Eigencone Problems for Odd and Even Orthogonal Groups

Emily Braley

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Approved by

Prakash Belkale

James Damon

Patricia Hersh

Shrawan Kumar

Robert Proctor

Abstract

EMILY BRALEY: Eigencone Problems for Odd and Even Orthogonal Groups
(Under the direction of Prakash Belkale)

In this work, we consider the eigenvalue problem for the even and odd orthogonal groups. We give an embedding of $SO(2n+1) \subset SO(2n+2)$ and consider the relationship between the intersection theories of homogeneous spaces for the two groups. We introduce the deformed product in cohomology, know as the BK-product, as a technical tool.

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Background and Introduction

Dating back to the nineteenth century, questions were asked about the eigenvalues of sums of Hermitian and real symmetric matrices. The classical Hermitian eigenvalue problem can be stated as follows:

Question 0.0.1. Given $n \times n$ Hermitian matrices A and B with eigenvalues $\alpha = (\alpha_1 \geq \ldots \geq \alpha_n)$ and $\beta = (\beta_1 \geq \ldots \geq \beta_n)$ respectively, what are the possible eigenvalues γ of a Hermitian matrix $C = A + B$?

A conjectural solution of the problem was given by Horn in 1962 [**Hor62**]. The development of the problem will be discussed in Section (0.1). We can generalize the Hermitian eigenvalue problem to an arbitrary complex semisimple algebraic group.

Let G be a connected simple complex algebraic group with maximal compact subgroup K and maximal torus H. Let $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of K and H respectively. K acts on $\mathfrak{k}$ via the adjoint action, and the orbits of this adjoint action are parameterized by the positive Weyl chamber in $\mathfrak{h}$. The analogue of the Hermitian eigenvalue problem is the problem of characterizing the conjugacy class of a sum $C = A + B$, given the conjugacy classes of $A, B \in \mathfrak{k}$. The eigencone of a group G is the set of such conjugacy classes, and it will be defined in (0.1.6). Notice that for $G = SL(n)$, its maximal compact subgroup is the special unitary group $SU(n)$. Traceless skew-Hermitian matrices form the Lie algebra $\mathfrak{su}(n)$, so this case specializes to the original statement of the Question (0.0.1).

In order to understand the relationship between the eigencone of $SO(2n+1)$ and the eigencone of $SO(2n+2)$, it is necessary to understand recent work that has been done on an analogous comparison of the eigencone of $Sp(2n)$ (resp $SO(2n+1)$) with the eigencone of $SL(2n)$ (resp. $SL(2n+1)$), where embeddings are induced from diagram automorphisms. In this introduction, the work comparing these eigencones is put in its proper context by explaining the genesis and the recent history of research done on eigencone problems. In Section (0.1) a historical background to the eigencone problems is given and recent work that has been done on eigencone problems is described. In Section (0.2) we explain the techniques from intersection theory and connections to Schubert calculus used to prove the desired results for the eigencones. In Section (0.3) we state the comparison results for $Sp(2n) \subset SL(2n)$ and $SO(2n+1) \subset SL(2n+1)$, and give a thumbnail sketch of the main body of the dissertation, introducing the setting for the comparison of the eigencones of $SO(2n+1)$ and $SO(2n+2)$.

0.1. Historical Context

In 1912, H. Weyl [**Wey12**] studied the conditions on a triple of eigenvalues that must be satisfied for the triple to appear as the eigenvalues of Hermitian matrices $C = A + B$. It wasn't until 1962 that A. Horn [**Hor62**] undertook a systematic study of the inequalities that α, β, and γ must satisfy. Horn conjectured that a particular set of index sets, parameterizing the Hermitian matrices, would give both the necessary and sufficient conditions for a triple (α, β, γ) to arise. In 1998, A. Klyachko [**Kly98**] used geometric invariant theory to connect the Hermitian eigenvalue problem with intersection theory. In fact, Klyachko's result made the important connection between the Saturation Conjecture and Schubert calculus for this homology of the Grassmannian. W. Fulton gives a summary of the work in a survey paper [**Ful00**].

Klyachko's result was generalized by B. Berenstein and R. Sjamaar [**BS00**] who gave a list of inequalities which characterize the space of eigenvalues. Then, A. Knuston and T. Tao [**KT99**] established the Saturation conjecture using a combinatorial argument with honeycombs. Together the works of Berenstein and Sjamaar, and Knuston and Tao proved Horn's conjecture, but the list of inequalities at that time were over-determined.

Following these works, others began working on generalizations of the Hermitian eigenvalue problem for arbitrary groups. Belkale [**Bel06**] determined an irredundant system as proved by Knuston, Tao, and Woodward [**KTW04**].

Then, P. Belkale gave a geometric proof of Horn's conjecture [**Bel07**]. In his work, Belkale used the fact that for general Schubert varieties intersecting in a point, they intersect transversally there by Kleiman's transversality. Conversely, one can determine if general Schubert varieties intersect by a tangent space calculation. In section (0.3) we state the eigencone comparison results for $Sp(2n) \subset SL(2n)$ and $SO(2n{+}1) \subset SL(2n{+}1)$ addressed by Belkale-Kumar [**BK10**].

0.1.1. Classical Work. Recall the Hermitian eigenvalue problem: If A, B, and C are n by n Hermitian matrices, we denote eigenvalues of A by

$$\alpha : \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$$

and similarly denote by β and γ the eigenvalues of B and C respectively. We can formally ask the question:

Question 0.1.1. What α, β, and γ can be the eigenvalues of n by n Hermitian matrices A, B, and C, with $C = A + B$?

The survey paper by Fulton [**Ful00**] describes the classical work on this problem. Initially, the approach was to take the diagonal matrix $A = D(\alpha)$, with diagonal entries $\alpha_1, \ldots, \alpha_n$, and similarly $B = D(\beta)$, and look for the eigenvalues of

$$D(\alpha) + UD(\beta)U^*,$$

as U varies over the unitary group $U(n)$. Another approach is to fix A and take B small, regarding C as a deformation of A. Neither of these approaches play a role in the discussion to follow, but are useful in initially understanding conditions on the eigenvalues that must be satisfied.

The most obvious condition is the trace condition:

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i. \tag{0.1}$$

H. Weyl gave other necessary conditions as early as 1912:

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \text{ whenever } i + j - 1 \leq n. \tag{0.2}$$

It was not until 1950 that V.B. Lidskii gave geometric conditions for α, β, and γ. Viewing $(\alpha_1, \ldots, \alpha_n)$ as a point in $\mathbb{R}^n$, Lidskii showed that γ must be in the convex hull of $\alpha + \beta_\sigma$, where σ varies over the symmetric groups S_n. Shortly thereafter H. Wielandt showed that this geometric condition is equivalent to the inequalities

$$\sum_{i \in I} \gamma_i \leq \sum_{i \in I} \alpha_i + \sum_{i=1}^{r} \beta_i, \tag{0.3}$$

for all I subset of $[n] = \{1, \ldots, n\}$ of cardinality r, and all $r < n$. All other necessary inequalities are of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j, \tag{0.4}$$

for I, J, and K subsets of $[n]$ of cardinality $r < n$. We will see later that it is convenient to write these index sets is increasing order:

$$I = \{i_1 < i_2 < \ldots < i_r\},$$

and will assume this convention from here on out.

0.1.2. Horn's conjecture. A. Horn undertook a systematic study of inequalities of the form (0.4). In 1962, Horn conjectured that a particular subset of triples (I, J, K) for which the inequalities in (0.4) hold along with the trace condition (0.1), would give both necessary and sufficient conditions for a triple (α, β, γ) to arise as the eigenvalues of Hermitian matrices A, B, and $C = A + B$. Horn gave a recursive formula to describe the subset of triples (I, J, K), which we will describe now.

Let U_r^n be the following set of triples, where $I, J, K \subset [n]$ are of cardinality r:

$$U_r^n := \Big\{(I, J, K) | \sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \frac{r(r+1)}{2}\Big\}.$$

Definition 0.1.2. Define a subset $H_r^n \subset U_r^n$, which we will call Horn's triples:

$$H_r^n := \Big\{(I, J, K) \in U_r^n | \text{ for all } p < r \text{ and all } (F, G, H) \in H_p^r,$$
$$\sum_{f\in F} i_f + \sum_{g\in G} j_g \leq \sum_{h\in H} k_h + \frac{p(p+1)}{2}\Big\}.$$

Conjecture 0.1.3. Horn's Conjecture: A triple (α, β, γ) occurs as the eigenvalues of $n \times n$ Hermitian matrices A, B, and C, with $C = A + B$, if and only if $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$ and the inequalities (0.4) hold for every $(I, J, K) \in H_r^n$ for all $r < n$.

0.1.3. Connections to representation theory and the saturation problem. An irreducible, finite-dimensional representation of the complex general linear group, $GL(n)$, is characterized by its highest weight which is a weakly decreasing sequence of integers:

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n).$$

We denote the irreducible representation with highest weight α by $V(\alpha)$. From representation theory we know that $GL(n, \mathbb{C})$ is reductive, meaning that any finite-dimensional holomorphic representation decomposes into a direct sum of irreducible representations. The number of times that a given irreducible representation $V(\gamma)$ appears in the sum is independent of the decomposition. In particular, the tensor product $V(\alpha) \otimes V(\beta)$ decomposes into a direct sum of representations $V(\gamma)$.

Define $c_{\alpha,\beta}^{\gamma}$ to be the number of copies of $V(\gamma)$ in an irreducible decomposition of $V(\alpha) \otimes V(\beta)$. The question of interest in this situation is:

Question 0.1.4. When does $V(\gamma)$ occur in $V(\alpha) \otimes V(\beta)$, i.e., when is $c_{\alpha,\beta}^{\gamma} > 0$?

In 1934, Littlewood and Richardson gave a combinatorial formula for the numbers $c_{\alpha,\beta}^{\gamma}$ and they became known as the Littlewood-Richardson coefficients. The answer to question (0.1.4) is that for some N, $c^{N\gamma}_{N\alpha,N\beta} > 0$ if and only if $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$ and the inequalities (0.4) hold for all triples $(I, J, K) \in H^n_r$ and all $r < n$. Equivalently, $c^{N\gamma}_{N\alpha,N\beta} > 0$ if and only if there is a triple of n by n Hermitian matrices A, B, and $C = A + B$, with eigenvalues α, β, and γ. The Littlewood-Richardson coefficients also appear as the structure coefficients for multiplication in the cohomology ring of the Grassmannian. In fact, they count the number of points in the intersection of Schubert varieties, as we will see in Lemma (0.2.1).

Conjecture 0.1.5. Saturation Conjecture: If there exists $N > 0$ such that

$$c^{N\gamma}_{N\alpha,N\beta} > 0, \text{ then } c^{\gamma}_{\alpha,\beta} > 0.$$

0.1.4. Algebraic groups. To generalize the Hermitian eigenvalue problem to complex algebraic groups, we first establish notation. Let G be a semi-simple algebraic group with maximal compact subgroup K. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Recall that K acts on $\mathfrak{k}$ by the adjoint action, and we parameterize the orbits of the action by elements from the positive Weyl chamber $\mathfrak{h}_+$.

The goal of the generalized Hermitian eigenvalue problem is to describe the conjugacy classes of $C = A + B$, given the conjugacy classes of $A, B \in \mathfrak{k}$. There is a bijection $c : \mathfrak{h}_+ \longrightarrow \mathfrak{k}/K$ where K acts on $\mathfrak{k}$ by the adjoint action.

We define the eigencone for any $s \geq 1$:

Definition 0.1.6. For a positive integer s, the eigencone of G is defined as the cone:

$$\Gamma(s, G) := \{(h_1, \ldots, h_s) \in (\mathfrak{h}_+)^s \mid \exists (k_1, \ldots, k_s) \in \mathfrak{k}^s, \text{ s.t.} \sum_{j=1}^{s} k_j = 0,\ c^{-1}(k_j) = h_j \ \forall j \in [s]\}.$$

The eigencone, which is a convex polyhedral cone, is the main object of study in the generalization of the of the Hermitian eigenvalue problem. The eigenvalue problem is to describe the system $\{(h_1, \ldots, h_s) \in (\mathfrak{h}_+)^s\}$ such that $(h_1, \ldots, h_s) \in \Gamma(s, G)$. In the next section, we recall a generalization of Klyachko's result from [**BS00**] which gives a description of the eigencone as a space of linear inequalities in $(\mathfrak{h}_+)^s$.

0.2. Schubert Calculus and Intersection Theory

We begin by recalling the intersection theory for the ordinary Grassmannian. The Grassmannian $Gr(r,n)$, is the variety of r-dimensional subspaces of $\mathbb{C}^n$ of dimension $r(n-r)$. A complete flag on $\mathbb{C}^n$ is an ascending chain of subspaces F_i where $\dim(F_i) = i$:

$$F_\bullet := 0 = F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^n.$$

For any index set $A_\alpha = \{a_1 < \ldots < a_r\}$ of cardinality $r \in [n]$, we define the shifted Schubert cell

(0.5)

$$\Omega_{A_\alpha}(F_\bullet) := \{X \in Gr(r,n) | \text{ for any } l \in [r] \text{ and any } a_l \le b < a_{l+1}, \dim(X \cap F_b) = l\}$$

where we set $a_0 = 0$ and $a_{r+1} = n+1$. It's closure in $Gr(r,n)$, denoted $\overline{\Omega}_{A_\alpha}(F_\bullet)$, is an irreducible subvariety of $Gr(r,n)$ of dimension $\sum(a_i - i)$.

Define $\sigma_{A_\alpha} = [\overline{\Omega}_{A_\alpha}(F_\bullet)]$ to be the cycle class in $H^{2|\alpha|}(Gr(r,n))$, where α is the partition defined by setting $\alpha_i = n - r + i - a_i$. To simplify notation, let $\sigma_{A_\alpha} = \sigma_\alpha$. The classes σ_α form a $\mathbb{Z}$ basis for the cohomology $H^*(Gr(r,n))$. It follows that for any two partitions α and β, there is a unique expression

$$\sigma_\alpha \cdot \sigma_\beta = \sum d^\gamma_{\alpha,\beta} \sigma_\gamma, \tag{0.6}$$

for integers $d^\gamma_{\alpha,\beta}$, summing over all γ with $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$.

Let ξ denote the partition consisting of the integer $n-r$ repeated r times. Note that we can identify the top class in cohomology of $Gr(r,n)$ with $\mathbb{Z}$:

$$H^{2r(n-r)}(Gr(r,n)) = \mathbb{Z} \cdot \sigma_\xi.$$

This bring us to an important fact from intersection theory:

Lemma 0.2.1. *For complete flags $F^k_\bullet$ in general position on $\mathbb{C}^n$, where $1 \le k \le 3$, choose index sets $A^k_\alpha \subset [n]$ of cardinality r, such that the codimensions of the Schubert varieties*

$\Omega_{A^k_\alpha}(F^k_\bullet)$ *sum to* $r(n-r)$. *Then the multiplicity of* σ_ξ *in* $\sigma_{A^1_\alpha} \cdot \sigma_{A^2_\alpha} \cdot \sigma_{A^3_\alpha}$ *is the number of points in the intersection* $\cap_{i=1}^3 \Omega_{A^k_\alpha}(F^k_\bullet)$.

This is a special case of a general fact in intersection theory that the intersection of classes of varieties has a representative on the intersection of the varieties. The converse is also true if the flags are in general position, which is a special case of Kleiman's Transversality Theorem [**Kle74**]. Kleiman's theorem asserts that if a complex reductive group acts transitively on smooth variety X, then general translates of subvarieties of X meet transversally. Consider $G = SL(n)$ and a maximal parabolic subgroup $P_r \subset G$, corresponding to the r^{th} fundamental weight for G. We identify the homogeneous space G/P_r with the Grassmannian $Gr(r,n)$, where P_r is the stabilizer of the span of the first r basis vectors of $\mathbb{C}^n$. It is clear that $Gr(r,n)$ has a transitive G action. So, Kleiman's theorem says that for general flags $F^1_\bullet, F^2_\bullet, F^3_\bullet$, the corresponding shifted Schubert cells $\Omega_{A_\alpha}(F^1_\bullet), \Omega_{B_\beta}(F^2_\bullet), \Omega_{C_\gamma}(F^3_\bullet)$ intersect transversally.

Now, the work of Klyachko and the generalization made by Berenstein and Sjaamar, connect the intersection theory of Schubert varieties in a homogeneous space G/P to the eigencone. Therefore, it is convenient to use the notation for the arbitrary group setting.

Again, let G be a semi-simple algebraic group with maximal compact subgroup K, and Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively. We denote the fundamental Weyl chamber by $\mathfrak{h}_+$. Let P be a maximal parabolic subgroup $P \subset G$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots for G, and $\Delta(P)$ denote the set of simple roots for the Levi subgroup of G containing P. We denote the Weyl group of P by W_P and the set of minimal length coset representatives in the quotient W/W_P by W^P. Denote the length of a word by $l(w)$. The fundamental weight associated to P, we denote ω_P. For any $w \in W^P$, define the shifted Schubert cell

$$\Lambda^P_w = w^{-1}BwP/P \subset G/P,$$

and let the cycle class of its closure be denoted $[\overline{\Lambda^P_w}] \in H^{2(\dim(G/P)-l(w))}(G/P)$. The following theorem gives a solution to the eigenvalue problem. We recall the theorem from Berenstein and Sjamaar [**BS00**]:

Theorem 0.2.2. *Let* $(h_1,\dots,h_s)\in\mathfrak{h}_+^s$. *Then, the following are equivalent:*

(1) $(h_1,\dots,h_s)\in\Gamma(s,G)$.

(2) For every standard maximal parabolic subgroup P *in* G *and every choice of s-tuples* $(w_1,\dots,w_s)\in(W^P)^s$ *such that*

$$[\overline{\Lambda}_{w_1}^P]\cdot\ldots\cdot[\overline{\Lambda}_{w_s}^P]=d[\overline{\Lambda}_e^P]\in H^*(G/P) \text{ for some nonzero } d,$$

the following inequality holds:

$$\omega_P(\sum_{j=1}^{s} w_j^{-1}h_j)\leq 0. \tag{0.7}$$

Here the product on the classes in cohomology is the classical cup product. As mentioned above, Belkale and Kumar [**BK06**] define a deformed product on cohomology known as the BK-product. We define this deformation here, and see in Theorem (0.2.4), that it gives a shorter list of inequalities describing the eigencone than Theorem (0.2.2). In this paper the deformation of cohomology is introduced as a technical tool, and we will return to the definition in Chapter 2.

The standard cup product is given by the structure coefficients $d_{\alpha,\beta}^{\gamma}$, as in (0.6). Let ρ denote half the sum of the positive roots of $\mathfrak{g}$ and let x_i be the basis of $\mathfrak{h}$ dual to the simple roots:

$$\alpha_i(x_j)=\delta_{i,j}.$$

For a standard parabolic subgroup P, let ρ^L be half the sum of the positive roots for the unique Levi subgroup $L\subset P$.

Definition 0.2.3. For a standard parabolic $P\subset G$, introduce the indeterminates τ_i for each $\alpha_i\in\Delta\setminus\Delta(P)$. Then define the BK-product, denoted $\odot_0$ as follows; first we define a deformed product $\odot$:

$$[\overline{\Lambda}_u^P]\odot[\overline{\Lambda}_v^P]=\sum_{w\in W^P}\Big(\prod_{\alpha_i\in\Delta\setminus\Delta(P)}\tau_i^{(\chi_w-(\chi_u+\chi_v))(x_i)}\Big)c_{u,v}^{w}[\overline{\Lambda}_w^P]$$

where $\chi_w \in \mathfrak{h}^*$ is the character $\chi_w = \rho - 2\rho^L + w^{-1}\rho$. To obtain the cohomology of G/P, $(H^*(G/P, \mathbb{Z}), \odot_0)$, set $\tau_i = 0$, to get $\odot_0$. As a $\mathbb{Z}$-module, this is the same as the singular cohomology $H^*(G/P, \mathbb{Z})$.

Notice that in the definition above, if you take $P \subset G$ to be a maximal parabolic, then you are introducing only one invariant τ. If you set $\tau = 1$, you recover the classical cup product.

Belkale and Kumar proved [**BK06**] that you can replace condition (2) in Theorem (0.2.2) with a statement in the deformed product. The proof of the theorem relies on geometric invariant theory, which relates the intersections of Schubert varieties in G/P to destabilizing one parameter subgroups for G. Then the Hilbert-Mumford criterion yields the inequalities appearing in (0.9). The intersection number one is a consequence of the uniqueness of maximally destabilizing one parameter subgroups:

Theorem 0.2.4. *Let $(h_1, \dots, h_s) \in (h)^s_+$. Then the following are equivalent:*

(A) $(h_1, \dots, h_s) \in \Gamma(s, G)$.

(B) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that

$$[\overline{\Lambda}^P_{w_1}] \odot_0 \cdots \odot_0 [\overline{\Lambda}^P_{w_s}] = [\overline{\Lambda}^P_e] \in (H^*(G/P, \mathbb{Z}), \odot_0), \tag{0.8}$$

the following inequality holds,

$$\omega_P(\sum_{j=1}^{s} w_j^{-1} h_j) \leq 0. \tag{0.9}$$

0.3. Conjecture and Results

For simple algebraic groups G_1 and G_2, suppose

$$G_1 \hookrightarrow G_2, \quad K_1 \hookrightarrow K_2, \quad \mathfrak{h}^{G_1}_+ \to \mathfrak{h}^{G_2}_+.$$

Then,

$$\begin{array}{ccc} \Gamma(G_1) & & \Gamma(G_2) \\ \big\downarrow & & \big\downarrow \\ (\mathfrak{h}_+^{G_1})^3 & \xrightarrow{\phi} & (\mathfrak{h}_+^{G_2})^3 \end{array}$$

It is known that $\phi(\Gamma(G_1)) \subset \Gamma(G_2)$, so we are interested in the following two questions:

Questions 0.3.1. (1) Is $\Gamma(G_1) = \phi^{-1}(\Gamma(G_2))$? (2) Is there any direct relation between the intersection theories of homogeneous spaces for G_1 and G_2?

