The General Quadruple Point Formula

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ABSTRACT

ROBERT MARANGELL: The General Quadruple Point Formula

(Under the direction of Richárd Rimányi)

Maps between manifolds $M^m \to N^{m+\ell}$ ($\ell > 0$) have multiple points, and more generally, multisingularities. The closure of the set of points where the map has a particular multisingularity is called the multisingularity locus. There are universal relations among the cohomology classes represented by multisingularity loci, and the characteristic classes of the manifolds. These relations include the celebrated Thom polynomials of monosingularities. For multisingularities, however, only the form of these relations is clear in general (due to Kazarian [**21**]), the concrete polynomials occurring in the relations are less known. In the present paper we prove the first general such relation outside the region of Morin-maps: the general quadruple point formula. We apply this formula in enumerative geometry by computing the number of 4-secant linear spaces to smooth projective varieties. Some other multisingularity formulas are also studied, namely 5, 6 and 7-tuple point formulas, and one corresponding to $\Sigma^2\Sigma^0$ multisingularities.

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CHAPTER 1

Introduction

For a map $f : X \to Y$ between manifolds one can consider so-called multisingularity loci in the source and in the target. The existence of universal identities relating cohomology classes represented by these mulitsingularity loci to the characteristic classes of X and Y has been known for decades. The study of such identities is what constitutes multisingularity theory.

These universal identities were popular in enumerative geometry and used effectively a few decades ago in the works of Kleiman, Colley, Le Barz and others. Until recently only a few sporadic formulas were known. However in the works of Rimányi, Fehér, Kazarian, Szenes and Bérczi, many new formulas were discovered, and the appropriate applications to enumerative problems were derived. However, hardly any *general* multisingularity formulas were known. By general we mean valid in any dimensional setting, i.e. the formula contains the dimensions of the manifolds as parameters. For example only the general double and triple point formulas were known, as well as that pertaining to the multisingularity called A_1A_0 (see Section 2.5). Partial results towards the generalized quadruple point formula are summarized in [**7**].

In this paper we use an approach from differential topology to calculate the general n-tuple point formulas for $n = 4, 5, 6$ and 7. Namely we use Rimányi's interpolation method to find the so-called residue polynomial corresponding to these multisingularities,

and then using the general form described by Kazarian, we deduce the general formulas. We also calculate the residue polynomial (and hence multisingularity formula) for the $III_{2,2}A_0$ multisingularity. We then go on to show how these formulas can be used in problems coming from enumerative geometry.

CHAPTER 2

Preliminaries from the Theory of Contact Singularities

2.1. Definitions and Examples

We wish to consider maps between complex manifolds on the local level. To that end we wish to reduce our field of study to the vector space of holomorphic map germs. Let X and Y be complex manifolds of (complex) dimension m and n respectively. Fix a point $q \in X$ and let $\mathcal{M}$ be the set of maps with range Y that are holomorphic on some open neighborhood of q. We define a set of equivalence classes on $\mathcal{M}$.

Definition 2.1.1. *Let U and V be open neighborhoods of the point q. Let f and g be holomorphic maps on U and V respectively. We say that $f : U \to Y$* is equivalent to *$g : V \to Y$ if there exists a neighborhood W of q on which $f\big|_W = g\big|_W$. A* map germ *at q is an equivalence class of such maps. We call an element of such an equivalence class a* representative *of a map germ. When there is no ambiguity, we will use the same notation f for a map germ as well as a representative.*

We will use the notation $f : (X, q) \to Y$ to denote a map germ at the point $q \in X$. If we wish to fix the target point as well, we will use the notation $f : (X, q) \to (Y, r)$ for germs mapping $q \mapsto r$.

Since complex manifolds are locally biholomorphic to $\mathbb{C}^m$ we can consider $X = \mathbb{C}^m$, $Y = \mathbb{C}^n$ and $q = 0$, the origin. Let $\mathcal{E}(m, n)$ denote the vector space of germs $f : (\mathbb{C}^m, 0) \to \mathbb{C}^n$. And let $\mathcal{E}^0(m, n)$ be the subspace consisting of map germs mapping $\mathbf{0}$

$\in \mathbb{C}^m$ to $\mathbf{0} \in \mathbb{C}^n$. We can locally coordinatize the spaces $\mathbb{C}^m$ and $\mathbb{C}^n$, and under these local coordinates, the map germ $f : (\mathbb{C}^m, 0) \to \mathbb{C}^n$ has the form $f : (x_1, \ldots, x_m) \mapsto (f_1, \ldots, f_n)$.

Let $\mathcal{D}_n$ denote the group of biholomorphic map germs $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ fixing the origin. Define

$$\mathcal{A}_{m,n} := \mathcal{A} := \mathcal{D}_m \times \mathcal{D}_n. \tag{2.1}$$

The group $\mathcal{A}$ acts on $\mathcal{E}^0(m, n)$ as follows. For $(\varphi, \psi) \in \mathcal{A}$,

$$(\varphi, \psi) \cdot f = \psi \circ f \circ \varphi^{-1}. \tag{2.2}$$

To consider orbits of this group action is to consider map germs up to local reparametrizations of the source and target.

Definition 2.1.2.

(1) We say that two germs are left-right equivalent *or* $\mathcal{A}$-equivalent *if they are in the same orbit of the action of* $\mathcal{A}$.

(2) We define a left-right singularity *or an* $\mathcal{A}$-singularity *to be an* $\mathcal{A}$*-equivalence class of germs.*

Consider the group $\mathcal{D}_{m+n}$ of germs of biholomorphims on the product space $\mathbb{C}^m \times \mathbb{C}^n$. In local coordinates, we let $\mathbf{x}$ represent the first m coordinates and $\mathbf{y}$ represent the last n coordinates.

Definition 2.1.3. *Let* $\mathcal{K}$*, the contact group, be defined as follows:*

$$\mathcal{K}_{m,n} := \mathcal{K} := \{H \in \mathcal{D}_{m+n} | H = (h(\mathbf{x}), H_1(\mathbf{x}, \mathbf{y})), \text{ where } h \in \mathcal{D}_m\}. \tag{2.3}$$

The group $\mathcal{K}$ acts on $\mathcal{E}^0(m,n)$ in the following way: $(H \cdot f)(\mathbf{x}) := H_1(h^{-1}(\mathbf{x}), f(h^{-1}(\mathbf{x})))$. This action can be thought of geometrically as follows: $\mathcal{K}$ is the subgroup of $\mathcal{D}_{m+n}$ such that for $H \in \mathcal{K}$, we have $H(\text{graph } (f)) = \text{graph } (H \cdot f)$.

Definition 2.1.4.

(1) We say that two germs are contact equivalent *or* $\mathcal{K}$-equivalent *if they are in the same orbit of the action of* $\mathcal{K}$.

(2) We define a contact singularity *or a* $\mathcal{K}$-singularity *to be a* $\mathcal{K}$*-equivalence class of germs.*

As we will primarily be concerned with $\mathcal{K}$-singularities in what follows, we will drop the $\mathcal{K}$ and refer to them as simply *singularities*.

We would like to introduce the notion of a local algebra. Let $\mathcal{E}(m) := \mathcal{E}(m,1)$ be the vector space of germs of holomorphic functions. Pointwise multiplication makes $\mathcal{E}(m)$ a local algebra with maximal ideal $\mathcal{E}^0(m,1) := \mathfrak{m}_m$. For a given $f \in \mathcal{E}^0(m,n)$ we have a pull-back ring homomorphism $f^* : \mathcal{E}(n) \to \mathcal{E}(m)$ given by composition with f.

Definition 2.1.5. *For an* $f \in \mathcal{E}^0(m,n)$ *we define the* local algebra of f *denoted* Q_f *as* $\mathcal{E}(m)/(f^*(\mathfrak{m}_n))$.

We now quote the following theorem.

Theorem 2.1.6. [**15**] *Map germs* f *and* $g \in \mathcal{E}^0(m,n)$ *are in the same orbit of* $\mathcal{K}$*'s action on* $\mathcal{E}^0(m,n)$ *if and only if* $Q_f \cong Q_g$.

From here on we will be concerning ourselves only with germs f whose local algebras are finite dimensional. Such germs are called *finite* and their local algebras have the following

description: let $f \in \mathcal{E}^0(m,n)$ be given in local coordinates $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$, then $Q_f \cong \mathbb{C}[[x_1, \ldots, x_m]]/(f_1, \ldots, f_n)$.

Remark 2.1.7. It should be noted that $\mathcal{A} \subsetneq \mathcal{K}$ and that the inclusion is indeed proper. For example let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be given by $(x, y) \mapsto (x^3, y)$ and $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ by $(x, y) \mapsto (x^3 + xy, y)$. These two germs are in the same $\mathcal{K}$ orbit as both of their local algebras are isomorphic to $\mathbb{C}[[x]]/(x^3)$, but they are not in the same $\mathcal{A}$-orbit. One way to see this is to examine the images of the points where the derivative drops rank in each case. With f we have that the derivative $df = \begin{pmatrix} 3x^2 & 0 \\ 0 & 1 \end{pmatrix}$ which drops rank on the (complex) line $(0, y)$. These points are mapped to the line $(0, y)$, while for g we have $dg = \begin{pmatrix} 3x^2 + y & x \\ 0 & 1 \end{pmatrix}$ which drops rank on the curve $y = -3x^2$ in the source. These points are mapped onto the curve $-27x^2 = 4y^3$ in the target. This is the so-called "semi-cubical parabola" and is not smooth at the origin.

2.2. Stability

We now move on to another important concept in singularity theory, the notion of stability. Let $\pi_X : E \to X$ be a complex vector bundle over a complex manifold. We denote the space of holomorphic sections of E by $\Gamma(E)$. By viewing each section as a map $s : X \to E$ we can consider the analogous ideas for germs: let $\Gamma(E, q)$ denote the space of holomorphic section germs of E at q. A map $f : X \to Y$ between manifolds induces a pullback $f^* : \Gamma(TY) \to \Gamma(f^*(TY))$ through precomposition with f (TY is the tangent bundle over the manifold Y).

Recall that for holomorphic maps $f : X \to Y$, between manifolds we can view the derivative $df : TX \to f^*(TY)$ as a holomorphic map between vector bundles, sending

the pair (x, v) to $(x, (df)_x(v))$. We thus get a map between sections of bundles, where if $s \in \Gamma(TX)$ is a section of TX, sending $x \mapsto (x, s(x))$, then $(df)(s) \in \Gamma(f^*(TY))$ is defined by $(df)(s)(x) := (x, (df)_x(s(x)))$.

Again, we can localize this idea to germs where if $s : (X, q) \to TX$ is a holomorphic section germ, and $f : (X, q) \to Y$ is a map germ, we can view $(df) : \Gamma(TX, q) \to \Gamma(f^*(TY), q)$ where $(df)(s) : (X, q) \to f^*(TY)$. Sometimes in the literature, $\Gamma(TX)$ is referred to as θ_X and $\Gamma(f^*(TY))$ by θ_f.

Definition 2.2.1. *We say a holomorphic map germ $f : (X, q) \to Y$ is* infinitesimally stable *if*

$$\Gamma(f^*(TY), q) = (df)(\Gamma(TX, q)) + f^*(\Gamma(TY, f(q))), \tag{2.4}$$

that is for every $\tau \in \Gamma(f^*(TY), q)$ we can find a $\zeta \in \Gamma(TX, q)$ and an $\eta \in \Gamma(TY, f(q))$, such that we can write $\tau = (df)(\zeta) + \eta \circ f$.

A map $f : X \to Y$ will be called *locally infinitesimally stable* at a point $q \in X$ if the map germ of f at q is infinitesimally stable. The motivation for Definition 2.2.1 comes from the following.

For a (finite dimensional) Lie group $\mathfrak{G}$ acting on a (finite dimensional connected) manifold M, each element $x_0 \in M$ gives the orbit map $\alpha_{x_0} : \mathfrak{G} \to M$ where $g \mapsto g \cdot x_0$. If we let e denote the identity of $\mathfrak{G}$, then the implicit function theorem implies that surjectivity of the derivative of the orbit map at e

$$(d\alpha_{x_0})_e : T_e\mathfrak{G} \to T_{x_0}M \tag{2.5}$$

is equivalent to the orbit of x_0 being open.

Now let $D(X)$ be the group of biholomorphisms of the manifold X. Let $C^\infty(X,Y)$ be the space of holomorphic maps between complex manifolds X and Y, and let $\mathfrak{G}$ be the product of the groups $D(X)$ and $D(Y)$. The group $\mathfrak{G}$ acts on $C^\infty(X,Y)$ by

$$(\varphi,\psi)\cdot f = \psi\circ f\circ\varphi^{-1} \tag{2.6}$$

The condition for local surjectivity of the derivative of the orbit map α_f for a point $f\in C^\infty(X,Y)$ at $e = id_X\times id_Y$ becomes

$$T_f C^\infty(X,Y) = (d\alpha_f)\mathfrak{G}_e \tag{2.7}$$

The identification of

$$T_f C^\infty(X,Y)\approx\Gamma(f^*(TY)) \text{ and}$$

$$(d\alpha_f)\mathfrak{G}_e\approx(d\alpha_f)(\Gamma(TX))+f^*\Gamma(TY)$$

leads to the prescribed condition for local infinitesimal stability.

Definition 2.2.2. *We will call a map germ* $f:(\mathbb{C}^m,0)\to\mathbb{C}^n$ stable *if it is infinitesimally stable.*

Remark 2.2.3. It is a fact that if f is a stable germ, then any other element of the $\mathcal{A}$-orbit of f is also stable. This is not the case with $\mathcal{K}$. Let f and g be as in Remark 2.1.7, that is $f,\ g:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $f:(x,y)\mapsto(x^3,y)$ and $g:(x,y)\mapsto(x^3+xy,y)$. Then we claim that f is not stable, while g is. First observe that we can identify

$$\Gamma(f^*(TY),0)\approx\Gamma(g^*(TY),0)\approx\Gamma(TX,0)\approx\Gamma(TY,0)\approx\mathcal{E}(2,2). \tag{2.8}$$

Stability of the germ f is then equivalent to being able to write

$$\mathcal{E}(2,2) = (df)\mathcal{E}(2,2)+f^*(\mathcal{E}(2,2)) \tag{2.9}$$

We claim that the map germ $(x, y) \mapsto (x, 0)$ is not contained in the right hand side of equation (2.9). Observe that we can view $\mathcal{E}(2,2)$ as an $\mathcal{E}(2)$-module, where if $h \in \mathcal{E}(2,2)$ in local coordinates has the form $h = (h_1, h_2)$ then for $f \in \mathcal{E}(2)$, $f \cdot h := (fh_1, fh_2)$. With this in mind, $(df)\mathcal{E}(2,2)$ is the submodule of $\mathcal{E}(2,2)$ generated by the derivatives of the coordinate functions of f. That is $(df)\mathcal{E}(2,2)$ is the submodule $\mathcal{E}(2) \cdot \{(3x^2, 0), (0, 1)\}$ so map germs $h \in (df)\mathcal{E}(2,2)$ take the form $h : (x, y) \mapsto h_1 \cdot (3x^2, 0) + h_2 \cdot (0, y)$ where $h_i \in \mathcal{E}(2)$ are (holomorphic) function germs. In particular, by writing out the Taylor expansion of $h_1 \cdot (3x^2, 0)$, we see that the lowest possible degree of the variable x in the first coordinate is 2. Next we examine $f^*(\mathcal{E}(2,2))$. For map germs $k \in \mathcal{E}(2,2)$, write k out locally as its coordinate Taylor expansions $k : (x, y) \mapsto (\sum a_{i,j} x^i y^j, \sum b_{i,j} x^i y^j)$, then precomposition with f implies that the lowest possible non-zero degree of x in the first coordinate is 3. So, in particular, the right hand side of equation (2.9) does not contain the map germ $(x, y) \mapsto (x, 0)$.

For the map g, we will show that all polynomial maps of the left hand side of equation (2.9) can be attained. For simplicity in what follows we will refer to a map germ by only its target value, e.g. the map germ $h : (x, y) \mapsto (x, 0)$ will be referred to as simply $(x, 0)$. Polynomials coming from $(dg)\mathcal{E}(2,2)$ will be finite sums of maps of the form

$$h_1 \cdot (3x^2 + y, 0) + h_2 \cdot (x, 1), \text{ where } h_i \in \mathcal{E}(2). \tag{2.10}$$

Polynomials coming from $g^*(\mathcal{E}(2,2))$ will be finite sums of maps of the form

$$k_1((x^3 + xy)^i y^j, 0) + k_2(0, (x^3 + xy)^i y^j), \text{ where } k_i \in \mathbb{C} \text{ and } i, j \in \mathbb{Z}_{\geq 0}. \tag{2.11}$$

Let $\mathcal{P}$ denote the set of polynomial map germs that are in $(dg)(\mathcal{E}(2,2)) + g^*(\mathcal{E}(2,2))$. The first observation that we make is that for $k \geq 1, (x^k, 0) \in \mathcal{P}$ implies that $(0, x^{k-1}) \in$

$\mathcal{P}$, since $(0, x^{k-1}) = x^{k-1}(x, 1) - (x^k, 0)$. The second observation we make is that if $(h_1(x, y), 0) \in \mathcal{P}$, where h_1 is a polynomial in x and y, then $(yh_1(x, y), 0) \in \mathcal{P}$ and if $(0, h_2(x, y)) \in \mathcal{P}$, then $(0, yh_2(x, y)) \in \mathcal{P}$ for h_2 a polynomial in x and y. These two observations reduce our task to showing that $(x^k, 0) \in \mathcal{P}$ for all $k \geq 1$. We proceed by induction on k. First, $(x, 0) = (x, 1) - (0, 1)$. Next assume that $(x^{k-1}, 0) \in \mathcal{P}$, and observe that $(x^k, 0) = \frac{1}{3}x^{k-1}(3x^2 + y, 0) - \frac{1}{3}(x^{k-1}y, 0)$. That the rest of the holomorphic germs in $\mathcal{E}(2, 2)$ are also in $(dg)(\mathcal{E}(2, 2)) + g^*(\mathcal{E}(2, 2))$, will follow from the next section.

