# Gravitational Radiation in $f(R)$ Gravity: A Geometric Approach

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**Abstract**

**ADAM SCOTT KELLEHER:** Gravitational Radiation in $f(R)$ Gravity: A Geometric Approach.
(Under the direction of Laura Mersini-Houghton.)

I summarize experimental and theoretical constraints on gravity theories. I explore metric f(R) gravity, and explore scalar field theory analogs. I present a different kind of mechanism to raise the effective scalar mass in f(R) gravity in environments with particular ranges of background scalar curvatures, and thus suppress scalar effects on solar system curvature scales, while allowing scalar effects at different curvature scales. I review the post-Newtonian and post-Minkowskian mathematical machinery for General Relativity, and generalize these expansions to metric f(R) gravity up to second order in small parameters.

Dedicated to Bill Walsh, a true brother.

## Acknowledgments

I would like to thank my brother for his support, Bart Dunlap and Greg Herschlag for useful conversations, and my advisor Laura Mersini-Houghton for her guidance. I thank the Department of Physics and Astronomy at UNC Chapel Hill for supporting this work.

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## List of Abbreviations and Symbols

| | |
|---|---|
| $i, j, k, \ldots$ | Spatial coordinates indices running from 1 to 3 |
| $\mu, \nu, \alpha, \ldots$ | Space time coordinates indices running from 0 to 3 |
| $g_{\mu\nu}$ | The components of the metric tensor |
| $R_{\mu\nu\alpha\beta}$ | The components of the Riemann tensor |
| $R_{\mu\nu}$ | The components of the Ricci tensor |
| $R$ | The scalar curvature |
| $\eta_{\mu\nu}$ | The Minkowski metric |
| $\phi, \varphi$ | Scalar field |
| $\overset{n}{h}_{\mu\nu}$ | The $n$th order part of $h_{\mu\nu}$ in some expansion |
| $\Gamma^{\alpha}_{\mu\nu}$ | The components of the Christoffel connection |
| $h_{\mu\nu}$ | A perturbation to the metric, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ |
| $g$ | The determinant of the metric tensor, $g_{\mu\nu}$ |
| $h$ | The approximate trace of the metric perturbation, $h = \eta^{\mu\nu} h_{\mu\nu}$ |
| $\mathbf{x}$ | A spatial 3-vector in a post-Newtonian coordinate system |
| $\hat{\mathbf{n}}$ | A spatial unit 3-vector in a post-Newtonian coordinate system |

# Chapter 1

## Introduction

General Relativity has been a great success in explaining phenomena that were not covered by Newtonian gravity. Its major historical successes included explaining the perihelion shift of Mercury, and properly calculating the lensing due to the Sun. It correctly reduces to Newtonian gravity in the low energy, low velocity limit, and thus agrees with past experiments in Newtonian physics.

As time went on, there were still more observations that could not be explained. As early as 1933, Zwicky [1] posed the "missing mass" problem for galaxy clusters, an early version of the dark matter problem. In 1975 the problem took on its well known form. The announcement was made at an AAS meeting that stars in spiral galaxies orbited with roughly the same speed, based on measurements with a new, more sensitive spectrograph [2]. The observations have been attributed to a yet-unobserved type of matter, now known as dark matter.

In addition to the dark matter problem, the universe was observed to be accelerating its expansion in 1998, by examining luminosity data of Type Ia supernovae [3][4]. This could be accounted for in the context of GR by introducing a cosmological constant, but that brought about a host of other problems. The "coincidence problem", for example, is the question"Why do we exist during the relatively brief period in the history of the universe when the vacuum energy density is comparable to the matter energy density?". Then, there is the magnitude problem. From particle theory considerations, the vacuum energy should be as much as 120 orders of magnitude larger than that observed [5][6]. Thus, the accelerated expansion of the universe can not yet be explained with particle theories in a way that is consistent with General Relativity (GR).

These observations lead to attempts at resolving them that typically involve either modifications to gravity (the geometric part of the action), or introducing new dynamic fields (the matter part of the action). By having a dark energy that can evolve with time, the coincidence problem can be avoided.

It is difficult to motivate the fact that extra fields should not couple to matter, as there is no symmetry to prevent this. It is also difficult to explain why the particle mass in dark energy theories should be so small. Thus, one could argue [7] that modified gravity theories are better motivated than theories involving new particle fields for explaining the nature of dark energy.

I prefer, then to look at modified gravity theories over particle theories. A particularly interesting class of theories is metric $f(R)$ gravity. While this class of theories hasn't been very successful at explaining dark matter[1], it has some promise for dark energy[2].

The theory effectively introduces a scalar field that only interacts with matter in combination with the gravitational field, so is rather similar to scalar field models of dark energy, but introduces no actual new matter fields. Thus, it has the strengths of particle theories of dark energy, but without the field-theory-based weaknesses. It is also theoretically interesting that other theories, or quantum corrections can introduce higher order curvature invariants [15, 16], and higher invariants can allow the theory of gravity to be renormalizable [17, 18]. All of this should be reason enough to motivate an investigation into $f(R)$ gravity, but we can do even better. It turns out that it really is the best option for generalizing GR.

One could ask what other types of modifications we could make to the gravitational action. It turns out that the form of the Lagrangian is very constrained. We'll go into detail on this when we address theoretical constraints on $f(R)$ gravity, but it turns out that we can only make the generalization that the action can contain some function of the Ricci scalar (hence the name, $f(R)$ gravity). Any other generalization would cause the theory to become unstable [19].

We are left, then, with the option to generalize gravity to $f(R)$ gravity. It should be done in a way that agrees with all past tests of GR and Newtonian gravity, and in a way that is theoretically sound and observationally distinguishable from other theories. From one perspective, you could talk about modifying gravity to be different from GR at particular curvature scales. For example, $f(R) = R + \alpha R^2$ is like GR with $f(R) = R$ as we move from $R = 0$ to larger $R$, but we pick up modifications when $R \sim 1/\alpha$. We could also produce modifications at low curvatures, like $f(R) = R + \alpha/R$. The $\alpha/R$ term dominates at lower $R$, but $f(R)$ transitions over to being dominated by the GR term when $R \sim \sqrt{\alpha}$.

In the next chapter, I will go into detail on the established properties of gravity theories, and give some understanding of how these properties were established. If a theory is viable, it should fit all of these constraints on the energy scales where these experiments were performed (typically, in systems with energy densities and gravitational potentials no greater than the Sun's). In the Chapter 3, I will

[1] Sotiriou [8] points out that the roughly quadratic modifications to the Einstein-Hilbert action found by fitting $f(R) = R^n$ based on galactic rotation curves [9, 10, 11, 12] contrast with bounds found by Barrow and Clifton[13, 14]

[2] Sotiriou shows that modifications to the action in $f(R)$ theory can come into the Friedman equations effectively as a perfect fluid. It is easy to choose $f(R)$ such that the equation of state parameter for this fluid is that of dark energy, $w_{eff} = -1$ [8].

give some theoretical background on a class of modifications to gravity called scalar-tensor theories. Here is where I will establish interesting physical phenomena that arise in these theories, many of which will have analogs in $f(R)$ gravity. I will also describe mechanisms that have been used in the past to make these theories agree with past observations, but which also render these theories rather uninteresting and untestable.

In the Chapter 4, I will go into detail on $f(R)$ gravity theories. With all of this set up, we will be ready to start finding solutions for gravitational phenomena (in particular, gravitational radiation) in $f(R)$ gravity. Thus, Chapter 5 introduces the mathematical machinery of series expansion solutions in GR, and explains the state of the art in $f(R)$ gravity (which has been minimal).

With all of the groundwork laid, I'll start working out solutions for $f(R)$ gravity: In the chapter 6, I describe new phenomena I've discovered in $f(R)$ gravity near critical points, where the theory approaches GR. I believe these new phenomena should lead to clear observable predictions. In Chapter 7, I develop the post-Newtonian series expansion for $f(R)$ gravity, and in Chapter 8 I develop the post-Minkowskian series expansion. I show that the post-Minkowskian expansion can be solved with an iterative procedure, as it can in GR.

# Chapter 2

# Properties of Gravity Theories

## 2.1 Equivalence Principles

There are certain properties of gravity theories that have been verified to high experimental precision, and should be included in any theory of gravity. Equivalence principles are one such feature. We will generally follow the excellent explanation of Will, [20], to explain the concept and the experiments supporting it. At it's most basic level, the weak equivalence principle (WEP) states that the property of matter that determines its response to an applied force, or "inertial mass", is equal to the quantity determining its acceleration due to gravity, or "passive gravitational mass". Phrased differently, we can say write Newton's second law as

$$F = m_I a, \tag{2.1.1}$$

and the law of gravitation as

$$F = m_P g. \tag{2.1.2}$$

Then the WEP states that $m_P = m_I$, so that $a = g$ for any body [21]. That is, bodies accelerate with the same acceleration in a gravitational field, independent of their masses or compositions. This has been confirmed experimentally to very high precision. The basic idea is that if some energy contributes differently to inertial and passive masses, then that difference can be parameterized approximately as

$$m_P = m_I + \sum_a \eta^a \frac{E^a}{c^2}, \tag{2.1.3}$$

where $\eta^a$ parameterizes the contribution from the $a^{th}$ type of internal energy, $E^a$. Typically we measure relative differences in acceleration, the "Eötvos ratio",

$$\eta \equiv 2\frac{|a_1 - a_2|}{|a_1 + a_2|} = \sum_a \eta^a \left( \frac{E_1^a}{m_1 c^2} - \frac{E_2^a}{m_2 c^2} \right). \tag{2.1.4}$$

Thus, given the various $E^a$ for a pair of bodies, measurements of $\eta$ put upper limits on the $\eta^a$. The best constraints on $\eta$ come from experiments at Princeton and Moscow [22], [23], which give results $\eta = 10^{-11}$ and $\eta = 10^{-11}$, respectively. We have various forms for the $E^a$ due to the strong, electromagnetic, and weak forces. For example, consider the electrostatic nuclear energy in platinum and aluminum. A semi-empirical formula for the approximate internal energy is

$$\frac{E^{ES}}{mc^2} = 7.6 \times 10^{-4} Z(Z-1)A^{-4/3}, \tag{2.1.5}$$

where $Z$ is the atomic number, and $A$ is the mass number. The difference in this quantity between platinum and aluminum is $2.5 \times 10^{-3}$. Simplifying the formula for $\eta$ to only include the internal electrostatic force, we can get an upper limit on $\eta^a$ (the case where the entire quantity $\eta$ is from this energy contribution) of $|\eta^{ES}| < 4 \times 10^{-10}$ [20]. Similar considerations can be applied to the other electromagnetic internal energy contributions, as well as the strong and weak forces.

With strong empirical support for the WEP, we should choose a theory of gravity where the WEP is either violated very little (small $\eta$), or we could choose it to be a guiding principle, restricting our consideration of viable theories to ones where $\eta \equiv 0$. Metric $f(R)$ gravity is a theory in the second group.

The WEP implies that all test bodies follow the same free fall paths, independent of their compositions. This suggests a kind of universality of motion in different reference frames: in any particular free-falling reference frame, the laws of physics are the same. One can postulate, as Einstein did, that we can treat these reference frames as inertial frames: they are as good as the constant-velocity frames of motion in special relativity. More precisely, the laws of physics (e.g. electromagnetism, mechanics, nuclear decay, etc.) inside of a freely falling elevator will be the same as those inside an orbiting spacecraft, or one floating in the vacuum of space. This postulate, together with the WEP, are known as the Einstein equivalence principle, or the EEP.

Since the laws of physics must be the same in different inertial frames, we should be able to transform from one frame to another at the same space-time point. Thus, Local Lorentz Invariance (LLI) should exist in any theory of gravity that obeys the EEP. It is difficult to find a "clean" test of LLI: could the apparent differences in forces be due to unknown physical phenomena? In the past, apparent LLI-violating effects have been confused with unknown particle interactions. For example, violation of 4-momentum conservation in beta decay was found to be caused by the emission of a new particle: the neutrino [20]. The Hughes-Drever experiment is one experiment where LLI has been tested in a "clean" way, as we will now describe.

The basic idea is that if there are changes to the quadrupole moment of the inertial mass of a particle due to its state of motion relative to some absolute frame, then that difference can be quantified as

$$\delta m_{ij} \sim \sum_a \delta^a \frac{E^a}{c^2}, \tag{2.1.6}$$

where $\delta m_{ij}$ represents the anisotropic part of the inertial mass quadrupole moment, $m_{ij}$, and $\delta^a$ parameterizes the contribution to the anisotropy due to the energy $E^a$ from the interaction $a$.

This experiment was performed using $^7$Li in the $J = 3/2$ ground state in an external magnetic field. In the absence of external perturbations, the state is split into 4 equally spaced energy levels. Perturbations with a non-vanishing quadrupole moment ruins the equality of the energy level spacing. Using nuclear magnetic resonance techniques, the change in line spacing was constrained to less than $1.7 \times 10^{-16} eV$. The result is that $\delta m_{ij} < 1.7 \times 10^{-16} eV$. This implies constraints on the various $\delta^a$ for different fundamental interactions of

$$\begin{aligned} |\delta^{strong}| &< 10^{-23} \\ |\delta^{electrostatic}| &< 10^{-22} \\ |\delta^{weak}| &< 5 \times 10^{-18}. \end{aligned}$$

The extremely high precision of these results has lead to the Hughes-Drever experiment being called the "most precise null experiment ever performed" [20]. This clearly suggests that LLI should be a requirement of any viable theory of gravity, and lends evidence that the EEP comprises a good set of postulates on which to build a theory of gravity.

One can make convincing arguments (see [20]) that the EEP implies that the correct theory of gravity must be a "metric" theory of gravity. We will now describe the properties of metric theories of gravity.

## 2.2 Metric Theories of Gravity

A metric theory of gravity is defined as one where

1. space-time is endowed with a metric, $\mathbf{g}$,
2. test bodies' worldlines are geodesics of $\mathbf{g}$, and
3. in local free-falling frames (local Lorentz frames), the laws of physics are those of special relativity.

GR is clearly a metric theory of gravity since it is Lorentz covariant, the Einstein equations describe how the presence of matter generates **g**, and the divergence of these equations produces equations of motion that reduce, in the case of test particles, to the equations for space-time geodesics. It's less obvious that gravity theories with other gravitational fields than the metric field are metric theories of gravity. The trick is in realizing that was long as matter doesn't couple independently to the extra fields, the metric alone determines the trajectories of matter. The role of extra gravitational fields is that of determining how matter generates curvature. Indeed, they do enter into the Einstein equations. The trajectories of test particles, however, are still determined by the geodesic equation for the metric. We can illustrate this point by looking at scalar-tensor theory, which is equivalent to a subset of metric $f(R)$ gravity theories. Examining the field equations in different conformal frames makes the point clear. The difference between the "Jordan" and "Einstein" frames amounts to a transformation of the scalar field space and conformal transformation of the metric. I should emphasize that a conformal transformation of the metric is a transformation to a physically different space-time. The use of performing a conformal transformation on a particular action is that it can be transformed into a form that is easily manipulated, and can be transformed back by reversing the conformal transformation.

The "Jordan frame" action is defined as the $\phi$-coordinate frame where the gravitational coupling to the $\phi$ field is of the form $\phi R$,

$$S_J = \kappa \int d^4x \sqrt{-\tilde{g}}(\phi \tilde{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_m(\psi, \tilde{g}_{\mu\nu})), \tag{2.2.1}$$

and the divergence of the Einstein field equations in this frame is

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0, \tag{2.2.2}$$

which amounts to the same equations of motion as in general relativity. This means that test particles follow the geodesics of the metric, $g_{\mu\nu}$.

The Einstein frame action is defined as the $\phi$-coordinates where the gravitational action takes the form $R$, and so the $\phi$ field doesn't couple to the scalar curvature in the action. It is written

$$S_E = \kappa \int d^4x \sqrt{-g^*}(R^* - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_m(\psi, A(\phi) g^*_{\mu\nu})), \tag{2.2.3}$$

and the divergence of the Einstein equations in this frame is [24]

$$\nabla^*_\nu T^{\mu\nu}_* = \alpha(\phi) T^* \nabla^\mu_* \phi. \tag{2.2.4}$$

Fujii and Maeda dedicate an appendix [25] to showing in detail that it is indeed the fact that $\phi$ enters into the matter action that causes the energy-momentum tensor to no longer be divergence-less. The basic idea is that if you add a coupling between the $\phi$ field and a matter current, $J$, to the Lagrangian, $L_m \to L_m + \phi J$, then you end up changing the covariant conservation law to $\nabla_\mu T^{\mu\nu} = \phi \nabla^\nu J$. Thus, we need $J = 0$ to have a covariant conservation law. The conservation law is obeyed if and only if particles follow geodesics, which are properties of space-time and not the particles. Thus, universality of free-fall is maintained, and the WEP is obeyed.

The situation is different in the Einstein frame. We lose the covariant conservation law, so particles no longer follow geodesics. Instead, they follow paths $u^\mu_*$ satisfying

$$\frac{Du^\mu_*}{D\tau_*} = \zeta \left[ (u^\lambda_* \partial_{\lambda\sigma}) u^\mu_* - g^{\mu\lambda}_* \partial_\lambda (u_{*\nu} u^\nu_*) \right] \tag{2.2.5}$$

where $\zeta^{-2} = 6 + 4\omega$, and $\sigma$ is defined by $\phi = 2\sqrt{\omega} e^{\zeta\sigma}$. While these paths are not space-time geodesics, they are still independent of the particle composition (e.g. particle mass is absent from this equation), so the WEP is still obeyed.

The coupling between the matter and the $\phi$ fields in the action (2.2.3) results in the apparent $\phi$ forces in the Einstein frame, as shown by eqn (2.2.4). Fortunately, the theory can be transformed into the Jordan frame, where the matter Lagrangian no longer contains the $\phi$ field, and the apparent $\phi$ forces disappear. The matter in this theory follows geodesics of the "physical" metric, $\tilde{g}_{\mu\nu}$, as is apparent in eqn (2.2.2).

## 2.3 PPN Formalism and Experimental Constraints

### 2.3.1 PPN Parameters

With all the possible metric theories of gravity, we need a systematic formalism for comparing the theories. If we had the most general possible metric, we could let the coefficients of each term to be parameterized, and thus construct any metric we like by setting the parameters to the values needed to create that metric. That is the basic idea behind the PPN formalism. This formalism does not attempt to describe the metric precisely. It only tries to describe the first post-Newtonian corrections to the metric. It would be correct to interpret the coefficients of the PPN solution for the metric for a particular theory of gravity as the series expansion coefficients for that metric. We will go into more detail on this later, when we systematically describe the post-Newtonian expansion. For now, we will simply construct the PPN metric.

Given a complete list of basic quantities that can enter into sources (e.g. density, pressure, shear, etc.), we can start combining those quantities to create possible terms that enter into the metric. Those terms should have the correct dimensions, and they should be reasonably "simple," a subjective requirement. Examples of such terms could be the Newtonian potential,

$$U = \int d^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

the mass current density

$$M^i = \int d^3x' \, \frac{\rho v^i}{|\mathbf{x} - \mathbf{x}'|},$$

and other combinations like

$$U^{ij} = \int d^3x' \, \frac{\rho(\mathbf{x} - \mathbf{x}')^i (\mathbf{x} - \mathbf{x}')^j}{|\mathbf{x} - \mathbf{x}'|},$$

all implicitly with the correct combinations of $G$ and $c$ to retain the proper dimensions. Integrals of some combinations of these terms are allowed, as well as first and second moments. More complex combinations than that are omitted, though they could in principle be present.

We can then write down the terms of the metric, $g_{00}$, $g_{0i}$, $g_{ij}$, which transform as a scalar, vector, and tensor, respectively, based on how those combinations of physical quantities transform. These quantities are given arbitrary coefficients, and thus we introduce a set of parameters describing a very generalized theory of gravity. A gauge transformation is performed to simplify the set of coefficients. This produces a "super-metric", parameterized by this set of coefficients. By fixing the coefficients to different values, one produces the post-Newtonian expansion of various theories of gravity. A full treatment of this formalism is beyond the scope of this paper. Instead, we will describe the two coefficients that will be mentioned in this paper, as they are the only two that are relevant given the theoretical constraints on gravity motivated by the above experimental constraints.

As long as a gravity theory conserves 4-momentum and has no preferred frame or location effects, the only non-zero PPN parameters are written as $\gamma$ and $\beta$. We are interested in these two parameters alone, since metric $f(R)$ gravity is a conservative theory with no preferred frame or location effects. Both enter into the metric as first corrections to Newtonian theory. They can be defined simply in terms of the metric components as

$$
\begin{aligned}
g_{00} &= -1 + 2U - 2\beta U^2 && (2.3.1)\\
g_{0j} &= -\frac{1}{2}(4\gamma + 3)V_j - \frac{1}{2}W_j && (2.3.2)\\
g_{jk} &= (1 + 2\gamma U)\delta_{jk}, && (2.3.3)
\end{aligned}
$$

where

$$U = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{2.3.4}$$

is the Newtonian gravitational potential, and $V_j$ and $W_j$ are defined by

$$V_j = \int d^3x' \frac{\rho v_j}{|\mathbf{x} - \mathbf{x}'|} \tag{2.3.5}$$

and

$$W_j = \int d^3x' \frac{\rho\left[\mathbf{v} \cdot (\mathbf{x} - \mathbf{x}')\right](x - x')_j}{(|\mathbf{x} - \mathbf{x}'|)^3}. \tag{2.3.6}$$

Here, $\gamma$ can be interpreted as the amount of spatial curvature produced per unit of mass. $\beta$ can be interpreted as the amount of non-linearity in the gravitational potential. The values for both of these parameters in GR are 1, which agrees very well with solar system experiments. This puts constraints on alternative gravity theories like metric $f(R)$ gravity. In particular, since these parameters have been measured precisely on solar system scales, metric $f(R)$ gravity must produce predictions that agree with these measurements on these scales. These constraints are described in more detail in the next section. Note that in each case, the experiment involves energy densities and gravitational potential wells no greater than that of the Sun.

### 2.3.2 Experimental Constraints on Gravity Theories

#### PPN Parameter Constraints

**Cassini Bound on $\gamma$** The strictest bounds on $\gamma$ have been achieved using the Cassini spacecraft, measuring the Doppler shift in a radio wave sent from the spacecraft to Earth on its way to Saturn. This shift is influenced by the gravitational field of a massive object that the wave passes by, in this case the Sun.

Writing the coordinate of a photon traveling in the direction $\hat{\mathbf{n}}$ as $x^j = \hat{n}^j(t - t_0) + x_p^j$, the un-deflected part of the path is accounted for by the first term, and the term $x_p^j$ accounts for the deflection. We take the vector $\hat{\mathbf{n}}$ to be normalized, $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$.

Inserting the expression for $x^j$ into the geodesic equation for the general PPN metric, as in [20], we get the equations for the deviation $\mathbf{x}_p$

$$\frac{d^2\mathbf{x}_p}{dt^2} = (1+\gamma)\left[\nabla U - 2\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \nabla U)\right] \tag{2.3.7}$$

$$\hat{\mathbf{n}} \cdot \frac{d\mathbf{x}_p}{dt} = -(1+\gamma)U \tag{2.3.8}$$

Now, we can define the perturbation components parallel and perpendicular to the unperturbed trajectory as

$$x_p(t)_\parallel = \hat{\mathbf{n}} \cdot \mathbf{x}_p(t) \tag{2.3.9}$$

$$\mathbf{x}_p(t)_\perp = \mathbf{x}_p(t) - \hat{\mathbf{n}}[\hat{\mathbf{n}} \cdot \mathbf{x}_p(t)]. \tag{2.3.10}$$

Then, with these definitions, equation (2.3.8) yields

$$\frac{dx_{p\parallel}}{dt} = -(1+\gamma)U, \tag{2.3.11}$$

where for a spherical source, the Newtonian potential is just $U = m/r$. Evaluating it along the particle trajectory gives $U = m/r(t)$, where $r(t) = |\mathbf{x}_0 + \hat{\mathbf{n}}(t - t_0)$ describes the path of the particle originating at coordinates $(t_0, \mathbf{x}_0)$ and traveling at unit velocity in the $\hat{\mathbf{n}}$ direction. Using this expression for the Newtonian potential, we can integrate equation (2.3.11) to get the the perturbation to the path length parallel to the particle trajectory,

$$x_p(t)_\parallel = -(1+\gamma)m\, ln\left[\frac{r(t) + \mathbf{x}(t) \cdot \hat{\mathbf{n}}}{r_0 + \mathbf{x}_0 \cdot \hat{\mathbf{n}}}\right]. \tag{2.3.12}$$

Now, we can use this expression to find the time difference for a particle to propagate in the perturbed vs. the un-perturbed space-time. Writing the photon trajectory as $\mathbf{x}(t) = \mathbf{x}_0 + \hat{\mathbf{n}}(t - t_0) + \mathbf{x}_p(t)$, we can dot both sides by the un-perturbed trajectory's unit vector, and rearrange to get

$$t - t_0 = |\mathbf{x} - \mathbf{x}_0| - \hat{\mathbf{n}} \cdot \mathbf{x}_p(t) \tag{2.3.13}$$

$$= |\mathbf{x} - \mathbf{x}_0| + (1+\gamma)m\, ln\left[\frac{r(t) + \mathbf{x}(t) \cdot \hat{\mathbf{n}}}{r_0 + \mathbf{x}_0 \cdot \hat{\mathbf{n}}}\right], \tag{2.3.14}$$

from equation (2.3.12). The Cassini experiment considers the round trip travel time when the

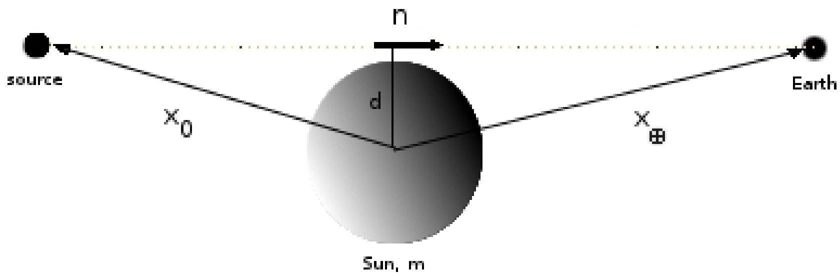


Figure 2.1: Our coordinate system is centered at the Sun. A particle on an un-perturbed trajectory travels along the dashed line.

spacecraft is on the far side of the Sun from Earth, see figure 2.3.2. In this case, to find the round trip travel time, we add together the travel time for each direction. We'll ignore the first term in equation (2.3.14), and just consider the part due to the perturbations. Taking $\mathbf{x}_0 = 0$, the denominator of the second term in (2.3.14) is just $d$. Then, for the trip from Earth the the spacecraft, we get $r(t) + \mathbf{x}(t) \cdot \hat{\mathbf{n}} = r_\oplus + \mathbf{x}_\oplus \cdot \hat{\mathbf{n}}$ and $r_e + \mathbf{x}_e \cdot \hat{\mathbf{n}} = d$, the impact parameter. For the return trip, $r(t) + \mathbf{x}(t) \cdot \hat{\mathbf{n}} = r_p + \mathbf{x}_p \cdot \hat{\mathbf{n}}$, where the $p$ subscript denotes the coordinates for the spacecraft relative to the Sun. Again, $r_e + \mathbf{x}_e \cdot \hat{\mathbf{n}} = d$. Since we're treating just the case where the spacecraft is approximately on the opposite side of the Sun from the Earth, we can approximate $\mathbf{x}_\oplus \cdot \hat{\mathbf{n}} \simeq r_\oplus$, $\mathbf{x}_p \cdot \hat{\mathbf{n}} \simeq -r_p$, and $d \simeq$ solarradius.