Belkale and Kumar [**BK10**] made a conjecture for a semisimple complex algebraic group G_2. Let σ be a diagram automorphism of G_2 with a fixed subgroup G_1. They conjectured that for any standard parabolic subgroup $P \subset G_2$ and Schubert cells $\Lambda_{w_1}^P$, $\Lambda_{w_2}^P$, $\Lambda_{w_3}^P$ in G_2/P, there exist elements $g_1, g_2, g_3 \in G_1$ such that the intersection $\cap_{i=1}^3 g_i \Lambda_{w_i}^P$ is proper. Belkale and Kumar show that this is true for $Sp(2n) \subset SL(2n)$ and $SO(2n{+}1) \subset SL(2n{+}1)$, where the embeddings are induced from diagram automorphisms. Specifically they answer (0.3.1) and use the intersection theory result to compare the eigencones for these pairs.

I have investigated questions (0.3.1) for a specific embedding $SO(2n{+}1) \hookrightarrow SO(2n{+}2)$, with the following results: Let $\mathfrak{h}_+^B$ denote the positive Weyl chamber for $SO(2n+1)$. Fix $r \leq n$ and take maximal parabolic subgroups $P_r^B \subset SO(2n+1)$ and $P_r^D \subset SO(2n+2)$. First we address the relationship between the intersection theories. The conjecture, as stated, is false. A counterexample is given in Chapter **??**Appendix) in (7.1).

There is a not direct analogue of the intersection theory result for $SO(2n+1) \subset SO(2n+2)$, but there is a statement which holds in the BK-product, when the classes multiply to the class of a point. Restricting to the BK-product gives an irredundant set of inequalities describing the eigencone for $SO(2n+1)$. In Chapter (1) we define sets of indices parameterizing Schubert varieties in $SO(2n+1)/P_r^B$ in (1.7) and sets of indices parameterizing Schubert varieties in $SO(2n+2)/P_r^D$ in (1.11) and (1.12). Then, in Section (3.2), we detail a correspondence between parameters for Schubert varieties in the these homogeneous spaces. The following theorem is a key result and will be proved in Chapter (5):

Theorem 0.3.2. *For parameters I^1, I^2, I^3 giving Schubert varieties in $SO(2n+1)/P_r^B$ and corresponding parameters J^1, J^2, J^3 giving Schubert varieties in $SO(2n+2)/P_r^D$, let σ_K denote the class in cohomology of the associated variety. Then,*

$$\sigma_{I^1} \odot_0 \sigma_{I^2} \odot_0 \sigma_{I^3} = \sigma_{pt} \in H^{2dim(OG(r,2n+1))}(OG(r,2n+1)) \Rightarrow$$

$$\sigma_{J^1} \odot_0 \sigma_{J^2} \odot_0 \sigma_{J^3} = d\sigma_{pt} \in H^{2dim(OG(r,2n+2))}(OG(r,2n+2)),$$

for some nonzero d.

A question that remains is whether $d = 1$ in Theorem (0.3.2). In the examples computed, including the example given in the final chapter, $d = 1$.

The following theorem is the main result of the paper comparing the eigencones of $SO(2n+1)$ and $SO(2n+2)$. It will be proved in Chapter (4):

Theorem 0.3.3. *For $(h_1, h_2, h_3) \in \mathfrak{h}_+^B$,*

$$(h_1, h_2, h_3) \in \Gamma(SO(2n+1)) \iff (h_1, h_2, h_3) \in \Gamma(SO(2n+2)).$$

By functoriality, it is clear that $\Gamma(SO(2n+1)) \subset \Gamma(SO(2n+2))$. Then for

$$(h_1, \ldots, h_s) \in \Big((\mathfrak{h}_+^B)^s \cap \Gamma(SO(2n+2)),$$

we must show that $(h_1, \ldots, h_s) \in \Gamma(SO(2n+1))$. So, the task is to take an inequality in the system describing $\Gamma(SO(2n+1))$ and show that it is implied by an inequality in the system describing $\Gamma(SO(2n+2))$.

Another key result used in comparing the eigencones is Theorem (0.3.2), which gives an intersection theoretic statement relating the intersection theories of $SO(2n+1)$ and $SO(2n+2)$. As mentioned above, this statement holds for Levi-movable intersections in $SO(2n+1)/P_r^B$. A numerical condition on parameters determining Levi-movable intersections is given in Theorem (2.2.6). One of the challenges in proving Theorem (0.3.2) was confirming a dimension formula for Schubert varieties on $SO(2n+2)/P_r^D$ that was comparable to known formulas for varieties in $SO(2n+1)/P_r^B$. The dimension formula and proof is given in Chapter (6). It is proved by constructing a chain of varieties based on a set of rules we call the bumping rules for $SO(2n+2)/P_r^D$. An example illustrating how the rules are used to create such a chain is given in Chapter 7 in (7.3).

CHAPTER 1

Notation and Preliminaries

In this chapter we establish notation for the rest of the paper and give details on the groups $SL(2n+1)$ in Section (1.2), $SO(2n+1)$ in Section (1.3), $SO(2n+2)$ in Section (1.4), and $Sp(2n)$ in Section (1.5). For a reader interested in the eigencone result, Sections (1.3) and (1.4) are the most important.

1.1. Algebraic Group G

Let G be a connected semisimple complex algebraic group. Fix a Borel subgroup B and a maximal torus $H \subset B$. Their Lie algebras are denoted by their respective Gothic characters: $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$. Let W denote the Weyl group of G defined by $W = N(H)/H$, where $N(H)$ is the normalizer of H in G. Let $R^+ \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{b}$, the positive roots, and denote the simple roots by $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset R^+$. Let $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the set of corresponding simple coroots and $\{s_1, \ldots, s_n\} \subset W$ be the set of corresponding simple reflections. The length of $w \in W$, denoted $l(w)$, is defined to be the minimum of the length of an expression for w as a product of simple reflections. Define the basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{h}$ dual to the basis $\{\alpha_i\}$ of $\mathfrak{h}^*$, so that $\alpha_j(x_i) = \delta_{i,j}$.

Fix a real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$, so that $\mathfrak{h}_{\mathbb{R}}$ is a real subspace of $\mathfrak{h}$ satisfying:

(1) $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}_{\mathbb{R}}$ and
(2) for any $1 \leq i \leq n$, $\alpha_i(\mathfrak{h}_{\mathbb{R}}) \subset \mathbb{R}$.

Define the positive Weyl chamber, or dominant chamber, $\mathfrak{h}_+ \subset \mathfrak{h}_{\mathbb{R}}$, to be:

$$\mathfrak{h}_+ = \{x \in \mathfrak{h}_{\mathbb{R}} \,|\, \alpha_i(x) \in \mathbb{R}_+ \text{ for all } \alpha_i\}.$$

For any $1 \leq i \leq n$, let $\omega_i \in \mathfrak{h}^*$ denote the i-th fundamental weight defined by

$$\omega_i(\alpha_j^\vee) = \delta_{i,j}.$$

A weight $\lambda = \sum c_i \omega_i$ is dominant if all c_i are non-negative integers.

Let $P \supset B$ be a standard parabolic subgroup. The parabolic admits a Levi decomposition with a unique Levi subgroup $L \supset H$. Let W_P be the Weyl group of P, which is the Weyl group of the Levi subgroup L. Let W^P denote the set of minimal-length coset representatives in the coset space W/W_P. For any $w \in W^P$, we define the Bruhat cell

$$\Lambda_w^P := BwP/P \subset G/P. \tag{1.1}$$

The Bruhat cell is a locally closed subset of G/P isomorphic to the affine space $\mathbb{A}^{l(w)}$. Its closure, denoted $\bar{\Lambda}_w^P$, is an irreducible projective variety of dimension $l(w)$. We denote the cycle class of this subvariety in the singular cohomology ring of G/P with integral coefficients by

$$[\bar{\Lambda}_w^P] \in H^{2(\dim_{\mathbb{C}}(G/P)-l(w))}(G/P). \tag{1.2}$$

The Bruhat decomposition gives an integral basis of $H^*(G/P)$ by the classes $\{[\bar{\Lambda}_w^P]\}_{w \in W^P}$.

In the following sections, we give a more detailed descriptions of the groups $SL(2n{+}1)$, $SO(2n{+}1)$ and $SO(2n{+}2)$. We will also introduce some of the structure and notation for $Sp(2n)$.

1.2. Special Linear Group $SL(2n{+}1)$

Let $V = \mathbb{C}^{2n+1}$ with basis $\{v_1, \ldots, v_{2n+1}\}$. Let $GL(2n{+}1)$ be the complex general linear group of all nonsingular complex $(2n{+}1 \times 2n{+}1)$ matrices. The complex special linear group $SL(2n{+}1) \subset GL(2n{+}1)$ is the subgroup consisting of matrices of determinant one. Fix a standard Borel subgroup B of $SL(2n{+}1)$, consisting of upper triangular matrices of determinant one. Let H denote the Cartan subgroup consisting of diagonal matrices of determinant one. We denote the Lie algebra of $SL(2n{+}1)$ by $\mathfrak{sl}(2n{+}1)$. Denote the

Lie algebras of the Borel and Cartan subgroups by $\mathfrak{b}$ and $\mathfrak{h}$ respectively, and define the dual to $\mathfrak{h}$ by

$$\mathfrak{h}^* = \{\mathfrak{t} = \mathrm{diag}(t_1, \ldots, t_{n+1}) \,|\, \sum t_i = 0\}.$$

Fixing a real form $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$, the positive Weyl chamber $\mathfrak{h}_+ \subset \mathfrak{h}$ is given by

$$\mathfrak{h}_+ = \{\mathfrak{t} \in \mathfrak{h}_{\mathbb{R}} \,|\, t_i \in \mathbb{R}_+ \text{ and } t_1 \geq \ldots \geq t_{n+1}\}.$$

The Lie group $SL(2n+1)$ has a root system of type A_{2n}. The Borel subgroup B determines the set of positive roots R^+ in the set of roots R. The set of simple roots is denoted $\Delta = \{\alpha_1, \ldots, \alpha_{2n}\} \subset R^+ \subset \mathfrak{h}^*$. For any $1 \leq i \leq 2n$, we define the simple roots, simple coroots, and fundamental weights:

$$\begin{aligned} \alpha_i(\mathfrak{t}) &= t_i - t_{i+1}, \\ \alpha_i^\vee &= \mathrm{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0), \text{ where the 1 is in the } i\text{th place and,} \\ \omega_i(\mathfrak{t}) &= t_1 + \ldots + t_i. \end{aligned}$$

Define the basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{h}$ dual to the basis $\{\alpha_i\}$ of $\mathfrak{h}^*$, so that $\alpha_j(x_i) = \delta_{i,j}$.

The Weyl group W of $SL(2n+1)$ can be identified with the symmetric group S_{2n+1} which acts by permuting the coordinates of $\mathfrak{t}$. Let $\{r_1, \ldots, r_{2n}\} \subset S_{2n+1}$ be the simple reflections corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_{2n}\}$ respectively. Then the simple reflections are given by the transpositions

$$r_i = (i, i+1). \tag{1.3}$$

For any $1 \leq r \leq 2n$, let P_r denote the standard maximal parabolic subgroup whose unique Levi subgroup L_r, has the associated set of simple roots $\Delta \setminus \{\alpha_r\}$. The Weyl group of P_r we denote W_{P_r}. The set of minimal length coset representatives in the quotient W/W_{P_r} is denoted W^{P_r}, and can be identified with the set of r-tuples

$$S(r, 2n+1) = \{A := 1 \leq a_1 < \ldots < a_r \leq 2n+1\}.$$

Any such r-tuple A corresponds to a permutation in its one-line notation:

$$w_A = (a_1, \ldots, a_r, a_{r+1}, \ldots, a_{2n+1}),$$

where $\{a_{r+1} < \ldots < a_{2n+1}\} = [2n+1] \setminus \{a_1, \ldots, a_r\}$.

We identify $SL(2n+1)/P_r$ with the Grassmannian of r-dimensional subspaces of V, $Gr(r, 2n+1)$.

Definition 1.2.1. We define a *complete flag on* V to be the chain of subspaces $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_{2n+1} = V$ where $\dim(F_i) = i$.

The standard complete flag on V, denoted $F_\bullet^\circ$, is the complete flag with $F_i = \langle v_1, \ldots, v_i \rangle$. Let $F_\bullet$ be a complete flag on V and $A \in S(r, 2n+1)$. We define the shifted Schubert cell in $Gr(r, 2n+1)$:

(1.4)

$$\Omega_A(F_\bullet) = \{M \in Gr(r, 2n+1) \mid \text{for any } 0 \leq l \leq r \text{ and any } a_l \leq b < a_{l+1}, \dim(M \cap F_b) = l\},$$

where $a_0 = 0$ and $a_{r+1} = 2n+1$.

The standard flag, denoted $F_\bullet^\circ$, has flag components $F_i^\circ = \text{span}\langle v_1, \ldots, v_i \rangle$. For the standard flag $F_\bullet^\circ$, we have $\Omega_A(F_\bullet^\circ) = \Lambda_{w_J}^{P_r}$. Otherwise, there is an element $g(F_\bullet) \in SL(2n+1)$, determined up to left multiplication by an element of B, such that $\Omega_A(E_\bullet) = g(F_\bullet)\Lambda_{w_J}^{P_r}$. The closure of a shifted Schubert cell in $Gr(r, 2n+1)$ is denoted $\overline{\Omega}_A(F_\bullet)$, and its cycle class in $H^*(Gr(r, 2n+1))$ is denoted $[\overline{\Omega}_A(F_\bullet)]$.

1.3. Special Orthogonal Group $SO(2n+1)$

Again, let $V = \mathbb{C}^{2n+1}$ with basis $\{v_1, \ldots, v_{2n+1}\}$. Equip V with a non-degenerate symmetric form $\langle , \rangle^B$ such that $\langle v_i, v_{2n+1-i} \rangle^B = 1$ for $1 \leq i \leq n$, $\langle v_{n+1}, v_{n+1} \rangle^B = 2$, and all other pairings are zero. Denote the $(2n+1 \times 2n+1)$ matrix of $\langle , \rangle^B$ by E_B. Let $v = \sum_{i=1}^{2n+1} t_i v_i$. Let Q^B be the quadratic form associated to $\langle , \rangle^B$, given by:

$$Q^B(v) = t_{n+1}^2 + \Sigma_{i=1}^n t_i t_{2n+2-i}. \tag{1.5}$$

Denote the special orthogonal group associated to Q^B by $SO(Q^B, 2n+1)$. It is the group defined by

$$SO(Q^B, 2n+1) = \{g \in SL(2n+1) | \langle gv, gw\rangle^B = \langle v, w\rangle^B \text{ for all } v, w \in V\}.$$

Since the choice of Q^B is clear, we will simplify the notation to $SO(2n+1) = SO(Q^B, 2n+1)$. The group $SO(2n+1)$ can be realized as the fixed point subgroup $SL(2n+1)^{\Theta}$ under the involution $\Theta : SL(2n+1) \to SL(2n+1)$, defined by $\Theta(A) = E_B^{-1}(A^T)^{-1}E_B$. The involution keeps both B and H stable. Moreover, $SO(2n+1)$ has Borel subgroup B^{Θ} and Cartan subgroup H^{Θ}. We will use the notation $B^B = B^{\Theta}$ to denote the Borel subgroup and $H^B = H^{\Theta}$ to denote the Cartan subgroup.

We denote the Lie algebra of $SO(2n+1)$ by $\mathfrak{so}(2n+1)$. The Cartan subgroup has Lie algebra $\mathfrak{h}^B$ given by

$$\mathfrak{h}^B = \{\mathfrak{t} = \text{diag}(t_1, \dots, t_n, 0, -t_n, \dots, -t_1) \,|\, t_i \in \mathbb{C}\}.$$

We fix a real form $\mathfrak{h}^B_{\mathbb{R}} \subset \mathfrak{h}^B$ and the positive Weyl chamber is given by

$$\mathfrak{h}^B_+ = \{\mathfrak{t} \in \mathfrak{h}^B_{\mathbb{R}} \,|\, t_i \in \mathbb{R} \text{ and } t_1 \geq \ldots \geq t_n \geq 0\}.$$

The Lie group $SO(2n+1)$ has a root system of type B_n. Fixing a Borel subgroup B^B determines a set of positive roots R_B^+ in the set of roots R_B. Let $\Delta^B = \{\delta_1, \dots, \delta_n\}$ be the set of simple roots. For any $i \in [n]$, $\delta_i = \alpha_i|_{\mathfrak{h}^B}$, where $\{\alpha_1, \dots, \alpha_{2n}\}$ are the simple roots of $SL(2n+1)$. The simple coroots $\{\delta_i^\vee\}$ are given by

$$\begin{aligned} \delta_i^\vee &= \alpha_i^\vee + \alpha_{2n+1-i}^\vee \text{ for } 1 \leq i < n \text{ and} \\ \delta_n^\vee &= 2(\alpha_n^\vee + \alpha_{n+1}^\vee). \end{aligned}$$

Define the basis $\{x_1^B, \dots, x_n^B\}$ of $\mathfrak{h}^B$ dual to the basis $\{\delta_i\}$ of $\mathfrak{h}^{*B}$, so that $\delta_j(x_i^B) = \delta_{i,j}$.

The Weyl group of $SO(2n+1)$ is denoted W_B. Recall that H^B is Θ-stable. Therefore there is an induced action of Θ on the S_{2n+1}, the Weyl group of $SL(2n+1)$. So, the Weyl group of $SO(2n+1)$ can be identified with the Θ-invariant subgroup of S_{2n+1}:

$$\{(a_1, \ldots, a_{2n+1}) \in S_{2n+1} \,|\, a_{2n+2-i} = 2n+2 - a_i, \text{ for all } 1 \le i \le 2n+1\}.$$

Thus, $w = (a_1, \ldots, a_{2n+1}) \in W_B$ is determined by $(a_1, \ldots, a_n)$ and $a_{n+1} = n + 1$.

Let $\{s_1, \ldots, s_n\}$ be the simple reflections corresponding to the simple roots $\{\delta_1, \ldots, \delta_n\}$. We can express the simple reflections s_i in terms of the simple reflections r_i given in (1.3):

$$\begin{aligned} s_i &= r_i r_{2n+1-i}, \text{ if } 1 \le i \le n-1, \text{ and} \\ s_n &= r_n r_{n+1} r_n. \end{aligned}$$

A subspace $U \subset V$ is called isotropic if $\langle u_1, u_2 \rangle = 0$ for all $u_1, u_2 \in U$. For $r \le n$, we define the subvariety of $Gr(r, 2n+1)$ containing all r-dimensional isotropic subspaces of V:

$$OG(r, 2n+1) := \{\Lambda \in Gr(r, 2n+1) | \langle u_1, u_2 \rangle^B = 0, \text{ for all } u_1, u_2 \in \Lambda\}. \tag{1.6}$$

This closed irreducible subvariety of $Gr(r, 2n+1)$ is of dimension $\frac{r}{2}(4n-3r+1)$. Notice that this space is defined for $r \le n$ since V does not contain an isotropic subspace of dimension $n+1$. Let P_r^B is the standard maximal parabolic subgroup with unique Levi subgroup L_r^B whose associated set of simple roots is $\Delta^B \setminus \{\delta_r\}$. The maximal parabolic subgroup P_r^B is the stabilizer of the isotropic subspace $\langle v_1, \ldots, v_r \rangle$. The orthogonal group $SO(2n+1)$ acts transitively on $OG(r, 2n+1)$, and we identify $OG(r, 2n+1)$ with the homogeneous space $SO(2n+1)/P_r^B$.

The set of minimal length coset representatives of $W_B/W_{P_r^B}$ is denoted W_r^B, and can be identified with the following subset of $S(r, 2n+1)$,

$$\mathfrak{S}(r, 2n+1) = \{J := 1 \le j_1 < \ldots < j_r \le 2n+1 \,|\, j_t \ne n+1 \text{ for any } t \text{ and } J \cap \bar{J} = \emptyset\}, \tag{1.7}$$

where $\bar{J} = \{2n{+}2 - j_r, \ldots, 2n{+}2 - j_1\}$. Any $J \in \mathfrak{S}^B(r, 2n{+}1)$ determines a minimal length coset representative in W_r^B:

$$w_J = (j_1, \ldots, j_r, \tilde{j}_1, \ldots, \tilde{j}_{2n+1-2r}, \bar{j}_1, \ldots, \bar{j}_r), \tag{1.8}$$

where

$$\tilde{J} = (\tilde{j}_1, \ldots, \tilde{j}_{2n+1-2r}) = [2n{+}1] \setminus (J \sqcup \bar{J}),$$

with elements written in increasing order.

Definition 1.3.1. A complete flag $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_{2n+1} = V$ is called *isotropic* if F_i is an isotropic subspace of V for all $i \in [n]$ and $F_i = F_{2n+1-i}^{\perp}$ for all i.

