_2 .\end{aligned} \tag{3.22}$$

PROOF. We shall first prove the boundaryless version of (3.22) where $\mathcal{K} = \emptyset$. We will prove this initial claim via induction on N where ν is fixed. The base case is trivial. To deal with the induction step, we first observe that if we integrate by parts and apply Cauchy-Schwarz, we get

$$\begin{aligned}\sum_{i,j=1}^n \|\partial_i \partial_j u(t,\cdot)\|_2^2 &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_i \partial_j u(t,x) \overline{\partial_i \partial_j u(t,x)}\, dx \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_i^2 u(t,x) \overline{\partial_j^2 u(t,x)}\, dx \\ &\lesssim \left(\sum_{j=1}^n \left\| \partial_j^2 u(t,\cdot) \right\|_2 \right)^2 \\ &\lesssim \|\Delta u(t,\cdot)\|_2^2 .\end{aligned}$$

Using this calculation, we see that for $N \geq 1$,

$$
\begin{aligned}
\sum_{|\alpha|\le N} \left\| L^\nu \partial^\alpha u'(t,\cdot)\right\|_2 &\lesssim \sum_{i,j=1}^{n} \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|\partial_i\partial_j (L^\mu \partial^\alpha u)(t,\cdot)\|_2 \\
&\quad + \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu\\ j\le 1}} \left\| L^\mu \partial^\alpha (\partial_t^j u)'(t,\cdot)\right\|_2 \\
&\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu\\ j\le 1}} \left\| L^\mu \partial^\alpha (\partial_t^j u)'(t,\cdot)\right\|_2 \\
&\quad + \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu \partial^\alpha \Box u(t,\cdot)\|_2 .
\end{aligned} \tag{3.23}
$$

Applying the induction hypothesis to the first term in the right hand side of the above inequality, the claim is proved. We will now prove (3.22) for $\mathcal{K} \neq \emptyset$ and for general N in the case $\nu = 0$. We shall first establish the bound on the region $\{|x| < 4\}$. We define the function

$$
s(N) = \begin{cases} \sum_{k=1}^{N} k^{-2}, & N > 0, \\ 0, & N = 0. \end{cases}
$$

We shall first prove inductively on N that for $R \geq 4$ the following inequality holds:

$$
\begin{aligned}
\sum_{|\alpha|\le N} \left\|\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<R)} &\lesssim \sum_{|\alpha|\le N-1} \|\partial^\alpha \Box u(t,\cdot)\|_{L^2(|x|<R+s(N))} \\
&\quad + \sum_{\substack{|\alpha|\le N-1\\ j\le 1}} \left\|\partial^\alpha (\partial_t^j u)'(t,\cdot)\right\|_{L^2(|x|<R+s(N))} .
\end{aligned} \tag{3.24}
$$

The base case $N = 0$ is trivial. To handle the induction step, we will suppose that (3.24) holds for N replaced by $N - 1$. By elliptic regularity, we see that for $R \geq 4$,

$$
\begin{aligned}
\sum_{|\alpha|\le N} \left\|\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<R)} &\lesssim \sum_{\substack{|\alpha|\le N-1\\ |\beta|=2}} \left\|\partial^\alpha \partial_x^\beta u(t,\cdot)\right\|_{L^2(|x|<R)} \\
&\quad + \sum_{\substack{|\alpha|\le N-1\\ j\le 1}} \left\|\partial^\alpha (\partial_t^j u)'(t,\cdot)\right\|_{L^2(|x|<R)} \\
&\lesssim \sum_{|\alpha|\le N-1} \|\partial^\alpha \Box u(t,\cdot)\|_{L^2(|x|<R+1/N^2)}
\end{aligned}
$$

$$+\sum_{\substack{|\alpha|\le N-1\\ j\le 1}} \left\|\partial^\alpha(\partial_t^j u)'(t,\cdot)\right\|_{L^2(|x|<R+1/N^2)}.$$

By the induction hypothesis (3.24) and because ∂_t preserves the Dirichlet boundary conditions, we have proved (3.24). Moreover, since $s(N)<2$ for all N, we have the inequality

$$\begin{aligned}\sum_{|\alpha|\le N} \left\|\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<R)} \lesssim &\sum_{|\alpha|\le N-1} \left\|\partial^\alpha \Box u(t,\cdot)\right\|_{L^2(|x|<R+2)}\\ &+\sum_{j\le N}\left\|\partial_t^j u'(t,\cdot)\right\|_{L^2(|x|<R+2)}.\end{aligned}\tag{3.25}$$

We shall now prove the bound in the region $\{|x|>4\}$. We shall prove via induction that

$$\begin{aligned}\sum_{|\alpha|\le N} \left\|\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|>4)} \lesssim &\sum_{|\alpha|\le N-1} \left\|\partial^\alpha \Box u(t,\cdot)\right\|_2\\ &+\sum_{j\le N}\left\|\partial_t^j u'(t,\cdot)\right\|_2\end{aligned}\tag{3.26}$$

holds for all N. We again observe that the $N=0$ case is obvious. Suppose that (3.26) holds for N replaced by $N-1$. Fix a cutoff $\rho\in C^\infty(\mathbb{R}^3)$ such that $\rho(x)=1$ when $|x|>4$ and zero when $|x|<3$. If we let $u_0=\rho u$, we see that u_0 solves $\Box u_0=\rho G-2\nabla_x\rho\cdot\nabla_x u-(\Delta\rho)u$ with vanishing initial data. Due to the fact that (3.22) holds when $\mathcal{K}=\emptyset$, we get

$$\begin{aligned}\sum_{|\alpha|\le N} \left\|\partial^\alpha u_0'(t,\cdot)\right\|_2 \lesssim &\sum_{j\le N}\left\|\partial_t^j u_0'(t,\cdot)\right\|_2\\ &+\sum_{|\alpha|\le N-1} \left\|\partial^\alpha \Box u_0(t,\cdot)\right\|_2.\end{aligned}\tag{3.27}$$

By Lemma 3.2 and the fact that $u_0=\rho u$, we observe that

$$\sum_{j\le N}\left\|\partial_t^j u_0'(t,\cdot)\right\|_2\lesssim\sum_{j\le N}\left\|\partial_t^j u'(t,\cdot)\right\|_2,$$

which is controlled by the right hand side of (3.22). By Lemma 3.2, the last term in the right hand side of (3.27) is controlled by

$$\sum_{|\alpha|\le N-1} \left\|\partial^\alpha \Box u(t,\cdot)\right\|_2+\sum_{|\alpha|\le N-1} \left\|\partial^\alpha u'(t,\cdot)\right\|_2.\tag{3.28}$$

Applying (3.25) and the induction hypothesis (3.26) to the second term in (3.28), we have proved (3.22) for the case $\nu=0$ for general N. Our previous work establishes the base case.

We shall now prove (3.22) holds for general ν via induction. To do this, we will assume that (3.22) holds for ν replaced by $\nu-1$ for any value of N. We observe that

$$\sum_{|\alpha|\le N} \left\|L^\nu \partial^\alpha u'(t,\cdot)\right\|_2 \lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} t^\mu \left\|\partial_t^\mu \partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)} + \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu \partial^\alpha u_0'(t,\cdot)\right\|_2 . \tag{3.29}$$

Since ∂_t^μ preserves the Dirichlet boundary conditions, we can apply (3.25) for $R=4$ to each summand in the first term in the right hand side of (3.29). Thus, we see that the first term on the right hand side of (3.29) is controlled by

$$\begin{aligned}\sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} t^\mu \left\|\partial_t^j (\partial_t^\mu u)'(t,\cdot)\right\|_{L^2(|x|<6)} &+ \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} t^\mu \left\|\partial^\alpha \Box(\partial_t^\mu u)(t,\cdot)\right\|_{L^2(|x|<6)} \\ \lesssim \sum_{j\le N} \left\|L^\nu \partial_t^j u'(t,\cdot)\right\|_{L^2(|x|<6)} &+ \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu-1}} \left\|L^\mu \partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<6)} \\ + \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu \partial^\alpha \Box u(t,\cdot)\right\|_{L^2(|x|<6)} .\end{aligned} \tag{3.30}$$

Applying the induction hypothesis to the second term in the right hand side of (3.30), we see that the right hand side of (3.30) is controlled by the right hand side of (3.22). Thus, it remains to control the second term on the right hand side of (3.29). Due to the fact that (3.22) holds for boundaryless wave equations, we see that

$$\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu \partial^\alpha u_0'(t,\cdot)\right\|_2 \lesssim \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu \partial_t^j u_0'(t,\cdot)\right\|_2 + \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu \partial^\alpha \Box u_0(t,\cdot)\right\|_2 . \tag{3.31}$$

Since $u_0 = \rho u$, Lemma 3.2 implies that the right hand side of (3.31) is controlled by the right hand side of (3.22). This completes the induction argument, which shows that (3.22) holds for all ν and N. $\square$

Before proving the estimates that we will use in Chapter 4 to control L^2 energy norms that involve a scaling vector field, we shall review the methods that Keel, Smith and Sogge used in [**28**] in the case that $\mathcal{K}$ is assumed to be star-shaped. Their main estimate utilized the same kind of argument involving the energy-momentum vector that was used in the proof of Theorem 3.1. From the proof of Theorem 3.1, however, it is clear that the boundary terms that arise from the divergence theorem are nontrivial when one allows for scaling to be applied to u. In particular, it is troublesome that $Lu(t,x)$ does not vanish when $x \in \partial\mathcal{K}$.

Keel, Smith and Sogge in [**28**] managed to overcome this problem by noting that even though the boundary terms are not zero, the most troublesome part of these terms has a favorable sign and can be ignored. However, the boundary terms seem to have this property only if a strong geometric condition is imposed on $\mathcal{K}$, such as when $\mathcal{K}$ is star-shaped. Thus, it is not clear how, in the general case, one could control the resulting boundary terms by applying this same method. Optimally one would hope to reduce the number of scaling vector fields appearing in the boundary terms in the right hand side of one's estimate. This is the case in the following estimate that was proved by Keel, Smith and Sogge in [**28**] for star-shaped $\mathcal{K}$.

Theorem 3.4. *Let $u \in C^\infty([0,T] \times \mathbb{R}^n \backslash \mathcal{K})$ solve*

$$\begin{cases} \Box_\gamma u(t,x) = G(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n \backslash \mathcal{K}, \\ u(t,x) = 0, \quad x \in \partial\mathcal{K}, \\ u(0,x) = \partial_t u(0,x) = 0, \end{cases} \tag{3.32}$$

where $\gamma^{ij,IJ}$ satisfy (3.3) and that, for $0 \le t \le T$,

$$\|\gamma(t,\cdot)\|_\infty \le \frac{\delta}{1+t}, \tag{3.33}$$

where δ is small. Also suppose that there is a uniform constant $C > 0$ that is independent of T such that

$$\int_0^T \left\|\gamma'(t,\cdot)\right\|_\infty \, dt \le C. \tag{3.34}$$

Suppose that for every fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Also assume $\mathcal{K}$ is star-shaped. Then it follows that for $0 \le t \le T$,

$$\begin{aligned} \left\|(Lu)'(t,\cdot)\right\|_2 \lesssim &\int_0^t \|\Box_\gamma Lu(s,\cdot)\|_2 \, ds \\ &+ \sum_{|\alpha|\le 2} \left\|\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^n\backslash\mathcal{K}:|x|<2\})}. \end{aligned} \tag{3.35}$$

Note that the scaling vector field only appears in the first term in the right hand side of (3.35) as desired. We should note that we will also need to prove a bound similar to (3.34) in the proof of Theorem 1.4, but for expository reasons we will postpone that proof until Chapter 4.

Proof. To prove (3.35), we shall use the energy-momentum vector e_j as defined in (3.5), (3.8), and (3.15) where u is replaced by Lu. We apply the divergence theorem in the same manner as in (3.18) to see that

$$\begin{aligned}
&\partial_0 \int_{\mathbb{R}^n\backslash\mathcal{K}} e_0(Lu)\,dx - \sum_{j=1}^{n}\int_{\partial\mathcal{K}} e_j(Lu)n_j\,d\omega \\
&= 2\int_{\mathbb{R}^n\backslash\mathcal{K}} \langle \partial_0 Lu, \Box_\gamma Lu\rangle\,dx + \int_{\mathbb{R}^n\backslash\mathcal{K}} R((Lu)',(Lu)')\,dx,
\end{aligned}\tag{3.36}$$

where R is the same remainder term that was defined in the proof of Theorem 3.1. Because Lu does not satisfy the Dirichlet boundary conditions for $\partial\mathcal{K}$, the boundary term in the left hand side of (3.36) does not vanish. However, we can rewrite this term so that the part of it that grows like t can be ignored. Because of the Dirichlet boundary conditions, we see that on $\partial\mathcal{K}$ we have the equality

$$\begin{aligned}
\partial_0 Lu^I &= \partial_0 u^I + t\partial_0^2 u^I + \sum_{k=1}^{n} x_k\partial_k\partial_0 u^I \\
&= \langle x, \nabla_x\rangle\,\partial_0 u^I \\
&= \langle \langle x,\vec{n}\rangle\,\vec{n}, \nabla_x\rangle\,\partial_0 u^I \\
&= \langle x,\vec{n}\rangle\,\partial_{\vec{n}}\partial_0 u^I,
\end{aligned}\tag{3.37}$$

where $\partial_{\vec{n}} = \langle \vec{n}, \nabla_x\rangle$ is differentiation with respect to the outward unit normal vector on $\mathcal{K}$. We also see that

$$\begin{aligned}
\sum_{k=1}^{n} n_k\partial_k Lu^I &= t\,\langle \vec{n},\nabla_x\rangle\,\partial_0 u^I + \langle \vec{n},\nabla_x\rangle\,\langle x,\nabla_x\rangle\,u^I \\
&= t\partial_{\vec{n}}\partial_0 u^I + \partial_{\vec{n}}(\langle x,\nabla_x\rangle\,u^I).
\end{aligned}\tag{3.38}$$

It follows that

$$-\sum_{k=1}^{n} e_k(Lu)n_k = 2\sum_{I=1}^{D}[tc_I^2 \langle x, \vec{n}\rangle (\partial_{\vec{n}}\partial_0 u^I)^2 + c_I^2 \langle x, \vec{n}\rangle \partial_{\vec{n}}\partial_0 u^I \partial_{\vec{n}}(\langle x, \nabla_x\rangle u^I) \tag{3.39}$$
$$- \langle x, \vec{n}\rangle \partial_{\vec{n}}\partial_0 u^I \sum_{k=1}^{n}\sum_{J=1}^{D}\sum_{j=0}^{n} \gamma^{jk,IJ} n_k \partial_j Lu^J].$$

Due to the fact that the perturbation terms $\gamma^{jk,IJ}$ satisfy the bound (3.33), we can rewrite the above equation as

$$-\sum_{k=1}^{n} e_k(Lu)n_k = 2\sum_{I=1}^{D} tc_I^2 \langle x, \vec{n}\rangle (\partial_{\vec{n}}\partial_0 u^I)^2 + F(u', u''),$$

where we have the following uniform bound for F:

$$|F(u', u'')| \lesssim \sum_{|\alpha|\le 1} \left|\partial^\alpha u'\right|^2,$$

where the implicit constant is independent of t. Thus, we can rewrite (3.36) as

$$\partial_0 \int_{\mathbb{R}^n\setminus\mathcal{K}} e_0(Lu)\, dx + 2\sum_{I=1}^{D}\int_{\partial\mathcal{K}} tc_I^2 \langle x, \vec{n}\rangle (\partial_n\partial_0 u^I)^2\, d\omega \tag{3.40}$$
$$= \int_{\partial\mathcal{K}} F(u', u'')\, d\omega + 2\int_{\mathbb{R}^n\setminus\mathcal{K}} \langle \partial_0 Lu, \Box_\gamma Lu\rangle\, dx + \int_{\mathbb{R}^n\setminus\mathcal{K}} R((Lu)', (Lu)')\, dx.$$

Since $\mathcal{K}$ is assumed to be star-shaped, the inner product $\langle x, \vec{n}\rangle > 0$, for $x \in \partial\mathcal{K}$, which means that the second quantity in the left hand side of (3.40) is positive. Applying Gronwall's inequality and (3.34), we see that

$$\left\|(Lu)'(t,\cdot)\right\|_2 \lesssim \int_0^t \left\|\Box_\gamma Lu(s,\cdot)\right\|_2\, ds \tag{3.41}$$
$$+ \left(\sum_{|\alpha|\le 1}\int_0^t \int_{\partial\mathcal{K}} \left|\partial^\alpha u'(s,x)\right|^2\, d\omega\, ds\right)^{1/2}.$$

Applying the trace theorem, we see that second term in the left hand side of (3.41) is controlled by

$$\sum_{|\alpha|\le 2} \left\|\partial^\alpha u'\right\|_{L^2([0,t]\times\{x\in\mathbb{R}^n\setminus\mathcal{K}: |x|<2\})}.$$

$\Box$

Due to the fact that we are not requiring that $\mathcal{K}$ be star-shaped, it is clear that the argument used to prove the previous theorem breaks down. However, Metcalfe and Sogge in [**45**] showed that the issue of controlling the boundary terms can still be circumvented using a similar line of reasoning. One uses the fact that scaling vector field decomposes into two terms,

$$L = t\partial_t + r\partial_r.$$

We first note that the coefficient of the second term $r\partial_r$ is uniformly bounded on $\partial\mathcal{K}$. Just as in the proof of the previous theorem, the quantities that result from this term can be easily controlled by applying the trace theorem. Another similarity to the star-shaped case is that the first term $t\partial_t$ is still the most problematic to control since its tangential component grows like t as $t \to \infty$. However, the fact that $t\partial_t$ preserves the Dirichlet boundary conditions should indicate that energy methods still might be useful. We consider the modified scaling operator:

$$\tilde{L} = t\partial_t + \eta(x) r\partial_r,$$

where $\eta \in C^\infty(\mathbb{R}^n)$ is a bump function such that $\eta(x) = 0$ for $x \in \mathcal{K}$ and $\eta(x) = 1$ for $|x| > 1$. This definition for the cut-off η makes sense due to our assumption that $\mathcal{K} \subset \{|x| < 1\}$. It is clear from the definition that this operator does in fact preserve the Dirichlet boundary conditions. The main idea will be to begin with estimates for the modified operator $\tilde{L}$. These will then give rise to useful L^2-estimates that involve the original scaling vector field L.

For the next lemma we will need the quantity:

$$X_{\nu,j} = \int e_0(\tilde{L}^\nu \partial_t^j u)(t,x)\, dx.$$

We shall also be concerned with how the invariant vectors fields commute with the perturbed wave operator $\Box_\gamma$. With a slight abuse of notation , we define

$$\left|[P, \gamma^{kl}\partial_k\partial_l]\ u\right| = \sum_{\substack{0\le k,l\le n\\ 1\le I,J\le D}} \left|[P, \gamma^{kl,IJ}\partial_k\partial_l]\ u^J\right|,$$

where $P = P(t, x, \partial_t, \partial_x)$ is a differential operator. We are now ready to state the lemma, which was originally proved by Metcalfe and Sogge in [**45**].

Lemma 3.5. *Let $u \in C^\infty([0,T] \times \mathbb{R}^n \backslash \mathcal{K})$ solve* (3.1) *and assume the perturbation terms γ^{ij} are as in* (3.3) *and* (3.4). *Also assume that δ in* (3.4) *is small. Also suppose that for every* *fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then the following inequality holds.*

$$\begin{aligned}
\partial_t X_{\nu,j} &\lesssim X_{\nu,j}^{1/2} \left\| \tilde{L}^\nu \partial_t^j \Box_\gamma u(t,\cdot) \right\|_2 + \left\| \gamma'(t,\cdot) \right\|_\infty X_{\nu,j} \\
&+ X_{\nu,j}^{1/2} \left\| \left[\tilde{L}^\nu \partial_t^j, \gamma^{kl} \partial_k \partial_l \right] u(t,\cdot) \right\|_2 \\
&+ X_{\nu,j}^{1/2} \sum_{\mu \le \nu - 1} \left\| L^\mu \partial_t^j \Box u(t,\cdot) \right\|_2 \\
&+ X_{\nu,j}^{1/2} \sum_{\substack{\mu + |\alpha| \le j + \nu \\ \mu \le \nu - 1}} \left\| L^\mu \partial^\alpha u'(t,\cdot) \right\|_{L^2(|x|<1)} .
\end{aligned} \tag{3.42}$$

Proof. Note that for all ν and j, we also have that $\tilde{L}^\nu \partial_t^j u(t,x) = 0$ for $x \in \partial\mathcal{K}$. Repeating the same argument in the proof of Theorem 3.1, we see that

$$\partial_t X_{\nu,j} \lesssim X_{\nu,j}^{1/2} \left\| \Box_\gamma \tilde{L}^\nu \partial_t^j u(t,\cdot) \right\|_2 + \left\| \gamma'(t,\cdot) \right\|_\infty X_{\nu,j}. \tag{3.43}$$

We then observe that

$$\begin{aligned}
\left| \Box_\gamma \tilde{L}^\nu \partial_t^j u \right| &\le \left| \tilde{L}^\nu \partial_t^j \Box_\gamma u \right| + \left| \left[\tilde{L}^\nu \partial_t^j, \gamma^{kl} \partial_k \ \partial_l \right] u \right| + \left| \left[\tilde{L}^\nu, \Box \right] \partial_t^j u \right| \\
&\le \left| \tilde{L}^\nu \partial_t^j \Box_\gamma u \right| + \left| \left[\tilde{L}^\nu \partial_t^j, \gamma^{kl} \partial_k \ \partial_l \right] u \right| + \left| \left[\tilde{L}^\nu - L^\nu, \Box \right] \partial_t^j u \right| \\
&+ \left| \left[L^\nu, \Box \right] \partial_t^j u \right| .
\end{aligned} \tag{3.44}$$

From (3.44), it follows that

$$\begin{aligned}
\left\| \Box_\gamma \tilde{L}^\nu \partial_t^j u(t,\cdot) \right\|_2 &\lesssim \left\| \tilde{L}^\nu \partial_t^j \Box_\gamma u(t,\cdot) \right\|_2 + \left\| \left[\tilde{L}^\nu \partial_t^j, \gamma^{kl} \partial_k \ \partial_l \right] u(t,\cdot) \right\|_2 \\
&+ \sum_{\substack{\mu + |\alpha| \le \nu + j \\ \mu \le \nu - 1}} \left\| L^\mu \partial^\alpha u'(t,\cdot) \right\|_{L^2(|x|<1)} + C \sum_{\mu \le \nu - 1} \left\| L^\mu \partial_t^j \Box u(t,\cdot) \right\|_2 ,
\end{aligned} \tag{3.45}$$

which follows from Lemma 3.2, the fact that $\nabla_x \eta(x) = 0$ for $|x| > 1$ and $[\Box, L] = 2\Box$. We see that (3.42) follows from this inequality and (3.43). □

Using (3.22), we can now prove the following estimate, which was proved in [**45**], in order to control L^2-energy norms that involve the scaling vector field.

Theorem 3.6. *Let $u \in C^\infty([0,T] \times \mathbb{R}^n \backslash \mathcal{K})$ solve* (3.1). *Assume the perturbation terms γ^{ij} are as in* (3.3) *and* (3.4) *and that the constant δ in* (3.4) *is small. Suppose that for any fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Also suppose that* (3.34) *holds with the uniform* *constant C being independent of T. Suppose further that*

$$\sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}}\left(\left\|\tilde{L}^\mu\partial_t^j\Box_\gamma u(t,\cdot)\right\|_2+\left\|[\tilde{L}^\mu\partial_t^j,\gamma^{kl}\partial_k\partial_l]u(t,\cdot)\right\|_2\right)$$
$$\le F(t)\sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}}\left\|\tilde{L}^\mu\partial_t^j u'(t,\cdot)\right\|_2+H_{\nu,N}(t), \tag{3.46}$$

where N and ν are fixed and $F\in C^\infty([0,T])$ satisfies the bound

$$\int_0^T F(s)\,ds\le C, \tag{3.47}$$

where C is independent of T. Then it follows that

$$\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}}\left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_2$$
$$\le C\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}}\|L^\mu\partial^\alpha\Box u(t,\cdot)\|_2+C\sum_{\substack{\mu+j\le N+\nu\\ \mu\le\nu}}X_{\mu,j}^{1/2}(0)$$
$$+C\left(\int_0^t\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu-1}}\|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2\,ds+\int_0^t H_{\nu,N}(s)\,ds\right) \tag{3.48}$$
$$+C\int_0^t\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu-1}}\left\|L^\mu\partial^\alpha u'(s,\cdot)\right\|_{L^2(|x|<1)}\,ds.$$

PROOF. We first reduce the proof to the case where we are dealing with $\tilde{L}$. We will show that

$$\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}}\left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_2\lesssim\sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}}\left\|\tilde{L}^\mu\partial_t^j u'(t,\cdot)\right\|_2$$
$$+\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}}\|L^\mu\partial^\alpha\Box u(t,\cdot)\|_2\,. \tag{3.49}$$

We shall prove this via induction on ν. Applying Proposition 3.3 proves (3.49) in the case $\nu = 0$. The general case is handled by noting that

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_2 &\lesssim \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial_t^j u'(t,\cdot)\right\|_2 \\ &\quad+ \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha\Box u(t,\cdot)\right\|_2 \\ &\lesssim \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} \left\|\tilde{L}^\mu\partial_t^j u'(t,\cdot)\right\|_2 \\ &\quad+ \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} \left\|(L^\mu-\tilde{L}^\mu)\partial_t^j u'(t,\cdot)\right\|_2 \\ &\quad+ \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha\Box u(t,\cdot)\right\|_2 \\ &\lesssim \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} \left\|\tilde{L}^\mu\partial_t^j u'(t,\cdot)\right\|_2 \\ &\quad+ \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu-1}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<1)} \\ &\quad+ \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha\Box u(t,\cdot)\right\|_2,\end{aligned} \tag{3.50}$$

and by applying the induction hypothesis to the second term in the right hand side. Because δ in (3.4) is small, it follows that

$$\begin{aligned}\left(5\max_I\{c_I,c_I^{-1}\}\right)^{-1} \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} X^{1/2}_{\mu,j}(t) &\le \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} \left\|\tilde{L}^\mu\partial_t^j u'(t,\cdot)\right\|_2 \\ &\le 5\max_I\{c_I,c_I^{-1}\} \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} X^{1/2}_{\mu,j}(t).\end{aligned} \tag{3.51}$$

Thus, it suffices to show that $\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} X_{\mu,j}^{1/2}(t)$ is controlled by the last four terms on the right hand side of (3.48). By (3.42) and (3.46) and (3.51), it follows that

$$
\begin{aligned}
\partial_t \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} X_{\mu,j}^{1/2}(t) \lesssim F(t) &\sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} X_{\mu,j}^{1/2}(t) + H_{\nu,N}(t)\\
&+ \left\|\gamma'(t,\cdot)\right\|_\infty \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} X_{\mu,j}^{1/2}(t)\\
&+ \sum_{\substack{j+\mu\le N+\nu-1\\ \mu\le\nu-1}} \left\|L^\mu \partial_t^j \Box u(t,\cdot)\right\|_2\\
&+ \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu-1}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<1)}.
\end{aligned}
\tag{3.52}
$$

Applying Gronwall's inequality, (3.34) and (3.47) completes the proof. □

To control the last term in the right hand side of (3.48), we will need to use the estimates from Lemma 2.9 in Metcalfe-Sogge [**45**]. Prior to proving this lemma, we will prove an estimate that uses elliptic regularity and local energy decay (1.6) to control local L^2 norms.