Putting these approximations in place, the term in the time change equation (2.3.14) due to the space perturbation effects becomes

$$\delta t = 2(1+\gamma)m\ln\left(\frac{4r_\oplus r_p}{d^2}\right) \tag{2.3.15}$$

This is the formula used by Bertotti et al. [26] to derive the Doppler shift in the Cassini experiment. In particular, they find, after converting from natural units and replacing $d = b$,

$$\gamma_{gr} = \frac{d\delta t}{dt} = -2(1+\gamma)\frac{Gm}{c^3 b}\frac{db}{dt} \tag{2.3.16}$$

for the frequency shift. Since the spacecraft is much farther from the Sun than the Earth is, using the approximation that the change in impact parameter is approximately the Earth's velocity, $db/dt \simeq v_{earth}$ , they find $\gamma_g r = 6 \times 10^{-10}$, implying $\gamma - 1 < 2.3 \cdot 10^{-5}$ [26].

Any proposed theory of gravity should fit this tight constraint within the scale of the solar system.

**Bounds on** $\beta$ The current best constraint on $\beta$ comes from the measurements of the perihelion shift of Mercury. In Newtonian physics, orbits in 2-body systems form an ellipse. In general, the orbits point of closest approach, the perihelion, can have some angular velocity, $\omega$.

One can calculate the perihelion shift from the 2-body equations of motion, as in Will [20]. We'll consider two bodies of masses $m_1$ and $m_2$, with total mass $m = m_1 + m_2$ and reduced mass $\mu = m_1 m_2/m$. The instantaneous eccentricity is $e$, semi-major axis is $a$, and semi-latus rectum is $p = a(1-e^2)$. The result, assuming a conservative gravity theory (which removes other PPN parameters), is a change of

$$\Delta\omega = \frac{6\pi m}{p}\left[\frac{1}{3}(2+2\gamma-\beta) + J_2\left(\frac{R^2}{2mp}\right)\right] \tag{2.3.17}$$

per orbit. Here, $J_2$ is the magnitude of the quadrupole moment of the Sun, defined by $J_2 = (C-A)/m_{sun}R^2$. $C$ and $A$ are the moments of inertia about the symmetry axis and equatorial axis, respectively. $R$ is the radius, and $m_{sun}$ the solar mass.

Most of the perihelion shift is due to perturbations from other planets. Once this is accounted for, the remaining perturbation should be given by inserting the astronomical values for the Sun and Mercury into (2.3.17),

$$\dot{\omega} = \frac{42.95''}{c}\left[\frac{1}{3}(2+2\gamma-\beta) + 3\times 10^{-4}\left(\frac{J_2}{10^{-7}}\right)\right] \tag{2.3.18}$$

and measurements give $\mid 2\gamma - \beta - 1 \mid < 3 \times 10^{-3}$ [27]. Since $\mid \gamma - 1 \mid < 10^{-5}$, this gives $\mid \beta - 1 \mid < 10^{-3}$.

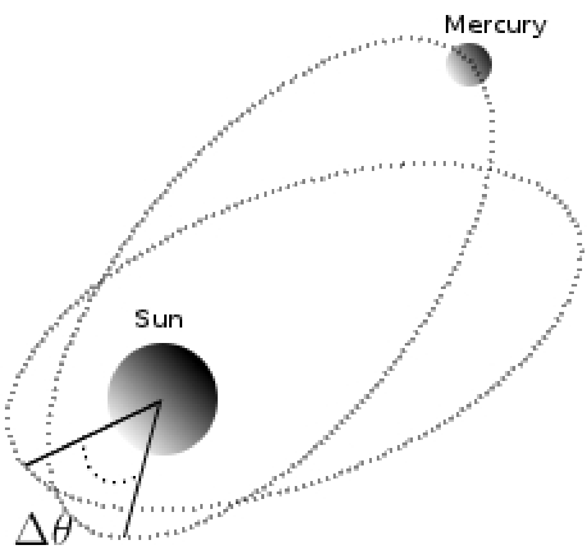


Figure 2.2: In Newtonian gravity, Mercury would follow an elliptical orbit around the Sun. In GR, the orbit can be described well by the parameters of an ellipse instantaneously, but the ellipse precesses. The change $\Delta\theta$ shown is the change in the perihelion in a time interval $\Delta t$. This precession can be described by the angular velocity $\omega$ given above.

### 2.4 Constraints from Pulsar Data

We will see in Chapter 3 that binary systems in scalar-tensor theories can emit dipole gravitational radiation. Consequently, observed binary systems put constraints on this class of theories. Eardley [28] showed that the dipole contribution to the change in the orbital period is proportional to the square of the difference in the object's sensitivities, $\mathcal{S}^2 = (s_1 - s_2)^2$. This difference is not appreciable for neutron star binary systems (NS-NS systems) like the double-pulsar PSR J0737-3039 [29], and so the dipole emission has little effect. The difference is significant with NS-white dwarf systems. Alsing et. al [30] examined these types of systems (as well as constraints from Cassini and the Nordtvedt effect) to give exclusion maps showing how suppressed the scalar should be (how large its mass must be) given a particular Brans-Dicke parameter ($\omega_{BD}$) for the scalar-tensor theory to be allowed by Pulsar observations. Observations of PSR J1141-6545 appear to just rule out $\omega_{BD} = 0$, but these calculations ignore the non-negligible eccentricity of this system, and so need some refinement.

# Chapter 3

## Tensor-Scalar Gravity

In this section, I will review an important class of metric theories of gravity: tensor-scalar theories. These theories belong to a simple class of extensions to gravity where modifications are described by a scalar field, $\phi$. By letting $\phi \to 1$, all modifications to GR vanish. The general class of these theories allows for multiple scalar fields, and provides a metric on the space of these fields. These models have been used, for example, in multi-field inflation. They include the double-field inflation model [31, 32, 33], and other multi-scalar inflationary models. Other more restricted cases typically involve just one scalar field. In the Lagrangian, this field may or may not have a potential term associated with it, and the coefficient of the kinetic term may or may not depend on the field. What distinguishes these theories from other theories that involve scalar matter fields is that the scalar field only enters into the matter part of the Lagrangian as a conformal factor for the metric. Thus, the scalar field only interacts with matter through its coupling with the metric field. In the rest of this section, I will go into some detail on the basic manipulations of these theories, and distinguish between similar theories that can lead to some confusion. I will then start with the most general case, when there are multiple scalar fields with potentials, and introduce successive simplifications. This leads to the classic Brans-Dicke scalar-tensor theories. Throughout the following sections, I will point out significant physical phenomena and relate them back to the motivation for my original research.

It is common to perform conformal transformations to make equations in scalar-tensor theories easier to work with. These transformations transform the metric $\tilde{g}_{\mu\nu} \to g^*_{\mu\nu} = A^2(\phi)\tilde{g}_{\mu\nu}$ from the physical conformal frame, in which the theory is defined, into a different conformal frame. Here, $A^2(\phi)$ is a positive real function of $\phi$. Note that this transformation is different from a general coordinate transformation. The physics in these two different frames can be very different: energy-momentum may or may not be conserved in either frame, and the gravitational constant may not actually be constant. Once the equations are transformed into a different frame, the equations are manipulated. Results in the the physical frame are finally obtained by transforming back to the original frame simply by inverting the transformation.

There are two particular frames that are commonly used. In Brans-Dicke theory, the physical frame is the one where the scalar field couples directly to the scalar curvature,

$$S_J = \kappa \int d^4x \sqrt{-\tilde{g}}(\phi \tilde{R} - \frac{\omega_0}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_m(\psi, \tilde{g}_{\mu\nu})) \tag{3.0.1}$$

Transforming to a different frame via a conformal transformation $\tilde{g}_{\mu\nu} = A(\phi) g^*_{\mu\nu}$, and changing the coordinates in $\phi$ space, we can arrive at the so-called Einstein frame. This frame gets its name from the fact that the gravitational action is the same as that in general relativity,

$$S_E = \kappa \int d^4x \sqrt{-g^*}(R^* - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_m(\psi, A(\phi) g^*_{\mu\nu})). \tag{3.0.2}$$

I will now outline this procedure, as presented in [25]. First, we find it convenient to change $\phi$ coordinates to give us a canonical kinetic term. We make the change $\phi = \frac{1}{8\omega}\varphi^2$, and rewrite (3.0.1) as

$$S_J = \kappa \int d^4x \sqrt{-\tilde{g}}(\frac{1}{8\omega} \varphi^2 \tilde{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) + \mathcal{L}_m(\psi, \tilde{g}_{\mu\nu})). \tag{3.0.3}$$

Next, we will perform the conformal transformation. First, we define $F(\varphi) = \varphi^2$. We focus on the first term in the Lagrangian, $\mathcal{L}_1 = \frac{1}{2}\sqrt{-\tilde{g}} F(\varphi) \tilde{R}$, and find the transformation that results in $\tilde{R}$ entering into the Lagrangian as it does in GR. We now need some basic properties of conformal transformations. For the transformation

$$\tilde{g_{\mu\nu}} \to g^*_{\mu\nu} = A^2(\varphi) \tilde{g}_{\mu\nu}$$

, we get

$$g^{\mu\nu} = A^2 g_*^{\mu\nu},$$

$$\sqrt{-\tilde{g}} = A^{-4} \sqrt{-g_*}$$

. Then, defining $f = \log A$ and $f_\mu = \partial_\mu f$ (this $f$ is not the be confused with the $f(R)$ from modified gravity theories, which is equivalent to $\frac{1}{2}F(\varphi)$), we find that

$$\tilde{R} = A^2(R_* + 6\Box_* f - 6 g_*^{\mu\nu} f_\mu f_\nu)$$

. Applying these to the first term of the Lagrangian, we get

$$\mathcal{L}_1 = \sqrt{-g_*} \frac{1}{2} F(\varphi) A^{-2} (R_* + 6\Box_* f - 6 g_*^{\mu\nu} f_\mu f_\nu). \tag{3.0.4}$$

Now, our goal is to get the term containing $R_*$ to enter into the Lagrangian as it does in GR, like $\sqrt{-g_*}\frac{1}{2}R_*$. To do this, we set $F(\varphi)A^{-2} = 1$, which gives us our definition for $A(\varphi)$, specifying the conformal transformation we were looking for as $g^{\mu\nu} = \varphi g_*^{\mu\nu}$.

We have the term we're looking for now, but have to finish transforming the action. The second term in (3.0.4) can be eliminated by integrating by parts, and $f_\mu = \frac{1}{2}\frac{F'}{F}\partial_\mu\varphi$ so (3.0.4) becomes

$$\mathcal{L}_1 = \sqrt{-g_*}\left(\frac{1}{2}R_* - \frac{3}{4}\left(\frac{F'}{F}\right)^2 g_*^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right).$$

Finally, the kinetic term, $-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$, is transformed into

$$-\frac{1}{2F}g_*^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi.$$

Then, since the second term of $\mathcal{L}_1$ is of the same form, we can combine terms to get

$$-\frac{1}{2\varphi^2}g_*^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi.$$

The final result is then

$$S_E = \int d^4x\sqrt{-g_*}\left(\frac{1}{2}R_* - \frac{1}{2\varphi^2}g_*^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \mathcal{L}_m\right).$$

## 3.1 Tensor Multi-Scalar Theory

### 3.1.1 Action and Field Equations

The most general tensor-scalar theory is one with multiple scalar fields, and with a potential for those fields. Allowing for coupling between the kinetic terms, we can write the action for this in the Einstein frame as [24], following Damour's notation,

$$S_{tot} = S_{g_*} + S_\varphi + S_m \tag{3.1.1}$$

where

$$S_{g_*} = \frac{c^4}{4\pi G_*}\int \frac{d^4x}{c}\sqrt{g_*}\frac{R_*}{4} \tag{3.1.2}$$

$$S_{\varphi} = \frac{c^4}{4\pi G_*}\int \frac{d^4x}{c}\sqrt{g_*}\left(\frac{1}{2}g_*^{\mu\nu}\gamma_{ab}\partial_\mu\varphi^a\partial_\nu\varphi^b + V(\varphi^a)\right) \tag{3.1.3}$$

$$S_m = S_m\left[\psi_m, A^2(\varphi^a)g^*_{\mu\nu}\right] \tag{3.1.4}$$

We've written this in terms of a metric, $g^*_{\mu\nu}$, called the Einstein metric. This is not the metric whose curvature is "felt" by matter. That metric is the one that enters into the matter action, as indicated in the definition of $S_m$, and we write it $\tilde{g}_{\mu\nu} = A^2(\varphi^a)g^*_{\mu\nu}$.

We now should look into what physical effects this generalization has. We are concerned, in particular, with how the radiation in this theory differs from general relativity.

### 3.1.2 Radiation Sources

In General Relativity, we can choose to examine perturbations to the Minkowski metric, $\eta_{\mu\nu}$, and write our metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in terms of the perturbation, $h_{\mu\nu}$. If we keep only terms in the Einstein equations linear in the perturbation, we will arrive at a wave equation for the metric perturbations. We will detail this procedure in section 5.3.1. We will find the dominant contribution to radiation to result from the second time derivative of the mass-quadrupole-moment of our source mass distribution.

It helps to look at a particular case. We'll examine two compact bodies orbiting each other with circular orbits. See the figure to clarify the coordinate system. They will have masses $m_1$ and $m_2$, total inertial mass $M = m_A^{(0)} + m_B^{(0)}$, and $\nu = m_A^{(0)}m_B^{(0)}/M^2$. Their coordinates will be $\mathbf{z}_A$ and $\mathbf{z}_B$, orbital radius $R = \|\mathbf{Z}\| = \|\mathbf{z}_A - \mathbf{z}_B\|$, separation unit vector $\mathbf{N} = \mathbf{Z}/R$, and their relative velocity $\mathbf{V} = d\mathbf{Z}/dt$. We can write the dominant term for power radiated in GR as

$$P^{GR}_{quadrupole} = \frac{8G_*}{15c^5}\left(\frac{G_{AB}M^2\nu}{R^2}\right)^2\left[12\mathbf{V}^2 - 11(\mathbf{N}\cdot\mathbf{V})^2\right] + O\left(\frac{1}{c^7}\right) \tag{3.1.5}$$

We can contrast this to the situation when scalar fields are introduced. In this case, we can have scalar wave radiation. We find this to be sourced in a multipole expansion by as low as the monopole term. The monopole, dipole, and quadrupole power emission in tensor-multi-scalar theory is, for

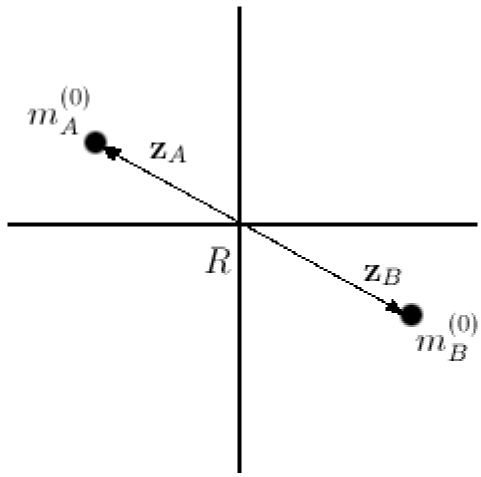


Figure 3.1: Our coordinates are in the center-of-mass frame for the two-body system.

the portion radiated via the scalar field, given by the expressions

$$\begin{aligned} P^{TMS}_{monopole} &= \frac{G_*}{c^5}\left(\frac{G_{AB}M^2\nu}{R^2}\right)^2 (\mathbf{N}\cdot\mathbf{V})^2 \left[\frac{5}{3}(\alpha^a_A+\alpha^a_B) \right. && (3.1.6) \\ &\left. -\frac{2}{3}(\alpha^a_A X_A+\alpha^a_B X_B)+\frac{\beta^a_{Ab}\alpha^b_B+\beta^a_{Bb}\alpha^b_A}{1+\alpha_A\alpha_B}\right]^2 + O(\frac{1}{c^7}) \\ P^{TMS}_{dipole} &= \frac{G_*}{3c^3}\left(\frac{G_{AB}M^2\nu}{R^2}\right)^2 [\alpha^a_A-\alpha^a_B]^2 + O(\frac{1}{c^5}) && (3.1.7) \\ P^{TMS}_{quadrupole} &= \frac{G_*}{30c^5}\left(\frac{G_{AB}M^2\nu}{R^2}\right)^2 \left[32\mathbf{V}^2-\frac{88}{3}(\mathbf{N}\cdot\mathbf{V})^2\right] && (3.1.8) \\ &\times[(\alpha^a_A+\alpha^a_B)-(\alpha^a_A X_A+\alpha^a_B X_B)]^2 + O\left(\frac{1}{c^7}\right) && (3.1.9) \end{aligned}$$

and matches the GR expression for the portion radiated, at quadrupole order, via the transverse-traceless metric perturbations. Here, $\alpha^a_A = \partial ln(m_A(\varphi))/\partial\varphi^a$, and is a property of $A^{th}$ body, and $\beta^A_{ab}(\varphi) = D_a D_b ln(m_A(\varphi))$, where $D_a$ is the covariant derivative with respect to the sigma model metric, $\gamma_{ab}$, and $X_A = m^{(0)}_A/M$.

In general relativity, there is no source for monopole or dipole radiation. Conservation of mass implies that there is no time-varying monopole moment for the mass distribution of a localized source. Conservation of momentum implies the same for a time-varying mass dipole moment. The existence of the former would imply the total mass-energy of a closed system were changing with time. The latter would imply that the center of mass of a closed system were oscillating, which can't happen without external forces. This is why gravitational radiation in GR is due to a time-varying quadrupole (and higher) moment.

In TMS theory, however, the source for radiation can be from the scalar field mass-energy. A single body that is increasing in density can have increasing scalar field energy-momentum. Two bodies orbiting each other with the same total mass-energy, but different contributions to that mass-energy from the scalar field will have a time-varying scalar field energy-momentum dipole moment. Thus, in TMS theory, we can have monopole and dipole radiation.

We notice, then, that in TMS theory we can have much greater total power radiation. The quadrupole radiation in both TMS and GR are suppressed relative to the TMS dipole term by a factor of order $(v/c)^2$. This makes scalar radiation a potentially interesting way of investigating TMS theory relative to GR.

Note that the dipole radiation vanishes for bodies with identical sensitivities, $\alpha_i^a$. Eardley [28] points out that black holes always have sensitivity of unity. Dwarfs and normal stars have negligible sensitivities of around $10^{-3}$ and $10^{-6}$, respectively, and neutron stars have sensitivities ranging from around 0.01 for $0.132M_\odot$ stars to 0.78 for $1.41M_\odot$ stars. Thus, systems with a star and black hole would make the best candidates for strong dipole radiation.

### 3.2 Tensor-Scalar Theory: Brans-Dicke with Potential and Variable $\omega$

If we simplify Tensor-Multi-Scalar theory by allowing only one scalar field, we get the action for a Brans-Dicke theory with potential and variable $\omega$. Restricting to one scalar, we drop the indices on the sigma model metric, $\gamma^{ab}$, and rewrite it as $\gamma = \gamma(\varphi)$. We redefine the $\varphi$ field, and perform a conformal transformation to the Jordan frame, and thus put the action in the form

$$S = \int d^4x\sqrt{-g}\left[\varphi R - \frac{\omega(\varphi)}{\varphi} g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi) + \mathcal{L}_m\right]. \tag{3.2.1}$$

Varying the action gives the equations of motion

$$\begin{aligned} G_{\mu\nu} &= -\frac{4\pi\tilde{G}}{\varphi}T^{(m)}_{\mu\nu} + \frac{\omega(\varphi)}{\varphi^2}\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right) \qquad (3.2.2)\\ &\quad + \frac{1}{\varphi}\left(\varphi_{;\mu\nu} - g_{\mu\nu}\Box\varphi\right) + \frac{1}{2\varphi}g_{\mu\nu}V(\varphi) \\ \Box\varphi &= \frac{1}{2\omega(\varphi)+3}\left(-4\pi\tilde{G}T^{(m)} + 2V(\varphi) + \varphi V'(\varphi) + \frac{d\omega(\varphi)}{d\varphi}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right) \qquad (3.2.3) \end{aligned}$$

Since the Jordan frame metric is the one that enters into the matter action, this frame is where physical effects are easiest to see. We'll use this to examine a few effects in these generalized Brans-Dicke theories.

### 3.2.1 Variation of Newton's Constant

One effect that results from the scalar field coupling to the Ricci curvature in the Jordan frame action is an effective variation of the gravitational constant, $G$, with $\varphi$. As $\varphi$ can vary throughout space and time, so also can the effective gravitational constant. This is apparent when we look at equation (3.2.2), and we see the place of $G$ in the GR Einstein equations taken by $G/\varphi$.

If the cosmological value of $\varphi$ changes with time, we can have a slow time-variation of the gravitational constant. The Viking Project [34] and binary pulsar data [35] constrain $\dot{G}/G < (0.2 \pm 0.4) \times 10^{-11}\,\text{years}^{-1}$ and $\dot{G}/G < (-0.06 \pm 0.2) \times 10^{-11}\,\text{years}^{-1}$, respectively [25].

It should be noted that in the Einstein frame, the lack of coupling to the scalar curvature results in the gravitational constant being constant [25].

### 3.2.2 Effect of Potential

If we restrict to the asymptotic region, where the scalar field takes its constant cosmological value, the effect of including a potential term for the scalar field becomes more clear. Equation (3.2.2) reduces to

$$G_{\mu\nu} \quad = \quad -\frac{4\pi\tilde{G}}{\varphi}T^{(m)}_{\mu\nu} + \frac{1}{2\varphi}g_{\mu\nu}V(\varphi), \tag{3.2.4}$$

and so we have an effective cosmological constant term.

Another effect of including the potential is that the field has only a finite range.

## 3.3 Brans-Dicke without Potential with Constant $\omega$

The least general class of TMS theories we'll look at is one with one scalar field, no scalar potential, and fixed $\omega$. This is of theoretical interest as a toy theory, and can be used to investigate the effects of the non-minimal coupling term, $\sqrt{-g}\varphi R$, in the Lagrangian. This theory reduces to GR in the limit $\omega \to \infty$. It can be shown that the presence of a non-minimal coupling term with no $\varphi$ field in the matter Lagrangian implies that the weak equivalence principle is obeyed [25].

Brans-Dicke theories imply immediate physical results involving an extra gravitational scalar force. This would imply measurements for PPN parameters that differ from GR. In particular,

$\gamma = \frac{1+\omega}{2+\omega}$, while $\beta = 1$ agrees with its value in GR. The parameter $\gamma$ only agrees exactly with the value in GR if $\omega \to \infty$.

### 3.4 The Chameleon Mechanism

Brans-Dicke theories, for small enough $\omega$, disagree with solar system experiments. Solar system experiments impose the constraint $\omega > 40000$ [36]. The generalized versions with potential can get around this constraint through non-linear effects that suppress the scalar field's effects at solar system scales, but still allow it to act at larger scales. This is known as the chameleon mechanism.

