ABSTRACT

JOHN F. JOSEPH. Application of Queuing Theory to Standpost Design. (With the Assistance of Dr. DONALD T. LAURIA)

Queuing theory was used to show that lengthy waiting lines can develop at standposts if World Health Organization guidelines are used for design. Line lengths were predicted as a function of population size, customer arrival rate, and other factors.

CONTENTS

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I thank God Almighty, who has a heart for the poor, and who was my primary inspiration for completing this report.

I thank Dr. Donald T. Lauria, who learned about queuing theory so that he could effectively help me write this report. I thank him for his guidance in making the abstract concepts concrete and more easily understood for the reader.

I thank Dr. Daniel A. Okun and Dr. Dale Whittington for their review and interest.

Chapter I

INTRODUCTION

A. Purpose

Hundreds of thousands of people live in the slums of Tegucigalpa, Honduras. Many of them have migrated from rural areas in the hope of finding a better life in the city, yet most struggle to obtain even the basic necessities. One such necessity is water. House connections to the public water supply are rare in the slums of Tegucigalpa. Residents must often travel to a standpost, and upon arriving there, they may have to wait in line for hours before filling their buckets.

The time these Tegucigalpans spend waiting in line for water is of concern. A mother waiting in line might otherwise care for her children or sell more tortillas so that her family could have more adequate meals. A child waiting in line might otherwise be in school. Time is precious for all, especially those struggling to survive.

In Ukunda, Kenya, a rural community, not only do the water consumers have to wait in line, but the water vendors do as well in order to fill containers prior to selling them door-to-door. If these vendors could spend less time waiting in line, they might be able to make better livings for themselves, serve more people, and possibly pass on the savings to the consumers.

The problem of poor people having to wait in line for water is common not only in Tegucigalpa and Ukunda but in many Third World countries. Wherever significant numbers of people are without house connections to a water system, there is a possibility that they will have to spend valuable time waiting in line.

The purpose of this paper is twofold:

1. Develop a fuller understanding of how waiting lines develop at public standposts.
2. Apply queuing theory to determine the standpost discharge capacity (i.e., the flowrate when the faucets are fully open) required to serve a given population size, or to determine the population size which can be served by a standpost of a given discharge capacity.

The problem of long waiting lines is caused by many factors, many of which are nontechnical (e.g., lack of funding for capital improvements). However, a technical understanding of how waiting lines develop and the design requirements for eliminating lengthy waiting lines are necessary steps towards solving the problem.

B. The Role of Queuing Theory

The World Health Organization has published guidelines for standpost design in <u>Public Standpost Water Supplies - A Design Manual - Technical Paper Series 14</u>. Annex A is an excerpt from this manual. Some items of interest pertaining to these WHO guidelines are the following:

1. The guidelines are based on the assumption that during 'peak hours' the number of customers which can be served per unit time should be equal to the expected number of arrivals per unit time.
2. The number of 'peak hours' may vary from 4 to 12, according to section 3.3 of the excerpt. Section 3.4 states that there is a certain "water collection pattern

during peak hours". 'Peak hours' in the excerpt represents the time period over which the arrivals at the standpost are relatively frequent. The arrivals during some hours within this time period are more frequent than the arrivals during other hours within this time period. Therefore, the 'peak demand' in the excerpt is actually an average demand over a period of several hours during which a time-varying demand may occur.

In regard to item #1, queuing theory can show that if the discharge capacity is such that the number of customers that can be served per unit time is equal to the expected number of arrivals per unit time, then, in situations where the number of users is sufficiently large, the length of the waiting line will tend to grow quite long as time progresses. Queuing theory suggests that discharge capacities should generally be higher than those indicated in the guidelines, if long waiting lines are to be avoided.

In regard to item #2, queuing theory can show that if demand follows a time-varying pattern, then the average length of the line will be longer than if demand were constant. The average length of the line increases with the extent to which demand varies with time. The degree of variation during 'peak hours' should be considered for design purposes.

Regardless of whether the guidelines in Annex A are satisfactory for consumers and vendors with small containers to fill, intuition suggests that standposts for water vendors who have carts or trucks should be designed for different discharge capacities. A truck or a cart may take much longer to fill than a personal container carried by hand. A waiting line of 3 persons may not be a cause for concern, but a waiting line of 3 vendors with trucks or carts would probably be of great concern to other users. Queuing theory can be applied to help determine required discharge capacities for standposts serving such vendors.

Finally, queuing theory is useful in developing an understanding of how waiting lines develop. Such understanding is helpful for design purposes.

Chapter II

LITERATURE REVIEW

Sule and Oni (1988) have applied queuing theory to standpost design, but make the following three assumptions:

1) The expected number of arrivals at the standpost per unit time is constant throughout the day.

2) The time required to serve customers is exponentially distributed among the customers.

3) The population is large enough to be considered infinite.

In regard to the first two assumptions, the arrival rate will in many cases vary throughout the day (Feachem, et al, 1972) and the time required to serve customers is generally not exponentially distributed. Queuing models which allow for a time-varying arrival rate have been presented by Koopman (1972) in his analysis of airplane queues at airports and by Stevenson (1971) in his study of emergencies requiring ambulances. These models use a step function to approximate the time-varying behavior, and also do not require that service times be exponentially distributed. (The models presented in Chapter IV generally follow the pattern of these models.) Yet, as in the work by Sule and Oni, the population is assumed to be infinite.

The assumption of an infinite population is often used in the literature because it allows for simplicity and flexibility. The literature does not clearly state how large a population must be if this assumption is to provide accurate results. It is suspected that in some cases the population served by a standpost may be too small to be considered infinite, and that a small-population model is necessary. WHO guidelines suggest that the population served by a single-faucet standpost be kept between 25 and 125 persons. A population of 25 is far less than any of the population sizes assumed to be infinite in the literature.

Unfortuneately, the amount of queuing literature for populations which are too small to be considered infinite is scanty. Peck and Hazelwood (1958) present equations and tables for the expected queue length without assuming the population is infinite, but some other rather restrictive assumptions are employed. Also, the solutions apply only when the queue length has reached equilibrium, and no indication is given of the time required to reach equilibrium.

In summary, the literature is helpful for populations which can be considered infinite, but it neither states how large in practical terms the population must be to be considered infinite nor what to do when it cannot be considered infinite.

Chapter III

WHO DESIGN GUIDELINES AND QUEUING THEORY SERVICE TIME

Based on the WHO guidelines in Annex A, the required discharge capacity per standpost, Q_{max}, in units of volume per hour, is given by the following equation:

$$Q_{max} = N \times C_d/24 \times P \times 1/(1 - w) \times 1/f \qquad (3.1)$$

where N = design population. The guidelines recommend limiting N to 100 to 250 people per standpost, not exceeding 500 people per standpost in any case. Also, the number of users per faucet should be in the range of 25 to 125.

C_d = average per capita daily demand

P = peak factor. P is supposed to account for the standpost being used more intensively during some hours than others. The guidelines state that P is normally in the range of 2 to 4, and that the number of peak hours per day is typically between 4 and 12. The guidelines state that P can be approximated as 24/t, in which t is the number of peak hours.

w = waste factor, or the fraction of water discharged at the standpost that is not carried away by customers. This portion includes water spilt as containers are being filled and also water used directly from the tap for purposes such as washing clothes. According to the guidelines, w is in the range of 0.1 to 0.4.

f = efficiency factor. This factor is supposed to account for the tap not flowing fully while it is being opened and closed. The guidelines state that an ordinary screw tap has an efficiency factor of 0.8 to 0.9, while a rapid closing ball valve has an efficiency factor of nearly 1.0.

The efficiency factor f and the waste factor w require careful consideration. The efficiency factor in equation 3.1 is necessary because of the time required to open and close the tap. This opening and closing leaves less time available for the standpost to discharge at its full capacity. The discharge capacity must therefore be increased by a factor of 1/f to compensate for the lost time. The inclusion of the efficiency factor in equation 3.1 would theoretically account for the closing and opening time only if this time were proportional to the time required to fill the container, which is not the case. The opening and closing time depends on the valve type and remains constant regardless of the time required to fill a container. However, throughout this paper, the efficiency factor is assumed to account for the valve opening and closing time.

The waste factor w includes all water which is discharged from the standpost but not hauled away in containers. The factor thus not only includes water spilt while containers are being filled, but also water used directly from the tap for purposes such as washing clothes and water wasted due to the tap being left open or leaking when the standpost is not in use. For the examples worked in this paper, it is assumed that taps are kept closed when the standpost is not in use and that the standpost can be used only for filling containers. Thus w is at the lower end of its range, about 0.1.

A basic parameter of queuing theory is tau (τ), the time required to serve a single customer. This service time is given by

$$\tau = V \times (c / Q_{max}) \times 1/(1 - w) \times 1/f \qquad (3.2)$$

where V = volume of water customers obtain per visit to the standpost

c = number of taps at the standpost

Notice that w in equation 3.2 includes only water wasted while containers are being filled. It does not include other water wasted (e.g., water being wasted by a tap left open when not in use) because such wasted water would not contribute to the service time.

Neither equation 3.2 nor the WHO guidelines account for the time required to position the container under the tap and remove it when full. This time is assumed to be negligible throughout this paper.

Chapter IV

LARGE-POPULATION MODELS

A key assumption for the models presented in this chapter is that the population served by the standpost is large enough to be considered infinite. This assumption makes modeling relatively easy and flexible. However, if the population is not sufficiently large, the assumption will produce erroneous results. Therefore, a small-population model is presented in the following chapter. Its results will be compared with those of the large-population model to determine how large a population must be to be considered infinite.

A. Basic Large-Population Model

The basic large-population model employs the following assumptions:

1. A standpost begins service with no one waiting in line when it opens.
2. The standpost has only one tap.
3. The time required to serve a customer, tau (τ), is the same for all customers. Units for tau are minutes or hours.
4. The expected (i.e., average) rate at which customers arrive, lambda (λ), does not vary with time. Units for lambda are persons/minute or persons/hour.

Model derivation consists of the following three steps:

1. Determining the probability density function (PDF) for the number of arrivals at the standpost.
2. Using the PDF determined in step 1 to derive the PDF of the number of people in line.

3. Using the PDF determined in step 2 to determine expected line lengths and waiting times.

To illustrate these steps and determine resulting line lengths when the WHO guidelines are used for design, assume the standpost has an operating period from 6:00 a.m. to 6:00 p.m., which roughly corresponds to daylight hours. The standpost has a single tap and serves a population of 120, with an average per capita demand of 12 gallons/day. The waste factor w is 0.11, and the efficiency factor f is 0.9. The expected arrival rate of customers does not vary between 6:00 a.m. and 6:00 p.m., so the number of peak hours is 12, resulting in a peak factor of 24/12 = 2. Based on WHO guidelines (equation 3.1), the required discharge capacity of the standpost is

$$
\begin{aligned}
Q_{max} &= 120 \times (12/24) \times 2 \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 150 \text{ gallons/hour} \\
&= 2.5 \text{ gallons/minute} \quad \text{(gpm)}
\end{aligned}
$$

Assume that all containers are 6 gallons, and that each person who visits the standpost carries only one container per visit, so that each member of the population takes an average of two trips to the standpost. The expected arrival rate, λ, is constant throughout the 12-hour period and is (120 x 2)/12 = 20 persons/hr. The service time, τ, is given by equation 3.2 to be

$$
\begin{aligned}
\tau &= 6 \times (1/2.5) \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 3.0 \text{ minutes}
\end{aligned}
$$

The service time τ of 3 minutes is equivalent to a service rate capacity, mu (μ), of 20 persons per hour; i.e., $\mu = 1/\tau$ with τ in hours. The expected arrival rate λ and the service rate capacity μ are therefore equal.

The above 3-step procedure can now be applied as follows.

1. Probability Density Function of the Number of Arrivals

Assuming that the potential number of standpost users during a time interval of duration t is large enough to be considered infinite, the Poisson equation can be employed to express the probability of the number of arrivals during t. Letting $p_k(t)$ be the probability that k arrivals occur during time interval t and letting λ be the expected number of arrivals per unit time, the Poisson equation is

$$p_k(t) = (\lambda t)^k e^{-\lambda t}/k! \qquad (4.1)$$

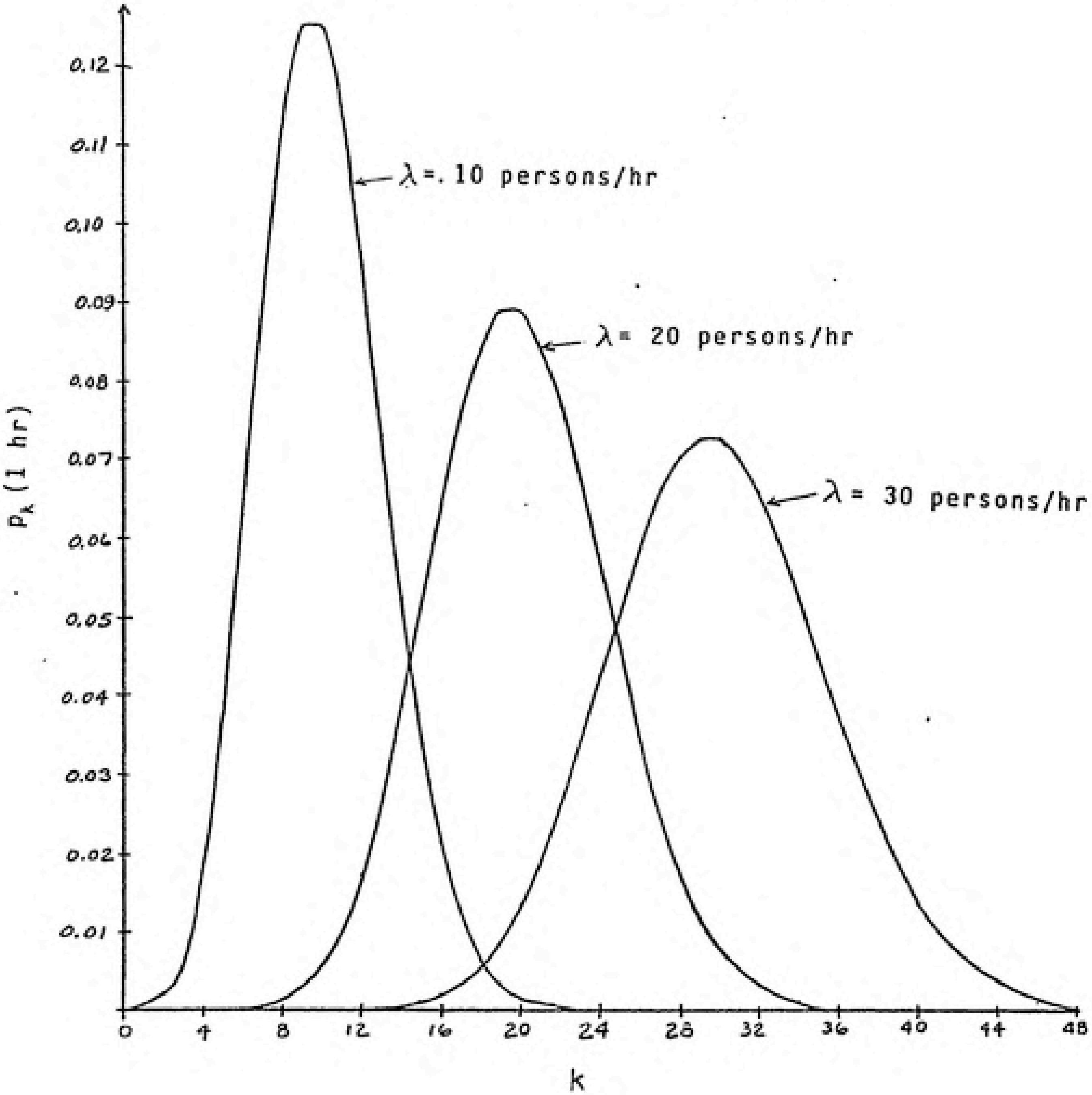
For example, if the expected arrival rate of customers (λ) is 20 per hour, then the probability that exactly 18 arrivals occur in an interval t of one hour is

$$p\ (1.0\ \text{hr}) = (20/\text{hr} \times 1.0\ \text{hr})^{18} e^{-20\times 1}/18!$$
$$= 0.084$$

or about 1 in 12. In other words, if the standpost were observed for 1000 1-hour time periods selected at random, it would be expected that 18 arrivals would occur in 84 of them.

Figure 4-1 shows how the probability of arrivals during a 1 hour period varies with k for expected arrival rates of 10 persons/hour, 20 persons/hour, and 30 persons/hour. Continuous curves are shown for exposition. However, the

FIGURE 4-1. Probability of k arrivals in 1 hour vs. k for expected arrival rates of 10 persons/hr, 20 persons/hr, and 30 persons/hr.



probability function is actually discrete.

2. Probability Density Function of the Number of People at the Standpost

The key to converting the PDF of arrivals into the PDF of customers at the standpost is to select a period of time t in the Poisson equation equal to that required to serve a single customer. Assume that all customers begin to be served at the start of a service period and finish being served at the end of it. Since the PDF of the number at the standpost is known at the beginning of the first interval (i.e., the probability that no one is initially at the standpost is 1, and the probability that 1 or more customers are at the standpost initially is 0 under the assumptions stated at the outset), the PDF of customers in the line at the beginning of the second and later intervals can be determined.

For the scenario presented on p.11, this procedure for converting the PDF of arrivals to the PDF of customers at the standpost is as follows. First, the Poisson equation is applied to determine the probability density function of the number of arrivals between 6:00 a.m and 6:03 a.m. (i.e., during the first interval) with t in the equation replaced by the service time τ, which in this case is 3 minutes (0.05 hour).

$$p_k(\tau) = (\lambda \times \tau)^k e^{-\lambda\tau} / k!$$

The probability that no one arrives during the first interval is

$$\begin{aligned} p_0(0.05 \text{ hr}) &= (20/\text{hr} \times 0.05 \text{ hr})^0 e^{20 \times 0.05} / 0! \\ &= 0.368 \end{aligned}$$

The probabilities for other arrivals are as follows.

$p_1(0.05 \text{ hr}) = 0.368$
$p_2(0.05 \text{ hr}) = 0.184$
$p_3(0.05 \text{ hr}) = 0.061$
$p_4(0.05 \text{ hr}) = 0.015$
$p_5(0.05 \text{ hr}) = 0.003$
$p_6(0.05 \text{ hr}) = 0.001$

The probabaility of 7 or more arrivals during the 0.05 hour interval is 0.000.

With this PDF of arrivals in the first interval, the PDF of the number at the standpost at 6:03 a.m. can be determined. For example, the probability that 0 people are at the standpost at 6:03 a.m., the beginning of the second interval is as follows:

joint probability that no one is in line at 6:00 a.m. and no arrivals occur between 6:00 and 6:03 a.m.	+	joint probability that 1 person is in line at 6:00 a.m. and no arrivals occur between 6:00 and 6:03 a.m.

The above two terms describe the only two possible ways for no one to be at the standpost at 6:03. If no one is at the standpost at 6:00 and no one arrives between 6:00 and 6:03, then no one will be at the standpost at 6:03. Also, if 1 person is at the standpost at 6:00 and no one arrives between 6:00 and 6:03, then no one will be at the standpost at 6:03 because the 1 person will have been served. The above two terms describe mutually exclusive events. The terms can therefore be added. Letting $v_j(n)$ be the probability that j people are at the standpost at the beginning of the nth service period, the above situation can be expressed as follows:

$$v_o(2) = v_o(1) \times p_o(0.05 \text{ hr}) + v_1(1) \times p_o(0.05 \text{ hr})$$

Since it has been assumed that no one is at the standpost when it opens (6:00 a.m.), $v_o(1) = 1.0$ and $v_1(1) = 0.0$. The above equation can thus be written as follows:

$$v_o(2) = 1.0 \times 0.368 + 0.0 \times 0.368 = 0.368$$

Similarly, the probability that 1 person is at the standpost at 6:03 is as follows:

joint probability that no one is in line at 6:00 and 1 arrival occurs during the first service period

+

joint probability that 1 person is in line at 6:00 and 1 arrival occurs during the first service period

+

joint probability that 2 persons are in line at 6:00 and no arrivals occur during the first service period

The first term results in 1 person at the standpost at 6:03 only if the arriving person must wait until the beginning of the second interval to be served, which was assumed on p.14. The second and third terms result in 1 person at the standpost at 6:03 because only 1 person present at 6:00 is served from 6:00 to 6:03, and anyone arriving during the first interval must wait until the second interval to be served. These three mutually exclusive events can be expressed symbollically as follows:

$$v_1(2) = v_0(1) \times p_1(0.05\ hr) + v_1(1) \times p_1(0.05\ hr) + v_2(1) \times p_0(0.05\ hr)$$

$$= 1.0 \times 0.368 + 0.0 \times 0.368 + 0.0 \times 0.368$$

$$= 0.368$$

Proceeding in a similar fashion,

$$v_2(2) = v_0(1) \times p_2(0.05\ hr) + v_1(1) \times p_2(0.05\ hr) + v_2(1) \times p_1(0.05\ hr) + v_3(1) \times p_0(0.05\ hr)$$

$$= 1.0 \times 0.184 + 0.0 \times 0.184 + 0.0 \times 0.368 + 0.0 \times 0.368$$

$$= 0.184$$

The other values of the PDF for customers at the standpost at 6:03 are as follows:

$$v_3(2) = 0.061$$
$$v_4(2) = 0.015$$
$$v_5(2) = 0.003$$
$$v_6(2) = 0.001$$
$$v_{j>6}(2) = 0.000$$

The $v_j(2)$ values can then be used to determine $v_j(3)$ values, i.e., the PDF at 6:06. For example, $v_1(3)$, the probability that one is at the standpost at the beginning of the third interval, is

$$v_1(3) = v_0(2) \times p_1(0.05\ hr) + v_1(2) \times p_1(0.05\ hr) + v_2(2) \times p_0(0.05\ hr)$$

$$= 0.368 \times 0.368 + 0.368 \times 0.368 + 0.184 \times 0.368$$

$$= 0.339$$

The $v_j(3)$ values can then be used to determine $v_j(4)$ values, and so on, until the PDF of the number in line is known for the beginning of all service periods.

The general form of the PDF for customers at the standpost at the beginning of any interval is as follows:

probability that j persons are in line at the beginning of period n + 1	=	joint probability that no one is in line at the beginning of period n and j persons arrive during period n
		+
		joint probability that 1 person is in line at the beginning of period n and j persons arrive during period n
		+
		joint probability that 2 persons are in line at the beginning of period n and j - 1 persons arrive during period n
		+
		joint probability that 3 persons are in line at the beginning of period n and j - 2 persons arrive during period n
		+
		.
		.
		+
		joint probability that j + 1 persons are in line at the beginning of period n and no persons arrive during period n

In mathematical symbols:

$$v_j(n + 1) = \sum_{i=0}^{j+1} (v_i(n) \times p_k(\tau)) \qquad (4.2)$$

where $k = j$ if $i = 0$, and $k = j - i + 1$ if $i > 0$.

Equation 4.2 indicates that if $i > 0$, then one customer is served during the nth interval. The number of arrivals, k, is therefore

1 more than the difference in the number at the standpost at the beginning of interval n and the number at the beginning of interval n + 1. However, if i = 0, then no departure occurs during the nth interval, and k = j - i = j. Another observation concerning equation 4.2 is that the number of right hand side terms is always j + 2.

One of the useful purposes of equation 4.2 is determining the probability that no one is at the standpost at any time. A computer program using equation 4.2 was written to determine this. The resulting curve for the illustrative example is shown in Figure 4-2. Although $v_o(n)$ decreases with time, it remains positive; there is always a possibility that no one is at the standpost.

3. Line Lengths and Waiting Times

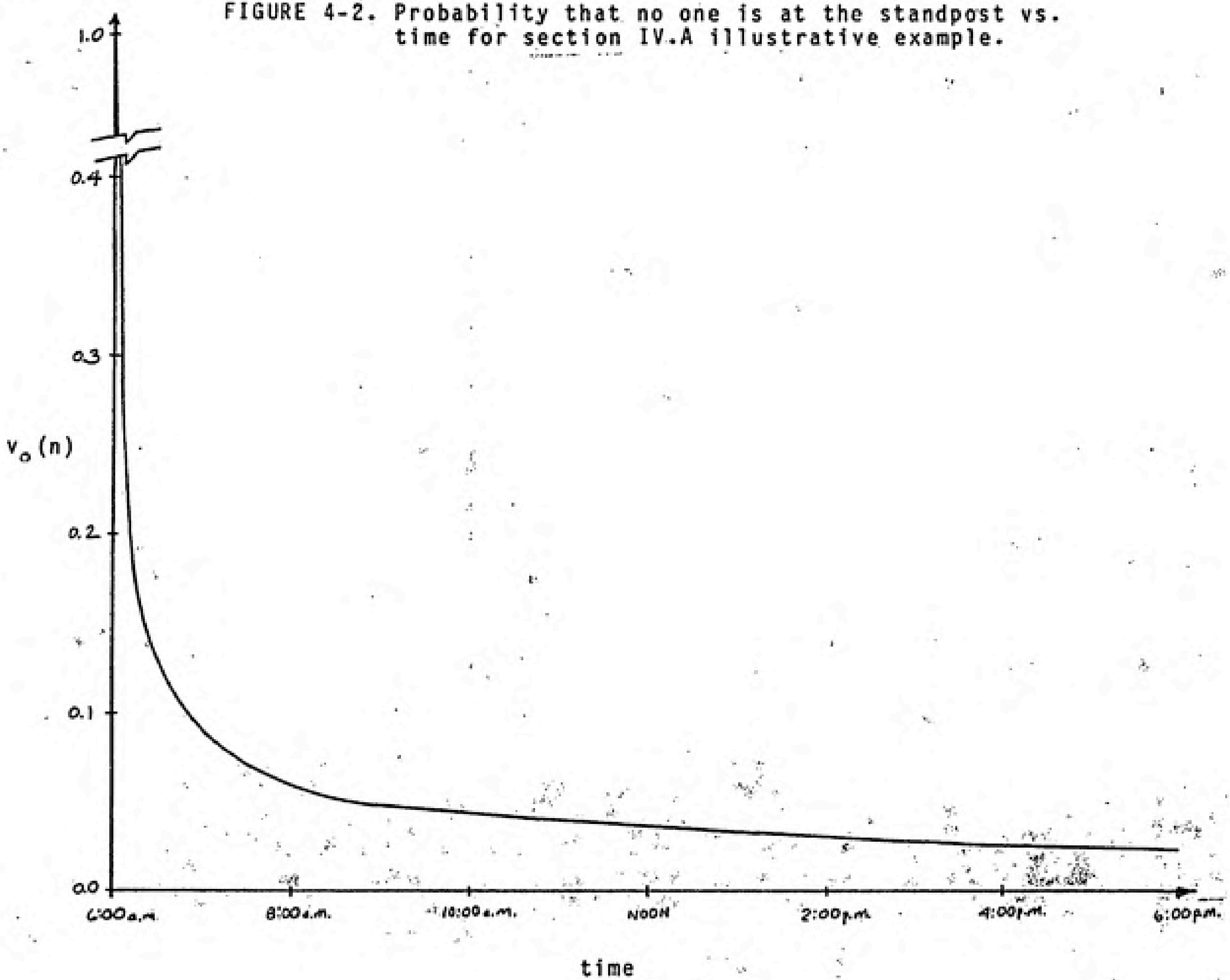
The PDF of the line length (which includes the person being served) at the beginning of the nth interval can be used to determine the expected (i.e., average) line length, L(n), as follows:

$$L(n) = \sum_{\text{all } j} j \times v_j(n) \qquad (4.3)$$

For the illustrative example, the expected number of people in line at the beginning of the second interval (6:03) is

$$\begin{aligned} L(2) &= 0 \times 0.368 + 1 \times 0.368 + 2 \times 0.184 + \\ &\quad 3 \times 0.061 + 4 \times 0.015 + 5 \times 0.003 + \\ &\quad 6 \times 0.001 \\ &= 1.00 \text{ person} \end{aligned}$$

FIGURE 4-2. Probability that no one is at the standpost vs. time for section IV.A illustrative example.



