TWO ESSAYS ON SEQUENTIAL AUCTIONS AND RESEARCH JOINT VENTURES

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ABSTRACT

OMUR CELMANBET: Two Essays on Sequential Auctions and Research Joint Ventures
(Under the direction of Claudio Mezzetti)

The dissertation consists of two essays. In "Choosing the Order of Sale in Multi-Unit Auctions" I examine a sequence of two English Auctions of two common-value, heterogeneous objects: a low-value and a high-value object. I ask the question: Should the auctioneer sell the low-value or the high-value first in order to maximize his revenue? Under the assumption that bidders' signals about the common-value are affiliated random variables, I study two models. In Model I, some bidders only want to buy the high-value item while the others only want the low-value item. In Model II, there are also some bidders that want both objects. In Model I, I show that: (1) It is optimal for the seller to auction the low-value item first, when the number of bidders competing for the high-value object is not greater than the number of bidders competing for the low-value object. (2) When there are more bidders in the auction of the high-value item than in the auction of the low-value item, then either order of sale can be optimal for the seller, depending on the relative values of the objects. In particular, if there are sufficiently more bidders competing for the high-value object, then it is optimal for the seller to auction the high-value item first. I also show that the main insights from Model I generalize to Model II.

In the second essay, "Voluntary Disclosure to Form a Research Joint Venture" I investigate why many potentially successful research joint ventures (RJV) do not start and suggest a way to remedy the problem. To inform potential partners about the value of the know-how that it would bring into an RJV, a firm must disclose some of this know-how. In a weak intellectual property rights environment, this creates the danger of exposing the firm to expropriation: The revealed know-how cannot be protected and the potential partners may use it to innovate themselves. Because of this fear of expropriation, many potentially successful RJVs cannot be formed. I introduce a contractual procedure that guarantees that firms disclose their know-how fully and encourages firms to form RJVs.

To my Dad, Seracettin Celmanbet and to my Mom, Neriman Celmanbet: I could not be here without you.

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TABLE OF CONTENTS

Chapter I

Choosing the Order of Sale in Multi-Unit Auctions

1 Introduction

Many auctions involve the simultaneous or sequential sale of different and differently valued goods. Examples include art, real estate, and government auctions to sell mineral rights, radio spectrum licenses, and other state owned assets. The design of efficient and revenue maximizing auctions of multiple objects has attracted considerable interest in the wake of the FCC spectrum auctions of 1994 (e.g., see Armstrong (2000), Ausubel and Milgrom (2002) and Ausubel (2002), among others). Most auctioneers, however, have continued to use standard auctions to sell multiple items. A natural question arises: which order of sale in a standard sequential auction maximizes the seller's revenue? This paper is an attempt to answer this fairly simple, yet unexplored question.

Governments all around the world have been using sequential auctions to privatize state owned firms. While some governments sell their most valuable firms early, others wait to auction them off after the sale of small enterprises. For example, Gupta et al. (2004) report that state owned firms were sold in the order of declining value in the Czech Republic. On the other hand, the Turkish Government is still in the process of selling Tupras, which is one of the most profitable state owned enterprises in Turkey, after almost 20 years of continued privatization.[1]

Sequential auctions are used not only in the privatization of government assets, but also in the creation of new markets. In the recent European UMTS/IMT-2000 spectrum auctions, the German and Austrian governments sold the unpaired spectrum after the paired spectrum, which is more valuable.[2] In contrast, in an auction to sell three wireless licences - two of them for 28 MHz block and one for 56 MHz - the Swiss goverment decided to auction off the largest (and hence most valuable) spectrum last.

Perhaps the most illustrative examples in which the sequence of sale matters come from art auctions. Suppose an auction house has to decide which of two paintings by Picasso should be auctioned first, a masterpiece or a minor work. Actual practioners of auctions often resort to psychological arguments to answer this question. Two factors seem to be primarily driving their decision: "warming up the room" and "establishing lively bidding". Warming up the room, it is argued, requires that the minor work goes first, in order to generate enthusiasm in the competition for the masterpiece, among bidders who are initially hesitant. Establishing lively bidding, on the other hand, requires that the masterpiece goes first, in order to exploit the enthusiasm that competition for the masterpiece generates in the subsequent sale of the minor work.[3] In a study based on data from Contemporary Art auctions at Christie's in London in 1980-1994 and Impressionist and Modern Art auctions in both Christie's and

[1] www.oib.gov.tr

[2] See Jehiel and Moldovanu (2001)

[3] See Cassady (1967) and Benoit and Krishna (2001) for discussions of these factors.

Sotheby's in New York and London in 1980-1990, Beggs and Graddy (1997) found evidence that the objects are ordered by declining value.

To study the optimal sequence of sale, I develop a simple common-value model of two sequential English auctions with an auctioneer who has two heterogeneous objects: a low-value and a high-value object.[4] The values of the objects are correlated. In particular, each object is tied to an unknown state of nature. This is a plausible modeling assumption, given the examples above. For instance, in the case of two paintings by Picasso, the state variable could be a summary of the current economic conditions of the art market and the resale value of Picasso paintings in general. Similarly, in the privatization of different firms by a government, the state of nature may represent the country's economic conditions, tax policy and corporate law.

In my model, there are three group of bidders: H-bidders, who want to buy only the high value object, L-bidders who want to buy only the low value object, and B-bidders who want to buy both the low and the high value items. Thus, H-bidders and L-bidders have only unit demand. One explanation for the existence of unit-demand bidders is that some bidders may not have the financial and technological resourses to buy and utilize both objects at the same time, as observed for example in the spectrum auctions by the Swiss government. I assume that each bidder has a private signal concerning the state variable and that the bidders' signals are affiliated.

Given these assumptions, I study two models. In Model I there are only H-bidders and L-bidders, in order to separate the issues arising from the presence of multi-demand bidders, while in Model II there are also B-bidders.

The main result of this paper is that in both models either selling the high-value good first or selling the low-value good first can be optimal depending on the relative values of the objects and the number of bidders of each type. My findings suggest an explanation of why in practice we observe that some sellers auction off their goods in the order of declining value, while others do the opposite. My results also provide guidelines for auctioneers in deciding which good to sell first to maximize revenue.

I begin with Model I by analyzing the auctions of the items when there are only unit-demand bidders. It is shown, as expected, that it is never optimal for the seller to hold simultaneous sales where no information is released between the auctions. The intuition follows from the Linkage principle of Milgrom and Weber (1982), which implies that the expected revenue increases if bidders are given information that is related to the true value of the object. While the expected revenue from the first sale is the same as the revenue from an independent auction of the same item, a sequential sale improves the seller's revenue in the second auction. Unlike in simultaneous sales, which are essentially two independent English auctions, in a sequential sale the information of the bidders from the first

[4] I chose to study the English auction because it is widely used and because, with affiliated values and risk neutral bidders, it raises the highest revenue among the common auction formats, as shown by Milgrom and Weber (1982).

auction is passed on to the bidders in the second sale. Since the revealed information is related to the state of nature, and hence to the value of the object to be sold in the second stage, the information disclosure leads to more aggressive bids and higher payoff to the seller in the second auction, as compared to the revenue in the auction of that item when the goods are sold simultaneously.

After establishing that the seller should sell the goods sequentially, I ask the main question of the paper: which of the items should the seller auction first in order to maximize total revenue? At first glance, the answer would seem to be that the auction of the high value item should be held later. After all, by revealing the signals of the bidders participating in the first auction, a sequential sale increases the seller's payoff only in the second sale. Since the contribution of a signal to the high value good is larger than to the low value item, one may expect a larger price increase from auctioning off the high value item later. However, it turns out that there is another effect present, which may conflict with the value effect. In an auction, a bidder's rent, or conversely the seller's loss, on the unknown part of the state of nature depends on the number of bidders competing in that auctions, and the rent decreases as the auction becomes more competitive. Thus, this "competition effect" suggests that the more competitive item should be auctioned first. When the number of L-bidders is greater than or equal to the number of H-bidders, the sale of the low value item is at least as competitive as the sale of more valuable object. In this case, both the value effect and the competition effect imply that the high value good should be sold later. On the other hand, when there are more H-bidders than L-bidders, that is the auction of the more valuable good is more competitive than the other auction, the value effect and the competition effect conflict with each other. Therefore, in this case the optimal sequence of sale depends on which effect is dominant. I find that when the number of H-bidders is sufficiently large relative to the number of L-bidders, the competition effect outweighs the value effect, implying that the high value good should be auctioned first.

I then consider Model II. The presence of multi-demand bidders complicates the situation, since B-bidders may not be willing to reveal their information in the first auction in order to buy the second good at a favorable price. This may lead to two types of equilibria in the first auction: a separating equilibrium, in which the bid of the bidders that want both objects is a strictly increasing function of their signals, and a pooling equilibrium, in which their bid is independent of their signals. A pooling equilibrium always exists, while a separating equilibrium may or may not exist. The fact that each sequence of sale may have multiple possible equilibrium outcomes makes it difficult to compare revenue. Neverthless, I show that the intuition and spirit of the results of Model I still hold. In particular, I find that if the number of H-bidders is sufficiently higher than the number of L-bidders and B-bidders, then selling the high value object first is optimal for the seller. If, on the other hand, the number of L-bidders is sufficiently larger than the number of H-bidders and B-bidders, then the auction of the low value object should be held first. The intuition is similar to the one in Model I. When there are sufficiently many H-bidders, the auction of the high value good becomes very competitive regardless of the equilibrium considered, implying that information disclosure will always be more valuable in the sale of the low value item, which should then be sold second.

This paper is related to a small literature on the order of sale of heterogeneous goods in sequential auctions. The effects of the sequence of sale on the auctioneer's revenue have been studied by Bernhardt & Scoones (1994), Benoit & Krishna (2001), Chakraborty, Gupta & Harbaugh (2003), Elmaghraby (2003), and Pitchik (2004).

Bernhardt & Scoones (1994) consider a private value, sequential second price auction of two heterogeneous goods with different distributions of buyer valuations. They show that it is optimal for the seller to auction the good with the highest variance first. Their paper differs from this paper in that I look at common value English auctions and, most importantly, unlike in their model, in my setting each bidder agrees about which object is more valuable than the other.

Benoit & Krishna (2001) study sequential sales of common value objects with complete information and budget constrained bidders. They conclude that the seller prefers to sell the high value object first. As they point out, it is not the desire to take advantage of information disclosure that is behind their result. The auctioneer's preference for selling the high value item first is due to the combination of two factors: the bidders' incentive to reduce the budgets of their rivals, and the seller's incentive to have the wealthiest bidder win the first auction. Pitchik (2004) also considers sequential auctions with budget constrained bidders, but in an incomplete information and private value setting. In her setting, there are two bidders and it is not the case that the bidders consider the same object as the more valuable. Neverthless, similar to Benoit & Krishna (20001), she finds that the auctioneer's profit is maximized whenever the first object sold goes to the bidder with the highest income. Elmaghraby (2003) extends her paper by allowing arbitrary number of bidders in a sequential procurement of two jobs with different costs, and he shows that it is optimal for the buyer to outsource the more costly task first.

Chakraborty, Gupta & Harbaugh (2003) share the most similarities with this paper, in that they also consider a sequential English auction of two common value heterogeneous goods with different values. However, their paper differs from mine in one important aspect: while the seller knows which of the objects is more valuable, bidders cannot distinguish between the values of the objects initially. Thus, in constrast to my setting, in their model there is strategic interaction between the seller and the bidders. They show that the revenue to the seller is higher if he sells the high value good later, but this order of sale may not be an equilibrium.

The rest of the paper is organized as follows. In Section 2, I describe the details of the model. In Section 3, I investigate the simpler case where there are only unit-demand bidders. After finding the seller's revenue in each sequence of sale, I compare them in order to determine the conditions which make one sequence better than the other from the seller's point of view. In Section 4, I consider the general model in which there are also some bidders who wish to buy both items. Section 5 concludes.

2 The Model

An auctioneer is to sell a low-value, L, and a high-value, H, object. There are $m+n+r$ risk neutral bidders. Each of the $m \geq 2$ bidders (H bidders) in the set $M = \{1, 2, ..., m\}$ only wants H, each of the $n \geq 2$ bidders (L bidders) in the set $N = \{m+1, m+2, ..., m+n\}$ only wants L, and each of the $r \geq 1$ bidders (B bidders) in the set $R = \{m+n+1, m+n+2, ..., m+n+r\}$ wants both H and L. The number of bidders of each type, and who wants which object(s) are common knowledge.

Each bidder $i \in I = M \cup N \cup R$ gets a private signal s_i from the set $S = \{0, 1\}$. I denote the signals, or signal profile, of the bidders in the set $A \subseteq I$ by $\mathbf{s}_A = (s_i)_{i \in A}$. Also, I denote by $\mathbf{s}_A(k)$ the signal profile of the bidders in the set A such that the first k bidders in A have signal 0 and the rest have signal 1. When I consider only the realized signal of bidder i, I write $s_i = 0$ or $s_i = 1$.

The objects have common values to all bidders, and the values of the objects depend on the realizations of all signals. Specifically[5],

$$v_L = \frac{\mu_L}{m+n+r} \sum_{i=1}^{m+n+r} s_i \qquad \text{and}$$

$$v_H = \frac{\mu_H}{m+n+r} \sum_{i=1}^{n+m+r} s_i \qquad \text{where } \mu_H > \mu_L$$

Let $p(\mathbf{s}_I)$ be the commonly known joint probability of all signals. Then, $p(\mathbf{s}_A)$ denotes the marginal probability of the signals in the set $A \subseteq I$. That is,

$$p(\mathbf{s}_A) = \sum_{\mathbf{s}_{I\setminus\{A\}} \in S^{|I\setminus\{A\}|}} p(\mathbf{s}_I)$$

Similarly, $p(\mathbf{s}_A|\mathbf{s}_B)$ denotes the probability of the signals in the set A conditional on the signals in the set B, where A, $B \subseteq I$.

I assume that the signals have a non-degenerate, symmetric, probability density; that is, for all $A \subseteq I$ we have $p(\mathbf{s}_A) > 0$ and $p(\mathbf{s}_A) = p(\mathbf{s}_{\sigma(A)})$ for all $\mathbf{s}_A \in \Pi_{i \in A} S$ and for all permutations σ on the set I.[6] Furthermore, I assume that the signals are affiliated. Affiliation has been a common

[5] The general model is as follows. Let $s_1, s_2, ..., s_{m+n+r} \in S$ be a random sample from a distribution with an unknown value of the parameter V. Suppose that v is the realization of V. The values of the objects are $v_L = \mu_L v$ and $v_H = \mu_H v$. In this paper, I consider the set of distributions such that the posterior distribution of V depends on the observed values $s_1, s_2, ..., s_{m+n+r}$ only through $\frac{1}{m+n+r} \sum_{i=1}^{m+n+r} s_i$. See DeGroot(1970) for more details on such class of distributions.

[6] Note that symmetry implies that $p(\mathbf{s}_A(k)) = p(\mathbf{s}_{\sigma(A)}(k))$ for all permutation σ and for all $k \leq |A|$.

assumption in the auction literature, starting with Milgrom and Weber (1982). For any $\mathbf{s}_A$ and $\widehat{\mathbf{s}}_A$ (where $A \subseteq I$) let $\overline{\mathbf{s}}_A$ and $\underline{\mathbf{s}}_A$ be the component-wise maximum and component-wise minimum of $\mathbf{s}_A$ and $\widehat{\mathbf{s}}_A$, respectively. Then, the variables are said to be affiliated if $p(\overline{\mathbf{s}}_A)p(\underline{\mathbf{s}}_A) \geq p(\mathbf{s}_A)p(\widehat{\mathbf{s}}_A)$ for all $\mathbf{s}_A$ and $\widehat{\mathbf{s}}_A$. If the inequality is strict, then the variables are said to be strictly affiliated.

The timing of the game is as follows. The auctioneer chooses which object to sell first in a sequence of two English auctions. Then, the first English auction starts. Bidders who want to buy the item being auctioned participate. After the first auction ends, the English auction for the other item starts, and bidders interested in that item participate. There are many versions of the English auction (see Klemperer (1999) and Milgrom and Weber (1982) among many others). In the English auction considered here, all bidders that wish to buy the item being sold are initially active at a price of zero. Bidders drop out of the auction as price increases until no or only one bidder remains; bidders cannot bid again once they have dropped out. The object is awarded to the winner(s) at the last drop-out price, and ties are broken by a random draw with equal probabilities among the winners. An important feature of an English auction is that the active bidders can observe at what prices inactive bidders have quit. Thus, at any point in time an active bidder can estimate the value of the object conditional on his signal and the drop-out prices of the inactive bidders. In other words, the early bids convey information to the bidders who are still active. I assume that the bidders in the second auction can also observe all the bids in the first auction, and use this information to update their beliefs about the value of the item to be sold second.

3 Model I: No B-bidders, $r = 0$

In this section, to simplify the initial analysis, I study the sale of H and L when there are no B-bidders. Finding the expected revenue to the seller in Model I will help me later, in Section 4, to find the seller's expected revenue in the general model with $r \geq 1$.

I begin by deriving the benchmark case when the auctioneer sells the objects simultaneously.

3.1 A Benchmark: Simultaneous Auctions

All of the m bidders participate in the auction of H and all of the n bidders compete in the auction of L. I assume that none of the H-bidders (L-bidders) can observe the bids in the auction of L (H). No information from one auction is passed on to the bidders in the other auction. Thus, the simultaneous sale of H and L corresponds to two independent English auctions.

Consider the auction of H. Following Milgrom and Weber (1982), I now derive the drop-out prices in the symmetric equilibrium. A bidder with signal 0 drops out first at the price $b^H(0)$, which is equal to the expected value of H conditional on him and the other $m-1$ bidders having signal 0. That is,

$$b^H(0) = E(v_H|\mathbf{s}_M(m)) = \frac{E(\sum_{j=m+1}^{m+n} s_j|\mathbf{s}_M(m))}{n+m}\mu_H. \quad (1)$$

After all the bidders with signal 0 drop out, a bidder with signal 1 observes the number of bidders, k, who have quit at $b^H(0)$ and updates his information about the other bidders. Thus, a bidder with signal 1 drops out at the price $b_k^H(1)$ which is equal to expected value of H conditional on k bidders having signal 0 and $m-k$ bidders having signal 1; that is,

$$b_k^H(1) = E(v_H|\mathbf{s}_M(k)) = \frac{E(\sum_{j=m+1}^{m+n} s_j|\mathbf{s}_M(k))}{n+m}\mu_H + \frac{m-k}{n+m}\mu_H, \quad (2)$$

where $k = 0, 1, 2, ..., m$. It is convienient to define $b_m^H(1) = b^H(0)$, even though $b_m^H(1)$ is not a bidding function.

The bidding functions for the auction of L are analogous.

