POLITICAL LOBBYING WITH PRIVATE INFORMATION

Justin Contat

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Economics.

Chapel Hill
2015

Approved by:

Sergio Parreiras

Gary Biglaiser

Fei Li

Peter Norman

Andrew Yates

© 2015
Justin Contat
ALL RIGHTS RESERVED

ABSTRACT

JUSTIN CONTAT: Political Lobbying With Private Information.
(Under the direction of Sergio Parreiras)

This dissertation presents original research on a game theoretic model of political lobbying. I model political lobbying as an all-pay auction with two players and private information, where each player/lobbyist submits a campaign contribution and the highest contribution wins political favor. In the first chapter I defend the use of using finite types in the lobbying game by showing that equilibrium behavior in a continuous type all-pay auction can be approximated (to any degree of accuracy) by equilibrium behavior in a finite-type all-pay auction, provided that the distributions of both games are close enough. In the second chapter I next show that introducing asymmetry between lobbyists in the private information game has unusual and counter-intuitive effects on the total campaign contributions. In particular, while the equilibrium behavior is robust to small changes in the information structure, the comparative statics of asymmetry are not. Introducing an arbitrarily small amount of private information can completely reverse the comparative statics conclusions of the complete information game. Finally in the third chapter I introduce maximum contribution limits on total campaign contributions. Contribution limits work to reverse the effects of asymmetry and change the relative probabilities of winning political favor. Since asymmetry may decrease total expected contributions, it is possible that by reversing these effects that imposing maximum contribution limits can *increase* total expected contributions. With complete information lobbying imposing maximum limits will increase total expected contributions if and only if lobbyists are asymmetric. With private information, however, it is possible that in expectation maximum contribution limits will not increase total expected contributions. Intuitively maximum contribution limits hurt lobbyists who are likely to donate a lot, but benefit lobbyists who normally do not contribute much. Total campaign contributions will increase only if the decrease in total

contributions from the stronger lobbyist is outweighed by the increase in contributions from the weaker lobbyist.

ACKNOWLEDGMENTS

First I would like to thank my advisor Sergio Parreiras, my committee members Gary Biglaiser, Fei Li, Peter Norman, Andrew Yates, and my colleague Ken Reddix for all of their patience, encouragement, and invaluable advice. I would also like to thank the entire faculty of the economics department at the University of North Carolina at Chapel Hill, in particular Helen Tauchen for all of her support throughout the years. I would like to thank the graduate students of the microeconomic theory graduate seminar for their feedback during my presentations. Lastly I would like to thank the graduate school at the University of North Carolina at Chapel Hill for the opportunity to pursue graduate studies in economics.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

SUMMARY

1.1 Introduction

In the past decades there has been growing concern about the influence of money in politics. Most recently in 2010, The Supreme Court of the United States of America ruled in *Citizens United vs. Federal Elections Commission* that as long as contributions are made to an independent third party (such as a Super PAC) that the contributions are expressions of free speech and thus protected by the First Amendment to the Constitution of the United States of America. Thus in certain circumstances, individuals are allowed to contribute unlimited amounts of money as long as the contribution is not direct. The main goal of this dissertation is to understand the impact of maximum contribution limits on the behavior of lobbyists relative to a lobbying environment without contribution limits.

Political lobbying has been studied in the economics literature for several years, often referred to as rent-seeking. Early examples include Becker (1983) and Baye, Kovenock, and de Vries (1993) among others. Most commonly political lobbying is modeled as a contest between lobbyists, each of whom submits a non-refundable contribution. It is commonly understood that the highest contribution is granted political favor by a local politician. Politicians cannot grant refunds to lobbyists, as this would clearly signal that the politician was in the business of selling political favors. Accusations of corruption, graft, and bribery would hastily be thrown. The politician is assumed to view the lobbyists as perfect substitutes in that it is only the lobbyist who contributes most that is the favorite of the politician. It is in this environment that we analyze political lobbying. Formally the lobbyists are risk-neutral players in an all-pay auction, where each valuations is drawn independently of the other from some probability distribution.

I show that the combined effects of private information and asymmetry between lobbyists are needed to fully understand changes in lobbying behavior and total contributions when contribution limits are enacted. Lobbyists have private information about the value of political favor. Each lobbyist does not exactly know the valuation of the other lobbyist, only the relative likelihoods of different valuations (i.e. the probability distribution of valuations). The lobbyists may differ in the relative likelihoods they place on each other's valuations. In other words, they are *ex-ante* asymmetric because their type space and type space distributions may differ. Relaxing either private information or asymmetry between the lobbyists will lead to different qualitative conclusions regarding lobbying behavior.

The first chapter of this dissertation provides a technical result that justifies the use of finite types in the lobbying game. With types drawn from an absolutely continuous distribution, equilibrium behavior is characterized as a solution to system of non-linear first-order differential equations. In all but the simplest distributions, an analytical solution is not possible. In contrast, equilibria under finite types are much easier to describe. The main result of the first chapter is to show that the equilibrium correspondence (in fact it is a function) is continuous with respect to the information structure. Small changes in the type space distribution lead to small changes in equilibrium behavior. While this result is known to hold in games with continuous payoff functions, the all-pay auction has payoff functions that are neither upper-semicontinuous nor lower-semicontinuous and so that standard results do not apply. The key driver of the results is the monotonicity of the all-pay auction: higher types contribute more with certainty.

Additionally this approach provides some insight for the equilibrium and its key properties that are more difficult to discover with continuous types: equilibrium strategies are increasing in valuation and there are decreasing returns to contributing more. What matters is not the modeling choice of the researcher (finite types or continuous types), but rather the distribution of types that each lobbyist faces. In the first chapter we introduce the model and notation that will be used in subsequent chapters. This includes the payoff functions, information structure (i.e. type spaces and probability distributions over type spaces), and equilibrium definition and characterization.

The second chapter provides another technical result. It demonstrates the complexity that arises when even a small amount of asymmetry is introduced into the game. In particular, total contributions can increase or decrease depending upon how a lobbyist is made stronger relative to the other. Nevertheless we provide conditions under which asymmetry will increase total contributions. Two effects capture the intuition of the results. The first effect, the *assimilation effect* , shows that lobbyists always play to the level of their competition. Weaker competition means less contributions while stronger competition means more contributions. The second effect, the *stacking effect*, exploits the monotonicity of the lobbying game. Whenever a type of a lobbyist would want to contribute more, so too must all of the lobbyists higher types. Monotonicity is the most important property of the lobbying game.

The third and final chapter is the main focus of the dissertation. Building on the results of the first two chapters, we show that contribution limits essentially work to reverse the effects of asymmetry. Thus to understand contribution limits, it is first necessary to understand how asymmetry changes equilibrium behavior. Lowering the maximal allowed contribution means that stronger lobbyists will not be able to contribute as much as they were previously (without the limit). Weaker lobbyists will see greater returns from contributing and these contribute more. The tension between these two effects determines whether revenues will increase or decrease. After proving existence of equilibrium in the private information environment, which is a natural extension of the complete information construction, I demonstrate that asymmetry is necessary for a maximum limit to be able to increase total expected contributions relative to the case with no-limits. Reversing the order, we see that unlimited contributions may not be revenue maximizing if the playing field is not level between lobbyists. However when lobbyists are evenly matched, unlimited contribution limits allow more surplus extraction by politicians by creating an arms race between political donors. This behavior is echoed in modern political life with wealthy donors on both sides of the politician spectrum trying to outspend each other to the benefit of the politician.

CHAPTER 2

EQUILIBRIUM CONVERGENCE IN THE ALL-PAY AUCTION

2.1 Introduction

In an all-pay auction (APA), each player chooses a costly action for a chance to win an indivisible prize. The player with the largest action wins the prize, but *all* players must *pay* a cost equal to their own action irrespective of who wins. Payments[1] are totally unconditional in that each player's payment does not depend on the actions of others or the allocation of the prize. In this paper I restrict attention to the standard[2] all-pay auction with two players.

My main result shows that despite the inherent payoff discontinuities in the APA, the equilibrium correspondence is continuous[3] with respect the information structure of the game. In fact, since each information structure generates a unique equilibrium it would be more appropriate to use the terminology "equilibrium function" instead of equilibrium correspondence. I show that as a sequence of finite type space distributions converge to a continuous distribution, then the sequence of unique equilibria in the finite games converge to the unique equilibrium of the continous-type game. This implies the equilibrium correspondence is both upper and lower hemi-continuous[4]. In particular, the main result implies that any equilibrium in a continuous-type APA can be approximated with an equilibrium of a finite-type APA, provided the type space distributions of the two

---

[1]Payments can also be interpreted as costs of acquiring effort, political donations, or investment in R&D for example. See Konrad (2009) for a more detailed introduction to the theory and application of APA's.

[2]In this dissertation the standard all-pay auction means risk neutral players with independent private values.

[3]It is possible to assign a metric to measure the distance between two probability distributions. The notion of continuity here would then correspond to the usual $\epsilon - \delta$ definition, where distance between two distributions is given by the information metric (such as the Levy-Prokhorov metric). See Shiryaev (1996) Chapter III Section 7 for a detailed definition.

[4] A correspondence $\Gamma : X \rightrightarrows Y$ is upper hemi-continuous if for any $x_n \to x$ and $y_n \in \Gamma(x_n)$ for each n, then there exists a subsequence of y_m that converges to some $y \in \Gamma(x)$. A correspondence $\Gamma : X \rightrightarrows Y$ is lower hemi-continuous if for any $x_n \to x$ and $y \in \Gamma(x)$, then there exists a sequence $y_n \in \Gamma(x_n)$ such that $y_n \to y$.

games are close. The mixed strategy equilibria of the finite-type game converge to the pure strategy equilibria of the continuous-type game as the type space becomes finer.

With complete information it is straightforward to show that APA equilibria converge as the valuations converge. In this sense we can say that the equilibrium correspondence is upper hemi-continuous in the information structure when there is complete information. The equilibrium correspondence is also trivially lower hemi-continuous. This chapter extends both of these results to the incomplete information APA in a very natural way. Siegel (2013) has the closest result to ours. He shows that for any sequence of distributions for player 1 there exist *some* sequence of distributions for player 2 such that the sequence of equilibria converge to that of the continuum game. In contrast I show that *any* sequence of equilibria converge provided their distributions converge as well.

In general it is not true that the Nash equilibrium correspondence is continuous (i.e. both lower and upper hemi-continuous), even for games of complete information. Engl (1995) shows that the equilibrium correspondence is in general not lower hemi-continuous, but is asymptotically lower hemi-continuous in that an equilibrium in the limit game can be approached by a sequence of ϵ_n equilibria. Milgrom and Weber (1985) show upper hemi-continuity of the equilibrium correspondence in a class of games with uncountable state spaces, though their results require uniformly continuous payoffs for all players, a condition which is not satisfied by the APA. Additionally they provide an example to show that in general continuity cannot be dropped.

A large insight of Milgrom and Weber (1985) is that it is only the expected *distribution* of players' actions that matters. In fact, the underlying monotonicity of the game means that in equilibrium players experience decreasing returns to their actions. This ensures concavity of what I label as "interim payoffs", which in turn ensures upper hemi-continuity. I also use their concept of distributional strategies as the appropriate object of convergence. This allows me to embed any incomplete information APA (finite-type or continuous-type), as well as the complete information APA, into a single environment. A distributional strategy for a player is a joint probability distribution over actions and types that agrees [5] with the player's type-space distribution.

This result complements the literature on equilibrium convergence in the APA. While others have focused on equilibria change as a finite *action* set is enlarged to that of the continuum, I study how equilibria change as a finite *type* set is enlarged. For a class of games which includes the complete information APA, Dasgupta and Maskin (1986) show that the limit of equilibria, as action sets become finer, is the equilibria of the limit. Similarly Athey (2001) shows that in a class of incomplete information games with certain monotonicity properties[6] (which includes the APA), equilibria in finite-action games converge to equilibria in continuous-action games when the action sets are made finer and finer and the type space is held fixed. My results, like those of Athey (2001), crucially depend upon the underlying monotonicity of the game.

Amann and Leininger (1996) have shown for continuum types and 2 players that the if the type spaces converge to a single point (i.e. complete information APA) then the associated sequence of equilibria converge as well. This paper complements theirs in that I show that for type spaces "in between" continuum types and a single type (i.e. complete information) the sequence of equilibria converge as well.

2.2 Model

2.2.1 Information Structure

There are two risk-neutral players who compete for a chance to win an indivisible prize worth $t_i > 0$ to player i. We restrict attention to strictly positive valuations, but this is without loss of generality because all zero-valuation types will not participate in equilibrium and hence will not affect any of the results. Each $t_i \in T_{i,n} \equiv \{t_{i,n}^1, t_{i,n}^2, \dots, t_{i,n}^m\} \subset \mathbb{R}_{++}$ is private information and independently distributed according to c.d.f. $F_{i,n}$. A typical finite-type distribution is shown on the left in Figure 2.1. Let $p_{i,n}^k \equiv F_{i,n}(t_{i,n}^k) - F_{i,n}(t_{i,n}^{k-1})$ denote the probability that i will have valuation $t_{i,n}^k$. Hence each $F_{i,n}(\cdot)$ is an increasing piece-wise constant function (i.e. a step function) that is

[5]The marginal on the distributional strategy must be equal to the probability distribution of type space.

[6]Her Single Crossing Condition (SCC) states that each player's best response is increasing in type whenever his opponents use increasing strategies themselves

discontinuous on $T_{i,n}$, a countable set. There will also be a limiting space $T_{i,\infty} = \lim_{n\to\infty} T_{i,n}$, described in detail in the next paragraph, which we will assume exists. This limiting space $T_{i,\infty}$ will be dense in some interval of strictly positive types $T_{i,*} \equiv [\underline{v_i}, \overline{v_i}]$. Hence the sequences of $T_{i,n}$ will converge and form a countable dense subset of some closed interval $T_{i,*}$.

Consider any sequence of finite-type c.d.f.'s $\{F_{i,n}\}_n$ that converges in distribution[7] to an absolutely continuous[8] distribution $F_{i,*}$ as in Figure 1. In general, each new term in the sequence $\{F_{i,n}\}_n$ of finite-type c.d.f.'s may have more or fewer types, where types may be the same or different. For example it could be the case that $T_{i,n} = \{1, 2, 3, 4, 5\}$, $T_{i,n+1} = \{2.1, 4.8\}$, and $T_{i,n+2} = \{2, 2.1, 3, 4, 4.8, 5, 7\}$. In addition, the probability of each type typically changes for every term in the sequence. Since I am interested in converging sequences of type spaces and their distributions, I will assume that $|T_{i,n}| = n$ for notational convenience[9]. We require the limiting c.d.f. $F_{i,*}$ to represent a continuous distribution. Though my result holds for any sequence of distributions, in particular we can think of adding types to a finite-type space, where at each iteration after a new type is added the probabilities are adjusted to make the new distribution closer to $F_{i,*}$.

In this way continuous-type spaces (i.e. those with absolutely continuous distributions) can be approximated to any desired degree of accuracy with finite-type spaces by choosing a large enough index in the sequence. The approximation procedure is analogous to the approximation of a measurable function by a sequence of simple functions. Take an absolutely continuous increasing function $F_{i,*}$ with bounded support $[\ \underline{t_i}, \overline{t_i}\ ]$. One sequence that approximates $F_{i,*}$ can be constructed by repeated bisection: for $k = 1, \ldots, n$

---

[7] $\{F_{i,n}\}_n$ converges in distribution to $F_{i,*}$, written $F_{i,n} \to^d F_{i,*}$, if $F_{i,n}(x) \to F_{i,*}(x)$ pointwise for all x where $F_{i,*}(x)$ is continuous. Since $F_{i,*}(x)$ is continuous, this means pointwise convergence (in fact uniform convergence) for all x.

[8] Absolutely continuous with respect to Lebesgue measure. The Radon-Nikodym Theorem then asserts the existence of a density function $f_{i,*}$ such that $F_{i,*}(x) = \int_{-\infty}^{x} f_{i,*}(t)dt$, where the integral is with respect to the Lebesgue measure.

[9] Though all that is needed is that $\lim_{n\to\infty} T_{i,n} = T_{i,\infty}$

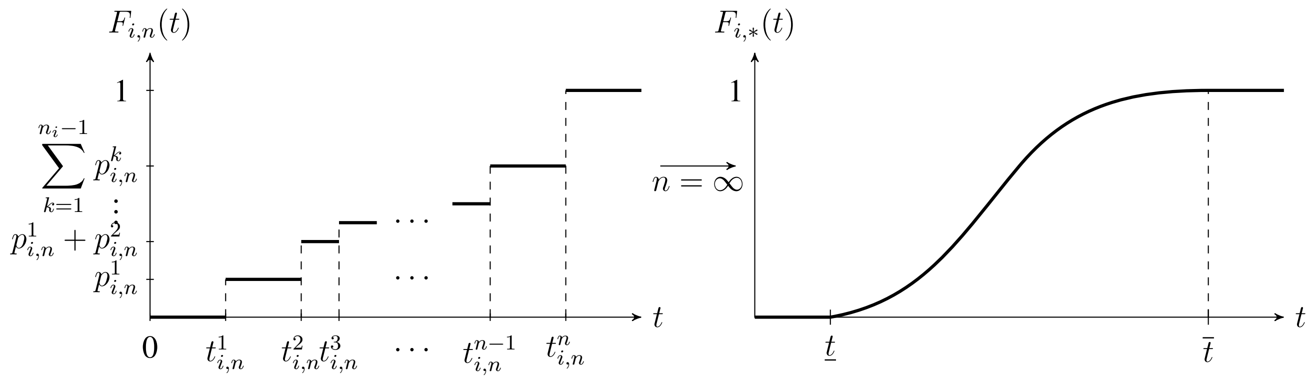

Figure 2.1: Finite-Type Space Distribution

$$F_{i,n}(x) = \begin{cases} F_{i,*}(\underline{t_i}) = 0 & \text{if } x \leq \underline{t_i} \\ F_{i,*}\left(\underline{t_i} + \frac{2^k}{2^n}(\overline{t_i} - \underline{t_i})\right) & \text{if } x \in \left(\underline{t_i} + \frac{2^k}{2^n}(\overline{t_i} - \underline{t_i}), \underline{t_i} + \frac{2^{k+1}}{2^n}(\overline{t_i} - \underline{t_i})\right) \\ F_{i,*}(\overline{t_i}) = 1 & \text{if } x \geq \overline{t_i}. \end{cases} \tag{2.1}$$

2.2.2 Payoffs

Each player i submits action $a_i \in A_i \equiv [0, \bar{a}_i]$ for some $\bar{a}_i$ sufficiently large[10]. The player with the highest action wins the prize, but each player i must incur a playing cost equal to a_i regardless of the allocation of the prize. Hence each player's *ex-post* payoffs $u_i(a_i, a_j|t_i)$ are discontinuous when the actions are equal:

$$u_i(a_i, a_j|t_i) \equiv \begin{cases} t_i - a_i & \text{if } a_i > a_j \\ \frac{t_i}{2} - a_i & \text{if } a_i = a_j \\ -a_i & \text{if } a_i < a_j. \end{cases} \tag{2.2}$$

[10] Letting $\bar{a}_i$ be greater than player i's largest possible valuation is sufficient. This is just to ensure that there are actions large enough that no player will choose. See Contat (2014) and Chapter 3 for a discussion of equilibrium existence and characterization when minimum and/or maximum constraints on the action space are imposed.

In fact payoffs are neither upper semi-continuous nor lower semi-continuous in each player's action a_i. I show later that despite this fact the equilibrium correspondence is in fact continuous for the APA.

With finite types, we can define a mixed strategy for player i as a finite vector of functions, where each coordinate corresponds to a different probability distributions used by a type: $G_{i,n}(x) \equiv \big(G_{i,n}(x|t^1_{i,n}), \ldots, G_{i,n}(x|t^n_{i,n})\big)$. Here $G_{i,n}(x|t_i)$ is the probability that type t_i chooses an action less than or equal to x.

Defining mixed strategies in this way introduces measurability problems for uncountable state spaces, where it is not clear how to define a mixed strategy in such an environment.[11] Additionally, in the APA equilibria with finite-types are in mixed strategies while equilibria with continuous-types will use pure strategies. To circumvent these technical difficulties, I use distributional strategies to embed both type spaces and strategies into a common environment. Formally, a distributional strategy $\mu_{i,n}$ for player i is a joint probability measure on $A_i \times T_{i,n}$ with the restriction that the marginal over the type space agrees with the player's type distribution: for all x, $\mu_{i,n}(A_i, (-\infty, x]) \equiv \lim_{a\to\infty} \mu_{i,n}((-\infty, a], (-\infty, x]) = F_{i,n}(x)$. In other words, $\mu_{i,n}(B, S)$ is the probability that player i will both place an action in $B \subset A_i$ and also have a type that is located in the set $S \subset T_{i,n}$, where it is understood that B and S are measurable sets.

The equilibrium construction with finite types is characterized in terms of c.d.f. behavior $G_{i,n}$. It is straightforward to translate this same behavior into the form of distributional strategies. I say that a distributional strategy $\mu_{i,n}$ and finite-type behavior $G_{i,n}$ are *consistent* with each other if for each measurable set T,

$$\mu_{i,n}((-\infty, x], T) = \sum_{t\in T\cap T_{i,n}} p^t_{i,n} G_{i,n}(x|t), \tag{2.3}$$

[11] See Aumann (1964) for a more detailed and thorough discussion of this idea.

where the measure is 0 if $T \cap T_{i,n} = \emptyset$. If $\mu_{i,n}$ is consistent with $G_{i,n}$ for each i, let $\sigma_{i,n}(a_i|\mu_{j,n}) \equiv \mu_{j,n}((-\infty, a_i], T_{j,n})$ be the associated probability that i wins with action a_i when player $j \neq i$ uses mixed strategies $G_{j,n}$. That is, it is the probability that all of j's types together (in expectation) will place an action less than a_i. This is the fundamental equilibrium object for the APA. We will see that monotonicity of the game ensures that this is a concave function, and hence continuous.

If each i uses distributional strategy $\mu_{i,n}$ consistent with some $G_{i,n}$ then i's *interim payoffs* $U_i(a_i|\mu_{j,n}, t_i)$ are what i would expect to receive in payoffs given that j is using $\mu_{j,n}$:

$$U_i(a_i|\mu_{j,n}, t_i) \equiv t_i \; \sigma_{i,n}(a_i|\mu_{j,n}) - a_i. \tag{2.4}$$

2.2.3 Equilibrium Characterization

Now I define an equilibrium for this game. The notion of equilibrium is the standard Bayesian Nash Equilibrium. I define the support of each $G_{i,n}(\cdot|t_i)$ as the closure of the actions where the c.d.f. is increasing.

Definition 1 *A pair of strategy profiles $(G_{1,n}, G_{2,n})$ consistent with $(\mu_{1,n}, \mu_{2,n})$ is an equilibrium for the APA with type-distributions $(F_{1,n}, F_{2,n})$ if for all i, for all $t_i \in T_{i,n}$, for all $\hat{a}_{t_i}$ in t_i's support, and all $a \in A_i$,*

$$U_i(\hat{a}_{t_i}|\mu_{j,n}, t_i) \geq U_i(a|\mu_{j,n}, t_i) \tag{2.5}$$

With a finite number of types, Siegel (2013) has shown that there exists a unique equilibrium in mixed strategies where types mix over disjoint intervals. This equilibrium is monotonic in that higher types mix over higher intervals and hence place higher actions with probability 1. My main result shows that these intervals shrink to single point for each type and monotonicity is preserved. I present his equilibrium existence result for completeness:

Proposition 1 (Siegel (2013)) *For every pair of finite-type distributions* $(F_{1,n}, F_{2,n})$*, there exists a unique equilibrium in mixed strategies* $(G_{1,n}, G_{2,n})$*, that is consistent with some* $(\mu_{1,n}, \mu_{2,n})$*, with the following properties:*

- ***Monotonicity****: If* $t_i > t_i'$*, then* $G_i(\cdot|t_i) \leq G_i(\cdot|t_i')$*, with equality only if both are equal to 0 or 1.*

- ***Absolute Continuity****: for each* $t_i \in T_{i,n}$*,* $G_{i,n}(x|t_i)$ *is absolutely continuous and piecewise-linear in* x *for all* x*. This implies* $U_i(a_i|\mu_{j,n}, t_i)$ *is continuous in* a_i*.*

- ***Concavity****: for all* i*,* $\sigma_{i,n}(a_i|\mu_{j,n})$ *is an increasing concave function of* a_i*. This implies* $U_i(a_i|\mu_{j,n}, t_i)$ *is concave in* a_i *and that* $U_i(a_i|\mu_{j,n}, t_i)$ *is strictly decreasing for large enough* a_i*.*

Note that for each type t_i, $U_i(a_i|\mu_{j,n}, t_i)$ either has a maximum at 0 or attains its maximum over an interval. In the limit the interim payoffs will be maximized either at 0 or a single action.