We will follow the work of Khoury and Weltman, [37], in presenting this theory. We start with the basic action in the Einstein frame,

$$S = \int d^4x\sqrt{-g^*}\left(\frac{M_{PL}^2}{2}R^* - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + S_{matter}(\psi, e^{2\beta\phi/M_{PL}}g^*_{\mu\nu}), \tag{3.4.1}$$

where Einstein frame quantities are denoted with a $*$.

To have a long-range, massless field in environments with no matter present, we require a potential of the runaway form with no minimum. It should satisfy the constraints

$$\lim_{\phi\to\infty} V = 0 \quad \lim_{\phi\to\infty}\frac{V_{,\phi}}{V} = 0 \quad \lim_{\phi\to\infty}\frac{V_{,\phi\phi}}{V_{,\phi}} = 0 \tag{3.4.2}$$

and

$$\lim_{\phi\to 0} V = \infty \quad \lim_{\phi\to 0}\frac{V_{,\phi}}{V} = \infty \quad \lim_{\phi\to 0}\frac{V_{,\phi\phi}}{V_{,\phi}} = \infty, \tag{3.4.3}$$

like and inverse power law potential, $V(\phi) = M^{4+n}\phi^{-n}$, for some mass scale $M$ and positive constant $n$. A schematic function $V(\phi)$ is pictured in figure 3.4

If we were to add to this a monotonically increasing function, we could introduce a minimum and thus cause the potential to pick up a mass. We will now see that the field does indeed experience an effective potential of this form in the Einstein frame. This is depicted in figure 3.4

The equation of motion for $\phi$ resulting from the action (3.4.1) is

$$\nabla^2\phi = V_{,\phi} - \frac{\beta}{M_{PL}}e^{4\beta\phi/M_{PL}}\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu}, \tag{3.4.4}$$

where a over a quantity denotes that it is in the Jordan frame. Working in an approximately

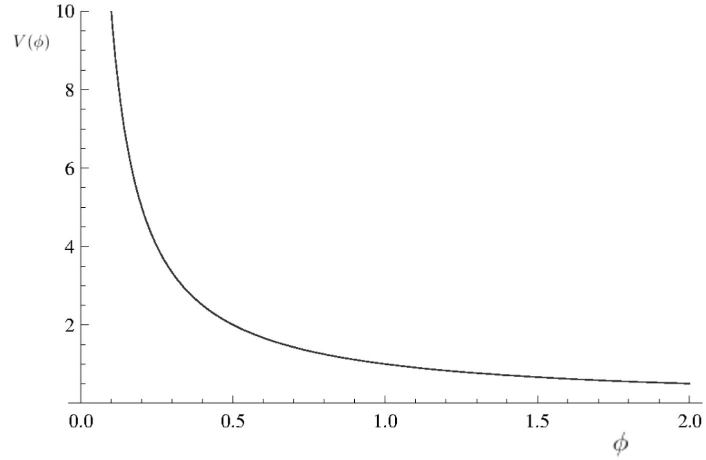


Figure 3.2: This is a schematic $V(\phi)$ satisfying all the constraints in equations 3.4.2 to 3.4.3

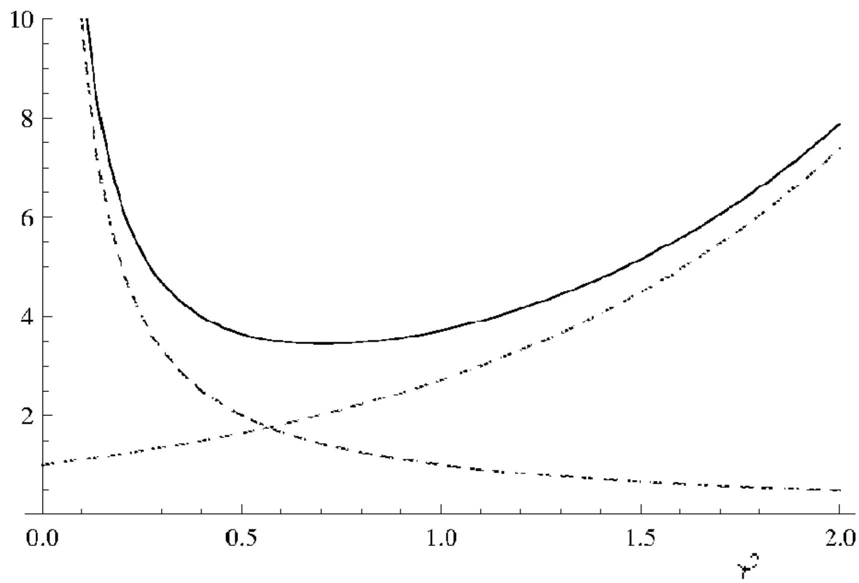


Figure 3.3: This is a schematic potential $V(\phi)$ together with the exponential term, resulting in an effective potential with a minimum at $\phi_{min}$, even though the original potential was monotonically decreasing.

Minkowski space, we approximate $g_{\mu\nu} \approx \eta_{\mu\nu}$, and so $\tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu} \approx -\tilde{\rho}$, where $\tilde{\rho}$ is the Jordan frame energy density. Following [37], we express this in terms of the Einstein frame conserved energy density, $\rho = \tilde{\rho}e^{3\beta\phi/M_{PL}}$ as

$$\nabla^2\phi = V_{,\phi} + \frac{\beta}{M_{PL}}\rho e^{\beta\phi/M_{PL}}, \tag{3.4.5}$$

and so we see that $\phi$ is governed by an effective potential of the type described above, $V_{eff} = V(\phi) + \rho e^{\beta\phi/M_{PL}}$. The value assumed by $\phi$ at the minimum of the effective potential can be found from

$$V_{,\phi_{min}} + \frac{\beta}{M_{PL}}\rho e^{\beta\phi_{min}/M_{PL}} = 0 \tag{3.4.6}$$

and, if we interpret the field as a particle in a quadratic well, the mass of fluctuations about this minimum is

$$m^2_{min} = V_{,\phi\phi}(\phi_{min}) + \frac{\beta^2}{M^2_{PL}}\rho e^{\beta\phi_{min}/M_{PL}}, \tag{3.4.7}$$

implying that the mass increases with local matter density. The presence of this effective mass limits the range of the force associated with the scalar field, and so scalar forces are suppressed by a Yukawa-like factor [1], $e^{-m_{min}r}/r$.

As shown in [38], there is also a non-linear shielding effect in the Jordan frame. For an object with an inertial mass $M$, the effective gravitational mass becomes $M_{eff} = M\frac{1+2\epsilon\alpha^2}{1+2\alpha^2}$. In effect, the gravitational mass is smaller than the inertial mass due to a screening effect. This screening effect operates when the gravitational well of an object, with mass $M$ and characteristic size $r_c$, is large compared with the background scalar field, $\varphi_0$. It is parameterized by $\epsilon \simeq \frac{\varphi_0}{2\alpha}\left(\frac{GM}{r_c}\right)^{-1}$ [38]. Here, $\alpha$ is a parameter that comes from producing the Jordan frame equations by a conformal transformation from the Einstein frame equations, using the conformal factor $A^2(\varphi) = 1 - 2\alpha\varphi$, and following the reverse of the procedure outlined at the beginning of Chapter 2. See figures 3.4,3.4

for an illustration of how this effect looks schematically in terms of the scalar field, given some mass distribution. Note that the scalar generally tries to approach a constant value in a constant background, but can't when the backgrounds gravitational well is not deep enough.

There is a related perspective on the scalar interaction between objects that are large compared to the range of the scalar field in their interiors. Consider a sphere of radius $R$ and energy density $\rho$ in a vacuum. If the range of the scalar field is small inside the sphere, and large in the vacuum outside of it, it's range must interpolate between the two in a thin shell near the surface of the sphere.

[1] as confirmed by numerical analyses, see [37]

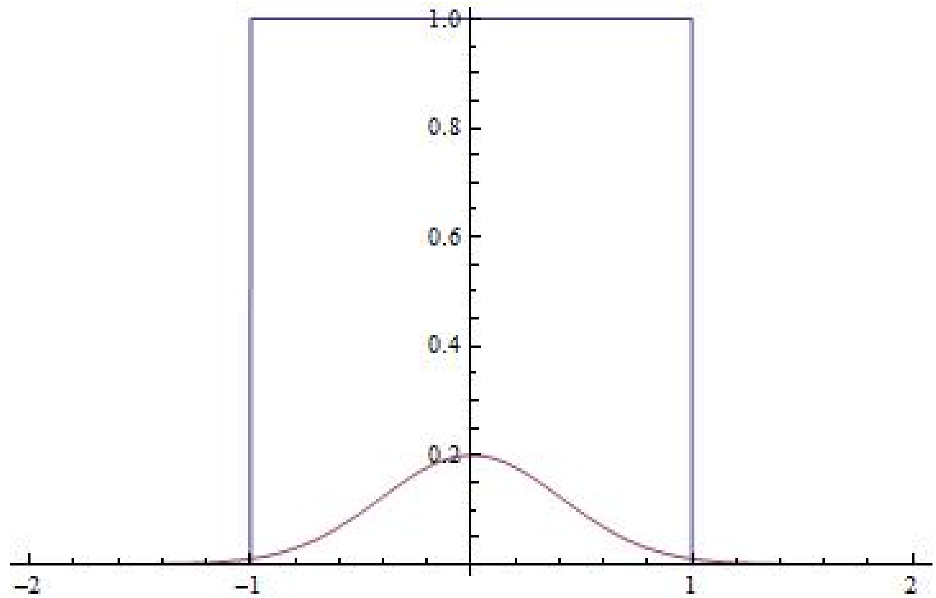


Figure 3.4: Here, the matter distribution (square curve) is plotted against distance along the central axis through a body in space. The gravity well of this body is weak, so elicits only a weak response (qualitatively depicted) from the scalar field, whose value relative, $\phi - \phi_0$, to its background value is depicted in the figure.

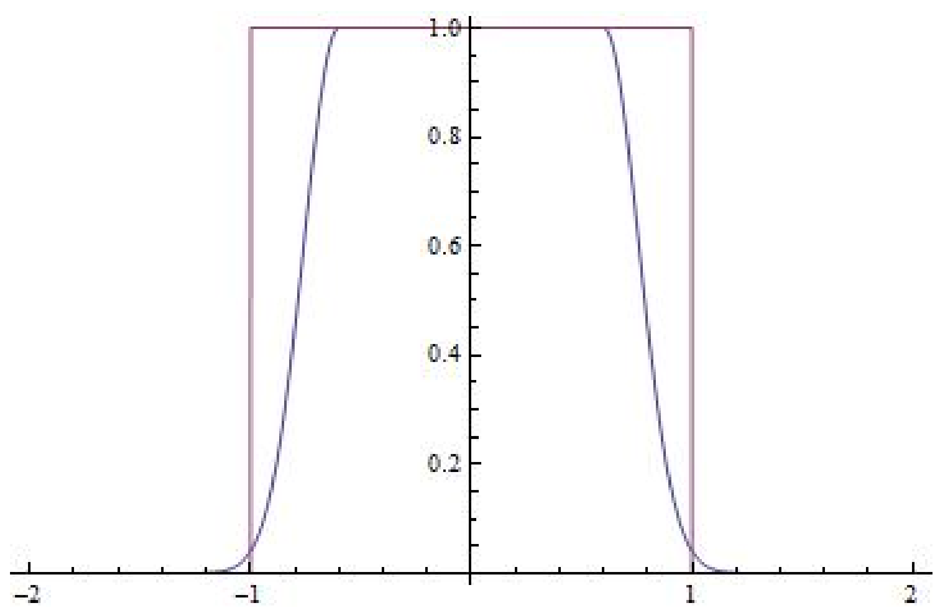


Figure 3.5: Here, the matter distribution (square curve) is plotted against distance along the central axis through a body in space. The gravity well of this body is rather strong, so elicits only a strong response (qualitatively depicted) from the scalar field, whose value relative, $\phi - \phi_0$, to its background value is depicted in the figure.

Khoury and Weltman show this analytically, and argue that this results in the $\phi$-force between the two objects being sourced only by a thin shell of material near their surfaces. This allows planetary systems, with long range $\phi$-forces between objects, to still pass solar system tests of gravity. Such a thin shell is qualitatively depicted in the transitional area in figure 3.4 where the field interpolates between its background value, and its value inside the matter distribution. Presumably, the value inside the matter distribution is such that $m_{eff}(\phi_{inside})$ is large compared its value at the external value of the scalar field.

# Chapter 4

## $f(R)$ Gravity

### 4.1 Basic Theory

The action for general relativity is the Einstein-Hilbert action,

$$S_{EH} = \int d^4x\sqrt{-g}R + S_{mat}(\psi, g_{\mu\nu}), \tag{4.1.1}$$

where $g_{\mu\nu}$ is the space-time metric, $g$ is its determinant, $R$ is its scalar curvature, and $S_{matt}$ is the matter action. The field $\psi$ represents all matter fields of the theory.

From general relativity, we know that we can vary this action with respect to $g_{\mu\nu}$ to produce the Einstein field equations. We also know that there are two equivalent approaches to doing this: the Einstein-Hilbert variational principle, or the Palatini variational principle. In the first case, the scalar curvature, $R$, is the curvature defined by the metric connection, $\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$. Varying the action with respect to the metric results in a term involving$\delta\Gamma^{\rho}_{\mu\nu} \neq 0$, resulting from varying the connection with respect to the metric. In the second case, the scalar curvature is defined by a connection,$\Gamma^{\alpha}_{\mu\nu}$, that is independent of the metric. This is considered a field in its own right, and need not have a physically interpretable metric associated with it. In this case, the scalar curvature produced using this connection is usually denoted $\mathcal{R}$, to distinguish it from the scalar curvature, $R$, produced using the metric's connection. This produces another set of field equations by variation with respect to the independent field $\Gamma^{\alpha}_{\mu\nu}$. Combining all these equations gives $R = \mathcal{R}$, and yields the Einstein field equations. When generalizing the action, we will see that these two approaches are not equivalent.

The basic action of $f(R)$ gravity is

$$S = \int d^4x\sqrt{-g}f(R) + S_{mat}(\psi, g_{\mu\nu}), \tag{4.1.2}$$

where $S_{mat}$ is the matter action, $\psi$ are the matter fields, and $g_{\mu\nu}$ is the space-time metric. The definition of the Jordan frame was made for scalar-tensor theories. It does not carry over directly to $f(R)$ gravity, but we will see that this metric is defined in an analogous frame. It will turn out that the modification to the Einstein-Hilbert action made above introduces, in many cases, a scalar degree of freedom. Other degrees of freedom could be introduced, but they turn out to introduce instabilities. We will go more into this later. The action given in (4.1.2) turns out to be the most general action we can use. This is why I chose to work with $f(R)$ gravity theories.

With the $f(R)$ gravity action, the two variational principles from GR are no longer equivalent. The Einstein-Hilbert variational principle produces a theory called "metric $f(R)$ gravity,", and the Palatini variational principle produces "Palatini $f(R)$ gravity". Palatini $f(R)$ gravity is not well-behaved, while metric $f(R)$ gravity works well for certain $f(R)$. We will now briefly go into some detail about the problems with Palatini $f(R)$ gravity, and then describe the constraints on $f(R)$ required by metric $f(R)$ gravity. In these sections, we will generally follow the review by Sotiriou, [8].

### 4.2 Problems with Palatini

There are several problems with Palatini $f(R)$ gravity, which we will now illustrate. After satisfying ourselves that Palatini $f(R)$ gravity is not viable, we will restrict further consideration to only metric $f(R)$ gravity.

#### 4.2.1 Metric algebraically related to energy-momentum

It has been shown that in Palatini $f(R)$ gravity, the Ricci curvature is related algebraically, and not differentially, to the matter distribution. We start from the field equations,

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.2.1}$$

$$\bar{\nabla}_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0, \tag{4.2.2}$$

where $\mathcal{R}$ is the Ricci curvature of the independent connection, $g_{\mu\nu}$ is the space-time metric, the bar indicates covariant differentiation with respect to the independent connection, and $(\mu\nu)$ indicates symmetrization over $\mu$ and $\nu$.

We can take the trace of (4.2.1) to get

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T. \tag{4.2.3}$$

Since $f(\mathcal{R})$ and $f'(\mathcal{R})$ are just functions of the scalar curvature, we see that there is an algebraic relationship between the curvature and the matter. As Sotiriou suggests [8], this is undesirable behavior. Consider the curvature sourced by a point particle, represented as a delta function. There is extremely large curvature, even with very little mass. Consider also a discontinuity in the matter distribution, like at the surface of a planet. This would introduce a discontinuity in the curvature.

While it is not a nice property, GR has this characteristic as well. In GR, by taking the trace of the field equations, we get $R$ is proportional to $T$. While it is nice to avoid this, it isn't a fatal flaw.

### 4.2.2 Background-dependent Newtonian Limit

One can also show that when $f''(\mathcal{R}) \neq 0$, the Newtonian limit of the metric depends on the background. The solution for the first post-Newtonian correction to the 0-0 component of the metric, for example, is [8]

$$h_{00}^{(1)}(t,x) = \frac{2G_{eff}M_{\odot}}{r} + \frac{V_0}{g\phi_0}r^2 + \Omega(T), \tag{4.2.4}$$

where $M_{\odot} = \phi_0 \int d^3x' \rho(t,x')/\phi$, $\phi$ is the gravitational scalar field, $\phi_0$ is the value of the scalar field far from any sources, $G_{eff}$ is an effective gravitational constant, $\rho$ is the energy density of the source, $V_0$ is the scalar potential evaluated at $\phi_0$, and $\Omega(T) = \log(\phi/\phi_0)$. While for some background densities this will produce a Newtonian limit with $\gamma = 1$, it is impossible for the full range of densities. Also, we again have the same problem of divergences and discontinuities in the metric in response to delta-function or discontinuous sources that we did with the scalar curvature. Consider what this means for the Newtonian force at the boundary of a planet!

The problems with Palatini $f(R)$ gravity are severe enough that we will not consider those theories any farther. We will instead turn consideration toward metric $f(R)$ gravity.

## 4.3 Properties of Metric $f(R)$ Gravity

Metric $f(R)$ gravity uses the action (4.1.2) where $R$ is the metric's Ricci curvature, and the Einstein-Hilbert variational principle is used. This produces the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right] f'(R) \quad = \quad \kappa T_{\mu\nu}. \tag{4.3.1}$$

Taking the trace of this equation yields

$$f'(R)R - 2f(R) + 3\Box f'(R) = \kappa T, \tag{4.3.2}$$

where $T$ is the trace of the matter energy-momentum tensor. With this, we can now consider what the vacuum solutions of metric $f(R)$ gravity look like. We will look for a maximally symmetric space-time, and require $R = C$ for some constant $C$. The field equations (4.3.1) reduce to

$$f'(C)R_{\mu\nu} - \frac{1}{2}f(C)g_{\mu\nu} = 0, \tag{4.3.3}$$

which gives $R_{\mu\nu} = g_{\mu\nu}C/4$, which is de Sitter or anti-de Sitter, depending on if C is positive or negative. Later, when we expand the metric in a series, we will treat our vacuum solution for the metric as approximately Minkowskian. It will be understood that we are taking appropriate time and distance scales for this to be a reasonable approximation. These will be, as in GR, much less than the Hubble time and Hubble radius.

Note that we can turn equation (4.3.1) into a much more familiar form. By using the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, we can rewrite (4.3.1) as

$$G_{\mu\nu} = \frac{\kappa}{f'}T_{\mu\nu} + \frac{1}{2f'}(f - Rf')g_{\mu\nu} + \frac{1}{f'}[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]f'.$$

Now compare this with the field equations for Jordan frame scalar-tensor gravity, equations (4.3.4), which I'll re-write here for convenience,

$$\begin{aligned} G_{\mu\nu} &= -\frac{4\pi\tilde{G}}{\varphi}T^{(m)}_{\mu\nu} + \frac{\omega(\varphi)}{\varphi^2}\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right) && (4.3.4)\\ &\quad + \frac{1}{\varphi}\left(\varphi_{;\mu\nu} - g_{\mu\nu}\Box\varphi\right) + \frac{1}{2\varphi}g_{\mu\nu}V(\varphi) \\ \Box\varphi &= \frac{1}{2\omega(\varphi)+3}\left(-4\pi\tilde{G}T^{(m)} + 2V(\varphi) + \varphi V'(\varphi) + \frac{d\omega(\varphi)}{d\varphi}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right). && (4.3.5) \end{aligned}$$

By taking $\omega(\varphi) = 0$, $V(\varphi) = f - Rf'$, and $\varphi = f'(R)$, we see that we're left with the equations for metric $f(R)$ gravity, equations (4.3.1). This is equivalent to an $\omega = 0$ Brans-Dicke theory with potential. To produce this from an action, however, we need that $f''(R) \neq 0$. We will now go into some detail on this point.

### 4.3.1 Equivalence of $f(R)$ and Scalar-Tensor Gravity

There is an equivalence, under appropriate circumstances, between metric and Palatini $f(R)$ gravity and a Brans-Dicke theory with potential. We will now show this equivalence, so we can apply results from analyses of scalar-tensor theories to metric $f(R)$ gravity.

We take the standard approach [8] and start with the metric $f(R)$ action, (4.1.2), and introduce an auxiliary field $\chi$. We produce the dynamically equivalent action,

$$S_{met} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ f(\chi) + f'(\chi)(R - \chi) \right] + S_m(g_{\mu\nu}, \psi). \tag{4.3.6}$$

If we vary with respect to $\chi$, we get

$$f''(\chi)(R - \chi) = 0. \tag{4.3.7}$$

This implies, as long as $f''(R) \neq 0$, that $R = \chi$. This reproduces the metric $f(R)$ action. This is where the equivalence breaks down when $f''(R) \neq 0$. Now, to put this in the form of a Brans-Dicke theory, we make some redefinitions. We define $\phi = f'(\chi)$, and $V(\phi) = \chi(\phi)\phi - f(\chi(\phi))$. Then, the action (4.3.6) becomes

$$S_{met} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \phi R - V(\phi) \right] + S_m(g_{\mu\nu}, \psi). \tag{4.3.8}$$

This is the action for a Brans-Dicke theory with potential with Brans-Dicke parameter $\omega_0 = 0$. This is the Jordan frame for the tensor-scalar theory equivalent to metric $f(R)$ gravity for $f''(R) \neq 0$. The field equations for this theory, after some manipulation, are

$$G_{\mu\nu} = \frac{\kappa}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right), \tag{4.3.9}$$

and

$$3\Box\phi + 2V(\phi) - \phi V'(\phi) = \kappa T. \tag{4.3.10}$$

We will see later that the condition $f''(R) \to 0$ causes the scalar mass to diverge. Another viewpoint is that it causes the coefficient of the kinetic term in (4.3.10) to be 0 for small perturbations to $R$, so the scalar perturbations have no dynamics.

It will be useful to have this in the Einstein frame as well. That can be achieved by the conformal transformation to the metric $\tilde{g}_{\mu\nu}$, defined by $\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}$, along with the scalar field redefinition,

$$\phi = f'(R) = e^{\sqrt{\frac{2\kappa}{3}}\tilde{\phi}}. \tag{4.3.11}$$

Then, the action (4.3.8) transforms to

$$S_{met} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2\kappa} - \frac{1}{2}\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - U(\phi)\right] + S_m(e^{-\sqrt{\frac{2\kappa}{3}}\tilde{\phi}}\tilde{g}_{\mu\nu}, \psi), \tag{4.3.12}$$

where

$$U(\tilde{\phi}) = \frac{Rf'(R) - f(R)}{2\kappa(f'(R))^2}, \tag{4.3.13}$$

and $R = R(\tilde{\phi})$. This form of the action will be useful for analyzing in the context of the chameleon mechanism.

### 4.3.2 Chameleon Mechanism in $f(R)$ Gravity

Sortiriou [8] shows that metric $f(R)$ gravity fails to satisfy experimental constraints on the post-Newtonian parameter $\gamma$ unless the effects of the scalar field are suppressed on solar system scales. The most interesting suppression, in the sense that there are still scalar field effects on large scales, is via the chameleon mechanism. As detailed in section 3.4, the chameleon mechanism requires the potential in the Einstein frame to satisfy the constraints (3.4.2) and (3.4.3). In the context of the previous section, the potential (4.3.13) should satisfy these same constraints. This translates to fairly complicated constraints on the form of $f(R)$.

There has been some work on theories satisfying these constraints, and they have been dubbed "vanilla $f(R)$ gravity", since they are observationally indistinguishable from GR with a cosmological constant.

Instead of taking this approach, I will ask what a theory might look like if $f''(R) \neq 0$ in certain environments. We can consider several cases of interest. First, if $f''(R) \neq 0$ in sufficiently dense environments, and $f''(R) = 0$ elsewhere. Second, where $f''(R) = 0$ in dense environments, and is non-zero elsewhere, and third, where $f''(R) = 0$ for some particular value of $R$, and is non-zero elsewhere. The PPN parameter was derived under the condition that $f''(R) \neq 0$. As long as $f''(R) = 0$ at solar system scales, we should satisfy solar system constraints on $\gamma$. If $f''(R) = 0$ for some particular $R$, we expect that we would need the chameleon mechanism to be effective for typical $R$ in our solar system.