With our choice of basis the standard flag, $F_\bullet^\circ$, is isotropic with respect to $\langle , \rangle^B$. For any isotropic flag $F_\bullet$, there exists a unique element, up to multiplication by an element of B^B, $g(F_\bullet) \in SO(2n{+}1)$ which takes the standard flag to $F_\bullet$:

$$g(F_\bullet)F_\bullet^\circ = F_\bullet.$$

Definition 1.3.2. For any $J \in \mathfrak{S}(r, 2n{+}1)$ and any isotropic flag $F_\bullet$ we define the shifted Schubert cell in $OG(r, 2n{+}1)$:

$$\Psi_J(F_\bullet) = \{M \in OG(r, 2n{+}1) \mid \text{for any } 0 \leq l \leq r, \text{ and any } j_l \leq b < j_{l+1}, \dim(M \cap F_b) = l\}.$$

where we set $j_0 = 0$ and $j_{r+1} = 2n{+}1$.

Set theoretically we have an equality:

$$\Psi_J(F_\bullet) = \Omega_J(F_\bullet) \cap OG(r, 2n{+}1).$$

In fact, this is a scheme theoretic equality [**Bel07**]. Moreover, there exists $g(F_\bullet) \in SO(2n{+}1)$ such that $\Psi_J(F_\bullet) = g(F_\bullet)\Lambda_{w_J^B}^{P_r^B}$, where w_J^B is the minimal length coset representative associated to J. Let $\overline{\Psi}_J(F_\bullet)$ denote the closure of $\Psi_J(F_\bullet) \in OG(r, 2n{+}1)$ and we denote its cycle class in cohomology by

$$[\overline{\Psi}_J(F_\bullet)] \in H^{2(\dim(SO(2n+1)/P^B_r)-l(w_J))}(OG(r,2n+1)).$$

Notice that for the standard flag, $F^\circ_\bullet$ on V, $\Psi_J(F^\circ_\bullet) = \Lambda^{P^B_r}_{w^B_J}$.

In Chapter (3), we realize $SO(2n+1)$ as a fixed point of a diagram automorphism of $SO(2n+2)$.

1.4. Special Orthogonal Group $SO(2n+2)$

Let $W = \mathbb{C}^{2n+2}$ with basis $\{w_1, \ldots, w_{2n+2}\}$, and equipped with a symmetric nondegenerate bilinear form $\langle , \rangle^D$. Denote the $(2n+2 \times 2n+2)$ matrix of $\langle , \rangle^D$ by E_D. Let $w = \sum_{i=1}^{2n+2} t_i w_i$. The associated quadratic form on W is given by

$$Q^D(w) = \Sigma_{i=1}^{2n+2} t_i t_{2n+3-i}. \tag{1.9}$$

Define the even special orthogonal group associated to Q^D by

$$SO(Q^D, 2n+2) = \{g \in SL(2n+2) | \langle gw_1, gw_2\rangle^D = \langle w_1, w_2\rangle^D\}.$$

Since the choice of the quadratic form is clear, we will simplify the notation to $SO(2n+2) = SO(Q^D, 2n+2)$.

Similar to the description of $SO(2n+1)$ as a fixed point subgroup of $SL(2n+1)$, $SO(2n+2)$ is the fixed point subgroup of an involution on $SL(2n+2)$:

$$\begin{aligned} \Theta' : SL(2n+2) &\rightarrow SL(2n+2), \\ A &\mapsto E_D(A^T)^{-1}E_D. \end{aligned}$$

Observe that Θ' keeps H and B stable. We denote the Cartan subgroup of $SO(2n+2)$ by $H^D = H^{\Theta'}$ and the standard Borel subgroup by $B^D = B^{\Theta'}$.

We denote the Lie algebra of $SO(2n{+}2)$ by $\mathfrak{so}(2n{+}2)$. The Cartan subgroup has Lie algebra $\mathfrak{h}^B$ given by

$$\mathfrak{h}^D = \{\mathfrak{t} = \mathrm{diag}(t_1, \dots, t_{n+1}, -t_{n+1}, \dots, -t_1) \,|\, t_i \in \mathbb{C}\}.$$

We fix a real form $\mathfrak{h}^D_{\mathbb{R}} \subset \mathfrak{h}^D$ and the positive Weyl chamber is given by

$$\mathfrak{h}^D_+ = \{\mathfrak{t} \in \mathfrak{h}^D_{\mathbb{R}} \,|\, t_i \in \mathbb{R} \text{ and } t_1 \geq \dots \geq t_{n+1} \geq 0\}.$$

The Lie group $SO(2n{+}2)$ has a root system of type D_{n+1}. Fixing a Borel subgroup B^D determines a set of positive roots R^+_D in the set of roots R_D. Let $\Delta^D = \{\vartheta_1, \dots, \vartheta_{n+1}\}$ be the set of simple roots. Then, for any $i \in [n+1]$, $\vartheta_i = \alpha_i|\mathfrak{h}^D$. Define the basis $\{x^D_1, \dots, x^D_{n+1}\}$ of $\mathfrak{h}^D$ dual to the basis $\{\vartheta_i\}$ of $\mathfrak{h}^{*D}$, so that $\vartheta_j(x^D_i) = \delta_{i,j}$.

The Weyl group of $SO(2n{+}2)$ is denoted W_D. Recall that H^D is Θ'-stable. So, there is an induced action of Θ' on S_{2n+2}. Therefore, the Weyl groups of $SO(2n{+}2)$ can be identified with the Θ'-invariant subgroup of S_{2n+2}:

$$\{(a_1, \dots, a_{2n+2}) \in S_{2n+2} \,|\, a_{2n+3-i} = 2n+3-a_i, \text{ for all } 1 \leq i \leq 2n{+}2\}.$$

So, $w' = (a_1, \dots, a_{2n+2}) \in W_D$ is determined by $(a_1, \dots a_{n+1})$.

Let $\{s'_1, \dots, s'_{n+1}\}$ be the simple reflections in W_D, the Weyl group for $SO(2n{+}2)$, corresponding to the simple roots $\{\vartheta_1, \dots, \vartheta_{n+1}\}$. We can express the simple reflections Δ^D in terms of simple reflections r_i, defined in (1.3):

$$\begin{aligned} s'_i &= r_i r_{2n+2-i}, \text{ for all } 1 \leq i \leq n, \text{ and} \\ s'_{n+1} &= r_{n+1} r_n r_{n+2} r_{n+1}. \end{aligned}$$

Recall that a subspace $U \subset W$ is called isotropic if $\langle u_1, u_2 \rangle^D = 0$ for all $u_1, u_2 \in U$.

The set of isotropic subspaces of W of dimension $n+1$ form two orbits under the action of $SO(2n{+}2)$. Therefore, the set of complete isotropic flags on $\mathbb{C}^{2n+2}$, defined in (1.4.3) below, form two orbits under the left action of $SO(2n{+}2)$.

Definition 1.4.1. Let S_1 and S_2 be two $n+1$ dimensional isotropic subspaces of W. We say that S_1 and S_2 are *in the same family* if $\dim(S_1 \cap S_2) \equiv (n+1)\text{mod}2$, otherwise we say that they are in the opposite family.

Definition 1.4.2. Define the *standard complete flag* on $\mathbb{C}^{2n+2}$, denoted $F^\circ_\bullet$, to be the flag with components $F^\circ_i = \text{span}\langle w_1, \ldots, w_i \rangle$.

Definition 1.4.3. A complete flag $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_{2n+2} = W$ is called *isotropic* if F_i is an isotropic subspace of W for all $i \in [n+1]$, $F_i = F^\perp_{2n+2-i}$ for all i, and F_{n+1} is in the same family as F°_{n+1}.

Definition 1.4.4. To each isotropic flag $F_\bullet$ we define a corresponding *alternate isotropic flag* $\tilde{F}_\bullet$ such that $\tilde{F}_i = F_i$ for all $i \leq n$, and $\tilde{F}_{n+1}$ in the opposite family from F_{n+1}.

For $r < n$, we define the subvariety of $Gr(r, 2n+2)$ containing all r-dimensional isotropic subspaces of W:

$$OG(r, 2n{+}2) := \{\Lambda \in Gr(r, 2n{+}2) \,|\, \langle w_1, w_2 \rangle^D = 0, \text{ for all } w_1, w_2 \in \Lambda\}. \tag{1.10}$$

This orthogonal Grassmannian is a closed irreducible subvariety of $Gr(r, 2n+2)$ is of dimension $\frac{r}{2}(4n - 3r + 3)$. For $r < n$, let P^D_r be the standard maximal parabolic subgroup for which $\Delta^D \setminus \{\vartheta_r\}$ is the set of simple roots of L^D_r, the unique Levi subgroup of P^D_r containing H^D. The parabolic P^D_r is the stabilizer of the span of the first r basis elements of W. So we make the identification of $OG(r, 2n+2)$ with the homogeneous space $SO(2n{+}2)/P^D_r$.

For $r = n+1$, the space of maximal isotropic subspaces of $\mathbb{C}^{2n+2}$ has two connected components, each isomorphic to the even orthogonal Grassmannian, $OG(n{+}1, 2n{+}2) = SO(2n{+}2)/P_{n+1}$. The maximal parabolic subgroup P_{n+1}, of $SO(2n{+}2)$, is the parabolic associated to a right end root of the Dynkin diagram of Type D_{n+1}. We denote the two connected components by $OG^{\pm}(n{+}1, 2n{+}2)$. Let $OG^{+}(n{+}1, 2n{+}2)$ denote the orbit containing the F°_{n+1}, the $n{+}1^{st}$ component of the standard flag $F'^\circ_\bullet$. Let $OG^{-}(n{+}1, 2n{+}2)$

denote the orbit containing all of the $n+1$-dimensional spaces in the opposite family. In Chapter (3), we will show that $OG^{+}(n{+}1, 2n{+}2)$ and $OG^{-}(n{+}1, 2n{+}2)$ are homeomorphic to the space of maximal isotropic subspaces $OG(n, 2n{+}1)$, defined in (1.6).

The set of minimal length coset representatives of the quotient space $W_D/W_{P_r^D}$ is denoted W_r^D. For $r < n$, W_r^D can be identified with the set

$$\mathfrak{S}^D(r, 2n{+}2) := \{J := 1 \leq j_1 < \ldots < j_r \leq 2n{+}2 \mid J \cap \bar{J} = \emptyset\}, \tag{1.11}$$

where $\bar{J} = \{2n+3-j_r, \ldots, 2n+3-j_1\}$. For $r = n+1$, we define two parameterizing sets. Let

$$\mathfrak{S}^D(n{+}1, 2n{+}2) := \{J := 1 \leq j_1 < \ldots < j_r \leq 2n{+}2 \mid J \cap \bar{J} = \emptyset\},$$

and define

$$\mathfrak{S}^{+}(n{+}1, 2n{+}2) = \{J \in \mathfrak{S}^D(n{+}1, 2n{+}2) | \#\{j_k \leq n+1\} \equiv 0 \mathrm{mod} 2\} \text{ and,} \tag{1.12}$$

$$\mathfrak{S}^{-}(n{+}1, 2n{+}2) = \{J \in \mathfrak{S}^D(n{+}1, 2n{+}2) | \#\{j_k \leq n+1\} \equiv 1 \mathrm{mod} 2\}. \tag{1.13}$$

The sets defined in (1.12) are parameterizing sets for Schubert varieties in $OG^{+}(n{+}1, 2n{+}2)$ and $OG^{-}(n{+}1, 2n{+}2)$ respectively.

For any isotropic flag $F_\bullet$, such that $F_{n+1} \subset OG^{+}(n{+}1, 2n{+}2)$, there is an element $g(F_\bullet) \in SO(2n{+}2)$ that takes the standard flag to the isotropic flag $F_\bullet$. Note that $g(F_\bullet)$ is determined up to right multiplication by an element of B^D.

Definition 1.4.5. For any $J \in \mathfrak{S}^D(r, 2n{+}2)$, and any isotropic flag $F_\bullet$ we define, for $r < n$ the shifted Schubert cell in $OG(r, 2n{+}2)$:

(1.14)

$$\Upsilon_J(F_\bullet) = \{M \in OG(r, 2n{+}2) \mid \text{ for any } 0 \leq l \leq r, \text{ and any } j_l \leq b < j_{l+1}, \dim(M \cap F_b) = l\}.$$

Set theoretically (also scheme theoretically) we have an equality:

$$\Upsilon_{J'}(F'_\bullet) = \Omega_{J'}(F'_\bullet) \cap OG(r, 2n{+}2). \tag{1.15}$$

For $r = n+1$, the definition of the Schubert cell (1.14) holds and for $J_+ \in \mathfrak{S}^+(n+1, 2n+2)$ and $J_- \in \mathfrak{S}^-(n+1, 2n+2)$, parameters for $\Upsilon_{J_+}(F_\bullet) \subset OG^+(n+1, 2n+2)$ and $\Upsilon_{J_-}(F_\bullet) \subset OG^-(n+1, 2n+2)$, respectively. The set theoretic intersection defined in (1.15) will also hold with these choices for $J_\pm$.

Any $J \in \mathfrak{S}^D(r, 2n+2)$ determines a word in W_r^D. Let $\tilde{J} = (\tilde{j}_1, \ldots, \tilde{j}_{2n+2-2r}) = [2n+2] \setminus (J \sqcup \bar{J})$, with the elements written in increasing order. For $r < n$,

$$(1.16) \qquad w_J = (j_1, \ldots, j_r, \tilde{j}_1, \ldots, \tilde{j}_{2n+2-2r}, \bar{j}_1, \ldots, \bar{j}_r),$$

if the first $n+1$ components of the word have the same number of integers greater than $n+1$ as less than $n+1$, and

$$(1.17) \qquad w_J = (j_1, \ldots, j_r, \tilde{j}_1, \ldots, \tilde{j}_{n-r}, \tilde{j}_{n+2-r}, \tilde{j}_{n+1-r}, \tilde{j}_{n+3-r}, \ldots, \tilde{j}_{2n+2-2r}, \bar{j}_1, \ldots, \bar{j}_r)$$

otherwise. When $r = n+1$, for $w \in W_r^D$, then

$$w \equiv w u_i (\text{mod} W_{P_{n-1}}), \text{ for } 0 \leq i \leq n+1 \text{ and } i \neq n-1,$$

where,

$$u_i = \begin{cases} s'_n & \text{if } i = n \\ id & \text{if } i = 0 \\ s'_i s'_{i+1} \cdots s'_{n-2} s'_n & \text{if } 1 \leq i \leq n-2. \end{cases}$$

1.5. Symplectic Group $Sp(2n)$

Although we will not focus on the symplectic group in this thesis, we will apply important results about the cohomology ring and the eigencone of the group and its relationship with $SO(2n+1)$. In Chapter (2) we will see that the system of irredundant inequalities describing the eigencone of $Sp(2n)$ coincides with the inequalities describing $SO(2n+1)$. We only refer to $Sp(2n)$ for this comparison to derive a numerical condition on the index parameters for Levi-movability in an orthogonal Grassmannian $OG(r, 2n+1)$. For completeness, we include some background information on $Sp(2n)$.

Let $U = \mathbb{C}^{2n}$ be equipped with a nondegenerate symplectic form $\langle , \rangle^C$ so that its matrix $(\langle u_i, u_j \rangle^C)_{1 \leq i,j \leq 2n}$ in the basis $\{u_1, \ldots, u_{2n}\}$, is given by

$$E_c = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

where J is the $n \times n$ matrix with 1 along the anti-diagonal, and zeroes elsewhere. $Sp(2n)$ can be realized as the fixed point subgroup under the involution $\sigma : SL(2n) \to SL(2n)$ defined by $\sigma(A) = E_c(A^T)^{-1}E_c^{-1}$. Let B and H be the Borel and Cartan subgroups of $SL(2n)$, then the involution σ keeps B and H stable, and we denote B^σ and H^σ by B^C and H^C respectively.

We denote the Lie algebra of $Sp(2n)$ by $\mathfrak{sp}(2n)$. The Lie algebra of H^C is the Cartan subalgebra,

$$\mathfrak{h}^C = \{\mathfrak{t} = \mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) \,|\, t_i \in \mathbb{C}\}.$$

Fix a real form of $\mathfrak{h}^C_{\mathbb{R}} \subset \mathfrak{h}^C$. Then, the fundamental Weyl chamber is given by

$$\mathfrak{h}^C_+ = \{\mathfrak{t} \in \mathfrak{h}^C_{\mathbb{R}} \,|\, t_i \in \mathbb{R}, \text{ and } t_1 \geq \ldots \geq t_n \geq 0\}.$$

The Lie group $Sp(2n)$ has a root system of type C_n. Fixing a Borel subgroup B^C determines a set of positive roots R_C^+ in the set of roots R_C. Denote by ρ^C half the sum of the positive roots. Let $\Delta^C = \{\beta_1, \ldots, \beta_n\}$ be the set of simple roots, such that for all $i \in [n]$, $\beta_i = \alpha_i|_{\mathfrak{h}^C}$ where $\{\alpha_1, \ldots, \alpha_{2n-1}\}$ are the simple roots of $SL(2n)$. The corresponding simple coroots are denoted $\{\beta_1^\vee, \ldots, \beta_n^\vee\}$, and are given by:

$$\begin{aligned} \beta_i^\vee &= \alpha_i^\vee + \alpha_{2n-i}^\vee, \text{ for } 1 \leq i < n \text{ and} \\ \beta_n^\vee &= \alpha_n^\vee. \end{aligned}$$

Define the basis $\{x_1^C, \ldots, x_n^C\}$ of $\mathfrak{h}^C$ dual to the basis $\{\beta_i\}$ of $\mathfrak{h}^{*C}$, so that $\beta_j(x_i^B) = \delta_{i,j}$.

The Weyl group of $Sp(2n)$ is denoted W_C. Recall that H^C is $\sigma-$stable, and therefore there is an induced action of σ on S_{2n}. Therefore, W_C can be identified with a subgroup of S_{2n}:

$$\{(a_1, \ldots, a_{2n}) \in S_{2n} \,|\, a_{2n+1-i} = 2n{+}1 - a_i, \text{ for all } 1 \le i \le 2n\}.$$

Let $\{s_1, \ldots, s_n\}$ be the simple reflections in the Weyl group W^C of $Sp(2n)$ corresponding to the simple roots $\{\beta_1, \ldots, \beta_n\}$ respectively.

For any $1 \le r \le n$, we let $IG(r, 2n)$ be the set of r-dimensional isotropic subspaces of V with respect to the form $\langle , \rangle^C$:

$$IG(r, 2n) := \{M \in Gr(r, 2n) | \langle v, v' \rangle^C = 0, \forall v, v' \in M\}.$$

This is the quotient space $Sp(2n)/P_r^C$ of $Sp(2n)$ by the standard maximal parabolic subgroup P_r^C with $\delta^C \setminus \{\beta_r\}$ as the set of simple roots of its Levi component L_r^C.

Define a class of index sets that we will identify with the set of minimal-length coset representatives of the quotient space $W_C/W_{P_r^C}$:

$$\mathfrak{S}(r, 2n) = \{I : 1 \le i_1 < \ldots < i_r \le 2n \text{ and } I \cap \bar{I} = \emptyset\}, \tag{1.18}$$

where $\bar{I} := \{2n{+}1 - i_r, \ldots, 2n{+}1 - i_1\}$.

For any $I = \{i_1 < \ldots < i_r\} \in \mathfrak{S}(r, 2n)$ we give a corresponding minimal length coset representative as the permutation

$$w_I = (i_1, \ldots, i_r, i_{r+1}, \ldots, i_{2n+1-2r}, 2n{+}1 - i_r, \ldots, 2n{+}1 - i_1) \in W^C \tag{1.19}$$

by taking $\{i_{r+1} < \ldots < i_{2n+1-2r}\} = [2n] \setminus (I \sqcup \bar{I})$.

Definition 1.5.1. We define a *complete isotropic flag* on $\mathbb{C}^{2n}$ to be a complete flag

$$G_\bullet : 0 = G_0 \subset G_1 \subset \ldots \subset E_{2n} = \mathbb{C}^{2n}$$

where $G_a^\perp = G_{2n-a}$ for $a \in [2n]$.

For an isotropic flag $G_\bullet$, there exists an element $k(G_\bullet) \in Sp(2n)$ which takes the standard flag $G_\bullet^\circ$ to the flag $G_\bullet$.

For any $I \in \mathfrak{S}(r, 2n)$ and any isotropic flag $G_\bullet$, we have the corresponding shifted Schubert Cell in $IG(r, 2n)$:

$$\Phi_{I^C}(G_\bullet^C) = \{M \in IG(r, 2n) | \text{ for any } 0 \leq l \leq r \text{ and any } i_l \leq a < i_{l+1}, \dim(M \cap G_a) = l\},$$

where we set $i_0 = 0$ and $i_{r+1} = 2n$. Set theoretically we have the equality:

$$\phi_I(G_\bullet) = \Omega_I(G_\bullet) \cap IG(r, 2n).$$

Notice that $k(G_\bullet)\Lambda_{w_I}^{P_r^C} = \phi_I(G_\bullet)$. Denote its closure in $IG(r, 2n)$ by $\overline{\phi}_I(G_\bullet)$ and denote its class in cohomology by $[\overline{\phi}_I(G_\bullet)] \in H^*(IG(r, 2n))$.

CHAPTER 2

Levi-Movability and Deformed Product on Cohomology

The aim in this chapter is to first introduce the definition of Levi-movability and recall from [**BK06**] the numerical condition for Levi-movable s-tuples. In Section (2.2), we compare the numerical condition for a Levi-movable triple in $Sp(2n)$ with that of $SO(2n{+}1)$ to give a numerical condition for Levi-movable Schubert varieties in $SO(2n{+}1)$ on the index sets parameterizing the varieties. In particular, this numerical condition is given in (2.7).

