

# THE Q-DEFORMED KNIZHNIK–ZAMOLODCHIKOV–BERNARD HEAT EQUATION

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ABSTRACT. We introduce a  $q$ -deformation of the genus one  $sl_2$  Knizhnik–Zamolodchikov–Bernard heat equation. We show that this equation for the dependence on the moduli of elliptic curves is compatible with the qKZB equations, which give the dependence on the marked points.

## 1. INTRODUCTION

The Knizhnik–Zamolodchikov–Bernard equations are a system of differential equations arising in conformal field theory on Riemann surfaces. For each  $g, n \in \mathbb{Z}_{\geq 0}$ , a simple complex Lie algebra  $\mathfrak{g}$ ,  $n$  highest weight  $\mathfrak{g}$ -modules  $V_i$ , and a complex parameter  $\kappa$ , we have such a system of equations. In the case of genus  $g = 1$ , they have the form

$$(1) \quad \kappa \partial_{z_j} v = - \sum_{\nu} h_{\nu}^{(j)} \partial_{\lambda_{\nu}} v + \sum_{l:l \neq j} r(z_j - z_l, \lambda)^{(j,l)} v.$$

The unknown function  $v$  takes values in the zero weight space  $V[0] = \bigcap_{x \in \mathfrak{h}} \text{Ker}(x)$  of the tensor product  $V = V_1 \otimes \cdots \otimes V_n$  with respect to a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . It depends on variables  $z_1, \dots, z_n \in \mathbb{C}$  and  $\lambda = \sum \lambda_{\nu} h_{\nu} \in \mathfrak{h}$ , where  $(h_{\nu})$  is an orthonormal basis of  $\mathfrak{h}$ , with respect to a fixed invariant bilinear form. The notation  $x^{(j)}$  for  $x \in \text{End}(V_j)$  or  $x \in \mathfrak{g}$  means  $1 \otimes \cdots \otimes x \otimes \cdots \otimes 1$ . Similarly  $x^{(i,j)}$  denotes the action on the  $i$ th and  $j$ th factor of  $x \in \text{End}(V_i \otimes V_j)$ .

The “ $r$ -matrix”  $r \in \mathfrak{g} \otimes \mathfrak{g}$  obeys

$$r(z, \lambda) + r(-z, \lambda)^{(2,1)} = 0, \quad [r(z, \lambda), h \otimes 1 + 1 \otimes h] = 0, \forall h \in \mathfrak{h},$$

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with  $(\sum_i x_i \otimes y_i)^{(21)} = \sum_i y_i \otimes x_i$ , and is a solution of the classical dynamical Yang–Baxter equation [FW] ( $r^{(1,2)} = r(z_1 - z_2, \lambda) \otimes 1 \in U\mathfrak{g}^{\otimes 3}$  etc.)

$$\sum_{\nu} \partial_{\lambda_{\nu}} r^{(1,2)} h_{\nu}^{(3)} + \sum_{\nu} \partial_{\lambda_{\nu}} r^{(2,3)} h_{\nu}^{(1)} + \sum_{\nu} \partial_{\lambda_{\nu}} r^{(3,1)} h_{\nu}^{(2)} \\ - [r^{(1,2)}, r^{(1,3)}] - [r^{(1,2)}, r^{(2,3)}] - [r^{(1,3)}, r^{(2,3)}] = 0.$$

As a consequence, the KZB equations (1) are compatible, meaning that if the equations are written as  $\nabla_j v = 0$ , then the differential operators  $\nabla_j$  commute with each other. The solutions of the classical dynamical Yang–Baxter equation arising in conformal field theory are parametrized by the modulus  $\tau$  in the upper half plane and can be expressed in terms of theta functions, see [FW, FV1].

A difference version of this story was proposed in [F]: Suppose that for an Abelian complex Lie algebra  $\mathfrak{h}$  we have  $\mathfrak{h}$ -modules  $V_i$ ,  $i = 1, \dots, n$  with a weight decomposition  $V_i = \bigoplus_{\mu \in \mathfrak{h}^*} V_i[\mu]$  into finite dimensional weight spaces  $V_i[\mu]$ . Then we say that meromorphic functions  $R_{ij}(z, \lambda)$  of  $z \in \mathbb{C}$  and  $\lambda \in \mathfrak{h}^*$  with values in  $\text{End}_{\mathfrak{h}}(V_i \otimes V_j)$ , ( $1 \leq i \neq j \leq n$ ) form a *system of dynamical R-matrices* if they obey the (quantum) dynamical Yang–Baxter equation

$$R_{ij}(z_1 - z_2, \lambda - 2\eta h^{(3)})^{(12)} R_{ik}(z_1 - z_3, \lambda)^{(13)} R_{jk}(z_2 - z_3, \lambda - 2\eta h^{(1)})^{(23)} \\ = R_{jk}(z_2 - z_3, \lambda)^{(23)} R_{ik}(z_1 - z_3, \lambda - 2\eta h^{(2)})^{(13)} R_{ij}(z_1 - z_2, \lambda)^{(12)},$$

in  $\text{End}(V_i \otimes V_j \otimes V_k)$  for all  $i < j < k$  and are “unitary”:

$$R_{ij}(z, \lambda) R_{ji}(-z, \lambda)^{(21)} = \text{Id}_{V_i \otimes V_j}.$$

We adopt a standard notation: for instance,  $R(z, \lambda - 2\eta h^{(3)})^{(12)}$  acts on a tensor  $v_1 \otimes v_2 \otimes v_3$  as  $R(z, \lambda - 2\eta \mu_3) \otimes \text{Id}$  if  $v_3$  has weight  $\mu_3$ .

The deformation parameter (“Planck’s constant”) is here  $\eta$ . If we have a family of dynamical  $R$ -matrices depending on  $\eta$  such that  $R_{ij} = \text{Id}_{V_i \otimes V_j} + 2\eta r_{ij} + O(\eta^2)$  as  $\eta \rightarrow 0$ , we recover the classical dynamical Yang–Baxter equation and the unitarity condition for  $r_{ij}$ , i. e. the properties that  $r$  obeys, viewed as an element of  $\text{End}(V_i \otimes V_j)$ .

If we have a system of dynamical  $R$ -matrices  $R_{ij}$  we can then construct a compatible system of difference equations, the qKZB equations for a function  $v(z_1, \dots, z_n, \lambda) \in V[0]$ . They are a dynamical version of the I. Frenkel–Reshetikhin qKZ equations [FR], and their semiclassical limit are the KZB equations. Their construction is reviewed in 2.1 below.

The main examples of solutions of the classical and of the quantum dynamical Yang–Baxter equations are associated with elliptic curves. In the quantum case, they can be viewed as intertwining operators between tensor products of representations of elliptic quantum groups taken in different orders [FV2]. In the rank one case (one-dimensional  $\mathfrak{h}$ ) explicit expressions for  $R$  matrices  $R_{\Lambda, M}$  depending on two “highest weights”  $\Lambda, M \in \mathbb{C}$  are known. They are associated to pairs of evaluation Verma modules [FV2] for the elliptic quantum group  $E_{\tau, \eta}(sl_2)$  and were computed using the functional realization of

these modules [FTV1]. If  $n$  highest weights  $\Lambda_1, \dots, \Lambda_n \in \mathbb{C}$  are given, then  $R_{ij} = R_{\Lambda_i, \Lambda_j}$  form a system of dynamical  $R$  matrices as described above.

Hypergeometric solutions of the corresponding qKZB equations were introduced and studied in [FTV1], [FTV2], [FV4]. See also [T] where similar equations are studied and solved. Special cases of these equations appear in the statistical mechanics of RSOS models. Form factors and correlation functions in the infinite volume limit are conjectured to obey qKZB equations. In these cases explicit formulae were proposed by Lukyanov and Pugai [LP].

The subject of this paper is a deformation of the KZB heat equation: in the classical case, additionally to the KZB equations above, that are associated to the variation of the marked points on the elliptic curve, one also has an equation associated to the variation of the modulus  $\tau$  of the elliptic curve. The function  $v$  also depends on  $\tau$  and one has an additional equation, compatible with the KZB equations, the KZB *heat equation*

$$4\pi i \kappa \partial_\tau v = \Delta_\lambda v + \frac{1}{2} \sum_{i,j} s(z, \lambda, \tau)^{(ij)} v.$$

for some  $s \in \mathfrak{g} \otimes \mathfrak{g}$ . Here  $\Delta_\lambda$  denotes the Laplacian of  $\mathfrak{h}$  corresponding to the invariant bilinear form. For example, if  $n = 1$  then this equation reduces to

$$4\pi i \kappa \partial_\tau v = (\Delta_\lambda - \sum_{\alpha \in \Delta} \wp(\alpha(\lambda), \tau) e_\alpha e_{-\alpha}) v,$$

where  $e_\alpha$  are properly normalized root vectors and  $\wp$  is the Weierstrass function.

In this paper we propose a discrete version of the KZB heat equation in the rank one case. The heat operator is an integral operator, whose kernel is given in terms of hypergeometric integral solutions of the qKZB equations of [FTV2].

In Sect. 2 we review the qKZB equations and their hypergeometric solutions. Then we introduce the elliptic Shapovalov form, which is an ingredient in the integral operator, and the qKZB heat equation in Sect. 3. We prove that it is compatible with the qKZB equations, discuss its properties and show in Sect. 4, in an illustrative example, that its semiclassical limit coincides with the KZB heat equation. Finally, in Sect. 5 we study the special case where the step of the difference equation is a negative integer multiple of the deformation parameter. In this case, the semiclassical limit gives the KZB equations with positive integer  $\kappa$ , the situation arising in conformal field theory. In this case the KZB equations are defined on sections of the finite dimensional vector bundle of conformal blocks, of which we describe a difference analogue in simple cases.

It is very likely that the hypergeometric solutions of the qKZB equations are also solutions of the qKZB heat equation. However, we were able to prove this only in the case where the sum of the highest weights is two. In this case the hypergeometric integrals are one-dimensional.

In a sequel to this paper, we show that integral operators of the kind introduced in this paper can also be used to describe the transformation properties of hypergeometric solutions under the modular group. In fact it turns out that the hypergeometric solutions,

at least if  $\sum \Lambda_i = 2$  obey remarkable identities under transformations of the modulus  $\tau$  and the step  $p$  by  $SL(3, \mathbb{Z})$  acting on  $\mathbb{C}P^2$  with affine coordinates  $\tau, p$ . These identities give both the solutions and the monodromy of the solutions. The whole picture results in a non-Abelian version of the properties of the elliptic gamma functions, which is a generalized Jacobi modular form for  $SL(3, \mathbb{Z}) \times \mathbb{Z}^3$  in the sense of [FV5].

