Localized Energy Estimates Of The Wave Equation On Higher Dimensional Hyperspherical Schwarzschild Spacetimes

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Abstract

PARUL LAUL: Localized Energy Estimates Of The Wave Equation On Higher Dimensional Hyperspherical Schwarzschild Spacetimes

(Under the direction of Jason Metcalfe)

The purpose of this dissertation is to discuss a robust way to measure dispersion of the linear wave equation on the $(n+1)$-dimensional Schwarzschild spacetime. One of the greater motivations for studying the higher dimensional Schwarzschild and Kerr spacetimes is to address the question of asymptotic stability of solutions to Einstein's equations. That is, if initial conditions are slightly perturbed, does the solution tend to the unperturbed solution. Even in the simplest case (Minkowski spacetime), establishing nonlinear stability proved to be highly nontrivial. This was originally shown by Christodoulou and Klainerman [**23**], and later simplified and generalized in [**31**] and [**4**], respectively.

In considering the Kerr solution, we ask whether solutions to small perturbations of Kerr initial data asymptotically approach perhaps a different member of the Kerr family. Decay estimates are fundamental tools in addressing this question and by studying the linear wave equation on Schwarzschild, we hope to gain some intuition in pursuing this problem.

In this thesis we will determine localized energy estimates of the inhomogeneous wave equation $\Box_g \phi = F$ on the $(n+1)$-dimensional Schwarzschild manifold, for $n \geq 4$. An inevitable loss in the estimate arises due to trapped rays on a surface known as the photon sphere. We then modify our technique and improve the estimate at this region.

To my father.

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CHAPTER 1

Introduction

In this dissertation we discuss a robust way to measure dispersion of waves on the higher dimensional hyperspherical Schwarzschild spacetime. The Schwarzschild manifold is the unique spherically symmetric, stationary solution to the vacuum Einstein Equations of General Relativity, and the simplest of which contains a black hole. Here we extend the $(3+1)$-dimensional localized energy estimates of [**34**] to the $(n+1)$-dimensional case for $n \geq 4$. We adopt the notation $d = n-3$ and write the $(d+4)$-dimensional Schwarzschild manifold as $\mathcal{M} = \mathbb{R} \times (r_s, \infty) \times \mathbb{S}^{d+2}$, with line element

$$ds^2 = -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) dt^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} dr^2 + r^2 d\omega^2. \tag{1.1}$$

Here $d\omega$ denotes the surface element of $\mathbb{S}^{d+2}$. The surface $r = r_s$ is known as the *event horizon* or *Schwarzschild radius*, and physically represents the boundary of the black hole.

When discussing localized energy estimates, one begins with the notion of a conserved energy quantity. We define the wave operator or d'Alembertian,

$$\Box_g := \nabla^\alpha \partial_\alpha = -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \partial_{tt} + \frac{1}{r^{d+2}} \partial_r \left[r^{d+2} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \partial_r \right] + \not\nabla \cdot \not\nabla,$$

where $\not\nabla = \frac{1}{r} \not\nabla_0$, and $\not\nabla_0$ denotes the gradient on $\mathbb{S}^{d+2}$. By exploiting its invariance under time translation we then define the conserved energy, $E[\phi](t)$, with respect to the Killing vector field ∂_t, as

$$E[\phi](t) = \int_{\mathbb{S}^{d+2}} \int_{r_s}^{\infty} \left[\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} (\partial_t \phi)^2(t,r,\omega) \right.$$
$$\left. + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r \phi)^2(t,r,\omega) + |\nabla\!\!\!/ \phi|^2(t,r,\omega) \right] r^{d+2} dr d\omega.$$

Energy conservation in this case implies for ϕ satisfying the homogeneous wave equation $\Box_g \phi = 0$, $E[\phi](t) = E[\phi](0)$ for all time t. In what follows we consider the inhomogeneous wave equation $\Box_g \phi = F$ and instead deduce the energy inequality,

$$E[\phi](T) \lesssim E[\phi](0) + \int_0^T \int_{\mathbb{S}^{d+2}} \int_{r \geq r_s} |\Box_g \phi||\partial_t \phi| r^{d+2} dr d\omega dt.$$

In keeping with notation similar to [**34**], we set LE_0 and LE_0^* to respectively describe the localized energy and dual localized energy spaces, and define their respective norms as,

$$||\phi||_{LE_0}^2 = \int_0^\infty \int_{\mathbb{S}^{d+2}} \int_{r_s}^\infty \left[c_t^0(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} (\partial_t \phi)^2(t,r,\omega) + \right.$$
$$\left. c_r^0(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r \phi)^2(t,r,\omega) + c_\omega^0(r) |\nabla\!\!\!/ \phi|^2(t,r,\omega) + c^0(r) \phi^2(t,r,\omega) \right] r^{d+2} dr d\omega dt,$$
$$||F||_{LE_0^*}^2 = \int_0^\infty \int_{\mathbb{S}^{d+2}} \int_{r_s}^\infty c_F^0(r) F^2 r^{d+2} dr d\omega dt.$$

Here, the coefficients are given by

$$c_r^0 = \frac{1}{r^{d+3}\left(1 - \log\left(\frac{r-r_s}{r}\right)\right)^2}, \quad c_\omega^0 = \frac{1}{r}\left(\frac{r-r_{ps}}{r}\right)^2, \quad c_F^0 = \frac{r^{d+3}\left(\frac{r-r_s}{r}\right)\left(1-\log\left(\frac{r-r_s}{r}\right)\right)^3}{\left(\frac{r-r_{ps}}{r}\right)^2}$$
$$c_t^0 = \frac{\left(\frac{r-r_{ps}}{r}\right)^2}{r^{d+3}\left(1 - \log\left(\frac{r-r_s}{r}\right)\right)^3} \quad c^0 = \frac{1}{r^3\left(\frac{r-r_s}{r}\right)\left(1-\log\left(\frac{r-r_s}{r}\right)\right)^4}.$$

We point out that the surface $r = r_{ps} := \left(\frac{d+3}{2}\right)^{\frac{1}{d+1}} r_s$ denotes the *photon sphere* and is of particular importance in our results. We elaborate on a few features below and refer the reader to Chapter 2 for a more detailed presentation.

With this setup, we state the main result of this dissertation:

Theorem 1.1. *Let ϕ satisfy the inhomogeneous wave equation $\Box_g\phi = F$ on the $(d+4)$-dimensional hyperspherical Schwarzschild manifold. Then for $d \geq 1$, we have the following estimate*

$$\sup_{t\geq 0} E[\phi](t) + ||\phi||^2_{LE_0} \lesssim E[\phi](0) + ||F||^2_{LE_0^*}. \tag{1.2}$$

A few remarks are in order. First note that for large r the Schwarzschild spacetime can be considered a slight perturbation of the Minkowski space. The results of [**37**] and [**38**] then give an improved estimate near infinity, whereby the factor r^{d+3} in the coefficients $c_r^0(r)$, $c_t^0(r)$ and $c_F^0(r)$ can be replaced with $r^{1+\delta}$, $\delta > 0$. Second, we point out the logarithmic singularity of the coefficients at $r = r_s$, an indication of trapped rays at the event horizon, is due to our choice of coordinates. It reinforces that in the given (t, r, ω) coordinate system, the estimate is only meaningful exterior to the black hole. An alternative coordinate choice and a non-degenerate energy quantity allows for estimates within the black hole as well. This has been done in the $(3+1)$-dimensional case in [**17**] and [**34**] and extended to higher dimensions in [**44**], [1] by exploiting the *gravitational redshift effect*, a phenomenon we discuss in Chapter 2. Finally, we point out that the coefficient of the angular derivative, c_ω^0, vanishes at the the photon sphere, an inevitable loss again due to trapping. This will be discussed in greater detail in the subsequent chapters. Here we point out that the quadratic loss can be improved to a logarithmic loss, giving the stronger estimate

Theorem 1.2. *Let ϕ satisfy the inhomogeneous wave equation $\Box_g\phi = F$ on the $(d+4)$-dimensional hyperspherical Schwarzschild manifold. Then for $d \geq 1$, we have that estimate* (1.2) *holds, where the coefficients c_t^0, c_ω^0 and c_F^0 are now given by*

$$c_t^0 = \frac{\left(1-\log\left|\frac{r-r_{ps}}{r}\right|\right)^{-2}}{r^{d+3}\left(1-\log\left(\frac{r-r_s}{r}\right)\right)^3}, \qquad c_\omega^0 = \frac{1}{r}\left(1-\log\left|\frac{r-r_{ps}}{r}\right|\right)^{-2}$$

$$c_F^0 = \frac{r^{d+3}\left(\frac{r-r_s}{r}\right)\left(1-\log\left(\frac{r-r_s}{r}\right)\right)^4}{\left(1-\log\left|\frac{r-r_{ps}}{r}\right|\right)^{-2}}.$$

[1] Upon submission of this thesis, we learned V. Schlue independently proved result (1.1) which later appeared in ArXiv:1012.5963v1

This thesis is divided as follows. In Chapter 2 we provide a condensed introduction to the relevant aspects of General Relativity. We discuss the geometry of the Minkowski, Schwarzschild and briefly, Kerr spacetimes, to establish the setting on which we consider the localized energy estimates. In Chapter 3 we provide an overview of the known localized energy estimates of the the wave equation in the $(3+1)$-dimensional case and in Chapter 4 we prove the main result of the dissertation. In Chapter 5 we show how the techniques in [**34**] can directly be applied to the higher dimensional case in order to obtain the improved estimate.

CHAPTER 2

General Relativity

General Relativity, or Einstein's theory of relativity, is the discipline describing gravitational influences via geometry. The theory is formulated on a Lorentzian manifold $(\mathcal{M}, g)$ called spacetime, and describes how the movement of particles is affected as a consequence of gravitational forces. A distinguishing feature from classical Newtonian physics is the fact that gravity is no longer considered a force, but instead an intrinsic feature of the spacetime. Induced by mass, gravity causes the space-time to bend and this bending is measured by curvature. The relationship between these two entities is governed by *Einstein's Field Equations*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the stress energy-momentum tensor of the surrounding matter distribution. The left hand side of (2.1), often denoted $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is termed *the Einstein tensor* and enjoys the divergence free property $\nabla_\nu G^{\mu\nu} \equiv 0$. Moreover, conservation of energy-momentum implies that $\nabla_\mu T^{\mu\nu} = 0$. The field equations are inspired by this relation, ie.$\nabla_\nu G^{\mu\nu} = \nabla_\mu T^{\mu\nu}$, thus proposing the tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ to be proportional.

In this exposition, the localized energy estimates considered are described on certain solutions to the $(n+1)$- dimensional *vacuum* equations, $T_{\mu\nu} = 0$. Equivalently, from (2.1) this amounts to determining the metric such that $R_{\mu\nu} \equiv 0$. By imposing constraints on the geometry of the spacetime, we discuss three different solutions, ordered by increasing generality. In addition to the metric, we will also analyze particularly relevant properties of the spacetimes including the structure of the geodesics and the notion of geodesic trapped rays. The geodesic equations are given by

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{2.2}$$

where λ represents an affine parameter. Geodesics represent the natural, unaccelerated course of a particle and hence are a fundamental tool in understanding the geometry of the spacetime.

In regards to the metric, we will choose coordinates so that we may represent the causal structure of events via a conformal diagram, as certain features of the geometry will be more discernible in this case. However, in the estimates discussed in Chapters 3, 4 and 5, Cartesian and polar coordinates will be the systems of choice. Elaborating slightly on the notion of causality, on a Lorentzian manifold (M, g), a tangent vector V can be classified as being either *time-like* if $g(V, V) < 0$, *light-like* or *null* if $g(V, V) = 0$, or *space-like* if $g(V, V) > 0$. A curve $x(\lambda)$ on $\mathcal{M}$ is said to be *time-like, null* or *space-like*, if the tangent vector at each point of the curve is respectively, *time-like, null* or *space-like*. At any given event P in spacetime, we define the set of all null vectors at P as the *light cone* of P. This divides spacetime in to three regions and illustrates the fact that physical particles can not move faster than the speed of light. The set of events that can be reached via a time-like curve passing through P is said to describe the *future light cone* of P, whereas those points which pass through P via a time like curve comprise of the *past light cone* of P. In our study of waves, we consider *causal curves*, that is geodesics that are either time-like or light-like. Finally, we denote i^+, i^-, and i^0 as *future timelike infinity*, *past timelike infinity* and *spatial infinity*, respectively, and $\mathcal{I}^+$, respectively, $\mathcal{I}^-$ as *future*, respectively, *past null infinity*.

We now describe the solutions to the vacuum equations considered in this exposition.

2.1. Minkowski Spacetime

The Minkowski spacetime $(\mathbb{R}^{n+1}, g_M)$ is the trivial solution to the vacuum equations. In terms of Cartesian co-ordinates, the metric has matrix representation $g_M =$

$diag(-1, 1, ..., 1)$, or re-written in polar coordinates, $ds^2 = -dt^2 + dr^2 + r^2 d\omega^2$ where $d\omega$ is the surface element on $\mathbb{S}^{n-1}$. The spacetime is described as being *flat* and thus, as expected, solving the geodesics equations (2.2) yields straight lines,

$$x(\lambda) = (a_0 + \lambda b_0, a_1 + \lambda b_1, .., a_n + \lambda b_n),$$

where $a_i, b_i \in \mathbb{R}$. The cartesian coordinate representation of Minkowski space does in fact produce light-like curves at 45° so that the causal structure of events is easy to determine. It is favourable, however, to compactify the space so that infinity is represented by a finite value and thus geodesics can be represented in their entirety. A series of co-ordinate transformations are conducted to achieve both properties. Define

$$\begin{array}{lllllll} 1. & u & = & t - r, & v & = & t + r \\ 2. & U & = & \arctan u, & V & = & \arctan v \\ 3. & T & = & V + U, & R & = & V - U \end{array}$$

The end result is the metric $ds^2 = (\cos T + \cos R)(-dT^2 + dR^2 + \sin^2 R d\omega^2)$ where $R \in [0, \pi)$, and $T \in (R - \pi, R + \pi)$, [**15**]. The spacetime is compactified with finite time and radial co-ordinates, and maintains the original 45° light cones that clearly describe the causal structure of events. This is shown in Figure 2.1. below.

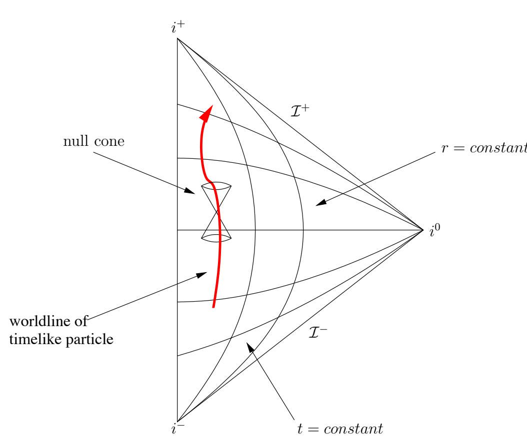

FIGURE 2.1. Penrose diagram of the Minkowski spacetime.

Observe that all geodesics escape to infinity or minus infinity. That is, the Minkowski space-time does not exhibit any trapped rays, a distinguishing feature from the spacetimes described below.

2.2. Schwarzschild Spacetime:

It is on this setting for which the results of this dissertation are based, and hence we provide greater details in formulating the solution. The Schwarzschild spacetime, $(\mathcal{M}, g)$ is the non trivial spherically symmetric, static solution to the vacuum equations. The term *static* refers to a spacetime which is stationary, that is, it admits a time-like Killing vector field, and that this Killing vector field is orthogonal to a family of space-like hypersurfaces. In terms of the metric in the traditional (t, x) coordinate system, *stationary* implies the metric is time-independent, and the *static* condition imposes that $g_{tj} = 0$, where $j = 1, ..., n$, ie. there are no cross-terms in the matrix representation of the metric. We point out that Birkhoff's theorem states that any spherically symmetric vacuum solution is necessarily static, and hence the Schwarzschild metric is the unique solution under radial symmetry.

2.2.1. Metric derivation: We begin the derivation of the solution with the initial claim that any spherically symmetric spacetime has line element of the form

$$ds^2 = g_{aa}(a,b)da^2 + g_{ar}(a,b)(dadb + dbda) + g_{bb}db^2 + r^2(a,b)d\omega^2, \tag{2.3}$$

where

$$d\omega^2 = \left[d\theta_1^2 + (\sin^2\theta_1)d\theta_2^2 + (\sin^2\theta_1 \sin^2\theta_2)d\theta_3^2 + ... + (\sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{n-2})d\theta_{n-1}^2\right]$$

is the line element of the $\mathbb{S}^{n-1}$ sphere. This is not obvious and the choice of the variable r is merely suggestive notation. Several texts describe the solution, [**15**], [**25**], [**40**], [**56**], as well the articles [**43**], [**49**]; we refer the reader to say, [**15**], for a more elaborate account and from which this derivation is based. A simple change of coordinates allows us to consider $b = b(a,r)$ so that (2.3) becomes

$$ds^2 = g_{aa}(a,r)da^2 + g_{ar}(a,r)(dadr + drda) + g_{rr}(a,r)dr^2 + r^2 d\omega^2. \tag{2.4}$$

To eliminate the cross terms, we look for a function $t(a,r)$ and note that t then satisfies

$$dt^2 = \left(\frac{\partial t}{\partial a}\right)^2 da^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(dadr + drda) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2. \tag{2.5}$$

We wish to write (2.4) in the form

$$ds^2 = m(t,r)dt^2 + n(t,r)dr^2 + r^2 d\omega^2 \tag{2.6}$$

for some functions m, n. Thus substituting (2.5) in to this equation and comparing with (2.4) imposes the conditions that $g_{aa} = m\left(\frac{\partial t}{\partial a}\right)^2, g_{ar} = m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)$, and $g_{rr} = m\left(\frac{\partial t}{\partial r}\right)^2$, a system of three equations to solve three unknowns $t(a,r)$, $m(a,r)$, $n(a,r)$. The assumption that our spacetime is Lorentzian forces either m or n to be negative and the other positive. The choice is not arbitrary and we refer the reader to general relativity texts, [**15**], [**40**], [**56**] for more detail. For our purposes, as the Schwarzschild solution approaches the Minkowski spacetime for large r, we choose m to be negative and write (2.6) as

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\omega^2. \tag{2.7}$$

To solve the vacuum equations, we require determining $\alpha(t,r)$ and $\beta(t,r)$ so that $R_{\mu\nu} \equiv 0$. This in turn requires a calculation of the non-vanishing Christofell symbols and components of the Riemann Tensor and Ricci Tensor. Denoting $I_k = \{k, ..., n-1\}$, we have:

Christofell symbols:

$$
\begin{aligned}
\Gamma^t_{tt} &= \partial_t \alpha(t,r) & \Gamma^t_{tr} &= \partial_r \alpha(t,r) \\
\Gamma^r_{tt} &= e^{2(\alpha-\beta)}\partial_r \alpha(t,r) & \Gamma^r_{tr} &= \partial_t \beta(t,r) \\
\Gamma^t_{rr} &= e^{2(\beta-\alpha)}\partial_t \beta(t,r) & \Gamma^r_{rr} &= \partial_r \beta(t,r) \\
\Gamma^{\theta_i}_{r\theta_i} &= \tfrac{1}{r}, i \in I_1, & \Gamma^{\theta_i}_{\theta_j\theta_i} &= \tfrac{\cos\theta_j}{\sin\theta_j}, i \in I_2, j < i \\
\Gamma^r_{\theta_1\theta_1} &= -re^{-2\beta} & \Gamma^r_{\theta_i\theta_i} &= -re^{-2\beta}\Pi_{j=1}^{i-1}\sin^2\theta_j, i \in I_2 \\
\Gamma^{\theta_{i-1}}_{\theta_i\theta_i} &= -\sin\theta_{i-1}\cos\theta_{i-1}, i \in I_2 & \Gamma^{\theta_j}_{\theta_i\theta_i} &= -\sin\theta_j\cos\theta_j\Pi_{k=j+1}^{i-1}\sin^2\theta_k, \\
& & & \quad i \in I_2, j < i-1
\end{aligned}
$$

Riemann Tensor components:

$$R^t_{rtr} = e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + [\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2]$$

$$
\begin{aligned}
R^{\theta_1}_{\theta_i\theta_1\theta_i} &= (1-e^{-2\beta})\Pi_{k=1}^{i-1}\sin^2\theta_k, & R^{\theta_j}_{\theta_i\theta_j\theta_i} &= -(e^{-2\beta} + \cos^2\theta_{j-1})\Pi_{k=1}^{i-1}\sin^2\theta_k, \\
&\quad i \in I_2 & &\quad i \in I_3, j < i \\
R^{\theta_i}_{\theta_1\theta_i\theta_1} &= 1 - e^{-2\beta}, \; i \in I_2 & R^{\theta_j}_{\theta_i\theta_j\theta_i} &= 1 - (e^{-2\beta} + \cos^2\theta_{i-1})\Pi_{k=1}^{i-1}\sin^2\theta_k, \\
& & &\quad i \in I_2, j > i \\
R^t_{\theta_1 t\theta_1} &= -re^{-2\beta}\partial_r\alpha & R^t_{\theta_i t\theta_i} &= -re^{-2\beta}\partial_r\alpha\Pi_{j=1}^{i-1}\sin^2\theta_j, i \in I_2 \\
R^t_{\theta_1 r\theta_1} &= -re^{-2\alpha}\partial_t\beta & R^t_{\theta_i r\theta_i} &= -re^{-2\alpha}\partial_t\beta\Pi_{j=1}^{i-1}\sin^2\theta_j i \in I_2 \\
R^r_{\theta_1 r\theta_1} &= re^{-2\beta}\partial_r\beta & R^r_{\theta_i r\theta_i} &= re^{-2\beta}\partial_r\beta\Pi_{j=1}^{i-1}\sin^2\theta_j, i \in I_2
\end{aligned}
$$

Ricci tensor components:

$$
\begin{aligned}
R_{tt} &= [\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \tfrac{2}{r}\partial_r\alpha] \\
R_{rr} &= -[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \tfrac{2}{r}\partial_r\beta] \\
R_{\theta_1\theta_1} &= e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - (n-2)] + (n-2) \\
R_{\theta_i\theta_i} &= R_{\theta_1\theta_1}\Pi_{j=1}^{i-1}\sin^2\theta_j, i \in I_2 \\
R_{tr} &= \tfrac{2}{r}\partial_t\beta
\end{aligned}
$$

The condition that $R_{\mu\nu} = 0$ in particular implies that $R_{tr} = \frac{2}{r}\partial_t\beta = 0$, so that $\beta = \beta(r)$. Moreover using $R_{\theta_1\theta_1} = 0$ and differentiating with respect to t, we may deduce that $\partial_t\partial_r\alpha(t,r) = 0$. Thus α takes the form $\alpha(t,r) = f(r) + g(t)$. Choosing t such that

$g(t) = 0$, and substituting these in to (2.7) yields

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\omega^2. \tag{2.8}$$

With our choice of coordinates and initial claim, we have shown that the spherically symmetric assumption on the solution necessarily implies the metric is static.