Equation 4.3 includes the possibility that no one is in the line (i.e., $j = 0$) to calculate the expected line length $L(n)$. However, when the line length is 0, no one is present to benefit from the "shortness" of the line. The expected line length experienced by customers must exclude time when the line is empty. Therefore, it is of interest to know $L'(n)$, the expected line length given that the line is not empty (i.e., the expected line length given that at least one person is at the standpost). At the beginning of interval n,

probability that j persons are in line	=	probability that the line is not empty	x	probability that j persons are in line given that the line is not empty

Referring to $v_j'(n)$ as the conditional probability that j people are in line given that the line is not empty, the above equation can be expressed in mathematical symbols as

$$v_j(n) = (1 - v_o(n)) \times v_j'(n)$$

Solving for $v_j'(n)$ yields

$$v_j'(n) = v_j(n)/(1 - v_o(n)) \tag{4.4}$$

For the illustrative example,

$$\begin{aligned} v_1'(2) &= 0.368/(1 - 0.368) = 0.582 \\ v_2'(2) &= 0.291 \\ v_3'(2) &= 0.097 \\ v_4'(2) &= 0.024 \\ v_5'(2) &= 0.005 \end{aligned}$$

Note that $v_j'(n)$ is always larger than $v_o(n)$ by the constant multiple $1/(1 - v_o(n))$. With the conditional probability of one or more persons at the standpost at any time, it is possible to calculate L'(n), the expected line length given that the line is not empty, as follows:

$$L'(n) = \sum_{\text{all } j>0} j \times v_j'(n) \qquad (4.5)$$

Combining equations 4.4 and 4.5,

$$L'(n) = \sum_{\text{all } j>0} j \times v_j(n)/(1 - v_o(n)) \qquad (4.6)$$

from which it follows that

$$L'(n) = L(n)/(1 - v_o(n)) \qquad (4.7)$$

At the beginning of the second service period for the illustrative example,

$$\begin{aligned} L'(2) &= 1.00/(1 - 0.368) \\ &= 1.58 \text{ persons} \end{aligned}$$

which is substantially greater than L(2) = 1.00.

In addition to calculating the expected line length at the beginning of any service interval, it is also possible to estimate the average amount of time persons arriving at the beginning of the nth interval will have to wait to be served. W(n), which is the expected waiting time for the last person in line at the beginning of the nth interval, is simply the product of the expected number of service periods the person must spend at the standpost, S(n), and the time required to serve each customer, . W(n) includes both the time the customer spends in the queue before beginning to be served and the time spent being served.

$$W(n) = S(n) \times \tau \qquad (4.8)$$

For a standpost with only one faucet, $S(n) = L'(n)$. At the beginning of the second service period for the illustrative example,

$$\begin{aligned} W(n) &= 1.58 \text{ persons} \times 3 \text{ minutes/person} \\ &= 4.74 \text{ minutes} \end{aligned}$$

The "τ = 3.00 minutes" curve in Figure 4-3 shows the increase in the expected line length $L'(n)$ with time when the expected arrival rate is 20 persons/hour, the service time is 3 minutes, and the discharge capacity is as determined by WHO guidelines as shown on p. 11. For this curve the value of the traffic intensity rho (ρ), which is the ratio of the expected arrival rate λ to the service rate capacity μ, is

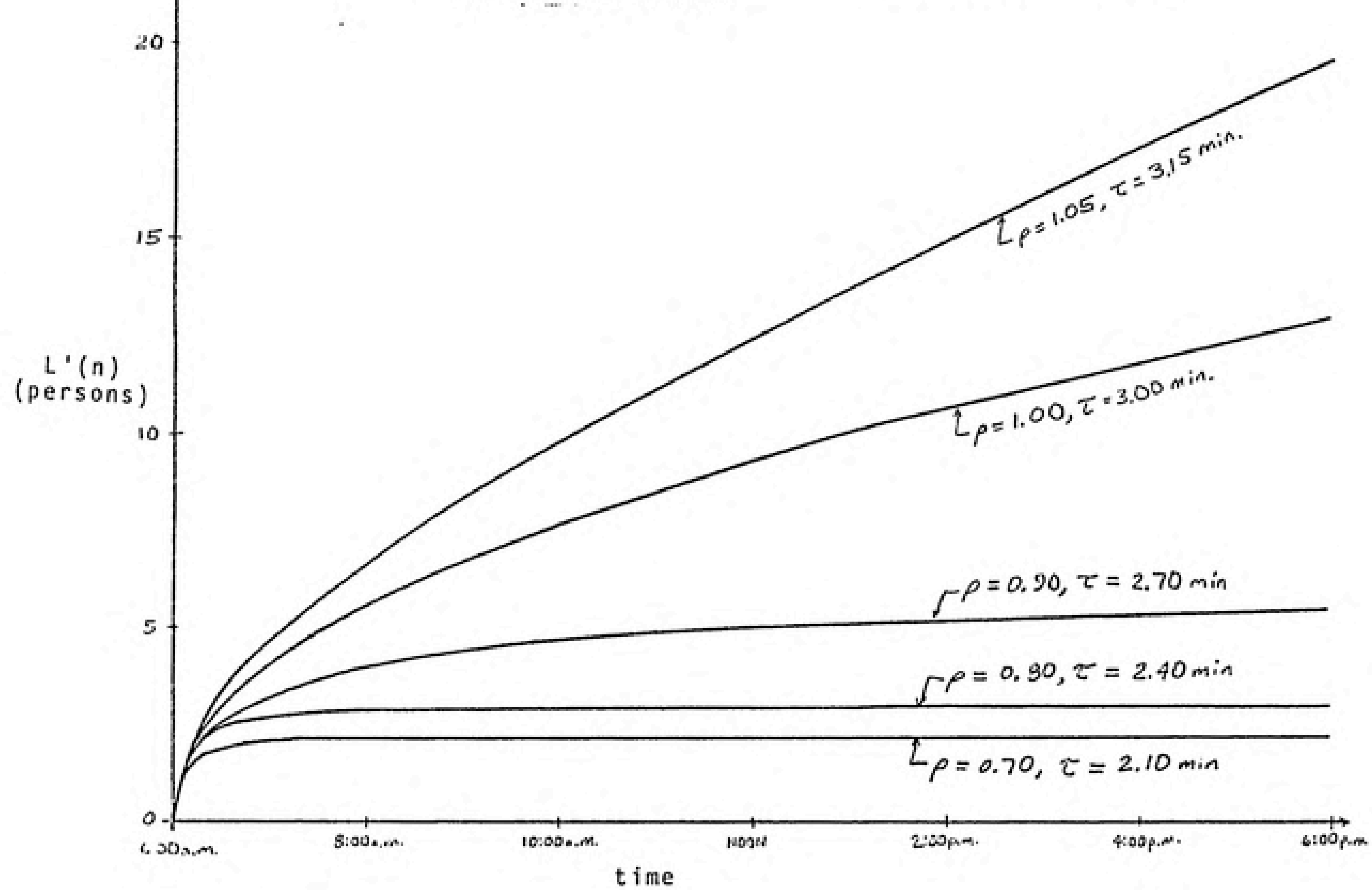
$$\rho = \lambda/\mu = \lambda/(1/\tau) = 20/\text{hr} \,/\, (1 \,/\, 3\text{min}) = 1.00$$

(The computer program used to determine the data points for the curve is shown in Annex C.1.) Intuition might suggest that no lengthy waiting lines would develop because the service rate capacity is equal to the expected arrival rate, but such is not the case. At 6:00 p.m., $L'(n)$, the average line length given that at least one person is at the standpost, is 13 persons. With a service time of 3 minutes, this length corresponds to a waiting time of 3 x 13 = 39 minutes.

An explanation of why $L'(n)$ increases monotonically is as follows:

1. With T being the time since the standpost began operating, the expected line length $L(T)$ is equal to the expected number of

FIGURE 4-3. Expected line length (given that the line is not empty) vs. time for various traffic intensities and service times.



arrivals during T minus the expected number of people served during T.

$$L(T) = \lambda T - \text{expected number served during } T$$

2. The expected number served during T is equal to the number of service intervals during T minus the expected number of service periods during which no one is being served.

$$\text{expected number served during } T = T/\tau - \text{expected number of service periods during which no one is served}$$

3. Combining equations from steps 1 and 2 and noting from p. 12 that $1/\tau$ is the service rate capacity μ ,

$$L(T) = \lambda T - \mu T + \text{expected number of service periods during which no one is served}$$

4. However on p.12 it was shown that $\lambda = \mu$, from which it follows that

$$L(T) = \text{expected number of service periods during which no one is served}$$

5. As shown in Figure 4-2, the probability that no one is in line is always greater than 0. The expected number of service periods during which no one is served during T thus increases with T. L(T) must therefore also increase with T.

6. Equation 4.7 expresses the relationship between L'(T) and L(T). Since L'(T) is always greater than L(T), L'(T) must also increase with T.

It is no coincidence that the traffic intensity $\rho = 1.0$ for the illustrative example. The WHO equation 3.1 sets the discharge capacity equal to the demand rate, with adjustments made only for waste and for valve opening and closing. The WHO guidelines assume that a service rate capacity equal to the expected arrival rate is adequate, and do not consider the possibility of lengthy lines shown by queuing theory.

Note that for the traffic intensity $\rho > 1$ the expected arrival rate is greater than the service rate capacity, and for $\rho < 1$ the reverse is true. The traffic intensity therefore indicates the number of persons that are expected to arrive during the time it takes to serve one person. It is of interest to know the sensitivity of L'(n) to changes in ρ. Figure 4-3 illustrates this for ρ values of 1.05, 1.00, 0.90, 0.80, and 0.70. Note that for ρ = 1.05 and with an expected arrival rate λ of 20 persons per hour, the service rate capacity μ is 19.05 persons per hour, which is equivalent to a service period τ of 0.0525 hr per person, or 3.15 minutes. Such a situation might exist for the illustrative example if time required to position and remove the container from under the faucet were not negligible, but were 0.15 minutes (9 seconds). For ρ values less than 1.0, the discharge capacity is increased to decrease the service time τ, thereby increasing μ. For example, for ρ = 0.80, the discharge capacity is increased to reduce the service period to 2.4 minutes, increasing the service rate capacity μ to 25 per hour. Figure 4-3 shows that if a standpost is designed such that service rate capacity is equal to or less than the expected arrival rate (i.e., for $\rho \geq 1.00$) then L'(n) may be undesireably long. When the service rate capacity significantly exceeds the expected arrival rate (e.g., ρ = 0.80), results tend to be satisfactory.

of $L'(n)$, is given by the following equation based on the literature (Hillier and Lieberman, 1980):

$$L' = 1 + \rho/[2(1 - \rho)] \tag{4.9}$$

This equation is applicable only when the service time does not vary among customers, the standpost has only one faucet, the expected arrival rate is constant, and the population is large enough to be considered infinite. This equation can be used to quickly determine the maximum expected line length (given that the line is not empty) when it is known that $L'(n)$ is essentially at steady-state before the standpost closes or the arrivals cease. Figure 4-3 shows that if ρ is adequately less than 1.0, steady-state is essentially reached very quickly, making equation 4.9 useful for such ρ values.

B. Variable Arrival Rates

Accounting for an expected arrival rate that changes over time is quite straightforward. λ is assumed to be constant for any particular service interval but is allowed to vary from one interval to the next. The variation is therefore approximated by a discrete function, which requires the application of equation 4.1 at each step. The model remains essentially the same as that presented in the previous section, except that new p_K values must be calculated for each service period having a different λ. This recalculation is necessary because p_K is a function of λ.

To illustrate the application of queuing theory to a time-varying expected arrival rate, λ is assumed to vary as shown in

Figure 4-4a but is approximated as a discrete function for the queuing model. The shape of this arrival pattern is similar to that for villages in Lesotho, Africa (Feachem, et al, 1978) and is believed to be not uncommon for urban areas. The average value of λ is 20 persons/hour, and is thus equal to that of the scenario presented in the previous section. The standpost has only one tap. Population size, average per capita demand, container size, the waste factor, and the efficiency factor are as in the example of the previous section. The designer has little data on how the water collection pattern varies throughout the day, but knows that there will be significant usage during every hour from 6:00 a.m. to 6:00 p.m. but no usage from 6:00 p.m. to 6:00 a.m. He therefore estimates the number of peak hours to be 12, and uses WHO guidelines to calculate P to be 24 hr / 12 hr = 2.0 (by definition of P on p.7) and the required discharge capacity to be

$$
\begin{aligned}
Q_{max} &= 120 \times (12/24) \times 2 \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 2.50 \text{ gpm}
\end{aligned}
$$

The service time is

$$
\begin{aligned}
\tau &= 6 \times (1/2.50) \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 3.0 \text{ minutes}
\end{aligned}
$$

This example is thus identical to that of the previous section except that the expected arrival rate λ varies with time.

The value of λ in persons per hour as a function of the service interval is given by the following equations:

FIGURE 4-4a. Expected arrival rate vs. time for section IV.B illustrative example.

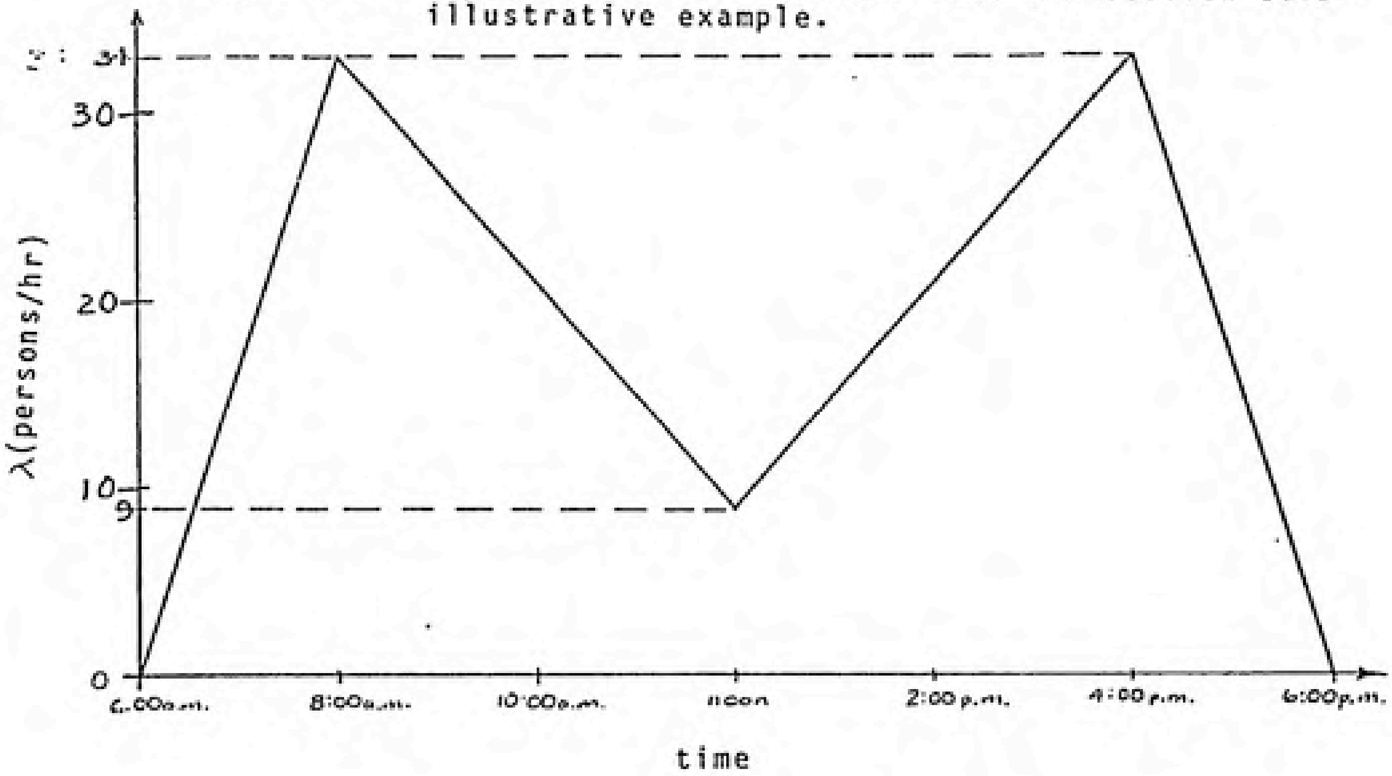
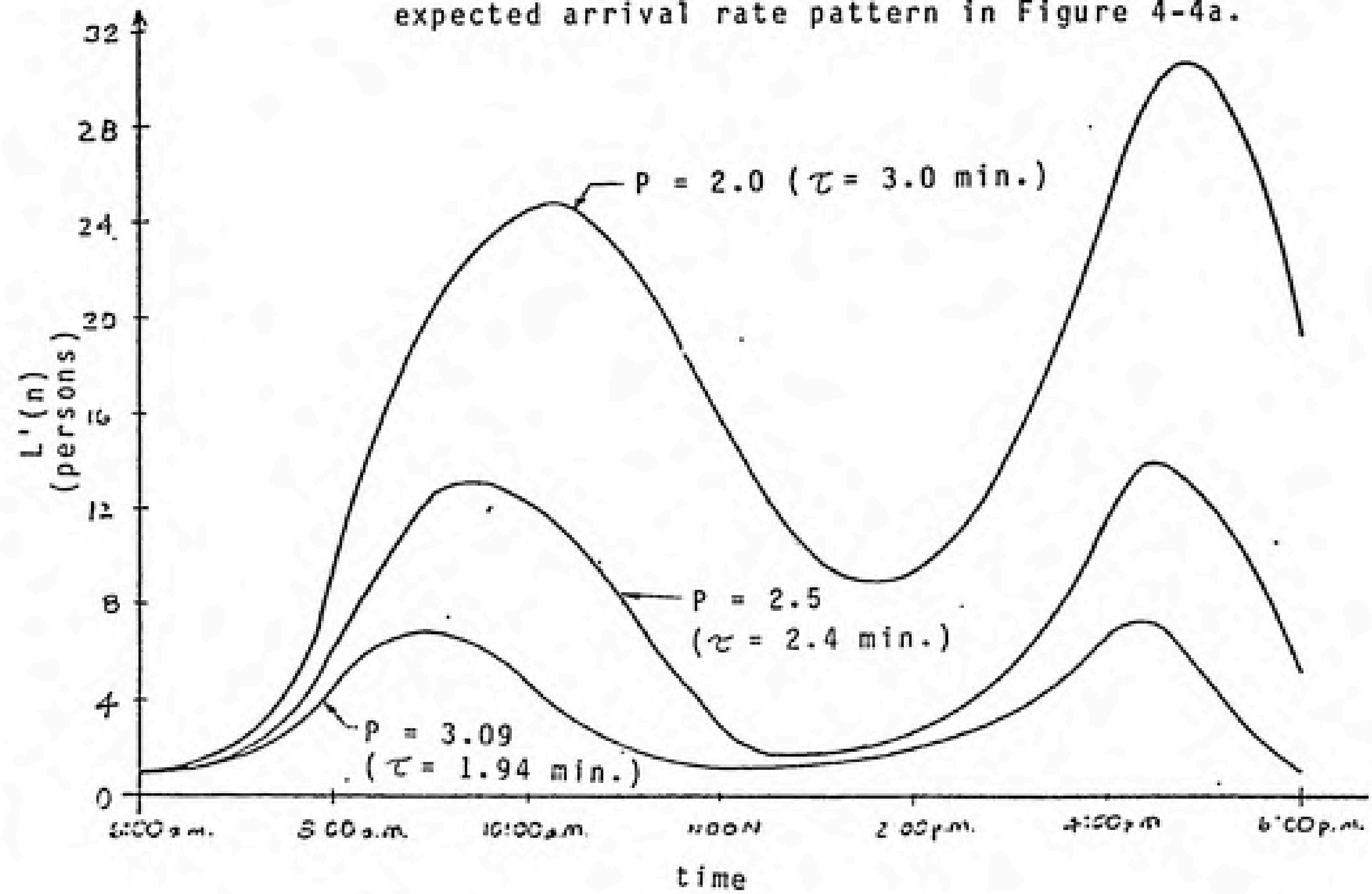


FIGURE 4-4b. Expected line length (given that the line is not empty) vs. time for various peak factors and the expected arrival rate pattern in Figure 4-4a.



$$\lambda(n) = 34 \times n/40 \qquad \text{for } 0 \leq n \leq 40 \text{ or 6:00 a.m. to 8:00 a.m.} \tag{4.10}$$

$$\lambda(n) = 46.5 \times (148.8 - n)/148.8 \qquad \text{for } 40 \leq n \leq 120 \text{ or 8:00 a.m. to noon} \tag{4.11}$$

$$\lambda(n) = 34 \times (n - 91.2)/108.8 \qquad \text{for } 120 \leq n \leq 200 \text{ or noon to 4:00 p.m.} \tag{4.12}$$

$$\lambda(n) = 34 \times (240 - n)/40 \qquad \text{for } 200 \leq n \leq 240 \text{ or 4:00 p.m. to 6:00 p.m.} \tag{4.13}$$

To determine, for example, the probability that the line at the standpost is empty at the beginning of the second service period, equation 4.2 is applied as in the constant expected arri-rate case to obtain

$$v_o(2) = v_o(1) \times p_o(\text{from 6:00 a.m. to 6:03 a.m.}) + v_1(1) \times p_o(\text{from 6:00 a.m. to 6:03 a.m.})$$

From equation 4.10, the λ value for determining p_o(from 6:00 a.m. to 6:03 a.m.) is

$$\lambda(1) = 34 \times 1/40 = 0.85 \text{ per hour}$$

and, by equation 4.1,

$$p_o(\text{from 6:00 a.m. to 6:03 a.m.}) = (0.85/\text{hr} \times 0.05\text{hr})^0 \, e^{-(0.85/\text{hr} \times 0.05\text{hr})} / 0! = 0.958$$

Given that $v_o(1) = 1.0$ and $v_1(1) = 0.0$,

$$v_o(2) = 1.0 \times 0.958 + 0.0 \times 0.958 = 0.958$$

Thus, the procedure for determining probability values $v_j(n)$, line lengths given that someone is at the standpost, and waiting times, is exactly the same as when the expected arrival rate is constant, except that $\lambda(n)$ must be calculated for each service period.

The resulting L'(n) values are shown by the P = 2.0 curve in Figure 4-4b. (The curve was calculated by the computer program in Annex C.4.) L'(n) reaches much greater values than shown by the ρ = 1.00 curve in Figure 4-3, even though the average ρ value for the time-varying case is also 1.0. For example, at about 4:50 p.m. L'(n) for the time-varying case is 31 persons, corresponding to an expected waiting time W(n) of 31 x 3 = 93 minutes, which is quite long. The L'(n) value from the ρ = 1.00 curve in Figure 4-3 at 4:50 is 12, corresponding to an expected waiting time of 36 minutes. The time variation in the expected arrival rate is obviously an important consideration in standpost design.

The designer may suspect that usage during the 12-hour period from 6:00 a.m. to 6:00 p.m. varies significantly, and although he may not know the extent of the variation, he may wish to chose a more conservative peak factor of 2.5 to account for the variation. The required discharge capacity by WHO guidelines would be

$$\begin{aligned} Q_{max} &= 120 \times (12/24) \times 2.5 \times (1/[1 - 0.11]) \times (1/0.9) \\ &= 187 \text{ gallons/day} \\ &= 3.1 \text{ gpm} \end{aligned}$$

The service time would be

$$\tau = 6 \times (1/3.1) \times (1/[1 - 0.11]) \times (1/0.9)$$
$$= 2.4 \text{ minutes}$$

The computer program used to determine L'(n) for P = 2.0 was modified to determine L'(n) for P = 2.5. The resulting curve is shown in Figure 4-4b. L'(n) reaches its maximum value of 14 persons at 4:30. The corresponding expected waiting time is 14 x 2.4 = 34 minutes, which is still rather long.

The designer may perceive that a thorough investigation of the water collection pattern is warranted before designing the standpost. If he collects data to estimate the usage during every hour between 6:00 a.m. and 6:00 p.m., he would find that peak hourly usage occurs during the hour from 8:00 a.m. to 9:00 a.m. and the hour from 3:00 p.m. to 4:00 p.m. He would find that the average number of arrivals during either of these hours is

$$\bar{\lambda} = (\underbrace{34 \text{ persons/hr}}_{\lambda \text{ at 8:00 a.m. from eqn. 4.10}} + \underbrace{27.75 \text{ persons/hr}}_{\lambda \text{ at 9:00 a.m. from eqn. 4.11}}) / 2 = 30.9 \text{ persons/hr}$$

The peak factor P would then be simply the ratio of this peak hourly expected arrival rate and the average expected arrival rate over the 24-hour period, or 30.9/10 = 3.09. The required discharge capacity by WHO guidelines would be 3.86 gpm. The service time would be 1.94 minutes. Computer program results, which are shown in the P = 3.09 curve in Figure 4-4b, show that L'(n) would reach its maximum value of 6.65 persons at 4:18 p.m. The corresponding expected waiting time is 6.65 x 1.94 = 12.9 minutes, which is a dramatic improvement over the P = 2.5 case.

The peak factor of 2.0 provides a service rate capacity of $\mu = 1/\tau$ = 1 person/3minutes = 20 persons/hour. This rate is equal to the expected arrival rate averaged over the peak period of 12 hours. The average value of ρ over this 12-hour period is therefore 1.0. The peak factor of 2.5 provides a service rate capacity of 25 persons/hour, which is equal to the expected arrival rate averaged over the peak period of 4 hours extending from 6:55 a.m. to 10:55 a.m. (or from 1:05 p.m. to 5:05 p.m. for the later of the twin peaks). When P = 2.5, the average ρ value over this 4-hour period is 1.0. The peak factor of 3.09 provides a service rate capacity of 30.9 persons/hour and an average ρ value of 1.0 over the 1.0-hour period extending from 8:00 a.m. to 9:00 a.m. (or from 4:00 p.m. to 5:00 p.m.).

The WHO guidelines state that the peak period is the time during which "the standpost is used more intensively than during the rest of the day", that it typically lasts between 4 and 12 hours, and that there is a time-varying water demand pattern during the peak period. Because the demand varies with time during the peak period, the peak demand estimate will tend to increase as the length of time selected for the peak decreases. This is clearly shown by the difference in P factors for the above examples. Peak periods of 12 hours, 4 hours, and 1 hour correspond to peak factors of 2.0, 2.5, and 3.09, respectively. The guidelines do not indicate how to select the length of the peak period, but indicate 4 to 12 hours to be the range of typical lengths. However, for the above example, using a peak period of 4 hours or 12 hours is unacceptable. Using a peak period of only 1 hour yields reasonable results (a maximum L'(n) value of 6.65 persons) but even this line length might be considered unacceptable if service times are not short. For example, if the service time is 5 minutes this length corresponds to an expected waiting time of 6.65 x 5 = 33 minutes.