Let $R_A(m,n)$ be the expected revenue to the seller in the isolated auction of $A \in \{H, L\}$, when there are m and n bidders competing for the items H and L, respectively. Before deriving the seller's revenue in the auctions of H and L, we need two definitions. Let

$$X_m = p(\mathbf{s}_M(m-1))\left[p(s_{m+1} = 1|\mathbf{s}_M(m-1)) - p(s_{m+1} = 1|\mathbf{s}_M(m))\right],$$

and

$$X_n = p(\mathbf{s}_N(n-1))\left[p(s_1 = 1|\mathbf{s}_N(n-1)) - p(s_1 = 1|\mathbf{s}_N(n))\right].$$

Proposition 1 *Assume that the auction of H and the auction of L are held simultaneously. Then,*

(i) *The expected revenue to the seller from the auction of H is*

$$R_H(m,n) = \mu_H p(s_1 = 1) - \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)) - \frac{mn}{m+n}\mu_H X_m$$

(ii) *The expected revenue to the seller from the auction of L is*

$$R_L(m,n) = \mu_L p(s_{m+1} = 1) - \frac{n}{m+n}\mu_L p(\mathbf{s}_N(n-1)) - \frac{mn}{m+n}\mu_L X_n$$

The first term in the formula for $R_H(m,n)$ corresponds to the unconditional expected value of H:

$$E(v_H) = \frac{\mu_H}{m+n}\sum_{i=1}^{m+n} E(s_i) = \mu_H p(s_1 = 1),$$

where the last equality follows from symmetry.

As we shall see, the two remaining terms in $R_H(m,n)$, which can be interpreted as a loss to the seller, correspond to the total expected payoff of the H-bidders,

$$L_H(m,n) = \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)) + \frac{mn}{m+n}\mu_H X_m.$$

Thus, we have that the expected revenue is $R_H(m,n) = E(v_H) - L_H(m,n)$.

To better understand the formula for $L_H(m,n)$, begin by noting that the true value of H, $v_H = \frac{\mu_H}{m+n}\sum_{i=1}^{m+n} s_i$, is the sum of the contribution of the H-bidders' signals, $v_H^H = \frac{\mu_H}{m+n}\sum_{i=1}^{m} s_i$, and of the L-bidders' signals, $v_H^L = \frac{\mu_H}{m+n}\sum_{i=m+1}^{m+n} s_i$: $v_H = v_H^H + v_H^L$.

In the symmetric equilibrium, all H-bidders with signal 0 drop out when the price equals the expected value of v_H conditional on all H-bidders having signal 0; that is, they quit at $b^H(0) = v_H^H(0) + v_H^L(0)$, where $v_H^H(0)$ is the expected value of v_H^H and $v_H^L(0)$ is the expected value of v_H^L, both conditional on all H-bidders having signal 0. Bidder $j \in M$ makes a positive profit if and only if he is the only H-bidder with signal 1, in which case he wins item H for sure and pays $b^H(0)$.[7] Thus, we can think that he pays

$$v_H^H(0) = E(v_H^H|\mathbf{s}_M(m)) = 0$$

for v_H^H, and

$$v_H^L(0) = E(v_H^L|\mathbf{s}_M(m)) = \frac{n}{m+n}\mu_H p(s_{m+1} = 1|\mathbf{s}_M(m))$$

for v_H^L. When bidder j is the only H-bidder with signal 1, the expected value of v_H^H is

$$E(v_H^H|\mathbf{s}_M(m-1)) = \frac{1}{m+n}\mu_H,$$

while the expected value of v_H^L is

$$E(v_H^L|\mathbf{s}_M(m-1)) = \frac{n}{m+n}\mu_H p(s_{m+1} = 1|\mathbf{s}_M(m-1)).$$

Since the probability that bidder j is the only H-bidder with signal 1 is $p(\mathbf{s}_M(m-1))$, and bidders are symmetric, it follows that the total expected payoff of the H-bidders from v_H^H is

$$m\left[E(v_H^H|\mathbf{s}_M(m-1)) - v_H^H(0)\right] p(\mathbf{s}_M(m-1)) = \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)),$$

which corresponds to the first term in $L_H(m,n)$, while the total expected payoff of the H-bidders from v_H^L is

$$m\left[E(v_H^L|\mathbf{s}_M(m-1)) - v_H^L(0)\right] p(\mathbf{s}_M(m-1)) = \frac{mn}{m+n}\mu_H X_m,$$

[7] In all the other cases, bidder j's payoff is zero because he either does not win or his bid is in a tie as one of the highest bid. In the latter case, if he wins, he pays his own bid, which is equal to the expected value of the object.

which corresponds to the second term in $L_H(m,n)$.

Analogous formulas hold for the auction of L. In particular, $R_L(m,n) = E(v_L) - L_L(m,n)$, and $\frac{mn}{m+n}\mu_L X_n$ is the expected total payoff of the L-bidders from v_L^H, the sum of the contribution of the H-bidders' signals to the true value of L.

The following lemma shows that the expected loss to the seller on v_H^L and v_L^H is indeed nonnegative.

Lemma 1 *X_m (X_n) is nonnegative for all m (n). If the signals are strictly affiliated, then X_m (X_n) is positive for all m (n).*

With simultaneous sales, the seller incurs a nonnegative (positive if the signals are strictly affiliated) loss because of the bidders whose signals affect the value of an item, but who do not compete for that item. In the next section, I will show that sequential sales reduce this loss.

3.2 The Sequential Sale of H and L

Now suppose that the seller auctions H and L sequentially. The first sale is identical to selling the first item in a one-shot isolated auction, whereas the second sale is not. Assume, for instance, that the seller auctions H first and L second (that is, the order is HL). Consider the auction of H. Each of the m H-bidders views the sale of H as an independent auction, because he does not compete for L. Thus, all H-bidders with signal 0 drop out at $b^H(0) = E(v_H|\mathbf{s}_M(m))$ and all H-bidders with signal 1 quit at $b_k^H(1) = E(v_H|\mathbf{s}_M(k))$ (where $k = 0, 1, 2, ..., m$ is the number of H-bidders who dropped out at $b^H(0)$), as in the simultaneous sales.

Now consider the auction of L. Since the equilibrium bidding function of the first auction is increasing in a bidder's signal, the vector of signals $\mathbf{s}_M$ is revealed to the n bidders participating in the auction of L (equilibrium in the auction of H is separating). The bidders use this information to update the expected value of L. Suppose that k bidders drop out at $b^H(0)$ and $m-k$ bidders drop out at $b_k^H(1)$ in the auction of H. Then, an L-bidder with signal 0 drops out first at the price $b_k^L(0)$ that is equal to the expected value of L conditional on k H-bidders having signal 0, $m-k$ H-bidders having signal 1 and him and the other $n-1$ L-bidders having signal 0. That is,

$$b_k^L(0) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_N(n)) = \frac{m-k}{m+n}\mu_L. \tag{3}$$

After all the bidders with signal 0 in the auction of L drop out at the price $b_k^L(0)$, an L-bidder with signal 1 observes the number t of L-bidders who have dropped out at $b_k^L(0)$ and updates his information about the other bidders. Thus, an L-bidder with signal 1 drops out at the price $b_{k,t}^L(1)$ which is equal to the expected value of L conditional on k H-bidders having signal 0, $m-k$ H-bidders having signal 1, t L-bidders having signal 0, and $n-t$ L-bidders having signal 1. That is,

$$b_{k,t}^L(1) = E(v_L|\mathbf{s}_M(k);\mathbf{s}_N(t)) = \frac{m-k}{m+n}\mu_L + \frac{n-t}{m+n}\mu_L, \tag{4}$$

where $t = 0, 1, ..., n$. It is convienient to define $b_{k,n}^L(1) = b_k^L(0)$, but recall that $b_{k,n}^L(1)$ is not a bidding function.

Let $R_{AB}(m,n)$ be the seller's expected revenue when the order is AB (where $A, B \in \{H, L\}$), there are m bidders competing for H, and n bidders competing for L.

Proposition 2 *(i) The expected payoff to the seller when object H is auctioned first and object L is auctioned second is*

$$R_{HL}(m,n) = R_H(m,n) + R_L(m,n) + \frac{mn}{m+n}\mu_L X_n$$

(ii) The expected payoff to the seller when object L is auctioned first and object H is auctioned second is

$$R_{LH}(m,n) = R_H(m,n) + R_L(m,n) + \frac{mn}{m+n}\mu_H X_m$$

I now compare the revenues to the seller in simultaneous and sequential sales. Recall from Section 3.1 that $R_H(m,n) + R_L(m,n)$ is the auctioneer's payoff in the simultaneous sales of the items. In Section 3.1, I also showed that X_m and X_n are nonnegative (positive if the signals are strictly affiliated). Thus, I have proved the following corollary.

Corollary 1 *The seller weakly prefers sequential sales to simultaneous sales. If the signals are strictly affiliated, then the seller strictly prefers sequential sales to simultaneous sales.*

The difference between the expected revenues to the seller in sequential and simultaneous sales can be explained by the "Linkage Principle," one of the fundamantal results in auction theory (Milgrom and Weber (1982)). The principle implies that the seller's expected revenue increases when bidders are provided with additional information that is related to the true value of the object. In sequential sales, the seller's expected revenue from the sale of the first item is the same as the expected revenue from the auction of that item when the objects are sold simultaneously. However, this is not true for the auction of the second item. Consider, for instance, the seller's revenue in the auction of L when the objects are sold simultaneously,

$$R_L(m,n) = \mu_L p(s_{m+1} = 1) - \frac{n}{m+n}\mu_L p(\mathbf{s}_N(n-1)) - \frac{mn}{m+n}\mu_L X_n$$

A part of the seller's loss, $\frac{mn}{m+n}\mu_L X_n$ is due to the L-bidders having to consider the expected values of the unknown vector of signals $\mathbf{s}_M$ while forming their bids. Recall that $\frac{mn}{m+n}\mu_L X_n$ is the expected total payoff to the L-bidder on v_L^H, the part of the value of L due to the signals of the H-bidders. Therefore, $\frac{mn}{m+n}\mu_L X_n$ is also the expected loss to the seller on v_L^H.

Now consider the auction of L when the order is HL. Since the equilibrium in the first auction is separating, the vector of realizations of the H-bidders' signals, $\mathbf{s}_M$, becomes known to the L-bidders. As a result, each L-bidder obtains a zero payoff on the part v_L^H of the true value of L. In other words, if item L is sold in the second auction, the seller avoids the loss on v_L^H that he incurs in the simultaneous sales. Therefore, the seller's expected revenue when the order is HL exceeds the revenue in the simultaneous sales by $\frac{mn}{m+n}\mu_L X_n$. Since $X_n \geq 0$ by Lemma 1, the seller weakly prefers to sell the items sequentially to simultaneous sales. If the signals are strictly affiliated, then $X_n > 0$, and the seller strictly prefers sequential sales.

In summary, when the objects are sold sequentially, the information of the bidders in the first auction becomes known to the bidders in the second auction. As a result, the bidders in the second auction bid more aggressively, increasing the price of the item to be sold second and the expected revenue of the seller.

3.3 The Optimal Order of Sale

After concluding that the seller should auction the items sequentially, I will now determine which order of sales is optimal from the seller's point of view.

Proposition 3 *Assume $m = n$. Then, the seller weakly prefers selling L before H. If the signals are strictly affiliated, then the seller strictly prefers selling L first.*

As I discussed in Section 3.2, if H is auctioned second, the auctioneer's revenue rises by $\frac{mn}{m+n}\mu_H X_m$ compared to simultaneous sales, and it increases by $\frac{mn}{m+n}\mu_L X_n$ if L is auctioned second. If $m = n$, then $X_m = X_n \geq 0$, with strict inequality if signals are strictly affiliated. When there is an equal number of bidders in each auction, the information content spilling over from the first auction into the second auction is the same, but information is more valuable in the sale of the high value object, since $\mu_H > \mu_L$. In other words, the seller suffers a higher loss in the sale of H due to lack of information about the L-bidders' signals, than in the sale of L due to lack of information about the H-bidders' signals. It is thus optimal to sell L first. More generally, we can think that a "value effect," the fact that object H is more valuable than L, favors the sale of the high value object later. Following the jargon of practitioners, we could call the "value effect" a "warming up the room" effect. Proposition 3 says that if competition for the two items is the same, then the seller should exploit the information spillovers from the sale of the less valuable object, in effect "warming up the room," to raise the price of the more valuable object.

The next lemma implies that $\frac{n}{m+n}\mu_H X_m$, the expected payoff of an H-bidder from v_H^L decreases with m, the number of bidders competing for H.

Lemma 2 *X_m (X_n) is decreasing in m (n). Moreover, if the signals are strictly affiliated, then X_m (X_n) is strictly decreasing in m (n).*

This lemma suggests that, along with the value effect, the seller should also take into account the "competition effect," when determining the optimal order of sale. The "competition effect" favors the more competitive auction to be held first, since there is less room for information disclosure to rise the price of that item.

When the number of bidders competing for H is less than the number of bidders competing for L, both the value and the competition effects suggest that L should be auctioned first. That this is indeed the case is established in the next Proposition.

Proposition 4 *If $m < n$ then the seller weakly prefers selling L first to selling H first. If the signals are strictly affiliated, then the seller strictly prefers selling L first to selling H first.*

The next corollary says that for each level of competition for the H item, if the auction of the L item is above a minimum level, then auctioning L first maximizes revenue.

Corollary 2 *For any m there exists n^* such that if $n \geq n^*$, then the seller weakly prefers selling L first to selling H first. Moreover, if the signals are strictly affiliated, then the seller strictly prefers selling L first to selling H first.*

When there are more H-bidders than L-bidders, either order can be optimal. While the competition effect suggests that H should be sold first, the value effect favors the opposite order. For instance, if the value of L is sufficiently close to the value of H, then it is more likely that the competition effect outweighs the value effect. In that case, the seller should auction H first. In contrast, if the value of L is really small compared to the value of H, then it is likely that the value effect outweighs the competition effect. In this case, the auction of H should be held first.

The following two lemmas will be used to determine which object should be auctioned first when there are more H-bidders than L-bidders.

Lemma 3 $\underset{m\to\infty}{lim}\, p(\mathbf{s}_M(m-1)) = 0.$

Lemma 4 $\underset{m\to\infty}{lim}\, X_m = 0.$

The next proposition shows that, for any number of L-bidders, there is always a threshold level of H-bidders above which revenue is maximized by auctioning the H item first. The reason is that the seller's loss in the auction of H due to the unknown signals of the L-bidder becomes small as the auction of H gets sufficiently competitive. This results seems to capture the effect that practitioners call "establishing lively bidding." If the competition for the high value item is sufficiently intense, then the seller should exploit the information spillovers from the sale of the more valuable item, in effect "establishing lively bidding," to raise the price of the less valuable object.

Proposition 5 *For any n there exists m^* such that if $m \geq m^*$ then the seller weakly prefers selling H first to selling L first. If the signals are strictly affiliated, then the seller strictly prefers selling H first.*

The next corollary simply says that if there is no value effect ($\mu_H = \mu_L$), then the seller should auction first the item for which competition is greatest.

Corollary 3 *Assume that the signals are strictly affiliated and $\mu_H = \mu_L$. Then, the seller strictly prefers selling H first to selling L first if and only if $m > n$.*

Under the assumption that the signals are strictly affiliated, I now show that as the number of H-bidders (L-bidders) becomes large, the seller finds it optimal to sell H (L) first, while he is indifferent between selling L (H) first and auctioning the items simultaneously.

Proposition 6 *Assume that the signals are strictly affiliated. Then,*

(*i*) *For any $n < \infty$ selling H first gives the seller the highest revenue while simultaneous sale of the objects and selling L first give the same revenue as $m \to \infty$.*

(*ii*) *For any $m < \infty$ selling L first gives the seller the highest revenue while simultaneous sale of the objects and selling H first give the same revenue as $n \to \infty$.*

The next result is that, if both the number of H-bidders and of L-bidders become large, then the seller is indifferent between either order of sequential sales and simultanoeus auctions.

Proposition 7 *Any order of sequential sales and simultaneous sales give the seller the same revenue as $(m, n) \to (\infty, \infty)$.*

4 Model II: A Positive Number of B-bidders, $r \geq 1$

In this section, I look at the more general model, Model II, in which there are also bidders, the B-bidders, that want both objects. As in the previous section, I start by presenting the benchmark case in which the auctioneer sells the items simultanously.

4.1 A Benchmark: Simultaneous Auctions

When the objects are sold simultaneosly, there is no information transmission between the auctions. Thus, the sale of H is an isolated English auction with $m + r$ bidders, who do not know the signals of the n L-bidders. Similarly, the sale of L is an isolated English auction with $n + r$ bidders, who do not know the signals of the m H-bidders.[8] The expected revenue of the seller follows as a corollary of Proposition 1.

Proposition 8 *The expected payoff to the seller with simultaneous auctions of H and L when there are m H-bidder, n L-bidders and r B-bidders is*

$$R(m,n,r) = R_H(m+r,n) + R_L(m,n+r)$$

4.2 Sequential Auctions

Suppose that item $A \in \{H, L\}$ is sold first. Then, all the bidders demanding only item A and the B-bidders, which demand both items, participate in the first auction. An A-bidder does not worry about revealing his signal to the bidders in the second auction, because he does not compete for the second item. Hence his drop out prices will be increasing in his signal. On the other hand, a B-bidder may not be willing to reveal his signal in the first auction, in order to buy the second item at a favorable price. Therefore, there are two types of potential equilibria of the first auction: a "pooling equilibrium," in which all B-bidders drop out at the same price, irrespective of their signals, and a "separating equilibrium," in which the drop out prices of a B-bidder is an increasing function of his signal.

4.2.1 Pooling Equilibrium

In a pooling equilibrium all B-bidders quit the first auction at the same time at price b. I assume that they quit before any unit-demand bidder drops out. For argument's sake, suppose H is auctioned first. Realizing that the signals of the B-bidders are not incorporated in the sale price, the H-bidders will bid as if they were in a single isolated auction of H with m bidders. That is, all H-bidders with signal 0 drop out at $b^H(0) = E(v_H|\mathbf{s}_M(m))$ and all H-bidders with signal 1 drop out at $b_k^H(1) = E(v_H|\mathbf{s}_M(k))$ where $k = 0, 1, 2, ..., m$ is the number of H-bidders who dropped out at $b^H(0)$.[9]

Now consider the auction of L. Since in the sale of H the bidding function of the H-bidders is increasing in their signals, while all B-bidders pool by quitting right away, the signal profile $\mathbf{s}_M$ is revealed, whereas $\mathbf{s}_R$ is not. Thus, the L-bidders and the B-bidders view the auction of L as a one-shot English auction with $n+r$ bidders having private signals. They use the realized signals of the

[8] I assume that each B-bidder has an agent in each auction and the agents cannot communicate with each other.

[9] In this equilibrium $b < b^H(0)$. There may be other equilibria in which all B-bidders pool and $b \geq b^H(0)$. The reason for focusing on this equilibrium is that it is the only symmetric pooling equilibrium in which all H-bidders bid as in the case in which there are no B-bidders. The reason is the following. First, $b^H(0)$ and $b_k^H(1)$ is always a best response of an H-bidder, no matter at what price all B-bidders quit. Second, a B-bidder with signal 0 gets a negative expected payoff by bidding $b \geq b^H(0)$ if all the other B-bidders also quit at b and all H-bidders drop out according to $b^H(0)$ and $b_k^H(1)$. This is because the B-bidder with signal 0 would end up winning H sometimes, but always at a price higher than the value of the object.