Given subsets I and J of $[m]$, we denote by $|I \geq J|$ the number of pairs (i,j) with $i\in I$, $j\in J$, and $i \geq j$. We set $|I \geq \emptyset| = 0$ and if $K = \{k\}$, then we abbreviate $|I \geq K|$ to $|I \geq k|$.

2.1. Levi-Movability

All of the notation in this section will be consistent with section (1.1). We let G be a connected semisimple complex algebraic group and $P \subset G$ a maximal parabolic subgroup. Recall the definitions of Λ_w^P in (1.1), and $[\overline{\Lambda}_w^P]$ in (1.2). We denote the structure coefficients under the regular cup product by $c_{u,v}^w$, and they are defined by

$$[\overline{\Lambda}_u^P] \cdot [\overline{\Lambda}_v^P] = \sum_{w\in W^P} c_{u,v}^w [\overline{\Lambda}_w^P]. \tag{2.1}$$

Recall from Lemma (0.2.1) that number $c_{u,v}^w$ counts the number of points in the intersection of shifted Schubert cells. For each $w \in W^P$, there exists a $w^\vee$, such that $w^\vee = w_0 w w_0^P$, where w_0 is the longest word in W and w_0^P is the longest word in the Weyl group W_P. So, for a generic triple $(g_1, g_2, g_3) \in G^3$, $c_{u,v}^w$ counts the number of points in

$$g_1\Lambda_u^P \cap g_2\Lambda_v^P \cap g_3\Lambda_{w^\vee}^P. \tag{2.2}$$

Note that if $c_{u,v}^w \neq 0$, this implies that $l(w) = l(v) + l(u)$.

Definition 2.1.1. Let $w_1, \ldots, w_s \in W^P$ be minimal coset representative such that

$$\sum_{j=1}^{s} \text{codim}\Lambda_{w_j}^P = \dim(G/P). \tag{2.3}$$

Then we call an s-tuple $(w_1, \ldots, w_s)$ *Levi-movable* if for generic $(l_1, \ldots, l_s) \in L^s$, the intersection

$$l_1\Lambda_{w_1}^P \cap \ldots \cap l_s\Lambda_{w_s}^P$$

is transverse at e.

Note that the definition above is independent of the coset representative chosen, so we have the notion of Levi-movability for any s-tuple $(w_1, \ldots, w_s) \in (W^P)^s$. Belkale and Kumar showed that inequalities corresponding to Levi-movable triples with $c_{u,v}^w = 1$, give a necessary and sufficient set of inequalities describing the eigencone for all types [**BK06**].

Belkale and Kumar give a numerical condition for Levi-movability [**BK06**], which we give in Theorem (2.1.2). Let ρ denote half the sum of the positive roots and ρ^L denote half the sum of the roots in R_L^+, the positive roots for the Levi Subgroup $L \subset P$.

Theorem 2.1.2. *Assume that $(w_1, \ldots, w_s) \in (W^P)^s$ satisfy (2.3). Then the following are equivalent:*

- (A) *The s-tuple $(w_1, \ldots, w_s) \in (W^P)^s$ is Levi-movable.*
- (B) $[\overline{\Lambda}_{w_1}^P] \cdot \ldots \cdot [\overline{\Lambda}_{w_s}^P] = d[\overline{\Lambda}_e^P] \in H^{2dim(G/P)}(G/P)$ *for some nonzero d and for each $\alpha_i \in \Delta \backslash \Delta(P)$, we have*

$$\Big((\sum_{j=1}^{s} \chi_{w_j}) - \chi_e\Big)(x_j) = 0,$$

where $\chi_w \in \mathfrak{h}^*$ is the character $\chi_w = \rho - 2\rho^L + w^{-1}\rho$, and χ_e is the character of the identity.

Notice that this numerical condition given in (2.1.2) is precisely the exponent on τ in the definition of the deformed product given in Definition (0.2.3).

2.2. Levi-Movable Intersection with Intersection Number One

In this section we give numerical criterion for a Levi-movable intersection with intersection number one. Recall the set of parameters $\mathfrak{S}(r, 2n)$ from (1.18). For $I \in \mathfrak{S}(r, 2n)$, recall the correspondence between a minimal length coset representative $w_I \in W^C$, given in (1.19). We establish a bijection between sets $I = \{i_1, \ldots, i_r\} \in \mathfrak{S}(r, 2n)$ and $J = \{j_1, \ldots, j_r\} \in \mathfrak{S}(r, 2n+1)$ by

$$i_k = \begin{cases} j_k & : i_k \leq n \\ j_k - 1 & : i_k > n. \end{cases}$$

After making this identification we refer to both index sets as I.

We denote half the sum of the positive roots in R_+^B by ρ^B, and similarly half the sum of the positive roots in R_+^C by ρ^C. Fix $I^1, \ldots, I^s \in \mathfrak{S}(r, 2n)$, and define functions $\theta^B, \theta^C : \mathfrak{S}(r, 2n) \to \mathbb{Z}$, by

$$\theta^B(I) = (\chi^B_{w_I} - \sum_{j=1}^{s} \chi^B_{w_{Ij}})(x^B_r) \text{ and,} \tag{2.4}$$

$$\theta^C(I) = (\chi^C_{w_I} - \sum_{j=1}^{s} \chi^C_{w_{Ij}})(x^C_r), \tag{2.5}$$

where $\chi^B_{w_I} = (\rho^B + w_I^{-1}\rho^B)$ and $\chi^C_{w_I}$ is defined similarly. From [**BK10**], we have the following lemma:

Lemma 2.2.1. *For $r < n$ and any $I \in \mathfrak{S}(r, 2n)$,*

$$\theta^C(I) = \theta^B(I) + |I \leq n| - \sum_{j=1}^{s} |I^j \leq n|,$$

and for $r = n$,

$$2\theta^C(I) = \theta^B(I) + |I \leq n| - \sum_{j=1}^{s} |I^j \leq n|.$$

There is a canonical Weyl group equivariant identification between the Cartan subalgebras $\mathfrak{h}_C$ and $\mathfrak{h}_B$ [**KLM03**], and also the Weyl groups W_C and W_B of $Sp(2n)$ and $SO(2n+1)$ respectively. We will make these identifications and denote the identified Weyl group by W and the identified fundamental Weyl chamber by $\mathfrak{h}$. Let $\mu(w)$ denote the number of times that the simple reflection s_n appears in any reduced decomposition of $w \in W$.

We recall the following theorem from [**BK10**]:

Theorem 2.2.2. *The map*

$$\phi : H^*(SO(2n+1)/B^B, \mathbb{C}) \to H^*(Sp(2n)/B^C, \mathbb{C})$$

given by

$$\phi([\overline{\Lambda}_w(B)]) = 2^{\mu(w)-n}[\overline{\Lambda}_w(C)]$$

for any $w \in W$, is an algebra homomorphism.

Theorem (2.2.2) gives us a relationship between the structure coefficients, which we state in Lemma (2.2.3). Take a triple $(u, v, w) \in W^3$.

$$[\overline{\Lambda}_u^{P^B}] \cdot [\overline{\Lambda}_v^{P^B}] = \sum_{w \in W} c_{u,v}^w(B)[\overline{\Lambda}_w^{P^B}]$$

and

$$[\overline{\Lambda}_u^{P^C}] \cdot [\overline{\Lambda}_v^{P^C}] = \sum_{w \in W} c_{u,v}^w(C)[\overline{\Lambda}_w^{P^C}].$$

The following corollary follows directly from Theorem (2.2.2):

Corollary 2.2.3. *In the regular cup product, we have*

$$c_{u,v}^w(C) = 2^{\mu(u^\vee)+\mu(v^\vee)-\mu(w^\vee)} c_{u,v}^w(B).$$

The following lemma is proved in [**BK10**]:

Lemma 2.2.4. *For $I \in \mathfrak{S}(r, 2n)$,*

$$\mu(w_I) = |I > n| = r - |I \leq n|.$$

The following theorem is due to Kumar:

Theorem 2.2.5. *A triple $(u, v, w) \in (W^B)^3$ is Levi-movable with $c_{u,v}^w(B) = 1$, if and only if $(u, v, w) \in (W^C)^3$ is Levi-movable with $c_{u,v}^w(C) = 1$.*

Proof. Notice that if $(u, v, w) \in (W^B)^3$ is Levi-movable with $c_{u,v}^w(B) = 1$, then $\theta^B(I_w) = 0$, where $I_w \in \mathfrak{S}(r, 2n+1)$ is the corresponding index set from (1.8). So we assume $(u, v, w) \in (W^B)^3$ is Levi-movable with $c_{u,v}^w(B) = 1$, and show that $\theta^B(I_w) = 0$. If $\theta^B(I_w) = 0$, then by (2.2.3),

$$c_{u,v}^w(C) = 2^{\mu(u^\vee)+\mu(v^\vee)-\mu(w^\vee)} \geq 1, \text{ which implies}$$

$$\mu(u^\vee) + \mu(v^\vee) - \mu(w^\vee) \geq 0.$$

Since $\theta^B(I_w) = 0$, we can rewrite the relationship between $\theta^C(I_w)$ as

$$\theta^C(I_w) = \begin{cases} \mu(w^\vee) - \mu(u^\vee) - \mu(v^\vee), \text{ for } r < n, \\ \frac{1}{2}\mu(w^\vee) - \frac{1}{2}\mu(u^\vee) - \frac{1}{2}\mu(v^\vee), \text{ for } r = n. \end{cases}$$

This implies $\theta^C(I_w) \leq 0$. From Theorem (2.2.2) and [**BK06**](Proposition 17), we know that $\theta^C(I_w) \geq 0$,

$$\theta^C(I) = 0.$$

To show the other direction, we assume $(u, v, w) \in (W^C)^3$ is Levi-movable with $c_{u,v}^w(C) = 1$. A similar argument shows that $\theta^C(I) = 0 \Rightarrow \theta^B(I) = 0$. □

The following theorem gives a numerical condition on index sets:

Theorem 2.2.6. *If an s-tuple $(w_1, \ldots, w_s) \in (W^B)^s$ is Levi-movable and*

$$[\overline{\Lambda}_{w_1}^{P_r}] \cdot \ldots \cdot [\overline{\Lambda}_{w_s}^{P_r}] = [\overline{\Lambda}_e^{P_r}] \tag{2.6}$$

then, for index sets $I_{w_1}, \dots, I_{w_s}$ corresponding to $w_1, \dots, w_s$ by (1.8), the following condition holds:

$$r = \sum_{i=1}^{s} |I_{w_i} \leq n|. \tag{2.7}$$

PROOF. Let $(w_1, \dots, w_s) \in (W^B)^s$ be Levi-movable and satisfy (2.6). By definition $\theta^B(I_e) = 0$. By Lemma (2.2.5), $\theta^C(I_e) = 0$. Then, by Lemma (2.2.1):

$$r = \sum_{i=1}^{s} |I_{w_i} \leq n|.$$

□

CHAPTER 3

Embedding $SO(2n+1)$ in $SO(2n+2)$

In this chapter we give an embedding of $V = \mathbb{C}^{2n+1}$ in $W = \mathbb{C}^{2n+2}$, and more importantly and embedding of $SO(2n+1)$ in $SO(2n+2)$. In the second section we establish a correspondence between parameters for Schubert varieties in homogeneous spaces $OG(r, 2n+1)$ and $OG(r, 2n+2)$, defined in (1.6) and (1.10) respectively. In the final section we consider a restriction of Schubert varieties from $OG(r, 2n+2)$ to $OG(r, 2n+1)$.

3.1. Embedding $SO(2n+1) \subset SO(2n+2)$

Recall the definitions of the quadratic forms Q^B and Q^D from (1.5) and (1.9) respectively. In this section we will detail a particular embedding of $V = \mathbb{C}^{2n+1}$ in $W = \mathbb{C}^{2n+2}$ so that V is invariant with respect to Q^D. Let $\{w_1, \ldots, w_{2n+2}\}$ be the basis of $W = \mathbb{C}^{2n+2}$ such that $\langle w_i, w_{2n+3-i}\rangle^D = 1$ and all other pairings are zero. Let V be the vector subspace spanned by:

$$v_i = w_i, \text{ for } 1 \leq i \leq n, \tag{3.1}$$

$$v_{n+1} = w_{n+1} + w_{n+2}, \text{ and} \tag{3.2}$$

$$v_j = w_{j+1} \text{ for } n+2 \leq j \leq 2n+1. \tag{3.3}$$

Then $\langle v_i, v_{2n+2-i}\rangle^D = 1$ for all $i \neq n+1$ and $\langle v_{n+1}, v_{n+1}\rangle^D = 2$, so that $Q^D|_V = Q^B$. We extend the subspace V to W with the vector $v_0 := w_{n+1} - w_{n+2}$.

This embedding agrees with the following embedding of $SO(2n+1)$ in $SO(2n+2)$. Notice that the Dynkin diagram for $SO(2n+2)$, has a symmetry interchanging the two right end nodes of the diagram. This corresponds to an outer automorphism of the Lie group, which has a fixed subgroup that stabilizes the non-isotropic vector v_0. The stabilizer of the $2n+1$-dimensional orthogonal complement of v_0 is the group $SO(2n+1)$. Recall the sets of simple roots Δ^B from section (1.3) and Δ^D from section (1.4). In this embedding, the short simple root δ_n of $SO(2n+1)$, is embedded into the direct sum of ϑ_n and ϑ_{n+1}.

On the level of Lie algebras, there is an embedding

$$\mathfrak{h}^B \hookrightarrow \mathfrak{h}^D \tag{3.4}$$

$$\mathrm{diag}(t_1, \ldots, t_n, 0, -t_n, \ldots, -t_1) \mapsto \mathrm{diag}(t_1, \ldots, t_n, 0, 0, -t_n, \ldots, -t_1). \tag{3.5}$$

In fact, $\mathfrak{h}^B_+ \hookrightarrow \mathfrak{h}^D_+$.

This induces a restriction map $\mathfrak{h}^{D*}_+ \rightarrow \mathfrak{h}^{B*}_+$. Let $\omega_i^D \in \mathfrak{h}^{D*}$ and $\omega_i^B \in \mathfrak{h}^{B*}$ denote the i^{th} fundamental weights. Then

$$\omega_i^D \mapsto \omega_i^B \text{ for all } i \leq n \text{ and} \tag{3.6}$$

$$\omega_{n+1}^D \mapsto \omega_n^B. \tag{3.7}$$

From the descriptions of the simple reflections in the Weyl groups W_B and W_D we give the natural map between Weyl groups:

$$\begin{aligned} W_B &\rightarrow W_D \\ s_i &\mapsto \begin{cases} s'_i & 1 \leq i \leq n-1 \\ s'_n s'_{n+1} & i = n. \end{cases} \end{aligned}$$

This embedding of the Lie groups allows a left action of $SO(2n+1)$ on the homogeneous space $OG(r, 2n+2)$.

Lemma 3.1.1. *Under the left action of $SO(2n+1)$, $OG(r, 2n+2)$ has two orbits, namely*

$$O_r = \{S \in OG(r, 2n+2) \mid S \subset V\} \text{ and,} \tag{3.8}$$

$$O_{r-1} = \{S \in OG(r, 2n+2) \mid dim(S \cap V) = r-1\}. \tag{3.9}$$

Proof. Let $T \in SO(2n+1) \subset SO(2n+2)$. Recall that $SO(2n+1)$ preserves V, and T has full rank. So,

$$\begin{aligned} \dim(T \cdot S \cap V) &= \dim(TS \cap TV) \\ &= \dim(T(S \cap V)) \\ &= \dim(S \cap V). \end{aligned}$$

Therefore, if $S \in O_r$, then $TS \subset V$, and if $S \in O_{r-1}$, then $\dim(TS \cap V) = r-1$. Therefore, $SO(2n+1)$ acts on each set, and it is clear that the action on O_r is transitive. We want to show that $SO(2n+1)$ acts transitively on O_{r-1}. We will fix a subspace $C \in O_{r-1}$ and show that for arbitrary $\Lambda \in O_r$ we can produce $A \in SO(2n+1)$ such that $A : C \mapsto \Lambda$.

Fix an r-dimensional isotropic subspace $C \in OG(r, 2n+2)$ spanned by basis vectors $\langle w_1, \ldots, w_{r-1}, w_{n+1} \rangle$. Let $\Lambda \in OG(r, 2n+2)$ be an arbitrary r-dimensional subspace such that $\Lambda \cap V = r-1$. Then Λ contains a 1-dimensional subspace of the form $u = \sum_{i=1}^{2n+2} u_i w_i$ where $u_{n+1} \neq u_{n+2}$. Take u as a basis vector of Λ. We can extend u to an orthogonal basis of Λ with respect to $\langle , \rangle^D$ so that the other $r-1$ basis vectors of Λ are in V (i.e. for a basis vector $b = \sum_{i=1}^{2n+2} b_i w_i$, $b_{n+1} = b_{n+2}$.) Call the basis $S = \{u, s_1, \ldots, s_{r-1}\}$. Define a vector $\tilde{u}$ by

$$\tilde{u} = (u_1, \ldots, u_n, u_{n+1} - 1, u_{n+2} + 1, u_{n+3}, \ldots, u_{2n+2}).$$

It is clear that u and $\tilde{u}$ are independent. Now, $\dim(\langle u, \tilde{u} \rangle \cap V) \in \{0, 1, 2\}$. The dimension cannot be 0 since $\langle u, \tilde{u} \rangle \cup V$ would have dimension $2n+3$. The dimension cannot be 2 since $\dim(u \cap V) = 0$ by construction. Therefore, $\dim(\langle u, \tilde{u} \rangle \cap V) = 1$. So complete $S \cup \langle u, \tilde{u} \rangle$ to an orthogonal basis of V with respect to $\langle , \rangle^D$. Then for a change of basis $A \in SO(2n+2)$ such that $A : C \mapsto \Lambda$, by the construction above, A will preserve V and $A \cdot v_0 = v_0$. So, $A \in SO(2n+1)$ as desired. $\square$

3.2. Identifying Index Sets

In this section we give a correspondence between index sets parameterizing Schubert varieties in $OG(r, 2n+1)$ and $OG(r, 2n+2)$.

Fix the following notation for Schubert varieties: Let $\Psi_I(E_\bullet)$ be a Schubert variety in $OG(r, 2n+1)$ where $I \in \mathfrak{S}^B(r, 2n+1)$ and $F_\bullet$ is a complete isotropic flag on $\mathbb{C}^{2n+1}$ as defined in (1.3.1). Similarly, let $\Upsilon_{J^D}(E^D_\bullet)$ be a Schubert variety in $OG(r, 2n+2)$ where $J \in \mathfrak{S}^D(r, 2n+2)$ and $E_\bullet$ is a complete isotropic flag on $\mathbb{C}^{2n+2}$ as defined in (1.4.3).

Question 3.2.1. Given a Schubert variety $\Upsilon_J(E_\bullet)$, does the action of $SO(2n+1)$ on $OG(r, 2n+2)$ break the variety up over the orbits O_r and O_{r-1}? In other words, can we describe $\Upsilon_J(E_\bullet) \cap O_r$?

First, we identify $OG(r, 2n+1)$ with the orbit O_r:

$$\begin{aligned} O_r &= \{\Lambda | \langle v_1, v_2 \rangle^D = 0 \text{ for all } v_1, v_2 \in \Lambda \text{ and } \Lambda \subset V\} \\ &= \{\Lambda | \langle v_1, v_2 \rangle^B = 0 \text{ for all } v_1, v_2 \in \Lambda\} \\ &= OG(r, 2n+1) \end{aligned}$$

Define a subset of $\mathfrak{S}(r, 2n+2)$ which is the parameterizing set for Schubert varieties in the orbit O_r,

$$\mathfrak{T}(r, 2n+2) := \{J \in \mathfrak{S}(r, 2n+2) \mid j_i \neq n+1, j_i \neq n+2 \text{ for any } i\}.$$

Then, the answer to Question 3.2.1 boils down to examining the relationship between index sets $J \in \mathfrak{T}(r, 2n + 2)$ and $I \in \mathfrak{S}(r, 2n+1)$.