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## 2. HYPERGEOMETRIC SOLUTIONS OF THE qKZB EQUATIONS

**2.1. The qKZB equations.** Fix  $\vec{\Lambda} = (\Lambda_1, \dots, \Lambda_n) \in \mathbb{R}^n$  such that  $m = \sum_{i=1}^n \Lambda_i/2$  is a nonnegative integer, and a complex number  $\eta$ . Unless stated otherwise, we will assume that these parameters are generic.

Let  $\tau$  and  $p$  be generic complex numbers in the upper half plane.

Let  $V_{\Lambda_j}$  be the vector space with basis  $e_0, e_1, \dots$  equipped with the action of an operator  $h$  given by  $he_k = (\Lambda_j - 2k)e_k$ . We view  $V_{\Lambda_i}$  as a representation of the Abelian Lie algebra  $\mathfrak{h} = \mathbb{C}h$ .

To these data is associated a system of dynamical  $R$ -matrices and thus a system of qKZB difference equations. The  $R$ -matrices  $R_{\Lambda_j, \Lambda_k}(z, \lambda, \tau)$  [FV2] of  $E_{\tau, \eta}(sl_2)$  are endomorphisms of  $V_{\Lambda_j} \otimes V_{\Lambda_k}$ . Let  $V_{\vec{\Lambda}} = V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_n}$ . The qKZB equations are equations for a meromorphic function  $v(\vec{z}, \lambda)$  of  $\vec{z} \in \mathbb{C}^n$  and  $\lambda \in \mathbb{C}$  taking its values in the zero weight subspace  $V_{\vec{\Lambda}}[0] = \text{Ker}(\sum_{i=1}^n h^{(i)})$  of  $V_{\vec{\Lambda}}$  (this subspace is nontrivial since  $\sum \Lambda_i/2$  is assumed to be a nonnegative integer). It will be more convenient to view  $v$  as a function  $v(\vec{z})$  taking values in the space  $\mathcal{F}(V_{\vec{\Lambda}}[0])$  of meromorphic functions of  $\lambda \in \mathbb{C}$  with values in  $V_{\vec{\Lambda}}[0]$ . Let  $\delta_j, j = 1, \dots, n$  be the standard basis of  $\mathbb{C}^n$ . Then the qKZB equations have the form

$$v(\vec{z} + p\delta_i) = K_i(\vec{z}, \tau, p)v(\vec{z}), \quad i = 1, \dots, n.$$

The qKZB operators  $K_i(\vec{z}, \tau, p)$  act on the space  $\mathcal{F}(V_{\vec{\Lambda}}[0])$  and are given by

$$\begin{aligned} K_j(\vec{z}, \tau, p) &= R_{j, j-1}(z_j - z_{j-1} + p, \tau) \cdots R_{j, 1}(z_j - z_1 + p, \tau) \\ &\quad \Gamma_j R_{j, n}(z_j - z_n, \tau) \cdots R_{j, j+1}(z_j - z_{j+1}, \tau). \end{aligned}$$

The operators  $R_{j, k}(z, \tau)$  are defined by the formula

$$R_{j, k}(z, \tau) v(\lambda) = R_{\Lambda_j, \Lambda_k}(z, \lambda - 2\eta \sum_{l=1, l \neq j}^{k-1} h^{(l)}, \tau) v(\lambda),$$

and  $(\Gamma_j v)(\lambda) = v(\lambda - 2\eta\mu)$  if  $h^{(j)}v(\lambda) = \mu v(\lambda)$  and is extended by linearity to  $\mathcal{F}(V_{\vec{\Lambda}}[0])$ .

The qKZB system of difference equations is compatible, i.e., we have

$$(2) \quad K_j(\vec{z} + p\delta_l, \tau, p) K_l(\vec{z}, \tau, p) = K_l(\vec{z} + p\delta_j, \tau, p) K_j(\vec{z}, \tau, p),$$

for all  $j, l$ , as a consequence of the dynamical Yang–Baxter equations satisfied by the  $R$ -matrices. We also consider the “mirror” qKZB operators

$$\begin{aligned} K_j^\vee(\vec{z}, p, \tau) &= R_{j,j+1}^\vee(z_j - z_{j+1} + \tau, p) \cdots R_{j,n}^\vee(z_j - z_n + \tau, p) \\ &\quad \Gamma_j R_{j,1}^\vee(z_j - z_1, p) \cdots R_{j,j-1}^\vee(z_j - z_{j-1}, p), \end{aligned}$$

with

$$R_{j,k}^\vee(z, p) v(\lambda) = R_{\Lambda_j, \Lambda_k}(z, \lambda - 2\eta \sum_{l=k+1, l \neq j}^n h^{(l)}, p) v(\lambda),$$

The corresponding system of qKZB equations

$$v(\vec{z} + \tau \delta_j) = K_j^\vee(\vec{z}, p, \tau) v(\vec{z}), \quad j = 1, \dots, n,$$

is also compatible. In fact, if we write  $\vec{x}^\vee = (x_n, \dots, x_1)$  for any  $\vec{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$  and let  $P : V_{\vec{\Lambda}} \rightarrow V_{\vec{\Lambda}^\vee}$  be the linear map sending  $v_1 \otimes \cdots \otimes v_n$  to  $v_n \otimes \cdots \otimes v_1$ , then we have, adding the dependence on  $\vec{\Lambda}$  in the notation,

$$K_i^\vee(\vec{z}, p, \tau; \vec{\Lambda}) = P^{-1} K_{n+1-i}(\vec{z}^\vee, p, \tau; \vec{\Lambda}^\vee) P.$$

**2.2. Hypergeometric solutions.** In [FTV2] we constructed a “universal hypergeometric function”, which is a projective solution of the qKZB equations: it is a function  $u(\vec{z}, \lambda, \mu, \tau, p)$ , defined for generic values of the parameters  $\eta, \vec{\Lambda}$ , taking values in  $V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$  and obeying the equations

$$\begin{aligned} (3) \quad u(\vec{z} + \delta_j p, \tau, p) &= K_j(\vec{z}, \tau, p) \otimes D_j u(\vec{z}, \tau, p), \\ u(\vec{z} + \delta_j \tau, \tau, p) &= D_j^\vee \otimes K_j^\vee(\vec{z}, p, \tau) u(\vec{z}, \tau, p), \\ u(\vec{z} + \delta_j, \tau, p) &= u(\vec{z}, \tau, p). \end{aligned}$$

Here we view  $u$  as taking values in the space of functions of  $\lambda$  and  $\mu$  with values in  $V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$ .  $K_j$  acts on the variable  $\lambda$  and  $K_j^\vee$  on the variable  $\mu$ . The operators  $D_j, D_j^\vee$  act by multiplication by diagonal matrices  $D_j(\mu), D_j^\vee(\lambda)$ , respectively. For our purpose, the most convenient description of these matrices is in terms of the function

$$\alpha(\lambda) = \exp(-\pi i \lambda^2 / 4\eta).$$

We have, for  $j = 1, \dots, n$ ,

$$\begin{aligned} D_j(\mu) &= \frac{\alpha(\mu - 2\eta(h^{(j+1)} + \cdots + h^{(n)}))}{\alpha(\mu - 2\eta(h^{(j)} + \cdots + h^{(n)}))} e^{\pi i \eta \Lambda_j (\sum_{l=1}^{j-1} \Lambda_l - \sum_{l=j+1}^n \Lambda_l)}, \\ D_j^\vee(\lambda) &= \frac{\alpha(\lambda - 2\eta(h^{(1)} + \cdots + h^{(j-1)}))}{\alpha(\lambda - 2\eta(h^{(1)} + \cdots + h^{(j)}))} e^{-\pi i \eta \Lambda_j (\sum_{l=1}^{j-1} \Lambda_l - \sum_{l=j+1}^n \Lambda_l)}. \end{aligned}$$

These operators are diagonal in the basis of  $V_{\vec{\Lambda}}[0]$  formed by tensor products  $e_I = e_{i_1} \otimes \cdots \otimes e_{i_n}$  of basis vectors of the  $V_{\Lambda_k}$  so that  $\sum (\Lambda_k - 2i_k) = 0$ .

From  $u$  one can construct projective solutions (eigenfunctions of the corresponding difference operators) by taking coefficients of the basis vectors  $e_I$ . If  $D_i(\mu)e_I = d_{i,I}(\mu)e_I$ ,

$d_{i,I}(\mu) \in \mathbb{C}$ , and  $u = \sum u^I \otimes e_I$  then for any fixed  $I$  and  $\mu$ , the function  $\tilde{v}(\vec{z}, \lambda) = u^I(\vec{z}, \lambda, \mu, \tau, p)$  obeys  $\tilde{v}(\vec{z} + p\delta_i) = d_{i,I}(\mu)K_i(\vec{z}, \tau, p)\tilde{v}(\vec{z})$ . It follows that

$$v(\vec{z}, \lambda) = \prod_{i=1}^n d_{i,I}(\mu)^{-z_i/p} \tilde{v}(\vec{z}, \lambda),$$

is a true solution to the qKZB equations. The parameters  $I$  and  $\mu$  determine the multipliers, as is easily seen from the explicit expression for  $u$  below:

$$(4) \quad v(\vec{z} + \delta_i, \lambda) = d_{i,I}(\mu)^{-1/p} v(\vec{z}, \lambda), \quad v(\vec{z}, \lambda + 1) = e^{-\frac{\pi i(\mu + 2\eta m)}{2\eta}} v(\vec{z}, \lambda).$$

The second system of equations in (3) gives the monodromy of these solutions, see [FTV2].