H-bidders to update the expected value of L. Suppose that k H-bidders quit at $b^H(0)$ and $m-k$ H-bidders quit at $b^H_k(1)$. Then, as in Model I, in the L auction all bidders with signal 0 drop out at $b^L_k(0) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_{N\cup R}(n+r))$, and all with signal 1 drop out at $b^L_{k,t}(1) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_{N\cup R}(t))$, where $t = 0, 1, 2, ...n+r$ is the number of bidders who dropped out at $b^L_k(0)$.

Incentives to deceive arise in sequential auctions with multi-unit demand bidders, when bidders have good news concernig the value of the objects.[10] In Model II, it is the B-bidders with signal 1 who have an incentive to hide their signals and bid as if they had signal 0. If a B-bidder with signal 1 bids aggressively in the first auction and thus reveals his signal, he ends up with no private information and hence makes zero profit in the second auction.

Let $R^p_{AB}(m, n, r)$ be the expected revenue to the seller in the pooling equilibrium when the order of sale is AB (where $A, B \in \{H, L\}$), and there are m bidders competing only for H, n bidders competing only for L, and r bidders competing for both H and L.

Proposition 9 *(i) A symmetric "pooling" equilibrium in which all B-bidders pool in the first auction always exists when H is auctioned first and L is auctioned second. The equilibrium payoff to the seller is*

$$R^p_{HL}(m, n, r) = R_{HL}(m, n+r)$$

(ii) A symmetric "pooling" equilibrium in which all B-bidders pool in the first auction always exists when L is auctioned first and H is auctioned second. The equilibrium payoff to the seller is

$$R^p_{LH}(m, n, r) = R_{LH}(m+r, n)$$

The seller's expected revenue in the pooling equilibrium can be expressed using the formulas for revenue from Model I. Suppose, for example, that H is auctioned first. Since all B-bidders drop out before any of the H-bidders, there are only m "effective" bidders (the H-bidders) who compete in the sale of H. On the other hand, since the signals of the B-bidders are not revealed in the first auction, there are $n+r$ effective bidders in the sale of L (the L-bidders plus the B-bidders). Therefore, the seller's revenue in the pooling equilibrium, when H is auctioned first and there are r B-bidders, is the same as the seller's revenue when H is sold first and there are m H-bidders, $n+r$ L-bidders and no B-bidders.

It may seem suprising that the pooling equilibrium, especially when H is auctioned first, always exists regardless of the relative values of the objects and the number of bidders of each type. In

[10] See, for example Hausch (1994), Weber (1983) and Ortega Reichert (1968).

equilibrium, a B-bidder with signal 1 makes zero profit in the auction of H and a positive profit in the auction of L. Naturally, one may wonder why he is willing to give up the profit that he could have made by bidding aggressively in the sale of the more valuable good, in order to make a positive profit in the sale of the less valuable good. To put it differently, why does he not bid agressively in the auction of H to make a gain on H at the cost of revealing his signal and get a zero payoff in the sale of L? The answer follows from the fact that bidders update their perceptions about the other bidders' information during an English auction. In a pooling equilibriums, H-bidders expect all B-bidders to drop out early in the sale of H. If a B-bidder does not drop out along with the other B-bidders, then all H-bidders will infer that he has signal 1 and will bid aggressively, leaving the B-bidder with a zero profit from the first auction. Having revealed his signal, the B-bidder would also make zero profit in the second auction. He thus has no incentive to bid aggressively in the sale of H.

4.2.2 Separating Equilibrium

In a "separating" equilibrium, the bidders that only demand the good auctioned first and the B-bidders reveal their signals in the first auction. I will focus on the symmetric separating equilibrium, even though there may be asymmetric equilibria in which the bidders demanding a single good and the B-bidders follow different strategies in the first auction.

Suppose, for argument's sake, that H is auctioned first. The equilibrium drop-out prices in the symmetric equilibrium of the first auction are the same as in an isolated auction of H with $m + r$ bidders. That is, all H and B-bidders with signal 0 drop out at $b^H(0) = E(v_H|\mathbf{s}_{M\cup R}(m+r))$, and all bidders with signal 1 drop out at $b^H_k(1) = E(v_H|\mathbf{s}_{M\cup R}(k))$, where $k = 0, 1, ..., m+r$ is the number of bidders who dropped out at $b^H(0)$.

Now consider the auction of L. Suppose that in the first auction there were k bidders with signal 0 and $m + r - k$ bidders with signal 1. Since the signals of all B-bidders are revealed in the first auction, the actual competition for L is between the n L-bidders. The L-bidders bid as if there were n bidders with private signals competing for L. That is, all L-bidders with signal 0 drop out at $b^L_k(0) = E(v_L|\mathbf{s}_{M\cup R}(k); \mathbf{s}_N(n))$ and all L-bidders with signal 1 drop out at $b^L_{k,t}(1) = E(v_L|\mathbf{s}_{M\cup R}(k); \mathbf{s}_N(t))$, where $t = 0, 1, 2, ..., n$ is the number of L-bidders who drop out at price $b^L_k(0)$. Now consider a B-bidder. Since his signal was revealed in the auction of H, he cannot win the item L and make a positive payoff. Thus, any bid $b \in [0, b^L_k(0)]$ is a best response to the other B-bidders dropping out at b and all L-bidders following the strategies $b^L_k(0)$ and $b^L_{k,t}(1)$.[11] Thus, $b^L_k(0)$, $b^L_{k,t}(1)$ and any $b \in [0, b^L_k(0)]$ form an equilibrium of the L auction.

[11] For a B-bidder quitting at $b > b^L_k(0)$ is not a best response to $b^L_k(0)$ and $b^L_{k,t}(1)$ when $r \geq 2$ and all the other B-bidders drop out at b. The reason is that a B-bidder may find himself the winner and paying b. In that case, his payoff would be $b^L_k(0) - b < 0$. When $r = 1$, on the other hand, an equilibrium with $b > b^L_k(0)$ may exist. Since I consider a general setting with $r \geq 1$, I focus on $b \leq b^L_k(0)$.

While a pooling equilibrium always exists, a symmetric separating equilibrium may or may not exist. Consider the order HL. In the first auction, the B-bidders bid as if they were in an independent sale of H with $m+r$ bidders. The equilibrium expected payoff is zero for a B-bidder with signal 0, and it is positive for a B-bidder with signal 1. Revealing his signal in the first auction costs nothing to a B-bidder with signal 0, because he cannot make a positive payoff in the second auction even if he conceals his signal. On the other hand, revealing his signal is costly to a B-bidder with signal 1, since he loses his chance of making a positive profit on L. Thus, such a bidder may attemp to bid as if he has signal 0. If he bids so, then the other bidders in the sale of L bid more conservatively, because they believe that he has a low signal. As a result, a B-bidder with signal 1 who bids as if he had signal 0 in the sale of H makes a positive payoff in the L auction. However, bidding like a bidder with a low signal in the first auction is costly, because either the bidder does not win H or he wins H with a tied bid with all the other bidders. Thus, a symmetric separating equilibrium exists when the order is HL if and only, for a B-bidder with signal 1, the cost of mimicking a B-bidder with signal 0 is higher than the gain.

I now provide a necessary and sufficient condition for the existence of a separating equilibrium when the order is HL:

Condition $\mathbf{S}_{\mathrm{HL}}$:

$$\frac{1}{m+n+r}\mu_L p(\mathbf{s}_N(n)|s_j=1) \leq \frac{m+r-1}{m+r}\mu_H\left[\frac{nX_{m+r}}{p(s_j=1)}+p(\mathbf{s}_{M\cup R\setminus\{j\}}(m+r-1)|s_j=1)\right]$$

Consider a B-bidder with signal 1. If he bids in the first auction as if his signal were 0, then the LHS of condition S_{HL} is the gain of such a bidder in the second auction, while the RHS is his cost in the first auction.

Similarly, the necessary and sufficient condition for the existence of separating equilibrium when the order is LH is

Condition $\mathbf{S}_{\mathrm{LH}}$:

$$\frac{1}{m+n+r}\mu_H p(\mathbf{s}_M(m)|s_j=1) \leq \frac{n+r-1}{n+r}\mu_L\left[\frac{mX_{n+r}}{p(s_j=1)}+p(\mathbf{s}_{N\cup R\setminus\{j\}}(n+r-1)|s_j=1)\right]$$

Let $R^S_{AB}(m,n,r)$ be the expected revenue to the seller in the symmetric separating equilibrium when the order is AB (where $A,B\in\{H,L\}$), there are m bidders who want to buy only H, n bidders who want to buy only L, and r bidders who wants to buy both H and L.

Proposition 10 *(i) A symmetric equilibrium with all B-bidders separating in the first sale exists when H is auctioned first if and only if condition S_{HL} holds.*

The expected payoff to the seller is

$$R^S_{HL}(m,n,r) = R_{HL}(m+r,n).$$

(ii) A symmetric equilibrium with all B-bidders separating in the first stage exists when L is auctioned first if and only if condition S_{LH} holds.

The expected payoff to the seller is

$$R^S_{LH}(m,n,r) = R_{LH}(m,n+r).$$

As in the pooling equilibrium, the seller's revenue in the separating equilibrium can be expressed by using the revenue formulas from Model I. The B-bidders compete aggresively in the first auction and then drop out early and make no profit in the second auction, because their signals have become public. Thus, the B-bidders only affect the sale price of the item that is auctioned first. If H is auctioned first, then the seller's revenue is the same as if there were no B-bidders, $m+r$ H-bidders in the first auction and n L-bidders in the second auction. If H is auctioned second, then the seller's revenue is the same as if there were no B-bidders, $n+r$ L-bidders in the first auction and m H-bidders in the second auction[12].

4.3 The Optimal Order of Sale

Even though the existence of multiple equilibria makes it more difficult for the seller to determine the optimal order of sale, I will now show that the main insights from Model I remain valid when there are B-bidders.

The seller's revenue in the pooling and in the separating equilibrium is $R_{HL}(m,n+r)$ and $R_{HL}(m+r,n)$, respectively, when the order of sale is HL; it is $R_{LH}(m+r,n)$ and $R_{LH}(m,n+r)$ when the order of sale is LH. Thus, the number of effective bidders in the sale of $A \in \{H,L\}$ is different in the two equilibria. Even though they wish to buy both H and L, B-bidders effectively compete only in one of the auctions. In the pooling equilibrium they compete in the second auction, while in the separating equilibrium they compete in the first auction.

In Model I, I showed that the seller should auction first the item with a sufficiently higher number

[12] Another possible scenario is that the seller auctions off the items only as a bundle. In that case, I find that there is no symmetric separating equilibrium in pure strategies. The intuition is as follows. First note that in such an equilibrium all types of bidders with signal 0 should drop out at the price of zero, and they all make a payoff of zero. Otherwise, a bidder with signal 0 may find himself a winner paying more than the worth of the object when all the other bidders have signal 0. Now suppose that instead of droping out at the price of zero, an H-bidder with signal 0 plays the following strategy: he waits until the price is ϵ. Then, he quits immediately if all the other bidders have quit already or there is at least one H or B-bidder who has not quit at the price of zero. If, on the other hand, all H-bidders have dropped out at the price of zero, then he quits right after all L-bidders drop out. With this strategy, an H-bidder with signal 0 makes either a payoff of zero or a positive payoff. Thus, he deviates.

of bidders. The reason was that the seller's loss on the unknown signals of the bidders who do not compete for an item gets smaller as the auction becomes more competitive. The counterpart of this result in the presence of B-bidders is that the auction of the object with a sufficiently higher number of effective bidders should be held first, since it is the number of effective bidders in an auction which determines the degree of competition.

If the number of H-bidders is sufficiently high relative to the number of B and L-bidders, then the auction of H becomes more competitive both in the pooling and the separating equilibrium. This is because H-bidders always compete in the sale of H, regardless of the type of equilibrium. As the next proposition shows, as the number of H-bidders increases, the seller's revenue in both equilibria when the order of sale is HL is greater than the revenues in the pooling and the separating equilibrium when the order is LH. Similarly, when the number of L-bidders becomes high relative to the number of H and B-bidders, the auction of L gets highly competitive in both equilibria and the order of sale LH yields the seller a higher revenue, no matter which equilibrium prevails.

Proposition 11 *Assume that the signals are strictly affiliated. Then,*

(i) For any finite pair $(m, r) < (\infty, \infty)$ there exists an n^* such that, for all $n \geq n^*$, selling L first yields the seller a higher revenue than selling H first, or selling the items simultaneously.

(ii) For any finite pair $(n, r) < (\infty, \infty)$ there exists an m^* such that, for all $m \geq m^*$, selling H first gives the seller a higher revenue than selling L first, or selling the items simultaneously.

5 Conclusions

Most auctioneers use standard auction formats to sell multiple non-identical goods sequentially. While some sellers auction their items in the order of declining value, others do the opposite. The current paper addresses and analyzes this real life phenomena. The main message of this research is that the sequence of sale affects the auctioneer's expected revenue through the information content spilling over from the early auction to the later one. I have found that in an environment with only unit demand bidders the auctioneer should sell the less valuable good first to "warm up the room," when the number of bidders who wish to buy the more valuable item is not greater than the number of bidders competing for the less valuable item. On the other hand, when the number of bidders who want to buy the high value item is sufficiently high, then the seller should auction the high value good first to "establish lively bidding." I have also found that when there are multi-unit demand bidders as well, similar results still hold. My findings suggest an explanation for why in practice we observe different orders of sales. They also provide guidelines to actual sellers about how to choose the order of sale.

Chapter II

Voluntary Disclosure to Form a Research Joint Venture

1 Introduction

Innovation is an uncertain process. A firm's success rate of inventing a new product typically increases as its know-how increases. It often happens that two or more competing firms try to develop the same innovation. A research joint venture (RJV) is a form of inter-firm cooperation to share know-how in order to increase the chances of successful innovation. They are widely observed in many industries such as software, computer hardware, biotechnology and telecommunications. In spite of the potential advantages of forming a RJV, there are many cases in which potentially successful RJV's are not formed. One important reason is that know-how is private information. Under asymmetric information, the parties may not reach an agreement on how the value created by the venture is going to be shared. In this case, pre-contract disclosures become necessary. In other words, a firm must disclose some of its know-how to inform a potential partner about the value of the intellectual property (IP) that it would bring into a RJV to obtain favorable transaction terms. However, by disclosing its knowledge a firm exposes itself to expropriation: disclosed know-how enhances the likely performance of the competitor and cannot be protected under weak or absent intellectual property rights regimes. Thus, the potential partner may appropriate the disclosed information, improving its bargaining position, and hence decreasing the firm's expected gains from cooperation.

The threat of expropriation not only prevents formation of RJVs but also causes them to be highly unstable. Even if the parties agree on the ownership shares, perhaps by the help of a disinterested third party or some default sharing arrangements such as 50:50, some RJVs collapse before completing their task. The reason is that partners may renege on their promises to share their know-how while learning the others' knowledge in order to use it for their own invention of the product. For example, Baker and Mezzetti (2001) mention that the venture between Advent Inc., a small R&D firm and Unisys, a computer company, to develop a marketable software document management systems broke up two years after it was formed because Unisys decided to develop the product itself.

The possibility of this type of opportunistic behavior may give the partners disincentives to disclose their IP to the venture, hence causing the venture to break down. In some cases, the fear that the other firm will act opportunistically is so severe that some potentially profitable RJVs are not even formed in the first place. Perez-Castrillo and Sandonis (1996) point out that a possible venture between US aircraft manufacturers and Japanese firms on the Boeing 767 did not start because each firm feared disclosing its know-how without learning the other's.

Disclosure contingent contracts may ease the problem of moral hazard associated with disclosure. However, they are not feasible in real life due to such contracts being incomplete and know-how being unverifiable by third parties.

The current paper deals with the problems of asymmetric information and moral hazard regarding know-how that RJVs face. In particular, I develop a decentralized procedure which does not rely on disclosure contingent contracts and the verifiability of know-how by third parties. In fact, my procedure alleviates the asymmetric information problem and encourages firms to form a RJV. Moreover, in my setting once the venture is established the problem of moral hazard disappears.

I analyze, in a stylized model, the tradeoff for a firm between disclosing its know-how to inform a potential partner of its research ability, and exposing itself to expropriation. In model, there are two firms, each having its own research lab. The firms have complementary know-how that is useful for a potential innovation. A firm's research knowledge cannot be observed by its potential partner, but firms may disclose some of their know-how to each other. Any revealed idea is perfectly appropriable by the receiver, and increases the receiver's stock of knowledge. In my framework, firms simultaneously disclose information in rounds. If both firms disclose in any round, firms are expected to disclose again in the next round. This process continues until one firm stops disclosing. The firm which made the last disclosure then makes a take-it-or-leave-it offer to the other firm to acquire the firm's research lab. If the offer is accepted, a RJV with two research labs is formed, and the party that made the offer becomes the sole owner.

I investigate a Perfect Bayesian Equilibrium in which a firm discloses at any round whenever it has any remaining undisclosed know-how. I show that such a "Full Disclosure" equilibrium exists. The reason that the firms keep disclosing to each other, in spite of the risk of being expropriated, is that each firm wants to make as low as possible an offer to the other party and it wants to receive as high as possible an offer from his opponent. A Full Disclosure equilibrium exists because the cost of disclosure is offset by the future gains from signaling that the firm is a strong competitor. Moreover, in the equilibrium whenever the RJV is formed, the firm with more knowledge becomes the owner since the party with lower IP runs out of know-how to disclose at some stage. Once the venture is established, there are no moral hazard issues associated with disclosure since only one firm becomes the residual claimant of the venture and the other firm which "sells" its research lab has already disclose everything it has.

Despite the many advantages of RJVs, including the higher success rate in achieving an innovation, many academics and policy makers are concerned that RJVs lead to the monopolization of markets. Therefore, I additionally consider the effects of RJVs on social welfare in my model. I show that the benefits of a higher success rate of innovation dominates the cost of monopoly, and hence RJVs are socially desirable.

Coordinating research through an exchange of know-how in RJVs has been investigated in the literature before. Several works approach the problem from a mechanism design point of view. Those models involve a planner and disclosure-contingent transfer payments. Bhattacharya, Glazer and Sappington (1992) develop licensing mechanisms to implement efficient sharing of know-how and efficient efforts in a RJV when each firm's research ability is private information. Unlike my setting, their model assumes that firms' know-how is Blackwell-ordered so only the most knowledgeable firm is pivotal. D' Aspremont, Bhattacharya and Gerard-Verat (1998) extends Bhattarcharya et al's work by including an arbitrary knowledge spillover function. In a two-firm setting, D' Aspremont, Bhattacharya and Gerard-Verat (2000) first consider a direct bargaining mechanism which induces full disclosure. Second, they show that full disclosure can be implemented via infinite horizon sequential bargaining. However, in their model there is one-sided incomplete information while my paper deals with two-sided asymmetric information. Also, unlike in my procedure, a firm can make a disclosure-contingent offer in their sequential bargaining game.

Another branch of the RJV literature addresses the instability of RJVs due to the problem of moral hazard associated with disclosing knowledge. In a complete information setting, Veugelers and Kesteloot (1994) show that a two-firm RJV with equal ownership is more likely to be stable when synergy effects are high and know-how of the venture is sufficiently proprietary. Instead of assuming 50:50 shares, Perez-Castrillo and Sandonis (1996) analyze possible incentive compatible and renegotiation proof contracts between two firms inducing full disclosure after the venture is formed. In contrast to my model, their setting assumes that the firms' research abilities are common knowledge so they focus on moral hazard issues only.