The guidelines also state that the peak factor is typically in the range of 2 to 4. When the designer selects 3.1 for the conditions of this example, the maximum expected waiting time is reasonable. However, in some cases even a peak factor of P = 4 will not provide adequate discharge capacities. Assume, for example, that the arrival pattern is identical to that of Figure 4-4a except that no arrivals occur after noon. People rely on another source in the afternoon because the standpost is closed. The daily demand from the standpost is thus reduced from 1440 gallons/day to 720 gallons/day. The average hourly demand from 6:00 a.m. to noon would be 720/6 = 120 gallons/hour, and the average hourly demand over the day would be 30 gallons/hour. If a peak factor of P = 120/30 = 4 is used, the resulting discharge capacity would be

$$
\begin{aligned}
Q_{max} &= 120 \times (6/24) \times 4 \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 150 \text{ gallons/day} \\
&= 2.5 \text{ gpm}
\end{aligned}
$$

The service time would be

$$
\begin{aligned}
\tau &= 6 \times (1/2.5) \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 3.0 \text{ minutes}
\end{aligned}
$$

The discharge capacity and the service time are identical to that for the example resulting in the P = 2.0 curve in Figure 4-4b. The L'(n) values would thus equal those shown by the P = 2.0 curve from 6:00 a.m. to noon. The maximum L'(n) value is 25 persons. The corresponding expected waiting time is 25 x 3 = 75 minutes, which is very long.

C. Multiple Faucets

Incorporating multiple faucets into the large-population model is accomplished by recognizing that the maximum number of departures which can occur during a service interval is no longer 1 as in the single faucet case, but is equal to the number of faucets. If, for example, there are two faucets, then equation 4.2 must be modified as follows:

$$v_j(n + 1) = \sum_{i=0}^{j+2} (v_i(n) \times p_k(\tau)) \qquad (4.14)$$

$$\text{where } k = j \quad \text{if} \quad i = 0 \text{ or } 1$$
$$k = j - i + 2 \quad \text{if} \quad i > 1$$

The derivation of this equation is analogous to that starting on p. 15 for the single faucet case. For example, the mutually exclusive events which can account for one person at the standpost at the beginning of the second service interval with two faucets is as follows, which can be compared with its counterpart on p. 17 for a single faucet.

$$v_1(2) = v_0(1) \times p_1(\tau) + v_1(1) \times p_1(\tau) + v_2(1) \times p_1(\tau) + v_3(1) \times p_0(\tau)$$

Equation 4.2 can also be modified for three or more faucets by taking into account that the maximum number of people served during a service interval equals the number of faucets.

Any even number of persons at a two-faucet standpost results in a number of service periods equal to one-half the number of persons. Three or any higher odd number of persons at a standpost with two faucets results in a number of service periods equal to one-half the difference between that number and 1. S(n), the expected number of service periods spent in line, is as follows for a two-faucet standpost.

$$
\begin{aligned}
S(n) = \; & 1 \times v_1'(n) \\
& + 0.5 \times 2 \times v_2'(n) + 0.5 \times (3 - 1) \times v_3'(n) \\
& + 0.5 \times 4 \times v_4'(n) + 0.5 \times (5 - 1) \times v_5'(n) \\
& + \ldots.
\end{aligned} \tag{4.15}
$$

The advantage of using S(n) over L'(n) is that it gives a better indication of how long a person is expected to wait when there is more than one faucet at the standpost. The expected waiting time is determined simply by multiplying S(n) by τ, the time required to serve one person. When the standpost has only one tap, S(n) = L'(n), but when there is more than one faucet, S(n) < L'(n).

To illustrate the queuing model for a two-faucet standpost, assume that the arrival pattern is as shown in Figure 4.4a and that the scenario is identical to that of the previous section. A peak period of 1 hour is used and, as shown in the previous section, this result in a peak factor P = 3.09 and a WHO required discharge capacity of 3.86 gpm. The discharge capacity per faucet is 3.86/2 = 1.93 gpm. The service time is

$$
\begin{aligned}
\tau &= 6 \times (2/3.86) \times (1/[1 - 0.11]) \times (1/0.9) \\
&= 3.9 \text{ minutes}
\end{aligned}
$$

The computer program which calculates S(n) and L'(n) is in Annex

C.6. The results are shown in Figure 4-5.

The maximum values of the expected line length (given that at least one person is at the standpost) and the expected number of service periods spent at the standpost are 7.2 and 3.9, respectively. The corresponding maximum expected waiting time is 3.9 x 3.9 = 15.2 minutes. In the previous section the single-faucet example with a peak factor of 3.09 resulted in a maximum expected waiting time of 12.9 minutes, which is a bit less than that of this two-faucet example. One might be surprised by this difference because the standpost discharge capacities are equal in these two examples. The service time at the two-faucet standpost is twice as great as at the single-faucet standpost because the flowrate through each of the faucets is half as great. Although the service time is twice as great, one would expect the number of service periods spent at the standpost to be halved because of the extra faucet, and that the waiting times for these two examples would be equal. However, when only one person is at the standpost the availability of a second faucet does not reduce that person's waiting time. Similarly, when some higher odd number of persons are in line, the number of service intervals required to serve all of them is the same as if one additional customer were in line. In general, if two standposts have equal discharge capacities, the one with the greater number of faucets will have slightly longer waiting times.

Figure 4-6 applies to constant expected arrival rates. The Figure shows S, the steady-state expected number of service intervals spent at the standpost vs. the traffic intensity ρ (the ratio of the expected arrival rate $\hat{\lambda}$ to the standpost service

FIGURE 4-5. Expected line length (given that the line is not empty) and expected number of service intervals spent at the standpost vs. time for section IV.C illustrative example.

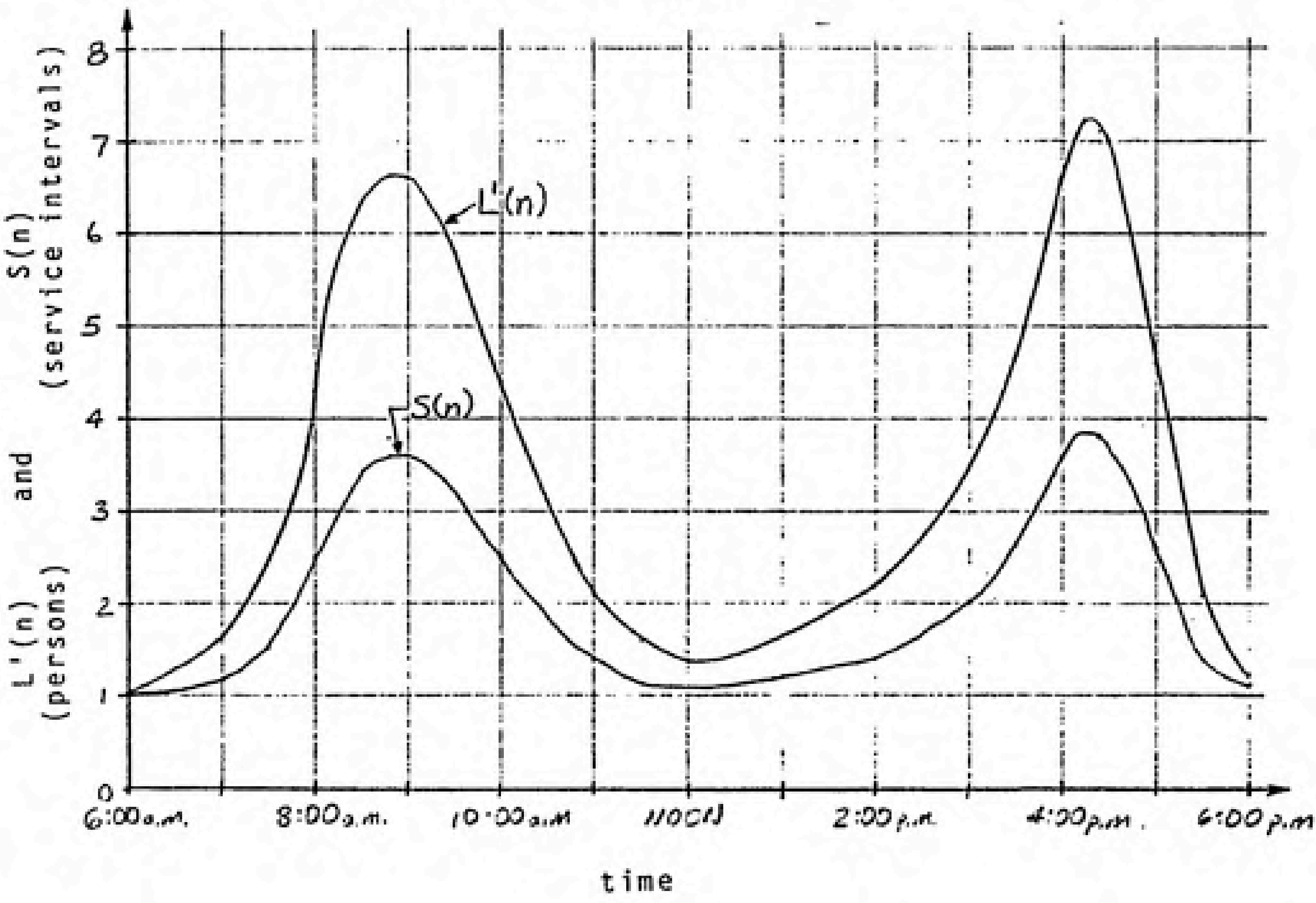
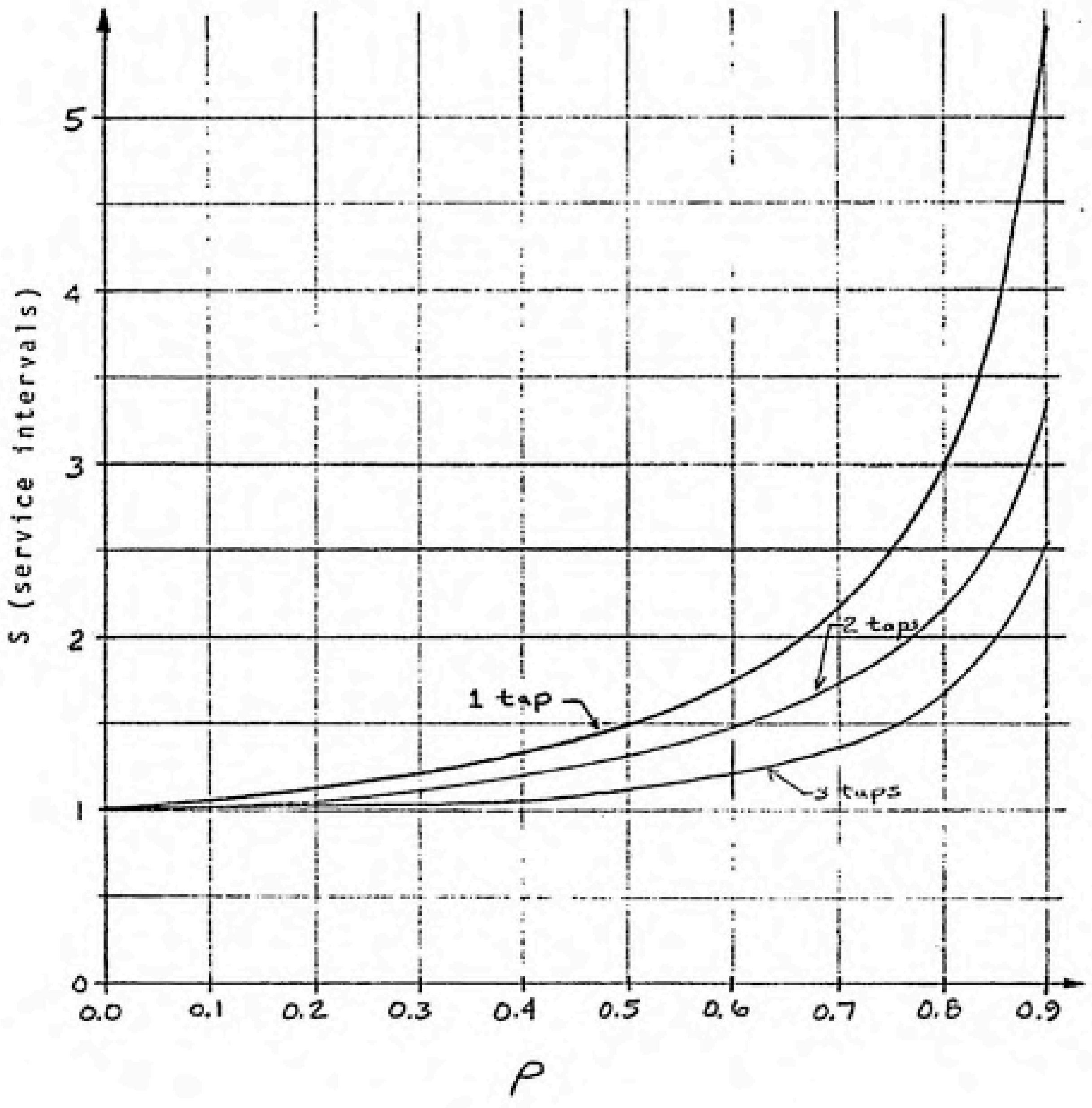


FIGURE 4-6. Steady-state expected number of service intervals spent at the standpost vs. traffic intensity for 1, 2, and 3 taps.



rate capacity $c\mu$, where c is the number of taps and μ is the service rate capacity per tap) for 1, 2, and 3 faucets. For the single-faucet curve, values were obtained by use of equation 4.9, with $S = L'$. For the multiple-faucet curves, values were obtained by the computer programs shown in Annexes C.2 and C.3. These computer programs do not actually calculate true steady-state values, but calculate values for 400 service periods, which is a long enough time for steady-state to be essentially reached when $\rho = 0.9$. Values for ρ close to but less than 1.0 are not included in Figure 4-6 because of the lengthy time required to reach near steady-state.

Figure 4-6 shows that at a given ρ value, S decreases as the number of faucets increases. For example, at $\rho = 0.8$, S = 3.00 for one faucet and S = 1.66 for three faucets. However, as discussed previously, for a given standpost discharge capacity the steady-state expected waiting time W increases as the number of faucets decreases. Assume for example that the standpost discharge capacity is 4.00 gpm, making the service time to be 2.00 minutes when there is one faucet. For three faucets the flow per faucet would be 1.33 gpm, making the service time to be 6.00 minutes. The resulting steady-state expected waiting time W at $\rho = 0.8$ would be 3.00 x 2.00 = 6.00 minutes for one faucet, and 1.66 x 6.00 = 9.96 minutes for three faucets.

D. Service Time Varying Among Users

The volume of containers that users fill will often vary, causing the service time to vary among users. This causes difficulty in applying the Poisson equation because the value of τ used in the equation is variable. This difficulty can be overcome by selecting probability distributions which are in a form which lend themselves to the numerical modeling of queues. However, such models are not presented in this paper for the following reasons:

1. For a constant expected arrival rate, the steady-state line length at an adequately designed standpost will be reached fairly quickly. Figure 4-3 shows how soon steady-state is reached when ρ (the traffic intensity, which is the ratio of the expected arrival rate to the service rate capacity) is adequately low. Reaching steady-state requires more time when ρ is higher. Although steady-state equations may not accurately calculate line lengths at underdesigned standposts (because ρ is too high and the standpost may close or arrivals may stop before steady-state is reached), they can be used to adequately design standposts, making numerical modeling unnecessary when the expected arrival rate is constant.

2. Koopman (1972) has shown that for time-varying expected arrival rates, the variation in service time among users does not seem to have a substantial impact on expected line lengths. Thus, when the expected arrival rate varies, the service time among users may be assumed constant and the queuing model of the previous section may be applied.

The following equation, which can be obtained from queuing theory texts (e.g., Hillier and Liebermann, 1980), determines L', the steady-state expected line length given that the line is not empty. The equation is applicable for any service time probability distribution, but the expected arrival rate must be constant, the standpost must have only one faucet, and the population must be large enough to be considered infinite.

$$L' = [\; + (\lambda^2 \sigma_\tau^2 + \rho^2)/[2\rho(1-\rho)] \qquad (4.16)$$

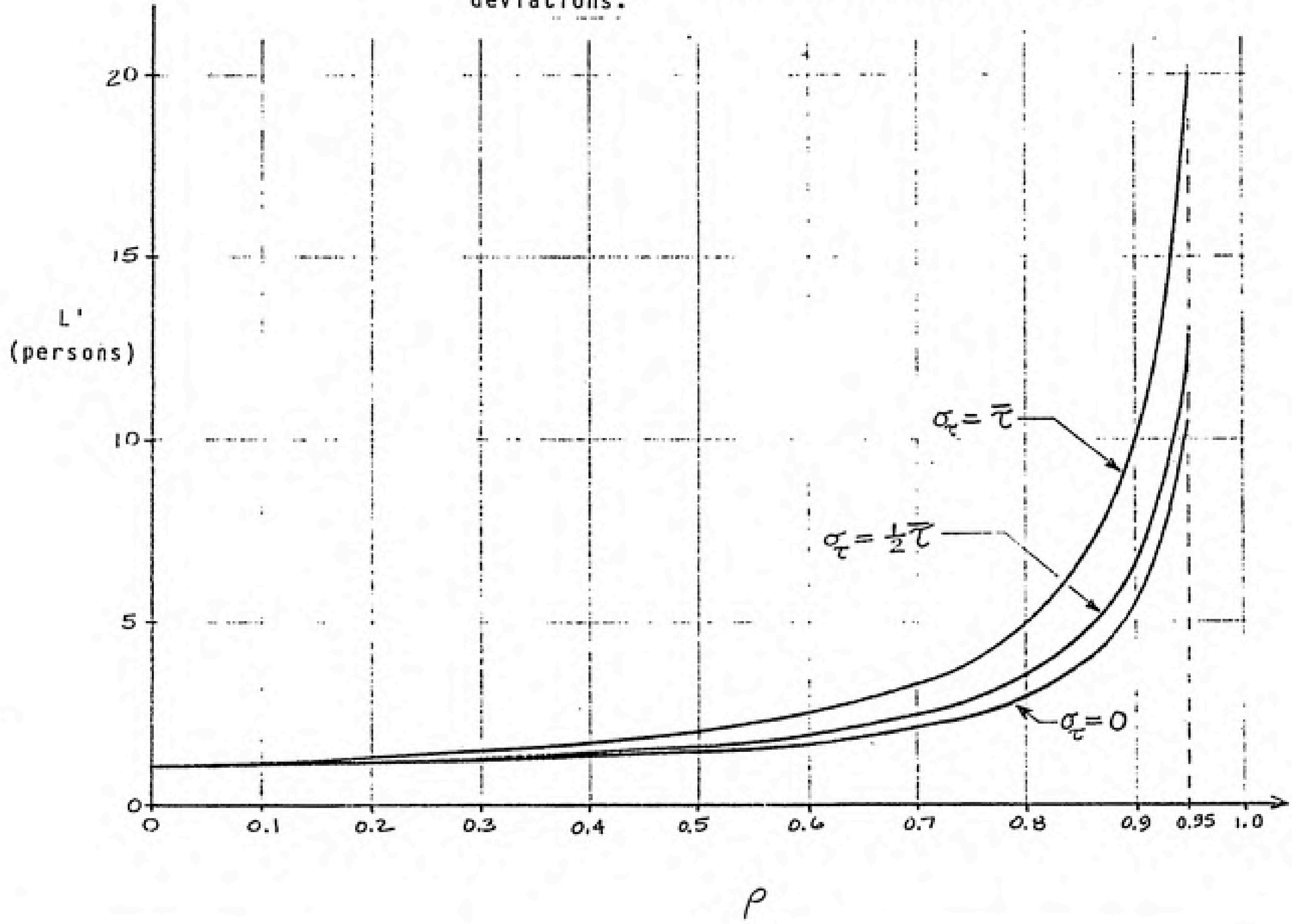
where σ_τ = standard deviation of the service time

Equation 4.16 is applicable only if ρ (the traffic intensity) is less than 1. If ρ is equal to or greater than 1, then steady-state is never reached and equation 4.16 is meaningless. Also, the equation should only be used if it is known that steady-state is reached before arrivals cease or the standpost closes.

Figure 4-7 shows L' (the steady-state expected line length given that someone is at the standpost) vs. ρ curves for various service time standard deviations. As ρ increases, the effect of the standard deviation on L' also increases. At $\rho = 0.2$ (which would correspond to, for example, an expected arrival rate of 20 persons/hour and a service rate capacity of 100 persons/hour), the standard deviation has essentially no effect on L'. At $\rho = 0.95$, L' when the standard deviation equals the average service time is nearly twice L' when the standard deviation is 0.

As an example of applying equation 4.16, assume that one-third of the customers are children with container size of 3 gallons (about 25 pounds) corresponding to a 1.2-minute service time τ; one-third of the customers are adults with container size of 6 gallons (about 50 pounds) and $\tau = 2.4$ minutes; and the other third are adults with container size of 9 gallons (about 75 pounds) and $\tau = 3.6$ minutes. The standard deviation of the service time is about one minute. The average service time $\overline{\tau}$ is 2.4 minutes, and the service rate capacity μ is $1/2.4 = 0.42$ persons/minute. Assuming an expected arrival rate of 20 persons/hour (0.33 persons/minute), $\rho = \lambda / \mu = 0.33/0.42 = 0.8$. The resulting steady-state expected line length given that at least one person is at the standpost is

FIGURE 4-7. Steady-state expected line length (given that the line is not empty) vs. traffic intensity for 1 tap and various service time standard deviations.



$$L' = 1 + (0.333^2 \times 0.98^2 + 0.8^2)/[2 \times 0.8 \times (1 - 0.8)]$$
$$= 3.33 \text{ persons}$$

If the standard deviation were 0, L' would be

$$L' = 1 + 0.8^2/(2 \times 0.8 \times [1 - 0.8])$$
$$= 3.00 \text{ persons}$$

The increase in L' due to the variation is service time is thus 100% x (3.33 - 3.00)/3.33 = 11% for this example.

By multiplying equation 4.16 by $\overline{\tau}$ (the average service time) and using $\rho = \lambda/\mu = \lambda\overline{\tau}$, the steady-state expected waiting time W for a single-faucet standpost and constant expected arrival rate is as follows:

$$W = \overline{\tau} + \lambda(\sigma_\tau^2 + \overline{\tau}^2)/(2 \times [1 - \lambda\overline{\tau}]) \qquad (4.17)$$

For the above example, equation 4.17 shows that the steady-state expected waiting time W when the service time standard deviation σ_τ is 0 is 7.20 minutes, and 8.00 minutes when the standard deviation is a minute. The increase in W due to the standard deviation is (8.00 - 7.20)/7.20 x 100% = 11%, which is of course the same percent increase the standard deviation causes in L'.

Based on a telephone operator staffing study done by Sze (1984), the percent increase in the steady-state expected waiting time W that the service time variation causes when there are multiple faucets is the same as the percent increase when there is a single faucet, assuming that the average of the varying service time equals the constant service time. For example, assume that ρ = 0.8, τ = 2.4 minutes, and the standpost has two taps. Based on Figure 4-6 W = 2.2 x 2.4 = 5.3 minutes. If the service time standard deviation were increased to 1.0 minute, as in the previous example, then the steady-state expected waiting time W would be increased by 11%. The new value of W would be 1.11 x 5.3 = 5.9 minutes.

Figures 4-8 and 4-9 show for the effect that the service time standard deviation σ_τ has on S (the steady-state expected number of service periods customers spend at the standpost) for two and three taps, respectively. The curves for $\sigma_\tau > 0$ were determined with the knowledge that S is directly proportional to W, and that the percent increase in S due to σ_τ is therefore the same for multiple faucets as it is for one faucet.

Understanding why the service time variation causes W to increase may prove useful. While customers having short service times are served, the service rate capacity is in effect increased, thereby decreasing the ρ value during that service time. While customers having long service times are served, the opposite occurs. Both long and short service times effectively change ρ temporarily. However, the relationship between W and ρ is concave-up (i.e., as ρ increases, the W vs. ρ curve becomes steeper). Therefore, longer service times cause a greater increase in W than do the shorter service times cause a decrease in W. The net effect is that W is greater when the service time varies among users than when service time is constant among users. As the variation is service time increases, so does W.

FIGURE 4-8. Steady-state expected number of service intervals spent at the standpost vs. traffic intensity for 2 taps and various service time standard deviations.

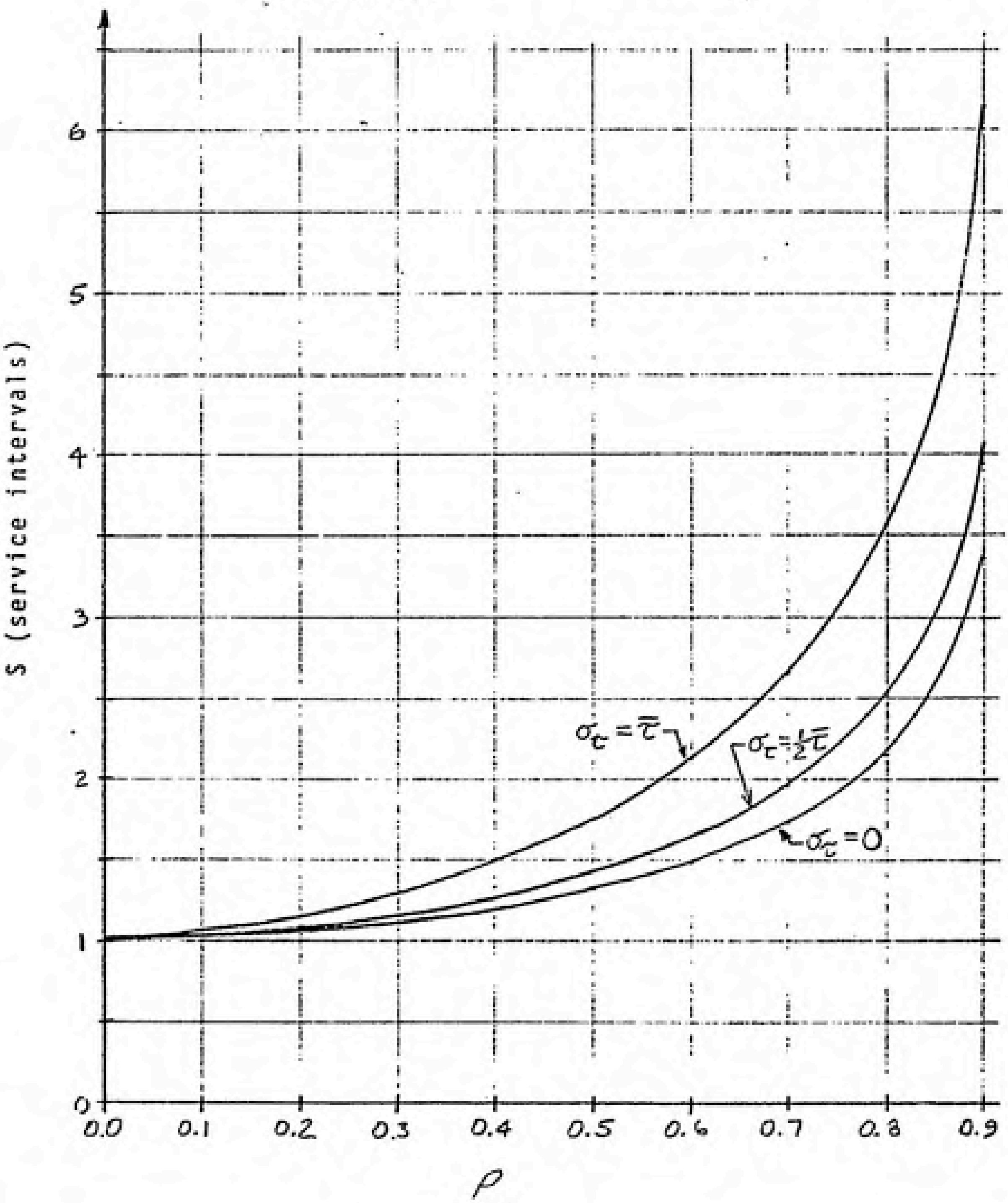
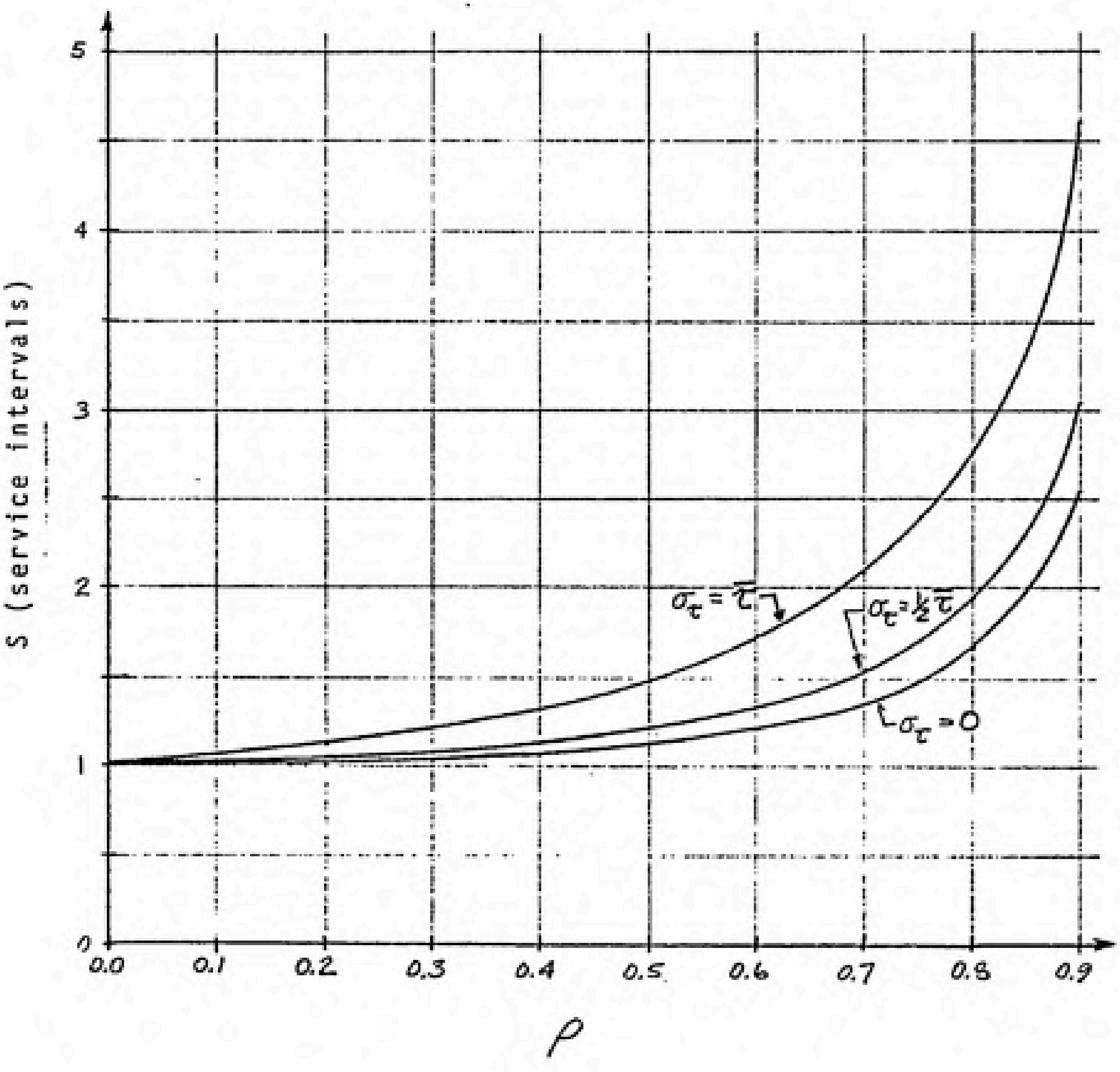


FIGURE 4-9. Steady-state expected number of service intervals spent at the standpost vs. traffic intensity for 3 taps and various service time standard deviations.