The explicit expression for  $u$  is given by the following formulas.

$$(5) \quad u(\vec{z}, \lambda, \mu, \tau, p) = e^{-\frac{\pi i \lambda \mu}{2\eta}} \int \prod_{i,k} \Omega_{\eta \Lambda_k}(t_i - z_k, \tau, p) \prod_{i < j} \Omega_{-2\eta}(t_i - t_j, \tau, p) \\ \sum_{I,J} \omega_I(t, \vec{z}, \lambda, \tau) \omega_J^\vee(t, \vec{z}, \mu, p) dt_1 \cdots dt_m e_I \otimes e_J.$$

The phase function  $\Omega$  has the product formula

$$\Omega_a(z, \tau, p) = \prod_{j,k=0}^{\infty} \frac{(1 - e^{2\pi i(z-a+j\tau+kp)})(1 - e^{2\pi i(-z-a+(j+1)\tau+(k+1)p)})}{(1 - e^{2\pi i(z+a+j\tau+kp)})(1 - e^{2\pi i(-z+a+(j+1)\tau+(k+1)p})}.$$

It is symmetric under exchanging  $\tau$  and  $p$  and obeys the functional equation

$$(6) \quad \Omega_a(z + p, \tau, p) = e^{2\pi i a \frac{\theta(z+a, \tau)}{\theta(z-a, \tau)}} \Omega_a(z, \tau, p).$$

The weight functions  $\omega_I$  are given by

$$\omega_{(i_1, \dots, i_n)}(t_1, \dots, t_m, \vec{z}, \lambda, \tau) = \prod_{i < j} \frac{\theta(t_i - t_j, \tau)}{\theta(t_i - t_j + 2\eta, \tau)} \sum_{I_1, \dots, I_n} \prod_{l=1}^n \prod_{i \in I_l} \prod_{k=1}^{l-1} \frac{\theta(t_i - z_k + \eta \Lambda_k, \tau)}{\theta(t_i - z_k - \eta \Lambda_k, \tau)} \\ \times \prod_{k < l} \prod_{i \in I_k, j \in I_l} \frac{\theta(t_i - t_j + 2\eta, \tau)}{\theta(t_i - t_j, \tau)} \prod_{k=1}^n \prod_{j \in I_k} \frac{\theta(\lambda + t_j - z_k - \eta \Lambda_k + 2\eta i_k - 2\eta \sum_{l=1}^{k-1} (\Lambda_l - 2i_l), \tau)}{\theta(t_j - z_k - \eta \Lambda_k, \tau)}.$$

The summation is over all  $n$ -tuples  $I_1, \dots, I_n$  of disjoint subsets of  $\{1, \dots, m\}$  such that  $I_k$  has  $i_k$  elements,  $1 \leq k \leq n$ . The theta function is here the first Jacobi theta function

$$\theta(t, \tau) = - \sum_{j \in \mathbb{Z}} e^{\pi i(j + \frac{1}{2})^2 \tau + 2\pi i(j + \frac{1}{2})(t + \frac{1}{2})}.$$

The ‘‘mirror’’ weight functions are related to the weight functions with the reversed order of factor. Indicating the dependence on the highest weights explicitly, we have  $\omega_I^\vee(t, \vec{z}, \mu, p, \vec{\Lambda}) = \omega_{I^\vee}(t, \vec{z}^\vee, \mu, p, \vec{\Lambda}^\vee)$ , where, as above,  $(x_1, \dots, x_n)^\vee = (x_n, \dots, x_1)$ .

The integral over  $t_1, \dots, t_m$  of the 1-periodic integrand in (5) is defined by analytic continuation from a region of the space of parameter where the rule  $\sum \Lambda_i = 2m$  does not hold: one starts from the region  $\text{Im}(\eta) < 0, \text{Im}(\tau), \text{Im}(p), \text{Im}(\eta\Lambda_i) > 0$  for which the integral is over the torus  $(\mathbb{R}/\mathbb{Z})^m$  and defines the integral in general by analytic continuation.

**2.3. Remark.** In [FTV2] we used only weight functions and no mirror weight functions. Then the qKZB equations (3) only involve qKZB operator and no mirror qKZB operators. The choices of this paper make the qKZB heat equation more transparent. The proof that  $u$  obeys the relations (3) is the same as the proof of Theorem 31 in [FTV2]. Note however that the conventions in the definitions of  $D_j$  are different there.

### 3. THE qKZB HEAT EQUATION

In this section we define a q-analogue of the KZB heat equation, prove that it is compatible with the other qKZB equations and show that the differential KZB heat equation arises in the semiclassical limit in the simplest non-trivial case.

The qKZB heat equation is an integral equation. The integration kernel is a contraction with the fundamental hypergeometric solution. The contraction is defined using the elliptic Shapovalov form.

**3.1. The elliptic Shapovalov form.** For  $j = 1, \dots, n, \mu, \tau \in \mathbb{C}, \text{Im} \tau > 0$ , let  $Q^{\Lambda_j}(\mu, \tau) : V_{\Lambda_j} \otimes V_{\Lambda_j} \rightarrow \mathbb{C}$  be the bilinear form on  $V_{\Lambda_j}$  with matrix elements  $Q^{\Lambda_j}(\mu, \tau)(e_k \otimes e_l) = \delta_{k,l} Q_k^{\Lambda_j}(\mu, \tau)$ ,

$$(7) \quad Q_k^{\Lambda_j}(\mu, \tau) = \left( \frac{\theta'(0, \tau)}{\theta(2\eta, \tau)} \right)^k \prod_{l=1}^k \frac{\theta(2\eta(\Lambda_j + 1 - l), \tau) \theta(2\eta l, \tau)}{\theta(\mu + 2\eta(\Lambda_j + 1 - k - l), \tau) \theta(\mu - 2\eta l, \tau)}.$$

Out of these bilinear forms we define a bilinear form  $Q(\mu, \tau) : V_{\bar{\Lambda}} \otimes V_{\bar{\Lambda}} \rightarrow \mathbb{C}$  on the tensor product  $V_{\bar{\Lambda}}$ :

$$Q(\mu, \tau) = Q^{\Lambda_1}(\mu, \tau)^{(1, n+1)} Q^{\Lambda_2}(\mu + 2\eta h^{(1)}, \tau)^{(2, n+2)} \dots Q^{\Lambda_n}(\mu + 2\eta \sum_{j=1}^{n-1} h^{(j)}, \tau)^{(n, 2n)},$$

and a bilinear form on the space of functions of  $\lambda$  with values in  $V_{\bar{\Lambda}}[0]$ : if  $f$  and  $g$  are holomorphic functions from  $\mathbb{C}$  to  $V_{\bar{\Lambda}}[0]$ , we set (if the integral converges)

$$Q_\tau(f \otimes g) = \int Q(\mu, \tau) f(\mu) \otimes g(-\mu) \alpha(\mu) d\mu.$$

The integration is on the path  $t \mapsto 2\eta t + \epsilon, -\infty < t < \infty$ . This bilinear form is called the elliptic Shapovalov form.

The main property of the elliptic Shapovalov form is that the  $R$ -matrix is in a certain sense symmetric with respect to it, see Lemma 3.9. In particular it is a sort of contravariant form for the action of the elliptic quantum group, see eq. (12) below.

**3.2. Notation.** To write the following formulae in the most transparent form we will use the following notational conventions. Let for  $k \in \mathbb{Z}_{\geq 1}$  and a complex vector space  $V$ ,  $\mathcal{F}_k(V)$  denote the space of meromorphic functions of  $k$  complex variables with values in  $V^{\otimes k}$ . For example  $u(\vec{z}, \tau, p) \in \mathcal{F}_2(V_{\vec{\lambda}}[0])$ . We also set  $\mathcal{F}_0(V) = \mathbb{C}$  and  $\mathcal{F}(V) = \mathcal{F}_1(V)$ . If  $f \in \mathcal{F}_j(V)$  and  $g \in \mathcal{F}_k(V)$ , we define  $f \otimes g \in \mathcal{F}_{j+k}$  by  $(f \otimes g)(\lambda_1, \dots, \lambda_{j+k}) = f(\lambda_1, \dots, \lambda_j) \otimes g(\lambda_{j+1}, \dots, \lambda_{j+k})$ . If  $A_i \in \text{End}(\mathcal{F}(V))$  are difference operators with meromorphic coefficients we write  $A_1 \otimes \dots \otimes A_r \in \text{End}(\mathcal{F}_r(V))$  to denote the composition of the operators  $A_i$ , each acting on the  $i$ th variable and the  $i$ th factor. We also use this notation if one of the  $A_i$  is an integral operator  $Q : D \subset \mathcal{F}_2(V) \rightarrow \mathbb{C}$  of the form  $f \mapsto \int Q(\mu) f(\mu, -\mu) d\mu$ ,  $Q(\mu) \in (V \otimes V)^*$ , defined on some subset  $D$ . Then  $A_1 \otimes \dots \otimes A_r$  is defined on some subset of  $\mathcal{F}_r(V)$  and maps to  $\mathcal{F}_{r-2}(V)$ .

**3.3. The qKZB heat equation.** Let  $T(\vec{z}, \tau, p)$  be the integral operator on  $V_{\vec{\lambda}}[0]$ -valued functions of one complex variable  $\lambda$

$$(8) \quad T(\vec{z}, \tau, p)v = (\alpha \otimes Q_{\tau+p})u(\vec{z}, \tau, \tau + p) \otimes v,$$

where  $\alpha$  is the operator of multiplication by the function  $\alpha(\lambda)$ . More explicitly, if  $\{e_I\}$  is a basis of  $V_{\vec{\lambda}}[0]$  consisting of tensor products of basis vectors and  $u = \sum_{I,J} u_{I,J} e_I \otimes e_J$ ,  $v = \sum v_I e_I$ ,  $Q(\mu, \tau)(e_I \otimes e_J) = Q_I(\mu, \tau) \delta_{I,J}$ , we have

$$(9) \quad T(\vec{z}, \tau, p)v(\lambda) = \sum_I \left( \alpha(\lambda) \sum_J \int u_{I,J}(\vec{z}, \lambda, \mu, \tau, \tau + p) Q_J(\mu, \tau + p) v_J(-\mu) \alpha(\mu) d\mu \right) e_I.$$

**Theorem 3.1.** *The equations*

$$(10) \quad v(\vec{z} + p\delta_j, \tau) = K_j(\vec{z}, \tau, p)v(\vec{z}, \tau), \quad j = 1, \dots, n,$$

$$v(\vec{z}, \tau) = T(\vec{z}, \tau, p)v(\vec{z}, \tau + p),$$

are compatible, i.e., we have, in addition to (2),

$$T(\vec{z} + p\delta_j, \tau, p)K_j(\vec{z}, \tau + p, p) = K_j(\vec{z}, \tau, p)T(\vec{z}, \tau, p).$$

The proof of this theorem is given in 3.6.