Lemma 3.7. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ solves*

$$
\begin{cases}
\Box u(t,x) = G(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^3\backslash\mathcal{K},\\
u(0,x) = \partial_t u(0,x) = 0,\\
u(t,x) = 0, \quad x\in\partial\mathcal{K}.
\end{cases}
\tag{3.53}
$$

Also suppose that $\mathcal{K}$ satisfies (1.6) and that σ, M are as in (1.6). If $\Box u(t,x) = 0$ for $|x|>10$, then it follows that for $0\le t\le T$,

$$
\begin{aligned}
\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)} \lesssim &\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u(t,\cdot)\|_2\\
&+ \int_0^t \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \langle t-s\rangle^{-2-\sigma+\mu}\|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2\, ds.
\end{aligned}
\tag{3.54}
$$

Proof. We first observe that

$$
\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)} \lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} t^\mu \left\|\partial_t^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)}.
\tag{3.55}
$$

Applying (3.25) where $R = 4$ to the right hand side of (3.55), we see that

$$\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu \partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<4)} \lesssim \sum_{\substack{j+\mu\le N+\nu\\ \mu\le\nu}} t^\mu \left\|\partial_t^{\mu+j} u'(t,\cdot)\right\|_{L^2(|x|<6)} + \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} t^\mu \left\|\partial^\mu\partial^\alpha \Box u(t,\cdot)\right\|_{L^2(|x|<6)}. \tag{3.56}$$

By Duhamel's principle and the local energy decay estimate (1.6), the first term in (3.56) is controlled by

$$\sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} t^\mu \int_0^t \langle t-s\rangle^{-2-\sigma} \left\|\partial_s^\mu \partial^\alpha \Box u(s,\cdot)\right\|_{L^2(|x|<10)}\, ds. \tag{3.57}$$

Since $\langle t\rangle \lesssim \langle t-s\rangle \langle s\rangle$ for $0\le s\le t$, it follows that (3.57) is bounded by

$$\sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \int_0^t \langle t-s\rangle^{-2-\sigma+\mu} \left\|\langle s\rangle^\mu \partial_s^\mu \partial^\alpha \Box u(s,\cdot)\right\|_{L^2(|x|<10)}\, ds.$$

The above quantity is controlled by the second term in (3.54). The second term in (3.56) is controlled by the first term in the right hand side of (3.54). □

The next estimate was originally proved in Metcalfe-Sogge [**45**] for 3-dimensional wave equations. Although we state the next estimate for 3 dimensions only, it should be noted that it is possible to obtain analogous estimates in dimensions $n \ge 4$ (see Metcalfe-Sogge [**46**], Lemma 5.2) by using methods that do not rely on sharp Huygens' principle.

THEOREM 3.8. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ solves (3.53) and that for any fixed t, $u(t,x)$ vanishes for sufficiently large $|x|$. Also assume that $\mathcal{K}$ satisfies (1.6) and that M,σ are as in*

(1.6). *It follows that for* $0 \le t \le T$,

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<2)}\\
&\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \Big[\int_0^t \langle t-s\rangle^{-2-\sigma+\mu}\,\|L^\mu\partial^\alpha G(s,\cdot)\|_{L^2(|x|<4)}\,ds\\
&\quad+\|L^\mu\partial^\alpha G(t,\cdot)\|_{L^2(|x|<4)}\Big]\\
&\quad+\sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \int_0^t \langle t-s\rangle^{-2-\sigma+\mu}\left(\int_0^s \|L^\mu\partial^\alpha G(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\right)ds\\
&\quad+\sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \int_0^t \|L^\mu\partial^\alpha G(s,\cdot)\|_{L^2(||x|-(t-s)|<10)}\,ds
\end{aligned}\tag{3.58}$$

and when $\nu=0$, *then it also follows that*

$$\begin{aligned}
&\int_0^t \sum_{|\alpha|\le N}\left\|\partial^\alpha u'(s,\cdot)\right\|_{L^2(|x|<2)}\,ds\\
&\lesssim \sum_{|\alpha|\le N+M}\int_0^t\left(\int_0^s \|\partial^\alpha G(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\right)ds\\
&\quad+\int_0^t \sum_{|\alpha|\le N+M}\|\partial^\alpha G(s,\cdot)\|_{L^2(|x|<4)}\,ds.
\end{aligned}\tag{3.59}$$

Proof. We shall consider two cases: (1) $G(t,x)$ vanishes when $|x|>3$ and (2) $G(t,x)$ vanishes when $|x|<2$. By (3.54), it follows that

$$\begin{aligned}
\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<2)} &\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu\partial^\alpha G(t,\cdot)\|_{L^2(|x|<3)}\\
&\quad+\int_0^t \langle t-s\rangle^{-2-\sigma+\mu}\sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \|L^\mu\partial^\alpha G(s,\cdot)\|_{L^2(|x|<3)}\,ds.
\end{aligned}\tag{3.60}$$

This handles case 1.

To deal with case 2, we fix a cutoff $\rho\in C^\infty(\mathbb{R}^3)$ such that $\rho(x)=1$ when $|x|<2$ and $\rho(x)=0$ when $|x|>3$. Let $u=u_0+u_r$ where u_0 solves the boundaryless wave equation $\Box u_0=G$ with vanishing initial data. If we let $w=\rho u_0+u_r$, notice that w solves $\Box w=$

$\rho G - 2\nabla_x\rho\cdot\nabla_x u_0 - (\Delta\rho)u_0$. Note that $\rho G = 0$ since ρ and G have disjoint supports. Since $\Box w(t,x) = 0$ when $|x| > 3$, it follows from case (1) that

(3.61)
$$\begin{aligned}
\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<2)} &= \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha w'(t,\cdot)\right\|_{L^2(|x|<2)} \\
&\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu\partial^\alpha \Box w(t,\cdot)\|_{L^2(|x|<3)} \\
&\quad + \int_0^t \langle t-s\rangle^{-2-\sigma+\mu} \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box w(s,\cdot)\|_{L^2(|x|<3)}\, ds.
\end{aligned}$$

One can then see that the right hand side is controlled by

(3.62)
$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left[\left\|L^\mu\partial^\alpha u_0'(t,\cdot)\right\|_{L^2(|x|<3)} + \|L^\mu\partial^\alpha u_0(t,\cdot)\|_{L^2(|x|<3)}\right] \\
&+ \int_0^t \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \langle t-s\rangle^{-2-\sigma+\mu} \left(\left\|L^\mu\partial^\alpha u_0'(s,\cdot)\right\|_{L^2(|x|<3)} + \|L^\mu\partial^\alpha u_0(s,\cdot)\|_{L^2(|x|<3)}\right) ds.
\end{aligned}$$

We will only bound the first two terms in the right hand side of (3.62) since the other terms can be bounded using an identical argument. Fixing t, we observe that on the set $\{|x| < 3\}$, $u_0(t,x)$ is equal to

$$u_1(t,x) = \int_0^t \int E(t-s, x-y) G_0(s,y)\, dy\, ds,$$

where E is the fundamental solution to the linear wave equation and G_0 is a smooth function such that $G_0(s,y) = G(s,y)$ when $|(t-s)-|y|| < 9$ and is equal to zero when $|(t-s)-|y|| > 10$. Using the fact that $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(|x| < 3) \hookrightarrow L^2(|x| < 3)$, we see that the first two terms in the right hand side of (3.62) are controlled by

$$\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u_1'(t,\cdot)\right\|_2,$$

which, by the energy inequality, is controlled by

$$\int_0^t \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \|L^\mu\partial^\alpha G(s,\cdot)\|_{L^2(|(t-s)-|y||<10)}\, ds.$$

This proves (3.58). To see how this implies inequality (3.59), one first integrates both sides of (3.58) with respect to t. Inequality (3.59) is obtained after applying Young's inequality to the first and third terms in the left hand side of (3.58). This proves the theorem. $\square$

We also will need a perturbed energy estimate that involves the full collection of admissible vector fields, including scaling, rotations and translations. This estimate, which was proved in earlier papers such as Keel, Smith and Sogge [**28**] and Metcalfe and Sogge [**45**], will also be proved by using the same energy methods that were used to prove Theorem 3.1. We shall also need to define the following quantity:

$$Y_{N_1,N_2,\nu}(u)(t) = \sum_{\substack{|\alpha|+\mu\leq N_1+\nu \\ |\beta|\leq N_2 \\ \mu\leq\nu}} \int e_0(L^\mu Z^\alpha \partial^\beta u)(t,x)\,dx. \tag{3.63}$$

THEOREM 3.9. *Fix N_1, N_2, ν and assume the perturbation terms $\gamma^{ij,IJ}$ are as in* (3.3) *and* (3.4). *Also assume that δ in* (3.4) *is small. Assume that $u \in C^\infty([0,T]\times\mathbb{R}^n\backslash\mathcal{K})$ solves* (3.1) *and that for every fixed t, $u(t,x) = 0$ for $|x|$ sufficiently large. Then it follows that*

$$\begin{aligned}
\partial_t Y_{N_1,N_2,\nu}(t) \lesssim Y^{1/2}_{N_1,N_2,\nu}(t) &\sum_{\substack{|\alpha|+\mu\leq N_1+\nu \\ |\beta|\leq N_2 \\ \mu\leq\nu}} \left\|\Box_\gamma L^\mu Z^\alpha \partial^\beta u(t,\cdot)\right\|_2 \\
&+ \left\|\gamma'(t,\cdot)\right\|_\infty Y_{N_1,N_2,\nu}(t) + \sum_{\substack{|\alpha|+\mu\leq N_1+N_2+\nu+1 \\ \mu\leq\nu}} \left\|L^\mu \partial^\alpha u'(t,\cdot)\right\|^2_{L^2(|x|<1)}.
\end{aligned} \tag{3.64}$$

PROOF. From the proof of Theorem 3.1, it follows that

$$\begin{aligned}
&\partial_t Y_{N_1,N_2,\nu}(t) - \sum_{k=1}^n \int_{\partial\mathcal{K}} e_k \nu_k \, d\omega \\
&\leq C Y^{1/2}_{N_1,N_2,\nu} \sum_{\substack{|\alpha|+\mu\leq N_1+\nu \\ |\beta|\leq N_2 \\ \mu\leq\nu}} \left\|\Box_\gamma L^\mu Z^\alpha \partial^\beta u(t,\cdot)\right\|_2 + C\left\|\gamma'(t,\cdot)\right\|_{L^\infty(\mathbb{R}^n\backslash\mathcal{K})} Y_{N_1,N_2,\nu}(t),
\end{aligned} \tag{3.65}$$

where $e_k = \sum_{\substack{|\alpha|+\mu\leq N_1+\nu \\ |\beta|\leq N_2 \\ \mu\leq\nu}} e_k(L^\mu Z^\alpha \partial^\beta u)(t,x)$, for $k = 1,\dots,n$, are the components of the energy-momentum vector defined in (3.8). Since $\mathcal{K} \subset \{|x| < 1\}$, it follows from the trace theorem that

$$\sum_{k=1}^{n}\int_{\partial\mathcal{K}} |e_k\nu_k| \ d\omega \le C\int_{\{x\in\mathbb{R}^n\backslash\mathcal{K}:|x|<1\}} \sum_{\substack{|\alpha|+\mu\le N_1+N_2+\nu+1\\ \mu\le\nu}} \left|L^\mu\partial^\alpha u'(t,x)\right|^2 \ dx. \tag{3.66}$$

This completes the proof. □

3.2. Weighted L^2 Estimates

In this section, we will extend the weighted L^2 estimates that were proved in Sections 2.2 and 2.3. The first estimate was originally proved by Keel, Smith and Sogge [**28**] for star-shaped obstacles with a different weight. They were reproved for exterior domains where local energy decays sufficiently rapidly with a possible loss in regularity by Metcalfe and Sogge [**45**].

THEOREM 3.10. *Let $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ solve* (3.53). *Also assume that $\mathcal{K}$ satisfies* (1.6), *that for any fixed t, $u(t,x)$ vanishes for $|x|$ sufficiently large, and that M is the integer appearing in* (1.6). *If $\nu = 0$ or 1, then it follows that*

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4}L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} &\lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2 \ ds\\ &+\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}.\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4}L^\mu Z^\alpha u'\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} &\lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2 \ ds\\ &+\int_0^T \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \|L^\mu Z^\alpha\Box u(s,\cdot)\|_2 \ ds\\ &+\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}.\end{aligned} \tag{3.68}$$

Before proving this theorem, we shall need to prove a lemma that will use the local energy decay estimate (1.6) to control the local L^2 norms.

LEMMA 3.11. *Let $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ solve (3.53). Also assume that $\mathcal{K}$ satisfies (1.6), that for any fixed t, $u(t,x)$ vanishes for $|x|$ sufficiently large, and that M is the integer appearing in (1.6). If $\nu = 0$ or 1, then it follows that*

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})} &\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\\ &\quad+\int_0^T \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2\,ds.\end{aligned}\tag{3.69}$$

PROOF. Using cutoffs, we split the proof into two cases: (1) $G(t,x)=0$ when $|x|>4$, and (2) $G(t,x)=0$ when $|x|<3$. The first case is handled by Lemma 3.7. For we see that by (3.54), we get

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(s,\cdot)\right\|^2_{L^2(|x|<4)} &\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2^2\\ &\quad+\left(\int_0^s \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \langle s-\tau\rangle^{-2-\sigma+\mu}\,\|L^\mu\partial^\alpha\Box u(\tau,\cdot)\|_2\,d\tau\right)^2.\end{aligned}\tag{3.70}$$

Integrating with respect to s and then applying Young's inequality in the second quantity in the right hand side, we have proved (3.69) for the first case.

To deal with the second case, fix $\rho\in C^\infty(\mathbb{R}^3)$ such that $\rho(x)=1$ when $|x|<2$ and $\rho(x)=0$ when $|x|>3$. Write $u=u_0+u_r$, where u_0 solves the *boundaryless* wave equation $\Box u_0(t,x) = G(t,x)$ with vanishing initial data. Let $w=\rho u_0+u_r$. Note that $\Box w = -2\nabla_x\rho\cdot u_0 - (\Delta\rho)u_0$ since G and ρ have disjoint supports and that $\Box u_r = 0$. Applying (3.70), we see that

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'(s,\cdot)\right\|^2_{L^2(|x|<2)} &= \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha w'(s,\cdot)\right\|^2_{L^2(|x|<2)}\\ &\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u_0'(s,\cdot)\right\|_2^2\\ &\quad+\sum_{\substack{|\alpha|+\mu\le N+\nu-1\\ \mu\le\nu}} \|L^\mu\partial^\alpha u_0(s,\cdot)\|_2^2\end{aligned}$$

$$+\left(\int_0^s \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \langle s-\tau\rangle^{-2-\sigma+\mu} \left\|L^\mu\partial^\alpha u_0'(\tau,\cdot)\right\|_2\,d\tau\right)^2$$

$$+\left(\int_0^s \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \langle s-\tau\rangle^{-2-\sigma+\mu} \left\|L^\mu\partial^\alpha u_0(\tau,\cdot)\right\|_2\,d\tau\right)^2.$$

Integrating both sides with respect to s, if we apply Young's inequality, we see that

$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'\right\|^2_{L^2([0,T]\times\{x\in\mathbb{R}^3\setminus\mathcal{K}:|x|<2\})} &\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u_0'\right\|^2_{L^2([0,T]\times\{|x|<3\})}\\ &+\sum_{\substack{|\alpha|+\mu\le N+\nu+M\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u_0\right\|^2_{L^2([0,T]\times\{|x|<3\})}.\end{aligned}\tag{3.71}$$

Applying Corollary 2.6, we see that this quantity is controlled by the left hand side of (3.69). $\square$

We are now ready to prove Theorem 3.10.

Proof. (Theorem 3.10) By Lemma 3.11, we only need to deal with the case $|x|>2$. We shall only prove (3.67) since it shall be clear that (3.68) also follows from the same argument. Fix a cutoff $\rho\in C^\infty(\mathbb{R}^3)$ such that $\rho(x)=0$ if $|x|<1$ and $\rho(x)=1$ if $|x|>2$. If we let $w=\rho u$, then w solves the boundaryless wave equation $\Box w=\rho G-2\nabla_x\rho\cdot\nabla_x u-(\Delta\rho)u$ with vanishing initial data. Write $w=w_1+w_2$ where $\Box w_1=\rho G$ with vanishing initial data. Applying (2.4) in Theorem 2.3, we see that

$$\langle T\rangle^{-1/4}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|\langle x\rangle^{-1/4} L^\mu\partial^\alpha w_1'\right\|_{L^2([0,T]\times\mathbb{R}^3)} \lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha G(s,\cdot)\right\|_2\,ds.$$

To deal with w_2, apply Lemma 2.7 and Lemma 3.2 to get

$$\begin{aligned}&\langle T\rangle^{-1/2}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left\|\langle x\rangle^{-1/4} L^\mu\partial^\alpha w_2'\right\|^2_{L^2([0,T]\times\mathbb{R}^3)}\\ &\lesssim \sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}} \left(\left\|L^\mu\partial^\alpha u'\right\|^2_{L^2([0,T]\times\{|x|<2\})}+\left\|L^\mu\partial^\alpha u\right\|^2_{L^2([0,T]\times\{|x|<2\})}\right)\end{aligned}$$

$$\lesssim \sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu}} \left\|L^\mu\partial^\alpha u'\right\|^2_{L^2([0,T]\times\{|x|<2\})}.$$

By (3.69), this quantity is controlled by the square of the left hand side of (3.67). This completes the proof. $\square$

We are now ready to prove one of our main L^2 estimates which allows us to bound the L^2 norm of the solution u without the spacetime gradient. This extends the L^2 estimates of Du and Zhou [**11**] to exterior domains where the obstacle $\mathcal{K}$ satisfies our local energy decay assumption (1.6).

THEOREM 3.12. *Let $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ solve* (3.53). *Also assume that $\mathcal{K}$ satisfies* (1.6), *that for any fixed t, $u(t,x)$ vanishes for $|x|$ sufficiently large, and that M is the integer appearing in* (1.6). *If $\nu = 0$ or 1, then it follows that for $0\leq t\leq T$,*

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu}} \sup_{0\leq t\leq T}\|L^\mu Z^\alpha u(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu}}\left\|\langle x\rangle^{-1/4}L^\mu Z^\alpha u\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\lesssim \sup_{0\leq t\leq T}\sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu}}\left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_2\\
&\quad+\int_0^T\sum_{\substack{|\alpha|+\mu\leq N+\nu+M\\ \mu\leq\nu}}\|L^\mu\partial^\alpha\Box u(s,\cdot)\|_2\,ds\\
&\quad+\int_0^T\sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu}}\left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha\Box u(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\,ds\\
&\quad+\sum_{\substack{|\alpha|+\mu\leq N+\nu-1\\ \mu\leq\nu}}\|L^\mu\partial^\alpha\Box u\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}.
\end{aligned}\tag{3.72}$$

PROOF. We shall first prove the case when $|x|<2$. We observe that

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu\\ |\alpha|\geq 1}}\|L^\mu Z^\alpha u(t,\cdot)\|_{L^2(|x|<2)} + \sum_{\substack{|\alpha|+\mu\leq N+\nu\\ \mu\leq\nu\\ |\alpha|\geq 1}}\|L^\mu Z^\alpha u\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}\\
&\lesssim \sum_{\substack{|\alpha|+\mu\leq N+\nu-1\\ \mu\leq\nu}}\left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<2)} + \sum_{\substack{|\alpha|+\mu\leq N+\nu-1\\ \mu\leq\nu}}\left\|L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}.
\end{aligned}$$

Applying Lemma 3.11, we see that the quantity in the right hand side is controlled by the right hand side of (3.72). Thus, it suffices to handle the case where $N = 0$. We see that

$$\sum_{\mu\le\nu}\|L^\mu u(t,\cdot)\|^2_{L^2(|x|<2)} \lesssim \sum_{\substack{|\alpha|+\mu\le\nu-1\\ \mu\le\nu-1}}\left\|t^\mu\partial_t^\mu\partial^\alpha u'(t,\cdot)\right\|^2_{L^2(|x|<2)} + \sum_{\mu\le\nu}\|t^\mu\partial_t^\mu u(t,\cdot)\|^2_{L^2(|x|<2)}.$$

Applying (3.21) to the second term in the right hand side of this inequality, we see that the left hand side is controlled by

$$\sum_{\substack{|\alpha|+\mu\le\nu\\ \mu\le\nu}}\left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|^2_{L^2(|x|<2)}.$$

This shows that

$$\begin{aligned}
&\sum_{\mu\le\nu}\Big(\|L^\mu u(t,\cdot)\|_{L^2(|x|<2)} + \|L^\mu u\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}\Big)\\
&\lesssim \sum_{\substack{|\alpha|+\mu\le\nu\\ \mu\le\nu}}\Big(\left\|L^\mu\partial^\alpha u'(t,\cdot)\right\|_{L^2(|x|<2)} + \left\|L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}\Big).
\end{aligned} \tag{3.73}$$

The first term in the right hand side of (3.73) is controlled by the first term in the right hand side of (3.72). By Lemma 3.11, the second term in the right hand side of (3.73) is bounded by the second and fourth terms in the right hand side of (3.72).

To deal with the case when $|x| > 2$, we fix a cutoff $\rho \in C^\infty(\mathbb{R}^3)$ such that $\rho(x) = 0$ when $|x| < 1$ and $\rho(x) = 1$ when $|x| > 2$. If we let $w = \rho u$, then w solves the boundaryless wave equation $\Box w = \rho G - 2\nabla_x\rho\cdot\nabla_x u - (\Delta\rho)u$ with vanishing initial data. Write $w = w_1 + w_2$ where $\Box w_1 = \rho G$ with vanishing initial data. Applying (2.39), we see that

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}}\|L^\mu Z^\alpha w_1(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\sum_{\substack{|\alpha|+\mu\le N+\nu\\ \mu\le\nu}}\left\|\langle x\rangle^{-1/4}L^\mu Z^\alpha w_1\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\lesssim \int_0^T\sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}}\|L^\mu\partial^\alpha G(s,\cdot)\|_{L^2(|x|<3)}\,ds\\
&+\int_0^T\sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}}\left\|\langle x\rangle^{-1/2}L^\mu Z^\alpha G(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\,ds.
\end{aligned}$$

To deal with w_2, apply Lemma 2.12 and Lemma 3.2 to get

(3.74)
$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}} \|L^\mu Z^\alpha w_2(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4} L^\mu Z^\alpha w_2\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\lesssim \sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})} + \sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}} \|L^\mu\partial^\alpha u\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}\\
&\lesssim \sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}.
\end{aligned}$$

To finish the proof, we apply Lemma 3.11 to see that the quantity

$$\sum_{\substack{|\alpha|+\mu\le\nu+N\\ \mu\le\nu}} \left\|L^\mu\partial^\alpha u'\right\|_{L^2([0,T]\times\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}$$

is controlled by the second and fourth terms in the right hand side of (3.72). □

3.3. L^1, L^∞ Estimates

We now prove the exterior domain analog of Hörmander's L^1, L^∞ estimate (see [**16**]) that was proved by Keel, Smith and Sogge in [**28**]. Using Proposition 2.15, we will prove the following analogous estimate in $\mathbb{R}^3\backslash\mathcal{K}$.

THEOREM 3.13. *Suppose $u \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ solves* (3.53). *Also suppose that $\mathcal{K}$ satisfies* (1.6) *and that M is as in* (1.6). *Fix α such that $|\alpha| = N$. Then it follows that for $0 \le t \le T$,*

(3.75)
$$\begin{aligned}
(1+t+|x|)|Z^\alpha u(t,x)| \lesssim &\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \left|L^\mu Z^\beta G(s,y)\right| \frac{dy\,ds}{|y|}\\
&+ \int_0^t \sum_{\substack{|\beta|+\mu\le N+3+M\\ \mu\le 1}} \left\|L^\mu\partial^\beta G(s,\cdot)\right\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<5\})}\,ds.
\end{aligned}$$

PROOF. We shall split the proof into two cases: (1) $|x| > 2$ and (2) $|x| < 2$. To deal with case 1, fix a cutoff $\rho \in C^\infty(\mathbb{R}^3)$ such that $\rho(x) = 1$ when $|x| > 2$ and $\rho(x) = 0$ when $|x| < 1$. If we let $w = \rho Z^\alpha u$, then w solves the boundaryless wave equation $\Box w = \rho\Box(Z^\alpha u) - 2\nabla_x\rho\cdot\nabla_x Z^\alpha u - (\Delta\rho)Z^\alpha u$ with vanishing initial data. Write $w = w_1 + w_2$ where w_1 solves $\Box w_1 = \rho\Box(Z^\alpha u)$ with vanishing initial data. Applying Proposition 2.15 to w_1 and recalling the commutator relations $[\Box, Z] = 0$, we can see that the resulting quantity is controlled by the first term in the right hand side of (3.75). Applying Lemma 2.16 to w_2, if we let $\Box w_2 = F$, then we obtain the following inequality:

$$|w_2(t,x)| \lesssim \frac{1}{|x|} \int_0^t \int_{||x|-(t-s)|}^{|x|+(t-s)} \sup_{|\theta|=1} |F(s, r\theta)| \; rdr \; ds. \tag{3.76}$$

We claim that (3.76) implies

$$|w_2(t,x)| \lesssim \frac{1}{|x|} \frac{1}{1+|t-|x||} \sup_{\substack{t-|x|-2 \leq s \leq t-|x|+2 \\ |y|<2}} (1+s) \left[|Z^\alpha u(s,y)| + |(Z^\alpha u)'(s,y)| \right]. \tag{3.77}$$

If (3.77) were to hold, then it would remain to prove (3.75) only for $|x| < 2$. Observe that F is supported on $\{(s, r\theta) \in [0,T] \times \mathbb{R}^+ \times S^2 : 1 \leq r \leq 2\}$. It follows that the integrand in the right hand side of (3.76) is nonzero only if

$$-2 \leq |x| - (t-s) \leq 2,$$

which implies that $F(s, r\theta)$ is nonzero only if

$$t - |x| - 2 \leq s \leq t - |x| + 2. \tag{3.78}$$

From this we see that the integral in (3.76) is nonzero only if $t - |x| \geq -2$. This implies that $|t-|x|| \leq \max\{2, t-|x|\}$. Combining this with (3.78), we get

$$1 + |t - |x|| \lesssim 1 + s. \tag{3.79}$$

By examining the support of the integrand in (3.76), we get the inequality:

$$|w_2(t,x)| \lesssim \frac{1}{|x|} \sup_{\substack{t-|x|-2 \leq s \leq t-|x|+2 \\ |y|<2}} \frac{1}{1+s} (1+s) \left[|Z^\alpha u(s,y)| + |(Z^\alpha u)'(s,y)| \right].$$

Applying (3.79) to the right hand side of the above inequality, we have proved (3.77).