Chapter V

SMALL-POPULATION MODEL

The large-population models of Chapter IV are based on the assumption that the population is large enough to be considered infinite. This assumption allows for model simplicity and flexibility. Populations should therefore be assumed infinite whenever they are large enough for the assumption not to cause significant inaccuracies. In this chapter a small-population model (i.e., a model that does not assume the population is infinite) will be derived. In Chapter VI its results will be compared with the results of the model presented in section IV.A. Minimum population sizes which can be assumed to be infinite without causing serious inaccuracies will thus be determined. The same four assumptions listed on page 10 for the large-population model are also assumed for the small-population model so that any difference in results may be attributed strictly to the limitation of population size.

The small-population model is similar to the large-population model in that expected queue lengths are calculated at times which are interger multiples of the service period, i.e., expected line lengths are calculated at the beginning of the second service period, the beginning of the third service period, etc. However, the derivation of the small population model is more complex because the number of arrivals during any particular service period affects the PDF of the number of arrivals during other service periods. The reason for this is that the number of arrivals that have occured may be a significant portion of the population, thus influencing the probabilities of future arrivals. The model derivation consists of the following steps:

1. Determining the PDF for the number of arrivals at the standpost during any service period.
2. Using the PDF in step 1 to develop the PDF for sequences of arrivals over a series of consecutive service periods. (One of the probabilities expressed by this PDF would be, for example, the probability that 2 arrivals occur in the first service period, 0 arrivals occur in the second service period, and 4 arrivals occur in the third service period.)
3. Using the PDF determined in step 2 to calculate expected line lengths and waiting times.

These steps are discussed below. Assume that a single-faucet standpost serves a population of 20 customers. All of the customers arrive at the standpost between 6:00 a.m. and 7:00 a.m.; employment away from home, school, etc., make other times inconvenient. The size of containers is assumed to be 6 gallons and the discharge capacity is 2.5 gpm. The waste factor w is 0.11 and the efficiency factor f is 0.9. The resulting service time τ is, by equation 3.2,

$$\begin{aligned}\tau &= 6 \times (1/2.5) \times (1/[1 - 0.11]) \times (1/0.9) \\ &= 3.0 \text{ minutes}\end{aligned}$$

The scenario is therefore identical to that of section IV.A except that the population is limited to 20 and arrivals occur only between 6:00 a.m. and 7:00 a.m.

1. Probability Density Function of the Number of Arrivals During a Service Period

The time period during which all customers arrive is defined as T, which in this example is 1 hour (6:00 a.m. to 7:00 a.m.).

Because the expected arrival rate is assumed not to vary with time, each member is just as likely to arrive between 6:00 a.m. and 6:03 a.m. as between, say, 6:30 a.m. and 6:33 a.m. The probability that a particular customer (i.e., an arbitrarily chosen customer out of the population of 20) arrives during some time period of length τ within T is simply τ/T. If T is the 1-hour period from 6:00 a.m. to 7:00 a.m. and τ is 3 minutes (0.05hr), then this probability is 0.05hr/1.0hr = 0.05. The probability that two particular customers arrive during a time period of length τ is $(\tau/T) \times (\tau/T)$. The customers act independently of each other and their probabilities of arriving during a period of length τ are therefore multiplied to determine the joint probability that both arrive during the period of length τ. In general, the probability that k particular customers arrive during a time period of length τ is $(\tau/T)^k$.

Similarly, if the population is N, the probability that N - k particular customers arrive outside of a time period of length τ but still within time period T is $([T - \tau]/T)^{N-k}$.

The probability that k particular customers arrive during a time period of length τ and N - k particular customers arrive outside of τ (i.e., during $T - \tau$) is

$$(\tau/T)^k \times ([T - \tau]/T)^{N-k}$$

The probability that <u>any</u> k customers arrive during a period of length τ and <u>any</u> N - k customers arrive during $T - \tau$ is an integer multiple of the above product. The integer is the number of possible sets of size k that can be selected from population

N, which is the permutation

$$N!/(k![N - k]!)$$

Therefore, defining $r_k(\tau)$ as the probability that k customers arrive during a period of length τ,

$$r_k(\tau) = (N!/(k![N - k]!))(\tau/T)^k([T - \tau]/T)^{N-k} \quad (5.1)$$

Equation 5.1 is a form of the binomial distribution. For example, the probability that 2 customers arrive between 6:00 a.m. and 6:03 a.m. is

$$\begin{aligned} r_2(0.05hr) &= (20!/2!18!)(0.05hr/1.0hr)^2(0.95hr/1.0hr)^{20-2} \\ &= 0.189 \end{aligned}$$

Because the entire population of 20 arrives in a 1-hour period and the expected arrival rate does not vary with time, the expected arrival rate is 20/hr. Table 5-1 compares the probabilities when applying the Poisson equation (equation 4.1) and the binomial equation (equation 5.1) over a 0.05hr period when the expected arrival rate λ = 20 persons/hr. The Poisson equation assumes the population is infinite, whereas the binomial equation assumes a population size of N, which in this example is 20. It is well known that as N increases, the results of the binomial equation approach those of the Poisson equation. Table 5-1 shows that when the number of arrivals k is equal to 1, which is the expected number of arrivals per service period, $r_K(0.05hr) >$ p (0.05hr). That is, the probability of 1 arrival during a time period of length τ is greater when the population is 20 than when the population is infinite. This same inequality is also true when the number of arrivals k = 2. However, in general, $r_k(\tau)$ is less than $p_k(\tau)$ for values of k different from the expected number of arrivals per service period.

TABLE 4-1
Comparison of Binomial (N = 20) and Poisson (N = ∞)
Equation Results for τ = 0.05 hr

k	r_k(0.05hr)	p_k(0.05hr)
0	0.358	0.368
1	0.377	0.368
2	0.189	0.184
3	0.060	0.061
4	0.013	0.015
5	0.002	0.003
6	0.000	0.001
7	0.000	0.000
.	.	.
.	.	.
.	.	.
20	0.000	0.000

2. Probability Density Function for Sequences of Arrivals During a Series of Consecutive Service Periods

With known PDF of arrivals during the first service period and given that no one is in line at 6:00 a.m., equation 4.2 can be used to determine the PDF for the number in line at the beginning of the second service period. For this purpose, the $r_k(\tau)$ values in Table 5-1 are used in place of $p_k(\tau)$ because the population is 20 instead of infinite. The PDF can then be used to calculate the expected line length at the beginning of the second service period.

Calculating the PDF and expected line length at the beginning of the third service period is more difficult due to the fact that arrival probabilities during the second service period depend on those during the first. For example, if two arrivals occur during the first period then the probability of three arrivals in the second period is different than if only one arrival had occured during the first period. It is therefore necessary to determine the probability of sequences of arrivals.

To determine the probabilities of sequences of arrivals, let the variables k_1 and k_2 represent the number of arrivals in the first and second periods, respectively. The probability of k_2 arrivals during the second period given that k_1 arrivals have already occured in the first period can be determined by equation 5.1, but with the population reduced to $N - k_1$ and T reduced to $T - \tau$. This probability is

$$([N-k_1]!/(k![N - k_1 - k_2]!))(\tau/[T - \tau])^{k_2}([T - \tau - \tau]/[T - \tau])^{N-k_1-k_2}$$

In the above expression, the probability of k_2 arrivals in the second interval is conditional on the probability of k_1 arrivals in the first interval. The product of the two probabilities is therefore equal to the joint probability that k_1 arrivals occur in the first interval and k_2 arrivals occur in the second. Multiplying the two probabilities yields

$$r_{k_1,k_2}(\tau) = [N!/(k_1!k_2!(N - k_1 - k_2)!)](\tau/T)^{k_1+k_2}((T - 2\tau)/T)^{N-k_1-k_2} \quad (5.2)$$

Similarly, the joint probability of k_1 arrivals in the first period, k_2 arrivals during the second period, and k_3 arrivals in the third period is

$$r_{k_1,k_2,k_3}(\tau) =$$
$$(N!/(k_1!k_2!k_3![N - k_1 - k_2 - k_3]!)(\tau/T)^{k_1+k_2+k_3}([T - 3\tau]/T)^{N-k_1-k_2-k_3} \quad (5.3)$$

Probabilities for sequences of arrivals over four or more service periods can similarly be determined.

3. Expected Line Lengths and Waiting Times

Once the probabilities of sequences of arrivals are determined, the PDF of the number in line can be determined. For example, to determine the probability that one person is in line at the beginning of the third service period, the probabilities of all possible arrival sequences causing one person to be at the standpost must be summed; that is

$v_1(3)$ = probability of no arrivals in the first service period and 1 arrival in the second period

+

probability of 1 arrival in the first period and 1 arrival in the second period

+

probability of 2 arrivals in the first period and no arrivals in the second period

The terms on the right would then be calculated by use of equation 4.2. The process would be repeated to determine $v_o(3)$, $v_2(3)$, $v_3(3)$, etc. until the entire PDF of the number in line at the beginning of the third interval is known. This PDF can then be used to determine L'(n) by applying equation 4.6.

Unfortunately, this process for determining L'(n) is quite cumbersome for populations greater than about ten and for service periods later than the fourth or fifth; development of the computer program is excessively laborious. This difficulty is due to the very large number of possible arrival sequences. However, as discussed below, it is possible to determine the expected line length if $v_o(n)$ is known for each n. The necessity of calculating all $v_j(n)$ values other than $v_o(n)$ is thus eliminated. The multitude of arrival sequences that must be considered is greatly reduced.

The derivation of the procedure for calculating L'(n) values based on $v_o(n)$ values begins with the following equation:

$$L(n + 1) = L(n) + \text{expected number of arrivals during period } n - \text{expected number of customers served during period } n \tag{5.4}$$

The expected number of arrivals during a service period is ρ, the traffic intensity, which for the small-population model is the ratio of the population size to the number of service intervals during which the standpost is open. Thus,

$$L(n + 1) = L(n) + \rho - \text{expected number of customers served during period } n \tag{5.5}$$

The number of customers served during a service interval must be either 0 or 1 for a single-faucet standpost. Therefore,

$$\begin{matrix}\text{probability} \\ \text{that no} \\ \text{customers} \\ \text{are served} \\ \text{during} \\ \text{period } n\end{matrix} + \begin{matrix}\text{probability} \\ \text{that 1} \\ \text{customer} \\ \text{is served} \\ \text{during} \\ \text{period } n\end{matrix} = 1$$

and

$$\begin{array}{l}\text{probability}\\ \text{that 1}\\ \text{customer}\\ \text{is served}\\ \text{during}\\ \text{period } n\end{array} \quad = \quad 1 \quad - \quad \begin{array}{l}\text{probability}\\ \text{that no}\\ \text{customers}\\ \text{are served}\\ \text{during}\\ \text{period } n\end{array} \tag{5.6}$$

Also,

$$\begin{array}{l}\text{expected}\\ \text{number of}\\ \text{customers}\\ \text{served}\\ \text{during}\\ \text{period } n\end{array} \quad = \quad \begin{array}{l}\text{probability}\\ \text{that no}\\ \text{customers}\\ \text{are served}\\ \text{during}\\ \text{period } n\end{array} \quad \times \quad 0 \quad + \quad \begin{array}{l}\text{probability}\\ \text{that 1}\\ \text{customer}\\ \text{is served}\\ \text{during}\\ \text{period } n\end{array} \quad \times \quad 1$$

$$= \quad \begin{array}{l}\text{probability}\\ \text{that 1}\\ \text{customer}\\ \text{is served}\\ \text{during}\\ \text{period } n\end{array} \tag{5.7}$$

Combining equations 5.6 and 5.7 yields

$$\begin{array}{l}\text{expected number}\\ \text{of customers}\\ \text{served during}\\ \text{period } n\end{array} \quad = \quad 1 \quad - \quad \begin{array}{l}\text{probability}\\ \text{that no}\\ \text{customers}\\ \text{are served}\\ \text{during}\\ \text{period } n\end{array} \tag{5.8}$$

Equations 5.5 and 5.8 yield

$$L(n + 1) = L(n) + \rho - \left(1 - \begin{array}{l}\text{probability that no}\\ \text{customers are served}\\ \text{during period } n\end{array}\right) \tag{5.9}$$

The probability that no one is served during period n is equal to the probability that no one is at the standpost at the beginning of period n. So, equation 5.9 can be rewritten as

$$L(n + 1) = L(n) + \rho - (1 - v_o(n))$$
$$= L(n) + \rho + v_o(n) - 1 \qquad (5.10)$$

To illustrate equation 5.10, L(2) is calculated for the scenario presented in this chapter. Substituting n = 1 in equation 5.10,

$$L(2) = L(1) + \rho + v_o(1) - 1$$

No one is waiting at the standpost when it opens at 6:00 a.m. Therefore, $L(1) = 0$ and $v_o(1) = 1$. Also, $\rho = 1$. The above equation thus reduces to

$$L(2) = 0 + 1 + 1 - 1 = 1$$

Hence, the expected number in line at the beginning of the second period (i.e, at 6:03 a.m.) is 1.

To calculate the expected number of customers at the beginning of the third interval,

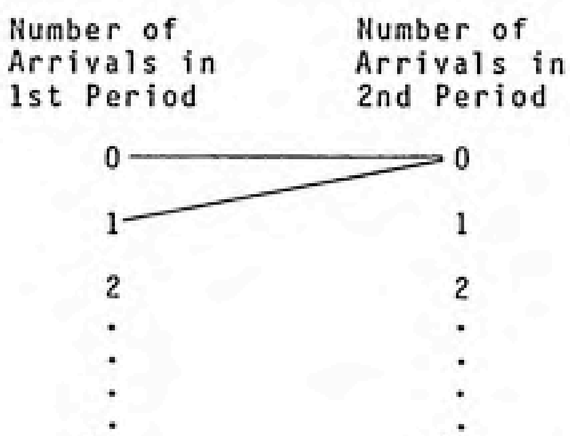
$$L(3) = L(2) + \rho + v_o(2) - 1$$

As shown before, $L(2) = 1$. $v_o(2)$, the probability that no customers are in line at the beginning of the second period, is, by equation 4.2, equal to the probability that no customers arrive during the first period. This latter probability is given by equation 5.1 and has been calculated in Table 4.1 to be 0.358. Thus,

$$L(3) = 1 + 1 + 0.358 - 1$$
$$= 1.358$$

Similarly, with L(3) known, L(4) can be determined after calculating $v_o(3)$. The line is empty at the beginning of the third period if no customers arrive in the first and second periods or if 1 customer arrives in the first interval and no customers arrive in the second period. The following diagram illustrates this. Lines interconnect numbers to indicate arrival sequences which cause no one to be at the standpost during the third service interval. For example, a line connects 1 in the first column with 0 in the second column because 1 arrival in the first period and 0 arrivals in the second period causes no one to be at the standpost in the third period. No line connects 2 in the first column with 2 in the second column, for example, because such an arrival sequence would cause someone to be at the standpost during the third interval.

Sequences of Arrivals for Which No one is at the Standpost at the Beginning of the Third Period



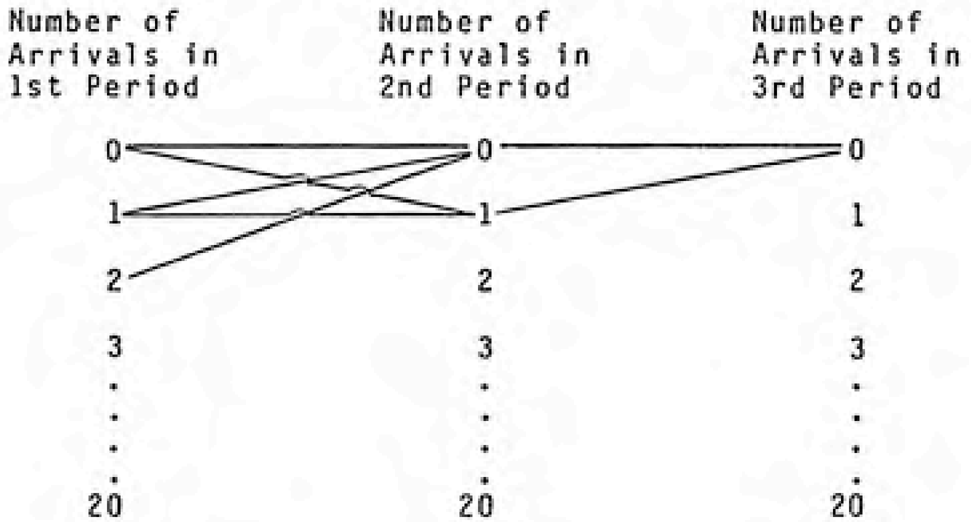
The probability that no customers are in the queue at the beginning of the third period is thus

$$v_o(3) = \text{probability that no customers arrive in the 1st and 2nd periods} + \text{probability that 1 customer arrives in the 1st period and no customers arrive in the 2nd period} \qquad (5.11)$$

Equation 5.11 is solved by application of equation 5.2 to each of the right hand side terms. The resulting L(4) is then calculated by equation 5.10.

Equation 5.10 can be applied in the above manner to each successive period. L(n + 1) can be determined once L(n) is known. However, calculating $v_o(n)$ becomes increasingly complicated as n increases. This is due to the increase in the number of possible arrival sequences resulting in an empty line. For example, at the beginning of the fourth period, no customers will be present if any of the arrival sequences shown in the following diagram occur. The number of possible sequences is 5, which is a 150% increase over the number of possible sequences causing an empty line at the beginning of the third period.

Sequences of Numbers of Arrivals for Which the Queue Is Empty at the Beginning of the Fourth Period



The computer program in Annex C.8 calculates $v_o(2)$, $v_o(3)$, $v_o(4)$, $v_o(5)$ and $v_o(6)$ for various values of ρ and N. The program does not calculate $v_o(n)$ for $n > 6$ becuase the list of possible arrival seqences for which no one is at the standpost at the beginning of period n is very large and complicated. Fortunately, an interesting phenomenon occurs at the beginning of period $n = N + 1$: the expected line length reaches steady-state. That is, $L(n + 1) = L(n)$ for $n \geq N + 1$*. By equation 5.10 this implies that $v_o(n) = 1 - \rho$ for $n \geq N + 1$. With calculated values of $v_o(n)$ for the early periods, a curve through them (on a plot of $v_o(n)$ versus n) can be extrapolated to the $v_o(N + 1)$ value of $1 - \rho$ at the beginning of period $N + 1$. Such a curve is shown in Figure 5-1 for the scenario of this chapter where $N = 20$. That is, $v_o(20 + 1) = 1 - \rho = 1 - 1 = 0$, which occurs at 7:00 a.m.

Values from the curve in Figure 5-1 were used to determine $L(n)$ values by equation 5.10. The $L'(n)$ values were then calculated by equation 4.5 and plotted in Figure 5-2. $L'(n)$ reaches a maximum value of 2.8 at the beginning of period $n = 21$, or 7:00 a.m. The expected waiting time at 7:00 a.m. is $W(21) = 2.8 \times 3$ minutes = 8.4 minutes, which is fairly short. However, if the service time were, for example, 20 minutes with vendors hauling water by the cart, then $W(21) = 2.8 \times 20$ minutes = 56 minutes, which is quite lengthy. In this case the 21st interval would begin $20 \times 20 = 400$ minutes = 6.7 hours after 6:00 a.m., or at 12:40 p.m.

* Recall that N is the total size of the population to be served.

FIGURE 5-1. Probability that no one is in line at the beginning of interval n vs. n and time for chapter V illustrative example.

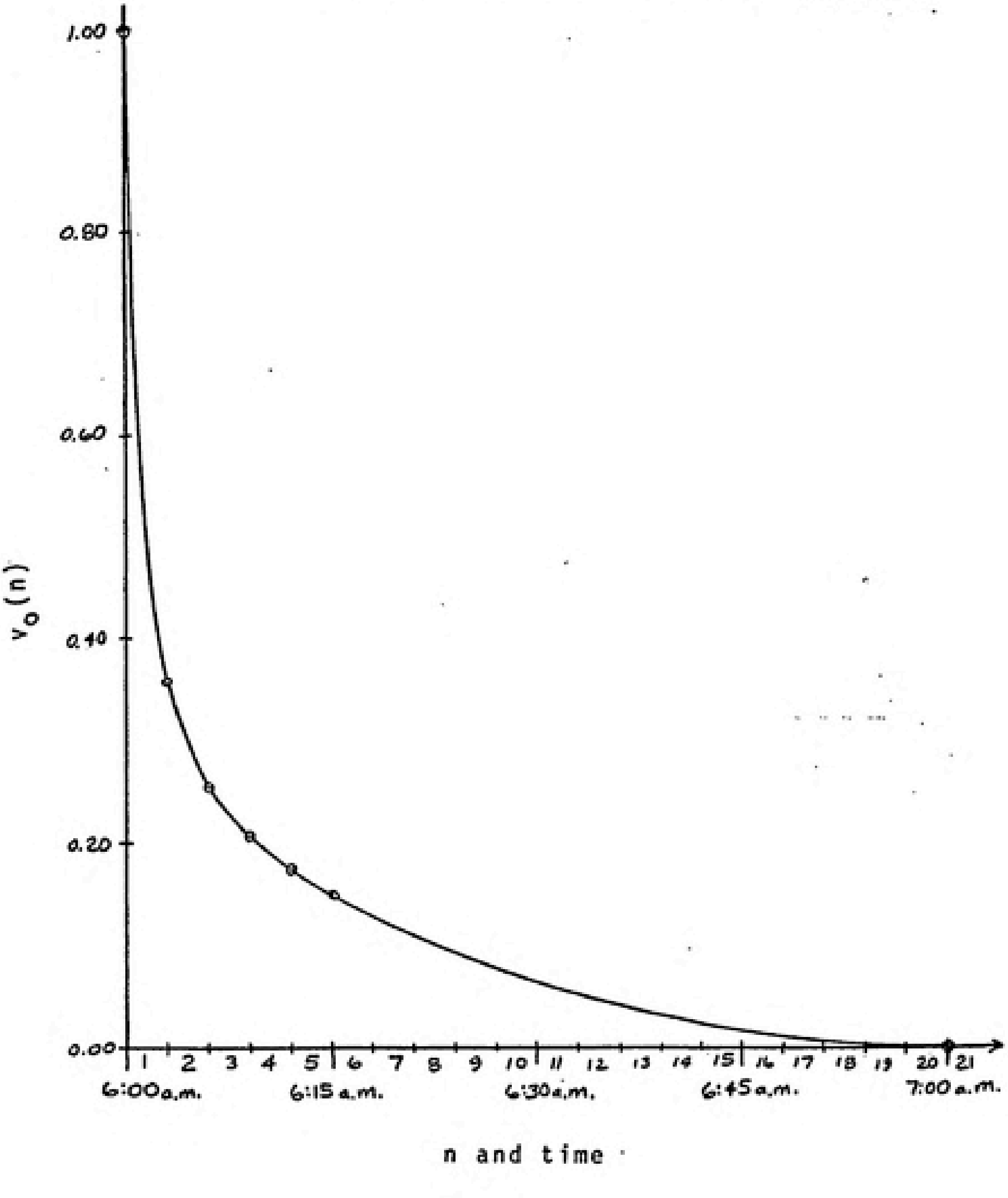
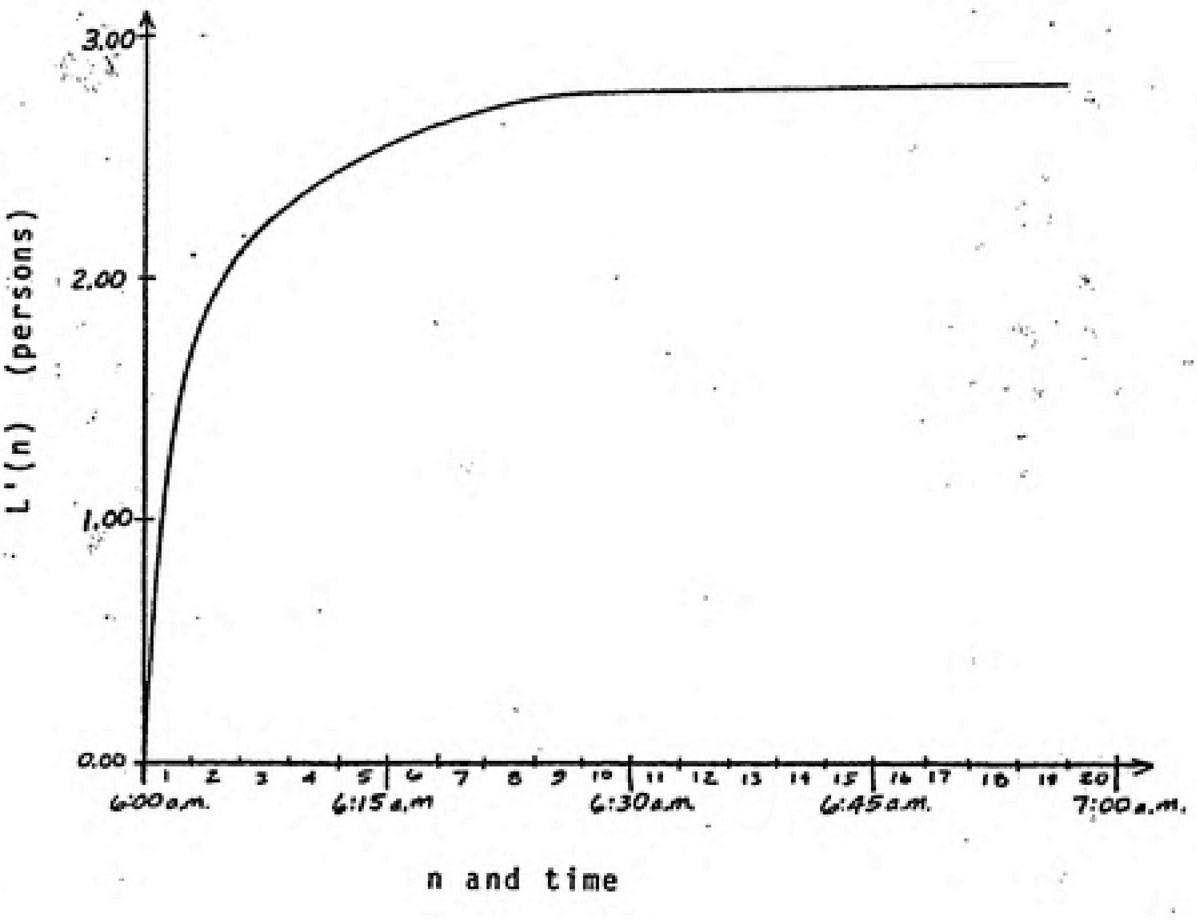


FIGURE 5-2. Expected line length (given that the line is not empty) vs. n and time for chapter V illustrative example.