The current paper is most closely related to the strand of the literature which analyzes pre-contract disclosure strategies of privately informed parties for the purpose of signaling the extent of their information. Bhattacharya and Ritter (1983) study how a firm can signal its value to the capital market in order to obtain favorable financing terms by publicly disclosing part of its valuable IP. The down-side of disclosure is disclosed knowledge becomes available to competitors increasing their innovation ability. Anton and Yao (1994) investigate whether pre-sale full disclosure is a signaling equilibrium in a model with an innovator who wants to sell his idea when the disclosed information can be appropriated by the potential buyers. In a similar setting, Anton and Yao (2002) consider the possibility of an equilibrium with partial disclosure. The sale of ideas with pre-transaction disclosures is also explored in Ielceanu (2003) when there are two sellers and one buyer. These works are similar to mine in that the party with private information may have an incentive disclose its knowledge to obtain favorable transaction terms despite of the threat of expropriation. However, there is one important difference: in contrast to those models, I focus on a two-sided incomplete information environment.

The rest of the paper is organized as follows. In Section 2, I describe the decentralized sequential disclosure game. In Section 3, I investigate whether "Full Disclosure" equilibrium of the game exists. I also check if the firms are willing to participate in this procedure. In Section 4, I analyze the effects of RJVs on social welfare. Finally, Section 5 concludes.

2 The Model

Two risk neutral firms, each having its own research lab, are trying to obtain the same product innovation. Firm i ($i = 1, 2$) has a stock of knowledge θ_i which is its private information. The value of θ_i is the probability that firm i succesfully invents. The know-how levels θ_1 and θ_2 are drawn independently from a probability density p with support $\Theta = \{0, \Delta, 2\Delta, ...T\Delta\}$ and this is common knowledge. For simplicity I assume $p(n\Delta) = \frac{1}{T+1}$ for all $n = 0, 1, ..., T$.

There are four possible outcomes of R&D stage: Both firms succeed, both fail, and only firm i achieves the invention. In the latter case, firm i becomes the monopolist in the product market and earns a profit of Π_m. On the other hand, if both firms obtain the innovation, each firm makes a profit of zero since I assume homogeneous product Bertrand Competition in the product market. An unsuccesful firm always earns a payoff of zero. Therefore, firm i 's expected payoff is $\Pi_m\theta_i(1 - \theta_j)$ when its know-how is θ_i and the opponent's is θ_j.

Each firm's knowledge can be partially or fully transferred to its competitor through a disclosure of information. Once a firm reveals its know-how, it can not take it back, and the competitor can freely use it. I assume that the firms have perfectly complementary technologies. That is, if firm i discloses $d \leq \theta_i$ to firm j, the probability of successful innovation by firm j becomes $d + \theta_j$[1].

Since the technologies are perfectly complementary, one may think that it may be in the firms' mutual interest to form a Research Joint Venture (RJV) and fully share their know-how. In fact, in my model there are always gains from forming a RJV. To see this first note that there would be two research labs in the RJV, each having a stock of knowledge $\theta_1 + \theta_2$ after full disclosure. So, the probability that the RJV obtains the innovation is $1 - [1 - (\theta_1 + \theta_2)]^2$ where the expression in the bracket is the probability of failure with one research lab. Thus, the value of the RJV is

$$\Pi_m(1 - [1 - (\theta_1 + \theta_2)]^2) = \Pi_m[2(\theta_1 + \theta_2) - (\theta_1 + \theta_2)^2]$$

Therefore, the gains from the RJV are

$$\begin{aligned}&\Pi_m[2(\theta_1 + \theta_2) - (\theta_1 + \theta_2)^2] - \Pi_m\theta_1(1 - \theta_2) - \Pi_m\theta_2(1 - \theta_1)\\&= \Pi_m(\theta_1 + \theta_2 - \theta_1^2 - \theta_2^2)\end{aligned}$$

which is always positive since $\theta_i \leq \frac{1}{2}$ for $i = 1, 2$.

I now describe a game which may lead to an efficient formation of a RJV. The rules are as follows. At any time t ($t = 1, ..., T - 1, T$) there is a simultaneous disclosure game in which each firm has two

[1] I assume $\Delta T \leq \frac{1}{2}$ to ensure that the probabilities do not exceed 1.

possible actions: disclose Δ and no disclosure. If both firms disclose at t, then a new disclosure game starts at $t+1$ (except for $t = T$). If no firm discloses at any round, then the game terminates, and each firm tries to obtain the innovation on its own. If, on the other hand, only firm i discloses at t, firm i makes a take-it-or-leave-it offer, a monetary transfer, to firm j in exchange for acquiring firm j's research lab. If the offer is accepted, the RJV is formed, and firm i becomes the owner. Otherwise, each firm tries to invent alone.

I now specify a rule for the case that both firms make a disclosure at $t = T$. In that case, I employ Nash Bargaining Solution at the last round. That is, the firms equally share the gains from the RJV

I solve this dynamic game of incomplete information for a Perfect Bayesian Equilibrium (PBE). A PBE consists of strategies and the beliefs such that the strategies are optimal given beliefs, and the beliefs are determined by Bayes rule and the players' equilibrium strategies whenever possible. In this game, strategic options for firm k ($k = 1, 2$) with type θ_k at time $T - i$ (where $i = 0, 1, ..., T - 1$ and $\theta_k \geq (T - i)\Delta)^2$ are as follows. Its disclosure strategy is given by $\sigma^k_{T-i} : \Theta \to \{\Delta, 0\}$ where Δ is short for "disclose Δ" and 0 is for "no disclosure". At the bargaining game firm k follows $O^k_{T-i} : \Theta \to \Re^+$ when it is to make an offer, and $R^k_{T-i} : \Theta \to \{accept, reject\}$ when it is to respond to an offer.

I focus on PBE in which the sequential disclosure strategy is "separating". That is, a firm discloses Δ at any stage t if it has an undisclosed Δ at t. I call this "Full Disclosure" equilibrium. Since the beliefs should be consistent with equilibrium strategies, the beliefs are the following: If a firm does not disclose at round t, the opponent believe that it is because the firm has already disclosed all its know-how through $t - 1$.

3 Full Disclosure Equilibrium

In this section, I derive the Full Disclosure equilibrium. I start by investigating the equilibrium of the bargaining game at any round. Then, I examine whether disclosure is an optimal strategy at any stage for a firm whenever it has undisclosed Δ.

I now formally define the beliefs that would support "Full Disclosure" equilibrium. Since firms are symmetric, w.l.o.g. I focus on firm 1. Let the probability $\mu_{T-i}(\theta_2 = (T - i - 1)\Delta|\sigma^2_{T-i} = 0)$ denote the firm 1's beliefs about firm 2's know-how θ_2 upon observing that firm 2 has not disclosed at stage $T - i$. Then, the beliefs on the equilibrium path are $\mu_{T-i}(\theta_2 = (T - i - 1)\Delta|\sigma^2_{T-i} = 0) = 1$ for all $i = 0, 1, ..., T - 1$. That is, after observing no disclosure by its opponent firm 1 believes that firm 2 has no remaining undisclosed information. In other words, firm 2's type is $(T - i - 1)\Delta$ for sure.

I begin the equilibrium analysis of the dynamic game by examining the equilibrium of the bargaing

[2] I represent the rounds in the declining order for simplicity of notation. Also, note that firm k can have an option to disclose Δ at round t if only if its know-how θ_k is at least $t\Delta$.

game at stage $T-i$ when the outcome of disclosure game at $T-i$ is $(\sigma^1_{T-i}=\Delta,\ \sigma^2_{T-i}=0)$.

Consider round $T-i$. Since firm 2 has not disclosed at $T-i$, firm 1 believes that its competitor's type is $\theta_2=(T-i-1)\Delta$ for sure. On the other hand, from firm 2's point of view firm 1 could be any type θ_1 in $\Theta_{T-i}=\{(T-i)\Delta,(T-i+1)\Delta,......,(T-1)\Delta,T\Delta\}$. Note that the firms have disclosed Δ to each other through the stage $T-i-1$, and firm 1 has disclosed Δ to firm 2 at $T-i$. Thus, know-how levels of firm 1 and firm 2 after the disclosure game at $T-i$ are $\theta_1+(T-i-1)\Delta$ and $\theta_2+(T-i)\Delta$, respectively. Therefore, firm 2's expected outside option after the disclosure game but before the bargaining at $T-i$ is

$$\begin{aligned}O_{T-i} &= E\left[\Pi_m(2T-2i-1)\Delta(1-\theta_1-(T-i-1)\Delta)|\theta_1\in\Theta_{T-i}\right]\\ &= \Pi_m(2T-2i-1)\Delta(1-\Delta(2T-\frac{3}{2}i-1))\end{aligned}$$

Note that O_{T-i} is the lowest amount that firm 2 with type $(T-i-1)\Delta$ would accept.

The following lemma describes an equilibrium of the bargaining game at $T-i$.

Lemma 1 *Assume that the beliefs are* $\mu_{T-i}(\theta_k=(T-i-1)\Delta|\sigma^l_{T-i}=0)=1$ *for* $k,l=1,2$, *and the outcome of the disclosure game at stage* $T-i$ *is* $(\sigma^1_{T-i}=\Delta,\ \sigma^2_{T-i}=0)$. *Then,* $(O_{T-i},\ R_{T-i}(\theta_2))$ *is an equilibrium of the bargaining game at* $T-i$ *for all* $i=0,1,...,T-1$ *where*

$$\begin{aligned}O_{T-i} &= \Pi_m(2T-2i-1)\Delta(1-\Delta(2T-\frac{3}{2}i-1))\\ R_{T-i}(\theta_2) &= \begin{cases}\textit{accept} & \textit{if the offer is at least } \Pi_m(\theta_2+(T-i)\Delta)(1-\Delta(2T-\frac{3}{2}i-1))\\ \textit{reject} & \textit{otherwise}\end{cases}\end{aligned}$$

In the equilibrium, when firm 1 discloses Δ and firm 2 does not disclose at round $T-i$, the payoff to firm 1 with type θ_1 at the end of the bargaining game is

$$P_{T-i}(\theta_1)=RJV_{T-i}(\theta_1)-O_{T-i}$$

where

$$RJV_{T-i}(\theta_1)=\Pi_m[2(\theta_1+(T-i-1)\Delta)-(\theta_1+(T-i-1)\Delta)^2]$$

is the value of the RJV to firm 1 when it is formed at stage $T-i$, and O_{T-i} is amount it pays firm 2.

Consider the disclosure game of stage $T-i$ $(i=1,2,...,T-1)$.[3] Assume $\theta_1=(T-j)\Delta$ where

[3] Since I employ Nash Bargaining solution when the game ever reaches round T and both firms disclose Δ at T, the formulas for the equilibrium payoffs in the last stage are different. Thus, I will investigate round T, hence the equilibrium behavior of type $T\Delta$ later.

$1 \leq j \leq i$.[4] Prior to the disclosure game firm 1 knows that its opponent's type θ_2 is in the set $\{(T-i-1)\Delta, (T-i)\Delta, (T-i+1)\Delta, ..., (T-1)\Delta, T\Delta\}$, and each type is equally likely.

Assume that firm 1 discloses Δ at stage $T-i$. Then, it makes the offer O^1_{T-k} at stage $T-k$ with probability $\frac{1}{i+2}$ where $k = j, j+1, ..., i$. Since in the equilibrium the opponent accepts O_{T-k}, the RJV is formed and firm 1 becomes the owner at stage $T-k$. Thus, firm 1 makes $P_{T-k}((T-j)\Delta) = RJV_{T-k}((T-j)\Delta) - O_{T-k}$ at $T-k$ with probability $\frac{1}{i+2}$ for all $k = j, j+1, ..., i$.

Now suppose that the game comes to stage $T-j+1$.[5] This means that firm 2's type is at least $(T-j)\Delta$. Assume that $\theta_2 = (T-j)\Delta$ which would happen with probability $\frac{1}{i+2}$ In this case the firms do not disclose, and each tries to innovate on its own. In other words, each firm gets its outside option at $T-j+1$. Thus the payoff to firm 1 with type $(T-j)\Delta$ is

$$OP_{T-j+1}((T-j)\Delta) = \Pi_m(2T-2j)\Delta(1-(2T-2j)\Delta)$$

Now assume that $\theta_2 > (T-j)\Delta$ which occurs with probability $\frac{j}{i+2}$. In this case, firm 1 receives the offer O_{T-j+1}, and accepts it. Thus, its payoff is

$$O_{T-j+1} = \Pi_m(2T-2(j-1)-1)\Delta(1-\Delta(2T-\frac{3}{2}(j-1)-1))$$

Therefore, if type $(T-j)\Delta$ discloses Δ at stage $T-i$, its expected payoff is

$$EP_{T-i}((T-j)\Delta) = \Pi_m[\{\sum_{k=j}^{i} \frac{1}{i+2} P_{T-k}((T-j)\Delta)\} + \frac{1}{i+2} OP_{T-j+1}((T-j)\Delta) + \frac{j}{i+2} O_{T-j+1}]$$

Instead assume that firm 1 with type $(T-j)\Delta$ does not disclose at stage $T-i$. Note at $T-i$ we have $\theta_2 = (T-i-1)\Delta$ with probability $\frac{1}{i+2}$ and $\theta_2 \geq (T-j)\Delta$ with probability $\frac{i+1}{i+2}$. If $\theta_2 = (T-i-1)\Delta$, firm 1 gets its outside option

$$\overline{OP}_{T-i}((T-j)\Delta) = \Pi_m(2T-j-i-1)\Delta(1-(2T-2i-2)\Delta)$$

since firm 2 does not disclose at $T-i$, either, and hence each firm tries to obtain the innovation alone. On the other hand, if $\theta_2 \geq (T-j)\Delta$, then firm 1 receives the offer $O_{T-i} = \Pi_m(2T-2i-1)\Delta(1-\Delta(2T-\frac{3}{2}i-1))$. However, it rejects the offer since its expected outside option $\Pi_m(2T-i-j)\Delta(1-\Delta(2T-\frac{3}{2}i-1))$ is greater than O_{T-i}.

Therefore, if type $(T-j)\Delta$ deviates to no disclosure at round $T-i$, its expected payoff is

[4] Note that only the types $(T-i)\Delta, (T-i+1)\Delta, ..., T\Delta$ of firm 1 have an undisclosed Δ at round $T-i$.

[5] $T-j+1$ is the last round the game can forward when $\theta_1 = (T-j)\Delta$.

$$D_{T-i}((T-j)\Delta) = \Pi_m[\frac{1}{i+2}(2T-j-i-1)\Delta(1-\Delta(2T-2i-2))+ \frac{i+1}{i+2}(2T-i-j)\Delta(1-\Delta(2T-\frac{3}{2}i-1))]$$

Thus, we have that type $(T-j)\Delta$ has no incentive to deviate to no disclosure at round $T-i$ if and only if $EP_{T-i}((T-j)\Delta) \geq D_{T-i}((T-j)\Delta)$.

The proof of next lemma shows that the deviation payoff is not greater than the equilibrium payoff to type $(T-j)\Delta$ at stage $T-i$ for $j = 1, 2, ...i$ and $i = 1, 2, ..T-1$. Therefore, disclosing Δ at any stage as long as it has undisclosed Δ is indeed the optimal strategy for type $(T-j)\Delta$ where $j = 1, 2, ...i$.

Lemma 2 *Type $(T-j)\Delta$ discloses Δ at stage $T-i$ where $j = 1, 2, ..., i$ and $i = 1, 2, .., T-1$. Its expected payoff at $T-i$ is*

$$EP_{T-i}((T-j)\Delta) = \Pi_m[\{\sum_{k=j}^{i} \frac{1}{i+2} P_{T-k}((T-j)\Delta)\} + \frac{1}{i+2} OP_{T-j+1}((T-j)\Delta) + \frac{j}{i+2} O_{T-j+1}] \quad (1)$$

I now examine equilibrium strategy of type $T\Delta$. Suppose that firm 1 with type $T\Delta$ finds itself at the last round T and both firm 1 and its opponent disclose Δ at T. This means that firm 2 has type $T\Delta$ for sure. Then, the value of the RJV if formed is $\Pi_m[4T\Delta - 4T^2\Delta^2]$. Note that each firm has an outside option of $\Pi_m 2T\Delta(1-2T\Delta)$ after both disclose at T. Thus, the gains from the RJV at round T is $\Pi_m 4T^2\Delta^2$. Since they split the surplus 50:50 (Nash Bargaining), the payoff to firm 1 of type $T\Delta$ at stage T is $\Pi_m[2T\Delta - 2\Delta^2T^2]$. Therefore, the equilibrium payoff to type $T\Delta$ at stage $T-i$ is

$$EP_{T-i}(T\Delta) = \Pi_m[\{\sum_{k=o}^{i} \frac{1}{i+2}(P_{T-k}(T\Delta)\} + \frac{1}{i+2}(2T\Delta - 2\Delta^2T^2)]$$

The lemma below shows that disclosure at each round is an optimal strategy for type $T\Delta$, too.

Lemma 3 *Type $T\Delta$ discloses Δ at stage $T-i$ where $i = 0, 1, 2, .., T-1$. Its expected payoff at $T-i$ is*

$$EP_{T-i}(T\Delta) = \Pi_m[\{\sum_{k=0}^{i} \frac{1}{i+2} P_{T-k}(T\Delta)\} + \frac{1}{2}(2T\Delta - 2\Delta^2T^2)] \quad (2)$$

Having establish that a firm discloses at any round whenever he has undisclosed knowledge, I now derive the equilibrium payoff to firm 1 with type θ_1. First consider $\theta_1 = (T-j)\Delta$ where $j = 1, 2, 3, ..., T-1$. We can use the formula (1) to find the firm's expected payoff at the first round.

By plugging $i = T-1$, the equilibrium expected payoff to firm 1 with type $\Delta(T - j)$ is

$$EP((T-j)\Delta) = \Pi_m[\frac{1}{T+1}\{\sum_{k=j}^{T-1} P_{T-k}((T-j)\Delta)\} + \frac{1}{T+1}\Delta(2T-2j)(1-\Delta(2T-2j))$$
$$+\frac{j}{T+1}\Delta(2T-2(j-1)-1)(1-\Delta(2T-\frac{3}{2}(j-1)-1))]$$

Now assume that $\theta_1 = T\Delta$. Similarly, by plugging $i = T-1$ in (2), we have that the equilibrium expected payoff to type $T\Delta$ is

$$EP(\Delta T) = \Pi_m[\frac{1}{T+1}\{\sum_{k=0}^{T-1} P_{T-k}(\Delta T)\} + \frac{1}{T+1}(2T\Delta - 2\Delta^2 T^2)]$$

The lemma below summarizes the findings:

Lemma 4 *The equilibrium payoff to type $(T-j)\Delta$ is*

$$EP((T-j)\Delta) = \Pi_m[\frac{1}{T+1}\{\sum_{k=j}^{T-1} P_{T-k}((T\Delta)\} + \frac{1}{T+1}\Delta(2T-2j)(1-\Delta(2T-2j))$$
$$+\frac{j}{T+1}\Delta(2T-2(j-1)-1)(1-\Delta(2T-\frac{3}{2}(j-1)-1))]$$

where $j = 1, 2, ..., T-1$. The equilibrium payoff to type $T\Delta$ is

$$EP(\Delta T) = \Pi_m[\frac{1}{T+1}\{\sum_{k=0}^{T-1} P_{T-k}(\Delta T)\} + \frac{1}{T+1}(2T\Delta - 2\Delta^2 T^2)]$$

At this point one may ask if the firms are willing to participate in this game once they receive their private information. To answer this question first note that type θ_1's outside option before the game starts is $O(\theta_1) = E(\Pi_m\theta_1(1-\theta_2)|\theta_2 \in \Theta)$. So, the game is individually rational if and only if the equilibrium payoff of each type θ_1 is not less than his outside option $O(\theta_1)$. The next lemma shows that it is indeed true.