Lemma 3.2.2. *For $r < n$ there is a correspondence of sets:*

$$\mathfrak{s} : \mathfrak{S}(r, 2n+1) \leftrightarrow \mathfrak{T}(r, 2n+2) \tag{3.10}$$

$$I \leftrightarrow J, \tag{3.11}$$

where $\mathfrak{s}$ *is a bijection defined by*

$$j_l = \begin{cases} i_l & \text{for } 1 \le i_l \le n \\ i_l + 1 & \text{for } n+3 \le i_l \le 2n{+}2. \end{cases}$$

Proof. It is clear that for $J = \mathfrak{s}(I)$, $J \cap \overline{J}$ is empty. Let $J \in \mathfrak{T}(r, 2n{+}2)$ such that $\mathfrak{s}^{-1}(J) = I$. We check that $I \in \mathfrak{S}(r, 2n{+}1)$. It is clear that $n+1 \notin I$, since $n+2 \notin J \in \mathfrak{T}(r, 2n{+}2)$ by definition. We check that $I \cap \overline{I} = \emptyset$. Let $j_l \in J$ and $i_l \in I$. First, if $j_l = i_l$ this implies that $j_l \le n$ and $2n+3-j_l \notin J$. We want to know if $2n{+}2-i_l$ is in I. Notice that $2n+2-i_l > n$ so if $i_l \in I$ it came from

$$2n{+}2 - i_l + 1 = 2n + 3 - i_l = 2n + 3 - j_l \notin J.$$

So, $2n{+}2 - i_l$ is not in I. If $i_l = j_l - 1$, so $j_l > n$, then $2n+3-j_l \le n$ is not in J. Again we want to know if $2n{+}2 - i_l$ is in I. Well,

$$2n{+}2 - i_l = 2n{+}2 - (j_l - 1) = 2n + 3 - j_l \notin J.$$

Therefore, $2n{+}2 - i_l \notin I$, and $I \in \mathfrak{S}^B(r, 2n{+}1)$. □

Lemma 3.2.3. *For* $r = n$, *define the correspondence between index sets :*

$$\mathfrak{s} : \mathfrak{S}(n, 2n{+}1) \;\leftrightarrow\; \mathfrak{S}(n{+}1, 2n{+}2)$$

$$I \;\leftrightarrow\; J,$$

where $\mathfrak{s} : I \mapsto J$ *is given by*

$$j_l = \begin{cases} i_l & \text{for } 1 \le j_l \le n \\ n+1 & \text{if } \#i_l \le n \cong 0 mod 2, \\ n+1 & \text{if } \#i_l \le n \cong 1 mod 2, \\ i_l + 1 & \text{for } n+3 \le j_l \le 2n{+}2, \end{cases}$$

PROOF. Let $I \in \mathfrak{S}(n, 2n+1)$ such that $\mathfrak{s}(I) = J$. It is clear that $J \cap \overline{J} = \emptyset$, since either $n+1 \in J$ or $n+2 \in J$, but not both. $\square$

Remark 3.2.4. Notice that in Lemma (3.2.2), the correspondence is well defined in either direction. In Lemma (3.2.3), an important choice must be made. Given $I \in \mathfrak{S}(n, 2n+1)$, to produce $J = \mathfrak{s}(I)$, one of the two indices $n+1$ or $n+2$, is inserted to complete J. Let $j_k \in J \in \mathfrak{S}(n+1, 2n+2)$. If $\#\{j_k \leq n+1\} \equiv 0 \bmod 2$, then J determines a Schubert variety in $OG^+(n+1, 2n+2)$, and if $\#\{j_k \leq n+1\} \equiv 1 \bmod 2$, then J determines a Schubert variety in $OG^-(n+1, 2n+2)$.

3.3. Correspondence of Schubert Varieties

It is clear that complete isotropic flags $F_\bullet$ on V and $E_\bullet$ on W can both be completely described by the first n components of the flag.

Definition 3.3.1. Given complete isotropic flags $F_\bullet$ on V and $E_\bullet$ on W, we call the flags *corresponding* if $F_i = E_i$ for all $i \leq n$.

Recall that given components $E_1, \ldots, E_n$ of $E_\bullet$, $E_n^\perp / E_n$ has two isotropic lines and by definition we adjoin to E_n the line the makes the maximal isotropic component E_{n+1} in the same family as the standard flag, as in section (1.4).

Lemma 3.3.2. *Let $r < n$. Let $E_\bullet$ be a complete isotropic flag on W, and $F_\bullet$ be the complete isotropic flag on V so that $F_\bullet$ and $E_\bullet$ are corresponding. If $J \in \mathfrak{T}^D(r, 2n+2)$, then $\Upsilon_J(E_\bullet) \subset O_r$ and*

$$\Upsilon_J(E_\bullet) \cap O_r = \Psi_I(F_\bullet), \tag{3.12}$$

where $I \in \mathfrak{S}(r, 2n+1)$ such that, $J = \mathfrak{s}(I)$.

PROOF. Let $J = (j_1, \ldots, j_r) \in \mathfrak{T}(r, 2n+2)$ and $E_\bullet$ a complete isotropic flag on W. Denote by $F_\bullet$ the corresponding isotropic flag in V, so that $E_i = F_i$ for all $i \leq n$. Let $T \in \Upsilon_J(E_\bullet)$. Notice for $T \subset O_r$, $T \cap E_i = T \cap F_i$, for all $i \leq n$. Since $w_{n+1} + w_{n+2} \in F_{n+1}$

we see that the following equalities hold:

$$T \cap E_{n+1} = T \cap F_n \tag{3.13}$$

$$T \cap E_{n+2} = T \cap F_{n+1} \text{ and,} \tag{3.14}$$

$$T \cap E_j = T \cap F_{j-1}, \text{ for all } j \geq n+3. \tag{3.15}$$

If $J \in \mathfrak{T}(r, 2n+2)$, then for any $T \in \Upsilon_J(E_\bullet)$ and for any $j_k \leq a < j_{k+1}$ we have $T \cap E_a = T \cap F_a$ which implies $\dim(T \cap O_r \cap E_a) = \dim(T \cap O_r \cap F_a) = k$, for $j_k \leq n \leq j_{k+1}$.

Similarly we can see for $j_k \geq n+3$ and $j_k \leq b < j_{k+1}$,

$$\dim(T \cap O_r \cap E_b) = \dim(T \cap O_r \cap F_{b-1}) = k-1.$$

So, $T \in \Psi_I(F_\bullet)$ where $\mathfrak{s}(I) = J$. Similarly, by considering the list (3.13) above, it is clear that If $T \in \Psi_I(F_\bullet)$, T satisfies the Schubert condition for $\Upsilon_J(E_\bullet)$. Therefore, $\Upsilon_J(E_\bullet) \cap O_r = \Psi_I(F_\bullet)$, as desired. □

Proposition 3.3.3. *The map*

$$\begin{array}{c} OG^{\pm}(n+1, 2n+2) \ni \Lambda \\ \big\downarrow \varphi^{\pm} \\ OG(n, 2n+1) \ni (\Lambda \cap V) \end{array}$$

is a homeomorphism,

$$OG(n, 2n+1) \cong OG^{\pm}(n+1, 2n+2). \tag{3.16}$$

Proof. As in section (3.1), $V \subset W$ is a $2n+1-$dimensional subspace of W such that $\langle , \rangle^D|_V$ is nondegenerate on V. Fix the component through the identity to be $OG^+(n+1, 2n+2)$. Observe that $\Lambda \nsubseteq V$, since V does not have an isotropic subspace of dimension $n+1$. Every maximal isotropic space $\Lambda \in OG^+(n+1, 2n+2)$ contains a unique n-dimensional isotropic subspace. Let $M \in OG(n, 2n+1)$. As in the Remark (3.2.4) above, to extend M to an $n+1$-dimensional isotropic space, you make a choice between two isotropic lines in $M^\perp \setminus M$, which determines which connected component of $OG^\pm(n+1, 2n+2)$ you extend to. This is a continuous choice. $\square$

CHAPTER 4

Comparison of Eigencones

To begin we recall the most relevant definitions from previous chapters. Let $V = \mathbb{C}^{2n+1}$ and $W = \mathbb{C}^{2n+2}$.

For $SO(2n{+}1)$, we define the homogeneous space $OG(r, 2n{+}1)$ in (1.6), and the set of parameters $\mathfrak{S}(r, 2n{+}1)$ defined in (1.7), giving Schubert varieties $\Psi_I(F_\bullet) \in OG(r, 2n{+}1)$, as in (1.3.2). For $SO(2n{+}2)$, we define the homogeneous space $OG(r, 2n{+}2)$ in (1.10), and the sets of parameters $\mathfrak{S}(r, 2n{+}1)$ defined in (1.11), giving Schubert varieties $\Upsilon_J(E_\bullet) \in OG(r, 2n{+}2)$, and $\mathfrak{S}^{\pm}(n{+}1, 2n{+}2)$ defined in (1.12), giving Schubert varieties $\Upsilon_J(E_\bullet) \in OG(n+1, 2n+2)$, as in (1.4.5). We give a bijection of parameters $\mathfrak{s} : \mathfrak{S}(r, 2n+1) \leftrightarrow \mathfrak{S}(r, 2n{+}2)$, defined in Lemma (3.2.2).

We denote the fundamental Weyl chamber for $SO(2n{+}1)$ by $\mathfrak{h}_+^B$ and the fundamental Weyl chamber for $SO(2n+2)$ by $\mathfrak{h}_+^D$. We give an embedding $\mathfrak{h}_+^B \hookrightarrow \mathfrak{h}_+^D$ in (3.4). We denote the fundamental weights for $SO(2n{+}1)$ by ω^B and the fundamental weights for $SO(2n{+}2)$ by ω^D.

Finally, we recall the definition of the eigencone. Let G be a connected semisimple group. Choose a maximal compact subgroup K of G with Lie algebra $\mathfrak{k}$. Recall from Chapter zero, there exists a homeomorphism

$$C : \mathfrak{k}/K \to \mathfrak{h}_+$$

where K acts on $\mathfrak{k}$ by the adjoint action, and $\mathfrak{h}_+$ is the positive Weyl chamber of G in $\mathfrak{h}$.

Definition 4.0.1. For a positive integer s, the eigencone of G is defined as the cone:

$$\Gamma(s, G) := \{(h_1, \ldots, h_s) \in (\mathfrak{h}_+)^s \,|\, \exists (k_1, \ldots, k_s) \in \mathfrak{k}^s, \text{ s.t.} \sum_{j=1}^{s} k_j = 0,\ C^{-1}(k_j) = h_j\ \forall j \in [s]\}.$$

For convenience we restate the main theorem (0.3.3):

Theorem 4.0.2. *For* $h_i \in \mathfrak{h}_+^B$,

$$(h_1,\dots,h_s)\in\Gamma(s,SO(2n+1)) \iff (h_1,\dots,h_s)\in\Gamma(s,SO(2n+2)).$$

4.1. Proof of Theorem (0.3.3)

Proof. It is clear that $\Gamma(s,SO(2n+1)) \subset \Gamma(s,SO(2n+2))$. We need to show the converse. That is, if $(h_1^B,\dots,h_s^B) \in (\mathfrak{h}_+^B)^s$ such that $(h_1^B,\dots,h_s^B) \in \Gamma(s,SO(2n+2))$, then $(h_1^B,\dots,h_s^B) \in \Gamma(s,SO(2n+1))$. So, the task is to take an inequality in the system describing $\Gamma(SO(2n+1))$ and show that it is implied by an inequality in the system describing $\Gamma(SO(2n+2))$.

From Theorem (0.2.4), we know that for the solution of the eigencone problem we can restrict to a smaller set of inequalities coming from the Levi-movable s-tuples with intersection number one. In Theorem (2.2.6) we showed that Levi-movable s-tuples, parameterized by index sets $I^1,\dots,I^s \in \mathfrak{S}(r,2n+1)$, with intersection number one, satisfying the following numerical condition, given in (2.7):

$$r = \sum_{k=1}^{s} |I^k \le n|. \tag{4.1}$$

We will separate the proof in two steps, first considering $1 \le r < n$, and then $r = n$.

For $1 \le r < n$, let $I^1,\dots,I^s \in \mathfrak{S}(r,2n+1)$ such that $\sum_{k=1}^{s}|I^k \le n| = r$ and

$$[\overline{\Psi}_{I^1}]\cdot\ldots\cdot[\overline{\Psi}_{I^s}] = [\overline{\Psi}_e] \in H_{\odot_0}^{2\dim(OG(r,2n+1))}(OG(r,2n+1)),$$

where $[\overline{\Psi}_e]$ is the class of the point. Then, by Theorem (5.2.1),

$$[\overline{\Upsilon}_{J^1}]\cdot\ldots\cdot[\overline{\Upsilon}_{J^s}] = d[\overline{\Upsilon}_e] \in H_{\odot_0}^{2\dim(OG(r,2n+2))}(OG(r,2n+2)), \tag{4.2}$$

for some nonzero d, and $J^k = \mathfrak{s}(I^k)$, for $\mathfrak{s}$ defined in (3.2.2).

Let $w_{J^k} \in W_r^D$ denote the minimal length coset representative corresponding to $J^k \in \mathfrak{S}(r, 2n+2)$ as in (1.16), and similarly $w_{I^k} \in W_r^B$ denote the minimal length coset representative corresponding to $I^k \in \mathfrak{S}(r, 2n+1)$.

Applying Theorem (0.2.2) to the intersection in (4.2) for $SO(2n{+}2)$ and for $1 \leq r \leq n$, we have

$$\omega_r^D(\sum_{k=1}^{s} w_{J^k}^{-1} h^{D_k}) \leq 0$$

where ω_r^D is the r^{th} fundamental weight of $SO(2n{+}2)$.

from (3.4) we recall the embedding:

$$\begin{aligned} \mathfrak{h}^B &\xrightarrow{\phi} \mathfrak{h}^D \\ h^B = (h_1^B, \ldots, h_n^B, 0, h_{n+2}^B, \ldots, h_{2n+1}^B) &\mapsto h^D = (h_1^D, \ldots, h_n^D, 0, 0, h_{n+3}^D, \ldots, h_{2n+2}^D). \end{aligned}$$

Then for $I^k = \{i_1^k, \ldots, i_r^k\}$, and $\mathfrak{s}(I^k) = J^k = \{j_1^k, \ldots, j_r^k\}$, we have $h_{j_l^k}^D = h_{i_l^k}^B$.

Then,

$$\omega_r^D(\sum_{k=1}^{s} w_{J^k}^{-1} h^{D_k}) \leq 0$$

implies

$$\sum_{k=1}^{s} \Big(\sum_{l=1}^{r} h_{j_l^k}^{D_k} \Big) = \sum_{k=1}^{s} \Big(\sum_{l=1}^{r} h_{i_l^k}^{B_k} \Big) \leq 0.$$

So,

$$\omega_r^B(\sum_{k=1}^{s} w_{I^k}^{-1} h^{B_k}) \leq 0.$$

Now, let $r = n$. We have homeomorphisms $OG(n, 2n{+}1) \cong OG^{\pm}(n{+}1, 2n{+}2)$. Every maximal isotropic subspace $M \in OG(n+1, 2n+2)$ contains a unique n-dimensional isotropic subspace with respect to $\langle , \rangle^B$. So, $M \cap V \in OG(n, 2n+1)$. Then $M \cap V$ can be extended continuously to $M^+ \in OG^+(n+1, 2n+2)$ or $M^- \in OG^-(n+1, 2n+2)$ determined by which isotropic line in $(M^\perp/M)$ one extends M by. Note that

$$\dim(OG(n, 2n+1)) = \dim(OG(n+1, 2n+2)) = \frac{n(n+1)}{2},$$

and for $J = \mathfrak{s}(I)$, $\dim(\Psi_I(F_\bullet)) = \dim(\Upsilon_I(E_\bullet))$. The intersection numbers will be preserved. Since the two connected components of $OG^{\pm}(n{+}1, 2n{+}2)$ are isomorphic, we will restrict our attention to the connected component $OG^{+}(n{+}1, 2n{+}2)$ containing the isotropic space $\langle w_1, \ldots, w_{n+1}\rangle$.

Following an argument similar to the $r < n$ case, we let $I^1, \ldots, I^s \in \mathfrak{S}(n, 2n{+}1)$ such that

$$[\overline{\Psi}_{I^1}] \cdot \ldots \cdot [\overline{\Psi}_{I^s}] = [\overline{\Psi}_e] \in H^{n(n+1)}_{\odot_0}(OG(n, 2n{+}1)).$$

Then, for $J^k = \mathfrak{s}(I^k)$

$$[\overline{\Upsilon}_{J^1}] \cdot \ldots \cdot [\overline{\Upsilon}_{J^s}] = [\overline{\Upsilon}_e] \in H^{n(n+1)}_{\odot_0}(OG(n+1, 2n{+}2)).$$

Let $w_{J^k} \in W^D_{n+1}$ denote the minimal length coset representative corresponding to $J^k \in \mathfrak{S}(n+1, 2n+2)$ as in (1.16), and similarly $w_{I^k} \in W^B_n$ denote the minimal length coset representative corresponding to $I^k \in \mathfrak{S}(n, 2n+1)$.

Applying Theorem (0.2.2) to the intersection in (4.2) for $SO(2n{+}2)$ and for $1 \leq r \leq n$, we have

$$\omega^D_{n+1}(\sum_{k=1}^{s} w^{-1}_{J^k} h^{D_k}) \leq 0$$

where ω^D_{n+1} is the $n+1^{st}$ fundamental weight of $SO(2n{+}2)$. Since $h^{D_k}_{n+1} = h^{D_k}_{n+2} = 0$ for all k, we have

$$\sum_{k=1}^{s} \Big(\sum_{l=1}^{r} h^{D_k}_{j^k_l} \Big) = \sum_{k=1}^{s} \Big(\sum_{l=1}^{r} h^{B_k}_{i^k_l} \Big) \leq 0,$$

and So,

$$\omega^B_n(\sum_{k=1}^{s} w^{-1}_{I^k} h^{B_k}) \leq 0.$$

Therefore applying Theorem (0.2.4) for $SO(2n{+}1)$, to the cases $r < n$ and $r = n$, we have $(h^{B_1}, \ldots, h^{B_s}) \in \Gamma(s, SO(2n{+}1))$. □

CHAPTER 5

Intersection Result

The aim of this chapter is to prove Theorem (0.3.2). This is a key result used in the comparison of eigencones for $SO(2n+1)$ and $SO(2n+2)$. From Theorem (0.2.4), we know that for the solution of the eigencone problem we can restrict to Levi-movable s-tuples with intersection number one to produce a smaller system of inequalities. From Theorem (2.2.6), we have a numerical condition on parameters for Levi-movable intersections with intersection number one, given in (2.7). We will restrict our interest to such s-tuples satisfying condition (2.7) here. Finally we give an important corollary of (0.3.2) which we state as Theorem (5.2.1).

To begin we recall the most relevant definitions from previous chapters. Let $V = \mathbb{C}^{2n+1}$ and $W = \mathbb{C}^{2n+2}$. For $SO(2n+1)$, we define the homogeneous space $OG(r, 2n+1)$ in (1.6), and the set of parameters $\mathfrak{S}(r, 2n+1)$ defined in (1.7), giving Schubert varieties $\Psi_I(F_\bullet) \in OG(r, 2n+1)$, as in (1.3.2). For $SO(2n+2)$, we define the homogeneous space $OG(r, 2n+2)$ in (1.10), and the sets of parameters $\mathfrak{S}(r, 2n+1)$ defined in (1.11), giving Schubert varieties $\Upsilon_J(E_\bullet) \in OG(r, 2n+2)$, and $\mathfrak{S}^{\pm}(n+1, 2n+2)$ defined in (1.12), giving Schubert varieties $\Upsilon_J(E_\bullet) \in OG(n+1, 2n+2)$, as in (1.4.5). Recall the definition of corresponding flags from Definition (3.3.1). Finally, we recall the bijection $\mathfrak{s} : \mathfrak{S}(r, 2n+1) \leftrightarrow \mathfrak{S}(r, 2n+2)$, defined in Lemma (3.2.2).

5.1. Intersection Result

Theorem 5.1.1. *Let $F^1_\bullet, F^2_\bullet, \ldots, F^s_\bullet$ be complete isotropic flags in general positions on $V = \mathbb{C}^{2n+1}$ and $E^1_\bullet, E^2_\bullet, \ldots, E^s_\bullet$ the corresponding flags on $W = \mathbb{C}^{2n+2}$. Let $I^1, I^2, \ldots, I^s$*

be index sets in $\mathfrak{S}(r, 2n{+}1)$*, such that*

$$\sum_{k=1}^{s} |I^k \leq n| = r, \tag{5.1}$$

and $J^1, J^2, \ldots, J^s$ *be the corresponding index sets in* $\mathfrak{S}(r, 2n{+}2)$ *such that* $\mathfrak{s}(J^k) = I^k$*, and*

$$\sum_{k=1}^{s} codim \Psi_{I^k}(F_\bullet^k) = dim(OG(r, 2n{+}1)).$$

Then the intersection of varieties

$$\cap_{k=1}^{s} \Upsilon_{J^k}(E_\bullet^k) \tag{5.2}$$

in $OG(r, 2n{+}2)$ *is proper. Moreover, (5.2) intersects in finitely many points.*

Proof. Choose isotropic flags $\{F_\bullet^k\}_{1 \leq k \leq s}$ on $\mathbb{C}^{2n+1}$ such that the intersection $\cap_{k=1}^{s} \Psi_{I^k}(F_\bullet^k)$ is dense in $\cap_{k=1}^{s} \overline{\Psi}_{I^k}(F_\bullet^k)$ for all $I^k \in \mathfrak{S}(r, 2n{+}1)$ and all $1 \leq r \leq n$. Let $F_\bullet^k$ on $\mathbb{C}^{2n+1}$ and $E_\bullet^k$ on $\mathbb{C}^{2n+2}$ be corresponding flags for all $1 \leq k \leq s$.

For any irreducible component $C \subset \cap_{k=1}^{s} \Upsilon_{J^k}(E_\bullet^k)$, the task is to show that

$$\dim(C) \leq \dim(OG(r, 2n{+}2)) - \sum_{k=1}^{s} \operatorname{codim}(\Upsilon_{J^k}(E_\bullet^k)), \tag{5.3}$$

for any $J^k \in \mathfrak{S}(r, 2n{+}2)$.

We have two cases to consider. Recall that $OG(r, 2n{+}2)$ has an $SO(2n{+}1)$ action with two orbits O_r and O_{r-1}, defined in (3.8), where O_{r-1} is an open orbit and O_r is closed.

The first case to consider is $C \cap O_{r-1} \neq \emptyset$. Then $(C \cap O_{r-1}) \subset \cap_{k=1}^{s}(O_{r-1} \cap \Upsilon_{J^k}(E_\bullet^k))$, and by Kleiman's Transversality Theorem (5.3) is satisfied.