Next, we consider the other components of the Ricci tensor. With α, β independent of time, and both $R_{tt}, R_{rr} = 0$, we obtain $e^{-2(\alpha-\beta)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta) = 0$. This shows the sum of α and β is a constant c, and (2.8) becomes $ds^2 = -e^{-2\beta}e^{2c}dt^2 + e^{2\beta}dr^2 + r^2 d\omega^2$. Changing variables so that $t \to e^{-c}t$, the metric takes the form

$$ds^2 = -e^{-2\beta}dt^2 + e^{2\beta}dr^2 + r^2 d\omega^2, \tag{2.9}$$

and we are left to determine the function β. The equation $R_{\theta_1\theta_1} = 0$ reduces to solving $e^{-2\beta}(2r\partial_r\beta - (n-2)) = -(n-2)$, or equivalently

$$\partial_r(r^{n-2}e^{-2\beta}) = (n-2)r^{n-3}. \tag{2.10}$$

Denoting the constant of integration as r_s^{n-2}, we deduce that $e^{-2\beta} = \left(1 - \frac{r_s^{n-2}}{r^{n-2}}\right)$, thus completing the derivation. Note that we have an overdetermined system for α and β. It can be readily verified that the other components of the Ricci tensor do in fact vanish with β defined above.

The physical interpretation of the constant of integration can be determined by the fact that in the limit $r \to \infty$, the Schwarzschild solution approaches the Minkowski spacetime and must coincide with Newtonian theory. In the weak field limit, ie. slight perturbations of the Minkowski solution, the g_{tt} component of the metric is found to be $-\left(1 - \frac{16\pi G}{(n-1)\omega_{n-1}}\frac{M}{r^{n-2}}\right)$, where M describes the relativistic mass of the body, $\omega_{\mathbb{S}^{n-1}}$ denotes the area of $\mathbb{S}^{n-1}$, and G is the gravitational constant, [**43**]. A direct correspondence with our solution above implies $r_s^{n-2} = \frac{16M\pi G}{(n-1)\omega_{n-1}}$, and thus we may interpret the constant r_s as the mass of the rotating object. In particular, setting $G = 1$ we obtain the well known feature that $r_s = 2M$ in the $(3+1)$-dimensional case . We emphasize that this mass parameter alone describes the family of Schwarzschild solutions.

In the remainder of this exposition it will be convenient to use the notation $d = n-3$ to represent the spatial dimension. In Schwarzschild coordinates, the Schwarzschild manifold $(\mathcal{M}, g)$ is then given by the space $\mathcal{M} = \mathbb{R} \times (r_s, \infty) \times \mathbb{S}^{d+2}$, with line element

$$ds^2 = -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) dt^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} dr^2 + r^2 d\omega^2. \tag{2.11}$$

Before performing a series of coordinate transformations so as to obtain a representation that easily depicts the causal structure of the spacetime, we highlight a few key points described by the metric at hand. The surface $r = r_s$, termed the *Schwarzschild radius*, or *event horizon*, represents the distance from the centre of the mass to the radius at which the gravitational escape velocity is the speed of light. As photons can not escape the gravitational pull once inside this radius, the region $r < r_s$ forms a black hole. Also note that from (2.11), the metric appears undefined at $r = r_s$. This turns out to merely be a consequence of the chosen coordinates and we will shortly see that many other coordinate systems do not exhibit this singularity. Finally, observe that the metric is also singular at $r = 0$. This is in fact a true singular point in the spacetime and represents the point of infinite curvature.

2.2.2. Geodesic equations/Trapped rays: To determine the geodesics of this spacetime, we choose the coordinate system $(r, t, \theta_1, \theta_2, ...\theta_{d+2})$, and seek curves $x(\lambda)$ satisfying (2.2). This set of coupled equations is greatly reduced once considering the symmetries of the spacetime. Since the metric components are independent of $t, \theta_1, ...\theta_{d+2}$ we deduce that $\partial_t, \partial_{\theta_1}, ...\partial_{\theta_{d+2}}$ are Killing vector fields. Moreover, rotational symmetries, correlating to conservation of angular momentum, imply that particles traverse in a plane. Thus, we may assume without loss of generality that $\theta_1 = \theta_2 = \cdots = \theta_{d+1} = \pi/2$. This gives the Killing vector fields

$$K = \partial_t \longleftrightarrow K_\mu = \left(-\left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right), 0, ..., 0 \right) \tag{2.12}$$

$$R = \partial_{\theta_{d+2}} \longleftrightarrow R_\mu = (0, ..., 0, r^2 \Pi_{i=1}^{d+1} \sin^2 \theta_i \big|_{\theta_i = \pi/2}) = (0, ..., 0, r^2). \tag{2.13}$$

Next, we claim that Killing vectors fields are constant along geodesic flows. Indeed, by definition, if ξ is Killing then $\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0$. Furthermore, if we let γ be a geodesic with tangent vector u^μ then by defintion $\nabla_\nu u^\mu = 0$ and hence $u^\nu \nabla_\nu u^\mu = 0$. Consider then

$$u^\delta \nabla_\delta (g(\xi, u)) = u^\delta \nabla_\delta (\xi_\mu u^\mu) = u^\delta u^\mu \nabla_\delta \xi_\mu + \xi_\mu u^\delta \nabla_\delta u^\mu.$$

The first term vanishes by the defining property of Killing vector fields, and the second vanishes since γ is a geodesic. Thus along geodesics, $\xi_\mu u^\mu = \textit{constant}$ as claimed. Applying this to the vectors K and R above, we have the conserved quantities

$$E := -K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) \frac{dt}{d\lambda} \tag{2.14}$$

$$L := R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\theta_{d+2}}{d\lambda}. \tag{2.15}$$

Physically, the time translation and rotational symmetry respectively correspond to conservation of energy and angular momentum. In addition to these conserved quantities, the geodesic equations $\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$ imply

$$-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0, \tag{2.16}$$

for null geodesics. Expanding (2.16) for $x(\lambda)$ yields,

$$-\left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) \left(\frac{dt}{d\lambda} \right)^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^{-1} \left(\frac{dr}{d\lambda} \right)^2 + r^2 \left(\frac{d\theta_{d+2}}{d\lambda} \right)^2 = 0,$$

or equivalently

$$-\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^2\left(\frac{dt}{d\lambda}\right)^2+\left(\frac{dr}{d\lambda}\right)^2+r^2\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\left(\frac{d\theta_{d+2}}{d\lambda}\right)^2=0.$$

We substitute (2.14) and (2.15) in to the above equation to obtain

$$-E^2+\left(\frac{dr}{d\lambda}\right)^2+\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\left(\frac{L^2}{r^2}\right)=0. \tag{2.17}$$

Let $V(r)=\frac{1}{2}\frac{L^2}{r^2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)$ denote the effective potential of the geodesic. We may then interpret (2.17) as describing the trajectory of a particle with energy $\frac{1}{2}E^2$ in a one dimensional potential, V, [**55**]. Specifically, this relationship can be written as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2+V(r)=\frac{1}{2}E^2. \tag{2.18}$$

As the angular momentum varies we obtain different trajectories governed by $V(r)$. Consider first the case where the angular momentum identically vanishes, $L\equiv 0$. This corresponds to radial null geodesics and equation (2.18) becomes $\left(\frac{dr}{d\lambda}\right)^2=\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^2\left(\frac{dt}{d\lambda}\right)^2$. By scaling λ we may assume $E=1$ so that $\frac{dr}{d\lambda}=\pm 1$, and hence $r=\pm\lambda$. In particular, $\frac{dt}{dr}=\pm\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}$ so that as $r\to r_s$ we have that $\frac{dt}{dr}\to\pm\infty$. That is, the slope of the light cones in the $r-t$ coordinate system become vertical. This suggests that null geodesics tangent to the surface at $r=r_s$ are trapped, neither escaping to infinity or the singularity, but instead are moving "vertically". However, due to the *red shift effect*, energy loss is exhibited, in fact, exponentially. This phenomenon essentially asserts that the stronger the gravitational influence, proper time of an object slows down. Thus frequency of waves radiated by objects near prominent gravitational fields lowers, shifting to the red end of the electromagnetic spectrum. The proportionality between energy and frequency thus implies energy loss. The red shift effect is apparent in the above situation. The fact that $\frac{dt}{dr}\to\pm\infty$ suggests that it is not possible for objects to reach, let alone, surpass the surface $r=r_s$. Objects falling in to the black hole do in fact cross $r=r_s$, however to an observer at infinity, it appears as if the object only asymptotically approaches this surface. This misleading observation can be rectified by choosing an alternate coordinate system.

Next consider the case where L is not identically 0. As above, we may set $E = 1$, and seek $r(\lambda)$ satisfying

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) = 1. \tag{2.19}$$

The behaviour of the particle is determined by the potential. At non-extreme points, the photons move along the potential until possibly reaching a turning point *ie.* $\left(\frac{dr}{d\lambda}\right)^2 = 0$, and then traverse in the opposite direction. Of our interest is when the potential attains extrema values, as this corresponds to particles moving in a circular orbit. Local maxima (resp. local minima) correspond to unstable (resp. stable) orbits. We calculate $\frac{dV}{dr} = -\frac{2L^2}{r^{d+4}}\left(r^{d+1} - \frac{(d+3)r_s^{d+1}}{2}\right) = 0$ and find this occurs at $r = r_{ps} := r_s\left(\frac{d+3}{2}\right)^{1/d+1}$, easily classified as a maximum of the function. The surface $r = r_{ps}$ is called the *photon sphere*, and like $r = r_s$, it is a trapped surface. It is of particular importance as null geodesics initially tangent to this surface will remain in a circular orbit, necessitating a loss in energy estimates. Nonetheless, as a maximum of the potential, the photons will be unstable here and hence slight perturbations will cause them to deviate from the orbit.

The above calculations classify the null geodesics of the Schwarzchild spacetime and in particular, reveal the trapping phenomenon that is absent in Minkowski. In the last section on Schwarzschild we re-write (2.11) and determine the Penrose representation.

2.2.3. Causal representation of Schwarzschild: We determine a suitable coordinate system to represent the causal structure of geodesics and compactify the Schwarzschild space-time. We write the sequence of coordinate transformations and refer the reader to say, [**15**], [**40**], [**56**] for a more detailed account on the motivation of each step. We begin by defining the Regge-Wheeler Tortoise co-ordinates, r^*,

$$r^*(r) = \int_{r_{ps}}^{r}\left(1 - \frac{r_s^{d+1}}{\rho^{d+1}}\right)^{-1} d\rho, \tag{2.20}$$

with corresponding Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)(dt^2 - dr^{*2}) + r^2 d\omega^2. \tag{2.21}$$

We highlight that in these coordinates as $r \to r_s$, $r^* \approx \ln(r - r_s)$; thus, $r^* \to -\infty$ as $r \to r_s$. We then define the Eddington-Finkelstein coordinates

$$\tilde{u} = t + r^*, \quad \tilde{v} = t - r^*$$

so that now

$$ds^2 = -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) d\tilde{u} d\tilde{v} + r^2 d\omega^2. \tag{2.22}$$

Let

$$u' = e^{\frac{\tilde{u}}{2r_s^{d+1}}} = e^{\frac{r^*+t}{2r_s^{d+1}}}, \quad v' = -e^{\frac{-\tilde{v}}{2r_s^{d+1}}} = -e^{\frac{r^*-t}{2r_s^{d+1}}},$$

with line element

$$ds^2 = 4r_s^{2(d+1)}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) e^{-\frac{r^*}{r_s^{d+1}}} du' dv' + r^2 d\omega^2. \tag{2.23}$$

Defining the higher dimensional analog of the Kruskal-Szekres coordinates, we have

$$u = \frac{1}{2}(u' + v') = e^{\frac{r^*}{r_s^{d+1}}} \cosh\left(\frac{t}{2r_s^{d+1}}\right), \quad v = \frac{1}{2}(u' - v') = e^{\frac{r^*}{r_s^{d+1}}} \sinh\left(\frac{t}{2r_s^{d+1}}\right), \tag{2.24}$$

where now the line element becomes

$$ds^2 = 4r_s^{2(d+1)}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) e^{-\frac{r^*}{r_s^{d+1}}} (du^2 - dv^2) + r^2 d\omega^2. \tag{2.25}$$

Observe that r^* and u, v satisfy $u^2 - v^2 = e^{\frac{r^*}{r_s^{d+1}}} \to 0$ as $r \to r_s$ so that the event horizon is described by the lines $v = \pm u$. The relationship between t and u, v is given by the

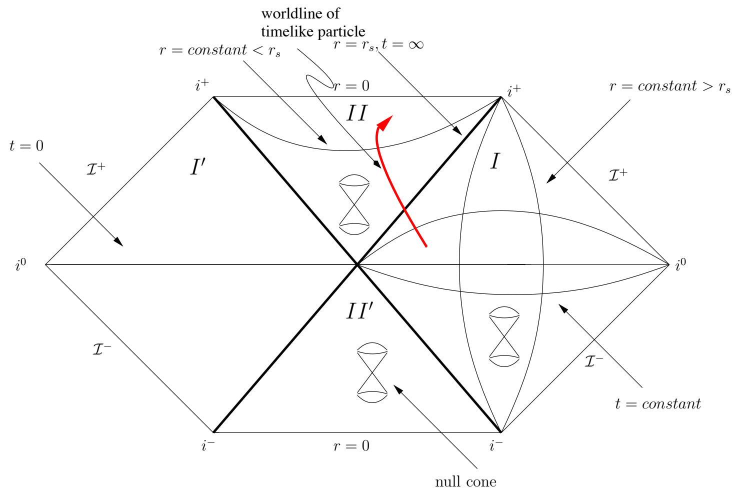

FIGURE 2.2. Penrose diagram of the Schwarzschild spacetime.

equation $\frac{v}{u} = \tanh\left(\frac{t}{2r_s^{d+1}}\right)$. To obtain a closed form of the space, define

$$u'' = \arctan\left(\frac{u'}{\sqrt{r_s^{d+1}}}\right), \quad v'' = \arctan\left(\frac{v'}{\sqrt{r_s^{d+1}}}\right) \tag{2.26}$$

where $u'', v'' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $-\pi < u'' + v'' < \pi$. Then

$$\tan u'' \tan v'' = \frac{1}{r_s^{d+1}} e^{\frac{r^*}{r_s^{d+1}}}, \quad \frac{\tan u''}{\tan v''} = e^{\frac{t}{r_s^{d+1}}}, \tag{2.27}$$

and the Penrose diagram is shown in Figure 2.2.

Regions I and II depict the exterior and interior of the blackhole, respectively. Regions I' and II' constitute the *symmetric universe*, and are physically unrealistic. Region II' is the so called *white hole*, representing a region in which objects can enter our universe but not the reverse whereas Region I' depicts the trajectories of space like, hence unphysical particles. The estimates presented in this dissertation are carried through solely in Region I.

2.3. Kerr Spacetime

We provide a very brief description of the higher dimensional Kerr solution to the vacuum equations, as this is the setting to which we hope to extend the results of this dissertation. The Kerr spacetime is an axial-symmetric, rotating black-hole, with angular momentum a. The solution to Einstein's equations in the Boyer-Lindquist coordinate system is given by

$$ds^2 = -\left(1 - \frac{r_s^{d+1}}{r^{d-1}\rho^2}\right) dt^2 + \frac{2r_s^{d+1}}{r^{d-1}\rho^2} a \sin^2\theta_1 dt d\theta_2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta_1^2$$
$$+ \left(r^2 + a^2 + \frac{r_s^{d+1}}{r^{d-1}\rho^2} a^2 \sin^2\theta_1\right) \sin^2\theta_1 d\theta_2^2 + r^2\cos^2\theta_1 d\omega_{\mathbb{S}^d}^2,$$

where, $\rho^2 = r^2 + a^2\cos^2\theta_1$ and $\Delta = r^2 + a^2 - \frac{r_s^{d+1}}{r^{d-1}}$, [**24**]. The apparent singularity at $\Delta = 0$ is merely due to the chosen coordinates whereas the surface defined by $\rho = 0$ is actually a true singularity, evidenced by the diverging Ricci scalar $R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}$ at this point. Both $r = 0$ and $\theta_1 = \frac{\pi}{2}$ are singular points so that unlike the Schwarzschild spacetime, the "point" of infinite curvature is now a ring. The Kerr metric is also undefined at the roots of the equation $\Delta = 0$. It is precisely these surfaces that determine the event horizon of a black hole and can be found by solving

$$y(r) = r^{d+1} + a^2 r^{d-1} - r_s^{d+1} = 0. \tag{2.28}$$

For $d = 0$, there are precisely two horizons, $r_\pm = \frac{r_s \pm \sqrt{r_s^2 - 4a^2}}{2}$; however for $d \geq 1$, $y(r)$ has only one positive root and thus the black hole necessarily has exactly one horizon [**24**]. A conformal representation of the $(3+1)$-dimensional ($d = 0$) Kerr spacetime is shown in Figure 2.3. Region I depicts the physical universe, while Region II denotes the region between the two horizons. Note that upon passing the first horizon (the boundary of the black hole and the equivalent of $r = r_s$ in Schwarzschild), the radial coordinate reverts from being *spacelike* to *timelike* thus causing the particle to move in the direction of decreasing r. Particles are then necessarily forced to pass the second horizon where r switches back to being *spacelike*. It is therefore possible to avoid the singularity $\rho^2 = 0$, and enter another universe (Region I), or pass the singularity to enter a so called *negative space*, Region III.

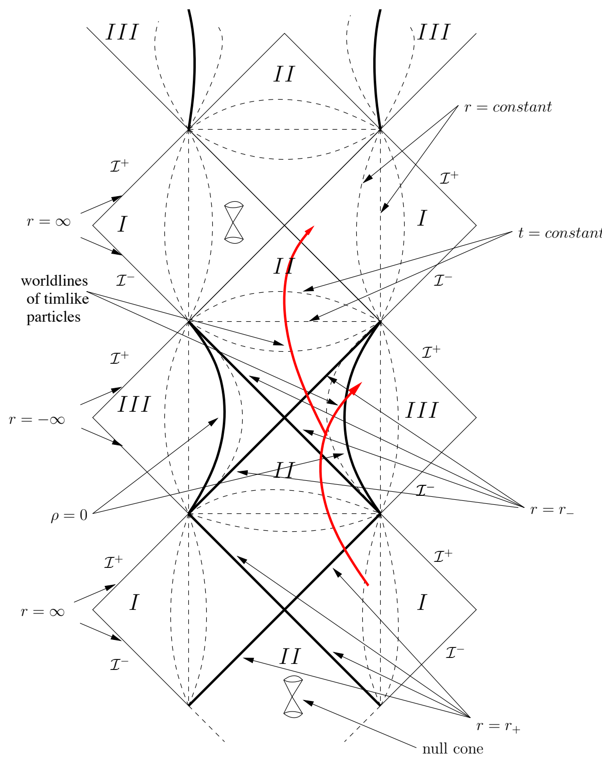

FIGURE 2.3. Conformal diagram of the $(3+1)$-dimensional Kerr spacetime.

Determining the trapped null geodesics of Kerr is far more complicated than in the Schwarzschild solution. It is now necessary to adhere to microlocal analysis, as characterizing these regions requires both space and frequency components. For $d = 0$, we refer the reader to [**53**] and references therein, where it is shown that all trapped null geodesics lie within $O(a)$ of the surface $r = 3M$, the photon sphere for $d = 0$.

CHAPTER 3

Localized Energy Estimates for the Wave Equation in (3 + 1)-dimensions

The localized energy estimates under consideration are the so-called Morwatez type estimates. In this section we summarize a few known results on the $(3+1)$-dimensional Minkowski, Schwarzschild and Kerr spacetimes.

3.1. Minkowski Spacetime

Estimates of this type were first introduced by Morwatez in studying the decay properties of the Klein-Gordon equation [**41**]. As a consequence, if ϕ solves the flat homogeneous wave equation $\Box\phi := \partial_{tt}^2\phi - \Delta\phi = \partial_{tt}^2\phi - \sum_{j=1}^{n}\partial_{jj}^2\phi = 0$, with initial conditions $\phi_0 = \phi(0)$ and $\phi_1 = \partial_t\phi(0)$, then for $n \geq 3$, the estimate

$$\int_0^T \int_{\mathbb{R}^n} \frac{1}{r}|\nabla\!\!\!/\phi(t,x)|^2 dxdt \lesssim ||\nabla\phi_0||_{L^2}^2 + ||\phi_1||_{L^2}^2 \tag{3.1}$$

can be used to show that the solution and the local energy decay to zero as T tends to infinity. Here, the implicit constant is independent of T, $r = |x|$ is the radial component of x and $\nabla\!\!\!/\phi$ denotes the angular derivative of ϕ, defined as $\nabla\!\!\!/_j = \partial_j - \frac{x_j}{r}\partial_r$. This can be proved via the *positive commutator* method, although we remark that Morawetz's original proof relied on clever energy identities. The commutator technique essentially requires determining an appropriate differential multiplier $X = f(r)\partial_r + \frac{n-1}{2}\frac{f(r)}{r}$ and analyzing the space-time integral $\int_0^T\int_{\mathbb{R}^n}(\Box\phi X\phi)dxdt$, [**47**]. Upon integrating by parts we determine conditions imposed on $f(r)$ necessary to obtain a meaningful estimate. In particular, the decay estimate (3.1) can be determined by choosing $f(r) \equiv 1$. By altering the multiplier, that is, choosing appropriate functions $f(r)$, we may generate other such estimates. For example, considering the function $f(r) = \frac{r}{r+2^j}$ and integrating over dyadic annuli $A_j = 2^{j-1} \leq |x| \leq 2^j$, gives

$$(3.2) \qquad ||\langle x\rangle^{-1/2}\nabla_{t,x}\phi\ ||^2_{L^2_{t,x([0,T]\times A_j)}} + ||\langle x\rangle^{-3/2}\phi||^2_{L^2_{t,x([0,T]\times A_j)}} \lesssim ||\nabla\phi_0||^2_{L^2} + ||\phi_1||^2_{L^2},$$

where $\langle x\rangle$ denotes the Japanese bracket $\sqrt{|x|^2+1}$. An estimate over all of $\mathbb{R}^n$ is achieved by including a summability factor, and hence comes at the cost of replacing the terms $\langle x\rangle^{-1/2}$ and $\langle x\rangle^{-3/2}$ by the weights $\langle x\rangle^{-1/2-\delta}$ and $\langle x\rangle^{-3/2-\delta}$, respectively. Otherwise, allowing for a logarithmic blow up in time, we obtain

$$(3.3) \quad (\log(2+T))^{-1}\left(||\langle x\rangle^{-1/2}\nabla_x\phi\ ||^2_{L^2_{t,x([0,T]\times\mathbb{R}^n)}} + ||\langle x\rangle^{-3/2}\phi||^2_{L^2_{t,x([0,T]\times\mathbb{R}^n)}}\right) \lesssim ||\nabla\phi_0||^2_{L^2} + ||\phi_1||^2_{L^2}.$$

By introducing an appropriate dual norm we may generalize the hypothesis and consider solutions to the inhomogeneous wave equation, $\Box\phi = F$. A more detailed account is provided in the proof of the dissertation result.