Lemma 5 *In the equilibrium each type gets at least its outside option $O(\theta_1) = E(\Pi_m\theta_1(1-\theta_2)|\theta_2 \in \Theta)$.*

We can now formally state the equilibrium. Consider the following strategy of firm k at stage $T-i$ where $i = 0, 1, ..., T-1$

$$\sigma^k_{T-i}(\theta) = \sigma_{T-i}(\theta) = \begin{cases} \Delta & \text{if } \theta \geq (T-i)\Delta \\ 0 & \text{otherwise} \end{cases}$$

$$O_{T-i}^{k} = O_{T-i} = \Pi_m(2T - 2i - 1)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1)) \text{ for } i = 0, 1, ..., T - 1$$

$$R_{T-i}^{k}(\theta) = R_{T-i}(\theta) = \begin{cases} accept & \text{if the offer} \geq \Pi_m(\theta + (T-i)\Delta)(1 - \Delta(2T - \frac{3}{2}i - 1)) \\ reject & \text{otherwise} \end{cases}$$

It is clear by lemmas 1-5, that the strategy profile $\{(\sigma_{T-i}^{k}; O_{T-i}^{k}; R_{T-i}^{k})\}_{0 \leq i \leq T-1}^{k=1,2}$ and the beliefs $\{\mu_{T-i}(\theta_k = (T - i - 1)\Delta | \sigma_{T-i}^{l} = 0) = 1\}_{0 \leq i \leq T-1}^{k,l=1,2}$ is an PBE of the dynamic game of incomplete information. I call this "Full Disclosure" equilibrium.

Proposition 1 *There is a "Full Disclosure" equilibrium.*

The intuition for Full Disclosure equilibrium is as follows. At any round there are two consequences for a firm associated with disclosing Δ. First, revealed Δ may win the firm a right to make an offer and signals the firm's privately observed know-how. Note that a RJV is always efficient in my model. Thus, being eligible to make a take-it-or-leave offer is highly rewarding for a firm because it can get all the residual gains from the RJV. Also, a firm with high level of knowledge wants to separate itself from those with low level of knowledge for two reasons: it wants to offer the competitior as little as possible when it makes an offer and wants to extract from the opponent as much as possible when it receives an offer. Second, disclosed Δ can be used by its competititor with no legal consequences, decreasing the firm's payoff when its competitor innovate outside of the RJV. The full Disclosure equilibrium exists because the cost of disclosing is offset by the future gains from being qualified to make an offfer and signaling that the firm is a strong competititor.

The equilibrium has interesting features. First, the value of the RJV, if formed, is shared somewhat depending on firms' stock of knowledge. In particular, the firm with higher know-how obtains most of the gains while the other firm gets its expected outside option. Thus, this game could be a solution for firms who want to cooperate but can not agree on what the "fair" shares are under incomplete information. The game ensures that whoever has more information gets more surplus. Second, there are no moral hazard issues in the RJV. This is because whenever the RJV is formed, only one firm has possibly undisclosed know-how, and that party is the owner.

4 Social welfare

Even though the success rate in achieving an innovation is higher with a RJV, there is a social cost of a RJV due to monopolization of the market. In order to evaluate social costs and social benefits of a RJV, I consider a demand function in the consumer good market $q = \frac{1}{P^a}$ where $a > 1$ is the constant elasticity of the demand. I also assume marginal cost of production is c.

Suppose that there is only one firm in the market. Then, the monopolist sets its price at $P_m = \frac{ca}{a-1}$. Thus, the monopolist's profit is

$$\begin{aligned}\Pi_m &= (P_m - c)q(P_m) \\ &= \frac{(a-1)^{a-1}}{c^{a-1}a^a}\end{aligned}$$

and the consumer surplus under monopoly is

$$CS_m = \int_{\frac{ca}{a-1}}^{\infty} \frac{1}{P^a} dP = \frac{(ca)^{1-a}}{(a-1)^{2-a}}$$

Suppose instead that there are two firms in the consumer product market. Since the firms are Bertrand competitiors, each makes a profit of zero while the consumer surplus is

$$CS_b = \int_{c}^{\infty} \frac{1}{P^a} dP = \frac{c^{1-a}}{a-1}$$

Assume that there is a RJV in the market[6]. Since the probability of succesfull innovation in a RJV is $2(\theta_1+\theta_2) - ((\theta_1+\theta_2)^2$ and there would be only one firm in the product market, the expected social welfare with a RJV becomes

$$W_{RJV} = E([2(\theta_1+\theta_2) - ((\theta_1+\theta_2)^2] \left[\Pi_m + CS_m\right] |\theta_1, \theta_2 \in \Theta) \tag{3}$$

Assume instead that there is no RJV. If only one firm invents, which happens with the probability $\theta_1(1-\theta_2)+\theta_2(1-\theta_1)$, the social welfare is $\Pi_m + CS_m$. If, on the other hand, both firms achieves the innovation, which occur with probability $\theta_1\theta_2$, the social surplus is CS_B. Thus, the expected social welfare without a RJV is

$$W_{no} = E(\{[\theta_1(1-\theta_2)+\theta_2(1-\theta_1)][\Pi_m + CS_m] + \theta_1\theta_2 CS_B\} | \theta_1, \theta_2 \in \Theta) \tag{4}$$

In order decide if RJVs are socially desirable, I compare the expressions (3) and (4). We have

$$W_{RJV} - W_{no} = (2E(\theta_1) - 2E(\theta_1^2)) \left[\Pi_m + CS_m\right] - E(\theta_1)^2 CS_B$$

Since $E(\theta_1) = \frac{\Delta}{T+1}\sum_{i=0}^{T} i = \frac{T\Delta}{2}$ and $E(\theta_1^2) = \frac{\Delta^2}{T+1}\sum_{i=0}^{T} i^2 = \frac{T\Delta^2(2T+1)}{6}$, we get

[6] Note that there is a positive probability that no RJV is formed. This occurs when the firms have exactly the same knowledge. Morever, as $\Delta \to 0$ the probability of no RJV approaches to zero.

$$W_{RJV} - W_{no} = (T\Delta - \frac{T\Delta^2(2T+1)}{3})\,[\Pi_m + CS_m] - \frac{T^2\Delta^2}{4}CS_B$$

So, we have

$$W_{RJV} - W_{no} > 0 \Leftrightarrow$$
$$\frac{(T\Delta - \frac{T\Delta^2(2T+1)}{3})}{\frac{T^2\Delta^2}{4}} > \frac{CS_B}{\Pi_m + CS_m} \tag{5}$$

The next lemma indicates that the RHS of inequality (5) is bounded. That is, we have $1 < \frac{CS_B}{\Pi_m+CS_m} < \frac{e}{2}$.

Lemma 6 *(i)* $\frac{CS_B}{\Pi_m+CS_m}$ *is increasing for all* $a > 1$*; (ii)* $\lim_{a\to 1} \frac{CS_B}{\Pi_m+CS_m} = 1$*; (iii)* $\lim_{a\to\infty} \frac{CS_B}{\Pi_m+CS_m} = \frac{e}{2}$.

Consider the LHS of (5). We have $\frac{(T\Delta - \frac{T\Delta^2(2T+1)}{3})}{\frac{T^2\Delta^2}{4}} = \frac{4}{T\Delta} - \frac{8}{3} - \frac{4}{3T}$, which takes its minimum at $T\Delta = \frac{1}{2}$ and $T = 3$. Thus, the RHS is at least $\frac{44}{9}$. This implies that the inequality (5) is always satisfied since $\frac{44}{9} > \frac{e}{2}$. Therefore, the social benefit of a RJV dominates the its social costs, and hence a RJV is socially desirable for all $a > 1$.

Even if the social planner puts different weights on the producer and consumer surplus, a RJV is still socially desirable. To see this, let Λ and $1-\Lambda$ be the weights on consumer and producer surplus, respectively. Then, the RHS of (5) becomes $\frac{\Lambda CS_b}{(1-\Lambda)\Pi_m+\Lambda CS_m}$. In this case, we have $\lim_{a\to 1} \frac{\Lambda CS_b}{(1-\Lambda)\Pi_m+\Lambda CS_m} = 1$ and $\lim_{a\to\infty} \frac{\Lambda CS_b}{(1-\Lambda)\Pi_m+\Lambda CS_m} = \Lambda e$. Thus, the expected social welfare with a RJV is still higher since the LHS of (5) is always greater than Λe for all $\Lambda \in [0,1]$.

Proposition 2 *A RJV increases social welfare for all* T*,* Δ*, and* $a > 1$*.*

5 Conclusions

When information about firms' know-how is private, firms need to disclose at least part of their know-how to inform each other about the value of the intellectual property they would bring into a RJV. However, by providing their intellectual property firms expose themselves to expropriation. In weak intellectual property rights regimes the revealed know-how cannot be protected and the potential partners may use it freely to innovate themselves. Because of this fear of expropriation, many potentially successful RJVs cannot be formed. In this paper I have suggested a sequential disclosure procedure which alleviates the asymmetric information problem and encourages firms to form RJVs. I have shown that the firms fully disclose their know-how gradually over time.

Despite the higher success rate in achieving an innovation in a RJV, many academics and policy makers are concerned that RJVs lead to the monopolization of markets, and hence may decrease the social welfare. However, I have found that benefits of an increasing success rate of innovation outweighs the cost of monopoly, and hence RJVs are socially desirable.

Appendix A: Appendix for Chapter I

The following lemma lists some useful equations.

Lemma 5

$$b_m^H(1) - b_{m-1}^H(1) = -\frac{1}{m+n}\mu_H - \frac{E(\sum_{j=m+1}^{m+n} s_j|\mathbf{s}_M(m-1)) - E(\sum_{j=m+1}^{m+n} s_j|\mathbf{s}_M(m))}{m+n}\mu_H \tag{5}$$

$$\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} p(s_M(k)) = 1 \textit{ for all } m \tag{6}$$

$$\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} p(s_M(k); s_N) = p(s_N) \textit{ for all } m \tag{7}$$

$$\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} kp(s_M(k)) = mp(s_1 = 0) \textit{ for all } m. \tag{8}$$

$$E\left(\sum_{j=m+1}^{m+n} s_j|s_M(k)) = np(s_{m+1} = 1|s_M(k)\right) \textit{ for all } m, n \textit{ and } k = 0, ..., m \tag{9}$$

Proof. Equation (5) follows from (2). Equation (6) follows from the property of a probability function, and equation (7) is the definition of $p(s_N)$. Consider equation (8). We have

$$\begin{aligned}\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} kp(\mathbf{s}_M(k)) &= m\sum_{k=1}^{m}\begin{pmatrix} m \\ k \end{pmatrix}\frac{k}{m}p(\mathbf{s}_M(k)) = m\sum_{k=1}^{m}\begin{pmatrix} m-1 \\ k-1 \end{pmatrix}p(\mathbf{s}_M(k)) \\ &= mp(s_1 = 0).\end{aligned}$$

Now consider equation (9). We have

$$\begin{aligned}E\left(\sum_{j=m+1}^{m+n} s_j|\mathbf{s}_M(k)\right) &= \sum_{j=m+1}^{m+n} E(s_j|\mathbf{s}_M(k)) = \sum_{j=m+1}^{m+n} p(s_j = 1|\mathbf{s}_M(k)) \\ &= np(s_{m+1} = 1|\mathbf{s}_M(k)),\end{aligned}$$

where the last equality follows from symmetry. ■

Proof of Proposition 1. (i) It is clear that all bidders with signal 0 drop out at $b^H(0) = E(v_H|\mathbf{s}_M(m))$ and all bidders with signal 1 quit at $b_k^H(1) = E(v_H|\mathbf{s}_M(k))$ (where $k = 0, 1, 2, ...m$ is the number of bidders who drop out at price $b^H(0)$) form an equilibrium of a single-shot English auction of H (see

Milgrom and Weber (1982)). The equilibrium price in the auction of H is $b_k^H(1)$ when there are k bidders with signal 0 and $m-k$ bidders with signal 1 for $k = 0, 1, ..., m-2$ and it is $b_m^H(1)$ when there is at most one bidder having signal 1. Thus, the expected equilibrium revenue to the seller in the auction of H is

$$R_H(m,n) = \sum_{k=0}^{m-2} \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) b_k^H(1) + \begin{pmatrix} m \\ m-1 \end{pmatrix} p(\mathbf{s}_M(m-1)) b_m^H(1) + \begin{pmatrix} m \\ m \end{pmatrix} p(\mathbf{s}_M(m)) b_m^H(1)$$

Using (5) and rearranging, we get

$$R_H(m,n) = \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) b_k^H(1) - \begin{pmatrix} m \\ m-1 \end{pmatrix} p(\mathbf{s}_M(m-1)) \frac{1}{m+n} \mu_H - \begin{pmatrix} m \\ m-1 \end{pmatrix} p(\mathbf{s}_M(m-1)) \left[\frac{E(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(m-1))}{m+n} - \frac{E(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(m))}{m+n} \mu_H \right]$$

Using (2) and $\begin{pmatrix} m \\ m-1 \end{pmatrix} = m$, we have

$$R_H(m,n) = \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) \left[\frac{E(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(k))}{n+m} \mu_H + \frac{m-k}{n+m} \mu_H \right] - \frac{m}{m+n} \mu_H p(\mathbf{s}_M(m-1)) - \frac{m}{m+n} \mu_H p(\mathbf{s}_M(m-1)) \left[\left(E \sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(m-1)) - E(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(m) \right) \right].$$

Rearranging, we get

$$R_H(m,n) = \frac{m}{m+n} \mu_H \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) - \frac{1}{m+n} \mu_H \sum_{k=0}^{m} k \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) + \frac{1}{m+n} \mu_H \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) E \left(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(k)) - \frac{m}{m+n} \mu_H p(\mathbf{s}_M(m-1) \right) - \frac{m}{m+n} \mu_H p(\mathbf{s}_M(m-1)) \left[E \left(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(m-1)) - E(\sum_{j=m+1}^{m+n} s_j | \mathbf{s}_M(m) \right) \right]$$

Using (6), (8), (9), we get

$$\begin{aligned}
R_H(m,n) &= \frac{m}{m+n}\mu_H - \frac{m}{m+n}\mu_H p(s_1=0) \\
&\quad + \frac{1}{m+n}\mu_H \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} p(\mathbf{s}_M(k)) n p(s_{m+1}=1|\mathbf{s}_M(k)) - \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)) \\
&\quad - \frac{mn}{m+n}\mu_H p(\mathbf{s}_M(m-1))\left[p(s_{m+1}=1|\mathbf{s}_M(m-1)) - p(s_{m+1}=1|\mathbf{s}_M(m))\right]
\end{aligned}$$

Since $np(s_M(k))p(s_{m+1}=1|s_M(k)) = np(s_M(k); s_{m+1}=1)$, using (7) we have

$$\begin{aligned}
R_H(m,n) &= \frac{m}{m+n}\mu_H - \frac{m}{m+n}\mu_H p(s_1=0) + \frac{n}{m+n}\mu_H p(s_{m+1}=1) \\
&\quad - \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)) - \frac{mn}{m+n}\mu_H p(\mathbf{s}_M(m-1))\left[p(s_{m+1}=1|\mathbf{s}_M(m-1))\right. \\
&\quad \left. - p(s_{m+1}=1|\mathbf{s}_M(m))\right]
\end{aligned}$$

Using symmetry and $p(s_1=1) = 1 - p(s_1=0)$, we get

$$R_H(m,n) = \mu_H p(s_1=1) - \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)) - \frac{mn}{m+n}\mu_H X_m$$

where

$$X_m = p(\mathbf{s}_M(m-1))\left[p(s_{m+1}=1|\mathbf{s}_M(m-1)) - p(s_{m+1}=1|\mathbf{s}_M(m))\right]$$

(ii) Similar to the proof of (i) ■

Proof of Lemma 1. Since the probability distribution is symmetric, w l.o.g., consider X_m. Recall that $X_m = p(s_M(m-1))\widetilde{X}_m$ where

$$\widetilde{X}_m = p(s_{m+1}=1|\mathbf{s}_M(m-1)) - p(s_{m+1}=1|\mathbf{s}_M(m)).$$

I will show that $\widetilde{X}_m \geq 0$. We have

$$\begin{aligned}
&p(s_{m+1}=1|\mathbf{s}_M(m-1)) - p(s_{m+1}=1|\mathbf{s}_M(m)) \geq 0 \Longleftrightarrow \\
&p(\mathbf{s}_M(m-1); s_{m+1}=1)p(\mathbf{s}_M(m)) \geq p(\mathbf{s}_M(m); s_{m+1}=1)p(\mathbf{s}_M(m-1)) \Longleftrightarrow \\
&p(\mathbf{s}_M(m-1); s_{m+1}=1)p(\mathbf{s}_M(m); s_{m+1}=0) + p(\mathbf{s}_M(m-1); s_{m+1}=1)p(\mathbf{s}_M(m); s_{m+1}=1) \geq \\
&\quad p(\mathbf{s}_M(m); s_{m+1}=1)p(\mathbf{s}_M(m-1); s_{m+1}=0) + p(\mathbf{s}_M(m); s_{m+1}=1)p(\mathbf{s}_M(m-1); s_{m+1}=1) \Longleftrightarrow \\
&p(\mathbf{s}_M(m-1); s_{m+1}=1)p(\mathbf{s}_M(m); s_{m+1}=0) \geq p(\mathbf{s}_M(m); s_{m+1}=1)p(\mathbf{s}_M(m-1); s_{m+1}=0),
\end{aligned}$$

which is true for any $m \geq 2$ since the signals are affiliated. Morever, $\widetilde{X}_m$ is positive if the signals are strictly affiliated. Recall that $p(s_M(m-1)) > 0$. Therefore, we have $X_m \geq 0$ for all m and if the signals are strictly affiliated, then we have $X_m > 0$ for all m. ■

Before proving the results for sequential sales of H and L, the following lemma lists some useful equations.