2.3. Stability and Unfoldings

Given a germ $f : (\mathbb{C}^m, 0) \to \mathbb{C}^n$, we would like to find a new germ (perhaps between spaces of different dimensions) which has the same local algebra as f, but is *stable* . It turns out that for *finite* germs this is always possible.

Definition 2.3.1. *An* (r-parameter) unfolding *of a map germ* $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ *is a map germ* $F : (\mathbb{C}^{m+r}, 0) \to (\mathbb{C}^{n+r}, 0)$ *where if* $\boldsymbol{x} := (x_1, \ldots, x_m)$ *are the first* m *coordinates and* $\boldsymbol{u} := (u_1, \ldots u_r)$ *are the last* r *coordinates, we have* $F(\boldsymbol{x}, \boldsymbol{u}) = (f_1(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{u})$ *where* $f_1(\boldsymbol{x}, 0) = f(\boldsymbol{x})$

Notice that an unfolding always exists, for we can always define the unfolding $F(\mathbf{x}, \mathbf{u}) = (f(\mathbf{x}), \mathbf{u})$. We can think of an unfolding as an r-parameter family of maps containing f, and parametrized by the coordinates of $\mathbb{C}^r$.

Definition 2.3.2. *Let* $F : (\mathbb{C}^{m+r}, 0) = (\mathbb{C}^m \times \mathbb{C}^r, (0, 0)) \to (\mathbb{C}^n \times \mathbb{C}^r, (0, 0))$ *be an* r*-parameter unfolding of a map germ* f*. Then* F *is* trivial *if there exist biholomorphic map germs* $G \in \mathcal{D}_{m+r}$ *and* $H \in \mathcal{D}_{n+r}$ *such that* G *and* H *are unfoldings of the identity*

in $\mathcal{D}_m$ and $\mathcal{D}_n$ respectively, and such that the following diagram commutes

$$\begin{array}{ccc} (\mathbb{C}^m \times \mathbb{C}^r, (0,0)) & \xrightarrow{F} & (\mathbb{C}^n \times \mathbb{C}^r, (0,0)) \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle H} \\ (\mathbb{C}^m \times \mathbb{C}^r, (0,0)) & \xrightarrow[f \times id_r]{} & (\mathbb{C}^n \times \mathbb{C}^r, (0,0)) \end{array}$$

where id_r is the identity element in $\mathcal{D}_r$.

If f is a finite map germ we can consider its local algebra Q_f. We then observe that any unfolding $F(\mathbf{x}, \mathbf{u}) = (f_1(\mathbf{x}, \mathbf{u}), \mathbf{u})$ has an isomorphic local algebra. Viewed as a map germ $F : (\mathbb{C}^{m+r}, 0) \to \mathbb{C}^{n+r}$ we can then ask whether or not F is stable, in the sense previously described.

Definition 2.3.3. *A stable unfolding of f for which the dimension of the parameter space is minimal will be called a* minimal unfolding *of f.*

We can determine a stable unfolding using the following process (for a more complete description see for example, [**6**]). For simplicity consider germs $f \in \mathcal{E}^0(m,n)$ as above, but with $df = 0$ at 0. We have that $\Gamma(f^*(T\mathbb{C}^n), 0) \approx \mathcal{E}(m,n)$. Recall that $\mathcal{E}^0(m,n)$, is the submodule of $\mathcal{E}(m,n)$ viewed as a module over $\mathcal{E}(m)$, of germs mapping 0 to 0. Let $I(f)^{(n)}$ be the $\mathcal{E}(m)$-submodule in $\mathcal{E}^0(m,n)$ generated by $(f^*(\mathfrak{m}_n))$, that is $I(f)^{(n)} := f^*(\mathfrak{m}_n) \cdot \mathcal{E}^0(m,n)$. Let D be the $\mathcal{E}(m)$-submodule of $\mathcal{E}^0(m,n)$ generated by $\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}\}$. Consider the $\mathcal{E}(m)$ module

$$N(f) = \mathcal{E}^0(m,n) \Big/ (D + I(f)^{(n)}). \tag{2.12}$$

If $\{\varphi_1, \ldots, \varphi_r\}$ with $\varphi_i \in \mathcal{E}^0(m,n)$ are germs that under projection span $N(f)$ over $\mathbb{C}$, then a stable unfolding $F(\mathbf{x}, \mathbf{u})$ of f is given by

$$F(\mathbf{x}, \mathbf{u}) = (f(\mathbf{x}) + \sum_{i=1}^{r} u_i \varphi_i, u_1, \ldots, u_r). \tag{2.13}$$

Remark 2.3.4. We note that if f is a finite map germ then $f^*(\mathfrak{m}_n)$ will always contain some power of the maximal ideal $\mathfrak{m}_m$. So $I(f)^{(n)}$ will then contain a power of the maximal ideal in each coordinate, and thus $N(f)$ will be finite dimensional as a $\mathbb{C}$ vector space, so there will always exist a stable unfolding of f in some parameter space.

Example 2.3.5. Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be given by $x \mapsto x^4$. The local algebra of f is $Q_f \approx \mathbb{C}[[x]]/(x^4)$. The space $\mathcal{E}^0(1,1)$ is the module of holomorphic function germs in one variable that have Taylor expansion at 0 which is without constant coefficients. The ideal $I(f)^{(1)}$ is the $\mathcal{E}(1)$ submodule generated by the pullback of the function f, this includes for example polynomials of the form x^{4k}. In this example D is the submodule generated by df, $D = \mathcal{E}(1) \cdot \{4x^3\}$. Thus we have that $N(f)$ is the (finite dimensional) quotient of $\mathcal{E}^0(1,1)$ spanned by representatives of the images of $\{x, x^2\}$ under the quotient map, so a stable unfolding of f is given by $F : \mathbb{C}^{1+2} \to \mathbb{C}^{1+2}$ where

$$F(x, u_1, u_2) = (x^4 + u_1 x^2 + u_2 x, u_1, u_2). \tag{2.14}$$

Example 2.3.6. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0)$ be given by $(x, y) \mapsto (x^2, xy, y^3, 0)$. The local algebra of f is $Q_f \approx \mathbb{C}[[x, y]]/(x^2, xy, y^3)$. The space $\mathcal{E}^0(2,4)$ is the module of holomorphic function germs in two variables in four coordinates, whose Taylor expansion at 0 is without constant coefficients. The submodule D is generated by the elements $\{(2x, y, 0, 0), (0, x, 3y^2, 0)\}$, and the submodule $I(f)^{(4)}$ by the coordinate functions of f in each of the four coordinates, we have that $N(f)$ as an $\mathcal{E}(2)$-module is spanned by the nine elements

$$\{(x,0,0,0), (y,0,0,0), (y^2,0,0,0), (0,x,0,0), (0,0,x,0),$$
$$(0,0,y,0), (0,0,0,x), (0,0,0,y), (0,0,0,y^2)\}.$$

So we can describe a stable unfolding of f as $F(x, y, u_1, \ldots, u_9) =$

$$(x^2 + u_1x + u_2y + u_3y^2, xy + u_4x, y^3 + u_5x + u_6y, u_7x + u_8y + u_9y^2, u_1, \ldots, u_9)$$

Remark 2.3.7. Notice that in both of the previous examples we produced stable unfoldings with the same *relative codimension* as the original map. That is the number $\ell = n - m$ was the same for the unfolding as it was for the orginal map. If we are given a finitely generated, commutative local algebra with identity Q and a minimal presentation (i.e. $Q \approx \mathbb{C}[[x_1, \ldots, x_a]]/(r_1, \ldots, r_b)$ where the number of variables x_i and relations r_j are minimal), and a fixed relative codimension $\ell \geq (b - a) \geq 0$, we can consider the so-called *genotype map germ* (or just the *genotype*) f of a finite singularity with local algebra Q and fixed relative codimension ℓ, given by $f : (\mathbb{C}^a, 0) \to (\mathbb{C}^{a+\ell}, 0)$, $(x_1, \ldots, x_a) \mapsto (r_1, \ldots r_b, \underbrace{0, \ldots, 0}_{\ell-(b-a) \text{ zeros}})$ Any stable unfolding of f will, by virtue of the process used, have the same relative codimension. If we let F denote a minimal stable unfolding of f, then we have the following theorem.

Theorem 2.3.8. [**26**] *Fix a finite dimensional, commutative local algebra with identity Q, and a minimal presentation $Q \approx \mathbb{C}[[x_1, \ldots, x_a]]/(r_1, \ldots r_b)$. Fix a relative codimension ℓ. Let f be the genotype map corresponding to the minimal presentation of Q and the relative codimension ℓ. Let F be a minimal unfolding of f. Then all stable $\mathcal{A}$-singularities with local algebra Q and relative codimension ℓ are trivial unfoldings of F.*

The map F in the previous theorem is referred to as the *prototype* of the singularity with local algebra Q and relative codimension ℓ. We also have the following theorem.

Theorem 2.3.9. [**26**] *Let f and $g \in \mathcal{E}^0(m, n)$ be two stable $\mathcal{K}$-equivalent germs with the same relative codimension. Then f and g are $\mathcal{A}$-equivalent.*

From now on all local algebras will be as in Theorem 2.3.8, that is finite dimensional, commutative and with unity. We are now able to list representatives ('normal forms') of all stable $\mathcal{A}$-singularities with a specified local algebra. We let ℓ vary, and then construct the minimal unfolding of the genotype map. For example if $Q \approx \mathbb{C}[[x]]/(x^3)$, and $\ell = 2$, then the genotype map $x \mapsto (x^3, 0, 0)$. A minimal unfolding is then $(x, u, v_1, w_1, v_2, w_2) \mapsto (x^3+ux, v_1x+w_1x^2, v_2x+w_2x^2)$. If $\ell = 3$, the genotype map is $x \mapsto (x^3, 0, 0, 0)$. A minimal unfolding is $(x, u, v_1, w_1, v_2, w_2, v_3, w_3) \mapsto (x^3 + ux, v_1x + w_2x^2, v_2x + w_2x^2, v_3x + w_3x^2)$.

Remark 2.3.10. Fix a local algebra Q, and a minimal presentation for it, $\mathbb{C}[[x_1, \ldots, x_a]]/(r_1, \ldots, r_b)$. It follows from Theorem 2.3.8 that the minimal relative codimension of a stable germ with local algebra Q is $(b - a)$. For each ℓ, the minimal source dimension for a stable germ with local algebra Q is $a + \dim(N(f))$, where f is the genotype map for such a local algebra and ℓ. Since we fixed the local algebra, we are left with a function of ℓ. The minimal source dimension for a stable germ with local algebra Q will turn out to be an important invariant of a given singularity. From here on we will refer to it as the *codimension* of the singularity.

For example let $Q \approx \mathbb{C}[[x]]/(x^3)$. The minimal value for ℓ is 0. When $\ell = 0$ we have that the dimension of $N(f) = 1$, so the codimension of the singularity is 2. Recall that when $\ell = 0$, the genotype f maps $x \mapsto x^3$, and we have that the image of $\{x\}$ is a basis for $N(f)$. When $\ell = 1$, we have a new genotype map (which we call by the same name as the previous case), f mapping $x \mapsto (x^3, 0)$, and the projections of$\{(x, 0), (0, x^2), (0, x)\}$ form a spanning set for $N(f)$. So the codimension is 4. Continuing on in this fashion shows that each time ℓ increases by 1, two new dimensions are added to $N(f)$ (as an $\mathbb{C}$

vector space). So we have that the codimension of the singularity with local algebra Q (as a function of ℓ) is $1 + 1 + 2\ell = 2\ell + 2$.

2.4. Jet Bundles and Multi-Jet Bundles

In this section we follow [**15**].

Definition 2.4.1. *Let M, N be holomorphic manifolds, and p a point in M. Suppose that $f, g : (M, p) \to (N, q)$ are holomorphic map germs.*

(1) The map germ f has first order contact *with g if $(df)_p = (dg)_p$ as mappings of $T_pM \to T_qN$.*

*(2) The map germ f has k*th order contact *with g if $(df) : TM \to TN$ has $(k-1)$st order contact with (dg) for every v in T_pN. This is written as $f \sim_k g$.*

*(3) Let $J^k(M, N)_{p,q}$ denote the set of equivalence classes under "$\sim_k$" of holomorphic map germs $g : (M, p) \to (N, q)$. Let $j^k g$ denote its equivalence class in $J^k(M, N)_{p,q}$. This is called the k-*jet *of the map germ g. We call p the* source *of $j^k g$ and q the* target.

(4) Let $J^k(M, N) = \bigcup_{(p,q)\in M\times N} J^k(M, N)_{p,q}$ (disjoint). We call the set $J^k(M, N)$ the jet space *of M and N, or the k-*jet space *of M and N.*

*(5) Let $f : M \to N$ be a holomorphic map. The k-*jet of f *is defined by $j^k f : M \to J^k(M, N), j^k f(p) = \{$ the equivalence class of the germ of f at p in $J^k(M, N)_{p,f(p)}\}$ for every p in M.*

The motivation for Definition 2.4.1 is the following

Proposition 2.4.2. [**15**] *Let U be an open subset of $\mathbb{C}^m$ containing the point p. The maps f and $g : U \to \mathbb{C}^n$ have k-th order contact at p if and only if the Taylor expansions of f and g up to and including order k are identical at p.*

Let U, V be open subsets of $\mathbb{C}^m, \mathbb{C}^n$ respectively. Let $J^k(m,n)$ denote n copies of the vector space of polynomials in m variables of degree less than or equal to k without constant term. The previous Proposition implies that we can give $J^k(U,V)$ a natural complex manifold structure that is biholomorphic to $U \times V \times J^k(m,n)$. Indeed, we can think of $J^k(U,V)$ as a vector bundle over $U \times V$ with fiber the vector space $J^k(m,n)$. This leads to the following theorem.

Theorem 2.4.3. [**15**] *Let M and N be complex manifolds with dimensions m and n respectively. Then,*

(1) $J^k(M,N)$ is a complex manifold of dimension $= m + n +$ dim $(J^k(m,n))$.

(2) $J^k(M,N) \to M \times N$ is a complex fiber bundle with fiber $J^k(m,n)$. Furthermore $J^k(M,N) \to M$ and $J^k(M,N) \to N$ are also complex fiber bundles. Let α and β denote the (holomorphic) projections onto M and N respectively.

(3) if $f : M \to N$ is holomorphic, then $j^k f : M \to J^k(M,N)$ is a holomorphic section.

The map $\alpha : J^k(M,N) \to M$ is called the *source map* and $\beta : J^k(M,N) \to N$ is called the *target map* .

Since we are interested in multisingularities, we must also introduce the concept of a multi-jet. Let M^s denote the s-fold product of the complex manifold M. Let $\Delta = \{(x_1, \ldots, x_s) \in M^s | x_i = x_j \text{ for some } i < j\}$ be the generalized diagonal of M^s. Denote $M \setminus \Delta$ by $M^{(s)}$. With $\alpha : J^k(M,N) \to M$ the source map as before, we let

α^s denote the projection $(J^k(M,N))^s \to M^s$. Then $J^k_s(M,N) := (\alpha^s)^{-1}(M^{(s)})$ is called the *s-fold k-jet bundle.* We note that $J^k_s(M,N)$ is a holomorphic manifold, and for any holomorphic map $f : M \to N$ we have a holomorphic map $j^k_s f : M^{(s)} \to J^k_s(M,N)$ called the *s-fold k-jet of f,* defined by $j^k_s f : (x_1,\ldots,x_s) \mapsto (j^k f(x_1),\ldots,j^k f(x_s))$. This bundle will be used to phrase the concept of admissibility for multisingularities (c.f. Section 2.6).

Finite Determinacy.

Definition 2.4.4. *Let f be a map germ in $\mathcal{E}^0(m,n)$. We say that f is* k-determined *if for any $g \in \mathcal{E}^0(m,n)$ such that $j^k g = j^k f$, the $\mathcal{K}-$orbit of f contains g. We will say that f is* finitely determined *if it is k-determined for some positive integer k.*

Recall that for a germ $f \in \mathcal{E}^0(m,n)$ we had the $\mathcal{E}(m)$ module $N(f) = \mathcal{E}^0(m,n)\big/(D + I(f)^{(n)})$ defined in equation(2.12). We have the following theorem of Mather.

Theorem 2.4.5 (Finite Determinacy Theorem). [**25**] *A necessary and sufficient condition for a map germ $f \in \mathcal{E}^0(m,n)$ to be finitely determined is that $N(f)$ be a finite dimensional $\mathbb{C}$ vector space.*

For a finite singularity η with local algebra Q_η we let $\eta^k_0 :=$ { the germs of the k-th order Taylor approximation of the germs in η}. Since locally the k-jet bundle, $J^k(M,N)$, is biholomorphic to $U \times V \times J^k(m,n)$, where $U \subseteq \mathbb{C}^m$ and $V \subseteq C^n$, and η^k_0 is invariant under the action of the structure group of this bundle (see Section 3.4), we can consider η^k_0 in each fiber. Thus we obtain a submanifold within this bundle, which we will call η^k.