If types are independently drawn from some absolutely continuous $F_{i,*}$, Amann and Leininger (1996) have shown that there exists a unique equilibrium in pure strategies. Further, the equilibrium is monotonic in that higher types place higher actions with probability 1.

Proposition 2 Amann and Leininger (1996) *For every pair of absolutely continuous (with respect to Lebegsue measure) distributions* (F_1^*, F_2^*)*, if there is an equilibrium, then it is unique and in pure strategies* $(a_1^*(\cdot), a_2^*(\cdot))$ *with the following properties:*

- ***Monotonicity****: if* $t < t'$ *then* $a_i^*(t) \leq a_i^*(t')$ *with equality only possible if* $a_i^*(t) = a_i^*(t') = 0$*.*

- ***Concavity****: the probability that* i *wins with* a_i *is strictly concave in* a_i*. Hence* interim payoffs $U_i(a_i|\mu_{j,n}, t_i)$ *are continuous in* a_i*.*

My main result shows that the equilibrium described in Proposition 1 converges to the equilibrium described in Proposition 2.

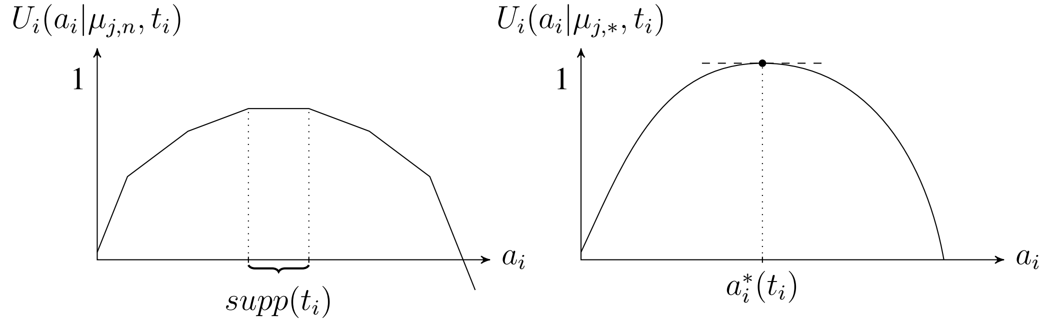

Figure 2.2: Concavity of $U_i(a|\mu_{j,n}, t_i)$

Indifference conditions require that for all actions a_i in t_i's support where $\sigma_{i,n}(a_i|\mu_{j,n})$ is differentiable that $\frac{d\sigma_{i,n}(a_i|\mu_{j,n})}{da_i} = \frac{1}{t_i}$. Monotonicity ensures that higher actions are chosen by higher types. Concavity of $\sigma_{i,n}(a_i|\mu_{j,n})$ follows immediately for both finite types and continuous types. Concavity implies that each type's marginal gain of increasing a_i is decreasing function, so that eventually the marginal cost of increasing a_i will be larger. The concavity of interim payoffs implies that interim payoffs are maximized over an interval for finite types and over a single point for continuous types. This is illustrated in Figure 2.

Hence each player's equilibrium distribution of behavior is determined in such a way as to make the *other* player indifferent in equilibrium. Suppose player j adopts equilibrium strategy $G_{j,n}$ consistent with some $\mu_{j,n}$. The problem player i then faces is to choose the $a \geq 0$ that maximizes the concave function $U_{i,n}(a|\mu_{j,n})$. Since this function is continuous, and the domain for a is compact, a maximum exists. There are many $G_{j,n}$ functions that can accomplish this. An equilibrium will require that the $G_{j,n}$ will be chosen in such a way that the induced behavior of player i, $G_{i,n}$, will simultaneously make j indifferent.

Using the insight of Milgrom and Weber (1985), we see that from player i's perspective it does not matter whether player j has a finite number of types or a continuum of types. What does matter for i is the expected distribution of j's actions, specifically the probability that i can expect to win with action a. Recall this is exactly the definition of $\sigma_{i,n}(a, \mu_{j,n})$. Each type t_i of i compares the marginal expected benefit from increasing his action $t_i \frac{d\sigma_i(a_i|\cdot)}{da_i}$ with the marginal expected cost of -1. The previous two propositions show that the cardinality of j's type space only affects how "smooth" $\sigma_i(a_i|\cdot)$ is. Figure 3 below illustrates this point. The left graph illustrates the finite-type APA while the right graph is the continuous-type case. In equilibrium type t_i maximizes his payoffs by choosing an a_i^* where $t_i \frac{d\sigma_i(a_i^*|\cdot)}{da_i} - 1 = 0$, or equivalently, where $\frac{d\sigma_i(a_i^*|\cdot)}{da_i} = \frac{1}{t_i}$.

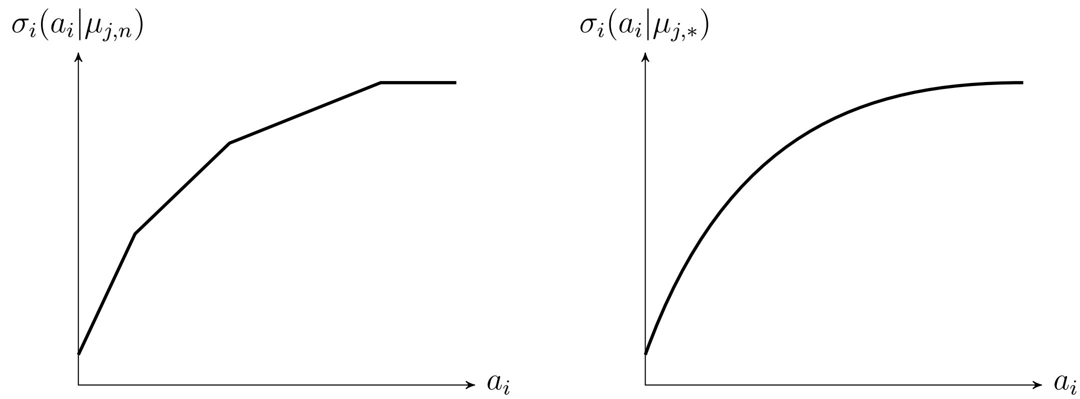

Figure 2.3: Concavity of $\sigma_i(a_i|\cdot)$

2.3 Convergence of Equilibria

Recall that a measure $\mu_{i,n}$ defined on measurable subsets of $\mathbb{R}^2$ converges in distribution to $\mu_{i,*}$ (also defined on measurable subsets of $\mathbb{R}^2$), written $\mu_{i,n} \to_d \mu_{i,*}$, if $\mu_{i,n}((-\infty, x_1], (-\infty, x_2]) \to \mu_{i,*}((-\infty, x_1], (-\infty, x_2])$ for all (x_1, x_2) where $\mu_{i,*}((-\infty, x_1], (-\infty, x_2])$ is continuous. In other words convergence in distribution means that the c.d.f.'s converge point-wise everywhere the limiting probability distribution is continuous.

For the finite game, define the equilibrium support of type $t_{i,n}^k$ as $supp_n(t_i^k) \equiv \{a \in A_i : g_{i,n}^k(a) \neq 0\}$. For finite type spaces in the APA this is well defined. This is just the equilibrium interval where $t_{i,n}^a$ places weight. In practice this depends upon the type space c.d.f.'s, but we suppress the notation as $\{F_{i,n}\}_n$ and $\{F_{j,n}\}_n$ are taken as given. Below is a graph that illustrates the equilibrium supports with the thick shaded lines. The support for each type has a piece-wise constant density associated with it. I later argue that the support in the limit is just a single point, which matches the equilibrium characterization given by Amann and Leininger (1996).

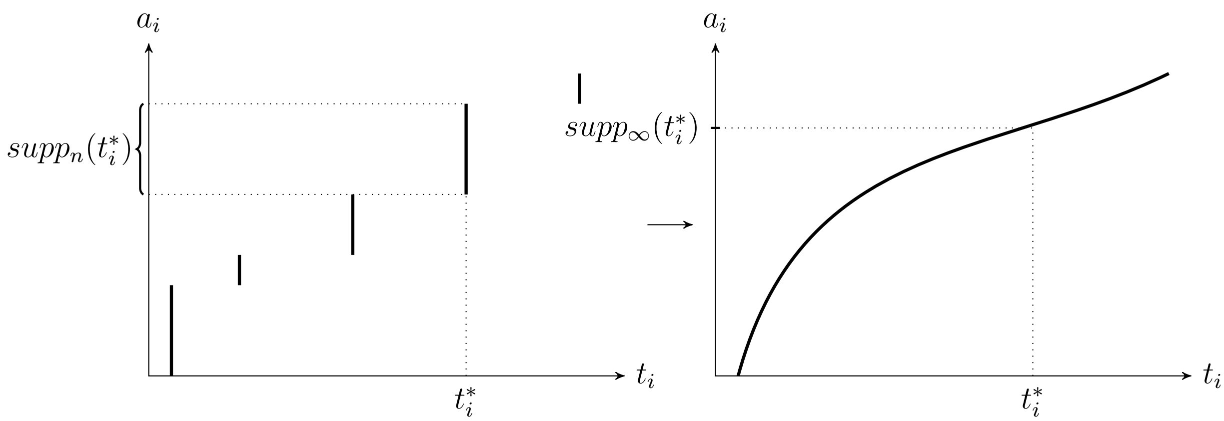

Figure 2.4: Convergence of Equilibrium Supports

2.3.1 Symmetric Convergence of Equilibria

Before moving to the general asymmetric case, I start with the simple case of symmetric players to illustrate the main features of the convergence. With symmetric type spaces, the k^{th} type of both players will mix uniformly with density $g_{1,n}^k(x) = g_{2,n}^k(x) = g_n^k(x) = \frac{1}{p_n^k v_n^k}$ over the interval $[b_n^{k-1}, b_n^k]$, where $b_n^0 = 0$ and $b_n^k = \sum_{r=1}^k p_n^k v_n^k$. As the number of types increases, necessarily we will have that $p_n^k \to 0$ and hence $g_n^k \to \infty$. This implies that each type's equilibrium support, if it converges, will converge to single point. Also note that for each n, types $t_{i,n}^k$ and $t_{j,n}^k$ are the only ones that are matched. Each player faces his mirror type in competition. This only holds with symmetric players. Thus monotonicity ensures that the probability that $t_{i,n}^k$ wins is the probability that $t_{i,n}^k$ meets types $t_{j,1n}^1, \ldots, t_{j,n}^k$.

Note that previously I have indexed types in order of greatest valuation, where $t_{i,n}^k$ is the k^{th} lowest valuation for player i. It will be more useful to describe the upper boundary of each type's equilibrium support in a slightly different manner. For each $t \in T_{i,n}$, define $T_{i,n}^{\leq t} = \{t' \in T_{i,n} | t' \leq t, t \in \mathbb{R}_+\}$ as the set of all types less than or equal to t. Then define $c_{i,n}(t)$ as the upper bound of the support for type t under type space T^n:

$$c_{i,n}(t) = \sum_{t' \in T_{i,n}^{\leq t}} t' p_{t'}^n. \tag{2.6}$$

If $t \notin T_{i,n}$, define $c_{i,n}(t) = \inf\{c_{i,n}(\tau) : \tau \geq t, t \in T_{i,n}^{\leq t}\}$. For symmetric types, if $t \in T_{i,n}^{\leq t}$ then $c_{i,n}(t) = \int_0^t x dF^n(x)$.

Fixing a type, we can construct the sequence of corresponding upper bounds for that type's equilibrium behavior, i.e. the upper bound of the interval. This sequence must have at least one convergent subsequence by the Bolzano-Weierstrauss Theorem. Hence it is possible to construct a sequence of type spaces so that equilibrium behavior does converge, as Siegel (2013) shows. One would just have to choose the right sub-sequence of type spaces for one of the players given the other. We prove the stronger statement that every subsequence (of the upper bounds for each type) converges to the same limit. In other words, for *any* sequence of type spaces the corresponding equilibria converge as well.

The first Helly-Bray Theorem[12] implies that for all continuous and bounded functions $h : \mathbb{R} \to \mathbb{R}$ that $\int h(x)dF^n(x) \to \int h(x)dF^*(x)$. Note that we can write $c_{i,n}(t)$ as the integral (with respect to Lebesgue-Stieltjes measure[13]) of a continuous function, namely the identity function, and so the Helly-Bray Theorem implies convergence since the measure of all types less than t converges:

$$c_{i,n}(t) = \sum_{t' \in T_{i,n}^{\leq t}} t' p_{t'}^n = \int_0^t x dF^n(x) \to \int_0^t x dF^*(x).$$

Hence each $c_{i,n}(t)$ converges, so that in the limit each type $t_i \in T^\infty \equiv \cup_{n=1}^\infty T^n$ chooses action $c_{i,*}(t) \equiv \lim_{n\to\infty} c_{i,n}(t) = a_i^*(t_i)$ with probability 1. By construction this also extends $c_{i,*}(t)$ to types in $T^* \setminus T^\infty$, since $T_{i,\infty}$ is dense in $T_{i,\infty}$. Monotonicity pins down the behavior of all types in $T_{i,*}$ from the behavior of just types in $T_{i,\infty}$. Formally, there exists a unique continuous extension which is monotonic.

All that remains to be shown is that the limiting behavior is in fact an equilibrium. Recall that for each n, "equilibrium payoffs" are concave and hence continuous in each player's actions, meaning

[12]See Athreya and Lahiri (2010) Theorem 9.2.2 for a detailed statement and proof of this theorem.

[13]See Athreya and Lahiri (2010) pg. 25-27 for a more detailed discussion.

that fixing the strategies of the other players' to be the equilibrium strategies, each player has continuous payoffs. The limit of a sequence of concave functions is concave. Further, in the limit we have shown that each type has a unique maximizer meaning in the limit interim payoffs are strictly concave. Hence it is an equilibrium, and since the equilibrium is unique is the limit game, it is THE equilibrium.

The asymmetric equilibrium converges in a similar manner, though the proof requires careful attention to detail. With symmetric types, each type is matched with only one type on equilibrium, namely its mirror image type of the other player. Further the matchups do not change over time. With asymmetric equilibrium, I must account for both of these possibilities. However in the limit I show later that each type must be matched up with exactly one opposing type of the other player due to monotonicity.

2.3.2 Asymmetric Convergence of Equilibria

We now prove the more general case for asymmetric type spaces. The result follows from three lemmas. The first lemma shows that for each type $t_i \in T_{i,\infty} \equiv \cup_{n\geq 1} T_{i,n} = \mathbb{Q} \cap T_{i,*}$, the equilibrium support converges to a single point.

Lemma 1 *Fix a sequence of* $\{F_{1,n}, F_{2,n}\}_n$. *For each* $t_i \in T_{i,\infty}$, $supp_\infty(t_i) \equiv \lim_{n\to\infty} supp_n(t_i) = \{a\}$ *for some* $a \in \mathbb{R}_+$.

The proof of Lemma 1 is somewhat involved due to the number of possible equilibrium configurations. Similar to the symmetric case proof, I know that the measure of the interval each type mixes over shrinks to 0. I also show that the support of each type can be written as an integral of a continuous function with respect to the type space distribution $F_{i,n}$. Convergence in distribution implies that these integrals converge. The sequence of upper bounds for each types interval is a bounded sequence and hence has a convergent subsequence. I show that in fact the sequence itself is convergent by showing whenever the type space is iterated each type's upper bound changes by at most ϵ. This precludes two different sub-sequential limits.

Proof 1 *This proof builds on the example in the online appendix of Siegel (2013), who shows there* *exist* some *sequence of finite-type space distributions. that converges to a continuous type equilibrium of Amann and Leininger (1996). I complete his proof by showing that for* any *sequence* $(F_{1,n}, F_{2,n})$ *of type space distributions, the associated equilibrium converge as well. The key property is the monotonicity of the equilibrium.*

Let $F_{i,n} \to_d F_{i,*}$*, where* $F_{i,*}$ *is absolutely continuous with respect to the Lebesgue measure. Further let the support of* $F_{i,*}$ *be* $[\underline{t_i}, \overline{t_i}]$*, where* $\underline{t_i} > 0$*. Thus there exists a density* $f_{i,*}$ *such that* $F_{i,*}(x) = \int_{\underline{t_i}}^{x} f_{i,*}(v)dv$*. Using the insights of Siegel (2013) and Athey (2001), we can partition* $[\underline{t_i}, \overline{t_i}]$*, with* $\{q_{i,n}(k)\}_{k=0}^{n}$*, where* $q_{i,n}(0) = \underline{t_i}$ *and* $q_{i,n}(n) = \overline{t_i}$*, such that*

$$\int_{q_{i,n}(k-1)}^{q_{i,n}(k)} f_{i,*}(v)dv = p_{i,n}^{k}. \tag{2.7}$$

Note that for each $t_i \in T_{i,*}$ *and for each* n*, there is exactly one* k *such that* $t_i \in [q_{i,n}(k-1), q_{i,n}(k)]$*. Note that* t_i *need not be a member of* $T_{i,n}$*. Hence with each* $t_i \in T_{i,*}$ *there exists a unique sequence of upper bounds of* $[q_{i,n}(k-1), q_{i,n}(k)]$ *which we denote by* $\{q_{i,n}(k_n)\}_{n=1}^{\infty}$*. Further this sequence must converge for any sequence of* $F_{i,n}$ *since* $\lim_{n\to\infty} T_{i,n} \equiv T_{i,\infty}$ *is dense in* $T_{i,*}$*.*

Additionally, for all $t_i \in [q_{i,n}(k-1), q_{i,n}(k)]$*, which includes* $t_{i,n}^{k}$*, we associate the upper bound of* $t_{i,n}^{k}$*'s support of actions as* $a_{i,n}(k)$*. Forming a new sequence* $\{a_{i,n}(k_n)\}_{n=1}^{\infty}$ *, we see that there is a one-to-one correspondence between* $q_{i,n}(k_n)$ *and* $a_{i,n}(k_n)$*. Since each* $t_i \in [\underline{t_i}, \overline{t_i}]$ *is associated with a unique limit* $\lim_{n\to\infty} q_{i,n}(k_n)$*, we see that* $\lim_{n\to\infty} a_{i,n}(k_n)$ *exists and is unique for each* t_i*. Hence each* t_i *is associated with a unique action in the limit.*

The second lemma shows that for each type $t_i \in T_{i,\infty}$ interim payoffs are maximized at the limiting action when the other player j's types are drawn from $T_{j,\infty}$ according to $F_{i,\infty} = F_{i,*}$.

Lemma 2 *Let* $a_i^{\infty}(t_i)$ *denote the action chosen with probability 1 in the limit for each* $t_i \in T_{i,\infty}$*. Then for all* $a \in A_i$*,*

$$U_i(a_i^{\infty}(t_i)|\mu_{j,\infty}, t_i) \geq U_i(a|\mu_{j,\infty}, t_i) \tag{2.8}$$

Proof 2 *The limit of a sequence of concave functions is concave. Further, we can say the function $U_i(a_i^\infty(t_i)|\mu_{j,\infty}, t_i)$ is strictly concave since we know it has a unique maximizer from Lemma 1. Thus the limit of both the upper and lower bound of $supp_n(G_{i,n}^k)$ converge to a single point. The continuity of the maximum follows from th Berge-Debreu Theorem of the Maximum. See Ok (2007) pg. 306 for a formal description and proof.*

The final lemma argues that since $T_{i,\infty}$ is dense in $T_{i,*}$, behavior for all types in $t_i \in T_{i,*} \setminus T_{i,\infty}$ is pinned down and constitutes equilibrium behavior.

Lemma 3 *For all types $t \in T_{i,*}$ define $a_i^*(t) \equiv \inf\{a_i^\infty(t')|t \geq t' \in T_{i,\infty}\}$ and let $\sigma_{i,*}(a|\mu_{j,*})$ be the associated winning probability for i when Then $a_i^*(t)$ is the equilibrium of the APA with type spaces $F_{1,*}$ and $F_{2,*}$. Specifically we have for all $t \in T_{i,*}$ and all $a \in A_i$ that:*

$$U_i(a_i^*(t)|\mu_{j,*}, t) \geq U_i(a|\mu_{j,*}, t). \tag{2.9}$$

Proof 3 *Note that $T_{i,\infty}$ is dense in $T_{i,*}$ so there exists a unique continuous extension of $a_i^\infty(t)$ defined on all of $T_{i,*}$. Since $a_i\infty(t)$ is increasing, so too will be $a_i*(t)$. In fact, for each $t_{i,\infty} \in T_{i,\infty}$ and each $t_{i,*} \in T_{i,*}$ the measures $\mu_{j,*}$ and $\mu_{j,\infty}$ are indistinguishable for player i.*

My main result, Theorem 1, now follows from the 3 Lemmas.

Theorem 1 *If $F_{i,n}(x) \rightarrow_d F_{i,*}(x)$ for all i, then the unique equilibrium distributional strategies $\mu_{i,n} \rightarrow_d \mu_{i,*}$.*

Proof 4 *Let $A \subset A_i$ and $T \subset T_i$ be measurable subsets with respect to Lebesgue measure on $\mathbb{R}$. Define the the c.d.f. of $\mu_{i,n}(A,T)$ as $H_{i,n}(a,t) = \mu_{i,n}((\infty, a], (\infty, t])$. To show $\mu_{i,n}(A,t) \rightarrow^d \mu_{i,*}^d(A,T)$, we need to show that for every fixed (a,t) that $H_{i,n}(a,t) \rightarrow H_{i,*}(a,t)$, where the convergence is the usual notion of point-wise convergence. Since $F_{i,n}(x) \rightarrow F_{i,*}(x)$, we know that $H_{i,n}(a,t) \rightarrow H_{i,*}(a,t)$ for sufficiently large a above the equilibrium support. We now extend the result to lower values of a.*

By assumption $F_{,i,n}(t) \to F_{i,*}(t)$ for all t. Further this convergence is uniform since $F_{i,*}$ is absolutely continuous (and hence continuous) in t.

Note that (with slight abuse of notation) I can rewrite $H_{i,n}(a,t)$ using conditional probability as

$$H_{i,n}(a,t) = \frac{\mu_{i,n}((\infty,a]|t_i \in (\infty,t])}{\mu_{i,n}((\infty,t])} = \frac{\sum_{\hat{t} \in T_{i,n} \cap (\infty,t]} p_{i,n}^{\hat{t}} G_{i,n}(a|\hat{t})}{F_{i,n}(t)} \tag{2.10}$$

The denominator in the above expression converges by the assumption of converging type space distribution. The numerator converges from Lemmas 1-3.

2.4 Conclusion

There is no qualitative difference between using finite types or continuum types. With the appropriate measures of the "closeness" of information structures, as long as a finite-type distribution is close to a continuous-type distribution, the differences in equilibrium behavior are negligible. Interesting future research might extend the results to more general classes of monotonic games with finite types. The technique adapted in the proof would require minor modifications. Additionally a simple extension would include showing that the finite-type to finite-type,and continuous-type to continuous-type convergence holds as well.

