1–D Isothermal Spinning Models for Liquid Crystalline Polymer Fibers *

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Abstract

A slender 1-D model for filaments of liquid crystalline polymers (LCPs) is applied to simulate isothermal fiber spinning of materials with internal orientation. The model and predictions focus on the hydrodynamic-orientation interactions in spinning flows, isolated from other significant spinline effects of temperature, crystallization and phase changes. An important practical feature of the model is that spun fiber orientation (in particular, birefringence) is deduced from first principles along with fiber diameter and velocity. A nonlinear stress-optical relation is provided by a constitutive law, which gives the fiber axial stress in terms of viscous Newtonian, orientational, and capillary contributions to stress. One result of our modeling and simulations is that, in isothermal spinning, the microstructure (orientation tensor) is weakly radially dependent and can be calculated along with the free surface fiber flow from 1-D models. Families of numerical steady state solutions are presented which predict the steady fiber diameter, velocity and LCP orientation profiles from upstream to the take-up location in isothermal spinning. Linearized stability of these LCP spinning states is computed to reveal upper bounds on throughput in terms of the *critical draw ratio*, above which the process is unstable. Interestingly, enhancement of the effects of LCP kinetic energy or relaxation can either stabilize or destabilize steady spinning solutions, depending on the relative balance of other physical effects. Enhanced anisotropic drag (i.e., increasing the ratio of frictional resistance encountered orthogonal versus parallel to the LCP molecular axis) destabilizes spinning states: the critical draw ratio is lowered on the order of a couple percent to as much as twenty percent in cases reported here which compare isotropic to highly anisotropic friction tensors. Evidence is given for a preferred degree of upstream LCP alignment at which the critical draw ratio achieves a maximum, indicating an important role played by near-spinneret conditions in increasing throughput.

1 Introduction

Many ultra-strength textile fibers, including Vectran and Kevlar, achieve their distinguished commercial properties (such as high tensile modulus) as a result of the interaction between the anisotropic molecular-scale structure of the melt or solution, the macroscopic hydrodynamics of spinning flows, and non-isothermal effects including radial heat transfer through the fiber free surface, temperature dependence of material properties, crystallization, and phase changes.

The aim of this paper is to isolate the orientation-hydrodynamic interactions in fiber spinning flows, uncoupled from the significant energy effects and phase changes. We thereby determine fiber properties and behavior due solely to the feedback and coupling of microstructure in a nearly elongational fiber flow. The isolation of a subset of dominant effects can serve important conceptual as well as calculational purposes, precisely due to the simplification of the resulting equations, physics, and numerical codes; often such a simplification is necessary since the theory for certain physical effects is not available. This has been the case with the evolution of orientation, or birefringence, in fibers. Absent of a first-principles description of orientation as it couples to fiber flows, the standard practice has been to calculate spinless stress from scientific models, and then to infer orientational microstructure (most notably, birefringence) from an empirical stress-optical law. Our goal in this study is to calculate microstructure as it couples to spinning flows from first principles, beginning from 3D Doi-type models. It seems appropriate as we look ahead to a full model of a “real spinning process” to isolate the evolution of microstructure in isothermal spinning flows, and to study these effects in detail.

This paper then serves as a platform from which to couple the energy equation, associated heat loss boundary conditions, and perhaps crystallization. We note in the text how non-isothermal effects may be coupled to this model. This sort of an advance is consistent with the historical development of thin filament models for fiber spinning applications. Newtonian fiber models were developed first in an isothermal setting (Matovich and Pearson, 1969), then extended to non-Newtonian constitutive laws (e.g., Denn, Petrie and Avenas, 1975) before energy considerations were included.

Indeed, the inability to compute the coupling between orientation and flow, and then to add other dominant effects, has led to various compromises in otherwise state-of-the-art modeling. For example, important phenomena like crystallization are triggered by some combination of orientation, stress, and temperature; since the work of Ziabicki (Ziabicki, 1974), it has been accepted that crystallization kinetics is strongly influenced by orientation. One cannot accurately address spinline crystallization from first principles until orientation is coupled from first principles. There are complementary studies to ours, each selecting a subset of dominant effects, in the current literature. We note in particular studies which

focus on:

- (i) Orientation-induced crystallization in a spinline, where the birefringence is inferred from a thermal prediction of stress in a Newtonian viscous model, followed by use of an empirical stress-optical law, and then the Ziabicki form (Ziabicki, 1974) for crystallization rates is utilized (Vassilatos et al., 1986);
- (ii) flow-induced crystallization, using an isothermal isotropic fluid model (Kulkarni and Beris, 1997);
- (iii) thermal transport in isotropic polymeric mixtures (Curtiss and Bird, 1997);
- (iv) thermal flow-induced crystallization in isotropic polymers (McHugh, 1997).

In each of these studies, anisotropic orientational effects are either completely neglected, or inferred from an empirical stress-optical law from the first principles calculation of spinline stress. Of course these studies have features that our model does not, namely the ability to model molecular extension, or a general conformation tensor. Some combination of both features appears optimal.

One of the issues addressed here is the limit on throughput; it is a practical reality that some products have to be produced faster to remain economically viable. An upper bound on throughput is predicted from modeling in terms of the maximum takeup speed at which a spinning process is stable—the *critical draw ratio*. Faster spinline leads to changes in fiber performance properties, often introduce new effects such as stress-induced crystallization, and approach the limits of stable spinning. Once more detailed behavior is coupled to this model, like entanglements, the bounds on throughput are likely to drop due to additional sources of nonuniformity or breaks.

It is widely accepted in the fiber community that limits on draw ratio, and sporadic spinline breaks, have two quite distinct root causes: either due to rheological factors (material impurities, polymer degradation, or phase changes) or because of dynamic instabilities in the spinning flow. (We refer to (Hull and Jones, 1996) for a thorough discussion and extensive literature citations.)

The former (rheological) factors require sophisticated and very accurate material characterization of the polymer chemistry and thermal history in order to address with modeling. The latter reason is accessible from transient models, and is addressed in this paper. In particular, physical intuition suggests that instabilities of LCP spinning states are either hydrodynamic or orientation-driven. Thus the focus on spinline breaks due to instability of streamwise fiber perturbations (initiated perhaps by cross-flow quench air) is reasonably addressed with isothermal models. Alternative instabilities which arise because of significant radial variations in temperature, which translate to significant radial orientation variations, are not addressed by this isothermal study. The practical rationale for the isothermal study is that we can derive a more tractable set of model equations, for the already complicated me-

chanical problem, and study the spinning steady states and their stability before embarking on the thermomechanical problem.

The 1-D model equations we employ are derived in (Forest et al., 1997) from the 3-D equations of (Bhave et al., 1993), assuming the Doi closure approximation. The latter is a significant assumption, which has been tested on model problems by (Bhave et al., 1993); further discussion of the errors inherent in this approximation may be found in (Larson, 1990). The utility of the Doi closure assumption is that one can entirely eliminate the orientation probability distribution function from the governing equations, and obtain an equation for the orientation tensor directly, as it couples to the hydrodynamics; the reader is encouraged to look ahead to equations (6), (19). We recognize there are errors in making this assumption, but the alternative of integrating the coupled 3-D equations with the probability distribution function is impractical for the applications and multi-parameter studies presented here; we refer to (Larson and Öttinger, 1991) for examples of largescale isothermal computations. The further complexity of an energy equation and heat loss boundary conditions renders the calculation quite challenging and more appropriate for a dedicated calculation aimed at a very specific process feature such as flow near the spinneret. We note that one of the authors (Wang, 1997) has pursued corrections to the straightforward Doi closure assumption, due to (Hinch and Leal, 1976), and the essential predictions of this paper and (Forest et al., 1997) are unaltered. We expect the qualitative trends and phenomena of this study to be useful; the quantitative details have not been refined together with experimental data.

For industrial purposes, however, first the thermal generalization of this model is necessary, then material constants and forms must be characterized, and finally experimental spinline data must be used to fit empirical correlations. The LCP parameters in our model equations are to be construed as effective material parameters for such effects as relaxation time, anisotropic drag, polymer density and molecular kinetic energy; the form function for the bulk free energy is again an effective potential function. The simplified models presented here are useful for large parameter studies to study trends in spinline performance due to various material and processing conditions, as we illustrate below. The qualitative phenomena we report below are robust to model details.

We also remark that reduced models have played a fundamental role in developing empirical correlations for spinline effects of air drag and heat loss (Kase and Matsuo, 1965, 1967; Matsui, 1976). Similarly, models of the type presented here are easily combined with more complex models (e.g., thermal, air drag) and then used as a basis to fine tune details of the microstructure modeling.

The model explored in this paper yields an approximate description of the flow-orientation interactions, and an assessment of the orientation influence on previous isothermal Newtonian (e.g., Schultz and Davis, 1982) and viscoelastic (e.g., Denn et al., 1975, Bechtel et al.,

1992) liquid fiber models. This paper extends the study in (Forest and Wang, 1994) for an upper convected Maxwell fluid model to the Doi-type rheological theory.

It is customary in 1-D spinning models, both isothermal and thermal, to assume weak velocity variations in the fiber cross-section. Indeed, this assumption has been validated both in 2-D model simulations and in experimental measurements. It is likewise plausible to assume, a priori, that in isothermal spinning the microstructure (orientation tensor) is weakly radially dependent. Here we test this assumption with our models, and one of our conclusions is that indeed the radial variations can be calculated and shown to remain higher order effects. This result supports a widely held opinion that any significant radial variation observed in spun thermotropic LCP fibers is likely due to thermally induced effects, such as significant heat loss at the fiber surface. It is also possible there is thermal enhancement of the weak radial variation induced by the orientation-hydrodynamic coupling. Crystallization effects further alter radial fiber properties, especially in the presence of rapid surface cooling and high-speed spinning where skin-core effects occur. Once more, however, one can make the case that the advance regarding orientation coupling lays the foundation for realistic thermal crystallization modeling.