As we noted earlier, this reduces matters to considering the case (2) when $|x| < 2$. Because the coefficients of Z are bounded when $|x| < 2$, it follows that we only need to show that for $|\gamma| \leq |\alpha| + 1 = N + 1$

$$t \sup_{|x|<2} |\partial^\gamma u(t,x)|$$

is controlled by the right hand side of (3.75). Applying the fundamental theorem of calculus, we see that for $|x| < 2$,

$$t\,|\partial^\gamma u(t,x)| \lesssim \int_0^t \sum_{j\le 1} \left|(s\partial_s)^j \partial^\gamma u(s,x)\right| \,ds.$$

Applying Sobolev embedding and (3.21), we see that the right hand side is controlled by

$$\int_0^t \sum_{\substack{|\beta|\le N+2\\ j\le 1}} \left\|(s\partial_s)^j \partial^\beta u'(s,\cdot)\right\|_{L^2(|x|<3)} \,ds \lesssim \int_0^t \sum_{\substack{|\beta|+\mu\le N+3\\ \mu\le 1}} \left\|L^\mu \partial^\beta u'(s,\cdot)\right\|_{L^2(|x|<3)} \,ds. \tag{3.80}$$

We now consider two separate subcases: (1) $G(s,y) = 0$ when $|y| > 5$ and (2) $G(s,y) = 0$ when $|y| < 4$. In the first subcase, if we apply Lemma 3.7 to the right hand side of (3.80), we see that resulting quantity is

$$\begin{aligned}
&\int_0^t \sum_{\substack{|\beta|+\mu\le N+2\\ \mu\le 1}} \left\|L^\mu \partial^\beta G(s,\cdot)\right\|_{L^2(|x|<5)} \,ds \\
&+ \int_0^t \int_0^s \sum_{\substack{|\beta|+\mu\le N+3+M\\ \mu\le 1}} \langle s-\tau\rangle^{-2-\sigma+\mu} \left\|L^\mu \partial^\beta G(\tau,\cdot)\right\|_{L^2(|x|<5)} \,d\tau ds.
\end{aligned}$$

Because $\langle s\rangle^{-2-\sigma+\mu}$ is integrable on $[0,\infty)$ for any $\sigma > 0$ and $\mu = 0, 1$, it follows that if we apply Young's inequality, this quantity is controlled by the second term in right hand side of (3.75). To deal with the subcase when $G(s,y) = 0$ when $|y| < 4$, we write $u = u_0 + u_r$ where u_0 solves the boundaryless wave equation $\Box u_0 = G$ with vanishing initial data. Fix a cutoff $\eta \in C^\infty(\mathbb{R}^3)$ where $\eta(y) = 0$ when $|y| > 4$ and $\eta(y) = 1$ when $|y| < 3$. Let $\tilde{u} = \eta u_0 + u_r$. Since $\eta G = 0$, it follows that $\Box\tilde{u} = -2\nabla_x \eta \cdot \nabla_x u_0 - (\Delta\eta)u_0$. Also observe that for $|x| < 3$, $u(t,x) = \tilde{u}(t,x)$. It follows from subcase 1 that the right hand side of (3.80) is controlled by

$$\begin{aligned}
\int_0^t \sum_{\substack{|\beta|+\mu\le N+3\\ \mu\le 1}} \left\|L^\mu \partial^\beta u'(s,\cdot)\right\|_{L^2(|x|<3)} \,ds &= \int_0^t \sum_{\substack{|\beta|+\mu\le N+3\\ \mu\le 1}} \left\|L^\mu \partial^\beta \tilde{u}'(s,\cdot)\right\|_{L^2(|x|<3)} \,ds \\
&\lesssim \int_0^t \sum_{\substack{|\beta|+\mu\le N+4+M\\ \mu\le 1}} \left\|L^\mu \partial^\beta u_0(s,\cdot)\right\|_{L^\infty(3<|x|<4)} \,ds.
\end{aligned} \tag{3.81}$$

Applying Lemma 2.16 and Sobolev embedding on S^2, we see that the left hand side of (3.81) is controlled by

$$\int_0^t \int_0^s \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \int_{|s-\tau-|y||\le 4} \Big| L^\mu Z^\beta G(\tau,\cdot)\Big| \frac{dy}{|y|}\, d\tau ds.$$

While the double integral in s and τ might seem troubling, this can be controlled by observing that the sets $C_s = \{(\tau, y) : 0 \le \tau \le s, |s-\tau-|y|| \le 5\}$ have the property that $C_j \cap C_k$ is empty for $|j-k| > 10$. If we let $[t]$ be the smallest integer that is less than or equal to t, then we see that

$$\begin{aligned}
&\int_0^t \int_0^s \int_{|s-\tau-|y||\le 4} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \Big| L^\mu Z^\beta G(\tau,y)\Big| \ \frac{dyd\tau}{|y|}\ ds \\
&\lesssim \sum_{k=0}^{[t]} \int_k^{k+1} \int_0^s \int_{|s-\tau-|y||\le 4} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \Big| L^\mu Z^\beta G(\tau,y)\Big| \ \frac{dyd\tau}{|y|}\ ds \\
&+ \int_{[t]}^t \int_0^s \int_{|s-\tau-|y||\le 4} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \Big| L^\mu Z^\beta G(\tau,y)\Big| \ \frac{dyd\tau}{|y|}\ ds \\
&\lesssim \sum_{k=0}^{[t]} \iint_{\{(\tau,y):0\le\tau\le k+1, |k-\tau-|y||\le 5\}} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \Big| L^\mu Z^\beta G(\tau,y)\Big| \ \frac{dyd\tau}{|y|} \\
&+ \iint_{\{(\tau,y):0\le\tau\le t, |t-\tau-|y||\le 5\}} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \Big| L^\mu Z^\beta G(\tau,y)\Big| \ \frac{dyd\tau}{|y|} \\
&\lesssim \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu\le N+6+M\\ \mu\le 1}} \Big| L^\mu Z^\beta G(\tau,\cdot)\Big| \ \frac{dyd\tau}{|y|}.
\end{aligned}$$

This completes the proof. □

CHAPTER 4

Proof of Main Theorem

4.1. Preliminaries

Now that we have proved our necessary estimates, we are ready to prove Theorem 1.4. To get started, we will use the following local existence result that follows from Theorems 9.4 and 9.5 in Keel, Smith and Sogge [**27**]. In the local existence theorem, we need to specify the spaces that contain the local solution. We define $L^\infty([0,T];H^N(\mathbb{R}^3\backslash\mathcal{K}))$ to be the space of functions that are bounded in the following norm:

$$\|h\|_{L^\infty([0,T];H^N(\mathbb{R}^3\backslash\mathcal{K}))} = \operatorname{ess}\sup_{t\in[0,T]} \|h(t,\cdot)\|_{H^N(\mathbb{R}^3\backslash\mathcal{K})}\,.$$

We also write $C^{0,1}([0,T];H^N(\mathbb{R}^3\backslash\mathcal{K}))$ to denote the inhomogeneous space of Lipschitz continuous functions whose topology is given by the norm:

$$\begin{aligned}\|h\|_{C^{0,1}([0,T];H^N(\mathbb{R}^3\backslash\mathcal{K}))} &= \sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}} \frac{\|h(t_1,\cdot)-h(t_2,\cdot)\|_{H^N(\mathbb{R}^3\backslash\mathcal{K})}}{|t_1-t_2|}\\ &\quad+\operatorname{ess}\sup_{t\in[0,T]} \|h(t,\cdot)\|_{H^N(\mathbb{R}^3\backslash\mathcal{K})}\,.\end{aligned}$$

We are now ready to state the local existence theorem that we will be using.

Theorem 4.1. *Suppose that the initial data (f,g) are as in Theorem 1.4 and that N in* (1.8) *is greater than 6. Then there is a $T>0$ such that the initial value problem* (1.4) *with f,g as initial data has a classical C^2 solution satisfying*

$$u\in L^\infty([0,T];H^N(\mathbb{R}^3\backslash\mathcal{K}))\cap C^{0,1}([0,T];H^{N-1}(\mathbb{R}^3\backslash\mathcal{K})).$$

The supremum of such T is equal to the supremum of all T such that the initial value problem has a C^2 solution with $\partial^\alpha u$ bounded for $|\alpha|\leq 2$. Also, one can take $T\geq 2$ if $\|f\|_{H^N}+\|g\|_{H^{N-1}}$ is sufficiently small.

Although this theorem was originally proved for diagonal single-speed systems, it also applies to multiple-speed, nondiagonal systems that satisfy the symmetry conditions (1.3) since the proof relied solely on energy estimates.

Standard arguments also show that our local solution is uniformly small in proportion to the size of the initial data (f, g). For the combinatorics in the proof of Theorem 1.4 to work out, we will fix a positive integer N_0 such that it satisfies the inequality $N_0 \geq [(N_0 + 42 + 6M)/2] + 2$, where $[k]$ denotes the largest integer that is less than or equal to k. Thus, if we take N in Theorem 4.1 to be equal to $N_0 + 42 + 6M$, then there exists an absolute constant $C_0 > 0$ such that

$$\sup_{t\in[0,2]} \sum_{|\alpha|\leq N_0+42+6M} \|\partial^\alpha u(t,\cdot)\|_2 \leq C_0\epsilon. \tag{4.1}$$

We will use Theorem 4.1 to simplify (1.4) by reducing to a quasilinear wave equation that has an additional forcing term and vanishing initial data. This will enable us to avoid dealing with the compatibility conditions on the initial data (f, g) in our Picard iteration. Let us fix a cutoff $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $t < 1$ and $\eta(t) = 0$ for $t > 2$. If u is the local solution that is provided in Theorem 4.1 above, we can set $u_0(t, x) = \eta(t)u(t, x)$. It follows that u_0 solves

$$\begin{cases} \Box u_0(t,x) = \eta Q(u, u', u'') + [\Box, \eta]u, \quad (t,x) \in [0,T] \times \mathbb{R}^3\backslash\mathcal{K}, \\ u_0(0,x) = f(x), \quad \partial_t u_0(0,x) = g(x), \\ u_0(t,x) = 0, \quad x \in \partial\mathcal{K}. \end{cases} \tag{4.2}$$

If we let $w = u - u_0$, then it follows that w solves

$$\begin{cases} \Box w(t,x) = (1-\eta) Q(u, u', u'') - [\Box, \eta]u, \quad (t,x) \in [0,T] \times \mathbb{R}^3\backslash\mathcal{K}, \\ w(0,x) = \partial_t w(0,x) = 0, \\ w(t,x) = 0, \quad x \in \partial\mathcal{K}. \end{cases} \tag{4.3}$$

By this argument, it follows that u is a solution to (1.4) on $[0,T] \times \mathbb{R}^3\backslash\mathcal{K}$ if and only if w is a solution to (4.3) on $[0,T] \times \mathbb{R}^3\backslash\mathcal{K}$. To define our Picard iteration, we set $w_0 = 0$ and recursively let w_k be the solution to

$$\begin{cases} \Box w_k(t,x) = (1-\eta) Q(u_{k-1}, u'_{k-1}, u''_k) - [\Box, \eta]u, \quad (t,x) \in [0,T] \times \mathbb{R}^3\backslash\mathcal{K}, \\ w_k(0,x) = \partial_t w_k(0,x) = 0, \\ w_k(t,x) = 0, \quad x \in \partial\mathcal{K}, \end{cases} \tag{4.4}$$

where $u_k = w_k + u_0$ for all $k \geq 1$. By standard existence theory for linear wave equations, we know that each $w_k(t, \cdot)$ exists for all $t \geq 0$. To show our solution w exists in the classical sense for our desired lifespan (1.9), we will first need to prove a uniform bound for all the functions w_k in our iteration. More specifically, we will show that in a certain normed vector space of functions X_T, there is a uniform constant B that is independent of k such that

$$\|w_k\|_{X_T} \leq B\epsilon, \tag{4.5}$$

for ϵ in (1.8) sufficiently small. We shall state the exact value for B at a later point in the proof. Afterwards, we shall demonstrate this uniform bound will imply that the sequence $\{w_k\}$ is Cauchy in a suitably chosen Banach space Y_T. We shall let $M_k(T) := \|w_k\|_{X_T}$ and also let

$$M_k(T) = I_k(T) + \cdots + V_k(T),$$

where $I_k(T), \ldots, V_k(T)$ are defined as follows.

$$\begin{aligned}
I_k(T) = & \sum_{|\alpha| \leq N_0+40+6M} \sup_{0 \leq t \leq T} \left\| \partial^\alpha w_k'(t, \cdot) \right\|_2, \\
& + \langle T \rangle^{-1/4} \sum_{|\alpha| \leq N_0+35+5M} \left\| \langle x \rangle^{-1/4} \partial^\alpha w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}, \\
II_k(T) = & \sum_{\substack{|\alpha| \leq N_0+30+4M \\ |\beta| \leq 2}} \sup_{0 \leq t \leq T} \left\| Z^\alpha \partial^\beta w_k(t, \cdot) \right\|_2 \\
& + \langle T \rangle^{-1/4} \sum_{\substack{|\alpha| \leq N_0+30+4M \\ |\beta| \leq 1}} \left\| \langle x \rangle^{-1/4} Z^\alpha \partial^\beta w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}, \\
III_k(T) = & \sum_{\substack{|\alpha|+\mu \leq N_0+26+3M \\ \mu \leq 1}} \sup_{0 \leq t \leq T} \left\| L^\mu \partial^\alpha w_k'(t, \cdot) \right\|_2 \\
& + \langle T \rangle^{-1/4} \sum_{\substack{|\alpha|+\mu \leq N_0+21+2M \\ \mu \leq 1}} \left\| \langle x \rangle^{-1/4} L^\mu \partial^\alpha w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}, \\
IV_k(T) = & \sum_{\substack{|\alpha|+\mu \leq N_0+11+M \\ |\beta| \leq 2 \\ \mu \leq 1}} \sup_{0 \leq t \leq T} \left\| L^\mu Z^\alpha \partial^\beta w_k(t, \cdot) \right\|_2 \\
& + \langle T \rangle^{-1/4} \sum_{\substack{|\alpha|+\mu \leq N_0+11+M \\ |\beta| \leq 1 \\ \mu \leq 1}} \left\| \langle x \rangle^{-1/4} L^\mu Z^\alpha \partial^\beta w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})},
\end{aligned}$$

$$V_k(T) = \sum_{|\alpha| \leq N_0} \sup_{0 \leq t \leq T} \langle t \rangle \left\| Z^\alpha w_k(t, \cdot) \right\|_\infty .$$

The logic behind the choice of norms follows informally from the "hierarchy" of combinations of vector fields that was discussed in the introduction to this paper. By inspecting the number of vector fields appearing in each quantity $I_k(T), \ldots, V_k(T)$, it should be apparent that collections of vector fields that appear higher in the hierarchy occur in larger quantities than those that are lower in the hierarchy. It is no coincidence that all the norms that involve the scaling vector field L are lower in the hierarchy since norms involving L will be the most difficult to control.

We shall prove (4.5) via induction. The base case, establishing the bound for w_1, follows from (4.1) and from the same arguments that will be made for the general case. This is due to the fact that $w_0 = 0$ and w_1 satisfies $\Box w_1 = (1-\eta)Q(u_0, u_0', (w_1 + u_0)'') - [\Box, \eta]u$. Thus, we assume that (4.5) holds for $k-1$, where $k \geq 2$, and will prove (4.5) holds for k. We shall first prove an estimate that will allow us to deal with the combinatorics that arise in applying the product rule for derivatives.

Proposition 4.2. *Let $p, q \in C^\infty$. If $\{V\}$ is a collection of vector fields and $|\alpha| = N$, then it follows that*

$$|V^\alpha(pq)| \lesssim \sum_{|\beta| \leq N} \left| V^\beta p \right| \times \sum_{|\gamma| \leq [N/2]} |V^\gamma q| + \sum_{|\beta| \leq [N/2]} \left| V^\beta p \right| \times \sum_{|\gamma| \leq N} |V^\gamma q| .$$

Proof. Applying the Leibniz rule, we see that $V^\alpha(pq)$ is a linear combination of terms of the form $V^\beta p V^\gamma q$, where $|\beta| + |\gamma| = N$. Thus, either $|\beta|$ or $|\gamma|$ must be less than or equal to $[N/2]$. □

We will implicitly use this lemma throughout the proof of Theorem 1.4. To shorten some of the notation, we will often write

$$Q_k = Q(u_{k-1}, u_{k-1}', u_k'').$$

Although $\Box w_k = (1-\eta)Q_k - [\Box, \eta]u$, estimates for Q_k will imply bounds for $(1-\eta)Q_k$. The terms that arise from $[\Box, \eta]u$ will often be dealt with separately. Using this notation, we will now state an important consequence of the previous proposition.

Lemma 4.3. *Suppose that $|\alpha| = P$, $|\beta| = R$. It follows that*

$$
\begin{aligned}
\left|Z^\alpha \partial^\beta Q_k\right| \lesssim \sum_{\substack{|\gamma|\le P\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \times \\
&\left(\sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le [R/2]}} \left|Z^\gamma \partial^\delta w_k''\right| + \sum_{|\gamma|\le P+R+2} \left|\partial^\gamma u_0\right| \right) \\
&+ \sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \\
&+ \sum_{\substack{|\gamma|\le P\\ |\delta|\le R}} \left|Z^\gamma \partial^\delta w_k''\right| \left(\sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \sum_{|\gamma|\le P+R+1} \left|\partial^\gamma u_0\right| \right) \\
&+ \sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le R}} \left|Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \\
&+ \sum_{\substack{|\gamma|\le P\\ |\delta|\le [R/2]}} \left|Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \left(\sum_{|\gamma|\le P+R+2} \left|\partial^\gamma u_0\right| \right)^2,
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
\left|LZ^\alpha\partial^\beta Q_k\right| \lesssim & \sum_{\substack{|\gamma|\le P\\|\delta|\le R+1}} \left|LZ^\gamma\partial^\delta w_{k-1}\right| \times \\
& \left(\sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le[R/2]+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| + \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le[R/2]}} \left|Z^\gamma\partial^\delta w_k''\right| + \sum_{|\gamma|\le P+R+3} |\partial^\gamma u_0|\right) \\
& + \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le R+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| \times \sum_{\substack{|\gamma|\le P\\|\delta|\le[R/2]+1}} \left|LZ^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le R+1}} \left|LZ^\gamma\partial^\delta w_{k-1}\right| \times \sum_{\substack{|\gamma|\le P\\|\delta|\le[R/2]+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le P\\|\delta|\le R}} \left|LZ^\gamma\partial^\delta w_k''\right| \left(\sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le[R/2]+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| + \sum_{|\gamma|\le P+R+2} |\partial^\gamma u_0|\right) \\
& + \sum_{\substack{|\gamma|\le P\\|\delta|\le R}} \left|Z^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le[R/2]+1}} \left|LZ^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le R}} \left|Z^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le P\\|\delta|\le[R/2]+1}} \left|LZ^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le P\\|\delta|\le[R/2]}} \left|Z^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le R+1}} \left|LZ^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le[R/2]}} \left|LZ^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le P\\|\delta|\le R+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le P\\|\delta|\le[R/2]}} \left|LZ^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le R+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le[P/2]\\|\delta|\le R}} \left|LZ^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le P\\|\delta|\le[R/2]+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right| \\
& + \left(\sum_{|\gamma|\le P+R+3} |\partial^\gamma u_0|\right)^2 .
\end{aligned}
\tag{4.7}
$$

Also if $\Box_\gamma$ is the operator defined in (3.2) *and the components of γ are defined to be*

$$\gamma^{ij,IJ}(t,x) = -B^{ij,IJ}(u_{k-1}, u'_{k-1}), \tag{4.8}$$

where $B^{ij,IJ}$ are as in (1.2)*, then it follows that*

$$\begin{aligned}
\left|\Box_\gamma Z^\alpha \partial^\beta u_k\right| \lesssim \sum_{\substack{|\gamma|\le P\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \times \\
\left(\sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \sum_{\substack{|\gamma|+|\delta|\le [P/2]+[R/2]-1\\ |\gamma|\le [P/2]\\ |\delta|\le [R/2]}} \left|Z^\gamma \partial^\delta w''_k\right| + \sum_{|\gamma|\le P+R+2} |\partial^\gamma u_0| \right) \\
+ \sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \\
+ \sum_{\substack{|\gamma|+|\delta|\le P+R-1\\ |\gamma|\le P\\ |\delta|\le R}} \left|Z^\gamma \partial^\delta w''_k\right| \left(\sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \sum_{|\gamma|\le P+R+1} |\partial^\gamma u_0| \right) \\
+ \sum_{\substack{|\gamma|+|\delta|\le [P/2]+R-1\\ |\gamma|\le [P/2]\\ |\delta|\le R}} \left|Z^\gamma \partial^\delta w''_k\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le [R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \\
+ \sum_{\substack{|\gamma|+|\delta|\le P+[R/2]-1\\ |\gamma|\le P\\ |\delta|\le [R/2]}} \left|Z^\gamma \partial^\delta w''_k\right| \times \sum_{\substack{|\gamma|\le [P/2]\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \left(\sum_{|\gamma|\le P+R+2} |\partial^\gamma u_0| \right)^2,
\end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
\left|\Box_\gamma L Z^\alpha \partial^\beta u_k\right| \lesssim \sum_{\substack{|\gamma|\le P\\ |\delta|\le R+1}} \left|L Z^\gamma \partial^\delta w_{k-1}\right| \times & \\
& \left(\sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le[R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \sum_{\substack{|\gamma|+|\delta|\le[P/2]+[R/2]-1\\ |\gamma|\le[P/2]\\ |\delta|\le[R/2]}} \left|Z^\gamma \partial^\delta w_k''\right| + \sum_{|\gamma|\le P+R+3} |\partial^\gamma u_0| \right) \\
& + \sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le[R/2]+1}} \left|L Z^\gamma \partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le R+1}} \left|L Z^\gamma \partial^\delta w_{k-1}\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le[R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|+|\delta|+\mu\le P+R\\ |\gamma|\le P\\ |\delta|\le R\\ \mu\le 1}} \left|L^\mu Z^\gamma \partial^\delta w_k''\right| \left(\sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le[R/2]+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| + \sum_{|\gamma|\le P+R+2} |\partial^\gamma u_0| \right) \\
& + \sum_{\substack{|\gamma|+|\delta|\le P+R-1\\ |\gamma|\le P\\ |\delta|\le R}} \left|Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le[R/2]+1}} \left|L Z^\gamma \partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|+|\delta|\le[P/2]+R-1\\ |\gamma|\le[P/2]\\ |\delta|\le R}} \left|Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le[R/2]+1}} \left|L Z^\gamma \partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|+|\delta|\le P+[R/2]-1\\ |\gamma|\le P\\ |\delta|\le[R/2]}} \left|Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le R+1}} \left|L Z^\gamma \partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|+|\delta|+\mu\le[P/2]+[R/2]\\ |\gamma|\le[P/2]\\ |\delta|\le[R/2]\\ \mu\le 1}} \left|L^\mu Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le P\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right| \\
& + \sum_{\substack{|\gamma|+|\delta|+\mu\le P+[R/2]\\ |\gamma|\le P\\ |\delta|\le[R/2]\\ \mu\le 1}} \left|L^\mu Z^\gamma \partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\le[P/2]\\ |\delta|\le R+1}} \left|Z^\gamma \partial^\delta w_{k-1}\right|
\end{aligned}
\tag{4.10}
$$

$$+ \sum_{\substack{|\gamma|+|\delta|\leq[P/2]+R-1 \\ |\gamma|\leq[P/2] \\ |\delta|\leq R}} \left|LZ^\gamma\partial^\delta w_k''\right| \times \sum_{\substack{|\gamma|\leq P \\ |\delta|\leq[R/2]+1}} \left|Z^\gamma\partial^\delta w_{k-1}\right|$$

$$+ \left(\sum_{|\gamma|\leq P+R+3} |\partial^\gamma u_0| \right)^2 .$$

PROOF. The first two inequalities are an immediate consequence of the previous proposition. The last two inequalities follow from the additional observation that if one fixes $\nu = 0$ or 1 and $|\alpha| = P$, $|\beta| = R$ the commutator

$$\left[\sum_{i,j=0}^{3} \gamma^{ij,IJ}\partial_i\partial_j, L^\nu Z^\alpha \partial^\beta \right]$$

is a linear combination of terms of the form

$$\sum_{i,j=0}^{3} L^{\mu_1} Z^{\alpha_1}\partial^{\beta_1}\gamma^{ij,IJ} L^{\mu_2} Z^{\alpha_2}\partial^{\beta_2}\partial_i\partial_j,$$

where $\mu_1 + \mu_2 \leq \nu$, $|\alpha_1| + |\alpha_2| \leq P$, $|\beta_1| + |\beta_2| \leq R$ and $|\alpha_2| + |\beta_2| + \mu_2 \leq P + R + \nu - 1$. The reason that L is not being applied to u_0 in any of the inequalities is because $u_0(t, x) = 0$ if $t > 2$ and $u_0(0, \cdot)$ is compactly supported. Thus, by finite propagation speed, one can see that

$$|Lu_0| \lesssim \sum_{|\alpha|\leq 1} |\partial^\alpha u_0|.$$

□

The most difficult terms to control in (4.6)-(4.10) shall tend to be the terms that are grouped in parentheses. These correspond to the terms where the maximum number of vector fields are being applied to w_{k-1} and w_k''.

4.2. Proof of Uniform Bound

Until it is noted otherwise, we will use C to denote a constant that depends only on B, C_0 and the implicit constants that occur in the estimates in this paper. C_1 will denote a constant that is independent of B and depends only on C_0 and the implicit constants that occur in the estimates in this paper. Both C and C_1 shall be allowed to vary from line to line.