The second case to consider is $C \cap O_{r-1} = \emptyset$, so $C \subset O_r$. By definition,

$$O_r = \{S \in OG(r, 2n{+}2) \mid S \subset V\} = OG(r, 2n{+}1).$$

By Lemma (3.3.2), we have $\Upsilon_{J^k}(E_\bullet^k) \cap O_r \subset OG(r, 2n{+}1)$, and furthermore, that

$$\Upsilon_{J^k}(E_\bullet^k) \cap O_r = \Psi_{I^k}(F_\bullet^k).$$

Therefore, $\dim(C) \leq \dim(O_r) - \sum_{k=1}^s \operatorname{codim}(\Psi_{I^k}(F_\bullet^k))$, and we need to show that the following is greater than or equal to zero:

$$[\dim(OG(r, 2n{+}2)) - \textstyle\sum_{k=1}^s (\operatorname{codim}(\Upsilon_{J^k}(E_\bullet^k)))] - \tag{5.4}$$

$$[\dim(OG(r, 2n{+}1)) - \textstyle\sum_{k=1}^s (\operatorname{codim}(\Psi_{I^k}(F_\bullet^k)))]. \tag{5.5}$$

In fact, we will show an equality.

Observe that

$$\begin{aligned} \dim(OG(r, 2n{+}2)) - \dim(OG(r, 2n{+}1)) &= \\ \tfrac{r}{2}(4n - 3r + 3) - \tfrac{r}{2}(4n - 3r + 1) &= r. \end{aligned}$$

So, we rewrite (5.4) as

$$r + \sum_{k=1}^{s} (\operatorname{codim}(\Psi_{I^k}(F_\bullet^k))) - \sum_{k=1}^{s} (\operatorname{codim}(\Upsilon_{J^k}(E_\bullet^k))). \tag{5.6}$$

We expand (5.6) using the dimension formula for $\Psi_{I^k}(F_\bullet^k)$, given in Lemma (6.1.1), and the dimension formula for $\Upsilon_{J^k}(E_\bullet^k)$, given in Lemma (6.2.1):

$$r - \sum_{k=1}^{s} \left[\tfrac{r}{2}(4n{-}3r{+}3) - |J^k > \tilde{J}^k| - \tfrac{1}{2}|J^k > \bar{J}^k| + \frac{1}{2}|J^k > n+1|\right] + \tag{5.7}$$

$$\sum_{k=1}^{s} \left[\frac{r}{2}(4n{-}3r{+}1) - |I^k > \tilde{I}^k| - \frac{1}{2}|I^k > \bar{I}^k| + \frac{1}{2}|I^k > n|\right] \tag{5.8}$$

(5.9)

By inspection we reduce (5.7):

$$r - sr + \sum_{k=1}^{s} |I^k > n| = r - \sum_{k=1}^{s} |I^k \leq n|.$$

By assumption, $r = \sum_{k=1}^{s} |I^k \leq n|$, and we confirm that (5.4) is equal to zero. Thus, (5.3) is satisfied and (5.4) equalling zero, implies $\cap_{k=1}^{s} \Upsilon_{J^k}(E_\bullet^k)$ intersects in finitely many points. □

5.2. Consequence in Cohomology

Theorem 5.2.1. *Let $1 \leq r \leq n$ and $I^1, I^2, \dots, I^s \in \mathfrak{S}(r, 2n+1)$ be such that*

$$\prod_{k=1}^{s} [\overline{\Psi}_{I^k}] = [\overline{\Psi}_e] \in H_{\odot_0}^{2dim(OG(r,2n+1))}(OG(r, 2n+1)).$$

Then, for indices $J^k = \mathfrak{s}(I^k)$,

$$\prod_{k=1}^{s} [\overline{\Upsilon}_{J^k}] = d[\overline{\Upsilon}_e] \in H_{\odot_0}^{2dim(OG(r,2n+2))}(OG(r, 2n+2)),$$

for some $d \neq 0$.

Before giving a proof of Theorem (5.2.1), we need the following preliminary work. Let $M \in \Psi_I(E_\bullet) \subset OG(r, 2n+1) \subset OG(r, 2n+2)$. Determine $M^{\perp^V}$ with respect to $\langle , \rangle^B$, and $M^{\perp^W}$ with respect to $\langle , \rangle^D$. To calculate the tangent spaces to $OG(r, 2n+1)$ and $OG(r, 2n+2)$, recall the embedding of V in W described in Chapter (3). Viewing $OG(r, 2n+1) = O_r \subset OG(r, 2n+2)$,

$$\mathrm{T}(OG(r, 2n+1))_M \subset \mathrm{T}(OG(r, 2n+2))_M \subset \mathrm{T}(Gr(r, 2n+2))_M = \mathrm{Hom}(M, W/M).$$

$M^{\perp^W}$ is a $2n+2-r$ dimensional subspace of W that contains M and there is a canonical isomorphism $W/M^{\perp^W} \cong M^*$ induced from the symmetric form. So, we have an exact sequence

$$0 \to \mathrm{Hom}(M, M^{\perp^W}/M) \to \mathrm{Hom}(M, W/M) \xrightarrow{\phi} \mathrm{Hom}(M, W/M^{\perp^W}) = \mathrm{Hom}(M, M^*) \to 0,$$

via the inclusions $M \subset M^{\perp^W} \subset W$. It is clear that $M^{\perp^W}/M$ is a $2n+2-2r$ dimensional space that possesses a nondegenerate symmetric form. Let P_M denote the stabilizer of M in $SO(2n+2)$ and $\wedge^2 M^*$ the space of skew-symmetric bilinear forms on M.

Lemma 5.2.2. *$T(OG(r,2n{+}2))_M = \phi^{-1}(\wedge^2 M^*)$ and there is an exact sequence*

$$0 \to Hom(M, M^{\perp^W}/M) \xrightarrow{\zeta} T(OG(r,2n{+}2))_M \xrightarrow{\phi} \wedge^2 M^* \to 0.$$

PROOF. Let $\varphi \in \mathrm{Hom}(M, W/M)$. Then, $\varphi(M)$ needs to be isotropic with respect to $\langle , \rangle^D$. So, for all $m, m' \in M$ we have

$$\langle m + \epsilon\varphi(m), m' + \epsilon\varphi(m')\rangle^D = 0.$$

Hence,

$$\langle m, \varphi(m')\rangle^D + \langle \varphi(m), m'\rangle^D = 0, \text{ or}$$
$$\langle m, \varphi(m')\rangle^D = -\langle m', \varphi(m)\rangle^D.$$

Therefore we have a skew-symmetric bilinear form $\phi(\varphi)(m, m') = \langle m, \varphi(m)'\rangle^D$. Hence, $\mathrm{T}(OG(r,2n{+}2)) \subseteq \phi^{-1}(\wedge^2 M^*)$. However,

$$\dim(\mathrm{T}(OG(r,2n{+}2))) = \dim(OG(r,2n{+}2)) = \frac{r}{2}(4n - 3r + 3) = \dim(\phi^{-1}(\wedge^2 M^*)),$$

implies, $T(OG(r,2n{+}2)) = \phi^{-1}(\wedge^2 M^*)$. □

Similarly, we have an exact sequence

$$0 \to \mathrm{Hom}(M, M^{\perp^V}/M) \to \mathrm{Hom}(M, V/M) \xrightarrow{\phi} \mathrm{Hom}(M, V/M^{\perp^V}) = \mathrm{Hom}(M, M^*) \to 0,$$

via the inclusions $M \subset M^{\perp^V} \subset V$. It is clear that $M^{\perp^V}/M$ is a $2n{+}1 - 2r$ dimensional space that possesses a nondegenerate symmetric form. Again, $\wedge^2 M^*$ the space of skew-symmetric bilinear forms on M. A nearly identical proof as given of Lemma 5.2.2 gives the following:

Lemma 5.2.3. *$T(OG(r,2n{+}1))_M = \phi^{-1}(\wedge^2 M^*)$ and there is an exact sequence*

$$0 \to Hom(M, M^{\perp^V}/M) \xrightarrow{\zeta} T(OG(r,2n{+}1))_M \xrightarrow{\phi} \wedge^2 M^* \to 0.$$

Now, we give the proof of Theorem (5.2.1).

PROOF. The assumption $\prod_{k=1}^{s}[\overline{\Psi}_{I^k}] = [\overline{\Psi}_e] \in H_{\odot_0}^{\dim(OG(r,2n+1))}(OG(r,2n+1))$ implies that for generic isotropic flags $F_\bullet^k$ on $\mathbb{C}^{2n+1}$, the corresponding Schubert cells intersect in a point:

$$\cap_{k=1}^{s}\Psi_{I^k}(F_\bullet^k) = \{e\}.$$

We assume that the varieties intersect at the identity without loss of generality. Otherwise, by Kleiman's theorem we can find $(g_1,\ldots,g_s)$ such that the translation of the intersection is transverse at $\{e\}$. This implies that for indices $J^k = \mathfrak{s}(I^k)$ and corresponding flags $E_\bullet$ on $\mathbb{C}^{2n+2}$,

$$\cap_{k=1}^{s}\Upsilon_{J^k}(E_\bullet^k) \neq \emptyset. \tag{5.10}$$

Moreover, from Theorem (5.1.1) we know that the intersection $\cap_{k=1}^{s}\Upsilon_{J^k}(E_\bullet^k)$ is finitely many points. Therefore, in the classical cup product,

$$\prod_{k=1}^{s}[\overline{\Upsilon}_{J^k}] = d[\overline{\Upsilon}_e] \text{ for some } d \neq 0. \tag{5.11}$$

It remains to show that the product in (5.11) can be replaced with the BK-product $\odot_0$, given in Definition (0.2.3). To simplify notation, denote $H_V = \mathrm{Hom}(M, M^{\perp^V}/M)$, and $H_W = \mathrm{Hom}(M, M^{\perp^W}/M)$, and $\mathrm{T}_M^I = \mathrm{T}(\Psi_I(F_\bullet))_M$ and $\mathrm{T}_M^J\mathrm{T}(\Upsilon_J(E_\bullet))_M$. Then we have the following inclusions of exact sequences:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
H_W \cap \mathrm{T}_M^J & \longrightarrow & H_W & \longleftarrow & H_V & \longleftarrow & H_V \cap \mathrm{T}_M^I \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{T}_M^J & \longrightarrow & \mathrm{T}(\mathrm{OG}(r,2n{+}2))_M & \longleftarrow & \mathrm{T}(OG(r,2n{+}1))_M & \longleftarrow & \mathrm{T}_M^I \\
\big\downarrow & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow \\
V_2^D & \longrightarrow & \wedge^2\mathrm{M}^* & \longleftarrow & \wedge^2 M^* & \longleftarrow & V_2^B \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}
$$

Where,

(5.12) $V_2^B \;:=\; \phi(\mathrm{T}_M^I) = \{\gamma \in \wedge^2 M^* \mid \gamma(F_b \cap M, F_{2n+1-b} \cap M) = 0,\ \forall b \in [2n{+}1]\}$

(5.13) $V_2^D \;:=\; \phi(\mathrm{T}_M^J) = \{\gamma \in \wedge^2 M^* \mid \gamma(E_b \cap M, E_{2n+2-b} \cap M) = 0,\ \forall b \in [2n{+}2]\}.$

Belkale and Kumar compute the dimension of V_2 in [BK1],

$$\dim(V_2^B) = \frac{1}{2}(|I > \overline{I}| - |I > n|). \tag{5.14}$$

A nearly identical calculation shows that V_2^D is of dimension

$$\dim(V_2^D) = \frac{1}{2}(|J > \overline{J}| - |J > n+1|).$$

From the identification of index sets given in (3.10), it is clear that

$$\dim(V_2^B) = \dim(V_2^D) \tag{5.15}$$

From the exact sequence given in Lemma (5.2.2), we have $\mathrm{T}(OG(r,2n+2))_M = H_W \oplus \wedge^2 M^*$. For J^k, define the following:

$$\begin{aligned} A(J^k) &= \dim(\mathrm{T}(\Upsilon_{J^k}(E^k_\bullet))_M \cap H_W), \\ B(J^k) &= \dim(\mathrm{T}(\Upsilon_{J^k}(E^k_\bullet))_M), \\ C(J^k) &= \dim(V_2^D). \end{aligned}$$

To simplify notation, let

$$\begin{aligned} A &= \mathrm{Hom}(M, M^\perp/M) \\ B &= T(OG(r,2n+2))_M \\ C &= \wedge^2 M^*. \end{aligned}$$

To show that $\cap_{k=1}^s \Upsilon_{J^k}(E^k_\bullet)$ is Levi-movable, we must show numerically that $\cap_{k=s}^s(\mathrm{T}(\Upsilon_{J^k}(E^k_\bullet))_M \cap H_W)$ meets transversally in and H_W and that $\phi(\mathrm{T}(\Upsilon_J(E_\bullet))_M)$ meets transversally in $\wedge^2\mathrm{M}^*$. That is, we must show the following equalities:

$$\dim(A) = \sum_{k=1}^{s} \Big(\dim(A) - A(J^k)\Big) \tag{5.16}$$

$$\dim(B) = \sum_{k=1}^{s} \Big(\dim(B) - B(J^k)\Big) \tag{5.17}$$

$$\dim(C) = \sum_{k=1}^{s} \Big(\dim(C) - C(J^k)\Big). \tag{5.18}$$

From above, $\dim(OG(r,2n+2)) = \sum_{k=1}^s(\Upsilon_{J^k}(E^k_\bullet))$, and (5.17) is satisfied. From the inclusion of sequences, given above (5.12), and the comparison (5.15), (5.18) is satisfied. From the dimension formula given in (6.2.1), and the inclusions given above (5.12), we have $C(J^k) = |J^k > \tilde{J}^k|$, and the difference of (5.17) and (5.16), implies that (5.18) is satisfied. Therefore, the intersection $\cap_{k=1}^s \Upsilon_{J^k}(E^k_\bullet)$ is Levi-movable.

□

CHAPTER 6

Computing Dimensions of Schubert Varieties in Orthogonal Grassmannians

The aim in this chapter is to give a known dimension formula for a Schubert variety in a homogeneous space for $SO(2n+1)$ in Lemma (6.1.1), and to confirm a dimension formula for Schubert varieties in a homogeneous space for $SO(2n+2)$ that is comparable to Lemma (6.1.1). This dimension formula for $SO(2n+2)$ is give in Lemma (6.2.1).

Given subsets I and J of $[m]$, we denote by $|I > J|$ the number of pairs (i, j) with $i \in I$, $j \in J$, and $i > j$. We set $|I > \emptyset| = 0$ and if $K = \{k\}$, then we abbreviate $|I > K|$ to $|I > k|$. For $1 \leq r \leq m$, and $I = \{i_1, \ldots, i_r\}$, we let $\overline{I} = \{m+1-i_r, \ldots, m+1-i_1\}$ and $\tilde{I} = [m] \setminus (I \sqcup \overline{I})$.

The notation in this chapter will be consistent with the notation in Chapter (1).

6.1. Dimension Formula in an Odd Orthogonal Grassmannian

Lemma 6.1.1. *For $I \in \mathfrak{S}(r, 2n+1)$ and an isotropic flag $F_\bullet$ on $\mathbb{C}^{2n+1}$,*

$$dim(\Psi_I(F_\bullet)) = \frac{1}{2}(|I > \bar{I}| - |I > n|) + |I > \tilde{I}|. \tag{6.1}$$

Proof. Recall the embedding from chapter 2:

$$\begin{aligned} W^B &\rightarrow S_{2n+1} \\ w_I &\mapsto \hat{w}_I = (i_1, \ldots, i_r, j_1, \ldots, j_{2n+1-2r}, 2n+2-i_r, \ldots, 2n+2-i_1) \end{aligned}$$

where $j_k \in \tilde{I}$. Let $l^{A_{2n}}(w_A)$ denote the length of a word w_A in a Weyl group of type A_{2n} and let $l^{B_n}(w_I)$ denote the length of a word w_I in a Weyl group of type B_n. Recall that the length of a word is given by $l^{A_{2n}}(w) = |\{i < j | w(i) > w(j)\}|$. So,

$$\begin{aligned} l^{B_n}(w_I) &= \frac{1}{2}(|I > \tilde{I}| + |I > \bar{I}| + |\tilde{I} > \bar{I}| - |I > n|) \\ &= \frac{1}{2}(|I > \tilde{I}| + |I > \bar{I}| + |I > \tilde{I}| - |I > n|) \\ &= \frac{1}{2}(|I > \bar{I}| - |I > n|) + |I > \tilde{I}|. \end{aligned}$$

□

6.2. Dimension Formula in an Even Orthogonal Grassmannian

Let $I \in \mathfrak{S}(r, 2n{+}2)$. Then for a complete isotropic flag $F_\bullet$ on $\mathbb{C}^{2n+2}$, $\Upsilon_I(F_\bullet)$ is a Schubert cell in $OG(r, 2n{+}2)$. The aim in this section is to confirm the following formula:

Lemma 6.2.1. *Given an index set $I \in \mathfrak{S}(r, 2n{+}2)$ and a complete isotropic flag $F_\bullet$ on $\mathbb{C}^{2n+2}$,*

$$dim(\Upsilon_I(F_\bullet)) = |I > \tilde{I}| + \frac{1}{2}(|I > \bar{I}| - |I > n+1|).$$

Before we prove this lemma, we need the following preliminary work.

Let

$$G(I) = |I > \tilde{I}| + \frac{1}{2}(|I > \bar{I}| - |I > n+1|).$$

The approach in proving Lemma (6.2.1) will be to first define a set of rules taking $\mathcal{B} : \mathfrak{S}(r, 2n{+}2) \to \mathfrak{S}(r, 2n{+}2)$ and show that for $\mathcal{B} : I \to J$, $G(J) = G(I) + 1$. We will produce a chain of varieties in this way, and then use an induction argument to show that $G(I) = \dim(\Upsilon_I(F_\bullet))$.

In the set of rules defined below, an index $I \in \mathfrak{S}(r, 2n{+}2)$ is mapped to an index $J \in \mathfrak{S}(r, 2n{+}2)$ by bumping at most two indices i_s, i_t to corresponding j_s, j_t and leaving all other indices fixed. For a given I, this process can be done systematically for each $i_t \in I$ and therefore we produce multiple such J.

Definition 6.2.2. We define *bumping rules* for $OG(r, 2n+2)$,

$$\mathcal{B} : \mathfrak{S}(r, 2n+2) \to \mathfrak{S}(r, 2n+2)$$
$$I = \{i_1, \ldots, i_r\} \mapsto J = \{j_1, \ldots, j_r\}.$$

The rules are given by phase (1) and phase (2). Given $I = \{i_1, \ldots, i_r\}$, we produce $\mathcal{B}(I) = J = \{j_1, \ldots, j_r\}$ such that:

(1) (a) If $i_k \neq n+1$, $i_k + 1 \notin \overline{I}$, and $i_k + 1 \lneq i_{k+1}$, then set $j_k = i_k + 1$. Complete J by setting $j_l = i_l$ for all $l \neq k$. If $i_k \in \overline{I}$, move to phase (2). If $i_k + 1 = i_{k+1}$, then terminate the process.

(b) If $i_k = n$ or $i_k = n + 1$, $i_k + 2 \notin \overline{I}$, and $i_k + 2 \lneq i_{j+1}$, then set $j_k = i_k + 2$. Complete J by setting $j_l = i_l$ for all $l \neq k$. If $i_k \in \overline{I}$, move to phase (2). If $i_k + 1 = i_{k+1}$, then terminate the process.

(2) We move to this phase if $j_k = i_k + 1 \in \overline{I}$ in phase (1a) or $j_k = i_k + 2 \in \overline{I}$ in phase (1b). There exists $i_l \in I$, such that $i_l + j_k = 2n+3$. Repeat phase (1a), and phase (1b) if applicable, for both j_k ($j_k = i_k + 1$ or $j_k = i_k + 2$) and i_l. In most cases, this will result in multiple valid index sets J.

Remarks 6.2.3. The following observations about the bumping rules may be useful in computing examples:

(1) Notice that when $n + 1 \in I$, to bump from $i_k = n + 1$, it skips over $n + 2$ to $n + 3$ when it bumps because n is allowed to bump to $n + 1$ or $n + 2$ as seen in rule (1b). We will see that this rule occurs because for parameters I and J such that $n + 1 \in I$ and $n + 2 \in J$, and all other indices i_k and j_k match, I and J give Schubert varieties of the same dimension. This is easy to see if you consider the k-strict partitions corresponding to I and J. These partitions and a discussion of this identification is given in Chapter 7 in (7.4).

(2) Moving to phase (2) sends you back to phase (1). It may be necessary to move to phase (2) again, but this will happen at most $r-1$ times, and with $r \leq n+1$, the process will terminate.

Lemma 6.2.4. *For $I \in \mathfrak{S}(r, 2n+2)$, and $\mathcal{B}(I)$, $\overline{\Upsilon}_I(F_\bullet) \subset \overline{\Upsilon}_{\mathcal{B}(I)}(F_\bullet)$.*

Proof. For a complete isotropic flag on $\mathbb{C}^{2n+2}$ Recall the definition of a Schubert variety defined in $OG(r, 2n+2)$:

$\overline{\Upsilon}_I(F_\bullet) = \{M \in OG(r, 2n+2) \mid$ for any $0 \leq l \leq r,$ and any $i_l \leq b < i_{l+1}$, $\dim(M \cap F_b) \geq l\}$.

Let $I = (i_1, \dots, i_r)$ and $\mathcal{B}(I) = (b_1, \dots, b_r)$. Then,

$$i_1 \leq b_1 \leq i_2 \leq b_2 \leq \dots \leq i_r \leq b_r$$

and $F_{i_l} \subseteq F_{b_l}$. So, for $M \in \overline{\Upsilon}_{\mathcal{B}(I)}(F_\bullet)$, $\dim(M \cap F_{b_l}) \geq \dim(M \cap F_{i_l}) \geq l$. □

The following lemma is the main tool that we use in the proof of Lemma (6.2.1).