Example 2.4.6. Consider $J^4(\mathbb{C},\mathbb{C})$. Let η be the singularity with local algebra $\mathbb{C}[[x]]/(x^3)$. Then η^4 is the set of fourth degree Taylor approximations to germs in

η. We view $J^4(\mathbb{C},\mathbb{C})$ as a $\mathbb{C}^4$ bundle over $\mathbb{C}\times\mathbb{C}$ with each fiber parametrized by the polynomials without constant term of degree less than or equal to four in one variable x. That is, each fiber is the vector space described by $\{a_1x+a_2x^2+a_3x^3+a_4x^4\}$ where the a_i's $\in\mathbb{C}$. We have that η^4 is the subbundle defined by the subspace $(0,0,a_3,a_4)$ in each fiber. Note that the codimension of η^4 in the jet space is equal to two.

Example 2.4.7. Now consider $J^4(\mathbb{C},\mathbb{C}^2)$. We consider the η with the same local algebra as in the previous example. This time the fibers of the jet bundle are parametrized by pairs of polynomials without constant term of degree less than or equal to four in one variable; i.e. each fiber can be described by $\{a_1x+a_2x^2+a_3x^3+a_4x^4, b_1x+b_2x^2+b_3x^3+b_4x^4\}$ where a_i and $b_i\in\mathbb{C}$. Here, we have that η^4 is the subbundle defined by the subspace $(0,0,a_3,a_4,0,0,b_3,b_4)$ in each fiber. Note that the codimension of η^4 in the jet space is equal to four $=2\times1+2$.

Codimension.

We note that had we considered $J^{k\geq4}(M,N)$ in Examples 2.4.6 and 2.4.7, the codimension of η^k in $J^k(M,N)$ would have been the same. It is a corollary of Theorem 2.4.5 that this always happens for finite germs.

Corollary 2.4.8. *Let η be a finite singularity, then for k large enough, the codimension of η^{k+i} in $J^{k+i}(M,N)$ is the same for all $i\geq0$.*

We will call this the *k-codimension of the singularity η.* For a finite singularity we also have a geometric interpretation of the codimension of $\eta\subseteq\mathcal{E}^0(m,n)$ as a $\mathcal{K}$ orbit (specifically the dimension of $T_f(\mathcal{E}^0(m,n))/T_f(\eta)$ for an $f\in\eta$). We will call this the *geometric codimension.* We thus have three notions of codimension for a finite singularity, that

which we defined in Remark 2.3.10, the k-codimension, and the geometric codimension. It is a corollary of Theorem 2.4.5 that these three codimensions are all the same.

COROLLARY 2.4.9. *For a finite singularity η, the codimension is equal to the geometric codimension which is equal to the k-codimension.*

From now on we will simply refer to all of these as the codimension of η denoted $codim(\eta)$.

2.5. Singularities and Multisingularities

Singularities.

REMARK 2.5.1. For fixed integers $m < n$, we have already stated that the group $\mathcal{K}_{m,n}$ contains $\mathcal{A}$, the group of holomorphic reparametrizations of the source $(\mathbb{C}^m, 0)$ and target spaces $(\mathbb{C}^n, 0)$. Hence for a map $f : M \to N$ between manifolds of dimension m and n respectively it makes sense to talk about the contact singularity of f at a point in M. So for a map $f : M \to N$ and a singularity $\eta \subset \mathcal{E}^0(m, n)$, we can define the singularity subset

$$\eta(f) = \{x \in M | \text{ the germ of } f \text{ at } x \text{ belongs to } \eta\}.$$

Theorem 2.3.8 implies that the classification of finite singularities is roughly the same as the classification of finite dimensional commutative local $\mathbb{C}$-algebras. Only 'roughly', because for a given m and n only algebras that can be presented by m generators and n relations turn up as local algebras of singularities of map germs $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$.

A natural approach is to try to classify singularities in the order of their codimensions in $\mathcal{E}^0(m,n)$. For large enough ℓ the classification of small codimensional singularities is as follows (see e.g. [**1**]).

codim	0	$\ell+1$	$2\ell+2$	$2\ell+4$	$3\ell+3$	$3\ell+4$	$3\ell+5$
Σ^0	A_0						
Σ^1		A_1	A_2		A_3		
Σ^2				$III_{2,2}$		$I_{2,2}$	$III_{2,3}$

Here we use the following notations: A_i means the singularity with local algebra $\mathbb{C}[[x]]/(x^{i+1})$; $I_{a,b}$ means the singularity with local algebra $\mathbb{C}[[x,y]]/(xy, x^a+y^b)$; and $III_{a,b}$ means the singularity with local algebra $\mathbb{C}[[x,y]]/(x^a, xy, y^b)$. The symbol Σ^r is a property of a singularity, it means that the derivative drops rank by r, equivalently, that the local algebra can be minimally generated by r generators. The $\Sigma^{\leq 1}$ singularities are called Morin singularities (a.k.a. corank 1, or curvilinear singularities). As one studies singularities of high codimension, they appear in moduli. However such singularities will not play a role in what follows.

Observe that we gave the classification independent of m and n, that is, we gave the same name for singularities for different dimension settings. E.g. the following are all A_2 germs: $x \mapsto x^3$ $(m=n=1)$, $(x,y) \mapsto (x^3,y)$ $(m=2, n=2)$, $(x,y) \mapsto (x^3+xy, y)$ $(m=n=2)$. The essential difference between these two as we have shown, is that the second one is *stable* (it is called cusp singularity), the first is not.

Singularity subsets stratify the source space of a map $f: M \to N$ between manifolds. We want to study a finer stratification — one which corresponds to multisingularities.

Multisingularities.

Consider contact singularities $\alpha_i \subset \mathcal{E}^0(m,n)$, with $m < n$.

Definition 2.5.2. *A* multisingularity $\underline{\alpha}$ *is a multi-set of singularities* $(\alpha_1, \ldots, \alpha_r)$ *together with a distinguished element, denoted* α_1.

We define the codimension of a multisingularity $(\alpha_1, \ldots, \alpha_r)$ by

$$codim(\underline{\alpha}) = (r-1)\ell + \sum codim(\alpha_i). \tag{2.15}$$

Observe that the codimension does not depend on the order of the monosingularities. The list of multisingularities of small codimension (when ℓ is large enough) is given in the following table.

codim	0	ℓ	ℓ+1	2ℓ	2ℓ+1	2ℓ+2	$2\ell+4$	3ℓ	$3\ell+1$	$3\ell+2$	$3\ell+3$	$3\ell+4$
Σ^0	A_0	A_0^2		A_0^3				A_0^4				
Σ^1			A_1		A_1A_0	A_2			$A_1A_0^2$	A_2A_0 A_1^2	A_3	
Σ^2							$III_{2,2}$					$III_{2,2}A_0$ $I_{2,2}$

Here we used the notation $\alpha_1\alpha_2\ldots$ for the multiset $(\alpha_1, \alpha_2, \ldots)$, and any of its permutations. We define Σ^i for a multisingularity to be the Σ^j where j is maximal in the list of Σ^js corresponding to the α_js in $\underline{\alpha}$.

Definition 2.5.3. *Let* $f : M \to N$ *be a holomorphic map of complex manifolds, and* $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ *a multisingularity. We define the following multisingularity loci in the source* M *and the target* N

$$M_{\underline{\alpha}} = \{x_1 \in M | f(x_1) \text{ has exactly } r \text{ pre-images } x_1, \ldots, x_r, \text{ and } f \text{ has singularity } \alpha_i \text{ at } x_i\},$$

and $N_{\underline{\alpha}} = f(M_{\underline{\alpha}})$.

If we permute the monosingularities in $\underline{\alpha}$, i.e. choose another singularity to be α_1, then $M_{\underline{\alpha}}$ may change, while $N_{\underline{\alpha}}$ does not.

2.6. Admissible Maps

We wish to study certain identities among cohomology classes represented by multisingularity loci. We can only expect such identities if the map satisfies certain transversality conditions. We will define these maps presently and call them 'admissible'.

For a holomorphic map $f : M \to N$ between complex manifolds, we can consider the holomorphic map $j^k f : M \to J^k(M, N)$. For a large enough k we have the submanifold η^k which has codimension $= codim(\eta)$ in $J^k(M, N)$. Moreover, Theorem 2.4.5 implies that the set $\eta(f)$ is the projection onto M of the intersection of η^k and the image of $j^k f(M)$. In particular if $j^k f$ is transversal to η^k then the codimension of $\eta(f)$ in M is the same as the codimension of η. We will call maps f satisfying such a transversality property *admissible* for the monosingularity η.

We need, however, the admissibility property for multisingularities as well. Again, let $f : M \to N$ be a holomorphic map between complex manifolds of relative codimension $\ell = n - m > 0$. Let $\underline{\alpha} := (\alpha_1, \ldots, \alpha_s)$ be a multisingularity. For each $\alpha_i \in \underline{\alpha}$ we have a submanifold $\alpha_i^k \subseteq J^k(M, N)$. For the manifold N^s let $B := \{(y_1, \ldots, y_s) | y_i = y_j$ for all $i, j\}$. Recall that for ordinary k-jet space we have a projection map $\beta : J^k(M, N) \to N$ onto the target manifold, and for the s-fold k-jet we have the map $\beta^s : J_s^k(M, N) \to N^s$ onto the s-fold product of N. The mulitsingularity submanifold of the s-fold k-jet space

corresponding to $\underline{\alpha}$ is defined as

$$A^k(\underline{\alpha}) := \left(\alpha_1^k \times \alpha_2^k \times \cdots \times \alpha_s^k\right) \bigcap (\beta^s)^{-1}(B) \bigcap J_s^k(M,N). \tag{2.16}$$

For large enough k the submanifold $A^k(\underline{\alpha}) \subseteq J^k(M,N)$ has codimension $= codim(\underline{\alpha}) + (s-1)n$ in $J_s^k(M,N)$. Just as in the mono-singularity case, when k is large enough, Theorem 2.4.5 implies that the set $M_{\underline{\alpha}}$ is the projection onto the first factor of $M^{(s)}$ of the intersection of $A^k(\underline{\alpha})$ and the image of $j_s^k(M^{(s)})$. So, in particular, if $j_s^k f$ is transversal to $A^k(\underline{\alpha})$, then the codimension of $M_{\underline{\alpha}}$ in M is the same as $codim(\underline{\alpha})$, and the codimension of $N_{\underline{\alpha}}$ is $codim(\underline{\alpha}) + \ell$. We will call such maps for which the s-fold k-jet is transversal to $A^k(\underline{\alpha})$, *admissible* for the multisingularity $\underline{\alpha}$.

CHAPTER 3

Thom Polynomials and Residue Polynomials

3.1. Cohomology Classes Represented by Multisingularity Submanifolds

Let $f : M \to N$ be an admissible map for the multinsingularity $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ between compact complex manifolds of (complex) dimensions $m < n$ respectively. Let ℓ denote the relative codimension $n - m$ of the map f. Denote by $\overline{m}_{\underline{\alpha}} = \left[\bar{M}_{\underline{\alpha}}\right] \in H^{2codim\underline{\alpha}}(M; \mathbb{Q})$ the cohomology class Poincaré dual to the closure of the $\underline{\alpha}$-multisingularity locus in the source, and $\overline{n}_{\underline{\alpha}} = \left[\bar{N}_{\underline{\alpha}}\right] \in H^{2(\ell+codim\underline{\alpha})}(N; \mathbb{Q})$, the Poincaré dual to the $\underline{\alpha}$-multisingularity locus in the target. Unless otherwise stated, all cohomologies are to be taken with coefficents in $\mathbb{Q}$.

We denote the Poincaré duality isomorphism for the manifold M, $D_M : H^k(M) \approx H_{n-k}(M)$. For a holomorphic map $f : M \to N$ between compact complex manifolds, we recall that the *Gysin map* $f_! : H^k(M) \to H^{k+\ell}(N)$, is defined as $f_! := D_N^{-1} \circ f_* \circ D_M$, where f_* is the ordinary push-forward in homology. The Gysin map observes the following *projection formula* [**11**]. For $x \in H^a(M), y \in H^b(N)$, let $f^* : H^*(N) \to H^*(M)$ denote the pull-back corresponding to the map f. We have,

$$f_!(xf^*(y)) = f_!(x)y. \tag{3.1}$$

Since it is often of use to consider these classes $\overline{m}_{\underline{\alpha}}, \overline{n}_{\underline{\alpha}}$ with their natural multiplicities, we let

$$m_{\underline{\alpha}} = \#Aut(\alpha_2, \ldots, \alpha_r)\overline{m}_{\underline{\alpha}}, \qquad n_{\underline{\alpha}} = \#Aut(\alpha_1, \alpha_2, \ldots, \alpha_r)\overline{n}_{\underline{\alpha}},$$

where $\#Aut(\alpha_1, \alpha_2, \ldots, \alpha_r) = \#Aut(\underline{\alpha})$ is the number of permutations $\sigma \in \mathfrak{S}_r$ such that $\alpha_{\sigma(i)} = \alpha_i$ for all i from 1 to r. So if $\underline{\alpha}$ contains k_1 singularities of type α_1, k_2 of type α_2, etc., then $\#Aut(\underline{\alpha}) = k_1!k_2!\ldots$.

On the restriction $f : M_{\underline{\alpha}} \to N_{\underline{\alpha}}$ is a holomorphic covering with the number of sheets equal to the number of α_1 singularities in $\underline{\alpha}$, hence we have the following relation

$$n_{\underline{\alpha}} = f_!(m_{\underline{\alpha}}). \tag{3.2}$$

3.2. Classes of Multisingularity Loci in Terms of Characteristic Classes

Given a complex vector bundle $E \to M$ over a complex manifold, we recall that the *ith Chern class of the bundle*, $c_i(E) \in H^{2i}(M)$ as being a cohomology class satisfying the following axioms:

(1) The class $c_0(E) = 1$ for all vector bundles $\pi : E \to M$. Also if $r > \mathrm{rk}(E)$, then $c_r(E) = 0$.

(2) (Naturality) For a bundle map $f : F \to E$, $f^*(c_i(F)) = c_i(f^*(F))$.

(3) (Whitney axiom) If $F \oplus E$ denotes the Whitney sum of two vector bundles, then $c_i(E \oplus F) = \sum_{k=0}^{i} c_k(F)c_{i-k}(E)$

(4) The first Chern class, c_1, of the tautological bundle over complex projective n space $\mathbb{P}^n$ is $-h$ where h is the class corresponding to a hyperplane in $\mathbb{P}^n$.

The *total Chern class* of the bundle, $c(E)$, is the sum of all the i-th Chern classes of the bundle $c(E) = 1 + c_1(E) + \cdots c_{\mathrm{rk}(E)}(E)$. We can thus re-write axiom (3) as $c(F \oplus E) = c(E)c(F)$. We define $c_i = 0$ for $i \leq 0$. For the proof of the existence of the Chern classes as well as more details concerning their properties, see for example [**4**], [**29**].

For a map $f : M \to N$ between complex manifolds, we define the *total Chern class of the map*, $c(f)$ as

$$c(f) := (c(TM))^{(-1)}c(f^*TN) = \frac{c(f^*TN)}{c(TM)}.$$

This is also described as the total Chern class of the *virtual normal* bundle, ν_f of the map f. The virtual normal bundle is the formal difference $f^*(TN) - TM$ of bundles over M. This is an actual bundle if f is an immersion and $c(f) = c(\nu(f))$.

A classical theorem of Thom [**35**] is that *mono*singularity loci in the source can be expressed as a polynomial (the Thom polynomial) of characteristic classes of the source and target. We summarize the ideas behind this in the following sections.

3.3. Equivariant Cohomology

Recall that a group G acts *freely* on a topological space X if for each $p \in X$, the stabilizer subgroup of p, G_p is the identity, e of the group.

Definition 3.3.1. *(Borel Construction) Let the group G act on the topological space X from the left. Let E be a contractible space on which G acts freely on the right. We define a left action on the product space $E \times X$ by $g \cdot (e, x) := (eg^{-1}, gx)$. We define the G-*equivariant cohomology *of the space X, $H^*_G(X)$, as the singular cohomology of the quotient of $E \times X$ by G's action i.e. $H^*_G(X) := H^*(E \times_G X) := H^*((E \times X)/G)$.*

We remark that $H^*_G(X)$ is functorial in both X and G [**18**]. Also if G acts freely on X then the projection $X \times E \to X$ induces a fibration of the quotients $(X \times_G E) \to X/G$ with fiber E, and since E is contractible, $H^*_G(X) \approx H^*(X/G)$.

We recall that a *principal G-bundle* $X \to Y$ over a (paracompact, Hausdorff) space Y, is a fiber bundle with right action on the total space by the group G which preserves

the fibers and acts freely transitively on them. This means, among other things, that the fibers of the bundle are G. Let EG be a contractible space on which G acts freely, then we can consider $EG \to EG/G$ as a principal G-bundle. Denote the quotient by $BG := EG/G$. We have the following classification theorem for principal G-bundles.

THEOREM 3.3.2. [**16**] *Let Y be a topological space and $X \to Y$ a principal G-bundle. Then there exists a map*

$$K_Y : Y \to BG$$

and an isomorphism of principal G-bundles

$$\Phi : X \approx K_Y^* EG$$

where K_Y^* is the pull-back of the bundle $EG \to BG$. Moreover K_Y and Φ are unique up to homotopy.

The EG in Theorem 3.3.2 is referred to as the *classifying bundle* of G and BG as the *classifying space*. We use definite articles because of the following corollary.

COROLLARY 3.3.3. [**16**] *Definition 3.3.1 is independant of the choice of E. That is if E_1 and E_2 are both contractible spaces on which G acts freely, then there exists G-equivariant homotopy equivalences $\phi : E_1 \to E_2$ and $\psi : E_2 \to E_1$.*

We also have the following.