Lemma 6

$$b^L_{k,n}(1) - b^L_{k,n-1}(1) = -\frac{1}{m+n}\mu_L \tag{10}$$

$$\sum_{t=0}^{n} \binom{n}{t} p(s_N(t)|s_M(k)) = 1 \quad \textit{for all } m, n \textit{ and } k = 0, 1, .., m \tag{11}$$

$$\sum_{t=0}^{n} t \binom{n}{t} p(s_N(t)|s_M(k)) = np(s_{m+1} = 0|s_M(k)) \textit{ for all } m, n \textit{ and } k = 0, 1, .., m \tag{12}$$

Proof. Equation (10) follows from (4) and equation (11) follows from the property of a probability density. Finally, equation (12) is similar to equation (8). ■

Proof of Proposition 2. (i) Following Milgrom and Weber (1982), it is clear that all H-bidders with signal 0 and with signal 1 drop out at $b^H(0) = E(v_H|\mathbf{s}_M(m))$ and $b^H_k(1) = E(v_H|\mathbf{s}_M(k))$, respectively and all L-bidders with signal 0 and with signal 1 quit at $b^L_k(0) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_N(n))$ and $b^L_{k,t}(1) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_N(t))$, respectively (where $k = 0, 1, 2, ...m$ is the number of H-bidders who drop out at price $b^H(0)$ and $t = 0, 1, 2, ...n$ is the number L-bidders who quit at the price $b^L_k(0)$) form an equilibrium of the two stage English auction when H is sold first and L is sold second. Notice that the equilibrium drop out prices in the sale of H in the order of HL are the same as the ones in the auction of H, when the items are sold simultaneously. Thus, the expected revenue to the seller from the auction of H when H is auctioned first is $R_{H/HL}(m, n) = R_H(m, n)$.

Now consider the auction of L. Suppose that there are k H-bidders with signal 0 and $m-k$ H-bidders with signal 1. Then, the equilibrium price in the auction of L is $b^L_{k,t}(1)$, when there are t L-bidders with signal 0 and $n-t$ L-bidders with signal 1 for $t = 0, 1, ..., n-2$, and it is $b^L_{k,n}(1)$ when there is at most one L-bidder with signal 1. Thus, the expected payoff to the seller in the auction of L is

$$R_{L/HL}(m, n) = \sum_{k=0}^{m} \binom{m}{k} p(\mathbf{s}_M(k)) \times \left[\sum_{t=0}^{n-2} \binom{n}{t} p(\mathbf{s}_N(t)|\mathbf{s}_M(k)) b^L_{k,t}(1) \right.$$
$$\left. \binom{n}{n-1} p(\mathbf{s}_N(n-1)|\mathbf{s}_M(k)) b^L_{k,n}(1) + \binom{n}{n} p(\mathbf{s}_N(n)|\mathbf{s}_M(k)) b^L_{k,n}(1) \right]$$

Using (10) and rearranging, we have

$$R_{L/HL}(m,n) = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p(\mathbf{s}_M(k)) \sum_{t=0}^{n} \left(\begin{array}{c} n \\ t \end{array} \right) p(\mathbf{s}_N(t)|\mathbf{s}_M(k)) b^L_{k,t}(1) \\ - \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p(\mathbf{s}_M(k)) p(\mathbf{s}_N(n-1)|\mathbf{s}_M(k)) \frac{n}{m+n} \mu_L$$

Since $p(s_M(k))p(s_N(n-1)|s_M(k)) = p(s_N(n-1); s_M(k))$, using (7) we get

$$R_{L/HL}(m,n) = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p(\mathbf{s}_M(k)) \sum_{t=0}^{n} \left(\begin{array}{c} n \\ t \end{array} \right) p(\mathbf{s}_N(t)|\mathbf{s}_M(k)) b^L_{k,t}(1) \\ - \frac{n}{m+n} \mu_L p(\mathbf{s}_N(n-1))$$

Using (4) and rearranging, we get

$$R_{L/HL}(m,n) = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p(\mathbf{s}_M(k)) \times \left[\frac{m+n-k}{m+n} \mu_L \sum_{t=0}^{n} \left(\begin{array}{c} n \\ t \end{array} \right) p(\mathbf{s}_N(t)|\mathbf{s}_M(k)) \right. \\ \left. - \frac{1}{m+n} \mu_L \sum_{t=0}^{n} t \left(\begin{array}{c} n \\ t \end{array} \right) p(\mathbf{s}_N(t)|\mathbf{s}_M(k)) \right] - \frac{n}{m+n} \mu_L p(\mathbf{s}_N(n-1))$$

Using (11) and (12), we have

$$R_{L/HL}(m,n) = \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p(\mathbf{s}_M(k)) \frac{m+n-k}{m+n} \mu_L \\ - \frac{1}{m+n} \mu_L \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) p(\mathbf{s}_M(k)) n p(s_{m+1}=0|\mathbf{s}_M(k)) - \frac{n}{m+n} \mu_L p(\mathbf{s}_N(n-1))$$

Since $np(s_M(k))p(s_{m+1}=0|s_M(k)) = np(s_M(k);\ s_{m+1}=0)$, using (6)-(8) we get

$$R_{L/HL}(m,n) = \mu_L - \frac{m}{m+n} \mu_L p(s_1=0) - \frac{n}{m+n} \mu_L p(s_{m+1}=0) - \frac{n}{m+n} \mu_L p(\mathbf{s}_N(n-1))$$

Using symmetry, we have

$$R_{L/HL}(m,n) = \mu_L p(s_{m+1}=1) - \frac{n}{m+n} \mu_L p(\mathbf{s}_N(n-1))$$

Since $R_L(m,n) = \mu_L p(s_{m+1} = 1) - \frac{n}{m+n}\mu_L p((s_N(n-1)) - \frac{mn}{m+n}\mu_L X_n$, we have

$$R_{L/HL}(m,n) = R_L(m,n) + \frac{mn}{m+n}\mu_L X_n.$$

Therefore, the total expected revenue to the seller when H is sold first is

$$R_{HL}(m,n) = R_{H/HL}(m,n) + R_{L/HL}(m,n) = R_H(m,n) + R_L(m,n) + \frac{mn}{m+n}\mu_L X_n$$

(ii) Similar to the proof of (i). ■

Proof of Proposition 3. Assume $m = n$. Then, we have $R_{LH}(m,m) \geq R_{HL}(m,m) \iff \mu_H X_m \geq \mu_L X_m$. Since $X_m \geq 0$ by Lemma 1 and $\mu_H > \mu_L$, we have $R_{LH}(m,m) \geq R_{HL}(m,m)$. Moreover, if the signals are strictly affiliated; that is, we have $X_m > 0$ by lemma 1, we get $R_{LH}(m,m) > R_{HL}(m,m)$.
■

Proof of Lemma 2. Since p is symmetric, w l.o.g., consider X_m. I show that $X_m \geq X_{m+1}$. Recall that

$$X_m = p(\mathbf{s}_M(m-1))\left[p(s_{m+1} = 1|\mathbf{s}_M(m-1)) - p(s_{m+1} = 1|\mathbf{s}_M(m))\right]$$

Define $\widetilde{M} = \{1, 2, ..., m+1\}$, $\widetilde{N} = \{m+2, ..., m+n+1\}$. Then, we have

$$X_{m+1} = p(\mathbf{s}_{\widetilde{M}}(m))\left[p(s_{m+2} = 1|\mathbf{s}_{\widetilde{M}}(m)) - p(s_{m+2} = 1|\mathbf{s}_{\widetilde{M}}(m+1))\right]$$

Thus, we have

$$X_m \geq X_{m+1} \iff p(\mathbf{s}_M(m-1); s_{m+1} = 1) - p(s_{m+1} = 1|\mathbf{s}_M(m))p(\mathbf{s}_M(m-1)) \geq$$
$$p(\mathbf{s}_{\widetilde{M}}(m); s_{m+2} = 1) - p(s_{m+2} = 1|\mathbf{s}_{\widetilde{M}}(m+1))p(\mathbf{s}_{\widetilde{M}}(m))$$

Since

$$p(\mathbf{s}_M(m-1); s_{m+1} = 1) = p(\mathbf{s}_M(m-1); s_{m+1} = 1; s_{m+2} = 1) + p(\mathbf{s}_M(m-1); s_{m+1} = 1; s_{m+2} = 0)$$

and p is symmetric, we have

$$X_m \geq X_{m+1} \iff p(\mathbf{s}_M(m-1); s_{m+1} = 1; s_{m+2} = 1) \geq$$
$$p(s_{m+1} = 1|\mathbf{s}_M(m))p(\mathbf{s}_M(m-1)) - p(s_{m+2} = 1|\mathbf{s}_{\widetilde{M}}(m+1))p(\mathbf{s}_{\widetilde{M}}(m))$$

$$\begin{aligned}
&\iff p(\mathbf{s}_M(m-1); s_{m+1}=1; s_{m+2}=1) \geq \\
&\frac{p(\mathbf{s}_M(m); s_{m+1}=1)p(\mathbf{s}_M(m-1))}{p(\mathbf{s}_M(m))} - \frac{p(\mathbf{s}_{\widetilde{M}}(m+1); s_{m+2}=1)p(\mathbf{s}_{\widetilde{M}}(m))}{p(\mathbf{s}_{\widetilde{M}}(m+1))} \\
&\iff p(\mathbf{s}_M(m-1); s_{m+1}=1; s_{m+2}=1)p(\mathbf{s}_M(m))p(\mathbf{s}_{\widetilde{M}}(m+1)) \\
&\geq p(\mathbf{s}_M(m); s_{m+1}=1)p(\mathbf{s}_M(m-1))p(\mathbf{s}_{\widetilde{M}}(m+1)) - p(\mathbf{s}_{\widetilde{M}}(m+1); s_{m+2}=1)p(\mathbf{s}_{\widetilde{M}}(m))p(\mathbf{s}_M(m))
\end{aligned}$$

Now set

$$\begin{aligned}
a &= p(\mathbf{s}_{\widetilde{M}}(m+1); s_{m+2}=0) \\
b &= p(\mathbf{s}_{\widetilde{M}}(m+1); s_{m+2}=1) \\
c &= p(\mathbf{s}_M(m-1); s_{m+1}=0; s_{m+2}=1) \\
d &= p(\mathbf{s}_M(m-1); s_{m+1}=1; s_{m+2}=1)
\end{aligned}$$

Then, we get

$$\begin{aligned}
p(\mathbf{s}_M(m)) &= a+2b+c \\
p(\mathbf{s}_{\widetilde{M}}(m+1)) &= a+b \\
p(\mathbf{s}_M(m); s_{m+1}=1) &= b+c \\
p(\mathbf{s}_M(m-1)) &= b+2c+d
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&X_m \geq X_{m+1} \\
&\quad \iff d(a+2b+c)(a+b) \geq (b+c)(b+2c+d)(a+b) - b(b+c)(a+2b+c) \\
&\quad \iff da^2 + a(2bd-2c^2-2bc) + (b^2d - bc^2 + b^3) \geq 0
\end{aligned}$$

Consider the LHS of the above inequality as a quadratic equation in the variable a. First note that the quadratic is convex since $d > 0$. Second, the discriminant is $\Delta_a = (2bd-2c^2-2bc)^2 - 4d(b^2d-bc^2+b^3) = -4(b+c)^2(bd-c^2)$. Since $bd - c^2 \geq 0$ by affiliation, we have $\Delta_a \leq 0$. If $\Delta_a = 0$, then the quadratic has only one zero. In this case, the LHS is always positive except at the zero because the quadratic is convex. If $\Delta_a < 0$, then there are no zeros of the quadratic, which implies that the quadratic is positive in this case, because it is convex. Thus, the LHS is always nonnegative. This implies that $X_m \geq X_{m+1}$. Moreover, if the signals are strictly affiliated, that is $bd - c^2 > 0$, then $\Delta_a < 0$. Thus, the LHS is always positive implying $X_m > X_{m+1}$. ■

Proof of Proposition 4. We have $R_{LH}(m,n) \geq R_{HL}(m,n) \iff \mu_H X_m \geq \mu_L X_n$. Assume that $m < n$. Then, $X_m \geq X_n \geq 0$ by Lemmas 1 and 2. Thus, we have $R_{LH}(m,n) \geq R_{HL}(m,n)$ since $\mu_H > \mu_L$. Moreover, if the signals are strictly affiliated; that is, we have $X_m > X_n > 0$ by Lemmas 1 and 2, then $R_{LH}(m,n) > R_{HL}(m,n)$. ■

Proof of Corollary 2. Set $n^* = m$. Propositions 3 and 4 together imply that the seller weakly (strictly if the signals are strictly affiliated) prefers selling L first to selling H first for all $n \geq n^*$. ■

Proof of Lemma 3. We have $0 < p(s_M(m-1)) < \frac{1}{m}$ since p is symmetric and non-degerate. Thus, by the sandwich theorem we get $\underset{m\to\infty}{lim} p(s_M(m-1)) = 0$. ■

Proof of Lemma 4. Recall that $X_m = p(s_M(m-1))\widetilde{X}_m$. Since $0 \leq \widetilde{X}_m \leq 1$ and $\underset{m\to\infty}{lim} p(s_M(m-1)) = 0$ by Lemma 3, we get $\underset{m\to\infty}{lim} X_m = 0$. ■

Proof of Proposition 5. Assume that $X_n = 0$. Then, choose $m^* = n$. Thus, we have $X_m = 0$ for all $m \geq m^*$ by Lemmas 1 and 2. Therefore, $R_{LH}(m,n) = R_{HL}(m,n)$ for all $m \geq m^*$. Now assume that $X_n > 0$. Set $\epsilon = \frac{\mu_L X_n}{\mu_H}$. Since $\underset{m\to\infty}{lim} X_m = 0$ by Lemma 4, there exists an integer m^* such that $X_m < \frac{\mu_L X_n}{\mu_H}$ for all $m \geq m^*$. Thus, we have $R_{LH}(m,n) < R_{HL}(m,n)$ for all $m \geq m^*$. Note that if the signals are strictly affiliated, Lemma 1 implies that $X_m > 0$. Therefore, for any n there exists m^* such that the seller strictly prefers selling H first to selling L first for all $m \geq m^*$ if the signals are strictly affiliated. ■

Proof of Corollary 3. Since $\mu_H = \mu_L$, we have $R_{HL}(m,n) > R_{LH}(m,n) \iff X_n > X_m$. Lemma 2 implies that $X_n > X_m \iff m > n$. Therefore, $R_{HL}(m,n) > R_{LH}(m,n) \iff m > n$. ■

The next two lemmas are needed in the proof of Proposition 6

Lemma 7 *i)* $\underset{m\to\infty}{lim} R_H(m,n) = \mu_H p(s_1 = 1)$ *and* $\underset{m\to\infty}{lim} R_L(m,n) = \mu_L p(s_{m+1} = 1) - n\mu_L X_n$ *for any* $n < \infty$.

ii) $\underset{n\to\infty}{lim} R_H(m,n) = \mu_H p(s_1 = 1) - m\mu_H X_m$ *and* $\underset{n\to\infty}{lim} R_H(m,n) = \mu_L p(s_{m+1} = 1)$ *for any* $m < \infty$.

Proof. Recall that

$$R_H(m,n) = \mu_H p(s_1 = 1) - \frac{m}{m+n}\mu_H p(\mathbf{s}_M(m-1)) - \frac{mn}{m+n}\mu_H X_m$$

$$R_L(m,n) = \mu_L p(s_{m+1} = 1) - \frac{n}{m+n}\mu_L p(s_N(n-1)) - \frac{mn}{m+n}\mu_L X_n.$$

(i) We have $\underset{m\to\infty}{lim} \frac{m}{m+n}\mu_H p(s_M(m-1)) = 0$ by Lemma 3 and $\underset{m\to\infty}{lim} \frac{mn}{m+n}\mu_H X_m = 0$ by Lemma 4. Also, note that $\underset{m\to\infty}{lim} \frac{n}{m+n}\mu_L p(s_N(n-1)) = 0$ and $\underset{m\to\infty}{lim} \frac{mn}{m+n}\mu_L X_n = n\mu_L X_n$. Therefore, we obtain $\underset{m\to\infty}{lim} R_H(m,n) = \mu_H p(s_1 = 1)$ and $\underset{m\to\infty}{lim} R_L(m,n) = \mu_L p(s_{m+1} = 1) - n\mu_L X_n$.

(ii) Similar to (i). ■

Lemma 8 *i) For any $n < \infty$, we have*

$$\lim_{m\to\infty} R_H(m,n) + R_L(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - n\mu_L X_n$$

$$\lim_{m\to\infty} R_{HL}(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1)$$

$$\lim_{m\to\infty} R_{LH}(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - n\mu_L X_n$$

ii) For any $m < \infty$, we have

$$\lim_{n\to\infty} R_H(m,n) + R_L(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - m\mu_H X_m$$

$$\lim_{n\to\infty} R_{HL}(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - m\mu_H X_m$$

$$\lim_{n\to\infty} R_{LH}(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1)$$

Proof of Proposition 6. Directly follows from Lemma 8. ■

The next two lemmas are needed in the proof of Proposition 7

Lemma 9 $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} \frac{mn}{m+n}\mu_H X_m = \lim\limits_{\substack{n\to\infty \\ m\to\infty}} \frac{mn}{m+n}\mu_L X_n = 0.$

Proof. To be added. ■

Lemma 10 *i)* $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} R_H(m,n) = \mu_H p(s_1 = 1).$

ii) $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} R_L(m,n) = \mu_L p(s_{m+1} = 1).$

Proof. (i) First note that $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} \frac{m}{m+n}\mu_H p(s_M(m-1)) = 0$ since $\lim\limits_{m\to\infty} p(s_M(m-1)) = 0$ by Lemma 3 and $0 < \frac{m}{m+n} < 1$. Second, we have $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} \frac{mn}{m+n}\mu_H X_m = 0$ by Lemma 9. Thus, we get $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} R_H(m,n) = \mu_H p(s_1 = 1)$.

(ii) Similar to (i). ■

Proof of Proposition 7. By Lemmas 9 and 10, we obtain $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} R_H(m,n) + R_L(m,n) = \lim\limits_{\substack{n\to\infty \\ m\to\infty}} R_{HL}(m,n) = \lim\limits_{\substack{n\to\infty \\ m\to\infty}} R_{LH}(m,n) = \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1)$. ■

Proof of Proposition 9. I show that the drop out prices $b^H(0) = E(v_H|\mathbf{s}_M(m))$, $b_k^H(1) = E(v_H|\mathbf{s}_M(k))$, $b < b^H(0)$, $b_k^L(0) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_{N\cup R}(n+r))$ and $b_{k,t}^L(1) = E(v_L|\mathbf{s}_M(k); \mathbf{s}_{N\cup R}(t))$ where $k = 0, 1, 2, ..., m$ and $t = 0, 1, 2, ...n + r$ form an equilibrium.

First notice that no H-bidder has an incentive to deviate. The reason is that quitting at $b^H(0)$ ($b_k^H(1)$) is a best response of an H-bidder with signal 0 (signal 1) given that all B-bidders drop out at $b < b^H(0)$ and all the other H-bidders follow the strategies $b^H(0)$ and $b_k^H(1)$.

Second note that if the drop out prices are $b < b^H(0)$, $b^H(0)$ and $b_k^H(1)$ in the sale of H, then $b_k^L(0)$ and $b_{k,t}^L(1)$ form an equilibrium of the auction of L. Thus, neither an L-bidder nor a B-bidder has an incentive to deviate in the second sale.