Variations of the above estimates can be found in [**27**], [**29**], [**28**], [**35**], [**45**],[**47**], [**48**]. For the variable coefficient wave equation, see for example [**1**], [**37**], [**38**], [**39**], and [**13**], [**14**], for time-dependent and independent perturbations, respectively.

The applications of localized energy estimates have been widespread, having played a role in establishing long-time existence to the semi- and quasi-linear wave equations, e.g. [**28**], [**37**], in proving global Strichartz estimates e.g. [**14**], [**36**], [**45**] and in scattering theory e.g. [**42**], [**48**].

3.2. Schwarzschild spactime

In developing the same type of theory for the black hole setting, localized energy estimates have also been studied on the $(3+1)$-dimensional Schwarzschild spacetime. Such estimates were originally established for the Schrödinger equation with radially symmetric data, [**32**], and thereafter for the wave equation, [**6**] -[**8**], [**18**], [**20**], [**34**]. Conformal Morawetz type estimates are shown in [**12**] and [**17**]; these are estimates analogous to Morawetz conformal estimates on Minkowski space-time, obtained by using the multiplier $(t^2 + r^2)\partial_t + 2tr\partial_r$.

The approach used in establishing localized energy estimates in [**6**], [**7**], [**17**], [**16**], [**32**] is to decompose the wave equation into spherical harmonics, determine a multiplier on each harmonic, l, and then sum over all l. In contrast, the technique exploited in [**20**] and [**34**] refrains from a spherical harmonic decomposition so as to avoid potential summability issues. Consequently, only one multiplier is required.

Recall from Chapter 2 that the event horizon, $r = r_s$, and the photon sphere, $r = r_{ps}$, a priori impose difficulties in measuring the dispersion of waves. Photons may potentially remain on these surfaces and not propagate out to infinity. On the photon sphere, the instability of the rays here does permit energy decay, albeit with a loss. Specifically, in [**6**] -[**8**] and [**17**], a quadratic polynomial loss is established at this region. However, it is shown in [**34**] that the estimate near the photon sphere can be improved to a logarithmic loss. By using Regge-Wheeler coordinates, decomposing the solution into spherical harmonics and performing the Fourier transform in the time variable, one obtains an ODE modeled by $\phi'' + \lambda^2(x^2 \pm \varepsilon)\phi = f$. A WKB approximation gives an asymptotic expansion of solutions to this ODE which in turn leads to the logarithmic energy loss at the photon sphere.

The trapped rays on the event horizon also pose a potential problem, however the red shift effect renders this trapping as negligible [**18**]. The logarithmic loss here, shown in [**6**] -[**8**] and [**17**], is improved in [**34**] whereby using a non-degenerate energy, changing coordinates and exploiting gravitational redshift [**17**], the estimate is extended past the event horizon, although not so far as to the singularity. These results are used in showing global Strichartz estimates of solutions to the wave equation on Schwarzschild. More recently, the localized energy estimates determined in [**34**] have been used in establishing the conjectured *Price's Law*, which states that solutions to the wave equation have a pointwise decay rate of $|t|^{-3}$, [**51**].

3.3. Kerr spacetime

Dispersive estimates on Kerr are very much a newly developing theory. Estimates for solutions with small angular momentum $a \ll M$ have been established, as here the Kerr spacetime is considered a small perturbation of Schwarzschild; see [**3**], [**19**]-[**22**], [**52**], [**53**]. The increased difficulty in the problem is due to the fact that a single differential multiplier is no longer sufficient to obtain an estimate, and one must resort to two tensors or pseudodifferential operators [**2**], [**3**], [**17**]. Furthermore, recall from Chapter 2 that the trapped sets are also more complicated; we must adhere to microlocal analysis, as both spatial and frequency components are necessary to classify them.

CHAPTER 4

Higher Dimensional Localized Energy Estimate for the Wave Equation on the Schwarzschild Spacetime

Here we prove the main result of the dissertation, Theorem 1.1, stated in the Introduction. The technique we pursue is the so called *Positive Commutator* method. Recall that the wave equation can be considered as the critical point of the Lagrangian functional $\int_0^T \int_{\mathbb{S}^{d+2}} \int_{r\geq r_s} L(\phi, g) r^{d+2} dr d\omega dt$, where L is the *Lagrange density* or *Lagrangian* given by $L = g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$. Associated to L, we may define the *stress-energy momentum tensor*, $Q_{\alpha\beta} := \frac{\partial L}{\partial g^{\alpha\beta}} - \frac{1}{2}g_{\alpha\beta}L$, which becomes

$$Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi \tag{4.1}$$

for the specific choice of L defined above. One of the most important features of this tensor is $\nabla^\alpha Q_{\alpha\beta} = \Box_g\phi X\phi$, and in particular, it is divergence free if $\Box_g\phi = 0$, [**50**]. Contracting $Q_{\alpha\beta}$ with the radial multiplier $X = \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) f(r)\partial_r$, we form the momentum density $P_\alpha[\phi, X] = Q_{\alpha\beta}[\phi]X^\beta$, with divergence

$$\begin{aligned}
\nabla^\alpha P_\alpha[\phi, X] &= \nabla^\alpha Q_{\alpha\beta}[\phi]X^\beta + Q_{\alpha\beta}[\phi]\nabla^\alpha X^\beta \\
&= \Box_g\phi X\phi + Q_{\alpha\beta}[\phi]\pi^{\alpha\beta} \\
&= \Box_g\phi X\phi + f'(r)\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)^2 (\partial_r\phi)^2 + \Big(\frac{r^{d+1} - r_{ps}^{d+1}}{r^{d+1}}\Big)\frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 \\
&\quad - \frac{1}{2}\Big[\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r\big(r^{d+2}f(r)\big)\Big]\partial^\gamma\phi\partial_\gamma\phi.
\end{aligned} \tag{4.2}$$

Here $\pi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)$ denotes the deformation tensor of X. The theorem will follow by taking the space time integral of the divergence of appropriate momentum density tensors. We will divide our analysis in to considering the spatial derivatives and lower order terms separate from the time derivative term in the theorem.

4.0.1. $(\partial_r\phi)^2, |\nabla\!\!\!/\phi|^2, \phi^2$ ***terms:.*** In equation (4.2), we observe that the last term consisting of the Lagrangian is unsigned. However, as the name of the technique suggests, we ideally seek positive terms. Consequently we define

$$\tilde{P}_\alpha[\phi, X] = P_\alpha[\phi, X] + \frac{1}{2}\Big[\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\Big]\phi\partial_\alpha\phi \\ - \frac{1}{4}\partial_\alpha\Big[\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\Big]\phi^2,$$

and recompute the divergence

$$\begin{aligned}\nabla^\alpha \tilde{P}_\alpha[\phi, X] = \Box_g\phi\Big[X\phi + \frac{1}{2}\Big\{\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r(f(r)r^{d+2})\Big\}\phi\Big] \\ +\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)^2 f'(r)(\partial_r\phi)^2 + \Big(\frac{r^{d+1} - r_{ps}^{d+1}}{r^{d+1}}\Big)\frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 \\ -\frac{1}{4}\nabla^\alpha\partial_\alpha\Big[\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)r^{-(d+2)}\partial_r(f(r)r^{d+2})\Big]\phi^2.\end{aligned} \tag{4.3}$$

With an appropriate choice of the radial function $f(r)$, the theorem follows by taking the space-time integral of equation (4.3) and bounding the resulting expression by the initial energy and dual localized energy norm. Specifically, upon integrating over the space-time slab $[0, T] \times (r_s, \infty) \times \mathbb{S}^{d+2}$ and applying the divergence theorem, (4.3) becomes

(4.4)

$$-\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s} f(r)\partial_t\phi\partial_r\phi\, r^{d+2}\, dr\, d\omega\Big|_0^T - \frac{1}{2}\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\frac{1}{r^{d+2}}\partial_r(f(r)r^{d+2})\phi\partial_t\phi\, r^{d+2}\, dr\, d\omega\Big|_0^T \\ - \int_0^T\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\Box_g\phi\left(X\phi + \frac{1}{2}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)r^{-(d+2)}\partial_r(f(r)r^{d+2})\phi\right)r^{d+2}drd\omega dt = \\ \int_0^T\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^2 f'(r)(\partial_r\phi)^2 + \left(\frac{r^{d+1} - r_{ps}^{d+1}}{r^{d+1}}\right)\frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 + l(f)\phi^2 r^{d+2}drd\omega dt,$$

where we have set

$$l(f) = -\frac{1}{4r^{d+2}}\partial_r\left(r^{d+2}\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)\partial_r\left(\frac{1}{r^{d+2}}\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)\partial_r(r^{d+2}f(r))\right)\right).$$

We remark that the above integration was conducted under the assumption that $f(r)$ is $C^2(\mathbb{R})$. Ideally, we would like to choose $f(r)$ such that in addition to this regularity condition, $f(r)$ is bounded and all three terms in the integrand of the right hand side of (4.4) are non-negative. It appears, however, that this is not possible and we sacrifice the regularity of $f(r)$. In doing so, we pick up boundary terms that must be controlled. Moreover, with the choice of $f(r)$ described below we find that $l(f)\phi^2$ is not signed. We instead show that $\int\int\int l(f)\phi^2 r^{d+2} dr d\omega dt$ is bounded below by a positive term minus a fraction of the $(\partial_r\phi)^2$ term and a small radial boundary term.

We naively begin our construction by defining

$$g(r) = \frac{r^{d+2} - r_{ps}^{d+2}}{r^{d+2}} \quad \text{and} \quad h(r) = \log\left(\frac{r^{d+1} - r_s^{d+1}}{\frac{d+1}{2} r_s^{d+1}}\right),$$

and setting the multiplier to equal

$$g(r) + \frac{A r_s^{d+1}}{r^{d+2}} h(r),$$

for a constant $A > 0$ to be determined. We find that in addition to $h(r)$ being unbounded at r_s, ensuring the appropriate signs for $f'(r)$ and $l(f)$ leads to contradicting restrictions on the choice of A. Despite these fallacies, we use this as a base in constructing the multiplier. Note that the main source of the issues arise from the logarithmic term, $\dfrac{A}{r^{d+2}} h(r)$ at r_s and at infinity. Consequently, the idea we pursue is to smooth out $h(r)$ at these two regions.

We set

$$a(x) = \begin{cases} -\frac{1}{\varepsilon}\frac{\varepsilon x+1}{\delta(\varepsilon x+1)-1} - \frac{1}{\varepsilon}, & x \le -\frac{1}{\varepsilon} \\ x, & -\frac{1}{\varepsilon} \le x \le 0 \\ x - \frac{2}{3\alpha^2}x^3 + \frac{1}{5\alpha^4}x^5, & 0 \le x \le \alpha \\ \frac{8\alpha}{15}, & x \ge \alpha \end{cases}$$

for $\alpha = 5-\delta_0$, and $0<\delta_0,\varepsilon \ll 1$. Note that except for the discontinuity of a'' at $x=-\frac{1}{\varepsilon}$ a is C^2. We use this as a smoothing function and define the bounded function

$$f(r) = g(r) + \frac{Ar_s^{d+1}}{r^{d+2}}a(h(r)), \tag{4.5}$$

where $A = \frac{d+2}{d+3}r_{ps}$. The choice of A and α will be described in the calculations below. To simplify notation, we let r_θ be the value of r such that $h(r_\theta)=\theta$, ie. $r_\theta = r_s\left(\frac{d+1}{2}e^\theta+1\right)^{\frac{1}{d+1}}$. Thus $f''(r)$ is discontinuous at $r_{-\frac{1}{\varepsilon}}$ so that in integrating (4.3), we must add the positive term $\frac{1}{4}r_{-\frac{1}{\varepsilon}}^{d+2}\left(1-\frac{r_s^{d+1}}{r_{-\frac{1}{\varepsilon}}^{d+1}}\right)^2\left(f''(r^-_{-\frac{1}{\varepsilon}})-f''(r^+_{-\frac{1}{\varepsilon}})\right)\int_0^T\int_{\mathbb{S}^{d+2}}\phi^2(t,r_{-\frac{1}{\varepsilon}},\omega)d\omega dt$ or equivalently,

$$\frac{Ar_s^{d+1}(d+1)^2}{2}\frac{\delta\epsilon}{r^2_{-1/\varepsilon}}\int_0^T\int_{\mathbb{S}^{d+2}}\phi^2(t,r_{-\frac{1}{\varepsilon}},\omega)d\omega dt \tag{4.6}$$

to the right hand side of equation (4.4).

The choice of a lends itself in to dividing the analysis in to four cases:

Case 1. $r\in(r_s, r_{-\frac{1}{\varepsilon}}]$

Case 2. $r\in[r_{-\frac{1}{\varepsilon}}, r_{ps}]$

Case 3. $r\in[r_{ps}, r_\alpha]$

Case 4. $r\in[r_\alpha,\infty)$.

For each case, we readily show

- $f(r)>0$ for $r>r_{ps}$ and $f(r)<0$ for $r<r_{ps}$
- $f'(r)>0$ for $r>r_s$.

The difficulty in our analysis comes primarily in the coefficient of the lower order term, ϕ^2. With additional effort, we prove

- in *Cases 2-4*, $l(f) \geq 0$
- in *Case 1*, for $0 < \kappa_1 < 1, \kappa_2 > 0$ and $p(r) > 0$,

$$\int\int\int l(f)\phi^2 r^{d+2} dr d\omega dt \geq \int\int\int p(r)\phi^2 - \kappa_1\Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)^2 f'(r)(\partial_r\phi)^2 r^{d+2} dr d\omega dt$$
$$- \kappa_2 \int\int \delta\epsilon\phi^2(r_{-1/\varepsilon}) d\omega dt,$$

where the radial boundary term above will be controlled by the commutator in *Case 2*.

This will show that the right hand side of (4.4) is non-negative as desired, and it will remain to show the time boundary terms are dominated by the energy $E[\phi](t)$. The latter will be proven in subsection 4.0.3. We proceed case by case:

Case 1: $r \in (r_s, r_{-1/\varepsilon}]$

The multiplier in this region was so constructed to smooth out the logarithmic blowup of $f(r)$ at the event horizon. For r in this region, $h(r) \leq -\frac{1}{\varepsilon}$ so that $a(h(r)) < 0$. Thus $f(r) < 0$ and

$$f'(r) = \frac{(d+2)r_{ps}^{d+2}}{r^{d+3}} - \frac{Ar_s^{d+1}(d+2)}{r^{d+3}} a(h(r)) + \frac{Ar_s^{d+1}}{r^{d+2}} \frac{(d+1)r^d}{\big(\delta(\epsilon h(r)+1)-1\big)^2(r^{d+1}-r_s^{d+1})} > 0 \tag{4.7}$$

as desired. Next we consider $l(f)$,

$$l(g(r)) + l\Big(\frac{Ar_s^{d+1}}{r^{d+2}}a(h(r))\Big) = \frac{d+2}{4r^{2d+5}}\Big(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}\Big) \tag{4.8}$$
$$+ \frac{Ar_s^{d+1}(d+1)(d+3)}{2}\frac{(r_{ps}^{d+1} - r^{d+1})}{r^{2d+6}}a'(h(r))$$
$$+ \frac{Ar_s^{d+1}(d+1)^2(d+5)}{4}\frac{1}{r^{d+5}}a''(h(r))$$
$$- \frac{Ar_s^{d+1}(d+1)^3}{4}\frac{1}{r^4\big(r^{d+1} - r_s^{d+1}\big)}a'''(h(r)).$$

Given that $a'(h(r)) = (\delta(\varepsilon h(r) + 1) - 1)^{-2} > 0$ and $r < r_{-1/\varepsilon} < r_{ps}$, the second term in the right is positive. The third term in the right of (4.8) also has the desired sign as $a''(h(r)) = -2\delta\varepsilon/(\delta(\varepsilon h(r) + 1) - 1)^3$ and $h(r) \leq -1/\varepsilon$ here. Observe that $l(g) < 0$ in this range; however, we will use its boundedness property to show it can be controlled for ϵ sufficiently small. We thus concentrate our efforts on the last term. Beginning with a straightforward application of the Fundamental Theorem of Calculus, we find

$$\int_{r_s}^{r_{-1/\epsilon}} \partial_r \left(\frac{2Ar_s^{d+1}\delta\epsilon(d+1)^2}{r^2(R(r))^3}\phi^2(r)\right) dr = -\frac{2Ar_s^{d+1}\delta\epsilon(d+1)^2}{r_{-1/\varepsilon}^2}\phi^2(r_{-1/\varepsilon}), \tag{4.9}$$

where we have introduced $R(r) = (\delta(\varepsilon h(r)+1)-1)$ and suppressed the t and ω dependence of ϕ, to simplify notation. Calculating the derivative of the integrand yields

$$\int_{r_s}^{r_{-1/\varepsilon}} \frac{6Ar_s^{d+1}\delta^2\varepsilon^2(d+1)^3}{r^2(R(r))^4}\frac{r^d}{r^{d+1} - r_s^{d+1}}\phi^2\, dr \tag{4.10}$$
$$= -\int_{r_s}^{r_{-1/\varepsilon}} \frac{4Ar_s^{d+1}\delta\varepsilon(d+1)^2}{r^3(R(r))^3}\phi^2\, dr + \int_{r_s}^{r_{-1/\varepsilon}} \frac{4Ar_s^{d+1}\delta\varepsilon(d+1)^2}{r^2(R(r))^3}\phi\partial_r\phi\, dr$$
$$+ \frac{2Ar_s^{d+1}\delta\varepsilon(d+1)^2}{r_{-1/\varepsilon}^2}\phi(r_{-1/\varepsilon})^2.$$

We apply the Cauchy Schwarz inequality to the second term to obtain

$$\int_{r_s}^{r_{-1/\varepsilon}} \frac{4Ar_s^{d+1}\delta\epsilon(d+1)^2}{r^2(R(r))^3}\phi\partial_r\phi dr \leq \frac{4}{3}\int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)(r^{d+1} - r_s^{d+1})}{r^{d+2}(R(r))^2}(\partial_r\phi)^2 dr \tag{4.11}$$
$$+ 3\int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}\delta^2\epsilon^2(d+1)^3}{r^2(R(r))^4}\frac{r^d}{r^{d+1} - r_s^{d+1}}\phi^2 dr.$$

In combining the results of (4.10) and (4.11), we find

$$\int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^3}{4} \frac{1}{r^4(r^{d+1}-r_s^{d+1})} a'''(h(r))\phi^2 r^{d+2} dr \leq \tag{4.12}$$
$$\int_{r_s}^{r_{-1/\varepsilon}} \frac{2Ar_s^{d+1}(d+1)}{3} \frac{(r^{d+1}-r_s^{d+1})}{r^{d+2}(R(r))^2} (\partial_r\phi)^2 dr + \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^2}{r^3} a''(h(r))\phi^2 dr$$
$$+ \frac{\delta\epsilon Ar_s^{d+1}(d+1)^2}{r_{-1/\varepsilon}^2} \phi^2(r_{-1/\varepsilon}),$$

so that for $\gamma > 0$,

$$\left(\frac{1}{4}+\gamma\right) \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^3}{r^4(r^{d+1}-r_s^{d+1})} a'''(h(r))\phi^2 r^{d+2} dr \leq \tag{4.13}$$
$$\left(\frac{2}{3}+\frac{8\gamma}{3}\right) \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)}{r^{d+2}} \frac{(r^{d+1}-r_s^{d+1})}{(R(r))^2} (\partial_r\phi)^2 dr$$
$$+ (1+4\gamma) \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^2}{r^3} a''(h(r))\phi^2 dr + (1+4\gamma) \frac{Ar_s^{d+1}\delta\epsilon(d+1)^2}{r_{-1/\varepsilon}^2} \phi^2(r_{-1/\varepsilon}).$$

Dropping the positive $a'(h(r))$ term in (4.8) and applying the results of (4.13) we have,

$$\int_{r_s}^{r_{-1/\varepsilon}} l(f)\phi^2 r^{d+2} dr d\omega dt \geq \tag{4.14}$$
$$\int_{r_s}^{r_{-1/\varepsilon}} l(g)\phi^2 r^{d+2} dr d\omega dt + \gamma \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^3}{r^4(r^{d+1}-r_s^{d+1})} a'''(h(r))\phi^2 r^{d+2} dr$$
$$+ \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^2}{r^3} \left(\frac{d+5}{4} - (1+4\gamma)\right) a''(h(r))\phi^2 dr$$
$$- \left(\frac{2}{3}+\frac{8\gamma}{3}\right) \int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)}{r^{d+2}} \frac{(r^{d+1}-r_s^{d+1})}{(R(r))^2} (\partial_r\phi)^2 dr$$
$$- (1+4\gamma) \frac{\delta\epsilon Ar_s^{d+1}(d+1)^2}{r_{-1/\varepsilon}^2} \phi^2(r_{-1/\varepsilon}).$$

We choose γ such that $\left(\frac{2}{3}+\frac{8\gamma}{3}\right) < 1 \Rightarrow \gamma < \frac{1}{8}$. As $\frac{d+5}{4} \geq \frac{6}{4}$, for $d \geq 1$, this choice of γ automatically implies that the a'' term yields a positive contribution. Moreover, observe that $a'''(h(r)) = \frac{6\delta^2\epsilon^2}{(R(r))^4}$ so that for ϵ sufficiently small, the corresponding term can be made large enough to control $l(g)$. For concreteness, we choose ϵ so that after absorbing $l(g)$, one half of the a''' term remains. That is,

(4.15)

$$\int_{r_s}^{r_{-1/\varepsilon}} l(f)\phi^2 r^{d+2} dr d\omega \geq -\int_{r_s}^{r_{-1/\varepsilon}} \left(\frac{2}{3}+\frac{8\gamma}{3}\right)\frac{Ar_s^{d+1}(d+1)}{r^{d+2}}\frac{(r^{d+1}-r_s^{d+1})}{(R(r))^2}(\partial_r\phi)^2 dr$$
$$+\frac{\gamma}{2}\int_{r_s}^{r_{-1/\varepsilon}} \frac{Ar_s^{d+1}(d+1)^3}{r^4(r^{d+1}-r_s^{d+1})}a'''(h(r))\phi^2 r^{d+2} dr - (1+4\gamma)\frac{\delta\epsilon A r_s^{d+1}(d+1)^2}{r_{-1/\varepsilon}^2}\phi^2(r_{-1/\varepsilon}).$$

γ was chosen such that the sum of the first term on the right of (4.4) with the first term on the right of (4.15) is positive. Thus, in combining the boundary term above with (4.6), it remains to control the term

$$-\left(\frac{1}{2}+4\gamma\right)\frac{Ar_s^{d+1}\delta\epsilon(d+1)^2}{r_{-1/\varepsilon}^2}\int_0^T\int_{\mathbb{S}^{d+2}}\phi^2(r_{-1/\varepsilon})d\omega dt. \tag{4.16}$$

Boundary term at $r_{-1/\varepsilon}$:

We show that (4.16) can be dominated by the right hand side of (4.4) for $r \in [r_{-1/\varepsilon}, r_{ps}]$, whose positivity we confirm immediately afterwards. For ε chosen in *Case (i)*, we choose a parameter δ sufficiently small so as to control (4.16).