A similar statement holds for the qKZB difference operators  $K_j^\vee$ : they obey the compatibility identities

$$K_j^\vee(\vec{z} + \tau\delta_l, p, \tau)K_l^\vee(\vec{z}, p, \tau) = K_l^\vee(\vec{z} + \tau\delta_j, p, \tau)K_j^\vee(\vec{z}, p, \tau),$$

for all  $j, l$ , and the operator

$$T^\vee(\vec{z}, p, \tau)v = (Q_{\tau+p} \otimes \alpha)v \otimes u(\vec{z}, \tau + p, p),$$



obeys

$$T^\vee(\vec{z} + \tau\delta_j, p, \tau)K_j^\vee(\vec{z}, p + \tau, \tau) = K_j^\vee(\vec{z}, p, \tau)T^\vee(\vec{z}, p, \tau),$$

implying the compatibility of the mirror qKZB equations

$$\begin{aligned} v(\vec{z} + \tau\delta_j, p) &= K_j^\vee(\vec{z}, p, \tau)v(\vec{z}, p), & j = 1, \dots, n, \\ v(\vec{z}, p) &= T^\vee(\vec{z}, p, \tau)v(\vec{z}, p + \tau). \end{aligned}$$

**Corollary 3.2.** *Let  $U(\vec{z}, \lambda, \mu, \tau, p)$  be the function*

$$U(\vec{z}, \tau, p) = \alpha \otimes Q_{\tau+p} \otimes \alpha(u(\vec{z}, \tau, \tau + p) \otimes u(\vec{z}, \tau + p, p)).$$

*Then  $U$  obeys the system of equations (3).*

**Conjecture 3.3.** *Let  $U(\vec{z}, \tau, p)$  be the function defined in Corollary 3.2. Then*

$$U(\vec{z}, \tau, p) = Cu(\vec{z}, \tau, p),$$

*for some constant  $C$ .*

This conjecture is proved in the case where  $\sum \Lambda_i = 2$  (with  $C = -e^{4\pi i\eta}/(2\pi\sqrt{4i\eta})$ ). This proof will be published elsewhere, [FV3].

Assuming the conjecture correct, we can obtain solutions to the full system (10) as in 3.3: for arbitrary  $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$  with  $\sum(\Lambda_k - 2i_k) = 0$  and  $\mu \in \mathbb{C}$ , let

$$v(\vec{z}, \lambda, \tau) = e^{-\frac{i\pi\mu^2\tau}{4\eta p}} \prod_{i=1}^n d_{i,I}(\mu)^{-z_i/p} u^I(\vec{z}, \lambda, \mu, \tau, p).$$

Then the function  $v(\vec{z}, \tau) : \lambda \mapsto v(\vec{z}, \lambda, \tau)$  is a solutions of (10). It also obeys (4) and  $v(\vec{z}, \lambda, \tau + 1) = e^{-\frac{i\pi\mu^2}{4\eta p}} v(\vec{z}, \lambda, \tau)$ .

**Remark.** The Shapovalov pairing  $Q_\tau$  contains an integration  $\int d\mu$  which we chose to be the integral on the path  $t \mapsto 2\eta t + \epsilon$ . This choice makes the integral convergent if  $\text{Im}(\eta) < 0$  for a large class of functions, thanks to the strong decay at infinity of the Gaussian function  $\alpha(\mu)$  on this path. This class contains in particular our hypergeometric solutions.

It should be however emphasized that the only properties of  $\int d\mu$  that are needed for this construction are that it be a linear form on functions of  $\mu$  invariant under translations by  $2\eta$  times weights of vectors in  $V_{\Lambda_i}$ , and that it be well defined on a suitable class of functions.

In particular, if the highest weights are rational with greatest common denominator  $d$ , we may replace the integral over  $\mu$  by the sum over the set  $\{\lambda + 2\eta k/d, k \in \mathbb{Z}\}$ , so that the heat equation may be viewed as a difference equation of infinite order.

**3.4. The case of integer highest weights.** If  $\Lambda \in \mathbb{Z}_{\geq 0}$ , let  $S_\Lambda$  be the  $\mathfrak{h}$ -submodule of  $V_\Lambda$  generated by  $e_{\Lambda+1}, e_{\Lambda+2}, \dots$ . The  $\Lambda + 1$ -dimensional quotient  $V_\Lambda/S_\Lambda$  is denoted by  $L_\Lambda$ . The classes  $\bar{e}_0, \dots, \bar{e}_\Lambda$  of  $e_0, \dots, e_\Lambda$  build a basis of  $L_\Lambda$ . The space  $L_\Lambda$  carries a one-dimensional family of representations of the elliptic quantum group  $E_{\tau,\eta}(sl_2)$ , see [FV2].

For integer highest weights  $\Lambda_i, \Lambda_j$ , the  $R$ -matrix  $R_{\Lambda_i, \Lambda_j}(z, \lambda, \tau)$  preserves  $S_{\Lambda_i} \otimes V_{\Lambda_j}$  and  $V_{\Lambda_i} \otimes S_{\Lambda_j}$  [FV2], [FTV1]. Therefore it induces an endomorphism, still denoted  $R_{\Lambda_i, \Lambda_j}(z, \lambda, \tau)$ , of  $L_{\Lambda_j} \otimes L_{\Lambda_i}$ . If  $\Lambda_1, \dots, \Lambda_n \in \mathbb{Z}_{\geq 0}$  we then have a system of dynamical  $R$ -matrices and thus a system of qKZB equations defined on  $L_{\vec{\Lambda}}[0] = (L_{\Lambda_1} \otimes \dots \otimes L_{\Lambda_n})[0]$ .

A universal hypergeometric function  $\hat{u}(\vec{z}, \lambda, \mu, \tau, \eta)$  taking values in the tensor product of finite dimensional modules  $L_{\vec{\Lambda}}[0] \otimes L_{\vec{\Lambda}}[0]$  and obeying (3) was found in [MV]. It is defined by  $\hat{u}(\vec{z}, \lambda, \mu, \tau, \eta) = \sum_{I,J} u_{I,J}(\vec{z}, \lambda, \mu, \tau, p, \eta) \bar{e}_I \otimes \bar{e}_J$ , where  $u_{I,J}$  are the analytic continuation of the components of the universal hypergeometric function for  $V_{\vec{\Lambda}}[0] \otimes V_{\vec{\Lambda}}[0]$  which are shown to exist for these values of  $I, J$ .

Then we can introduce a heat operator  $\hat{T}(\vec{z}, \tau, p)$  acting on functions with values in  $L_{\vec{\Lambda}}[0]$  by the same formula (8).

**Theorem 3.4.** *Suppose that  $\Lambda_1, \dots, \Lambda_n$  are non-negative integers. Then the equations*

$$(11) \quad v(\vec{z} + p\delta_j, \tau) = K_j(\vec{z}, \tau, p)v(\vec{z}, \tau), \quad j = 1, \dots, n,$$

$$v(\vec{z}, \tau) = \hat{T}(\vec{z}, \tau, p)v(\vec{z}, \tau + p),$$

for a function  $v$  taking values in  $L_{\vec{\Lambda}}[0]$  are compatible, i.e., we have, in addition to (2),

$$\hat{T}(\vec{z} + p\delta_j, \tau, p)K_j(\vec{z}, \tau + p, p) = K_j(\vec{z}, \tau, p)\hat{T}(\vec{z}, \tau, p).$$

The proof of this Theorem is contained in 3.7 below.

**3.5. Rational  $\eta$ .** A particularly interesting case is the case of integer highest weights and rational  $\eta$ . Let us for instance assume that  $2N\eta = 1$  for some positive integer  $N$  and suppose that the highest weights  $\Lambda_1, \dots, \Lambda_n$  are positive integers.

If  $N$  is large enough, then the qKZB operators may still be defined. Indeed we have:

**Lemma 3.5.** *Let  $\Lambda_1, \Lambda_2$  be positive integers, and  $N$  be a large enough integer. Then the matrix elements of the  $R$ -matrix  $R_{\Lambda_1, \Lambda_2}(z, \lambda, \tau; \eta) \in \text{End}(L_{\Lambda_1} \otimes L_{\Lambda_2})$  with respect to the basis  $\{\bar{e}_i \otimes \bar{e}_j\}$  are regular functions of  $\eta$  at  $2\eta = 1/N$  for fixed generic values of  $z, \lambda, \tau$ .*

*Proof:* The  $R$  matrix  $R_{\Lambda_1, \Lambda_2}(z_1 - z_2, \lambda, \tau; \eta)$ , for generic  $\eta$ , is uniquely determined up to normalization by an intertwining condition for tensor products of  $E_{\tau,\eta}(sl_2)$ -modules  $L_{\Lambda_i}(z_i)$ , see [FV2]. The  $E_{\tau,\eta}(sl_2)$  module  $L_\Lambda(z)$  may be realized for integer  $\Lambda$  as a symmetric tensor product of two-dimensional modules [FV2], so that the matrix elements of  $R_{\Lambda_i, \Lambda_j}$ , for a basis consisting of symmetrized tensor products of basis vectors, can be expressed as polynomials in the matrix elements of  $R_{1,1}$ . The latter matrix elements

are known explicitly and are regular as functions of  $\eta$ . For  $i = 1, \dots, \Lambda$ , the basis vector  $\bar{e}_i$  of  $L_\Lambda$  is proportional to the symmetrized tensor products of basis vectors of the two dimensional modules. The proportionality constant is an elliptic factorial  $\prod_{j=1}^i \theta(2\eta j)/\theta(2\eta)$  which is regular and non-zero at  $2\eta N = 1$  as long as  $N > \lambda$ .

Thus if  $N > \max(\Lambda_1, \dots, \Lambda_n)$ , the matrix elements of the  $R$ -matrix are regular at  $2\eta N = 1$ .  $\square$

Fix some generic complex number  $\epsilon$ . Then we may consider the qKZB equations as equations for functions  $v(\vec{z}, \lambda)$ , where the dynamical parameter  $\lambda$  runs over the finite set  $\{k/N + \epsilon \mid k \in \mathbb{Z}/2N\mathbb{Z}\}$ . Indeed, the coefficients of the qKZB operators are 1-periodic functions of  $\lambda$  and the shifts of  $\lambda$  in the difference operators  $\Gamma_i$  are integer multiples of  $2\eta = 1/N$ . The shift by the generic number  $\epsilon$  ensures that on this finite set no poles of the qKZB operators are encountered. Thus, we get:

**Proposition 3.6.** *Suppose that  $N = (2\eta)^{-1}$  and  $\Lambda_1, \dots, \Lambda_n$  are positive integers, with  $N$  large enough. Fix a generic complex number  $\epsilon$ . Let  $F_N(\epsilon)$  be the space of functions  $f: \frac{1}{N}\mathbb{Z} \rightarrow V_{\vec{\Lambda}}[0]$  so that  $f(\lambda+2) = f(\lambda)$ . Then the qKZB operators  $K_i(\vec{z}, \tau, p)$ ,  $K_i^\vee(\vec{z}, p, \tau)$  are well-defined endomorphisms of  $F_N(\epsilon)$ .*

In this situation we thus have a truly holonomic system, i.e., a compatible system of difference equations for functions taking values in a finite dimensional vector space  $F_N(\epsilon)$ . In order to define the heat equation we have to worry about the fact that the universal hypergeometric function  $\hat{u}$  is not defined for all values of the parameters. Recall that  $\hat{u}(\vec{z}, \lambda, \mu, \tau, p)$  is also a meromorphic function of  $\eta$ . Let us say that  $\eta$  is a regular point for  $\hat{u}$  if  $\hat{u}$  is regular at this point for all  $\lambda, \mu$  and all generic  $\vec{z}, \tau, p$ .