Lemma 6.2.5. *Given an index set $I \in \mathfrak{S}(r, 2n+2)$ and a valid index set $J = \mathcal{B}(I)$, produced from a bumping rule for $OG(r, 2n+2)$,*

$$G(I) + 1 = G(J).$$

The proof of Lemma (6.2.5) requires a comparison of the parts of the formulas $G(I)$ and $G(J)$. We first give the proof of the lemma of interest, Lemma (6.2.1):

Recall that the formula to show is

$$\dim(\Upsilon_I(F_\bullet)) = |I > \tilde{I}| + \frac{1}{2}(|I > \bar{I}| - |I > n+1|). \tag{6.2}$$

Proof. Given I, $\overline{\Upsilon}_I(F_\bullet) \subsetneq \overline{\Upsilon}_{\mathcal{B}(I)}(F_\bullet)$, by Proposition (6.2.4). Consider the formula $G(I)$ for the indices parameterizing the top class in cohomology and the zero class in cohomology. For $I_{top} = \{1, \dots, r\}$, $G(I_{top}) = 0 + \frac{1}{2}(0-0) = 0$, and for $I_0 = \{2n+3-$

$r, \ldots, 2n+2\}$,

$$G(I_0) = (2nr + 2r - 2r^2) + \frac{1}{2}(r^2 - r) = \frac{r}{2}(4n - 3r + 3) = \dim(OG(r, 2n{+}2)).$$

So in these two cases, $\dim(\Upsilon_{I_0}(F_\bullet)) = G(I_0)$, and $\dim(\Upsilon_{I_{top}}(F_\bullet)) = G(I_{top})$.

Now, for I_{top}, there is only one valid shift, which is to bump $i_r = r$ to $j_r = r+1$, so $J = \{1, \ldots, r-1, r+1\}$. J parameterizes the only one dimensional Schubert variety $\overline{\Upsilon}_J(F_\bullet)$. So, $\overline{\Upsilon}_{I_{top}}(F_\bullet) \subset \overline{\Upsilon}_J(F_\bullet)$ and $G(J) = G(I_{top}) + 1$. Let $K \in \mathfrak{S}(r, 2n+2)$ and $G(K) = l$ for some $l \geq 1$. Now, we can repeatedly bump and create a chain

$$\overline{\Upsilon}_K(F_\bullet) \subset \overline{\Upsilon}_{\mathcal{B}(K)}(F_\bullet) \subset \overline{\Upsilon}_{\mathcal{B}(\mathcal{B}(K))}(F_\bullet) \subset \ldots \subset \overline{\Upsilon}_{I_0}(F_\bullet).$$

We can similarly work back through the bumping rules to create a chain from $\overline{\Upsilon}_{I_{top}}(F_\bullet)$ to $\overline{\Upsilon}_K(F_\bullet)$, so we have a full chain:

$$\overline{\Upsilon}_{I_{top}}(F_\bullet) \subset \overline{\Upsilon}_J(F_\bullet) \subset \ldots \subset \overline{\Upsilon}_K(F_\bullet) \subset \overline{\Upsilon}_{\mathcal{B}(K)}(F_\bullet) \subset \ldots \subset \overline{\Upsilon}_{I_0}(F_\bullet).$$

So we were able to build a chain of $\frac{r}{2}(4n - 3r + 3) + 1$ Schubert varieties. We know that the formula $G(I)$ agrees with the dimension of $\Upsilon_I(F_\bullet)$ at the top and the bottom of this chain. It is clear that since $\overline{\Upsilon}_K(F_\bullet) \subset \overline{\Upsilon}_{\mathcal{B}(K)}(F_\bullet)$, that $\dim(\overline{\Upsilon}_K(F_\bullet)) < \dim(\overline{\Upsilon}_{\mathcal{B}(K)}(F_\bullet))$. So the dimension formula agrees: $\dim(\Upsilon_I(F_\bullet)) = |J > \tilde{J}| + \frac{1}{2}(|J > \bar{J}| - |J > n+1|)$. □

Finally, we give the proof of Lemma (6.2.5).

Proof. We will prove each rule separately and in each case we let $I \in \mathfrak{S}(r, 2n{+}2)$ and $J = \mathcal{B}(I)$. In each case we will compare each of the pieces of the formulas $G(I)$ and $G(J)$:

(1) (a) For all $t \neq k$, $i_t = j_t$. Since $i_k \neq n+1$, we have $|I > n+1| = |J > n+1|$. Since we are only shifting one index we have

$$\overline{i_r} = \overline{j_r} < \overline{i_{r-1}} = \overline{j_{r-1}} < \ldots < \overline{i_{k+l}} = \overline{j_{k+1}} < \overline{i_k} < \overline{j_k} < \overline{i_{k-1}} = \overline{j_{k+1}} < \ldots < \overline{i_1} = \overline{j_1},$$

and it is clear that

$$\begin{aligned} i_k < j_k < \overline{j_k} < \overline{i_k} \quad & \text{if } i_k < n+1 \text{ and} \\ \overline{j_k} < \overline{i_k} < i_k < j_k \qquad & \text{if } i_k > n+1. \end{aligned}$$

In either case, $|I > \overline{I}| = |J > \overline{J}|$ since J and $\overline{J}$ are disjoint,

$$|j_k > \overline{J}| = |i_k + 1 > \overline{J}|.$$

Finally we compare $|I > \tilde{I}|$ and $|J > \tilde{I}|$. Note that $j_k = i_k + 1 \in \tilde{I}$. Let $\tilde{i}_p = i_k + 1 \in \tilde{I}$. Then, we have

$$\tilde{i}_1 = \tilde{j}_1 < \ldots < \tilde{i}_{p-1} = \tilde{j}_{p-1} < \tilde{j}_p < \tilde{i}_p < \tilde{j}_{p+1} = \tilde{i}_{p+1} < \ldots < \tilde{j}_{2n+2-2r} < \tilde{i}_{2n+2-2r},$$

and it is clear that whether $i_k > n+1$ or $i_k < n+1$,

$$i_k = \tilde{j}_p < j_k = \tilde{i}_p.$$

So,

$$|j_k > \tilde{J}| = |i_k > \tilde{I}| + 1,$$

but for any $l \neq k$,

$$|j_l > \tilde{J}| = |i_l > \tilde{I}|.$$

Therefore, $|J > \tilde{J}| = |I > \tilde{I}| + 1$, and $G(J) = G(I) + 1$.

(b) This is a more specific rule dealing with two particular shifts, when $i = n$ and when $i = n+1$.

Let $i_k = n$. Then,

$$i_1 < \ldots < i_{k-1} < n < i_{k+1} < \ldots < i_r$$

$$j_1 < \ldots < j_{k-1} < n+2 < j_{k+1} < \ldots < j_r.$$

Therefore, $|J > n+1| = |J > n+1| + 1$. Following the same argument in part 1(a), we see that since

$$i_k < \overline{j_k} < j_k < \overline{i_k},$$

$|J > \overline{J}| = |I > \overline{I}| + 1$.

Let $i_k = n+1$. Then,

$$i_1 < \ldots < i_{k-1} < n+1 < i_{k+1} < \ldots < i_r$$

$$j_1 < \ldots < j_{k-1} < n+3 < j_{k+1} < \ldots < j_r.$$

Therefore, $|J > n+1| = |J > n+1| + 1$. Following the same argument in part 1(a), we see that since

$$\overline{j}_k < i_k < \overline{i}_k < j_k,$$

$|J > \overline{J}| = |I > \overline{I}| + 1$.

So, in either of the cases $i_k = n$ or $i_k = n+1$:

$$\frac{1}{2}(|I > \overline{I}| - |I > n|) =$$
$$\frac{1}{2}(|I > \overline{I}| + 1 - |I > n| - 1) =$$
$$\frac{1}{2}(|J > \overline{I}| - |J > n|).$$

Finally, to compare $|J > \tilde{J}|$ and $|I > \tilde{I}|$. The following chart that shows the relationships between the indices:

$i_k = n$	I	J		$i_k = n+1$	I	J
n	i_k	$\tilde{j}_p$		n	$\tilde{i}_p$	$\overline{j_k}$
$n+1$	$\tilde{i}_p$	$\bar{j}_k$		$n+1$	i_k	$\tilde{j}_p$
$n+2$	$\tilde{i}_{p+1}$	j_k		$n+2$	$\bar{i}_k$	$\tilde{j}_{p+1}$
$n+3$	$\bar{i}_k$	$\tilde{j}_{p+1}$		$n+3$	$\tilde{i}_{p+1}$	j_k

Therefore, for $i = n$, or $i = n+1$, in either case $|j_k > \tilde{J}| = |i_k > \tilde{I}| + 1$, and $G(J) = G(I) + 1$.

Before we move to phase (2), observe that if $(i_k, i_{k+1}) \in I$, then there exists $J = \mathcal{B}(I)$ such that $(n+2, n+3) \in J$. To make the consideration of phase (2) easier, we consider this case separately:

(c) In this case we are shifting exactly two indices:

$$i_1 < \ldots < i_{k-1} < n < n+1 < i_{k+2} < \ldots < i_r$$

$$j_1 < \ldots < j_{k-1} < n+2 < n+3 < j_{k+2} < \ldots < j_r.$$

It is immediate that $|J > n+1| = |I > n+1| + 2$, and $|J > \tilde{J}| = |I > \tilde{I}|$. Consider the following table:

	I	J
n	i_k	$\bar{j}_{k+1}$
$n+1$	i_{k+1}	$\bar{j}_k$
$n+2$	$\bar{i}_{k+1}$	j_k
$n+3$	$\bar{i}_k$	j_{k+1}

So, $|J > \overline{J}| = |I > \overline{I}| + 4$. Therefore,

$$\begin{aligned} G(J) &= |I > \tilde{I}| + \frac{1}{2}(|I > \overline{I}| + 4 - (|I > n+1| + 2)) \\ &= |I > \tilde{I}| + \frac{1}{2}(|I > \overline{I}| - |I > n+1|) + 1 = G(I) + 1. \end{aligned}$$

(2) For this rule, we must consider both possible valid shifts that can occur. Let

$$J_1 = (i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_{t-1}, i_t + 1, \ldots, i_r)$$

$$J_2 = (i_1, \ldots, i_{k-1}, i_k + 2, i_{k+1}, \ldots, i_{t-1}, i_t, \ldots, i_r)$$

We will consider these two cases separately and keep in mind that both can produce multiple index sets.

(J_1) We have three cases to consider: $i < n$, $i = n$, and $i > n$. If $i_k + 1 < n + 1$, then $n + 1 < i_t < i_{t+1}$. If $i_k + 1 = n + 1$, so $i_t = n + 2$, and $i_t + 1 > n + 2$. If $i_k + > n+1$, then $i_t + 1 < n+1$. So in all three cases, $|I > n+1| = |J_1 > n+1|$. Now, consider the following

I	J_1
$i_k \in I$	$i_k \in \overline{J_1}$
$i_k + 1 \in \overline{I}$	$i_k + 1 \in J_1$
$i_t \in I$	$i_t \in \overline{J_1}$
$i_t + 1 \in \overline{I}$	$i_t + 1 \in J_1$

Note that the chart above does not assume whether $i_k > n+1$ or $i_k < n+1$, since making this change simply reverses the roles of i_k and i_t. Also note that $i_k = n + 1$ is not considered here since if $n + 1 \in I$, then $n \in I$ and $n + 2, n + 3 \in \overline{I}$. Therefore, this case will be dealt with in (1c). So it is clear that $|J_1 > \overline{J}_1| = |I > \overline{I}| + 2$. Referring to the same chart, and knowing that there will be no elements that change from $\tilde{I}$ to $\tilde{J}_1$, we see that $|J_1 > \tilde{J}_1| = |I > \tilde{I}|$. Finally, we see

$$G(J_1) = |I > \tilde{I}| + \frac{1}{2}(|I > \overline{I}| + 2 + |I > n+1|) = G(I) + 1.$$

(J_2) We again have cases to consider: $i_k < n$, $i_k = n$, $i_k = n+1$, $i_k = n+1$, and $i_k > n+1$. First note that we dismiss two cases; when $i_k = n$, then $i_k + 2 = n+2$, but this was taken care of in (1c). Similarly, when $i_k = n+1$, $i_{k+2} = n+3$, but if $n+3 \in \overline{I}$, then $n \in I$ and this case is handled in (1c). If $i_k < n$, then $i_k + 2 \leq n+1$. If $i_k > n+1$, then $i_k + 2 > n+3$. So, in both of these cases, $|I > n+1| = |J_2 > n+1|$. Let $i_t = 2n+3 - i + k$. Now, for $i_k < n$, we have the following indices changing:

I	J_2
$i_k \in I$	$i_k \in \tilde{J}_2$
$i_k + 1 \in \overline{I}$	$i_k + 1 \in \overline{J}_2$
$i_k + 2 \in \tilde{I}$	$i_k + 2 \in J_2$
$i_t - 2 \in \tilde{I}$	$i_t - 2 \in \overline{J}_2$
$i_t - 1 \in I$	$i_t - 1 \in J_2$
$i_t \in \overline{I}$	$i_t \in \tilde{J}_2$

So,

$$G(J_2) = |I > \tilde{I} + \frac{1}{2}(|I > \overline{I}| + 2 - |I > n+1|) = G(I) + 1. \tag{6.3}$$

Again, let $i_t = 2n+3-i+k$, and consider $i_k > n$. So, we have the following indices changing:

I	J_2
$i_t - 2 \in \tilde{I}$	$i_t - 2 \in \overline{J}_2$
$i_t - 1 \in I$	$i_t - 1 \in J_2$
$i_t \in \overline{I}$	$i_t \in \tilde{J}_2$
$i_k \in I$	$i_k \in \tilde{J}_2$
$i_k + 1 \in \overline{I}$	$i_k + 1 \in \overline{J}_2$
$i_k + 2 \in \tilde{I}$	$i_k + 2 \in J_2$

and equation (6.3) holds here.

Now, phase two may be repeated if $i_k + 2 \in \overline{I}$ or $i_t + 1 \in \overline{I}$, but it is easy to see that this would be necessary at most $r - 1$. For any number of times that this step occurs it is easy to see how the calculation will change. For $i_k < n$:

I	J_2
$i_k \in I$	$i_k \in \tilde{J}_2$
$i_k + 1 \in \overline{I}$	$i_k + 1 \in \overline{J}_2$
$\vdots$	$\vdots$
$i_k + m \in \overline{I}$	$i_k + m \in \overline{J}_2$
$i_k + m + 1 \in \tilde{I}$	$i_k + m + 1 \in J_2$
$i_t - 2 \in \tilde{I}$	$i_t - 2 \in \overline{J}_2$
$i_t - m \in I$	$i_t - m \in J_2$
$\vdots$	$\vdots$
$i_t - 1 \in I$	$i_t - 1 \in J_2$
$i_t \in \overline{I}$	$i_t \in \tilde{J}_2$

Restricting our view to the indices in the chart above and noting $|J > n+1| = |I > n+1|$, we compare:

$$|J > \tilde{J}| - |I > \tilde{I}| = m + 1 - 2m = -m + 1$$

$$|J > \overline{J}| - |I > \overline{J}| = m(m+1) - m^2 = m$$

Taking the sum of these differences, we get 1. Note, again that $|J > n+1| = |I > n+1|$. Therefore, $G(J) = G(I) + 1$. This calculation is for $i_k < n$, and a nearly identical argument holds for $i_k > n$.

□

CHAPTER 7

Examples

7.1. Counterexample

Let $n = 3$ and $r = 2$, and let $E^j_\bullet$ for $j \in \{1, 2, 3\}$ be three complete flags in general position in $V = \mathbb{C}^7$. Let $I^1 = (37)$, $I^2 = (37)$, and $I^3 = (36)$ in $\mathfrak{S}(2, 7)$. Let

$$\sigma_{I^k} = [\bar{\Psi}_{I^k}(E^k_\bullet)] \in H^*(OG(2, 7)).$$

Let $F^j_\bullet$ for $j \in \{1, 2, 3\}$ be three complete flags in general position in $W = \mathbb{C}^8$. Recall the map $\mathfrak{s} : \mathfrak{S}^B(r, 2n{+}1) \leftrightarrow \mathfrak{S}(r, 2n{+}2)$ from (3.10). We have $J^1 = \mathfrak{s}((37)) = (38)$, $J^2 = (38)$, and $J^3 = \mathfrak{s}((36)) = (37)$. Let

$$\sigma'_{J^k} = [\bar{\Upsilon}_{J^k}(F^k_\bullet)] \in H^*(OG(2, 8)).$$

Then,

$$\sigma_{(37)} \cdot \sigma_{(37)} \cdot \sigma_{(36)} = \sigma_{(37)}[\sigma_{(15)} + \sigma_{(23)}] = \sigma_{(12)} \in H^{14}(OG(2, 2n{+}1)) \text{ and,}$$

$$\sigma'_{(38)} \cdot \sigma'_{(38)} \cdot \sigma'_{(37)} = \sigma'_{(38)}[\sigma'_{(23)} + \sigma'_{(14)} + \sigma'_{(15)}] = 0 \in H^{18}(OG(2, 2n{+}2)).$$

7.2. D4 and B3

In this section we use the standard Bourbaki numbering of the nodes of the Dynkin diagram. Let $h^B_1, h^B_2, h^B_3 \in {\mathfrak{h}_+}^B$ where

$$h^B_i = (x_i, y_i, z_i, 0, -z_i, -y_i, -z_i),$$

and, under the embedding given in (3.4), $h^B_i \mapsto h^D_i$, where

$$h^D_i = (x_i, y_i, z_i, 0, 0, -z_i, -y_i, -z_i).$$

For the first node of the Dynkin Diagrams we are considering $SO(7) \subset SO(8)$, and the homogeneous spaces $OG(1,7)$ and $OG(1,8)$. Their dimensions are $\dim OG(1,7) = 5$ and $\dim OG(1,8) = 6$. the following two tables

	$OG(1,7)$			$OG(1,8)$	
Index	Min Coset Rep	Word	Index	Min Coset Rep	Word
(1)	(1,2,3,4,5,6,7)	idt	(1)	(1,2,3,4,5,6,7,8)	idt
(2)	(2,1,3,4,5,7,6)	s_1	(2)	(2,1,3,4,5,6,8,7)	s_1
(3)	(3,1,2,4,6,7,5)	s_2s_1	(3)	(3,1,2,4,5,7,8,6)	s_2s_1
			(4)	(4,1,2,3,6,7,8,5)	$s_3s_2s_1$
			(5)	(5,1,2,6,3,7,8,4)	$s_4s_2s_1$
(5)	(5,1,2,4,6,7,3)	$s_3s_2s_1$	(6)	(6,1,2,5,4,7,8,3)	$s_3s_4s_2s_1$
(6)	(6,1,3,4,5,7,2)	$s_2s_3s_2s_1$	(7)	(7,1,3,5,4,6,8,2)	$s_2s_3s_4s_2s_1$
(7)	(7,2,3,4,5,6,1)	$s_1s_2s_3s_2s_1$	(8)	(8,2,3,5,4,6,7,1)	$s_1s_2s_3s_4s_2s_1$

The following list give the Levi-movable triples with intersection number one and an example of a corresponding inequality in the eigencones $\Gamma(3, SO(7))$ and $\Gamma(3, SO(8))$. It should be noted that all of the intersection numbers in $OG(1,8)$ will also have intersection number one. The inequalities produced from permuting the order of the intersection (i.e. (8)(7)(2) vs. (7)(2)(8),) will yield different inequalities in the cone.

Triple in $OG(1,8)$	Triple in $OG(1,7)$	Corresponding Inequality
(8)(8)(1)	(7)(7)(1)	$-x_1 - x_2 + x_3$
(8)(7)(2)	(7)(6)(2)	$-x_1 - y_2 + y_3$
(8)(6)(3)	(7)(5)(3)	$-x_1 - z_2 + z_3$
(7)(7)(3)	(6)(6)(3)	$-y_1 - y_2 + z_3$
(7)(6)(6)	(6)(5)(5)	$-y_1 - z_2 - z_3$

For the second node of the diagram, we are considering $OG(2,7$ and $OG(2,8)$. Their dimensions are $\dim OG(2,7) = 7$, and $\dim OG(2,8) = 9$. We give all the parameters in each space, and list the sets that correspond under the $\mathfrak{s}$ side by side.