CHAPTER 3

COMPARATIVE STATICS IN THE ALL-PAY AUCTION

3.1 Introduction

In this chapter I show that asymmetry has different qualitative effects on equilibrium behavior, even when an arbitrarily small amount of incomplete information is introduced. Here asymmetry means differences in the distributions over the players' type spaces. These differences can be thought of as differences in valuations or differences in probabilities over those valuations. I provide a characterization when revenues will increase for these different comparative statics. With complete information, Hillman and Riley (1989) show that similar asymmetry *must* decrease revenues. Hence asymmetry has different effects of behavior when there is private information. The main focus of this chapter is on understanding this discontinuity of revenues with respect to the information structure.

Adding asymmetry in valuations induces what I label an *assimilation effect*: players bid more when facing tougher competition and bid less when facing weaker competition. Asymmetry changes the competition that every type faces with incomplete information while with complete information the matchup is fixed. Asymmetry in probabilities induces slightly different comparative statics. I do not focus heavily on this type of comparative static because the next chapter uses the intuition gained from valuation comparative statics only. Generally speaking, for both types of asymmetry the auctioneer will collect more from higher types and less from lower types. In a sense, lower types are discouraged from bidding as much while higher types are encouraged to bid more when equality between the players is lessened. If the revenue-increasing high types are more likely than the revenue-decreasing low types, then expected revenues will tend to increase. Each player bids to the level of his opponent and no more.

The assimilation effect is present in the complete information game, though in partial form. Asymmetry in the complete information game is analogous to asymmetry in the lowest possible valuation for the players. In both cases revenues must decrease because no matchups between the players can change. Only in the incomplete information game does there exist a *stacking effect*, whereby monotonicity of the equilibrium implies that *ceteris paribus* if low types bid more on average than so must high types. Alternatively stated, if lower types bid more on average than the bids of higher players are bumped up.

This chapter also contributes to a small literature on the comparative statics of APAs. In this chapter, I slightly perturb symmetric type spaces and determine the effect on each player's expected bid. Kirkegaard (2013) and Fibich, Gavious, and Sela (2004) perform similar comparative statics. Both find that revenues must increase when one player is made stronger. Kirkegaard (2013) considers a perturbation that increases the support of valuations[1]. We show that his assumption of the lower bound of the support being zero is important in determining comparative statics, as a negative term in the change in revenues disappears, making revenues increase.

Fibich et al. (2004) considers perturbations where the upper support of valuations is fixed[2]. In contrast, with finite types I show that performing the same comparative static may actually *decrease* revenues, as in the complete information case. I show in Section 3.4 that even with two types, performing the same comparative statics as Fibich et al. (2004) gives the opposite predictions. I conjecture the discrepancy arises because of differentiability assumptions[3]. This is a surprising result since recently Contat (2013b) (and also Chapter 1 of this dissertation) has shown that equilibrium bidding behavior in any continuum type model (including those of Kirkegaard (2013) and Fibich et al. (2004)) can be approximated to any degree of accuracy by finite-type equilibrium where type spaces are sufficiently close.

---

[1]Specifically the type space of a bidder is changed by a particular first order stochastic shift: $F_i(v) = F(\frac{v}{\bar{v}_i})$ for some F common to both players. The comparative static is on $\bar{v}_i$.

[2]Specifically, each has CDF type distribution of the form $F_i(v) = F(v) + \epsilon H_i(v)$, where F and F_i are both over a common support $[\underline{v}, \bar{v}]$.

[3]We only require right differentiability, but Fibich et al. (2004) requires differentiability.

I also show that the comparative statics of APAs are not robust to incomplete information, despite the equilibrium strategies themselves being robust. Our notion of robustness requires that adding *any* small amount of incomplete information not change equilibrium strategies too much[4]. Kirkegaard (2013) finds a similar result, where asymmetry always causes revenues to increase, the opposite conclusion of the complete information case. I show there always exists arbitrarily small amounts of information where revenues decrease. Unlike Kirkegaard (2013) however, we show that there always exist type spaces where revenues may decrease as well. This shows the incomplete information case is not always a "knife-edge" case. The way in which incomplete information is added matters. I show that if the players believe higher types are more likely than lower types, incomplete information has qualitatively similar results to the complete information case. Incomplete information will generate more revenue as long as bidders are optimistic, in the sense that adding incomplete information lowers the expected valuation of each player. For example, suppose both bidders have valuation v_m with probability $1-\epsilon$ for some arbitrarily small $\epsilon > 0$. Depending upon how bidders assign valuations to the remaining weight ϵ, an auctioneer might be hurt or helped by asymmetry in v_m between the bidders. If the remaining weight ϵ is placed on some $v_h > v_m$ common to both bidders, asymmetry in v_m will always reduce revenues. However if the weight ϵ is placed on some $v_l < v_m$, asymmetry in v_m can increase revenues. Hence the effects of asymmetry on incomplete information games depend very crucially upon the information structure.

The general finding is that while asymmetry *always* reduces revenues when there is complete information, asymmetry can actually increase revenues when there is incomplete information. The reason is that it matters in which direction one goes when one introduces small amounts of uncertainty. Alternatively stated, it matters which sequence of type space distributions you approach the complete information game with. Despite the fact that equilibrium behavior will converge for all such sequences, it is not the case that the *changes* in revenues due to asymmetry will converge.

---

[4] See Kajii and Morris (1997) for a more detailed motivation for this definition. A game is robust if adding arbitrarily small amounts of uncertainty ensures equilibrium behavior doesn't vary too much, for *any* way incomplete information can be added.

3.2 Complete Information Comparative Statics

First I present and further develop the results of the seminal paper Hillman and Riley (1989), who analyze the complete information APA. There are two risk neutral players, each of whom submits a non-refundable bid. The highest bid wins the indivisible prize. The valuations of the prize for both players are common knowledge.

3.2.1 Symmetric Equilibrium Benchmark

If the players have the same valuation v, then Hillman and Riley (1989) show that in equilibrium each player mixes uniformly with density $g(x) = \frac{1}{v}$ over the interval $[0, v]$. The auctioneer expects to collect $\int_0^v \frac{1}{v} x dx = \frac{v}{2}$ from each player, and hence v overall. There will be full surplus extraction by running an APA when there are symmetric players and complete information. Relaxing either symmetry between players or complete information[5] will cause this result not to hold. Additionally the APA is trivially *ex-post* efficient since both players are identical and one of them receives the prize with certainty.

3.2.2 The Assimilation Effect: A First Look

Now I introduce asymmetry into the complete information environment. Now the valuation of player i is v_i ($i = 1, 2$), where without loss of generality let $v_1 < v_2$. Hillman and Riley (1989) were the first to show that in equilibrium players 1 and 2 will both mix uniformly over $[0, v_1]$ with densities $g_1(x) = \frac{1}{v_2}$ and $g_2(x) = \frac{1}{v_1}$ respectively. Exactly like the all-pay auction where types are drawn from some absolutely continuous distribution, in the all-pay auction with finite types asymmetry between the players causes inefficiencies. With positive probability the player with the lower valuation wins, though the probability of this happening converges to 0 as asymmetry is increased.

In equilibrium, each player i's mixing density is set equal to the reciprocal of the opponent's valuation v_j: $g_i(x) = \frac{1}{v_j}$. This ensures that player i is indifferent between increasing or decreasing his bid over the equilibrium support $[0, \min\{v_1, v_2\}]$. The marginal benefit of bidding more is equal

[5] With private information, the auctioneer will collect an *ex-ante* full surplus, i.e. the total expected revenues will be the sum of each player's expected valuation. However, *ex-post* there will be types that are paid positive information rents with positive probability.

to the marginal cost for bids over this region. When a player becomes stronger, *he* will not change his behavior but *his opponent* will change her behavior. I will later show that with incomplete information asymmetry will cause both player's *lower* types to face different competition. So with incomplete information, there also exist feedback effects that change behavior for both players.

Note that the total weight expended by the weaker player (i.e. player 1) is strictly less than 1. Hence player 1 has "leftover" weight of $(1-v_1)\times\frac{1}{v_2}=\frac{v_2-v_1}{v_2}$. This residual weight must be placed as an atom at 0 in equilibrium. Increasing the degree of asymmetry between the players causes the weaker player to shift more weight from an interval over positive bids to an atom at 0. I say that in this case, due to the presence of increased competition one of the players is discouraged from bidding as much as before. This will be precisely the opposite intuition as the incomplete information case, where stronger competition induces players to bid more. The reason is that with only one type a piece, equilibrium matchups (there is only one) are fixed. The stronger player realizes that the weaker player is not willing to bid more and hence leaves his behavior unchanged. The weaker player therefore cannot bid more and is thus forced to not bid at all with positive probability.

There are slightly different comparative statics if one of the players is made weaker *ceteris paribus*. The equilibrium support decreases. Not only will the weaker player increase the size of her atom at 0 (thus decreasing revenues), but the stronger player will also on average bid less since he is mixing uniformly over a smaller interval. In this paper I focus on the less intuitive comparative static of making one player stronger, though the case of making a player weaker follows easily from my results.

It follows from the above that asymmetry of any form, meaning an increase in v_2-v_1 keeping one of the valuations fixed, will decrease expected revenues. The weaker player (with valuation v_1) will generate revenues of $Rev_2^C\equiv\frac{v_2-v_1}{v_2}\times 0+\int_0^{v_1}\frac{1}{v_2}xdx=\frac{v_1^2}{2v_2}<\frac{v_1}{2}$. The stronger player (with valuation $v_2>v_1$) will generate revenues of $Rev_1^C\equiv\int_0^{v_1}\frac{1}{v_1}xdx=\frac{v_1}{2}<\frac{v_2}{2}$. Hence for any valuations v_1 and v_2, with $0<v_1<v_2$, the total expected revenues collected from both players under a complete information all-pay auction is $Rev^C(v_1,v_2)\equiv Rev_1^C(v_1,v_2)+Rev_2^C(v_1,v_2)=$

$\frac{v_1^2}{2v_2} + \frac{v_1}{2} = \frac{v_1(v_1+v_2)}{2v_2}$.

Note that $Rev^C(v_1, v_2) < \frac{v_1+v_2}{2}$, meaning that the auctioneer will not be able to extract full surplus out of the APA when there is asymmetry between the players. Also note that $Rev^C(v_1, v_2) < v_2$, where v_2 would be the expected revenues collected *if* both players have identical valuations of v_2. Hence if one of two identical players with valuation v_2 is made weaker (i.e. now has valuation $v_1 < v_2$), then *ceteris paribus* total expected revenues must decrease. This is not a surprising result as lower valuations would suggest the player is not willing to exert as much costly effort and so will bid less on average. The surprising result is that $Rev^C(v_1, v_2) < v_1$ as well, meaning if one of two identical players with valuation v_1 is made stronger (i.e. one now has valuation $v_2 > v_1$), revenues will actually decrease. I collect these results in the following proposition.

Proposition 3 *Let players have valuations $v_1 = v$ and $v_2 = v + a$, where $a \in \mathbb{R}$. Then as a function of a, total expected revenues $Rev^C(v, v + a)$ have a global maximum at $a = 0$. Further $Rev^C(v, v + a)$ is increasing in v for a fixed a.*

Proof 5 *Suppose first that initially that both players have valuation v, and then player 2's valuation increases from $v \to v + a$ for some $a > 0$, holding v_1 constant at $v_1 = v$. The equilibrium support will not change for both players. Only the density for the weaker player will change, namely it will decrease. The residual weight will of course be placed as an atom at 0. Total revenues in this case can be computed from to be $Rev_C(v, v + a) = \frac{v^2}{2(v+a)} + \frac{v}{2}$, which are strictly decreasing and convex in a when $a \in [0, \infty]$. If on the other hand player 2 is made weaker by $a > 0$, i.e. $v_1 = v$ but $v_2 = v - a$, revenues will be $Rev_C(v, v - a) = \frac{v-a}{2} + \frac{(v-a)^2}{2v}$, which are also in strictly decreasing and convex when $a > 0$. Hence revenues decrease at an increasing rate in a.*

These results are illustrated in Figure 3.1 . Also note from Figure 3.1 that revenues are not differentiable at $\epsilon = 0$. It is precisely this "kink" in the revenues that leads to unusual comparative statics when incomplete information is added. Kirkegaard (2013) has pointed out the fact a similar "knife-edge" feature of information: adding a small amount of incomplete information may sharply change the comparative statics of revenues. Here holding information constant, we

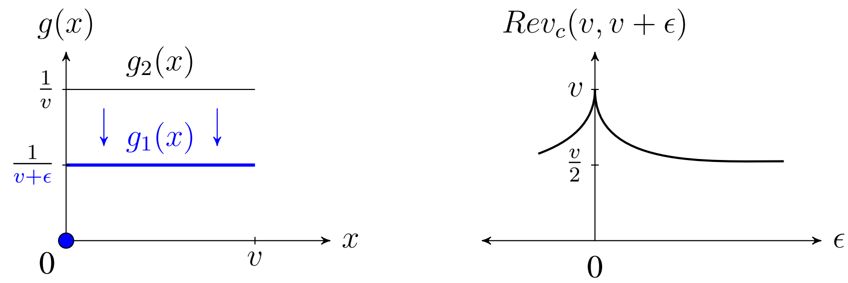

Figure 3.1: Complete Information Equilibrium and Revenues

see that asymmetry in the value of a causes a sharp change in revenues. Adding incomplete information may also sharply change expected revenues, but in a different way. There exist families of type space distributions , arbitrarily close to the (degenerate) complete information distributions, such that the change in revenues in the family of these distributions is uniformly bounded above some positive constant. Yet with complete information distributions, asymmetry must mean that the change in revenues is negative. Hence it is possible to construct a sequence of type space distributions that converge to the complete information distribution, yet the effects of asymmetry on revenues do not converge. It is in this sense that I mean the comparative statics are not continuous with respect to information. Small amounts of incomplete information may completely change how the auctioneer feels about asymmetry.

In summary, asymmetry is unambiguously bad for revenues in the complete information APA. Making the weaker player even weaker discourages the player from placing weight on positive bids. With positive probability asymmetry causes the weaker player to “give up”, i.e. bid nothing. The stronger player never increases his bid because he will only bid up to what his opponent is willing to bid. Asymmetry may actually reduce the expected bid of the stronger player.

3.3 Private Information Comparative Statics (Valuations)

Now consider the private information environment as presented in Chapter 2. I present the easier symmetric case first to highlight the general characteristics of equilibria before moving on to the equilibrium and comparative statics of the more general asymmetric case. The equilibrium characterization with private information is a simple extension of the complete information case. The key property, as was mentioned in Chapter 2, is the monotonicity of equilibria: higher types mix over higher disjoint intervals. This implies that if a lower type's interval is shifted up, and hence bids more on average, then a higher type's interval is also shifted up and hence bids higher, *ceteris paribus*.

Without loss of generality order the types of player i from smallest to largest: $T_i \equiv \{t_{i,1}, t_{i,2}, \ldots, t_{i,n}\}$ with corresponding probabilities $\{p_{i,1}, p_{i,2}, \ldots, p_{i,n}\}$. In other words, $t_{i,1} \leq t_{i,2} \leq \ldots \leq t_{i,n}$. Note that I assume players that can have an equal number of types, but this has no bearing on the results. As long as each player has a *finite* number of types, which includes the complete information case, all of the qualitative results will go through. I can also assume that each player can have only strictly positive valuation (i.e. $t_{i,1} > 0$) because any type with a valuation of zero has a strictly dominant strategy to bid 0 with probability 1. These types will have no bearing on the behavior of the other types nor the comparative statics which we develop later.

3.3.1 Review of Equilibrium Structure

Recall from chapter 1 that in equilibrium each type of each player will mix (piecewise) uniformly over an interval. Each change in uniform density corresponds to a new matchup, i.e. a new type of the other player that is mixing over the type's interval. If type $v_{1,k}$ is matched up with type $v_{2,m}$, the equilibrium densities are pinned down so that each type is indifferent over their interval of bids. The density of player 1 is chosen so that player 2 is indifferent and vice-versa. In equilibrium these densities are

$$g_{1,k}(x) = \frac{1}{p_{1,k}v_{2,m}} \qquad g_{2,m}(x) = \frac{1}{p_{2,m}v_{1,k}}. \tag{3.1}$$

A player's density changes only when either his opponent's valuation changes or the player's own probability changes. Thus it will matter whether a player becomes stronger by means of an increased valuation (which will not *directly* affect his bidding behavior) or by means of an increased valuation (which has a direct effect). Recall that these densities are part of a larger equilibrium construction. Increasing the level of asymmetry between the players will also change equilibrium matchups, which will change both player's densities. One density will increase, though this density is not always the stronger player's density. The other density must decrease. If a density increases, the length of the interval over which the player is mixing must compensate and decrease in order that probability is well defined. This is ultimately the cause of the assimilation effect, whereby a player responds to asymmetry by increasing his density and ultimately "compressing" his equilibrium interval of bids. Since the lower bound of bids is fixed at 0, this effect tends to decrease revenues. If a density decrease, the player must "stretch" his interval of equilibrium bids, thus increasing expected bids.

3.3.2 Symmetric Equilibrium Benchmark

Suppose that players are symmetric, i.e. that $v_{1,k} = v_{2,k} = v_k$ and $p_{1,k} = p_{2,k} = p_k$ for all $k = 1, \ldots, n$. If player i's realized type is v_k, then he will mix uniformly over $[b_{k-1}, b_k]$ with density $g_k(x) = \frac{1}{p_k v_k}$, where $b_0 = 0$ and $b_k = \sum_{r=1}^{k} p_r v_r$. Expected revenues from each player are then $\sum_{k=1}^{n} p_k \left(\int_{b_{k-1}}^{b_k} g_k(x) x dx \right) = \sum_{k=1}^{n} p_k \left(\frac{b_k + b_{k-1}}{2} \right) = \sum_{k=1}^{n} p_k \left(\frac{p_k v_k}{2} + \sum_{r=1}^{k-1} p_r v_r \right)$.

3.3.3 Asymmetric Equilibrium: The Assimilation and Stacking Effects

With asymmetric players the (unique) equilibrium in an incomplete information APA will not have a neatly written closed form in general. Fortunately the equilibrium construction is easy to describe. Szech (2012) was among the first to extend Hillman and Riley (1989) to private information games, though only with two types. Later Siegel (2013) extended the results for an arbitrary number of finite types (and also inter-dependent valuations, which are not considered here). I use the equilibrium construction of Siegel (2013) to study the comparative statics of the underlying type spaces.

As in chapter 2, we say that type $v_{1,k}$ of player 1 and type $v_{2,m}$ of player 2 are matched up if the intersection of their equilibrium supports is non-empty. More easily stated, each type mixes over an interval in equilibrium. If the intervals of $v_{1,k}$ and $v_{2,m}$ overlap we say they are matched up. The equilibrium is constructed in the following manner. The highest types of both players are matched up. This pins down their densities. Typically one type will be able to "fill up their weight" first. The densities can be thought of as the rate that each type fills weight/probability. With different rates, the total length required to expend a probability weight of 1 will vary, so that in general one type fills up before another (with probability 1). Since each type cannot expend more than 1 in weight/probability, there is nothing more this type can do. Thus one highest type will typically have residual weight that is used against the second highest type of the other player. When a matchup is formed one type has exactly a weight of 1 to fill and the other has a weight less than or equal to 1. The distribution of the type space pins down the densities for the matched up types, where in general types will have different densities and hence "fill" at different rates. One type (say $t_{i,k}$) fills up its weight first[6], then a new matchup is formed with the filling player's next highest type ($t_{i,k-1}$ and $t_{j,m}$) , now $t_{i,k-1}$ has a weight of 1 to fill and $t_{j,m}$ has some weight strictly less than one to fill, etc. Monotonicity is crucial to the construction, and as I show later, to the comparative statics of the equilibrium.

A comparative static in this chapter means a change in a type space distribution, specifically a shift in the c.d.f. F_i of a player's type space. In this dissertation I restrict attention to changes in F_i where one player is made unambiguously stronger in a first-order stochastic dominance[7] sense. Thus I rule out situations where a player's minimum valuation is increased (and hence made stronger) but also the same player's maximum valuation is decreased (and hence made weaker).

With finite types, there are only three ways that a player can experience a first order stochastic shift in his type space (and hence be made stronger) *ceteris paribus* : (1) a player's valuation may increase, (2) the probability of the player having high valuations may increase(and thus the probability of having a low valuation decreases), or (3) some combination or compounding of the first two effects. In the upcoming sections, I consider each of these comparative statics on type space distributions that are initially identical. I show that even the simplest of asymmetries between players can have complex effects on behavior. In a later section I will discuss the qualitative changes that will take place when more than one parameter in the type space has changed.

---

[6]It could be that the types fill up exactly together, though this happens with probability 0. Even if this were to happen the construction can still be applied though.

[7]Thus I consider only $\hat{F}_i$'s where $\hat{F}_i(x) \leq F_i(x)$ for all x, or for the weaker player $\hat{F}_j(x) \geq F_j(x)$ for all x.

First I consider changes in equilibrium behavior when one of a player's valuations is increased, *ceteris paribus*. It is useful to separate the cases into increasing the lowest valuation and increasing any other valuation. Increasing a player's lowest valuation in the private information environment, *ceteris paribus*, will have qualitatively the same features as the complete information environment. Increasing other (higher) valuations, however, will introduce new features that are not seen in the complete information environment.

Increased Lowest Valuation $\approx$ Complete Information Comparative Statics

Suppose that the players are initially symmetric but then $v_{1,1}$ is increased by some small $\epsilon > 0$ *ceteris paribus*, so that $v_{1,1} = v_1 + \epsilon$ while $v_{2,1} = v_1$. Following the equilibrium construction, the matchups and densities (and hence interval lengths) will remain unchanged for all but the lowest types of both players. The densities for the lowest types will be $g_{1,1}(x) = \frac{1}{p_l v_l}$ and $g_{2,1}(x) = \frac{1}{p_l(v_l+\epsilon)}$ for players 1 and 2 respectively. The length of the interval corresponding to this matchup does not change (since $v_{1,1}$ can still fill up his weight first with the same length of $p_1 v_1$), and hence the bidding intervals for all types do not change. Since no more (lower) types of player 1 remain with which $v_{2,1}$ can be matched, $v_{2,1}$'s extra weight must be placed as an atom at 0.

Hence all types of both players behave exactly the same after the asymmetry is introduced with the exception of the lowest type of the weaker player ($v_{2,1}$), who bids strictly less. Overall total expected revenues decrease, as in the complete information case. The stronger player does not increase his bidding even though he desires the object more because he doesn't have to. The weaker player is unwilling to bid more as well and so in equilibrium the stronger player will not bid more. Revenues also decrease when one of the players is made weaker. I summarize these results in the following proposition.

Proposition 4 *Let bidders be symmetric except that* $v_{1,1} = v_1 + a$ *and* $v_{2,1} = v_1$*, where* $a \in [-v_1, v_1]$*. Then as a function of* a*, total expected revenues are strictly increasing in* a *when* $a < 0$ *and strictly decreasing when* $a > 0$*. Hence as a function of* a*, where* $a \in [-v_1, v_1]$*, total expected revenues are maximized when* $a = 0$*.*

Increased Non-Lowest Valuation $\not\approx$ Complete Information Comparative Statics

Now consider the more general case of increasing a valuation for a type that is not the lowest for a player. Suppose that players are symmetric except that $v_{1,k}$ is increased by $\epsilon > 0$ for player 1 *ceteris paribus*, where $k > 1$. Using the equilibrium construction, none of the matchups and interval lengths for the types greater than $v_{1,k}$ will be changed. However, it will be the case that even though the *lengths* of the intervals for these larger types is unchanged, their *absolute position* will be changed[8]. Now consider the matchup of $v_{1,k}$ against $v_{2,k}$. The densities for this matchup are again pinned down to be $g_{1,k}(x) = \frac{1}{p_k v_k}$ and $g_{2,k}(x) = \frac{1}{p_k(v_k+\epsilon)}$. Type $v_{1,k}$ will be able to fill his weight in an interval of length $p_k v_k$. Now $v_{2,k}$ will have some leftover weight of $1 - (p_k v_k)\frac{1}{p_k(v_k+\epsilon)} = \frac{\epsilon}{v_k+\epsilon}$. Unlike the previous case however, there *is* a type of player 1, namely $v_{1,k-1}$, towards which $v_{2,k}$ can apply her remaining weight in a new matchup.