### 4.3.3 Theoretical Constraints

We have talked a little about the theoretical constraints on scalar field theories. Of course, if $f(R)$ theories are to be viable, then they must obey these whenever there is a scalar degree of freedom. In addition, there are also certain theoretical constraints on these theories.

**Ostrogradski Instability**

The Ostrogradski instability is so fundamental that it can be illustrated in very simple classical systems, like a harmonic oscillator in classical physics [19]. The problem arises when a Lagrangian depends on higher than $1^{\text{st}}$ order time derivatives of a coordinate $q$, like $L(q, \dot{q}, \ddot{q})$, in a way that can't be eliminated by integration by parts. The property of not being able to get rid of higher derivatives through integration by parts is called non-degeneracy. The result is that the system has both positive and negative energy degrees of freedom (not just positive and negative frequency!), and so can decay into a highly excited state of negative and positive energy quanta [19].

Introducing the curvature scalar, $R$, into the Lagrangian introduces second time derivatives of a component of the metric. This escapes the Ostrogradski instability by violating the assumption of non-degeneracy. The key is that only one component of the metric, the 0-0 component, enters into the Lagrangian with second derivatives. Because we have gauge freedom, we can fix this component using the other components and fields. This avoids the instability. Note that we can't have higher derivatives of the Ricci scalar: this would introduce even higher derivatives, and we have no more freedom to remove those instabilities. We also can't have different contractions of the Riemann tensor, as this would put higher time derivatives on other components of the metric. The only allowed modifications to the action involve functions of the Ricci scalar. This is a powerful constraint on generalizations of Einstein's action.

**Ricci Stability**

We can develop the instability of curvature perturbations in metric $f(R)$ gravity following the work of Faraoni [39]. Parameterize deviations for GR as $f(R) = R + \epsilon\varphi(R)$. Here, $\epsilon$ is a small parameter. It has dimensions of mass squared so that $\varphi$ is dimensionless. We can perturb the scalar field equation of motion using

$$f'(R) = 1 + \epsilon\varphi' \tag{4.3.14}$$

$$\Box f'(R) = \epsilon\varphi''\Box R \tag{4.3.15}$$

and we find that equation (4.3.2) becomes

$$\Box R + \frac{\varphi'''}{\varphi''}\nabla^\alpha R \nabla_\alpha R + \frac{\epsilon\varphi' - 1}{3\epsilon\varphi''}R = \frac{\kappa T}{3\epsilon\varphi''} + \frac{2\varphi}{3\varphi''} \tag{4.3.16}$$

We assume that $f''(R) \neq 0$, as otherwise these equations are not well-defined. Next, we need to look at the perturbations for the scalar curvature and metric. After all, the scalar curvature perturbations are where the stability lies. Continuing, we choose to perturb the metric around Minkowski space-time, $\eta_{\mu\nu}$. We perturb the scalar curvature around the GR value, $R_{GR} = -\kappa T$. Following this, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.3.17}$$

$$R = -\kappa T + R_1 \tag{4.3.18}$$

and assume that the perturbations are small compared to the GR values, $|R_1/\kappa T| \ll 1$. Traceless matter can be treated separately, and the results will be the same. That procedure is omitted for brevity. Then, keeping only terms in (4.3.16) that are first order in $R_1$, and dropping covariant notation, we get

$$\begin{aligned} &\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa\varphi'''}{\varphi''}\dot{T}\dot{R}_1 + \frac{2\kappa\varphi'''}{\varphi''}\nabla T \cdot \nabla R_1 \\ &+\frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right) R_1 = \kappa\ddot{T} - \kappa\nabla^2 T - \frac{\kappa T\varphi' + 2\varphi}{3\varphi''} \end{aligned} \tag{4.3.19}$$

Now, the coefficient for $R_1$ on the left hand side of this equation is a mass term. Since $\epsilon$ must be very small for this theory to agree with observations, the mass term is dominated by $(3\epsilon\varphi'')^{-1}$. The theory is stable only if this effective mass, $m_{eff} \simeq (3\epsilon\varphi'')^{-1}$ is positive. This is true as long as $\varphi'' = f''(R) \geq 0$. Thus, we get the stability constraint that $f''(R) \geq 0$.

Sotiriou points out a nice way of viewing this instability, due to Faraoni [8], [40]. If we look at the effective gravitational constant in metric $f(R)$ gravity, we see $G_{eff} \equiv G/f'(R)$. If we differentiate this, we get

$$\frac{dG_{eff}}{dR} = -\frac{f''(R)G}{(f'(R))^2}. \tag{4.3.20}$$

Take the case where this is positive. Then, as R increases, so does the gravitational coupling. This drives the curvature to even higher values, and we have a runaway effect. If instead this value is negative, then an increase in curvature results in a decrease in the gravitational coupling. It takes ever higher curvatures to increase the coupling, and so the runaway effect is suppressed.

The take away from all this is that in order for the scalar curvature to remain stable, we require that $f''(R) \geq 0$.

# Chapter 5

## Series Expansion Solutions in Gravity

### 5.1 Overview

The Einstein equations in GR are very non-linear 2nd-order coupled partial differential equations for the metric, $g_{\mu\nu}$, and matter fields. Together with the Euler equations for the matter, and an equation of state, they determine the dynamics of space-time and matter. For all but very symmetric space-times and matter distributions, these equations are generally solved numerically.

Fortunately, series expansion solutions have been developed. These allow for approximate analytical solutions to the daunting problem of solving the aforementioned system of equations. This computational technology started with weak-field and slow-moving approximations, where gravitational effects from weak sources were treated, and space-time was approximately Minkowskian. The Newtonian limit of GR is developed in this scheme, and post-Newtonian corrections can be derived at higher orders. This work was done by Einstein and others in 1916 [41].

Linearizations of the Einstein equations were developed (see, e.g. [42], for a standard treatment), allowing for fast-moving treatments of weak sources. This scheme is not quite a series expansion solution, but rather an approximation scheme for solving the field equations that omit non-linear effects. The generalization of this scheme, the post-Minkowskian expansion, takes greater non-linearities into account at higher orders.

All of these schemes were integrated into each other to find analytic solutions for gravitational radiation from localized sources. The post-Newtonian expansion is used to find a particular solution near the source, and the post-Minkowskian expansion provides a general solution far from the source. The two solutions are matched to each other in the region where both expansions are valid, and so the particular radiative solution for a localized source can be found.

There has been some work done toward generalizing this method to $f(R)$ gravity. Damour has done it for a tensor-multiscalar theory[24], but the work does not apply to the $f''(R) = 0$ case. Berry and Gair have worked out linearized gravity [43] for metric $f(R)$ gravity, and the Newtonian limit. Corda and Capoziello have worked out the linearized solution for tensor-scalar gravity [44].

We will soon describe in more detail the procedures and approximations made in the various series

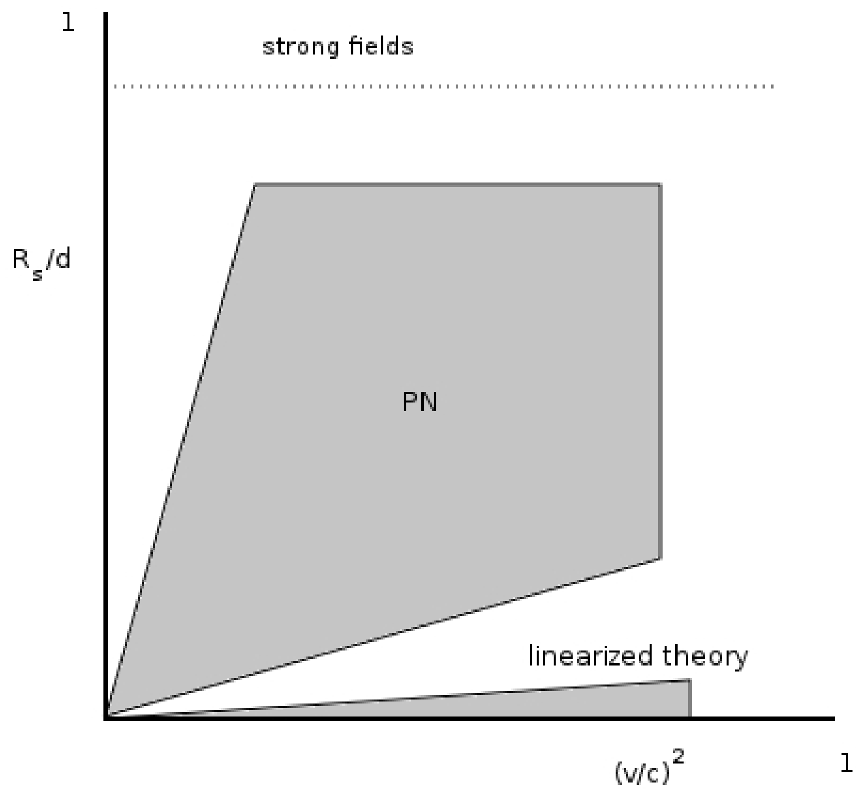


Figure 5.1: The domains of validity for the post-Newtonian (PN) series and linearized gravity in the domain comparing the system's Schwarzschild radius, $R_s$ to its size, and the system's typical velocity to the speed of light.

expansions. The goal is to lay the groundwork for generalizing these methods to metric $f(R)$ gravity for general choices of $f(R)$. First, I'll give a basic picture of how all of this work fits together. Then, we start by describing the post-Newtonian expansion in GR. We continue into linearized gravity, and describe the work that has been done in scalar-tensor theory and $f(R)$ gravity. Finally, we describe the post-Minkowskian expansion.

### 5.1.1 Series Solutions and Matching

Each series solution has its own domain of validity. Consider compact source with characteristic size $d$ and speed $v$. Then Maggiore [45] provides a convenient visualization for the domains of validity of the different series expansions near a source, as I've re-created in figure (5.1.1).

Note that the post-Minkowskian series isn't pictured. It is used in vacuum, away from sources.

As long as the energy density of gravity waves is small enough to keep curvatures low, the expansion is valid. So now we understand the types of sources that can be treated with the different expansions. What about their regions of validity? We should not expect a solution in Newtonian gravity, which treats fields as having instantaneous sources, to remain valid when relativistic wave effects (retarded sources) become significant. Maggiore points out that solutions to the wave equations can be given in terms of arbitrary left- and right-moving functions, so we can write $h_{\mu\nu} = \frac{1}{r}F_{\mu\nu}(t - r/c)$. The post-Newtonian series tries to reconstruct this function using instantaneous terms $F^{(}i)_{\mu\nu}(t)$. An expansion in small retardation effects ($r/c \ll t$) gives

$$\frac{1}{r}F_{\mu\nu}(t - r/c) = \frac{1}{r}F_{\mu\nu}(t) - \frac{1}{c}\dot{F}_{\mu\nu}(t) + \frac{r}{2c^2}\ddot{F}_{\mu\nu}(t) - \cdots \tag{5.1.1}$$

which blows up when retardation effects get large, at large $r$. Thus, the terms in a post-Newtonian series blow up when we get far from our sources. This is why we need the post-Minkowskian expansion. It is valid far from sources, and can describe gravitational field effects in the vacuum. To find these effects as they are caused by a source, we have to match the two solutions in a domain where they're both valid. It has taken some considerable mathematical technology to do this. The literature is extensive, and was finally streamlined in a way that's easy to follow, as we'll see shortly, in Maggiore's book [45] in 2008. The basic idea is to define certain regions of validity. For the post-Newtonian expansion, it's where the radial coordinate $r$ is less than the reduced wavelength of radiation emitted, $\lambda = (c/v)d$, which is much larger than the size of the system, $d$. The post-Minkowskian is valid anywhere in vacuum where there are weak fields, so for $r > d$. Thus, there is considerable overlap ($d < r < cd/v$) where the two series are valid.

In the region of overlap, the two series can be matched together. This allows the general radiative solution in the far zone to acquire information about the local source. The procedure is complicated. The two series are written as general multipole expansions in terms of a set of algorithmic moments (later to be matched to mass, current, and spin moments), and then the post-Newtonian expansion is expanded in $d/r$ (the post-Minkowskian way), while the post-Minkowskian (PM) expansion is expanded in $v/c$ (the post-Newtonian way). It works out that the $n^{th}$ term in the PM expansion expands up to order $n$ in the post-Newtonian expansion, so that we can truncate both series after a finite number of terms to do the matching. There is considerably more detail, but for the purposes of my work, this is not required. We will be satisfied to work out the PN and PM expansions for $f(R)$ gravity, and leave the matching problem for further study.

To get an idea of the literature, this procedure was pioneered by Blanchet, Damour, and Iyer in the late 1980s and early 1990s. The formalism is reviewed in [46]. General ideas are outlined in [47, 48]. There is a nice treatment of the multipole expansion in [49]. The 1PN results for gravity waves are computed in [50]. The next few breakthroughs involved bringing the results to further PN orders, and allowing for spin in the sources. The 1PN expression for matching the spin moments was found in 1991 in [51] and the 2PN expressions for mass and current moments were found in 1995 by Blanchet [52]. This result was then immediately applied to binary systems in [53]. The matching problem gets more complicated at 2.5PN order, where the moments start mixing, and was worked out in 1996 [54], and the 3.5PN order was found in 2002 [55]. The fully general matching is found in [52, 56].

### 5.2 The Post-Newtonian Expansion

The essence of the post-Newtonian expansion is to expand results in GR in small retardation effects. Newtonian gravity acts instantaneously over distance. GR modifies this by taking wave effects into account. The more relativistic a source is, the larger the corrections due to GR. The source can be relativistic in terms of its dynamics, pressure, or in terms of how it generates curvature. Thus, the small parameters that form the variables we expand with are the velocity relative to the speed of light, $v/c$, the Newtonian potential, the ratio of pressure to density, $p/\rho$, and the ratio of stress to density $T_{ij}/\rho$. See [57] for approximate values for these parameters in the solar system. If we define $\epsilon$ to be the smallest number that is larger than all of these quantities, we can talk about expanding by orders in $\epsilon$. Another perspective is that these requirements all enforce the requirement that velocities stay very small. The stress/density requirement makes sure sound wave velocities are small. The pressure/density requirement keeps fluid velocities small.

For this expansion to be valid, we require $\epsilon << 1$. We will now follow the presentation by Maggiore [45], and develop the Newtonian approximation to GR and the first post-Newtonian corrections.

We start by writing the metric, $g_{\mu\nu}$, as an expansion by order in $\epsilon$. We will denote a term of order $\epsilon^n$ by $\overset{n}{g}_{\mu\nu}$. First, note that a classical system subject to only conservative forces should be invariant under time reversal. Note also that $g_{00}$ and $g_{ij}$ are even under time reversal, while $g_{0j}$ and $v/c$ is odd. It then follows that $g_{00}$ and $g_{ij}$ can contain only even powers of $v/c$, and $g_{0j}$ can contain only odd powers. Thus, we can write

$$g_{00} = -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \dots \tag{5.2.1}$$

$$g_{0j} = \overset{3}{g}_{0\mathrm{j}} + \overset{5}{g}_{0\mathrm{j}} + \dots \tag{5.2.2}$$

$$g_{ij} = \delta_{ij} + \overset{2}{g}_{\mathrm{ij}} + \overset{4}{g}_{\mathrm{ij}} + \dots \tag{5.2.3}$$

With similar reasoning, we can expand the energy-momentum tensor:

$$T^{00} = \overset{0}{T}{}^{00} + \overset{2}{T}{}^{00} + \dots \tag{5.2.4}$$

$$T^{0i} = \overset{1}{T}{}^{0\mathrm{i}} + \overset{3}{T}{}^{0\mathrm{i}} + \dots \tag{5.2.5}$$

$$T^{ij} = \overset{2}{T}{}^{\mathrm{ij}} + \overset{4}{T}{}^{\mathrm{ij}} + \dots \tag{5.2.6}$$

Now, we consider the relative sizes of derivatives. Since our source moves with velocity $v$, our time derivatives will be a factor $\mathcal{O}(v) \sim \epsilon$ smaller than spatial derivatives. That is, $\partial_0 \sim \epsilon \partial_i$. This lets write an approximation for the d'Alembertian operator, $\Box = -(\partial_0)^2 + \partial_i \partial^i = (1 + \mathcal{O}(\epsilon^2))\partial_i \partial^i \sim \partial_i \partial^i$, which is just the Laplacian.

Now, we're ready to start building up the post-Newtonian corrections to the metric. We continue to follow Maggiore's explanation here. At Newtonian order, we have Newtonian gravity, and only the Newtonian potential is taken into account. If we look at the acceleration of a test particle in a weak field described by the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and write the geodesic equation

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, \tag{5.2.7}$$

the leading term on the RHS in $\mathcal{O}(\epsilon)$ is

$$\frac{d^2 x^i}{d\tau^2} = -c^2 \Gamma^i_{00} \simeq \frac{c^2}{2} \partial^i h_{00}. \tag{5.2.8}$$

This is just Newton's law of gravitation, identifying $\phi = -c^2 \frac{h_{00}}{2}$ the Newtonian gravitational potential. This identification can be confirmed by solving the approximation to Einstein's equations at the appropriate order. This potential is $\mathcal{O}(\epsilon^2)$, so the Newtonian approximation corresponds to

$$h_{00} = -1 + \overset{2}{h}_{00} \tag{5.2.9}$$

$$h_{0i} = 0 \tag{5.2.10}$$

$$h_{ij} = \delta_{ij}, \tag{5.2.11}$$

which is just the Minkowski solution with $\mathcal{O}(\epsilon^2)$ corrections to $h_{00}$.

To go beyond this order, we need to take the next order corrections into account. This corresponds to $g_{00}$ to $\mathcal{O}(\epsilon^4)$, $g_{0\mathrm{i}}$ to $\mathcal{O}(\epsilon^3)$, and $g_{\mathrm{ij}}$ to $\mathcal{O}(\epsilon^2)$.

Proceeding, we will impose the harmonic gauge condition, $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$. We can plug our expansions into the Einstein equations, and use this gauge condition to simplify the result. Identifying $\kappa$ as $\mathcal{O}(\epsilon^2)$, we get the vacuum Minkowski solution at $\mathcal{O}(\epsilon^0)$. At $\mathcal{O}(\epsilon^2)$ we get the Newtonian result,

$$\nabla \overset{2}{g}_{00} = \kappa \overset{0}{T}_{00} \tag{5.2.12}$$

The first post-Newtonian corrections come from matching terms at the next order. We find

$$\nabla^2 \overset{2}{g}_{\mathrm{ij}} = \kappa\delta_{ij} \overset{0}{T}_{00} \tag{5.2.13}$$

$$\nabla^2 \overset{3}{g}_{0\mathrm{j}} = 2\kappa \overset{1}{T}_{0j} \tag{5.2.14}$$

$$\nabla^2 \overset{4}{g}_{00} = \partial_0^2(\overset{2}{g}_{00}) + \overset{2}{g}_{\mathrm{ij}}\, \partial_i\partial_j \overset{2}{g}_{00} - \partial_i(\overset{2}{g}_{00})\partial_i(\overset{2}{g}_{00}) \tag{5.2.15}$$

$$-\kappa \left( \overset{2}{T}^{00} + \overset{2}{T}^{ii} - 2 \overset{2}{g}_{00} \overset{0}{T}^{00} \right). \tag{5.2.16}$$

This equations can be solved the usual way, by applying the Green's function for the Laplacian.

This work can be taken to higher orders, but as early as the second post-Newtonian correction (2PN order), divergences start to appear. This is due to the fact that the integral solutions are over fields defined over all space, and is really an issue of boundary conditions. It can be worked around using a well-defined regularization method to give finite results at all orders. The fullest generalization, to 2.5PN order was done by Blanchet, Poujade, and others[58, 59, 60], where it was also matched to an external radiative solution.

Some work has been done to generalize this approach to metric $f(R)$ gravity. In [61], they make the assumptions that $m_\phi r \ll 1$, that $f''(R_0) \neq 0$ on the background value $R_0$, and the Taylor expansion of $f(R)$ and $f'(R)$ is well defined near $R_0$. Under these assumptions, they find $\gamma = 1/2$.

This is in strong contrast with solar system tests. The way around this is to break the assumptions. This is often done by employing the Chameleon mechanism, as described earlier.

### 5.3 Linearized Gravity

We will now go into some detail on linearized gravity for GR and scalar-tensor theory. We will be able to apply the logic and results of this analysis later to the post-Minkowskian expansions for GR and metric $f(R)$ gravity.

#### 5.3.1 In General Relativity

We will generally follow the treatment of Wald [42]. We start by choosing a background around which to expand our metric. We will use Minkowski space-time, with metric $\eta_{\mu\nu}$, as this background, so we have a very simple vacuum solution to the Einstein equations around which to expand. We will allow "small" perturbations, $h_{\mu\nu}$, to this background, in the sense that $|h_{\mu\nu}| << 1$ in some global inertial coordinate system of $\eta_{\mu\nu}$. Taking this, we write the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{5.3.1}$$

We raise and lower indices with $\eta_{\mu\nu}$ in order not to hide $g_{\mu\nu}$, but use $g^{\mu\nu}$ to denote the inverse metric, and not $\eta^{\mu\alpha}\eta^{\nu\beta}g_{\alpha\beta}$. We also write $\partial$ to denote the derivative associated with $\eta_{\mu\nu}$. Thus, to linear order,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{5.3.2}$$

Now, we want to plug this metric into the Einstein equations. If we keep terms of only linear order, we will have a linear differential equation for $h_{\mu\nu}$. We start by writing the Christoffel symbols,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right). \tag{5.3.3}$$

Then, also to linear order, the Ricci tensor is

$$\begin{aligned} R^{(1)}_{\alpha\beta} &= \partial_\mu \Gamma^{(1)\mu}_{\alpha\beta} - \partial_\alpha \Gamma^{(1)\mu}_{\mu\beta} \\ &= \partial^\mu \partial_{(\beta} h_{\alpha)\mu} - \frac{1}{2}\partial^\mu\partial_\mu h_{\alpha\beta} - \frac{1}{2}\partial_\alpha\partial_\beta h, \end{aligned} \tag{5.3.4}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$, and $(\mu\nu)$ denotes symmetrization. Then, contracting this with and forming the

Einstein tensor, keeping terms to linear order, we get

$$G^{(1)}_{\alpha\beta} = \partial^\rho \partial_{(\beta} h_{\alpha)\rho} - \frac{1}{2}\partial^\rho \partial_\rho h_{\alpha\beta} - \frac{1}{2}\partial_\alpha \partial_\beta h - \frac{1}{2}\eta_{\alpha\beta}\left(\partial^\mu \partial^\nu h_{\mu\nu} - \partial^\rho \partial_\rho h\right). \tag{5.3.5}$$

We can redefine our metric variable to

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{5.3.6}$$

to simplify $G^{(1)}_{\mu\nu}$ to

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\nu} \bar{h}_{\mu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}, \tag{5.3.7}$$

and then we change coordinates to satisfy the gauge condition

$$\partial^\alpha \bar{h}_{\beta\alpha} = 0. \tag{5.3.8}$$

We then arrive at

$$\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{5.3.9}$$

which has wave solutions for $\bar{h}_{\mu\nu}$.

Peters and Matthews used this result to calculate the gravitational radiation from two point masses in a Keplerian orbit in 1963-4 [62, 63].

### 5.3.2 In Scalar-Tensor Gravity

The linearized theory has been studied in vacuum for the case of massless and small mass gravity waves by Corda and Capozziello [44]. It is useful for calculating the effects of scalar gravity waves on interferometers, but not on how sources might generate them. This work is covered in analytically in [64, 65, 66]. In [67], the interferometer response is calculated numerically for spherical collapse, and this type of series approximation isn't used.

Eardley [28] calculates the change in orbital period for a binary system in a Newtonian approximation, which is analogous with some results of Peters and Matthews, but he does not compute any radiative effects.

### 5.3.3 In f(R) Gravity

I began working out the linearized version of $f(R)$ gravity during the Spring of 2011. Before finishing, Berry and Gair printed their excellent paper, [43], on the subject.

They approximate $f(R)$ as a MacLauren series, $f(R) = a_0 + R^{(1)}$ in the linearized Ricci scalar, $R^{(1)}$, with $f'(R) = 1 + a_2 R^{(1)}$. They restrict to a flat background, and set $a_0 = 0$. Linearizing the trace equation,

$$Rf' + 3\Box f' - 2f = 0 \tag{5.3.10}$$

using the usual linearization $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gives $\Box R^{(1)} + \Upsilon^2 R^{(1)} = 0$, where $\Upsilon^2 = -1/3a_2$.