	$OG(2,7)$			$OG(2,8)$	
Index	Min Coset Rep	Word	Index	Min Coset Rep	Word
(1,2)	(1,2,3,4,5,6,7)	idt	(1,2)	(1,2,3,4,5,6,7,8)	idt
(1,3)	(1,3,2,4,6,5,7)	s_2	(1,3)	(1,3,2,4,5,7,6,8)	s_2
			(1,4)	(1,4,2,3,6,7,5,8)	s_3s_2
			(1,5)	(1,5,2,6,3,7,4,8)	s_4s_2
(1,5)	(1,5,2,4,6,3,7)	s_3s_2	(1,6)	(1,6,2,5,4,7,3,8)	$s_3s_4s_2$
(1,6)	(1,6,3,4,5,2,7)	$s_2s_3s_2$	(1,7)	(1,7,3,5,4,6,2,8)	$s_2s_3s_4s_2$
(2,3)	(2,3,1,4,7,5,6)	s_1s_2	(2,3)	(2,3,1,4,5,8,7,6)	s_1s_2
			(2,4)	(2,4,1,3,6,8,5,7)	$s_3s_1s_2$
			(2,5)	(2,5,1,6,3,8,4,7)	$s_3s_4s_2$
(2,5)	(2,5,1,4,7,3,6)	$s_1s_3s_2$	(2,6)	(2,6,1,5,4,8,3,7)	$s_3s_1s_4s_2$
(2,7)	(2,7,3,4,5,1,6)	$s_1s_2s_3s_2$	(2,8)	(2,8,3,5,4,6,1,7)	$s_1s_2s_3s_4s_2$
			(3,4)	(3,4,1,2,7,8,5,6)	$s_2s_1s_3s_2$
			(3,5)	(3,5,1,7,2,8,4,6)	$s_2s_1s_4s_2$
(3,6)	(3,6,1,4,7,2,5)	$s_2s_3s_1s_2$	(3,7)	(3,7,1,5,4,8,2,6)	$s_2s_3s_1s_4s_2$
(3,7)	(3,7,2,4,6,1,5)	$s_1s_2s_3s_1s_2$	(3,8)	(3,8,2,5,4,7,1,6)	$s_1s_2s_3s_1s_4s_2$
			(4,6)	(4,6,1,7,2,8,3,5)	$s_3s_2s_1s_4s_2$
			(4,7)	(4,7,1,6,3,8,2,5)	$s_2s_3s_2s_1s_4s_2$
			(4,8)	(4,8,2,6,3,7,1,5)	$s_1s_2s_3s_2s_1s_4s_2$
			(5,6)	(5,6,1,2,7,8,3,4)	$s_4s_2s_1s_3s_2$
			(5,7)	(5,7,1,3,6,8,2,4)	$s_2s_4s_2s_1s_3s_2$
			(5,8)	(5,8,2,3,6,7,1,4)	$s_1s_2s_4s_2s_1s_3s_2$
(5,6)	(5,6,1,4,7,2,3)	$s_3s_2s_3s_1s_2$	(6,7)	(6,7,1,4,5,8,2,3)	$s_3s_2s_4s_2s_1s_3s_2$
(5,7)	(5,7,2,4,6,1,3)	$s_1s_3s_2s_3s_1s_2$	(6,8)	(6,8,2,4,5,7,1,3)	$s_3s_1s_2s_4s_2s_1s_3s_2$
(6,7)	(6,7,3,4,5,1,2)	$s_2s_1s_3s_2s_3s_1s_2$	(7,8)	(7,8,3,4,5,6,1,2)	$s_2s_3s_1s_2s_4s_2s_1s_3s_2$

The following list give the Levi-movable triples with intersection number one and an example of a corresponding inequality in the eigencones $\Gamma(3, SO(7))$ and $\Gamma(3, SO(8))$. It should be noted that all of the intersection numbers in $OG(2,8)$ will also have intersection number one. The inequalities produced from permuting the order of the intersection (i.e. (78)(78)(12) vs. (78)(12)(78),) will yield different inequalities in the cone.

Triple in $OG(2,8)$	Triple in $OG(2,7)$	Corresponding Inequality
(78)(78)(12)	(67)(67)(12)	$-y_1 - x_1 - y_2 - x_2 + x_3 + y_3$
(78)(68)(13)	(67)(57)(13)	$-y_1 - x_1 - z_2 - x_2 + x_3 + z_3$
(78)(67)(23)	(67)(56)(23)	$-y_1 - x_1 - z_2 - y_2 + y_3 + z_3$
(78)(38)(16)	(67)(37)(15)	$-y_1 - x_1 + z_2 - x_2 + x_3 - z_3$
(78)(28)(17)	(67)(27)(16)	$-y_1 - x_1 + y_2 - x_2 + x_3 - y_3$
(78)(37)(26)	(67)(36)(25)	$-y_1 - x_1 + z_2 - y_2 + y_3 - z_3$
(68)(68)(23)	(57)(57)(23)	$-z_1 - x_1 - z_2 - x_2 + y_3 + z_3$
(68)(38)(26)	(57)(37)(25)	$-z_1 - x_1 + z_2 - x_2 + y_3 - z_3$
(68)(38)(17)	(57)(37)(16)	$-z_1 - x_1 + z_2 - x_2 + x_3 - y_3$
(68)(37)(28)	(57)(36)(27)	$-z_1 - x_1 + z_2 - y_2 + y_3 - x_3$
(67)(38)(37)	(56)(37)(36)	$-z_1 - y_1 + z_2 - x_2 + z_3 - y_3$
(67)(38)(28)	(56)(37)(27)	$-z_1 - y_1 + z_2 - x_2 + y_3 - x_3$

	$OG(3,7)$		
Index	Min Coset	Word	Index
(1,2,3)	(1,2,3,4,5,6,7)	idt	(1,2,3,4)
(1,2,5)	(1,2,5,4,3,6,7)	s_3	(1,2,5,6)
(1,3,6)	(1,3,6,4,2,5,7)	s_2s_3	(1,3,5,7)
(1,5,6)	(1,5,6,4,2,3,7)	$s_3s_2s_3$	(1,4,6,7)
(2,3,7)	(2,3,7,4,1,5,6)	$s_1s_2s_3$	(2,3,5,8)
(2,5,7)	(2,5,7,4,1,3,6)	$s_1s_3s_2s_3$	(2,4,6,8)
(3,6,7)	(3,6,7,4,1,2,5)	$s_2s_1s_3s_2s_3$	(3,4,7,8)
(5,6,7)	(5,6,7,4,1,2,3)	$s_3s_2s_1s_3s_2s_3$	(5,6,7,8)

	$OG^{\pm}(3,7)$	$OG^{-}(4,8)$	$OG^{+}(4,8)$
Index	Min Coset	Word	Word
(1,2,3,4)	(1,2,3,4,5,6,7,8)	idt	idt
(1,2,5,6)	(1,2,5,6,3,4,7,8)	s_3	s_4
(1,3,5,7)	(1,3,5,7,2,4,6,8)	s_2s_3	s_2s_4
(1,4,6,7)	(1,4,6,7,2,3,5,8)	$s_4s_2s_3$	$s_3s_2s_4$
(2,3,5,8)	(2,3,5,8,1,4,6,7)	$s_1s_2s_3$	$s_1s_2s_4$
(2,4,6,8)	(2,4,6,8,1,3,5,7)	$s_1s_4s_2s_3$	$s_1s_3s_2s_4$
(3,4,7,8)	(3,4,7,8,1,2,5,6)	$s_2s_1s_4s_2s_3$	$s_2s_1s_3s_2s_4$
(5,6,7,8)	(5,6,7,8,1,2,3,4)	$s_3s_2s_1s_4s_2s_3$	$s_4s_2s_3s_1s_2s_4$

The following list give the Levi-movable triples with intersection number one and an example of a corresponding inequality in the eigencones $\Gamma(3, SO(7))$ and $\Gamma(3, SO(8))$. It should be noted that all of the intersection numbers in $OG(2,8)$ will also have intersection number one. The inequalities produced from permuting the order of the intersection (i.e.

(5678)(5678)(1234) vs. (5678)(1234)(5678),) will yield different inequalities in the cone.

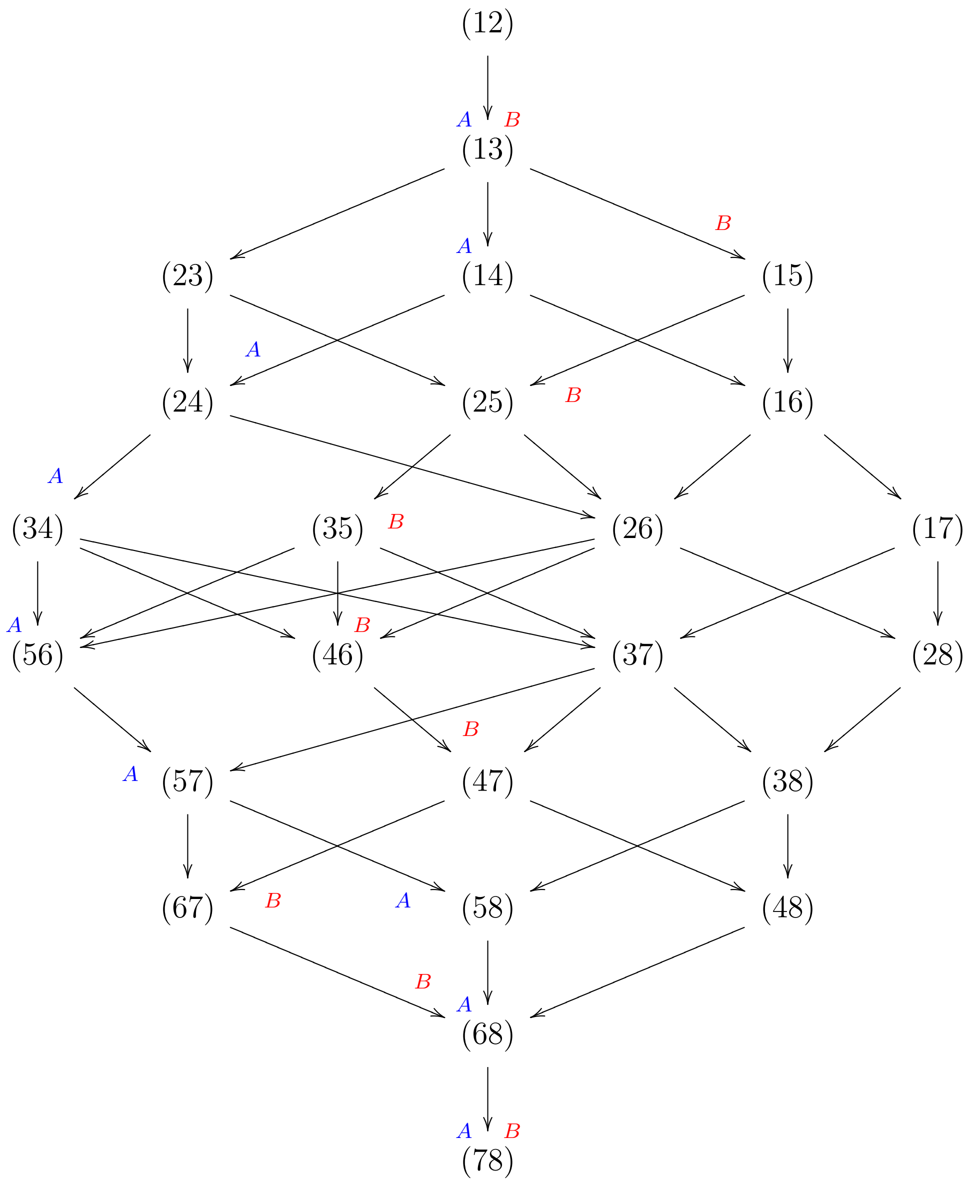
Triple in $OG(2,8)$	Triple in $OG(2,7)$	Corresponding Inequality
(5678)(5678)(1234)	(567)(567)(123)	$-z_1 - y_1 - x_1 - z_2 - y_2 - x_1 + x_3 + y_3 + z_3$
(5678)(3478)(1256)	(567)(367)(125)	$-z_1 - y_1 - x_1 + z_2 - y_2 - x_2 + x_3 + y_3 - z_3$
(5678)(2468)(1357)	(567)(257)(136)	$-z_1 - y_1 - x_1 + y_2 - z_2 - x_2 + x_3 + z_3 - y_3$
(3478)(3478)(1357)	(367)(367)(136)	$z_1 - y_1 - y_1 + z_2 - y_2 - x_2 + x_3 + z_3 - y_3$
(5678)(1467)(2358)	(567)(156)(237)	$-z_1 - y_1 - x_1 + x_2 - z_2 - y_2 + y_3 + z_3 - x_3$
(3478)(2468)(2358)	(367)(257)(237)	$z_1 - y_1 - x_1 + y_2 - z_2 - x_2 + x_3 - z_3 - y_3$
(3478)(2468)(1467)	(367)(257)(156)	$z_1 - y_1 - x_1 + y_2 - z_2 - x_2 + y_3 + z_3 - x_3$

7.3. Bump Example

Consider the homogeneous space $OG(2,8)$, so $n = 3$ and $r = 2$. The table below includes the valid index sets in $\mathfrak{S}(2,8)$ and gives the corresponding formula components for $G(I)$ and the dimension of the associated Schubert variety:

I	$\dim(\Upsilon_I)$	$\lvert I > n+1 \rvert$	$\lvert I > \overline{I} \rvert$	$\lvert I > \tilde{I} \rvert$	$G(I)$
(1,2)	0	0	0	0	0
(1,3)	1	0	0	1	1
(1,4)	2	0	0	2	2
(1,5)	3	1	1	2	3
(1,6)	3	1	1	3	3
(1,7)	4	1	1	4	4
(2,3)	2	0	0	2	2
(2,4)	3	0	0	3	3
(2,5)	3	1	1	3	3
(2,6)	4	1	1	4	4
(2,8)	5	1	3	4	5
(3,4)	4	0	0	4	4
(3,5)	4	1	1	4	4
(3,7)	5	1	3	4	5
(3,8)	6	1	3	5	6
(4,6)	5	1	3	4	5
(4,7)	6	1	3	5	6
(4,8)	7	1	3	6	7
(5,6)	5	2	4	4	5
(5,7)	6	2	4	5	6
(5,8)	7	2	4	6	7
(6,7)	7	2	4	6	7
(6,8)	8	2	4	7	8
(7,8)	9	2	4	8	9

Below is a diagram of the possible chains that run from the zero-dimensional class of a point, to the nine-dimensional class, which is the top dimension:



Will will follow the two highlighted chains from the figure above, and see which rules apply:

$A : (12) \xrightarrow{1} (13) \xrightarrow{2} (14) \xrightarrow{3} (24) \xrightarrow{4} (34) \xrightarrow{5} (56) \xrightarrow{6} (57) \xrightarrow{7} (58) \xrightarrow{8} (68) \xrightarrow{9} (78)$

$B : (12) \xrightarrow{1} (13) \xrightarrow{2} (15) \xrightarrow{3} (25) \xrightarrow{4} (35) \xrightarrow{5} (46) \xrightarrow{6} (47) \xrightarrow{7} (67) \xrightarrow{8} (68) \xrightarrow{9} (78)$

In the table below we show how each step is taken to build the chains above. The notation we use will indicate a phase and a step in a phase. So, (2)(1b) means that phase (2) was required and then in phase (1), shift (b) was performed.

A bump	Rule	Other valid bumps
$(12) \xrightarrow{1} (13)$	(1a)	none
$(13) \xrightarrow{2} (14)$	(1a)	to (15) by (1b), to (23) by (1a)
$(14) \xrightarrow{3} (24)$	(1a)	to (16) by (1b)
$(24) \xrightarrow{4} (34)$	(1a)	to (26) by (1b)
$(34) \xrightarrow{5} (56)$	(2)(1b)	to (37) by (2)(1a), to (46) by (2)(1a)
$(56) \xrightarrow{6} (57)$	(1a)	none
$(57) \xrightarrow{7} (58)$	(1a)	to (67) by (1a)
$(58) \xrightarrow{8} (68)$	1(a)	none
$(68) \xrightarrow{9} (78)$	1(a)	none

B bump	Rule	Other valid bumps
$(12) \xrightarrow{1} (13)$	(1a)	none
$(13) \xrightarrow{2} (15)$	(1b)	to (14) by (1a), to (23) by (1a)
$(15) \xrightarrow{3} (25)$	(1a)	to (16) by (1a)
$(25) \xrightarrow{4} (35)$	1(a)	to (26) by (1a)
$(35) \xrightarrow{5} (46)$	(2)(1a)	to (37) by (2)(1a), to (56) by (2)(1b)
$(46) \xrightarrow{6} (47)$	(1a)	none
$(47) \xrightarrow{7} (67)$	(1b)	to (48) by (1a)
$(67) \xrightarrow{8} (68)$	(1a)	none
$(68) \xrightarrow{9} (78)$	(1a)	none

7.4. Parameters in an Even Orthogonal Grassmannian

In this section, we give another way of parameterizing Schubert Varieties in an even orthogonal Grassmannian is by typed k-strict partitions. As in [**BKT**], we define a typed

k-strict partition λ and then give a recipe for producing an index set $P(\lambda) \in \mathfrak{S}(r, 2n{+}2)$. The goal here is to motivate the definition of the bumping rules for $OG(r, 2n+2)$, by making it clear by examining the typed k-strict partitions, that two index sets, one containing $n+1$, and the other containing $n+2$, will have the same dimension.

Definition 7.4.1. A k-strict partition is an integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that all parts λ_i greater than k are distinct.

Definition 7.4.2. A typed k-strict partition is a pair consisting of a k-strict partition λ together with an integer, *type*$(\lambda) \in \{0, 1, 2\}$.

In the Grassmannian $OG(r, 2n{+}2)$, we set $k = n + 1 - r$. For every typed k-strict partition we define an index set $P(\lambda) = \{p_1 < \ldots < p_r\} \subset [2n{+}2]$, by

$$p_j = n \quad + \quad k - \lambda_j + \#\{i < j | \lambda_i + \lambda_j \leq 2k - 1 + j - i\} \tag{7.1}$$

$$+ \begin{cases} 1 & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n + j + \textit{type}(\lambda) \text{ is even.} \\ 2 & \text{otherwise.} \end{cases} \tag{7.2}$$

The benefit to parameterizing with typed k-strict partitions is the containment of Schubert varieties becomes a question of partition inclusion. Recall the following definition from [**Ful97**]:

Definition 7.4.3. We say a partition λ is included in a partition μ and write $\lambda \subset \mu$ if the Young diagram of λ is contained in the Young diagram of μ, or equivalently, $\lambda_i \leq \mu_i$ for all i.

It is clear from the definition that the typed k-strict partitions containing a k part correspond to two index sets. In particular they correspond to two index sets that are the same up to trading $n+1$ for $n+2$. Therefore it is clear that for $I_1 = \{i^1_1, \ldots, i^1_r\}$ and $I_2 = \{i^2_1, \ldots, i^2_r\}$ k-strict partitions such that

$$i^1_1 = i^2_1 < \ldots < i^1_{m-1} = i^2_{m-1} < i^1_m = n+1 < i^2_m = n+2 < i^1_{m+1} = i^2_{m+1} < \ldots < i^1_r = i^2_r,$$

the corresponding Schubert varieties will have the same dimensions. This is clear in table (7.4) below.

Table (7.4) also shows the different ways that you can count the dimension (or codimension) of a D-type Schubert variety. It is the length of the word, $l(w_I)$, the complement of the number of boxes in the Young diagram given by $\lambda(I)$, in this case $9 - |\lambda(I)|$, and there is a third way to determine the dimension by considering the planar interpretation of the reduced decompositions as in [**Che84**] and [**CS08**].

Index I	Partition $\lambda(I)$	Permutation $s(w_I)$	Word w_I	Dimension $\dim(\Upsilon_I)$
(78)	$\{0,0\}$ type 0	(7,8,3,4,5,6,1,2)	$s_2s_3s_1s_2s_4s_2s_1s_3s_2$	9
(68)	$\{1,0\}$ type 0	(6,8,2,4,5,7,1,3)	$s_3s_1s_2s_4s_2s_1s_3s_2$	8
(67)	$\{1,1\}$ type 0	(6,7,1,4,5,8,2,3)	$s_3s_2s_4s_2s_1s_3s_2$	7
(58)	$\{2,0\}$ type 2	(5,8,2,3,6,7,1,4)	$s_1s_2s_4s_2s_1s_3s_2$	7
(48)	$\{2,0\}$ type 1	(4,8,2,3,6,7,1,5)	$s_1s_2s_3s_2s_1s_4s_2$	7
(57)	$\{2,1\}$ type 2	(5,7,1,3,6,8,2,4)	$s_2s_4s_2s_1s_3s_2$	6
(47)	$\{2,1\}$ type 1	(4,7,1,3,6,8,2,5)	$s_2s_3s_2s_1s_4s_2$	6
(38)	$\{3,0\}$ type 0	(3,8,2,4,5,7,1,6)	$s_1s_2s_3s_1s_4s_2$	6
(56)	$\{2,2\}$ type 2	(5,6,1,2,7,8,3,4)	$s_4s_2s_1s_3s_2$	5
(46)	$\{2,2\}$ type 1	(4,6,1,2,7,8,3,5)	$s_3s_2s_1s_4s_2$	5
(37)	$\{3,1\}$ type 0	(3,7,1,4,5,8,2,6)	$s_2s_1s_3s_4s_2$	5
(28)	$\{4,0\}$ type 0	(2,8,3,4,5,6,1,7)	$s_1s_2s_3s_4s_2$	5
(35)	$\{3,2\}$ type 2	(3,5,1,2,7,8,4,6)	$s_2s_1s_4s_2$	4
(34)	$\{3,2\}$ type 1	(3,4,1,2,7,8,5,6)	$s_2s_1s_3s_2$	4
(26)	$\{4,1\}$ type 0	(2,6,1,4,5,8,3,7)	$s_3s_1s_4s_2$	4
(17)	$\{5,0\}$ type 0	(1,7,3,4,5,6,2,8)	$s_2s_3s_4s_2$	4
(25)	$\{4,2\}$ type 2	(2,5,1,3,6,8,4,7)	$s_3s_4s_2$	3
(24)	$\{4,2\}$ type 1	(2,4,1,3,6,8,5,7)	$s_3s_1s_2$	3
(16)	$\{5,1\}$ type 0	(1,6,2,4,5,7,3,8)	$s_3s_4s_2$	3
(23)	$\{4,3\}$ type 0	(2,3,1,4,5,8,6,7)	s_1s_2	2
(15)	$\{5,2\}$ type 2	(1,5,2,3,6,7,4,8)	s_4s_2	2
(14)	$\{5,2\}$ type 1	(1,4,2,3,6,7,5,8)	s_3s_2	2
(13)	$\{5,3\}$ type 0	(1,3,2,4,5,7,6,8)	s_2	1
(12)	$\{5,4\}$ type 0	(1,2,3,4,5,6,7,8)	e	0

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