Before proceeding further in the equilibrium construction, some notation is needed. If type k of player 1 and type m of player 2 are matched up, the length of the interval where they are matched up will be denoted as $L_{k,m}$. Hence so far in the equilibrium construction we know that $L_{r,r} = p_r v_r$ for all $r \geq k$.

A new matchup is now formed between $v_{1,k-1}$ and $v_{2,k}$. Previously only each type's mirrored image was his matchup. As we will shortly see, the pattern now will be that all of player 1's remaining types face the mirrored type and also the next largest type. In other words, type $v_{1,r}$ will face $v_{2,r}$ and $v_{2,r+1}$ when $r < k$. Hence all of player 1's remaining types are matched up with stronger competition. Similarly each type of player 2 now faces the same mirror type, and also weaker competition.

I consider a few iterations of the equilibrium construction to emphasize the changing matchups. This is crucial to understanding the encouragement and discouragement effects. The densities for this new matchup are $g_{1,k-1}(x) = \frac{1}{p_{k-1}v_k}$ and $g_{2,k}(x) = \frac{1}{p_k v_{k-1}}$. As before, the length of the interval corresponding to this matchup will depend upon how long of an interval $v_{2,k}$ needs

[8]This is analogous to the two intervals $[1, 2]$ and $[5, 6]$. But have length 1, but the lower and upper bounds have changed.

to fill the remaining weight from the $v_{1,k}$ matchup [9]. To fill a weight of $\frac{\epsilon}{v_k+\epsilon}$ with a density of $g_{2,k}(x) = \frac{1}{p_k v_{k-1}}$, a length of $\frac{\epsilon p_k v_{k-1}}{v_k+\epsilon}$ is needed. Hence the length of the new interval from the matchup of $v_{1,k-1}$ and $v_{2,k}$ is $L_{k-1,k} = \frac{\epsilon p_k v_{k-1}}{v_k+\epsilon}$.

In the next matchup of $v_{1,k-1}$ and $v_{2,k-1}$, it will be the case that $v_{1,k-1}$ has expended some of his weight. He has weight of $1 - \frac{\epsilon p_k v_{k-1}}{v_k+\epsilon}\frac{1}{p_{k-1}v_k}$ when he faces $v_{2,k-1}$. Since these types have the same densities $g_{1,k-1}(x) = g_{2,k-1}(x) = \frac{1}{p_{k-1}v_{k-1}}$, but $v_{1,k-1}$ has less weight to fill, $v_{1,k-1}$ will be able to fill first. The length of the interval required for this matchup is $L_{k-1,k-1} = p_{k-1}v_{k-1} - \frac{\epsilon p_k (v_{k-1})^2}{(v_k+\epsilon)v_k}$. The first term represents the length of the interval in the symmetric case. Hence the length of this interval for the matchup of $v_{1,k-1}$ and $v_{2,k-1}$ has been decreased (since $v_{1,k-1}$ has already used up some of his weight).

We consider one more iteration of this process before generalizing the results. Now that $v_{1,k-1}$ has filled up his weight against $v_{2,k-1}$, a new matchup of $v_{1,k-2}$ and $v_{2,k-1}$ is formed. Now $v_{2,k-1}$ has remaining weight of $1 - L_{k-1,k-1} \times \frac{1}{p_{k-1}v_{k-1}} = \frac{\epsilon p_k v_{k-1}}{(v_k+\epsilon)p_{k-1}v_k}$. Densities for this new matchup are pinned down to be $g_{1,k-2} = \frac{1}{p_{k-2}v_{k-1}}$ and $g_{2,k-1} = \frac{1}{p_{k-1}v_{k-2}}$, where we have suppressed the argument (x) of the density because the density is constant. If ϵ is small enough, then $v_{2,k-1}$ will indeed be able to fill up her weight before $v_{1,k-2}$ does. This will require a length of $L_{k-2,k-1} = \frac{\epsilon p_k v_{k-1} v_{k-2}}{(v_k+\epsilon)v_k}$.

Now $v_{1,k-2}$ will be matched up with $v_{2,k-2}$, where $v_{1,k-2}$ has a weight of $1 - L_{k-2,k-1} \times \frac{1}{p_{k-2}v_{k-1}} = 1 - \frac{\epsilon p_k v_{k-2}}{(v_k+\epsilon)v_k p_{k-2}}$. Both densities are the same at $g_{1,k-2} = g_{2,k-2} = \frac{1}{p_{k-2}v_{k-2}}$. Of course $v_{1,k-2}$ will be able to fill up his weight first since they have the same density but $v_{1,k-2}$ has weight strictly less than 1 to fill. The length required to fill up his weight against $v_{2,k-2}$ is $L_{k-2,k-2} = p_{k-2}v_{k-2} - \frac{\epsilon p_k (v_{k-2})^2}{(v_k+\epsilon)v_k}$.

Continuing in this manner for all the rest of the types we see that the asymmetry will have an effect that ripples through all of the remaining (i.e. lower) types matchups and densities. The equilibrium construction will continue until $v_{1,1}$ fills up his weight against $v_{2,1}$, leaving $v_{2,1}$ with

[9] For very small values in ϵ, $v_{2,k}$ will be able to fill her weight since the weight will also be arbitrarily small

leftover weight to be placed as an atom at zero. In general, if $r < k$ then type $v_{1,r}$ will be matched up with types $v_{2,r+1}$ and $v_{2,r}$. The lengths of these intervals will be $L_{r,r+1} = \frac{\epsilon p_k v_r v_{r+1}}{(v_k+\epsilon)v_k}$ and $L_{r,r} = p_r v_r - \frac{\epsilon p_k (v_r)^2}{(v_k+\epsilon)v_k}$. Intuitively a small asymmetry changes the equilibrium in a small manner in that the size of the interval of the new matchups are $L_{r,r+1} < \epsilon$. Since there are a finite number of types, this implies that for small enough $\epsilon > 0$ the change in revenues can be made arbitrarily small. One final piece of notation is required before generalizing the change in the stronger player's behavior. For types $r < k$, define $\overline{b_r} = \sum_{t=1}^{r} L_{t,t} + L_{t,t+1}$ as the upper bound of type $v_{1,r}$'s equilibrium support.

Corollary 1 *For $r < k$,*

$$\frac{d}{d\epsilon} L_{r,r}(\epsilon) = -\frac{p_k (v_r)^2}{(v_k+\epsilon)^2} < 0 \tag{3.2}$$

$$\frac{d}{d\epsilon} L_{r,r+1}(\epsilon) = \frac{v_r v_{r+1} p_k}{(v_k+\epsilon)^2} > 0 \tag{3.3}$$

$$\frac{d}{d\epsilon} \overline{b_r}(\epsilon) = \frac{p_k}{(v_k+\epsilon)^2} \sum_{t=1}^{r} v_t (v_{t+1} - v_t) > 0. \tag{3.4}$$

The above corollary shows that the interval over which all of player 1's types mix is shifted upwards while only the intervals of player 2's highest types are shifted upwards. The lowest types of player 2 compress their equilibrium intervals. I now summarize the behavior of the stronger player after the asymmetry is introduced.

Proposition 5 *Let players be symmetric except $v_{1,k} = v_k + \epsilon$ and $v_{2,k} = v)k$, where $k > 1$ and $\epsilon > 0$. Equilibrium behavior of player 1 (the stronger player) is characterized by:*

Types $r < k$ *Type $v_{1,r}$ will mix over $[\overline{b_{r-1}}, \overline{b_{r-1}} + L_{r,r}]$ with density $g_{1,r} = \frac{1}{p_r v_r}$ and over $[\overline{b_{r-1}} + L_{r,r}, \overline{b_r}]$ with density $g_{1,r} = \frac{1}{p_r v_{r+1}}$.*

Type k *The perturbed type, $v_{1,k}$, will mix over $[\overline{b_{k-1}}, \overline{b_{k-1}} + p_k v_k]$ with density $g_{1,k} = \frac{1}{p_k v_k}$.*

Types $r > k$ *Type $v_{1,r}$ will mix over $[\overline{b_{k-1}} + \sum_{t=k}^{r-1} p_t v_t, \overline{b_{k-1}} + \sum_{t=k}^{r} p_t v_t]$ with density $g_{1,t} = \frac{1}{p_t v_t}$.*

Calculation of expected revenues for each type of player 1 is tedious but straightforward. Similar to the complete information environment, revenues are not differentiable at $\epsilon = 0$, so we focus instead on the right hand side derivative, which does exist. Since $\epsilon > 0$, by construction we are implicitly considering comparative statics when players become stronger. If $\epsilon < 0$ the equilibrium would be qualitatively different in that there would be different matchups, interval lengths ($L_{k,m}$'s), and densities. When I write the derivative of revenues, $\frac{dRev_{i,r}(0)}{d\epsilon}$, I mean $\frac{dRev_{i,r}(0)}{d\epsilon} \equiv \lim_{\epsilon \searrow 0} \frac{Rev_{i,r}(\epsilon) - Rev_{i,r}(0)}{\epsilon}$. In this context, it is equivalent to differentiating the revenues with respect to ϵ and then take the limit as $\epsilon \to 0$.

Let $Rev_{i,r}(\epsilon)$ be the product of the conditional expected revenue collected (conditional on type r realized for player 1) and the probability of the type. In other words, $Rev_{i,r}(\epsilon)$ is type $v_{i,r}$'s contribution to expected revenues. The sum $\sum_{t=1}^{n} Rev_{1,t}(\epsilon)$ then represents the total expected revenues collected from player 1. All of the comparative statics for the strong player are collected in the following proposition.

Proposition 6 *Let players be symmetric except that $v_{1,k} = v_k + \epsilon$ and $v_{2,k} = v_k$ for some $\epsilon > 0$. The rate of changes in revenues for player 1 are*

$$\frac{dRev_{1,t}(\epsilon)}{d\epsilon} = \frac{p_k p_t}{v_k^2} \sum_{r=1}^{t-1} v_r (v_{r+1} - v_r) > 0 \quad \textit{if } t \leq k \tag{3.5}$$

$$\frac{dRev_{1,t}(0)}{d\epsilon} = \frac{p_t p_k}{v_k^2} \sum_{r=1}^{k-1} v_r (v_{r+1} - v_r) > 0 \quad \textit{if } t > k \tag{3.6}$$

Proof 6

$$Rev_{1,r}(\epsilon) = p_r \left(\int_{\overline{b_{1,r-1}}}^{\overline{b_{1,r-1}}+L_{r,r}} \frac{1}{p_r v_r} x dx + \int_{\overline{b_{1,r-1}}+L_{r,r}}^{\overline{b_{1,r}}} \frac{1}{p_r v_{r+1}} x dx \right) \tag{3.7}$$

$$= \frac{L_{r,r}(2\overline{b_{1,r-1}} + L_{r,r})}{2v_r} + \frac{L_{r,r+1}(2\overline{b_{1,r}} - L_{r,r+1})}{2v_{r+1}} \tag{3.8}$$

$$\frac{dRev_{1,r}(\epsilon)}{d\epsilon} = \frac{1}{v_r} \left[L_{r,r} \frac{d\overline{b_{r-1}}}{d\epsilon} + \overline{b_{r-1}} \frac{dL_{r,r}}{d\epsilon} + L_{r,r} \frac{dL_{r,r}}{d\epsilon} \right] \tag{3.9}$$

$$+ \frac{1}{v_{r+1}} \left[L_{r,r+1} \frac{d\overline{b_r}}{d\epsilon} + \overline{b_r} \frac{dL_{r,r+1}}{d\epsilon} - L_{r,r+1} \frac{dL_{r,r+1}}{d\epsilon} \right] \tag{3.10}$$

$$\frac{dRev_{1,r}(0)}{d\epsilon} = \frac{1}{v_r} \left[p_r v_r (\frac{p_k}{v_k^2}) \sum_{t=1}^{r-1} v_t (v_{t+1} - v_t) + (\sum_{t=1}^{r-1} p_t v_t)(-\frac{p_k v_r^2}{v_k^2}) + p_r v_r (-\frac{p_k v_r^2}{v_k^2}) \right] \tag{3.11}$$

$$+ \frac{1}{v_{r+1}} \left[0 + (\sum_{t=1}^{r} p_t v_t) \frac{v_r v_{r+1} p_k}{v_k^2} - 0 \right] \tag{3.12}$$

$$= \frac{p_r p_k}{v_k^2} \sum_{t=1}^{r-1} v_t (v_{t+1} - v_t) - \frac{p_k v_r}{v_k^2} \left(\sum_{t=1}^{r} p_t v_t - p_r v_r \right) - \frac{p_r p_k v_r^2}{v_k^2} + \frac{v_r p_k}{v_k^2} \left(\sum_{t=1}^{r} p_t v_t \right) \tag{3.13}$$

$$= \frac{p_k p_r}{v_k^2} \sum_{t=1}^{r-1} v_t (v_{t+1} - v_t) > 0 \tag{3.14}$$

Now consider the revenues of the perturbed type $v_{1,k}$, who mixes over $[\overline{b_{k-1}}, \overline{b_{k-1}} + p_k v_k]$ with density $g_{1,k} = \frac{1}{p_k v_k}$. Revenues (again premultiplied by the probability) for this type are

$$Rev_{1,k}(\epsilon) = p_k \int_{\overline{b_{k-1}}}^{\overline{b_{k-1}}+p_k v_k} \frac{1}{p_k v_k} x dx = \frac{p_k^2 v_k}{2} + p_k \overline{b_{k-1}} \tag{3.15}$$

Directly differentiating we get

$$\frac{dRev_{1,k}(\epsilon)}{d\epsilon} = p_k \frac{d\overline{b_{k-1}}}{d\epsilon} = p_k \left(\frac{p_k}{(v_k + \epsilon)^2} \sum_{t=1}^{k-1} v_t (v_{t+1} - v_t) \right) > 0. \tag{3.16}$$

For the higher types $r > k$, revenues will increase because the densities are the same, but the *lower and upper bounds of the intervals are both increased:*

$$Rev_{1,r}(\epsilon) = p_r \int_{\overline{b_{k-1}}+\sum_{t=k}^{r-1} p_t v_t}^{\overline{b_{k-1}}+\sum_{t=k}^{r} p_t v_t} \frac{1}{p_r v_r} x dx = \frac{1}{2v_r} p_r v_r (2\overline{b_{k-1}} + 2\sum_{t=k}^{r-1} p_t v_t + p_r v_r) \tag{3.17}$$

$$\frac{dRev_{1,r}(0)}{d\epsilon} = p_r \frac{d\overline{b_{k-1}}}{d\epsilon} = \frac{p_r p_k}{v_k^2} \sum_{t=1}^{k-1} v_r (v_{r+1} - v_r) > 0. \tag{3.18}$$

Hence the lower types of the player that is made stronger all bid *more* on average. This is the *assimilation effect*. Player 1's lower types see tougher competition and respond by stretching their equilibrium intervals and ultimate bidding more. In contrast player 1's higher types see the same competition but bid more because of the *stacking effect*. Since the lower types are bidding more, the higher types must bid more as well. Hence we can say that expected revenues collected from every type of the stronger player (i.e. player 1) will increase, but for different reasons.

Now consider the weaker player, i.e. player 2. Following a similar approach we partition the types of player 2 into the kth type, types below k, and types above k. The matchups and densities for types above k will not change for bidder 2. Hence all of the results that are true for bidder 1's types above k are also true for bidder 2's types above k. The largest types of the weaker player will bid more also because of the *stacking effect*. The first matchup to change will be type $v_{2,k}$. In response to $v_{1,k}$ becoming stronger, $v_{2,k}$ responds by decreasing her density to $g_{2,k} = \frac{1}{p_k(v_k+\epsilon)}$. In equilibrium, type $v_{2,k}$ will mix over $[\overline{b_{k-1}} - L_{k-1,k}, \overline{b_{k-1}}]$ with density $g_{2,k} = \frac{1}{p_k v_{k-1}}$ and over $[\overline{b_{k-1}}, \overline{b_{k-1}} + L_{k,k}]$ with density $g_{2,k} = \frac{1}{p_k(v_k+\epsilon)}$.

Proposition 7 *Suppose initially that bidders are symmetric, i.e.* $v_{1,t} = v_{2,t} = v_t$ *and* $p_{1,t} = p_{2,t} = p_t$ *for all* $t = 1, \ldots, n$*. If for some* k*,* $v_{1,k}$ *is increased by* $\epsilon > 0$ *to* $v_{1,k} = v_k + \epsilon$*, then the lowest types generate less revenue while the highest types generate more revenues. In particular, the right hand side derivatives are given by:*

$$\frac{dRev_{2,t}(0)}{d\epsilon} = \frac{p_t p_k}{v_k^2}\left(\sum_{r=1}^{t-1} v_r(v_{r+1}-v_r) - v_t^2\right) < 0 \quad \textit{if } t < k \tag{3.19}$$

$$\frac{dRev_{2,t}(0)}{d\epsilon} = \frac{p_k^2}{v_k^2}\left(\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r) - \frac{v_k^2}{2}\right) < 0 \quad \textit{if } t = k \tag{3.20}$$

$$\frac{dRev_{2,t}(0)}{d\epsilon} = \frac{p_t p_k}{v_k^2}\left(\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) > 0 \quad \textit{if } t > k \tag{3.21}$$

We present a useful Lemma before characterizing revenues for player 2.

Lemma 4 *Let $0 \leq v_1 < v_2 < \cdots < v_m$, where $m \geq 2$. For a fixed v_m, the sum $\sum_{r=1}^{m-1} v_r(v_{r+1}-v_r)$ is maximized when $v_r = \frac{r}{m}v_m$ for $r = 1, \ldots, m-1$. Further, at the maximum $\sum_{r=1}^{m-1} v_r(v_{r+1} - v_r) \leq \frac{v_m^2}{2}$.*

Proof 7 *To show that $\sum_{r=1}^{m-1} v_r(v_{r+1}-v_r)$ is maximized when $v_r = \frac{r}{m}v_m$ for $r = 1, \ldots, m-1$, I show that the sum has a strict local maximum by taking first and second order conditions, showing that all of the leading principal minors of the Hessian matrix alternate in sign, with the first being negative. The first order conditions of the sum are*

$$v_1: \quad v_2 - 2v_1 = 0 \tag{3.22}$$

$$v_2: \quad v_1 + v_3 - 2v_2 = 0 \tag{3.23}$$

$$v_3: \quad v_2 + v_4 - 2v_3 = 0 \tag{3.24}$$

$$\vdots \tag{3.25}$$

$$v_r: \quad v_{r-1} + v_{r+1} - 2v_r = 0 \tag{3.26}$$

$$\vdots \tag{3.27}$$

$$v_{m-1}: \quad v_{m-1} + v_{m+1} - 2v_m = 0 \tag{3.28}$$

Solving this system of equations yields solutions of the form $v_r = \frac{r}{m}v_k$. The Hessian matrix is given by

$$\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots & \vdots \\ 0 & \cdots & & & \cdots & 0 & 1 & 0 \\ 0 & \cdots & & & \cdots & 1 & -2 & 1 \\ 0 & \cdots & & & \cdots & 0 & 1 & -2 \end{bmatrix}$$

The first leading principal minor is -2, the second is 3, the third is -4, and so on. Hence we have a strict local maximum and the largest the sum could be is

$$\max \sum_{r=1}^{m-1} v_r(v_{r+1} - v_r) = \sum_{r=1}^{m-1} \left(\frac{r}{m} v_m\right) (\frac{v_m}{m}) = \frac{v_m^2}{m^2} \sum_{r=1}^{m-1} r = \frac{v_m^2}{m^2} \frac{(m-1)m}{2} = \frac{v_m^2}{2} \frac{m-1}{m} \leq \frac{v_m^2}{2}. \tag{3.29}$$

Proof 8 *(Proof of Proposition 7) Elementary calculus is tedious, but directly yields the results.*

$$Rev_{2,k}(\epsilon) = p_k \int_{\overline{b_{k-1}}-L_{k-1,k}}^{\overline{b_{k-1}}} \frac{1}{p_k v_{k-1}} x dx + p_k \int_{\overline{b_{k-1}}}^{\overline{b_{k-1}}+L_{k,k}} \frac{1}{p_k(v_k+\epsilon)} x dx \tag{3.30}$$

$$= \frac{1}{v_{k-1}} \int_{\overline{b_{k-1}}-L_{k-1,k}}^{\overline{b_{k-1}}} x dx + \frac{1}{v_k+\epsilon} \int_{\overline{b_{k-1}}}^{\overline{b_{k-1}}+L_{k,k}} x dx \tag{3.31}$$

$$= \frac{1}{2v_{k-1}} (L_{k-1,k})(2\overline{b_{k-1}} - L_{k-1,k}) + \frac{1}{2(v_k+\epsilon)} (L_{k,k})(2\overline{b_{k-1}} + L_{k,k}) \tag{3.32}$$

$$= \frac{1}{2v_{k-1}} \left(2L_{k-1,k}\overline{b_{k-1}} - (L_{k-1,k})^2\right) + \frac{1}{2(v_k+\epsilon)} \left(2L_{k,k}\overline{b_{k-1}} + (L_{k,k})^2\right) \tag{3.33}$$

$$\frac{dRev_{2,k}}{d\epsilon}(\epsilon) = \frac{1}{2v_{k-1}} \left(2L_{k-1,k}\frac{d\overline{b_{k-1}}}{d\epsilon} + 2\frac{dL_{k-1,k}}{d\epsilon}\overline{b_{k-1}} - 2L_{k-1,k}\frac{dL_{k-1,k}}{d\epsilon}\right) \tag{3.34}$$

$$+ \frac{1}{2(v_k+\epsilon)} \left(2L_{k,k}\frac{d\overline{b_{k-1}}}{d\epsilon} + 2\frac{dL_{k,k}}{d\epsilon}\overline{b_{k-1}} + 2L_{k,k}\frac{dL_{k,k}}{d\epsilon}\right) \tag{3.35}$$

$$- \frac{1}{2(v_k+\epsilon)^2} \left(2L_{k,k}\overline{b_{k-1}} + (L_{k,k})^2\right) \tag{3.36}$$

$$\frac{dRev_{2,k}}{d\epsilon}(0) = \frac{1}{2v_{k-1}} \left(0 + 2\left(\frac{v_{k-1}v_k p_k}{v_k^2}\right)\left(\sum_{t=1}^{k-1} p_t v_t\right) - 0\right) \tag{3.37}$$

$$+ \frac{1}{2v_k} \left[2p_k v_k \left(\frac{p_k}{v_k^2} \sum_{t=1}^{k-1} v_t(v_{t+1}-v_t)\right) + 0 + 0\right] - \frac{1}{2v_k^2} \left(2p_k v_k \left(\sum_{t=1}^{k-1} p_t v_t\right) + p_k^2 v_k^2\right) \tag{3.38}$$

$$= \frac{p_k}{v_k}\left(\sum_{t=1}^{k-1} p_t v_t\right) + \frac{p_k^2}{v_k^2}\left(\sum_{t=1}^{k-1} v_t(v_{t+1}-v_t)\right) - \frac{p_k}{v_k}\left(\sum_{t=1}^{k-1} p_t v_t\right) - \frac{p_k^2}{2} \tag{3.39}$$

$$= \frac{p_k^2}{v_k^2}\left(\sum_{t=1}^{k-1} v_t(v_{t+1}-v_t)\right) - \frac{p_k^2}{2} = \frac{p_k^2}{v_k^2}\left(\sum_{t=1}^{k-1} v_t(v_{t+1}-v_t) - \frac{v_k^2}{2}\right) < 0, \tag{3.40}$$

where the last inequality follows from Lemma 1.