Define the smooth cut-off function $\beta(r) = \begin{cases} 1, & r \in (r_s, r_{-1/\varepsilon}] \\ 0, & ,r \geq r_{ps}. \end{cases}$

Then by the Fundamental Theorem of Calculus and applications of Cauchy Schwarz, the following sequence of equations holds for $r \leq r_{-1/\varepsilon}$:

$$\begin{aligned}
\phi(r)^2 = &\left(\int_r^{r_{ps}} \partial_s(\beta(s)\phi(s))ds\right)^2 = \left(\int_r^{r_{ps}} \beta'\phi ds + \int_r^{r_{ps}} \beta\phi' ds\right)^2 \\
&\leq 2\left(\int_r^{r_{ps}} |\beta'| ds\right)\left(\int_r^{r_{ps}} |\beta'|\phi^2(s) ds\right) + \\
&\qquad 2\left(\int_r^{r_{ps}} \frac{\beta(s)}{s^{d+1}-r_s^{d+1}} ds\right)\left(\int_r^{r_{ps}} \beta(s)(s^{d+1}-r_s^{d+1})(\partial_s\phi)^2 ds\right) \\
&\lesssim \int_r^{r_{ps}} |\beta'|\phi^2 ds - h(r)\int_r^{r_{ps}} \beta(s)(s^{d+1}-r_s^{d+1})(\partial_s\phi)^2 ds.
\end{aligned}$$

Applying this to $r = r_{-1/\varepsilon}$ and using equation (4.3), we have

$$\frac{Ar_s^{d+1}\delta\epsilon}{r_{-1/\varepsilon}^2}\phi^2(r_{-1/\varepsilon}) \lesssim \delta\int_{r_{-1/\varepsilon}}^{r_{ps}} \nabla^\alpha \bar{P}_\alpha[\phi, X]dr,$$

where the implicit constant is independent of ε. We will show next that the commutator in the region $[r_{-1/\varepsilon}, r_{ps}]$ is in fact positive. Thus, integrating over $[0,T]\times\mathbb{S}^{d+2}$, we find that the boundary term can be bootstrapped into the terms in *Case 2* for δ sufficiently small.

Case 2: $r \in [r_{-1/\varepsilon}, r_{ps}]$

In this region, the smoothing function is the identity and we have

$$\begin{aligned} f(r) &= \frac{r^{d+2} - r_{ps}^{d+2}}{r^{d+2}} + \frac{Ar_s^{d+1}}{r^{d+2}}\ln\left(\frac{r^{d+1} - r_s^{d+1}}{\frac{d+1}{2}r_s^{d+1}}\right) \quad \text{and} \\ f'(r) &= \frac{(d+2)r_{ps}^{d+2}}{r^{d+3}} + \frac{Ar_s^{d+1}(d+1)}{r^{d+2}}\frac{r^d}{r^{d+1} - r_s^{d+1}} - \frac{Ar_s^{d+1}(d+2)}{r^{d+3}}\ln\left(\frac{r^{d+1} - r_s^{d+1}}{\frac{d+1}{2}r_s^{d+1}}\right). \end{aligned}$$

The logarithmic term is non-positive for $r \leq r_{ps}$, thus we easily see that $f(r) \leq 0$ and $f'(r) \geq 0$. It remains to show that $l(f)$ is positive. Evaluating this expression we find

$$\begin{aligned} l(f) = \frac{(d+2)}{4r^{2d+5}}&(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2(d+1)}) \\ &- \frac{Ar_s^{d+1}}{4r^{2d+6}}(d+1)(d+3)(2r^{d+1} - (d+3)r_s^{d+1}). \end{aligned}$$

We apply $r \leq r_{ps}$ to the second term to give

$$\begin{aligned} l(f) \geq \frac{1}{4r^{2d+5}}\Bigg(&(d+2)(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2(d+1)}) \\ &-\frac{Ar_s^{d+1}}{r_{ps}}(d+1)(d+3)(2r^{d+1} - (d+3)r_s^{d+1})\Bigg). \end{aligned} \tag{4.17}$$

Let $\beta \in [r_s, r_{ps}]$ denote the positive root of $dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}$. Then for $r > \beta$, $l(g) > 0$. Moreover, if $r \in [r_s, \beta]$, the right hand side of (4.17) is non-negative if

$$A \geq \sup_{[r_s,\beta]} \frac{-r_{ps}(d+2)\left(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}\right)}{-(d+3)(d+1)r_s^{d+1}\left(2r^{d+1} - (d+3)r_s^{d+1}\right)}$$

or equivalently,

(4.18)

$$A \geq \sup_{[r_s,\beta]} \frac{(d+2)r_{ps}}{(d+3)(d+1)r_s^{d+1}} \left[\frac{d}{2}r^{d+1} + \frac{(d+2)(d+3)r_s^{d+1}}{4} + \frac{(d+2)(d+1)^2 r_s^{2d+2}}{4(2r^{d+1} - (d+3)r_s^{d+1})}\right].$$

We find the derivative of the quantity in brackets to be $\frac{(d+1)r^d}{2}\left(d - \frac{(d+1)^2(d+2)r_s^{2d+2}}{(2r^{d+1}-(d+3)r_s^{d+1})^2}\right) < 0$. Thus the right hand side of (4.18) is decreasing so that the supremum occurs at $r = r_s$. This implies $A \geq \frac{d+2}{d+3} r_{ps}$; the lower bound of this inequality is our choice of A.

Case 3: $r \in [r_{ps}, r_\alpha]$

In this region, $a(x) = x - \frac{2}{3\alpha^2}x^3 + \frac{1}{5\alpha^4}x^5$ and f and f' are respectively,

$$f(r) = \frac{r^{d+2} - r_{ps}^{d+2}}{r^{d+2}} + \frac{Ar_s^{d+1}}{r^{d+2}} \frac{h(r)}{15\alpha^4}\left(3h(r)^4 - 10h(r)^2\alpha^2 + 15\alpha^4\right) \text{ and} \tag{4.19}$$

$$f'(r) = \frac{(d+2)r_{ps}^{d+2}}{r^{d+3}} - \frac{A(d+2)r_s^{d+1}}{r^{d+3}} a(h(r)) + \frac{Ar_s^{d+1}(d+1)}{r^2\left(r^{d+1} - r_s^{d+1}\right)} \frac{(h(r)^2 - \alpha^2)^2}{\alpha^4}. \tag{4.20}$$

$f(r)$ can easily seen to be non-negative when we view the quantity in parenthesis as a non-negative polynomial in $h(r)$. Moreover, the range of r corresponds to $0 \leq h(r) \leq \alpha$, so that using that a has a maximum value of $\frac{8\alpha}{15}$, the sum of the first two terms in (4.20) is bounded below by $\frac{(d+2)r_s^{d+1}r_{ps}}{r^{d+3}}\left(\frac{d+3}{2} - \frac{8\alpha}{15}\frac{d+2}{d+3}\right)$. It is straightforward to verify this is positive for $\alpha < 5 \leq \frac{15}{16}\frac{(d+3)^2}{d+2}$; the last inequality follows from observing that $\frac{15}{16}\frac{(d+3)^2}{d+2}$ is smallest at $d = 1$. Hence the choice $\alpha = 5 - \delta_0$, for $0 < \delta_0 \ll 1$.

Finally we verify $l(f) > 0$. We begin by calculating

$$\begin{aligned} l(f) = \frac{d+2}{4r^{2d+5}}\Big(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}\Big) \\ + \frac{(d+2)(d+1)r_s^{d+1}r_{ps}}{4(d+3)r^{2d+6}(r^{d+1}-r_s^{d+1})}\Big(-2(d+3)(r^{d+1}-r_{ps}^{d+1})(r^{d+1}-r_s^{d+1})a'(h(r)) \\ + (d+1)(d+5)r^{d+1}(r^{d+1}-r_s^{d+1})a''(h(r)) \\ - (d+1)^2 r^{2d+2}a'''(h(r))\Big). \end{aligned}$$

Establishing positivity of the above expression is equivalent to showing

$$p(r) + n_1(r) + n_2(r) + n_3(r) > 0$$

where,

$$\begin{aligned} p(r) &= r(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}) \\ n_1(r) &= -r_{ps}r_s^{d+1}(d+1)(2r^{d+1} - (d+3)r_s^{d+1})\frac{(h(r)^2-\alpha^2)^2}{\alpha^4} \\ n_2(r) &= r_{ps}r_s^{d+1}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}\frac{4h(r)(h(r)^2-\alpha^2)}{\alpha^4} \\ n_3(r) &= r_{ps}r_s^{d+1}\frac{(d+1)^3}{d+3}\frac{r^{2d+2}}{r^{d+1}-r_s^{d+1}} \cdot 4\frac{\alpha^2-3h(r)^2}{\alpha^4}. \end{aligned}$$

The dominant term is $p(r)$, and we shall show

$$\frac{1}{3}p(r) + n_1(r) > 0, \tag{4.21}$$

$$\frac{1}{2}p(r) + n_2(r) \geq 0, \tag{4.22}$$

$$\frac{1}{6}p(r) + n_3(r) \geq 0. \tag{4.23}$$

In all three cases we use that $r \geq r_{ps}$, and for the first two inequalities, that $|h(r)^2 - \alpha^2|$ is maximized when $h(r) = 0$.

Proof of (4.21)*:*

$$\begin{aligned}\frac{1}{3}p(r)+n_1(r) \geq \frac{1}{3}r_{ps}(dr^{2d+2}+(d+3)r_s^{d+1}r^{d+1}-(d+2)^2r_s^{2d+2})\\ -(d+1)r_{ps}r_s^{d+1}(2r^{d+1}-r_s^{d+1}(d+3))\\ =\frac{1}{3}r_{ps}\left(dr^{2d+2}-(5d+3)r_s^{d+1}r^{d+1}+(2d^2+8d+5)r_s^{2d+2}\right)\\ =\frac{1}{3}r_{ps}\left(d\left(r^{d+1}-\frac{5d+3}{2d}r_s^{d+1}\right)^2+\frac{1}{4d}(d+1)^2(8d-9)r_s^{2d+2}\right).\end{aligned}$$

This last quantity is easily seen to be positive for $d>1$. For the case $d=1$,

$$\begin{aligned}\frac{1}{3}p(r)+n_1(r) &\geq \frac{1}{3}r(r^4+4r_s^2r^2-9r_s^4)-2\sqrt{2}r_s^3(2r^2-4r_s^2)\\ &=\frac{1}{3}\Big(3\sqrt{2}r_s^5-13r_s^4(r-\sqrt{2}r_s)+20\sqrt{2}r_s^3(r-\sqrt{2}r_s)^2+24r_s^2(r-\sqrt{2}r_s)^3+\\ &\qquad 5\sqrt{2}r_s(r-\sqrt{2}r_s)^4+(r-\sqrt{2}r_s)^5\Big)\\ &\geq \frac{1}{3}\left(3\sqrt{2}r_s^5-13r_s^4(r-\sqrt{2}r_s)+20\sqrt{2}r_s^3(r-\sqrt{2}r_s)^2\right)\end{aligned}$$

where we have done a Taylor expansion about the photon sphere, $\sqrt{2}r_s$, to the right hand side of the first inequality. The above quadratic has no real roots thus proving the result for $d=1$.

Proof of (4.22)*:*

(4.24)

$$\begin{aligned}\frac{1}{2}p(r)+n_2(r) \geq&\frac{1}{2}r_{ps}(dr^{2d+2}+(d+3)r_s^{d+1}r^{d+1}-(d+2)^2r_s^{2d+2})-\\ &\qquad \frac{4r_{ps}r_s^{d+1}}{\alpha^2}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}h(r)\\ =&\frac{1}{2}r_{ps}\Big(d(r^{d+1}-r_s^{d+1})^2+3(d+1)r_s^{d+1}(r^{d+1}-r_s^{d+1})-(d+1)^2r_s^{2d+2}\\ &\qquad -\frac{8r_s^{d+1}}{\alpha^2}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}h(r)\Big).\end{aligned}$$

Setting $x = h(r)$, or equivalently, $r^{d+1} - r_s^{d+1} = \frac{d+1}{2} r_s^{d+1} e^x$, we have that for $0 \leq x \leq \alpha$, the right hand side (4.24) becomes

$$q(x) := \frac{1}{2} r_{ps} r_s^{2d+2} (d+1)^2 \left[\frac{d}{4} e^{2x} + \frac{3}{2} e^x - 1 - \frac{4}{\alpha^2} \frac{(d+5)(d+1)}{d+3} x e^x - \frac{8}{\alpha^2} \frac{d+5}{d+3} x \right].$$

Note that $q(0) = \frac{r_{ps} r_s^{2d+2} (d+1)^2 (d+2)}{8}$, thus it suffices to show that $q'(x) > 0$ for $x \in [0, \alpha]$. Calculating this quantity for $\alpha = 5$, we find

$$\begin{aligned} q'(x) &= \frac{1}{2} r_{ps} r_s^{2d+2} (d+1)^2 \left[\frac{1}{2} e^x (3 + d e^x) - \frac{4}{\alpha^2} \frac{(d+5)}{d+3} \Big(2 + (d+1) e^x (1+x) \Big) \right] \\ &\geq \frac{1}{2} r_{ps} r_s^{2d+2} (d+1)^2 \left[\frac{1}{2} e^x (3 + d e^x) - \frac{6}{25} \Big(2 e^x + (d+1) e^{2x} \Big) \right] \\ &= \frac{1}{2} r_{ps} r_s^{2d+2} (d+1)^2 \left[\left(\frac{13d - 12}{50} \right) e^{2x} + \left(\frac{3}{2} - \frac{12}{25} \right) e^x \right] > 0. \end{aligned}$$

where the inequality follows from using $\frac{d+5}{d+3} \leq \frac{3}{2}$ for $d \geq 1$ and $1 + x \leq e^x$. By continuity, the inequality also holds for $\alpha = 5 - \delta_0$ for some $\delta_0 > 0$, thus completing the proof.

Proof of (4.23)*:*

(4.25)

$$\begin{aligned} \frac{1}{6} p(r) + n_3(r) \geq \frac{1}{6} r_{ps} \Big[& d r^{2d+2} + (d+3) r_s^{d+1} r^{d+1} - (d+2)^2 r_s^{2d+2}) + \\ & \frac{24 r_s^{d+1}}{\alpha^2} \frac{(d+1)^3}{d+3} \left(1 - \frac{3}{\alpha^2} h(r)^2 \right) \frac{r^{2d+2}}{r^{d+1} - r_s^{d+1}} \Big] \\ = \frac{1}{6} r_{ps} \Big[& d (r^{d+1} - r_s^{d+1})^2 + 3(d+1) r_s^{d+1} (r^{d+1} - r_s^{d+1}) - (d+1)^2 r_s^{2d+2} + \\ & \frac{24 r_s^{d+1}}{\alpha^2} \frac{(d+1)^3}{d+3} \left(1 - \frac{3}{\alpha^2} h(r)^2 \right) \left((r^{d+1} - r_s^{d+1}) + 2 r_s^{d+1} + \frac{r_s^{2d+2}}{r^{d+1} - r_s^{d+1}} \right) \Big] \\ \geq \frac{1}{6} r_{ps} \Bigg(& d (r^{d+1} - r_s^{d+1})^2 + 3(d+1) r_s^{d+1} (r^{d+1} - r_s^{d+1}) - (d+1)^2 r_s^{2d+2} \\ & + \frac{24 r_s^{d+1}}{\alpha^2} \frac{(d+1)^3}{d+3} \left(1 - \frac{3}{\alpha^2} h(r)^2 \right) \left((r^{d+1} - r_s^{d+1}) + 2 r_s^{d+1} \right) - \frac{144 r_s^{2d+2}}{\alpha^4} \frac{(d+1)^2}{d+3} h(r)^2 \Bigg), \end{aligned}$$

where in the last inequality we have used that

$$
\begin{aligned}
\frac{24r_s^{d+1}}{\alpha^2}\frac{(d+1)^3}{d+3}\Big(1-\frac{3}{\alpha^2}(h(r))^2\Big)\frac{r_s^{2d+2}}{r^{d+1}-r_s^{d+1}} &\geq -\frac{72r_s^{d+1}}{\alpha^4}\frac{(d+1)^3}{d+3}(h(r))^2\frac{r_s^{2d+2}}{r^{d+1}-r_s^{d+1}} \\
&\geq -\frac{144}{\alpha^4}r_s^{2d+2}\frac{(d+1)^2}{d+3}(h(r))^2,
\end{aligned}
$$

as $-\frac{1}{r^{d+1}-r_s^{d+1}}$ is minimized at the photon sphere.

We proceed as above and use the change variables $x = h(r)$, where $x \in [0, \alpha]$. Then the last line of (4.25) can be re-written as

$$
\begin{aligned}
s(x) := \frac{r_{ps}r_s^{2d+2}(d+1)^2}{6}\Bigg[\frac{d}{4}e^{2x} + \left(\frac{3}{2}+\frac{12}{\alpha^2}\frac{(d+1)^2}{d+3}\right)e^x - \frac{36}{\alpha^4}\frac{(d+1)^2}{d+3}x^2e^x \\
- \frac{144}{\alpha^4}\frac{d+2}{d+3}x^2 + \left(\frac{48}{\alpha^2}\frac{d+1}{d+3}-1\right)\Bigg].
\end{aligned}
\tag{4.26}
$$

Then $s(0) = \frac{r_{ps}r_s^{2d+2}(d+1)^2}{24}\left(d+2+\frac{48(d+1)(d+5)}{\alpha^2(d+3)}\right) > 0$ and a calculation of $s'(x)$ gives

$$
\begin{aligned}
s'(x) &= \frac{1}{6}r_{ps}r_s^{2d+2}(d+1)^2\Bigg[\frac{d}{2}e^{2x} + \left(\frac{3}{2}+\frac{12}{\alpha^2}\frac{(d+1)^2}{d+3}\right)e^x \\
&\qquad - \frac{288}{\alpha^4}\frac{d+2}{d+3}x - \frac{36}{\alpha^4}\frac{(d+1)^2}{d+3}xe^x(x+2)\Bigg] \\
&\geq \frac{1}{6}r_{ps}r_s^{2d+2}(d+1)^2\Bigg[\frac{d}{2}e^{2x} + \left(\frac{3}{2}+\frac{12}{\alpha^2}\frac{(d+1)^2}{d+3}\right)e^x \\
&\qquad - \frac{288}{\alpha^4}\frac{d+2}{d+3}e^x - \frac{36}{\alpha^4}\frac{(d+1)^2}{d+3}e^{2x}(5+2)\Bigg] \\
&= \frac{1}{6}r_{ps}r_s^{2d+2}(d+1)^2\Bigg[\left(\frac{d}{2}-\frac{252}{\alpha^2}\frac{(d+1)^2}{d+3}\right)e^{2x} \\
&\qquad + \left(\frac{3}{2}+\frac{12}{\alpha^4}\frac{(\alpha^2(d+1)^2-24(d+2))}{d+3}\right)e^x\Bigg].
\end{aligned}
$$

It is rather straight forward to verify the coefficients of the exponential terms are positive for $d \geq 1$ and $\alpha = 5$. Again by continuity, the positivity remains for $\alpha = 5 - \delta_0$, for some $\delta_0 > 0$.

Case 4: $r \in [r_\alpha, \infty)$

The calculations in this case are quite simple. The smoothing function is constant, and we have

$$
\begin{aligned}
f(r) &= \frac{r^{d+2}-r_{ps}^{d+2}}{r^{d+2}} + \frac{Ar_s^{d+1}}{r^{d+2}}\frac{8\alpha}{15}, \quad \text{and} \\
f'(r) &= \frac{(d+2)r_{ps}^{d+2}}{r^{d+3}} - \frac{A8\alpha r_s^{d+1}}{15}\frac{(d+2)}{r^{d+3}} = \frac{(d+2)r_{ps}r_s^{d+1}}{r^{d+3}}\left(\frac{d+3}{2} - \frac{8\alpha}{15}\frac{d+2}{d+3}\right).
\end{aligned}
$$

f is easily seen to be positive, and as was described in *Case 3*, α was precisely chosen so that $f' > 0$. Finally,

$$
l(f) = l(g) = \frac{(d+2)}{4r^{2d+5}}\left(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2(d+1)}\right)
$$

where the quadratic term in parenthesis has one root $\beta \in (r_s, r_{ps})$; thus $l(f)$ is indeed positive.

In the cases above, we have shown that the multiplier with $f(r)$ defined in (4.5) satisfies the criteria we imposed. It can also be readily verified that the coefficients of the $(\partial_r\phi)^2$, $|\nabla\!\!\!/\phi|^2$ and ϕ^2 terms are respectively bounded below by $c_r(r), c_\omega(r)$ and $c_0(r)$. Thus to show Theorem 1.1, we require bounding the $(\partial_t\phi)^2$ term in the localized energy norm, the time boundary terms and the forcing term. These are proved in section 4.0.2, section 4.0.3, and section 4.0.4, respectively.