THEOREM 3.3.4. [**16**] *Classifying bundles (and classifying spaces) exist for all compact Lie Groups.*

Proposition 3.3.5. *(Associtated Bundles) Let V be a complex vector space on which G acts. Let $F \to M$ be a fiber bundle with fiber V and structure Group G. Then $F \approx E \times_G V \to M$, where $E \to M$ is a principal G bundle, and the (right) action of G on the product space $E \times V$ is defined as $g \cdot (e, v) = (eg^{-1}, gv)$.*

Proof. We construct the principal G bundle E (associated to the bundle F) as follows. Let $\{U_j \times V\}$ be a cover of trivializing neighborhoods. A condition for G being the structure group means that we must have a map $U_i \cap U_j \to G$ on the intersections of two neighborhoods in the base space. We replace $U_i \times V$ with $U_i \times G$ and the gluing is done via left multiplication with the targets of the maps $U_i \cap U_j \to G$. Moreover the right action of G on itself commutes with left multiplication, and we have a well defined action on the total space E.

On the product space $E \times V$ with the(right) action defined as above, we then take the quotient by G, $E \times_G V$ and observe that this is isomorphic to the bundle $F \to M$, in the sense that the fibers, trivializing neighborhoods and transition functions are all the same. $\square$

Given a fiber bundle $\pi_M : F \to M$ with fiber V and structure group G, Propositision 3.3.5 and Theorem 3.3.2 enable us to construct a universal bundle with fiber V and structure group G, and classifying maps K_M and K_F which are unique up to homotopy, with the following commutative diagram

$$\begin{tikzcd} F \arrow[r, "K_F"] \arrow[d, "\pi_M"'] & EG \times_G V \arrow[d, "\pi_G"] \\ M \arrow[r, "K_M"] & BG. \end{tikzcd} \tag{3.3}$$

We thus obtain a map $K_M^* : H^*(BG) \to H^*(M)$ on the cohomology level.

G-Invariant Subvarieties.

Let G act on the (complex) vector space V. Let $\eta \subseteq V$ be a G-invariant subvariety. We have the notion of a G - *fundamental class*, $[\eta] \in H^*_G(V) \approx H^*_G(pt) \approx H^*(BG)$. This will also be referred to as the *Thom polynomial* of η, Tp_η or the *equivariant Poincaré dual* of η. If ξ is a G orbit, then by Tp_ξ we will mean $Tp_{\bar{\xi}}$, the Thom polynomial of the closure of ξ in V. The construction of such classes can be approached in various ways, using either spectral sequences as in [**20**], and [**28**] or through a limiting process of associated bundles as in [**13**].

We will outline a few properties that such a class has, particularly those which mimic properties belonging cohomology classes dual to closed subvarieties in the singular cohomology of compact complex manifolds. For proofs see the previous references.

For a fiber bundle $\pi : F \to M$ over a complex compact manifold M, with fiber a complex vector space V and structure group G, let K_M and K_F denote the classifying maps with the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{K_F} & EG \times_G V \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_G} \\ M & \xrightarrow{K_M} & BG. \end{array} \tag{3.4}$$

For a G-invariant subvariety $\eta \subseteq V$, we can construct a subvariety $\eta(F)$ of the total space by considering η in each fiber. Moreover, if we have a section $s : M \to F$ which is transversal to $\eta(f)$, then $s^{-1}(\eta(F))$ is a smooth subvariety of M of codimension = the codimension of $\eta(F)$ in F.

THEOREM 3.3.6. [**13**] [**20**] [**28**] *Let everything be as in diagram 3.4. Let η be a G-invariant subvariety of V. The image of $[\eta]$ under the map K^*_M, $K^*_M([\eta]) = [s^{-1}(\eta(F))]$ in $H^*(M)$.*

Theorem 3.3.6 gives some geometric intuition behind the classes defined by closures of G-orbits. We would like to extend the idea of Chern classes to equivariant cohomology.

If $F \to M$ is an equivariant complex G-bundle, i.e. if $F \to M$ is a fiber bundle with fiber the vector space V and the total space F admits a G-action which preserves fibers, then we can talk about the *i-th equivariant Chern class* of the bundle, $c_i^G(F) \in H_G^{2i}(M)$. Indeed, we follow the Borel construction to get a bundle $EG \times_G F \to EG \times_G M$ and we can take $c_i^G(F)$ to be the i-th Chern class of this bundle (again, see [**13**], [**20**], and [**28**] for a more detailed construction of such a class).

If we are in the situation where a group G acts on a vector space V, and we consider an orbit η with its closure $\overline{\eta}$, we let $sing(\overline{\eta})$ denote the singular locus of $\overline{\eta}$. We have that the normal bundle to $\overline{\eta} \setminus sing(\overline{\eta})$ in V is an equivariant complex G-bundle. Denote this bundle by ν. We have the following theorem.

THEOREM 3.3.7. [**13**] [**20**] [**28**] *Let everything be as in the above paragraph.*

(1) The natural inclusion $\imath : V \setminus \eta \to V$ induces a map on the (equivariant) cohomologies $\imath^* : H_G^*(V) \to H_G^*(V \setminus \eta)$ called restriction *and under this restriction $[\eta] \mapsto 0$.*

(2) Under the restriction homomorphism induced by $\imath : \overline{\eta} \setminus sing(\overline{\eta}) \hookrightarrow V$, the class represented by $\overline{\eta}$, maps to the equivariant Euler class (top Chern class) of ν. i.e. $\imath^* \ : \ H_G^*(V) \to H_G^*(\overline{\eta} \setminus sing(\overline{\eta}))$ with $[\overline{\eta}] \mapsto e^G(\nu)$.

We close with a useful reduction theorem in equivariant cohomology.

THEOREM 3.3.8. [**18**] *Let G be a compact connected Lie group. Let T be a maximal torus, let $N(T)$ be the normalizer of T in G, and W be the Weyl group acting as an*

automorphism group of T. Then

$$H^*(BG) \approx H^*(BN(T)) \approx H^*(BT)^W,$$

where $H^*(BT)^W$ are the elements invariant under the induced action of W on $H^*(BT)$.

3.4. Thom Polynomials of Singularities

We apply the ideas of equivariant cohomology to the jet bundles described earlier.

We can consider the map $j^k : \mathcal{E}^0(m,n) \to J^k(m,n)$ taking each germ to its k-jet. We also have a natural inclusion $i_k : J^k(m,n) \to \mathcal{E}^0(m,n)$, by considering each element of $J^k(m,n)$ as a polynomial map of degree less than or equal to k in m variables. In Section 2.1 we defined an action of $\mathcal{K}$ on $\mathcal{E}^0(m,n)$, and we use i_k and j^k to define an action of $\mathcal{K}$ on $J^k(m,n)$, as follows. For a $p \in J^k(m,n)$, and an $H \in \mathcal{K}$, let $H \cdot p := j^k(H \cdot i_k(p))$.

For a $\mathcal{K}$ orbit $\eta \subseteq \mathcal{E}^0(m,n)$ we can consider its closure $\overline{\eta}$ and then its j^k image in $J^k(m,n)$, which we denote by $\overline{\eta}_0^k$. The subgroup $U(m) \times U(n) \leq \mathcal{K}$ will leave $\overline{\eta}_0^k$ invariant as $\mathcal{K}$ does.

Definition 3.4.1. *Let η be a k-determined singularity. We define the* Thom polynomial *of the singularity η, $Tp_\eta := [\overline{\eta}_0^k] \in H^*_{U(m)\times U(n)}(J^k(m,n))$.*

Given a map $f : M \to N$ between compact complex manifolds we have a bundle ϑ, over the graph of f

$$J^k(M,N) \to \text{graph}(f)$$

(see Section 2.4) with fiber $J^k(m,n)$ and structure group $\mathcal{A}^k := \{$the k-th order Taylor polynomials of elements in $\mathcal{A}\}$. For a given singularitiy η, we can consider the subvariety of $J^k(M,N)$ coresponding to $\overline{\eta}_0^k$ in each fiber, $\overline{\eta}(J^k(M,N))$. We also have a natural

section $j^k f : \text{graph}(f) \to J^k(M,N)$. Theorem 3.3.6 together with the fact that $\mathcal{A}^k$ is homotopy equivalent to $U(m) \times U(n)$ implies that within $H^*(\text{graph}(f)) \approx H^*(M)$, if $j^k f$ is transversal to $\overline{\eta}(J^k(M,N))$ we have:

$$[(j^k f)^{-1}(\overline{\eta}(J^k(M,N)))] = Tp_\eta(\vartheta).$$

Finally we have that Theorem 2.4.5 implies that for large enough k,

$$(j^k f)^{-1}(\overline{\eta}(J^k(M,N))) = \overline{\eta(f)}$$

with $\overline{\eta(f)}$ as defined in Section 2.5.

It has been shown that Thom polynomials of singularities are indeed polynomials in the Chern classes of maps [**5**]. The corresponding theory for multisingularities was developed by Kazarian.

3.5. Residue Polynomials

THEOREM 3.5.1 (Kazarian [**21**]). *For multisingularities $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $J \subset \{1, \ldots, r\}$, let $\bar{J} = \{1, \ldots, r\} \setminus J$. There exist unique polynomials $R_{\underline{\alpha}}$ in the Chern classes of the map f, called* residue (or residual) polynomials*, satisfying*

$$m_{\underline{\alpha}} = R_{\underline{\alpha}} + \sum_{1 \in J \subsetneq \{1,\ldots,r\}} R_{\underline{\alpha}_J} f^*(n_{\underline{\alpha}_{\overline{J}}}) \tag{3.5}$$

for admissible maps. Here the sum is taken over all possible subsets of $\{1, \ldots, r\}$ containing 1. Moreover the residue polynomials are independent of the order of the monosingularities α_i in $\underline{\alpha}$.

In particular, if $\underline{\alpha} = (\alpha)$ is a monosingularity, then (3.5) yields $m_\alpha = R_\alpha$, hence R_α is the Thom polynomial Tp_α of the given singularity written in the Chern classes of f. For example, for $n = m + 1$ we have $R_{A_2} = c_2^2 + c_1 c_3 + 2c_4$; and this means that the

cohomology class represented by points in M where the map has singularity A_2, is equal to $c_2^2 + c_1c_3 + 2c_4$ of the virtual normal bundle of the map.

Observe that in the last example we did not specify m and n, only their difference. This is a classical fact about Thom polynomials: the Thom polynomial of singularities having the same local algebra and the same relative codimension ℓ (but maybe living in different vector spaces $\mathcal{E}^0(m,n)$) are the same when expressed as a polynomial in the Chern classes of the virtual normal bundle [**5**].

We can set $S_{\underline{\alpha}} = f_!(R_{\underline{\alpha}})$, and putting (3.5) together with (3.2) and the projection formula for the Gysin map yields

$$n_{\underline{\alpha}} = S_{\underline{\alpha}} + \sum_{1 \in J \subsetneq \{1,\ldots,r\}} S_{\underline{\alpha}_J} n_{\underline{\alpha}_{\overline{J}}}. \tag{3.6}$$

The proof of equations (3.5) and (3.6) relies on understanding the cohomology of the so-called classifying space for multi-singularities see for example, [**32**] and [**21**].

CHAPTER 4

Calculation of Residue Polynomials via Interpolation

Different calculational techniques for residue polynomials of monosingularities, that is, Thom polynomials of contact singularities, have been studied for decades. An effective technique, which also generalizes to residue polynomials of multisingularities was invented by Rimányi. It is called the interpolation method, and we summarize it in this chapter. For more details and proofs see [**30**], [**9**].

4.1. Maximal Compact Subgroups of Symmetry Groups

For a stable representative of a finite singularity $\xi : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ with local algebra Q_ξ and fixed relative codimension ℓ, we want to consider the equivariant cohomology associated to the stabilizer subgroup Aut_ξ of ξ within $\mathcal{A}$. However, in general, we do not have a topology on $\mathcal{A}$, and so we will not have a topology on these stabilizers either. But we still would like these groups to share some properties with Lie groups. Namely, we want to establish a notion of *compactness* for certain subgroups and show that such subgroups satisfy similar properties to those of compact Lie groups.

In this section we follow [**31**].

Definition 4.1.1. *A subgroup G of Aut_ξ is called* compact *if it is conjugate to a compact linear group.*

We claim some fundamental properties of compact subgroups of $\mathcal{A}$.

THEOREM 4.1.2. [**31**] *Let ξ be a stable representative of a finite singularity.*

(1) Every compact subgroup of Aut_ξ is contained in a maximal compact subgroup.

(2) Any two maximal compact subgroups are conjugate in Aut_ξ

Lastly, it was shown in [**31**] that the quotient of Aut_ξ with a maximal compact subgroup is contractible in a generalized sense.

4.2. Interpolation

Within the context of equivariant cohomology, the interpolation method can be seen as putting some constraints on a given Thom polynomial. It turns out that in many situations (including the ones we want to consider), these constraints completely determine the Thom polynomial.

Given a maximal compact subgroup $G_\xi \leq Aut_\xi \leq \mathcal{A} \approx \mathcal{D}_m \times \mathcal{D}_{m+\ell}$ of the stabilizer of ξ within $\mathcal{A}$, we have that $G_\xi \leq GL_m \times GL_{m+\ell}$, and we get induced linear actions on the source and target spaces via projection. Let us call these linear actions on the source and target $\lambda_1(\xi)$ and $\lambda_2(\xi)$ respectively. The following diagram commutes

$$\begin{array}{ccc} (\mathbb{C}^m, 0) & \xrightarrow{\xi} & (\mathbb{C}^{m+\ell}, 0) \\ {\scriptstyle \lambda_1(\xi)}\downarrow & & \downarrow{\scriptstyle \lambda_2(\xi)}\,. \\ (\mathbb{C}^m, 0) & \xrightarrow{\xi} & (\mathbb{C}^{m+\ell}, 0). \end{array}$$

Such an action determines vector bundles $(EG_\xi \times_{G_\xi} \mathbb{C}^m) \to BG_\xi$ and $(EG_\xi \times_{G_\xi} \mathbb{C}^{m+\ell}) \to BG_\xi$ with fibers $\mathbb{C}^m, \mathbb{C}^{m+\ell}$ respectively, and with structure groups G_ξ. By $c(\lambda_1)$ we mean $c((EG_\xi \times_{G_\xi} \mathbb{C}^m) \to BG_\xi) \in H^*(BG_\xi)$ and by $c(\lambda_2)$ we mean $c((EG_\xi \times_{G_\xi} \mathbb{C}^{m+\ell}) \to BG_\xi) \in H^*(BG_\xi)$.

DEFINITION 4.2.1. *Let $\ell > 0$ and let $\xi : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ be a stable representative of a singularity with local algebra Q_ξ. Let G_ξ be a maximal compact subgroup, and*

let its representations (projections) on the source and target space be $\lambda_1(\xi)$ *and* $\lambda_2(\xi)$ *respectively. The* total Chern class *of the singularity with local algebra* Q_ξ *is defined as*

$$c(\xi) := \frac{c(\lambda_2(\xi))}{c(\lambda_1(\xi))} \in H^{\Pi}(BG_\xi).$$

Let the Euler class $e(\xi) \in H^{2codim\xi}(BG_\xi)$ *be the highest Chern class of* $\lambda_1(\xi)$.

The ring $H^{\Pi}(X)$ is the ring consisting of all formal infinite series $z = z_0 + z_1 + \cdots$ with $z_i \in H^{2i}(X)$ see, [**29**].

The main theorem of interpolation gives the value of a Thom polynomial under certain substitutions.

Theorem 4.2.2 (Interpolation). [**30**] *Let* ξ, ζ *be stable singularities.*

$$Tp_\xi\Big(c(\zeta)\Big) = \begin{cases} e(\zeta) & \text{if } \zeta = \xi \\ 0 & \text{if } codim(\zeta) \leq codim(\xi) \text{and } \zeta \neq \xi. \end{cases}$$

The proof of the Interpolation Theorem uses Theorems 3.3.6 and 3.3.7.

If necessary, we may also want to work within a subgroup $G'_\xi \leq G_\xi$. We can define $c'(\xi)$ and $e'(\xi)$ in the same way as in Definition 4.2.1. We have the following corollary to the Theorem 4.2.2.

Corollary 4.2.3. [**30**]

$$Tp_\xi\Big(c'(\zeta)\Big) = \begin{cases} e'(\zeta) & \text{if } \zeta = \xi \\ 0 & \text{if } codim(\zeta) \leq codim(\xi) \text{and } \zeta \neq \xi \end{cases}$$

For any subgroup $G'_\xi \leq G_\xi$.

Remark 4.2.4. Theorem 4.2.2 means that the defining relation of Thom polynomials $[\eta(f)] = Tp_\eta(c(f))$ holds for stable singularities (in the equivariant cohomology of the

maximal compact subgroups of the appropriate stabilizer subgroup). In fact the defining relation for residue polynomials, equation ([**?**]), also holds for stable singularities in equivariant cohomology.

When we apply a cohomology formula to a stable singularity, we say that we *restrict* the formula to, or *evaluate* the formula on the sinsingularity. The motivation for these words comes from the classifying space of multisingularities.