Chapter VI

COMPARISON OF LARGE- AND SMALL-POPULATION MODELS

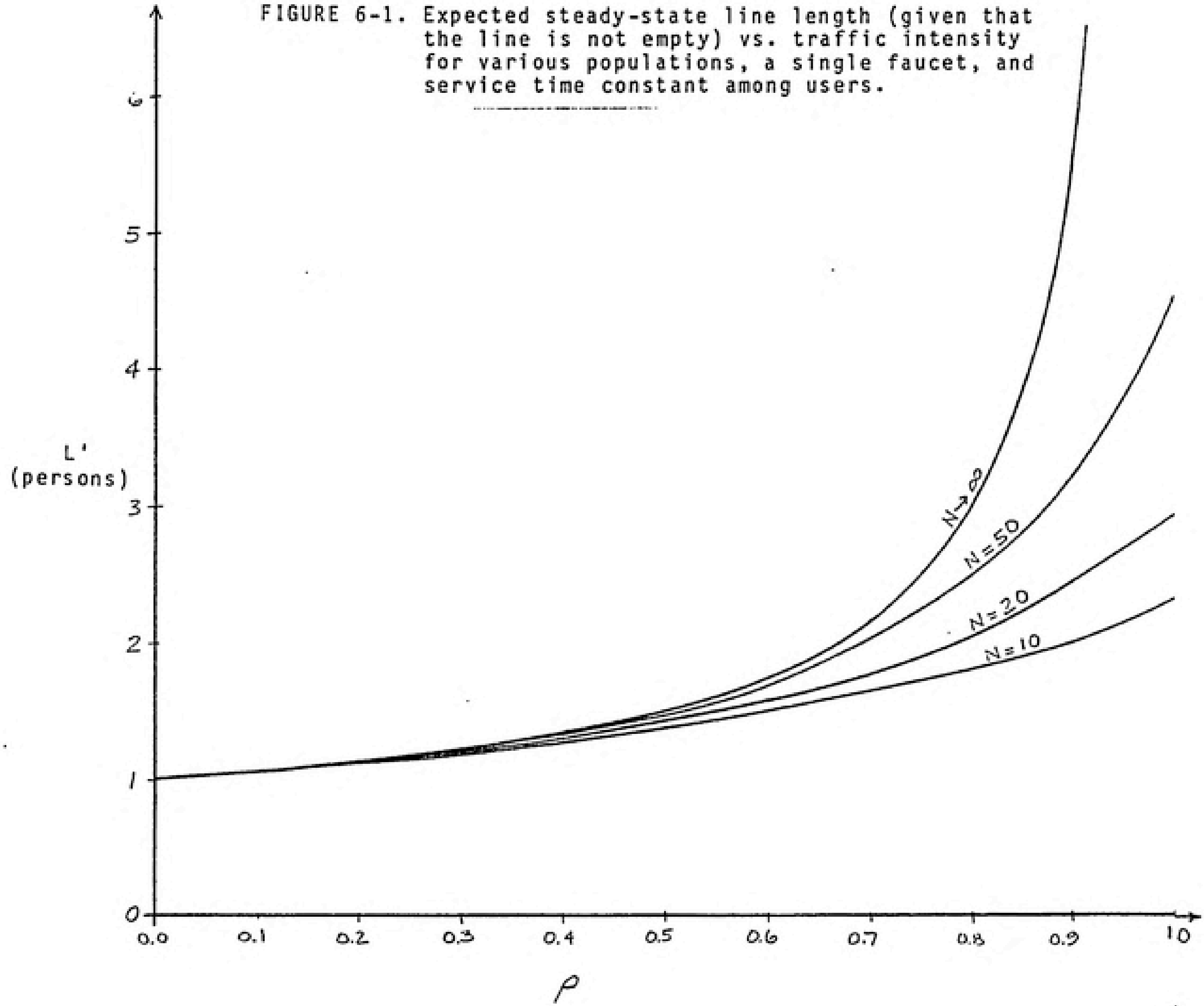
A. Constant Arrival Rates

The curves in Figure 6-1 show L' (the steady-state expected line length given that the line is not empty) vs. the traffic intensity ρ when the expected arrival rate λ and service time τ are constant and the standpost has only one faucet. The curves for populations of N = 10, 20, and 50 were obtained by the procedure described in chapter V. Recall that this procedure assumes that each member of the population makes one and only one trip to the standpost (p. 49). The $N = \infty$ curve was obtained by equation 4.9.

The curves show that if the traffic intensity ρ is near 0.8, and each member of the population makes one and only one trip to the standpost, then a population as small as 50 can be assumed infinite without causing serious error. If ρ is near 0.5 then a population as small as 10 can be assumed infinite without causing serious error. As the ratio of the expected arrival rate to the service rate capacity decreases, so does the minimum population size which can be assumed infinite.

In many cases members of the population will make more than one trip to the standpost. WHO guidelines suggest that the average per capita daily demand may sometimes be as high as 60 liters, or 130 pounds of water, which would not easily be carried in a single trip. This is not considered in the small-population model of Chapter V or its resulting curves in Figure 6-1. Also, the small-population model and curves are for only one faucet, while a standpost may often have multiple faucets. For these two

FIGURE 6-1. Expected steady-state line length (given that the line is not empty) vs. traffic intensity for various populations, a single faucet, and service time constant among users.



reasons it is necessary to define N' as the effective population size, or the product of the actual population size and the average number of trips made per person divided by the number of taps at the standpost. N' can be used in the following procedure to determine S, the steady-state expected number of service periods spent at the standpost, which in turn can be multiplied by the service time if the waiting time is to be known:

1. Assume the population is infinite and apply the approaches described in chapter IV to determine S. (Recall that for a single-faucet standpost, S is numerically equal to L'.)

2. Determine the effective population size N'. For example, if a population of 75 has a per capita demand of 12 gallons and 6 gallons is obtained per trip, then the effective population size is 75 x (12/6)/3 = 50 persons per faucet.

3. Multiply S determined in step 1 by the ratio of L' in Figure 6-1 for the effective population size to L' in Figure 6-1 for $N = \infty$. For example, assume L' for a three-faucet standpost and a varying service time is determined in step 1 to be 3.20 at $\rho = 0.8$. If the effective population size N' is 50 as determined in step 2, the ratio obtained from Figure 6-1 from the N = 50 and $N = \infty$ curves at $\rho = 0.8$ is 2.5/3.0 = 0.83. The resulting S is thus 3.20 x 0.83 = 2.7 periods.

Although the accuracy of this technique is not rigorously proven, it is believed to be reasonably accurate.

B. Variable Arrival Rates

The error due to assuming an infinite population when the expected arrival rate is constant is shown in Figure 6-1, but the error when the expected arrival rate varies with time is more difficult to quantify exactly.

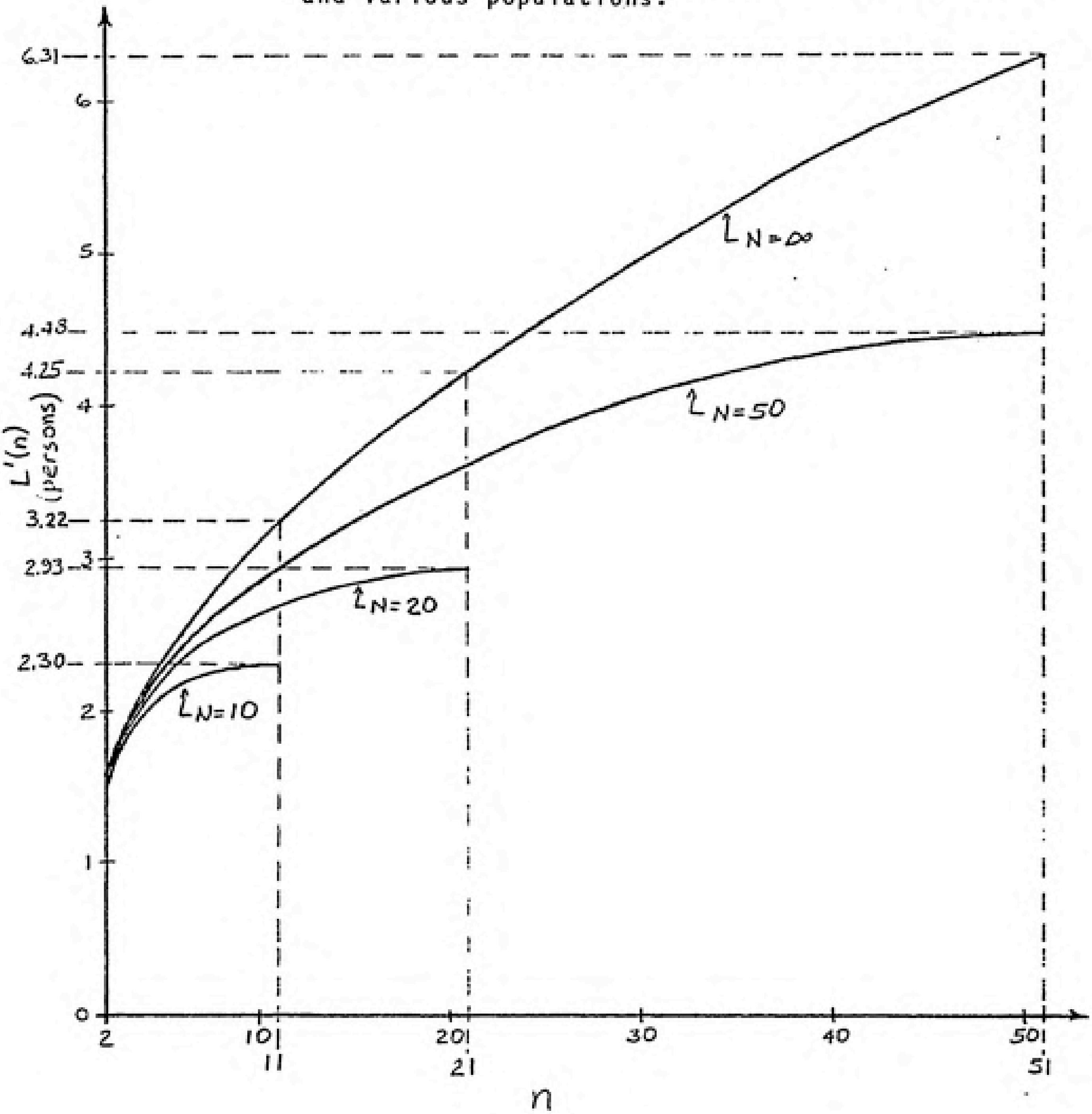
A rough estimate of the error can be obtained by considering

Figure 6-2. This figure shows L'(n) vs. n for population of 10, 20 and 50 and an infinite population, given that the standpost has one faucet and expected arrival rates and service times are constant. The curves are for ρ (the ratio of the expected arrival rate to the service rate capacity) equal to 1.0. Notice that the ratio of L'(11) (the expected line length at the beginning of the 11th service period given that at least one person is at the standpost) for population N = 10 to L'(11) for $N = \infty$ is 2.30/3.22 = 0.71. Also, the ratio of L'(21) for N = 20 to L'(21) for $N = \infty$ is 2.93/4.25 = 0.69, and the ratio of L'(51) for N = 50 to L'(51) for $N = \infty$ is 4.48/6.31 = 0.71. Regardless of the population size, the maximum expected line length (given that the line is not empty) reached at the beginning of interval N + 1 is approximately 70% of what the value would be if the population were infinite.

Notice also that for $n < N + 1$, the ratio of L'(n) for N = 10, 20, or 50 to L'(n) for $N = \infty$ is greater than 0.70. The ratio increases as n decreases, and nearly equals 1.0 for small n values. When $\rho = 1$, the beginning of interval N + 1 is when arrivals cease, either because the standpost closes or people have finished collecting water for the day. So, for $\rho = 1$, ratio of the limited-population L'(n) to the infinite-population L'(n) reaches is minimum value of about 0.7 when arrivals cease. The ratio at earlier times in the day will be higher.

A set of curves similar to that shown in Figure 6-2 could also be prepared for ρ values other than 1.0. Such sets of curves would show that the limited-population L'(n) to infinite-population L'(n) ratios always reach their minimum value when

FIGURE 6-2. Expected line length (given that the line is not empty) vs. n for a traffic intensity of 1.0, a single faucet, service time constant among users, and various populations.



arrivals cease, but that this minimum value is always greater than 0.7. (Note: For ρ values other than 1, arrivals cease at the beginning of interval $(N + 1)/\rho$.) This minimum ratio value which occurs when arrivals cease is smallest for $\rho = 1$, and increases as ρ differs from 1. For example, for a population of $N = 50$ and $\rho = 0.8$, the ratio is 0.83 when arrivals cease. For a population of $N = 50$ and $\rho = 0.6$ the ratio is 0.91 when arrivals cease. For $\rho = 0.4$ the ratio when arrivals cease is 0.98. This same behavior is also demonstrated for $N = 10$ for which the ratios are slightly lower but still always exceed 0.7 and increase as ρ differs from 1. The curves would also indicate that, as in the $\rho = 1$ case, the ratio of the limited-population $L'(n)$ to the infinite-population $L'(n)$ increases as n decreases.

In summary, the above observations indicate that if the expected arrival rate λ is constant and $\rho = 1.0$, then $L'(n)$ when arrivals cease is about 70% of what it would be if the population were infinite. This percentage increases as ρ differs from 1. Also, as n decreases, the limited-population $L'(n)$ to infinite-population $L'(n)$ ratio increases.

In light of these observations, conclusions can be drawn for time-varying expected arrival rates. When the expected arrival rate varies with time, such as shown in Figure 4-4a, ρ is near 1.0 for only a small percentage of the time. Thus $L'(n)$ when arrivals cease for a limited population must be greater than 70% of what it would be if the population were infinite. Furthermore, because the limited-population $L'(n)$ to infinite-population $L'(n)$ ratio increases as n decreases, and because the maximum $L'(n)$ value for a time-varying expected arrival rate typically occurs well before arrivals cease, the maximum L'(n) value when λ varies for a limited population must be fairly close to that value when the population is infinite. The former is believed to be generally between 85% and 95% of the latter. Thus, the large-population model usually provides acceptable accuracy for determining maximum L'(n) values when λ varies.

Chapter VII

ROLE OF QUEUING THEORY IN STANDPOST DESIGN

A. General

Queuing theory can be used for standpost design to ensure that the maximum expected waiting time is kept below some fixed value, or to meet other such waiting time criteria. Basically, the two methods of applying queuing theory are as follows:

1. With the population to be served by the standpost known, use queuing theory to determine the standpost discharge capacity required to meet the waiting time criterion.
2. With the standpost discharge capacity known, use queuing theory to determine the maximum population which can be served by the standpost while meeting waiting time criterion.

These methods are discussed in detail in the following sections.

The WHO has recommended limitations on populations served per standpost (Annex A). Theoretically, queuing theory can be used to successfully design standposts for populations exceeding WHO limitations. However, the designer should keep WHO guidelines in mind, realizing that the use of queuing theory for standpost design is not yet field-proven.

B. Fixed Population and Constant Arrival Rate

If the population to be served by a standpost is fixed, the customers arrive throughout the day at a constant rate, the standpost is to have only one faucet, and a maximum expected waiting time criterion is to be met, then the expected line length is assumed to reach near steady-state before the standpost closes or arrivals cease. The maximum expected waiting time thus equals the steady-state expected waiting time, allowing for the use of equation 4.17 to determine the required service time. The procedure is as follows:

1. Estimate the total number of trips to the standpost made by the population, the average volume of water obtained per trip and its standard deviation, the waste factor w, and the efficiency factor f.

2. Determine the expected rate at which the customers arrive at the standpost. Since this rate is constant, simply divide the total number of trips made to the stanpost by the number of hours over which arrivals occur. For example, if the population is 100 with each person making an average of 2 trips to the standpost between 6:00 a.m. and 6:00 p.m., then the expected arrival rate λ is 16.7 persons/hour.

3. Express the standard deviation of the volume obtained per trip as the standard deviation of the service time. Equation 3.2 shows that the service time is proportional to the volume obtained. If the standard deviation of the volume is 2.1 gallons and the average volume is 7.5 gallons, then the standard deviation of the service time is 2.1/7.5 x average service time, or $\sigma_{\tau} = 0.28\bar{\tau}$.

4. Use equation 4.17 to solve for the average service time. For example, if it is desired to keep the maximum expected waiting time (which is also the steady-state expected waiting time) below 10 minutes, and the expected arrival rate and the standard deviation of the service time are as in the above steps, then $\bar{\tau}$ = 2.94 minutes. (Note: Equation 4.17 is solved for $\bar{\tau}$ by trial-and-error because cannot be isolated to the left side of the equation.)

5. Determine the required discharge capacity by use of equation 3.2. If the waste factor w is 0.1 and the efficiency factor f is 0.9, then the required discharge capacity by rearranging equation 3.2 is

   Q_{max} = 7.5 x (1/2.94) x 1/(1 - 0.1) x 1/0.9 = 3.15 gpm

6. Determine the service rate capacity μ, which is the inverse of the average service time, or 1/2.94 = 0.34 persons/minute.

7. Determine ρ, which is the ratio of the expected arrival rate λ to the service rate capacity μ, or 0.278/0.34 = 0.82.

8. Use Figure 6-1 to see whether assuming that the population is large enough to be considered infinite causes a significant error. The effective population size is $N' = 200$ (100 people x 2 visits per person at a single-faucet standpost). Figure 6-1 shows that if an $N = 200$ curve were interpolated between the $N = 50$ and the $N = \infty$ curves, it would be very close to the $N = \infty$ curve at $\rho = 0.82$. Therefore, the assumption of an infinite population causes no significant error, and the discharge capacity determined in step 5 is appropriate.

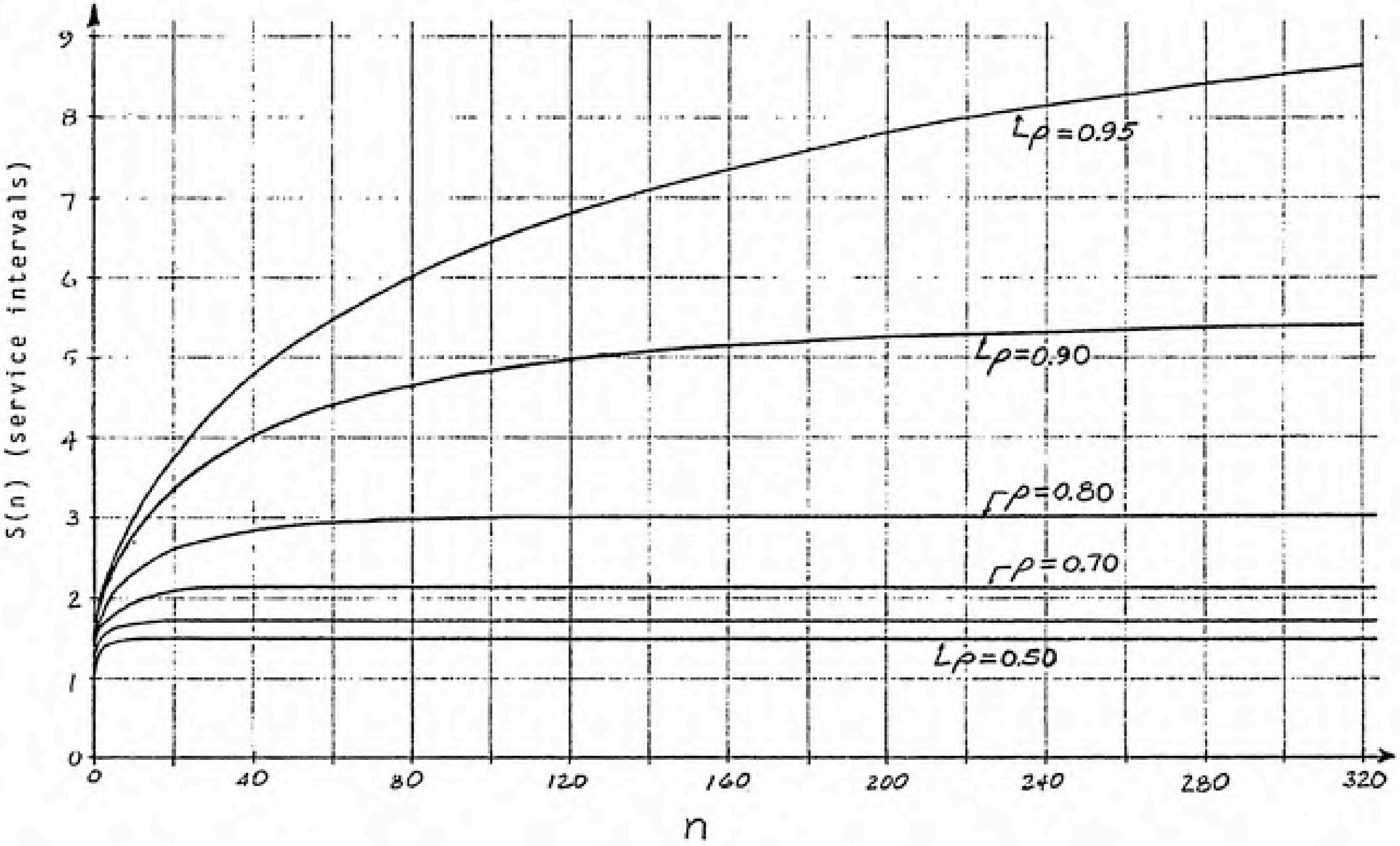
   If Figure 6-1 had shown that the assumption of an infinite population causes significant error, then the 10-minute waiting time criterion would have to be ficticiously increased slightly to, say, 10.5 minutes. Then steps 4 through 7 would have to be reworked. The ρ value that would result from applying these steps would be 0.83. The ratio of S from the interpolated $N = 200$ curve at $\rho = 0.83$ to S from the $N = \infty$ curve at $\rho = 0.83$ would then be multiplied by 10.5 minutes. If the result would be other than 10.0 minutes, then a new fictitious waiting time criterion would again have to be selected and steps 4 through 7 again repeated.

The designer should verify that steady-state is essentially reached by considering the curves in Figure 7-1 and the number of service intervals that will have elapsed when arrivals cease. In the unlikely event that steady-state is not reached, the designer can modify and apply the program in Annex C.1, and use it as a substitute for equation 4.17 in the above 8-step procedure.

As discussed in section IV.C, for a given standpost discharge capacity, the waiting time decreases as the number of faucets decreases (and the discharge per faucet increases). Thus a standpost being designed should first be assumed to have one faucet. If the resulting capacity cannot be provided by a single faucet, then the standpost should be assumed to have two or more faucets. The following procedure should be used for standposts with more than one faucet:

1. Do steps 1 through 3 described above.

FIGURE 7-1. Expected number of service intervals spent at the standpost vs. n for various traffic intensities, a single faucet, and service time constant among users.



2. Assume an average service time $\bar{\tau}$ of, say, 6.0 minutes, and determine the service rate capacity per faucet, which is the inverse of the average service time, or 1/6.0 = 0.167 persons/minute. Determine the traffic intensity ρ , which is the ratio of the expected arrival rate λ to the standpost service rate capacity $c\mu$. If the standpost has two faucets, ρ = 0.278/(2 x 0.167) = 0.83.

3. For ρ determined in step 2, use Figure 4-8 (or Figure 4-9 if the standpost were to have 3 faucets) to determine S. If the service time standard deviation σ_τ = $0.5\bar{\tau}$, then the corresponding S value at ρ = 0.83 is 2.8 intervals. (Note: Interpolate if the standard deviation is other than 0, $0.5\bar{\tau}$, or $\bar{\tau}$.) The resulting expected waiting time W is 2.8 x 6.0 = 16.8 minutes.

4. Apply step 8 above to see if assuming an infinite population causes error, and to adjust W if necessary.

5. If the adjusted W from step 4 is different from the waiting time criterion, assume a new value of the average service time and rework steps 2 through 4.

The designer can verify that steady-state is essentially reached by considering Figure 7-1 and the number of elapsed service intervals. The programs in Annexes C.2 and C.3 can be modified and applied as necessary if steady-state is not yet reached.

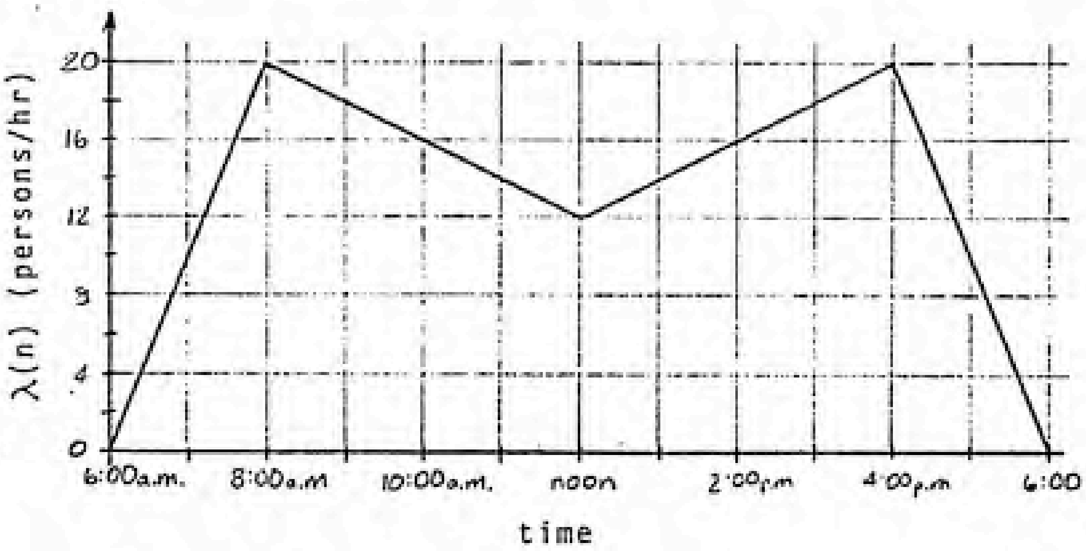
C. Fixed Population and Time-varying Expected Arrival Rate

If the population served per standpost is fixed and the expected rate at which customers arrive varies with the time of day, then queuing theory can be used to determine the required discharge capacity as follows:

1. Estimate the average volume of water customers obtain per standpost visit, the waste factor w, and the efficiency factor f. Estimate the expected rate at which the customers arrive at the standpost as a function of the time of day.