4.2.1. Term I Bounds. To control the first term in $I_k(T)$, we will be applying Theorem 3.1. To do this, we let γ be as in (4.8). Since we will want to apply our perturbed energy estimates from Chapter 3, such as Theorem 3.1, we must show that (3.3), (3.4) are satisfied. (3.3) is an immediate consequence of the symmetry condition (1.3). By the induction hypothesis, we see that for $0 \le t \le T_\epsilon$,

$$\left\|\gamma^{ij,IJ}(u_{k-1}, u'_{k-1})(t,\cdot)\right\|_\infty \le M_{k-1}(T_\epsilon) \le B\epsilon. \tag{4.11}$$

By the above inequality, if ϵ is sufficiently small, then (3.4) is satisfied. It will also be useful to note that (3.34) holds for our choice of $\gamma^{ij,IJ}$. From the induction hypothesis, we see that for $0 \le s \le T$,

$$\left\|\gamma'(s,\cdot)\right\|_\infty \le \frac{B\epsilon}{1+s}. \tag{4.12}$$

Integrating both sides of (4.12), we get

$$\begin{aligned}\int_0^{T_\epsilon} \left\|\gamma'(s,\cdot)\right\|_\infty \, ds &\le B\epsilon \int_0^{T_\epsilon} \frac{ds}{1+s} \\ &\le B\epsilon \log(1+T_\epsilon) \\ &\le B\epsilon \log(1+c/\epsilon^2),\end{aligned} \tag{4.13}$$

where T_ϵ and c are the constants appearing in (1.9). If we take $\epsilon < 1$ and $c < 1$, then this quantity is bounded above by a uniform constant that is independent of k, c, T_ϵ and ϵ. Furthermore, we can make ϵ_0, which initially appeared in the statement of Theorem 1.4, sufficiently small such that if $\epsilon \le \epsilon_0$, then it follows that an even stronger version of (4.13) holds:

$$\int_0^{T_\epsilon} \left\|\gamma'(s,\cdot)\right\|_\infty \, ds \le 1. \tag{4.14}$$

We also observe that u_k solves

$$\Box_\gamma u_k = A(u_{k-1}, u'_{k-1}), \tag{4.15}$$

where the components of $A(u_{k-1}, u'_{k-1})$ are defined in (1.2) and $\Box_\gamma$ is the operator defined in (3.2). We now apply Proposition 3.3 and use (4.1) to see that

$$
\begin{aligned}
\sum_{|\alpha|\le N_0+40+6M} \sup_{0\le t\le T} \left\|\partial^\alpha w'_k(t,\cdot)\right\|_2 &\lesssim \sum_{j\le N_0+40+6M} \sup_{0\le t\le T} \left\|\partial_t^j u'_k(t,\cdot)\right\|_2 \\
&\quad + \sum_{|\alpha|\le N_0+39+6M} \sup_{0\le t\le T} \left\|\partial^\alpha \Box w_k(t,\cdot)\right\|_2 \\
&\quad + \sum_{j\le N_0+40+6M} \sup_{0\le t\le 2} \left\|\partial_t^j u'_0(t,\cdot)\right\|_2 \\
&\lesssim \sum_{j\le N_0+40+6M} \sup_{0\le t\le T} \left\|\partial_t^j u'_k(t,\cdot)\right\|_2 \\
&\quad + \sum_{|\alpha|\le N_0+39+6M} \sup_{0\le t\le T} \left\|\partial^\alpha \Box w_k(t,\cdot)\right\|_2 + C_1\epsilon.
\end{aligned}
\tag{4.16}
$$

By the definition of $\Box w_k$, Sobolev embedding, (4.1) and the induction hypothesis, we see that the second term on the right hand side of (4.16) is controlled by

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\sum_{|\alpha|\le N_0+39+6M} \left\|\partial^\alpha w'_{k-1}(t,\cdot)\right\|_2 + \|w_{k-1}(t,\cdot)\|_2 + \sum_{|\alpha|\le N_0+41+6M} \|\partial^\alpha u_0(t,\cdot)\|_2 \right) \times \\
&\sup_{0\le t\le T}\left(\sum_{|\alpha|\le N_0+39+6M} \left\|\partial^\alpha w'_{k-1}(t,\cdot)\right\|_2 + \|w_{k-1}(t,\cdot)\|_2 \right. \\
&\qquad \left. + \sum_{|\alpha|\le N_0+42+6M} \|\partial^\alpha u_0(t,\cdot)\|_2 + \sum_{|\alpha|\le N_0+39+6M} \left\|\partial^\alpha w''_k(t,\cdot)\right\|_2 \right) \\
&\qquad + \sum_{|\alpha|\le N_0+40+6M} \sup_{0\le t\le 2} \|\partial^\alpha [\Box,\eta] u(t,\cdot)\|_2 \\
&\le C\epsilon^2 + C\epsilon M_k(T) + C_1\epsilon.
\end{aligned}
\tag{4.17}
$$

It remains to control the first term on the right hand side of (4.16). We set

$$E_N(t) = E_N(u_k)(t),$$

using the notation from (3.6). By (4.11), just as in (3.20), for $0\le t\le T_\epsilon$, we have the inequality

$$
(5\max_I\{c_I^2, c_I^{-2}\})^{-1} E_N^{1/2}(t) \le \sum_{j\le N} \left\|(\partial_t^j u_k)'(t,\cdot)\right\|_2 \le 5\max_I\{c_I^2, c_I^{-2}\} E_N^{1/2}(t).
\tag{4.18}
$$

By Theorem 3.1, it follows that for $0 \le t \le T_\epsilon$,

$$\partial_t \left[E^{1/2}_{N_0+40+6M}(t) \right] \lesssim \sum_{j=0}^{N_0+40+6M} \left\| \Box_\gamma \partial_t^j u_k(t,\cdot) \right\|_2 + \left\| \gamma'(t,\cdot) \right\|_\infty E^{1/2}_{N_0+40+6M}(t). \tag{4.19}$$

Applying Gronwall's inequality and (4.14) to (4.19), it follows that for $0 \le t \le T_\epsilon$,

$$E^{1/2}_{N_0+40+6M}(t) \lesssim E^{1/2}_{N_0+40+6M}(0) + \int_0^{T_\epsilon} \sum_{j \le N_0+40+6M} \left\| \Box_\gamma \partial_s^j u_k(s,\cdot) \right\|_2 \, ds. \tag{4.20}$$

By (1.8), (4.18) and the compatibility conditions, we see that

$$E^{1/2}_{N_0+40+6M}(0) \le C_1 \epsilon. \tag{4.21}$$

Applying Lemma 4.3, we see that

$$\begin{aligned} &\sum_{j \le N_0+40+6M} \left| \Box_\gamma \partial_t^j u_k \right| \lesssim \left(\sum_{|\alpha| \le N_0+40+6M} \left| \partial^\alpha w'_{k-1} \right| + |w_{k-1}| + \sum_{|\alpha| \le N_0+42+6M} |\partial^\alpha u_0| \right) \times \\ &\left(\sum_{|\alpha| \le [(N_0+40+6M)/2]+1} |\partial^\alpha w_{k-1}| + \sum_{|\alpha| \le [(N_0+40+6M)/2]} \left| \partial^\alpha w''_k \right| + \sum_{|\alpha| \le [(N_0+40+6M)/2]+2} |\partial^\alpha u_0| \right) \\ &+ \sum_{|\alpha| \le N_0+39+6M} \left| \partial^\alpha w''_k \right| \left(\sum_{|\alpha| \le [(N_0+40+6M)/2]+1} |\partial^\alpha w_{k-1}| + \sum_{|\alpha| \le [(N_0+40+6M)/2]+1} |\partial^\alpha u_0| \right). \end{aligned} \tag{4.22}$$

Before dealing with the terms in the right hand side of (4.22), we will prove a useful lemma.

Lemma 4.4. *Suppose that $v_1, v_2 \in C^\infty([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})$. Then it follows that for $0 \le t \le T$,*

$$\int_0^T \|v_1(t,\cdot) v_2(t,\cdot)\|_2 \, dt \lesssim \log(2+T) \sup_{0 \le t \le T} \|v_1(t,\cdot)\|_2 \sup_{0 \le t \le T} \langle t \rangle \|v_2(t,\cdot)\|_\infty , \tag{4.23}$$

and

$$\|v_1 v_2\|_{L^2([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})} \lesssim \sup_{0 \le t \le T} \|v_1(t,\cdot)\|_2 \times \sup_{0 \le t \le T} \langle t \rangle \|v_2(t,\cdot)\|_\infty . \tag{4.24}$$

Proof. The left hand side of (4.23) is controlled by

$$\int_0^T \|v_1(t,\cdot)\|_2 \|v_2(t,\cdot)\|_\infty \, dt \lesssim \int_0^T \langle t \rangle^{-1} \, dt \times \sup_{0 \le t \le T} \|v_1(t,\cdot)\|_2 \sup_{0 \le t \le T} \langle t \rangle \|v_2(t,\cdot)\|_\infty .$$

Noting that $\int_0^T \langle t\rangle^{-1}\ dt \lesssim \log(2+T)$, this proves (4.23). To prove (4.24), we apply the same argument to see that

$$\int_0^T \|v_1(t,\cdot)\|_2^2 \|v_2(t,\cdot)\|_\infty^2\ dt \lesssim \int_0^T \langle t\rangle^{-2}\ dt \times \left(\sup_{0\le t\le T} \|v_1(t,\cdot)\|_2\right)^2 \times \left(\sup_{0\le t\le T} \langle t\rangle \|v_2(t,\cdot)\|_\infty\right)^2.$$

Since $\int_0^T \langle t\rangle^{-2}\ dt \lesssim 1$, this completes the proof. □

We will first deal with the term that appears in the third line of the right hand side of (4.22). If we apply Lemma 4.4, we get, for $T \le T_\epsilon$, the inequality

(4.25)

$$\begin{aligned}
&\int_0^T \sum_{\substack{|\alpha|\le N_0+39+6M \\ |\beta|\le[(N_0+40+6M)/2]+1}} \left\|\partial^\alpha w_k''(s,\cdot)\right\|_2 \times \\
&\quad \left(\left\|\partial^\beta w_{k-1}(s,\cdot)\right\|_\infty + \left\|\partial^\beta u_0(s,\cdot)\right\|_\infty\right)\ ds \\
&\lesssim \log(2+T)\left(\sum_{|\alpha|\le N_0+39+6M} \sup_{0\le s\le T} \left\|\partial^\alpha w_k''(s,\cdot)\right\|_2\right) \times \\
&\left(\sum_{|\alpha|\le[(N_0+40+6M)/2]+1} \sup_{0\le s\le T} \langle s\rangle \|\partial^\alpha w_{k-1}(s,\cdot)\|_\infty + \sum_{|\alpha|\le[(N_0+40+6M)/2]+3} \sup_{0\le s\le T} \|\partial^\alpha u_0(s,\cdot)\|_2\right),
\end{aligned}$$

where Sobolev embedding was applied to the u_0 term in the last step. Applying the induction hypothesis and that $[(N_0+40+6M)/2]+1 \le N_0$, we see that the left hand side of (4.25) is controlled by $C\log(2+T)\epsilon M_k(T)$. Applying a similar argument, we can deal with the remaining terms in the right hand side of (4.22):

$$\begin{aligned}
&\int_0^T \left(\sum_{|\alpha|\le N_0+40+6M} \left\|\partial^\alpha w_{k-1}'(s,\cdot)\right\|_2 + \|w_{k-1}(s,\cdot)\|_2 + \sum_{|\alpha|\le N_0+42+6M} \|\partial^\alpha u_0(s,\cdot)\|_2\right) \times \\
&\quad \left(\sum_{|\beta|\le[(N_0+40+6M)/2]+1} \left\|\partial^\beta w_{k-1}(s,\cdot)\right\|_\infty\right. \\
&\quad \left. + \sum_{|\beta|\le[(N_0+40+6M)/2]} \left\|\partial^\beta w_k''(s,\cdot)\right\|_\infty + \sum_{|\beta|\le[(N_0+40+6M)/2]+2} \left\|\partial^\beta u_0(s,\cdot)\right\|_\infty\right)\ ds \\
&\le C\log(2+T)\epsilon^2 + C\log(2+T)\epsilon M_k(T),
\end{aligned} \tag{4.26}$$

for $T \leq T_\epsilon$. Combining this with (4.17), (4.20), (4.21) and (4.25), it follows that

$$\sum_{|\alpha|\leq N_0+40+6M} \sup_{0\leq t\leq T} \left\|\partial^\alpha w_k'(t,\cdot)\right\|_2 \leq C\log(2+T)\epsilon^2 + C\log(2+T)\epsilon M_k(T) + C_1\epsilon, \tag{4.27}$$

provided $T \leq T_\epsilon$.

To control the second term in $I_k(T)$, we apply the first inequality in Theorem 3.10 to see that

$$\begin{aligned}\sum_{|\alpha|\leq N_0+35+5M} \langle T\rangle^{-1/4} \left\|\langle x\rangle^{-1/4}\partial^\alpha w_k'\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} &\lesssim \int_0^T \sum_{|\alpha|\leq N_0+35+6M} \|\partial^\alpha Q_k(s,\cdot)\|_2 \, ds \\ &+ \sum_{|\alpha|\leq N_0+35+5M} \|\partial^\alpha Q_k\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\ &+ \sum_{|\alpha|\leq N_0+35+6M} \sup_{0\leq t\leq 2} \|\partial^\alpha[\Box,\eta]u(t,\cdot)\|_2 \, .\end{aligned} \tag{4.28}$$

By (4.1), the last term in (4.28) is controlled by $C_1\epsilon$. To control the first term in the right hand side (4.28), we apply Lemma 4.3 to see that

$$\begin{aligned}\sum_{|\alpha|\leq N_0+35+6M} |\partial^\alpha Q_k| \lesssim &\left(\sum_{|\alpha|\leq N_0+35+6M} \left|\partial^\alpha w_{k-1}'\right| + |w_{k-1}| + \sum_{|\alpha|\leq N_0+37+6M} |\partial^\alpha u_0|\right) \times \\ &\left(\sum_{|\alpha|\leq[(N_0+35+6M)/2]+1} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\leq[(N_0+35+6M)/2]} \left|\partial^\alpha w_k''\right| + \sum_{|\alpha|\leq[(N_0+35+6M)/2]+2} |\partial^\alpha u_0|\right) \\ &+ \sum_{|\alpha|\leq N_0+35+6M} \left|\partial^\alpha w_k''\right| \left(\sum_{|\alpha|\leq[(N_0+35+6M)/2]+1} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\leq[(N_0+35+6M)/2]+1} |\partial^\alpha u_0|\right).\end{aligned} \tag{4.29}$$

Using the same arguments that were used to obtain (4.27) from (4.25) and (4.26), it follows from (4.29) that the first term in the right hand side of (4.28) is controlled by $C\log(2+T)\epsilon^2 + C\log(2+T)\epsilon M_k(T) + C_1\epsilon$. By applying (4.29) and Lemma 4.4, one see that the second term in (4.28) is also controlled by $C\epsilon^2 + C\epsilon M_k(T) + C_1\epsilon$. Before stating our final bound for term $I_k(T)$, we make an observation to ease exposition. We note that $\log(2+T) \lesssim \langle T\rangle^{1/2}$. Thus, (4.27)-(4.29) demonstrate that for $T \leq T_\epsilon$ and ϵ sufficiently small, we get the bound

$$I_k(T) \leq C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon. \tag{4.30}$$

4.2.2. Term II Bounds. To control term $II_k(T)$, we shall start with the case that $1 \le |\beta| \le 2$. The case where $|\beta| = 0$ will require a different argument. Using the notation of (3.63), we set

$$Y_{N_1,N_2,\nu}(t) = Y_{N_1,N_2,\nu}(u_k)(t). \tag{4.31}$$

By (3.20), for ϵ sufficiently small and $0 \le t \le T_\epsilon$, we have the inequality

$$(5\max_I\{c_I^2, c_I^{-2}\})^{-1} Y_{N,1,0}^{1/2}(t) \le \sum_{\substack{|\alpha|\le N\\|\beta|\le 1}} \left\|(Z^\alpha\partial^\beta u_k)'(t,\cdot)\right\|_2 \le 5\max_I\{c_I^2, c_I^{-2}\} Y_{N,1,0}^{1/2}(t). \tag{4.32}$$

By Theorem 3.9, we see that for, $0 \le t \le T_\epsilon$, we have the bound

(4.33)

$$\partial_t Y_{N_0+30+4M,1,0}(t) \le C Y_{N_0+30+4M,1,0}^{1/2}(t) \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} \left\|\Box_\gamma Z^\alpha \partial^\beta u_k(t,\cdot)\right\|_2$$
$$+ C\left\|\gamma'(t,\cdot)\right\|_\infty Y_{N_0+30+4M,1,0}(t) + C\sum_{|\alpha|\le N_0+31+4M} \left\|\partial^\alpha u_k'(t,\cdot)\right\|_{L^2(|x|<1)}^2.$$

Applying Gronwall's inequality, for $T \le T_\epsilon$, we get

(4.34)

$$\sup_{0\le t\le T} Y_{N_0+30+4M,1,0}(t) \le C\int_0^T \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} Y_{N_0+30+4M,1,0}(s)^{1/2} \left\|\Box_\gamma Z^\alpha\partial^\beta u_k(s,\cdot)\right\|_2 \, ds$$
$$+ C\sum_{|\alpha|\le N_0+31+4M} \left\|\partial^\alpha u_k'\right\|_{L^2([0,T]\times\{|x|<1\})}^2 + CY_{N_0+30+4M,1,0}(0).$$

The first term in the right hand side of (4.34) is controlled by

$$\frac{1}{2}\sup_{0\le t\le T} Y_{N_0+30+4M,1,0}(t) + C\left(\int_0^T \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} \left\|\Box_\gamma Z^\alpha\partial^\beta u_k(s,\cdot)\right\|_2 \, ds\right)^2. \tag{4.35}$$

After bootstrapping the first term in (4.35) back into the left hand side of (4.34), for $T \le T_\epsilon$, we get

$$
\begin{aligned}
\sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} &\sup_{0\le t\le T} \left\| Z^\alpha \partial^\beta w_k'(t,\cdot)\right\|_2 \\
&\lesssim \sup_{0\le t\le T}\left(Y^{1/2}_{N_0+30+4M,1,0}(t) + \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} \left\| Z^\alpha \partial^\beta u_0'(t,\cdot)\right\|_2\right) \\
&\lesssim \int_0^T \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} \left\| \Box_\gamma Z^\alpha \partial^\beta u_k(s,\cdot)\right\|_2 \, ds \\
&\quad + \sum_{|\alpha|\le N_0+31+4M} \left\|\partial^\alpha u_k'\right\|_{L^2([0,T]\times\{|x|<1\})} \\
&\quad + Y_{N_0+30+4M,1,0}(0)^{1/2} \\
&\quad + \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} \sup_{0\le t\le T} \left\| Z^\alpha \partial^\beta u_0'(t,\cdot)\right\|_2 .
\end{aligned}
\tag{4.36}
$$

By (4.32), (1.8) and the compatibility conditions, it follows that

$$
Y_{N_0+30+4M,1,0}(0)^{1/2} + \sum_{\substack{|\alpha|\le N_0+30+4M\\|\beta|\le 1}} \left\| Z^\alpha \partial^\beta u_0'(t,\cdot)\right\|_2 \le C_1\epsilon. \tag{4.37}
$$

Also note that the second term in the right hand side of (4.36) is controlled by the same quantities that bound term $I_k(T)$. To control the term involving $\Box_\gamma$, we apply Lemma 4.3 to see that

(4.38)

$$\sum_{\substack{|\alpha|\le N_0+30+4M\\ |\beta|\le 1}} \left|\Box_\gamma Z^\alpha\partial^\beta u_k\right| \lesssim \left(\sum_{\substack{|\alpha|\le N_0+30+4M\\ |\beta|\le 2}}\left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+31+4M}|Z^\alpha u_0|\right)\times$$
$$\left(\sum_{\substack{|\alpha|\le[(N_0+30+4M)/2]\\ |\beta|\le 2}}\left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{\substack{|\alpha|\le[(N_0+30+4M)/2]\\ |\beta|\le 2}}\left|Z^\alpha\partial^\beta w_k\right| + \sum_{|\alpha|\le N_0+32+4M}|Z^\alpha u_0|\right)$$
$$+\sum_{\substack{|\alpha|\le N_0+30+4M\\ |\beta|\le 2}}\left|Z^\alpha\partial^\beta w_k\right|\left(\sum_{\substack{|\alpha|\le[(N_0+30+4M)/2]\\ |\beta|\le 2}}\left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+31+4M}|Z^\alpha u_0|\right).$$

We will first demonstrate how to deal with the terms in third line of (4.38). Using that fact that $[(N_0+30+4M)/2]\le N_0$, we apply Lemma 4.4 and Sobolev embedding to get

$$\int_0^T \sum_{\substack{|\alpha|\le N_0+30+4M\\ |\beta|\le 2}}\left\|Z^\alpha\partial^\beta w_k(s,\cdot)\right\|_2\times$$
$$\left(\sum_{\substack{|\alpha|\le[(N_0+30+4M)/2]\\ |\beta|\le 2}}\left\|Z^\alpha\partial^\beta w_{k-1}(s,\cdot)\right\|_\infty + \sum_{|\alpha|\le N_0+31+4M}\|Z^\alpha u_0(s,\cdot)\|_\infty\right)\,ds$$

(4.39)
$$\lesssim \log(2+T)\left(\sup_{0\le t\le T}\sum_{\substack{|\alpha|\le N_0+30+4M\\ |\beta|\le 2}}\left\|Z^\alpha\partial^\beta w_k(t,\cdot)\right\|_2\right)\times$$
$$\left(\sum_{\substack{|\alpha|\le[(N_0+30+4M)/2]\\ |\beta|\le 2}}\sup_{0\le t\le T}\langle t\rangle\left\|Z^\alpha\partial^\beta w_{k-1}(s,\cdot)\right\|_\infty + \sum_{|\alpha|\le N_0+33+4M}\sup_{0\le t\le T}\|Z^\alpha u_0(s,\cdot)\|_2\right)$$
$$\le C\log(2+T)\epsilon M_k(T).$$

Applying the same argument to the remaining terms in (4.38), we see that

$$
\begin{aligned}
&\int_0^T \left(\sum_{\substack{|\alpha|\le N_0+30+4M \\ |\beta|\le 2}} \left\| Z^\alpha \partial^\beta w_{k-1}(s,\cdot) \right\|_2 + \sum_{|\alpha|\le N_0+31+4M} \left\| \partial^\alpha u_0(s,\cdot) \right\|_2 \right) \times \\
&\quad \left(\sum_{\substack{|\alpha|\le [(N_0+30+4M)/2] \\ |\beta|\le 2}} \left\| Z^\alpha \partial^\beta w_{k-1}(s,\cdot) \right\|_\infty + \sum_{\substack{|\alpha|\le [(N_0+30+4M)/2] \\ |\beta|\le 2}} \left\| Z^\alpha \partial^\beta w_k(s,\cdot) \right\|_\infty \right. \\
&\quad \left. + \sum_{|\alpha|\le [(N_0+30+4M)/2]+2} \left\| \partial^\alpha u_0(s,\cdot) \right\|_\infty \right) \, ds \\
&\le C \log(2+T)\epsilon^2 + C\epsilon \log(2+T) M_k(T).
\end{aligned} \tag{4.40}
$$

Thus, by (4.36)-(4.40), we obtain the bound

$$
\sup_{0\le t\le T} \sum_{\substack{|\alpha|\le N_0+30+4M \\ |\beta|\le 1}} \left\| Z^\alpha \partial^\beta w_k'(t,\cdot) \right\|_2 \le C\log(2+T)\epsilon^2 + C\log(2+T)\epsilon M_k(T) + C_1\epsilon, \tag{4.41}
$$

provided that $T \le T_\epsilon$. We now turn our attention to the weighted term in $II_k(T)$ in the case that $|\beta| = 1$. We apply Theorem 3.10 to see that

$$
\begin{aligned}
\sum_{|\alpha|\le N_0+30+4M} \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} Z^\alpha w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} &\lesssim \int_0^T \sum_{|\alpha|\le N_0+30+5M} \left\| \partial^\alpha \Box w_k(s,\cdot) \right\|_2 \, ds \\
&+ \int_0^T \sum_{|\alpha|\le N_0+30+4M} \left\| Z^\alpha \Box w_k(s,\cdot) \right\|_2 \, ds \\
&+ \sum_{|\alpha|\le N_0+30+4M} \left\| \partial^\alpha \Box w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}.
\end{aligned} \tag{4.42}
$$

First observe that the first term and third terms in the right hand side of (4.42) are controlled by the right hand side of (4.28), which is in turn bounded by (4.30). Applying Lemma 4.3, we see that

$$
\sum_{|\alpha|\le N_0+30+4M} |Z^\alpha \Box w_k|
$$

is controlled by the right hand side of (4.38), it follows from (4.39)-(4.40) that

$$\begin{aligned}\sum_{|\alpha|\le N_0+30+4M} \langle T\rangle^{-1/4} \left\|\langle x\rangle^{-1/4} Z^\alpha w_k'\right\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})} \\ \le C\log(2+T)\epsilon^2 + C\log(2+T)\epsilon M_k(T) + C_1\epsilon.\end{aligned} \tag{4.43}$$

To handle the case $|\beta|=0$, we will split the terms into two pieces:

$$\begin{aligned}&\sum_{|\alpha|\le N_0+30+4M}\left[\sup_{0\le t\le T}\|Z^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4} Z^\alpha w_k\right\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\right] \\ &= \sum_{|\alpha|\le N_0+29+4M}\left[\sup_{0\le t\le T}\|Z^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4} Z^\alpha w_k\right\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\right] \\ &+ \sum_{|\alpha|= N_0+30+4M}\left[\sup_{0\le t\le T}\|Z^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4} Z^\alpha w_k\right\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\right].\end{aligned} \tag{4.44}$$

We will deal with the first term in parentheses on the right hand side. Applying Theorem 3.12, we see that

$$\begin{aligned}&\sum_{|\alpha|\le N_0+29+4M}\left[\|Z^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4} Z^\alpha w_k\right\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\right] \\ &\lesssim \sum_{|\alpha|\le N_0+29+5M}\sup_{0\le t\le T}\left\|\partial^\alpha w_k'(t,\cdot)\right\|_2 \\ &+ \int_0^T \sum_{|\alpha|\le N_0+29+5M}\|\partial^\alpha \Box w_k(s,\cdot)\|_2\, ds \\ &+ \int_0^T \sum_{|\alpha|\le N_0+29+4M}\left\|\langle x\rangle^{-1/2} Z^\alpha \Box w_k(s,\cdot)\right\|_{L^1_r L^2_\omega(|x|>2)}\, ds \\ &+ \sum_{|\alpha|\le N_0+28+4M}\|\partial^\alpha\Box w_k\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}.\end{aligned} \tag{4.45}$$

The first term on the right hand side of (4.45) is bounded by the same quantities that bound term $I_k(T)$. The second and fourth terms on the right hand side of (4.45) are controlled by the right hand side of (4.28), which is controlled by (4.30). To control the remaining term in (4.45), we will prove the following lemma.