Given this motivation, we proceed to the content of this paper. We now briefly recall some preliminaries from our previous paper (Forest et al., 1997), along with minor additions, before presenting the numerical fiber spinning solutions and stability results.

2 3-Dimensional Formulation

2.A. Nematic LCP Orientation Tensor

The macroscopic, or average, internal orientation properties of nematic liquid crystals are defined in terms of a second order *orientation (or structure) tensor* $\mathbf{Q}$,

$$\mathbf{Q} = \langle \mathbf{m}\mathbf{m} \rangle - \mathbf{I}/3, \quad (1)$$

where

$$\langle (\bullet) \rangle = \int_{|\mathbf{m}|=1} (\bullet) f(\mathbf{m}, \mathbf{x}, t) d\mathbf{m}, \quad (2)$$

and $f(\mathbf{m}, \mathbf{x}, t)$ is the normalized orientation distribution function corresponding to the probability that an arbitrary dumbbell or rod-like LCP molecule is in direction $\mathbf{m}$ at time t and location $\mathbf{x}$. (Bhave et al., 1993), extending earlier results of (Doi, 1980, 1981), derived a diffusion equation for $f(\mathbf{m}, \mathbf{x}, t)$,

$$\frac{\partial f}{\partial t} = \frac{\sigma}{6\lambda} \left(\frac{\partial}{\partial \mathbf{m}} \cdot \frac{\partial}{\partial \mathbf{m}} f \right) - \frac{\partial}{\partial \mathbf{m}} \cdot [[\nabla \mathbf{v} \cdot \mathbf{m} - \nabla \mathbf{v} : \mathbf{m}\mathbf{m}\mathbf{m}] f] - \frac{\sigma}{6kT\lambda} \left(\frac{\partial}{\partial \mathbf{m}} \Phi \right) f, \quad (3)$$

which accounts for polymers in a Newtonian solvent, subject to anisotropic hydrodynamic drag (with drag coefficient σ), and a polymer-polymer mean-field interaction with Maier-Saupe potential,

$$\Phi = -\frac{3}{2}NkT(\mathbf{m}\mathbf{m} - \mathbf{I}/3) : \mathbf{Q}. \quad (4)$$

The *drag or friction parameter* σ takes on values between zero and one; $\sigma = 1$ is the isotropic case where the friction tensor is proportional to the identity. As σ decreases, this corresponds to increasing the ratio of resistance encountered perpendicular to the dumbbell axis versus resistance along the axis; the limit $\sigma = 0$ is the most anisotropic limit in which the friction is zero along the molecular axis. The parameter λ is the *relaxation time* of the LCP molecules associated with rotation of the dumbbell molecules, N is a *dimensionless measure of the LCP density* and characterizes the strength of the intermolecular potential, k is the *Boltzmann constant*, and T is *absolute temperature*.

We remark that T is constant for this study, but here lies the source of direct coupling to temperature for thermotropic LCP modeling, along with the material parameter dependence on temperature.

2.A.1. Orientation-dependent constitutive law

Note from the diffusion equation for f and the intermolecular potential Φ that f and $\mathbf{Q}$ are coupled. That is, if one tries to derive a dynamical equation for $\mathbf{Q}$, averages of fourth order tensor products couple to $\mathbf{Q}$. A Doi-type closure approximation, that averages of fourth order tensor products of $\mathbf{m}$ are given by products of second order averages,

$$(\bullet) :< \mathbf{m}\mathbf{m}\mathbf{m}\mathbf{m} > = ((\bullet) :< \mathbf{m}\mathbf{m} >) < \mathbf{m}\mathbf{m} >, \quad (5)$$

allows one to eliminate the probability distribution function f from the analysis, as well as all higher order tensor product averages. This has two profoundly simplifying effects: first, the orientation dynamics (given below in equation (19)) involves only $\mathbf{Q}$ and the velocity field $\mathbf{v}$; second, the constitutive equation for stresses couples only the orientation tensor $\mathbf{Q}$ to the usual Navier-Stokes (Newtonian) relation:

Constitutive equation for stresses (Bhave et al., 1993):

$$\begin{aligned} \tau &= -p\mathbf{I} + \hat{\tau}, \\ \hat{\tau} &= 2\eta\mathbf{D} + 3ckT[(1 - N/3)\mathbf{Q} - N(\mathbf{Q} \cdot \mathbf{Q}) + N(\mathbf{Q} : \mathbf{Q})(\mathbf{Q} + \mathbf{I}/3) + \\ &\quad 2\lambda(\nabla\mathbf{v} : \mathbf{Q})(\mathbf{Q} + \mathbf{I}/3)], \end{aligned} \quad (6)$$

where $\mathbf{v}$ is the *velocity field*, $\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + \nabla\mathbf{v}^T]$, $\nabla\mathbf{v}$ has i, j component $\frac{\partial v_i}{\partial x_j}$ in Cartesian coordinates, p is the scalar *pressure*, η is the *solvent viscosity*, c is the *number of polymer*

molecules per unit volume, while the other parameters are described above.

2.A.2. Properties of the nematic orientation tensor $\mathbf{Q}$

In (Forest et al., 1997) a representation for the orientation tensor $\mathbf{Q}$ is developed which is compatible with the nearly elongational flows of fiber spinning. The utility of this representation is that we describe $\mathbf{Q}$ in terms of three scalar functions, each of which has a clear role in describing the LCP microstructure.

Let μ_i be the eigenvalues and $\mathbf{n}_i$ the corresponding orthonormal eigenvectors of the symmetric tensor $\mathbf{Q}$. The biaxial representation for $\mathbf{Q}$ is given by

$$\mathbf{Q} = s(\mathbf{n}_3\mathbf{n}_3 - \mathbf{I}/3) + \beta(\mathbf{n}_2\mathbf{n}_2 - \mathbf{I}/3), \quad (7)$$

where s and β are two independent nematic LCP order parameters which carry nonlinear average information about the angles ϕ_i the polymer molecule $\mathbf{m}$ makes with the eigenvectors $\mathbf{n}_i$ of $\mathbf{Q}$:

$$\begin{aligned} s &= \langle (\mathbf{n}_3 \cdot \mathbf{m})^2 \rangle - \langle (\mathbf{n}_1 \cdot \mathbf{m})^2 \rangle, \\ \beta &= \langle (\mathbf{n}_2 \cdot \mathbf{m})^2 \rangle - \langle (\mathbf{n}_1 \cdot \mathbf{m})^2 \rangle. \end{aligned} \quad (8)$$

The range of values for (s, β) is a closed triangular region in the space (s, β) with vertices $(1, 0)$, $(0, 1)$ and $(-1, -1)$. Simple manipulations yield the direction cosines as linear combinations of the order parameters s, β :

$$\begin{aligned} \langle \cos^2(\phi_3) \rangle &= \langle (\mathbf{m} \cdot \mathbf{n}_3)^2 \rangle = \frac{1}{3}(2s - \beta + 1), \\ \langle \cos^2(\phi_2) \rangle &= \langle (\mathbf{m} \cdot \mathbf{n}_2)^2 \rangle = \frac{1}{3}(2\beta - s + 1), \\ \langle \cos^2(\phi_1) \rangle &= \langle (\mathbf{m} \cdot \mathbf{n}_1)^2 \rangle = \frac{1}{3}(1 - s - \beta), \end{aligned} \quad (9)$$

and eigenvalues μ_i as linear functions of the direction cosines:

$$\mu_i = \langle (\mathbf{m} \cdot \mathbf{n}_i)^2 \rangle - 1/3, i = 1, 2, 3. \quad (10)$$

If either $s\beta = 0$ or $s = -\beta$, the nematic liquid crystal is called *uniaxial*, in which case two eigenvalues of $\mathbf{Q}$ are equal and the unit eigenvector corresponding to the third eigenvalue is a distinguished direction, referred to as the *optical axis*, or orientation axis of symmetry, or the *director* of the uniaxial nematic LCP. Otherwise, the nematic liquid crystal is called *biaxial*, where all the eigenvector directions are equally important.

We restrict to biaxial representations in a neighborhood of the particular uniaxial limit defined by $\beta = 0$. In this limit, $s = \frac{3}{2}\mu_3$, and the corresponding eigenvector $\mathbf{n}_3$ is the

distinguished optical axis or uniaxial director. From the formulas above, we arrive at the particular uniaxial representation corresponding to $\beta = 0$:

$$\mathbf{Q} = s(\mathbf{n}_3\mathbf{n}_3 - \mathbf{I}/3), \quad (11)$$

where the scalar *uniaxial order parameter* s is related to the polymeric direction $\mathbf{m}$ by

$$s = \frac{3}{2}\langle(\mathbf{m} \cdot \mathbf{n}_3)^2\rangle - \frac{1}{2} = \frac{3}{2}\langle\cos^2(\phi_3)\rangle - \frac{1}{2}, \quad -1/2 \leq s \leq 1. \quad (12)$$

The order parameter s describes the average *degree of orientation* between the molecular direction $\mathbf{m}$ and $\mathbf{n}_3$. The projection of $\mathbf{m}$ onto the plane orthogonal to $\mathbf{n}_3$ is isotropic. When $0 < s \leq 1$, the liquid crystal is said to exhibit prolate uniaxial symmetry; when $-1/2 \leq s < 0$, one infers oblate uniaxial symmetry; $s = -1/2$ corresponds to all molecules aligned in the plane orthogonal to $\mathbf{n}_3$; $s = 1$ corresponds to parallel alignment of $\mathbf{n}_3$ and $\mathbf{m}$; finally, $s = 0$ corresponds to an isotropic state in which the structure tensor vanishes and orientation preference is lost.