2. Assume that the standpost is to have only one faucet and assume a service time. A good service time to assume is one that causes the service rate capacity to be equal to the peak expected arrival rate. If, for example, the designer determines the expected arrival pattern to be as shown in Figure 7-2, then assume the service rate capacity is 20 persons/hour and that the service time is the inverse of this, or 0.05 hours = 3.0 minutes.

FIGURE 7-2. Expected arrival rate pattern for section VII.C example.



3. Determine the required discharge capacity by use of equation 3.2. If the volume obtained per visit is 6 gallons, the waste factor w is 0.1 and the efficiency factor f is 0.9, then the required discharge capacity by rearranging equation 3.2 is

$$Q_{max} = 6 \times (1/3.0) \times 1/(1 - 0.1) \times 1/0.9 = 2.5 \text{ gpm}$$

If the designer believes that this discharge cannot be provided by a single faucet without causing excessive splashing or waste, he should assume that the standpost has two faucets instead of one, with the service time being doubled and the discharge per faucet being 2.5/2 = 1.25 gpm. In this example, the discharge is appropriate for a single faucet.

4. Express the variation in the expected arrival rate shown in Figure 7-2 as a function of the service period, as follows:

$\lambda(n) = 0.5n$ $\qquad 0 \leq n \leq 40$ (i.e., from 6:00 a.m. to 8:00 a.m.)

$\lambda(n) = 24 - 0.1n$ $\qquad 40 \leq n \leq 120$ (i.e., from 8:00 a.m. to noon)

$\lambda(n) = 0.1n$ $\qquad 120 \leq n \leq 200$ (i.e., from noon to 4:00 p.m.)

$\lambda(n) = 120 - 0.5n$ $\qquad 200 \leq n \leq 240$ (i.e., from 4:00 to 6:00 p.m.)

5. Modify the computer program in Annex C.4 by substituting in the service time of 0.05 hours and the above expected arrival rate function. This modified program is shown in Annex C.5. Running the program to obtain results shows that the maximum expected waiting time is 14.3 minutes and occurs at about 4:10 p.m.

6. If the waiting criterion requires that the maximum expected waiting time not exceed 10 minutes, then the waiting time determined in step 5 is too long. A shorter service time must be selected and steps 3 through 5 must be reworked. Repeated attempts indicate that a service time of 0.046 hours (2.7 minutes) ensures that the maximum expected waiting time does not exceed 10 minutes. The corresponding discharge capacity is 2.7 gpm.

D. Fixed Discharge Capacity and Constant Expected Arrival Rate

If the standpost discharge capacity is fixed and the expected arrival rate is constant, then queuing theory can be used to determine the maximum population size which can be served while meeting a maximum expected waiting time criterion. The line length is assumed to reach its steady-state, maximum value before arrivals cease. The steps of the procedure are as follows:

1. Estimate the average volume of water obtained per visit to the standpost and its standard deviation, the waste factor w, and the efficiency factor f.

2. Use equation 3.2 to determine the corresponding average service time. For this example, assume the discharge capacity is 5.0 gpm to serve vendors who obtain an average of 40 gallons per visit. If the waste factor is 0.1 and the efficiency factor is 0.95, then the average

service time is

$$\bar{\tau} = 40 \times (1/5.0) \times 1/(1 - 0.1) \times 1/0.95$$

$$= 9.36 \text{ minutes}$$

3. Determine the standard deviation of the service time. If the standard deviation of the volume of water obtained per trip is 10 gallons and the average volume is 6 gallons, then based on the proportionality of equation 3.2, the standard deviation of the service time is 10/40 x 9.36 = 2.34 minutes.

4. Use equation 4.17 to solve for the expected arrival rate λ. For example, if it is desired to keep the maximum expected waiting time (which is also the steady-state expected waiting time) below 10 minutes, then equation 4.17 can be rearranged to yield

$$\lambda = 2(W - \bar{\tau})/(\sigma_{\tau}^2 + \bar{\tau}^2 + 2[W - \bar{\tau}]\bar{\tau})$$

$$= 2 \times (15 - 9.36)/(2.34^2 + 9.36^2 + 2 \times [15 - 9.36] \times 9.36)$$

$$= 0.0568 \text{ persons/minute} = 3.41 \text{ persons/hour}$$

(Note: If the standpost has more than 1 faucet, then Figure 4-8 or 4-9 must be used instead of equation 4.17. The value of ρ at S = 15/9.36 intervals must be found from the appropriate curve, and then λ found from $\lambda = \rho/\bar{\tau}$.)

5. Determine the population based on the expected arrival rate found in step 4. If the average number of trips made to the standpost per person is 2, and the standpost is open for 12 hours, then the population is 3.41 x 12/2 = 20 persons.

6. Determine the service rate capacity μ, which is the inverse of the average service time, or 0.107 persons/minute.

7. Determine the traffic intensity ρ, which is the ratio of the expected arrival rate λ to the service rate capacity μ, or 0.0568/0.107 = 0.53.

8. Use Figure 6-1 to see whether assuming that the population is large enough to be considered infinite causes a significant error. The effective population size is N'= 40 (20 vendors x 2 visits per vendor at a single-faucet standpost). Figure 6-1 shows that if an N = 40 curve were interpolated between the N = 20 and the N = 50 curves, the resulting L' at ρ = 0.53 would be very close to the L' value from the N = ∞ curve. Therefore, the assumption of an infinite population causes no significant error, and the population determined in

step 5 is appropriate.

If Figure 6-1 had shown that the assumption of an infinite population causes significant error, then a a higher effective population size N' would have to be assumed and its curve interpolated between the curves in Figure 6-1. The value of ρ at L' = 15/9.36 persons would be selected from the curve and the corresponding expected arrival rate λ determined from ρ . λ would then be used to calculate an effective population size. If this calculated N' equals the assumed N', then the assumed N' is correct. The actual population size N can then be determined from N'. However, if the assumed N' does not equal the calculated N', then a new N' must be assumed and the process repeated.

The number of standposts required in the standpost system can be found by dividing the total population by the population determined in the above procedure.

As discussed in section VII.B, the designer can verify that steady-state is essentially reached by considering Figure 7-1, and can modify and apply the programs in Annexes C.1, C.2, C.3.

E. Fixed Discharge Capacity and Time-varying Expected Arrival Rate

If the standpost discharge capacity is fixed and the expected customer arrival rate varies throughout the day, then queuing theory can be used to determine the population size which can be served while meeting waiting time criterion. The procedure is as follows:

1. Estimate the average volume of water obtained per visit to the standpost, the waste factor w, and the efficiency factor f.

2. Use equation 3.2 to determine the corresponding average service time. For this example, assume the standpost has two faucets and a discharge capacity of 5.0 gpm, or 2.5 gpm per faucet. If the average volume obtained per visit is 8 gallons, the waste factor is 0.1, and the efficiency factor is 0.9, then the average service time

is

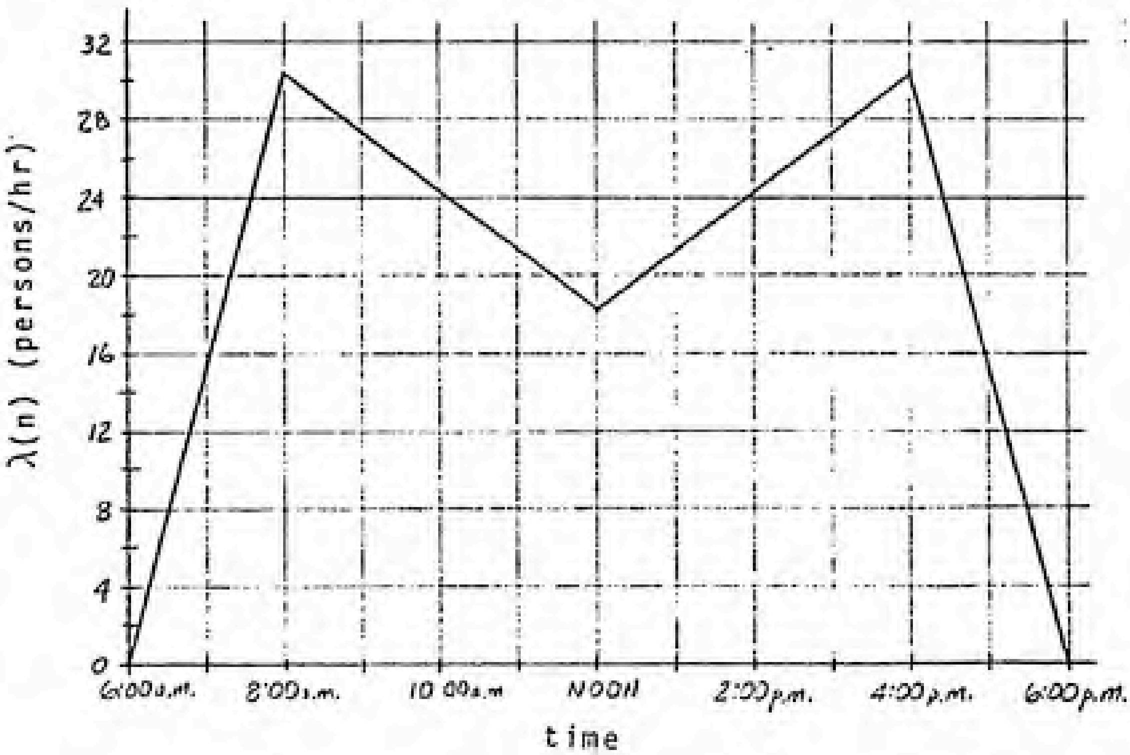
$$\bar{\tau} = 8 \times (1/2.5) \times 1/(1 - 0.1) \times 1/0.9$$

$$= 3.95 \text{ minutes}$$

3. Based on collected field data or literature, determine the geometry of the expected arrival rate pattern throughout the day. If the shape of the expected arrival rate pattern is similar to that shown in Figure 7-2, then note that the peak expected arrival rate is 20/12 = 1.67 times the minimum expected arrival rate. Also note the times at which the peak and minimum values occur and that increases and decreases in the expected arrival rate are linear.

4. The expected arrival rate during peak standpost usage must be determined by trial and error. As a first guess in this example assume that it is equal to the standpost service rate capacity, which is $c\mu = c/\tau$ = 2/3.95 = 0.506 persons/minute = 30.4 persons/hour. Use this as the peak expected arrival rate value and to determine other key expected arrival rate values that define the geometry of the expected arrival rate pattern. For this example, the minimum expected arrival rate is 12/20 times the peak, or 12/20 x 30.4 = 18.2 persons/hour. Thus, the arrival pattern is as shown below.

Expected arrival rate pattern for section VII.E example.



5. Express the variation in the expected arrival rate shown above as a function of the service period, as follows:

$\lambda(n) = n \qquad 0 \le n < 30$ (i.e., from 6:00 a.m. to 8:00 a.m.)

$\lambda(n) = 36.5 - 0.201n \qquad 30 \le n \le 91$ (i.e., from 8:00 a.m. to noon)

$\lambda(n) = 0.201n - 0.1 \qquad 91 \le n \le 152$ (i.e., from noon to 4:00 p.m.)

$\lambda(n) = 182 - n \qquad 152 < n \le 182$ (i.e., from 4:00 to 6:00 p.m.)

6. Modify the computer program in Annex C.6 by substituting in the service time of 0.0658 hours and the above expected arrival rate function. The resulting program is in Annex C.7. If the waiting time criterion is not met, assume a new peak expected arrival rate and rework steps 4 through 6. For this example, it is assumed that the waiting time criterion is met when the peak expected arrival rate is 30.4 arrivals/hour as shown in step 4.

7. Determine the population size from the expected arrival rate pattern shown in step 4. The area under the curve is

$$2(0.5 \times 2\text{hr} \times 30.4\text{person/hr} + 4\text{hr} \times 18.2\text{person/hr} + 0.5 \times 4\text{hr} \times (30.4 - 18.2)\text{person/hr}) = 255 \text{ persons.}$$

If an average of 2 trips is made per person, then the population size is 255/2 = 128 persons.

The number of standposts required in the standpost system can be found by dividing the total population by the population determined in the above procedure.

F. Optimal Design

Ultimately, queuing theory can be used to optimally design a standpost system. One way would be to assign a money value to time spent waiting at the standpost and the distance walked to the standpost. This money value could be added to the price the customers are to pay to cover the cost of the system (if there is a charge), and the objective would be to minimize the sum while providing customers with their required amount of water.

Chapter VIII

CONCLUSIONS AND RECOMMENDATIONS

Queuing theory has important implications for standpost design. The following conclusions apply to time periods during which the expected customer arrival rate at a standpost does not vary:

1. If the expected rate at which customers arrive at a single-faucet standpost is equal to the standpost's service rate capacity (i.e., the rate at which customers can be served) then the expected line length will increase monotonically until the number of service periods which has elapsed is equal to the total number of arrivals that occur at the standpost per day. For example, if users start arriving at the standpost at 8:00 a.m., the time required to serve a customer is 5 minutes and a total of 30 trips are made to the standpost, then the expected line length will increase monotonically until 5 x 30 = 150 minutes have elapsed, or until 10:30 a.m.

   The WHO guidelines in Annex A are based on the assumption that the service rate capacity should equal the arrival rate averaged over the peak hours, and that the number of peak hours per day is typically in the range of 4 to 12. Over such a long time period it is possible for rather lengthy waiting lines to develop.

2. If the expected rate at which customers arrive at a standpost is less than the service rate capacity, the expected line length will increase monotonically until the number of service periods which has elapsed is equal to the number of arrivals that occur at the standpost per day. In this sense, the line length increases as in conclusion #1 above. However, if the expected arrival rate is significantly less than the service rate capacity, then the expected line length will essentially reach its maximum value (but will not exactly equal its maximum value) long before such a number of time periods has elapsed.

   When designing standposts, the discharge capacity should be high enough so that the service rate capacity is significantly more than the expected arrival rate. This will ensure that the line nearly reaches its maximum value rather quickly, and will not grow to an undesireably long length.

3. As the ratio of the expected arrival rate to the service rate capacity increases, the steady-state expected line length increase. For ratios higher than about 0.8, the expected line length increases greatly for small increases in the ratio.

   The standpost service rate capacity should be at least roughly 1/0.8 = 1.25 times greater than the expected arrival rate. This will ensure that waiting lines are kept fairly short, even if there is a slight error in estimates of the system pressure or other factors influencing the line length.

4. As the time required to serve customers varies among customers (because customers have different container sizes, or fill different numbers of containers), the line length increases.

   WHO guidelines in no way take into account the variation in the volume of water customers obtain per trip to the standpost. Queuing theory shows that to meet expected waiting time or line length criteria, the standpost discharge capacity must increase as this variation increases.

5. As the ratio of the expected arrival rate to the service rate capacity decreases, the population size which can be safely assumed to be infinite also decreases. For example, if the ratio is 0.5, then a population as small as 10 can be assumed to be infinite without causing significant error; whereas if the ratio is 0.9 then the population must be a few hundred to avoid significant error.

   The assumption of an infinite population allows for model simplicity and flexibility. However, if the population is too small to be assumed infinite, then the techniques presented in Chapters V and VI can be used to estimate line lengths.

The following conclusions apply to time periods during which the expected arrival rate varies:

6. As discussed in Chapter VI, the assumption that the population served by a standpost is infinite provides reasonably accurate estimates of the maximum expected waiting time.

7. The variation in the volume customers obtain per trip to the standpost does not have a significant effect on the waiting line length. The volume can therefore be

assumed to be constant and equal to the average volume.

8. If a standpost's service rate capacity equals the expected arrival rate during the period of peak usage, then waiting times will not necessarily be of satisfactory shortness. This is especially true if the peak period is long, such as 4 to 12 hours, which is stated to be a typical length in the WHO guidelines. If the expected arrival rate actually varies during a so-called peak period, then setting the service rate capacity equal to this arrival rate will lead to even longer waiting times.

The following conclusion applies regardless of whether the expected arrival rate varies or is constant:

9. For a given standpost discharge capacity, the expected time spent at the standpost increases slightly as the number of faucets increases. For example, a two-faucet standpost with a discharge capacity of 3.0 gallons/minute (1.5 gallons/minute from each faucet) will provide slightly longer waiting times than a single-faucet standpost having a 3.0 gallons/minute discharge capacity.

   When designing a standpost, the number of faucets should initially be assumed to be one unless evidence indicates otherwise. If the resulting required discharge capacity is too great for a single faucet (because, for example, excessive splashing or waste may result) then the stand-should be designed again with two or more faucets.

B. Recommendations for Further Study

Queuing theory is useful for standpost design. Additional research may make its application easier and more successful.

The time required to serve a customer is a key parameter of queuing theory. Data concerning the amount of water wasted when filling a customer's container and the time required to position and remove containers from under the faucet may prove useful in accurately estimating the service time. As discussed in chapter III, the time required to open and close the tap is generally not proportional to the volume of the container to be filled. Use of the efficiency factor f in equations 3.1 and 3.2 is theoretically incorrect; the possibility of resulting service times being erroneous may be worth investigating. An appropriate alternative should be adopted if necessary.

A broad data base of how water collection varies throughout the day for various scenarios (e.g., urban area where the standpost is open only for a few hours in the morning, rural areas where the standpost is always open, etc.) may prove useful. From such a data base common patterns would perhaps emerge. Computer programs similar to those in Annex C could then be applied to each scenario, and relatively simple equations or tables relating discharge capacity, population size, and waiting time could be developed. Designers would then not need to spend much time on data collection and computer modeling.

ANNEX A. Excerpt from Public Standpost Water Supplies - A Design Manual - Technical Series Paper 14, W.H.O. International Reference Center for Community Water Supply, 1979.

3. THE REQUIRED DISCHARGE CAPACITY PER STANDPOST

An important factor in the design of a public standpost water supply system, is the required maximum discharge capacity (Q_{max}) *per standpost.*
This parameter is the basis for the hydraulic calculation and the determination of the dimensions of the service pipe and of the type and number of taps.

This Q_{max} is determined by the design population (N), the average demand per capita (C_d), the peak factor (P), the number of standposts (S), the waste factor (w) and an efficiency factor (f).
The relationship between the factors is presented by the formula:

$$Q_{max} = N \times \frac{1}{S} \times \frac{C_d}{24} \times P \times \frac{1}{1-w} \times \frac{1}{f}$$

Each of the six factors that constitute this formula have to be determined separately before the maximum discharge capacity can be calculated.

3.1. <u>Design Population</u> (N)

This factor depends on the initial magnitude of the population to be served, the growth rate of the population, and the design period.

The *initial number of users* (N_o) in the area to be served by the prospective standposts should be counted or estimated.

An indication of the expected *annual growth rate* (r) of the population can be obtained from historical statistical data on the area concerned. If these are not available, one may substitute figures related to a similar area or to the country as a whole.

The standpost must have sufficient capacity to meet the users' water demands during a period of several years. The *design period* (T) is determined by various technical and economic factors; a period of ten years is usually sufficient, but there may be local reasons that make a shorter or longer period desirable.

The *Design Population* (N) can now be calculated according to:

$$N = N_o * (1+r)^T$$

One may prefer to work with a growth factor: $m = (1+r)^T$. In that case use can be made of a table giving the growth factor as a function of a fixed annual growth rate and the design period. In general the growth factor will be in the range of 1.0-1.6 (see Annex 3).

3.2. Demand per Capita (C_d)

The average volume of water drawn from public standposts is typically between 20 and 60 litres per capita per day* (lcd). Local habits related to domestic water use will have to be studied in order to establish an accurate figure, particularly in cases where there has been no previous experience with standposts or where other sources are available. Normally, a provision is also made for livestock watering.

Future increases in demand should be provided for. It is advisable to provide excess capacity for a design period of at least 10 years and that allows for an increased consumption per person.

In the design of the distribution network, allowance may also be made for uses other than domestic, such as small industries, etc.

3.3. Peak Factor (P)

During some hours the standpost will be used more intensively than during the rest of the day. The water supply system should be capable of dealing with this peak demand. Therefore, a peak factor representing the ratio between peak and average demand, is introduced in the calculation Q_{max}. Normally, this peak factor is in the range of 2 to 4, a typical average being 3.

The peak factor can be approximated by $P = 24/t$, in which t is the number of peak hours (normally in the range of 4 to 12 per day). The result of this method tends, however, to be on the high side, as the method presupposes that no water at all is drawn outside the peak hours.

The peak factor should be determined with great care, as it has a considerable influence on the Q_{max}. It is recommended that the local water collection pattern should always be studied thoroughly.

3.4. The Number of Standposts (S)

The required number of standposts is based on two other design criteria, the maximum walking distance to the standpost and the maximum number of users per tap. These two criteria are directly related to the intended "level of service", which results from the consultations between the users and the planners of the public standpost scheme.

The Maximum Walking Distance

In general it is advisable to limit the walking distance to 200 m and in densely populated areas it is often possible to limit the distance to 100 m. In very sparsely populated areas a walking distance of up to 500 m may sometimes be acceptable.

In densely populated areas, however, the application of this criterion may result in too many users per standpost and per tap. Therefore, a second criterion that relates to the population density in a particular area, is required.

The Maximum Number of Users per Standpost

It is advisable to limit the number of users per standpost to 100-250; in no case should this number exceed 500. The number of users per tap should preferably be in the range of 25-125. This criterion is directly related to the maximum discharge capacity of the taps and to the water collection pattern during peak hours.

Proceeding from more or less evenly spaced standposts in a given area (A) and a set maximum walking distance (R), the number of standposts can be assessed using the formula:

$$S = A/\pi R^2$$

A = total area of the scheme in m^2

S = the number of standposts

R = the service radius in m

π = 3.14

It should be noted that the service radius does not exactly equal the maximum walking distance (see Annex 4).

In case the afore-mentioned approximation of the number of standposts (S) leads to a too high average number of users per standpost, one may increase the number of taps per standpost, in order to meet the criterion set for the maximum number of users per tap. However, the number of taps per standpost should be limited to avoid the crowding of too many people near one standpost: it is advisable not to install more than four taps per standpost.
If by providing more than one tap the criterion of the maximum number of users per tap can still not be met, the number of standposts in the area under consideration is to be increased.

In practice, the actual local geographical and demographical circumstances will determine the siting and the exact number of standposts and taps, as well as the actual number of users for each standpost. However, in most cases, as a first assessment, the above method gives satisfactory results.

3.5. Waste Factor (w)

Part of the water is inevitably spilt by users when filling their containers or drawing water for immediate use at the standpost. To express this numerically, the waste factor (w) is introduced. This factor can be determined by calculating the ratio of the amount of water actually taken away by the users and the total amount of water discharged through the taps.
Spillage and wastage depend on the way in which containers are filled, the type and condition of the taps, the height of the taps above the bucket-stand, the water pressure, and on whether or not the standpost is supervised.

The waste factor can best be estimated from data obtained in other schemes. It should only include the spillage and wastage related to the collection and use of water at the tap and not the leakage of water in the main and branch pipes of the distribution network, as this leakage water is not discharged through the taps.

The waste factor is in the range of 0.1 to 0.4 which means that 10 - 40% of the total discharge at the tap is spilt or wasted. As the waste factor has a considerable influence on the Q_{max} careful assessment of the factor is required.
It is important to decide whether water used for purposes, such as washing of utensils and personal cleaning at the tap should be considered as part of the per capita demand or is to be regarded in terms of spillage or wastage.

3.6. The Efficiency Factor (f)

The efficiency factor is introduced to take into account that the suppliers rating of the capacity of a tap is usually based on continuous discharge at 10 mhw with the tap fully open. In practice the pressure will never be exactly 10 mhw and discharge is usually not continuous. The closing and opening of the tap will make the actual discharge smaller than the theoretical maximum.

Depending on the type of tap, the efficiency factor (f) can range from almost 1.0 for a ball valve (rapid closing tap), to 0.9 and 0.8 for an ordinary screw tap, and 0.7 for spring-loaded taps.

3.7. Calculation Example

The calculation of the required maximum discharge capacity of standposts can best be illustrated by an example; assume the following situation:

N_o	=	initial number of users = 12 000
r	=	population growth rate = 2% per year
T	=	design period = 10 years
A	=	area of scheme = 100 ha = $1.10^6 m^2$
S	=	number of standposts
C_d	=	average demand per capita = 40 lcd
t	=	number of peak hours = 4.5 h
w	=	wastage factor = 0.2
f	=	efficiency factor = 0.9

The design population can be calculated as:

$N = N_o * (1+r)^T = 12000 * 1.02^{10} = 12000 * 1.22 = 14640$ persons.

If the service radius is set as R=200 m and the number of users per standpost is limited to 250, the *number of standposts* can be determined as follows:

a. $S > A/\pi R^2 = 10^6/3.14 * (200)^2 = 7.96$ and

b. $S > N/250 = 14640/250 = 58.56$

Consequently, the maximum number of users per standpost is the decisive factor in this case (b). If the number of standposts is now set at 60, the average number of users per standpost is: 14640/60 = 244 persons, and the service radius about 73 m.

The *required discharge capacity per standpost* can be calculated as:

$$Q_{max} = \frac{N}{S} * \frac{C_d}{24} * P * \frac{1}{1-w} * \frac{1}{f} =$$

$$\frac{14640}{60} * \frac{40}{24} * \frac{24}{4,5} * \frac{1}{(1-0,2)} * \frac{1}{0,9} = 3012 \text{ l/h}$$

If the area had been 1000 ha instead of 100 ha, the result of the equation $S \geq A/\pi R^2$, for $R = 200$, would have been 80 standposts and the criterion of the maximum walking distance would have been decisive (a). The average number of users per standpost would then have been: 14640/80 = 183 and the Q_{max} equal to 2260 l/h per standpost; assuming that population and standposts are evenly distributed.

However, in practice, and particularly in larger areas, the *population will almost never be evenly distributed.* Therefore, the number of standposts and their location will always have to be determined on the basis of the local geographical cicumstances, including the variation in population density in the area concerned. This may lead to a variation in the number of users per standpost and subsequently to a difference in the required discharge capacity of the various standposts.

The above serves to show that, generally speaking, the maximum number of users per standpost will be the criterion in densely populated areas, whilst in sparsely populated areas the maximum walking distance will be the decisive factor.

Finally, it should be pointed out that Q_{max}, as calculated in this Chapter, only relates to water that is actually discharged via standposts. The total amount of water that is pumped into the distribution network is often reported to be 10-50% higher than the total discharge via standposts and house connections. This is due to *leakage in the pipe system.* This factor is *NOT* included in the calculations in this publication, as this publication only deals with the discharge from standposts.

<u>All the afore-mentioned values have been assumed</u>. The designer should substitute his own data depending on the characteristics of the system he is working on.

ANNEX B. Probability Fundamentals

Modeling queuing processes requires use of probability theory. Some useful definitions are the following:

* The probability that an event occurs is the % chance of it happening divided by 100%.

* Independent events are a group of events such that the occurence of any member(s) of the group does not change the probability that any of the other member(s) will occur. For example, suppose a group of 3 events consists of 3 coin tosses, each resulting in "heads". Each event of the group has a probability of 0.5. Regardless of the result of, say, the first coin toss, each of the other 2 events still has a probability of 0.5 because the 3 events are independent. The result of one coin toss in no way changes the probability of "heads" on another coin toss.