**Theorem 3.7.** *Let  $\eta$ ,  $\vec{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ ,  $\epsilon$  be as in Proposition 3.6 and assume that  $\eta$  is a regular point for  $\hat{u}$ . Then the heat operator*

$$\hat{T}_N(\vec{z}, \tau, p)v(\lambda) = e^{-\frac{Ni\pi\lambda^2}{2}} \sum_{k=0}^{2N-1} (1 \otimes Q(\mu_k, \tau + p)) \hat{u}(\vec{z}, \lambda, \mu_k, \tau, \tau + p) \otimes v(-\mu_k) e^{-\frac{Ni\pi\mu_k^2}{2}},$$

with  $\mu_k = -\epsilon + k/N$ , maps  $F_N(\epsilon)$  to itself. Moreover, the equations for  $v(\vec{z}, \tau) \in F_N(\epsilon)$

$$\begin{aligned} v(\vec{z} + p\delta_j, \tau) &= K_j(\vec{z}, \tau, p)v(\vec{z}, \tau), \quad j = 1, \dots, n, \\ v(\vec{z}, \tau) &= \hat{T}_N(\vec{z}, \tau, p)v(\vec{z}, \tau + p), \end{aligned}$$

are compatible, i.e., we have, in addition to (2),

$$\hat{T}_N(\vec{z} + p\delta_j, \tau, p)K_j(\vec{z}, \tau + p, p) = K_j(\vec{z}, \tau, p)\hat{T}_N(\vec{z}, \tau, p),$$

on  $F_N(\epsilon)$ .

The proof of this Theorem is contained in 3.7 below.

The description of the set of regular points for  $\hat{u}$  will be studied elsewhere. Here we only remark that in the case  $n = 1$ ,  $\Lambda_1 = 2$ , the point  $\eta = 1/2N$  is a regular point for

all  $N \geq 3$ , as can easily be checked since  $\hat{u}$  is given by a one-dimensional integral. In this case, Conjecture 3.3 holds, see [FV3], namely, we have

$$\begin{aligned} \hat{u}(\vec{z}, \lambda, \nu, \tau, p) &= C_N e^{-\frac{Ni\pi(\lambda^2 + \nu^2)}{2}} \\ &\times \sum_{k=0}^{2N-1} (1 \otimes Q(\mu_k, \tau + p) \otimes 1)(\hat{u}(\vec{z}, \lambda, \mu_k, \tau, \tau + p) \otimes \hat{u}(\vec{z}, -\mu_k, \nu, \tau + p, p)) e^{-\frac{Ni\pi\mu_k^2}{2}}, \end{aligned}$$

for all  $\lambda \in \epsilon + \frac{1}{N}\mathbb{Z}$ ,  $\nu \in \frac{1}{N}\mathbb{Z}$ . The constant is  $C_N = \frac{i\epsilon^{2\pi i/N}}{S(N)}$ , with the Gauss sum

$$S(N) = \sum_{k=0}^{2N-1} e^{-\frac{\pi i k^2}{2N}} = (1 - i)\sqrt{N}.$$

**3.6. Proof of Theorem 3.1.** The proof is based on some identities involving  $R$ -matrices,  $Q$  and  $D_j$ . As above, we set  $\alpha(\lambda) = \exp(-\pi i \lambda^2 / 4\eta)$

**Lemma 3.8.** *For any  $\Lambda, M$ ,*

$$\frac{\alpha(\lambda - 2\eta(h^{(1)} + h^{(2)}))}{\alpha(\lambda - 2\eta h^{(2)})} R_{\Lambda, M}(z + \tau, \lambda, \tau) = e^{-2\pi i \eta \Lambda, M} R_{\Lambda, M}(z, \lambda, \tau) \frac{\alpha(\lambda - 2\eta h^{(1)})}{\alpha(\lambda)}.$$

*Proof:* One way to prove this lemma is to use the functional realization (see [FTV1]): The matrix  $R_{\Lambda, M}$  relate two bases of the same space of functions. The basis elements are products of ratios of theta functions and have therefore well-behaved transformation properties under shifts of  $z$  by  $\tau$ . The computation is straightforward and will not be reproduced here.  $\square$

**Lemma 3.9.** *Let  $\Lambda, M \in \mathbb{C}$  and  $v, w \in V_\Lambda \otimes V_M$ . Then*

$$\begin{aligned} &\langle Q^\Lambda(\mu + 2\eta h^{(2)}, \tau) \otimes Q^M(\mu, \tau) v, R_{\Lambda, M}(z, -\mu, \tau) w \rangle = \\ &= \langle Q^\Lambda(\mu, \tau) \otimes Q^M(\mu + 2\eta h^{(1)}, \tau) R_{\Lambda, M}(z, \mu + 2\eta(h^{(1)} + h^{(2)}), \tau) v, w \rangle. \end{aligned}$$

*Proof:* We first prove a version of this identity for  $L$ -operators. Let  $L(\zeta, \lambda) \in \text{End}(\mathbb{C}^2 \otimes V_\Lambda(z))$  be the  $L$ -operator of the evaluation Verma module  $V_\Lambda(z)$ . We claim that, for any  $v_1, v_2 \in \mathbb{C}^2 \otimes V_\Lambda(z)$ ,

$$\begin{aligned} &\langle Q^1(\mu + 2\eta h^{(2)}, \tau) \otimes Q^\Lambda(\mu, \tau) v_1, L(\zeta, -\mu, \tau) v_2 \rangle = \\ (12) \quad &= \langle Q^1(\mu, \tau) \otimes Q^\Lambda(\mu + 2\eta h^{(1)}, \tau) L(\zeta, \mu + 2\eta(h^{(1)} + h^{(2)}), \tau) v_1, v_2 \rangle. \end{aligned}$$

We have  $Q_0^1(\mu, \tau) = 1$  and  $Q_1^1(\mu, \tau) = \theta(2\eta)\theta(\mu - 2\eta)^{-1}\theta(\mu)^{-1}$ . Define the matrix elements of  $L$  by  $L(\zeta, \mu)e_j \otimes v = \sum_{k=0,1} e_k \otimes L_j^k(\zeta, \mu)v$ . Then the claim is equivalent

to

$$\begin{aligned}
& Q_k^\Lambda(\mu, \tau) \langle e_k, L_0^0(\zeta, -\mu) e_k \rangle \\
&= Q_k^\Lambda(\mu + 2\eta, \tau) \langle L_0^0(\zeta, \mu + 2\eta(\Lambda - 2k + 1)) e_k, e_k \rangle, \\
& Q_k^\Lambda(\mu, \tau) \langle e_k, L_1^0(\zeta, -\mu) e_{k-1} \rangle \\
&= Q_1^1(\mu, \tau) Q_{k-1}^\Lambda(\mu - 2\eta, \tau) \langle L_1^1(\zeta, \mu + 2\eta(\Lambda - 2k + 1)) e_k, e_{k-1} \rangle, \\
& Q_1^1(\mu + 2\eta(\Lambda - 2k), \tau) Q_k^\Lambda(\mu, \tau) \langle e_k, L_0^1(\zeta, -\mu) e_{k+1} \rangle \\
&= Q_{k+1}^\Lambda(\mu + 2\eta, \tau) \langle L_1^0(\zeta, \mu + 2\eta(\Lambda - 2k - 1)) e_k, e_{k+1} \rangle, \\
& Q_1^1(\mu + 2\eta(\Lambda - 2k), \tau) Q_k^\Lambda(\mu, \tau) \langle e_k, L_1^1(\zeta, -\mu) e_k \rangle \\
&= Q_1^1(\mu, \tau) Q_k^\Lambda(\mu - 2\eta, \tau) \langle L_1^1(\zeta, \mu + 2\eta(\Lambda - 2k - 1)) e_k, e_k \rangle.
\end{aligned}$$

These identities follow immediately from the explicit expressions given in [FV2] for the operators  $L_k^j$  (called  $a, b, c, d$  in [FV2]).

We now extend this result to the general case. We use the intertwining property of the  $R$ -matrix: let  $\mathcal{L}_\Lambda, \mathcal{L}_M$  be the  $L$ -operators of  $V_\Lambda(z_1), V_M(z_2)$ , respectively. Then<sup>1</sup>  $R_{\Lambda, M}(z_1 - z_2, \mu) \in \text{End}(V_\Lambda(z_1) \otimes V_M(z_2))$  is uniquely determined up to a factor by the relation

$$\mathcal{L}(\zeta, \mu) R_{\Lambda, M}(z_1 - z_2, \mu - 2\eta h^{(1)})^{(23)} = R_{\Lambda, M}(z_1 - z_2, \mu)^{(23)} \mathcal{L}'(\zeta, \mu).$$

The operators  $\mathcal{L}$  and  $\mathcal{L}'$  (giving the action of the quantum group on the tensor product by using the coproduct and the opposite coproduct, respectively) are defined by

$$\begin{aligned}
\mathcal{L}(\zeta, \mu) &= \mathcal{L}_\Lambda(\zeta, \mu - 2\eta h^{(3)})^{(12)} \mathcal{L}_M(\zeta, \mu)^{(13)}, \\
\mathcal{L}'(\zeta, \mu) &= \mathcal{L}_M(\zeta, \mu - 2\eta h^{(2)})^{(13)} \mathcal{L}_\Lambda(\zeta, \mu)^{(12)}.
\end{aligned}$$

The  $R$ -matrix normalized by the condition  $R_{\Lambda, M}(z_1 - z_2, \mu) e_0 \otimes e_0 = e_0 \otimes e_0$ .