4.0.2. ***$(\partial_t\phi)^2$ term:.*** Here we define an alternate one tensor, $\bar{P}_\alpha[\phi, X]$, to obtain $(\partial_t\phi)^2$ terms in the divergence. We set

$$
\bar{P}_\alpha[\phi, X] = c_t(r)\phi\partial_\alpha\phi - \frac{1}{2}(\partial_\alpha c_t(r))\phi^2,
$$

whose divergence is

$$
\nabla^\alpha \bar{P}_\alpha[\phi, X] = c_t(r)\partial^\alpha\phi\partial_\alpha\phi + c_t(r)\phi\nabla^\alpha\partial_\alpha\phi - \frac{1}{2}\nabla^\alpha(\partial_\alpha c_t(r))\phi^2. \tag{4.27}
$$

Analogous to the case in controlling the spatial derivatives and lower order terms, we integrate (4.27) over the space time block $[0,T]\times(r_s,\infty)\times\mathbb{S}^{d+2}$ and apply the divergence theorem. This gives

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{S}^{d+2}} \int_{r\geq r_s} \left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} c_t(r)(\partial_t\phi)^2 r^{d+2} dr d\omega dt = \\
&\qquad \int_{\mathbb{S}^{d+2}} \int_{r\geq r_s} \left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} c_t(r)\phi\partial_t\phi\Big|_0^T r^{d+2} dr d\omega + \\
&\int_0^T \int_{\mathbb{S}^{d+2}} \int_{r\geq r_s} \Bigg[\Box_g\phi\,(c_t(r)\phi) + \left(\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)(\partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\right) c_t(r) \\
&\qquad + \frac{1}{2r^{d+2}}\partial_r\left(r^{d+2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\partial_r(c_t(r))\right)\phi^2\Bigg] r^{d+2} dr d\omega dt.
\end{aligned} \tag{4.28}
$$

Using that $c_t(r)$ is bounded, the first term on the right can be controlled by $E[\phi](t)$ via Cauchy Schwarz and a Hardy inequality which we describe in the next section. Moreover, $c_t(r) \leq c_r(r), c_\omega(r)$ by construction, so that we may apply the bounds of the radial and angular derivative terms in the previous section to the second expression. In calculating the coefficient of the lower order term, we find

$$
\begin{aligned}
&\frac{1}{2r^{d+2}}\partial_r\left(r^{d+2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\partial_r(c_t(r))\right) = \\
&\quad \frac{1}{2r^{d+2}}\left((d+1)r_s^{d+1}c_t'(r) + (d+2)(r^{d+1}-r_s^{d+1})c_t'(r)\right) + \frac{1}{2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)c_t''(r).
\end{aligned} \tag{4.29}
$$

Given that as $r \to r_s$, $\begin{cases} c_t'(r) & \approx \frac{1}{(r-r_s)\left(1-\log\left(\frac{r-r_s}{r}\right)\right)^4}, \\ c_t''(r) & \approx \frac{1}{(r-r_s)^2\left(1-\log\left(\frac{r-r_s}{r}\right)\right)^5} \end{cases}$ and as $r\to\infty$, $\begin{cases} c_t'(r) & \approx \frac{1}{r^{d+4}} \\ c_t''(r) & \approx \frac{1}{r^{d+5}} \end{cases}$ we may conclude that $\frac{1}{2r^{d+2}}\partial_r\left(r^{d+2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\partial_r(c_t(r))\right) \lesssim c_0(r)$ thus controlling the lower order term.

4.0.3. A Hardy Inequality and Time Boundary Terms: In this section we control the time boundary terms of equations (4.4) and (4.28) by the energy quantity. The bound on the first time boundary term of (4.4) follows quite readily by exploiting the boundedness of $f(r)$ and via an application of the Cauchy-Schwarz inequality. Indeed,

$$\int\limits_{\mathbb{S}^{d+2}} \int\limits_{r \geq r_s} f(r)\partial_t\phi\partial_r\phi r^{d+2} dr d\omega \lesssim$$

$$\int\limits_{\mathbb{S}^{d+2}} \int\limits_{r\geq r_s} \left[\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} (\partial_t\phi)^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)(\partial_r\phi)^2 \right] r^{d+2} dr d\omega \lesssim E[\phi](t).$$

To control the second time-boundary term, we again apply Cauchy-Schwarz so that

(4.30)

$$\int\limits_{\mathbb{S}^{d+2}} \int\limits_{r\geq r_s} \frac{1}{2r^{d+2}} \partial_t\phi\partial_r(r^{d+2}f(r))\phi r^{d+2} dr d\omega \lesssim \int\limits_{\mathbb{S}^{d+2}} \int\limits_{r\geq r_s} (\partial_t\phi)^2 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} r^{d+2} dr d\omega$$

$$+ \int\limits_{\mathbb{S}^{d+2}} \int\limits_{r\geq r_s} \left(\frac{1}{r^{d+2}}\partial_r(r^{d+2}f(r))\right)^2 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \phi^2 r^{d+2} dr d\omega.$$

The first term on the right of (4.30) is trivially bounded by $E[\phi](t)$. For the second term we claim

(4.31)

$$\int_{r\geq r_s} \left(\frac{1}{r^{d+2}}\partial_r(r^{d+2}f(r))\right)^2 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \phi^2 r^{d+2} dr \lesssim \int_{r\geq r_s} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r\phi)^2 r^{d+2} dr$$

and point out that the coefficient of ϕ^2 in the left integrand is

$$O\left(\frac{1}{(\log(r^{d+1} - r_s^{d+1}))^4 (r^{d+1} - r_s^{d+1})}\right) \text{ as } r \to r_s \text{ and } O\left(\frac{1}{r^2}\right) \text{ as } r \to \infty. \tag{4.32}$$

It will be convenient to revert to the $(d+2)$-dimensional Regge-Wheeler coordinates, $r^*(r) = \int_{r_{ps}}^r \left(1 - \frac{r_s^{d+1}}{\rho^{d+1}}\right)^{-1} d\rho$. Of particular relevance is that as $r \to r_s$, $r^* \approx \log(r - r_s)$, while as $r \to \infty$, $r^* \approx r$. Then the theorem will follow via the Hardy-type inequality

$$\int_{r_s}^{\infty} \frac{r^2}{(1+|r^*|)^4} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \phi^2 r^{d+2} dr \lesssim \int_{r_s}^{\infty} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r\phi)^2 r^{d+2} dr, \tag{4.33}$$

and verifying the left hand side of (4.33) is bounded below by the left hand side of (4.31) at the event horizon and at infinity, (since it is bounded elsewhere). This last fact is an immediate consequence of (4.32).

We now set out to prove (4.33), or equivalently

$$\int_{-\infty}^{\infty} \frac{r^2}{(1+|r^*|)^4}\phi^2 r^{d+2} dr^* \lesssim \int_{-\infty}^{\infty} (\partial_{r^*}\phi)^2 r^{d+2} dr^*. \tag{4.34}$$

We follow the proof of estimate (52) in [**17**].

<u>*Proof* (4.34):</u> Define $t(r^*)$ such that $t'(r^*) = \frac{r^2}{(1+|r^*|)^4} r^{d+2}$ and $t(-\infty) = 0$. That is, set

$$t(r^*) = \int_{-\infty}^{r^*} \frac{r(\rho)^2}{(1+|\rho|)^4} r(\rho)^{d+2} d\rho. \tag{4.35}$$

Observe that the coefficient of the left integrand of (4.34) is exactly $t'(r^*)$. Thus we calculate

$$\begin{aligned}
\int t'(r^*)\phi^2 r^{d+2} dr^* &= -2\int t(r_*)\phi\partial_{r_*}\phi\, r^{d+2} dr^* \\
&\leq 2\Big(\frac{t(r^*)^2}{t'(r^*)}(\partial_{r^*}\phi)^2\, r^{d+2} dr^*\Big)^{1/2}\Big(\int t'(r^*)\phi^2\, r^{d+2} dr^*\Big)^{1/2} \\
&= 2\Big(\int \frac{(1+|r^*|)^4}{r^2} t(r^*)^2 (\partial_{r^*}\phi)^2\, r^{d+2} dr^*\Big)^{1/2}\Big(\int t'(r^*)\phi^2\, r^{d+2} dr^*\Big)^{1/2}.
\end{aligned}$$

Dividing through by the term $(\int t'(r^*)\phi^2\, r^{d+2} dr^*)^{1/2}$ gives

$$\Big(\int t'(r^*)\phi^2\, r^{d+2} dr^*\Big)^{1/2} \leq 2\Big(\int \frac{(1+|r^*|)^4}{r^2} t(r^*)^2 (\partial_{r^*}\phi)^2\, r^{d+2} dr^*\Big)^{1/2}. \tag{4.36}$$

Moreover, as $r^* \to -\infty$, $t(r^*) \sim \frac{1}{|r^*|^3}$, and as $r^* \to \infty$, $t(r^*) \sim \frac{1}{|r^*|}$ so that in both cases $\frac{(1+|r_*|)^4}{r^2} t(r_*)^2 \lesssim r^{d+2}$. Thus,

$$\int_{-\infty}^{\infty} \frac{r^2}{(1+|r^*|)^4}\phi^2 r^{d+2} dr^* \lesssim \int_{-\infty}^{\infty} (\partial_{r^*}\phi)^2 r^{d+2} dr^*$$

proving (4.34) as desired.

Finally, we show the boundary terms in (4.28) are bounded above by $E[\phi](t)$. Another application of Cauchy Schwarz yields

$$\begin{aligned}(4.37)\quad &\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} c_t(r)\phi\partial_t\phi r^{d+2}drd\omega \lesssim \\ &\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}(\partial_t\phi)^2 r^{d+2}drd\omega + \int_{\mathbb{S}^{d+2}}\int_{r\geq r_s} c_t^2(r)\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)^{-1}\phi^2 r^{d+2}drd\omega,\end{aligned}$$

where the first term on the right is clearly bounded by $E[\phi](t)$ and the integrand of the second satisfies (4.32). We can then use (4.33) and the energy inequality to complete the proof.

4.0.4. ***Inhomogeneous term.*** Recall that the energy inequality gives

$$E[\phi](t) \lesssim E[\phi](0) + \int_0^t\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\Box_g\phi\partial_t\phi r^{d+2}drd\omega dt. \tag{4.38}$$

Moreover, in combining the results of the previous three sections, we find

$$\begin{aligned}(4.39)\quad ||\phi||^2_{LE_0} \lesssim |E[\phi](t)-E[\phi](0)| + \int_0^\infty\int_{\mathbb{S}^{d+2}}\int_{r\geq r_s}\Bigg(|\Box_g\phi||X\phi + c_t(r)\phi \\ + \left(\frac{1}{2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)r^{-(d+2)}\partial_r(f(r)r^{d+2})\right)\phi|\Bigg)r^{d+2}drd\omega dt.\end{aligned}$$

We thus require showing the integrals on the right of (4.38) and (4.39) are controlled by the dual localized energy norm. Combining the two integrals and applying Cauchy Schwarz yields the upper bound

$$\begin{aligned}(4.40)\quad \int_{t>0}\frac{1}{\epsilon}||c_F^{\frac{1}{2}}F||^2_{L^2}dt + \int_{t>0}\epsilon||c_F^{-\frac{1}{2}}|X\phi + c_t(r)\phi + c_1\partial_t\phi \\ + \frac{1}{2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)r^{-(d+2)}\partial_r(f(r)r^{d+2})\phi|||^2_{L^2}dt.\end{aligned}$$

The second term is then controlled by

$$\epsilon \int_{t>0} \int_{\mathbb{S}^{d+2}} \int_{r \geq r_s} c_F^{-1} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^2 (\partial_r \phi)^2 + c_F^{-1} c_t^2 \phi^2 + c_F^{-1} c_1^2 (\partial_t \phi)^2 \\ + c_F^{-1} \left(\frac{1}{2r^{d+2}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \partial_r (r^{d+2} f(r)) \right)^2 \phi^2 r^{d+2} dr d\omega dt. \tag{4.41}$$

Observe that $c_F^{-1} c_t^2 = \frac{\left(\frac{r - r_{ps}}{r}\right)^6}{r^{3d+9}\left(\frac{r - r_s}{r}\right)\left(1 - \log\left(\frac{r - r_s}{r}\right)\right)^6} \leq c_0(r)$. Moreover, $\left(\frac{1}{2r^{d+2}}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)\partial_r(r^{d+2}f(r))\right)^2$ satisfies (4.32) so that
$c_F^{-1}\left(\frac{1}{2r^{d+2}}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)\partial_r(r^{d+2}f(r))\right)^2 \lesssim c_0(r)$. Finally, the weighted factor $c_F(r)$ was conveniently chosen to satisfy $c_F^{-1} \lesssim c_r, c_\omega, c_t$, thus we may bootstrap the second term in to the localized energy norm by choosing ϵ sufficiently small.

This now completes the proof of Theorem 1.1.

CHAPTER 5

Logarithmic Loss at the Photon Sphere

In this section we show the improved estimate, Theorem 1.2, whereby the second degree polynomial loss at the photon sphere is replaced by a logarithmic loss. The proof of the theorem follows almost directly from the corresponding theorem in [**34**], ie. the $(3+1)$-dimensional case.

To prove the theorem we localize near the photon sphere and define the smooth cutoff function $\chi_{ps}(r)$ supported in a neighbourhood of $r = r_{ps}$. We strengthen the LE_0 norm and weaken the dual norm LE_0^*, and define the LE and LE^* spaces such that

$$||\phi||^2_{LE} = ||\phi||^2_{LE_0} + ||\chi_{ps}\phi||^2_{LE_{ps}} \tag{5.1}$$

$$||\phi||^2_{LE^*} = ||\phi||^2_{LE_0^*} + ||\chi_{ps}\phi||^2_{LE^*_{ps}}. \tag{5.2}$$

Postponing the definition of the LE_{ps} and LE^*_{ps} spaces, the estimate we then wish to prove is

Theorem 5.1. *Let ϕ satisfy the inhomogeneous wave equation $\Box_g\phi = F$ on the $(d+4)$-dimensional hyperspherical Schwarzschild manifold. Then for $d \geq 1$, we have*

$$\sup_{t\geq 0} E[\phi](t) + ||\phi||^2_{LE} \lesssim E[\phi](0) + ||F||^2_{LE^*}. \tag{5.3}$$

However, given the results of Chapter 4, it suffices to show

Theorem 5.2. *Let ϕ and F be functions supported in a neighbourhood of r_{ps} such that $\Box_g\phi = F$. Then for $d \geq 1$,*

$$||\phi||^2_{LE_{ps}} \lesssim ||F||^2_{LE^*_{ps}}. \tag{5.4}$$

In the analysis to follow we will determine the inequalities

$$
\begin{aligned}
&||\phi||_{LE_{ps}} \gtrsim |||\ln|r^*||^{-1}\nabla_{t,x}\phi||_{L^2} \\
&||F||_{LE^*_{ps}} \lesssim |||\ln|r^*||F||_{L^2},
\end{aligned} \tag{5.5}
$$

where r^* refers to the Regge-Wheeler coordinate defined in Chapter 2. Consequently, pending the proof of Theorem 5.2, we deduce the desired result:

Corollary 5.3. *Let ϕ satisfy the inhomogeneous wave equation $\Box_g\phi = F$ on the $(d+4)$-dimensional Schwarzschild manifold. Then for $d \geq 1$, the conclusion of Theorem 1.2 holds.*

We begin by setting up the necessary background to establish (5.5) so that Corolloary 5.3 is indeed validated.

It is convenient to convert to Regge-Wheeler coordinates where the photon sphere is now translated to $r^* = 0$. Observe that the spherically symmetric nature of the trapped set, and the fact that losses are only an issue in the high frequency limit makes it favourable to appeal to pseudo-differential operators and to decompose $\Delta_{\mathbb{S}^{d+2}}$ into spherical harmonics. With this we introduce some notation that will be used below. Let λ be the eigenvalue associated to the λ-th harmonic in the expansion of $\sqrt{-\Delta_{\mathbb{S}^{d+2}}}$, and Π_λ the projection of the operator $\sqrt{-\Delta_{\mathbb{S}^{d+2}}}$ on the λ-th eigenspace. Due to spherical symmetry, our desired pseudo-differential operator, A, is independent of $\omega \in \mathbb{S}^{d+2}$ so that we may fix λ and consider estimates on each λ-th harmonic. Specifically, to the symbol $a_{ps}(r^*, \xi, \lambda)$, we fix λ and denote the corresponding one dimensional weyl operator as $a^w_{ps}(\lambda)$, where $a^w_{ps}(\lambda)(f) = \int\int a\left(\frac{x+y}{2}, \xi, \lambda\right) e^{i\xi(x-y)} f(y)dyd\xi,$. We can then write

$$
A_{ps} := \sum_\lambda a^w_{ps}(\lambda)\Pi_\lambda,
$$

and by Plancherel's theorem, determine the estimates in question on individual projections.

To construct the desired symbol, we begin with defining the smooth increasing functions γ_0, γ_1, such that

$$\gamma_0 : \mathbb{R} \to \mathbb{R}^+, \qquad \gamma_1 : \mathbb{R}^+ \to \mathbb{R}^+,$$
$$\gamma_0(y) = \begin{cases} 1, & y < 1, \\ y, & y \geq 2 \end{cases} \qquad \gamma_1(y) = \begin{cases} \sqrt{y}, & y < 1/2, \\ 1, & y \geq 1. \end{cases}$$

and the smooth function $\gamma : \mathbb{R}^2 \to \mathbb{R}^+$, satisfying

$$\gamma(y, z) = \begin{cases} 1, & z < C \\ \gamma_0(y), & y < \sqrt{z/2}, z \geq C, \\ \sqrt{z}\gamma_1(y^2/z), & y \geq \sqrt{z/2}, z \geq C \end{cases}$$

where C is a fixed large constant. We then define the symbol a_{ps},

$$a_{ps}(r^*, \xi, \lambda) = \gamma(-\ln(r^{*2} + \lambda^{-2}\xi^2), \ln\lambda)$$
$$= \begin{cases} 1, & \ln\lambda < C \\ \begin{cases} 1, & r^{*2} + \lambda^{-2}\xi^2 > e^{-1} \\ -\ln(r^{*2} + \lambda^{-2}\xi^2), & e^{-\sqrt{\frac{\ln\lambda}{2}}} < r^{*2} + \lambda^{-2}\xi^2 < e^{-2} \\ \sqrt{\ln\lambda}, & r^{*2} + \lambda^{-2}\xi^2 < e^{-\sqrt{\ln\lambda}} \end{cases} & , \ln\lambda \geq C \end{cases}$$

with reciprocal $a_{ps}^{-1} = 1/a_{ps}$. Denote the corresponding Weyl operator, $A_{ps} = \sum_\lambda a_{ps}^w(\lambda)\Pi_\lambda$, and approximate inverse $A_{ps}^{-1} = \sum_\lambda (a_{ps}^{-1})^w(\lambda)\Pi_\lambda$. With this definition we record a few boundedness properties of the symbols and operators as they are frequently used in the analysis below.

Observe that for λ sufficiently small, $\ln\lambda < C$ so that $a_{ps} = a_{ps}^{-1} = 1$ and for λ large, the following inequalities hold

$$1 \leq a_{ps}(r^*, \xi, \lambda) \leq a_{ps}(r^*, 0, \lambda) \leq (\ln\lambda)^{1/2} \tag{5.6}$$

$$(\ln\lambda)^{-1/2} \leq a_{ps}^{-1}(r^*, 0, \lambda) \leq a_{ps}^{-1}(r^*, \xi, \lambda) \leq 1. \tag{5.7}$$

Moreover, a calculation of the derivatives of the symbols leads to the bounds

$$
\begin{aligned}
&|\partial_{r^*}^{\alpha}\partial_{\xi}^{\beta}\partial_{\lambda}^{\nu}a_{ps}(r^*,\xi,\lambda)| \leq c_{\alpha,\beta,\nu}\lambda^{-\beta-\nu}(r^{*2}+\lambda^{-2}\xi^2+e^{-\sqrt{\ln\lambda}})^{-\frac{\alpha+\beta}{2}} \\
&|\partial_{r^*}^{\alpha}\partial_{\xi}^{\beta}\partial_{\lambda}^{\nu}a_{ps}^{-1}(r^*,\xi,\lambda)| \leq c_{\alpha,\beta,\nu}\lambda^{-\beta-\nu}a_{ps}^{-2}(r^*,\xi,\lambda)(r^{*2}+\lambda^{-2}\xi^2+e^{-\sqrt{\ln\lambda}})^{-\frac{\alpha+\beta}{2}}
\end{aligned}
\tag{5.8}
$$

where $\alpha+\beta+\nu > 0$, and we have used the fact that $y^2 > z \Rightarrow r^{*2}+\lambda^{-2}\xi^2 < e^{-\sqrt{\ln\lambda}}$. It follows from the equations in (5.8) that for λ fixed, $|\partial_{r^*}^{\alpha}\partial_{\xi}^{\beta}\partial_{\lambda}^{\nu}a(r^*,\xi,\lambda)| \leq C_{\alpha,\beta,\nu}\langle|\xi|\rangle^{\delta-(|\beta|+|\nu|)}$. A similar estimate holds for a_{ps}^{-1} and we can then conclude both $a_{ps}, a_{ps}^{-1} \in S_{1,0}^{\delta}$, for $\delta > 0$. It follows that

$$
||a_{ps}^w(\lambda)(a_{ps}^{-1})^w(\lambda) - I||_{L^2} \lesssim \lambda^{-1}e^{\sqrt{\ln\lambda}}. \tag{5.9}
$$

With this setup, we define the LE_{ps} norm

$$
||\phi||_{LE_{ps}} = ||A_{ps}^{-1}\phi||_{H_{t,x}^1} \approx ||A_{ps}^{-1}\nabla_{t,x}\phi||_{L^2} \tag{5.10}
$$

with respective dual norm

$$
||F||_{LE_{ps}^*} = ||A_{ps}F||_{L^2}. \tag{5.11}
$$

From the bounds in (5.6) and (5.7), we respectively have, $||a_{ps}^w(\lambda)F||_{L^2} \lesssim ||a_{ps}(r^*,0,\lambda)F||_{L^2}$ and $||(a_{ps}^{-1})^w(\lambda)\phi||_{L^2} \gtrsim ||a_{ps}^{-1}(r^*,0,\lambda)\phi||_{L^2}$. Moreoever, combining these with the definitions of $a_{ps}(r^*,0,\lambda)$ and $a_{ps}^{-1}(r^*,0,\lambda)$, we can deduce the bounds

$$
||a_{ps}^w(\lambda)F||_{L^2} \lesssim ||a_{ps}(r^*,0,\lambda)F||_{L^2} \lesssim ||\big|\ln|r^*|\big|F||_{L^2} \tag{5.12}
$$

$$
||(a_{ps}^{-1})^w(\lambda)\phi||_{L^2} \gtrsim ||a_{ps}^{-1}(r^*,0,\lambda)\phi||_{L^2} \gtrsim ||\big|\ln|r^*|\big|^{-1}\phi||_{L^2}. \tag{5.13}
$$

These imply (5.5) and we are now ready to prove Theorem 5.2.