Because of the specific biaxial representation presented here, with a distinguished limit identified by $\beta = 0$, *we hereafter refer to s as the uniaxial order parameter, and to β as the biaxial order parameter.*

In the uniaxial limit, we further wish to achieve some crude estimate, for intuitive and graphical purposes, of the angle ϕ_3 between the LCP molecular direction $\mathbf{m}$ and the optical axis $\mathbf{n}_3$. Since $\langle\cos^2(\phi_3)\rangle = \frac{1}{3}(2s + 1)$, we cannot infer direct information about ϕ_3 . However, if we make the strong approximation that averaging commutes with the nonlinear function $\cos^2$, we can infer a crude estimate of $\langle\phi_3\rangle$:

$$\langle\phi_3\rangle \sim \pm \cos^{-1}(\sqrt{\frac{1}{3}(2s + 1)}). \quad (13)$$

This rough estimate yields a cone of most likely LCP molecular directions $\mathbf{m}$ associated with the uniaxial optical axis $\mathbf{n}_3$ at each point in the fiber. The optical axis is the axis of symmetry of the cone, while the cone angle is $\langle\phi_3\rangle$. We emphasize that this heuristic cone of molecular directions is for graphical purposes only; we do not use this approximation in any computation of solutions. The picture becomes more accurate near the limit of perfect alignment, $s = 1$, as the molecules align with the optical axis.

One may refer ahead to Figure 2 where the cone information is superimposed inside the free surface of a typical steady state LCP fiber solution. The cone shape at each point is computed solely from the order parameter s ; however, the orientation of the cone axis of symmetry (the optical axis) relative to the fiber axis $\mathbf{e}_z$ requires eigenvector information from $\mathbf{Q}$, which we turn to next.

The order parameters depict the average molecular orientation relative to the three eigenvector directions. We next need information to place the orthogonal frame associated to $\mathbf{Q}$

relative to the fixed cylindrical polar coordinate system in the fluid domain. The eigenvectors $\mathbf{n}_i$ of $\mathbf{Q}$ are parameterized by three angles in any orthogonal coordinate system; if we restrict to flows with zero angular velocity (sometimes called torsionless), then two of these angles may be set identically to zero (Forest et al., 1997), simplifying to:

$$\begin{aligned}\mathbf{n}_1 &= (0, -1, 0), \\ \mathbf{n}_2 &= (-\cos \Psi, 0, \sin \Psi), \\ \mathbf{n}_3 &= (\sin \Psi, 0, \cos \Psi),\end{aligned}\tag{14}$$

where these vectors have coordinates with respect to the ordered orthonormal basis $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$, respectively; for example, $\mathbf{n}_1 = -\mathbf{e}_\theta$. (In these coordinates, a flow is torsionless if $\mathbf{v} \cdot \mathbf{e}_\theta = 0$.)

The orientation tensor $\mathbf{Q}$ is therefore completely determined by three scalar functions, the *angle parameter* Ψ and the two order parameters s, β . The following constraints imposed by the fiber geometry and flow are recalled from (Forest et al., 1997).

Centerline conditions on $\mathbf{Q}$:

$$\Psi(r = 0, z, t) = 0,\tag{15}$$

which guarantees that at the fiber centerline the LCP is uniaxial with optical axis parallel with the axis of symmetry,

$$\mathbf{n}_3(r = 0, z, t) = \mathbf{e}_z.\tag{16}$$

Further we assume

$$\beta(r = 0, z, t) = \frac{\partial \beta}{\partial r}(r = 0, z, t) = 0,\tag{17}$$

which corresponds to perfect uniaxial symmetry at the centerline and a weak biaxial symmetry for small r . This condition is also consistent with the behavior for small r of the normal stress differences that arise in the constitutive equation (6).

Notice that at the fiber centerline, $\mathbf{Q}$ admits a special uniaxial form with the optical axis parallel to the centerline ($\mathbf{n}_3(r = 0, z) = \mathbf{e}_z$), with a single free parameter (degree of orientation) given by s . The order parameter s is uniform to leading order across the fiber cross-section, indicating that the statistical average orientation of LCP molecules relative to the optical axis is nearly uniform in the fiber cross-section. Thus the orientation order parameter s is similar in this respect to the fiber axial velocity.

Furthermore, in the plane orthogonal to the fiber axis, the special uniaxial symmetry at $r = 0$ is first broken at order $O(r)$ to yield a general uniaxial structure with optical axis, $\mathbf{n}_3$, tilted away from the flow axis of symmetry, $\mathbf{e}_z$, as computed from the angle Ψ . This angle

of tilt between the optical and flow axes is completely macroscopic, averaged information. Therefore, with regard to Figure 2 below, once we have crudely identified a cone of molecular directions $\mathbf{m}$ centered at $\mathbf{n}_3$, the eigenvector information captured by the variable Ψ allows us to lay the cones down in the fluid domain with the cone axis $\mathbf{n}_3$ tilted by angle Ψ away from the fiber axis $\mathbf{e}_z$.

Finally, the uniaxial symmetry is broken at order $O(r^2)$ to yield a weakly biaxial structure, as computed from the second order parameter β which first appears at order $O(r^2)$. The introduction of weak biaxial effects, which are most prominent near the fiber free surface, is not represented in Figure 2.

2.B. 3-D dynamical equations

Given our posited representation for $\mathbf{Q}$, and the constitutive law (6), we now present the remaining 3-D macroscopic equations and then proceed to 1-D models derived from them.

2.B.1. Conservation of momentum

$$\rho \frac{d}{dt} \mathbf{v} = \nabla \cdot \boldsymbol{\tau} + \mathbf{g}, \quad (18)$$

where ρ is the *density of the polymeric liquid*, $\mathbf{v}$ is the velocity with radial component v_r and axial component v_z , $\boldsymbol{\tau}$ is the *total stress tensor*, $\mathbf{g}$ is the external force due to gravity given by $\mathbf{g} = g\mathbf{e}_z$ in the cylindrical coordinate, and $\frac{d}{dt}(\bullet)$ denotes the material derivative $\frac{\partial}{\partial t}(\bullet) + \mathbf{v} \cdot \nabla(\bullet)$.

2.B.2. Orientation tensor equation

$$\begin{cases} \frac{d}{dt} \mathbf{Q} - (\nabla \mathbf{v} \cdot \mathbf{Q} + \mathbf{Q} \cdot \nabla \mathbf{v}^T) = F(\mathbf{Q}) + G(\mathbf{Q}, \nabla \mathbf{v}), \\ F(\mathbf{Q}) = -\sigma/\lambda \{ (1 - N/3)\mathbf{Q} - N(\mathbf{Q} \cdot \mathbf{Q}) + N(\mathbf{Q} : \mathbf{Q})(\mathbf{Q} + \mathbf{I}/3) \}, \\ G(\mathbf{Q}, \nabla \mathbf{v}) = \frac{2}{3}\mathbf{D} - 2(\nabla \mathbf{v} : \mathbf{Q})(\mathbf{Q} + \mathbf{I}/3). \end{cases} \quad (19)$$

Note that F characterizes the orientation dynamics independent of flow, whereas G describes the flow-orientation interaction.

2.B.3. Free surface and corresponding boundary conditions

The *axisymmetric free surface* is given by

$$r = \phi(z, t). \quad (20)$$

The *kinematic boundary condition* is

$$\frac{d}{dt}(r - \phi(z, t)) = 0, \quad (21)$$

i.e., the free surface convects with the flow; the usual *kinetic boundary conditions* are

$$\begin{aligned} (\tau - \tau_a)\mathbf{n}_f &= -\sigma_s\kappa\mathbf{n}_f, \\ \kappa &= \phi^{-1}(1 + \phi_z^2)^{-\frac{1}{2}} - \phi_{zz}(1 + \phi_z^2)^{-\frac{3}{2}}, \end{aligned} \tag{22}$$

where $\mathbf{n}_f$ is the unit outward normal of the free surface (20), σ_s is the surface tension coefficient, κ is the mean curvature of the free surface, and τ_a is the ambient stress tensor. (Note that the orientation contribution to the stress tensor τ offers a means by which the LCP microstructure can strongly influence the dynamics at the free surface.)

We assume that the ambient stress τ_a is a constant pressure, i.e.,

$$\tau_a = -p_a\mathbf{I}. \tag{23}$$

3 Asymptotic 1-D Model for Thin LCP Filaments

In (Forest et al., 1997), we developed a slender longwave perturbation theory for the 3-D orientation, free surface flow equations presented above. We will recall basic elements of the resulting models so that their application to fiber spinning is clarified.

The asymptotic parameter is the *slenderness ratio*, ϵ , which is the aspect ratio of typical radial r_0 to axial z_0 lengthscales:

$$0 < \epsilon = \frac{r_0}{z_0} \ll 1. \tag{24}$$

This geometric condition presumes that typical axial lengthscales are much longer than the scale of radial variations; this condition is not valid, for example, for the flow at or very near the spinneret.

The identification of these (and other) characteristic scales is particular to the experiment or process. For fiber spinning applications, the beginning upstream location, $z = 0$, is a fixed location approximately two spinneret diameters downstream where the flow has become nearly elongational. Then, r_0 is defined to be the free surface radius at $z = 0$, and z_0 is the distance (L) from $z = 0$ to the take-up location where downstream conditions on axial velocity are applied. In nondimensional coordinates, the axial domain is normalized with $0 \leq \tilde{z} \leq 1$.

In order to impose the slenderness constraint, we have to: (1) nondimensionalize the full 3-D flow-orientation equations; (2) posit an asymptotic expansion for $\mathbf{v}, p, \phi$ and $\mathbf{Q}$; and (3) posit the relative balance among all the physical effects we have incorporated in the 3-D problem.

3.A. Nondimensionalization

All coordinates and variables are nondimensionalized with respect to the following: t_0 is a characteristic timescale; r_0, z_0 are the characteristic radial and axial lengthscales, respectively, defined above; f_0 is a characteristic force scale which we use to nondimensionalize both pressure and the extra stress. (For example, $f_0 = \rho r_0^2 v_0^2$ is the inertial force.) Naturally, the orientation structure tensor $\mathbf{Q}$ is dimensionless. The scaled coordinates are then

$$\tilde{z} = \frac{z}{z_0}, \quad \tilde{r} = \frac{r}{r_0}, \quad \tilde{t} = \frac{t}{t_0}. \quad (25)$$

To build in the centerline conditions (15), (16), (17), we define an *angle variable*, ψ , associated to Ψ and a *biaxial order parameter variable*, $\bar{\beta}$, associated to β by:

$$\Psi = r\psi, \quad \beta = r^2\bar{\beta}, \quad (26)$$

where ψ has units of $length^{-1}$, and $\bar{\beta}$ has units of $length^{-2}$. We pick the lengthscale for ψ and $\bar{\beta}$ to be z_0 .