Hence the revenues collected from the kth type will decrease. Now we consider the revenues collected from type $r < k$, which can be written as

$$
\begin{aligned}
Rev_{2,r}(\epsilon) =& p_r \int_{\overline{b_{r-1}}-L_{r-1,r}}^{\overline{b_{r-1}}} \frac{1}{p_r v_{r-1}} x dx + p_r \int_{\overline{b_{r-1}}}^{\overline{b_{r-1}}+L_{r,r}} \frac{1}{p_r v_r} x dx && (3.41)\\
=& \frac{1}{2v_{r-1}} L_{r-1,r}(2\overline{b_{r-1}} - L_{r-1,r}) + \frac{1}{2v_r} L_{r,r}(2\overline{b_{r-1}} + L_{r,r}) && (3.42)\\
=& \frac{1}{2v_{r-1}} \left(2L_{r-1,r}\overline{b_{r-1}} - (L_{r-1,r})^2\right) + \frac{1}{2v_r}\left(2L_{r,r}\overline{b_{r-1}} + (L_{r,r})^2\right) && (3.43)\\
\frac{dRev_{2,r}(\epsilon)}{d\epsilon} =& \frac{1}{v_{r-1}} \left(L_{r-1,r} \frac{d\overline{b_{r-1}}}{d\epsilon} + \frac{dL_{r-1,r}}{d\epsilon}\overline{b_{r-1}} - L_{r-1,r}\frac{dL_{r-1,r}}{d\epsilon} \right) && (3.44)\\
&+ \frac{1}{v_r}\left(L_{r,r}\frac{d\overline{b_{r-1}}}{d\epsilon} + \frac{dL_{r,r}}{d\epsilon}\overline{b_{r-1}} + L_{r,r}\frac{dL_{r,r}}{d\epsilon} \right) && (3.45)\\
\frac{dRev_{2,r}(0)}{d\epsilon} =& \frac{1}{v_{r-1}} \left(0 + (\sum_{t=1}^{r-1} p_t v_t) \frac{v_{r-1} v_r p_k}{v_k^2} - 0 \right) && (3.46)\\
&+ \frac{1}{v_r}\left(p_r v_r \frac{p_k}{v_k^2} \sum_{t=1}^{r-1} v_t(v_{t+1}-v_t) + (\sum_{t=1}^{r-1} p_t v_t)(-\frac{p_k v_r^2}{v_k^2}) + p_r v_r (-\frac{p_k v_r^2}{v_k^2}) \right) && (3.47)\\
=& \frac{p_r p_k}{v_k^2} \left(\sum_{t=1}^{r-1} v_t(v_{t+1}-v_t) - v_r^2 \right) < 0. && (3.48)
\end{aligned}
$$

As with the type k case, the last inequality above follows from Lemma 1. Hence revenues collected from bidder 2's lower types also contribute less. We now collect the results of the changes in bidder 2's revenues.

Combining the results of the previous propositions, I now characterize changes in total expected revenues due to the asymmetry.

Proposition 8 *Suppose that players are initially symmetric and then player 1's kth valuation is increased by some small $\epsilon > 0$* ceteris paribus . *The changes in revenues ΔRev for each type and each player are given in the table below.*

ΔRev	*types $t < k$*	*type $t = k$*	*types $t > k$*	*all types*
Player 1	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
Player 2	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow, =, \downarrow$
Both Players	$\downarrow$	$\uparrow, =, \downarrow$	$\uparrow$	$\uparrow, =, \downarrow$

Specifically the (right-handed) derivative of total expected revenues is given by:

$$\sum_{t=1}^{n} \frac{dRev_{1,t}(0)}{d\epsilon} + \frac{dRev_{2,t}(0)}{d\epsilon} = \frac{2p_k}{v_k^2}\left[\sum_{t=1}^{k-1} p_t \left(-\frac{v_t^2}{2} + \sum_{r=1}^{t-1} v_r(v_{r+1}-v_r)\right)\right. \tag{3.49}$$

$$+\ p_k \left(-\frac{v_k^2}{4} + \sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.50}$$

$$\left. +\ \sum_{t=k+1}^{n} p_t \left(\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right)\right]. \tag{3.51}$$

Proof 9

$$\sum_{t=1}^{n} \frac{dRev_{1,t}(0)}{d\epsilon} = \sum_{t=1}^{k-1} \frac{p_t p_k}{v_k^2}\left(\sum_{r=1}^{t-1} v_r(v_{r+1}-v_r)\right) + \sum_{t=k}^{n} \frac{p_t p_k}{v_k^2}\left(\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.52}$$

$$\sum_{t=1}^{n} \frac{dRev_{2,t}(0)}{d\epsilon} = \sum_{t=1}^{k-1} \frac{p_t p_k}{v_k^2}\left(-v_t^2 + \sum_{r=1}^{t-1} v_r(v_{r+1}-v_r)\right) + \frac{p_k^2}{v_k^2}\left(-\frac{v_k^2}{2} + \sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.53}$$

$$+\ \sum_{t=k+1}^{n} \frac{p_t p_k}{v_k^2}\left(\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.54}$$

Hence

$$\sum_{t=1}^{n} \frac{dRev_{1,t}(0)}{d\epsilon} + \frac{dRev_{2,t}(0)}{d\epsilon} = \sum_{t=1}^{k-1} \frac{p_t p_k}{v_k^2}\left(-v_t^2 + 2\sum_{r=1}^{t-1} v_r(v_{r+1}-v_r)\right) \tag{3.55}$$

$$+\ \frac{p_k^2}{v_k^2}\left(-\frac{v_k^2}{2} + 2\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.56}$$

$$+\ \sum_{t=k+1}^{n} \frac{p_t p_k}{v_k^2}\left(2\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.57}$$

$$= \frac{2p_k}{v_k^2}\left[\sum_{t=1}^{k-1} p_t \left(-\frac{v_t^2}{2} + \sum_{r=1}^{t-1} v_r(v_{r+1}-v_r)\right)\right. \tag{3.58}$$

$$+\ p_k \left(-\frac{v_k^2}{4} + \sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right) \tag{3.59}$$

$$\left. +\ \sum_{t=k+1}^{n} p_t \left(\sum_{r=1}^{k-1} v_r(v_{r+1}-v_r)\right)\right] \tag{3.60}$$

The net change in the revenues can be broken apart into 3 groups of types. The higher types of both players will all generate more revenue, conditional on their type being realized. This is due to monotonicity. The lower types of the stronger player will bid more due to the encouragement effect. The lower types of the weaker player will bid less due to the discouragement effect. There is thus a tension between the first two groups of types and the third group of types. Ultimately the likelihood of being a high type versus a low type determines which type has the greater magnitude. Informally, if there are more revenue increasing types than revenue decreasing types then expected revenues will increase.

Before moving on to the other type of comparative static (changing probabilities), it is useful to briefly introduce the comparative statics on type spaces that are already asymmetric. These types of comparative statics will either reverse the asymmetry or exacerbate the asymmetry, depending upon whether the asymmetry causes the atom at 0 of the weaker player to increase or decrease. I postpone this discussion until Chapter 4, where I show that maximal limits on bids are a type of "asymmetry-reversing" comparative static. For now I will just note that asymmetry is a bit like bending an elastic rod at different lengths and then measuring the total length of the rod. Bending the rod in different directions may cancel the effects of each other while twice bending the rod in the same direction serves to amplify the effect of the initial asymmetry.

Generally speaking, the same pattern will hold for other types of asymmetry rather than just asymmetry relative to the symmetric type space. Suppose that players are initially symmetric and then type k is increased for player 1. My previous results show that revenues will be increased only if the probability of types above k is sufficiently large enough. Now suppose that player 1's m-type, where m is WLOG smaller than k. This means that player 1 is even more likely to see tougher competition, so that the intuition from the assimilation and stacking effects carry over. If the probability of high types is great enough, revenues will be increased. If however instead of player 1's mth type it is player 2's mth type that is increased, then relative to symmetric type space revenues can increase if the probability of high types is small enough, since this change would allow player 2 to remove some of her atom at 0 and reverse the asymmetry. Other forms of asymmetry can be stacked upon in this way.

Corollary 2 *For more than one comparative static,*

- *If player 1 is stronger in more than 1 type, the assimilation and stacking effects will be amplified so that revenues will increase only if the probability of high types is large enough.*

- *If player 1 is stronger in one type, but player 2 in another, revenues will tend to be closer to the symmetric case because the effects tend to cancel each other out. It depends upon the relative size of the asymmetries and the relative size of the types above the asymmetric types.*

3.4 Private Information Comparative Statics (Probabilities)

Changing valuations are just one of the ways in which a player can experience a first order stochastic shift in his or her type space distribution. A player may also become stronger by means of increasing the probability of his being a higher type. In this section the set of possible valuations T_i is fixed for each player but the the *probabilities* over these types are changing. The difficulty in such a comparative static is that there are too many ways probability weight can be taken from lower types and transferred to higher types. For example, with 3 types (v_1, v_2, v_3) there are 7 cases[10], and each of these cases has sub-cases depending upon the relative amounts of probability taken from each lower type. With n possible types it is infeasible to describe, as the number of cases would be on the order of $n!$. As a result I consider the comparative statics of type spaces with two and three types and then proceed to generalize qualitatively to larger type spaces. I show that it is necessary to consider at least 3 types to fully capture the effects of changing probabilities on behavior.

With changing probabilities, there is a type of dis-assimilation effect whereby stronger player's high types increase their densities, compress their equilibrium interval, and tend to bid less. Weaker types tend to decrease their densities and bid more. These two effects almost nullify the stacking effect, but tend to produce an increased equilibrium support. However it is possible for this type of comparative static to increase or decrease total expected revenues.

[10]For example probability from type 1 could be transferred to type 2 only, or to type 3 only, to both other types, or to none. In the first of these 3 cases, probability from type 2 may or may not be transferred to type 3.

3.4.1 Two Types

When each player has only two types, there is only one way in which the probability of a player's high type can be increased: weight must be equaled transferred from p_1 to p_2. Specifically let players initially be symmetric so that $v_{1,1} = v_{2,1} = v_1$, $v_{1,2} = v_{2,2} = v_2$, $p_{1,1} = p_{2,1} = p_1$, and $p_{1,2} = p_{2,2} = p_2 = 1 - p_1$. Now increase the probability of player 1 being a high type by ϵ: $p_{1,1} = p_1 - \epsilon$ and $p_{1,2} = p_2 + \epsilon$. Applying the equilibrium construction, the matchup between the highest types will furnish densities $g_{1,2} = \frac{1}{(p_2+\epsilon)v_2}$ and $g_{2,2} = \frac{1}{p_2 v_2}$. After $v_{2,2}$ fills her weight in an interval of length $L_{2,2} = p_2 v_2$, then $v_{1,2}$ will have remaining weight of $1 - (L_{2,2} \times \frac{1}{(p_2+\epsilon)v_2}) = \frac{\epsilon}{p_2+\epsilon}$. This will force a new matchup of $v_{1,2}$ and $v_{2,1}$ with densities $g_{1,2} = \frac{1}{(p_2+\epsilon)v_1}$ and $g_{2,1} = \frac{1}{p_1 v_2}$. If ϵ is small enough, then $v_{1,2}$ will be able to fill his weight in an interval of length $L_{2,1} = \epsilon v_1$. This will cause $v_{2,1}$ to have remaining weight of $1 - \frac{\epsilon v_1}{p_1 v_2}$. Now $v_{1,1}$ and $v_{2,1}$'s matchup will generate densities of $g_{1,1} = \frac{1}{(p_1-\epsilon)v_1}$ and $g_{2,1} = \frac{1}{p_1 v_1}$ respectively. Since $v_1 < v_2$ by assumption, it will be the case that $v_{1,1}$ will be able to fill his weight first, despite $v_{2,1}$ starting the matchup already having expended some weight. This is because $v_{1,1}$ has increased his density and will fill at a faster rate. The length required for this last/lowest interval is $L_{1,1} = (p_1 - \epsilon)v_1$.

Define the equilibrium interval bounds as $b_1 \equiv p_1 - \epsilon$, $b_2 \equiv b_1 + L_{2,1} = p_1 v_1$, and $\bar{b} \equiv p_1 v_1 + p_2 v_2$. Notice that the total equilibrium support of bids is the same since $\bar{b}$ does not depend upon ϵ. What does change is the distribution of bids over the equilibrium support. Asymmetry in this form actually causes the stronger player to mix using a distribution that is first order stochastically dominated by his previous (symmetric) density. Hence becoming stronger has decreased the expected amount of bidding. Then the expected revenues collected by each type (again multiplied by their probabilities) are summarized by the following proposition.

Proposition 9 *Suppose that there are two symmetric players with two possible types. Now suppose that p_2 is increased by ϵ and p_1 is decreased by ϵ for one of the players. Then the associated comparative statics (in terms of ϵ) are*

$$\begin{aligned}
\frac{dRev_{1,1}(\epsilon)}{d\epsilon} =& - v_1(p_1 - \epsilon) < 0, && (3.61)\\
\frac{dRev_{1,2}(\epsilon)}{d\epsilon} =& v_1(p_1 - \epsilon) > 0, && (3.62)\\
\frac{dRev_{2,1}(\epsilon)}{d\epsilon} =& (p_1 - \epsilon)v_1(-1 + 2\frac{v_1}{v_2}) \gtreqqless 0, && (3.63)\\
\frac{dRev_{2,2}(\epsilon)}{d\epsilon} =& 0. && (3.64)
\end{aligned}$$

Proof 10 *The resulting revenues are given by:*

$$\begin{aligned}
Rev_{1,1}(\epsilon) =& (p_1 - \epsilon)\int_0^{b_1} \frac{1}{(p_1 - \epsilon)v_1} x dx = \frac{v_1}{2}(p_1 - \epsilon)^2, && (3.65)\\
Rev_{1,2}(\epsilon) =& (p_2 + \epsilon)\int_{b_1}^{b_2} \frac{1}{(p_2 + \epsilon)v_1} x dx + (p_2 + \epsilon)\int_{b_2}^{\bar{b}} \frac{1}{(p_2 + \epsilon)v_2} x dx = \epsilon p_1 v_1 - \frac{\epsilon^2 v_1}{2}, && (3.66)\\
Rev_{2,1}(\epsilon) =& p_1\int_0^{b_1} \frac{1}{p_1 v_1} x dx + p_2\int_{b_1}^{b_2} \frac{1}{p_1 v_2} x dx = \frac{(p_1 - \epsilon)^2 v_1}{2} + \frac{\epsilon v_1}{v_2}(2p_1 v_1 - \epsilon v_1), && (3.67)\\
Rev_{2,2}(\epsilon) =& p_2\int_{b_2}^{\bar{b}} \frac{1}{p_2 v_2} x dx = \frac{p_2}{2}(2p_1 v_1 + p_2 v_2). && (3.68)
\end{aligned}$$

Elementary calculus yields the result.

Corollary 3 *For small $\epsilon > 0$, total expected revenues are strictly increasing in ϵ iff $\frac{v_1}{v_2} > \frac{1}{2}$.*

Recall that $Rev_{i,k}$ is the probability of player i being a k type multiplied by the expected revenues conditional on that type being realized. The above proposition shows that this product is decreasing for $v_{1,1}$ but increasing for $v_{1,2}$. However, conditional on type, both types of player 1 generate less revenue. Informally, there are less $v_{1,1}$'s around and more $v_{1,2}$'s around. Both types bid less on average. But since there are more $v_{1,2}$'s, the overall contribution to revenues from these types will increase. This explains why $\frac{dRev_{1,2}(\epsilon)}{d\epsilon} > 0$. Despite experiencing a first order stochastic shift where he is unambiguously made stronger, each type of player 1 bids less on average. This feature is absent in complete information, where the stronger player will always bid the same on average. With incomplete information, this example shows that the stronger player may actually bid less on average.

While the auctioneer will expect to collect less from the lower type of player 1, the higher type of player 1 will generate more revenues on average. These two effects will exactly offset so that player 1, on average, will bid the same amount. In contrast the high type of player 2 will contribute the same amount, but the change in revenues for the lower type can be positive or negative. If $\frac{v_1}{v_2} > \frac{1}{2}$, so that the two different possible types of the players are not too dissimilar, then expected revenues will increase.

Note that this result does not at all depend upon the values of p_1 and p_2. Therefore, adding even arbitrarily small amounts of incomplete information to the game may yield qualitatively different effects on the changes in revenues (when asymmetry is introduced). If both players valuations are common knowledge, then asymmetry lowers total expected revenues for all pairs of valuations. If, however, it is common knowledge that the players have the same valuations as the previous sentence with probability 0.99 and with probability 0.01 have some other valuation, then asymmetry can increases revenues if each player's types are sufficiently close to each other. This represents a type of discontinuity in the information structure. Even though small changes in the information structure change equilibrium behavior by a small amount, the rates of change of expected revenues may change discontinuously.

Proposition 10 *There exists a family of symmetric type space distributions arbitrarily close to the complete information (degenerate) distribution where asymmetry strictly increases revenues.*

Proof 11 *Let $\epsilon > 0$ and $\frac{v_1}{v_2} > \frac{1}{2}$. Also let both players have valuation v_2 with probability $1 - \epsilon$ and v_1 with probability ϵ. Then if $p_{1,2}$ is increased by any arbitrarily small $\delta > 0$, revenues strictly increase for any $\epsilon > 0$.*

My results agree with Kirkegaard (2013), who was the first to point out the "knife-edge" case of complete information in determining comparative statics. His comparative static involves taking weight from all existing types and placing the weight on new higher types. This comparative static will increase revenues. My results imply his. Additionally, our results imply those of Fibich et al. (2004), who considers a similar comparative static and shows that revenues *must* increase.

However the comparative static they perform holds the support of the type space fixed. My results allow for changing supports. Both of these papers investigate comparative statics where infinite numbers of valuations and probabilities are changed simultaneously. My results help disentangle the effects of changing valuations and changing probabilities. Ultimately my results can explain any type of by properly choosing the order of successive comparative statics. I postpone this discussion until the more general 3 types case.

3.4.2 Three Types

With three types, there are several ways that a player can be made stronger by way of increasing the probability of higher types. I consider only one type of comparative static to bring out the differences between changing valuations and changing probabilities. Suppose first that weight from the lowest type is transferred to the highest type. Specifically, let there be two *ex-ante* symmetric players with $n = 3$ types each. The comparative static I perform is analyzing changes in revenues when weight is passed from the lowest type only to the highest type. This corresponds to changing player 1's prior probabilities to $p_{1,1} = p_1 - \epsilon$ and $p_{1,3} = p_3 + \epsilon$ for some small $\epsilon > 0$. Qualitatively, what is important about this case is the fact that there is a middle type (whose probability is unchanged) over whom weight is transferred.

Hence I do not consider other cases such as having weight is moved from the middle type to the high type or having weight added from the low type to the middle type. In the conclusion section I will discuss the qualitative aspects of other comparative statics. The general intuition from the single case is enough to understand this type of comparative static.

Here is a sketch of the equilibrium construction of the first of our three cases. Let $p_{1,1} = p_1 - \epsilon$ and $p_{1,3} = p_3 + \epsilon$ for some small $\epsilon > 0$. Starting with the matchup between the two highest types, the equilibrium densities will be $g_{1,3} = \frac{1}{(p_3+\epsilon)v_3}$ and $g_{2,3} = \frac{1}{p_3 v_3}$. Now it is the *stronger* player who is unable to fill his weight. After a length of $L_{3,3} = p_3 v_3$, $v_{1,3}$ will have remaining weight of $1 - (L_{3,3} \times \frac{1}{(p_3+\epsilon)v_3})$. Hence $v_{2,2}$ will now face stronger competition and increase her density. After $v_{1,3}$ fills up his weight in a new interval, $v_{2,2}$ will face $v_{1,2}$ where the former is able to fill up first. The pattern continues until the type where weight was removed from. In this case, the stronger player's density actually increases in such a way so that the total bidding support remains constant.

What is qualitatively different is that it is primarily the stronger player's behavior that is changed. His densities are decreases for larger types, thus expanding the interval he requires to fill all of his weight. On the other hand the densities for smaller types are increased, thus contracting the intervals required for filling all of his weight.

Proposition 11 *Let $n = 3$ and players be symmetric. Then let $p_{1,3}$ is increased by some small $\epsilon > 0$ and $p_{1,1}$ is decreased by ϵ. The table below summarizes the changes in "expected revenues", where expected revenues are the probability of each type multiplied by the expected revenue conditional on that type.*

	v_1	v_2	v_3	Total
Player 1	↓	↓	↑	↑
Player 2	↓	↓	↑	↑↓
Both Players	↓	↓	↑	↑↓

Figure 3.2: ΔRev: $p_{1,1} = p_1 - \epsilon$ and $p_{1,3} = p_3 + \epsilon$

Conditional on type *all types of both players except for $v_{2,3}$ will bid less on average. Note the table above gives revenues conditional on type multiplied by probabilities.*

This type of type space distribution change may or may not increase revenues. With changing probabilities the ratios of the valuations is the key factor. When either $\frac{v_1}{v_2}$, $\frac{v_1}{v_3}$, or $\frac{v_2}{v_3}$ is large (*ceteris paribus*) revenues will be more likely to decrease. Else revenues are more likely to increase when valuations increase at an increasing rate: $v_3 >> v_2 >> v_1$. For types spaces with a sufficiently large n, this result suggests that revenues will be likely to be decreasing as the ratio between consecutive types approaches 1.

As weight is transferred from the lowest type to the highest type, then the stronger player as a whole will contribute more to revenues while the weaker player's effect is ambiguous. Overall revenues will be likely to increase if types are far away from one another with respect to the Euclidean metric.

3.4.3 Summary of Changing Probabilities

When weight is transferred from the lowest type to the highest type, revenues may increase or decrease. Consider first the stronger player. There exists a cutoff type such that prior probability from types below the cutoff is transferred to types above the cutoff. For the higher types, there will be stretching due to smaller densities. This will increase revenues. For the lower types, there will be compressing due to larger densities. This will decrease revenues. The net effect will be positive. Similarly the weaker player will stretch at the top and compress on the bottom but the overall effect on revenues is ambiguous. Hence it is possible for asymmetry in the form of probabilities to decrease revenues. This result stands in contrast to Fibich et al. (2004), who find that the same comparative static *must* increase revenues for small changes.

3.5 Conclusion

With complete information, there is only one way to make a player stronger relative to another (i.e. increase asymmetry between players). One of the player's known valuations must change. Even if the asymmetry is in the form of player 1 having arbitrarily large (but finite) valuation, asymmetry will always decrease revenues. Thus any auctioneer interested in maximizing total expected revenues must ultimately eliminate all asymmetry when valuations of both players are common knowledge. In the next chapter we will see one such way to equalize the players: imposing maximum bidding limits.

With incomplete information (i.e. private information in our context), comparative statics are not so clear. First it matters *how* a player is made stronger. A player can become stronger by either increasing the valuation associated with one of his types *ceteris paribus*, or by increasing the probability of a high valuation *ceteris paribus*. These two changes have qualitatively very different effects on revenues. In some instances, some of the "stronger" player's types may actually bid less on average.