Example 4.2.5. Consider the stable germ $\xi : (x, y) \mapsto (x^2, xy, y)$, called Whitney umbrella. The group $G = U(1) \times U(1)$ is a group of symmetries of ξ with the representations

$$(\alpha, \beta)\cdot(x, y) = (\alpha x, \beta y), \qquad (\alpha, \beta)\cdot(u, v, w) = (\alpha^2 u, \alpha\beta v, \beta w), \qquad \big((\alpha, \beta) \in U(1)\times U(1)\big)$$

on the source and target spaces respectively. Indeed,

$$\xi(\alpha x, \beta y) = (\alpha^2, \alpha\beta, \beta)\xi(x, y).$$

Using the notation $H^*BG = \mathbb{Z}[a, b]$ (where a and b are the universal Chern classes corresponding to the first and second factor of $U(1) \times U(1)$), we have that

$$c(\xi) = \frac{(1 + 2a)(1 + a + b)(1 + b)}{(1 + a)(1 + b)} = 1 + (2a + b) + (ab) + (-a^2 b) + \dots.$$

The closure of the double point locus (in the source space) of this map is $\{y = 0\}$. Its cohomology class is therefore b, the equivariant Euler class of its normal bundle. The cohomology class represented by the image of this map can be calculated to be $2(a + b)$ (see lemma 5.1.1 below). The pullback map ξ^* is an isomorphism (as for all germs), hence the pullback of the cohomology class of the image of ξ is $2(a + b)$. One of Kazarian's

formulas (3.5) (for maps from 2 dimensions to 3 dimensions) states that the difference of these two multisingularity classes is $-R_{A_0^2}$. Hence we get that

$$b - 2(a+b) = R_{A_0^2}(c_1 = 2a+b, c_2 = ab, \ldots) \in \mathbb{Z}[a,b].$$

This has only one solution, $R_{A_0^2} = -c_1$.

Conditions obtained from stable singularities often determine uniquely the residue polynomials as follows. Let α be a multisingularity of codimension d, and suppose that there are only finitely many monosingularities ξ with codimension $\leq d$. For each ξ we can consider the maximal compact symmetry group G_ξ of a stable representative. It is explained in [**10**] that G_ξ acts on the normal bundle of $\xi \subset \mathcal{E}^0(m,n)$. We have the following theorem.

THEOREM 4.2.6. [**10**] *Suppose the G_ξ-equivariant Euler class of the normal bundle of the embedding $\xi \subset \mathcal{E}^0(m,n)$ is not a 0-divisor for all the finitely many singularities ξ with codim$\xi \leq d$. If formula (3.5) holds for stable representatives of all the finitely many ξ with codimension $\leq d$ (in G_ξ equivariant cohomology), then formula (3.5) holds for all admissible maps.*

Strictly speaking this theorem is proved in [**10**] only for monosingularities (since that was the object of the paper). However, what is proved there is that the map

$$\mathbb{Q}[c_1, c_2, \ldots] \to \oplus H^*(BG_\xi),$$

whose component functions are the evaluations of Chern classes at the stable representatives of the ξ's with codim $\leq d$, is injective in degrees $\leq d$. This implies the result for multisingularities as well.

Mather [**27**] determined the codimensions in which moduli of singularities occur: for large ℓ moduli occurs in codimension $6\ell+9$. Calculations show that the condition in the theorem on the Euler classes of the monosingularities of codimension $\leq 6\ell+8$ also hold.

4.3. A Sample Thom Polynomial Calculation

We will show how Theorem 4.2.6 can be used to find the Thom polynomial of A_1 (a classical result, due to Giambelli, Whitney, Thom in various disguises). We will carry out the calculation for general ℓ.

The codimension of the A_1 singularity is $\ell+1$, hence Tp_{A_1} is a degree $\ell+1$ polynomial in Chern classes c_i where the degree of $c_i = i$, such that

$$[\overline{A_1(f)}] = Tp_{A_1}(c(f)) \tag{4.1}$$

for any admissible map f. There are only two singularities with codimension $\leq \ell+1$, namely: A_0 and A_1. Hence from Theorem 4.2.6 we can deduce two constraints on the Tp_{A_1}. It turns out that the constraint coming from A_0 is redundant, hence we will only consider the constraint coming from A_1 itself. For this we need to choose a stable representative of the singularity A_1. We obtain the following germ $f : (\mathbb{C}^{\ell+1}, 0) \to (\mathbb{C}^{2\ell+1}, 0)$:

$$(x, y_1, \ldots, y_\ell) \mapsto (x^2, xy_1, \ldots, xy_\ell, y_1, \ldots, y_\ell).$$

The general procedure to find the maximal compact symmetry group is described in [**31**]. For our germ we obtain $G_f = U(1) \times U(\ell)$ with the representations

$$\rho_1 \oplus (\overline{\rho_1} \otimes \rho_\ell), \qquad \rho_1^2 \oplus \rho_\ell \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces, where ρ_1 and ρ_ℓ are the standard representations of $U(1)$ and $U(\ell)$. It is easier to understand the representations of the maximal torus $U(1) \times U(1)^\ell$, so we proceed as follows. Let $(\alpha, \beta_1, \ldots, \beta_\ell) \in U(1) \times U(1)^\ell$. The diagonal actions given by

$$(\alpha, \bar{\alpha}\beta_1, \ldots, \bar{\alpha}\beta_\ell), \qquad \text{and} \qquad (\alpha^2, \beta_1, \ldots, \beta_\ell, \bar{\alpha}\beta_1, \ldots, \bar{\alpha}\beta_\ell)$$

is clearly a symmetry of the germ above.

Hence, when we apply formula (4.1) to the germ f, we obtain an equation in $H^*(B(U(1)\times U(\ell)))$. By abuse of language we denote the Chern roots of $U(1)$ and $U(\ell)$ by α and $\beta_1, \ldots, \beta_\ell$. Let b_i be the i'th elementary symmetric polynomial of the β_i's, that is the universal Chern classes of the group $U(\ell)$. Then the total Chern class of f is

$$c(f) = \frac{(1+2\alpha)\prod^\ell(1+\beta_i)\prod^\ell(1+\beta_i-\alpha)}{(1+\alpha)\prod^\ell(1+\beta_i-\alpha)} = \frac{(1+2\alpha)\prod^\ell(1+\beta_i)}{(1+\alpha)} =$$

$$= 1 + (b_1+\alpha) + (b_2 + b_1\alpha - \alpha^2) + (b_3 + b_2\alpha - b_1\alpha^2 + \alpha^3) + \ldots,$$

that is, $c_1(f) = b_1 + \alpha$, $c_2(f) = b_2 + b_1\alpha - \alpha^2$, etc.

Now we need the left hand side of formula (4.1) for our germ f. The A_1 locus of the germ f is only the origin, hence $[A_1(f)]$ is the class represented by the origin. By definition the class represented by the origin in the equivariant cohomology of a vector space is the Euler class (a.k.a. top Chern class) of the representation. In our case it is

$$\alpha \prod^\ell (\beta_i - \alpha).$$

Hence formula (4.1) reduces to

$$\alpha \prod^\ell (\beta_i - \alpha) = Tp_{A_1}(c_1 = b_1 + \alpha, c_2 = b_2 + b_1\alpha - \alpha^2, \ldots).$$

It is simple algebra to show that the polynomials $b_1+\alpha, b_2+b_1\alpha-\alpha^2, \ldots$ (up to the degree $\ell+1$) are algebraically independent in $\mathbb{Z}[\alpha, b_1, b_2, \ldots, b_\ell]$, and that $c_{\ell+1} = \alpha \prod^{\ell}(\beta_i - \alpha)$. This yields that $Tp_{A_1} = c_{\ell+1}$.

4.4. The Known Residue Polynomials

Infinitely many Thom polynomials can be named at the same time, due to certain stabilization properties that they satisfy. At present we are concerned with two such stabilizations. The first was already mentioned, namely that the Thom polynomial only depends on ℓ, not on m and n (for the same local algebra). The second—Theorem 4.4.1 below—concerns the Thom polynomial as ℓ varies (while not changing the local algebra). To phrase Theorem 4.4.1 we need some notions.

Let Q be a local algebra of a singularity. In singularity theory one considers three integer invariants of Q as follows: (i) $\delta = \delta(Q)$ is the complex dimension of Q, (ii) the *defect* $d = d(Q)$ of Q is defined to be the minimal value of $b - a$ if Q can be presented with a generators and b relations; (iii) the definition of the third invariant $\gamma(Q)$ is more subtle, see [**27**]. The existence of a *stable* singularity $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ with local algebra Q is equivalent to the conditions $\ell \geq d$, $\ell(\delta - 1) + \gamma \leq m$ (see Section 2.3). Under these conditions the codimension of the contact singularity with local algebra Q in $\mathcal{E}^0(m, n)$ is $\ell(\delta - 1) + \gamma$.

Theorem 4.4.1. [**10**] *Let Q be a local algebra of singularities. Assume that the normal Euler classes of the singularities in $\mathcal{E}^0(m, n)$ with local algebra Q are not 0. Then associated with Q there is a formal power series (*Thom series*) Ts_Q in the variables $\{d_i | i \in \mathbb{Z}\}$, of degree $\gamma(Q) - \delta(Q) + 1$, such that all of its terms have $\delta(Q) - 1$ factors, and*

the Thom polynomial of $\eta \subset \mathcal{E}^0(m,n)$ with local algebra Q is obtained by the substitution $d_i = c_{i+(n-m+1)}$ (here the variable d_i has degree i).

Even though there are powerful methods by now to compute individual Thom polynomials (i.e. finite initial sums of the Ts), finding closed formulas for these Thom series remains a subtle problem. Here are some examples.

$\mathbf{A}_0$: $Q = \mathbb{C}$ (*embedding*). Here $\delta = 1$, $\gamma = 0$, and

$$Ts = 1.$$

$\mathbf{A}_1$: $Q = \mathbb{C}[x]/(x^2)$ (e.g. *fold, Whitney umbrella*). Here $\delta = 2$, $\gamma =1$, and

$$Ts = d_0 = c_{\ell+1}.$$

$\mathbf{A}_2$: $Q = \mathbb{C}[x]/(x^3)$ (e.g. *cusp*). Here $\delta = 3$, $\gamma = 2$, and [**33**]

$$Ts = d_0^2 + d_{-1}d_1 + 2d_{-2}d_2 + 4d_{-3}d_3 + 8d_{-4}d_4 + \dots$$

$\mathbf{A}_3$: $Q = \mathbb{C}[x]/(x^4)$. Here $\delta = 4$, $\gamma = 3$, and [**2**, Thm.4.2] [**3**]

$$Ts = \sum_{i=0}^{\infty} 2^i d_{-i}d_0d_i + \frac{1}{3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2^i 3^j d_{-i}d_{-j}d_{i+j} + \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}d_{-i-j}d_id_j,$$

where $a_{i,j}$ is defined by the formal power series

$$\sum_{i,j} a_{i,j}u^iv^j = \frac{u\frac{1-u}{1-3u} + v\frac{1-v}{1-3v}}{1-u-v}.$$

Although we used formal power series to describe Thom polynomials, of course, the Thom polynomials themselves are polynomials, since only finitely many terms are nonzero for any concrete ℓ. For example from the Thom series of A_2 above it follows that for $\ell = 1$ the Thom polynomial is $c_2^2 + c_1c_3 + 2c_4$, for $\ell = 2$ the Thom polynomial is $c_3^2 + c_2c_4 + 2c_1c_5 + 4c_6$, etc.

There are other Thom series known in *iterated residue form*: Bérczi and Szenes found the Thom series of A_i singularities for $i \leq 6$ [**3**]. In an upcoming paper [**8**], Fehér and Rimányi calculate the Thom series corresponding to several non-Morin singularities. In [**23**], Fehér and Kőműves calculate the Thom series of some second order Thom-Boardman singularities.

However, all the mentioned results are Thom polynomials, that is residue polynomials of monosingularities, rather than multisingularities. Several individual multisingularity residue polynomials are calculated for small ℓ in [**21**] and [**20**]. However, the methods used there do not easily extend to find formulas for all ℓ. For example it was known that

$$R_{A_0^4} = -6(c_1^3 + 3c_1c_2 + 2c_3) \quad \text{for } \ell = 1, \tag{4.2}$$

$$R_{A_0^4} = -6({c_2}^3 + 3\,c_1c_2c_3 + 7\,c_2c_4 + 2\,{c_1}^2c_4 + 10\,c_1c_5 + 12\,c_6 + {c_3}^2) \quad \text{for } \ell = 2, \tag{4.3}$$

but no $R_{A_0^4}$ formula was known for all ℓ. In other words the *residue series*, i.e. a formula containing ℓ as a parameter is known only for a very few multisingularities. Here is a complete list of those:

THEOREM 4.4.2. [**34**] *For admissible maps $f : M \to N$ with relative codimension ℓ, we have*

$$m_{A_0^2} = f^*(n_{A_0}) - c_\ell(f).$$

That is, the residue polynomial of the multisingularity A_0^2 is $-c_\ell$.

THEOREM 4.4.3. [**7**] *For admissible maps $f : M \to N$ we have*

$$m_{A_0^3} = f^*(n_{A_0^2}) - 2c_\ell f^*(n_{A_0}) + 2\Big(c_\ell^2 + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i}c_{\ell+1+i}\Big).$$

That is, the residue polynomial of the multisingularity A_0^3 is

$$R_{A_0^3} = 2\Big(c_\ell^2 + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i} c_{\ell+1+i}\Big).$$

THEOREM 4.4.4. [**19**] *For admissible maps $f : M \to N$ we have*

$$m_{A_1 A_0} = f^*(n_{A_0}) - 2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i} c_{\ell+2+i}\Big),$$

$$\begin{aligned} m_{A_0 A_1} &= f^*(n_{A_1}) - 2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i} c_{\ell+2+i}\Big) \\ &= f^*(f_!(c_{\ell+1})) - 2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i} c_{\ell+2+i}\Big). \end{aligned}$$

That is, the residue polynomial of the multisingularity $A_1 A_0$ is

$$R_{A_0 A_1} = -2\Big(c_\ell c_{\ell+1} + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i} c_{\ell+2+i}\Big).$$

There are basically two main reasons why the calculation of other residue polynomials is more difficult.

First, no transparent geometric meaning of residue polynomials of multisingularities has been found so far. While Thom polynomials (residue polynomials of monosingularities) are equivariant classes represented by geometrically relevant varieties in $\mathcal{E}^0(m,n)$ (i.e. they are part of equivariant cohomology), the residue polynomials of multisingularities do not seem to have this property. The cohomology ring of the classifying space of singularities is a ring of characteristic classes, while the cohomology ring of the classifying space of multisingularities contains Landweber-Novikov classes (see more details in [**21**]). Hence powerful techniques of equivariant cohomology (e.g. localization) can not be used directly for multisingularities.

The second reason can be seen in the diagram of multisingularities in Section 2.5. The codimension of the multisingularities considered in the above three theorems are smaller than the codimension of any non-Morin, (i.e. $\Sigma^{\geq 2}$) singularity. Therefore, non-Morin singularities can be disregarded when studying those three multisingularities. As the table shows, we will have "competing" non-Morin singularities for any other multisingularity.

The main result of this present paper is the calculation of residue polynomials in such non-Morin cases, namely the residue polynomial $R_{A_0^i}$ for all ℓ and $i \leq 7$.

CHAPTER 5

The General Quadruple Point Formula

In order to emphasize the relative codimension, let $R_{\underline{\alpha}}(\ell)$ denote the residue polynomial of the multisingularity $\underline{\alpha}$ for maps of relative codimension ℓ. We are now ready to state the main theorem.

THEOREM 5.0.1. *For $i \leq 6$ we have*

$$R_{A_0^{i+1}}(\ell) = (-1)^i i! R_{A_i}(\ell - 1).$$

Since the polynomial R_{A_i} is known for $i \leq 6$ [**3**] this theorem calculates the polynomial $R_{A_0^i}$, hence determines e.g. the general quadruple point formula. After some preparations, the proof for the case $i = 3$ will be given in Section 5.3. The cases $i = 4, 5, 6$ follow similarly, see Section 5.4.

5.1. Multiple Point Formulas for Germs

In what follows let us set the cohomology classes in the source and target of the set of j-tuples of points of a map f as $\bar{m}_j(f)$, and $\bar{n}_j(f)$ respectively. That is, $\bar{m}_j(f) = \bar{m}_{A_0^j}(f)$ and $\bar{n}_j(f) = \bar{n}_{A_0^j}(f)$. We also use n_1 for $\bar{n_1}$. Using these notations the defining equations

(3.5) of $R_{A_0^i}$'s can be brought to the following form

$$
\begin{aligned}
(5.1) \quad \bar{m}_2(f) &= f^*(\bar{n}_1(f)) + R_{A_0^2}(\ell) \\
(5.2) \quad \bar{m}_3(f) &= f^*(\bar{n}_2(f)) + R_{A_0^2}(\ell) f^*(\bar{n}_1(f)) + \frac{1}{2} R_{A_0^3}(\ell) \\
(5.3) \quad \bar{m}_4(f) &= f^*(\bar{n}_3(f)) + R_{A_0^2}(\ell) f^*(\bar{n}_2(f)) + \frac{1}{2} R_{A_0^3}(\ell) f^*(\bar{n}_1(f)) + \frac{1}{6} R_{A_0^4}(\ell).
\end{aligned}
$$

We want to apply the method of interpolation from Section 4.2, hence we want to apply equations (5.1)-(5.3) for stable germs with relative codimension ℓ, whose codimensions do not exceed the codimension of the relevant $\bar{m}_i$. For stable germs, however, more information is available for some of the ingredients.

Lemma 5.1.1. *Let f be a stable germ with relative codimension ℓ; and let G be a symmetry group of f with representations ρ_1 and ρ_2 on the source and target spaces respectively. For a G-representation ρ let $e(\rho)$ denote the G-equivariant Euler class of ρ, that is, the product of the weights of ρ. Then in G-equivariant cohomology we have*

- *f^* is isomorphism;*
- *$f^*(n_1)e(\rho_1) = f^*(e(\rho_2))$;*
- *$f^*(\bar{n_r}) = \frac{1}{r}\bar{m}_r f^*(n_1)$.*

Proof. The map f is equivariantly homotopic to the map of a one point space to a one point space, hence $f^* : H^*(BG) \to H^*(BG)$ is the identity map.