LEMMA 4.5. *Suppose that $v_1, v_2 \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$. Then it follows that*

$$
\begin{aligned}
&\int_0^T \left\| \langle x\rangle^{-1/2}\, v_1(s,\cdot)v_2(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\, ds\\
&\lesssim \left\|\langle x\rangle^{-1/4}\, v_1\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \times \sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4}\, Z^\alpha v_2\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}.
\end{aligned}
\tag{4.46}
$$

PROOF. Taking the supremum in the ω variable, we see that

$$
\begin{aligned}
&\int_0^T \left\| \langle x\rangle^{-1/2}\, v_1(s,\cdot)v_2(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\, ds\\
&\lesssim \int_0^T\int_2^\infty \left\|\langle r\rangle^{-1/4}\, v_1(s,r\cdot)\right\|_{L^2(S^2)}\left\|\langle r\rangle^{-1/4}\, v_2(s,r\cdot)\right\|_{L^\infty(S^2)}\, r^2 dr\, ds.
\end{aligned}
$$

If we apply Sobolev embedding on S^2 and Cauchy-Schwarz, we are done. □

We first see that the third term in the right hand side of (4.45) is bounded by

$$
\begin{aligned}
&\int_0^T \sum_{|\alpha|\le N_0+29+4M}\left\|\langle x\rangle^{-1/2}\, Z^\alpha \Box w_k(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\, ds\\
&\lesssim \int_0^T \sum_{|\alpha|\le N_0+29+4M}\left\|\langle x\rangle^{-1/2}\, Z^\alpha Q_k(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\, ds\\
&\quad+\int_0^2 \sum_{|\alpha|\le N_0+29+4M}\left\|\langle x\rangle^{-1/2}\, Z^\alpha [\Box,\eta]u(s,\cdot)\right\|_{L^1_rL^2_\omega(|x|>2)}\, ds.
\end{aligned}
\tag{4.47}
$$

By (4.1) and the fact that we are assuming the initial data (f,g) are compactly supported, the second term on the right hand side of the preceding inequality is controlled by $C_1\epsilon$. If we apply Lemma 4.3, we see that

$$
\begin{aligned}
&\sum_{|\alpha|\le N_0+29+4M}|Z^\alpha Q_k| \lesssim \Bigg(\sum_{\substack{|\alpha|\le N_0+29+4M\\|\beta|\le 1}}\left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+30+4M}|Z^\alpha u_0|\Bigg)\times\\
&\Bigg(\sum_{\substack{|\alpha|\le[(N_0+29+4M)/2]\\|\beta|\le 1}}\left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{\substack{|\alpha|\le[(N_0+29+4M)/2]\\|\beta|\le 2}}\left|Z^\alpha\partial^\beta w_{k}\right| + \sum_{|\alpha|\le N_0+31+4M}|Z^\alpha u_0|\Bigg)\\
&+\sum_{\substack{|\alpha|\le N_0+29+4M\\|\beta|\le 2}}\left|Z^\alpha\partial^\beta w_{k}\right|\times\Bigg(\sum_{\substack{|\alpha|\le[(N_0+29+4M)/2]\\|\beta|\le 1}}\left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+31+4M}|Z^\alpha u_0|\Bigg).
\end{aligned}
\tag{4.48}
$$

If we note that $[(N_0+29+4M)/2]+2 \le N_0$ and apply Lemma 4.5 and inequality (4.48), we see that the first term on the right hand side of (4.47) is controlled by

$$
\begin{aligned}
&\Bigg(\sum_{\substack{|\alpha|\le N_0+29+4M \\ |\beta|\le 1}} \left\| \langle x\rangle^{-1/4} Z^\alpha \partial^\beta w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sum_{|\alpha|\le N_0+30+4M} \left\| \langle x\rangle^{-1/4} Z^\alpha u_0 \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \Bigg) \times \\
&\Bigg(\sum_{\substack{|\alpha|\le [(N_0+29+4M)/2]+2 \\ |\beta|\le 1}} \left\| \langle x\rangle^{-1/4} Z^\alpha \partial^\beta w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sum_{\substack{|\alpha|\le [(N_0+29+4M)/2]+2 \\ |\beta|\le 2}} \left\| \langle x\rangle^{-1/4} Z^\alpha \partial^\beta w_{k} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sum_{|\alpha|\le N_0+33+4M} \left\| \langle x\rangle^{-1/4} Z^\alpha u_0 \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \Bigg) \\
&+ \sum_{\substack{|\alpha|\le N_0+29+4M \\ |\beta|\le 2}} \left\| \langle x\rangle^{-1/4} Z^\alpha \partial^\beta w_{k} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \times \\
&\Bigg(\sum_{\substack{|\alpha|\le [(N_0+29+4M)/2]+2 \\ |\beta|\le 1}} \left\| \langle x\rangle^{-1/4} Z^\alpha \partial^\beta w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sum_{|\alpha|\le N_0+33+4M} \left\| \langle x\rangle^{-1/4} Z^\alpha u_0 \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \Bigg) \\
&\le C\langle T\rangle^{1/2} \epsilon M_k(T) + C\langle T\rangle^{1/2}\epsilon^2.
\end{aligned}
\tag{4.49}
$$

Thus, it follows from (4.45)-(4.49) that

$$
\begin{aligned}
&\sum_{|\alpha|\le N_0+29+4M} \left[\sup_{0\le t\le T} \|Z^\alpha w_k(t,\cdot)\|_2 + \left\| \langle x\rangle^{-1/4} Z^\alpha w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right] \\
&\qquad \le C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon.
\end{aligned}
\tag{4.50}
$$

We should note that reason we dealt with the first term in the right hand side of (4.44) on its own is because the argument we just used will not work for the second term in the right hand side of (4.44). If one were to try to use the above argument to bound the case where

$|\alpha| = N_0 + 30 + 4M$, one of the resulting terms would be

$$\sum_{\substack{|\alpha|\le N_0+30+4M \\ |\beta|\le 2}} \left\| \langle x\rangle^{-1/4} Z^\alpha \partial^\beta w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}, \tag{4.51}$$

which just misses being bounded by term $II_k(T)$ by one derivative. The way we will avoid having quantities such as (4.51) appearing in the right hand side of our estimates will be to bound the term where all the vector fields Z^α are being applied to u_k using the divergence form estimate from Section 2.6. The remaining terms will be bounded by using the same argument used to prove (4.50). This issue is also present in the existence argument given by Du-Zhou [**11**], although it is not addressed in their paper. The bound we are about to prove repairs their argument as well as shows that their lifespan bound applies to a larger class of geometries.

To get started on the case where $|\alpha| = N_0 + 30 + 4M$, it is easier if we work with a fixed index α. We will consider the cases: $|x| < 3$ and $|x| > 2$. In the first case, we observe that

$$\begin{aligned}
&\sum_{|\alpha|=N_0+30+4M} \left[\sup_{0\le t\le T} \|Z^\alpha w_k(t,\cdot)\|_{L^2(|x|<3)} + \left\| \langle x\rangle^{-1/4} Z^\alpha w_k \right\|_{L^2([0,T]\times\{|x|<3\})} \right] \\
&\lesssim \sum_{|\alpha|\le N_0+29+4M} \left[\sup_{0\le t\le T} \left\|\partial^\alpha w_k'(t,\cdot)\right\|_{L^2(|x|<3)} + \left\| \langle x\rangle^{-1/4} \partial^\alpha w_k' \right\|_{L^2([0,T]\times\{|x|<3\})} \right] \\
&\lesssim I_k(T),
\end{aligned} \tag{4.52}$$

which is controlled by the right hand side of (4.30).

In the case that $|x| > 2$, we split the forcing term $\Box w_k$ into different pieces and apply different estimates to each piece. First we fix a cutoff $\rho \in C^\infty(\mathbb{R}^3)$ such that $\rho(x) = 0$ when $|x| < 1$ and $\rho(x) = 1$ when $|x| > 2$. If we let $v_k = \rho w_k$, it follows that v_k solves the following boundaryless wave equation

$$\begin{aligned}
\Box_{c_I} v_k^I = \rho(1-\eta)A^I(u_{k-1}, u_{k-1}') + \rho(1-\eta) \sum_{1\le J\le D} \sum_{i,j=0}^{3} B^{ij,IJ}(u_{k-1}, u_{k-1}')\partial_i\partial_j u_k^J \\
- \rho[\Box, \eta]u^I - 2\nabla_x\rho\cdot\nabla_x w_k^I - (\Delta\rho) w_k^I,
\end{aligned} \tag{4.53}$$

with vanishing initial data. We see that $Z^\alpha v_k^I$ solves the boundaryless wave equation

$$\begin{aligned}\Box_{c_I}(Z^\alpha v_k^I) &= R^I_{\alpha,k} \\ &+ \sum_{J=1}^{D}\sum_{i=0}^{3}\sum_{j=0}^{3}\partial_j\left(\rho(1-\eta)B^{ij,IJ}(u_{k-1},u'_{k-1})\partial_i Z^\alpha u_k^J\right) \\ &+ Z^\alpha[-2\nabla_x\rho\cdot\nabla_x w_k^I-(\Delta\rho)w_k^I],\end{aligned} \tag{4.54}$$

with vanishing initial data. In the above equation, one should view $R^I_{\alpha,k}$ as a remainder term consisting of quantities that are easier to control. One can see that $R^I_{\alpha,k}$ is a linear combination of terms of the form

$$\begin{aligned}&\sum_{J=1}^{D}\sum_{i,j=0}^{3} Z^\beta\left(\rho(1-\eta)B^{ij,IJ}(u_{k-1},u'_{k-1})\right)Z^\gamma\partial_i\partial_j u_k^J, \quad |\beta|+|\gamma|=|\alpha|, |\gamma|<|\alpha|, \\ &Z^\alpha\left(\rho(1-\eta)A^I(u_{k-1},u'_{k-1})\right), \\ &\sum_{J=1}^{D}\sum_{i,j=0}^{3}\partial_j\left(\rho(1-\eta)B^{ij,IJ}(u_{k-1},u'_{k-1})\right)Z^\alpha\partial_i u_k^J, \\ &Z^\alpha(\rho[\Box_{c_I},\eta]u^I).\end{aligned} \tag{4.55}$$

We write $Z^\alpha v_k = v_{1,k}+v_{2,k}+v_{3,k}$, where $v_{1,k}$ solves the boundaryless wave equation $\Box v_{1,k} = R_{\alpha,k}$ with vanishing initial data and $v_{3,k}$ solves $\Box v_{3,k} = Z^\alpha[-2\nabla_x\rho\cdot\nabla_x w_k-(\Delta\rho)w_k]$ with vanishing initial data. To handle, $v_{1,k}$, from (4.55), we see that

(4.56)

$$
\begin{aligned}
|R_{\alpha,k}| \lesssim & \left(\sum_{\substack{|\beta|\le N_0+30+4M\\ |\gamma|\le 1}} \left| Z^\beta \partial^\gamma w_{k-1}\right| + \sum_{|\beta|\le N_0+31+4M} \left| Z^\beta u_0 \right| \right) \times \\
& \left(\sum_{\substack{|\beta|\le [(N_0+30+4M)/2]\\ |\gamma|\le 1}} \left| Z^\beta \partial^\gamma w_{k-1}\right| + \sum_{|\beta|\le [(N_0+30+4M)/2]} \left| Z^\beta w_k'' \right| + \sum_{|\beta|\le N_0+32+4M} \left| Z^\beta u_0 \right| \right) \\
& + \sum_{\substack{|\beta|\le N_0+30+4M\\ |\gamma|\le 1}} \left| Z^\beta \partial^\gamma w_k \right| \times \\
& \left(\sum_{\substack{|\beta|\le [(N_0+30+4M)/2]\\ |\gamma|\le 2}} \left| Z^\beta \partial^\gamma w_{k-1}\right| + \sum_{|\beta|\le N_0+31+4M} \left| Z^\beta u_0 \right| \right) \\
& + \sum_{|\beta|\le N_0+30+4M} \left| Z^\beta [\Box, \eta] u \right|.
\end{aligned}
$$

One then applies Corollary 2.11 to $v_{1,k}$. By (4.1), the term involving $[\Box,\eta]u$ in the right hand side of (4.56) is controlled by

$$
\begin{aligned}
& \int_0^T \sum_{|\beta|\le N_0+30+4M} \left\| Z^\beta [\Box,\eta] u(s,\cdot) \right\|_{L^2(|x|<2)} \, ds \\
(4.57) \qquad & + \int_0^T \sum_{|\beta|\le N_0+30+4M} \left\| \langle x\rangle^{-1/2} Z^\beta [\Box,\eta] u(s,\cdot) \right\|_{L^1_r L^2_\omega(|x|>2)} \, ds \le C_1 \epsilon
\end{aligned}
$$

since the initial data are compactly supported. By previous arguments made in (4.45)-(4.50), the remaining terms in the right hand side of (4.56) are controlled by the right hand side of (4.50).

Observe that for any fixed $1 \le I, J \le D$ and $0 \le i, j \le 3$, there are constants $C^{ij,IJK}, B_\ell^{ij,IJK}$ such that

$$
(4.58) \qquad B^{ij,IJ}(u_{k-1}, u'_{k-1}) \partial_i Z^\alpha u_k^J = \sum_{K=1}^{D} (C^{ij,IJK} u_{k-1}^K + \sum_{\ell=0}^{3} B_\ell^{ij,IJK} \partial_\ell u_{k-1}^K) \partial_i Z^\alpha u_k^J.
$$

Consider the following boundaryless wave equations for fixed i, ℓ, I, J, K,

$$\begin{cases} \Box_{c_I} v_{2,k}^{i,IJK}(t,x) = \sum_{j=0}^{3} C^{ij,IJK}\partial_j(\rho(1-\eta)u_{k-1}^K\partial_i Z^\alpha u_k^J), & (t,x) \in [0,T]\times\mathbb{R}^3, \\ v_{2,k}^{i,IJK}(0,x) = \partial_t v_{2,k}^{i,IJK}(0,x) = 0, \end{cases} \tag{4.59}$$

and

$$\begin{cases} \Box_{c_I} v_{2,k}^{i,IJK,\ell}(t,x) = \sum_{j=0}^{3} B_\ell^{ij,IJK}\partial_j(\rho(1-\eta)\partial_\ell u_{k-1}^K\partial_i Z^\alpha u_k^J), & (t,x) \in [0,T]\times\mathbb{R}^3, \\ v_{2,k}^{i,IJK,\ell}(0,x) = \partial_t v_{2,k}^{i,IJK,\ell}(0,x) = 0. \end{cases} \tag{4.60}$$

Observe that

$$v_{2,k}^I = \sum_{J,K=1}^{D}\sum_{i=0}^{3}\left(v_{2,k}^{i,IJK} + \sum_{\ell=0}^{3} v_{2,k}^{i,IJK,\ell}\right). \tag{4.61}$$

We will deal with the terms of the form $v_{2,k}^{i,IJK}$ in (4.61). The remaining terms can be bounded similarly. If we apply Theorem 2.18, we get

(4.62)

$$\begin{aligned} \sup_{0\le t\le T}\left\|v_{2,k}^{i,IJK}(t,\cdot)\right\|_2 + \left\|v_{2,k}^{i,IJK}\right\|_{L^2([0,T]\times\mathbb{R}^3)} &\lesssim \int_0^T \left\|\Box_{c_I} v_{2,k}^{i,IJK}(s,\cdot)\right\|_2\, ds \\ &\lesssim \int_0^T \|u_{k-1}(s,\cdot)\|_\infty \sum_{|\beta|\le N_0+30+4M}\left\|Z^\beta u_k'(s,\cdot)\right\|_2\, ds \\ &\lesssim \log(2+T)\sup_{0\le s\le T}\langle s\rangle\|u_{k-1}(s,\cdot)\|_\infty \times \\ &\quad \sup_{0\le s\le T}\sum_{|\beta|\le N_0+30+4M}\left\|Z^\beta u_k'(s,\cdot)\right\|_2 \\ &\lesssim \log(2+T)\epsilon M_k(T) + C_1\epsilon. \end{aligned}$$

It should be noted that one can carry out these arguments even when the $B^{ij,IJ}$ are replaced by arbitrary smooth functions. This more general case can be dealt with by writing

$$B^{ij,IJ} = B_{\text{lin}}^{ij,IJ} + O(|u_{k-1}, u_{k-1}'|^2),$$

where $B_{\text{lin}}^{ij,IJ}$ is the linear component of $B^{ij,IJ}$ in its Taylor expansion. The linear term can dealt with using the argument carried out above. The remainder term can be bounded using Corollary 2.11 and the observation that

$$\int_0^T \left\||x|^{-1/2}u(s,\cdot)v(s,\cdot)w(s,\cdot)\right\|_{L_r^1L_\omega^2(|x|>2)}\, ds \lesssim \int_0^T \|u(s,\cdot)v(s,\cdot)\|_2\, ds \sup_{0\le s\le T}\|w(s,\cdot)\|_2$$

$$\lesssim \log(2+T) \sup_{0\le s\le T} \langle s\rangle \|u(s,\cdot)\|_\infty$$
$$\times \sup_{0\le s\le T} \|v(s,\cdot)\|_2 \sup_{0\le s\le T} \|w(s,\cdot)\|_2 .$$

Since all of the norms in the right hand side correspond to terms that are easier to bound in our norm $M_k(T)$. We are done with bounding the remainder term.

Thus, it remains to control

$$\sup_{0\le t\le T} \|v_{3,k}(t,\cdot)\|_{L^2(|x|>2)} + \langle T\rangle^{-1/4} \|v_{3,k}\|_{L^2([0,T]\times\{|x|>2\})} . \tag{4.63}$$

By Lemma 2.12, it follows that (4.63) is controlled by

$$\begin{aligned} &\sum_{|\beta|\le N_0+30+4M} \left(\left\|\partial^\beta w_k'\right\|_{L^2([0,T]\times\{|x|<3\})} + \left\|\partial^\beta w_k\right\|_{L^2([0,T]\times\{|x|<3\})} \right) \\ &\lesssim \sum_{|\beta|\le N_0+30+4M} \left\|\partial^\beta w_k'\right\|_{L^2([0,T]\times\{|x|<3\})} , \end{aligned} \tag{4.64}$$

which is controlled by the bounds established for term $I_k(T)$.

Thus, the arguments made in (4.52)-(4.64) show that for fixed $|\alpha| = N_0 + 30 + 4M$, the following bound holds:

$$\begin{aligned} &\sup_{0\le t\le T} \|Z^\alpha w_k(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \langle T\rangle^{-1/4} \left\|\langle x\rangle^{-1/4} Z^\alpha w_k\right\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})} \\ &\le C\langle T\rangle^{1/2} \epsilon^2 + C\langle T\rangle^{1/2} \epsilon M_k(T) + C_1\epsilon. \end{aligned} \tag{4.65}$$

Therefore, it follows from (4.41), (4.50) and (4.65) that for $T \le T_\epsilon$ and ϵ sufficiently small, we have the bound

$$II_k(T) \le C\langle T\rangle^{1/2} \epsilon^2 + C\langle T\rangle^{1/2} \epsilon M_k(T) + C_1\epsilon. \tag{4.66}$$

4.2.3. Term III Bounds. To control term $III_k(T)$, we shall first bound

$$\sum_{\substack{|\alpha|+\mu\le N_0+26+3M\\ \mu\le 1}} \sup_{0\le t\le T} \left\|L^\mu \partial^\alpha w_k'(t,\cdot)\right\|_2 . \tag{4.67}$$

Since (4.1) implies that

$$\sum_{\substack{|\alpha|+\mu\le N_0+26+3M\\ \mu\le 1}} \sup_{0\le t\le T} \left\|L^\mu \partial^\alpha u_0'(t,\cdot)\right\|_2 \le C_1\epsilon \tag{4.68}$$

and $u_k = w_k + u_0$, it follows that it suffices to bound (4.67) with w_k replaced by u_k. To do this, a few preliminary calculations are needed, which we will state in the following lemma.

LEMMA 4.6. *Suppose that* $v_1, v_2 \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$. *Then it follows that for* $0 \le t \le T$,

$$\|v_1(t,\cdot)v_2(t,\cdot)\|_2 \lesssim \left\|\langle x\rangle^{-1/4} v_1(t,\cdot)\right\|_2 \sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_2(t,\cdot)\right\|_2, \tag{4.69}$$

and

$$\|v_1 v_2\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \lesssim \left\|\langle x\rangle^{-1/4} v_1\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \sup_{0\le s\le T}\sum_{|\alpha|\le 2}\|Z^\alpha v_2(s,\cdot)\|_2. \tag{4.70}$$

PROOF. As (4.69) follows from Sobolev embedding when $|x|<2$, we shall only prove (4.69) when $|x|>1$. We see that by applying Proposition 2.14, we get

$$\begin{aligned}&\sum_{j=0}^\infty\left(\left\|\langle x\rangle^{-1/4} v_1(t,\cdot)\right\|_{L^2(2^j<|x|<2^{j+1})}\left\|\langle x\rangle^{1/4} v_2(t,\cdot)\right\|_{L^\infty(2^j<|x|<2^{j+1})}\right)\\ &\lesssim \sum_{j=0}^\infty\left(\left\|\langle x\rangle^{-1/4} v_1(t,\cdot)\right\|_{L^2(2^j<|x|<2^{j+1})}\sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_2(t,\cdot)\right\|_{L^2(2^{j-1}<|x|<2^{j+2})}\right).\end{aligned}$$

Applying Cauchy-Schwarz, we have proved (4.69). We see that (4.70) follows from (4.69) by seeing that

$$\begin{aligned}&\int_0^T \left\|\langle x\rangle^{-1/4} v_1(s,\cdot)\right\|_2^2 \sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_2(s,\cdot)\right\|_2^2\,ds\\ &\le \sup_{0\le s\le T}\sum_{|\alpha|\le 2}\|Z^\alpha v_2(s,\cdot)\|_2^2 \times \left\|\langle x\rangle^{-1/4} v_1\right\|^2_{L^2[0,T]\times\mathbb{R}^3\backslash\mathcal{K}}.\end{aligned}$$

This completes the proof. □

We should note that, due to the R^{-1} weight in the right hand side of (2.41), the above lemma also holds when the weights in the right hand sides of (4.69) and (4.70) are replaced with $\langle x\rangle^{-1/2}$. However, our restriction on the kinds of weighted L^2 norms we can use prevents us from utilizing the full decay given by Proposition 2.14.

We will now prepare to apply Theorem 3.6. Letting $\gamma^{ij,IJ}$ be as in (4.8), we see that $\sum_{\substack{\mu+j\le N_0+26+3M\\ \mu\le 1}} \tilde{L}^\mu\partial_t^j\Box_\gamma u_k$ is a linear combination of terms of the form

$$L^{\mu_1}\partial^\alpha u_{k-1}L^{\mu_2}\partial^\beta u_{k-1}, \quad |\alpha|+\mu_1\le N_0+26+3M, \mu_1\le 1, \tag{4.71}$$
$$|\beta|+\mu_2\le [(N_0+26+3M)/2]+1, \mu_2\le 1.$$

We also see that $\sum_{\substack{\mu+j\le N_0+24+3M\\ \mu\le 1}} [\tilde{L}^\mu\partial_t^j, \Box-\Box_\gamma]u_k$ is a linear combination of terms of the form

$$\begin{aligned}
&\tilde{L}^\mu\partial_t^j\partial_i\partial_j u_k\partial^\alpha u_{k-1}, && \mu+j\le N_0+25+3M, \mu\le 1,\\
& && |\alpha|\le [(N_0+26+3M)/2]+1,\\
&\tilde{L}^{\mu_1}\partial^\alpha\partial_i\partial_j u_k\tilde{L}^{\mu_2}\partial^\beta u_{k-1}, && |\alpha|+\mu_1\le [(N_0+26+3M)/2], \mu_1\le 1,\\
& && |\beta|+\mu_2\le N_0+26+3M, \mu_2\le 1\\
& && \mu_1+\mu_2\le 1.
\end{aligned}\tag{4.72}$$

From (4.71) and (4.72), it follows that

$$\begin{aligned}
&\sum_{\substack{\mu+j\le N_0+26+3M\\ \mu\le 1}} \left(\left|\tilde{L}^\mu\partial_t^j\Box_\gamma u_k\right| + \left|[\tilde{L}^\mu\partial_t^j, \Box-\Box_\gamma]u_k\right|\right)\\
&\lesssim \left(\sum_{\substack{j+\mu\le N_0+26+3M\\ \mu\le 1}} \left|\tilde{L}^\mu\partial_t^j u_k'\right| + \sum_{\substack{j+\mu\le N_0+25+3M\\ \mu\le 1}} \left|\tilde{L}^\mu\partial_t^j u_k''\right|\right)\times\\
&\qquad \sum_{|\alpha|\le[(N_0+26+3M)/2]+1} |\partial^\alpha u_{k-1}|\\
&\quad + \sum_{\substack{|\alpha|+\mu\le N_0+26+3M\\ \mu\le 1}} |L^\mu\partial^\alpha u_{k-1}| \sum_{|\alpha|\le[(N_0+26+3M)/2]} \left|\partial^\alpha u_k''\right|\\
&\quad + \sum_{|\alpha|\le N_0+26+3M} |\partial^\alpha u_{k-1}| \sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]\\ \mu\le 1}} \left|L^\mu\partial^\alpha u_k''\right|\\
&\quad + \sum_{|\alpha|\le N_0+26+3M} |\partial^\alpha u_{k-1}| \sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]\\ \mu\le 1}} \left|L^\mu\partial^\alpha u_{k-1}\right|.
\end{aligned}\tag{4.73}$$

Thus, it follows that from the fact that $[(N_0+26+3M)/2] \leq N_0$, (4.73) and Sobolev embedding that