Additionally, the underlying shape of the type space distribution determines whether asymmetry will increase or decrease revenues. Specifically the relative likelihoods of lower types versus higher types are the key determinant for valuation-comparative statics. The relative ratios of the types are the key determinant for probability-comparative statics. In the next chapter of the dissertation, I show that maximum bidding limits have similar effects as valuation comparative statics, so I will focus on those here. The comparative statics of the private information APA can be summarized by two effects, the *assimilation effect* and the *stacking effect*. Only the assimilation effect is present in the complete information game, though only partially. The comparative statics of asymmetry on revenues are not robust to incomplete information.

Asymmetry alters the equilibrium because it changes the matchups between the players. When a player meets tougher competition, he will respond by randomizing over a larger set of bids. This tends to increase the total expected bid *ceteris paribus* since the lower bound of bids is fixed at 0. When a player meets weaker competition, she will respond by randomizing over a smaller set of bids and bid less on average. Hence each player tends to assimilate to the level of their competition. Stronger competition induces players to bid more while weaker competition induces players to bid less. This is the assimilation effect. Each player only bids as much as their competition is willing to bid.

The “stacking effect” is a direct consequence of monotonicity. When lower types bid more, then *ceteris paribus* higher types will bid more as well because higher types must bid more than the lower types. This effect is trivially absent from the complete information case since each player has only 1 type. These two effects determine whether revenues will increase or not. In short, the auctioneer makes money off the high types and loses money off of the low types. If high types are sufficiently more likely, the auctioneer will expect to collect more money. The single type in the complete information game is treated as a low type, and hence revenues can never increase with asymmetry.

CHAPTER 4

MAXIMUM CONTRIBUTION LIMITS

4.1 Introduction

In 2010 the Supreme Court of the United States (SCOTUS) ruled in *Citizens United v. FEC* that certain types of political spending should be unlimited. After the decision contributions increased dramatically. This has been the concern of citizens of all political affiliations as it seems that special interests now have an even larger voice and influence with politicians. That this concern is credible is due to the fact[1] that the average time spent campaigning and talking with donors and lobbyists by a senator, representative, or federal official with authority is 60%. This chapter provides a theoretical understanding of this decision and of contribution limits in general. I consider the problem in reverse, first assuming no limits and then seeing the changes the maximum limits create in the equilibrium behavior. My results imply that enacting a policy with no contribution limits will increase the total expected contribution as long as the likelihood of people who are willing to donate more than the maximum, and ultimately the amounts which each person is willing to donate more than the maximum, is great enough. I use the words lobbyist and donor interchangeably.

Maximum limits reverse the effects of asymmetry in the lobbying game. Trivially if the maximum limits is 0, then all donors are equal as no influence-garnering contributions are allowed. Increasing the maximum limit from 0 dissuades low valuation donors from contributing because they know they will be out-contributed by higher valuation donors.

At the heart of my analysis is the importance of both private information and asymmetry between the lobbyists. To illustrate the individual effects of each, I compare and contrast 4 different environments: 1) complete information with symmetric lobbyists, 2) complete information with

[1] See Baumgartner, Berry, Hojnacki, Kimball, and Leech (2009) for more summary statistics of this type.

asymmetric lobbyists, 3) private information with symmetric lobbyists, and 4) private information with asymmetric lobbyists. Regardless of the information structure, complete or private, maximum contribution limits can increase total expected revenues (relative to the no limit case) when lobbyists are asymmetric.

Che and Gale (1998) have shown that with complete information between lobbyists it may be possible to have total expected contributions be larger when maximum limits are imposed precisely because the low valuation donor will be induced to contribute a larger amount so much that it offsets the loss in revenues from high valuation donors. My main result is the private information extension of their environment, i.e. the all-pay auction with risk-neutral 2 players who have independently drawn types.

The closest known work to this one is Sahuget (2006), who shows that with uniform distributions that there exist maximum limits that can increase total expected revenues only if the players are symmetric. The drawback of his approach is that he uses types drawn from absolutely continuous distributions. The equilibria in these games are characterized by the solutions of a system of a first order ordinary differential equations and do not lend themselves well to comparative statics. Further, tractable solutions exist only for trivial type space distributions like the uniform distributions. I use finite types to circumvent this difficulty and directly illustrate the effects of contribution limits on all types of both players, thus enabling me to keep track of how the distribution of contributions changes with the maximum allowed contribution. The first chapter of this dissertation justifies the use of finite types.

My main result says that imposing maximum limits may or may not increase total expected contributions. What matters is the likelihood of each lobbyist being a low versus a high valuation types. If high types are likely, then contribution limits will decrease total expected contributions. Intuitively if all lobbyists are willing to contribute more than a million dollars with probability 99%, then a maximum limit below a million dollars will only increase contributions in the event of a low valuation donor existing, which happens with probability 1%.

The plan of this chapter is as follows. First I introduce the notation and characterize the equilibrium when contributions are unrestricted, i.e. when no maximum limit is imposed. I briefly consider the complete information lobbying game, which is a special case[2] of the private information environment. I then move on to private information games, starting with symmetric lobbyists and then ending with asymmetric lobbyists. I will be able to disentangle the effects of asymmetry and the effects of private information by proceeding in this way. Finally I conclude and provide suggestions for future research.

4.2 Model and Equilibrium Characterization without Contribution Limits

I model political lobbying as an all-pay auction between two risk-neutral players. Each lobbyist, unaware of his opponent's value of political clout, pledges a non-refundable contribution towards a politician's campaign. It is understood that the highest contributor wins political clout with the politician. The key feature of this game is that contributions are non-refundable. A lobbyist cannot ask for a donation back if he or she did not receive political favor.

There are two lobbyists indexed by $i = 1, 2$. Lobbyist i has valuation v_i drawn from some finite set $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$ according to some cumulative distribution function F_i with corresponding probabilities $\{p_{i,1}, p_{i,2}, \ldots, p_{i,n}\} \in \Delta^n$. Note that I assume the number of types to be the same for both lobbyists. Allowing different cardinality of type spaces does not qualitatively change any of the main results. Without loss of generality I can assume that all types are strictly positive and that types are ordered from smallest to largest: $0 < v_{i,1} < v_{i,2} < \ldots < v_{i,n}$.

Lobbyist i contributes $c_i \in [0, \overline{C}]$, the maximum allowable contribution $\overline{C}$ is common knowledge and the same for both players. The lobbyist with the highest contribution wins political favor, but each lobbyist must consider the contributions as sunk costs. Cast in terms of auction theory, the transfers from the players to the auctioneer are unconditional in that they do not depend upon the realization of the allocation. Note that each lobbyist i's *ex-post* payoffs, $i \neq j$, $u_i(c_i, c_j | v_i)$ are discontinuous when the actions are equal. For $c_i \in [0, \overline{C}]$,

---

[2]The complete information game is equivalent to $n = 1$ in my model, where the notation is introduced in the next section.

$$u_i(c_i, c_j|v_i) = \begin{cases} v_i - c_i & \text{if } c_i > c_j \\ \frac{v_i}{2} - c_i & \text{if } c_i = c_j \\ -c_i & \text{if } c_i < c_j. \end{cases} \tag{4.1}$$

In fact ex-post payoffs are neither upper semi-continuous nor lower semi-continuous. A final note of clarification is needed. The above payoffs are valid only if the contributions fall in the allowed set. If lobbyist i contributes $c_i > \overline{C}$, we set $u_i(c_i, c_j|v_i) = -1$ for all c_j, to ensure contributing nothing would strictly dominate this strategy. Hence we assume some legal structure sufficient enough to monitor contribution limits so that that the politician cannot accept very large contributions ($c_i \not> \overline{C}$).

Each type $v_{i,k}$ randomizes over contributions using c.d.f. $G_{i,k}(x)$. Let the support[3] of this distribution be denoted $supp(G_{i,k})$. A strategy profile for lobbyist i is then a vector of c.d.f.'s $G_i = (G_{i,1}, \ldots, G_{i,n})$, where each $G_{i,k}$ is the c.d.f. that describes type $v_{i,k}$'s behavior. Note that this description of a strategy profile is quite general. In particular, it allows for smooth mixing over intervals as well as atoms to be placed anywhere. With contribution limits both of these features are present in equilibrium behavior. If j uses strategy profile G_j, we can then write i's *interim* expected payoffs $U_i(c_i, G_j|v_i)$ if he has type v_i and contributes c_i as

$$U_i(c_i, G_j|v_i) = v_i\sigma_i(c_i) - c_i \ , \tag{4.2}$$

where $\sigma_i(c_i) \equiv \sum_{m=1}^{n_j} p_{j,m}G_{j,m}(c_i)$ is the probability that lobbyist i wins against lobbyist j with contribution c_i. I now define an equilibrium of this game, which is the standard Bayesian

[3] In equilibrium, each $G_{i,k}$ will be absolutely continuous (with respect to the Lebesgue measure) so that we can write its corresponding density $g_{i,k}$. I define the support of a set S is closure of the set of points with positive density: $supp(G) = cl(\{x|g(x) > 0\})$. Since the players mix over an interval (and hence a connected set), the closure of the equilibrium contributions will simply be the interval itself. The endpoints are probability 0 events and do not factor into payoff considerations.

Nash Equilibrium definition.

Definition 2 *An equilibrium is a pair* (G_1, G_2) *s.t.* $\forall i$, $\forall v_{i,k} \in V_i$, $\forall \hat{c} \in supp(G_{i,k})$, *and for all* $c \in [0, \overline{C}]$

$$U_i(\hat{c}, G_j|v_{i,k}) \geq U_i(c, G_j|v_{i,k}).$$

Siegel (2013) has shown unique existence if lobbyists are free to contribute any non-negative amount, i.e. if $\overline{C} = \infty$. Existence of equilibria for all type spaces and for all $\overline{C}$ will be proved later by construction. The proof I present builds off of the results of the no-limit case. I present the no-limit results of Siegel (2013) in the following lemma to highlight the key features of the game that are present with or without maximum limits imposed.

Lemma 5 Siegel (2013): *Let* $\overline{C} = \infty$. *For all finite type distributions* F_1 *and* F_2, *there exists a unique equilibrium* (G_1, G_2) *in mixed strategies such that for all lobbyists* i *and types* k*:*

- ***no non-zero atoms****:* $G_{i,k}(x)$ *is continuous for all* $x > 0$ *(and is of course right continuous at* $x = 0$*),*

- ***piecewise-uniformity****:* $G_{i,k}(x)$ *is piecewise linear over some interval, where*

$$g_{i,k}(x) = \begin{cases} \frac{1}{p_{i,k} v_{j,m}} \textit{if } x \in supp(G_{i,k}) \cap supp(G_{j,m}) \\ 0 \textit{ else} \end{cases} \tag{4.3}$$

- ***monotonicity****:* $k < k' \quad \Rightarrow \quad \sup supp(G_{i,k}) \leq \inf supp(G_{i,k'})$, *with equality only possible if* $G_{i,k} = 0$.

An immediate consequence of monotonicity is that the probability that a lobbyist wins is increasing in the amount of the contribution, but at a decreasing rate.

Corollary 4 $\sigma_i(c_i) \equiv \sum_{m=1}^{n_j} p_{j,m} G_{j,m}(c_i)$ *is a (continuous) concave function of* c_i *for all* i, *and hence differentiable at all but a countable number of points. Where it is differentiable, the derivative is given by*

$$\frac{d\sigma_i(c_i)}{dc_i} = \frac{1}{v_{i,k}}, \tag{4.4}$$

if $c_i \in supp(G_{i,k})$.

From the perspective of each lobbyist, lobbying is a lottery. Each lobbyist makes a non-refundable payment for a chance to win political favor. Unknown to each lobbyist is the valuation or contribution of the other lobbyist. All that is known is the distribution of the other lobbyist. Monotonicity implies that in equilibrium larger donations have greater chances of winning *ceteris paribus*, but the gains from doing so decrease as the donation is made larger. Only lobbyists who value political favor sufficiently much will contribute large amounts. The reason is that the extra likelihood in winning is only desirable to lobbyists who stand to gain a lot from winning. This feature is present in the equilibrium with limits as well.

4.3 Maximum Limits with Complete Information

The complete information environment is a special case of the private information environment in which $n = 1$. Thus equilibrium existence (when there are no maximum or minimum limits) follows immediately, as does all of the other equilibrium properties. For completeness I summarize the equilibrium in the following Proposition and then illustrate the equilibrium in Figure 4.1. Note this is the equilibrium existence result without limits as first presented by Hillman and Riley (1989).

Proposition 12 (Hillman and Riley (1989)) *In the complete information all-pay auction with valuations* $v_1 > 0$ *and* $v_2 > v_1$*, there exists a unique equilibrium* (G_1, G_2)*, which is in mixed strategies, where players mix uniformly over* $[0, v_1]$*, with densities* $g_1(x) = \frac{1}{v_2}$ *and* $g_2(x) = \frac{1}{v_1}$ *respectively, and the weaker player (player 1) contributes 0 with probability* $\frac{v_2 - v_1}{v_2}$.

Proof 12 *See Hillman and Riley (1989) or take* $n = 1$ *in Lemma 5.*

Let $v > 0$ be the valuation of lobbyist 2 and $v + \epsilon$ be the valuation of lobbyist 1, for some $\epsilon > 0$. Hence ϵ is measure of the asymmetry between the players. Increasing ϵ results in the

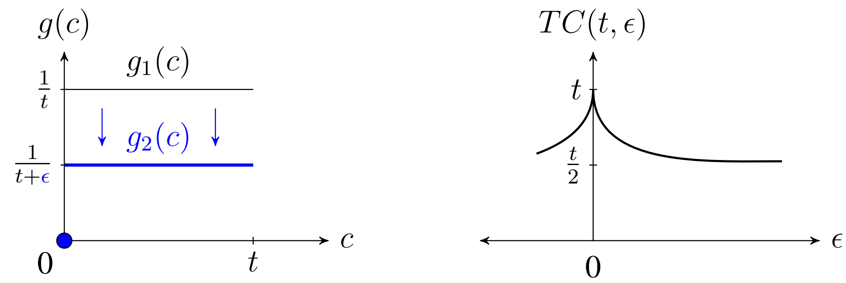

Figure 4.1: Complete Information Equilibrium, $\overline{C} = \infty$

weaker lobbyist contributing 0 with greater probability while the stronger lobbyist is unaffected. Thus total expected contributions collected between the players must decrease. A similar result holds when ϵ is lowered below 0: both players will randomize their donation over a set of smaller possible contributions. Hence for a fixed v, total expected contributions are maximized when $\epsilon = 0$.

Corollary 5 *Let $v_1 = v + a$ and $v_2 = v$ for some $a \in \mathbb{R}$. Then total expected revenues are maximized at $a = 0$.*

Proof 13 *See Hillman and Riley (1989).*

Hence a politician interested in maximizing total expected contributions will prefer a more level playing field for the lobbyists for a given minimum valuation between the two lobbyists. This is the intuition behind the exclusion principle of Baye et al. (1993), where it is shown that excluding strong bidders from an $n > 2$ player game from participating may increase total expected revenues. In the all-pay auction large expected contributions will not be possible unless *both* players are willing to contribute a lot. The reason is that the stronger lobbyist only has to contribute what the weaker lobbyist is willing to contribute. In the previous chapter this was referred to as the *assimilation effect*.

Che and Gale (1998) were the first to point that in the complete information environment, imposing maximum limits may actually *increase* total expected contributions precisely because they reduce the asymmetry between the players. For large limits $\overline{C}$, behavior will be largely unaffected

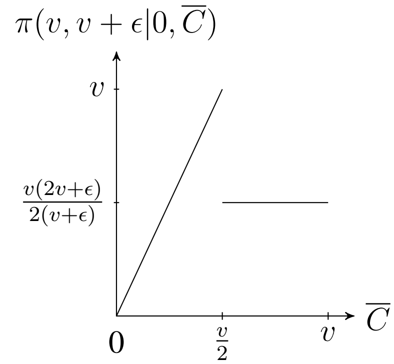

Figure 4.2: Che and Gale (1998)'s Complete Information Revenues

and so revenues will trivially remain the same. For example if $\overline{C} > v$, behavior will be unaffected by the imposition of the limit since no lobbyist is willing to contribute more than his valuation. If, on the other hand, the maximum limit is very small, then behavior will change. For example, if $\overline{C} = 0$ then revenues will trivially be zero as lobbyists are effectively banned from contributing. The surprising result is that intermediate values of maximum limits *must increase* total expected revenues if and only if the players are asymmetric. Here the increase should be clarified as an increase relative to the total expected contributions with no limit. If lobbyist 1 has valuation $v + \epsilon$ and lobbyist 2 has valuation v, then ϵ is a measure of the asymmetry between the players. When $\epsilon \neq 0$, Che and Gale (1998) pointed out an important kink in the total expected contributions. Formally let $\pi(v_1, v_2|\overline{C})$ be the total expected contributions collected from players with valuations v_1 and v_2 with maximum limit $\overline{C}$. Figure 4.2 illustrates pi as a function of $\overline{C}$, for a fixed v_1 and v_2.

Proposition 13 *(Che and Gale (1998)). Suppose that $v_1 = v + \epsilon$ and $v_2 = v$, where $\epsilon > 0$, and* $\underline{C} = 0$.

(Large Maximum Limits) *If $\overline{C} \in (\frac{v}{2}, v]$, both lobbyists will mix over some subinterval $[0, C']$ and place an atom of positive weight at $\overline{C}$, where $C' = 2\bar{C} - v < \overline{C}$. Behavior over $[0, C']$ is given by*

$$G_1(c) = \frac{c}{v} \qquad \textit{and} \qquad G_2(c) = \frac{\epsilon}{v+\epsilon} + \frac{c}{v+\epsilon}. \tag{4.5}$$

(Smaller Maximum Limits) *If $\overline{C} \in [0, \frac{v}{2}]$, both lobbyists will contribute $\overline{C}$ with probability 1.*

Proof 14 *See Che and Gale (1998).*

Note the equilibrium switches form once the maximum limit is lowered past the threshold of $\frac{v}{2}$. For $\overline{C}$ above the threshold, players substitute the weight formerly at the top of their equilibrium interval as an atom at the maximum bid. There must be a gap between C' and $\overline{C}$ in order to make the lobbyists indifferent between the two contributions as placing any positive weight on $\overline{C}$ causes the player to experience a discontinuous change in payoffs. The value C' is chosen precisely to make this change just acceptable for each player. As the corollary shows below, this is a revenue neutral change in behavior from the politician's standpoint. With private information the associated change in $\overline{C}$ will not be revenue neutral. But with complete information, both lobbyists contribute the same amount (on average) they did before the maximum limit was enacted, though the shape of the distribution is different. Specifically total expected revenues experience a second-order stochastic shift[4]. Alternatively stated, the distribution of bids without a maximum limit is a mean-preserving spread of the distribution of bids with large maximum limits. Thus a risk-averse politician would benefit from imposing a maximum limit while a risk-neutral politician would be indifferent.

If on the other hand $\overline{C} < \frac{v}{2}$, the politician will collect $2\overline{C}$ with certainty. Without any limits, total expected revenues are $\pi(v, v+\epsilon|0,\infty) = \frac{v}{2} + \frac{v(2v+\epsilon)}{2(v+\epsilon)}$. Simple algebra shows that for values of $\overline{C} \in (\frac{v(2v+\epsilon)}{4(v+\epsilon)}, \frac{v}{2})$, total expected contributions will be increased (relative to the game with no limits) as long as $\epsilon > 0$. If $\epsilon = 0$, so that the lobbyists are symmetric, then when $\overline{C}$ is lowered past the threshold total expected revenues will start to strictly decrease.

Corollary 6 *Let $v_1 = v + \epsilon$, and $v_2 = v$. If $\epsilon = 0$, then total expected revenues are (weakly) increasing in $\overline{C}$ for all $\overline{C}$. If $\epsilon > 0$, then total expected revenues are strictly increasing in $\overline{C}$ for $\overline{C} \in [0, \frac{v}{2}]$, and then take a discontinuous jump downward where they remain constant for all $\overline{C} \geq \frac{v}{2}$.*

---

[4] If $F(x)$ is the distribution for revenues when $\overline{C} > v$ and $G(x)$ is the distribution when $\overline{C} \in (\frac{v}{2}, v]$, then G second-order stochastic dominates F in that $\int u(x)dG(x) \geq \int u(x)dF(x)$ for every concave $u(x)$.

Proof 15 *See Che and Gale (1998)and Figure 4.2.*

Thus imposing maximum contribution limits will increase the total expected contributions of the politician if and only if the maximum limit is neither too small nor too large and the lobbyists are asymmetric. If the lobbyists are symmetric, then full surplus extraction is already taking place without maximum limits. Lowering $\overline{C}$ can only decrease total expected revenues in this case. However, if the lobbyists are asymmetric then full surplus extraction is generally not possible with or without maximum limits. But imposing a maximum contribution limit allows the politician to get closer to full surplus extraction by partially reversing the effects of asymmetry between the lobbyists. The politician can expect to collect v in total expected contributions, which is less than full surplus extraction. Thus, when valuations are commonly known, maximum limits can be used as an instrument of the politician to increase revenues when there are asymmetric lobbyists.

Before moving on to the private information environment, I should note that the maximum limits tend to equalize the winning probabilities of the lobbyists. Suppose that $v_1 = v + \epsilon$ and $v_2 = v$. If $\overline{C} > \frac{v}{2}$, which includes the case of no maximum limit, the stronger lobbyist wins with probability $1 - \frac{v}{2(v+\epsilon)} > \frac{1}{2}$. Asymmetry between lobbyists implies that the stronger lobbyist will win more often than the weaker lobbyist. Once the maximum limit is lowered enough ($\overline{C} < \frac{v}{2}$), the winning probabilities for both lobbyists are the same. Thus not only would the the mean of total expected contributions increase and the risk associated with expected contributions decrease, maximum limits would allow the politician to appear more fair in that the weaker lobbyist gains influence more often.

4.4 Maximum Limits with Private Information

Before moving on to the more general asymmetric case I characterize the symmetric case to illustrate the importance of asymmetry on the ability of the politician to increase total expected contributions by imposing a maximum limit.

4.4.1 Symmetric Lobbyists

Suppose now that the valuations for political favor are private information to each lobbyist according to the model presented in Section 4.2 . In this section I further assume the distributions are identical for the players, i.e. that $V_1 = V_2$ and $p_{1,k} = p_{2,k}$ for each type k. If the maximum limit $\overline{C}$ is sufficiently high, the unique equilibrium is unaffected by its imposition and is hence given from Lemma 5 . This symmetric equilibrium $G = (G_1, \ldots, G_n)$ has each type v_k mixing over $[B_{k-1}, B_k]$ with density $g_k(x) = \frac{1}{p_k v_k}$, where $B_0 = 0$ and $B_k = \sum_{r=1}^{k} p_r v_r$. Thus any $\overline{C} > B_n$ has no effect on the equilibrium. However, once $\overline{C}$ is lowered to a value slightly smaller than B_n, the equilibrium behavior changes qualitatively. I now construct the unique equilibrium of this game for all $\overline{C}$.

If $\overline{C} \in (\frac{B_{n-1}+B_n}{2}, B_n)$, then type v_n will mix over $[B_{n-1}, C'_n]$ for some $C'_n \in [B_{n-1}, B_n]$. This is similar to the complete information case where the player mixes over a smaller sub-interval and places positive weight as an atom at the maximum limit $\overline{C}$. This will be a revenue neutral change. Once $\overline{C} < \frac{B_{n-1}+B_n}{2}$, then types v_n will contribute $\overline{C}$ with probability 1. I repeat that this is analogous to the complete information case.

As $\overline{C}$ is lowered even further, this pattern continues for all of the remaining (lower) types, though with slight different cutoff values for the changes in behavior. For each type there corresponds three intervals which I call “large”, “intermediate”, and “small”. In general, for “large” values of $\overline{C}$, behavior for types v_k is unaffected. That is, types v_k behave as if there was no maximum limit, even though larger types behavior is affected by the imposition of the limit. For “intermediate” values of $\overline{C}$, types v_k shift the weight at top of the interval from which they were previously mixing and place it as an atom at the maximum limit. Specifically, they will mix over a smaller sub-interval $[B_{k-1}, C'_k]$, for some $C'_k \in (B_{k-1}, B_k)$ which depends upon $\overline{C}$, and place some positive weight at $\overline{C}$. For “small” values of $\overline{C}$, types v_k will simply contribute $\overline{C}$ with probability 1. Now I summarize the equilibrium characterization with symmetric lobbyists.