* Mutually exclusive events are events such that the occurrence of one makes the occurrence of another impossible. If, for example, each of the numbers 1 through 10 are written on 10 pieces of paper which are placed in a hat, then getting 4 on the first draw and 4 on the second draw are mutually exclusive events. The paper with 4 can be drawn only once.

Useful principles of probability theory are:

* The probability of a group of independent events is the product of the probability of each separate event. Using

the above coin toss example, the probability that all 3 tosses result in "heads" is 0.5 x 0.5 x 0.5 = 0.125. That is, only in about 12 times out of 100 that the experiment of tossing a coin 3 times is run will "heads" be obtained on all 3 tosses.

* The probability of an event that consists of a group of mutually exclusive events is the sum of the probabilities of each event in that group. Using the above numbers-in-the-hat example, the probability that the number on the first draw is a 2 or a 3 is 1/10 + 1/10 = 0.2. That is, for the experiment of drawing a number from a hat, about 20 out of a 100 times the number will be either 2 or 3.

ANNEX C. Computer Programs for Large and Small Population Models

The computer programs contained in this annex are written in Fortran IV.

Lines which begin with the letter "C" are comment lines to aid the reader.

The computer language does not allow for use of the subscript "0" for elements of vectors and matrices. For this reason, "1" is used when "0" is required and all higher subscripts have "1" added to them. For example, to indicate the probability of no one in the waiting line the subscript "1" is used, and to indicate the probability of one person in the waiting line the subscript "2" is used. For vectors and matrices having no element with a "0" subscript this adjustment is not necessary.

Variables in the computer programs with their corresponding symbols in the text are as follows:

Fortran IV	text
EAR	λ
EL	L
ELP	L'
PK	p_K
VO	v_o
VI	v_i
VJ	v_j

The Fortran IV symbols are followed by subscripts written in parenthesis, e.g., PK(I). These subscripts refer to the number in line when used with VI and VJ. For example, VI(4) is the probability that 3 (add "+1" to convert to Fortran IV) persons are in the queuing system at the beginning of interval n. When used with PK, the subscripts refer to the number of arrivals that occur during interval n. For example, PK(11) is the probability that 10 arrivals occur during interval n. The subscripts with EL, ELP, S, and W refer to the interval number. For example, EL(50) is the expected line length at the beginning of interval 50.

The variables A, B, C, D, and F have no direct significance in queuing theory, but are used as aids in the calculation process. The variable FACT(I) represents a vector of factorials used in calculating probabilities.

Equations 4.3, 4.5, and 4.6 are theoretically to be applied through $j = \infty$. For the purpose of calculation, an upper limit of j = 70 is assumed. If the line length is greater than 70, anyone arriving will decide not to join the line because of its great length. A new variable, P70(K), is introduced to be consistent with the limit of 70 being placed on j. P70(K) is the probability that the number of arrivals occurring during a service interval is equal to or greater than the number that would cause 70 persons to be in line at the end of the interval. $v_j(n)$ and $v_j'(n)$ for j = 70 are negligible and the resulting L(n) and L'(n) values are accurate unless the waiting line is extremely long.

Some variables are unique to the small population model. SASP3, SASP4, SASP5, and SASP6 are the probabilities that 0 departures occur in the third, fourth, fifth, and sixth intervals. For each of these intervals, there is more than one way that 0 departures will occur. For example, the probvability that 0 departures will occur in the third interval is the probability that 0 arrivals occur in the first interval and 0 arrivals occur in the second, plus the probability that 1 arrival occurs in the first and 0 in the second. In the computer program, these latter probabilities are given the symbol ASP_ with a subscript to distinguish them from each other. For example, SASP3 the sum of the ASP3's. The subscripts associated with each of the ASP's represent the number of arrivals (plus "+1" to convert to Fortran IV) in the order in which they occur. For example, ASP5(3,1,2) is the probability that 2 arrivals occur in the first interval, 0 in the second, and 1 in the third. Notice that the number of subscripts required will always be 2 less than the interval number of concern. In the case of ASP5, the number of subscripts required is 3. This is because the number of arrivals in the fourth interval is always 0 and the number of arrivals in the fifth interval is irrelevant.

```
C     ANNEX C.1.     Large-population model for determining vo(n)
C                    and L'(n) when lambda and tau are constant
C                    and the standpost has a single tap.  See sec-
C                    tion IV.A for details of the scenario being
C                    modeled.
C
C     The sizes of vectors are established.  Variables are
C     "DOUBLE PRECISION" so that high factorials can be
C     calculated.  The file "ANNC1.TXT" is opened for storage
C     of results.
C
      DIMENSION PK(70),P70(71),VI(71),VJ(71),VO(241),EL(241),
     &ELP(241),FACT(70)
      DOUBLE PRECISION A, B, C, D, F, FACT, VO, EL, ELP,
     &PK, P70, VI, VJ
      OPEN(1,FILE='ANNC1.TXT')
C
C     Factorials are calculated.  These factorials will be used
C     for calculating elements of the PK vector.
C
      F=1.0
      DO 001 I=1,70
      F=REAL(I)*F
  001 FACT(I)=F
C
C     Column headings are written in file ANNC1.TXT.
C
      WRITE(1,205)
  205 FORMAT(2X,'n',2X,'v (n)',3X,'L''(n)',9X,'n',2X,'v (n)',
     &3X,'L''(n)')
C
C     The PDF of the number in line at the beginning of the
C     first period is established.  No one is at the standpost
C     at this time.  Tau and lambda are assigned their values.
C
      VI(1)=1.0
      DO 002 I=2,71
  002 VI(I)=0.0
      TAU=0.05
      EAR=20
C
C     The PK and P70 vectors are calculated.
C
  004 PK(1)=1.0/EXP(EAR*TAU)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(EAR*TAU)**REAL(I-1)/(FACT(I-1)*EXP(EAR*TAU))
      A=PK(I)+A
      K=I+1
  005 P70(K)=1.0-A
C
C     n is set equal to 1 and increases by 1 for each service
C     period.  The following 5-step process is applied to each
C     value of n to determine vo(n) and L'(n).
```

```
C
      N=1
  003 N=N+1
C
C    Step 1: VJ(1) is calculated.
C
      VJ(1)=VI(1)*PK(1)+VI(2)*PK(1)
C
C    Step 2: VJ(2) through VJ(70) are calculated.
C
      DO 011 J=2,70
      B=VI(1)*PK(J)
      DO 010 I=2,J+1
      K=J-I+2
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C    Step 3: VJ(71) is calculated.
C
      C=VI(1)*P70(71)
      DO 015 I=2,71
      K=71-I+2
  015 C=VI(I)*P70(K)+C
      VJ(71)=C
C
C    Step 4: vo(n), L(n) and L'(N) are determined for the
C    beginning of period n.
C
      VO(N)=VJ(1)
      EL(N)=0.0
      DO 020 J=2,71
  020 EL(N)=EL(N)+VJ(J)*REAL(J-1)
      ELP(N)=EL(N)/(1.0-VJ(1))
C
C    Step 5:  VJ's become VI's so that the 5-step process may
C    be repeated.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
      IF(N.LT.241)GOTO 003
C
C    Results are written in file "ANNC1.TXT".
C
      VO(1)=1.0
      ELP(1)=0.0
      WRITE(1,210)(I,VO(I),ELP(I),I=1,241)
  210 FORMAT(I3,2X,F5.3,3X,F6.3,7X,I3,2X,F5.3,3X,F6.3)
      END
```

```
C     ANNEX C.2.   This is a large-population model with constant
C                  rho values, two taps, and service time constant
C                  among users.  The expected number of service
C                  intervals customers spend at the standpost is
C                  calculated as a function of the interval number.
C
C     The sizes of vectors are established.  Variables are
C     "DOUBLE PRECISION" so that high factorials can be
C     calculated.  The file "ANNC2.TXT" is opened for storage
C     of results.
C
      DIMENSION PK(70),P70(71),VI(71),VJ(73),S(400),FACT(70)
      DOUBLE PRECISION A, B, C, D, F, FACT, S, PK, P70,
     &VI, VJ, RHO
      OPEN(1,FILE='ANNC2.TXT')
C
C     Factorials are calculated.  These factorials will be used
C     for calculating elements of the PK vector.
C
      F=1.0
      DO 001 I=1,70
      F=REAL(I)*F
  001 FACT(I)=F
C
C     The program is run for rho values starting at 0.20.  The
C     rho value increases by increments of 0.1 until reaching 1.0
C     (See the second to last line of the program.)
C
      RHO=0.1
  100 RHO=RHO+0.1
C
C     The rho value and column headings are written in "ANNC2.TXT".
C
      WRITE(1,201)RHO
  201 FORMAT(27X,'RHO=',F3.1)
      WRITE(1,205)
  205 FORMAT(2X,'n',2X,'S(n)',7X,'n',2X,'S(n)',7X,'n',2
     &X,'S(n)',7X,'n',2X,'S(n)',7X,'n',2X,'S(n)')
C
C     The probability distribution of the number at the standpost
C     at the beginning of the first interval is established.  No
C     one is at the standpost at this time.
C
      VI(1)=1.0
      DO 002 I=2,71
  002 VI(I)=0.0
C
C     The PK and P70 vectors are calculated.
C
  004 PK(1)=1.0/EXP(2.0*RHO)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(2.0*RHO)**REAL(I-1)/(FACT(I-1)*EXP(2.0*RHO))
      A=PK(I)+A
```

```
      K=I+1
  005 P70(K)=1.0-A
C
C     The expected number of service intervals customers spend at
C     the standpost is calculated at the beginning of each of 400
C     service intervals.  This process involves 5 steps, as follows:
C
      DO 040 N=1,400
C
C     Step 1: VJ(1) and VJ(2) are calculated.
C
      VJ(1)=(VI(1)+VI(2)+VI(3))*PK(1)
      VJ(2)=(VI(1)+VI(2)+VI(3))*PK(2)+VI(4)*PK(1)
C
C     Step 2: VJ(3) through VJ(69) are calculated.
C
      DO 011 J=3,69
      B=(VI(1)+VI(2)+VI(3))*PK(J)
      DO 010 I=4,J+2
      K=J-I+3
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C     Step 3: VJ(70) and VJ(71) are calculated.
C
      VJ(70)=(VI(1)+VI(2)+VI(3))*PK(70)
      DO 013 I=4,71
      K=73-I
  013 VJ(70)=VI(I)*PK(K)+VJ(70)
      VJ(71)=(VI(1)+VI(2))*P70(71)
      DO 015 I=3,71
      K=74-I
  015 VJ(71)=VI(I)*P70(K)+VJ(71)
C
C     Step 4: S(N), the expected number of service period spent
C     at the standpost at the beginning of the nth interval is
C     calculated.
C
      S(N)=(VJ(2)+VJ(3)+VJ(4))*1.0
      DO 020 I=2,35
      J=2*I
      DO 020 K=J,J+1
  020 S(N)=S(N)+VJ(K)*REAL(J)/2.0
      S(N)=S(N)/(1.0-VJ(1))
C
C     Step 5:  VJ's at the end of the nth interval become VI's
C     for the beginning of interval n+1 so that the 5-step
C     process may be repeated for interval n+1.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
  040 CONTINUE
C
C     Results are written in file "ANNC2.TXT".
```

```
C
      WRITE(1,210)(I,S(I),I=1,10)
      WRITE(1,210)(10*I,S(10*I),I=1,40)
  210 FORMAT(I3,1X,F6.3,4X,I3,1X,F6.3,4X,I3,1X,F6.3,4X,I3,1X,
     &F6.3,4X,I3,1X,F6.3)
C
C     If rho is less than 1.0, then rho is increased by 0.1 and
C     the entire process is repeated.
C
      IF(RHO.LT.1.0)GOTO 100
      END
```

```
C     ANNEX C.3.    This is a large-population model with constant
C                   rho values, three taps, and the service time
C                   constant among users.  The expected number of
C                   service intervals customers spend at the stand-
C                   post is calculated at the beginning of each
C                   interval.
C
C     The sizes of vectors are established.  Variables are
C     "DOUBLE PRECISION" so that high factorials can be
C     calculated.  The file "ANNC3.TXT" is opened for storage
C     of results.
C
      DIMENSION PK(70),P70(71),VI(71),VJ(71),S(300),FACT(70)
      DOUBLE PRECISION A, B, C, D, F, FACT, S, PK, P70,
     &VI, VJ, RHO
      OPEN(1,FILE='ANNC3.TXT')
C
C     Factorials are calculated.  These factorials will be used
C     for calculating elements of the PK vector.
C
      F=1.0
      DO 001 I=1,70
      F=REAL(I)*F
  001 FACT(I)=F
C
C     The program is run for rho values starting at 0.20.  The
C     rho value increases by increments of 0.1 until reaching
C     1.0.  (See the second to last line of the program.)
C
      RHO=0.1
  100 RHO=RHO+0.1
C
C     The rho value and column headings are stored in "ANNC3.TXT".
C
      WRITE(1,201)RHO
  201 FORMAT(27X,'RHO=',F3.1)
      WRITE(1,205)
  205 FORMAT(2X,'n',2X,'L''(n)',6X,'n',2X,'L''(n)',6X,'n',2
     &X,'L''(n)',6X,'n',2X,'L''(n)',6X,'n',2X,'L''(n)')
C
C     The probability distribution of the number at the standpost
C     at the beginning of the first interval is established.  No
C     one is at the standpost at this time.
C
      VI(1)=1.0
      DO 002 I=2,71
  002 VI(I)=0.0
C
C     The PK and P70 vectors are calculated.
C
  004 PK(1)=1.0/EXP(3.0*RHO)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(3.0*RHO)**REAL(I-1)/(FACT(I-1)*EXP(3.0*RHO))
```

```
      A=PK(I)+A
      K=I+1
  005 P70(K)=1.0-A
C
C    The expected number of service intervals customer spend at
C    the standpost is calculated at the beginning of each of 300
C    intervals.  This process involves 5 steps, as follows:
C
      DO 040 N=1,300
C
C    Step 1: VJ(1), VJ(2), and VJ(3) are calculated.
C
      VJ(1)=(VI(1)+VI(2)+VI(3)+VI(4))*PK(1)
      VJ(2)=(VI(1)+VI(2)+VI(3)+VI(4))*PK(2)+VI(5)*PK(1)
      VJ(3)=(VI(1)+VI(2)+VI(3)+VI(4))*PK(3)+VI(5)*PK(2)+VI(6)*PK(1)
C
C    Step 2: VJ(4) through VJ(68) are calculated.
C
      DO 011 J=4,68
      B=(VI(1)+VI(2)+VI(3))*PK(J)
      DO 010 I=4,J+3
      K=J-I+4
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C    Step 3: VJ(69), VJ(70), AND VJ(71) are calculated.
C
      VJ(69)=(VI(1)+VI(2)+VI(3)+VI(4))*PK(69)
      DO 012 I=5,71
      K=73-I
  012 VJ(69)=VI(I)*PK(K)+VJ(69)
      VJ(70)=(VI(1)+VI(2)+VI(3)+VI(4))*PK(70)
      DO 013 I=5,71
      K=74-I
  013 VJ(70)=VI(I)*PK(K)+VJ(70)
      C=(VI(1)+VI(2)+VI(3))*P70(71)
      DO 015 I=4,71
      K=75-I
  015 C=VI(I)*P70(K)+C
      VJ(71)=C
C
C    Step 4: S(N), the expected number of service periods spent
C    at the standpost at the beginning of the nth interval is
C    calculated.
C
      S(N)=(VJ(2)+VJ(3)+VJ(4))*1.0
      DO 020 I=2,24
      J=3*I-1
      DO 020 K=J,J+2
  020 S(N)=S(N)+VJ(K)*REAL(J+1)/3.0
      S(N)=S(N)/(1.0-VJ(1))
C
C    Step 5:  VJ's at the end of the nth interval become VI's
C    for the beginning of interval n+1 so that the 5-step
```

```
C    process may be repeated for interval n+1.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
  040 CONTINUE
C
C    Results are written in file "ANNC3.TXT".
C
      WRITE(1,210)(I,S(I),I=1,10)
      WRITE(1,210)(10*I,S(10*I),I=1,30)
  210 FORMAT(I3,1X,F6.3,4X,I3,1X,F6.3,4X,I3,1X,F6.3,4X,I3,1X,
     &F6.3,4X,I3,1X,F6.3)
C
C    If rho is less than 1.0, then rho is increased by 0.1 and
C    the entire process is repeated.
C
      IF(RHO.LT.1.0)GOTO 100
      END
```

```
C     ANNEX C.4.      This is a large-population model for deter-
C                     mining L'(n) when lambda varies with time,
C                     the standpost has a single tap, and the
C                     service time is constant among users.  See
C                     section IV.B for details of the scenario
C                     being modeled.
C
C     The sizes of vectors are established.  Variables are
C     "DOUBLE PRECISION" so that high factorials can be
C     calculated.  The file "ANNC4.TXT" is opened for storage
C     of results.
C
       DIMENSION PK(70),P70(71),VI(71),VJ(71),EL(241),ELP(241),
      &FACT(70)
       DOUBLE PRECISION A, B, C, D, F, FACT, EL, ELP,
      &PK, P70, VI, VJ
       OPEN(1,FILE='ANNC4.TXT')
C
C     Factorials are calculated.  These factorials will be used
C     for calculating elements of the PK vector.
C
       F=1.0
       DO 001 I=1,70
       F=REAL(I)*F
   001 FACT(I)=F
C
C     Column headings are written.
C
       WRITE(1,205)
   205 FORMAT(2X,'n',2X,'L''(n)',5X,'n',2X,'L''(n)',
      &5X,'n',2X,'L''(n)',5X,'n',2X,'L''(n)',5X,'n',2X,'L''(n)')
C
C     The PDF of the number at the standpost at the beginning of the
C     first period is established.  No one is at the standpost
C     at this time.  Tau is assigned its value.
C
       VI(1)=1.0
       DO 002 I=2,71
   002 VI(I)=0.0
       TAU=0.05
C
C     Lambda is determined for period n.
C
       N=1
   003 IF(N.GT.40)GOTO 401
       EAR=0.85*REAL(N)
       GOTO 004
   401 IF(N.GT.120)GOTO 402
       EAR=46.5-0.3125*REAL(N)
       GOTO 004
   402 IF(N.GT.200)GOTO 403
       EAR=0.3125*REAL(N)-28.5
       GOTO 004
   403 EAR=204.0-0.85*REAL(N)
C
```

```
C     The PK and P70 vectors are calculated.
C
  004 PK(1)=1.0/EXP(EAR*TAU)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(EAR*TAU)**REAL(I-1)/(FACT(I-1)*EXP(EAR*TAU))
      A=PK(I)+A
      K=I+1
  005 P70(K)=1.0-A
C
C     The expected line length given that the line is not empty is
C     calculated at the beginning of each interval.  This process
C     involves 5 steps, as follows:
C
      N=N+1
C
C     Step 1: VJ(1) is calculated.
C
      VJ(1)=VI(1)*PK(1)+VI(2)*PK(1)
C
C     Step 2: VJ(2) through VJ(70) are calculated.
C
      DO 011 J=2,70
      B=VI(1)*PK(J)
      DO 010 I=2,J+1
      K=J-I+2
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C     Step 3: VJ(71) is calculated.
C
      C=VI(1)*P70(71)
      DO 015 I=2,71
      K=71-I+2
  015 C=VI(I)*P70(K)+C
      VJ(71)=C
C
C     Step 4: L(n) and L'(n) are calculated for the beginning of
C     period n.
C
      EL(N)=0.0
      DO 020 J=2,71
  020 EL(N)=EL(N)+VJ(J)*REAL(J-1)
      ELP(N)=EL(N)/(1.0-VJ(1))
C
C     Step 5:  VJ's become VI's so that the 5-step process may
C     be repeated.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
      IF(N.LT.241)GOTO 003
C
C     Results are written in file "ANNC4.TXT".
```

```
C
      ELP(1)=0.0
      WRITE(1,210)(I,ELP(I),I=1,241)
  210 FORMAT(I3,F7.2,I6,F7.2,I6,F7.2,I6,F7.2,I6,F7.2)
      END
```

```
C        ANNEX C.5.  Large-population model for determining W(n)
C                    when lambda varies with time, the standpost
C                    has a single tap, and the service time is
C                    constant among users.  See section VII.C for
C                    details of the scenario being modeled.
C
C    The sizes of vectors are established.  Variables are
C    "DOUBLE PRECISION" so that high factorials can be
C    calculated.  The file "ANNC5.TXT" is opened for storage
C    of results.
C
      DIMENSION PK(70),P70(71),VI(71),VJ(71),EL(241),ELP(241),
     &W(241),FACT(70)
      DOUBLE PRECISION A, B, C, D, F, FACT, EL, ELP, W,
     &PK, P70, VI, VJ
      OPEN(1,FILE='ANNC5.TXT')
C
C    Factorials are calculated.  These factorials will be used
C    for calculating elements of the PK vector.
C
      F=1.0
      DO 001 I=1,70
      F=REAL(I)*F
  001 FACT(I)=F
C
C    Column headings are written.
C
      WRITE(1,205)
  205 FORMAT(2X,'n',3X,'W(n)',5X,'n',3X,'W(n)',5X,'n',3X,'W(n)',
     &5X,'n',3X,'W(n)',5X,'n',3X,'W(n)')
C
C    The PDF of the number at the standpost at the beginning of the
C    first period is established.  No one is at the standpost
C    at this time.  Tau is assigned its value.
C
      VI(1)=1.0
      DO 002 I=2,71
  002 VI(I)=0.0
      TAU=0.05
C
C    Lambda is determined for period n.
C
      N=1
  003 IF(N.GT.40)GOTO 401
      EAR=0.5*REAL(N)
      GOTO 004
  401 IF(N.GT.120)GOTO 402
      EAR=24.0-0.1*REAL(N)
      GOTO 004
  402 IF(N.GT.200)GOTO 403
      EAR=0.1*REAL(N)
      GOTO 004
  403 EAR=120.0-0.5*REAL(N)
C
C    The PK and P70 vectors are calculated.
```

```
C
  004 PK(1)=1.0/EXP(EAR*TAU)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(EAR*TAU)**REAL(I-1)/(FACT(I-1)*EXP(EAR*TAU))
      A=PK(I)+A
      K=I+1
  005 P70(K)=1.0-A
C
C    The expected waiting time is calculated at the beginning of
C    each interval.  This process involves 5 steps, as follows:
C
      N=N+1
C
C    Step 1: VJ(1) is calculated.
C
      VJ(1)=VI(1)*PK(1)+VI(2)*PK(1)
C
C    Step 2: VJ(2) through VJ(70) are calculated.
C
      DO 011 J=2,70
      B=VI(1)*PK(J)
      DO 010 I=2,J+1
      K=J-I+2
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C    Step 3: VJ(71) is calculated.
C
      C=VI(1)*P70(71)
      DO 015 I=2,71
      K=71-I+2
  015 C=VI(I)*P70(K)+C
      VJ(71)=C
C
C    Step 4: L(n), L'(n), and W(n) are calculated for the beginning
C    of period n.  A factor of 60 is used to express W(n) in
C    minutes.
C
      EL(N)=0.0
      DO 020 J=2,71
  020 EL(N)=EL(N)+VJ(J)*REAL(J-1)
      ELP(N)=EL(N)/(1.0-VJ(1))
      W(N)=60.0*TAU*ELP(N)
C
C    Step 5:  VJ's become VI's so that the 5-step process may
C    be repeated for the next interval.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
      IF(N.LT.241)GOTO 003
C
C    Results are written in file "ANNC5.TXT".
```

```
C
      W(1)=0.0
      WRITE(1,210)(I,W(I),I=1,241)
  210 FORMAT(I3,F7.2,I6,F7.2,I6,F7.2,I6,F7.2,I6,F7.2)
      END
```

```
C        ANNEX C.6.   Large-population model for determining S(n)
C                     when lambda varies with time, the standpost
C                     has two taps, and the service time is con-
C                     stant among users.  See section IV.C for
C                     details of the scenario being modeled.
C
C     The sizes of vectors are established.  Variables are
C     "DOUBLE PRECISION" so that high factorials can be
C     calculated.  The file "ANNC6.TXT" is opened for storage
C     of results.
C
      DIMENSION PK(70),P70(71),VI(71),VJ(71),EL(185),ELP(185),
     &FACT(70),S(241)
      DOUBLE PRECISION A, B, C, D, F, FACT, EL, ELP, S,
     &PK, P70, VI, VJ
      OPEN(1,FILE='ANNC6.TXT')
C
C     Factorials are calculated.  These factorials will be used
C     for calculating elements of the PK vector.
C
      F=1.0
      DO 001 I=1,70
      F=REAL(I)*F
  001 FACT(I)=F
C
C     Column headings are written.
C
      WRITE(1,205)
  205 FORMAT(2X,'n',3X,'W(n)',5X,'n',3X,'W(n)',5X,'n',3X,'W(n)',
     &5X,'n',3X,'W(n)',5X,'n',3X,'W(n)')
C
C     The PDF of the number in line at the beginning of the
C     first period is established.  No one is at the standpost
C     at this time.  Tau is assigned its value.
C
      VI(1)=1.0
      DO 002 I=2,71
  002 VI(I)=0.0
      TAU=0.06478
C
C     Lambda is determined for period n.
C
      N=1
  003 IF(N.GT.31)GOTO 401
      EAR=1.1014*REAL(N)
      GOTO 004
  401 IF(N.GT.93)GOTO 402
      EAR=46.5-0.4049*REAL(N)
      GOTO 004
  402 IF(N.GT.154)GOTO 403
      EAR=0.4049*REAL(N)-28.5
      GOTO 004
  403 EAR=204.0-1.1014*REAL(N)
C
C     The PK and P70 vectors are calculated for period n.
```

```
C
  004 PK(1)=1.0/EXP(EAR*TAU)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(EAR*TAU)**REAL(I-1)/(FACT(I-1)*EXP(EAR*TAU))
      A=PK(I)+A
      K=I+1
  005 P70(K)=1.0-A
C
C    The expected number of service intervals customers spend at
C    the standpost is calculated at the beginning of each interval.
C    This process involves 5 steps, as follows:
C
      N=N+1
C
C    Step 1: VJ(1) and VJ(2) are calculated.
C
      VJ(1)=(VI(1)+VI(2)+VI(3))*PK(1)
      VJ(2)=(VI(1)+VI(2)+VI(3))*PK(2)+VI(4)*PK(1)
C
C    Step 2: VJ(3) through VJ(69) are calculated.
C
      DO 011 J=3,69
      B=(VI(1)+VI(2)+VI(3))*PK(J)
      DO 010 I=4,J+2
      K=J-I+3
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C    Step 3: VJ(70) and VJ(71) are calculated.
C
      VJ(70)=(VI(1)+VI(2)+VI(3))*PK(70)
      DO 013 I=4,71
      K=73-I
  013 VJ(70)=VI(I)*PK(K)+VJ(70)
      VJ(71)=(VI(1)+VI(2))*P70(71)
      DO 015 I=3,71
      K=74-I
  015 VJ(71)=VI(I)*P70(K)+VJ(71)
C
C    Step 4A: L(n) and L'(n) are calculated for the beginning of
C    period n.
C
      EL(N)=0.0
      DO 020 J=2,71
  020 EL(N)=EL(N)+VJ(J)*REAL(J-1)
      ELP(N)=EL(N)/(1.0-VJ(1))
C
C    Step 4B: S(n) is calculated for the beginning of period n.
C
      A=(VJ(2)+VJ(3))*1.0
      DO 021 I=2,35
      J=2*I
```

```
      DO 021 K=J,J+1
  021 A=A+VJ(K)*REAL(J)/2.0
      S(N)=A/(1.0-VJ(1))
C
C    Step 5:  VJ's become VI's so that the 5-step process may
C    be repeated for the next service interval.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
      IF(N.LT.185)GOTO 003
C
C    Results are written in file "ANNC6.TXT".
C
      ELP(1)=0.0
      S(1)=0.0
      WRITE(1,210)(I,ELP(I),S(I),I=1,185)
  210 FORMAT(I3,F7.3,I6,F7.3,I6,F7.3,I6,F7.3,I6,F7.3)
      END
```

```
C     ANNEX  C.7.     Large-population model for determining W(n)
C                     when lambda varies with time, the standpost
C                     has two taps, and the service time is con-
C                     stant among users.  See section VII.E for
C                     details of the scenario being modeled.
C
C     The sizes of vectors are established.  Variables are
C     "DOUBLE PRECISION" so that high factorials can be
C     calculated.  The file "ANNC7.TXT" is opened for storage
C     of results.
C
      DIMENSION PK(70),P70(71),VI(71),VJ(71),EL(182),ELP(182),
     &FACT(70),W(182)
      DOUBLE PRECISION A, B, C, D, F, FACT, EL, ELP, W,
     &PK, P70, VI, VJ
      OPEN(1,FILE='ANNC7.TXT')
C
C     Factorials are calculated.  These factorials will be used
C     for calculating elements of the PK vector.
C
      F=1.0
      DO 001 I=1,70
      F=REAL(I)*F
  001 FACT(I)=F
C
C     Column headings are written.
C
      WRITE(1,205)
  205 FORMAT(2X,'n',3X,'W(n)',5X,'n',3X,'W(n)',5X,'n',3X,'W(n)',
     &5X,'n',3X,'W(n)',5X,'n',3X,'W(n)')
C
C     The PDF of the number in line at the beginning of the
C     first period is established.  No one is at the standpost
C     at this time.  Tau is assigned its value.
C
      VI(1)=1.0
      DO 002 I=2,71
  002 VI(I)=0.0
      TAU=0.0658
C
C     Lambda is determined for period n.
C
      N=1
  003 IF(N.GT.29)GOTO 401
      EAR=1.0*REAL(N)
      GOTO 004
  401 IF(N.GT.91)GOTO 402
      EAR=36.5-0.201*REAL(N)
      GOTO 004
  402 IF(N.GT.152)GOTO 403
      EAR=0.201*REAL(N)-0.1
      GOTO 004
  403 EAR=182.3-1.0*REAL(N)
C
C     The PK and P70 vectors are calculated for period n.
```

```
C
  004 PK(1)=1.0/EXP(EAR*TAU)
      A=PK(1)
      P70(2)=1.0-A
      DO 005 I=2,70
      PK(I)=(EAR*TAU)**REAL(I-1)/(FACT(I-1)*EXP(EAR*TAU))
      A=PK(I)+A
      K=I+1
  005 P70(K)=1.0-A
C
C    The expected waiting time is calculated at the beginning of
C    each interval.  This process involves 5 steps, as follows:
C
      N=N+1
C
C    Step 1: VJ(1) and VJ(2) are calculated.
C
      VJ(1)=(VI(1)+VI(2)+VI(3))*PK(1)
      VJ(2)=(VI(1)+VI(2)+VI(3))*PK(2)+VI(4)*PK(1)
C
C    Step 2: VJ(3) through VJ(69) are calculated.
C
      DO 011 J=3,69
      B=(VI(1)+VI(2)+VI(3))*PK(J)
      DO 010 I=4,J+2
      K=J-I+3
  010 B=VI(I)*PK(K)+B
      VJ(J)=B
  011 CONTINUE
C
C    Step 3: VJ(70) and VJ(71) are calculated.
C
      VJ(70)=(VI(1)+VI(2)+VI(3))*PK(70)
      DO 013 I=4,71
      K=73-I
  013 VJ(70)=VI(I)*PK(K)+VJ(70)
      VJ(71)=(VI(1)+VI(2))*P70(71)
      DO 015 I=3,71
      K=74-I
  015 VJ(71)=VI(I)*P70(K)+VJ(71)
C
C    Step 4A: L(n) and L'(n) are calculated for the beginning of
C    period n.
C
      EL(N)=0.0
      DO 020 J=2,71
  020 EL(N)=EL(N)+VJ(J)*REAL(J-1)
      ELP(N)=EL(N)/(1.0-VJ(1))
C
C    Step 4B: W(n) is calculated for the beginning of period n.
C
      A=(VJ(2)+VJ(3))*1.0
      DO 021 I=2,35
      J=2*I
      DO 021 K=J,J+1
```

```
  021 A=A+VJ(K)*REAL(J)/2.0
      W(N)=60.0*TAU*A/(1.0-VJ(1))
C
C    Step 5:  VJ's become VI's so that the 5-step process may
C    be repeated for the next interval.
C
      DO 030 I=1,71
  030 VI(I)=VJ(I)
      IF(N.LT.182)GOTO 003
C
C    Results are written in file "ANNC7.TXT".
C
      W(1)=0.0
      WRITE(1,210)(I,W(I),I=1,182)
  210 FORMAT(I3,F7.2,I6,F7.2,I6,F7.2,I6,F7.2,I6,F7.2)
      END
```

```
C     ANNEX C.8.    This is a small-population model with constant
C                   rho values, a single tap, and service time con-
C                   stant among users.  The probabilities that no
C                   customers are served during intervals 2, 3, 4,
C                   5, and 6 are calculated for values of rho and
C                   population size.  Results are in Annex C.9.
C
C     The sizes of matrices are established.  Variables are "DOUBLE
C     PRECISION" so that high factorials can be calculated.  The file
C     "ANNC9.TXT" is opened for storage of results.
C
       DIMENSION ASP3(2),ASP4(3,2),ASP5(4,3,2),ASP6(5,4,3,2),
     &FACT(51)
       DOUBLE PRECISION ASP2,ASP3,ASP4,ASP5,ASP6,FACT,
     &F,SASP2,SASP3,SASP4,SASP5,SASP6
       OPEN(1,FILE='ANNC9.TXT')
C
C     Factorials are calculated.  These factorials will be used in
C     calculating ASP's.
C
       F=1.0
       DO 001 I=1,51
       F=REAL(I)*F
   001 FACT(I)=F
C
C     The program is run for rho values starting at 0.10 and increas-
C     ing in increments of 0.10.  The population size starts at 10
C     and increases in increments of 5.
C
       RHO=0.0
   300 RHO=RHO+0.10
       N=5
   400 N=N+5
       A=RHO/REAL(N)
C
C     The probability that no customers are served during the
C     second interval is calculated.
C
       ASP2=FACT(N)*REAL(N+1)/(FACT(N+1))*((1.0-A)**REAL(N))
       SASP2=ASP2
C
C     The probability that no customers are served during the
C     third interval interval is calculated.
C
       SASP3=0.0
       DO 030 I=1,2
       ASP3(I)=FACT(N)*REAL(I)*REAL(N+2-I)/(FACT(I)*FACT(N+2-I))*
     &(A**REAL(I-1))*((1.0-2.0*A)**REAL(N+1-I))
   030 SASP3=ASP3(I)+SASP3
C
C     The probability that no customers are served during the
C     fourth interval is calculated.
C
       SASP4=0.0
C
```

```
      DO 040 J=1,2
      I=1
      ASP4(I,J)=FACT(N)*REAL(I)*REAL(J)*REAL(N+3-I-J)/
     &(FACT(I)*FACT(J)*FACT(N+3-I-J))*
     &(A**REAL(I+J-2))*((1.0-3.0*A)**REAL(N+2-I-J))
  040 SASP4=ASP4(1,J)+SASP4
      DO 041 I=2,3
      DO 041 J=1,4-I
      ASP4(I,J)=FACT(N)*REAL(I)*REAL(J)*REAL(N+3-I-J)/
     &(FACT(I)*FACT(J)*FACT(N+3-I-J))*
     &(A**REAL(I+J-2))*((1.0-3.0*A)**REAL(N+2-I-J))
  041 SASP4=ASP4(I,J)+SASP4
C
C    The probability that no customers are served during the
C    fifth interval is calculated.
C
      SASP5=0.0
C
      DO 050 K=1,2
      I=1
      J=1
      ASP5(I,J,K)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(N+4-I-J-K)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(N+4-I-J-K))*
     &(A**REAL(I+J+K-3))*((1.0-4.0*A)**REAL(N+3-I-J-K))
  050 SASP5=ASP5(1,1,K)+SASP5
C
      DO 051 J=2,3
      DO 051 K=1,4-J
      I=1
      ASP5(I,J,K)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(N+4-I-J-K)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(N+4-I-J-K))*
     &(A**REAL(I+J+K-3))*((1.0-4.0*A)**REAL(N+3-I-J-K))
  051 SASP5=ASP5(1,J,K)+SASP5
C
      DO 052 K=1,2
      I=2
      J=1
      ASP5(I,J,K)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(N+4-I-J-K)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(N+4-I-J-K))*
     &(A**REAL(I+J+K-3))*((1.0-4.0*A)**REAL(N+3-I-J-K))
  052 SASP5=ASP5(2,1,K)+SASP5
C
      DO 053 J=2,3
      DO 053 K=1,4-J
      I=2
      ASP5(I,J,K)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(N+4-I-J-K)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(N+4-I-J-K))*
     &(A**REAL(I+J+K-3))*((1.0-4.0*A)**REAL(N+3-I-J-K))
  053 SASP5=ASP5(2,J,K)+SASP5
C
      DO 054 I=3,4
      DO 054 J=1,5-I
      DO 054 K=1,6-I-J
      ASP5(I,J,K)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(N+4-I-J-K)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(N+4-I-J-K))*
```

```
     &(A**REAL(I+J+K-3))*((1.0-4.0*A)**REAL(N+3-I-J-K))
  054 SASP5=ASP5(I,J,K)+SASP5
C
C    The probability that no customers are served during the
C    sixth interval is calculated.
C
      SASP6=0.0
C
      DO 060 L=1,2
      I=1
      J=1
      K=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  060 SASP6=ASP6(1,1,1,L)+SASP6
C
      DO 061 K=2,3
      DO 061 L=1,4-K
      I=1
      J=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  061 SASP6=ASP6(1,1,K,L)+SASP6
C
      DO 062 L=1,2
      I=1
      J=2
      K=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  062 SASP6=ASP6(1,2,1,L)+SASP6
C
      DO 063 K=2,3
      DO 063 L=1,4-K
      I=1
      J=2
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  063 SASP6=ASP6(1,2,K,L)+SASP6
C
      DO 064 J=3,4
      DO 064 K=1,5-J
      DO 064 L=1,6-J-K
      I=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
```

```
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
 064 SASP6=ASP6(1,J,K,L)+SASP6
C
      DO 065 L=1,2
      I=2
      J=1
      K=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
 065 SASP6=ASP6(2,1,1,L)+SASP6
C
      DO 066 K=2,3
      DO 066 L=1,4-K
      I=2
      J=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
 066 SASP6=ASP6(2,1,K,L)+SASP6
C
      DO 067 L=1,2
      I=2
      J=2
      K=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
 067 SASP6=ASP6(2,2,1,L)+SASP6
C
      DO 068 K=2,3
      DO 068 L=1,4-K
      I=2
      J=2
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
 068 SASP6=ASP6(2,2,K,L)+SASP6
C
      DO 069 J=3,4
      DO 069 K=1,5-J
      DO 069 L=1,6-J-K
      I=2
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
 069 SASP6=ASP6(2,J,K,L)+SASP6
C
      DO 610 L=1,2
      I=3
```

```
      J=1
      K=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  610 SASP6=ASP6(3,1,1,L)+SASP6
C
      DO 611 L=1,2
      I=3
      J=1
      K=2
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  611 SASP6=ASP6(3,1,2,L)+SASP6
C
      I=3
      J=1
      K=3
      L=1
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
      SASP6=ASP6(3,1,3,1)+SASP6
C
      DO 612 J=2,3
      DO 612 K=1,4-J
      DO 612 L=1,5-J-K
      I=3
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  612 SASP6=ASP6(3,J,K,L)+SASP6
C
      DO 613 I=4,5
      DO 613 J=1,6-I
      DO 613 K=1,7-I-J
      DO 613 L=1,8-I-J-K
      ASP6(I,J,K,L)=FACT(N)*REAL(I)*REAL(J)*REAL(K)*REAL(L)*
     &REAL(N+5-I-J-K-L)/
     &(FACT(I)*FACT(J)*FACT(K)*FACT(L)*FACT(N+5-I-J-K-L))*
     &(A**REAL(I+J+K+L-4))*((1.0-5.0*A)**REAL(N+4-I-J-K-L))
  613 SASP6=ASP6(I,J,K,L)+SASP6
C
C    Results are written in file "ANNC9.TXT"
C
      WRITE(1,099)RHO,N
  099 FORMAT('RHO=',F3.1,5X,'POPULATION=',I2)
      WRITE(1,100)SASP2,SASP3,SASP4,SASP5,SASP6
  100 FORMAT('SASP2=',F5.4,3X,'SASP3=',F5.4,3X,'SASP4=',F5.4,3X,
     &'SASP5=',F5.4,3X,'SASP6=',F5.4)
```

```
C
C     If the population is less than 50 or rho is less than
C     1.0, then the process is repeated for a larger population
C     and/or a different rho value.
C
      IF(N.LT.50)GOTO 400
      IF(RHO.LT.1.0)GOTO 300
      CONTINUE
      END
```