Now all unknowns are nondimensionalized as follows, with $v_0 = \frac{z_0}{t_0}$:

$$\begin{aligned} \phi &= r_0 \tilde{\phi}, & \mathbf{v} &= v_0 \tilde{\mathbf{v}}, & p - p_a &= f_0 r_0^{-2} \tilde{p}, \\ \tau &= f_0 r_0^{-2} \tilde{\tau}, & \Psi &= \epsilon \tilde{r} \tilde{\psi}, & \beta &= \epsilon^2 \tilde{r}^2 \tilde{\beta}. \end{aligned} \quad (27)$$

Upon nondimensionalizing the full set of 3-D equations in cylindrical coordinates the following collection of dimensionless parameters arises:

$$\begin{aligned} \epsilon &= \frac{r_0}{z_0}, \quad 1/R = \frac{\eta t_0}{\rho z_0^2}, \quad 1/W = \frac{\sigma_s}{\rho r_0 v_0^2}, \quad 1/F = \frac{g t_0^2}{z_0}, \\ \tilde{\lambda} &= \frac{\lambda}{t_0}, \quad N, \quad \tilde{\alpha} = \frac{3ckT}{\rho v_0^2}, \quad \sigma, \end{aligned} \quad (28)$$

where R, W, F are the *Reynolds, Weber and Froude numbers*, respectively, which parametrize viscosity, surface tension and gravity relative to inertia, respectively; $\tilde{\alpha}$ parametrizes the *molecular kinetic energy per unit volume relative to inertial energy per unit volume*; from the constitutive law one sees also that $\tilde{\alpha}$ characterizes the molecular stress relative to inertial stress. The combination $\tilde{\alpha}\tilde{\lambda}$ characterizes the relaxational stress due to molecular rotation relative to inertial stress, $\sigma/\tilde{\lambda}$ parametrizes anisotropic drag on polymer molecular motion relative to the solvent, and N is a polymer density parameter that enters prominently in the intermolecular excluded-volume potential.

Note: Beginning now, and hereafter, all tilde overbars will be dropped. All coordinates, variables and parameters from now on are nondimensional unless otherwise specified. All nondimensional parameters have been posited as order 1 quantities, corresponding to Regime

1 of (Forest et al., 1997).

3.B. The 1-D flow-orientation model for axisymmetric LCP filaments

We represent the leading order 1-D quantities by $\phi(z, t)$, the free surface radius; $v(z, t)$, the axial velocity; $u(z, t)$, the radial velocity; $p(z, t)$, the pressure; $s(z, t)$, the uniaxial order parameter; $\psi(z, t)$, which determines the angle of tilt between the optical axis $\mathbf{n}_3$ and the fiber axis $\mathbf{e}_z$; and $\beta(z, t)$, the weakly biaxial order variable. The governing equations for these quantities are (Forest et al., 1997):

$$\left\{ \begin{array}{l} (\phi^2)_t + (v\phi^2)_z = 0, \\ (\phi^2 v)_t + (\phi^2 v^2)_z = \frac{1}{F}\phi^2 + \frac{1}{W}\phi_z + [\phi^2(R_{eff}^{-1}(s)v_z + \alpha U(s))]_z, \\ s_t + vs_z = v_z(1-s)(2s+1) - \frac{\sigma}{\lambda}U(s), \\ u = -v_z/2, \\ p = \frac{1}{W\phi} - R^{-1}v_z - \frac{1}{3}\alpha U(s) + \frac{2}{3}\alpha\lambda s(1-s)v_z, \\ s(\psi_t + v\psi_z) + \frac{R}{3}[(3R^{-1} + 2\alpha\lambda s^2(1-s))v_z + \alpha(1-s)U(s)]\psi \\ = \frac{R}{3}(1-s)\phi_z\phi^{-1}[R_{eff}^{-1}(s)v_z + \alpha U(s)] - sv_{zz}/2, \\ \beta_t + v\beta_z + [2sv_z + \frac{\sigma}{\lambda}(\frac{U(s)}{s} + Ns)]\beta = \frac{R}{3}(1-s)\{R_{eff}^{-1}(s)\psi^2v_z + \\ 2\alpha\lambda s^2\psi^2v_z - 2R_{eff}^{-1}(s)\psi\phi_zv_z\phi^{-1} - 2\alpha\psi U(s)(\phi_z\phi^{-1} - \psi) - \frac{1}{4R}v_{zzz} \\ + \frac{1}{2}[\alpha U(s)\psi + 2\alpha\lambda s^2\psi v_z - \phi_z\phi^{-1}(\alpha U(s) + R_{eff}^{-1}(s)v_z)]_z\}, \end{array} \right. \quad (29)$$

where R is the *Newtonian Reynolds number*, $R_{eff}(s)$ is an *effective 1-D flow-orientation Reynolds number*,

$$R_{eff}^{-1}(s) = 3R^{-1} + 2\alpha\lambda s^2, \quad (30)$$

consisting of a Newtonian contribution and an orientation contribution ($2\alpha\lambda s^2$); W is the *Weber number*; λ is the *scaled LCP relaxation time or Deborah number*; σ is the *anisotropic drag parameter*; N is the *polymer concentration parameter*; α parametrizes the *LCP kinetic energy relative to inertia*; and

$$U(s) = s(1 - \frac{N}{3}(1-s)(2s+1)) \quad (31)$$

defines the *effective 1-D intermolecular potential*, $\int U(s)ds$.

Amplification. Equation (29b) is the axial momentum balance, which naturally contains the axial gradient of the axial force, F_{axial} , which in dimensionless variables is ϕ^2 times the

axial stress, τ_{axial} , where

$$\tau_{axial} = \tau_{zz} - p. \quad (32)$$

From the constitutive law (6), together with (29e),

$$p = \tau_{rr} + W^{-1}\phi^{-1}, \quad (33)$$

so we deduce a relationship between the axial stress on the fiber, the orientation order parameter s , and the fiber radius and velocity gradient:

$$\tau_{axial} = R_{eff}^{-1}(s)v_z + \alpha U(s) - W^{-1}\phi^{-1}. \quad (34)$$

This constitutive relationship is a *nonlinear stress optical relation* that yields a fundamental relation between the axial fiber stress and the “optical” quantity s . Indeed, one should think of s as the normalized birefringence,

$$s = Bi/Bi_{max}, \quad (35)$$

where Bi is the birefringence of the LCP melt and Bi_{max} is the maximum birefringence possible for the amorphous LCP melt. For small values of s much less than 1, if we retain only linear terms in s , the stress optical law (34) reduces to the form of the linear stress optical law employed in the textile fiber industry:

$$Bi = (\text{Stress optical coefficient})[\tau_{axial} + \text{Correction}], \quad (36)$$

where the term “Correction” is the departure from the simple linear stress-optical relation of rubber network theory, and these expressions are given by

$$\text{Stress optical coefficient} = [\alpha(1 - N/3)]^{-1}Bi_{max}, \quad (37)$$

$$\text{Correction} = W^{-1}\phi^{-1} - 3R^{-1}v_z. \quad (38)$$

This development indicates that the standard linear stress-optical law used to infer birefringence from a calculation of the spinline stress near the takeup location is consistent with a fundamental constitutive equation for fluids with internal orientation. However, rather than applying this relation in postprocessing from models which ignore the evolution of orientation, our model calculates birefringence all along the spinline as it couples to the hydrodynamics of fiber spinning. The stress-optical law then affords a mechanism for predicting spinline stress from the first principles solutions for the fiber radius, axial velocity and birefringence.

In all solutions presented below we fix $N = 4$, corresponding to a LCP density for which the 1-D potential energy has three critical points, given by the zeroes of $U(s)$, $s_j, j = -1, 0, 1$,

with $s_0 = 0, s_{\pm 1} = (1 \pm \sqrt{3})/4$. In the absence of gravity ($1/F = 0$) the zeroes of U correspond to order parameter equilibria associated with constant cylindrical LCP filaments, the stability of which are characterized in full generality in (Forest et al., 1997) for various regimes; in the gravity-driven flows considered in this paper there are no equilibria. Nonetheless, the zero gravity results reveal how the various orientation and hydrodynamic effects couple in thin cylindrical fiber flows, and may significantly influence the Rayleigh capillary instability. The present study explores the LCP-flow coupling in a very different class of flows, with boundary conditions imposed at either end of a finite fixed length, as opposed to cylindrical jets of infinite extent.

4 Steady State Solutions

We begin our analysis of solutions with nontrivial steady states satisfying fiber spinning boundary conditions. Our model is well-posed with upstream conditions (at $z = 0$) on the fiber radius ϕ , the axial velocity v , and the orientation variables s, ψ, β , and with downstream conditions (at $z = 1$) on the fiber velocity. These conditions are compatible with fiber spinning processes.

Consistent with our earlier nondimensionalization, the upstream conditions on fiber radius and velocity have been fixed:

$$\phi(0) = 1, v(0) = 1, \quad (39)$$

while *the remaining boundary conditions are free processing parameters*:

$$s(0), \psi(0), \beta(0), v(1). \quad (40)$$

The outline for this section is to first display a typical steady solution (a specific base steady state) followed by qualitative variations of this steady state with respect to free parameters in the model.