I should note that my work instead treats $f(R)$ with a Taylor expansion, so equates the term $a_0$ with $f''(R_0)$, where $R_0$ is the background value. Next, they examine the linearized Einstein equation,

$$\mathcal{G}^{(1)}_{\mu\nu} = R^{(1)} - \partial_\mu\partial_\nu(a_2 R^{(1)}) + \eta_{\mu\nu}\Box(a_2 R^{(1)}) - \frac{R^{(1)}}{2}\eta_{\mu\nu}. \tag{5.3.11}$$

In GR, we try performing a transformation $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu}$, which eliminates the mixed partial terms in (5.3.11). That transformation won't work here, so you must use an ansatz to find the new transformation that does the job. You write down all the possible tensors that can be part of $h_{\mu\nu}$, and give them arbitrary coefficients. Plug that in to equation (5.3.11), and make the coefficients of the mixed partial term 0. Doing this, you find that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \left(a_2 R^{(1)} + \frac{\bar{h}}{2}\right)\eta_{\mu\nu} \tag{5.3.12}$$

does the trick. This is how far I got in solving this problem before I discovered Berry and Gair's work. Plugging this identity into the trace equation, and then plugging the identity and the trace equation into the linearized Einstein equation, (5.3.11), yields the linearized wave equation for $f(R)$ gravity,

$$-\frac{1}{2}\Box\bar{h}_{\mu\nu} = \mathcal{G}^{(1)}_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.3.13}$$

## 5.4 The Post-Minkowskian Expansion

If we examine the space-time region far away from any sources, then $T_{\mu\nu} = 0$. If we assume further that the sources are weak enough that non-linearities in the gravitational field are small, then we are describing a situation very much like linearized gravity. This is the setting for the post-Minkowskian approach to solving the Einstein field equations. In particular, we assume the region far from sources is approximately Minkowskian, and then take more and more non-linear corrections into account. At first order, this approach reduces to linearized gravity. At higher orders, it can be looked at as a generalization to it.

In general relativity, there is an iterative procedure that takes increasingly non-linear terms into account. I will present this procedure for General Relativity, and go on to describe work that has been done toward developing a similar system in $f(R)$ gravity. It has been worked out to first order by Berry and Gair, and simultaneously by myself in unpublished work. I go on to show how I generalize the iterative procedure from GR to $f(R)$ gravity.

### 5.4.1 In General Relativity

We will follow the presentation of Maggiore in [45]. We start by re-writing the metric exactly as

$$\mathbf{h}^{\alpha\beta} \equiv (-g)^{1/2} g^{\alpha\beta} - \eta^{\alpha\beta}, \tag{5.4.1}$$

where we are making no approximations at this point. In particular, we do not assume that $g^{\alpha\beta}$ is approximately Minkowskian (though we will shortly), and we do not assume that $\mathbf{h}^{\alpha\beta}$ is small. Assuming there exists a harmonic coordinate system, we impose the gauge condition

$$\partial_\beta \mathbf{h}^{\alpha\beta} = 0. \tag{5.4.2}$$

Then, we can write the Einstein equations in the form

$$\Box \mathbf{h}^{\alpha\beta} = \frac{16\pi G}{c^4} \tau^{\alpha\beta}, \tag{5.4.3}$$

where $\Box \equiv -\partial_t^2 + \nabla^2$ is the flat space d'Alembertian. It is now more clear that the terms non-linear in $\mathbf{h}^{\alpha\beta}$ and its derivatives are in the right-hand side of this equation. The choice of gauge got rid of terms linear in $\mathbf{h}^{\alpha\beta}$ and its derivatives except for the d'Alembertian. This feature will make this form necessary for establishing an iterative procedure later. Continuing to define terms, we can write $\tau^{\alpha\beta}$ as

$$\tau^{\alpha\beta} \equiv -g T^{\alpha\beta} + \frac{c^4}{16\pi G} \Lambda^{\alpha\beta} \tag{5.4.4}$$

and $\Lambda^{\alpha\beta}$ is defined as

$$\Lambda^{\alpha\beta} = \frac{16\pi G}{c^4} (-g) t_{LL}^{\alpha\beta} + \partial_\mu \partial_\nu \chi^{\alpha\beta\mu\nu} \tag{5.4.5}$$

where

$$\frac{16\pi G}{c^4}(-g)t_{LL}^{\alpha\beta} = g_{\lambda\mu}g^{\nu\rho}\partial_\nu h^{\alpha\lambda}\partial_\rho h^{\beta\mu} + \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\partial_\rho h^{\lambda\nu}\partial_\nu h^{\rho\mu} \tag{5.4.6}$$
$$-g_{\mu\nu}(g^{\lambda\alpha}\partial_\rho h^{\beta\nu} + g^{\lambda\beta}\partial_\rho h^{\alpha\nu})\partial_\lambda h^{\rho\mu}$$
$$+\frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\partial_\lambda h^{\nu\tau}\partial_\mu h^{\rho\sigma}$$

and

$$\chi^{\alpha\beta\mu\nu} = \frac{c^4}{16\pi G}(h^{\alpha\mu}h^{\beta\nu} - h^{\mu\nu}h^{\alpha\beta}). \tag{5.4.7}$$

The key point of this is that in vacuum, we're left with the wave equation

$$\Box\mathbf{h}^{\alpha\beta} = \Lambda^{\alpha\beta}, \tag{5.4.8}$$

which involves only geometric terms. Furthermore, the right-hand side of (5.4.8) contains only terms quadratic or higher order in $\mathbf{h}^{\alpha\beta}$ and its derivatives.

Now, if we write the metric as a series expansion in powers of the gravitational constant G as

$$\mathbf{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n \mathbf{h}_n^{\alpha\beta} \tag{5.4.9}$$

in the same way as our post-Newtonian expansion uses powers of c, then we have to first order

$$\Box\mathbf{h}_1^{\alpha\beta} = 0, \tag{5.4.10}$$

and we can expand the right side of 5.4.8 to get the higher order parts. For example,

$$\Box\mathbf{h}_2^{\alpha\beta} = N^{\alpha\beta}[\mathbf{h}_1, \mathbf{h}_1] \tag{5.4.11}$$

where

$$N^{\alpha\beta}[\mathbf{h}_1, \mathbf{h}_2] = -\mathbf{h_1}^{\mu\nu}\partial_\mu\partial_\nu\mathbf{h_2}^{\alpha\beta} + \frac{1}{2}\partial^\alpha\mathbf{h_1}_{\mu\nu}\partial^\beta\mathbf{h_2}^{\mu\nu} - \frac{1}{4}\partial^\alpha\mathbf{h_1}\partial^\beta\mathbf{h_2} \tag{5.4.12}$$
$$-\partial^\alpha\mathbf{h_1}_{\mu\nu}\partial^\mu\mathbf{h_2}^{\beta\nu} - \partial^\beta\mathbf{h_1}_{\mu\nu}\partial^\mu\mathbf{h_2}^{\alpha\nu} + \partial_\nu\mathbf{h_1}^{\alpha\mu}(\partial^\nu\mathbf{h_2}_\mu^\beta + \partial_\mu\mathbf{h_2}^{\beta\nu})$$
$$\eta^{\alpha\beta}\left[-\frac{1}{4}\partial_\rho\mathbf{h_1}_{\mu\nu}\partial^\rho\mathbf{h_2}^{\mu\nu} + \frac{1}{8}\partial_\mu\mathbf{h_1}\partial^\mu\mathbf{h_2} + \frac{1}{2}\partial_\mu\mathbf{h_1}_{\nu\rho}\partial^\nu\mathbf{h_2}^{\mu\rho}\right],$$

and indices are raised and lowered by the Minkowski metric, $\eta^{\mu\nu}$, and $\mathbf{h} = \eta_{\alpha\beta}\mathbf{h}^{\alpha\beta}$ is the trace of $\mathbf{h}^{\alpha\beta}$ using the Minkowski metric.

Generally, the $n^{th}$ term equation can be written

$$\Box\mathbf{h}_n^{\alpha\beta} = \Lambda_n(\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{n-1}^{\alpha\beta}), \tag{5.4.13}$$

so that we can solve through these equations starting from the first and going forward as far as we want, and also using the gauge conditions

$$\partial_\beta \mathbf{h}_n^{\alpha\beta} = 0. \tag{5.4.14}$$

The general solution to this set of equations is detailed in [45], and I will not present it here. I should emphasize that it depends on having a hierarchy of equations that can be solved starting from the lowest order and using that solution at higher orders. This is the property we will be looking for later when asking if the same procedure can apply in metric $f(R)$ gravity.

### 5.4.2 In f(R) Gravity

In the past, work toward finding post-Minkowskian solutions for metric $f(R)$ gravity focused on an approach that is similar to linearized gravity. In [68], they expand $f(R)$ in a Taylor expansion. They proceed to treat the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and work out corrections to the Minkowski metric. This has been done for the scalar field theory equivalent, as well, lately by Corda and Capoziello [44] and earlier by others [64, 65, 66]. These analyses are useful, as in the case of GR, for finding how wave phenomena in $f(R)$ gravity compare with that in GR. In particular, we get an extra scalar wave that isn't present in GR. Unfortunately, these approaches typically examine plane waves, which are not generated (except approximately) by realistic sources. More precise solutions would be found by a systematic post-Minkowskian series solution like that described for GR in the previous section. This systematic expansion has not yet been developed for $f(R)$ gravity, so I have worked this out. When my results are reduced to the linear approximation, the results in [68] and [44] should provide a consistency check. I will now briefly summarize the work of Corda and Capoziello, since theirs frames the results in a way that is easy to interpret physically.

**Post-Minkowskian Scalar Perturbations**

We start with the action

$$S = \int d^4x\sqrt{-g}f(R) + S_m \tag{5.4.15}$$

To follow [44], we have to make the definitions

$$W(\varphi) = -\varphi R(\varphi) + f(R(\varphi)), \tag{5.4.16}$$

$$\omega(\varphi) = \epsilon\frac{f(\varphi)}{2f'(\varphi)}. \tag{5.4.17}$$

Then rewrite the action as

$$S = \int d^4x\sqrt{-g}[\varphi R - \frac{\omega(\varphi)}{\varphi}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} - W(\varphi)] + S_m \tag{5.4.18}$$

From here, varying the action with respect to $g_{\mu\nu}$ gives

$$\begin{aligned} G_{\mu\nu} &= -\frac{4\pi G}{\varphi}T^{(m)}_{\mu\nu} + \frac{\omega(\varphi)}{\varphi^2}(\varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\varphi_{;\alpha}\varphi_{;\beta}) \\ &\quad +\frac{1}{\varphi}(\varphi_{;\mu\nu} - g_{\mu\nu}\Box\varphi) + \frac{1}{2\varphi}g_{\mu\nu}W(\varphi) \end{aligned} \tag{5.4.19}$$

We want the case where $W(\varphi) = f(\varphi) - \varphi R$ and $\epsilon = 0$, which yields metric f(R) gravity. This simplifies the field equations to the familiar form,

$$G_{\mu\nu} = -\frac{4\pi G}{\varphi}T^{(m)}_{\mu\nu} + \frac{1}{\varphi}(\varphi_{;\mu\nu} - g_{\mu\nu}\Box\varphi) + \frac{1}{2\varphi}g_{\mu\nu}(f - \varphi R) \tag{5.4.20}$$

and

$$\Box\varphi = \frac{1}{3}(-4\pi GT^{(m)} + 2W(\varphi) + \varphi W'(\varphi)) \tag{5.4.21}$$

where $T^{(m)}_{\mu\nu}$ is the matter stress-energy tensor. Now we have the equations to linearize and solve for the gravity waves.

From here, Capoziello and Corda expand about a scalar field value, $\varphi_0$, that is a minimum of $W(\varphi)$. Doing this lets us assume $W \sim \delta\varphi^2$, and gets rid of some terms in the wave equation for $\varphi$ when we linearize. It leaves us with a simple mass term on the RHS of the wave equation, (5.4.21).

Defining perturbations about a Minkowski background, $h_{\mu\nu}$, in the standard way by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and about the background (Minkowski) value, $\varphi_0$, of the scalar field as $\varphi = \varphi_0 + \delta\varphi$, and keeping terms to linear order, we get (all to linear order)

$$G_{\mu\nu} = \partial_\mu \partial_\nu \xi - \eta_{\mu\nu} \Box \xi \tag{5.4.22}$$

and

$$\Box \xi = -m^2 \xi, \tag{5.4.23}$$

where

$$\xi = \frac{\delta \phi}{\varphi_0} \tag{5.4.24}$$

$$m^2 = \frac{\alpha \varphi_0}{3} \tag{5.4.25}$$

Thus, we see that the scalar degree of freedom added an extra massive scalar field, $\xi$, to the theory. Interestingly, this scalar field enters into the Einstein equations differently than a normal scalar field would. In particular, the effective energy momentum tensor for this scalar field, $T_{\mu\nu} = \partial_\mu \partial_\nu \xi + m^2 \eta_{\mu\nu} \xi$, doesn't even obey the weak energy condition. More precisely, one can show that for a non-spacelike path $u^\mu$, $u^\mu u^\nu T_{\mu\nu} \geq 0$. This is easy to show using the timelike 4-vector $u^\mu = (1, 0, 0, 0)$, and the solution to (5.4.23), $\xi = Ae^{\mathbf{i}k_\mu x^\mu}$. We get the dispersion relation $\omega^2 - k^2 = m^2$ from the wave equation, and can use this to show $u^\mu u^\nu T_{\mu\nu} = -k^2 \xi$ in this frame.

# Chapter 6

## New Phenomena Near Critical Points

Before we get into the details of series expansion solutions, and how they I will modify them for the case of $f(R)$ gravity, we should first talk about why we would want to do that. After all, tensor-scalar gravity "looks like" $f(R)$ gravity for a broad range of choices for $f(R)$. In particular, as long as $f''(R) \neq 0$ (for all $R$), the theories are equivalent. What do we gain by relaxing that restriction? I will now go into some detail on this point.

First, we note that the differential equation $f''(R) = 0$ has the solution $f(R) = R + const.$. Thus, as long as $f''(R) = 0$, metric $f(R)$ gravity is equivalent to general relativity with a cosmological constant. Now, if we imagine the case where for some $R$, $f''(R) \neq 0$, we can argue that the theory "acts like" general relativity wherever $f''(R) = 0$, and "acts like" modified gravity whenever $f''(R) \neq 0$. Thus, we can modify gravity at certain curvature scales. This makes more rigorous the procedure alluded to in the introduction, where we modify gravity at low curvatures by adding a term that is dominant at low curvatures, but suppressed at higher curvatures (e.g. $f(R) = R + \alpha/R$). This is the type of modification we might use if we want the scalar degree of freedom to behave like dark energy, since we need it to dominate at lower energy densities. We can modify gravity not just by focusing on which term in $f(R)$ is dominant, but by watching whether $f''(R)$ is small. To make the point more concrete, I will take a particular example for $f(R)$ with a well-defined point $R_0$ where the theory changes from GR to modified gravity. I will then introduce a physical scenario where we look at the theory as we move in one direction through a gas with slowly decreasing density. As this density decreases, so does the scalar curvature in the region it occupies. Thus, we can see the behavior caused by changing $f(R)$ by moving through space.

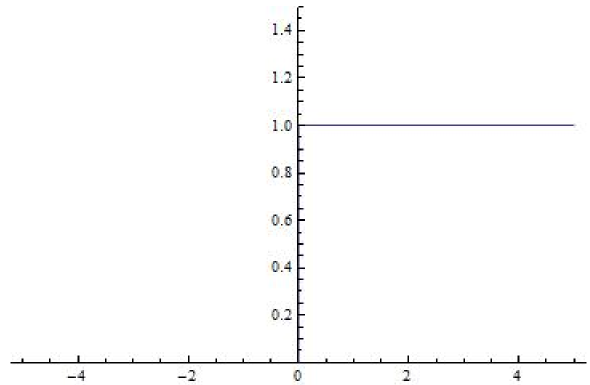


Figure 6.1: A displaced $\theta$ function, $\theta(R_0 - R)$.

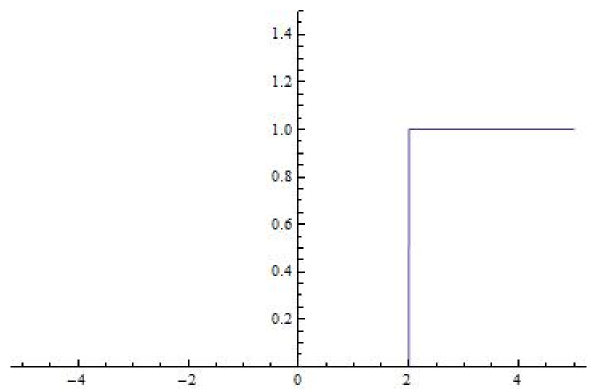


Figure 6.2: A displaced $\theta$ function, $\theta(R - R_0)$.

### 6.1 Discontinuous Case

First, recall the Heaviside $\theta$ function, $\theta(x)$,

$$\begin{aligned} \theta(x) = 1 &\quad x > 0 \\ 0 &\quad x \leq 0 \\ . & \end{aligned} \tag{6.1.1}$$

I'll review some basic manipulations, since we'll go into some depth with them. The graph of a displaced $\theta$ function, $\theta(R - R_0)$ is shown in figure 6.1. If we reverse the sign, it reflects the function across $R = R_0$, as shown in figure 6.1.

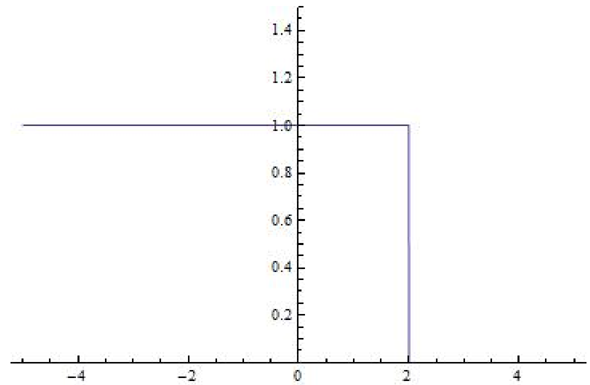


Figure 6.3: A reflected, displaced $\theta$ function, $\theta(R_0 - R)$.

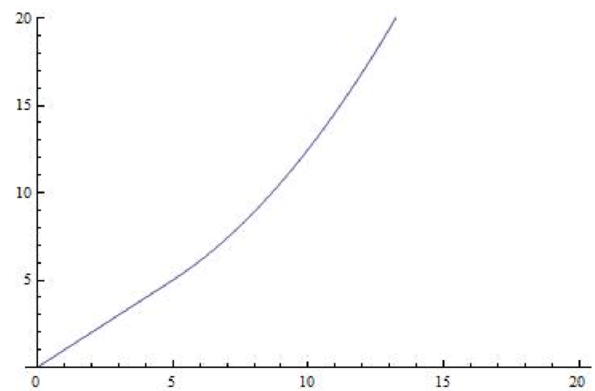


Figure 6.4: The function $f(R)$, where below $R_0$ the theory is like GR. Above $R_0$, there is another scalar degree of freedom.

For $f(R)$, we will choose the function

$$f(R) = R + \theta(R - R_0)(\mathcal{F}(R) - R), \tag{6.1.2}$$

whose graph is depicted in figure 6.1.

Examining the above function, we see $R < R_0$ implies $f(R) = R$. Also, $R \geq R_o$ implies $f(R) = \mathcal{F}(R)$. We have imposed a cutoff at $R_o$ where our theory abruptly changes from GR to modified gravity, and $f''(R) = 0$ for $R > R_o$, but is left general otherwise. Before getting into details about continuity and differentiability, lets say exactly what we mean by "acts like GR (or modified gravity)" for certain values of R. We can reverse this and put the modified gravity effects at higher curvatures, while keeping GR at lower curvatures, by flipping the sign of the $\theta$ function. The resulting $f(R)$ is shown in figure 6.1

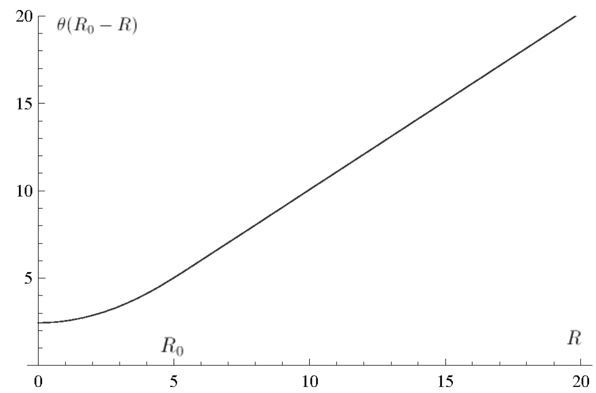


Figure 6.5: The function $f(R)$, where above $R_0$ the theory is like GR. Below $R_0$, there is another scalar degree of freedom.

Let's examine the field equations for metric $f(R)$ gravity. We will write them again here for convenience,

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box]\, f'(R) \quad = \quad \kappa T_{\mu\nu}. \tag{6.1.3}$$

When $f(R) = R + const.$, these field equations reduce to GR with cosmological constant. This is the situation implied when $f''(R) = 0$. When this constraint is met over some range of $R$, then the field equations are effectively the GR field equations in that range of curvatures.

This point becomes more clear when we examine the trace equation,

$$f'(R)R - 2f(R) + 3\Box f'(R) = \kappa T, \tag{6.1.4}$$

. and recall that when we make the equivalence with scalar-tensor theory, this is the wave equation for the field $\phi = f'(R)$. If we imagine that these scalar waves are a perturbation to the background curvature, $R_b$ (i.e. that $R = R_b + \delta R$), then we can write $\Box f'(R) \simeq f''(R_b)\Box\delta R$, and it becomes clear that the constraint $f''(R) = 0$ being true in some range of $R$ implies that there are no dynamic scalar wave perturbations to that range of background curvatures.

## 6.2 Continuous Case

Now, we can confront the issue of continuity and differentiability. Because of the Heaviside function used to switch between theories, we can get ugly artifacts. In particular, if $f(R)$ isn't differentiable, we can't include $f'(R)$ in our field equations. We must have well-defined field equations,

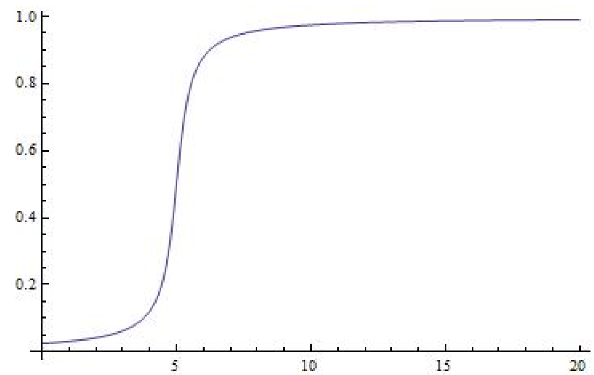


Figure 6.6: The function $g(R)$, which is a smoothed displaced, reflected $\theta$ function, whose transition is at $R_0$.

so we need $f(R)$ to be differentiable. I will now show that, as the reader might expect, this problem is not fundamental to the choice $f''(R) = 0$, but is simply an artifact of our use of the Heaviside function.

Let us use a somewhat more complicated function in place of the Heaviside function. We replace $\theta(R_0 - R)$ with $g(R)$, defined by

$$g(R) = \frac{1}{2}\left[1 - \frac{2}{\pi}\,\mathrm{atan}\left(\frac{R/R_0 - 1}{\epsilon}\right)\right] \tag{6.2.1}$$

, which is graphed in figure 6.2.

Notice that as $x = (R/R_o - 1)/\epsilon \to \infty$, $g(R)$ approaches 0. As $x \to -\infty$, $g(R)$ approaches 1. As $\epsilon \to 0$, $g(R)$ simply becomes the Heaviside function, $\theta(R_o - R)$. We can use this to get rid of the discontinuity in our choice of $f(R)$, while still suppressing modifications to gravity in certain ranges of $R$. Let's examine the asymptotics for $g(R)$.

First, we note the asymptotics for atan$(x)$. For $x \to -\infty$, we find $\mathrm{atan}(x) = -\frac{\pi}{2} + \frac{1}{x} - \frac{1}{3x^3} + \cdots$. Similarly, as $x \to \infty$ we find $\mathrm{atan}(x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \cdots$. Plugging these in for $g(R)$ for small and large $x = (R/R_o - 1)/\epsilon$, we find for large $x$,

$$g(x \gg 1) = \frac{\epsilon}{\pi(R/R_o - 1)} - \frac{\epsilon^3}{3\pi(R/R_o - 1)^3} + \cdots \tag{6.2.2}$$

and for $x \ll 1$,

$$g(x \ll -1) = 1 - \frac{\epsilon}{\pi(R/R_o - 1)} + \frac{\epsilon^3}{3\pi(R/R_o - 1)^3} - \cdots \tag{6.2.3}$$

.

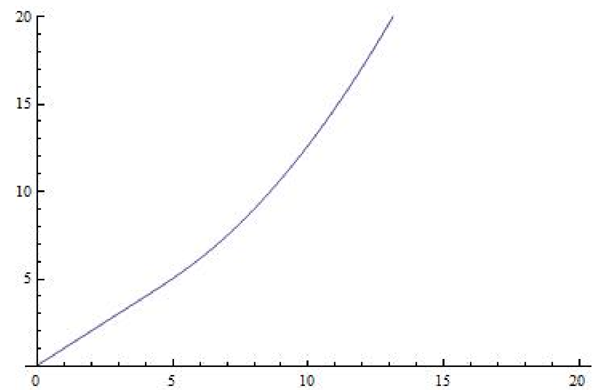


Figure 6.7: The differentiable function, $f(R)$.

Thus, as we increase the sharpness of our cutoff (smaller $\epsilon$), or get farther from it, we approach the Heaviside case.