Proposition 14 *Let lobbyists be symmetric and define* $B_k = \sum_{r=1}^{k} p_r v_r$ *and* $C'_k = 2\overline{C} - B_k - v_k(\sum_{r=k+1}^{n} p_r)$ *for each* $k = 1, \ldots, n$. *Recall the no-limit equilibrium mixing density and c.d.f.* $g_k(x) = \frac{1}{p_k v_k}$ *and* $G_{i,k}(x) = \frac{x - B_{n-1}}{p_k v_k}$ *respectively for each* k. *Also let* $L_k = \frac{B_{k-1}+B_k}{2} + \frac{v_k}{2}(\sum_{r=k+1}^{n} p_r)$ *and* $U_k = B_k + \frac{v_k}{2}(\sum_{r=k+1}^{n} p_r)$. *I say behavior is unaffected if it is the same behavior as the no-limit* ($\overline{C} = \infty$) *equilibrium. Then there is a unique equilibrium given by:*

- *If $\overline{C} > B_n$, then behavior is unaffected for all types.*
- *if $\overline{C} \in (U_k, L_{k+1})$*
  - *types $v > v_k$ contribute $\overline{C}$ with probability 1*
  - *types $v \leq v_k$ are unaffected*
- *if $\overline{C} \in (L_k, U_k)$*
  - *types $v > v_k$ contribute $\overline{C}$ with probability 1*
  - *type v_k mixes over $[B_{k-1}, C'_k]$ with density g_k and places an atom at $\overline{C}$ of size $1 - G_k(C'_k) = \frac{2B_k - 2\overline{C} + v_k(\sum_{r=k+1}^{n} p_r)}{p_k v_k}$*
  - *types $v < v_k$ are unaffected*
- *if $\overline{C} \in (U_{k-1}, L_k)$*
  - *types $v \geq v_k$ contribute $\overline{C}$ with probability 1*
  - *types $v < v_k$ are unaffected*

Proof 16 *First note that monotonicity of equilibrium ensures that when type v_k places an atom at $\overline{C}$, then so too will all types $v > v_k$ since this is the highest possible contribution and types v earn positive payoffs. Define $B_k = \sum_{r=1}^{k} p_r v_r$. Define $g_k(c) = \frac{1}{p_k v_k}$ and $G_k(c) = \frac{c - B_{k-1}}{p_k v_k}$. The proof here just uses simple indifference conditions and monotonicity to derive densities and argue that they are optimal.*

For type v_n, we can mimic the complete information construction. I claim that types $v < v_n$ will be unaffected for values of $\overline{C} > \frac{B_{n-1} + B_n}{2}$. I prove this later when constructing the behavior for v_{n-1} by showing that v_{n-1} would earn higher payoffs by not changing behavior. Now v_n will mix over $[\frac{B_{n-1} + B_n}{2}, C'_n]$ and place an atom at $\overline{C}$. Indifference conditions force that for all $c \in [B_{n-1}, C'_n]$:

$$v_n\left(\sum_{r=1}^{n-1} p_r * 1 + p_n G_n(c)\right) - c = v_n\left(\sum_{r=1}^{n-1} p_r + p_n G_n(C'_n) + p_n(1-G_n(C'_n)) * \frac{1}{2}\right) - \overline{C} \quad (4.6)$$

$$\overline{C} - c = \frac{p_n v_n}{2}\left(-2G_n(c) + 1 + G_n(C'_n)\right). \quad (4.7)$$

The value $1 - G_n(c'_n)$ is the size of the atom that i_n places at $\overline{C}$. Evaluating this at $c = C'_n$ pins down C'_n:

$$\overline{C} - C'_n = \frac{p_n v_n}{2}\left(1 - G_n(C'_n)\right) \quad (4.8)$$

$$2(\overline{C} - C'_n) = p_n v_n\left(1 - \frac{C'_n - \sum_{r=1}^{n-1} p_r v_r}{p_n v_n}\right) \quad (4.9)$$

$$2(\overline{C} - C'_n) = p_n v_n - C'_n + \sum_{r=1}^{n-1} p_r v_r \quad (4.10)$$

$$C'_n = 2\overline{C} - \sum_{r=1}^{n} p_r v_r = 2\overline{C} - B_n \quad (4.11)$$

This is only defined when $C'_n \geq B_{n-1}$, which is equivalent to $\overline{C} \geq \frac{B_{n-1}+B_n}{2}$. Thus whenever $\overline{C}$ is in the upper half of n's regular (i.e. without maximum limits) interval, he will mix over $[B_{n-1}, C'_n]$ with density $g_n(c) = \frac{1}{p_n v_n}$ and place an atom of size $1 - G_n(C'_n) = \frac{2(B_n - \overline{C})}{p_n v_n}$ at $\overline{C}$. Once $\overline{C} < \frac{B_{n-1}+B_n}{2}$, type v_n will contribute $\overline{C}$ with probability 1.

Now consider v_{n-1} types. I now solve for the value of $\overline{C}$ where the v_{n-1} types start mixing over $[B_{n-2}, C'_{n-1}]$ and placing an atom at $\overline{C}$. Let both players adopt this strategy when they are v_{n-1} types.

Indifference conditions are

$$v_{n-1}\left(\sum_{r=1}^{n-2} p_r + p_{n-1}G_{n-1}(c)\right) - c \tag{4.12}$$

$$= \tag{4.13}$$

$$v_{n-1}\left(\sum_{r=1}^{n-2} p_r + p_{n-1}\left(G_{n-1}(C'_{n-1}) + (1 - G_{n-1}(C'_{n-1})) * \frac{1}{2}\right) + \frac{1}{2}p_n\right) - \overline{C} \tag{4.14}$$

$$C'_{n-1} = 2\overline{C} - B_{n-1} - p_n v_{n-1} \tag{4.15}$$

Note that this is defined only when $C'_{n-1} \geq B_{n-2}$, which is equivalent to $\overline{C} \geq \frac{B_{n-2}+B_{n-1}}{2} + \frac{1}{2}p_n v_{n-1}$. Additionally it is only defined when $C'_{n-1} \leq B_{n-1}$, which is equivalent to $\overline{C} \leq B_{n-1} + \frac{1}{2}p_n v_{n-1}$. These are the boundaries for which behavior changes for the v_{n-1} types. Hence when $\overline{C} \geq B_{n-1}+\frac{1}{2}p_n v_{n-1}$, behavior for v_{n-1} types is unaffected. When $\overline{C} \in (\frac{B_{n-2}+B_{n-1}}{2}+\frac{1}{2}p_n v_{n-1}, B_{n-1}+\frac{1}{2}p_n v_{n-1})$, then v_{n-1} types will mix over $[B_{n-2}, C'_{n-1}]$ and place an atom at $\overline{C}$. Once $\overline{C} < \frac{B_{n-2}+B_{n-1}}{2} + \frac{1}{2}p_n v_{n-1}$, then v_{n-1} types will contribute $\overline{C}$ with probability 1.

By construction, v_{n-1} will not want to place any weight on until $\overline{C}$ until $\overline{C} < B_{n-1} + \frac{1}{2}p_n v_{n-1}$, provided the other v_{n-1} type of the other player does the same. Thus this is an equilibrium strategy.

We can generalize this for lower values of $\overline{C}$. Adapting the previous steps, we see that type $v_k < v_n$ will mix over $[B_{k-1}, B_k]$, i.e. not change his behavior from the no-limit case, when $\overline{C}$ is sufficiently high. Once $\overline{C}$ is low enough (I provide the exact value shortly), then v_k will be indifferent between mixing over $[B_{k-1}, C'_k]$ and placing an atom at $\overline{C}$ if

$$v_k\left(\sum_{r=1}^{k-1} p_r + p_k G_k(c)\right) - c \tag{4.16}$$

$$= v_k\left(\sum_{r=1}^{k-1} p_r + p_k\left[G_k(C'_k) + \frac{1}{2}(1 - G_k(C'_k))\right] + \frac{1}{2}\sum_{r=k+1}^{n} p_r\right) - \overline{C} \tag{4.17}$$

$$C'_k = 2\overline{C} - B_k - v_k(\sum_{r=k+1}^{n} p_r) \tag{4.18}$$

Using the fact that $C'_k \in [B_{k-1}, B_k]$*, we get that* $\overline{C} \in [\frac{B_{k-1}+B_k}{2} + \frac{1}{2}v_k(\sum_{r=k+1}^{n} p_r)\ ,\ B_k + \frac{1}{2}v_k(\sum_{r=k+1}^{n} p_r)]$.

All types who contribute $\overline{C}$ with probability 1 will continue to do so if $\overline{C}$ is lowered. Hence there is always a force from higher types that tends to lower total expected contributions when the maximum limit is lowered. If imposing maximum limits is to be profitable for the politician, it must be the case that there is some opposing force on contributions from the lower types that is larger in magnitude. With symmetric lobbyists I show that this opposite force is not enough to raise total expected contributions. With asymmetric lobbyists, a new opposing force exists. This force is that of reversing the asymmetry that higher types impose upon lower types through changing the matchups of these lower types.

To see this note that once $\overline{C}$ is lowered past U_k, there will be a discontinuous change in behavior from the v_k type. Weight is transferred from $[C'_k, B_n]$ to an atom of size $\frac{2B_k - 2\overline{C} + v_k(\sum_{r=k+1}^{n} p_r)}{p_k v_k}$ at $\overline{C}$. When $\overline{C}$ is just slightly lower than U_k, then $\overline{C} > B_k$. In other words, with positive probability type v_k will contribute an amount (i.e. $\overline{C}$) that was previously larger than the upper bound of v_k's no-limit equilibrium support (i.e. B_k). Necessarily this increases expected contributions collected from v_k types. From Proposition 14 we see that the lower types $v < v_k$ will be unaffected and hence revenue neutral. What I show later is that the rate at which total expected contributions increase from v_k transferring his atom to $\overline{C} > B_k$ is exactly the rate at which total expected contributions decrease from types $v > v_k$ contributing a lower $\overline{C}$ with probability 1.

Intuitively, lowering $\overline{C}$ limits the ability of high types to win as often as they would without limits. For low enough $\overline{C}$, high types $v > v_k$ will eventually donate the maximum amount with certainty. At this contribution level, the marginal benefit of contributing exceeds the marginal costs, but the high types are not permitted to donate more. The reduced competition from higher types allows middle types to see higher returns from contributing the maximum amount. Hence the middle type v_k responds by contributing the maximum amount, provided it is low enough. Low types do not change their behavior. In expectation I show below that these two effects offset so that net contributions are the same whenever $\overline{C} \in (L_k, U_k)$. However once this transferring stops taking place, total expected contributions must decrease. Hence total expected contributions can never be increased by imposing a maximum limit.

With private information, we can define $\pi(\overline{C}; F) \equiv \sum_{k=1}^{n} p_k \pi_k(\overline{C}; F)$ as the total expected contributions collected from each lobbyist when maximum limit $\overline{C}$ is imposed and both lobbyists have type space distribution F. Here $\pi_k(\overline{C}; F)$ is the contribution collected from type v_k, i.e. expected contributions conditional upon a type realization of v_k. Proposition 15 below summarizes this discussion. First I summarize the effect that $\overline{C}$ has on each type's total expected contributions in Lemma 6.

Lemma 6 *As a function of $\overline{C}$, each $\pi_k(\overline{C}; F)$ is continuous. Further $\pi_k(\overline{C}; F)$ strictly increasing when $\overline{C} \in [0, L_k]$, strictly decreasing when $\overline{C} \in (L_k, U_k)$, and constant when $\overline{C} \geq U_k$.*

Proof 17 *Using Proposition 14, when $\overline{C} \leq L_k$ type v_k contributes $\overline{C}$ with certainty and hence contributes more when $\overline{C}$ increases. When $\overline{C} \in (L_k, U_k)$, type v_k will remove weight from $\overline{C}$ and mix over a smaller interval with that weight. Hence contributions must decrease in this range. Once $\overline{C} > U_k$, type v_k is unaffected by the maximum limit and hence contributes the same amount. Hence $\pi_k(\overline{C}; F)$ is strictly increasing, then strictly decreasing, then constant. To show continuity note that the size of the atom at $\overline{C}$ is a continuous function of $\overline{C}$.*

Hence there does exist a region (L_k, U_k) where lower types can be induced to contribute more by limiting the ability of higher types to contribute a lot. However this increase in total expected contributions will exactly be offset by the loss of total expected contributions of the high types. Thus, the ability of maximum limits to increase total expected contributions (relative to contributions without any limits) is intimately linked with the ability of the maximum limit to change the matchups of the lower types. With symmetric lobbyists, matchups are always the same for lower types and so cannot be changed by a maximum limit. With asymmetric lobbyists, we will see that the matchups will be affected. This ultimately will allow the politician to potentially increase by imposing a limit, though this need always be the case.

Proposition 15 *Define B_k, U_k, and L_k as in Proposition 14. Holding the symmetric type space c.d.f. F fixed, $\pi(\overline{C}; F)$ is (weakly) increasing and continuous in $\overline{C}$. Specifically $\pi(\overline{C}; F)$ is strictly increasing when $\overline{C} \in (U_{k-1}, L_k)$ and constant when $\overline{C} \in [L_k, U_k]$ for each $k = 1, \ldots, n$.*

Proof 18 *Using Lemma 6, we see that monotonicity simplifies the behavior of $\pi(\overline{C}; F)$. Note that lobbyists are symmetric and so too is the equilibrium. I focus on the behavior of just a single lobbyist. Recall v_k's no-limit equilibrium behavior is summarized by the c.d.f., G_k, where*

$$G_k(c) = \begin{cases} 0 & \text{if } c < B_{k-1} \\ \frac{c - B_{k-1}}{p_k v_k} & \text{if } c \in [B_{k-1}, B_k] \\ 1 & \text{if } c > B_k \end{cases} \tag{4.19}$$

Let $\overline{C} = U_k - \delta$ for some positive $\delta > 0$. The net contribution from all of the types $v > v_k$ to total expected revenues will be $-\delta \overline{C} \sum_{r=k+1}^{n} p_r$. The net contribution for type v_k will be $p_k \left((1 - G_k(C_{k'}))\overline{C} - \int_{C'_k}^{B_k} \frac{1}{p_k v_k} x dx \right) = \delta(\sum_{r=k+1}^{n} p_r)$. The first term represents the extra expected contributions from contributing $\overline{C} > B_k$ with positive probability. Since types $v < v_k$ do not change their behavior, the overall change in total expected contributions is

$$\Delta\pi(\overline{C}, F)|_{\overline{C} \in [L_k, U_k]} \underbrace{\delta(\sum_{r=k+1}^{n} p_r)}_{\text{type } v_k} - \underbrace{\delta(\sum_{r=k+1}^{n} p_r)}_{\text{types } v > v_k} = 0 \tag{4.20}$$

This shows that $\pi(\overline{C}; F)$ is constant when $\overline{C} \in [L_k, U_k]$ for each k. Using Lemma 6 we see that when $\overline{C} \in (U_k, L_{k+1})$ we see that all types $v \leq v_k$ are unaffected, but increasing $\overline{C}$ will allow types $v > v_k$ to contribute more since they are contributing $\overline{C}$ with probability 1 in this range. The result follows.

Corollary 7 *Lowering $\overline{C}$ can never increase the total expected contributions $\pi(\overline{C}; F)$ if lobbyists are symmetric.*

Figure 4.3 illustrates the results of Proposition 15 for the case of $n = 3$ types. Compare Figure 4.2, which illustrates total expected contributions with complete information, with Figure 4.3, which illustrates the private information case with symmetric lobbyists. There is smooth concave pattern of expected contributions: increasing $\overline{C}$ will increase $\pi(\overline{C}, F)$ continuously, but at a decreasing rate. With asymmetric lobbyists there will "saw-tooth" shape, where $\pi(\overline{C}, \cdot)$ is increasing locally over sub-intervals, but then has discontinuous gains and drops when $\overline{C}$ crosses one of a finite number of thresholds.

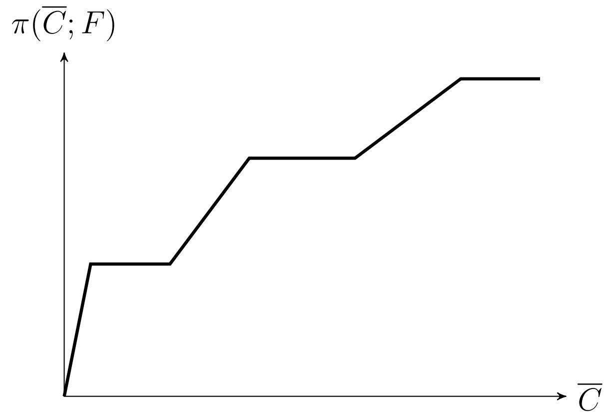

Figure 4.3: Expected Contributions for Symmetric Lobbyists

4.4.2 Asymmetric Lobbyists

Recall that in the complete information case, lowering the maximum contribution limit will allow the politician to increase total expected contributions only if the lobbyists are asymmetric. Asymmetry is detrimental to total expected contributions. Maximum contribution limits level the playing field and partially offset the effects of asymmetry. In fact, lowering the maximum limit to intermediate levels *must* increase total expected contributions. When private information is introduced we see that lowering the maximum limit must increase the expected contribution of some type, but also decreases the expected contribution of other types. Since total expected contributions are weighted by the prior probabilities of these types, the net effect of lowering the contribution may be that total expected contributions decrease. This happens if the probability or "weight" of the revenue decreasing types is larger than that of the revenue increasing types. This section characterizes the revenue increasing and revenue decrease types in terms of the asymmetry of the underlying type space. In general, the ability of a maximum limit to raise revenues is reduced as $\overline{C}$ is reduced further and further. This is because the increase in revenue from the revenue increasing types must offset a great mass of types that contribute a smaller $\overline{C}$ with probability 1.

The equilibrium construction closely follows the complete information construction given by Che and Gale (1998). Since I have already characterized this equilibrium in an earlier section, here I just review the most salient properties. A full derivation of Che and Gale (1998)'s results are provided in the Appendix. Trivially when $\overline{C}$ is sufficiently large, equilibrium behavior will be unaffected since the set of all possible contributions without limits is bounded. With private information, this true only for values of $\overline{C}$ in the upper half of the largest types' equilibrium supports. Lowering $\overline{C}$ well above type $v_{i,k}$'s support will in general affect the behavior of $v_{i,k}$. This highlights the importance of introducing asymmetry, as this feature is not present with symmetric lobbyists.

Further, when $\overline{C}$ is low enough, both players will contribute $\overline{C}$ with probability 1. With private information there will exist cutoffs L_k such that if $\overline{C} < L_k$ both player's kth largest types will contribute $\overline{C}$ with probability 1. However for intermediate values of $\overline{C}$, lobbyists will contribute more on average if and only if the lobbyists are asymmetric. There will be a discontinuous increase in total expected contributions when $\overline{C}$ is lowered past the midpoint of the normal equilibrium support. With private information, lowering $\overline{C}$ below L_k will cause a discontinuous change in payoffs that is proportional to the probability of v_k for each lobbyist. Hence there will be a "saw-tooth" graph as illustrated in Figure 4.4, which illustrates a typical equilibrium. Note however that whether the maximum of $\pi(\overline{C}, F_1, F_2)$ is larger or greater than $\pi(\infty, F_1, F_2)$ is determined by the underlying type space. In other words it can vary. I formally state this later. If player i has type space c.d.f. F_i, let $\pi(\overline{C}, F_1, F_2)$ be the total revenues the politician would expect to collect from both lobbyists when maximum limit $\overline{C}$ is imposed.

With arbitrary type space distributions, there are simply too many possible equilibrium matchups. Hence I present my results only in terms of the possibilities of what contribution limits can affect.

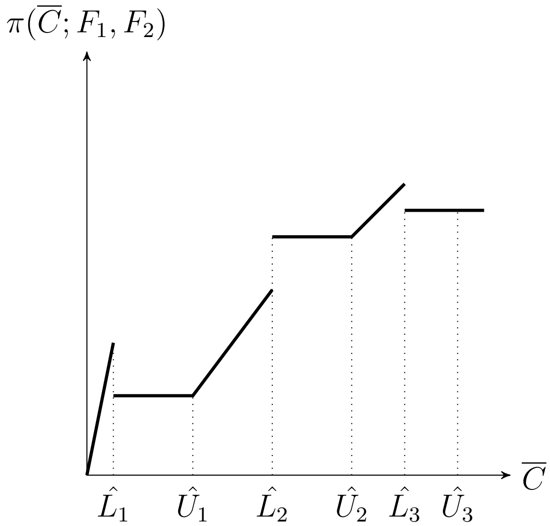

Figure 4.4: Expected Contributions with Asymmetric Lobbyists

The key insight is that for a low enough $\overline{C}$, types above $v_{1,k}$ and $v_{2,k}$ will contribute $\overline{C}$ with certainty. Thus using the equilibrium construction it is *as if* the lobbyists had only types $v_{1,1}, \ldots, v_{1,k}$ and $v_{2,1}, \ldots, v_{2,k}$ for low enough $\overline{C}$.

In other words, both types $v_{1,k}$ and $v_{2,k}$ start with fresh weight. Normally one of these types would face up against one of the other lobbyist's higher types. Now the weight must be expended on lower types as well. This causes the equilibrium support to decrease overall for these types since one of the k-types meets weaker competition. Whether or not this effect increases revenues depends upon whether the asymmetry from the $(k+1)$-types originally increased or decreased revenues.

Hence with asymmetric types there are three tensions affecting total expected revenues. The first tension is that by lowering $\overline{C}$, the politician will trivially lower the revenues collected from the high types who are contributing $\overline{C}$ with certainty. The second tension is that by lowering $\overline{C}$, the largest types not already contributing $\overline{C}$ (i.e. the k types in the previous paragraph) will contribute more. These two effects cancel with symmetric types .

The third tension of total expected revenues, which is unique to asymmetric lobbyists, is that by lowering $\overline{C}$, the matchups of the remaining lower types change. Suppose that $\overline{C}$ is low enough so that both players' $k+1, k+2, \ldots, n$ types contribute $\overline{C}$ with probability 1. Then $v_{1,k}$ will be matched up with $v_{2,k}$, and in general one of the these types will fill up its weight faster than the other. Suppose WLOG that $v_{1,k}$ fills up his weight first, so that $v_{2,k}$ will need to use her remaining weight on $v_{1,k-1}$ and possibly other lower types of player 1. The chapter on comparative statics in my dissertation illustrates that this asymmetry will increase total expected revenues if and only if the probability of types higher than k is large enough. Hence the third tension will be positive if the probability of types higher than k is low enough, since the limit effectively makes types above k symmetric.

The following proposition characterizes the equilibrium with maximum limits. The construction is very similar to the symmetric case, though with care taken for the fact that the behavior of the lower types is continually changing as $\overline{C}$ is lowered.