Now recall the projection formula for the Gysin map $f_!$ (which holds for any proper map, therefore for any stable map germ too):

$$f_!(f^*(x)y) = x f_!(y).$$

Applying f^* to this formula, and writing z for $f^*(x)$, and substituting $y = 1$ we obtain

$$f^*(f_!(z)) = zf^*(n_1), \tag{5.4}$$

where we also used that $f_!(1)$ is n_1. Since f^* is an isomorphism (hence surjective) this formula holds for any z.

Observe that $f_!(e(\rho_1)) = e(\rho_2)$. Indeed, the Poincaré dual of $e(\rho_1)$ is the homology class of 0 in the source, its homology push-forward is the homology class 0 in the target, whose Poincaré dual is then $e(\rho_2)$. Therefore substituting $z = e(\rho_1)$ in (5.4) we obtain the second statement of the lemma.

Observe that $f_!(\bar{m}_r) = r\bar{n}_r$. Therefore substituting $z = \bar{m}_r$ into (5.4) we obtain the third statement. □

Remark 5.1.2. Since f^* is an isomorphism for germs, we will sometimes suppress it from the notation. Observe that if $e(\rho_1) \neq 0$ then the second statement can be rewritten as $f^*(n_1) = e(\nu(f))$, the equivariant Euler class of the virtual normal bundle. The divisibility of $e(\rho_2)$ with $e(\rho_1)$ is a remarkable property of stable germs. For instance it does not hold for the non-proper blow-up map $(x, y) \to (x, xy)$ with group $U(1) \times U(1)$ acting via $\rho_1 = \alpha \oplus \beta$, $\rho_2 = \alpha \oplus (\alpha \otimes \beta)$.

Using the statements of Lemma 5.1.1 we can bring formulas (5.1)-(5.3) to the forms

$$\bar{m}_2(f) = R_{A_0^2}(\ell) + n_1, \tag{5.5}$$

$$\bar{m}_3(f) = \frac{1}{2}R_{A_0^3}(\ell) + n_1\big(\dots\big), \tag{5.6}$$

$$\bar{m}_4(f) = \frac{1}{6}R_{A_0^4}(\ell) + n_1\big(\dots\big), \tag{5.7}$$

where $n_1(\ldots)$ stands for a term divisible by n_1.

We will use these formulas to calculate certain substitutions of residue polynomials. The variables of these polynomials are $c_1, c_2, \ldots$. We will use the following notation for polynomials p with those variables: $p(1 + x_1 + x_2 + \ldots)$ will denote the substitution $c_1 = x_1$, $c_2 = x_2, \ldots$. Furthermore, the series $1 + x_1 + x_2 + \ldots$ will be usually given by (the Taylor series of) a rational function. For example,

$$p\left(\frac{1+2\alpha}{1+\alpha}\right)$$

means the polynomial p with substitution $c_1 = \alpha$, $c_2 = -\alpha^2$, $c_3 = \alpha^3$, etc.

5.2. Some Stable Singularities and Their Symmetries

Along the way of proving Theorem 5.0.1 we will need the following stable singularities.

- A stable A_1 singularity is $f_{A_1} = f_{A_1}(\ell) : \mathbb{C}^{\ell+1}, 0 \to \mathbb{C}^{2\ell+1}, 0$:

$$f_{A_1} : (x, y_1, \ldots, y_\ell) \mapsto (x^2, xy_1, \ldots, xy_\ell, y_1, \ldots, y_\ell).$$

Just like in Section 4.3, we consider its maximal compact symmetry group $G = U(1) \times U(\ell)$ with the representations

$$\rho_1 \oplus (\overline{\rho_1} \otimes \rho_\ell), \qquad \rho_1^2 \oplus \rho_\ell \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces. For the G-equivariant cohomology ring we have,

$$H^*(BG) \leq \mathbb{Q}[\alpha, \beta_1, \ldots, \beta_\ell],$$

where α, and β_i's are the Chern roots of the groups $U(1)$, and $U(\ell)$ (see Section 3.3). Using this notation the total Chern class of the virtual normal bundle of f_{A_1} is

$$c(f_{A_1}) = \frac{(1+2\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}.$$

Below is the list of the analogous data for singularities A_2, $III_{2,2}$, and A_3.

- A stable A_2 singularity is $f_{A_2} = f_{A_2}(\ell) : \mathbb{C}^{2\ell+2}, 0 \to \mathbb{C}^{3\ell+2}, 0$:

$$f_{A_2} : (x, a, y_1, \ldots, y_\ell, z_1, \ldots, z_\ell) \mapsto (x^3+xa, x^2y_1+xz_1, \ldots, x^2y_\ell+xz_\ell, a, y_1, \ldots, y_\ell, z_1, \ldots z_\ell).$$

Its maximal compact symmetry group $G = U(1) \times U(\ell)$ acts by the the representations

$$\rho_1 \oplus \rho_1^2 \oplus (\overline{\rho_1}^2 \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell) \qquad \rho_1^3 \oplus \rho_\ell \oplus \rho_1^2 \oplus (\overline{\rho_1}^2 \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces. For the G-equivariant cohomology ring we have,

$$H^*(BG) \leq \mathbb{Q}[\alpha, \beta_1, \ldots, \beta_\ell],$$

where α, and β_i's are the Chern roots of the groups $U(1)$, and $U(\ell)$. Using this notation the total Chern class of the virtual normal bundle of f_{A_2} is

$$c(f_{A_2}) = \frac{(1+3\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}.$$

- A stable $III_{2,2}$ singularity is $f_{III_{2,2}} = f_{III_{2,2}}(\ell) : \mathbb{C}^{2\ell+4}, 0 \to \mathbb{C}^{3\ell+4}, 0$:

$$f_{III_{2,2}} : (x_1, x_2, a, b, c, d, y_1, \ldots, y_{\ell-1}, z_1, \ldots, z_{\ell-1}) \mapsto$$

$$(x_1x_2, x_1^2+cx_1+ax_2, x_2^2+bx_1+dx_2, y_1x_1+z_1x_2, \ldots, y_{\ell-1}x_1+z_{\ell-1}x_2, a, b, c, d, y_1, \ldots, y_{\ell-1}, z_1, \ldots, z_{\ell-1})$$

We consider its symmetry group $G = U(1) \times U(1) \times U(\ell-1)$ with the representations

$$\rho_1 \oplus \rho_2 \oplus (\rho_1^2 \otimes \overline{\rho_2}) \oplus (\overline{\rho_1} \otimes \rho_2^2) \oplus \rho_1 \oplus \rho_2 \oplus (\rho_\ell \otimes \overline{\rho_1}) \oplus (\rho_\ell \otimes \overline{\rho_2}),$$

$$(\rho_1 \otimes \rho_2) \oplus \rho_1^2 \oplus \rho_2^2 \oplus \rho_\ell \oplus (\rho_1^2 \otimes \overline{\rho_2}) \oplus (\overline{\rho_1} \otimes \rho_2^2) \oplus \rho_1 \oplus \rho_2 \oplus (\rho_\ell \otimes \overline{\rho_1}) \oplus (\rho_\ell \otimes \overline{\rho_2})$$

on the source and target spaces. For the G-equivariant cohomology ring we have,

$$H^*(BG) \leq \mathbb{Q}[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_{\ell-1}],$$

where α_i, and β_j's are the Chern roots of the respective $U(1)$ groups, and of the group $U(\ell-1)$. Using this notation the total Chern class of the virtual normal bundle of $f_{III_{2,2}}$ is

$$c(f_{III_{2,2}}) = \frac{(1+2\alpha_1)(1+2\alpha_2)(1+(\alpha_1+\alpha_2))\prod^{\ell-1}(1+\beta_i)}{(1+\alpha_1)(1+\alpha_2)}.$$

- A stable A_3 singularity is $f_{A_3} = f_{A_3}(\ell) : \mathbb{C}^{3\ell+3}, 0 \to \mathbb{C}^{4\ell+3}, 0$:

$$f_{A_3} : (x, a, b, w_1, \ldots, w_\ell, y_1, \ldots, y_\ell, z_1, \ldots, z_\ell) \mapsto$$

$$(x^4+x^2a+xb, x^3w_1+x^2y_1+xz_1, \ldots, x^3w_\ell+x^2y_\ell+xz_\ell, a, b, w_1, \ldots, w_\ell, y_1, \ldots, y_\ell, z_1, \ldots z_l).$$

We consider its symmetry group $G_f = U(1) \times U(\ell)$ with the representations

$$\rho_1 \oplus \rho_1^2 \oplus \rho_1^3 \oplus (\overline{\rho_1}^3 \otimes \rho_\ell) \oplus (\overline{\rho_1^2} \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

$$\rho_1^4 \oplus \rho_\ell \oplus \rho_1^2 \oplus \rho_1^3 \oplus (\overline{\rho_1}^3 \otimes \rho_\ell) \oplus (\overline{\rho_1^2} \otimes \rho_\ell) \oplus (\overline{\rho_1} \otimes \rho_\ell)$$

on the source and target spaces. The G-equivariant cohomology ring

$$H^*(BG) \leq \mathbb{Q}[\alpha, \beta_1, \ldots, \beta_\ell],$$

where α, and β_i's are the Chern roots of the groups $U(1)$, and $U(\ell)$. Using this notation the total Chern class of the virtual normal bundle of f_{A_3} is

$$c(f_{A_3}) = \frac{(1+4\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}.$$

5.3. Proof of Theorem 5.0.1

Now we prove Theorem 5.0.1 for $i = 3$, that is

$$R_{A_0^4}(\ell) = -6R_{A_3}(\ell - 1). \tag{5.8}$$

Proof. Consider the stable singularity of type A_1 with relative codimension ℓ from Section 5.2. This map f_{A_1} has no quadruple point, hence $\bar{m}_4 = 0$ for it. This can be checked directly, or using the fact from singularity theory that the highest multiple points of a stable singularity with local algebra Q are the δ-tuple points, where δ is the dimension of the local algebra Q.

Lemma 5.1.1 above shows that for f_{A_1} we have

$$n_1(f_{A_1}) = \frac{2\alpha \prod^{\ell} \beta_i \prod^{\ell}(\beta_i - \alpha)}{\alpha \prod^{\ell}(\beta_i - \alpha)} = 2\prod^{\ell} \beta_i.$$

Thus, for f_{A_1} equation (5.7) becomes

$$0 = \frac{1}{6} R_{A_0^4}(\ell)\big(\frac{(1+2\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}\big) + \big(2\prod^{\ell}\beta_i\big)\big(\dots\big).$$

Plugging in $\beta_\ell = 0$ we obtain

$$0 = R_{A_0^4}(\ell)\big(\frac{(1+2\alpha)\prod^{\ell-1}(1+\beta_i)}{(1+\alpha)}\big). \tag{5.9}$$

We repeat the above arguments for the stable singularity of type A_2 with relative codimension ℓ, and we obtain

$$0 = R_{A_0^4}(\ell)\big(\frac{(1+3\alpha)\prod^{\ell-1}(1+\beta_i)}{(1+\alpha)}\big). \tag{5.10}$$

The argument for the $III_{2,2}$ singularity is similar. We have

$$n_1(f_{III_{2,2}}) = \frac{2\alpha_1 2\alpha_2(\alpha_1+\alpha_2)\prod^{\ell-1}\beta_i \prod^{\ell-1}(\beta_i-2\alpha_1)(\beta_i-2\alpha_2)(\beta_i-\alpha_1-\alpha_2)}{\alpha_1\alpha_2\prod^{\ell-1}(\beta_i-2\alpha_1)(\beta_i-2\alpha_2)(\beta_i-\alpha_1-\alpha_2)} = 4(\alpha_1+\alpha_2)\prod^{\ell-1}\beta_i.$$

Thus for $f_{III_{2,2}}$ equation (5.7) becomes

$$0 = \frac{1}{6}R_{A_0^4}(\ell)\big(\frac{(1+2\alpha_1)(1+2\alpha_2)(1+(\alpha_1+\alpha_2))\prod^{\ell-1}(1+\beta_i)}{(1+\alpha_1)(1+\alpha_2)}\big) + 4(\alpha_1+\alpha_2)\prod^{\ell-1}\beta_i\big(\dots\big).$$

Substituting $\beta_{\ell-1}=0$ we obtain

$$0 = R_{A_0^4}(\ell)\big(\frac{(1+2\alpha_1)(1+2\alpha_2)(1+(\alpha_1+\alpha_2))\prod^{\ell-2}(1+\beta_i)}{(1+\alpha_1)(1+\alpha_2)}\big). \tag{5.11}$$

Now consider the stable singularity of type A_3 with relative codimension ℓ from Section 5.2. The closure of the quadruple point set of f_{A_3} in the source space is $\{w_i = 0, y_i = 0, z_i = 0\}$. Hence for f_{A_3} we have $\bar{m}_4$ = Euler class of the normal bundle to $\{w_i = 0, y_i = 0, z_i = 0\}$. That is

$$\bar{m}_4 = \prod^{\ell}(\beta_i-\alpha)(\beta_i-2\alpha)(\beta_i-3\alpha).$$

Lemma 5.1.1 above shows that for f_{A_3} we have

$$n_1(f_{A_3}) = 4\prod^{\ell}\beta_i.$$

Thus, for f_{A_3} equation (5.7) becomes

$$\prod^{\ell}(\beta_i-\alpha)(\beta_i-2\alpha)(\beta_i-3\alpha) = \frac{1}{6}R_{A_0^4}(\ell)\big(\frac{(1+4\alpha)\prod^{\ell}(1+\beta_i)}{(1+\alpha)}\big) + 4\prod^{\ell}\beta_i\big(\dots\big).$$

Plugging in $\beta_\ell = 0$ we obtain

$$-6\alpha^3\prod^{\ell-1}(\beta_i-\alpha)(\beta_i-2\alpha)(\beta_i-3\alpha) = \frac{1}{6}R_{A_0^4}(\ell)\big(\frac{(1+4\alpha)\prod^{\ell-1}(1+\beta_i)}{(1+\alpha)}\big). \tag{5.12}$$

Observe that formulas (5.9), (5.10) (5.11) and (5.12) mean that the polynomial $-\frac{1}{6}R_{A_0^4}(\ell)$ satisfies the following properties: (i) it vanishes when applied to $f_{A_1}(\ell-1)$, $f_{A_2}(\ell-1)$, $f_{III_{2,2}}(\ell-1)$; (ii) it gives the Euler class of the source space when applied to $f_{A_3}(\ell-1)$. These are exactly the properties of the polynomial $R_{A_3}(\ell-1)$ applied to these four singularities. According to Theorem 4.2.6, these properties determine $R_{A_3}(\ell-1)$, hence we have proven that $R_{A_0^4}(\ell) = -6R_{A_3}(\ell-1)$. $\square$

In summary we obtained the *general quadruple point formula*:

$$m_4 = f^*(n_3) - 3c_\ell f^*(n_2) + 6\Big(c_\ell^2 + \sum_{i=0}^{\infty} 2^i c_{\ell-1-i}c_{\ell+1+i}\Big) f^*(n_1) + p(c_i) \tag{5.13}$$

where

$$p(c_i) = R_{A_0^4} = -6\left(\sum_{i=0}^{\infty} 2^i c_{\ell-i}c_\ell c_{\ell+i} + \frac{1}{3}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 2^i 3^j c_{\ell-i}c_{\ell-j}c_{\ell+i+j} + \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}c_{\ell-i-j}c_{\ell+i}c_{\ell+j}\right)$$

where $c_0 = 1, c_{<0} = 0$ and with the $a_{i,j}$'s defined as for the Thom polynomial of A_3, that is:

$$\sum_{i,j} a_{i,j}u^i v^j = \frac{u\frac{1-u}{1-3u} + v\frac{1-v}{1-3v}}{1-u-v}.$$

Remark 5.3.1. Another way of viewing the $a_{i,j}$'s is as the entries of the following, modified Pascal's triangle

$$\begin{array}{ccccccccccc}
 & & & & a_{0,0} & & & & & & \\
 & & & a_{1,0} & & a_{0,1} & & & & & \\
 & & a_{2,0} & & a_{1,1} & & a_{0,2} & & & & \\
 & a_{3,0} & & a_{2,1} & & a_{1,2} & & a_{0,3} & & & \\
a_{4,0} & & a_{3,1} & & a_{2,2} & & a_{1,3} & & a_{0,4} & &
\end{array}
=
\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 1 & & 1 & & & & & \\
 & & 3 & & 2 & & 3 & & & & \\
 & 9 & & 5 & & 5 & & 9 & & & \\
27 & & 14 & & 10 & & 14 & & 27 & &
\end{array}$$

where the rule for the i,jth entry remains the same, but we have placed powers of 3 on the edges instead of 1's.

5.4. Higher Multiple Point Formulas

The proof of Theorem 5.0.1 for $i = 4, 5, 6$ goes along the same line as for $i = 3$. One considers the finitely many monosingularities whose codimensions are less than $(i+1)\ell$, as well as the monosingularity $A_i(\ell)$. Applying the defining relation of $R_{A_0^{i+1}}$ for these monosingularities (in equivariant cohomology) results in certain formulas for different evaluations of $R_{A_0^{i+1}}$. Plugging in 0 for the "last Chern root", just like in (5.9), one obtains some shorter, simpler formulas, which turn out to mean that the residue polynomial $(-1)^i/i! \cdot R_{A_0^{i+1}}(\ell)$ satisfies the exact same substitutions as $R_{A_i}(\ell - 1)$. Using the statement that these substitutions determine $R_{A_i}(\ell - 1)$ (Theorem 4.2.6) we conclude that $R_{A_0^{i+1}}(\ell) = (-1)^i i! R_{A_i}(\ell - 1)$.