In particular, if  $v = w = e_0 \otimes e_0$ , the claim of the lemma is correct for trivial reasons. We prove the general case by induction: let us suppose that the lemma is proved for  $v, w$  of weight  $\Lambda + M - 2j$ ,  $j = 0, \dots, k-1$ ,  $k \geq 1$ . Now it is known, see [FV2], that, for generic parameters, the weight space  $V_\Lambda(z_1) \otimes V_M(z_2)[\Lambda + M - 2k]$  is spanned by vectors of the form  $\mathcal{L}_1^0(\zeta, \lambda) x$  (or  $\mathcal{L}_1^0(\zeta, \lambda) x$ ),  $\zeta \in \mathbb{C}$ ,  $x$  of weight  $\Lambda + M - 2(k-1)$ , and any fixed generic  $\lambda$ . Indeed, if these vectors did not span the weight space, they would be part of a proper submodule, contradicting the irreducibility of the tensor product.

By iterating (12), we obtain

$$\begin{aligned}
& \langle Q^1(\mu + 2\eta(h^{(2)} + h^{(3)})) \otimes Q^\Lambda(\mu + 2\eta h^{(3)}) \otimes Q^M(\mu) v_1, \mathcal{L}(\zeta, -\mu) v_2 \rangle \\
&= \langle Q^1(\mu) \otimes Q^\Lambda(\mu + 2\eta(h^{(1)} + h^{(3)})) \otimes Q^M(\mu + 2\eta h^{(1)}) \\
&\quad \mathcal{L}'(\zeta, \mu + 2\eta(h^{(1)} + h^{(2)} + h^{(3)})) v_1, v_2 \rangle.
\end{aligned}$$

---

<sup>1</sup>Here we omit the argument  $\tau$  to shorten the notation

In particular, if  $v_1 = e_0 \otimes v$ ,  $v_2 = e_1 \otimes w$ , one has

$$\begin{aligned} & \langle Q^\Lambda(\mu + 2\eta h^{(2)}) \otimes Q^M(\mu) v, \mathcal{L}_1^0(\zeta, -\mu) w \rangle \\ &= Q_1^1(\mu) \langle Q^\Lambda(\mu + 2\eta(-1 + h^{(2)})) \otimes Q^M(\mu - 2\eta) \mathcal{L}'_0(\zeta, \mu + 2\eta(1 + h^{(1)} + h^{(2)})) v, w \rangle. \end{aligned}$$

We turn to the proof of the induction step. Suppose that  $v, w$  have weight  $\Lambda + M - 2k$ , and write  $w = \mathcal{L}'_1{}^0(\zeta, \mu)x$ . Let us set  $z = z_1 - z_2$ . Then

$$\begin{aligned} & \langle Q^\Lambda(\mu + 2\eta h^{(2)}) \otimes Q^M(\mu) v, R_{\Lambda, M}(z, -\mu) w \rangle \\ &= \langle Q^\Lambda(\mu + 2\eta h^{(2)}) \otimes Q^M(\mu) v, R_{\Lambda, M}(z, -\mu) \mathcal{L}'_1{}^0(\zeta, -\mu) x \rangle \\ &= \langle Q^\Lambda(\mu + 2\eta h^{(2)}) \otimes Q^M(\mu) v, \mathcal{L}'_1{}^0(\zeta, -\mu) R_{\Lambda, M}(z, -\mu + 2\eta) x \rangle \\ &= Q_1^1(\mu) \langle Q^\Lambda(\mu + 2\eta(-1 + h^{(2)})) \otimes Q^M(\mu - 2\eta) \mathcal{L}'_0(\zeta, \mu + 2\eta(1 + h^{(1)} + h^{(2)})) v, \\ & \quad R_{\Lambda, M}(z, -\mu + 2\eta) x \rangle \\ &= Q_1^1(\mu) \langle Q^\Lambda(\mu - 2\eta) \otimes Q^M(\mu + 2\eta(-1 + h^{(1)})) \\ & \quad R_{\Lambda, M}(z, \mu + 2\eta(-1 + h^{(1)} + h^{(2)})) \mathcal{L}'_0(\zeta, \mu + 2\eta(1 + \Lambda + M - 2k)) v, x \rangle. \end{aligned}$$

In the last step, we used the induction hypothesis. The calculation continues by commuting  $R$  with  $\mathcal{L}'$ , and then by bringing  $\mathcal{L}$  to the right. This last part is similar to the above calculation read backwards, and will not be reproduced in detail. One finally obtains, as desired,

$$\langle Q^\Lambda(\mu) \otimes Q^M(\mu + 2\eta h^{(1)}) R_{\Lambda, M}(z, \mu + 2\eta(h^{(1)} + h^{(2)})) v, w \rangle.$$

This completes the induction step and thus the proof of the lemma.  $\square$

**Lemma 3.10.**

$$\begin{aligned} \alpha (D_j^\vee)^{-1} K_j(\vec{z}, \tau, p + \tau) &= K_j(\vec{z}, \tau, p) \alpha e^{-\pi i \eta \Lambda_j(\sum_{l \neq j} \Lambda_l)} \\ \alpha D_j^{-1} K_j^\vee(\vec{z}, p, \tau + p) &= K_j^\vee(\vec{z}, p, \tau) \alpha e^{-\pi i \eta \Lambda_j(\sum_{l \neq j} \Lambda_l)}. \end{aligned}$$

*Proof:* This is a straightforward consequence of the definition of the difference operators  $K_j, K_j^\vee$  and of Lemma 3.8.  $\square$

**Lemma 3.11.** *Let  $f, g$  be holomorphic functions from  $\mathbb{C}$  to  $V_\Lambda$ .*

$$\begin{aligned} Q_{\tau+p}(f, K_j(\vec{z}, \tau + p, p)g) &= Q_{\tau+p}(D_j^{-1} K_j^\vee(\vec{z} + p\delta_j, \tau + p, \tau)f, g) e^{\pi i \eta \Lambda_j(\sum_{l \neq j} \Lambda_l)} \\ Q_{\tau+p}(K_j^\vee(\vec{z}, \tau + p, \tau)f, g) &= Q_{\tau+p}(f, (D_j^\vee)^{-1} K_j(\vec{z} + \tau\delta_j, \tau + p, p)g) e^{\pi i \eta \Lambda_j(\sum_{l \neq j} \Lambda_l)}. \end{aligned}$$

*Proof:* The proof of the first identity is given by using Lemma 3.9 to bring the  $R$ -matrices in  $K_j$  to the left, the translation invariance of the integral to bring  $\Gamma_j$  to the left, and 3.10 to commute the resulting  $K_j^\vee(\vec{z} + p\delta_j, \tau + p, -p)$  with  $\alpha$ . The proof of the second identity is similar.  $\square$

We can now complete the proof of Theorem 3.1. Let  $C_j = e^{\pi i \eta \Lambda_j (\sum_{l \neq j} \Lambda_l)}$  and  $v$  a function from  $\mathbb{C}$  to  $V_{\vec{\Lambda}}[0]$ .

$$\begin{aligned}
& T(\vec{z} + p\delta_j, \tau, p) K_j(\vec{z}, \tau + p, p) v = \\
&= (\alpha \otimes Q_{\tau+p}) u(\vec{z} + p\delta_j, \tau, p + \tau) \otimes K_j(\vec{z}, \tau + p, p) v \\
&= C_j (\alpha \otimes Q_{\tau+p}) (1 \otimes D_j^{-1} K_j^\vee(\vec{z} + p\delta_j, \tau + p, \tau) u(\vec{z} + p\delta_j, \tau, p + \tau)) \otimes v \\
&= C_j (\alpha \otimes Q_{\tau+p}) ((D_j^\vee)^{-1} \otimes D_j^{-1} u(\vec{z} + (p + \tau)\delta_j, \tau, \tau + p)) \otimes v \\
&= C_j (\alpha \otimes Q_{\tau+p}) ((D_j^\vee)^{-1} K_j(\vec{z}, \tau, \tau + p) \otimes 1 u(\vec{z}, \tau, \tau + p)) \otimes v \\
&= K_j(\vec{z}, \tau, p) T(\vec{z}, \tau, p) v.
\end{aligned}$$

**3.7. Proof of Theorem 3.4 and Theorem 3.7.** Theorem 3.4 can be proven in the same way as Theorem 3.1.

There is however an apparent difficulty: the heat operator involves the Shapovalov form which contains a sum over all components of  $u$ , including those that a priori do not have a limit for integer highest weights. The solution is provided by Theorem 2 of [MV]: let us say that  $I = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$  is admissible for  $\vec{\Lambda}$  if  $i_a \leq \Lambda_a$  for all  $a = 1, \dots, n$ . Then (a special case of) Theorem 2 states that the components  $u_{I,J}(\vec{z}, \lambda, \mu, \tau, p)$  such that  $I$  or  $J$  is admissible, are regular functions of the highest weights at  $\vec{\Lambda}$  for generic values of the other variables. Moreover,  $Q_k^\Lambda(\mu, \tau)$  vanishes if  $k \geq \Lambda$ , cf. (7), so that the sum appearing in the compatibility condition is effectively restricted to admissible indices.

Theorem 3.7 is proven in the same way as Theorem 3.1 and 3.4. In fact the only property of the integral over  $\mu$  that is used in the proof is the translation invariance. So the same proof gives the compatibility relation in this case provided the function of  $\mu_k$  on the right-hand side is periodic in  $k$  with period  $2N$ . Now  $u(\vec{z}, \lambda, \mu, \tau, p)$  is  $\exp(-\pi i N \lambda \mu)$  times a 2-periodic function of  $\lambda$  and  $\mu$ . So the exponential factors combine into the expression

$$e^{-\frac{i\pi N(\lambda + \mu_k)^2}{2}} = e^{-\frac{i\pi N(\lambda - \epsilon + k/N)^2}{2}}.$$

If  $\lambda \in \epsilon + \frac{1}{N}\mathbb{Z}$ , this expression is periodic in  $k$  with period  $2N$ . The same argument shows that  $T_N(\vec{z}, \tau, p)v(\lambda)$  is 2-periodic in  $\lambda$  for  $\lambda \in \epsilon + \frac{1}{N}\mathbb{Z}$ .  $\square$

#### 4. SEMICLASSICAL LIMIT

We consider here the semiclassical limit of our quantum heat equation in the simplest non-trivial case and show that we do recover the KZB heat equation in this limit. The case we consider is  $n = 1$ , with  $\Lambda_1 = 2$ . The qKZB equations for the dependence of  $z_1$  are trivial in this case and we can assume that  $z_1 = 0$ . The zero weight space is one-dimensional, and we identify it with  $\mathbb{C}$  using the basis  $e_1$ . Suppose that  $v_\eta(\lambda, \tau)$  is a family of solutions of the qKZB equations with parameters  $\tau, p = -2\kappa\eta, \tau, \eta$ , parametrized by  $\eta$  around zero. Assume that  $v_\eta$  has an asymptotic expansion  $v_\eta(\lambda, \tau) = v_0(\lambda, \tau) + O(\eta)$