Proposition 16 *Let F_1 and F_2 be any finite type c.d.f.'s for lobbyists 1 and 2 respectively. WLOG*[5] *assume that if only types $1, \ldots, k$ existed, $p_{2,k}v_{1,k} < p_{1,k}v_{2,k}$ so that type $v_{2,k}$ would be able to fill up her weight first against $v_{1,k}$ if both start an equilibrium construction. If k-types are the largest type, define $B_{k,k-1}$ and $B_{k,k}$ as the lower and upper bound of the interval where they are matched up, where the length of this interval is given by $L_{k,k} \equiv B_{k,k} - B_{k,k-1} = p_{2,k}v_{1,k}$. Define $\hat{L}_k = \frac{B_{k,k-1}+B_{k,k}}{2} + \frac{1}{2}v_{1,k}\sum_{r=k+1}^{n} p_{2,r}$ and $\hat{U}_k = B_{k,k} + \frac{1}{2}v_{1,k}\sum_{r=k+1}^{n} p_{2,r}$. Lastly define $C'_k = 2\overline{C} - B_{k,k} - v_{1,k}\sum_{r=k+1}^{n} p_{2,r}$. Then there is a unique equilibrium characterized in the following manner as a function of $\overline{C}$:*

- *If $\overline{C} > B_{n,n} = \hat{U}_n$, then the equilibrium is unaffected by the imposition of the maximum limit $\overline{C}$.*
- *If $\overline{C} \in (\hat{U}_k, L_{k+1})$, then all $k+1, k+2, \ldots, n$ types for both players contribute $\overline{C}$ with probability 1. Types $1, \ldots, k$ behave as if the largest types were $v_{1,k}$ and $v_{2,k}$ and there is no limit.*
- *If $\overline{C} \in (\hat{L}_k, \hat{U}_k)$, then types $k+1, \ldots, n$ contribute $\overline{C}$ with probability 1. Types $v_{1,k}$ and $v_{2,k}$*

[5]This simply pins down the interval over which k-types are matched up to be $[B_{k,k-1}, B_{k,k}]$ rather than $[B_{k-1,k}, B_{k,k}]$.

mix over $(B_{k,k-1}, C'_k)$, with densities $g_{1,k}(c) = \frac{1}{p_{1,k}v_{2,k}}$ and $g_{2,k}(c) = \frac{1}{p_{2,k}v_{1,k}}$ respectively, and place an atom at $\overline{C}$. Types $1, \ldots, k-1$ behave as if the largest types were $v_{1,k}$ and $v_{2,k}$ and there is no limit.

- *If $\overline{C} \in (\hat{U}_{k-1}, \hat{L}_k)$, then types $k, k+1, \ldots, n$ for both players contribute $\overline{C}$ with probability 1. Types $1, \ldots, k-1$ behave as if there is no maximum limit and $k-1$-types are the largest type.*

Proof 19 *The construction of the equilibrium will proceed by starting with the largest types. If types $v_{1,k}$ and $v_{2,k}$ are matched up in equilibrium, define $B_{k,m}$ as the upper bound of the interval where the types are matched and $L_{k,m}$ as the corresponding length of that interval. Asymmetry introduced a new problem not seen with symmetric types. At each iteration in the equilibrium construction the matchups, and hence the values of $B_{k,m}$ and $L_{k,m}$ change. During the proof we will only need to note that both of these variables decrease.*

Let $\overline{B}$ be the upper bound of $v_{1,n}$'s (and hence also that of $v_{2,n}$) equilibrium interval when there are no limits (i.e. the value of $B_{n,n}$ in the equilibrium without limits). If $\overline{C} \geq \overline{C}$, then the maximum limit has no effect.

If $\overline{C} < \overline{B}$, then the equilibrium behavior must change. Replicating the construction for the complete information case, we see that both players will mix over a smaller subinterval and then place an atom at the maximum limit. Behavior of types $1, \ldots, n-1$ is unaffected since the atom of the weaker player is unchanged when $\overline{C} \in (B_{n,n} - \frac{1}{2}L_{n,n}, B_{n,n})$, i.e. when the maximum limit is in the top half of the top ($v_{1,n}$ vs. $v_{2,n}$) interval. When $\overline{C}$ is in this range, types $v_{1,n}$ and $v_{2,n}$ will mix over $[B_{n,n} - L_{n,n}, C'_n]$ with densities $g_{1,n}(c) = \frac{1}{p_{1,n}v_{2,n}}$ and $g_{2,n}(c) = \frac{1}{p_{2,n}v_{1,n}}$. Both types will also place an atom at $\overline{C}$. In addition, one of the two players will have extra weight which is used on lower types. As mentioned earlier, this leftover weight does not vary with $\overline{C}$ when $\overline{C} \in (B_{n,n} - \frac{1}{2}L_{n,n}, B_{n,n})$. This ensures the intuition for the complete information case can be used. This will be a revenue neutral change. The indifference conditions for $v_{1,n}$ is

$$v_{1,n}\left(\sum_{r=1}^{n-1} p_{2,r} + p_{2,n}G_{2,n}(c)\right) \tag{4.21}$$

$$= v_{1,n}\left(\sum_{r=1}^{n-1} p_{2,r} + p_{2,n}\left[G_{2,n}(C'_n) + \frac{1}{2}(1 - G_{2,n}(C'_n))\right]\right) - \overline{C}, \tag{4.22}$$

with an analogous equality for $v_{2,n}$. Suppose without loss of generality that $g_{1,n}(c) = \frac{1}{p_{1,n}v_{2,n}} < \frac{1}{p_{2,n}v_{1,n}} = g_{2,n}(c)$ so that player 2 fills up her weight first. This ensures that $L_{n,n} = p_{2,n}v_{1,n}$ and $B_{n,n-1} + L_{n,n} = B_{n,n}$, which implies $G_{2,n}(c) = \frac{c - B_{n,n-1}}{p_{2,n}v_{1,n}}$. The indifference condition reduces to

$$C'_n = 2\overline{C} - B_{n,n-1} - L_{n,n} \tag{4.23}$$

$$= 2\overline{C} - B_{n,n-1} - p_{2,n}v_{1,n}. \tag{4.24}$$

This is defined only when $C'_n \in (B_{n,n-1}, B_{n,n})$ which is equivalent to to $\overline{C} \in (B_{n,n-1} + \frac{1}{2}p_{2,n}v_{1,n}, B_{n,n})$. Once $\overline{C} < B_{n,n-1} + \frac{1}{2}p_{2,n}v_{1,n}$, then both $v_{1,n}$ and $v_{2,n}$ will start contributing $\overline{C}$ with probability 1. This of course means that $v_{1,n}$ is no longer matched up with $v_{2,n}$. Now the equilibrium is as if $v_{1,n-1}$ and $v_{2,n-1}$ were the largest types. In particular, from $v_{2,n-1}$'s perspective, $v_{2,n-1}$ now faces weaker competition for more of her weight, so that $v_{2,n-1}$ responds by increasing her density and contracting the overall size of her interval. In other words, although $B_{n-1,n-1}$ will increase when the n-types start contributing $\overline{C}$ with probability 1, $L_{n,n-1} \to 0$ so that the overall length of the equilibrium of all types up to and including $(n-1)$-types must decrease.

The significance of the above statement is that it ensures there is a gap between the mixing interval of the $(n-1)$-types (and hence all lower types) and the contributing behavior of the n-types, who contribute $\overline{C}$ with probability 1. This gap is necessary for the equilibrium construction in order to make the lower types not want to enjoy the discontinuous gain in payoff from contributing $\overline{C}$.

This general pattern is continued for all lower types. Once $\overline{C}$ is low enough, both players'

$k+1, k+2, \ldots, n$-types will contribute $\overline{C}$ with probability 1. This means $v_{1,k}$ and $v_{2,k}$ are first matched up with each other. Lowering $\overline{C}$ slightly from the threshold where $(k+1)$-types starting contributing $\overline{C}$ will not change the behavior of the newly matched up $v_{1,k}$ and $v_{2,k}$. There is a thus a small range of $\overline{C}$ where behavior will not change. However eventually $\overline{C}$ will be low enough so that k-types will both mix over $[B_{k,k}-L_{k,k}, C'_k]$ and place positive weight at $\overline{C}$, while all remaining lower types are unaffected. Further lowering $\overline{C}$ will eventually have both k-types contributing $\overline{C}$ with probability 1. Once this takes place, a new matchup of $v_{1,k-1}$ and $v_{2,k-1}$ takes place. The process continues until the lowest type of one of the players expends all weight.

Specifically, types $v_{1,k}$ and $v_{2,k}$ will be indifferent contributing $c \in (B_{k,k}-L_{k,k}, C'_k)$ when

$$v_{1,k}\left(\sum_{r=1}^{k-1} p_{2,r} + p_{2,k}G_{2,k}(c)\right) - c \tag{4.25}$$

$$= v_{1,k}\left(\sum_{r=1}^{k-1} p_{2,r} + p_{2,k}\left[G_{2,k}(C'_k) + \frac{1}{2}(1-G_{2,k}(C'_k))\right] + \frac{1}{2}\sum_{r=k+1}^{n} p_{2,r}\right) - \overline{C} \tag{4.26}$$

Suppose without loss of generality that player 2 would normally be the one to fill up weight first. In other words suppose that $g_{1,k}(c) = \frac{1}{p_{1,k}v_{2,k}} < \frac{1}{p_{2,k}v_{1,k}} = g_{2,k}(c)$. This means that $L_{k,k} = p_{2,k}v_{1,k}$ and also $B_{k,k-1} + L_{k,k} = B_{k,k}$. This implies that $G_{2,k}(c) = \frac{c-B_{k,k-1}}{p_{2,k}v_{1,k}}$ in this region of c. Hence the indifference condition for $v_{1,k}$ pins down the value of C'_k:

$$C'_k = 2\overline{C} - B_{k,k} - v_{1,k}\sum_{r=k+1}^{n} p_{2,r} \tag{4.27}$$

which is defined only when $C'_k \in (B_{k,k-1}, B_{k,k})$. Using the definition of C'_k this is equivalent to $\overline{C} \in \left(\frac{B_{k,k-1}+B_{k,k}}{2} + \frac{1}{2}v_{1,k}\sum_{r=k+1}^{n} p_{2,r}, B_{k,k} + \frac{1}{2}v_{1,k}\sum_{r=k+1}^{n} p_{2,r}\right) \equiv (\hat{L}_k, \hat{U}_k)$.

Hence when $\overline{C} > \hat{U}_k$, types $v_{1,k}$ and $v_{2,k}$ will mix over $[B_{k,k-1}, B_{k,k}]$ with player 2 being able to fill all her weight up here but $v_{1,k}$ needing to be matched up against $v_{2,k-1}$. All types above k for both players contribute $\overline{C}$ with probability 1. When $\overline{C} \in (\hat{L}_k, \hat{U}_k)$, lower types ($< k$) will again be unaffected while the k-types mix over $(B_{k,k-1}, C'_k)$ and place some weight at $\overline{C}$. Higher *types ($> k$) continue to contribute $\overline{C}$ with probability 1. When $\overline{C} < \hat{L}_k$, then both k-types (and all higher types) will contribute $\overline{C}$ with probability 1. Once this change in behavior occurs, matchups will change for the lower types, but the length of the interval that types $1, \ldots, k-1$ mix over must shrink in response. The pattern then continues for all remaining lower types.*

Note that this is the same qualitative equilibrium behavior as displayed in Sahuget (2006), who considers asymmetric lobbyists with types continuously drawn from uniform distributions. My result is the finite type version that allows for any type space distributions (not just uniform distributions).

Lowering $\overline{C}$ will eventually cause all types to contribute $\overline{C}$ with probability 1. As it is lowered from an arbitrarily large value, first the n-types start shifting weight from the top half of their interval to an atom at $\overline{C}$, while all other types are unaffected. Lowering $\overline{C}$ further will have both n-types contributing $\overline{C}$ with probability 1. Note that means that matchups (and hence equilibrium behavior) for the lower types is changed. This will contract the length of the interval that lower types contribute over. This has a discontinuous effect on behavior. Hence the asymmetric equilibrium has nearly the same form as the symmetric equilibrium.

The key difference is that with symmetric types, once the k-types start contributing $\overline{C}$ with probability 1, k-types will contribute more on average while all higher types contribute less. These two effects exactly offset. With asymmetric types, these two tensions can increase or decrease total expected contributions. The reasoning is that the behavior of the lower types change only when $\overline{C} = \hat{U}_k$ for every $k < n$. When $\overline{C} \in (U_k, U_{k+1})$, the atom of the weakest k type doesn't change from the same reasoning as the complete information game. Hence equilibrium behavior will be unaffected when $\overline{C}$ is in this range. At $\overline{C} = \hat{U}_k$, the asymmetry imposed upon the $1, \ldots, k-1$ types from the k-types will disappear, thus increasing or decreasing total expected revenues for the same reasoning as the complete information case. The proposition below summarizes this fact.

Proposition 17 *Consider the game with types $k+1, \ldots, n$ removed and $\overline{C} = \infty$. Total expected contributions have a discontinuous decrease at $\overline{C} = \hat{U}_k$ if the asymmetry in the k-types in the* *modified game decreased total expected revenues. Thus $\pi(\overline{C}, F_1, F_2)$ is increasing at $\overline{C} = \hat{U}_k$ if the probability of types larger than k is relatively large compared to $p_{1,k}$ and $p_{2,k}$.*

Corollary 8 *The size of the discontinuous changes in $\pi(\overline{C}, F_1, F_2)$ approaches 0 as $n \to \infty$. Thus total expected contributions will be continuous and potentially non-monotonic when $n \to \infty$*

Proof 20 *When $\overline{C} \in (\hat{L_k}, \hat{U_k})$ using the same logic as the complete information case and the symmetric private information, we se that total expected contributions are the same since the behavior of the lower types is unaffected. When $\overline{C} \in (\hat{U_{k-1}}, \hat{L_k}$, contributions will strictly decrease when $\overline{C}$ is lowered as behavior of the lower types is the same but all higher types contribute a smaller $\overline{C}$ with probability 1. When $\overline{C} = U_{k-1}$, $\pi(\overline{C}, F_1, F_2)$ is discontinuous and can increase or decrease.*

Hence depending upon the form of asymmetry, imposing maximum contribution limits may or may not increase total expected contributions. This contrasts with the complete information lobbying game, where intermediate values of maximum limits *must* increase total expected contributions. The intuition is simple. With private information, the gains in total expected contributions can only come from the lower types contributing more. These gains are discounted by the measure of these types, so that if the measure of the revenue-increasing types is larger than the measure of the revenue-decreasing types, total expected revenues increase. As the maximum limit is further lowered, the measure of revenue-decreasing types gets larger so that for low enough values of the maximum limit total expected revenues must decrease.

It may also be the case that in reversing the effects of asymmetry, lowering the maximum contribution limit may never increase the total expected contributions relative to the no limit case. While there must be *local* increases in total expected contributions when $\overline{C}$ is lowered, it may not be the case that *global* total expected revenues increase.

4.5 Conclusion

Contribution limits can limit the influence of large donors. Lowering the maximum limit will change the behavior of all the types for both lobbyists. Using the results from Chapter 2, we see that asymmetry in a particular type may serve to increase or decrease the total contributions collected from the lobbyists relative to the total contributions the politician can expect to collect when no limit is present. The higher types will contribute less, the medium types contribute more, and the lower types may contribute more or less as the maximum limit is decreased. Private information allows for the possibility that no maximum limit can increase total expected revenues, something which is impossible with complete information. In general, imposing a maximum contribution limit. Hence the perverse result of Che and Gale (1998) disappears when private information is added. Total expected contributions can increase after imposing a maximum limit only if the probability of low types is large enough.

Intuitively, when the limit is lowered the lower types are given a greater chance of winning and respond by increasing their contributions. Higher types contribute a smaller maximum allowable amount when the limit is lowered. The net effect ultimately depends upon the relative likelihood of the types. While equalizing the probability of winning between the lobbyists, maximum limits also equalize the probability of winning across types of the same lobbyist. If fairness is a concern, where fairness could mean equal winning probabilities, then setting $\overline{C} = 0$ is optimal. If maximizing total expected contributions is instead the goal, either no limits or some intermediate limit is optimal.

To apply this work to the real world, it is useful to consider a world with contribution limits that is suddenly changed by eliminating those limits. Typically total expected contributions will increase unless the likelihood of large donors is very small. Since this is not the case, my model can explain the increase in total expected contributions as due to the likely presence of high valuation donors. Thus one would expect the *Citizen's United vs. FEC* decision, which eliminated contribution limits, to increase due to the presence of lobbyists with lots of money to spend. The average citizen, with a low valuation for donating, is effectively displaced from the political conversation.

Future work might incorporate ideological bias of the politician, who might favor one of the lobbyist's policies *ext-ante* that would come with political favor. It might not be in the best interest of the politician to award political favor to the unpopular lobbyist if the unpopular lobbyist was able to donate more because doing so would alienate many voters in the politician's constituency. In other words the politician might trade off the benefit of a larger contribution with the ability of the contribution to generate more votes . This might be an interesting avenue to investigate transparency and the role of voters knowing the politician's political dispositions. Alternatively, the politician has a limited attention span and must decide which issues to let lobbyists compete over. It is not clear that issues with the most importance to the voters would always get priority as a minor issue might induce more contributions if the asymmetry between the players wills it so.

Designing the optimal contribution mechanism seems promising, where optimal can be defined in several ways (revenue maximizing, efficient, fair, etc.). This dissertation assumes that the contribution limits are the same for both lobbyists for example, but this may not be optimal for the politician. It would be interesting to investigate whether implementation of the optimal mechanism involves some of the typical contest designs found in practice: the use of head-starts, handicaps, reserve prices, and quotas. The optimal lobbying mechanism is an indirect mechanism with a report to mechanism designer being the payment. Essentially it would provide a converse approach, which is necessarily less optimal in terms of revenue maximizing, to the Myerson optimal auction (MOA). Recall in the MOA that only the winner pays and the payment is determined by the allocation rule. In my context, I would take the payments as given, where everyone pays, and determine the allocation rule that maximizes the payments.

REFERENCES

AMANN, E. AND W. LEININGER (1996): "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case," *Games and Economic Behavior*, 14, 1–18.

ATHEY, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861–889.

ATHREYA, K. B. AND S. N. LAHIRI (2010): *Measure Theory and Probability*, Springer.

AUMANN, R. J. (1964): "Mixed and Behavior Strategies in Infinite Extensive Games," *Advances in Game Theory*, 627–650.

——— (1987): "Correlated equilibrium as an expression of Bayesian rationality," *Econometrica*, 55, 1–18.

BAUMGARTNER, F. R., J. M. BERRY, M. HOJNACKI, D. C. KIMBALL, AND B. LEECH (2009): *Lobbying and Policy Change*, The University of Chicago Press.

BAYE, M. R., D. KOVENOCK, AND C. G. DE VRIES (1993): "Rigging the Lobbying Process: an application of the all-pay auction," *American Economic Review*, 83, 289–294.

——— (1996): "The all-pay auction with complete information," *Economic Theory*, 8, 291–305.

BECKER, G. (1983): "A Theory of Competition Among Pressure Groups for Political Influence," *Quarterly Journal of Economics*, 98, 371–400.

BLACKWELL, D. (1953): "Equivalent Comparisons of Experiments," *The Annals of Mathematical Statistics*, 24, 265–272.

BULOW, J. AND J. LEVIN (2006): "Matching and Price Competition," *American Economic Review*, 96, 653–668.

CHE, Y.-K. AND I. L. GALE (1998): "Caps on Political Lobbying," *American Economic Review*, 96, 643–651.

CONTAT, J. (2013a): "Minimum and Maximum Bets in Incomplete Information Political Lobbying," *Working Paper*.

——— (2013b): "A Note on the Convergence of Finite-Type All-Pay Auctions with Private Independent Values," *Working Paper*.

——— (2014): "Minimum and Maximum Contribution Limits in Political Lobbying," *Working Paper*.

DASGUPTA, P. AND E. MASKIN (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *The Review of Economic Studies*, 53, 1–26.

DAVIS, D. D. AND R. J. REILLY (1998): "Do too many cooks always spoil the stew? An experimental analysis of rent-seeking and the role of a strategic buyer," *Public Choice*, 95, 89–115.

ENGL, G. (1995): "Lower Hemicontinuity of the Nash Equilibrium Correspondence," *Games and Economic Behavior*, 9, 151–160.

FEHR, D. AND J. SCHMID (2011): "Exclusion in the All-Pay Auction: An Experimental Investigation," *Working Paper*.

FIBICH, G., A. GAVIOUS, AND A. SELA (2004): "Revenue Equivalence in asymmetric auctions," *Journal of Economic Theory*, 115, 309–321.

FUDENBERG, D. AND D. LEVINE (1986): "Limit Games and Limit Equilibria," *Journal of Economic Theory*, 38, 261–279.

GAVIOUS, A., B. MOLDOVANU, AND A. SELA (2002): "Bid Costs and Endogenous Bid Caps," *RAND Journal of Economics*, 96, 709–722.

HILLMAN, A. L. AND J. G. RILEY (1989): "Politically contestable rents and transfers," *Economic Politics*, 1, 17–40.

JAMISON, D. D. AND G. GORE, eds. (2012): *Super PACs (Political Action Committees): The New Force in Federal Elections*, Nova Science Publishers.

KAJII, A. AND S. MORRIS (1997): "The Robustness of Equilibria to Incomplete Information," *Econometrica*, 65, 1283–1309.

KAMENICA, E. AND M. GENTZKOW (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

——— (2012a): "Competition in Persuasion," *Working Paper*.

——— (2012b): "Disclosure of Endogenous Information," *Working Paper*.

——— (2014): "Costly Persuasion," *Working Paper*.

KIRKEGAARD, R. (2013): "Incomplete Information and rent dissipation in deterministic contests," *International Journal of Industrial Organization*, 31, 261–266.

KOLMOGOROV, A. AND S. FOMIN (1970): *Introductory Real Analysis*, New York: Dover Press, english ed.

KONRAD, K. A. (2009): *Strategy and Dynamics in Contests*, Oxford University Press.

LEBRUN, B. (1998): "Comparative Statics in First Price Auctions," *Games and Economic Behavior*, 25, 97–110.

MALBIN, M. J., ed. (2006): *The Election After Reform: Money, Politics, and the Bipartisan*

Campaign Reform Act, Rowma and Littlefield Publishers.

MASKIN, E. AND J. RILEY (2000a): "Asymmetric Auctions," *Review of Economic Studies*, 67, 413–438.

——— (2000b): "Equilibrium in Sealed High Bid Auctions," *Review of Economic Studies*, 67, 439–454.

MATEJKA, M., S. ONDERSTAL, AND A. D. WAEGENAERE (2002): "The Effectiveness of Caps on Political Lobbying," *Cent ER Discussion Paper*.

MILGROM, P. AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

MILGROM, P. R. AND R. J. WEBER (1985): "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, 10, 619–632.

MÜLLER, W. AND A. SCHOTTER (2010): "Workaholics and Dropouts in Organizations," *Journal of the European Economic Association*, 8, 717–743.

NOUSSAIR, C. AND J. SILVER (2006): "Behavior in All-Pay Auctions Under Incomplete Information," *Games and Economic Behavior*, 55, 189–206.

OK, E. A. (2007): *Real Analysis with Economic Applications*, Princeton University Press.

PARREIRAS, S. O. AND A. RUBINCHIK (2010): "Contests with three or more heterogeneous agents," *Games and Economic Behavior*, 69, 703–715.

PAVLOV, G. (2013): "Correlated equilibria and communication equilibria in all-pay auctions," *Working Paper*.

ROCKAFELLAR, R. (1970): *Convex Analysis*, Princeton University Press.

SAHUGET, N. (2006): "Caps in asymmetric all-pay auctions with incomplete information," *Working Paper*.

SHIRYAEV, A. N. (1996): *Probability (Graduate Texts in Mathematics)*, Springer-Verlag, second ed.

SIEGEL, R. (2013): "Asymmetric Contests with Interdependent Valuations," *Working Paper*.

SOBEL, J. (2010): "Giving and Receiving Advice," *Econometric Society 10th World Congress*.

SZECH, N. (2012): "Asymmetric All-Pay Auctions with Two Types," *Working Paper*.

VIVES, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305–321.

ZANDT, T. V. AND X. VIVES (2007): "Monotone equilibria in Bayesian games of strategic complementarities," *Journal of Economic Theory*, 134, 339–360.