ANNEX C.9. Results of small-population model with constant rho, single tap, and service time constant among users. (Computer program is in Annex C.8)

```
RHO= .1      POPULATION=10
SASP2=.9044   SASP3=.9004   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=15
SASP2=.9045   SASP3=.9005   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=20
SASP2=.9046   SASP3=.9005   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=25
SASP2=.9047   SASP3=.9005   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=30
SASP2=.9047   SASP3=.9006   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=35
SASP2=.9047   SASP3=.9006   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=40
SASP2=.9047   SASP3=.9006   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=45
SASP2=.9047   SASP3=.9006   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .1      POPULATION=50
SASP2=.9047   SASP3=.9006   SASP4=.9001   SASP5=.9000   SASP6=.9000

RHO= .2      POPULATION=10
SASP2=.8171   SASP3=.8033   SASP4=.8007   SASP5=.8002   SASP6=.8000

RHO= .2      POPULATION=15
SASP2=.8176   SASP3=.8037   SASP4=.8009   SASP5=.8002   SASP6=.8001

RHO= .2      POPULATION=20
SASP2=.8179   SASP3=.8039   SASP4=.8010   SASP5=.8003   SASP6=.8001

RHO= .2      POPULATION=25
SASP2=.8181   SASP3=.8040   SASP4=.8010   SASP5=.8003   SASP6=.8001

RHO= .2      POPULATION=30
SASP2=.8182   SASP3=.8040   SASP4=.8011   SASP5=.8003   SASP6=.8001

RHO= .2      POPULATION=35
SASP2=.8183   SASP3=.8041   SASP4=.8011   SASP5=.8003   SASP6=.8001

RHO= .2      POPULATION=40
SASP2=.8183   SASP3=.8041   SASP4=.8011   SASP5=.8003   SASP6=.8001

RHO= .2      POPULATION=45
SASP2=.8184   SASP3=.8041   SASP4=.8011   SASP5=.8003   SASP6=.8001
```

ANNEX C.9 (continued)

RHO= .2	POPULATION=50			
SASP2=.8184	SASP3=.8042	SASP4=.8011	SASP5=.8003	SASP6=.8001
RHO= .3	POPULATION=10			
SASP2=.7374	SASP3=.7105	SASP4=.7033	SASP5=.7010	SASP6=.7003
RHO= .3	POPULATION=15			
SASP2=.7386	SASP3=.7115	SASP4=.7040	SASP5=.7014	SASP6=.7005
RHO= .3	POPULATION=20			
SASP2=.7391	SASP3=.7120	SASP4=.7043	SASP5=.7017	SASP6=.7006
RHO= .3	POPULATION=25			
SASP2=.7395	SASP3=.7123	SASP4=.7045	SASP5=.7018	SASP6=.7007
RHO= .3	POPULATION=30			
SASP2=.7397	SASP3=.7125	SASP4=.7047	SASP5=.7019	SASP6=.7008
RHO= .3	POPULATION=35			
SASP2=.7399	SASP3=.7126	SASP4=.7048	SASP5=.7020	SASP6=.7008
RHO= .3	POPULATION=40			
SASP2=.7400	SASP3=.7127	SASP4=.7049	SASP5=.7020	SASP6=.7009
RHO= .3	POPULATION=45			
SASP2=.7401	SASP3=.7128	SASP4=.7049	SASP5=.7020	SASP6=.7009
RHO= .3	POPULATION=50			
SASP2=.7401	SASP3=.7129	SASP4=.7050	SASP5=.7021	SASP6=.7009
RHO= .4	POPULATION=10			
SASP2=.6648	SASP3=.6233	SASP4=.6094	SASP5=.6038	SASP6=.6015
RHO= .4	POPULATION=15			
SASP2=.6667	SASP3=.6252	SASP4=.6110	SASP5=.6051	SASP6=.6023
RHO= .4	POPULATION=20			
SASP2=.6676	SASP3=.6262	SASP4=.6119	SASP5=.6057	SASP6=.6028
RHO= .4	POPULATION=25			
SASP2=.6682	SASP3=.6268	SASP4=.6124	SASP5=.6061	SASP6=.6031
RHO= .4	POPULATION=30			
SASP2=.6685	SASP3=.6271	SASP4=.6127	SASP5=.6064	SASP6=.6034
RHO= .4	POPULATION=35			
SASP2=.6688	SASP3=.6274	SASP4=.6130	SASP5=.6066	SASP6=.6035
RHO= .4	POPULATION=40			
SASP2=.6690	SASP3=.6276	SASP4=.6131	SASP5=.6068	SASP6=.6036

RHO= .4 POPULATION=45
SASP2=.6691 SASP3=.6278 SASP4=.6133 SASP5=.6069 SASP6=.6037

RHO= .4 POPULATION=50
SASP2=.6692 SASP3=.6279 SASP4=.6134 SASP5=.6070 SASP6=.6038

RHO= .5 POPULATION=10
SASP2=.5987 SASP3=.5424 SASP4=.5205 SASP5=.5100 SASP6=.5047

RHO= .5 POPULATION=15
SASP2=.6014 SASP3=.5456 SASP4=.5236 SASP5=.5129 SASP6=.5071

RHO= .5 POPULATION=20
SASP2=.6027 SASP3=.5472 SASP4=.5252 SASP5=.5143 SASP6=.5084

RHO= .5 POPULATION=25
SASP2=.6035 SASP3=.5481 SASP4=.5262 SASP5=.5152 SASP6=.5091

RHO= .5 POPULATION=30
SASP2=.6040 SASP3=.5487 SASP4=.5268 SASP5=.5158 SASP6=.5097

RHO= .5 POPULATION=35
SASP2=.6043 SASP3=.5492 SASP4=.5272 SASP5=.5162 SASP6=.5100

RHO= .5 POPULATION=40
SASP2=.6046 SASP3=.5495 SASP4=.5276 SASP5=.5165 SASP6=.5103

RHO= .5 POPULATION=45
SASP2=.6048 SASP3=.5498 SASP4=.5278 SASP5=.5168 SASP6=.5106

RHO= .5 POPULATION=50
SASP2=.6050 SASP3=.5500 SASP4=.5280 SASP5=.5170 SASP6=.5107

RHO= .6 POPULATION=10
SASP2=.5386 SASP3=.4684 SASP4=.4379 SASP5=.4215 SASP6=.4119

RHO= .6 POPULATION=15
SASP2=.5421 SASP3=.4730 SASP4=.4430 SASP5=.4267 SASP6=.4169

RHO= .6 POPULATION=20
SASP2=.5438 SASP3=.4753 SASP4=.4456 SASP5=.4293 SASP6=.4194

RHO= .6 POPULATION=25
SASP2=.5448 SASP3=.4766 SASP4=.4471 SASP5=.4308 SASP6=.4209

RHO= .6 POPULATION=30
SASP2=.5455 SASP3=.4775 SASP4=.4480 SASP5=.4318 SASP6=.4219

RHO= .6 POPULATION=35
SASP2=.5460 SASP3=.4782 SASP4=.4487 SASP5=.4326 SASP6=.4226

RHO= .6 POPULATION=40
SASP2=.5463 SASP3=.4786 SASP4=.4493 SASP5=.4331 SASP6=.4231

RHO= .6 POPULATION=45
SASP2=.5466 SASP3=.4790 SASP4=.4497 SASP5=.4335 SASP6=.4236

RHO= .6 POPULATION=50
SASP2=.5468 SASP3=.4793 SASP4=.4500 SASP5=.4339 SASP6=.4239

RHO= .7 POPULATION=10
SASP2=.4840 SASP3=.4014 SASP4=.3628 SASP5=.3401 SASP6=.3253

RHO= .7 POPULATION=15
SASP2=.4883 SASP3=.4076 SASP4=.3701 SASP5=.3482 SASP6=.3338

RHO= .7 POPULATION=20
SASP2=.4904 SASP3=.4105 SASP4=.3737 SASP5=.3521 SASP6=.3379

RHO= .7 POPULATION=25
SASP2=.4917 SASP3=.4123 SASP4=.3758 SASP5=.3544 SASP6=.3403

RHO= .7 POPULATION=30
SASP2=.4925 SASP3=.4135 SASP4=.3772 SASP5=.3559 SASP6=.3419

RHO= .7 POPULATION=35
SASP2=.4931 SASP3=.4143 SASP4=.3781 SASP5=.3570 SASP6=.3431

RHO= .7 POPULATION=40
SASP2=.4935 SASP3=.4149 SASP4=.3789 SASP5=.3578 SASP6=.3439

RHO= .7 POPULATION=45
SASP2=.4939 SASP3=.4154 SASP4=.3794 SASP5=.3584 SASP6=.3446

RHO= .7 POPULATION=50
SASP2=.4941 SASP3=.4158 SASP4=.3799 SASP5=.3589 SASP6=.3451

RHO= .8 POPULATION=10
SASP2=.4344 SASP3=.3415 SASP4=.2958 SASP5=.2672 SASP6=.2470

RHO= .8 POPULATION=15
SASP2=.4395 SASP3=.3491 SASP4=.3054 SASP5=.2784 SASP6=.2597

RHO= .8 POPULATION=20
SASP2=.4420 SASP3=.3528 SASP4=.3099 SASP5=.2837 SASP6=.2656

RHO= .8 POPULATION=25
SASP2=.4435 SASP3=.3550 SASP4=.3126 SASP5=.2868 SASP6=.2690

RHO= .8 POPULATION=30
SASP2=.4445 SASP3=.3564 SASP4=.3144 SASP5=.2888 SASP6=.2713

ANNEX C.9 (continued)

```
RHO= .8       POPULATION=35
SASP2=.4452   SASP3=.3574   SASP4=.3156   SASP5=.2902   SASP6=.2729

RHO= .8       POPULATION=40
SASP2=.4457   SASP3=.3582   SASP4=.3166   SASP5=.2913   SASP6=.2740

RHO= .8       POPULATION=45
SASP2=.4461   SASP3=.3588   SASP4=.3173   SASP5=.2921   SASP6=.2750

RHO= .8       POPULATION=50
SASP2=.4464   SASP3=.3592   SASP4=.3179   SASP5=.2928   SASP6=.2757

RHO= .9       POPULATION=10
SASP2=.3894   SASP3=.2883   SASP4=.2371   SASP5=.2038   SASP6=.1792

RHO= .9       POPULATION=15
SASP2=.3953   SASP3=.2973   SASP4=.2488   SASP5=.2179   SASP6=.1958

RHO= .9       POPULATION=20
SASP2=.3982   SASP3=.3016   SASP4=.2543   SASP5=.2245   SASP6=.2034

RHO= .9       POPULATION=25
SASP2=.3999   SASP3=.3042   SASP4=.2575   SASP5=.2283   SASP6=.2078

RHO= .9       POPULATION=30
SASP2=.4010   SASP3=.3059   SASP4=.2596   SASP5=.2308   SASP6=.2106

RHO= .9       POPULATION=35
SASP2=.4018   SASP3=.3071   SASP4=.2611   SASP5=.2326   SASP6=.2126

RHO= .9       POPULATION=40
SASP2=.4024   SASP3=.3079   SASP4=.2622   SASP5=.2339   SASP6=.2141

RHO= .9       POPULATION=45
SASP2=.4029   SASP3=.3086   SASP4=.2631   SASP5=.2349   SASP6=.2152

RHO= .9       POPULATION=50
SASP2=.4032   SASP3=.3092   SASP4=.2638   SASP5=.2357   SASP6=.2161

RHO=1.0       POPULATION=10
SASP2=.3487   SASP3=.2416   SASP4=.1868   SASP5=.1505   SASP6=.1230

RHO=1.0       POPULATION=15
SASP2=.3553   SASP3=.2518   SASP4=.2001   SASP5=.1670   SASP6=.1429

RHO=1.0       POPULATION=20
SASP2=.3585   SASP3=.2567   SASP4=.2064   SASP5=.1746   SASP6=.1517

RHO=1.0       POPULATION=25
SASP2=.3604   SASP3=.2595   SASP4=.2101   SASP5=.1789   SASP6=.1568

RHO=1.0       POPULATION=30
SASP2=.3617   SASP3=.2614   SASP4=.2125   SASP5=.1818   SASP6=.1601
```

ANNEX C.9 (continued)

RHO=1.0	POPULATION=35			
SASP2=.3626	SASP3=.2628	SASP4=.2142	SASP5=.1838	SASP6=.1624
RHO=1.0	POPULATION=40			
SASP2=.3632	SASP3=.2638	SASP4=.2154	SASP5=.1853	SASP6=.1641
RHO=1.0	POPULATION=45			
SASP2=.3638	SASP3=.2646	SASP4=.2164	SASP5=.1864	SASP6=.1654
RHO=1.0	POPULATION=50			
SASP2=.3642	SASP3=.2652	SASP4=.2172	SASP5=.1874	SASP6=.1664

ANNEX D. Validity of Small-Population Model

The large-population models presented in chapter IV are similar to widely accepted models found in the literature. However, the small-population model in chapter V is not found in the literature. Its validity must be verified.

The small-population model determines the expected line length (given that the line is not empty) with a rather roundabout method. The method is necessary because the list of arrival sequences that must be considered quickly becomes cumbersome when the population is larger than a handful. To verify the validity of the method, the writer derived a more direct method. However, this method can be applied only to very small populations because its sequence lists are even more cumbersome than those of the chapter V method. This more direct method calculates the probability of each possible combination of numbers of arrivals. For each combination, the resulting line length is calculated and then multiplied by the probability of the combination occurring. The sum of these products equals the expected line length. As an example of this procedure, the expected line length at the beginning of the third interval when the population size is 2 is determined by first calculating the probability of 0 arrivals in the first and second intervals, the probability of 0 arrivals in the first interval and 1 arrival in the second, the probability of 0 in the first and 2 in the second, the probability of 1 in the first and 0 in the second, the probability of 1 in the first and 1 in the second, and the probability of 2 in the first and 0 in the second. Each of these probabilities is multiplied by its corresponding line length at the beginning of the third interval. (For example, if 0 arrivals occur in the first interval and 2 arrivals occur in the second interval, the line length at the beginning of the third interval would be 2 persons. Notice that, as in all other numerical models presented in this paper, the assumption is made that arrivals must wait at least until the beginning of the next interval to be served.) The sum of the products is the expected line length at the beginning of the third interval.

This model and the model presented in chapter V were run for a population size of 4. The results were identical. Also, as discussed at the end of chapter V, steady-state is attained at the beginning of the 5th interval, which is interval N+1.

The chapter V model was also verified by substituting the Poisson distribution for the binomial distribution. The results of this modified model are identical to the large-population model presented in chapter IV.

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