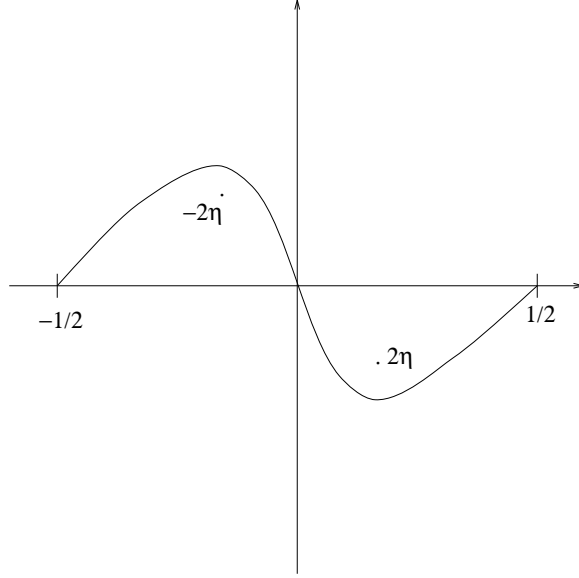


FIGURE 1. The integration cycle  $\gamma$ . The points  $\pm 2\eta$  are the singularities of the integrand

at  $\eta = 0$ . We want to find the equations satisfied by  $v_0$ . For this we expand the qKZB heat equation

$$(13) \quad v_\eta(\lambda, \tau) = \frac{-1}{4\pi\sqrt{i\eta}} e^{-\frac{i\pi\lambda^2}{4\eta}} \int u(\lambda, \mu, \tau, p, \eta) \frac{\theta(4\eta, \tau+p)\theta'(0, \tau+p)}{\theta(\mu+2\eta, \tau+p)\theta(\mu-2\eta, \tau+p)} e^{-\frac{i\pi\mu^2}{4\eta}} v_\eta(-\mu, \tau+p) d\mu,$$

around  $\eta = 0$ , setting  $p = -2\kappa\eta$  and keeping  $\tau, \kappa, \lambda$  fixed. The dependence of  $\eta$  of the constant in front of the integral was chosen in such a way that the semiclassical limit exists. The integration path is  $t \mapsto \mu = \eta t$  ( $t \in \mathbb{R}$ ). The hypergeometric solution  $u$  is independent of  $z$  in this case and is given by the formula:

$$u(\lambda, \mu, \tau, p, \eta) = e^{-\frac{i\pi\lambda\mu}{2\eta}} \int_\gamma \Omega_{2\eta}(t, \tau, \tau+p) \frac{\theta(\lambda+t, \tau)}{\theta(t-2\eta, \tau)} \frac{\theta(\mu+t, \tau+p)}{\theta(t-2\eta, \tau+p)} dt.$$

The integration cycle  $\gamma$  is depicted in Fig. 1.

**Theorem 4.1.** *Suppose that  $v_\eta(\lambda, \tau)$  is a family of solutions of (13) with an asymptotic expansion  $v_\eta(\lambda, \tau) = v_0(\lambda, \tau) + \eta v_1(\lambda, \tau) + \dots$ , then  $v(\lambda, \tau) = v_0(\lambda, \tau)/\theta(\lambda, \tau)$  obeys the KZB heat equation*

$$2\pi i \kappa \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \lambda^2} - 2\wp(\lambda, \tau)v + c(\tau)v,$$

for some  $c(\tau)$  independent of  $\lambda$ .



*Proof:* The integral on the right-hand side of (13) has the form

$$I_\eta = \frac{i}{\sqrt{4i\eta}} \int_{-\eta\infty}^{\eta\infty} e^{-\frac{i\pi}{4\eta}(\lambda+\mu)^2} g(\lambda, -\mu, \eta) d\mu.$$

This integral has the asymptotic expansion as  $\eta \rightarrow 0$

$$I_\eta = g(\lambda, \lambda, 0) + \eta \left( \frac{1}{i\pi} \frac{\partial^2}{\partial \mu^2} \Big|_{\mu=\lambda} g(\lambda, \mu, 0) + \frac{\partial}{\partial \eta} \Big|_{\eta=0} g(\lambda, \lambda, \eta) \right) + O(\eta^2).$$

To compute the various terms of this expression, we first notice that the integration cycle in  $u$  is pinched by the singularities as  $\eta \rightarrow 0$ . The integral defining  $u$  can then be expressed as a divergent (as  $\eta \rightarrow 0$ ) part given by  $2\pi i$  times the residue at  $t = 2\eta$  plus the integral on a cycle  $\bar{\gamma}$  which stays away from the singularities.

To compute the residue we introduce  $\tilde{\Omega}$  by

$$\begin{aligned} \Omega_{2\eta}(t, \tau, \tau + p) &= \frac{1 - e^{2\pi i(t-2\eta)}}{1 - e^{2\pi i(t+2\eta)}} \tilde{\Omega}_{2\eta}(t, \tau, \tau + p) \\ &= (t - 2\eta) \frac{2\pi i}{e^{8\pi i\eta} - 1} \tilde{\Omega}_{2\eta}(2\eta, \tau, \tau + p) + O((t - 2\eta)^2). \end{aligned}$$

As  $\eta \rightarrow 0$ ,  $\tilde{\Omega}_{2\eta}(2\eta, \tau, \tau - 2\kappa\eta)$  is regular and converges to 1.

We then have

$$\begin{aligned} g(\lambda, \mu, \eta) &= \frac{2\pi i}{e^{8\pi i\eta} - 1} \tilde{\Omega}_{2\eta}(2\eta, \tau, \tau + p) \frac{\theta(\lambda + 2\eta, \tau + p)\theta(4\eta, \tau + p)v_\eta(\mu, \tau + p)}{\theta'(0, \tau)\theta(\mu + 2\eta, \tau + p)} \\ &\quad - \frac{1}{2\pi i} \int_{\bar{\gamma}} \Omega_{2\eta}(t, \tau, \tau + p) \frac{\theta(\lambda + t, \tau)}{\theta(t - 2\eta, \tau)} \frac{\theta(\mu + t, \tau + p)}{\theta(t - 2\eta, \tau + p)} dt \\ &\quad \times \frac{\theta(4\eta, \tau + p)\theta'(0, \tau + p)}{\theta(\mu + 2\eta, \tau + p)\theta(\mu - 2\eta, \tau + p)} v_\eta(\mu, \tau + p). \end{aligned}$$

From these formulae we can compute the various terms:

$$g(\lambda, \lambda, 0) = v_0(\lambda, \tau),$$

$$\frac{\partial^2}{\partial^2 \mu} \Big|_{\mu=\lambda} g(\lambda, \mu, 0) = \theta(\lambda, \tau) \partial_\lambda^2 \left( \frac{v_0(\lambda, \tau)}{\theta(\lambda, \tau)} \right).$$

Finally

$$\begin{aligned} \frac{\partial}{\partial \eta} \Big|_{\eta=0} g(\lambda, \lambda, \eta) &= C_1(\tau) v_0(\lambda, \tau) + \eta \frac{\partial}{\partial \eta} \Big|_{\eta=0} v_\eta(\lambda, \tau) \\ &\quad - 2\kappa \theta(\lambda, \tau) \frac{\partial}{\partial \tau} \left( \frac{v_0(\lambda, \tau)}{\theta(\lambda, \tau)} \right) \\ &\quad - \frac{2}{\pi i} \int_{\bar{\gamma}} \frac{\theta(t + \lambda, \tau)\theta(t - \lambda, \tau)}{\theta(t, \tau)^2 \theta(\lambda, \tau)^2} dt v_0(\lambda, \tau). \end{aligned}$$

Here  $C_1(\tau)$  is some scalar function independent of  $\lambda$ . Using the identity

$$\frac{\theta(t + \lambda, \tau)\theta(t - \lambda, \tau)}{\theta(t, \tau)^2\theta(\lambda, \tau)^2} = \frac{1}{\theta'(0, \tau)^2} (\wp(\lambda, \tau) - \wp(t, \tau)),$$

we see that the right-hand side of (13) is

$$\begin{aligned} & v_0(\lambda, \tau) + \eta \frac{\partial}{\partial \eta} \Big|_{\eta=0} v_\eta(\lambda, \tau) \\ & + \eta \theta(\lambda, \tau) \left( \frac{1}{i\pi} \frac{\partial^2}{\partial \lambda^2} - 2\kappa \frac{\partial}{\partial \tau} - \frac{2}{\pi i} \wp(\lambda, \tau) + c(\tau) \right) \frac{v_0(\lambda, \tau)}{\theta(\lambda, \tau)} + O(\eta^2), \end{aligned}$$

for some function  $c(\tau)$  independent of  $\lambda$ .

Since the first two terms also appear on the left-hand side, the proof is complete.  $\square$

## 5. CONFORMAL BLOCKS

In this section, we introduce, in the simplest case of one marked point, a difference analogue the vector bundle of conformal blocks. We begin by reviewing the differential case. The vector bundle of conformal blocks is, in this case, a vector bundle on the moduli space  $\mathcal{M}_{1,1}$  of genus one curves with one marked point. The projectivization of this vector bundle carries a connection given by the KZB differential operator. We then give a difference analogue of this vector bundle. It has a (discrete) connection, which is now given by the qKZB heat operator  $T$ .

**5.1. The differential case.** Let  $\mathfrak{g}$  be a simple complex Lie algebra with Cartan subalgebra  $\mathfrak{h}$  and root space decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ . Let the non-degenerate invariant bilinear form  $(\ , \ )$  on  $\mathfrak{g} \simeq \mathfrak{g}^*$  be normalized so that the highest root  $\theta$  obeys  $(\theta, \theta) = 2$ . Let  $\kappa$  be an integer larger than or equal to the dual Coxeter number  $h^\vee$  of  $\mathfrak{g}$ , and  $\Lambda \in \mathfrak{h}^*$  be a dominant integral weight, so that  $(\theta, \Lambda) \leq \kappa - h^\vee$ . Denote by  $L_\Lambda$  the irreducible  $\mathfrak{g}$ -module of highest weight  $\Lambda$ .