Going farther, we can see where the corrections to the field equations come in. We can define a new, continuous $f(R)$ with this new $g(R)$, as shown in figure 6.2. First, we calculate the large and small $x$ asymptotics of $f(R)$,

$$f(x \ll -1) = \mathcal{F}(R) + (\mathcal{F}(R) - R)\left(-\frac{\epsilon}{\pi(R/R_o - 1)} + \frac{\epsilon^3}{3\pi(R/R_o - 1)^3} - \cdots\right) \tag{6.2.4}$$

and

$$f(x \gg 1) = R + (\mathcal{F}(R) - R)\left(\frac{\epsilon}{\pi(R/R_o - 1)} - \frac{\epsilon^3}{3\pi(R/R_o - 1)^3} + \cdots\right) \tag{6.2.5}$$

.

These are both corrections to the Heaviside version at the order $1/x$ for $|x| \gg 1$. Continuing, we calculate asymptotics for $f'(R)$ as

$$f'(x \ll -1) = \mathcal{F}'(R) + (\mathcal{F}'(R) - 1)\left(-\frac{\epsilon}{\pi(R/R_o - 1)} + \frac{\epsilon^3}{3\pi(R/R_o - 1)^3} - \cdots\right) \tag{6.2.6}$$

and

$$f'(x \gg 1) = 1 - \frac{\epsilon}{\pi R_o}\left(\frac{1}{1+x^2}\right)(\mathcal{F}(R) - R) + \left(\frac{1}{\pi x} - \frac{1}{3\pi x}\right)(\mathcal{F}'(R) - 1) + \cdots \tag{6.2.7}$$

We can put all of this into our expression for the field equations, and see that corrections come in at order $1/x$. For example, we can examine the trace equation. This will show us at what order the dynamics for $\delta R$ start to come in. Examining (6.2.7), we see that the wave operator term is suppressed by at least $1/x$. By increasing the sharpness of our cutoff, or by moving sufficiently far from it, we can suppress this term to arbitrarily small values. This effectively makes the mass of these perturbations too large to ever hope to excite one.

### 6.3 New Phenomena Near $f''(R) = 0$

I've found that something very interesting can happen when the scalar field theory equivalent to $f(R)$ gravity is examined near a region where $f''(R) \to 0$. I'm not aware of this having been noticed before. In this region, it has been known that the mass of the scalar can be large, since $m_\phi \propto (f''(R))^{-1}$. This has been examined before in the context of the Chameleon mechanism, where scalar effects are suppressed near massive bodies. Consider a case where the scalar curvature is only slightly perturbed from its GR value, $R = -\kappa T + R_1$, and for low energies $T$ is approximately proportional to the energy density, $T = -\rho$. Now, consider a form of $f(R)$ where $f''(R) \geq 0$ for $R \geq R_0$, but $f''(R) < \delta$ for $R < R_0$, for some small parameter $\delta$, as depicted in figure 6.3. If we examine some energy distribution $\rho$ where $R = \kappa\rho + R_1 \geq R_0$ in some sphere with radius $r < r_0$, and $R = \kappa\rho + R_1 < R_0$ for $r \geq r_0$, we see something very interesting happening. Inside the sphere, the scalar mass is some value away from 0. Outside, however, the scalar mass gets large very fast. In this region, the scalar is not able to propagate, and we get reflection at the boundary $r = r_0$. We can see this clearly if we examine the perturbation equations for the scalar degree of freedom, equation (4.3.19). I'll rewrite it here for convenience. Under the approximations

$$\begin{aligned} f'(R) &= 1 + \epsilon\varphi' \\ \Box f'(R) &= \epsilon\varphi''\Box R \\ g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ R &= -\kappa T + R_1, \end{aligned}$$

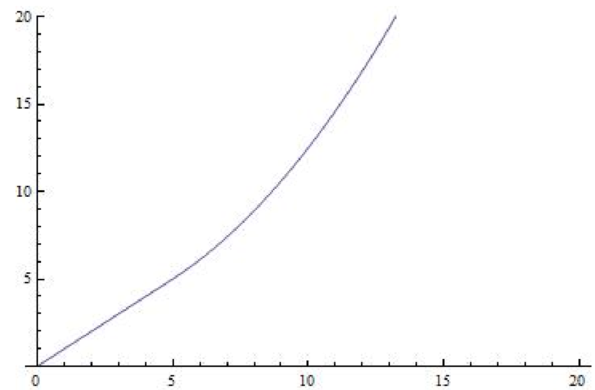


Figure 6.8: The differentiable function, $f(R)$, used to illustrate scalar wave effects. As the scalar wave moves from a region of high curvature to a region of lower curvature, it can be reflected at the boundary where $R \to R_0$.

we find that the equations of motion for the scalar degree of freedom simplify to

$$\begin{aligned} &\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa\varphi'''}{\varphi''}\dot{T}\dot{R}_1 + \frac{2\kappa\varphi'''}{\varphi''}\nabla T \cdot \nabla R_1 \\ &+\frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right) R_1 = \kappa\ddot{T} - \kappa\nabla^2 T - \frac{\kappa T\varphi' + 2\varphi}{3\varphi''}, \end{aligned}$$

where $\varphi$ and its derivatives are evaluated at $R = -\kappa T$.

Since we're really only interested in a region close to the boundary, we can choose somewhere where $T$ varies only very slightly. We will still get very abrupt reflections effects. We can also choose to work with a static background. This makes derivatives of $T$ all vanish. To simplify to the essence of the effect, let's restrict to one dimension. This reduces our equations of motion to a one-dimensional wave equation,

$$\frac{\partial^2 R_1}{\partial t^2} - \frac{\partial^2 R_1}{\partial x^2} + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right) R_1 = -\frac{\kappa T\varphi' + 2\varphi}{3\varphi''} \tag{6.3.1}$$

Here we run into a (surmountable) issue. Naively looking for small wave effects, and small changes from GR values puts us in an awkward position when we expect large values. This is the case when we're asking what the background scalar field value is in a region of uniform density. In such a region, the scalar curvature has a constant added to it due to a contribution from the scalar field. This makes sense, as we know that the scalar field will tend to a constant background value in the presence of matter. When I realized this, I modified the approach taken in [39] to allow for this extra contribution from the scalar field. I generalized equations (6.3.1) to

$$f'(R) = 1 + \epsilon\varphi' \tag{6.3.2}$$

$$f(R) = -\kappa T + R_s + R_1 + \epsilon\varphi \tag{6.3.3}$$

$$R = -\kappa T + R_s + R_1 \tag{6.3.4}$$

Here, $R_s$ represents the static contribution to the Ricci curvature due to the scalar field, and $R_1$ will end up representing wave perturbations. To find our value for $R_s$, we have to examine the equation of motion in $f(R)$ gravity representing the scalar degree of freedom. We will look at it in the geometric form,

$$3\Box f'(R) + f'(R)R - 2f(R) = \kappa T. \tag{6.3.5}$$

Plugging in my new perturbation scheme, we get

$$(1 + \epsilon\varphi')(-\kappa T + R_s + R_1) - 2(-\kappa T + R_s + R_1 + \epsilon\varphi) = \kappa T \tag{6.3.6}$$

which simplifies to

$$\varphi'\kappa T + 2\varphi = \left(\varphi' - \frac{1}{\epsilon}\right) R_s. \tag{6.3.7}$$

We can divide through by $-1/3\varphi''$ to put this in the much more useful form,

$$-\frac{\kappa T\varphi' + 2\varphi}{3\varphi''} = \left(\frac{1}{\epsilon} - f'(R)\right) \frac{1}{3\varphi''} R_s \tag{6.3.8}$$

Now it is probably apparent that this term will end up eliminating the source term for our wave equation. If we apply the new perturbation scheme to the original wave equation, we get

$$\frac{\partial^2 R_1}{\partial t^2} - \frac{\partial^2 R_1}{\partial x^2} + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right)(R_s + R_1) = -\frac{\kappa T\varphi' + 2\varphi}{3\varphi''}. \tag{6.3.9}$$

Notice that the left hand side of this equation appears in (6.3.8). If we make the substitution on the right hand side of (6.3.9), we end up cancelling off all appearance of $R_s$, and eliminating the source term on the right hand side of our wave equation. The final result is

$$\frac{\partial^2 R_1}{\partial t^2} - \frac{\partial^2 R_1}{\partial x^2} + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right) R_1 = 0, \tag{6.3.10}$$

which is easy to solve numerically.

We can consider a wave moving from the left side of the transition point, $x < x_0$, toward the right. I use $R_1 = e^{-(x-t)^2}$ for this initial wave, but this choice is arbitrary. We want a form for $\rho$ near the boundary so that when $x > x_0$, $\kappa\rho < R_0$. We also want to make sure $R_1$ is much less than the change in $R$ over the domain we're considering. We'll denote the domain boundaries by $(x_L, x_R)$, so we require $|\frac{R_1}{\kappa(\rho(x_L)-\rho(x_R))}| \ll 1$. Choosing $\rho$ to decrease by about 5% from $x = 0$ to $x = x_0$ gives $\rho = -0.1x/6 + 5.1$.

We these definitions, we're ready to choose a form for $f(R)$. I'll use the analytic one in the motivation section. To choose a good function for $\mathcal{F}(R)$ we have to be careful that the slope of $\mathcal{F}(R)$ at $R_0$ is less than 1 in order to make sure the smoothed $f(R)$ has $f''(R) \geq 0$ as we approach $R_0$. This ensures that the scalar mass remains positive, and so also makes sure that no instability arises as $R \to R_0$. A form for $\mathcal{F}(R)$ that works is

$$\mathcal{F}(R) = \frac{R_0}{1+w}\left(1 + w\left(\frac{R+\varepsilon}{R_0+\varepsilon}\right)^2\right),$$

where the term $\varepsilon = 0.04$ is used to suppress the negative second derivative of our function $g(R)$, and keep the total second derivative of $f(R)$ positive, $w = 100$ just helps make the second derivative of the quadratic term higher, to also help with this. With these definitions, we can work out the effective mass term in equation (6.3.1), $\frac{1}{3\varphi''}\left(\frac{1}{\epsilon-\varphi'}\right)$.

We choose to reverse the effective theta function, $g(R)$, so that the $\varphi$ field is turned on at higher curvatures, and off at lower curvatures. Then, our function $f(R)$ looks essentially the same as figure 6.3.

This results in an effective mass that increases toward smaller $R$ values, as depicted in figure 6.3 , so as a scalar wave moves to the right, the mass begins increasing, and we get partial reflection as we continue moving right. When we reach the boundary where the mass gets very large, the rest of the wave reflects. This is illustrated in figures 6.3 through 6.3.

Thus, we see that the waves can't pass from a high curvature region into a low curvature region. The same argument applies in reverse if we reverse the theta function in $f(R)$, and cause the scalar field to be active in low curvature regions. The result is, in the first case, if a wave is produced by a source in a high energy environment, then it cannot escape to a lower energy environment. It is reflected by the boundary where $f''(R) = 0$.

It is important to note that I have not been rigorous in effectively treating $f(R)$ with a Taylor

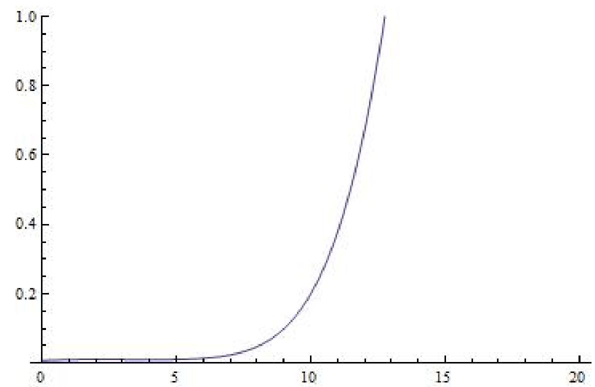


Figure 6.9: Effective mass for the scalar field. It gets large past $x_0$, and the scalar can't propagate past this point.

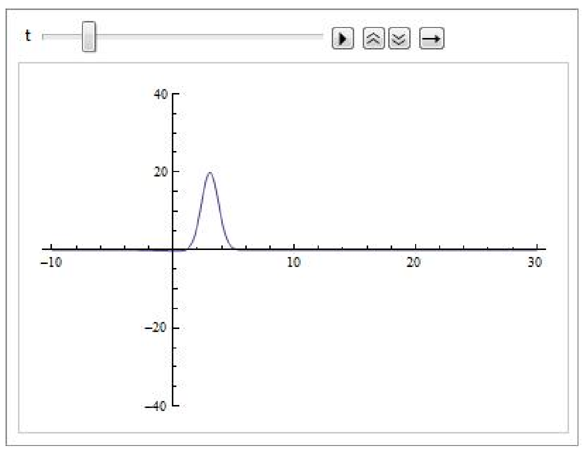


Figure 6.10: A right-moving wave heads toward the boundary, near $x = 10$.

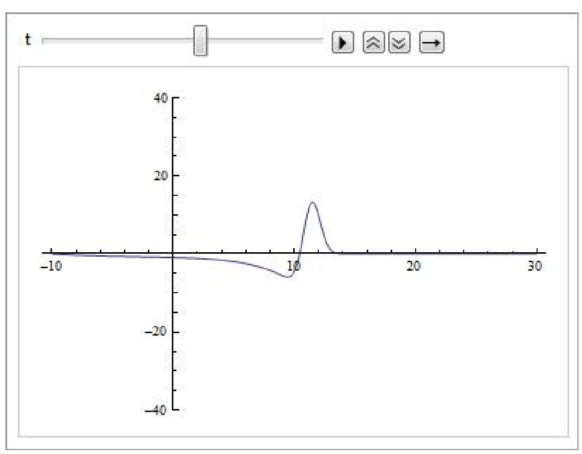


Figure 6.11: The wave gets partially deflected as it approaches the boundary.

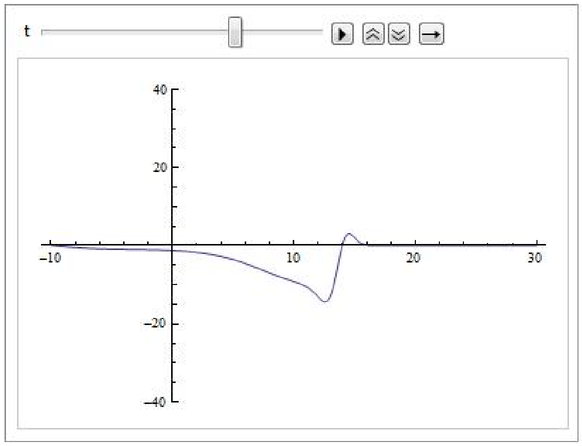


Figure 6.12: The wave is deflected, and is now deformed and moving to the left.

expansion in the above analysis. The function $f(R)$, since it uses the $arctan(x)$ function, inherits a finite radius of convergence within $|x| < 1$. Expanding around $R = R_0$, we find the radius of convergence to be $R < R_0(1+\epsilon)$, or $-\epsilon R_0 < R < R_0(2+\epsilon)$. The density $\rho$ is such that $R_0$ stays well within these limits, but the perturbation I have chosen as an illustration is too large. If I took into consideration how this perturbation changed the background $R_0$, then the Taylor expansion I have used would be invalid.

### 6.4 Conclusions

What do we achieve from the above analysis? I have shown that the effects of modifying gravity can, in principle, be suppressed at certain curvature scales. Above, I have used an example where modifications to gravity have effects at small curvature scales, but the effects are suppressed at larger curvatures. Previously, the chameleon mechanism was used to suppress modifications to gravity on solar system scales, while allowing it on larger scales. This allowed modified gravity to escape solar system constraints on post-Newtonian parameters, but had the unfortunate result that it was observationally indistinguishable from GR with cosmological constant. It was dubbed "vanilla $f(R)$ gravity" to express its uninteresting nature.

By my avoiding use of the chameleon mechanism to suppress gravity in certain ranges of scalar curvatures, we regain hope that metric $f(R)$ gravity could be used to explain the accelerated expansion of the universe and remain observationally distinguishable from GR with cosmological constant. We can imagine a scenario, for example, where the transition scale $R_o$ is somewhere in between the background curvature of the interstellar medium and that of the intergalactic medium. Then, we can have scalar gravitational waves propagating in the intergalactic medium, and also explain the accelerated expansion of the universe without use of a cosmological constant.

One can imagine another scenario where gravity is modified in very high curvature regions, like in the very high density environments of neutron stars. This is distinct from the chameleon mechanism, which required the theory to be like GR in all high-density environments. In this case, we could see scalar effects in the higher effective gravitational coupling of these objects (and, for example, a modification to the Chandrasekhar mass), as well as a difference in their orbits due to the production of scalar gravitational waves. Now, we will go into detail in developing the equations necessary to calculate the radiation emitted from a system in metric $f(R)$ gravity.

# Chapter 7

## Post-Newtonian Equations

All treatments of metric $f(R)$ gravity either make use of the equivalence to a scalar field theory (possible when $f''(R) \neq 0$), explicitly restrict that $f''(R) \neq 0$, or put several restrictions (beyond those required for the theory to be stable) on the form of $f(R)$. Interesting classes of theories have thus been excluded. Allowing $f''(R) = 0$ in some interval of $R$ gives a theory that behaves like GR where $f''(R) = 0$, but like a massive brans-dicke theory where $f''(R) \neq 0$. Below, I develop the first order post-Newtonian corrections for such a theory for general $R$. The expansion goes in powers of $1/c$, where the number above a term indicates how many powers of $1/c$ it contains, and we use the notation that $f \sim \mathcal{O}(n)$ to indicate that $f$ is of order $(1/c)^n$.

### 7.1 Set-Up

The field equations in metric $f(R)$ gravity are

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)\, g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box]f'(R) = \kappa\, T_{\mu\nu} \tag{7.1.1}$$

$$\tag{7.1.2}$$

The trace of this is useful,

$$3\Box f' + f'R - 2f = \kappa T \tag{7.1.3}$$

We can rewrite these, adding and subtracting $g_{\mu\nu}R/2$ to both sides, and dividing through by $f'(R)$, as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa}{f'}\left(T_{\mu\nu} + T_{\mu\nu}^{(eff)}\right) \tag{7.1.4}$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor, and

$$T_{\mu\nu}^{(eff)} = \frac{1}{\kappa}\left[\frac{f - Rf'}{R}g_{\mu\nu} + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'\right] \tag{7.1.5}$$

In this form, we can follow the approach from (Weinberg, 1972) on the left-hand side of (7.1.4). Results of that approach use here are in the appendix.

We can expand the metric in different order correction terms as

$$g_{00} = -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \ldots \tag{7.1.6}$$

$$g_{0\mathrm{i}} = \overset{3}{g}_{0\mathrm{i}} + \overset{5}{g}_{0\mathrm{i}} + \ldots \tag{7.1.7}$$

$$g_{\mathrm{ij}} = \delta_{ij} + \overset{2}{g}_{\mathrm{ij}} + \overset{4}{g}_{\mathrm{ij}} + \ldots, \tag{7.1.8}$$

and given the orders for the Ricci tensor (see Appendix),

$$R_{00} = \overset{2}{R}_{00} + \overset{4}{R}_{00} + \ldots \tag{7.1.9}$$

$$R_{0i} = \overset{3}{R}_{0i} + \overset{5}{R}_{0i} + \ldots \tag{7.1.10}$$

$$R_{ij} = \overset{2}{R}_{ij} + \overset{4}{R}_{ij} + \ldots \tag{7.1.11}$$

$$\tag{7.1.12}$$

we can compute the order of the Ricci Scalar and the terms entering at each order:

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} && (7.1.13)\\
&= (-1 + \overset{2}{g}^{00} + \overset{4}{g}^{00} + \ldots)(\overset{2}{R}_{00} + \overset{4}{R}_{00} + \ldots) && (7.1.14)\\
&\quad +(\delta_{ij} + \overset{2}{g}^{\mathrm{ij}} + \overset{4}{g}^{\mathrm{ij}} + \ldots)(\overset{2}{R}_{ij} + \overset{4}{R}_{ij} + \ldots) && (7.1.15)\\
&\quad +2(\overset{3}{g}^{0\mathrm{i}} + \overset{5}{g}^{0\mathrm{i}} + \ldots)(\overset{3}{R}_{0i} + \overset{5}{R}_{0i} + \ldots) && (7.1.16)\\
&= (-\overset{2}{R}_{00} + \delta_{ij}\overset{2}{R}_{ij}) + (-\overset{4}{R}_{00} + \overset{2}{g}^{00}\overset{2}{R}_{00} + \delta_{ij}\overset{4}{R}_{ij} + \overset{2}{g}^{\mathrm{ij}}\overset{2}{R}_{ij}) + \ldots && (7.1.17)\\
&= \overset{2}{R} + \overset{4}{R} + \ldots && (7.1.18)
\end{aligned}$$

Since $f(R)$ replaces $R$ in the action, it should start at the same order as $R$. Since $R$ is $\mathcal{O}(2)$, $f(R)$ is also $\mathcal{O}(2)$ and $f'(R) = df/dR$ is $\mathcal{O}(0)$.

Now, starting with the trace equation, we can find the equations for $f(R)$ and $f'(R)$. Writing the trace equation,

$$3\Box f\text{'} + f\text{'}\, R - 2f = \kappa\, T, \tag{7.1.19}$$

We need the intermediate results that, in the harmonic gauge where $\partial_\mu(\sqrt{-g}g^{\mu\nu})$,

$$\Box f' = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu f') \tag{7.1.20}$$
$$= g^{\mu\nu}\partial_\mu\partial_\nu f' \tag{7.1.21}$$

### 7.2 Solution

Now, to proceed, we can use $T = \overset{0}{T} + \overset{2}{T} + \ldots$ to write

$$3\left[(\overset{2}{\partial_0})^2 - \nabla + \ldots\right](\overset{0}{f'} + \overset{2}{f'} + \ldots) + (\overset{0}{f'} + \overset{2}{f'} + \ldots)(\overset{2}{R} + \overset{4}{R} + \ldots) - 2(\overset{2}{f} + \overset{4}{f} + \ldots) = \kappa(\overset{0}{T} + \overset{2}{T} + \ldots) \tag{7.2.1}$$

Then, matching the terms order by order we find

$$-3\nabla^2 \overset{0}{f'} = 0 \tag{7.2.2}$$
$$3(-(\overset{2}{\partial_0})^2 \overset{0}{f'} + \nabla^2 \overset{2}{f'}) + \overset{0}{f'}\overset{2}{R} - 2\overset{2}{f} = \kappa \overset{0}{T} \tag{7.2.3}$$

We can solve (7.2.2) with the homogeneous solution for the Laplace equation subject to boundary conditions. Our boundary conditions can come from cosmological boundary conditions on $R$ through the explicit form of $f'(R)$ at $\mathcal{O}(0)$, or from matching with the post-Minkowskian solution in the induction zone.

To find the next order for $f'$, we need to plug the first order into (7.2.3) and also find and plug in $\overset{2}{f}$. To find $\overset{2}{f}$, we can look at a Taylor expansion for $f(R)$ around some value $R_0$.

$$f(R) = f(R_0) + f'(R_0)(R - R_0) + \ldots \tag{7.2.4}$$

and truncate this at $\mathcal{O}(2)$.

$$\overset{2}{f}(R) = f(R_0) + f'(R_0)(\overset{2}{R} - R_0) + \ldots \tag{7.2.5}$$

Then, for $R$ close enough to $R_0$ such that the difference is $\mathcal{O}(1)$, we can truncate the series at the first two terms. If the difference is not $\mathcal{O}(1)$, we can always truncate after the $n^{th}$ term such that $(R^2 - R_0)^n \sim \mathcal{O}(1)$. Now, we if we can find $R^2$, we can finish working out the Poisson equation for $f'^2$ and bring our solution to the next iterative order. To do this, we need to work out the field equations for the metric. From the metric to order $\mathcal{O}(2)$ we can find $R^2$.

We continue from (7.1.4) by establishing the order of the terms in $T^{(eff)}_{\mu\nu}$. We will also have to consider the coefficient, $1/f$', when we bring everything back together into the field equations.

First, examining $T^{(eff)}_{\mu\nu}$ from (7.1.5),

$$T^{(eff)}_{\mu\nu} = \frac{1}{\kappa}\left(\frac{(\overset{2}{f} + \overset{4}{f} + \ldots) - (\overset{2}{R} + \overset{4}{R} + \ldots)(\overset{0}{f'} + \overset{2}{f'} + \ldots)}{2}\, g_{\mu\nu} \; + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\,\Box)(\overset{0}{f'} + \overset{2}{f'} + \ldots)\right), \tag{7.2.6}$$

we can break this up into components as

$$\overset{-2}{T}{}^{(eff)}_{00} = \frac{1}{\kappa}\nabla^2 \overset{0}{f'} = 0 \tag{7.2.7}$$

$$\overset{-1}{T}{}^{(eff)}_{0i} = \frac{1}{\kappa}\,\partial_0\,\partial_i \overset{0}{f'} = 0 \tag{7.2.8}$$

$$\overset{-2}{T}{}^{(eff)}_{ij} = \frac{1}{\kappa}\partial_i \partial_j \overset{0}{f'} = 0 \tag{7.2.9}$$

$$\overset{0}{T}{}^{(eff)}_{00} = \frac{1}{\kappa}\left(-\left(\frac{\overset{2}{f} - \overset{2}{R}\overset{0}{f'}}{2}\right) + \nabla^2 \overset{2}{f'}\right) \tag{7.2.10}$$

$$\overset{0}{T}{}^{(eff)}_{0i} = 0 \tag{7.2.11}$$

$$\overset{0}{T}{}^{(eff)}_{ij} = \frac{1}{\kappa}\left(\left(\frac{\overset{2}{f} - \overset{2}{R}\overset{0}{f'}}{2}\right)\delta_{ij} + \partial_i\partial_j \overset{2}{f'} - \delta_{ij}\left(-(\overset{2}{\partial}_0)^2\, \overset{0}{f'}\right)\right) \tag{7.2.12}$$

$$\overset{2}{T}{}^{(eff)}_{00} = \frac{1}{\kappa}\left[-\left(\frac{\overset{4}{f} - \overset{2}{R}\overset{2}{f'} - \overset{4}{R}\overset{0}{f'}}{2}\right) + \nabla^2 \overset{4}{f'}\right] \tag{7.2.13}$$

$$\overset{1}{T}{}^{(eff)}_{0i} = \partial_0 \partial_i \overset{2}{f'} \tag{7.2.14}$$

This first term vanishes. The second term we get from solving (7.2.2). It turns out we only need the first one to get the metric to $\mathcal{O}(2)$.