One naturally conjectures that $R_{A_0^{i+1}}(\ell) = (-1)^i i! R_{A_i}(\ell - 1)$ holds for all i. We found reasons supporting this conjecture, but no proof. The method this article uses certainly does not work for $i > 6$. The reason is a 1-dimensional family of singularities that together form a codimension $6\ell + 9$ variety in $\mathcal{E}^0(n, n+\ell)$ (for large ℓ) [**27**]. Hence, beyond codimension $6\ell + 8$ we can not apply Theorem 4.2.6.

CHAPTER 6

Applications to Problems from Enumerative Geometry

The idea of applying multisingularity formulas to problems in enumerative geometry is well established, see for example [**19**], [**22**] and the references therein. As an application of the quadruple point formula we find the number of 4-secant planes to smooth projective varieties. The method can be tailored to find the number of 4-secant (or 5-, 6-, or 7-secant) linear spaces of other dimensions. In fact, the method can be tailored to find the number of incident singularities for a wide range of problems. In order to better illustrate the method we begin with the known computation of bitangent lines to a sufficiently generic smooth projective plane curve of degree ≥ 2.

6.1. Bitangents to a Smooth Projective Plane Curve

Let $C \subseteq \mathbb{P}^2$ be a nonsingular generic (c.f. Remark 6.1.2) projective plane curve of degree $d \geq 2$. We will show that the number of bitangent lines (i.e. lines that are tangent to C at two (distinct) points) to C is

$$\frac{1}{2}d(d-2)(d-3)(d+3). \tag{6.1}$$

Let $\mathbf{G} := Gr_1\mathbb{P}^2 = Gr_2\mathbb{C}^3$ be the Grassmannian of lines in $\mathbb{P}^2$. We consider the following incidence varieties:

$$\begin{aligned} \mathbf{B} &:= \{(x, \Lambda) \in C \times \mathbf{G} | x \in \Lambda\} \\ \mathbf{F} &:= \{(x, \Lambda) \in \mathbb{P}^2 \times \mathbf{G} | x \in \Lambda\}. \end{aligned}$$

We let $p : \mathbf{B} \to C$ and $q : \mathbf{F} \to \mathbb{P}^2$ denote projections onto the first factors. Moreover the restriction of q to B is just p. If $\pi : \mathbf{F} \to \mathbf{G}$ denotes the projection onto the second factor of $\mathbf{F}$, then we have the following diagram (superscripts indicate dimension):

$$\begin{array}{ccccc} \mathbf{B}^2 & \xrightarrow{j} & \mathbf{F}^3 & \xrightarrow{\pi} & \mathbf{G}^2 \\ \downarrow p & & \downarrow q & & \\ C^1 & \xrightarrow{i} & \mathbb{P}^2. & & \end{array} \tag{6.2}$$

We let $f := \pi \circ j$, and observe that points corresponding to the A_1A_1 multisingularity locus of f (that is points x where f has exactly $d-2$ pre-images with singularity A_1 at x and exactly one other point and singularity A_0 at the rest) are in bijective correspondance with the bitangent lines to C. The codimension of the A_1A_1 multisingularity $= 2 =$ the dimension of B, so we in fact have that the A_1A_1 locus in the source is a finite set of points. Thus $f_!(m_{A_1A_1}) = n_{A_1A_1}$ will also be dual to a finite set of points. We will count the number of these points.

If the map f is admissible then the number of bitangents $\mathbf{N}$ to C can be calculated as

$$\mathbf{N} = \frac{1}{2}\int_{\mathbf{G}} n_{A_1A_1}(f) \tag{6.3}$$

$$= \frac{1}{2}\int_{\mathbf{G}} f_!(R_{A_1A_1}) + f_!(R_{A_1})^2 \tag{6.4}$$

where the polynomials $R_{A_1A_1}, R_{A_1}$ are evaluated at the Chern classes of the virtual normal bundle of the map f, ν_f. We will show how the integral in (6.4) can be calculated.

In fact the polynomials $R_{A_1A_1}$ and R_{A_1} are known for this relative codimension (one can use interpolation for example). Strictly speaking we originally only considerd multisingularity loci for maps with strictly positive relative codimension, but in certain cases, (in particular the map f here) all of the necessary ideas can be extended to a map with

relative codimension $= 0$. So in particular, in terms of the Chern classes c_i of ν_f we have:

$$R_{A_1A_1} = -4c_1^2 - 2c_2 \tag{6.5}$$

$$R_{A_1} = c_1 \tag{6.6}$$

We observe that $\nu_f = \nu_j \oplus p^*(\nu_\pi) = p^*(\nu_i) \ominus j^*(\kappa)$, where κ is the fiberwise tangent bundle to the fibration π. Let k be the first Chern class of κ and let n be the first Chern class of ν_i. The total Chern class of v_f is therefore:

$$c(\nu_f) = \frac{1+p^*(n)}{1+j^*(k)} = 1 + \underbrace{[p^*(n) - j^*(k)]}_{c_1(\nu_f)} + \underbrace{[j^*(k)^2 - p^*(n)j^*(k)]}_{c_2(\nu_f)} + \cdots \tag{6.7}$$

Substituting these values in to the poloynomials in (6.5) and (6.6) and then into integrand in (6.4) gives

(6.8)

$$\mathbf{N} = \frac{1}{2}\int_G -4f_!(p^*n^2) + 10f_!(j^*kp^*n) - 6f_!(j^*k^2) + [f_!(p^*n)]^2 - 2f_!(p^*n)f_!(j^*k) + [f_!(j^*k)]^2.$$

We can further reduce equation (6.8) with the observation that

$$\pi_! j_!(j^*(k^a)p^*(n^b)) = \pi_!(k^a \cdot j_! p^*(n^b)) = \pi_!(k^a \cdot q^*(i_!(n^b))) \qquad (a, b \in \mathbb{N}).$$

The cohomology classes $i_!(n^b)$ for $b = 0, 1$ are geometric invariants of $C \subseteq \mathbb{P}^2$. We want to calculate the number of bitangents to C in terms of these cohomological invariants. These classes can be encoded by integers as follows.

Definition 6.1.1. *Let h be the class represented by a hyperplane in $H^*(\mathbb{P}^2)$, hence $H^*(\mathbb{P}^2) \approx \mathbb{Q}[h]/(h^3)$. Let χ_0 and χ_1 be coefficients of the appropriate power of h in $i_!(1)$ and $i_!(n)$ respectively.*

We note here that for this particular calculation $n \in H^2(C)$ which means that $n^2 = 0$ for codimension reasons, which is why we did not specify a χ_2 in the above definition (it would necessarily be $= 0$). Though for any embedded subvariety $V \subseteq \mathbb{P}^n$ we can define for any monomial in the Chern classes of the normal bundle to the embedding, an analogous χ_I for a multi-index I (see Definition 6.2.1).

Using this notation the integrand in (6.8) can be written as a combination of terms of the form $\pi_!(k^a \cdot q^*(h^c))$ $(c \in \mathbb{N})$ with coefficients depending on the invariants χ_0 and χ_1.

Let S and Q be the universal sub and quotient bundles over G. The space $\mathbf{F}$ is the projectivization of the bundle S. We thus have a tautological exact sequence of bundles $0 \to l \to \pi^*S \to \pi^*S/l \to 0$ over F. Moreover as κ is the fiberwise tangent bundle we have that $\kappa = l^* \otimes \pi^*S/l$, which means in this case that $c(\kappa) = 1 + 2c_1(l^*) + c_1(\pi^*S)$. We also note that $q^*(h) = c_1(l) = -c_1(l^*)$. Putting all this together we have that the integrand can be written as a combination of terms of the form

$$\pi_!(c_1(l)^a c_1(\pi^*S)^b) = c_1(S)^b \pi_!(c_1(l)^a) \qquad (a, b \in \mathbb{N}).$$

The cohomology ring of G is $\mathbb{Q}[c_i(S), c_i(Q)]/(c(S)c(Q) = 1) \approx \mathbb{Q}[c_1(Q)]/(c_1(Q)^3)$. We also have that $\pi_!(c(l)^b) = c_{b-1}(Q)$ (for both of these facts see e.g. [**12**]). This enables us to compute the integrand in (6.8) as

$$\mathbf{N} = \frac{1}{2}\int_G (18\chi_0 - 10\chi_1 + \chi_0^2 + \chi_1^2 - 2\chi_0\chi_1)c_1^2(Q) \tag{6.9}$$

$$= \frac{1}{2}(18\chi_0 - 10\chi_1 + \chi_0^2 + \chi_1^2 - 2\chi_0\chi_1). \tag{6.10}$$

Making the substitution of $\chi_0 = d$, the degree of the plane curve, and $\chi_1 = d^2$ gives the number of bitangents in the form described in formula (6.1).

The justification for this last substitution can be given as follows. Let $h \in H^*(\mathbb{P}^2)$ represent the dual class to a hyperplane H, then $\chi_0 =$ the coefficient of h under $i_!(1)$ which is the number of times the curve C will intersect a generic hyperplane, i.e. d. In a similar fashion, χ_1 can be thought of as the number of times that the curve will intersect a perturbation of itself, i.e. d^2.

Remark 6.1.2. In formula (6.4) we required that the map be admissible for the multisingularity A_1A_1. This can be phrased in terms of genericity conditions on the curve C. For example if the curve is sufficiently generic that the dual curve contain only nodes and ordinary cusps then f will be admissible, though this will not always be a necessary condition for the formula in (6.10) (for example in the case $d = 4$ and we have a nonsingular plane quartic formula (6.10) holds even if the dual curve has a tacnode). For more explicit descriptions see [**17**] ch. IV.

6.2. 4-Secants to Smooth Projective Varieties

We return to finding the number of 4-secants to smooth projective varieties.

Let $i : V^a \subset \mathbb{P}^{4a+2}$ be a smooth projective variety, and let $\mathbf{G} = \mathrm{Gr}_2\,\mathbb{P}^{4a+2} = \mathrm{Gr}_3\,\mathbb{C}^{4a+3}$ denote the Grassmannian of projective 2-planes in $\mathbb{P}^{4a+2}$. Consider the following incidence varieties:

$$\begin{aligned} \mathbf{B} &:= \{(x, P) \in V \times \mathbf{G} \mid x \in P\}, \\ \mathbf{F} &:= \{(x, P) \in \mathbb{P}^{4a+2} \times \mathbf{G} \mid x \in P\}. \end{aligned}$$

The two projections of $\mathbf{F}$ to $\mathbb{P}^{4a+2}$ and $\mathbf{G}$ will be denoted by q and π. Both are fibrations with fibers $\mathrm{Gr}_2\,\mathbb{C}^{4a+2}$ and $\mathbb{P}^2$, respectively. The restriction of q to the variety V is the fibration $p : \mathbf{B} \to V$. Hence we obtain the following diagram.

$$
\begin{tikzcd}
\mathbf{B}^{9a} \arrow[r, "j"] \arrow[d, "p"] & \mathbf{F}^{12a+2} \arrow[r, "\pi"] \arrow[d, "q"] & \mathbf{G}^{12a} \\
V^a \arrow[r, "i"] & \mathbb{P}^{4a+2}, &
\end{tikzcd}
\tag{6.11}
$$

where the superscripts indicate dimension. Observe that the quadruple points of the map $f = \pi \circ j$ correspond bijectively to planes intersecting V exactly four times, ie. 4-secant planes.

We will make the assumption that the map f is admissible (cf. Remark 6.2.3). Hence, the number N_a of 4-secant planes to V is calculated as $N_a =$

$$
\begin{aligned}
(6.12) = \quad & \frac{1}{4!}\int_{\mathbf{G}} n_{A_0^4}(f) \\
(6.13) = \quad & \frac{1}{4!}\int_{\mathbf{G}} f_!(R_{A_0})^4 + 6f_!(R_{A_0^2})f_!(R_{A_0})^2 + 3f_!(R_{A_0^2})^2 + 4f_!(R_{A_0^3})f_!(R_{A_0}) + f_!(R_{A_0^4}),
\end{aligned}
$$

where the polynomials $R_{A_0^i}$ are evaluated at the Chern classes of the virtual normal bundle ν_f of f. In the rest of this section we show how this integral can be calculated.

First observe that $\nu_f = \nu_j \oplus j^*\nu_\pi = p^*(\nu_i) \ominus j^*(\kappa)$, where κ is the fiberwise tangent bundle to the fibration π. Let the Chern classes of κ be k_1, k_2, and let the Chern classes of ν_i be $n_1, \ldots, n_a$.

Since the the polynomials $R_{A_0^i}$ are explicitly known (see Section 4.4), the integrand in (6.13) is an explicit polynomial, whose terms are of the form $f_!(j^*(k)p^*(n))$, where k is a monomial in k_1, k_2, and n is a monomial in $n_1, \ldots, n_a$. This can be further re-written as:

$$
\pi_! j_!(j^*(k)p^*(n)) = \pi_!(k j_! p^*(n)) = \pi_!(k q^*(i_!(n))).
$$

The cohomology classes $i_!(n)$ are geometric invariants of the variety $V^a \subset \mathbb{P}^{4a+2}$— we want to calculate the number of 4-secant planes in terms of these invariants. These classes can be encoded by integers, as follows.

Definition 6.2.1. *Let h be the class represented by a hyperplane in $H^*(\mathbb{P}^{4a+2})$, hence $H^*(\mathbb{P}^{4a+2}) = \mathbb{Q}[h]/(h^{4a+3})$. For a multiindex $u = (u_1, u_2, \ldots, u_a)$ let χ_u be the coefficient of the appropriate power of h in $i_!(n_1^{u_1} n_2^{u_2} \cdots n_a^{u_a})$. (For example $\chi_{(0,\ldots,0)}$ is the degree of the embedding $V \subset \mathbb{P}^{4a+2}$.)*

Using this notation, we obtain that our integrand can be written as a combination of terms of the form $\pi_!(k \cdot q^*(h^w))$ ($w \in \mathbb{N}$), with coefficients depending on the invariants χ_u.

Let S and Q be the universal sub and quotient bundles over $\mathbf{G}$. The space $\mathbf{F}$ is the projectivization of the bundle S. Corresponding to this fact, we have the tautological exact sequence of bundles $0 \to l \to \pi^* S \to \pi^* S/l \to 0$ over F. Moreover, κ being the fiberwise tangent bundle, we have $\kappa = l^* \otimes \pi^* S/l$. Using the fact that $q^*(h)$ is the first Chern class of l, we obtain that the integrand can further be written as a combination of terms of the form

$$\pi_!(c_1(l)^w c_I(\pi^*(S))).$$

Here c_I is any Chern monomial, and w is a non-negative integer. This term is further equal to

$$c_I(S)\pi_!(c_1(l)^w).$$

The cohomology ring of $\mathbf{G}$, together with the $\pi_!$-image of powers of $c_1(l)$ can be found in for example [**12**]:

$$H^*(\mathbf{G}) = \mathbb{Q}[c_i(S), c_i(Q)]/(c(S)c(Q) = 1),$$

$$\pi_!(c_1(l)^w) = c_{w-2}(Q).$$

Hence our integrand is an explicit class in $H^*(\mathbf{G})$. Integration can be utilized in any computer algebra package. The results we obtain this way are as follows.