Now I show that no B-bidder j (where $j \in R$) has an incentive to deviate in the first sale. Note that bidder j makes a zero payoff both in the auction of H irrespective of his signal and in the auction of L if he has signal 0. However, if he has signal 1, his payoff in the auction of L is $\frac{\mu_L}{m+n+r} p(s_{N \cup R \setminus \{j\}}(n + r - 1)|s_j = 1) > 0$.[7]

Assume that all the other bidders believe that bidder j has signal 1 if he does not quit at b. Note that any deviation by bidder j is observable to all bidders since $b < b^H(0)$. Thus, when bidder j does not drop out at b, all H-bidders, all L-bidders and the other B-bidders will update their bids: All H-bidders with signal 0 and with signal 1 quit at $\widetilde{b}^H(0) = E(v_H|s_M(m); s_j = 1)$ and $\widetilde{b}_k^H(1) = E(v_H|s_M(k); s_j = 1)$, respectively, in the auction of H (where $k = 0, 1, ..., m$ is the number of H-bidders who drop out at the price $\widetilde{b}^H(0)$). Also, in the auction of L, all L and B-bidders (other than bidder j) with signal 0 and with signal 1 drop out at $\widetilde{b}_k^L(0) = E(v_L|s_M(k); s_{N \cup R}(n + r - 1))$ and $\widetilde{b}_{k,t}^L(1) = E(v_L|s_M(k); s_{N \cup R}(t))$, respectively (where $t = 0, 1, 2, ...n + r - 1$ is the number of bidders - other than the bidder j - who drop out at price $\widetilde{b}^L(0)$).

Suppose that bidder j has signal 0 and he does not quit at b. Then, it is clear that he cannot get the item L in the second auction with a postive payoff. Thus, he gets zero in the auction of L. Now consider the auction of H. If he wins the object H, he pays either $\widetilde{b}^H(0)$ or $\widetilde{b}_k^H(1)$. Consider the case he pays $\widetilde{b}^H(0)$. This implies that all H-bidders have signal 0. Then, the expected value of H to bidder j with signal 0 is $E(v_H|s_M(m); s_j = 0)$. But we have $E(v_H|s_M(m); s_j = 0) < \widetilde{b}^H(0)$ by the proof of Lemma 1. Now consider the case he pays $\widetilde{b}_k^H(1)$. This implies that there are k H-bidders with signal 0 and $m - k$ H-bidders with signal 1. Then, the expected value of H to the bidder j with signal 0 is $E(v_H|s_M(k); s_j = 0)$. But we have $E(v_H|s_M(k);\ s_j = 0) < \widetilde{b}_k^H(1)$, by the proof of Lemma 1. Therefore, bidder j with signal 0 has no incentive to deviate.

Now suppose that bidder j has signal 1 and he does not quit at b. Consider the auction of L. If he wins the object at $\widetilde{b}_k^L(0)$, then all the other bidders in the second sale have signal 0. In that case, the object's expected value is $E(v_L|s_M(k); s_{N \cup R}(n + r - 1))$ which is the same as $\widetilde{b}_k^L(0)$. If he wins the object at $\widetilde{b}_{k,t}^L(0)$, then there are t bidders with signal 0 and $n + r - t$ bidders with signal 1. In that case, the expected value of L is $E(v_L|s_M(k); s_{N \cup R}(t))$ which is the same as $\widetilde{b}_{k,t}^L(1)$. Thus, his payoff is zero in the auction of L. Now consider the auction of H. If he wins the object H, he pays either $\widetilde{b}^H(0)$ or $\widetilde{b}_k^H(1)$. Consider the case he pays $\widetilde{b}^H(0)$. This implies that all H-bidders have signal 0. Then, the

[7] To see this note that his payoff is postive in the auction of L if all the other $n+r-1$ bidders in the second auction have signal 0. Otherwise his payoff is zero. Thus, his expected payoff is $[E(v_L|\mathbf{s}_M(k); \mathbf{s}_{N\cup R}(n+r-1)) - E(v_L|\mathbf{s}_M(k); \mathbf{s}_{N\cup R}(n+r))]p(\mathbf{s}_{N\cup R\setminus\{j\}}(n + r - 1)|s_j = 1) =$
$\frac{\mu_L}{m+n+r} p(\mathbf{s}_{N\cup R\setminus\{j\}}(n + r - 1)|s_j = 1)$.

expected value of H to him is $E(v_H|s_M(m); s_j = 1)$ which is equal to $\widetilde{b}^H(0)$. Now consider the case he pays $\widetilde{b}^H_k(1)$. This implies that there are k H-bidders with signal 0 and $m-k$ H-bidders with signal 1. Then, the expected value of H to him is $E(v_H|s_M(k); s_j = 1)$ which is equal to $\widetilde{b}^H_k(1)$. Thus, his payoff is zero in the auction of H. Therefore, bidder j with signal 1 has no incentive to deviate.

Therefore, the drop out prices $b^H(0)$, $b^H_k(1)$, $b < b^H(0)$, $b^L_k(0)$ and $b^L_{k,t}(1)$ where $k = 0, 1, 2, ..., m$ and $t = 0, 1, 2, ...n + r$ form an equilibrium.

Notice that in the equilibrium there are actually m bidders competing for H and $n + r$ bidders competing for L. Therefore, we have $R^P_{HL}(m, n, r) = R_{HL}(m, n + r)$ by Proposition 2.

(ii) Similar to the proof of (i). ■

Proof of Proposition 10. (i) I show that $b^H(0) = E(v_H|\mathbf{s}_{M\cup R}(m + r))$, $b^H_k(1) = E(v_H|\mathbf{s}_{M\cup R}(k))$, $b^L_k(0) = E(v_L|\mathbf{s}_{M\cup R}(k); \mathbf{s}_N(n))$, $b^L_{k,t}(1) = E(v_L|\mathbf{s}_{M\cup R}(k); \mathbf{s}_N(t))$ and any $b \leq b^L_k(0)$, where $k = 0, 1, 2, ..., m + r$ and $t = 0, 1, 2, ...n$ form an equilibrium.

First note that no H-bidder has an incentive to deviate given that all B-bidders and the other H-bidders quit according to $b^H(0)$ and $b^H_k(1)$.

Second, neither an L-bidder nor a B-bidder has an incentive to deviate in the second auction since $b^L_k(0)$, $b^L_{k,t}(1)$ and $b \leq b^L_k(0)$ form an equilibrium of the auction of L.

Now consider bidder j (where $j \in R$). Assume that all the other bidders believe that bidder j has signal 1 if he does not drop out at $b^H(0)$ in the sale of H.

Suppose that bidder j has signal 0. It is clear that he has no incentive to deviate since his equilibrium payoff is zero.

Now suppose that bidder j has signal 1. Assume that k bidders drop out at price $b^H(0)$. Then, in the equilibrium he drops out at $b^H_k(1)$ and makes a positive payoff if he is the only bidder with signal 1 among the $m + r$ bidders. Otherwise he makes zero. Thus, his expected equilibrium payoff is

$$\begin{aligned}\Pi &= \left[E(v_H|\mathbf{s}_{M\cup R}(m + r - 1)) - E(v_H|\mathbf{s}_{M\cup R}(m + r))\right] p(\mathbf{s}_{M\cup R\setminus\{j\}}(m + r - 1)|s_j = 1) \\ &= \left[\frac{nX_{m+r}}{p(s_j = 1)} + p(\mathbf{s}_{M\cup R\setminus\{j\}}(m + r - 1)|s_j = 1)\right] \mu_H\end{aligned}$$

Now I show that bidder j with signal 1 has no incentive to deviate if and only if condition $\mathrm{S_{HL}}$ holds. Suppose that he quits at $b^H(0)$. Then, he wins the object H if and only if all the other $(m+r-1)$ bidders have signal 0. In that case, he'll get his equilibrium payoff Π with probability $\frac{1}{m+r}$. Later, in the auction of L all the other B-bidders and all n L-bidders believe that he has signal 0. Thus, they expect bidder j to drop out at $\frac{m+r-k}{m+n+r}\mu_L$ in the sale of L along with all the other B-bidders and the L-bidders with signal 0 (where k is the number of bidders who quit at $b^H(0)$). If he does not drop at that price , then all the remaining L-bidders can update their beliefs and bid accordingly. The only way for bidder j with signal 1 to make positive profit in the second sale by mimicking a bidder with signal 0 in the first sale is that all L-bidders must have signal 0 so they drop out at $\frac{m+r-k}{m+n+r}\mu_L$. In that case, which occurs with probability $p(s_N(n)|s_j = 1)$, he gets $\frac{\mu_L}{m+n+r}$. Therefore, bidder j with signal 1 has no incentive to deviate if and only if

$$\begin{aligned}\frac{1}{m+r}\Pi + \frac{\mu_L}{m+n+r}p(\mathbf{s}_N(n)|s_j = 1) \leq \Pi \Longleftrightarrow \\ (m+r)\frac{\mu_L}{m+n+r}p(\mathbf{s}_N(n)|s_j = 1) \leq (m+r-1)\Pi\end{aligned}$$

That is,

$$\frac{1}{m+n+r}\mu_L p(\mathbf{s}_N(n)|s_j = 1) \leq \frac{m+r-1}{m+r}\mu_H \left[\frac{nX_{m+r}}{p(s_j = 1)} + p(\mathbf{s}_{M\cup R\setminus\{j\}}(m+r-1)|s_j = 1)\right].$$

Notice that in the equilibrium there are actually $m+r$ bidders competing for H and n bidders competing for L. Thus, we have $R^S_{HL}(m,n,r) = R_{HL}(m+r,n)$ by Proposition 2. ■

I will now investigate the asymptotic properties of the equilibria in Model II. This will help me to determine the optimal order of sales.

The next three lemmas are needed in the proof of Proposition 11. The first two follow directly from Lemmas 7 and 8.

Lemma 11 *For any pair of* $(m,r) < (\infty,\infty)$*, we have*

$$\begin{aligned}\lim_{n\to\infty} R(m,n,r) &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - m\mu_H X_{m+r} \\ \lim_{n\to\infty} R^S_{HL} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - (m+r)\mu_H X_{m+r} \\ \lim_{n\to\infty} R^s_{LH} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) \\ \lim_{n\to\infty} R^p_{HL} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - m\mu_H X_m \\ \lim_{n\to\infty} R^p_{LH} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1)\end{aligned}$$

Lemma 12 *For any pair of* $(n,r) < (\infty,\infty)$*, we have*

$$\begin{aligned}\lim_{m\to\infty} R(m,n,r) &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - n\mu_L X_{n+r} \\ \lim_{m\to\infty} R^s_{HL} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) \\ \lim_{m\to\infty} R^s_{LH} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - (n+r)\mu_L X_{n+r} \\ \lim_{m\to\infty} R^p_{HL} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) \\ \lim_{m\to\infty} R^p_{LH} &= \mu_H p(s_1 = 1) + \mu_L p(s_{m+1} = 1) - n\mu_L X_n\end{aligned}$$

Lemma 13 *Let $\{x\}_j$ and $\{y\}_j$ be two sequences. Then, if $\lim_{j\to\infty}\{x\}_j > \lim_{j\to\infty}\{y\}_j$, then there exists an integer j^* such that $x_j > y_j$ for all $j \geq j^*$.*

Proof. Set $\lim_{j\to\infty} x_j = x$ and $\lim_{j\to\infty} y_j = y$ and assume $x > y$. Suppose that there is no such j^*. Then, there exist a subsequence $\{x\}_{j_k}$ of $\{x\}_j$ and a subsequence $\{y\}_{j_k}$of $\{y\}_j$ such that $x_{j_k} \leq y_{j_k}$ for all k. Thus, we have $\lim_{k\to\infty}\{x\}_{j_k} = x \leq \lim_{k\to\infty}\{y\}_{j_k} = y$ which is a contradiction. ■

Proof of Proposition 11. (i) Assume that the signals are strictly affiliated. Note that we have $R^p_{LH} > R^s_{HL}$ for all $n \geq m + r$ by Propositions 3 and 4. Now compare R^p_{LH} with R^p_{HL} and R^p_{LH} with $R(m, n, r)$. Lemma 11 implies that $\lim_{n\to\infty} R^P_{LH} > \lim_{n\to\infty} R^p_{HL}$ and $\lim_{n\to\infty} R^P_{LH} > \lim_{n\to\infty} R(m, n, r)$. Then, by Lemma 13 there exist integers n^*_2 and n^*_3 such that $R^p_{LH} > R^p_{HL}$ for all $n \geq n^*_2$ and $R^p_{LH} > R(m, n, r)$ for all $n \geq n^*_3$. Now choose $n^*_1 = \max\{m + r, n^*_2, n^*_3\}$. Thus, for any pair of $(m, r) < (\infty, \infty)$ there exists an integer n^*_1 such that $R^p_{LH} > R^s_{HL}$, $R^p_{LH} > R^p_{HL}$ and $R^p_{LH} > R(m, n, r)$ for all $n \geq n^*_1$.

Similarly, we have $R^s_{LH} > R^p_{HL}$ for all $n \geq m - r$ by Propositions 3 and 4. Also, Lemma 11 implies that $\lim_{n\to\infty} R^S_{LH} > \lim_{n\to\infty} R^S_{HL}$ and $\lim_{n\to\infty} R^S_{LH} > \lim_{n\to\infty} R(m, n, r)$. By lemma 13, there exist integers n^*_5 and n^*_6 such that $R^s_{LH} > R^s_{HL}$ for all $n \geq n^*_5$ and $R^S_{LH} > R(m, n, r)$ for all $n \geq n^*_6$. Choose $n^*_4 = \max\{m - r, n^*_5, n^*_6\}$. Thus, for any pair of $(m, r) < (\infty, \infty)$ there exists an integer n^*_4 such that $R^s_{LH} > R^p_{HL}$, $R^s_{LH} > R^s_{HL}$ and $R^s_{LH} > R(m, n, r)$ for all $n \geq n^*_4$.

Therefore, we have $R^p_{LH} > \max\{R^s_{HL},\ R^p_{HL},\ R(m, n, r)\}$ and $R^s_{LH} > \max\{R^s_{HL},\ R^p_{HL},\ R(m, n, r)\}$ for all $n \geq n^* = \max\{n^*_1,\ n^*_4\}$.

(ii) Similar to the proof of (i). ■

Appendix B: Appendix for Chapter II

Proof of Lemma 1. Suppose that the outcome of the disclosure game at round $T-i$ is (O_{T-i}, $R_{T-i}(\theta_2)$) and the beliefs are $\mu_{T-i}(\theta_k = (T-i-1)\Delta | \sigma^l_{T-i} = 0) = 1$ for $k, l = 1, 2$. Assume that firm 1 offers O_{T-i} and firm 2 plays $R_{T-i}(\theta_2)$ at the bargaining game of $T-i$. Then, in the equilibrium firm 2 accepts the offer and firm 1 becomes the owner of the RJV. The value of the RJV at $T-i$ for firm 1 with θ_1 is

$$RJV_{T-i}(\theta_1) = \Pi_m \left[2(\theta_1 + (T-i-1)\Delta) - (\theta_1 + (T-i-1)\Delta)^2\right]$$

Thus, its payoff is

$$\begin{aligned} P_{T-i}(\theta_1) &= RJV_{T-i}(\theta_1) - O_{T-i} \\ &= \Pi_m \left[2(\theta_1 + (T-i-1)\Delta) - (\theta_1 + (T-i-1)\Delta)^2\right] \\ &- \Pi_m (2T-2i-1)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1)) \end{aligned}$$

Let $OP_{T-i}(\theta_1)$ be the outside option of firm 1 at the bargaining game of stage $T-i$. That is, if it decides not to make an offer or its offer is rejected, firm 1's payoff would be $OP_{T-i}(\theta_1)$. We have

$$OP_{T-i}(\theta_1) = \Pi_m(\theta_1 + (T-i-1)\Delta)(1 - (2T-2i-1)\Delta)$$

I now compare $P_{T-i}(\theta_1)$ and $OP_{T-i}(\theta_1)$. We have

$$\begin{aligned} P_{T-i}(\theta_1) - OP_{T-i}(\theta_1) &= \Pi_m [(\theta_1 + (T-i-1)\Delta) - (2T-2i-1)\Delta \\ &+ (2T-2i-1)(2T - \frac{3}{2}i - 1)\Delta^2 + \\ &+ (\theta_1 + (T-i-1)\Delta)(2T-2i-1)\Delta - (\theta_1 + (T-i-1)\Delta)^2] \\ &= \Pi_m [(\theta_1 - (T-i)\Delta) - (\theta_1 + (T-i-1)\Delta)(\theta_1 - (T-i)\Delta) \\ &+ (2T-2i-1)(2T - \frac{3}{2}i - 1)\Delta^2] \\ &= \Pi_m[(\theta_1 - (T-i)\Delta)(1 - \theta_1 - (T-i-1)\Delta) \\ &+ (2T-2i-1)(2T - \frac{3}{2}i - 1)\Delta^2] \end{aligned}$$

Since $\theta_1 \geq T-i$ and θ_1, $(T-i-1)\Delta \leq \frac{1}{2}$, we get $P_{T-i}(\theta_1) > OP_{T-i}(\theta_1)$ for all $\theta_1 \in \Theta_{T-i}$ and $i = 0, 1, ..., T-1$. This implies that firm 1 has no incentive to deviate at $T-i$ given

$\mu_{T-i}(\theta_2 = (T-i-1)\Delta|\sigma^2_{T-i} = 0) = 1$ and firm 2 plays $R_{T-i}(\theta_2)$. Moreover, it is clear that $R_{T-i}(\theta_2)$ is a best response of firm 2 given that firm 1 offers O_{T-i}. Therefore, $(O_{T-i}, R_{T-i}(\theta_2))$ is an equilibrium of the bargaining game of round $T-i$ for all $i = 0, 1, ..., T-1$. ■

Proof of Lemma 2. I show that $EP_{T-i}((T-j)\Delta) - D_{T-i}((T-j)\Delta) \geq 0$ for $i = 1, ...T-1$ and $j = 1, ..i$.

We have

$$\begin{aligned}
EP_{T-i}((T-j)\Delta) - D_{T-i}((T-j)\Delta) = \Pi_m[\{\sum_{k=j}^{i} \frac{1}{i+2}(2(2T-j-k-1)\Delta - (2T-j-k-1)^2\Delta^2 \\
-(2T-2k-1)\Delta(1-\Delta(2T-\frac{3}{2}k-1)))\} + \frac{1}{i+2}(2T-2j)\Delta(1-\Delta(2T-2j)) \\
+\frac{j}{i+2}(2T-2(j-1)-1)\Delta(1-\Delta(2T-\frac{3(j-1)}{2}-1)) \\
-\frac{1}{i+2}(2T-j-i-1)\Delta(1-\Delta(2T-2i-2)) - \frac{i+1}{i+2}(2T-i-j)\Delta(1-\Delta(2T-\frac{3}{2}i-1))] \\
= \Pi_m[\frac{\Delta^2(1+i-j)(24+36i^2+35j+20j^2+49i+32ij-78Ti-96T-54Tj+48T^2}{12(2+i)} \\
+\frac{\Delta(1+i-j)12i}{12(2+i)}]
\end{aligned}$$

<u>Case I</u>: $j = 2, ...i$ and $i = 2, ...T-1$. Set

$$A(i,j) = 24 + 36i^2 + 35j + 20j^2 + 49i + 32ij - 78Ti - 96T - 54Tj + 48T^2$$

<u>Subcase I</u>: Assume that $A(i,j) \geq 0$.Then we have

$$EP_{T-i}((T-j)\Delta) - D_{T-i}((T-j)\Delta) = \Pi_m \frac{\Delta(1+i-j)12i + \Delta^2(1+i-j)A(i,j)}{12(2+i)} \geq 0$$

since $j \leq i$.