4.A. A typical steady state solution

We begin with a base steady state solution (selected to be linearly stable as confirmed later), depicted in Figure 1. Order one values are chosen for all parameters; refer to the figure caption. From Figure 1 we observe the following features:

(i) The qualitative behavior of the fiber radius and axial velocity is consistent with solutions of non-LCP slender filament models. *This suggests that the orientation coupling does not grossly influence the qualitative hydrodynamic behavior.*

(ii) The orientation order parameter, s , is set upstream with a value $s(0) = 0.5$, corresponding to a moderate degree of orientation between the LCP molecules and the uniaxial optical axis, $\mathbf{n}_3$. *The filament flow evolves to nearly complete alignment at the take-up location, with $s(1) \sim 1$.*

Recall from the end of Section 2.A.2 that the angle variable ψ measures the angle of tilt between the LCP optical axis and the fiber axis. Since the pde for ψ is linear and nonhomogeneous, in Figure 1d.i we isolate the *necessary*, unavoidable tilt angle with the particular nonhomogeneous solution ψ_p , with $\psi_p(0) = 0$; Figure 1d.ii depicts the tilt angle that results from upstream boundary conditions given by the homogeneous steady solution, denoted by ψ_h , with initial data $\psi_h(0) = 1$. The unique steady solution for ψ is then $\psi = \psi(0)\psi_h + \psi_p$, depicted in Figure 1d.iii with $\psi(0) = 1$.

(iii) From Figure 1d.ii, one observes that the effect of the initial data decays to zero near the take-up location. Thus, the terminal downstream value, $\psi(1)$, is determined from ψ_p . Recall (14) and (27): the actual angle between the optical axis and fiber axis in the cross-section $z = z_0$ is $\epsilon r \psi(z_0)$. This corresponds to a linear radial tilt of the optical axis away from the fiber axis, with perfect alignment at the centerline and a tilt angle of $\epsilon \phi(z_0) \psi(z_0)$ at the fiber surface.

For heuristic purposes only, we now invoke the approximation alluded to earlier, (13), which allows us to give an intuitive, graphical display (Figure 2) of the LCP steady state orientation associated with the base steady state solution, Figure 1. This picture only uses the uniaxial information afforded by s and ψ ; biaxial effects contained in β are only felt at quadratic order in the fiber radius and so are negligible in this picture.

Recall that $s(z)$ at each z_0 determines a cone of LCP directions $\mathbf{m}$ centered about the optical axis $\mathbf{n}_3$, and this cone is uniform across the fiber cross-section with cone angle $\langle \phi_3 \rangle$ approximated by (13). Since the angle between the cone axis and fiber axis is $\epsilon r \psi(z)$ to leading order, we find that while the cones are uniform in each cross-section $z = z_0$, they tilt relative to the fiber centerline with tilt angle $\epsilon r \psi(z_0)$.

Notice from Figure 2 that the fiber begins with rather wide cones of possible LCP molecular directions at $z = 0$, corresponding to the presumed moderate degree of upstream orientation $s(0) = 0.5$; the angle of tilt in the plane $z = 0$ has radian measure $\epsilon r \psi(0)$, ranging from perfect alignment at the centerline to a tilt angle of $\epsilon \psi(0)$ radians at the fiber surface. The remaining evolution of the cone of molecular directions, the cone axis of symmetry (optical axis), and the fiber free surface, are determined from the steady state solution of the two-point boundary value problem.

We note in particular that this picture confirms the notion of a converging flow of LCP rod-like molecules. As the flow develops downstream, one observes a contraction of the cone angle corresponding to the uniaxial order parameter s converging toward $s \sim 1$, together

with the cone axes (the optical axes) converging to a terminal radial distribution as $z \sim 1$. The negative value, $\psi(1) \sim -1.053$, at the take-up location translates to an inward tilt of the optical axes across the fiber cross-section.

(iv) The above solutions ϕ, v, s, ψ determine the uniaxial resolution of the fiber, which then drives the weakly biaxial order parameter variable, β . The equation for β is also linear and nonhomogeneous, so again (Figure 1e.i) we distinguish the *necessary* departure from uniaxial symmetry, $\beta_p(z)$, by imposing zero initial data, $\beta_p(0) = 0$ on the nonhomogeneous equation. This forced response in β is non-monotone and rather weak with a maximum absolute value of approximately .033, culminating in a downstream approach to the uniaxial limit of $\beta = 0$. The boundary-condition-generated behavior of β is also shown in Figure 1e.ii through the homogeneous solution, $\beta_h(z)$, with $\beta_h(0) = 1$. The plot of β_h is similar to ψ_h , as their equations suggest.

To summarize, β is initially driven away from zero, but eventually relaxes to near zero at the take-up location, independent of the initial value. *Thus, the terminal orientation of the fiber is effectively uniaxial.*

4.B. Qualitative changes in the base steady state solution due to parameter variations

Now we consider processing and material parameter variations of the typical base steady state in Figure 1. In particular, in fiber spinning and drawing processes a major goal is to achieve downstream alignment (as quantified by orientation or birefringence) from an initially less-oriented material. Birefringence is a fundamental fiber property from which other characteristics, such as tenacity, are inferred. For this reason we explore behavior associated with variations in upstream orientation conditions; the initial orientation, $s(0)$, is used here to parametrize the degree of orientation below the spinneret as the flow becomes nearly elongational, and therefore is a reflection of spinneret design and the ambient process conditions nearby. As well we study the steady state dependence on the downstream take-up speed, or draw ratio; finally we consider the influence of material parameters.

4.B.1. Effect of variation in upstream degree of orientation, $s(0)$

We consider $s(0) = s_0$ in the prolate range $.05 \leq s_0 \leq .95$, in increments of .05. The results are depicted in Figure 3, from which we make the following observations:

(i) There are *uniform responses* in ϕ, v, s to changes in s_0 . Physically, one concludes that a lower upstream degree of orientation s_0 yields steady processes in which the filaments are thinner and faster at each interior spinline location. The most significant variation is in $s(z)$,

since it is that function's initial value being varied. Note that the variation at downstream locations z due to s_0 washes out by the takeup location $z = 1$.

In fiber spinning, there is typically a target property of the spun fiber; moreover, spun fiber properties in general are locked-in upstream of the takeup location since the fibers solidify in actual processes. Therefore, it is desirable to know the location, which we denote by z^* , where a particular process reaches a pre-set target property.

(ii) Here we select the targeted fiber property to be the degree of LCP orientation, as measured by s ; we set a target value of $s = .95$ that the spun fiber must achieve. (This would correspond to the amorphous LCP fibers achieving 95% of the maximum possible birefringence for that particular LCP.) We calculate the critical location, z^* , where $s(z^*) = .95$ for the base steady state solution of Figure 1 which satisfies $s(0) = 0.5$. We then determine how this location varies due to s_0 . Figure 4a depicts the curve $z^*(s_0)$, defined implicitly by $s(z^*; s_0) = .95$, and deduced from the solutions in Figure 3. This curve implies an essentially linear behavior of z^* over the range $0 < s_0 < 0.7$. However, for highly oriented s_0 the slope of z^* steepens indicating the target orientation value is reached very early in the spinline.

These results are indicative of how one might use such models to explore the effects of spinneret design or upstream processing conditions. While the model will change to accommodate thermotropic LCP spinning effects, these sorts of calculations are appropriate for the more general model.

(iii) After an early detour, the unavoidable forced tilt of the optical axis relative to the fiber axis, measured by $\psi_p(z; s_0)$, approaches a terminal negative value, $\psi_p(z = 1; s_0)$, for each s_0 . This negative value indicates an inward tilt of the optical axis toward the fiber centerline. Figure 4b provides the surface tilt angle, $\epsilon\phi(1; s_0)\psi_p(1; s_0)$, which is only slightly greater for initially more oriented flows.

(iv) The forced weakly biaxial effect, computed from the biaxial order variable, $\beta_p(z; s_0)$, Figure 3e, shows an early spinline response with maximum initial effects corresponding to weaker s_0 . This behavior washes out downstream, as $\beta_p(z = 1; s_0)$ is nearly zero for all s_0 , so that the microstructure is essentially uniaxial by the takeup location.

One infers from these parameter variations on the base steady state that the qualitative variation due to upstream orientation is rather weak. Nonetheless, *the following trends are noteworthy: less upstream orientation induces the LCP filament to thin quicker, to speed up faster, to have more radial variation in the optical axis upstream, but by the takeup location to have less radial variation.*

4.B.2. Variation in draw ratio

Here we consider an important process parameter, the draw ratio, which in our nondimensional model coincides with the take-up velocity. The base solution is now explored for qualitative dependence on the take-up velocity, considering a range from slow spinning ($Dr = 2$) to fast spinning ($Dr = 40$). (This range of take-up speeds surpasses the critical draw ratio for the parameter values of the base state, as discussed in the next section.)

Refer now to Figure 5, from which we draw the following inferences:

- (i) One observes that a higher take-up speed leads to higher axial velocity, thinner fiber radius, and higher degree of orientation for all $z > 0$.
- (ii) The behavior of the uniaxial order parameter, $s(z; Dr)$, indicates that higher speed spinning leads to: (a) more rapid approach to downstream orientation between the optical axis and the LCP molecules; and (b) higher final orientation. These predictions agree with intuition and practice.

Analogous to Figure 4, in Figure 6a we post-process from these solutions the spinline location, z^* , where the steady state for each Dr achieves the target degree of orientation, $s(z^*; Dr) = 0.95$. The prediction is that the location z^* is far more sensitive to Dr in lowspeed spinning. For example, z^* moves nearly 5% of the spinline length as Dr increases from 20 to 30, whereas z^* moves nearly 50% of the spinline length as Dr increases from 3.7 to 10.

- (iii) The necessary radial tilt of the optical axis, calculated from $\psi_p(z; Dr)$, Figure 5d, shows that higher speeds yield a larger negative terminal angle, $\psi_p(z = 1; Dr)$. This is a weak effect, but the upshot is the radial nonuniformity is heightened in faster spinline. To illustrate, the terminal surface angle, with radian measure $\epsilon\phi(z = 1; Dr)\psi(z = 1; Dr)$, between the optical axis, $\mathbf{n}_3$, and the fiber axis, $\mathbf{e}_z$, is depicted in Figure 6b for this range of draw ratios; the graph converts to degrees. Figure 6b also provides the angle at the fiber surface between the tangent direction to the fiber surface and the optical axis.