Proof of Theorem 5.2:

We begin by setting $u = r^{\frac{d+2}{2}}\phi$, $g = \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) r^{\frac{d+2}{2}}F$ and by re-writing the d'Alembertian in terms of r^*, which we denote by $\Box_{RW}$. This yields

$$r^{\frac{d+2}{2}}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\Box_g\left(\frac{u}{r^{\frac{d+2}{2}}}\right)=\Box_{RW}u \tag{5.14}$$

where $\Box_{RW}=-\partial_t^2+\partial_{r^*}^2+\frac{1}{r^2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\Delta_{\mathbb{S}^{d+2}}+V(r)$, and $V(r)$ is the potential given by,

$$V(r)=\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\left(\frac{d+2}{2}\right)\frac{1}{r^{d+3}}\left[\frac{d}{2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)r^{d+1}+r_s^{d+1}(d+1)\right].$$

We then have that u solves $\Box_{RW}u=g$.

It will also be convenient to simplify the energy expressions. In a compact neighbourhood of $r^*=0$, observe that

$$E[\phi](t)\approx\int\int(\partial_t\phi)^2+(\partial_{r^*}\phi)^2+|\nabla\!\!\!/_0\phi|^2drd\omega. \tag{5.15}$$

and the initial localized energy norms, LE_0, LE_0^* can be expressed as

$$\begin{aligned}||\phi||_{LE_0}^2&\approx\int\int\int(\partial_{r^*}\phi)^2+r^{*2}((\partial_t\phi)^2+|\nabla\!\!\!/_0\phi|^2+\phi^2drd\omega dt\\ ||F||_{LE_0^*}^2&\approx\int\int\int r^{*-2}F^2drd\omega dt.\end{aligned} \tag{5.16}$$

Moroever, for ϕ and F supported in a neighbourhood of $r^*=0$,

$$||\phi||_{LE_{ps}}\approx||u||_{LE_{ps}}\text{ and }||F||_{LE_{ps}^*}\approx||g||_{LE_{ps}^*}$$

so that instead of (5.4) we may prove the bound

$$||u||_{LE_{ps}}^2\lesssim||g||_{LE_{ps}^*}^2. \tag{5.17}$$

The approach taken by [**34**] and [**54**] is to do a Fourier transform in time and to decompose the solution u in to spherical harmonics. That is, let Y_λ^i be an orthonormal basis for the space $\{Y_\lambda|\Delta_{\mathbb{S}^{d+2}}Y_\lambda=-\lambda^2Y_\lambda\}$, and let $u=\sum_{\lambda,i}u_{\lambda,i}Y_\lambda^i$, where $u_{\lambda,i}(\tau,r)\in\mathbb{R}$. Then we have

$$\sum_{\lambda,i}\left(\tau^2 u_{\lambda,i}Y^i_\lambda + \partial^2_{r_*}u_{\lambda,i}Y^i_\lambda + V(r)u_{\lambda,i}Y^i_\lambda - \frac{\lambda^2\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)}{r^2}u_{\lambda,i}Y^i_\lambda\right) = \sum_{\lambda,i} g_{\lambda,i}Y^i_\lambda. \tag{5.18}$$

We restrict the analysis to each harmonic, λ and each component i, and by an abuse of notation, set $u_{\lambda,i} = u$ and $g_{\lambda,i} = g$. This yields

$$\partial^2_{r^*}u + V_{\lambda,\tau}(r^*)u = g \tag{5.19}$$

where $V_{\lambda,\tau}(r^*) = \tau^2 - \frac{1}{r^2}\left(1-\frac{r_s^{d+1}}{r^{d+1}}\right)\lambda^2 + V$.

By Plancherel's theorem and (5.19), estimate (5.17) will follow from showing

$$||\partial_{r^*}u||_{L^2} + (|\tau| + |\lambda|)||(a_{ps}^{-1})^w(\lambda)u||_{L^2} \lesssim ||a^w_{ps}(\lambda)g||_{L^2}. \tag{5.20}$$

The analysis is broken down in to four cases depending on the relative sizes of $|\lambda|$ and $|\tau|$. For the first three, it will suffice to prove

$$||\partial_{r^*}u||_{L^2} + (|\tau| + |\lambda|)||u||_{L^2} \lesssim ||g||_{L^2}. \tag{5.21}$$

Indeed, this estimate has no loss at the photon sphere and hence is stronger than (5.4). We proceed case by case. In what follows, we let the support of u and u_{r^*} be contained in the interval $[r_-, r_+]$

Case 1: $\lambda, \tau \lesssim 1$.

In this case note that $|V_{\lambda,\tau}(r^*)| \lesssim 1$, thus we wish to show

$$||\partial_{r^*}u||_{L^2} + ||u||_{L^2} \lesssim ||g||_{L^2}. \tag{5.22}$$

Define the positive definite energy functional

$$E[u](r^*) = (\partial_{r^*}u)^2 + u^2 \tag{5.23}$$

and observe that

$$\begin{aligned} \partial_{r^*} E[u](r^*) &\lesssim |\partial_{r^*} u||g - V_{\lambda,\tau} u| + |\partial_{r^*} u u| \\ &\lesssim E[u](r^*) + g^2 \end{aligned} \tag{5.24}$$

where in the last bound we have used that $|V_{\lambda,\tau}(r^*)| \lesssim 1$. This gives

$$E[u](r^*) = \int_{r_-}^{r_+} E\, dr^* + ||g||_{L^2}^2 \tag{5.25}$$

so that by Gronwall's Inequality, $E^{1/2} \leq ||g||_{L^2}$, and hence (5.22) follows.

<u>*Case 2:* $\lambda \ll \tau$</u>.

Here we wish to show $||u||_{L^2} + |\tau|||\partial_{r^*} u||_{L^2} \lesssim ||g||_{L^2}$. We define the positive definite energy functional

$$E[u](r^*) = (\partial_{r^*} u)^2 + V_{\lambda,\tau}(r^*) u^2, \tag{5.26}$$

and as above calculate the derivative to obtain

$$\begin{aligned} \partial_{r^*} E[u](r^*) &\lesssim |\partial_{r^*} u g| + |\partial_{r^*} V_{\lambda,\tau}(r^*) u^2| \\ &\lesssim E[u](r) + g^2 \end{aligned}$$

where we have used that $V_{\lambda,\tau}(r^*) \approx \partial_{r^*} V_{\lambda,\tau}(r^*) \approx \tau^2$ and hence $\partial_{r^*} V_{\lambda,\tau}(r^*) u^2 \lesssim E[u](r^*)$. An application of Gronwall's Inequality gives the desired bound for $\lambda \ll \tau$.

<u>*Case 3:* $\tau \ll \lambda$</u>.

In this case, we consider the Dirichlet boundary value problem

$$\begin{cases} \partial_{r^*}^2 u + V_{\lambda,\tau}(r^*) u = g \\ u(r^*_-) = u(r^*_+) = 0. \end{cases} \tag{5.27}$$

Upon multiplying equation (5.27) by u and integrating by parts we obtain

$$-\int_{r_-}^{r_+} (\partial_{r^*} u)^2 dr^* + \int_{r_-}^{r_+} V_{\lambda,\tau}(r^*) u^2 dr^* = \int_{r_-}^{r_+} g u\, dr^*.$$

Observe that in this case $V_{\lambda,\tau} \approx -\lambda^2$ so that an application of Cauchy Schwarz to the right side yields the desired result.

Case 4: $\lambda \approx \tau \gg 1$.

We prove (5.20) as (5.21) is no longer true. That is, we wish to show

$$||\partial_{r^*} u||_{L^2} + \lambda||(a_{ps}^{-1})^w(\lambda)u||_{L^2} \lesssim ||a_{ps}^w(\lambda)g||_{L^2}, \tag{5.28}$$

which will follow through a series of reductions. The nature of the trapped rays arise in high frequencies so that in is this case, we consider λ and $|\xi|$ large. The goal will be to control the second term on the left in the above inequality, as $||\partial_{r*}u||_{L^2}$ does not exhibit a loss near the photon sphere. In fact, we have already shown in the previous section that it satisfies the desired bound. However, it will be advantageous to retain this term in proving (5.28) as it will be used to control $\lambda||(a_{ps}^{-1})^w(\lambda)u||_{L^2}$ for $|\xi| \gg \lambda$. When $|\xi| \ll \lambda$, the second term dominates and we need to show $\lambda||(a_{ps}^{-1})^w(\lambda)u||_{L^2} \lesssim ||a_{ps}^w(\lambda)g||_{L^2}$.

In the first reduction we re-characterize (5.19). First note that in this case we may consider V a small perturbation and thus write $V_{\lambda,\tau} \approx \tilde{V}_{\lambda,\tau} := \tau^2 - \frac{1}{r^2}\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)\lambda^2$ with derivative $\frac{2}{r^3}\left(\frac{r^{d+1} - r_{ps}^{d+1}}{r^{d+1}}\right)\lambda^2$. It immediately follows that $\tilde{V}'_{\lambda,\tau}(r_{ps}) = 0$, and $\tilde{V}''_{\lambda,\tau}(r_{ps}) > 0$. Thus for a smooth function $W(r^*)$ with nondegenerate minimum at $r^* = 0$, and for $|\epsilon| \lesssim 1$ we may write (5.19) as

$$\partial_{r^*}^2 u + ((W(r^*) + \epsilon)\lambda^2)u = g. \tag{5.29}$$

Then we claim:

Theorem 5.4. *Let W be a smooth function satisfying $W(0) = W'(0) = 0, W''(0) > 0$ and let $|\epsilon| \lesssim 1$. Suppose u and g are supported near $r^* = 0$ and satisfy*

$$(\partial_{r^*}^2 + \lambda^2(W(r^*) + \epsilon))u(r^*) = g. \tag{5.30}$$

Then (5.28) *holds. i.e.* $||\partial_{r^*} u||_{L^2} + \lambda||(a_{ps}^{-1})^w(\lambda)u||_{L^2} \lesssim ||a_{ps}^w(\lambda)g||_{L^2}$.

We would like to replace the norm $||a^w_{ps}(\lambda)g||_{L^2}$ with $||a_{ps}(r^*,0,\lambda)g||_{L^2}$ as this would allow us to avoid weyl calculus analysis. However recall from (5.12), the latter norm is larger, and in fact Theorem 5.4 is no longer true. Instead, we claim that we can decompose g into two functions, one of which will allow us to use the low frequency norm $||a_{ps}(r^*,0,\lambda)g||_{L^2}$. Specifically we have,

Lemma 5.5. *For each function $g \in L^2$ supported near the photon sphere, there exists functions g_1 and g_2 supported near the photon sphere such that $g = g_1 + \lambda^{-2}\partial^2_{r^*}g_2$ and*

$$||a_{ps}(r^*,0,\lambda)g_1||_{L^2} + |||r^{*2} + e^{-\sqrt{\ln\lambda}}|^{1/8}g_2||_{L^2} + \lambda^{-2}||\partial_{r^*}g_2||_{L^2} \lesssim ||a^w_{ps}(\lambda)g||_{L^2}. \tag{5.31}$$

The proof will be presented following the proof of Proposition 5.8. With the Lemma at hand, it will then suffice to show

$$\begin{aligned}(5.32)\quad ||\partial_{r^*}u||_{L^2} + \lambda||(a^{-1}_{ps}(\lambda))^w u||_{L^2} \lesssim \\ ||a_{ps}(r^*,0,\lambda)g_1||_{L^2} + |||r^{*2} + e^{-\sqrt{\ln\lambda}}|^{1/8}g_2||_{L^2} + \lambda^{-2}||\partial_{r^*}g_2||_{L^2}.\end{aligned}$$

We continue to reduce the problem further. Let g_1 and g_2 be described as above, and define

$$u = \lambda^{-2}g_2 + \tilde{u},$$

with derivative

$$\partial_{r^*}u = \lambda^{-2}\partial_{r^*}g_2 + \partial_{r^*}\tilde{u}.$$

By (5.32) we require showing

$$\begin{aligned}||\lambda^{-2}\partial_{r^*}g_2||_{L^2} + \lambda||\lambda^{-2}(a^{-1}_{ps}(\lambda))^w g_2||_{L^2} + ||\partial_{r^*}\tilde{u}||_{L^2} + \lambda||(a^{-1}_{ps}(\lambda))^w\tilde{u}||_{L^2} \\ \lesssim ||a_{ps}(r^*,0,\lambda)g_1||_{L^2} + |||r^{*2} + e^{-\sqrt{\ln\lambda}}|^{1/8}g_2||_{L^2} + \lambda^{-2}||\partial_{r^*}g_2||_{L^2}.\end{aligned}$$

The bounds in (5.13) imply $\lambda||\lambda^{-2}(a^{-1}_{ps}(\lambda))^w g_2||_{L^2} \lesssim \lambda||\lambda^{-2}g_2||_{L^2} \lesssim |||r^{*2}+e^{-\sqrt{\ln\lambda}}|^{1/8}g_2||_{L^2}$, so that Lemma 5.5 gives the desired bound for g_2. We thus need to show the appropriate bounds for $||\partial_{r^*}\tilde{u}||_{L^2}$ and $\lambda||(a^{-1}_{ps}(\lambda))^w\tilde{u}||_{L^2}$. First note that if u satisfies (5.30), then $\tilde{u}$

solves

$$\partial_{r^*}^2 \tilde{u} + \lambda^2(W+\epsilon)\tilde{u} = \tilde{g} \quad \text{where } \tilde{g} = g_1 - (W+\epsilon)g_2. \tag{5.33}$$

For $\tilde{u}$ then, we claim that

Lemma 5.6. *For each $\lambda^{-1} < \sigma < 1$ and each function $\tilde{u}$ with compact support,*

$$\lambda||(a_{ps}^{-1})^w(\lambda)\tilde{u}||_{L^2} \lesssim ||(\sigma + |W+\epsilon|)^{-1/4}\partial_{r^*}\tilde{u}||_{L^\infty} + \lambda||(\sigma + |W+\epsilon|)^{1/4}\tilde{u}||_{L^\infty}. \tag{5.34}$$

Furthermore, for the $\tilde{g}$ term described in (5.33) we claim that

$$||(\lambda^{-1} + |W+\epsilon|^{-1/4})g_1||_{L^1} + ||(\lambda^{-1} + |W+\epsilon|^{-1/4})(W+\epsilon)g_2||_{L^1} \lesssim ||a_{ps}^w(\lambda)g||_{L^2}. \tag{5.35}$$

Indeed, observe that

$$||(\lambda^{-1} + |W+\epsilon|^{-1/4})(W+\epsilon)g_2||_{L^1} \lesssim |||r^{*2} + e^{-\sqrt{\ln\lambda}}|^{1/8}g_2||_{L^2}$$

where we have used $||(\lambda^{-1} + |W+\epsilon|^{-1/4})(W+\epsilon)|r^{*2} + e^{-\sqrt{\ln\lambda}}|^{-1/8}||_{L^2} < \infty$ for a compact region about $r^* = 0$. Applying Lemma 5.5 then gives equation (5.35) for g_2. For g_1 we will show

Lemma 5.7. *For g, g_1 defined above, the following weighted L^1 bound holds for g_1,*

$$||(\lambda^{-1} + |W+\epsilon|^{-1/4})g_1||_{L^1} \lesssim ||a_{ps}(r^*, 0, \lambda)g||_{L^2}. \tag{5.36}$$

In this case, using (5.12), the right side of equation (5.36) is bounded above by $||a_{ps}^w(r^*, 0, \lambda)g_1||_{L^2}$, and again, appealing to Lemma 5.5, equation (5.35) follows. The proof of above two lemmas follows the proof of Lemma 5.5 at the end of the section. Summarizing the above reductions, we find that in proving (5.20) it will suffice to show the following proposition:

Proposition 5.8. *Given functions u, g supported in a neighbourhood of $r^* = 0$, and $|\epsilon| \lesssim 1$ satisfying*

$$(\partial_{r^*}^2 + \lambda^2(W+\epsilon))u = g, \tag{5.37}$$

there exists $\lambda^{-1} < \sigma < 1$ such that

$$||(\sigma + |W+\epsilon|)^{-1/4}\partial_{r^*}u||_{L^\infty} + \lambda||(\sigma + |W+\epsilon|)^{1/4}u||_{L^\infty} \lesssim ||(\lambda^{-1} + |W+\epsilon|)^{-1/4}g||_{L^1}. \tag{5.38}$$

The first term on the left gives the L^2 bound for u_{r*} in (5.20). Indeed,

$$\int |u_{r*}|^2 dr^* \lesssim ||(\sigma + |W+\epsilon|)^{-1/4}\partial_{r^*}u||_{L^\infty}^2 \int (|\sigma + |W+\epsilon|)^{1/2} dr^*, \tag{5.39}$$

where over a compactly supported region, $(\sigma + |W+\epsilon|)$ is integrable. We thus concentrate on the remainder of (5.20) and proceed with the proof of Proposition 5.8.

Proof of Proposition 5.8:
We are in the midst of proving *Case 4* of (5.17), so that $\lambda \approx \tau \gg 1$. We consider three subcases depending on the size of ϵ. The method of proof follows in the spirit of *Cases 1-3*, where we consider an energy functional, determine bounds on its derivative and apply Gronwall's inequality. Using that W has a non-degenerate minimum at $r^* = 0$, in what follows we write $|W+\epsilon| \approx |r^{*2} + \epsilon|$. We define the energy functional

$$E[u(r^*)] = \lambda^2(W+\epsilon)^{1/2}u^2 + (W+\epsilon)^{-1/2}u_{r^*}^2 + \frac{1}{2}W_{r^*}(W+\epsilon)^{-3/2}uu_{r^*}, \tag{5.40}$$

determined by doing a WKB analysis on (5.30). This will be slightly modified in each of the cases below.

Case (i): $\epsilon \gg \lambda^{-1}$:
Here we choose $\sigma = \epsilon$ and show:

$$\lambda(\epsilon + r^{*2})^{1/4}|u| + (\epsilon + r^{*2})^{-1/4}|\partial_{r*}u| \lesssim ||(\epsilon + r^{*2})^{-1/4}g||_{L^1}. \tag{5.41}$$

We define the energy as in (5.40) and claim that E is in fact positive. Indeed,

$$\frac{1}{2}W_{r^*}(W+\epsilon)^{-3/2}uu_{r^*} \geq -\frac{1}{4\delta}(W+\epsilon)^{-1/2}u_{r^*}^2 - \frac{\delta}{4}\left(W_{r*}^2(W+\epsilon)^{-5/2}u^2\right). \tag{5.42}$$

For $\delta > 0$ chosen appropriately, ie. $\delta > \frac{1}{4}$, the first term on the right can be absorbed in to the second summand of the energy functional, (5.40). For the second term, choosing $\delta = 1$ we assert that $\frac{1}{4}W_{r*}^2|W+\epsilon|^{-5/2}u^2 \leq \lambda^2|W+\epsilon|^{1/2}u^2$ or equivalently,

$$|W_{r^*}|^2 \leq 4\lambda^2|W+\epsilon|^3. \tag{5.43}$$

Since $W \approx r^{*2} \Rightarrow |W_{r*}| \approx |W|^{1/2}$ it is sufficient to show

$$|W| \ll \lambda^2|W+\epsilon|^3, \tag{5.44}$$

however this follows trivially since $\epsilon \gg \lambda^{-1}$. Thus the second term of (5.42) can also be bootstrapped in to (5.40) and validates positivity.

A computation of the derivative of $E[u(r^*)]$ yields

$$\begin{aligned}\frac{d}{dr^*}E[u(r^*)] = \frac{1}{2}uu_{r^*}\bigg(W_{r*r*}(W+\epsilon)^{-3/2} - \frac{3}{2}W_{r*}^2(W+\epsilon)^{-5/2}\bigg) + \\ (W+\epsilon)^{-1/4}g\bigg(2(W+\epsilon)^{-1/4}u_{r*} + \frac{1}{2}(W+\epsilon)^{-5/4}W_{r*}u\bigg).\end{aligned} \tag{5.45}$$

Using the bound $|W_{r^*r^*}| \lesssim 1$, the first and second cross terms respectively satisfy

$$\begin{aligned}\frac{1}{2}uu_{r^*}W_{r*r*}(W+\epsilon)^{-3/2} &\lesssim \frac{1}{\lambda}(W+\epsilon)^{-3/2}\bigg(\lambda^2(W+\epsilon)^{1/2}u^2 + (W+\epsilon)^{-1/2}u_{r^*}^2\bigg) \\ &\lesssim \frac{1}{\lambda}(W+\epsilon)^{-3/2}E[u(r^*)],\end{aligned}$$

$$\begin{aligned}W_{r^*}^2(W+\epsilon)^{-5/2}uu_{r^*} &\lesssim \frac{|W|}{\lambda(W+\epsilon)^{5/2}}\left(\lambda^2u^2(W+\epsilon)^{1/2} + u_{r^*}^2(W+\epsilon)^{-1/2}\right) \\ &\lesssim \frac{1}{\lambda}(W+\epsilon)^{-3/2}E[u(r^*)].\end{aligned}$$

Furthermore, the second term of (5.45) is bounded above by

$$
\begin{aligned}
&(W+\epsilon)^{-1/4}|g|\left(2(W+\epsilon)^{-1/4}|u_{r*}| + \frac{1}{2}|W+\epsilon|^{1/2}|W+\epsilon|^{-5/4}|u|\right)\\
&\lesssim |W+\epsilon|^{-1/4}|g|\left(|W+\epsilon|^{-1/4}|u_{r*}| + (\lambda^{-1}|W+\epsilon|^{-1})\lambda|W+\epsilon|^{1/4}|u|\right)\\
&\lesssim |W+\epsilon|^{-1/4}E^{1/2}[u(r^*)]|g|,
\end{aligned}
$$

where we have used that $|W_{r^*}| \approx |W|^{1/2}$ and $\lambda^{-1}|W+\epsilon|^{-1} \lesssim 1$. Combining these gives the estimate

$$\frac{d}{dr^*}E[u(r^*)] \lesssim \lambda^{-1}(W+\epsilon)^{-3/2}E + E^{1/2}(W+\epsilon)^{-1/4}|g|. \tag{5.46}$$

We apply this to $\frac{d}{dr^*}(E^{1/2})$ to give

$$
\begin{aligned}
\left|\frac{d}{dr^*}(E^{1/2})\right| &\leq \frac{CE^{-1/2}}{2}\left(\lambda^{-1}(W+\epsilon)^{-3/2}E + E^{1/2}(W+\epsilon)^{-1/4}|g|\right)\\
&= \frac{C}{2}\left(\lambda^{-1}(W+\epsilon)^{-3/2}E^{1/2} + (W+\epsilon)^{-1/4}|g|\right).
\end{aligned}
$$

It follows that $E^{1/2} \leq \frac{C}{2}\left(\int \lambda^{-1}(W+\epsilon)^{-3/2}E^{1/2}dr^* + \int (W+\epsilon)^{-1/4}g dr^*\right)$, so that we may apply Gronwall's inequality to obtain

$$E^{1/2} \leq ||(W+\epsilon)^{-1/4}g||_{L^1} \exp\left(\int \frac{C}{2}\lambda^{-1}(W+\epsilon)^{-3/2}ds\right). \tag{5.47}$$

This establishes (5.41) as $\int \frac{C}{2}\lambda^{-1}(W+\epsilon)^{-3/2}ds < \infty$ on a compact region.