(4.74)
$$\begin{aligned}
&\sum_{\substack{\mu+j\leq N_0+26+3M\\ \mu\leq 1}} \Big(\Big\|\tilde{L}^\mu\partial_t^j\Box_\gamma u_k(t,\cdot)\Big\|_2 + \Big\|[\tilde{L}^\mu\partial_t^j,\Box-\Box_\gamma]u_k(t,\cdot)\Big\|_2\Big)\\
&\lesssim \sum_{|\alpha|\leq[(N_0+26+3M)/2]+1} \|\partial^\alpha u_{k-1}(t,\cdot)\|_\infty \times\\
&\left(\sum_{\substack{j+\mu\leq N_0+26+3M\\ \mu\leq 1}} \Big\|\tilde{L}\partial_t^j u_k'(t,\cdot)\Big\|_2 + \sum_{\substack{j+\mu\leq N_0+25+3M\\ \mu\leq 1}} \Big\|\tilde{L}\partial_t^j u_k''(t,\cdot)\Big\|_2\right)\\
&+ \sum_{\substack{|\alpha|+\mu\leq N_0+26+3M\\ \mu\leq 1}} \|L^\mu\partial^\alpha u_{k-1}(t,\cdot)\|_2 \sum_{|\alpha|\leq[(N_0+26+3M)/2]} \big\|Z^\alpha u_k''(t,\cdot))\big\|_\infty\\
&+ \sum_{|\alpha|\leq N_0+26+3M} \Big\|\langle x\rangle^{-1/4}\,\partial^\alpha u_{k-1}(t,\cdot)\Big\|_2 \sum_{\substack{|\alpha|+\mu\leq[(N_0+26+3M)/2]+2\\ \mu\leq 1}} \Big\|\langle x\rangle^{-1/4}\,L^\mu Z^\alpha u_k''(t,\cdot)\Big\|_2\\
&+ \sum_{|\alpha|\leq N_0+26+3M} \Big\|\langle x\rangle^{-1/4}\,\partial^\alpha u_{k-1}(t,\cdot)\Big\|_2 \sum_{\substack{|\alpha|+\mu\leq[(N_0+26+3M)/2]+2\\ \mu\leq 1}} \Big\|\langle x\rangle^{-1/4}\,L^\mu Z^\alpha u_{k-1}(t,\cdot)\Big\|_2.
\end{aligned}$$

If we apply one of our elliptic regularity estimates (Proposition 3.3), we see that

$$\begin{aligned}
\sum_{\substack{j+\mu\leq N_0+25+3M\\ \mu\leq 1}} \Big\|\tilde{L}^\mu\partial_t^j u_k''(t,\cdot)\Big\|_2 &\lesssim \sum_{\substack{j+\mu\leq N_0+25+3M\\ \mu\leq 1}} \Big\|L^\mu\partial_t^j u_k''(t,\cdot)\Big\|_2\\
&+ \sum_{\substack{j+\mu\leq N_0+25+3M\\ \mu\leq 1}} \Big\|(L-\tilde{L})^\mu\partial_t^j u_k''(t,\cdot)\Big\|_2\\
&\lesssim \sum_{\substack{\mu+j\leq N_0+26+3M\\ \mu\leq 1}} \Big\|\tilde{L}^\mu\partial_t^j u_k'(t,\cdot)\Big\|_2 \qquad (4.75)\\
&+ \sum_{|\alpha|\leq N_0+26+3M} \big\|\partial^\alpha u_k'(t,\cdot)\big\|_{L^2(|x|<2)}\\
&+ \sum_{\substack{|\alpha|+\mu\leq N_0+25+3M\\ \mu\leq 1}} \|L^\mu\partial^\alpha\Box u_k(t,\cdot)\|_2.
\end{aligned}$$

To satisfy the hypotheses of Theorem 3.6, we set

$$F(t) := \sum_{|\alpha|\leq[(N_0+26+3M)/2]+1} \|\partial^\alpha u_{k-1}(t,\cdot)\|_\infty$$

By the arguments used in (4.12)-(4.14), it follows that F satisfies the bound (3.47) that is required to apply Theorem 3.6. From the induction hypothesis, we also know that F satisfies the bound

$$F(t) \le \frac{V_{k-1}(T)}{1+t} \le \frac{C\epsilon}{1+t}, \tag{4.76}$$

for $0 \le t \le T$. It follows from (4.75) that the right hand side of (4.74) is controlled by

(4.77)
$$\begin{aligned}
&F(t) \sum_{\substack{\mu+j\le N_0+26+3M \\ \mu\le 1}} \left\| \tilde{L}^\mu \partial_t^j u_k'(t,\cdot)\right\|_2 \\
&+ F(t)\left(\sum_{|\alpha|\le N_0+26+3M} \left\|\partial^\alpha u_k'(t,\cdot)\right\|_{L^2(|x|<2)} + \sum_{\substack{|\alpha|+\mu\le N_0+25+3M \\ \mu\le 1}} \|L^\mu \partial^\alpha \Box u_k(t,\cdot)\|_2 \right) \\
&+ \sum_{\substack{|\alpha|+\mu\le N_0+26+3M \\ \mu\le 1}} \|L^\mu\partial^\alpha u_{k-1}(t,\cdot)\|_2 \sum_{|\alpha|\le[(N_0+26+3M)/2]} \left\|Z^\alpha u_k''(t,\cdot)\right\|_\infty \\
&+ \sum_{|\alpha|\le N_0+26+3M} \left\|\langle x\rangle^{-1/4}\,\partial^\alpha u_{k-1}(t,\cdot)\right\|_2 \sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]+2 \\ \mu\le 1}} \left\|\langle x\rangle^{-1/4}\, L^\mu Z^\alpha u_k''(t,\cdot)\right\|_2 \\
&+ \sum_{|\alpha|\le N_0+26+3M} \left\|\langle x\rangle^{-1/4}\,\partial^\alpha u_{k-1}(t,\cdot)\right\|_2 \sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]+2 \\ \mu\le 1}} \left\|\langle x\rangle^{-1/4}\, L^\mu Z^\alpha u_{k-1}(t,\cdot)\right\|_2 .
\end{aligned}$$

We set $H_{1,N_0+25+3M}(t)$ equal to

(4.78)
$$\begin{aligned}
&F(t)\left(\sum_{|\alpha|\le N_0+26+3M} \left\|\partial^\alpha u_k'(t,\cdot)\right\|_{L^2(|x|<2)} + \sum_{\substack{|\alpha|+\mu\le N_0+25+3M \\ \mu\le 1}} \|L^\mu \partial^\alpha \Box u_k(t,\cdot)\|_2 \right) \\
&+ \sum_{\substack{|\alpha|+\mu\le N_0+26+3M \\ \mu\le 1}} \|L^\mu\partial^\alpha u_{k-1}(t,\cdot)\|_2 \sum_{|\alpha|\le[(N_0+26+3M)/2]} \left\|Z^\alpha u_k''(t,\cdot)\right\|_\infty \\
&+ \sum_{|\alpha|\le N_0+26+3M} \left\|\langle x\rangle^{-1/4}\,\partial^\alpha u_{k-1}(t,\cdot)\right\|_2 \sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]+2 \\ \mu\le 1}} \left\|\langle x\rangle^{-1/4}\, L^\mu Z^\alpha u_k''(t,\cdot)\right\|_2 \\
&+ \sum_{|\alpha|\le N_0+26+3M} \left\|\langle x\rangle^{-1/4}\,\partial^\alpha u_{k-1}(t,\cdot)\right\|_2 \sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]+2 \\ \mu\le 1}} \left\|\langle x\rangle^{-1/4}\, L^\mu Z^\alpha u_{k-1}(t,\cdot)\right\|_2 ,
\end{aligned}$$

and

$$X_{\nu,j}(t) := \int e_0(\tilde{L}^\nu \partial_t^j u_k)(t,x)\, dx, \tag{4.79}$$

where $e_0(u)$ is defined in (3.5). Thus, from Theorem 3.6, we see that for any $0 \le t \le T_\epsilon$, if ϵ is sufficiently small, then

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu \le N_0+26+3M \\ \mu \le 1}} \left\| L^\mu \partial^\alpha u_k'(t,\cdot) \right\|_2 \\
&\lesssim \sum_{\substack{|\alpha|+\mu \le N_0+25+3M \\ \mu \le 1}} \| L^\mu \partial^\alpha \Box u_k(t,\cdot)\|_2 + \sum_{\substack{\mu+j \le N_0+26+3M \\ \mu \le 1}} X_{\mu,j}^{1/2}(0) \\
&\quad + \left(\int_0^t \sum_{|\alpha| \le N_0+25+3M} \|\partial^\alpha \Box u_k(s,\cdot)\|_2 \, ds + \int_0^t H_{1,N_0+25+3M}(s)\, ds \right) \\
&\quad + \int_0^t \sum_{|\alpha| \le N_0+26+3M} \left\| \partial^\alpha u_k'(s,\cdot) \right\|_{L^2(|x|<1)} \, ds.
\end{aligned} \tag{4.80}$$

By (4.1) and (3.20), the second term in (4.80) is bounded by $C_1\epsilon$. By (4.76), we also see that the term in (4.80) that is enclosed in parentheses is controlled by

$$
\begin{aligned}
&\int_0^t \frac{C\epsilon}{1+s}\Bigg(\sum_{|\alpha|\le N_0+26+3M}\left\|\partial^\alpha u_k'(s,\cdot)\right\|_{L^2(|x|<2)}+\sum_{\substack{|\alpha|+\mu\le N_0+25+3M\\ \mu\le 1}}\|L^\mu\partial^\alpha\Box u_k(s,\cdot)\|_2\Bigg)\\
&\quad+\sum_{|\alpha|\le N_0+25+3M}\|\partial^\alpha\Box u_k(s,\cdot)\|_2\\
&\quad+\sum_{\substack{|\alpha|+\mu\le N_0+26+3M\\ \mu\le 1}}\|L^\mu\partial^\alpha u_{k-1}(s,\cdot)\|_2\times\\
&\qquad\sum_{|\alpha|\le[(N_0+26+3M)/2]}\left\|Z^\alpha u_k''(s,\cdot)\right\|_\infty\\
&\quad+\sum_{|\alpha|\le N_0+26+3M}\left\|\langle x\rangle^{-1/4}\partial^\alpha u_{k-1}(s,\cdot)\right\|_2\times\\
&\qquad\sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]+2\\ \mu\le 1}}\left\|\langle x\rangle^{-1/4}L^\mu Z^\alpha u_k''(s,\cdot)\right\|_2\\
&\quad+\sum_{|\alpha|\le N_0+26+3M}\left\|\langle x\rangle^{-1/4}\partial^\alpha u_{k-1}(s,\cdot)\right\|_2\times\\
&\qquad\sum_{\substack{|\alpha|+\mu\le[(N_0+26+3M)/2]+2\\ \mu\le 1}}\left\|\langle x\rangle^{-1/4}L^\mu Z^\alpha u_{k-1}(s,\cdot)\right\|_2\, ds,
\end{aligned}
\tag{4.81}
$$

which is bounded by

$$
\begin{aligned}
&C\epsilon\log(2+T)\times\\
&\Bigg(\sup_{0\le s\le T}\sum_{|\alpha|\le N_0+26+3M}\left\|\partial^\alpha u_k'(s,\cdot)\right\|_{L^2(|x|<2)}+\sup_{0\le s\le T}\sum_{\substack{|\alpha|+\mu\le N_0+25+3M\\ \mu\le 1}}\|L^\mu\partial^\alpha\Box u_k(s,\cdot)\|_2\Bigg)\\
&+\langle T\rangle^{1/2}C\epsilon M_k(T)+C\langle T\rangle^{1/2}\epsilon^2+C_1\epsilon.
\end{aligned}
\tag{4.82}
$$

If we apply Lemma 4.3, we see that

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N_0+25+3M\\ \mu\le 1}} |L^\mu \partial^\alpha \Box u_k| \\
&\lesssim \left(\sum_{\substack{|\alpha|+\mu\le N_0+25+3M\\ \mu\le 1}} \left|L^\mu \partial^\alpha w'_{k-1}\right| + \sum_{\mu\le 1} |L^\mu w_{k-1}| + \sum_{|\alpha|\le N_0+26+3M} |\partial^\alpha u_0| \right) \times \\
&\quad \left(\sum_{|\alpha|\le N_0+26+3M} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+25+3M} \left|\partial^\alpha w''_k\right| + \sum_{|\alpha|\le N_0+27+3M} |\partial^\alpha u_0| \right) \\
&\quad + \left(\sum_{|\alpha|\le N_0+26+3M} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+26+3M} |\partial^\alpha u_0| \right) \times \\
&\quad \left(\sum_{\substack{|\alpha|+\mu\le N_0+26+3M\\ \mu\le 1}} |L^\mu \partial^\alpha w_{k-1}| + \sum_{\substack{|\alpha|+\mu\le N_0+25+3M\\ \mu\le 1}} \left|L^\mu \partial^\alpha w''_k\right| + \sum_{|\alpha|\le N_0+27+3M} |\partial^\alpha u_0| \right).
\end{aligned} \tag{4.83}
$$

Using the induction hypothesis (4.5) and Sobolev embedding, the above inequality implies that the term in parentheses in (4.82) is controlled by $C\log(2+T)\epsilon^2 + C\log(2+T)\epsilon M_k(T) + C_1\epsilon$. This handles the term in parentheses in inequality (4.80). If one also applies (4.83) to the first term in the right hand side of (4.80), it follows that this term is controlled by $C\epsilon^2 + C\epsilon M_k(T) + C_1\epsilon$.

Thus, it remains to consider the last term on the right hand side of (4.80). By Theorem 3.8 and (4.1), it follows that the last term in the right hand side of (4.80) is controlled by

$$
\begin{aligned}
&\sum_{|\alpha|\le N_0+26+4M} \int_0^t \left(\int_0^s \|\partial^\alpha \Box w_k(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\ d\tau \right)\ ds \\
&+ \int_0^t \sum_{|\alpha|\le N_0+26+4M} \|\partial^\alpha \Box w_k(s,\cdot)\|_{L^2(|x|<4)}\ ds + C_1\epsilon.
\end{aligned} \tag{4.84}
$$

It can easily be seen that the second term in the right hand side of this inequality satisfies the same bounds as $I_k(T)$, which means that it is controlled by the right hand side of (4.30). To deal with the first term in (4.84), will prove the following lemma.

Lemma 4.7. *Suppose that $v_1, v_2 \in C^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$. Then it follows that for $0\le t\le T$,*

$$\int_0^t\int_0^s \|v_1(\tau,\cdot)v_2(\tau,\cdot)\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\,ds \sum_{|\alpha|\le 2} \lesssim \left\|\langle x\rangle^{-1/4} Z^\alpha v_1\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\times \left\|\langle x\rangle^{-1/4} v_2\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}. \tag{4.85}$$

Proof. From Proposition 2.14, we see that the left hand side of (4.85) is controlled by

$$\begin{aligned}
&\int_0^t\int_0^s\sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_1(\tau,\cdot)\right\|_{L^2(||x|-(s-\tau)|<20)}\left\|\langle x\rangle^{-1/4} v_2(\tau,\cdot)\right\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\,ds\\
&\quad\lesssim \sum_{k=0}^{[t]}\int_k^{k+1}\int_0^s\sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_1(\tau,\cdot)\right\|_{L^2(||x|-(s-\tau)|<20)}\times\\
&\qquad\left\|\langle x\rangle^{-1/4} v_2(\tau,\cdot)\right\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\,ds\\
&\quad+\int_{[t]}^t\int_0^s\sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_1(\tau,\cdot)\right\|_{L^2(||x|-(s-\tau)|<20)}\times\\
&\qquad\left\|\langle x\rangle^{-1/4} v_2(\tau,\cdot)\right\|_{L^2(||x|-(s-\tau)|<10)}\,d\tau\,ds.
\end{aligned}\tag{4.86}$$

Just as in the proof to Theorem 3.13, we note that the sets $C_s=\{(\tau,x): 0\le\tau\le s, ||x|-(s-\tau)|<21\}$ have the property that $C_{j_1}\cap C_{j_2}$ is empty if $|j_1-j_2|>50$. Applying Cauchy-Schwarz and the aforementioned observation, we see that the right hand side of (4.86) is controlled by

$$\begin{aligned}
&\sum_{|\alpha|\le 2}\Bigg(\sum_{k=0}^{[t]}\left\|\langle x\rangle^{-1/4} Z^\alpha v_1\right\|^2_{L^2([0,k+1]\times\{||x|-((k+1)-\tau)|<21\})}\\
&\qquad+\left\|\langle x\rangle^{-1/4} Z^\alpha v_1\right\|^2_{L^2([0,t]\times\{||x|-(t-\tau)|<21\})}\Bigg)^{1/2}\times\\
&\qquad\Bigg(\sum_{k=0}^{[t]}\left\|\langle x\rangle^{-1/4} v_2\right\|^2_{L^2([0,k+1]\times\{||x|-((k+1)-\tau)|<11\})}\\
&\qquad+\left\|\langle x\rangle^{-1/4} v_2\right\|^2_{L^2([0,t]\times\{||x|-(t-\tau)|<11\})}\Bigg)^{1/2}\\
&\qquad\lesssim\sum_{|\alpha|\le 2}\left\|\langle x\rangle^{-1/4} Z^\alpha v_1\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\left\|\langle x\rangle^{-1/4} v_2\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}.
\end{aligned}\tag{4.87}$$

This completes the proof. $\square$

Due to the fact that

$$\sum_{|\alpha|\le N_0+26+4M} |\partial^\alpha \Box w_k| \lesssim \left(\sum_{|\alpha|\le N_0+27+4M} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+28+4M} |\partial^\alpha u_0| \right) \times$$

$$\left(\sum_{|\alpha|\le N_0+27+4M} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+26+4M} \left|\partial^\alpha w_k''\right| + \sum_{|\alpha|\le N_0+28+4M} |\partial^\alpha u_0| \right) \tag{4.88}$$

$$+ \sum_{|\alpha|\le N_0+26+4M} |\partial^\alpha [\Box, \eta] u| \, ,$$

if we apply Lemma 4.7, then the first term in (4.84) is controlled by

(4.89)

$$\left(\sum_{|\alpha|\le N_0+27+4M} \left\| \langle x\rangle^{-1/4} \partial^\alpha w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+28+4M} \|\partial^\alpha u_0(s,\cdot)\|_2 \right) \times$$

$$\left(\sum_{|\alpha|\le [N_0+29+4M} \left\| \langle x\rangle^{-1/4} Z^\alpha w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|\le N_0+28+4M} \left\| \langle x\rangle^{-1/4} Z^\alpha w_k'' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right.$$

$$\left. + \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+30+4M} \|\partial^\alpha u_0(s,\cdot)\|_2 \right) + \sum_{|\alpha|\le N_0+26+4M} \|\partial^\alpha [\Box,\eta] u(s,\cdot)\|_2$$

$$\le C\epsilon \langle T\rangle^{1/2} M_k(T) + C \langle T\rangle^{1/2} \epsilon^2 + C_1\epsilon.$$

Hence, for $0 \le t \le T_\epsilon$ and ϵ sufficiently small, it follows from (4.75)-(4.89) that

$$\sup_{0\le t\le T} \sum_{\substack{|\alpha|+\mu\le N_0+26+3M \\ \mu\le 1}} \left\| L^\mu \partial^\alpha w_k'(t,\cdot) \right\|_2 \le C \langle T\rangle^{1/2} \epsilon^2 + C\epsilon \langle T\rangle^{1/2} M_k(T) + C_1\epsilon. \tag{4.90}$$

To handle the second term in $III_k(T)$, we apply Theorem 3.10 to see that

$$\sum_{\substack{|\alpha|+\mu\le N_0+21+2M \\ \mu\le 1}} \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} L^\mu \partial^\alpha w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}$$

$$\lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+21+3M \\ \mu\le 1}} \|L^\mu \partial^\alpha \Box w_k(s,\cdot)\|_2 \; ds \tag{4.91}$$

$$+ \sum_{\substack{|\alpha|+\mu\le N_0+21+2M \\ \mu\le 1}} \|L^\mu \partial^\alpha \Box w_k\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \, .$$

Applying Lemma 4.3, we see that

(4.92)

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N_0+21+3M\\ \mu\le 1}} |L^\mu \partial^\alpha \Box w_k| \\
&\lesssim \left(\sum_{\substack{|\alpha|\le N_0+21+3M\\ \mu\le 1}} \left|L^\mu \partial^\alpha w'_{k-1}\right| + \sum_{\mu\le 1} |L^\mu w_{k-1}| + \sum_{|\alpha|\le N_0+22+3M} |\partial^\alpha u_0| \right) \times \\
&\left(\sum_{|\alpha|\le[(N_0+21+3M)/2]+1} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\le[(N_0+21+3M)/2]} \left|\partial^\alpha w''_k\right| + \sum_{|\alpha|\le[(N_0+21+3M)/2]+2} |\partial^\alpha u_0| \right) \\
&+ \left(\sum_{|\alpha|\le N_0+21+3M} \left|\partial^\alpha w'_{k-1}\right| + |w_{k-1}| + \sum_{|\alpha|\le N_0+22+3M} |\partial^\alpha u_0| \right) \times \\
&\Bigg(\sum_{\substack{|\alpha|+\mu\le[(N_0+21+3M)/2]+1\\ \mu\le 1}} |L^\mu \partial^\alpha w_{k-1}| + \sum_{\substack{|\alpha|+\mu\le[(N_0+21+3M)/2]\\ \mu\le 1}} \left|L^\mu \partial^\alpha w''_k\right| \\
&\quad + \sum_{|\alpha|\le[(N_0+21+3M)/2]+2} |\partial^\alpha u_0| \Bigg) \\
&+ \sum_{|\alpha|\le N_0+21+3M} \left|\partial^\alpha w''_k\right| \left(\sum_{\substack{|\alpha|+\mu\le[(N_0+21+3M)/2]+1\\ \mu\le 1}} |L^\mu \partial^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+23+3M} |\partial^\alpha u_0| \right) \\
&+ \sum_{\substack{|\alpha|+\mu\le N_0+21+3M\\ \mu\le 1}} \left|L^\mu \partial^\alpha w''_k\right| \times \\
&\left(\sum_{|\alpha|\le[(N_0+21+3M)/2]+1} |\partial^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+23+3M} |\partial^\alpha u_0| \right) \\
&+ \sum_{|\alpha|\le N_0+21+3M} |\partial^\alpha [\Box, \eta] u| \,.
\end{aligned}
$$

Observing that $[(N_0+21+3M)/2]+1 \le N_0$, if we apply (4.92) and (4.1), then we see that first term on the right hand side of (4.91) is controlled by

$$
\begin{aligned}
&\Bigg(\sum_{\substack{|\alpha|+\mu\le N_0+21+3M \\ \mu\le 1}} \sup_{0\le s\le T} \left\| L^\mu \partial^\alpha w'_{k-1}(s,\cdot)\right\|_2 + \sum_{\mu\le 1} \sup_{0\le s\le T} \|L^\mu w_{k-1}(s,\cdot)\|_2 \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+22+3M} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg) \times \\
&\log(2+T) \Bigg(\sum_{|\alpha|+\mu\le [(N_0+21+3M)/2]+3} \sup_{0\le s\le T} \langle s\rangle \|Z^\alpha w_{k-1}(s,\cdot)\|_\infty \\
&+ \sum_{|\alpha|\le [(N_0+21+3M)/2]+2} \sup_{0\le s\le T} \langle s\rangle \left\|Z^\alpha w''_k(s,\cdot)\right\|_\infty \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le [(N_0+21+3M)/2]+4} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg) \\
&+ \Bigg(\sum_{|\alpha|\le N_0+21+3M} \left\| \langle x\rangle^{-1/4} \partial^\alpha w'_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \left\| \langle x\rangle^{-1/4} w_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+22+3M} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg) \times \qquad (4.93) \\
&\Bigg(\sum_{\substack{|\alpha|+\mu\le [(N_0+21+3M)/2]+3 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\alpha w_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sum_{\substack{|\alpha|+\mu\le [(N_0+21+3M)/2]+2 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\alpha w''_k\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le [(N_0+21+3M)/2]+4} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg) \\
&+ \sum_{|\alpha|\le N_0+21+3M} \left\| \langle x\rangle^{-1/4} \partial^\alpha w''_k\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \times \\
&\Bigg(\sum_{\substack{|\alpha|+\mu\le [(N_0+21+3M)/2]+3 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\alpha w_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+25+3M} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg)
\end{aligned}
$$

$$+\log(2+T)\sum_{\substack{|\alpha|+\mu\le N_0+21+3M\\ \mu\le 1}}\sup_{0\le t\le T}\left\|L^\mu\partial^\alpha w_k''(t,\cdot)\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\times$$
$$\left(\sum_{|\alpha|\le[(N_0+21+3M)/2]+1}\sup_{0\le t\le T}\langle s\rangle\left\|Z^\alpha w_{k-1}(t,\cdot)\right\|_\infty+\sup_{0\le s\le T}\sum_{|\alpha|\le N_0+25+3M}\left\|\partial^\alpha u_0(s,\cdot)\right\|_2\right)$$
$$+\sup_{0\le s\le T}\sum_{|\alpha|\le N_0+21+3M}\left\|\partial^\alpha[\Box,\eta]u(s,\cdot)\right\|_2$$
$$\le C\epsilon\langle T\rangle^{1/2}M_k(T)+C\epsilon^2\langle T\rangle^{1/2}+C_1\epsilon.$$

To control the second term in (4.91), we note that

$$\sum_{\substack{|\alpha|+\mu\le N_0+21+2M\\ \mu\le 1}}\left|L^\mu\partial^\alpha\Box w_k\right|$$
$$\lesssim\left(\sum_{\substack{|\alpha|\le N_0+21+2M\\ \mu\le 1}}\left|L^\mu\partial^\alpha w_{k-1}'\right|+\sum_{\mu\le 1}\left|L^\mu w_{k-1}\right|+\sum_{|\alpha|\le N_0+22+2M}\left|\partial^\alpha u_0\right|\right)\times$$
$$\left(\sum_{\substack{|\alpha|+\mu\le[(N_0+21+2M)/2]+1\\ \mu\le 1}}\left|L^\mu\partial^\alpha w_{k-1}\right|+\sum_{\substack{|\alpha|+\mu\le[(N_0+21+2M)/2]\\ \mu\le 1}}\left|L^\mu\partial^\alpha w_k''\right|\right.$$
$$\left.+\sum_{|\alpha|\le[(N_0+21+2M)/2]+2}\left|\partial^\alpha u_0\right|\right)\tag{4.94}$$
$$+\sum_{|\alpha|\le N_0+21+2M}\left|\partial^\alpha w_k''\right|\left(\sum_{\substack{|\alpha|+\mu\le[(N_0+21+2M)/2]+1\\ \mu\le 1}}\left|L^\mu\partial^\alpha w_{k-1}\right|+\sum_{|\alpha|\le N_0+23+2M}\left|\partial^\alpha u_0\right|\right)$$
$$+\sum_{\substack{|\alpha|+\mu\le N_0+21+2M\\ \mu\le 1}}\left|L^\mu\partial^\alpha w_k''\right|\left(\sum_{|\alpha|\le[(N_0+21+2M)/2]+1}\left|\partial^\alpha w_{k-1}\right|+\sum_{|\alpha|\le N_0+23+2M}\left|\partial^\alpha u_0\right|\right)$$
$$+\sum_{|\alpha|\le N_0+21+2M}\left|\partial^\alpha[\Box,\eta]u\right|.$$