Notice that both the fiber tangent direction and the optical axis at the fiber surface are tilted inward toward the fiber axis of symmetry, indicative of a converging flow in the fluid particle paths and the local optical axes. Note further that the final fiber surface tangent is always tilted further inward than the optical axis, so that the optical axis is always pointed exterior to the fluid domain. In the high speed limit, the surface angles of the fiber tangent and the optical axis approach similar limits (approximately -4.2 degrees).

- (iv) The forced, unavoidable, weakly biaxial effects, measured by $\beta_p(z; Dr)$, are depicted in Figure 5e; $\beta(z; Dr)$ remains bounded in absolute value by approximately 0.08 for all z and Dr , and approaches zero near the takeup location. *The conclusion is the downstream microstructure is essentially uniaxial.*

4.B.3. Variation in standard hydrodynamic parameters: Reynolds (R), Weber (W), and

Froude (F) numbers

We have calculated figures similar to Figure 5 which depict the base state response to variations in Reynolds number, Weber number, and Froude number. They indicate qualitative robustness in the typical base steady state with regard to these dimensionless viscosity, surface tension, and gravity parameters. They are omitted to save space.

Sensitivity to these hydrodynamic parameters *will* become evident, however, when we analyze stability of these steady states.

4.B.4. Variation in LCP parameters: anisotropic drag coefficient, σ , LCP kinetic energy relative to inertial energy, α , and dimensionless relaxation time, λ

Just as with the hydrodynamic parameters, as we explore dependence on the LCP parameters there is very little visible change in the base steady state radius, axial velocity, or orientation tensor. We again omit the figures to conserve space.

One concludes a very robust qualitative response of the base steady spinning state to all available model parameters; the Figures 1–6 are representative from a very exhaustive numerical sampling.

5 Linearized Stability of Steady State Solutions

For the remainder of this paper we focus on linearized temporal stability of steady states to superimposed spatial (streamwise) perturbations. The following roadmap is representative of how one might use such a spinline model to infer ways to increase throughput or to identify limiting factors to faster spinline. The reader should think of the base steady state, Figure 1, as an existing process—uniquely specified by these processing and material specifications.

The first subsection 5.A indicates how *process parameter variations* affect stability of the base steady state. Recall Figures 3 and 5 depict the qualitative variations in the base state due to variations in upstream orientation, $s(0) = s_0$ (Figure 3), and in downstream draw ratio, Dr (Figure 5). We now evaluate the stability of these families of steady states (Figures 7a,b). A primary focus is a detailed understanding of the transitions in parameter space from stable to unstable steady states. This issue leads to calculating the critical draw ratio for every initial upstream orientation condition (Figure 7c); this gives the boundary between stable and unstable steady state processes for the entire two-parameter family indexed by (s_0, Dr) , with all other values fixed at the base state. We restrict computations to the prolate ranges of s_0 as the likely conditions once the flow is nearly elongational and this model is

applicable.

At this point, we then have a full two-parameter variation of the base state, focussed on processing parameters only, yielding stability information all the way from the robust, stable base state to the transition to unstable processes.

From this new region of parameter space, we then (subsection 5.B) calculate how *material parameter variations* affect the stability of steady states. Naively, one might suspect that an increase in a certain parameter consistently either stabilizes or destabilizes steady state solutions. This scenario holds true for standard hydrodynamic parameters; for example, an increase in the surface tension parameter (reciprocal Weber number) is consistently destabilizing. However, an important finding reported below is that some LCP material parameters reverse their effect on stability from one region of parameter space to another, within the class of fiber spinning steady states.

With this preview, we move to the linearized stability analysis. The nonlinear pdes, and therefore the linearized equations for ϕ, v and s about an arbitrary steady state, are decoupled from the hyperbolic equations for ψ and β . Indeed, since the ψ and β pdes are first order hyperbolic (convection) equations, all perturbations propagate beyond the downstream location in finite time. Therefore these orientation variables do not provide a source of linearized instability for fiber spinning steady states.

The stability of steady states is therefore determined completely by the behavior of ϕ, v , and s , whose linearized equations have z -dependent coefficients, determined by the particular steady state solution whose stability is in question. The stability analysis consists first of (numerically) finding eigenvalues ω_n and eigenfunctions $\mathbf{f}_n(z)$, $n = 1, 2, \dots$ of the linearized equations with zero boundary conditions on the linearized variables: $(\delta\phi, \delta v, \delta s)(z = 0) = (0, 0, 0)$, $\delta v(1) = 0$. The real part of the eigenvalues ω_n , denoted $Re(\omega_n) = \nu_n$, is the *linearized growth rate*, since the corresponding linearized pde solution is $\exp(\omega_n t)\mathbf{f}_n(z)$. One deduces *linearized instability* of the steady state $(\phi, v, s)(z)$ if $\nu_{max} = \max_{n \geq 1} \{\nu_n\} > 0$, and *linearized stability* if $\nu_{max} < 0$. The *neutral stability condition* corresponds to $\nu_{max} = 0$. *Note that the stability ($\nu_{max} < 0$) or instability ($\nu_{max} > 0$) of any fixed steady state is completely characterized by the single diagnostic ν_{max} .* Therefore, it is this diagnostic that is reported in all figures to follow. For example, the *critical draw ratio* is, by definition, the draw ratio at which $\nu_{max} = 0$.

Our numerical stability calculations are carried out by a shooting method for the spatial eigenfunctions and associated eigenvalues, whose real parts determine the linearized growth rates. The number of spatial grid points used in each computation is 100.

5.A. Stability with respect to variations in processing parameters

5.A.1. *Stability with respect to variations in initial degree of orientation, $s(0)$*

As alluded to earlier, the base steady state of Figure 1 is linearly stable. We determine this result within a stability analysis of the one-parameter family of steady states from Figure 3, parametrized by $s(0) = s_0$. Figure 7a displays the maximum growth rate curves for the s_0 -parametrized family of Figure 3, from which we conclude:

This entire family of steady states is linearly stable; and a *weaker* upstream degree of orientation leads to *more stable* steady states.

Natural follow-up questions are: Does lowering s_0 always make the steady state more stable, i.e., make ν_{max} more negative? From a stable steady state, how much can the process be speeded up and still maintain stability? (To answer, we calculate Dr^* by solving for $\nu_{max}(Dr) = 0$.)

5.A.2. *Stability with respect to variations in draw ratio, Dr*

The maximum linearized growth rate as a function of Dr , $\nu_{max}(Dr)$, for the steady state family of Figure 5 is plotted in Figure 7b, from which we deduce:

- (i) The critical draw ratio, Dr^* , for the base steady state of Figure 1 is approximately $Dr^* = 29.8$.
- (ii) Faster processes are less stable in this region of parameter space. This might seem intuitively obvious, but we refer to a study (Forest and Wang, 1994) of isothermal Maxwell fluid models where the maximum growth rate curve oscillates about zero with varying Dr , creating alternating windows of stable and unstable draw speeds.

Since this family has a fixed prolate upstream degree of orientation, $s_0 = .5$, it is natural to ask how the critical draw ratio varies with s_0 .

5.A.3. *Critical draw ratio as a function of initial orientation, $Dr^*(s_0)$*

Rather than give a surface of maximum growth rates, we focus on the transition to instability: Figure 7c gives the critical draw ratio, or neutral stability curve, for every prolate value of s_0 ($0 < s_0 \leq 1$).

- (i) The critical draw ratios for the full s_0 -dependent family of Figure 3 all lie between $Dr = 28$ and $Dr = 30.4$. This explains why Figure 7a, with $Dr = 10$ far below the unstable drawing speed, has maximum linearized growth rates that are negative and well below zero.
- (ii) The salient feature of Figure 7c is *there is a preferred initial degree of orientation, here $s_0 \sim 0.25$, for which the critical draw ratio is maximum*. This information suggests that spinneret design and/or upstream flow controls play an important role for processes which

approach the critical draw ratio. The order of magnitude difference in Dr^* between the optimal s_0 and the worst ($s_0 = 1$) is on the order of 10% in this region of parameter space. We shall discover below that this measure of sensitivity to upstream degree of orientation varies considerably across parameter space.

(iii) For practical purposes, this prediction of our model, regarding a preferred upstream degree of orientation, needs first to be checked for robustness within this model, then within non-isothermal models, and then against physical spinline experiments. Here we can only address the first issue.

5.B. Stability of steady states with respect to variations in Newtonian material properties (Reynolds number, R) and surface tension parameter (Weber number, W)

5.B.1. Reynolds number variations

Figure 8a provides the maximum linearized growth rate for the indicated Reynolds number variations on the base steady state; recall from Figure 7a the base state is quite stable (when $R = 0.20$, $\nu_{max} \sim -2.6$).

(i) We deduce stability for all Reynolds number variations of the base state in the range $0.1 \leq 1/R \leq 10$.

That these solutions remain stable is not surprising since the maximum growth rate was negative and far below zero. The significant information is contained in the growth rate trends.

(ii) The maximum growth rate falls with increasing Reynolds number, and moreover the slope increases rapidly as R approaches unit value. This behavior suggests a significant stabilizing influence from low viscosity LCP solutions in this parameter regime.

From the preliminary indications of Figure 8a, we are led to pursue the critical draw ratio curve, $Dr^*(s_0)$, at several disparate Reynolds numbers. In Figure 9a we depict results for lower Reynolds numbers.

(iii) From Figure 8a, we expect lower critical draw ratios at lower Reynolds numbers, and Figure 9a yields consistent results: the range of critical draw ratios drops on the order of 15 – 20% when R is lowered from .20 to .10. The neutral stability curves maintain their shape, indicating a preferred prolate initial degree of orientation for each fixed R . The curves shift downward and to the right for lower R , indicating the preferred value of s_0 shifts toward a higher, but intermediate degree of orientation.