Now, we can use the results of (Weinberg, 1972) for the left-hand side of (7.1.4) along with the above results for $T^{(eff)}_{\mu\nu}$ to start iterating a post-Newtonian solution. To $\mathcal{O}(0)$, the $1/f$' that modifies the right hand side is just $1/\overset{0}{f}$'. We start with

$$\nabla^2 \overset{2}{g}_{00} = -\frac{\kappa}{\overset{0}{f}'}(\overset{0}{T}_{00} + \overset{0}{T}{}^{(eff)}_{00}) \tag{7.2.15}$$

$$\begin{aligned}\nabla^2 \overset{4}{g}_{00} &= \Bigg( (\overset{2}{\partial}_0)^2 \overset{2}{g}_{00} + \overset{2}{g}_{ij}\, \partial_i \partial_j \overset{2}{g}_{00} - \left(\partial_i \overset{2}{g}_{00}\, \partial_j \overset{2}{g}_{00}\right) \\ &\quad - \frac{\kappa}{\overset{0}{f}'}\left( \overset{2}{T}_{00} + \overset{2}{T}{}^{(eff)}_{00} - 2\overset{2}{g}_{00}(\overset{2}{T}_{00} + \overset{2}{T}{}^{(eff)}_{00} + \overset{2}{T}_{ii} + \overset{2}{T}{}^{(eff)}_{ii})\right)\Bigg)\end{aligned} \tag{7.2.16}$$

$$\nabla^2 \overset{3}{g}_{0i} = 2\frac{\kappa}{\overset{0}{f}'}\left(\overset{1}{T}_{0i} + \overset{1}{T}{}^{(eff)}_{0i}\right) \tag{7.2.17}$$

$$\nabla^2 \overset{2}{g}_{ij} = -\frac{\kappa}{\overset{0}{f}'}(\overset{0}{T}_{00} + \overset{0}{T}{}^{(eff)}_{00}) \tag{7.2.18}$$

and $\overset{2}{g}_{0i} = 0$. Notice that once we find the $\mathcal{O}(2)$ corrections, we have to find $\overset{2}{R}$ to get $\overset{2}{f}$ and $\overset{2}{f}'$ before we can find the $\mathcal{O}(4)$ corrections.

To solve the equation for ${g_{00}}^2$, we need to re-arrange it. We'll do this by plugging in the trace equation and using

$$\overset{2}{R} = -\overset{2}{R}_{00} + \delta_{ij}\overset{2}{R}_{ij} \tag{7.2.19}$$

and

$$\overset{2}{R}_{00} = \frac{1}{2}\nabla^2 \overset{2}{g}_{00} \tag{7.2.20}$$

$$\overset{2}{R}_{ij} = \frac{1}{2}\nabla^2 \overset{2}{g}_{ij} \tag{7.2.21}$$

so to $\mathcal{O}(2)$, after dividing the $\nabla^2 \overset{2}{g}_{00}$ and $\nabla^2 \overset{2}{g}_{ij}$ equations to relate them, $\overset{2}{R} = \nabla^2 \overset{2}{g}_{00}$

After doing this, we find

$$\frac{1}{2}\nabla^2 \overset{2}{g}_{00} + \frac{1}{3}\overset{2}{R} - \frac{1}{6}\frac{\overset{2}{f}}{\overset{0}{f}'} = -\frac{\kappa}{\overset{0}{f}'}\left[(\overset{0}{T}_{00} + \frac{1}{3}\overset{0}{T}) - \frac{({\partial_0}^2)^2 f^0}{\kappa}\right] \tag{7.2.22}$$

$$\frac{1}{2}\nabla^2 \overset{2}{g}_{ij} + (\frac{1}{3}\overset{2}{R} - \frac{1}{6}\frac{\overset{2}{f}}{\overset{0}{f}'})\delta_{ij} = -\delta_{ij}\frac{\kappa}{\overset{0}{f}'}\left[(\overset{0}{T}_{00} + \frac{1}{3}\overset{0}{T}) - \frac{({\partial_0}^2)^2 f^0}{\kappa}\right] \tag{7.2.23}$$

Notice that plugging in $f(R) = R$ recovers the correct 1PN limit of GR.

# Chapter 8

## Post-Minkowskian Equations

The standard approach to finding the gravitational radiation in the far zone emitted by some localized source requires two solutions to two sets of equations. The first set are the post-Newtonian equations. They are solved in the near-zone and induction-zone. The second set is a general solution in the radiation and induction zones. In GR, this is done as a post-Minkowskian expansion. We will generalize that approach here.

### 8.1 Set-Up

We work to second order, since our goal is to show that standard methods can be generalized metric $f(R)$ gravity, and high precision results along with the regularization methods required to achieve them are beyond the scope of this work. Similar work has been done by Berry and Gair [43], where they linearize metric $f(R)$ gravity. After a change of variables to simplify the first set of post-Minkowskian equations we find, our results will be equivalent at the linear level. This will serve as a consistency check.

Our problem, then, is to solve the relaxed Einstein Equations in a region with no source ( $T_{\mu\nu} = 0$), far away from a localized system ( $r >> R_s$ ). These equations are the geometric analog of (5.4.20). We'll choose to expand around Minkowski space, since it is the simplest choice of vacuum solution. That is, we want to solve

$$G_{\mu\nu} = \frac{1}{f'}\left(g_{\mu\nu}\left[\frac{f-Rf'}{2}\right] + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f'\right) \tag{8.1.1}$$

$$3\,\Box f' = 2f - f'R \tag{8.1.2}$$

$$\partial_\mu\, h^{\mu\nu} = 0 \tag{8.1.3}$$

with a series expansion in powers of $h_{ab}$, where $G_{\mu\nu}$ is the Jordan frame Einstein tensor, $g_{\mu\nu}$ is the Jordan frame metric, $R$ is it's Ricci scalar, and $h_{ab}$ is defined by

$$g_{ab} = \sqrt{-g}(\eta_{ab} - h_{ab} + h_{ac}h_{db}\eta^{cd}) \tag{8.1.4}$$

and

$$g^{ab} = \frac{1}{\sqrt{-g}}(\eta^{ab} + h^{ab}) \tag{8.1.5}$$

up to $O(h^2)$. We will denote by $g$ the determinant of the metric, and by $h$ the trace of $h_{\mu\nu}$.

### 8.1.1 Identities

To start, we can work out, to the same order,[1]

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{ab}h^{ab} \tag{8.1.6}$$

and, expanding the determinant, then the square root and fraction with Taylor series,

$$\frac{1}{\sqrt{-g}} = 1 - \frac{1}{2}h + \frac{1}{8}h^2 + \frac{1}{4}h_{ab}h^{ab}. \tag{8.1.7}$$

## 8.2 Solution

### 8.2.1 Einstein Tensor

We now work out the Einstein tensor for the metric $g_{\mu\nu}$, keep terms up to quadratic order in $h_{ab}$, and group terms by order in $h_{ab}$ (see Appendix for details). We find

$$\overset{1}{G}_{\sigma\nu} = \frac{1}{2}\Box h_{\sigma\nu} \tag{8.2.1}$$

$$\begin{aligned}\overset{2}{G}_{\sigma\nu} = & -h^{ab}\partial_a\partial_b h_{\sigma\nu} + \frac{1}{2}\partial_\sigma h_{ab}\partial_\nu h^{ab} - \frac{1}{4}\partial_\sigma h\partial_\nu h - \partial_\sigma h_{ab}\partial^a h^b_\nu \\ & -\partial^\nu h_{ab}\partial^a h^{\sigma b} + \partial_a h^b_\sigma\partial^a h_{\nu b} + \partial_a h^b_\sigma\partial_b h^a_\nu \\ & +\eta_{\sigma\nu}(-\frac{1}{4}\partial_\rho h_{\mu\lambda}\partial^\rho h^{\mu\lambda} + \frac{1}{8}\partial_a h\partial^a h + \frac{1}{2}\partial_a h_{b\rho}\partial^b h^{a\rho}).\end{aligned} \tag{8.2.2}$$

### 8.2.2 Effective Source

Now, we need to work out the right-hand side of Einstein's Equations in metric $f(R)$ theory, equation (8.1.1). To start, we can replace $\Box f'$ in (8.1.1) using (8.1.2). After simplifying, this gives

$$G_{\mu\nu} = \frac{1}{f'}\left(-\frac{g_{\mu\nu}}{6}(f + Rf') + \nabla_\mu\nabla_\nu f'\right) \tag{8.2.3}$$

[1] The trace of $h_{\mu\nu}$ is just the sum of components of $h_{\mu\nu}$, which will all be taken to be small. This makes $h$ small as well.

We'll start from here, and then expand each piece in different powers of $h_{ab}$. In what follows, we require either the background scalar curvature, $R_0$, in the post-Minkowskian expansion region (the far zone and induction zone) be small ($O(h)$) or zero. We also consider the case where the background can be approximated by the Minkowski background. This will be necessary to keep our procedure of splitting our field equations by power in $h_{ab}$ consistent, and keeping the equations from getting too complicated. We also require $f(R_0) \sim R_0 \sim O(h)$, again to make our iterative procedure consistent.

Into (8.2.3) we put the expansions

$$f = \overset{1}{f} + \overset{2}{f} + \dots \tag{8.2.4}$$

$$f' = \overset{0}{f'} + \overset{1}{f'} + \overset{2}{f'} + \dots \tag{8.2.5}$$

$$R = \overset{1}{R} + \overset{2}{R} + \dots \tag{8.2.6}$$

in powers of $h_{\mu\nu}$. Separating the equations by order, we get

$$\overset{1}{G}_{\mu\nu} = \frac{-\eta_{\mu\nu}}{6\overset{0}{f'}}(\overset{1}{f} + \overset{1}{R}\overset{0}{f'}) + \frac{1}{\overset{0}{f'}}(\partial_{\mu\nu}\overset{1}{f'} - \frac{\overset{1}{f'}}{\overset{0}{f'}}\partial_{\mu\nu}\overset{0}{f'}) \tag{8.2.7}$$

$$\begin{aligned}\overset{2}{G}_{\mu\nu} = {} & \frac{h_{\mu\nu}}{6\overset{0}{f'}}(-\overset{1}{f} - \overset{1}{R}\overset{0}{f'}) \\ & + \frac{\eta_{\mu\nu}}{6\overset{0}{f'}}(-\overset{2}{f} - \overset{1}{R}\overset{1}{f'} - \overset{2}{R}\overset{0}{f'} + \frac{\overset{1}{f'}}{\overset{0}{f'}}\overset{1}{f} + \frac{\overset{1}{f'}}{\overset{0}{f'}}\overset{1}{R}\overset{0}{f'}) \\ & + \frac{1}{\overset{0}{f'}}(\partial_{\mu\nu}\overset{2}{f'} - \frac{\overset{1}{f'}}{\overset{0}{f'}}\partial_{\mu\nu}\overset{1}{f'} + \frac{\overset{1}{f'}}{\overset{0}{f'}}\overset{1}{f'}\overset{0}{f'}\partial_{\mu\nu}\overset{0}{f'} - \frac{\overset{2}{f'}}{\overset{0}{f'}}\partial_{\mu\nu}\overset{0}{f'})\end{aligned} \tag{8.2.8}$$

(8.2.9)

### 8.2.3 Trace Equation

Now, to finish out our set of equations, we also need the wave equation for $f'$. We start with

$$3\Box f' = 2f - f' R \tag{8.2.10}$$

and again make substitutions (8.2.5 - 8.2.6). We find

$$3\Box \overset{1}{f'} = 2\overset{1}{f} - \overset{1}{R} - \overset{0}{f'}\,\overset{1}{R} + \overset{1}{f'}\,R_0 \tag{8.2.11}$$

$$3\Box \overset{2}{f'} = 2\overset{2}{f} - \overset{2}{R} + \overset{0}{f'}\,\overset{2}{R} + \overset{1}{f'}\,\overset{1}{R} \tag{8.2.12}$$

Our goal is to find a solution for $\overset{1}{f'}$ that we can plug into (8.2.7 - 8.2.8) to get our wave equation for $h_{\mu\nu}$ with a source that we can integrate. We plug in the expansions

$$\overset{0}{f'} = f'(R_0) \tag{8.2.13}$$

$$\overset{1}{f'} = f''(R_0)\,\overset{1}{R} \tag{8.2.14}$$

$$\overset{2}{f'} = f''(R_0)\,\overset{2}{R} + \frac{f'''}{2}(\overset{1}{R})^2 \tag{8.2.15}$$

to get wave equations for $\overset{1}{R}$. These are of the form

$$(\Box + m^2)A = S \tag{8.2.16}$$

which can be solved with Green's functions. We can then use (8.2.14) to write out $\overset{1}{f'}$ and put it into (8.2.7) to finish writing our system of equations for $h_{\mu\nu}$.

### 8.2.4 Field Re-definition

While this resulting set of equations is solvable (in principle), we want to drastically simplify the system in analog with the usual post-Minkowskian approach to GR. In particular, we want something of the form

$$\overset{1}{G}_{\mu\nu} = \Box\, h_{\mu\nu} = RHS - \overset{2}{G}_{\mu\nu} - \overset{3}{G}_{\mu\nu} - \ldots, \tag{8.2.17}$$

where RHS is defined in (8.2.7 - 8.2.8).

To get this, we'll need to change our definition of the metric. In particular, we'll want to add a linear piece that will cancel the linear terms arising in (8.2.7). To do this, we take a lesson from Berry and Gair's work. They find that adding a term to the definition of $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \eta_{\mu\nu}(f_0''\,\overset{1}{R} + \frac{\bar{h}}{2}), \tag{8.2.18}$$

they cancel off the linear terms involving derivatives of $\overset{1}{R}$ (note: they define $h_{ab}$ with opposite sign).

To make this new substitution, we have to relate this back to our metric, and figure out the term to add. In other words, we want to add a new term to the definition of $h^{ab}$, (8.1.5), such that when we expand it to second order in $h^{ab}$ we end up with eqn (8.2.18) to first order. We find that adding a term $-\eta^{ab} f_0'' \overset{1}{R}$ does the job. We continue by noticing that this term is just the 1st order part of a Taylor series expansion of $\eta^{ab}(1-f')$. Putting this into our definition of $h^{ab}$, we get

$$g^{\mu\nu} = \frac{1}{\sqrt{-g}}(\eta^{\mu\nu} + h^{\mu\nu} + \eta^{\mu\nu}(1-\ f')), \tag{8.2.19}$$

Taylor expanding, and grouping terms by order, we define

$$Q = 1 - f' = -f_0'' \overset{1}{R} - \left( f_0'' \overset{2}{R} + \frac{f_0'''}{2} (\overset{1}{R})^2 \right) + O(3), \tag{8.2.20}$$

and then make the substitution $h^{ab} \to h^{ab} + \eta^{ab} Q$ into our previous results. We find, to second order,

$$\begin{aligned} g_{ab} \quad = \quad & \eta_{ab} - h_{ab} + \frac{1}{2}\, \eta_{ab} h^{\beta}{}_{\beta} - \overset{1}{R}\, \eta_{ab} f_0'' \\ & - \overset{2}{R}\, \eta_{ab} f_0'' - \frac{1}{2}\, (\overset{1}{R})^2 \eta_{ab} f_0'' + \eta^{cd} h_{ac} h_{db} \\ & - \frac{1}{2}\, h_{ab} h^{\beta}{}_{\beta} + \frac{1}{4}\, \eta_{ab} h_{\beta}{}^{\beta} h_{\nu}{}^{\nu} - \overset{1}{R}\, \eta_{ab} f_0'' h_{\nu}{}^{\nu} \\ & - \frac{1}{4}\, \eta_{ab} h_{\beta\nu} h^{\beta\nu} \end{aligned} \tag{8.2.21}$$

and

$$\begin{aligned} g^{ab} \quad = \quad & \eta^{ab} + h^{ab} - \frac{1}{2}\, \eta^{ab} h^{\beta}{}_{\beta} + \overset{1}{R}\, \eta^{ab} f_0'' + \overset{2}{R}\, \eta^{ab} f_0'' \\ & + \frac{1}{2}\, \overset{1}{R}\, \overset{1}{R}\, \eta^{ab} f_0'' - \frac{1}{2}\, h^{\beta}{}_{\beta} h^{ab} + 2\, \overset{1}{R}\, f_0'' h^{ab} \\ & + \overset{1}{R}\, \overset{1}{R}\, \eta^{ab} f_0'' f_0'' + \frac{1}{4}\, \eta^{ab} h_{\beta\nu} h^{\beta\nu} \\ & + \frac{1}{8}\, \eta^{ab} h^{\beta}{}_{\beta} h^{\nu}{}_{\nu} - \overset{1}{R}\, \eta^{ab} f_0'' h^{\nu}{}_{\nu} \end{aligned} \tag{8.2.22}$$

### 8.2.5 Results

Plugging these into the Einstein Tensor, and keeping terms to linear order in $h_{ab}$, we get

$$\begin{aligned}\overset{1}{G}_{\sigma\nu} \quad = \quad & \partial_{\nu\sigma}\overset{1}{R}f_0'' + \partial_\nu \overset{1}{R}\partial_\sigma f_0'' + \partial_\sigma \overset{1}{R}\partial_\nu f_0'' \\ & + \overset{1}{R}\,\partial_{\nu\sigma}f_0'' + \frac{1}{2}\,\partial^\alpha{}_\alpha h_{\nu\sigma} - \eta_{\nu\sigma}\partial^\alpha{}_\alpha \overset{1}{R}f_0'' \\ & - \eta_{\nu\sigma}\partial^\alpha \overset{1}{R}\partial_\alpha f_0'' - \eta_{\nu\sigma}\partial^\alpha f_0''\partial_\alpha \overset{1}{R} \\ & - \overset{1}{R}\,\eta_{\nu\sigma}\partial^\alpha{}_\alpha f_0''.\end{aligned} \tag{8.2.23}$$

Setting this result equal to the right-hand side, (8.2.7), and plugging equations (8.2.13 - 8.2.15) into (8.2.7), we see that the mixed partials cancel off when $\overset{0}{f}' = 1$, as it should. Continuing, we can work out $\overset{2}{G}_{\sigma\nu}$ in these new coordinates. We find

$$
\begin{aligned}
\overset{2}{G}_{\sigma\nu} \;=\;& \eta_{\nu\sigma}(-\partial^{\alpha}_{\alpha}\overset{2}{R}f_0'' - \partial^{\alpha}\overset{2}{R}\partial_{\alpha}f_0'' - \partial^{\alpha}f_0''\partial_{\alpha}\overset{2}{R} - \overset{2}{R}\,\partial^{\alpha}_{\alpha}f_0'' - \partial^{\alpha}\overset{1}{R}\partial_{\alpha}\overset{1}{R}f_0'' - \overset{1}{R}\,\partial^{\alpha}_{\alpha}\overset{1}{R}f_0'' \\
& - \overset{1}{R}\,\partial^{\alpha}\overset{1}{R}\partial_{\alpha}f_0'' - \overset{1}{R}\,\partial^{\alpha}f_0''\partial_{\alpha}\overset{1}{R} - \frac{1}{2}\,\overset{1}{R}\,\overset{1}{R}\,\partial^{\alpha}_{\alpha}f_0'' - \frac{3}{4}\partial^{\alpha}\overset{1}{R}\partial_{\alpha}\overset{1}{R}f_0''f_0'' - \overset{1}{R}\,\partial^{\alpha}_{\alpha}\overset{1}{R}f_0''f_0'' \\
& -\frac{7}{4}\,\overset{1}{R}\,\partial^{\alpha}\overset{1}{R}\partial_{\alpha}f_0''f_0'' - \frac{7}{4}\,\overset{1}{R}\,\partial^{\alpha}f_0''\partial_{\alpha}\overset{1}{R}f_0'' - \overset{1}{R}\,\overset{1}{R}\,\partial^{\alpha}_{\alpha}f_0''f_0'' - \frac{3}{4}\,\overset{1}{R}\,\overset{1}{R}\,\partial^{\alpha}f_0''\partial_{\alpha}f_0'' + \frac{1}{8}\partial^{\alpha}h^{\beta\mu}\partial_{\alpha}h_{\beta\mu} \\
& -\partial^{\alpha}\overset{1}{R}\partial^{\beta}f_0''h_{\alpha\beta} + \frac{1}{8}\partial^{\alpha}h^{\beta}_{\beta}\partial_{\alpha}\overset{1}{R}f_0'' + \frac{1}{8}\,\overset{1}{R}\,\partial^{\alpha}h^{\beta}_{\beta}\partial_{\alpha}f_0'' + \frac{1}{8}\partial^{\alpha}\overset{1}{R}\partial_{\alpha}h^{\beta}_{\beta}f_0'' + \frac{1}{8}\,\overset{1}{R}\,\partial^{\alpha}f_0''\partial_{\alpha}h^{\beta}_{\beta} \\
& -1/16\partial^{\alpha}h^{\beta}_{\beta}\partial_{\alpha}h^{\mu}_{\mu} - \frac{3}{4}\partial^{\alpha\beta}h^{\mu}_{\alpha}h_{\beta\mu} - 5/8\partial^{\alpha}h^{\beta\mu}\partial_{\beta}h_{\alpha\mu} - \frac{1}{2}\partial^{\alpha\beta}\overset{1}{R}f_0''h_{\alpha\beta} \\
& -\frac{1}{2}\,\overset{1}{R}\,\partial^{\alpha\beta}f_0''h_{\alpha\beta} + \frac{1}{2}\,\overset{1}{R}\,\partial^{\alpha\beta}h_{\alpha\beta}f_0'' + \frac{1}{8}\partial^{\alpha}\overset{1}{R}\partial^{\beta}h_{\alpha\beta}f_0'' + \frac{1}{8}\,\overset{1}{R}\,\partial^{\alpha}f_0''\partial^{\beta}h_{\alpha\beta} \\
& -1/16\partial^{\alpha}h^{\beta}_{\beta}\partial^{\mu}h_{\alpha\mu} + \frac{3}{8}\partial^{\alpha}h^{\beta\mu}\partial_{\mu}h_{\alpha\beta}) + \partial_{\nu\sigma}\overset{2}{R}f_0'' + \partial_{\nu}\overset{2}{R}\partial_{\sigma}f_0'' \\
& +\partial_{\sigma}\overset{2}{R}\partial_{\nu}f_0'' + \overset{2}{R}\,\partial_{\nu\sigma}f_0'' + \partial_{\nu}\overset{1}{R}\partial_{\sigma}\overset{1}{R}f_0'' + \overset{1}{R}\,\partial_{\nu\sigma}\overset{1}{R}f_0'' \\
& + \overset{1}{R}\,\partial_{\nu}\overset{1}{R}\partial_{\sigma}f_0'' + \overset{1}{R}\,\partial_{\sigma}\overset{1}{R}\partial_{\nu}f_0'' + \frac{1}{2}\,\overset{1}{R}\,\overset{1}{R}\,\partial_{\nu\sigma}f_0'' + \frac{1}{2}\partial^{\nu}_{\alpha}h^{\beta}_{\alpha}h_{\beta\sigma} \\
& +\frac{1}{2}\partial_{\nu}h^{\alpha\beta}\partial_{\alpha}h_{\beta\sigma} - \frac{1}{2}\partial^{\nu}_{\alpha}\overset{1}{R}f_0''h_{\alpha\sigma} - \frac{1}{2}\partial_{\nu}\overset{1}{R}\partial^{\alpha}f_0''h_{\alpha\sigma} - \partial^{\alpha}\overset{1}{R}\partial_{\nu}f_0''h_{\alpha\sigma} \\
& -\frac{1}{2}\,\overset{1}{R}\,\partial^{\nu}_{\alpha}f_0''h_{\alpha\sigma} - \frac{1}{2}\partial^{\alpha}\overset{1}{R}\partial_{\nu}h_{\alpha\sigma}f_0'' - \frac{1}{2}\,\overset{1}{R}\,\partial^{\alpha}f_0''\partial_{\nu}h_{\alpha\sigma} + \frac{1}{2}\partial_{\nu}f_0''\partial^{\alpha}\overset{1}{R}h_{\alpha\sigma} \\
& -\frac{1}{2}\partial_{\nu}h^{\alpha}_{\sigma}\partial_{\alpha}\overset{1}{R}f_0'' - \frac{1}{2}\,\overset{1}{R}\,\partial^{\nu}_{\alpha}h_{\alpha\sigma}f_0'' - \frac{1}{2}\,\overset{1}{R}\,\partial_{\nu}h^{\alpha}_{\sigma}\partial_{\alpha}f_0'' + \frac{3}{2}\partial_{\nu}\overset{1}{R}\partial_{\sigma}\overset{1}{R}f_0''f_0'' \\
& + \overset{1}{R}\,\partial_{\nu\sigma}\overset{1}{R}f_0''f_0'' + \frac{5}{2}\,\overset{1}{R}\,\partial_{\nu}\overset{1}{R}\partial_{\sigma}f_0''f_0'' + \frac{5}{2}\,\overset{1}{R}\,\partial_{\sigma}\overset{1}{R}\partial_{\nu}f_0''f_0'' + \overset{1}{R}\,\overset{1}{R}\,\partial_{\nu\sigma}f_0''f_0'' \\
& +\frac{3}{2}\,\overset{1}{R}\,\overset{1}{R}\,\partial_{\nu}f_0''\partial_{\sigma}f_0'' + \partial_{\nu}h^{\alpha}_{\sigma}\partial_{\alpha}\overset{1}{R}f_0'' + \overset{1}{R}\,\partial_{\nu}h^{\alpha}_{\sigma}\partial_{\alpha}f_0'' - \frac{1}{4}\partial_{\sigma}h^{\alpha\beta}\partial_{\nu}h_{\alpha\beta} \\
& +\frac{1}{2}\partial^{\sigma}_{\alpha}h^{\beta}_{\alpha}h_{\beta\nu} - \frac{1}{2}\partial^{\sigma}_{\alpha}\overset{1}{R}f_0''h_{\alpha\nu} - \frac{1}{2}\,\overset{1}{R}\,\partial^{\sigma}_{\alpha}f_0''h_{\alpha\nu} - \frac{1}{2}\,\overset{1}{R}\,\partial^{\sigma}_{\alpha}h_{\alpha\nu}f_0'' \\
& -\frac{1}{2}\partial^{\alpha}_{\alpha}h^{\beta}_{\nu}h_{\beta\sigma} - \frac{1}{4}\partial^{\alpha}h^{\beta}_{\nu}\partial_{\alpha}h_{\beta\sigma} - \frac{1}{2}\partial^{\alpha}_{\alpha}h^{\beta}_{\sigma}h_{\beta\nu} - \frac{1}{2}\partial^{\alpha}h^{\beta}_{\sigma}\partial_{\alpha}h_{\beta\nu} \\
& +\frac{3}{2}\partial^{\alpha}_{\alpha}\overset{1}{R}f_0''h_{\nu\sigma} + \frac{3}{2}\partial^{\alpha}\overset{1}{R}\partial_{\alpha}f_0''h_{\nu\sigma} + \frac{1}{2}\partial^{\alpha}\overset{1}{R}\partial_{\alpha}h_{\nu\sigma}f_0'' + \frac{3}{2}\partial^{\alpha}f_0''\partial_{\alpha}\overset{1}{R}h_{\nu\sigma} \\
& +\frac{3}{2}\,\overset{1}{R}\,\partial^{\alpha}_{\alpha}f_0''h_{\nu\sigma} + \frac{1}{2}\,\overset{1}{R}\,\partial^{\alpha}f_0''\partial_{\alpha}h_{\nu\sigma} + \frac{1}{2}\,\overset{1}{R}\,\partial^{\alpha}_{\alpha}h_{\nu\sigma}f_0'' + \frac{1}{2}\partial^{\alpha\beta}h_{\nu\sigma}h_{\alpha\beta} \\
& -\frac{1}{4}\partial_{\sigma}\overset{1}{R}\partial_{\nu}h^{\alpha}_{\alpha}f_0'' - \frac{1}{4}\,\overset{1}{R}\,\partial_{\sigma}f_0''\partial_{\nu}h^{\alpha}_{\alpha} - \frac{1}{4}\partial_{\nu}\overset{1}{R}\partial_{\sigma}h^{\alpha}_{\alpha}f_0'' \\
& -\frac{1}{4}\,\overset{1}{R}\,\partial_{\nu}f_0''\partial_{\sigma}h^{\alpha}_{\alpha} + \frac{3}{8}\partial_{\nu}h^{\alpha}_{\alpha}\partial_{\sigma}h^{\beta}_{\beta} + \frac{3}{8}\partial_{\sigma}h^{\alpha}_{\nu}\partial_{\alpha}h^{\beta}_{\beta} + \frac{3}{4}\partial^{\alpha}h^{\beta}_{\nu}\partial_{\sigma}h_{\alpha\beta} - \frac{1}{2}\partial^{\alpha}\overset{1}{R}\partial_{\sigma}f_0''h_{\alpha\nu} \\
& -\frac{1}{2}\partial_{\sigma}\overset{1}{R}\partial^{\alpha}f_0''h_{\alpha\nu} - \frac{1}{4}\partial_{\sigma}h^{\alpha}_{\alpha}\partial_{\nu}h^{\beta}_{\beta} - \frac{3}{8}\partial^{\alpha}h^{\beta}_{\beta}\partial_{\sigma}h_{\alpha\nu} - \frac{1}{4}\partial_{\nu}h^{\alpha\beta}\partial_{\beta}h_{\alpha\sigma} \\
& -\frac{1}{4}\partial^{\alpha}h^{\beta}_{\sigma}\partial_{\beta}h_{\alpha\nu} - \frac{1}{4}\partial_{\sigma}h^{\alpha\beta}\partial_{\beta}h_{\alpha\nu} + \frac{1}{4}\partial^{\alpha}h^{\beta}_{\sigma}\partial_{\nu}h_{\alpha\beta} + \frac{1}{4}\partial^{\alpha}h^{\beta}_{\sigma}\partial_{\alpha}h_{\beta\nu} \\
& -\frac{1}{4}\partial^{\alpha}h^{\beta}_{\nu}\partial_{\beta}h_{\alpha\sigma}
\end{aligned}
\tag{8.2.24}
$$