<u>*Case (ii):* $|\epsilon| \lesssim \lambda^{-1}$</u>

Here we choose $\sigma = \lambda^{-1}$, thus $\sigma + |W+\epsilon| \approx \lambda^{-1} + r^{*2}$. Then we show:

$$\lambda(\lambda^{-1}+r^{*2})^{1/4}|u| + (\lambda^{-1}+r^{*2})^{-1/4}|\partial_{r*}u| \lesssim ||(\lambda^{-1}+r^{*2})^{-1/4}g||_{L^1}. \tag{5.48}$$

The analysis is divided in to two subcases: (a) $|r^*| \gg \lambda^{-1/2}$ and (b) $|r^*| \lesssim \lambda^{-1/2}$.

Case (ii)(a): $|r^*| \gg \lambda^{-1/2}$. We consider the energy functional (5.40) but note that E is now positive for $W \gg \lambda^{-1}$. A proof of this is analogous to *Case (i)* with W now dominating λ^{-1} as opposed to $\epsilon \gg \lambda^{-1}$.

We again compute the derivative of E, show the bound akin to (5.46), apply Gronwall's Inequality and hence obtain (5.48) for $|r^*| \gg \lambda^{-1/2}$.

Case (ii) (b): $|r^*| \lesssim \lambda^{-1/2}$. Using this relation, showing (5.48) reduces to establishing

$$\lambda^{3/4}|u| + \lambda^{1/4}|u_{r*}| \lesssim ||\lambda^{1/4} g||_{L^1}. \tag{5.49}$$

In the given region, $(\partial^2_{r^*} + \lambda^2(W+\epsilon))u = g$ is a small perturbation of $(\partial^2_{r*} + \lambda)u = g$. Via the multiplier method, we can then derive the energy functional

$$E[u(r^*)] = \lambda^{3/2}|u|^2 + \lambda^{1/2}|u_{r*}|^2,$$

whose derivative is given by

$$\frac{d}{dr^*}E[u(r^*)] = 2\lambda^{3/2}uu_{r*} + 2\lambda^{1/2}u_{r*}g - 2\lambda^{5/2}(W+\epsilon)uu_{r^*}. \tag{5.50}$$

Since we are in a region where both $|W|$, $|\epsilon| \lesssim \lambda^{-1}$, equation (5.50) satisfies

$$\begin{aligned}\left|\frac{d}{dr^*}E[u(r^*)]\right| &\lesssim \lambda^{3/2}|uu_{r*}| + \lambda^{1/2}|u_{r*}g| \lesssim (\lambda^2|u|^2 + \lambda|u_{r*}|^2) + \lambda^{1/2}|u_{r*}g| \\ &\lesssim \lambda^{1/2}E[u(r^*)] + E^{1/2}\lambda^{1/4}|g|.\end{aligned}$$

As above $E^{1/2} \leq \frac{C}{2}\left(\int \lambda^{1/2}E[u(r^*)]^{1/2}dr^* + \int \lambda^{1/4}g\, dr^*\right)$ so that by Gronwall's Inequality

$$E^{1/2} \lesssim ||\lambda^{1/4}g||_{L^1} \exp\left(\int_{-\lambda^{-\frac{1}{2}}}^{r^*} \frac{C}{2}\lambda^{1/2}ds\right).$$

This completes the proof of *Case (ii)(b)*.

Case (iii): $-\epsilon \gg \lambda^{-1}$.

Let $\sigma = |\epsilon|^{1/3}\lambda^{-2/3}$. We must show the pointwise bound

$$\begin{aligned}\lambda(|W+\epsilon| + |\epsilon|^{1/3}\lambda^{-2/3})^{1/4}|u| + (|W+\epsilon| + |\epsilon|^{1/3}\lambda^{-2/3})^{-1/4}|\partial_{r*}u| \\ \lesssim ||(|W+\epsilon| + |\epsilon|^{1/3}\lambda^{-2/3})^{-1/4}g||_{L^1}.\end{aligned} \tag{5.51}$$

The analysis is divided in to three regions:

(a) $W+\epsilon \gg \lambda^{-\frac{2}{3}}|\epsilon|^{\frac{1}{3}}$

(b) $|W+\epsilon| \lesssim \lambda^{-\frac{2}{3}}|\epsilon|^{\frac{1}{3}}$

(c) $[r'_-, r'_+] = \{W+\epsilon < -C\lambda^{-\frac{2}{3}}\epsilon^{\frac{1}{3}}\}$.

We remark that the technique to prove the first two cases closely resembles *Cases (i)* and *(ii)* whereas *Case (iii)(c)* uses a slightly different approach.

Beginning with *Case (iii)(a),* we define the energy functional from (5.40)

$$E[u(r^*)] = \lambda^2(W+\epsilon)^{1/2}u^2 + (W+\epsilon)^{-1/2}u_{r^*}^2 + \frac{1}{2}W_{r^*}(W+\epsilon)^{-3/2}uu_{r^*} \tag{5.52}$$

and claim it is positive definite, ie. (5.44) is satisfied. Indeed,

$$\begin{aligned}|W| &\leq |W+\epsilon|^3\left(|W+\epsilon|^{-2} + |\epsilon||W+\epsilon|^{-3}\right)\\ &\ll |W+\epsilon|^3(\lambda^{4/3}|\epsilon|^{-2/3} + |\epsilon|\lambda^2|\epsilon|^{-1}) \ll \lambda^2|W+\epsilon|^3,\end{aligned}$$

where in the second inequality we applied the region of consideration and in the last inequality, we used $-\epsilon \gg \lambda^{-1}$. Applying (5.46) and Gronwall's inequality then gives

$$E^{1/2} \leq ||(W+\epsilon)^{-1/4}g||_{L^1}\exp\left(\int_{\tilde{r}}^{r^*}\frac{C}{2}\lambda^{-1}(W+\epsilon)^{-3/2}ds\right). \tag{5.53}$$

Since $W+\epsilon \gg \lambda^{-2/3}|\epsilon|^{1/3}$, (5.51) immediately follows.

Next we prove the estimate for *Case (iii) (b)*, a symmetric region about the zeroes of $W+\epsilon$. As in *Case (ii) (b)*, since $|W+\epsilon| \lesssim \lambda^{-2/3}|\epsilon|^{1/3}$, (5.37) is a small perturbation of $(\partial_{r*}^2 + \lambda^{4/3}|\epsilon|^{1/3})u = g$. We can therefore define a corresponding energy functional to the latter and obtain

$$E[u(r^*)] = \lambda^2(\lambda^{-1/3}|\epsilon|^{1/6})u^2 + (\lambda^{1/3}|\epsilon|^{-1/6})u_{r*}^2.$$

We proceed via arguments similar to *Case (ii) (b)*:

$$
\begin{aligned}
\frac{d}{dr^*}E[u](r^*) &\leq 2\lambda^{5/3}|\epsilon|^{1/6}|uu_{r^*}| + 2|u_{r^*}g|\lambda^{1/3}|\epsilon|^{-1/6} + 2|uu_{r^*}||W+\epsilon|\lambda^{7/3}|\epsilon|^{-1/6} \\
&\lesssim \lambda^{5/3}|\epsilon|^{1/6}|uu_{r^*}| + 2|u_{r^*}g|\lambda^{1/3}|\epsilon|^{-1/6}
\end{aligned}
$$

where we have used $|W+\epsilon| \lesssim \lambda^{-2/3}|\epsilon|^{1/3}$. Then

$$
\left|\frac{d}{dr^*}E[u](r^*)\right| \lesssim \lambda^{2/3}|\epsilon|^{1/6}E + \lambda^{1/6}|\epsilon|^{-1/12}E^{1/2}g.
$$

Applying Gronwall's inequality, and noting that $\int_{|W+\epsilon|\leq C\lambda^{-2/3}|\epsilon|^{1/3}} \lambda^{2/3}|\epsilon|^{1/6}dr^* < \infty$ for an arbitrary constant C, gives the desired bound.

Finally we consider *Case (iii) (c)*. Let $\omega = |W+\epsilon| + |\epsilon|^{1/3}\lambda^{-2/3}$ and consider the bounds for $|u|$ and $|\partial_{r^*}u|$ separately. For $|u|$, we require showing

$$
\lambda^2||\omega^{1/4}u||^2_{L^\infty} \lesssim ||\omega^{-1/4}g||^2_{L^1}.
$$

Multiplying equation (5.37) by $-\lambda u$ and integrating by parts yields

$$
\int_{r'_-}^{r'_+} (\lambda|\partial_{r*}u|^2 + \lambda^3|W+\epsilon||u|^2)\, dr^* = \int_{r'_-}^{r'_+} -\lambda ug dr^* + \lambda uu_{r*}\Big|_{r'_-}^{r'_+}, \tag{5.54}
$$

where the positivity of the second term on the left follows since $W+\epsilon<0$. Note that the integral on the right satisfies the bound

(5.55)
$$
\int_{r'_-}^{r'_+} \lambda ug\, dr^* \lesssim \lambda||\omega^{1/4}u||_{L^\infty(r'_-,r'_+)} \int_{r'_-}^{r'_+} \omega^{-1/4}|g|dr^* \lesssim \lambda||\omega^{1/4}u||_{L^\infty(r'_-,r'_+)}||\omega^{-1/4}g||_{L^1(r'_-,r'_+)},
$$

and that the boundary terms fall in to the region considered in *Case (iii) (b)*. Thus we have,

$$
\begin{aligned}
&\left|\int_{r'_-}^{r'_+} (\lambda|\partial_{r*}u|^2 + \lambda^3|W+\epsilon||u|^2)dr^*\right| \lesssim \\
&\qquad \lambda||\omega^{1/4}u||_{L^\infty(r'_-,r'_+)}||\omega^{-1/4}g||_{L^1(r'_-,r'_+)} + ||\omega^{-1/4}g||^2_{L^1(r'_-,r'_+)}.
\end{aligned} \tag{5.56}
$$

To complete the proof for $|u|$ it remains to show

$$\lambda^2||\omega^{1/4}u||^2_{L^\infty} \lesssim \int_{r_-}^{r_+} (\lambda|\partial_{r*}u|^2 + \lambda^3|W+\epsilon||u|^2)dr^*.$$

We proceed via the Fundamental Theorem of Calculus and write

(5.57)
$$\lambda^2|\omega^{1/4}u|^2 = \lambda^2\left|\int_{r'_-}^{r^*} \partial_{r*}(\omega^{1/2}u^2)dr^*\right| = \lambda^2\left|\int_{r'_-}^{r^*} \left(\frac{1}{2}\omega^{-1/2}\partial_{r*}\omega u^2 + 2\omega^{1/2}uu_{r*}\right) dr^*\right|.$$

Using that $\partial_{r^*}\omega \approx |W_{r^*}| \lesssim \lambda|W+\epsilon|^{3/2}$ and $\omega^{-1/2} \leq |W+\epsilon|^{-1/2}$, for the first term on the right of (5.57) we have

$$\lambda^2\left|\int_{r'_-}^{r^*} (\frac{1}{2}\omega^{-1/2}\partial_{r*}\omega)u^2 dr^*\right| \lesssim$$
$$\int_{r'_-}^{r^*} \lambda^2|W+\epsilon|^{-1/2}(\lambda|W+\epsilon|^{3/2})|u|^2 dr^* = \int_{r'_-}^{r^*} \lambda^3|W+\epsilon||u|^2 dr^*,$$

while the second integral satisfies the following bound

$$\lambda^2\left|\int_{r'_-}^{r^*} 2\omega^{1/2}uu_{r*}dr^*\right| \lesssim \lambda^3\int_{r'_-}^{r^*} \omega|u|^2 dr^* + \lambda\int_{r'_-}^{r^*} |u_{r^*}|^2 dr^*$$
$$\lesssim \int_{r'_-}^{r^*} \lambda^3|W+\epsilon||u|^2 dr^* + \lambda\int_{r'_-}^{r_+} |u_{r*}|^2 dr^*.$$

where we have used that $|\epsilon|^{1/3}\lambda^{-2/3} \lesssim |W+\epsilon|$. Combining the above two inequalities gives

$$\lambda^2||\omega^{1/4}u||^2_{L^\infty} \lesssim \int_{r'_-}^{r^*} \lambda^3|W+\epsilon||u|^2 dr^* + \int_{r'_-}^{r^*} \lambda|\partial_{r*}u|^2 dr^*. \tag{5.58}$$

We thus have

$$\lambda^2||\omega^{1/2}u||^2_{L^\infty(r'_-,r'_+)} \lesssim \lambda||\omega^{1/4}u||_{L^\infty(r'_-,r'_+)}||\omega^{-1/4}g||_{L^1} + ||\omega^{-1/4}g||^2_{L^1}.$$

We use the fact that $a^2 \leq ab + b^2 \leq \delta a^2 + \frac{1/\delta}{b}^2 + b^2 \Rightarrow a^2 \lesssim b^2$ to obtain the desired bound on $|u|$

$$\lambda \omega^{1/4}|u| \lesssim ||\omega^{-1/4} g||_{L^1}. \tag{5.59}$$

Finally we show the pointwise bound on u_{r^*},

$$||\omega^{-1/4} u_{r^*}||_{L^\infty} \lesssim ||\omega^{-1/4} g||_{L^1}. \tag{5.60}$$

Define the energy functional

$$\bar{E}[u](r^*) = -\lambda^2 |W+\epsilon|^{1/2} u^2 + |W+\epsilon|^{-1/2} (\partial_{r^*} u)^2 - \frac{1}{2} W_{r^*} |W+\epsilon|^{-3/2} u u_{r^*} \tag{5.61}$$

where $\bar{E}[u] + 2\lambda^2 |W+\epsilon|^{1/2} u^2 = E[u]$ and

$$E[u] = \lambda^2 |W+\epsilon|^{1/2} u^2 + |W+\epsilon|^{-1/2} u_{r^*}^2 + \frac{1}{2} W_{r^*} |W+\epsilon|^{-3/2} u u_{r^*},$$

similar to the energy defined in (5.40). Note the negative coefficient of u^2 prevents E from being positive definite; however, this will not affect obtaining the desired bound for u_{r^*}. As in the previous cases, we calculate the derivative of $\bar{E}[u]$,

$$\begin{aligned} \frac{d}{dr^*} \bar{E}[u](r^*) = \frac{1}{2} u u_{r^*} (|W+\epsilon|^{-3/2} W_{r^* r^*} + \frac{3}{2} W_{r^*}^2 |W+\epsilon|^{-5/2}) - \\ |W+\epsilon|^{-1/4} (u_{r^* r^*} - \lambda^2 |W+\epsilon| u)(|W+\epsilon|^{-1/4} u_{r^*} + \frac{1}{2} |W+\epsilon|^{-5/4} W_{r^*} u), \end{aligned} \tag{5.62}$$

using $u_{r^* r^*} - \lambda^2 |W+\epsilon| u = g$, and the analysis of *Case (i)* we obtain the derivative bound

$$\left| \frac{d}{dr^*} \bar{E}[u](r^*) \right| \lesssim \lambda^{-1} |W+\epsilon|^{-3/2} E + E^{1/2} |W+\epsilon|^{-1/4} g. \tag{5.63}$$

We point out that the energy used on the right hand side is that defined in (5.40) to attain positivity. Integrating out the above equation and adding $2\lambda^2 |W+\epsilon|^{1/2} u^2$ to both sides yields

$$(5.64)\quad E[u](r^*) \lesssim \int \lambda^{-1}|W+\epsilon|^{-3/2}Edr^* + \int E^{1/2}|W+\epsilon|^{-1/4}gdr^* + 2\lambda^2|W+\epsilon|^{1/2}u^2.$$

We apply Gronwall's Inequality to the first two terms and use that on the region under consideration $|W+\epsilon| \approx \omega$ on the third so that

$$(5.65)\qquad E[u](r^*) \lesssim \lambda^2 ||\omega^{1/4}u||^2_{L^\infty(r'_-,r'_+)} + ||\omega^{-1/4}g||^2_{L^1(r'_-,r'_+)}.$$

This gives the desired estimate bound $||\omega^{-1/4}u_{r^*}||_{L^\infty} \lesssim ||\omega^{-1/4}g||_{L^1(r_-,r_+)}$.

With the exception of proving the technical lemmas involved in the reduction, this completes the proof of Proposition 5.8 and hence demonstrates the logarithmic loss at $r = r_{ps}$.

Proof of Lemma 5.5: Recall that

$$a_{ps}(r^*,\xi,\lambda) = \begin{cases} 1, & \ln\lambda < C \\ \begin{cases} 1, & r^{*2}+\lambda^{-2}\xi^2 > e^{-1} \\ -\ln(r^{*2}+\lambda^{-2}\xi^2), & e^{-\sqrt{\frac{\ln\lambda}{2}}} < r^{*2}+\lambda^{-2}\xi^2 < e^{-2} \\ \sqrt{\ln\lambda}, & r^{*2}+\lambda^{-2}\xi^2 < e^{-\sqrt{\ln\lambda}} \end{cases} & ,\ln\lambda \geq C \end{cases}$$

so that $a_{ps}(r^*,\xi,\lambda) \approx a_{ps}(r^*,0,\lambda)$ if

$$(5.66)\qquad \ln(r^{*2}+e^{-\sqrt{\ln\lambda}}) \approx \ln(r^{*2}+e^{-\sqrt{\ln\lambda}}+\lambda^{-2}\xi^2).$$

In particular, (5.66) includes the case where $\lambda^{-2}\xi^2 < (r^{*2}+e^{-\sqrt{\ln\lambda}})^{1/8}$. We characterize this set as the region $D = \left\{\ln(\lambda^{-2}\xi^2) < \frac{1}{8}\ln(r^{*2}+e^{-\sqrt{\ln\lambda}})\right\}$. Define a smooth cut off function, $\chi(x) = \begin{cases} 1, & x\in(-\infty,-1] \\ 0, & x\in[0,\infty) \end{cases}$ and a smooth characteristic function of D, $\chi_D(r^*,\xi,\lambda) = \chi(\ln(\lambda^{-2}\xi^2) - \frac{1}{8}\ln(r^{*2}+e^{-\sqrt{\ln\lambda}}))$. Upon computing $\partial_\xi^\alpha\partial_x^\beta\chi_D$ we find that $|\partial_\xi^\alpha\partial_{r^*}^\beta\chi_D(r^*,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{-\alpha+\delta\beta}$, so that for $\delta>0$, $\chi_D \in S^0_{1,\delta}$.

Define the symbol

$$q(r^*,\xi,\lambda)=\lambda^2\xi^{-2}(1-\chi_D)$$

and the components of g,

$$g_1 := (1+\lambda^{-2}D_{r^*}^2 q^w)g \qquad (5.67)$$
$$g_2 := q^w g,$$

where $D_{r^*}=\frac{1}{i}\partial_{r^*}$. It follows immediately that $g=g_1+\lambda^{-2}\partial_{r^*}^2 g_2$. As constructed, g_1 and g_2 are not necessarily supported near the photon sphere, $r^*=0$. We can define new functions,

$$\tilde{g}_1 := \chi_1(r^*)g_1$$
$$\tilde{g}_2 := \chi_1(r^*)g_2,$$

where χ_1 is a smooth compactly supported cutoff function, identically equal to 1 on the support of g. As will be evident below, lemma 5.5 still holds in this case. We thus omit the latter construction and define g_1 and g_2 as in (5.67). The analysis is divided in showing the estimate for g_1 and g_2 separately. Beginning with g_2, we are required to show

$$|||r^{*2}+e^{-\sqrt{\ln\lambda}}|^{1/8}g_2||_{L^2}+\lambda^{-2}||\partial_{r^*}g_2||_{L^2}\lesssim ||a_{ps}^w(\lambda)g||_{L^2}. \qquad (5.68)$$

However, using that $(a_{ps}^{-1})^w(\lambda)$ is an approximate inverse for $a_{ps}^w(\lambda)$, we can equivalently let $g=(a_{ps}^{-1})^w(\lambda)f$, and show

$$||(r^{*2}+e^{-\sqrt{\ln\lambda}})^{1/8}q^w(a_{ps}^{-1})^w(\lambda)f||_{L^2}+\lambda^{-2}||\partial_{r^*}(q^w(a_{ps}^{-1})^w(\lambda)f)||_{L^2}\lesssim ||f||_{L^2}. \qquad (5.69)$$

Consider the first term on the left hand side of (5.69). In calculating the operator product $(r^{*2}+e^{-\sqrt{\ln\lambda}})^{1/8}q^w(a_{ps}^{-1})^w(\lambda)$ it is sufficient to consider the principal symbol since the remainder lies in the smoother space $OPS^{-1+\delta}_{1,\delta}$. We show that the product of the symbols is bounded as this would enable us to use L^2-L^2 estimates. Note that a_{ps}^{-1}

is bounded by (5.7). For $q(r^*, \xi, \lambda)$, if $|\xi| \gtrsim \lambda$ than both $\frac{\lambda^2}{\xi^2}$ and $(1 - \chi_D)$ are bounded. For $|\xi| \lesssim \lambda$ note that the support of q corresponds to the region where $\chi_D \neq 1$, or equivalently, $\frac{\lambda^{-2}\xi^2}{(r^{*2}+e^{-\sqrt{\ln \lambda}})^{1/8}} > e^{-1}$ so that $q \lesssim (r^{*2} + e^{-\sqrt{\ln \lambda}})^{-1/8}$. It then follows that

$$||(r^{*2} + e^{-\sqrt{\ln \lambda}})^{1/8} q^w (a_{ps}^{-1})^w(\lambda) f||_{L^2} \lesssim ||f||_{L^2}.$$

For the second term, by Plancherel we consider $|\lambda^{-2}\xi q a^{-1}| = \frac{1}{|\xi|}|1 - \chi_D||a^{-1}|$. If $\lambda \lesssim |\xi|$ all three terms of the product are trivially bounded. For $\lambda \gtrsim |\xi|$, we use that on the support of χ_D, $\frac{\lambda^{-2}e}{(r^{*2}+e^{-\sqrt{\ln \lambda}})^{1/8}} > \frac{1}{\xi^2}$. This then establishes (5.69).