<u>Subcase II</u>: Assume that $A(i,j) < 0$. I show that

$$\Delta(1+i-j)12i + \Delta^2(1+i-j)A(i,j) \geq 0$$

Note that

$$\Delta(1+i-j)12i + \Delta^2(1+i-j)A(i,j) \geq 0 \Longleftrightarrow 12i + \Delta A(i,j) \geq 0$$

Since $\Delta \leq \frac{1}{2T}$, we have

$$12i + \Delta A(i,j) \geq 12i + \frac{1}{2T}A(i,j)$$

Now consider

$$12i + \frac{1}{2T}A(i,j) = \frac{48T^2 + T(-54i - 96 - 54j) + (24 + 26i^2 + 35j + 20j^2 + 49i + 32ji)}{2T}$$

Set

$$48T^2 + T(-54i - 96 - 54j) + (24 + 26i^2 + 35j + 20j^2 + 49i + 32ji) = 0$$

and solve for T. The discriminant is $X = (-54i-96-54j)^2 - 196(24+26i^2+35j+20j^2+49i+32ji) = 12(384-173i^2+304j-77j^2+80i-26ji)$. Since $\frac{\partial X}{\partial j} = 12(304-154j-26i) < 0$ for all $j = 2,3,...,i$ and for all $i = 3,...,T-1$, we have $X \leq 684-173i^2+28i < 0$ for all $j = 2,3,...,i$ and for all $i = 3,...,T-1$. Thus, the quadratic equation $48T^2 + T(-54i - 96 - 54j) + (24 + 26i^2 + 35j + 20j^2 + 49i + 32ji) \geq 0$ all $j = 2,3,...,i$ and for all $i = 3,...,T-1$. This implies that $EP_{T-i}((T-j)\Delta) - D_{T-i}((T-j)\Delta) \geq 0$ for $i = 3,...,T-1$ and $j = 2,...,i$.

Now I prove that $EP_{T-i}((T-j)\Delta) - D_{T-i}((T-j)\Delta) \geq 0$ when $j = 2$ and $i = 2$. Note that $EP_{T-2}((T-2)\Delta) - D_{T-2}((T-2)\Delta) = \Pi_m \frac{\Delta+\Delta^2(2T-15T+21)}{2}$. Since $2T - 15T + 21 \geq 0$ for all $T \geq 6$, we have $EP_{T-2}((T-2)\Delta) - D_{T-2}((T-2)\Delta) \geq 0$ for all $T \geq 6$.

If $T = 3$, then $EP_{T-2}((T-2)\Delta) - D_{T-2}((T-2)\Delta) = \Pi_m \frac{(1-6\Delta)\Delta}{2} \geq 0$ since $\Delta \leq \frac{1}{2T} = \frac{1}{6}$.

If $T = 4$, then $EP_{T-2}((T-2)\Delta) - D_{T-2}((T-2)\Delta) = \Pi_m \frac{(1-7\Delta)\Delta}{2} \geq 0$ since $\Delta \leq \frac{1}{2T} = \frac{1}{8}$.

If $T = 5$, then $EP_{T-2}((T-2)\Delta) - D_{T-2}((T-2)\Delta) = \Pi_m \frac{(1-4\Delta)\Delta}{2} \geq 0$ since $\Delta \leq \frac{1}{2T} = \frac{1}{10}$.

Therefore, $EP_{T-i}((T-j)\Delta) - D_{T-i}((T-j)\Delta) \geq 0$ for $i = 2,...,T-1$ and $j = 2,...,i$.

Case II: $j = 1$ and $i = 1,2,,,T-1$. Then, we have

$$EP_{T-i}((T-1)\Delta) - D_{T-i}((T-1)\Delta) = \Pi_m \frac{\Delta 12i^2 + \Delta^2 i(79 + 26i^2 + 81i - 78Ti - 150T + 48T^2)}{12(2+i)}$$

Set

$$A(i,1) = i(48T^2 + T(-150 - 78i) + 79 + 26i^2 + 81i)$$

Subcase I: Assume that $A(i,1) \geq 0$. Then, we have

$$EP_{T-i}((T-1)\Delta) - D_{T-i}((T-1)\Delta) = \Pi_m \frac{\Delta 12i^2 + \Delta^2 A(i,1)}{12(2+i)} \geq 0.$$

Subcase II: Assume that $A(i,1) < 0$. I will show that

$$\Delta 12i^2 + \Delta^2 A(i,1) \geq 0$$

Note that

$$\Delta 12i^2 + \Delta^2 A(i,1) \geq 0 \Longleftrightarrow 12i^2 + \Delta A(i,1) \geq 0$$

Since $\Delta \leq \frac{1}{2T}$, we have

$$12i^2 + \Delta A(i,1) \geq 12i^2 + \frac{1}{2T}A(i,1) = \frac{48iT^2 + T(-54i^2 - 150i) + 79i + 26i^3 + 81i^2}{2T}$$

Set

$$48iT^2 + T(-54i^2 - 150i) + 79i + 26i^3 + 81i^2 = 0$$

and solve for T. The discriminant is $X = (-54i^2 - 150i)^2 - 192i(79i + 26i^3 + 81i^2) = 12i^2(611 + 54i - 173i^2)$. Since $X < 0$ for all $i \geq 3$, we have $\Delta 12i^2 + \Delta^2 A(i,1) \geq 0$ for all $T \geq 3$ and for all $i \geq 3$.

If $i = 2$, then $EP_{T-2}((T-1)\Delta) - D_{T-2}((T-1)\Delta) = \Pi_m \frac{\Delta 8 + \Delta^2(115 - 102T + 16T^2)}{8} \geq 0$ for all $T \geq 3$.

If $i = 1$,then $EP_{T-1}((T-1)\Delta) - D_{T-1}((T-1)\Delta) = \Pi_m \frac{\Delta 2 + \Delta^2(31 - 38T + 8T^2)}{6} \geq 0$ for all $T \geq 3$.

Therefore, $EP_{T-i}((T-1)\Delta) - D_{T-i}((T-1)\Delta) \geq 0$ for $i = 1, 2, ..., T-1$. ■

Proof of Lemma 3. Suppose that firm 1 with type $T\Delta$ does not disclose at stage $T - i$. Then, with probability $\frac{1}{i+2}$ the other firm does not disclose, either since it is type $(T - i - 1)\Delta$). In that case, firm 1 will get its outside option $\Pi_m(2T - i - 1)\Delta(1 - \Delta(2T - 2i - 2))$. In all the other cases, which occur with probability $\frac{i+1}{i+2}$ the other firm has at least $(T - i)\Delta$, so firm 1 will recieve the offer $O_{T-i} = \Pi_m(2T - 2i - 1)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1))$. However, it will reject this offer since firm 1's expected outside option $\Pi_m(2T - i)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1))$ is greater than O_{T-i}. Thus, if firm 1 with type $T\Delta$ does not disclose stage $T - i$, it will get

$$D_{T-i}(T\Delta) = \Pi_m[\frac{1}{i+2}(2T - i - 1)\Delta(1 - \Delta(2T - 2i - 2)) + \frac{i+1}{i+2}(2T - i)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1))]$$

I now show that $EP_{T-i}(T\Delta) - D_{T-i}(T\Delta) \geq 0$ for all $i = 0, 1, ..., T-1$.

We have

$$\begin{aligned}
EP_{T-i}(T\Delta) - D_{T-i}(T\Delta) &= \Pi_m[\{\sum_{k=0}^{i} \frac{1}{i+2} P_{T-k}(T\Delta)\} + \frac{1}{2}(2T\Delta - 2T^2\Delta^2) \\
&\quad - \frac{1}{i+2}(2T - i - 1)\Delta(1 - \Delta(2T - 2i - 2)) - \frac{i+1}{i+2}(2T - i)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1))] \\
&= \Pi_m[\{\sum_{k=1}^{i} \frac{1}{i+2}(2(2T - i - 1)\Delta - (2T - i - 1)^2\Delta^2 - (2T - 2i - 1)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1)))\} \\
&\quad + \frac{2}{i+2}(\frac{1}{2}(2T - 1)\Delta + \frac{1}{2}(2T\Delta - 2\Delta^2 T^2)) - [\frac{1}{i+2}(2T - i - 1)\Delta(1 - \Delta(2T - 2i - 2)) \\
&\quad + \frac{i+1}{i+2}(2T - i)\Delta(1 - \Delta(2T - \frac{3}{2}i - 1)] \\
&= \Pi_m \frac{\Delta(12i + 12i^2) + \Delta^2(26i^3 + 75i^2 - 78Ti^2 + 24 - 96T + 72T^2 + 73i - 174Ti + 48iT^2)}{12(2+i)}
\end{aligned}$$

Set

$$A(i) = 26i^3 + 75i^2 - 78Ti^2 + 24 - 96T + 72T^2 + 73i - 174Ti + 48iT^2$$

Case I: Assume that $A(i) \geq 0$.Then we have

$$EP_{T-i}(T\Delta) - D_{T-i}(T\Delta) = \Pi_m \frac{\Delta(12i + 12i^2) + \Delta^2 A(i)}{12(2+i)} \geq 0$$

Case II: Assume that $A(i) < 0$. I show that

$$\Delta(12i + 12i^2) + \Delta^2 A(i) \geq 0$$

Notice that

$$\Delta(12i + 12i^2) + \Delta^2 A(i) \geq 0 \Longleftrightarrow (12i + 12i^2) + \Delta A(i) \geq 0$$

Since $\Delta \leq \frac{1}{2T}$, we have

$$(12i + 12i^2) + \Delta A(i) \geq (12i + 12i^2) + \frac{1}{2T} A(i)$$

Now consider

$$(12i + 12i^2) + \frac{1}{2T} A(i) = \frac{T^2(72 + 48i) + T(-54i^2 - 96 - 150i) + 26i^3 + 75i^2 + 24 + 73i}{2T}$$

Set

$$T^2(72 + 48i) + T(-54i^2 - 96 - 150i) + 26i^3 + 75i^2 + 24 + 73i = 0$$

and solve for T. Notice that this quadratic equation has no solution since the discriminant $X = (-54i^2 - 96 - 150i)^2 - 4(72 + 48i)(26i^3 + 75i^2 + 24 + 73i) = -12(-192 - 264i + 229i^2 + 474i^3 + 173i^4)$ is negative for all $i = 1, 2, ..., T-1$. Thus, $T^2(72+48i)+T(-54i^2-96-150i)+26i^3+75i^2+24+73i \geq 0$ for all $i = 1, ..., T-1$ and for all T. Also, we have to show that firm 1 with type $T\Delta$ has no incentive to deviate at the last round, either. To see this set $i = 0$. Then, we have

$$EP_T(T\Delta) - D_T(T\Delta) = \Pi_m \frac{\Delta^2(24 - 96T + 72T^2)}{24} \geq 0$$

Therefore, we have $EP_{T-i}(T\Delta) - D_{T-i}(T\Delta) \geq 0$ for all $i = 0, 1, ..., T-1$. ■

Proof of Lemma 5. First note that the expected outside option for firm 1 with type θ_1 before the game starts is

$$O(\theta_1) = E(\Pi_m \theta_1 (1 - \theta_2) | \theta_2 \in \Theta) = \Pi_m \theta_1 (1 - \frac{1}{2}\Delta T)$$

I show that $EP(\theta_1) - O(\theta_1) \geq 0$ for all $\theta_1 \in \Theta$.

(i) $\theta_1 = T\Delta$

Consider

$$EP(T\Delta) - O(T\Delta) = \Pi_m \frac{12T^2\Delta - \Delta^2(5T + 3T^2 + 4T^3)}{12(1+T)}$$

We have

$$EP(T\Delta) - O(T\Delta) \geq 0 \Longleftrightarrow 12T^2 - \Delta(5T + 3T^2 + 4T^3) \geq 0$$

Set

$$A(0) = 5T + 3T^2 + 4T^3$$

If $A(0) < 0$, then $E(\Delta T) - O(\Delta T) \geq 0$. Suppose that $A(0) \geq 0$. Then, we have

$$12T^2 - \Delta(5T + 3T^2 + 4T^3) \geq 12T^2 - \frac{1}{2T}(5T + 3T^2 + 4T^3) = 10T^2 - \frac{3}{2}T - \frac{5}{2} \geq 0$$

for all $T \geq 3$.

Therefore, $EP(T\Delta) - O(T\Delta) \geq 0$.

(ii) $\theta_1 = (T-1)\Delta$

Consider

$$EP((T-1)\Delta) - O((T-1)\Delta) = \Pi_m \frac{\Delta(12T^2 - 12T + 12) - \Delta^2(4T^3 + 45T^2 - 61T + 24)}{12(1+T)}$$

We have

$$EP((T-1)\Delta) - O((T-1)\Delta) \geq 0 \Longleftrightarrow (12T^2 - 12T + 12) - \Delta(4T^3 + 45T^2 - 61T + 24) \geq 0$$

Set

$$A(1) = 4T^3 + 45T^2 - 61T + 24$$

If $A(1) < 0$, then $E(\Delta(T-1)) - O(\Delta(T-1)) \geq 0$. Suppose that $A(1) \geq 0$. Then, we have

$$\begin{gathered}(12T^2 - 12T + 12) - \Delta(4T^3 + 45T^2 - 61T + 24) \geq \\ (12T^2 - 12T + 12) - \frac{1}{2T}(4T^3 + 45T^2 - 61T + 24) = 10T^2 - \frac{69}{2}T - \frac{85}{2} - \frac{12}{T} \geq 0\end{gathered}$$

for all $T \geq 3$.

Therefore, $EP((T-1)\Delta) - O((T-1)\Delta) \geq 0$.

(iii) $\theta_1 = (T-j)\Delta$ where $j = 2, 3, ..., T-1)$

Consider

$$EP((T-j)\Delta) - O((T-j)\Delta) = \Pi_m[\frac{\Delta(12T^2+24T-12Tj-12j)}{12(1+T)}$$
$$-\frac{\Delta^2(4T^3+27T^2+5T+18T^2j-24Tj-42Tj^2+20j^3+3j^2+j)}{12(1+T)}]$$

We have

$$E(\Delta(T-j)) - O(\Delta(T-j)) \geq 0 \Longleftrightarrow (12T^2+24T-12Tj-12j)$$
$$-\Delta(4T^3+27T^2+5T+18T^2j-24Tj-42Tj^2+20j^3+3j^2+j) \geq 0$$

First note that $(12T^2+24T-12Tj-12j) > 0$ (it is decreasing in j so $12T^2+24T-12Tj-12j \geq 12T > 0$). Set

$$A(j) = 4T^3+27T^2+5T+18T^2j-24Tj-42Tj^2+20j^3+3j^2+j$$

If $A(j) < 0$, then $E(\Delta(T-j)) - O(\Delta(T-j)) \geq 0$. Suppose that $A(j) \geq 0$. Then we have

$$\begin{aligned}
&(12T^2+24T-12Tj-12j) - \Delta(4T^3+27T^2+5T+18T^2j-24Tj-42Tj^2+20j^3+3j^2+j) \\
&\geq (12T^2+24T-12Tj-12j) - \frac{1}{2T}(4T^3+27T^2+5T+18T^2j-24Tj-42Tj^2+20j^3+3j^2+j) \\
&= 10T^2 + T(\frac{21}{2}-21j) + 21j^2 - \frac{5}{2} - \frac{20j^3+3j^2+j}{2T}
\end{aligned}$$

Now consider

$$10T^2 + T(\frac{21}{2}-21j) + 21j^2 - \frac{5}{2} - \frac{20j^3+3j^2+j}{2T}$$

Note that

$$10T^2 + T(\frac{21}{2}-21j) + 21j^2 - \frac{5}{2} - \frac{20j^3+3j^2+j}{2T} \geq 0 \Longleftrightarrow$$
$$20T^3 + T^2(21-42j) + 42Tj^2 - 5T - 20j^3 - 3j^2 - j \geq 0$$

Now we have

$$20T^3+T^2(21-42j)+42Tj^2-5T-20j^3-3j^2-j = (T-j)(20T^2+20j^2-22Tj)+21T^2-5T-3j^2-j$$

Notice that the second term is nonnegative for all $j = 2, ..., T$ and $T \geq 3$.

Now I show that the first term is nonnegative, too. Solve $20T^2+20j^2-22Tj$ for T given j. The quadratic equation has no real root since $(22j)^2 - (40j)^2 < 0$. Thus, $20T^2+20j^2-22Tj \geq 0$.

Therefore, $EP((T-j)\Delta) - O((T-j)\Delta) \geq 0$ for all $j = 2, ..., T$ and $T \geq 3$.

(iv) $\theta_1 = 0$

Firm 1 with type 0 has outside option of zero before the game starts. If it participates, the opponent disloses Δ at the first stage with probability $\frac{T}{T+1}$. Thus, its expected outside option becomes $E(\Pi_m \Delta(1-\theta_2)|\theta_2 \in \Theta_1) = \Delta(1 - \Delta(\frac{T}{2} + \frac{1}{2}))$ which is greater than zero. ∎

Proof of Proposition 1. By lemmas (1)-(5). ∎

Proof of Lemma 6. We have

$$\frac{CS_B}{\Pi_m + CS_m} = \frac{\frac{c^{1-a}}{a-1}}{\frac{(a-1)^{a-1}}{c^{a-1}a^a} + \frac{(ca)^{1-a}}{(a-1)^{2-a}}}$$

By simplifying we get

$$\frac{CS_B}{\Pi_m + CS_m} = \frac{(a-1)^{1-a}a^a}{2a-1}$$

Thus, we obtain $\lim\limits_{a\to 1} \frac{CS_B}{\Pi_m + CS_m} = \lim\limits_{a\to 1} \frac{(a-1)^{1-a}a^a}{2a-1} = 1$ and $\lim\limits_{a\to\infty} \frac{CS_B}{\Pi_m + CS_m} = \lim\limits_{a\to\infty} \frac{(a-1)^{1-a}a^a}{2a-1} = \frac{e}{2}$.

To see that $\frac{CS_B}{\Pi_m + CS_m}$ is increasing for all $a > 1$ first consider the derivative of $\frac{CS_B}{\Pi_m + CS_m}$. We have $\left(\frac{CS_B}{\Pi_m + CS_m}\right)' = \frac{(2a-1)\ln\frac{a}{a-1} - 2}{(2a-1)^2(a-1)^{2a-2}}$. Note that the denominator is postive for all $a > 1$. I now show that the numerator is non-negative for all $a > 1$. Set $f(a) = \ln\frac{a}{a-1} - \frac{2}{2a-1}$. Consider $f'(a) = -\frac{1}{a(a-1)} + \frac{4}{(2a-1)^2}$, which is always negative. This implies that $f(a)$ is decreasing. Thus, for all $a > 1$ we have $f(a) \geq \lim\limits_{a\to\infty} f(a) = 0$ ∎

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