# Chapter 9

# Conclusions

## 9.1 Summary

In the first few sections of this paper, I have reviewed the experimental and theoretical constraints on gravity theories. I've reviewed phenomena in scalar-tensor gravity, which provides intuition for scalar curvature effects in metric $f(R)$ gravity. There is a region where the scalar field theory analog breaks down, and I have covered new physical phenomena that happen in this region. This new work is related to past work in the sense that it has been recognized that increasing the scalar mass via the chameleon mechanism allows the theory to avoid solar system constraints, and also that allowing $f''(R)$ to get small increases the scalar mass. I have not, however, seen any past work regarding the scattering effects when a scalar wave is incident upon such a region. I have also noted that the fact that $f''(R)$ can approach zero in some region is sufficient to suppress scalar wave effects, and act as an effective chameleon mechanism. While there is some literature suggesting that $f''(R)$ getting small will suppress scalar effects, I have not seen it suggested that this phenomenon replace the chameleon mechanism. Before, the chameleon mechanism implied that if scalar tensor gravity agreed with solar system constraints, then the scalar effects must be suppressed at terrestrial energy scales, and so also at higher scales. This excludes interesting radiative effects, since they'd be produced by even higher energy systems, like neutron star binaries. I've outlined that with the new critical point phenomena, scalar field effects can be suppressed at terrestrial energy scales, but are allowed at higher energy scales. This should allow radiative effects in high energy systems.

I have gone on to start addressing the matching problem in metric $f(R)$ gravity. The state of the field is such that the starting point is finding the post-Newtonian and post-Minkowskian series expansions for metric $f(R)$ gravity, which I have done in the final sections of this paper. There has been basic work before, which found the first order post-Minkowskian and post-Newtonian terms in $f(R)$ gravity. I have worked out these expansions a little farther, to second order. I also started independently deriving linearized gravity in metric $f(R)$ gravity, but did not finish before Berry and Gair published their work on the subject.

### 9.2 Further Research

To continue this work, there are a few avenues to follow. The first major avenue lies in constraining the range of $R$ values over which $f(R)$ can deviate from $f(R) = R$. There should exist constraints on what curvature scales are allowed to have strong radiative effects, based on observations of binary pulsars such as the Hulse-Taylor binary[1] [69]. We could also look at how the mass dependence of binding energy might effect different systems, or continue searching for time- and space- variation of constants like $G$.

The second major avenue involves continuing to work out the matching problem for $f(R)$ gravity. I've laid out the iterative procedures for the PN and PM expansions, and expect that the regularization procedure established in the literature, because of its need arising from boundary conditions, should still apply to this problem. After verifying that, I'll next need to apply the multipolar expansion to my results, and then the PM expansion of the PN result, and vice versa. I would plan to build up from the simple case in the literature, and perform the matching first at the 1PN order. If we examine the case where $f''(R) = 0$ in the matching region, and then for greater values of $r$, we will have extra complications due to the fact that the coefficients of our second derivatives vanish, limiting the domain over which our solution is defined. This is the case in general with 2nd order differential equations.

[1] Since the masses of these neutron stars are so similar, at around $1.4 M_\odot$, their sensitivities should be similar, and thus dipole radiation should be suppressed

## Appendix A

## Einstein Tensor in $f(R)$ Gravity to 2nd Order

To start, we've expanded the metric and inverse metric, respectively, to second order in $h_{ab}$, as

$$g_{ab} = \sqrt{-g}(\eta_{ab} - h_{ab} + h_{ac}h_{db}\eta^{cd}) \tag{A.0.1}$$

and

$$g^{ab} = \frac{1}{\sqrt{-g}}(\eta^{ab} + h^{ab}). \tag{A.0.2}$$

Taylor expanding the determinant of the metric around $g_{ab} = \eta_{ab}$, we get

$$-g = 1 + h_{ab}\eta^{ab} + \frac{1}{2}(h^a_a h^b_b - h_{ab}h^{ab}). \tag{A.0.3}$$

Now, expanding the square root $\sqrt{1+x}$ around small x, and using $x = h_{ab}\eta^{ab} + \frac{1}{2}(h^a_a h^b_b - h_{ab}h^{ab})$, we get

$$\begin{aligned} \sqrt{-g} &= \sqrt{1+x} && \text{(A.0.4)} \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} && \text{(A.0.5)} \\ &= 1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h^{ab}h_{ab}}{4} && \text{(A.0.6)} \\ & && \text{(A.0.7)} \end{aligned}$$

and using the same process,

$$\frac{1}{\sqrt{-g}} = 1 - \frac{h}{2} + \frac{h^2}{8} + \frac{h^{ab}h_{ab}}{4} \tag{A.0.8}$$

Now we're ready to calculate the Christoffel symbols,

$$\Gamma^{\mu}_{\nu\alpha} = \frac{1}{2}g^{\mu\gamma}(\partial_\nu g_{\alpha\gamma} + \partial_\alpha g_{\nu\gamma} - \partial_\gamma g_{\nu\alpha}). \tag{A.0.9}$$

We plug in our expansions for $g_{ab}$ and $g^{ab}$, drop terms past second order, and find

$$\begin{aligned}
\Gamma^{\mu}_{\nu\alpha} \quad = \quad & -\frac{1}{2}\,\eta^{\mu\beta}\partial_\nu h_{\alpha\beta} + \frac{1}{2}\,\eta^{\mu\beta}\eta^{\gamma\lambda}\partial_\nu h_{\alpha\gamma}h_{\lambda\beta} + \frac{1}{2}\,\eta^{\mu\beta}\eta^{\gamma\lambda}\partial_\nu h_{\lambda\beta}h_{\alpha\gamma} + \frac{1}{4}\,\eta_{\alpha\beta}\eta^{\mu\beta}\partial_\nu h^{\gamma}{}_{\gamma} \\
& -\frac{1}{4}\,\eta^{\mu\beta}\partial_\nu h^{\gamma}{}_{\gamma}h_{\alpha\beta} - \frac{1}{8}\,\eta_{\alpha\beta}\eta^{\mu\beta}\partial_\nu h_{\gamma\lambda}h^{\gamma\lambda} - \frac{1}{8}\,\eta_{\alpha\beta}\eta^{\mu\beta}\partial_\nu h^{\gamma\lambda}h_{\gamma\lambda} - \frac{1}{2}\,\eta^{\mu\beta}\partial_\alpha h_{\nu\beta} \\
& +\frac{1}{2}\,\eta^{\mu\beta}\eta^{\gamma\lambda}\partial_\alpha h_{\nu\gamma}h_{\lambda\beta} + \frac{1}{2}\,\eta^{\mu\beta}\eta^{\gamma\lambda}\partial_\alpha h_{\lambda\beta}h_{\nu\gamma} + \frac{1}{4}\,\eta_{\nu\beta}\eta^{\mu\beta}\partial_\alpha h^{\gamma}{}_{\gamma} - \frac{1}{4}\,\eta^{\mu\beta}\partial_\alpha h^{\gamma}{}_{\gamma}h_{\nu\beta} \\
& -\frac{1}{8}\,\eta_{\nu\beta}\eta^{\mu\beta}\partial_\alpha h_{\gamma\lambda}h^{\gamma\lambda} - \frac{1}{8}\,\eta_{\nu\beta}\eta^{\mu\beta}\partial_\alpha h^{\gamma\lambda}h_{\gamma\lambda} + \frac{1}{2}\,\eta^{\mu\beta}\partial_\beta h_{\nu\alpha} - \frac{1}{2}\,\eta^{\mu\beta}\eta^{\gamma\lambda}\partial_\beta h_{\nu\gamma}h_{\lambda\alpha} \\
& -\frac{1}{2}\,\eta^{\mu\beta}\eta^{\gamma\lambda}\partial_\beta h_{\lambda\alpha}h_{\nu\gamma} - \frac{1}{4}\,\eta_{\nu\alpha}\eta^{\mu\beta}\partial_\beta h^{\gamma}{}_{\gamma} + \frac{1}{4}\,\eta^{\mu\beta}\partial_\beta h^{\gamma}{}_{\gamma}h_{\nu\alpha} + \frac{1}{8}\,\eta_{\nu\alpha}\eta^{\mu\beta}\partial_\beta h_{\gamma\lambda}h^{\gamma\lambda} \\
& +\frac{1}{8}\,\eta_{\nu\alpha}\eta^{\mu\beta}\partial_\beta h^{\gamma\lambda}h_{\gamma\lambda} - \frac{1}{2}\,\partial_\nu h_{\alpha\beta}h^{\mu\beta} + \frac{1}{4}\,\eta_{\alpha\beta}\partial_\nu h^{\gamma}{}_{\gamma}h^{\mu\beta} - \frac{1}{2}\,\partial_\alpha h_{\nu\beta}h^{\mu\beta} \\
& +\frac{1}{4}\,\eta_{\nu\beta}\partial_\alpha h^{\gamma}{}_{\gamma}h^{\mu\beta} + \frac{1}{2}\,\partial_\beta h_{\nu\alpha}h^{\mu\beta} - \frac{1}{4}\,\eta_{\nu\alpha}\partial_\beta h^{\gamma}{}_{\gamma}h^{\mu\beta}
\end{aligned} \tag{A.0.10}$$

Now, we can use this result to calculate the Ricci tensor, $R_{\nu\sigma}$,

$$R_{\nu\sigma} \quad = \quad \partial_\rho\Gamma^{\rho}_{\nu\sigma} - \partial_\sigma\Gamma^{\rho}_{\nu\rho} + \Gamma^{\rho}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\rho}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} \tag{A.0.11}$$

to second order in h. By plugging in the Christoffel symbols, dropping terms beyond second order, and using the gauge condition $\partial^\mu h_{\mu\nu} = 0$, we get

$$\begin{aligned}
R_{\nu\sigma} &= \frac{1}{2}\,\partial^{\alpha}{}_{\nu}h_{\sigma}{}^{\beta}h_{\beta\alpha} + \frac{1}{4}\,\partial_{\nu}h^{\alpha\beta}\partial_{\beta}h_{\sigma\alpha} - \frac{1}{4}\,\partial_{\sigma\nu}h^{\alpha}{}_{\alpha} - \frac{1}{4}\,\partial^{\alpha}{}_{\nu}h^{\beta}{}_{\beta}h_{\sigma\alpha} \\
&+ \frac{5}{4}\,\partial_{\sigma\nu}h^{\alpha\beta}h_{\alpha\beta} + \frac{3}{4}\,\partial_{\nu}h^{\alpha\beta}\partial_{\sigma}h_{\alpha\beta} + \frac{1}{4}\,\partial_{\sigma}h^{\alpha\beta}\partial_{\nu}h_{\alpha\beta} + \frac{1}{2}\,\partial^{\alpha}{}_{\sigma}h_{\nu}{}^{\beta}h_{\beta\alpha} \\
&+ \frac{1}{4}\,\partial_{\nu\sigma}h^{\alpha}{}_{\alpha} - \frac{1}{4}\,\partial^{\alpha}{}_{\sigma}h^{\beta}{}_{\beta}h_{\nu\alpha} - \frac{1}{4}\,\partial_{\nu\sigma}h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{2}\,\partial^{\alpha}{}_{\alpha}h_{\nu\sigma} \\
&- \frac{1}{2}\,\partial^{\alpha}{}_{\alpha}h_{\nu}{}^{\beta}h_{\beta\sigma} - \frac{1}{2}\,\partial^{\alpha}h_{\nu}{}^{\beta}\partial_{\alpha}h_{\beta\sigma} - \frac{1}{2}\,\partial^{\alpha}{}_{\alpha}h^{\beta}{}_{\sigma}h_{\nu\beta} - \frac{1}{4}\,\partial^{\alpha}h^{\beta}{}_{\sigma}\partial_{\alpha}h_{\nu\beta} \\
&- \frac{1}{4}\,\eta_{\nu\sigma}\partial^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta} + \frac{1}{4}\,\partial^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h_{\nu\sigma} + \frac{1}{8}\,\partial^{\alpha}h^{\beta}{}_{\beta}\partial_{\alpha}h_{\nu\sigma} + \frac{1}{8}\,\eta_{\nu\sigma}\partial^{\alpha}{}_{\alpha}h^{\beta}{}_{\mu}h_{\beta}{}^{\mu} \\
&+ \frac{1}{8}\,\eta_{\nu\sigma}\partial^{\alpha}h^{\beta}{}_{\mu}\partial_{\alpha}h_{\beta}{}^{\mu} + \frac{1}{8}\,\eta_{\nu\sigma}\partial^{\alpha}{}_{\alpha}h^{\beta\mu}h_{\beta\mu} + \frac{1}{8}\,\eta_{\nu\sigma}\partial^{\alpha}h^{\beta\mu}\partial_{\alpha}h_{\beta\mu} - \frac{1}{2}\,\partial^{\alpha}{}_{\nu}h_{\sigma}{}^{\beta}h_{\alpha\beta} \\
&+ \frac{1}{4}\,\partial^{\alpha}{}_{\nu}h^{\beta}{}_{\beta}h_{\alpha\sigma} - \frac{1}{2}\,\partial^{\alpha}{}_{\sigma}h_{\nu}{}^{\beta}h_{\alpha\beta} + \frac{1}{4}\,\partial^{\alpha}{}_{\sigma}h^{\beta}{}_{\beta}h_{\alpha\nu} + \frac{1}{2}\,\partial^{\alpha\beta}h_{\nu\sigma}h_{\alpha\beta} \\
&- \frac{1}{4}\,\eta_{\nu\sigma}\partial^{\alpha\beta}h^{\mu}{}_{\mu}h_{\alpha\beta} - \partial_{\sigma\nu}h^{\alpha\beta}h_{\beta\alpha} - \frac{1}{2}\,\partial_{\sigma}h^{\alpha\beta}\partial_{\nu}h_{\beta\alpha} - \frac{3}{4}\,\partial_{\nu}h^{\alpha\beta}\partial_{\sigma}h_{\beta\alpha} \\
&+ \frac{3}{8}\,\partial_{\nu}h^{\alpha}{}_{\alpha}\partial_{\sigma}h^{\beta}{}_{\beta} - \frac{1}{2}\,\partial_{\sigma}{}^{\alpha}h_{\nu}{}^{\beta}h_{\beta\alpha} + \frac{1}{4}\,\partial_{\sigma}{}^{\alpha}h^{\beta}{}_{\beta}h_{\nu\alpha} + \frac{3}{8}\,\partial_{\sigma}h_{\nu}{}^{\alpha}\partial_{\alpha}h^{\beta}{}_{\beta} \\
&- \frac{1}{4}\,\partial_{\sigma}h^{\alpha}{}_{\alpha}\partial_{\nu}h^{\beta}{}_{\beta} + \frac{1}{2}\,\partial_{\sigma}{}^{\alpha}h_{\nu}{}^{\beta}h_{\alpha\beta} + \frac{3}{4}\,\partial^{\alpha}h_{\nu}{}^{\beta}\partial_{\sigma}h_{\alpha\beta} - \frac{1}{4}\,\partial_{\sigma}{}^{\alpha}h^{\beta}{}_{\beta}h_{\alpha\nu} \\
&- \frac{3}{8}\,\partial^{\alpha}h^{\beta}{}_{\beta}\partial_{\sigma}h_{\alpha\nu} - \frac{1}{16}\,\eta_{\nu\sigma}\partial^{\alpha}h^{\beta}{}_{\beta}\partial_{\alpha}h^{\mu}{}_{\mu} - \frac{1}{4}\,\partial^{\alpha}h_{\sigma}{}^{\beta}\partial_{\beta}h_{\nu\alpha} + \frac{1}{4}\,\partial^{\alpha}h_{\nu}{}^{\beta}\partial_{\alpha}h_{\sigma\beta} \\
&- \frac{1}{8}\,\partial^{\alpha}h^{\beta}{}_{\beta}\partial_{\alpha}h_{\sigma\nu} - \frac{1}{4}\,\partial_{\sigma}h^{\alpha\beta}\partial_{\beta}h_{\nu\alpha} + \frac{1}{4}\,\partial^{\alpha}h^{\beta}{}_{\sigma}\partial_{\nu}h_{\alpha\beta} - \frac{1}{4}\,\partial^{\alpha}h_{\nu}{}^{\beta}\partial_{\beta}h_{\alpha\sigma} \\
&+ \frac{1}{16}\,\eta_{\sigma\nu}\partial^{\alpha}h^{\beta}{}_{\beta}\partial_{\alpha}h^{\mu}{}_{\mu}
\end{aligned} \tag{A.0.12}$$

Now, we can contract $R_{\mu\nu}$ with $g^{\mu\nu}$ to form the Ricci scalar, $R$,

$$\begin{aligned}
R &= \partial^{\alpha\beta}h_{\beta}{}^{\mu}h_{\mu\alpha} - \frac{1}{2}\,\partial^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta} - \partial^{\alpha\beta}h^{\mu}{}_{\mu}h_{\beta\alpha} + \frac{1}{2}\,\partial^{\alpha}{}_{\alpha}h^{\beta\mu}h_{\beta\mu} \\
&+ \frac{1}{4}\,\partial^{\alpha}h^{\beta\mu}\partial_{\alpha}h_{\beta\mu} + \frac{1}{4}\,\partial^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h^{\mu}{}_{\mu} + \frac{1}{8}\,\partial^{\alpha}h^{\beta}{}_{\beta}\partial_{\alpha}h^{\mu}{}_{\mu} - \partial^{\alpha\beta}h_{\beta}{}^{\mu}h_{\alpha\mu} \\
&+ \frac{1}{2}\,\partial^{\alpha\beta}h^{\mu}{}_{\mu}h_{\alpha\beta} - \frac{1}{2}\,\partial^{\alpha\beta}h_{\alpha}{}^{\mu}h_{\mu\beta} + \frac{1}{2}\,\partial^{\alpha\beta}h_{\alpha}{}^{\mu}h_{\beta\mu} + \frac{3}{4}\,\partial^{\alpha}h^{\beta\mu}\partial_{\beta}h_{\alpha\mu} \\
&- \frac{1}{4}\,\partial^{\alpha}h^{\beta\mu}\partial_{\mu}h_{\beta\alpha}
\end{aligned} \tag{A.0.13}$$

From these, we finally construct the Einstein tensor,

$$\overset{1}{G}_{\sigma\nu} = \frac{1}{2}\Box h_{\sigma\nu} \tag{A.0.14}$$

$$\begin{aligned}
\overset{2}{G}_{\sigma\nu} &= -h^{ab}\partial_{a}\partial_{b}h_{\sigma\nu} + \frac{1}{2}\partial_{\sigma}h_{ab}\partial_{\nu}h^{ab} - \frac{1}{4}\partial_{\sigma}h\partial_{\nu}h - \partial_{\sigma}h_{ab}\partial^{a}h^{b}_{\nu} \\
&-\partial^{\nu}h_{ab}\partial^{a}h^{\sigma b} + \partial_{a}h^{b}_{\sigma}\partial^{a}h_{\nu b} + \partial_{a}h^{b}_{\sigma}\partial_{b}h^{a}_{\nu} \\
&+\eta_{\sigma\nu}\left(-\frac{1}{4}\partial_{\rho}h_{\mu\lambda}\partial^{\rho}h^{\mu\lambda} + \frac{1}{8}\partial_{a}h\partial^{a}h + \frac{1}{2}\partial_{a}h_{b\rho}\partial^{b}h^{a\rho}\right).
\end{aligned} \tag{A.0.15}$$

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