Next we show the desired bound for g_1 in lemma 5.5,

$$||a_{ps}(r^*, 0, \lambda) g_1||_{L^2} \lesssim ||a_{ps}^w(\lambda) g||_{L^2}.$$

Again, as in the case of g_2 we show an equivalent bound:

$$||a_{ps}(r^*, 0, \lambda)(1 + \lambda^{-2} D_{r^*}^2 q^w)(a_{ps}^{-1})^w(\lambda) f||_{L^2} \lesssim ||f||_{L^2}.$$

We have already shown that for $\delta > 0$, $a_{ps}(r^*, 0, \lambda)$, $(a_{ps}^{-1})^w(\lambda)$ are in $S^{\delta}_{1,\delta}$. Moreoever $(1 + \lambda^{-2} D_{r^*}^2 q^w) \in S^0_{1,\delta}$ so that the product of the three lies in $S^{\delta}_{1,\delta}$. As above, it suffices to show that the principal symbol of the product, namely,

$$a_{ps}(r^*, 0, \lambda)\chi_D a_{ps}^{-1}(r^*, \xi, \lambda) = \frac{\gamma(-\ln r^{*2}, \ln \lambda)}{\gamma(-\ln(r^{*2} + \lambda^{-2}\xi^2), \ln \lambda)} \chi_D \tag{5.70}$$

is bounded. However, on the support of χ_D, (5.66) holds from which it easily follows that (5.70) is bounded. This completes the proof of lemma 5.5.

Proof of Lemma 5.6: We must show for every σ satisfying $\lambda^{-1} < \sigma < 1$ and compactly supported $\tilde{u}$, estimate (5.34) holds. Recall that we are assuming $W(0) = W'(0) = 0$ and $W''(0) > 0$ so that for r^* in a neighbourhood of the origin, $W \geq 0$. Thus if $\epsilon + \sigma > 0$, $\sigma + |W + \epsilon| \approx \sigma + |\epsilon| + W$. This allows us to replace the pair (ϵ, σ) with $(0, \sigma + |\epsilon|)$. We can then assume that either $\epsilon = 0$ or $\epsilon < -\sigma$ as the $\epsilon = 0$ case implies $\epsilon + \sigma > 0$. The proof is divided in to three cases.

Case 1: $|\epsilon|,\ \sigma < e^{-\sqrt{\ln\lambda}}$. Divide $\tilde{u}$ in to an almost orthogonal dyadic decomposition in the spatial r^* variable. That is, we divide the interval $(-1,1)$ and consider u on the regions $|r^*| < s_0$ and $|r^*| > s_0$, for s_0 specified below. Specifically,

$$\tilde{u} = \tilde{u}_{<s_0} + \sum_{s_0 \le s < 1} \tilde{u_s}, \quad s_0 = e^{-\frac{1}{2}\sqrt{\ln\lambda}},$$

where u_s is supported on the annulus I_s, and $|I_s| \approx s$. For each u_s, we fix r^* in a_{ps}^{-1}. The pseudo-differential operator reduces to a Fourier multiplier and we can write

$$\begin{aligned} &||(a_{ps}^{-1})^w(\lambda)\tilde{u_s}||_{L^2} \approx ||a_{ps}^{-1}(s,D,\lambda)\tilde{u}_s||_{L^2} \\ &||(a_{ps}^{-1})^w(\lambda)\tilde{u}_{<s_0}||_{L^2} \approx ||a_{ps}^{-1}(0,D,\lambda)\tilde{u}_{<s_0}||_{L^2}. \end{aligned} \tag{5.71}$$

We first consider the region $s > s_0$, so that $a_{ps}^{-1} \approx -(\ln(r^{*2} + \lambda^{-2}\xi^2))^{-1}$. When $\lambda^{-2}\xi^2 < s^{2\delta}$, we have $a_{ps}^{-1} \lesssim \ln|s|^{-1}$. On the other hand, for $\lambda^{-2}\xi^2 > s^{2\delta}$, we have that $a_{ps}^{-1} \le 1$. This gives

$$\lambda a_{ps}^{-1}(s,\xi,\lambda) \lesssim \lambda|\ln s|^{-1} + s^{-\delta}|\xi|, \quad \delta > 0. \tag{5.72}$$

It follows from (5.71) and Plancherel that

$$\lambda||(a_{ps}^{-1})^w(\lambda)\tilde{u_s}||_{L^2} \lesssim |\ln s|^{-1}\lambda||\tilde{u_s}||_{L^2} + s^{-\delta}||\partial_{r^*}\tilde{u_s}||_{L^2}. \tag{5.73}$$

We next show that the right hand side of (5.73) is bounded above by the right hand side of (5.34). In applying Cauchy Schwarz to each term, we respectively obtain,

$$\begin{aligned} \left(\int_{I_s} |\tilde{u_s}|^2 dr^*\right)^{1/2} &\lesssim ||(\sigma + |W+\epsilon|)^{1/4}\tilde{u_s}||_{L^\infty} \left(\int_{I_s} (\sigma + |W+\epsilon|)^{-1/2} dr^*\right)^{1/2} \\ \left(\int_{I_s} |\partial_{r^*}\tilde{u_s}|^2 dr^*\right)^{1/2} &\lesssim s^{1/2}||(\sigma + |W+\epsilon|)^{-1/4}\partial_{r^*}\tilde{u_s}||_{L^\infty} \left(\int_{I_s} s^{-1}(\sigma + |W+\epsilon|)^{1/2} dr^*\right)^{1/2}. \end{aligned} \tag{5.74}$$

Note that in the case under consideration, $|\epsilon|$, $\sigma < e^{-\sqrt{\ln\lambda}}$ and $s > s_0$, thus $\sigma + |W+\epsilon| \approx W$. In particular, on the support of $\tilde{u_s}$, $\sigma + |W+\epsilon| \approx s^2$. Substituting this in to the above integrals and using that $|I_s| = s$ gives

(5.75)
$$\lambda||(a_{ps}^{-1})^w(\lambda)\tilde{u_s}||_{L^2} \lesssim |\ln s|^{-1}\lambda||(\sigma+|W+\epsilon|)^{1/4}\tilde{u}||_{L^\infty} + s^{1-\delta}||(\sigma+|W+\epsilon|)^{-1/4}\partial_{r^*}\tilde{u}||_{L^\infty},$$

where we have replaced u_s with u on the right hand side. The almost L^2 orthogonal nature of the functions $(a_{ps}^{-1})^w(\lambda)\tilde{u}_s$ gives

$$\lambda\Big|\Big|\sum_{s_0\le s<1}(a_{ps}^{-1})^w(\lambda)\tilde{u_s}\Big|\Big|_{L^2}^2 \lesssim \lambda\sum_{s_0\le s<1}||(a_{ps}^{-1})^w(\lambda)\tilde{u_s}||_{L^2}^2. \tag{5.76}$$

Indeed, in calculating the kernel of $(a_{ps}^{-1})^w$ the cross terms exhibit rapid decay and can therefore be bootstrapped in to the square terms as desired. We apply this to (5.75) and note that both terms on the right are in fact summable. For the first term, letting $s \approx 2^{-j^2} \Rightarrow |(\ln|s|)^{-1}| \approx j^{-2}$ we have convergence as a p series, whereas in the second term $s \approx |2^{-j^2}|^{1-\delta}$ so that convergence is established via a geometric series. This gives the desired estimate

$$\lambda||(a_{ps}^{-1})^w(\lambda)\tilde{u}||_{L^2} \lesssim \lambda||(\sigma+|W+\epsilon|)^{1/4}\tilde{u}||_{L^\infty} + ||(\sigma+|W+\epsilon|)^{-1/4}\partial_{r^*}\tilde{u}||_{L^\infty}. \tag{5.77}$$

To complete the proof for this case, we consider the region where $|r^*| < s_0$. An analysis identical to the one above yields the bound

$$\lambda||(a_{ps}^{-1})^w(\lambda)\tilde{u}_{<s_0}||_{L^2} \lesssim |\ln s_0|^{-1}\lambda||\tilde{u}_{<s_0}||_{L^2} + s_0^{-\delta}||\partial_{r^*}\tilde{u}_{s_0}||_{L^2}. \tag{5.78}$$

In applying Cauchy-Schwarz as in (5.74) we obtain the bound

$$\int_{|r^*|<s_0}(\sigma+|W+\epsilon|)^{-1/2}dr^* \lesssim \ln\lambda, \tag{5.79}$$

so that we pick up the weaker estimate

(5.80)
$$\lambda||(a_{ps}^{-1})^w(\lambda)\tilde{u}_{<s_0}||_{L^2} \lesssim \lambda||(\sigma+|W+\epsilon|)^{1/4}\tilde{u}||_{L^\infty} + s_0^{1-\delta}||(\sigma+|W+\epsilon|)^{-1/4}\partial_{r^*}\tilde{u}||_{L^\infty}.$$

Despite losing the logarithmic weight the desired bound is obtained and hence completes the proof of *Case 1*.

Case 2: $\epsilon = 0$, $\sigma \geq e^{-\sqrt{\ln\lambda}}$. As before, we decompose u as

$$\tilde{u} = \tilde{u}_{<s_0} + \sum_{s_0 \leq s < 1} \tilde{u_s}, \quad s_0 = \sqrt{\sigma}.$$

The proof is analogous to *Case 1* for $r^* > s_0$. For $r^* < s_0$, the integral on the left of (5.79) is now integrable making the analysis easier.

Case 3: $\epsilon < -e^{-\sqrt{\ln\lambda}}, \sigma < -\epsilon$. Again we decompose u,

$$\tilde{u} = \tilde{u}_{<s_0} + \sum_{s_0 \leq s < 1} \tilde{u_s}, \quad s_0 = \sqrt{-\epsilon},$$

and follow the analysis done in the previous cases for $|r^*| > s_0$. As in *Case 1*, there is a singularity in the weight $(\sigma + |W+\epsilon|)^{-1/2}$ in the region $|r^*| < s_0$. However, here note that (5.78) satisfies

(5.81) $$\int_{|r^*|<s_0} (\sigma+|W+\epsilon|)^{-1/2} dr^* \lesssim \int_{|r^*|<s_0} |W+\epsilon|^{-1/2} dr^* \leq \int_{|r^*|<s_0} 2|\epsilon|^{-1/2} dr^* \lesssim 1.$$

This completes the proof of Lemma 5.6.

Proof of lemma (5.7)*:* We wish to show

(5.82) $$||(\lambda^{-1}+|W+\epsilon|)^{-1/4} g_1||_{L^1} \lesssim ||a_{ps}(r^*,0,\lambda)g||_{L^2}$$

with g_1, g defined above. Applying Cauchy-Schwarz to the left hand side, and denoting I to be the support of g_1, it would suffice to show

$$\int_I (\lambda^{-1} + |r^{*2} + \epsilon|)^{-1/2} a_{ps}^{-2}(r^*, 0, \lambda) dr^* < \infty. \tag{5.83}$$

Let $G(\epsilon, r^*)$ denote the integrand of (5.83) and consider the cases $\epsilon \geq 0$ and $\epsilon < 0$, separately. For $\epsilon \geq 0$, we let

$$\begin{aligned} I_1 &= \{|r^*| < \lambda^{-1/2}\} \\ I_2 &= \{\lambda^{-1/2} < |r^*| < e^{-\sqrt{\ln \lambda}}\} \\ I_3 &= \{I \cap \{e^{-\sqrt{\ln\lambda}} < |r^*|\}\}. \end{aligned}$$

Then,

$$\int_{I_1} G(\epsilon, r^*) dr^* \lesssim \int_{I_1} (\lambda^{-1})^{-1/2} dr^* \leq \lambda^{1/2}(2\lambda^{-1/2}) \lesssim 1.$$

$$\int_{I_2} G(\epsilon, r^*) dr^* \lesssim \int_{I_2} \frac{1}{|r^{*2}|^{1/2}} \frac{1}{(\sqrt{\ln\lambda})^2} dr^* \lesssim 1, \text{ since } a_{ps} \approx \sqrt{\ln\lambda} \text{ for } |r^*| < e^{-\sqrt{\ln\lambda}}.$$

$$\begin{aligned} \int_{I_3} G(\epsilon, r^*) dr^* \lesssim & \int_{I \cap \{e^{-\sqrt{\ln\lambda}} < |r^*| < e^{-1}\}} \frac{1}{|r^{*2}|^{1/2}} \frac{1}{(\ln r^*)^2} dr^* \\ & + \int_{I \cap \{e^{-1} < |r^*|\}} G(\epsilon, r^*) dr^* \lesssim 1 + |I| \lesssim 1 \end{aligned}$$

where in this last case we have used that $a_{ps}(r^*, 0, \lambda) \approx \begin{cases} \ln|r^*|, & \{e^{-\sqrt{\ln\lambda}} < |r^*| < e^{-2}\} \\ 1, & \{e^{-1} < |r^*|\} \end{cases}$

and that G is bounded away from $r^* = 0$.

For the case when $\epsilon < 0$, let

$$\begin{aligned} I_1 &= |r^*| < |\epsilon|^{1/2} \\ I_2 &= I \cap \{|\epsilon|^{1/2} < |r^*|\} \end{aligned}$$

Then,

$$\int_{I_1} G(\epsilon, r^*)dr^* \lesssim \int_{I_1} \frac{1}{(-r^{*2}+\epsilon)^{1/2}}dr^* \leq \int_{|r^*|<|\epsilon|^{1/2}} |\epsilon|^{-1/2}dr^* \lesssim 1,$$

$$\begin{aligned}
\int_{I_2} G(\epsilon, r^*)dr^* &\lesssim \int_{I\cap\{\sqrt{\epsilon}<|r^*|<e^{-1}\}} \frac{1}{(\ln r^*)^2(r^{*2}-\epsilon+\lambda^{-1})^{1/2}}dr^* + \int_{I\cap\{e^{-1}<|r^*|\}} G(\epsilon, r^*)dr^* \\
&\lesssim \int_{\sqrt{\epsilon}<|r^*|<e^{-1}} \frac{1}{(\ln r^*)^2 r^*}dr^* + |I| \lesssim 1
\end{aligned}$$

where again as above, we have used $a_{ps}(r^*, 0, \lambda) \approx \ln|r^*|$ in the first integral, and that $G(\epsilon, r^*)$ is bounded in the second.

This completes the proof of Lemma (5.7) and hence the proof of Proposition (5.8).

References

1. S. Alinhac: *On the Morawetz–Keel-Smith-Sogge inequality for the wave equation on a curved background*. Publ. Res. Inst. Math. Sci. **42** (2006), 705–720.

2. S. Alinhac: *Energy multipliers for perturbations of the Schwarzschild metric*. Comm. Math. Phys. **288** (2009), 199–224.

3. L. Andersson and P. Blue: *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, preprint. (arXiv: 0908.2265)

4. L. Bieri and N. Zipser, *Extensions of the stability theorem of the Minkowski space in general relativity*. AMS/IP Studies in Advanced Mathematics, **45**. American Mathematical Society, Providence, RI; International Press, Cambridge MA, 2009. 491pp.

5. P. Blue: *Decay of the Maxwell field on the Schwarzschild manifold*. J. Hyperbolic Diff. Equ. **5** (2008), 807–856.

6. P. Blue and A. Soffer: *Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates*. Adv. Differential Equations **8** (2003), 595–614.

7. P. Blue and A. Soffer: *The wave equation on the Schwarzschild metric II: Local decay for the spin-2 Regge-Wheeler equation*. J. Math. Phys. **46** (2005), 9pp.

8. P. Blue and A. Soffer: *Errata for "Global existence and scatttering for the nonlinear Schrödinger equation on Schwarzschild manifolds", "Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates", and "The wave equation on the Schwarzschild metric II: Local decay for the spin 2 Regge Wheeler equation"*, preprint. (ArXiv: gr-qc/0608073)

9. P. Blue and A. Soffer: *Phase space analysis on some black hole manifolds*. J. Funct. Anal. **256** (2009), 1–90.

10. P. Blue and A. Soffer: *Improved decay rates with small regularity loss for the wave equation about a Schwarzschild black hole*, preprint. (ArXiv: math/0612168)

11. P. Blue and A. Soffer: *A space-time integral estimate for a large data semi-linear wave equation on the Schwarzschild manifold*. Lett. Math. Phys. **81** (2007), 227–238.

12. P. Blue and J. Sterbenz: *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space*. Comm. Math. Phys. **268** (2006), 481–504.

13. J.-F. Bony and D. Häfner: *The semilinear wave equation on asymptotically Euclidean manifolds*, preprint. (ArXiv: 0810.0464)

14. N. Burq: *Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge "Global Strichartz estimates for nontrapping perturbations of the Laplacian"*. Comm. Partial Differential Equations **28** (2003), 1675–1683.

15. S. Carroll: *An Introduction to General Relativity Spacetime and Geometry*. Pearson Education Inc. San Francisco, CA (2004).

16. M. Dafermos and I. Rodnianski: *Small-amplitude nonlinear waves on a black hole background*. J. Math. Pures Appl. **84** (2005), 1147–1172.

17. M. Dafermos and I. Rodnianski: *The red-shift effect and radiation decay on black hole spacetimes*. Comm. Pure Appl. Math. **62** (2009), 859–919.

18. M. Dafermos and I. Rodnianski: *A note on energy currents and decay for the wave equation on a Schwarzschild background*, preprint. (ArXiv: 0710.017)

19. M. Dafermos and I. Rodnianski: *Lectures on black holes and linear waves*, preprint. (arXiv: 0811.0354)

20. M. Dafermos and I. Rodnianksi: *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, preprint. (ArXiv: 0910.4957)

21. M. Dafermos and I. Rodnianski: *The black hole stability problem for linear scalar perturbations*, preprint. (ArXiv:1010.5137)

22. M. Dafermos and I. Rodniaski: *Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| << M$ or axisymmetry*, preprint. (ArXiv: 1010.5132)

23. D. Christodoulou and S. Klanierman, *The global nonlinear stability o f the Minkowski space*. Princeton Mathematical Series, **41**. Princeton University Prss, Princeton, NJ, 1993. 514pp.

24. X. Dianyan: *Exact solutions of Einstein and Einstein-Maxwell equations in higher-dimensional spacetime* Class. Quantum Grav. **5** (1988) 871.

25. R. d'Inverno: *Introducing Einstein's Relativity* Oxford University Press, New York. 1992.

26. R. Gregory and R. Laflamme: *Hypercylindrical black holes*. Phys. Rev. D **37** (1988), 305–308.

27. K. Hidano and K. Yokoyama: *A remark on the almost global existence theorems of Keel, Smith, and Sogge*. Funkcial. Ekvac. **48** (2005), 1–34.

28. M. Keel, H. Smith, and C. D. Sogge: *Almost global existence for some semilinear wave equations*. J. Anal. Math. **87** (2002), 265–279.

29. C. E. Kenig, G. Ponce, and L. Vega: *On the Zakharov and Zakharov-Schulman systems*. J. Funct. Anal. **127** (1995), 204–234.

30. S. Klainerman: *The null condition and global existence to nonlinear wave equations,* Lectures in Applied Math. **23** (1986), 293 -326.

31. H. Lindblad and I. Rodnianksi, *The global stability of the Minkowski space-time in harmonic gauge,* preprint. (arXiv:math/0411109v2)

32. I. Laba and A. Soffer. *Global existence and scattering for the nonlinear Schrodinger equation on Schwarzschild manifolds*. Helv. Phys. Acta 72 (1999), 274-294.

33. J. Luk: *Improved decay for solutions to the linear wave equation on a Schwarzschild black hole*, preprint. (ArXiv: 0906.5588)

34. J. Marzuola, J. Metcalfe, D. Tataru, and M. Tohaneanu: *Strichartz estimates on Schwarzschild black hole backgrounds*. Comm. Math. Phys. **293** (2010), 37–83.

35. J. Metcalfe: *Global existence for semilinear wave equations exterior to nontrapping obstacles*. Houston J. Math. **30** (2004), 259–281.

36. J. Metcalfe: *Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle,* Trans. Amer. Math. Soc. 356 (2004), 4839-4855

37. J. Metcalfe and C. D. Sogge: *Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*. SIAM J. Math. Anal. **38** (2006), 391–420.

38. J. Metcalfe and D. Tataru: *Global parametrices and dispersive estimates for variable coefficient wave equations*. Math. Ann., to appear.

39. J. Metcalfe and D. Tataru: *Decay estimates for variable coefficient wave equations in exterior domains*. Advances in Phase Space Analysis of Partial Differential Equations, In Honor of Ferruccio Colombini's 60th Birthday, Progress in Nonlinear Differential Equations and Their Applications, Vol 78, 2009. p. 201–217.

40. C. W. Misner, K. S. Thorne, and J. A. Wheeler: *Gravitation*. San Francisco, CA: W. H. Freeman and Co., 1973.

41. C. Morawetz: *Time decay for the nonlinear Klein-Gordon equations*. Proc. Roy. Soc. Ser. A. **306** (1968), 291–296.

42. C. Morawetz and W. Strauss: *Decay and scattering of solutions of a nonlinear relativistic wave equation* Comm. Pure Appl. Math. **25** (1972), 1-31.

43. R. C. Myers and M. J. Perry: *Black holes in higher dimensional space-times*. Ann. Physics **172** (1986), 304–347.

44. V. Schlue: *Linear waves on higher dimensional Schwarzschild black holes*, preprint.

45. H. F. Smith and C. D. Sogge: *Global Strichartz estimates for nontrapping perturbations of the Laplacian*. Comm. Partial Differential Equations **25** (2000), 2171–2183.

46. C. D. Sogge and C. Wang: *Concerning the wave equation on asymptotically Euclidean manifolds*, preprint. (ArXiv: 0901.0022)

47. J. Sterbenz: *Angular regularity and Strichartz estimates for the wave equation.* With an appendix by I. Rodnianski. Int. Math. Res. Not. **2005**, 187–231.

48. W. A. Strauss: *Dispersal of waves vanishing on the boundary of an exterior domain.* Comm. Pure Appl. Math. **28** (1975), 265–278.

49. F. R. Tangherlini: *Schwarzschild field in n dimensions and the dimensionality of space problem*. Nuovo Cimento **27** (1963), 636–651.

50. T. Tao *Nonlinear Dispersive Equations: Local and Global Analysis* , CBMS Regional Conference Series in Mathematics , vol. 106. 2006

51. D. Tataru: *Local decay of waves on asymptotically flat stationary space-times*, preprint. (ArXiv: 0910.5290)

52. D. Tataru and M. Tohaneanu: *Local energy estimate on Kerr black hole backgrounds*, preprint. (ArXiv: 0810.5766).

53. M. Tohaneanu: *Local energy estimates and Strichartz estimates on Schwarzschild and Kerr black hole backgrounds.* University of California, Berkeley, (2009). Thesis.

54. M. Tohaneanu: *Strichartz estimates on Kerr black hold backgrounds*. Preprint. (ArXiv: 0910.1545).

55. G. Weinstein:General Relativity: The geodesics of the Schwarzschild metric. Lecture notes: http://www.math.uab.edu/weinstei/notes/geo.pdf.

56. R. M. Wald: *General relativity.* University of Chicago Press, Chicago, IL, 1984.