Applying Lemma 4.6, (4.94) and (4.1), we also see that the second term on the right hand side of (4.91) is controlled by

(4.95)

$$\begin{aligned}
&\Bigg(\sum_{\substack{|\alpha|+\mu\le N_0+21+2M\\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu \partial^\alpha w_{k-1}' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \sum_{\mu\le 1} \left\| \langle x\rangle^{-1/4} L^\mu w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+22+2M} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg) \times \\
&\Bigg(\sup_{0\le s\le T} \sum_{\substack{|\alpha|+\mu\le [(N_0+21+2M)/2]+3\\ \mu\le 1}} \|L^\mu Z^\alpha w_{k-1}(s,\cdot)\|_2 \\
&+ \sup_{0\le s\le T} \sum_{\substack{|\alpha|+\mu\le [(N_0+21+2M)/2]+2\\ \mu\le 1}} \left\| L^\mu Z^\alpha w_k''(s,\cdot)\right\|_2 \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le [(N_0+21+2M)/2]+4} \|\partial^\alpha u_0(s,\cdot)\|_2 \Bigg) \\
&+ \sum_{\substack{|\alpha|+\mu\le N_0+21+2M\\ \mu\le 1}} \sup_{0\le t\le T} \left\| L^\mu \partial^\alpha w_k''(t,\cdot)\right\|_2 \\
&\times \Bigg(\sum_{|\alpha|\le [(N_0+21+2M)/2]+3} \left\| \langle x\rangle^{-1/4} Z^\alpha w_{k-1}\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \sup_{0\le t\le T} \sum_{|\alpha|\le N_0+25+2M} \|\partial^\alpha u_0(t,\cdot)\|_2 \Bigg) \\
&+ \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+23+2M} \|\partial^\alpha [\Box,\eta] u(s,\cdot)\|_2 \\
&+ \sum_{|\alpha|\le N_0+20+2M} \left\| \langle x\rangle^{-1/4} \partial^\alpha w_k''\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\times \Bigg(\sum_{|\alpha|+\mu\le [(N_0+21+2M)/2]+3} \sup_{0\le t\le T} \|L^\mu Z^\alpha w_{k-1}(t,\cdot)\|_2 + \sup_{0\le t\le T} \sum_{|\alpha|\le N_0+25+2M} \|\partial^\alpha u_0(t,\cdot)\|_2 \Bigg) \\
&\le C\epsilon \langle T\rangle^{1/2} M_k(T) + C\epsilon^2 \langle T\rangle^{1/2} + C_1\epsilon.
\end{aligned}$$

Therefore, from (4.90)-(4.95), for $T \le T_\epsilon$ and for ϵ sufficiently small, we see that

$$III_k(T) \le C\langle T\rangle^{1/2} \epsilon^2 + C\epsilon \langle T\rangle^{1/2} M_k(T) + C_1\epsilon. \tag{4.96}$$

4.2.4. Term IV Bounds. To control term $IV_k(T)$, we shall apply an argument that is almost identical to the one used to control $II_k(T)$. We shall again first consider the case where $|\beta| = 1, 2$ for the first term in $IV_k(T)$. By (3.20), it follows that for ϵ sufficiently small and $0 \le t \le T_\epsilon$, we get the following inequality:

$$(5 \max_I \{c_I^2, c_I^{-2}\})^{-1} Y_{N,1,1}^{1/2}(t) \le \sum_{\substack{|\alpha|+\mu \le N \\ |\beta| \le 1 \\ \mu \le 1}} \left\| L^\mu Z^\alpha \partial^\beta u_k'(t, \cdot) \right\|_2 \le 5 \max_I \{c_I^2, c_I^{-2}\} Y_{N,1,1}^{1/2}(t). \tag{4.97}$$

By Theorem 3.9, we see that

$$\begin{aligned} \partial_t Y_{N_0+11+M,1,1}(t) \le{}& C Y_{N_0+11+M,1,1}^{1/2}(t) \sum_{\substack{|\alpha|+\mu \le N_0+11+M \\ |\beta| \le 1 \\ \mu \le 1}} \left\| \Box_\gamma L^\mu Z^\alpha \partial^\beta u(t, \cdot) \right\|_2 \\ &+ C \left\| \gamma'(t, \cdot) \right\|_\infty Y_{N_0+11+M,1,1}(t) + C \sum_{\substack{|\alpha|+\mu \le N_0+12+M \\ \mu \le 1}} \left\| L^\mu \partial^\alpha u'(t, \cdot) \right\|_{L^2(|x|<1)}^2 . \end{aligned} \tag{4.98}$$

Applying Gronwall's inequality, for $T \le T_\epsilon$, we get

$$\begin{aligned} \sup_{0 \le t \le T} Y_{N_0+11+M,1,1}(t) \le{}& C \int_0^T \sum_{\substack{|\alpha|+\mu \le N_0+11+M \\ |\beta| \le 1 \\ \mu \le 1}} Y_{N_0+11+M,1,0}(s)^{1/2} \left\| \Box_\gamma L^\mu Z^\alpha \partial^\beta u_k(s, \cdot) \right\|_2 \, ds \\ &+ C \sum_{\substack{|\alpha|+\mu \le N_0+12+M \\ \mu \le 1}} \left\| L^\mu \partial^\alpha u_k' \right\|_{L^2([0,T] \times \{|x|<1\})}^2 + C Y_{N_0+11+M,1,1}(0). \end{aligned} \tag{4.99}$$

The first term in the right hand side of (4.34) is controlled by

$$\frac{1}{2} \sup_{0 \le t \le T} Y_{N_0+11+M,1,1}(t) + C \left(\int_0^T \sum_{\substack{|\alpha|+\mu \le N_0+11+M \\ |\beta| \le 1 \\ \mu \le 1}} \left\| \Box_\gamma L^\mu Z^\alpha \partial^\beta u_k(s, \cdot) \right\|_2 \, ds \right)^2 . \tag{4.100}$$

The first term in (4.100) can be bootstrapped back into the left hand side of (4.99). Thus, we get

(4.101)
$$\begin{aligned}\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}}\Big\|L^\mu Z^\alpha\partial^\beta w_k'(t,\cdot)\Big\|_2 &\lesssim Y^{1/2}_{N_0+11+M,1,1}(t)+\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}}\Big\|L^\mu Z^\alpha\partial^\beta u_0'(t,\cdot)\Big\|_2\\
&\lesssim Y_{N_0+11+M,1,1}(0)^{1/2}\\
&+\int_0^T\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}}\Big\|\Box_\gamma L^\mu Z^\alpha\partial^\beta u_k(s,\cdot)\Big\|_2\,ds\\
&+\sum_{\substack{|\alpha|+\mu\le N_0+12+M\\ \mu\le 1}}\big\|L^\mu\partial^\alpha u_k'\big\|_{L^2([0,T]\times\{|x|<1\})}\\
&+\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}}\Big\|L^\mu Z^\alpha\partial^\beta u_0'(t,\cdot)\Big\|_2.\end{aligned}$$

By (4.97) and (4.1), it follows that

$$Y_{N_0+11+M,1,1}(0)^{1/2}+\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}}\Big\|L^\mu Z^\alpha\partial^\beta u_0'(t,\cdot)\Big\|_2\le C_1\epsilon. \tag{4.102}$$

If we apply Lemma 3.11 to third term on the right hand side of (4.101), we see that this term satisfies the same bounds as term $III_k(T)$.

then by (4.1) and (4.102), it follows that the right hand side of (4.101) is controlled by To control the term involving $\Box_\gamma$, we apply Lemma 4.3 to see that

(4.103)

$$\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}} \left|\Box_\gamma L^\mu Z^\alpha \partial^\beta u_k\right| \lesssim \sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 2\\ \mu\le 1}} \left|L^\mu Z^\alpha \partial^\beta w_{k-1}\right| \times\\
&\left(\sum_{\substack{|\alpha|\le[(N_0+11+M)/2]+2\\|\beta|\le 1}} \left|Z^\alpha \partial^\beta w_{k-1}\right| + \sum_{\substack{|\alpha|\le[(N_0+11+M)/2]+3\\|\beta|\le 1}} \left|Z^\alpha \partial^\beta w_k\right| + \sum_{|\alpha|\le N_0+14+M} |Z^\alpha u_0|\right)\\
&+\left(\sum_{\substack{|\alpha|\le N_0+11+M\\|\beta|\le 2}} \left|Z^\alpha \partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+13+M} |Z^\alpha u_0|\right) \times\\
&\left(\sum_{\substack{|\alpha|+\mu\le[(N_0+11+M)/2]+3\\|\beta|\le 1\\ \mu\le 1}} \left|L^\mu Z^\alpha \partial^\beta w_{k-1}\right| + \sum_{\substack{|\alpha|+\mu\le[(N_0+11+M)/2]+4\\|\beta|\le 1\\ \mu\le 1}} \left|L^\mu Z^\alpha \partial^\beta w_k\right|\right.\\
&\left.+\sum_{|\alpha|\le N_0+14+M} |Z^\alpha u_0|\right)\\
&+\sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 2\\ \mu\le 1}} \left|L^\mu Z^\alpha \partial^\beta w_k\right| \times\\
&\left(\sum_{\substack{|\alpha|\le[(N_0+11+M)/2]+2\\|\beta|\le 1}} \left|Z^\alpha \partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+13+M} |Z^\alpha u_0|\right)\\
&+\sum_{\substack{|\alpha|\le N_0+13+M\\|\beta|\le 1}} \left|Z^\alpha \partial^\beta w_k\right| \sum_{\substack{|\alpha|+\mu\le[(N_0+11+M)/2]+1\\|\beta|\le 1\\ \mu\le 1}} \left|L^\mu Z^\alpha \partial^\beta w_{k-1}\right|.
\end{aligned}$$

We will first demonstrate how to deal with the term in the sixth and seventh lines of (4.103). If we apply Lemma 4.4 and Sobolev embedding, it follows that

$$\begin{aligned}
&\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 2\\ \mu\le 1}} \left\| L^\mu Z^\alpha \partial^\beta w_k(s,\cdot)\right\|_2 \\
&\times \left(\sum_{\substack{|\alpha|\le[(N_0+11+M)/2]\\|\beta|\le 2}} \left\| Z^\alpha \partial^\beta w_{k-1}(s,\cdot)\right\|_\infty + \sum_{|\alpha|\le N_0+13+M} \| Z^\alpha u_0(s,\cdot)\|_\infty \right) ds \\
&\le C\log(2+T) \sup_{0\le s\le T} \sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 2\\ \mu\le 1}} \left\| L^\mu Z^\alpha \partial^\beta w_k(s,\cdot)\right\|_2 \times \\
&\left(\sup_{0\le s\le T} \sum_{\substack{|\alpha|\le[(N_0+11+M)/2]\\|\beta|\le 2}} \langle s\rangle \left\| Z^\alpha \partial^\beta w_{k-1}(s,\cdot)\right\|_\infty + \sup_{0\le s\le T} \sum_{|\alpha|\le N_0+15+M} \| Z^\alpha u_0(s,\cdot)\|_2 \right) \\
&\le C\log(2+T)\epsilon M_k(T).
\end{aligned} \tag{4.104}$$

Using the same argument, we see that the terms in the first and second lines of (4.103) result in quantities that are controlled by $C\log(2+T)\epsilon^2 + C\epsilon M_k(T)$. If we apply Lemma 4.6 to the last line of (4.103), we see that these terms can be controlled by $C\langle T\rangle^{1/2}\epsilon M_k(T)$. Using the same argument, it also follows that the remaining terms in the right hand side of (4.103) result in quantities that are controlled by $C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T)$. Combining these arguments with (4.101)-(4.104), we obtain the bound

$$\sup_{0\le t\le T} \sum_{\substack{|\alpha|+\mu\le N_0+11+M\\|\beta|\le 1\\ \mu\le 1}} \left\| L^\mu Z^\alpha \partial^\beta w_k'(t,\cdot)\right\|_2 \le C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon, \tag{4.105}$$

for $T\le T_\epsilon$.

We now turn our attention to the weighted term in $IV_k(T)$ in the case that $|\beta| = 1$. We apply Theorem 3.10 to see that

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\leq N_0+11+M\\ \mu\leq 1}} \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} L^\mu Z^\alpha w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\leq N_0+11+2M\\ \mu\leq 1}} \|L^\mu \partial^\alpha \Box w_k(s,\cdot)\|_2 \; ds \\
&+ \int_0^T \sum_{\substack{|\alpha|+\mu\leq N_0+11+M\\ \mu\leq 1}} \|L^\mu Z^\alpha \Box w_k(s,\cdot)\|_2 \; ds \\
&+ \sum_{\substack{|\alpha|+\mu\leq N_0+11+M\\ \mu\leq 1}} \|L^\mu \partial^\alpha \Box w_k\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} .
\end{aligned}
\tag{4.106}
$$

First observe that the first and third terms in the right hand side of (4.106) are controlled by the right hand side of (4.91), which we proved is controlled by (4.96). Since the quantity

$$
\sum_{\substack{|\alpha|+\mu\leq N_0+11+M\\ \mu\leq 1}} |L^\mu Z^\alpha \Box w_k|
$$

is also controlled by the right hand side of (4.103), it follows from the preceding arguments that the second term in the right hand side of (4.106) is controlled by the right hand side of (4.105). Thus, we conclude that that

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\leq N_0+11+M\\ \mu\leq 1}} \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} L^\mu Z^\alpha w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\qquad \leq C\langle T\rangle^{1/2} \epsilon^2 + C\langle T\rangle^{1/2} \epsilon M_k(T) + C_1\epsilon.
\end{aligned}
\tag{4.107}
$$

To handle the case $|\beta| = 0$, we observe that due to arguments used to bound term $II_k(T)$, it suffices to consider only the terms in $IV_k(T)$ where $\mu = 1$. We split the terms into two pieces:

$$
\begin{aligned}
&\sum_{|\alpha|\leq N_0+10+M} \left[\sup_{0\leq t\leq T} \|LZ^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} LZ^\alpha w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right] \\
&= \sum_{|\alpha|\leq N_0+9+M} \left[\sup_{0\leq t\leq T} \|LZ^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} LZ^\alpha w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right] \\
&+ \sum_{|\alpha|= N_0+10+M} \left[\sup_{0\leq t\leq T} \|LZ^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} Z^\alpha w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right]
\end{aligned}
\tag{4.108}
$$

We will deal with the first term on the right hand side of (4.108). Applying Theorem 3.12, we see that it is controlled by

$$
\begin{aligned}
&\sum_{\substack{|\alpha|+\mu\le N_0+10+2M\\ \mu\le 1}} \sup_{0\le t\le T} \left\| L^\mu \partial^\alpha w_k'(t,\cdot)\right\|_{L^2(|x|<3)} \\
&+\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+10+2M\\ \mu\le 1}} \|L^\mu\partial^\alpha \Box w_k(s,\cdot)\|_2 \, ds \\
&+\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ \mu\le 1}} \left\| \langle x\rangle^{-1/2} L^\mu Z^\alpha \Box w_k(s,\cdot)\right\|_{L^1_r L^2_\omega(|x|>2)} \, ds \\
&+\sum_{\substack{|\alpha|+\mu\le N_0+9+M\\ \mu\le 1}} \|L^\mu \partial^\alpha \Box w_k\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\,.
\end{aligned}
\tag{4.109}
$$

We see that the first term on the right hand side of (4.109) is bounded by term $III_k(T)$. We can also see that the second and fourth terms on the right hand side of (4.109) are controlled by the right hand side of (4.91). Since both (4.80) and (4.90) are controlled by (4.96), it remains to bound the third term in (4.109). We see that

$$
\begin{aligned}
&\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ \mu\le 1}} \left\| \langle x\rangle^{-1/2} L^\mu Z^\alpha \Box w_k(s,\cdot)\right\|_{L^1_r L^2_\omega(|x|>2)} \, ds \\
&\lesssim \int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ \mu\le 1}} \left\| \langle x\rangle^{-1/2} L^\mu Z^\alpha Q_k(s,\cdot)\right\|_{L^1_r L^2_\omega(|x|>2)} \, ds \\
&+\int_0^2 \sum_{|\alpha|\le N_0+10+M} \left\| \langle x\rangle^{-1/2} \partial^\alpha [\Box,\eta] u(s,\cdot)\right\|_{L^1_r L^2_\omega(|x|>2)} \, ds.
\end{aligned}
\tag{4.110}
$$

By (4.1) and the fact that we are assuming the initial data (f,g) are compactly supported, the second term on the right hand side of the preceding inequality is controlled by $C_1\epsilon$. If we apply

Lemma 4.3, we see that

(4.111)

$$\sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ \mu\le 1}} |L^\mu Z^\alpha Q_k| \lesssim \sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ |\beta|\le 1\\ \mu\le 1}} \left|L^\mu Z^\alpha \partial^\beta w_{k-1}\right| \times$$

$$\left(\sum_{\substack{|\alpha|\le[(N_0+10+M)/2]\\ |\beta|\le 1}} \left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{\substack{|\alpha|\le[(N_0+10+M)/2]\\ |\beta|\le 2}} \left|Z^\alpha\partial^\beta w_k\right| + \sum_{|\alpha|\le N_0+12+M} |Z^\alpha u_0|\right)$$

$$+ \left(\sum_{\substack{|\alpha|\le N_0+10+M\\ |\beta|\le 1}} \left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+12+M} |Z^\alpha u_0|\right) \times$$

$$\left(\sum_{\substack{|\alpha|+\mu\le[(N_0+10+M)/2]\\ |\beta|\le 1\\ \mu\le 1}} \left|L^\mu Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{\substack{|\alpha|+\mu\le[(N_0+10+M)/2]\\ |\beta|\le 2\\ \mu\le 1}} \left|L^\mu Z^\alpha\partial^\beta w_k\right|\right.$$

$$\left. + \sum_{|\alpha|\le N_0+13+M} |Z^\alpha u_0|\right)$$

$$+ \sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ |\beta|\le 2\\ \mu\le 1}} \left|L^\mu Z^\alpha\partial^\beta w_k\right| \times \left(\sum_{\substack{|\alpha|\le[(N_0+10+M)/2]\\ |\beta|\le 1}} \left|Z^\alpha\partial^\beta w_{k-1}\right| + \sum_{|\alpha|\le N_0+12+M} |Z^\alpha u_0|\right)$$

$$+ \sum_{\substack{|\alpha|\le N_0+10+M\\ |\beta|\le 2}} \left|Z^\alpha\partial^\beta w_k\right| \sum_{\substack{|\alpha|+\mu\le[(N_0+10+M)/2]+1\\ |\beta|\le 1\\ \mu\le 1}} \left|L^\mu Z^\alpha\partial^\beta w_{k-1}\right|.$$

Applying Lemma 4.5 and (4.111) in the same manner used to obtain (4.49), we see that the first term on the right hand side of (4.110) is controlled by $C\langle T\rangle^{1/2}\epsilon\, M_k(T) + C\langle T\rangle^{1/2}\epsilon^2$. Thus, it follows from (4.109)-(4.111) that

$$\sum_{|\alpha|\le N_0+9+M}\left[\sup_{0\le t\le T}\|LZ^\alpha w_k(t,\cdot)\|_2 + \langle T\rangle^{-1/4}\left\|\langle x\rangle^{-1/4} LZ^\alpha w_k\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\right] \tag{4.112}$$
$$\le C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon.$$

To deal with the case where $|\alpha| = N_0 + 10 + M$, we consider the cases: $|x| < 3$ and $|x| > 2$. Since it is the case that

$$\sum_{|\alpha|=N_0+10+M} \left[\sup_{0\le t\le T} \|LZ^\alpha w_k(t,\cdot)\|_{L^2(|x|<3)} + \left\| \langle x\rangle^{-1/4} LZ^\alpha w_k \right\|_{L^2([0,T]\times\{|x|<3\})} \right]$$
$$\le \sum_{\substack{|\alpha|+\mu\le N_0+10+M\\ \mu\le 1}} \left[\sup_{0\le t\le T} \left\|L^\mu \partial^\alpha w_k'(t,\cdot)\right\|_{L^2(|x|<3)} + \left\| \langle x\rangle^{-1/4} L^\mu \partial^\alpha w_k' \right\|_{L^2([0,T]\times\{|x|<3\})} \right] \tag{4.113}$$
$$+ \sum_{\mu\le 1} \left[\sup_{0\le t\le T} \|t^\mu \partial_t^\mu w_k(t,\cdot)\|_{L^2(|x|<3)} + \left\| \langle x\rangle^{-1/4} t^\mu \partial_t^\mu w_k \right\|_{L^2([0,T]\times\{|x|<3\})} \right],$$

if one applies Lemma 3.2 to the term in brackets on the last line of (4.113), one can see that the left hand side of (4.113) is controlled by $III_k(T)$.

In the case $|x| > 2$, we proceed just as in the arguments used in (4.53)-(4.65). First we fix a cutoff $\rho \in C^\infty(\mathbb{R}^3)$ such that $\rho(x) = 0$ when $|x| < 1$ and $\rho(x) = 1$ when $|x| > 2$. If we let $v_k = \rho w_k$, it follows that v_k solve the boundaryless wave equation given in (4.53) with vanishing initial data. If we fix α where $|\alpha| = N_0 + 10 + M$, we see that

$$\begin{aligned} \Box_{c_I}(LZ^\alpha v_k^I) &= \overline{R}^I_{\alpha,k} \\ &+ \sum_{J=1}^{D} \sum_{i=0}^{3} \sum_{j=0}^{3} \partial_j \left(\rho(1-\eta) B^{ij,IJ}(u_{k-1}, u_{k-1}') \partial_i (LZ^\alpha u_k^J) \right) \\ &+ LZ^\alpha[-2\nabla_x\rho \cdot \nabla_x w_k^I - (\Delta\rho) w_k^I], \end{aligned} \tag{4.114}$$

where $\overline{R}^I_{\alpha,k}$ is a remainder term that includes some of the quantities that are easier to bound. Just as in the arguments used to bound term $II_k(T)$, we note that $\overline{R}^I_{\alpha,k}$ is a linear combination of terms of the form

$$\sum_{J=1}^{D} \sum_{i,j=0}^{3} L^{\mu_1} Z^\beta \left(\rho(1-\eta) B^{ij,IJ}(u_{k-1}, u_{k-1}') \right) L^{\mu_2} Z^\gamma \partial_i \partial_j u_k^J, \tag{4.115}$$

where $|\beta| + |\gamma| = |\alpha|, |\gamma| < |\alpha|$, and $\mu_1 + \mu_2 = 1$, and of the form

$$\begin{aligned} &LZ^\alpha \left(\rho(1-\eta) A^I(u_{k-1}, u_{k-1}') \right), \\ &\sum_{J=1}^{D} \sum_{i,j=0}^{3} \partial_j \left(\rho(1-\eta) B^{ij,IJ}(u_{k-1}, u_{k-1}') \right) LZ^\alpha \partial_i u_k^J, \\ &LZ^\alpha(\rho[\Box_{c_I}, \eta] u^I). \end{aligned} \tag{4.116}$$

We write $LZ^\alpha v_k = \overline{v}_{1,k} + \overline{v}_{2,k} + \overline{v}_{3,k}$, where $\overline{v}_{1,k}$ solves the boundaryless wave equation $\Box \overline{v}_{1,k} = \overline{R}_{\alpha,k}$ with vanishing initial data and $\overline{v}_{3,k}$ solves $\Box \overline{v}_{3,k} = LZ^\alpha[-2\nabla_x \rho \cdot \nabla_x w_k - (\Delta\rho) w_k]$ with vanishing initial data.

To handle, $\overline{v}_{1,k}$, we apply (4.115) and (4.116) to see that

$$
\begin{aligned}
\left|\overline{R}_{\alpha,k}\right| \lesssim & \left(\sum_{\substack{|\beta|+\mu \le N_0+11+M \\ |\gamma|\le 1 \\ \mu \le 1}} \left| L^\mu Z^\beta \partial^\gamma w_{k-1} \right| + \sum_{|\beta| \le N_0+12+M} \left| Z^\beta u_0 \right| \right) \times \\
& \left(\sum_{\substack{|\beta|+\mu \le [(N_0+11+M)/2] \\ |\gamma|\le 2 \\ \mu \le 1}} \left| L^\mu Z^\beta \partial^\gamma w_{k-1} \right| \right. \\
& \left. + \sum_{\substack{|\beta|+\mu \le [(N_0+11+M)/2] \\ |\gamma|\le 2 \\ \mu \le 1}} \left| L^\mu Z^\beta \partial^\gamma w_{k} \right| + \sum_{|\beta| \le N_0+12+M} \left| Z^\beta u_0 \right| \right) \\
& + \sum_{\substack{|\beta|+\mu \le N_0+11+M \\ |\gamma|\le 1 \\ \mu \le 1}} \left| L^\mu Z^\beta \partial^\gamma w_{k} \right| \times \\
& \left(\sum_{\substack{|\beta|+\mu \le [(N_0+11+M)/2]+1 \\ |\gamma|\le 2 \\ \mu \le 1}} \left| L^\mu Z^\beta \partial^\gamma w_{k-1} \right| \sum_{|\beta| \le N_0+12+M} \left| Z^\beta u_0 \right| \right) \\
& + \sum_{|\beta| \le N_0+12+M} \left| Z^\beta [\Box, \eta] u \right|.
\end{aligned} \tag{4.117}
$$

One then applies Corollary 2.11 and (4.117) to $\overline{v}_{1,k}$. By (4.1) and Sobolev embedding, the term involving $[\Box,\eta]u$ in (4.117) is controlled by

$$
\begin{aligned}
& \int_0^T \sum_{|\beta| \le N_0+14+M} \left\| \partial^\beta [\Box,\eta] u(s,\cdot) \right\|_{L^2(|x|<2)} \, ds \\
& + \int_0^T \sum_{|\beta| \le N_0+14+M} \left\| \langle x \rangle^{-1/2} \, \partial^\beta [\Box,\eta] u(s,\cdot) \right\|_{L^1_r L^2_\omega(|x|>2)} \, ds \le C_1 \epsilon
\end{aligned} \tag{4.118}
$$

since the initial data are compactly supported. If we apply Lemma 4.5, we see that the remaining terms that result from (4.117) are controlled by

(4.119)

$$\left(\sum_{\substack{|\beta|+\mu\le N_0+11+M \\ |\gamma|\le 1 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\beta \partial^\gamma w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\beta|\le N_0+12+M} \sup_{0\le t\le T} \left\| Z^\beta u_0(t,\cdot) \right\|_2 \right) \times$$

$$\left(\sum_{\substack{|\beta|+\mu\le[(N_0+11+M)/2]+3 \\ |\gamma|\le 2 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\beta \partial^\gamma w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right.$$

$$\left. + \sum_{\substack{|\alpha|+\mu\le[(N_0+11+M)/2]+3 \\ |\gamma|\le 2 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\beta \partial^\gamma w_{k} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\beta|\le N_0+14+M} \sup_{0\le t\le T} \left\| Z^\beta u_0(t,\cdot) \right\|_2 \right)$$

$$+ \sum_{\substack{|\beta|+\mu\le N_0+11+M \\ |\gamma|\le 1 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\beta \partial^\gamma w_{k} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \times$$

$$\left(\sum_{\substack{|\beta|+\mu\le[(N_0+10+M)/2]+3 \\ |\gamma|\le 2 \\ \mu\le 1}} \left\| \langle x\rangle^{-1/4} L^\mu Z^\beta \partial^\gamma w_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right.$$

$$\left. + \sum_{|\beta|\le N_0+14+M} \sup_{0\le t\le T} \left\| Z^\beta u_0(t,\cdot) \right\|_2 \right),$$

which is controlled by $C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T)$.