(iv) We note that dramatically higher critical draw ratios are achieved at moderate to high Reynolds numbers. The figures are omitted to save space. Indeed, we observe stable drawing

speeds of several hundred for $R = 2$, and thus do not fit on the same scale with Figure 9a. We also note that an isotropic upstream orientation is clearly preferred at moderate to high Reynolds numbers, and the variation in Dr^* between $s_0 = 0$ and $s_0 = 1$ is very large, often an order of magnitude.

These are intriguing predictions, especially since LCP melts have significantly lower apparent viscosities than isotropic polymer melts. While the trends are reasonable, without orientation dependence of material parameters the order of this effect is questionable. Furthermore, experimental measurements yield the apparent viscosity, while these calculations vary only the Newtonian contribution to viscosity.

5.B.2. *Weber number influence on stability*

The Weber number influence on stability for states near the base state is very similar to the Reynolds number discussion; refer to Figures 8b and 9b which indicate:

- (i) An increase in surface tension is destabilizing (ν_{max} grows toward zero), as expected.

In Figure 9b we fix three disparate Weber numbers while holding other parameters at the base state values, and study how fast the process can be run before becoming unstable. The salient conclusions are:

- (ii) The critical draw ratio curves shift upward as surface tension is lowered. The draw ratio curve at high surface tension ($1/W = 10$) is approximately 50% lower than the zero surface tension curve.
- (iii) For each Weber number, there is a preferred upstream degree of orientation, around $s_0 = .25$, which does not appear to vary with W .
- (iv) The relative difference due to variation in s_0 between the maximum and minimum critical draw ratios is 5 to 10% for each of the fixed Weber number curves.

5.C. **Stability with respect to variations of LCP material parameters**

5.C.1. *Stability with respect to variations in the anisotropic drag parameter, σ*

Figure 8c provides $\nu_{max}(\sigma)$ for σ variations of the base state; all solutions are linearly stable. Figure 9c depicts the critical draw ratio curves, $Dr^*(s_0)$, for three disparate values of σ .

Recall $\sigma = 1$ is the isotropic friction condition, whereas $\sigma = 0$ is the highly anisotropic limit. These figures suggest:

- (i) From Figure 8c, *an increase in the anisotropic drag parameter is weakly stabilizing near the base state*, yielding around a 5% change in maximum linearized growthrate, ν_{max} , over

the unit range of σ .

(ii) The critical draw ratio curves for a fixed σ , Figure 9c, are consistent with Figure 8c: an increase in drag coefficient (toward a more isotropic friction tensor) allows for faster take-up speeds. The curves change qualitatively as the drag parameter increases toward the isotropic limit $\sigma = 1$: the critical draw ratio (Dr^*) shifts upward; the preferred upstream orientation condition moves toward higher s_0 ; and the curves flatten around the peaks so a wider range of orientation conditions are equally preferred.

(iii) *Contact with the stability of constant LCP filaments without gravity.* In our previous paper (Forest et al., 1997), we investigated the Rayleigh capillary instability of LCP cylindrical filaments in the absence of gravity. In particular, we identified the influence of orientation on the classical Rayleigh instability. In that study, we showed analytically that the highly anisotropic drag limit ($\sigma = 0$) is stabilizing, measurably lowering the growth rates for all unstable spatial modes. In contrast, here we find that for fiber spinning steady states (obviously a very different class of solutions), the isotropic drag limit yields the highest critical draw ratios. Thus it appears that some LCP material properties may have quite different stability effects depending on boundary conditions, e.g., extrusion versus takeup conditions.

5.C.2. *Stability variations with respect to the scaled LCP kinetic energy parameter, α , and relaxation parameter, λ*

Figures 8d,e provide the maximum linearized growth rate for α and λ variations on the base state of Figure 1, while Figures 9d,e provide the critical draw ratio curves versus s_0 for three values of these parameters. The predictions are intriguing:

(i) An increase in the LCP kinetic energy parameter, α , or the relaxation parameter, λ , is stabilizing for variations near the base state (Figures 8d,e), but the opposite conclusion is drawn from Figures 9d,e for steady states near the transition to unstable draw speeds. Namely, the maximum stable draw speed drops with increased α in Figure 9d; the critical draw ratio drops with increased λ for most s_0 , and actually can oscillate near the isotropic range of s_0 .

Amplification. The above observation yields one of the most striking predictions of our study with isothermal LCP spinning models. Namely, certain LCP material properties may reverse their influence on process stability depending on subtle factors, e.g., the relative order of magnitude of other physical effects, or whether the process is in a stable throughput regime or running near the maximum stable draw speed. If these model predictions can be realized, then the role of microstructure in spinning flows cannot be capsulized into simple rules of thumb that hold for all processes; this scenario would support more need for accurate models.

(ii) At high fixed Deborah numbers, an isotropic upstream degree of orientation is preferred, and the variation due to s_0 is most pronounced. For $\lambda = 10$, the critical draw ratio at $s_0 = 0$ is over 40% higher than at $s_0 = 1$. For low fixed Deborah numbers the dependence on s_0 is diminished.

(iii) At fixed s_0 , the variation in critical draw ratio due to small versus large Deborah number is significant (Figure 9), with a speedup of around 20% for $s_0 = 0$ that steadily grows to around 70% for $s_0 = 1$, in each instance achieved for low λ .

5.D. Parameter sensitivity near the transition to unstable steady states

We choose $Dr = 29$ and hold all other parameters at the base steady state values; we know the critical draw ratio for the base state is just above this draw ratio, so we are near the instability transition. We now consider several one-parameter families beginning from this steady state solution, and study their maximum linearized growth rates.

Consistent with previous evidence, Figures 10a,b,c indicate that increases in the Reynolds number (R), Weber number (W) and anisotropic drag parameter (σ) lower the maximum linearized growth rate, and therefore are stabilizing in this sense.

As suggested by Figures 9d,e at $s_0 = .5$, Figures 10d,e confirm that increases in α and λ are indeed destabilizing for states near the transition to instability, whereas the opposite effect occurs in Figures 8d,e. By varying s_0 , we can even make these curves oscillate, so the upshot is that these LCP material parameters have quite subtle influences on stability of steady spinning states near the critical draw speed.

Our final figure depicts a closeup of the transition to instability. Growth rate curves are monitored to see when the curves first yield zero growth rates, and which values of upstream orientation destabilize first. The information is gathered in Figure 11, which provides the maximum linearized growth rates versus s_0 for several fixed draw ratios. These are chosen since for $DR \leq 27.7$, all s_0 yield stable steady states; for $DR \geq 30.3$, all s_0 yield unstable steady states.

The lowest curve, $Dr = 27.7$, indicates that the highly oriented s_0 conditions are the first to destabilize. A band of unstable s_0 near $s_0 = 1$ grows as DR increases; for example, at the slightly higher draw ratio of $Dr = 29$, all $s_0 \geq .75$ yield unstable steady states.

The next feature is from the $Dr = 29.5$ curve, for which the steady state with an isotropic ($s_0 = 0$) upstream orientation condition becomes unstable. For $Dr \geq 29.5$, a band of unstable, positive s_0 grows, leaving an intermediate, shrinking, band of stable prolate s_0 : at $Dr = 30$, the stable band is approximately $.08 \leq s_0 \leq .48$. Finally, near $Dr = 30.3$, the stable band shrinks to a single value of $s_0 \sim .24$. For all higher draw ratios, all s_0 yield unstable steady states.

Similar curves to Figure 11 are observed when we explore the onset of linearized instability for any of the families reported above. By varying over parameter space, it is likely that we can find parameter regimes where the isotropic initial orientation condition destabilizes first, or where either the band starting at $s_0 = 0$ or the band starting at $s_0 = 1$ grows to consume the entire prolate range before another unstable band of s_0 opens at the opposite end.

6 Conclusion

We have applied a 1-D thin filament model for liquid crystalline polymers to study the interplay between macroscopic hydrodynamics and internal orientation in an axisymmetric, isothermal fiber spinning flow. We have chosen a general physical regime for this study, in which the effects of surface tension, viscosity, gravity and inertia couple to LCP orientation effects of anisotropic drag, relaxation time, an intermolecular potential, and polymer kinetic energy. The fundamental utility of this study is a calculation of spun fiber orientation, e.g., birefringence, as it evolves in an isothermal spinline; this feature replaces the standard practice of computing fiber stresses during spinning and then applying an empirical stress-optical law in post-processing to infer spun fiber birefringence. Our model therefore yields a first-principles prediction for fiber radius, velocity, stress, and orientation (birefringence) along an isothermal spinline. Spun fiber properties such as tenacity and elongation-to-break are deduced in *post-processing* from this output, from empirical formulas which use the first-principles solution; e.g., tenacity is given by an exponential function of birefringence. This advance is independent of thermal properties, and therefore transfers to future non-isothermal models.

Uniaxial and weakly biaxial radial orientation effects are calculated for various fiber spinning steady states. The tilt angle of the optical axis with the fiber centerline at the take-up location reveals a distribution consistent with a converging flow. We find that biaxial effects are generated near the upstream location but then are effectively washed out by the take-up location. The conclusion drawn from these simulations is that the orientation tensor may consistently be approximated by a weakly radially dependent form in isothermal spinning flows. The midstream biaxial orientation behavior should be studied in thermal models to see if this phenomenon contributes to radial variations in microstructure.

Qualitative trends in steady state solutions due to all available model parameters are presented first, followed by stability of multiple parameter families of steady states. It is noteworthy that the steady state behavior does not foreshadow the subsequent stability results. We close with a recall of some of the more interesting predictions of this study.

The degree of orientation at the upstream (initial) axial location, s_0 , is a variable process parameter, dependent upon spinneret design and upstream flow controls. We considered the spinline response for a range of initial degrees of orientation, in various regions of parameter space. The stability calculations of Figures 7a, 9 and 11 indicate a *preferred degree of upstream orientation*—processes can be run faster for these preferred orientation conditions; the actual preferred upstream orientation is robust to surface tension, for example (Figure 9b), whereas the dependence on LCP parameters may vary significantly (Figures 9c,d,e). The optimal choice may be anywhere from isotropic to fully aligned.