THEOREM 6.2.2. *Let $V^a \subset \mathbb{P}^{4a+2}$ be a smooth variety such that the associated map $f : \mathbf{B} \to \mathbf{G}$ defined in (6.11) is admissible. Let χ_u be the invariants of the embedding. Then for the number N_a of 4-secant planes to V^a we have*

$$4!N_1 = {\chi_0}^4 + 24\chi_1\chi_0 - 6\chi_1{\chi_0}^2 - 208{\chi_0}^2 + 24{\chi_0}^3 + 3{\chi_1}^2 + 1008\chi_0 - 174\chi_1,$$

$$\begin{aligned} 4!N_2 = {} & -36\chi_{1,0}\chi_{0,1} + 64\chi_{2,0}\chi_{0,0} - 3156\chi_{1,0}\chi_{0,0} + {\chi_{0,0}}^4 + 36\chi_{1,0}{\chi_{0,0}}^2 \\ & -6\chi_{0,1}{\chi_{0,0}}^2 - 126{\chi_{0,0}}^3 + 12075{\chi_{0,0}}^2 + 286\chi_{0,0}\chi_{0,1} - 1356\chi_{2,0} \\ & -1944\chi_{0,1} - 200838\chi_{0,0} + 3{\chi_{0,1}}^2 + 108{\chi_{1,0}}^2 + 42174\chi_{1,0}, \end{aligned}$$

$$
\begin{aligned}
4!N_3 \;=\;& -1728\chi_{1,0,0}\chi_{0,1,0} + 91200\chi_{0,0,0}\chi_{1,0,0} - 6\chi_{0,0,1}{\chi_{0,0,0}}^2 + 384\chi_{1,1,0}\chi_{0,0,0} \\
& -48\chi_{0,1,0}\chi_{0,0,1} - 144\chi_{0,0,0}\chi_{0,0,1} - 4352\chi_{2,0,0}\chi_{0,0,0} - 26004\chi_{1,1,0} + \\
& 3{\chi_{0,0,1}}^2 - 9523080\chi_{1,0,0} + {\chi_{0,0,0}}^4 + 42058080\chi_{0,0,0} + 614880\chi_{0,1,0} \\
& -23934\chi_{0,0,1} - 3156\chi_{3,0,0} - 448320{\chi_{0,0,0}}^2 + 437400\chi_{2,0,0} + 3888{\chi_{1,0,0}}^2 \\
& +192{\chi_{0,1,0}}^2 + 720{\chi_{0,0,0}}^3 - 5120\chi_{0,0,0}\chi_{0,1,0} + 216\chi_{1,0,0}\chi_{0,0,1} \\
& -216\chi_{1,0,0}{\chi_{0,0,0}}^2 + 48\chi_{0,1,0}{\chi_{0,0,0}}^2,
\end{aligned}
$$

$$
\begin{aligned}
4!N_4 \;=\;& 1280\chi_{1,0,1,0}\chi_{0,0,0,0} - 1320\chi_{0,0,0,1}\chi_{1,0,0,0} + 853550\chi_{0,1,0,0}\chi_{0,0,0,0} \\
& +1320\chi_{1,0,0,0}{\chi_{0,0,0,0}}^2 - 33024\chi_{1,1,0,0}\chi_{0,0,0,0} + 60\chi_{0,0,1,0}{\chi_{0,0,0,0}}^2 \\
& -72600\chi_{0,1,0,0}\chi_{1,0,0,0} - 330\chi_{0,1,0,0}{\chi_{0,0,0,0}}^2 - 3300\chi_{0,0,1,0}\chi_{0,1,0,0} \\
& -60\chi_{0,0,1,0}\chi_{0,0,0,1} + 6466\chi_{0,0,0,1}\chi_{0,0,0,0} - 4290{\chi_{0,0,0,0}}^3 \\
& -6721080\chi_{1,0,0,0}\chi_{0,0,0,0} - 92436\chi_{0,0,1,0}\chi_{0,0,0,0} + 247040\chi_{2,0,0,0}\chi_{0,0,0,0} \\
& +512\chi_{0,2,0,0}\chi_{0,0,0,0} + 330\chi_{0,0,0,1}\chi_{0,1,0,0} - 309768\chi_{0,0,0,1} \\
& +2126696220\chi_{1,0,0,0} + 13200\chi_{0,0,1,0}\chi_{1,0,0,0} + 300{\chi_{0,0,1,0}}^2 \\
& +1272924\chi_{3,0,0,0} + 9382770\chi_{0,0,1,0} - 9023984640\chi_{0,0,0,0} + 10379016\chi_{1,1,0,0} \\
& +3{\chi_{0,0,0,1}}^2 - 104832\chi_{0,2,0,0} + 145200{\chi_{1,0,0,0}}^2 - 158489298\chi_{0,1,0,0} \\
& -292860\chi_{1,0,1,0} + 9075{\chi_{0,1,0,0}}^2 + {\chi_{0,0,0,0}}^4 - 81576\chi_{2,1,0,0} \\
& -113973552\chi_{2,0,0,0} + 24962795{\chi_{0,0,0,0}}^2 - 6\chi_{0,0,0,1}{\chi_{0,0,0,0}}^2.
\end{aligned}
$$

We note that the expression for N_2 has appeared in [**36**] and [**24**], in the language of Hilbert schemes (and in the variables $d = \chi_{0,0}$, $\pi = \chi_{1,0} - 11\chi_{0,0}$, $\kappa = \chi_{2,0} - 22\chi_{1,0} +$

$121\chi_{0,0}$, $e = -\chi_{0,1} + \chi_{2,0} - 11\chi_{1,0} + 55\chi_{0,0}$), but we believe that N_3 and beyond are new results. Expressions for $N_{>4}$, as well as formulas counting 4-secant linear spaces of higher dimensions can be obtained similarly.

Remark 6.2.3. Theorem 6.2.2 contains the unpleasant condition that the associated map is admissible. Looking through the literature on enumerative geometry using topological methods we find that authors explicitly or implicitly suppose similar admissibility properties. Namely, the following seems to be a general belief: when starting with a geometric situation one associates a map between parameter spaces, and the map is not a Legendre or Lagrange map (e.g. its relative codimension is > 1), then the map is admissible, provided some genericity condition holds. We are not able to phrase (let alone prove) such a genericity condition, under which the admissibility property of the associated map holds.

CHAPTER 7

Another Multisingularity Formula

The interpolation method described in Section 4.2 can be applied to find finite initial sums of the series describing the general multisingularity polynomials. If the multisingularity is complicated enough, recognizing and proving the pattern in such initial sums quickly becomes intractable. An exception is given by the theorem below. We will use the following versions of Schur polynomials

$$s(i,j,k) = \det \begin{pmatrix} c_i & c_{i+1} & c_{i+2} \\ c_{j-1} & c_j & c_{j+1} \\ c_{k-2} & c_{k-1} & c_k \end{pmatrix}, \qquad s(i,j) = \det \begin{pmatrix} c_i & c_{i+1} \\ c_{j-1} & c_j \end{pmatrix}. \tag{7.1}$$

THEOREM 7.0.1. *The general $III_{2,2}A_0$-multisingularity residue polynomial for maps of relative dimension ℓ is*

$$R_{III_{2,2}A_0} = -\sum_{i=1}^{\infty} 2^{i+1} s(\ell+1+i, \ell+2, \ell+1-i). \tag{7.2}$$

PROOF. Let us denote the right hand side of equation (7.2) by $\mathcal{R}$. We will show that $\mathcal{R}$ satisfies the defining relation of the residue polynomial $R_{III_{2,2}A_0}$, that is, we will show

$$m_{III_{2,2}A_0} = \mathcal{R} + R_{III_{2,2}} n_{A_0} \tag{7.3}$$

for all admissible maps. The Giambelli-Thom-Porteous formula states that $R_{III_{2,2}} = s(\ell+2, \ell+2)$. Theorem 4.2.6 asserts that if (7.3) holds for stable representatives of A_0, A_1, A_2, A_3, $I_{2,2}$, and $III_{2,2}$ singularities (in equivariant cohomology with respect to

the maximal compact symmetry group of the particular singularity), then (7.3) holds for admissible maps. Below we prove these statements.

Restriction to A_r singularities.

Stable representatives of ℓ relative codimensional A_r singularities are universal unfoldings of germs $\mathbb{C} \to \mathbb{C}^{\ell+1}$

$$(x) \mapsto (x^{r+1}, 0, \ldots, 0).$$

Their maximal compact symmetry group is $U(1) \times U(\ell)$. The formal difference of the representation on the target and the source is

$$\rho_1^{r+1} \oplus \rho_\ell - \rho_1,$$

where ρ_1 and ρ_ℓ are the standard representations of the $U(1)$ and $U(\ell)$ factors. Therefore the Chern classes c_i of the stable representative of A_r are obtained by

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{1-(r+1)at}{1-at} \sum_{i=0}^{\ell} d_i t^i, \qquad (d_0 = 1) \tag{7.4}$$

where $-a$ is the first Chern class of $U(1)$, and d_i are the Chern classes of $U(\ell)$. Observe that relation (7.4) implies $c_{j+1} = ac_j$ for $j \geq \ell + 2$. Therefore the first two rows of each term of $\mathcal{R}$ are linearly dependent, making each determinant 0. Hence $\mathcal{R} = 0$ applied to any A_r singularity.

Since $III_{2,2}$ is a Σ^2 singularity, and all A_r's are Σ^1 singularities, near an A_r singularity there are no $III_{2,2}$ or $III_{2,2}A_0$ (multi)singularities. This implies that $R_{III_{2,2}}(= m_{III_{2,2}})$ and $m_{III_{2,2}A_0}$ applied to stable representatives of all A_r singularities are both 0. Therefore we proved that (7.3) holds for all A_r singularities.

Restriction to $I_{2,2}$ singularities.

Stable singularities of type $I_{2,2}$ of relative codimension ℓ are universal unfoldings of the germ $\mathbb{C}^2 \to \mathbb{C}^{\ell+2}$

$$(x, y) \mapsto (x^2, y^2, 0, \ldots, 0).$$

The maximal compact symmetry group of this germ is $U(1)^2 \times U(\ell)$, and the formal difference of the representations of this group on the target and on the source is:

$$\rho_1^2 \oplus \rho_1'^2 \oplus \rho_\ell - (\rho_1 \oplus \rho_1').$$

Here ρ_1 and ρ_1' are the standard representations of the two $U(1)$ factors, and ρ_ℓ is the standard representation of $U(\ell)$. Therefore the Chern classes c_i of the stable representative of $I_{2,2}$ are obtained by

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{(1-2at)(1-2bt)}{(1-at)(1-bt)} \sum_{i=0}^{\ell} d_i t^i, \tag{7.5}$$

where $-a$ and $-b$ are the first Chern classes of the two $U(1)$ factors, and d_i are the Chern classes of $U(\ell)$. We need the following lemma.

Lemma 7.0.2. *Let e_i and h_i denote the elementary, and complete symmetric polynomials of the variables a and b (e.g. $e_2 = ab$, $h_2 = a^2 + ab + b^2$). Suppose the variables c_i are expressed in terms of a, b, and $d_1, \ldots, d_\ell$ as defined in (7.5). We use the convention that $d_0 = 1$, $d_{<0} = 0$ and $d_{>\ell} = 0$. Then*

$$s(\alpha, \beta, \gamma) = e_2^{\beta-\ell-2} \cdot h_{\alpha-\beta} \cdot s(\ell+2, \ell+2) \cdot (d_\gamma - 2e_1 d_{\gamma-1} + 4e_2 d_{\gamma-2}),$$

for $\alpha \geq \beta \geq \gamma$, $\beta \geq \ell + 2$, and $\gamma \leq \ell$.

Proof. The Factorization Formula for Schur polynomials (e.g. [**14**]) claims that substituting

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{\sum_{i=0}^{\ell+2} D_i t^i}{(1-at)(1-bt)}$$

into $s(\alpha, \beta, \gamma)$ yields $e_2^{\beta-\ell-2} h_{\alpha-\beta} s(\ell+2, \ell+2) D_\gamma$. Carrying out the further substitution $\sum_{i=0}^{\ell+2} D_i t^i = (\sum_{i=0}^{\ell} d_i t^i)(1-2at)(1-2bt)$ gives the statement of the lemma. $\square$

A special case of this lemma claims that for $j \geq 1$ we have

$$s(\ell+1+j, \ell+2, \ell+1-j) = h_{j-1} \cdot s(\ell+2, \ell+2) \cdot (d_{\ell+1-j} - 2e_1 d_{\ell-j} + 4e_2 d_{\ell-1-j}).$$

Plugging this into the formula for $\mathcal{R}$ we obtain a linear function of the d_i variables. The coefficient of $d_{\ell-k}$ for $k > 0$ is

$$-2^k \cdot 4e_2 h_{k-2} - 2^{k+1}(-2e_1) h_{k-1} - 2^{k+2} h_k.$$

Dividing this expression by -2^{k+2} we obtain $e_2 h_{k-2} - e_1 h_{k-1} + h_k$, which is the k'th coefficient of the power series

$$(1 - e_1 t + e_2 t)(1 + h_1 t + h_2 t^2 + \ldots) = \frac{(1-at)(1-bt)}{(1-at)(1-bt)} = 1,$$

hence it is 0. We obtain that substituting (7.5) into the expression $\mathcal{R}$ is $-4d_\ell \cdot s(\ell+2, \ell+2)$. Lemma 5.1.1 implies that $n_{A_0} = (-2a)(-2b)d_\ell/((-a)(-b)) = 4d_\ell$ for the germ $I_{2,2}$. Thus we proved that formula (7.3) holds for stable representatives of $I_{2,2}$ singularities.

Restriction to $III_{2,2}$ singularities.

Stable singularities of type $III_{2,2}$ of relative codimension ℓ are universal unfoldings of the germ $\mathbb{C}^2 \to \mathbb{C}^{\ell+2}$

$$(x, y) \mapsto (x^2, y^2, xy, 0, \ldots, 0).$$

The maximal compact symmetry group of this germ is $U(1)^2 \times U(\ell-1)$, and the formal difference of the representations of this group on the target and on the source is:

$$\rho_1^2 \oplus \rho_1'^2 \oplus (\rho_1 \otimes \rho_1') \oplus \rho_{\ell-1} - (\rho_1 \oplus \rho_1').$$

Here ρ_1 and ρ_1' are the standard representations of the two $U(1)$ factors, and $\rho_{\ell-1}$ is the standard representation of $U(\ell-1)$. Therefore, the Chern classes c_i of the stable representative of $I_{2,2}$ are obtained by

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{(1-2at)(1-2bt)(1-(a+b)t)}{(1-at)(1-bt)} \sum_{i=0}^{\ell-1} d_i t^i, \tag{7.6}$$

where $-a$ and $-b$ are the first Chern classes of the two $U(1)$ factors, and d_i are the Chern classes of $U(\ell-1)$.

This shows that substituting (7.6) into $\mathcal{R}$ can be obtained by first substituting (7.5) into $\mathcal{R}$, then plugging in $d_\ell = -(a+b)$. The same holds for the other terms of (7.3) as well, hence the satisfaction of formula (7.3) for substitution (7.6) follows from the fact that it is satisfied for the substitution (7.5).

The proof of Theorem 7.0.1 is complete. □

Remark 7.0.3. One can consider applications of the $III_{2,2}A_0$-formula in enumerative geometry along the lines of Chapter 6. The outcome of such a calculation is then the number (or cohomology class) of k-planes in $\mathbb{P}^N$ that have two common points with a fixed smooth projective variety $V \subset \mathbb{P}^N$; one common point is a transversal intersection, and the other is a singular one, with singularity $III_{2,2}$.

Remark 7.0.4. (An Extension to Theorem 7.0.1)

In fact something more than just the form of the residue polynomial in Theorem 7.0.1 can be shown. In [**10**], Fehér and Rimányi show that the residue polynomial for the $III_{2,3}$ monosingularity has the form

$$R_{III_{2,3}} = \sum_{i=1}^{\infty} 2^i s(\ell+1+i, \ell+2, \ell+2-i). \tag{7.7}$$

In [**10**] Fehér and Rimányi describe the so-called *flat operator* $\flat : \mathbb{Q}[c_1, c_2, \ldots] \to \mathbb{Q}[c_1, c_2, \ldots]$. Here we extend the idea to what we call the *Schur-flat operator* $S_\flat$, where if $\mathcal{W}$ is the vector space in $\mathbb{Q}[c_1, c_2, \ldots]$ spanned by Schur functions with at most three parameters, then $S_\flat : \mathcal{W} \to \mathcal{W}$ with $S_\flat(s(a_1)) = S_\flat(s(a_1, a_2)) = 0$ and $S_\flat(s(a_1, a_2, a_3)) = s(a_1, a_2, a_3 - 1)$. Under this notation we can then write

$$R_{III_{2,2}A_0} = -\frac{1}{2} S_\flat(R_{III_{2,3}}). \tag{7.8}$$

CHAPTER 8

Conclusion

We showed in Theorem 5.0.1 that for $i \leq 6$, $R_{A_0^{i+1}}(\ell) = (-1)^i i! R_{A_i}(\ell - 1)$. We conjecture that such a pattern is true for all i, that is: the residue polynomial for the $i+1$-tuple point formula is a multiple of the Thom polynomial for the A_i singularity for maps of one less relative codimension for all i. Some calcultions support this conjecture; however, proving that the two general Thom series are the same cannot be done using interpolation due to the existence of moduli in the orbit space of $\mathcal{K}$'s action on $\mathcal{E}^0(m,n)$.

Equation (7.8) seems to suggest the existence of a deeper relationship between all of the residue polynomials, those for mono- and multi-singularities alike. Moreover the form that the residue polynomial for A_0A_1 takes is very similar to that of the Thom polynomial for A_2 (and consequently for A_0^2). In fact their relationship can be phrased in terms of the flat operator (see Remark 7.0.3). This leads us to the conjecture that for a particular value of $\mu := \delta(Q) - 1$ (see Section 4.4, this is also the number of terms in each monomial in the Thom Series), there are only a limited number of 'master' Thom series for residue polynomials with that μ, and all other residue polynomials can be obtained from these via purely algebraic machinations. However, at present, we are only able to phrase such relationships for a limited number of Thom/Residue series. We hope that the present paper is the first step towards a more general reduction in multisingularity theory.

The fact that residue polynomials for multisingularities share certain algebraic properties with those of monosingularities leads us to hope that certain techniques in equivariant cohomology could be tailored to work for them, (the prime example being localization). However, so far nothing along these lines has been accomplished. The chief impediment to accomplishing such a goal is that to date no definite geometric meaning has been found that accurately reflects how residue polynomials behave. Moreover, only the residue polynomials themselves seem to be part of equivariant cohomology, while their defining relations (i.e. the classes dual to multi-singularity loci), are in terms of so-called Ladweber-Novikov classes, which are the basis for the cohomology of the classifying space of multisingularities.

The main advantage to our techniques with respect to the enumerative geometry questions is that they generalize very easily. In the past various techniques in algebraic geometry have been applied (for example: techniques involving Hilbert Schemes). However these techniques have been fairly difficult to generalize to varieties (and k-planes) of arbitrary dimension, and often new techniques have been required for a problem of the same type with different dimensional constraints. Our technique does not suffer from this drawback. We are able to count the number (or cohomology class) of 4-secants (or 5-secants, 6-secants or 7-secants) of any k-plane with any sufficiently generic smooth projective variety of any complex dimension in any given projective space. The main limiting factors seem to be computing power and the dimensional settings where moduli of singularities occur. Moreover, once a general residue polynomial is known, one can answer the same types of enumerative questions (see Remark 7.0.3) with the same degree of generality as in the $4, 5, 6$ and 7-secant case.

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