For any fixed $1\le I,J\le D$ and $0\le i,j\le 3$, there are constants $C^{ij,IJK}, B_\ell^{ij,IJK}$ be the same constants as in (4.58). Consider the following boundaryless wave equations for fixed i,ℓ,I,J,K,

(4.120)

$$\begin{cases} \Box_{c_I} \overline{v}_{2,k}^{i,IJK}(t,x) = \sum_{j=0}^3 C^{ij,IJK}\partial_j(\rho(1-\eta)u_{k-1}^K \partial_i(LZ^\alpha u_k^J)), \quad (t,x)\in[0,T]\times\mathbb{R}^3, \\ \overline{v}_{2,k}^{i,IJK}(0,x) = \partial_t \overline{v}_{2,k}^{i,IJK}(0,x) = 0, \end{cases}$$

and

(4.121)
$$\begin{cases} \Box_{c_I} \overline{v}_{2,k}^{i,IJK,\ell}(t,x) = \sum_{j=0}^{3} B_\ell^{ij,IJK} \partial_j(\rho(1-\eta)\partial_\ell u_{k-1}^K \partial_i(LZ^\alpha u_k^J)), \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ \overline{v}_{2,k}^{i,IJK,\ell}(0,x) = \partial_t \overline{v}_{2,k}^{i,IJK,\ell}(0,x) = 0. \end{cases}$$

We note that

$$\overline{v}_{2,k}^{I} = \sum_{J,K=1}^{D} \sum_{i=0}^{3} \left(\overline{v}_{2,k}^{i,IJK} + \sum_{\ell=0}^{3} \overline{v}_{2,k}^{i,IJK,\ell} \right). \tag{4.122}$$

We will deal with the terms of the form $v_{2,k}^{i,IJK}$ in (4.122). The remaining terms can be bounded similarly. If we apply Theorem 2.18, we get

$$\begin{aligned}
&\left\| v_{2,k}^{i,IJK}(t,\cdot) \right\|_2 + \langle T \rangle^{-1/4} \left\| \langle x \rangle^{-1/4} v_{2,k}^{i,IJK} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\lesssim \int_0^T \left\| \Box_{c_I} v_{2,k}^{i,IJK}(s,\cdot) \right\|_2 \, ds \\
&\lesssim \int_0^T \|u_{k-1}(s,\cdot)\|_\infty \sum_{\substack{|\beta|+\mu \le N_0+11+M \\ \mu \le 1}} \left\| L^\mu Z^\beta u_k'(s,\cdot) \right\|_2 \, ds \\
&\lesssim \log(2+T) \sup_{0\le s\le T} \langle s \rangle \|u_{k-1}(s,\cdot)\|_\infty \times \\
&\sup_{0\le s\le T} \sum_{\substack{|\beta|+\mu \le N_0+11+M \\ \mu \le 1}} \left\| L^\mu Z^\beta u_k'(s,\cdot) \right\|_2 \\
&\lesssim \log(2+T)\epsilon^2 + \log(2+T)\epsilon M_k(T).
\end{aligned} \tag{4.123}$$

To control the terms involving $\overline{v}_{3,k}$,

$$\sup_{0\le t\le T} \|\overline{v}_{3,k}(t,\cdot)\|_{L^2(|x|<3)} + \langle T \rangle^{-1/4} \|\overline{v}_{3,k}\|_{L^2([0,T]\times\{|x|<3\})}, \tag{4.124}$$

we use the exact same argument used to control (4.63). It follows that

$$\begin{aligned}
&\sup_{0\le t\le T} \|Z^\alpha \overline{v}_{3,k}(t,\cdot)\|_{L^2(|x|<3)} + \langle T \rangle^{-1/4} \|Z^\alpha v_{3,k}\|_{L^2([0,T]\times\{|x|<3\})} \\
&\lesssim \sum_{|\beta|+\mu \le N_0+10+M} \left\| L^\mu \partial^\beta w_k' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})},
\end{aligned} \tag{4.125}$$

which is controlled by term $III_k(T)$. Thus, the arguments made in (4.113)-(4.125) show that if the index α is fixed so that $|\alpha| = N_0 + 10 + M$, then we get the bound:

$$\begin{aligned}
&\sup_{0\le t\le T} \|Z^\alpha w_k(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} Z^\alpha w_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\le C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon.
\end{aligned} \tag{4.126}$$

Therefore, by (4.107), (4.112) and (4.126) it follows that for $T \le T_\epsilon$ and ϵ sufficiently small, we have

$$IV_k(T) \le C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon. \tag{4.127}$$

4.2.5. Term V Bounds. To bound $V_k(T)$, we shall apply Theorem 3.13 to see that

$$\begin{aligned}
V_k(T) \lesssim &\int_0^T \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\alpha|+\mu\le N_0+6+M\\ \mu\le 1}} |L^\mu Z^\alpha Q_k| \ \frac{dy\, ds}{|y|} \\
&+ \int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+3+M\\ \mu\le 1}} \|L^\mu \partial^\alpha Q_k\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}\ ds \\
&+ \int_0^2 \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{|\alpha|\le N_0+6+M} |\partial^\alpha[\Box,\eta]u(s,y)| \ \frac{dy\, ds}{|y|} \\
&+ \int_0^2 \sum_{|\alpha|\le N_0+3+M} \|\partial^\alpha[\Box,\eta]u\|_{L^2(\{x\in\mathbb{R}^3\backslash\mathcal{K}:|x|<2\})}\ ds.
\end{aligned} \tag{4.128}$$

By (4.1) and the fact that the initial data are compactly supported, it follows that the last two terms in this inequality are bounded by $C_1\epsilon$. Thus, it remains to bound the first two terms in (4.128). We apply Lemma 4.3 to see that

$$\begin{aligned}
\sum_{\substack{|\alpha|+\mu\le N_0+6+M\\ \mu\le 1}} |L^\mu Z^\alpha Q_k| \lesssim \Bigg(&\sum_{\substack{|\alpha|+\mu\le N_0+7+M\\ \mu\le 1}} |L^\mu Z^\alpha w_{k-1}| + \sum_{|\alpha|\le N_0+8+M} \left|Z^\alpha w_k''\right| \\
&+ \sum_{\substack{|\alpha|+\mu\le N_0+8+M\\ \mu\le 1}} |L^\mu Z^\alpha u_0| \Bigg)^2.
\end{aligned} \tag{4.129}$$

Applying Cauchy-Schwarz, the terms that result from the right hand side of (4.129) are controlled by $C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T)$. We observe that the second term in (4.128) is controlled by the first term in the right hand side of (4.91), which is bounded by the right hand side of (4.96). Therefore, we see that for $T \leq T_\epsilon$ and ϵ sufficiently small, we get the bound

$$V_k(T) \leq C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon. \tag{4.130}$$

4.2.6. $M_k(T)$ **Bound.** For the remainder of the proof, we will assume that C, C_1 in (4.131) are fixed constants that do not vary from line to line. The reason we have made the distinction between C and C_1 throughout this proof is that we need to insure that our constant B is independent of k. We will now prove that for ϵ and c sufficiently small, $B = 2C_1$ is a constant for which (4.5) is true for all k. This would guarantee that B is independent of k.

If we use the uniform bound on the local solution (4.1) and the induction hypothesis (4.5), then it follows from our previous arguments that for $k \geq 1$,

$$M_k(T) \leq C\langle T\rangle^{1/2}\epsilon^2 + C\langle T\rangle^{1/2}\epsilon M_k(T) + C_1\epsilon. \tag{4.131}$$

We note that $\langle T\rangle^{1/2}\epsilon \leq \epsilon + \sqrt{c}$, where c is the constant appearing in (1.9). If we set

$$c \leq \min\{1, C_1\}/(1024C^2), \tag{4.132}$$

and let

$$\epsilon \leq \min\{1, C_1\}/(32C), \tag{4.133}$$

then we see that $C\langle T\rangle^{1/2}\epsilon \leq \min\{1, C_1\}/16$, where C is the same constant appearing in (4.131). Thus, from the preceding argument and the fact that we assumed that $\epsilon < 1$, the following inequality holds:

$$M_k(T) \leq \frac{C_1\epsilon}{16} + \frac{M_k(T)}{16} + C_1\epsilon.$$

This implies that

$$M_k(T) \leq \frac{17C_1\epsilon}{15} \leq 2C_1\epsilon = B\epsilon.$$

This completes the proof of (4.5) for all k.

4.3. Conclusion

Now that we have shown that the sequence $\{w_k\}$ is uniformly bounded in X_T, we are ready to prove existence in a slightly larger Banach space Y_T such that $X_T \hookrightarrow Y_T$. We define Y_T to be the space of functions whose topology is given by the following norm:

$$\|v\|_{Y_T} := \sum_{|\alpha|\le 10+M} \sup_{0\le t\le T} \|\partial^\alpha v(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\,. \tag{4.134}$$

We also define the quantity

$$\begin{aligned} A_k(T) := &\sum_{|\alpha|\le 10+M} \sup_{0\le t\le T} \|\partial^\alpha w_k(t,\cdot) - \partial^\alpha w_{k-1}(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \\ &+ \sum_{|\alpha|\le 2} \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4}\,\partial^\alpha (w_k - w_{k-1})\right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}\,. \end{aligned}$$

We shall show that for ϵ sufficiently small that for $k\ge 2$,

$$A_k(T) \le \frac{1}{2} A_{k-1}(T). \tag{4.135}$$

One can easily see that since $A_1(T)\le M_1(T)\le B\epsilon$. If one were to prove (4.135) as well, then one could show that $A_k(T)\le (1/2)^{k-1}B\epsilon$ for all k. Consequently, this would mean that the quantity

$$\|w_j - w_k\|_{Y_T}$$

could be made arbitrarily small for values of j,k that are sufficiently large. Thus, $\{w_k\}$ would converge in Y_T.

To begin proving (4.135), let us observe that there exist constants $A^{IJK}, A^{l,IJK}, A^{ij,IJK}$ for $1\le J,K\le D$ and $0\le i,j\le 3$ such that

$$A^I(v,v') = \sum_{J,K=1}^{D} \left(A^{IJK} v^J v^K + \sum_{l=0}^{3} A^{l,IJK} v^J \partial_l v^K + \sum_{i,j=0}^{3} A^{ij,IJK} \partial_i v^J \partial_j v^K \right),$$

where A^I are the quadratic forms appearing in (1.2). It should be mentioned that while these calculations are done specifically for trunicated quasilinear terms, they can be done for general quasilinear functions $Q(u,u',u'')$ that are smooth, vanishing to second order and are linear in u''. This is due to the fact that the higher order terms that are not present in our calculations are $O(\epsilon^2)$ and are, thus, easier to control. It follows that

$$\begin{aligned}
&A^I(u_k,u_k') - A^I(u_{k-1},u_{k-1}')\\
&= \sum_{J,K=1}^{D}\left(A^{IJK}u_k^J u_k^K + \sum_{l=0}^{3} A^{l,IJK}u_k^J\partial_l u_k^K + \sum_{i,j=0}^{3} A^{ij,IJK}\partial_i u_k^J \partial_j u_k^K\right)\\
&- \sum_{J,K=1}^{D}\left(A^{IJK}u_{k-1}^J u_{k-1}^K + \sum_{l=0}^{3} A^{l,IJK}u_{k-1}^J\partial_l u_{k-1}^K + \sum_{i,j=0}^{3} A^{ij,IJK}\partial_i u_{k-1}^J \partial_j u_{k-1}^K\right)\\
&= \sum_{J,K=1}^{D}\Big[A^{IJK}\left((u_k^J-u_{k-1}^J)u_k^K + u_{k-1}^J(u_k^K-u_{k-1}^K)\right)\\
&+\sum_{l=0}^{3} A^{l,IJK}\left((u_k^J-u_{k-1}^J)\partial_l u_k^K + u_{k-1}^J\partial_l(u_k^K-u_{k-1}^K)\right)\\
&+\sum_{i,j=0}^{3} A^{ij,IJK}\left(\partial_i(u_k^J-u_{k-1}^J)\partial_j u_k^K + \partial_i u_{k-1}^J\partial_j(u_k^K-u_{k-1}^K)\right)\Big].
\end{aligned}\tag{4.136}$$

Hence, it follows that for $N \geq 0$,

$$\begin{aligned}
&\sum_{|\alpha|\leq N}\left|\partial^\alpha(A(u_k,u_k')-A(u_{k-1},u_{k-1}'))\right|\\
&\lesssim \sum_{|\alpha|\leq N+1}|\partial^\alpha(u_k-u_{k-1})| \times \sum_{|\beta|\leq N+1}\left(|\partial^\beta u_k| + |\partial^\beta u_{k-1}|\right).
\end{aligned}\tag{4.137}$$

Secondly, we compute $\Box_{c_I}(w_k - w_{k-1})$:

$$\begin{aligned}
\Box_{c_I}(w_k^I - w_{k-1}^I) &= (1-\eta)[Q^I(u_{k-1},u_{k-1}',u_k'') - Q^I(u_{k-2},u_{k-2}',u_{k-1}'')]\\
&= (1-\eta)[A^I(u_{k-1},u_{k-1}') - A^I(u_{k-2},u_{k-2}')]\\
&+ (1-\eta)\sum_{J=0}^{D}\sum_{i,j=0}^{3} B^{ij,IJ}(u_{k-1},u_{k-1}')\partial_i\partial_j u_k^J\\
&- (1-\eta)\sum_{J=0}^{D}\sum_{i,j=0}^{3} B^{ij,IJ}(u_{k-2},u_{k-2}')\partial_i\partial_j u_{k-1}^J\\
&= (1-\eta)[A^I(u_{k-1},u_{k-1}') - A^I(u_{k-2},u_{k-2}')]\\
&+ (1-\eta)\sum_{J=0}^{D}\sum_{i,j=0}^{3} B^{ij,IJ}(u_{k-1}-u_{k-2},u_{k-1}'-u_{k-2}')\partial_i\partial_j u_k^J\\
&+ (1-\eta)\sum_{J=0}^{D}\sum_{i,j=0}^{3} B^{ij,IJ}(u_{k-2},u_{k-2}')\partial_i\partial_j (u_k^J-u_{k-1}^J).
\end{aligned}\tag{4.138}$$

We will again be needing to use the energy estimate for perturbed wave equations from Theorem 3.1. If we set

$$\gamma^{ij,IJ}(t,x) = -(1-\eta)B^{ij,IJ}(u_{k-2}, u'_{k-2}),$$

then it follows from (4.138) that

$$\begin{aligned}\Box_\gamma(u^I_k - u^I_{k-1}) &= (1-\eta)[A^I(u_{k-1}, u'_{k-1}) - A^I(u_{k-2}, u'_{k-2})] \\ &\quad + (1-\eta)\sum_{J=0}^{D}\sum_{i,j=0}^{3} B^{ij,IJ}(u_{k-1}-u_{k-2}, u'_{k-1}-u'_{k-2})\partial_i\partial_j u^J_k.\end{aligned} \tag{4.139}$$

Thus, it follows that for $N \geq 0$,

$$\begin{aligned}&\sum_{|\alpha|\leq N} |\Box\partial^\alpha(u_k - u_{k-1})| \\ &\lesssim \sum_{|\alpha|\leq N+2} |\partial^\alpha(u_k-u_{k-1})| \times \sum_{|\beta|\leq N+1}\Big(|\partial^\beta u_k| + |\partial^\beta u_{k-1}|\Big) \\ &+ \sum_{|\alpha|\leq N+1} |\partial^\alpha(u_{k-1}-u_{k-2})| \times \sum_{|\beta|\leq N+2}\Big(|\partial^\beta u_k| + |\partial^\beta u_{k-1}|\Big),\end{aligned} \tag{4.140}$$

and

$$\begin{aligned}&\sum_{|\alpha|\leq N} |\Box_\gamma\partial^\alpha(u_k - u_{k-1})| \\ &\lesssim \sum_{|\alpha|\leq N+1} |\partial^\alpha(u_k-u_{k-1})| \times \sum_{|\beta|\leq N+1}\Big(|\partial^\beta u_k| + |\partial^\beta u_{k-1}|\Big) \\ &+ \sum_{|\alpha|\leq N+1} |\partial^\alpha(u_{k-1}-u_{k-2})| \times \sum_{|\beta|\leq N+2}\Big(|\partial^\beta u_k| + |\partial^\beta u_{k-1}|\Big).\end{aligned} \tag{4.141}$$

If we apply Proposition 3.3, we see that

$$\begin{aligned}\sup_{0\leq t\leq T}\sum_{|\alpha|\leq 9+M}\big\|\partial^\alpha(u_k-u_{k-1})'(t,\cdot)\big\|_2 &\lesssim \sum_{j\leq 9+M}\sup_{0\leq t\leq T}\Big\|\partial_t^j(u_k-u_{k-1})'(t,\cdot)\Big\|_2 \\ &\quad + \sum_{|\alpha|\leq 8+M}\sup_{0\leq t\leq T}\|\partial^\alpha\Box(w_k-w_{k-1})(t,\cdot)\|_2\,.\end{aligned} \tag{4.142}$$

If we apply (4.140) for $N = 8+M$ and Sobolev embedding, we see that

$$
\begin{aligned}
&\sum_{|\alpha|\le 8+M} \|\partial^\alpha \Box (u_k - u_{k-1})(t,\cdot)\|_2 \\
&\le C_3 \sum_{|\alpha|\le 10+M} \|\partial^\alpha (u_k - u_{k-1})(t,\cdot)\|_2 \times \sum_{|\beta|\le 11+M} \left(\left\|\partial^\beta u_k(t,\cdot)\right\|_2 + \left\|\partial^\beta u_{k-1}(t,\cdot)\right\|_2 \right) \\
&+ C_3 \sum_{|\alpha|\le 9+M} \|\partial^\alpha (u_{k-1} - u_{k-2})(t,\cdot)\|_2 \times \sum_{|\beta|\le 12+M} \left(\left\|\partial^\beta u_k(t,\cdot)\right\|_2 + \left\|\partial^\beta u_{k-1}(t,\cdot)\right\|_2 \right) \\
&\le C_3 \epsilon A_{k-1}(T) + C_3 \epsilon A_k(T),
\end{aligned}
\tag{4.143}
$$

where C_3 is a positive constant that is allowed to vary from line to line. Thus, the second term in (4.142) is controlled by $C_3\epsilon A_{k-1}(T) + C_3\epsilon A_k(T)$. We set

$$E_N(t) = E_N(w_k - w_{k-1})(t),$$

using the notation from (3.6). From our uniform bound (4.5), for ϵ sufficiently small, just as in (3.20), we see that

$$(5\max_I\{c_I, c_I^{-1}\})^{-1} E_N^{1/2}(t) \le \sum_{j\le N} \left\|\partial_t^j (u_k - u_{k-1})'(t,\cdot)\right\|_2 \le 5\max_I\{c_I, c_I^{-1}\} E_N^{1/2}(t). \tag{4.144}$$

By Theorem 3.1, it follows that

$$
\begin{aligned}
\partial_t \left[E_{9+M}^{1/2}(t) \right] &\le C_3 \sum_{j=0}^{9+M} \left\| \Box_\gamma \partial_t^j (u_k - u_{k-1})(t,\cdot) \right\|_2 \\
&\quad + C_3 \left\|\gamma'(t,\cdot)\right\|_\infty E_{9+M}^{1/2}(t).
\end{aligned}
\tag{4.145}
$$

By our uniform bound (4.5), we know that by (4.14), for $T \le T_\epsilon$,

$$\int_0^T \left\|\gamma'(t,\cdot)\right\|_\infty \le C_3,$$

which is independent of T_ϵ. Applying Gronwall's inequality and (4.14), we see that

$$E_{9+M}^{1/2}(t) \lesssim \int_0^t \sum_{j=0}^{9+M} \left\| \Box_\gamma \partial_t^j (u_k - u_{k-1})(s,\cdot) \right\|_2 . \tag{4.146}$$

By (4.141), (4.144), (4.146) and Lemma 4.4, it follows that

$$\sum_{j\le 9+M} \left\|\partial_t^j (u_k - u_{k-1})'(t,\cdot)\right\|_2 \le C_3 \epsilon \log(2+T) A_k(T) + C_3\epsilon \log(2+T) A_{k-1}(T). \tag{4.147}$$

To deal with the weighted terms where at least one derivative is being applied to $u_k - u_{k-1}$, if we apply Theorem 3.10, we get the bound

(4.148)

$$\sum_{|\alpha|\le 1} \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} \partial^\alpha (u_k - u_{k-1})' \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \lesssim \int_0^T \sum_{|\alpha|\le M+1} \|\partial^\alpha \Box (u_k - u_{k-1})(t,\cdot)\|_2 \, dt$$
$$+ \sum_{|\alpha|\le 1} \|\partial^\alpha \Box (u_k - u_{k-1})\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} .$$

Applying (4.140) and Lemma 4.4, the first term in the right hand side of (4.148) is controlled by $C_3\epsilon \log(2+T) A_{k-1}(T) + C_3 \log(2+T)\epsilon A_k(T)$. Applying (4.140) where $N = 1$ and Lemma 4.4, we see that the second term in the right hand side of (4.148) is controlled by $C_3\epsilon A_{k-1}(T) + C_3\epsilon A_k(T)$. To control the terms where no derivatives are being applied to $u_k - u_{k-1}$, we apply Theorem 3.12 to see that

(4.149)

$$\sup_{0\le t\le T} \|(u_k - u_{k-1})(t,\cdot)\|_2 + \langle T\rangle^{-1/4} \left\| \langle x\rangle^{-1/4} (u_k - u_{k-1}) \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})}$$
$$\lesssim \sup_{0\le t\le T} \sum_{|\alpha|\le M} \|\partial^\alpha \Box (u_k - u_{k-1})(t,\cdot)\|_2$$
$$+ \int_0^T \sum_{|\alpha|\le M} \|\partial^\alpha \Box (u_k - u_{k-1})(t,\cdot)\|_2 \, dt + \int_0^T \left\| \langle x\rangle^{-1/2} \Box (u_k - u_{k-1})(t,\cdot) \right\|_{L^1_r L^2_\omega(|x|>2)} dt$$
$$+ \|\Box (u_k - u_{k-1})\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} .$$

The first term on the right hand side of (4.149) is controlled by the left hand side of (4.143), which is bounded by $C_3\epsilon A_k(T) + C_3\epsilon A_{k-1}(T)$. Applying (4.140) and Lemma 4.4, the second and fourth terms in the right hand side of (4.149) is controlled by $C_3\epsilon A_{k-1}(T) + C_3\epsilon \log(2 + T)A_{k-1}(T) + C_3\epsilon A_k(T) + C_3\epsilon \log(2+T) A_k(T)$. If we apply (4.140) and Lemma 4.5, it follows that the third term in the right hand side of (4.149) is controlled by

$$\int_0^T \left\| \langle x\rangle^{-1/2} \Box (u_k - w_{k-1})(s,\cdot) \right\|_{L^1_r L^2_\omega(|x|>2)} ds$$
$$\le C_3 \sum_{|\alpha|\le 2} \left\| \langle x\rangle^{-1/4} \partial^\alpha (u_k - u_{k-1}) \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \times$$
$$\sum_{|\beta|\le 3} \left(\left\| \langle x\rangle^{-1/4} Z^\beta u_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \left\| \langle x\rangle^{-1/4} Z^\beta u_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right)$$

$$
\begin{aligned}
&+ C_3 \sum_{|\alpha|\le 2} \left\| \langle x\rangle^{-1/4} \partial^\alpha (u_{k-1} - u_{k-2}) \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \times \\
&\sum_{|\beta|\le 4} \left(\left\| \langle x\rangle^{-1/4} Z^\beta u_k \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} + \left\| \langle x\rangle^{-1/4} Z^\beta u_{k-1} \right\|_{L^2([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \right) \\
&\le C_3\epsilon \langle T\rangle^{1/2} \epsilon A_k(T) + C_3\epsilon \langle T\rangle^{1/2} \epsilon A_{k-1}(T).
\end{aligned}
$$

Recalling that ϵ is small enough so that $\log(2+T) \le \langle T\rangle^{1/2}$, it follows that

$$A_k(T) \le C_3\epsilon \langle T\rangle^{1/2} A_k(T) + C_3\epsilon \langle T\rangle^{1/2} A_{k-1}(T).$$

To finish the argument, we now fix C_3 so that it no longer varies from line to line. We also make c, ϵ possibly smaller by requiring that c, ϵ must also satisfy

$$c \le 1/(1024 C_3^2),$$

and

$$\epsilon \le 1/(32 C_3).$$

We get

$$C_3\epsilon \langle T\rangle^{1/2} \le 1/4.$$

From this bound, we see that

$$A_k(T) \le \frac{1}{4} \left(A_{k-1}(T) + A_k(T) \right),$$

which implies that

$$A_k(T) \le \frac{1}{2} A_{k-1}(T).$$

Therefore, we have proved (4.135). From this, we conclude that $\{w_k\}$ converges in Y_T. If $\mathcal{D}'([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ is the space of distributions on $[0,T]\times\mathbb{R}^3\backslash\mathcal{K}$, then it follows that $u_k \to u$ in $\mathcal{D}'([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$ implies that $\Box u_k \to \Box u$ in $\mathcal{D}'([0,T]\times\mathbb{R}^3\backslash\mathcal{K})$. Thus, to see that u actually solves (4.3) in the classical sense, it suffices to show that

$$Q(u_{k-1}, u'_{k-1}, u''_k) \to Q(u, u', u''), \quad \text{in } C([0,T]; L^2(\mathbb{R}^3\backslash\mathcal{K})).$$

This follows from the fact that Q is smooth in its arguments and from the boundedness of $\{u_k\}$ and u in Y_T. Using standard local existence theory (see Theorems 9.4 and 9.5 in Keel, Smith and Sogge [**27**]), it follows from the fact that the initial data (f, g) are smooth that $u \in C^\infty([0,T] \times \mathbb{R}^3 \backslash \mathcal{K})$.

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