The effects on stability due to LCP parameter variations offer some consistent predictions along with evidence of very subtle interactions among these physical effects. On the predictable side, decreases in Reynolds number and Weber number are uniformly stabilizing. Increases in the anisotropic drag coefficient, σ , which corresponds to a more isotropic friction tensor, make the base spinning state more stable (Figure 8c), lead to higher critical draw ratios for each value of upstream degree of orientation (Figure 9c), and can stabilize an unstable steady state whose draw ratio is just above critical (Figure 10c). These predictions are consistent in all regions of solution space. Moreover, the more stabilizing, isotropic drag conditions ($\sigma = 1$) select highly aligned upstream conditions ($s_0 = 1$), while highly anisotropic drag conditions ($\sigma = 0$) prefer isotropic upstream alignment ($s_0 = 0$).

We remark that in our previous paper (Forest et al., 1997) we showed that constant radius, cylindrical LCP filaments in the absence of gravity are more stable in the highly anisotropic drag limit ($\sigma = 0$), precisely the opposite of effects noted here for fiber spinning steady states. Thus we have evidence that the effect of anisotropic drag on stability depends strongly on boundary conditions.

The stability effects due to the LCP scaled kinetic energy parameter, α , and the LCP scaled relaxation time (Deborah number), λ , are intriguing. From Figures 8d,e, increases in these parameters are stabilizing to the base, typical steady state solution of Figure 1. In stark contrast, the computations in Figures 9d,e and 10d,e indicate that an increase in either α or λ leads to lower critical draw ratios for sufficiently high upstream degree of orientation (e.g., $s_0 \geq .2$), while there can be an oscillation in critical draw ratio versus α or λ for nearly isotropic upstream degree of orientation. Thus these LCP parameters may be stabilizing or destabilizing, within the same class of fiber spinning steady states.

The predictions presented here provide a foundation for more realistic thermotropic LCP spinning models for LCP fibers, which necessarily must couple temperature and an energy equation, together with temperature-dependent behavior of material parameters.

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Figure 1: A typical fiber spinning steady state solution. All parameter values are fixed at order one values: $1/R = \alpha = 5, 1/W = 1, 1/F = 1, N = 4, \lambda = 1, \sigma = 0.5$. Boundary conditions are: $\phi(0) = v(0) = 1, s(0) = .5, Dr = 10 = v(1)$ for figures 2a,b,c; in figure 2d, $\psi_p(0) = 0, \psi_h(0) = 1$, with $\psi = \psi_p + \psi_h$; in figure 2e, $\beta_p(0) = 0, \beta_h(0) = 1$, with $\beta = \beta_p + \beta_h$.

Figure 2: A schematic portrait of orientation evolution associated with the typical fiber spinning steady state of Figure 1. The cones in this picture are defined at each point in the fiber, and constructed from uniaxial data consisting of the order parameter s and the angle variable ψ . The cone axis of symmetry at each point is the optical axis, $\mathbf{n}_3(r, z)$, whose angle of tilt with respect to the fiber axis $\mathbf{e}_z$ is $\epsilon r \psi(z)$. The surface of each cone is an approximation of the most likely LCP molecular directions, $\mathbf{m}$, at each location (r, z) ; the angle between the cone axis and the cone surface is inferred from the order parameter, $s(z)$. Since s is uniform in r to leading order, the cone shapes are uniform in each axial cross-section; the cone (optical) axes, however, tilt relative to the fiber axis $\mathbf{e}_z$ in each cross-section $z = z_0$ proportional to $\psi(z_0)$. From Figure 1d, with $\psi(0) = 1$ the cones are initially tilted outward from $\mathbf{e}_z$, then evolve through a parallel configuration when $\psi = 0$ just below $z = .3$, beyond which they tilt inward toward the takeup location, consistent with the notion of a converging flow. Meanwhile, the cone shapes evolve steadily downstream toward the perfect alignment limit, $s = 1$, indicated by the cone angle steadily contracting toward zero. We have set the slenderness ratio in this graph at $\epsilon = 0.2$.

Figure 3: Steady solutions reflecting variations in the base state, Figure 1, due to changes in the initial degree of orientation, $s(0)$. All material parameter values are identical to the prototype; the initial/boundary data on the fiber radius and axial velocity, ϕ and v , are also the same. We vary $s(0) = s_0$ from .05 to .95 in increments of .05. Arrows indicate the direction of increasing s_0 . Figures 3a,b,c display the family of solutions, $\phi(z; s_0)$, $v(z; s_0)$, $s(z; s_0)$, respectively. Figures 3d,e present the forced responses in the angle parameter, $\psi_p(z; s_0)$, and biaxial order variable, $\beta_p(z; s_0)$, for zero initial data, for each value of s_0 .

Figure 4: Detailed information post-processed from Figure 3. Figure 4a represents the spinline location z^* where the degree of fiber orientation reaches the value .95, $s(z^*; s_0) = 0.95$, for each value of the upstream orientation s_0 . Figure 4b depicts a nonuniform radial fiber property: the s_0 -dependence on the terminal surface angle, $\epsilon\phi(1; s_0)\psi_p(1; s_0)$ (converted to degrees in the graph) between the optical axis $\mathbf{n}_3$ and the fiber centerline axis $\mathbf{e}_z$. The angle of tilt in the final cross-section consists of a linear radial interpolation between this surface angle and the parallel alignment at the fiber axis.

Figure 5: Variations in the base state (Figure 1) due to changes in draw ratio, Dr . All other parameter values and initial data are the same as the base state, with the initial data on ψ_p, β_p set to zero to highlight the unavoidable, forced behavior in these radial measures of orientation. $Dr = v(1)$ is varied from 2 to 40, in unit increments.

Figure 6: Detailed information post-processed from Figure 5, regarding variations in draw ratio. Figure 6a shows the spinline location, z^* , where a target fiber degree of orientation, $s(z^*; Dr) = 0.95$, is achieved, for each solution in Figure 5. Figure 6b provides surface angle information at the takeup location, both of the optical axis and of the fiber surface tangent. Recall the radian measure of the angle between the optical axis $\mathbf{n}_3$ and the fiber axis $\mathbf{e}_z$ at $z = 1, r = \phi$ is $\epsilon\phi(1)\psi_p(1)$; the degree measure of this angle versus Dr is the solid curve in Figure 6b. The radian measure of the angle formed by the tangent to the fiber surface and the fiber centerline is $\tan^{-1}(\epsilon\frac{\partial\phi(1;Dr)}{\partial z})$; the dotted curve in Figure 6b depicts this angle in degrees for all Dr . We use $\epsilon = 0.2$ in these figures.

Figure 7: Linearized stability information. Figure 7a concerns the s_0 -parametrized family of Figure 3; Figure 7b concerns the Dr -parametrized family of Figure 5; Figure 7c summarizes the neutral stability boundary for a full two-parameter variation of the base steady state. Each data point in Figures 7a,b depicts the maximum real part, ν_{max} (vs. s_0 in Figure 7a, vs. Dr in Figure 7b), of all linearized growth rates for that respective steady state; a negative value of ν_{max} indicates linearized stability. From Figure 7a we deduce the entire Figure 3 family of steady states (including the base state) is linearly stable, and that the steady states become less stable with increased upstream orientation; recall each of these states has $Dr = 10$. From Figure 7b, with all parameters except Dr set to the base state values, in particular $s_0 = .5$, we deduce the critical draw ratio is approximately 29.8, above which the steady states are linearly unstable. Figure 7c then provides the full two-parameter neutral stability curve: the critical draw ratio, $Dr^*(s_0)$ for each s_0 , above which the steady state is linearly unstable. We deduce that the critical draw ratio for all steady states of Figure 3 is between $Dr = 27.5$ and $Dr = 30.5$. Most notable, however, is the shape of this neutral stability curve, indicating there is a *preferred initial degree of orientation* (around $s_0 = 0.25$) associated with the maximum stable drawing speed (around $Dr = 30.4$) for this two-parameter family of steady states.

Figure 8: Stability properties: the maximum real part of all linearized growth rates for several parameter variations of the base steady state solution. In each one-parameter family of steady states, all other parameters and boundary conditions are identical to the base state of Figure 1. In Figure 8a, $1/R = 0.1$ to 10; Figure 8b, $1/W = 0$ to 10; Figure 8c, $\sigma = 0.1$ to 0.95; Figure 8d, $\alpha = 0.1$ to 10; Figure 8e, $\lambda = 0.1$ to 10.

Figure 9: Critical draw ratio as a function of s_0 corresponding to selected parameter values from Figure 8. The salient features are the order of magnitude of the critical draw ratio as the respective material parameters vary, and the initial orientation value at which the maximum stable drawing speed is attained.

Figure 10: Stability properties near the critical draw ratio. Depicted are the maximum real parts of all linearized growth rates for several parameter variations of the steady state solution with $Dr = 29$ and all other parameters identical to the prototype (in particular, $s_0 = 0.5$). This base solution we vary from is weakly stable, near the transition to instability. In Figure 10a, $1/R = 0.1$ to 10; Figure 10b, $1/W = 0$ to 10; Figure 10c, $\sigma = 0.1$ to 10; Figure 10d, $\alpha = 0.1$ to 1; Figure 10e, $\lambda = 0.1$ to 10.

Figure 11: Stability properties near the transition to instability of the prototype. The bottom curve depicts the onset of an unstable band of initial order parameter values, emanating from $s_0 = 1$, at the critical draw ratio just above 27.7. This band of unstable upstream degrees of orientation grows as Dr increases. By $Dr = 29.5$, the highly aligned unstable band of s_0 has expanded nearly to .6, at which point another unstable band of s_0 emerges from the isotropic upstream condition, $s_0 = 0$. These two bands grow toward one another for larger Dr , leaving an interior stable band of prolate s_0 ; this stable band shrinks to a single value, $s_0 \sim .24$ at $Dr \sim 30.3$.