To these data one associates a holomorphic vector bundle of conformal blocks on the moduli space  $\mathcal{M}_{1,1}$  of genus one complex curves with one marked point [TUY]. Its projectivization carries a canonical flat connection. The fiber over a point may be defined as a space of coinvariants for the Lie algebra of  $\mathfrak{g}$ -valued rational functions on the curve whose poles are at the marked point, acting on the irreducible affine Kac–Moody Lie algebra module of highest weight  $\Lambda$  and level  $\kappa - h^\vee$ .

An explicit description [FW] of this bundle, which for our purposes can be taken as a definition, may be obtained by viewing  $\mathcal{M}_{1,1}$  as the quotient of the upper half plane  $H_+$  by  $\mathrm{SL}(2, \mathbb{Z})$ . We may then regard the vector bundle  $E_{\kappa, \Lambda}$  of conformal blocks as an  $\mathrm{SL}(2, \mathbb{Z})$ -equivariant vector bundle over  $H_+$ . Let  $L_\Lambda[0] = \{v \in L_\Lambda \mid \mathfrak{h}v = 0\}$  be the zero weight space space of  $L_\Lambda$ . It carries a natural linear action of the Weyl group  $W$  of  $\mathfrak{g}$ . The fiber of  $E_{\kappa, \Lambda}$  over  $\tau \in H_+$  is then to the space of holomorphic maps  $v : \mathfrak{h} \rightarrow L_\Lambda[0]$  such that

- (i)  $v(\lambda + q_1 + q_2\tau) = \exp(-\pi i\kappa(q_2, q_2)\tau - 2\pi i\kappa(q_2, \lambda))v(\lambda)$ , for all  $\lambda \in \mathfrak{h}$  and  $q_1, q_2$  in the coroot lattice  $Q^\vee$ .
- (ii)  $v(w \cdot \lambda) = \epsilon(w)w \cdot v(\lambda)$  for all  $w \in W$ , where  $\epsilon : W \mapsto \{\pm 1\}$  is the homomorphism sending reflections to  $-1$ .
- (iii) For all roots  $\alpha$ ,  $x \in \mathfrak{g}_\alpha$ , and integers  $l \geq 0, r, s$ , the map  $v$  obeys the vanishing condition

$$x^l v(\lambda) = O((\alpha(\lambda) - r - s\tau)^{l+1}),$$

as  $\alpha(\lambda) \rightarrow r + s\tau$ .

The action of  $\mathrm{SL}(2, \mathbb{Z})$  on the base may be lifted to an action on the bundle: let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$  act on  $H_+$  by  $\tau \mapsto g \cdot \tau = (a\tau + b)/(c\tau + d)$ . Then we have isomorphisms  $\psi_g(\tau) : E_{\kappa, m}(\tau) \rightarrow E_{\kappa, m}(g \cdot \tau)$  given by

$$\psi_g(\tau)v(\lambda) = e^{\frac{\pi i \kappa}{2} c(c\lambda + d)\lambda^2} v((c\tau + d)\lambda),$$

obeying the cocycle condition  $\psi_{gh}(\tau) = \psi_g(h \cdot \tau)\psi_h(\tau)$ . Denote by  $\pi : H_+ \rightarrow \mathcal{M}_{1,1}$  the canonical projection. Local holomorphic sections of the vector bundle of conformal blocks on an open set  $U \subset \mathcal{M}_{1,1}$  are then the same as holomorphic sections  $v$  of  $E_{\kappa, m}$  on  $\pi^{-1}(U)$  so that  $v(g \cdot \tau) = \psi_g(\tau)^{-1}v(\tau)$ . In other words, they are holomorphic functions  $v(\lambda, \tau)$  on  $\mathbb{C} \times p^{-1}(U)$  obeying (i)-(iii) for each fixed  $\tau$  and such that

$$v\left(\frac{\lambda}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = e^{-\frac{\pi i \kappa c \lambda^2}{2(c\tau + d)}} v(\lambda, \tau).$$

The projectivization of this vector bundle carries a holomorphic connection, and horizontal sections may be constructed by an elliptic version of hypergeometric integrals [FV1].

We describe here the connection in the case of  $sl(2, \mathbb{C})$ . If  $\mathfrak{g} = sl(2, \mathbb{C})$  and  $\Lambda = m\alpha$ ,  $m = 0, 1, \dots$ , then  $L_\Lambda[0]$  is one dimensional. Let us chose a basis of  $L_\Lambda[0]$  and identify  $\mathfrak{h} \simeq \mathfrak{h}^*$  with  $\mathbb{C}$  via the basis  $\alpha/2$ . Then  $E_{\kappa, \Lambda}(\tau) = E_{\kappa, 2m}(\tau)$  consists of holomorphic functions  $v(\lambda)$  on the complex plane so that (i)  $v(\lambda + 2r + 2s\tau) = \exp(-2\pi i\kappa(s^2\tau + s\lambda))v(\lambda)$ , (ii)  $v(-\lambda) = (-1)^{m+1}v(\lambda)$ , (iii)  $v$  is divisible by  $\theta(\lambda, \tau)^{m+1}$  in the ring of holomorphic functions.

If  $\kappa \geq 2m + 2$ , we have  $E_{\kappa, 2m}(\tau) = \theta(\lambda, \tau)^{m+1}\Theta_{\kappa-2m-2}(\tau)^W$ , where  $\Theta_\kappa(\tau)^W$  is the  $\kappa + 1$ -dimensional space of holomorphic even functions obeying (i). Otherwise  $E_{\kappa, 2m}(\tau)$  is trivial.

It follows that

$$\dim(E_{\kappa, 2m}(\tau)) = \begin{cases} \kappa - 2m - 1, & \text{if } \kappa \geq 2m + 2, \\ 0, & \text{otherwise.} \end{cases}$$

The connection on  $E_{\kappa,2m}$  is defined by its covariant derivative  $\Gamma(U, E_{\kappa,2m}) \rightarrow \Gamma(U, E_{\kappa,2m}) \otimes \Omega^1(U)$  on local holomorphic sections:

$$\nabla v(\lambda, \tau) = \left( \partial_\tau - \frac{1}{2\pi i \kappa} (\partial_\lambda^2 - m(m+1)\wp(\lambda, \tau)) - \eta(\tau)^{-1} \partial_\tau \eta(\tau) \right) v(\lambda, \tau) d\tau.$$

Here  $\wp$  is the Weierstrass elliptic function with periods 1 and  $\tau$  and

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{j=1}^{\infty} (1 - e^{2\pi i j \tau})$$

is the Dedekind  $\eta$ -function.<sup>2</sup> In spite of the poles of the  $\wp$  function, this connection is well-defined on  $E_{\kappa,2m}$  as can be seen by noticing that the poles cancel in the expression of the induced connection  $\theta^{-m-1} \circ \nabla \circ \theta^{m+1}$  on  $\Theta_{\kappa-2m-2}^W$ . The fact that  $\nabla$  preserves (i) and (ii) is easily checked.

The connection  $\nabla$  is  $\mathrm{SL}(2, \mathbb{Z})$ -equivariant, in the following sense: if  $U \subset H_+$  is an  $\mathrm{SL}(2, \mathbb{Z})$ -invariant open set, and  $g \in \mathrm{SL}(2, \mathbb{Z})$ , we have the pull-back  $g^* : \Gamma(U, E_{\kappa,2m}) \rightarrow \Gamma(U, E_{\kappa,2m})$ , sending a section  $v(\tau)$  to  $\psi_g(\tau)^{-1} v(g \cdot \tau)$ . We may extend  $g^*$  to  $\Gamma(U, E_{\kappa,2m}) \otimes \Omega^1(U)$  by tensoring with the pull-back of differential forms. Then  $g^* \circ \nabla = \nabla \circ g^*$ . Therefore the connection is well-defined on the vector bundle of conformal blocks on  $\mathcal{M}_{1,1}$ .

*Example.* If  $m = 0$ ,  $\nabla$  is essentially the differential operator of the heat equation. The theta functions

$$\theta_{j,\kappa}(\lambda, \tau) = \sum_{r \in \mathbb{Z} + j/2\kappa} e^{2\pi i \kappa (r^2 \tau + r \lambda)}, \quad j \in \mathbb{Z}/2\kappa\mathbb{Z},$$

form a basis of  $\Theta_\kappa(\tau)$  for fixed  $\tau$ , and obey the heat equation  $2\pi i \kappa \partial_\tau \theta_{j,\kappa} = \partial_\lambda^2 \theta_{j,\kappa}$ . Moreover, we have  $\theta_{j,\kappa}(-\lambda, \tau) = \theta_{-j,\kappa}(\lambda, \tau)$ . It follows that the functions

$$(14) \quad v_j(\lambda, \tau) = \eta(\tau)^{-1} (\theta_{j+1,\kappa}(\lambda, \tau) - \theta_{-j-1,\kappa}(\lambda, \tau)), \quad j = 0, 1, \dots, \kappa - 2,$$

form a basis of the space of horizontal sections.

See [FV1] for the case of arbitrary  $m$ .

**5.2. The difference case.** Let us turn to the difference case (for  $sl(2, \mathbb{C})$ ). We describe a difference analogue of  $E_{\Lambda,2m}$ , a holomorphic vector bundle  $E_{\Lambda,2m,\eta}$  on  $H_+$  which is preserved by the qKZB heat operator. We fix a generic  $\eta$  in the lower half plane. Guided by the semiclassical analysis of Sect. 4, we suppose that  $-p/2\eta = \kappa$  is an integer  $\geq 2$  and consider the qKZB heat operator (8) for  $n = 1$ ,  $z_1 = 0$ ,  $\Lambda_1 = 2m$ .

We start with the somewhat trivial but instructive case  $m = 0$ , and write  $T_{\kappa,0}(\tau) = T(z = 0, \tau, p = -2\eta\kappa)$ . Here the qKZB heat operator is

$$T_{\kappa,0}(\tau)v(\lambda) = \int_{2\eta\mathbb{R}} e^{-\frac{\pi i}{4\eta}(\lambda+\mu)^2} v(-\mu) d\mu.$$

---

<sup>2</sup>The connection, being on the projectivization, is really defined up to adding a multiple of the identity. We have chosen it here so that it defines a connection on the vector bundle over  $\mathcal{M}_{1,1}$

The integral is over the path  $t \mapsto 2\eta t$ ,  $-\infty < t < \infty$ .

We define  $E_{\kappa,2m=0,\eta} = E_{\kappa,0}$  to be the holomorphic vector bundle of odd theta functions, as in the differential case: the fiber over  $\tau \in H_+$  is  $E_{\kappa,0,\eta}(\tau) = \{f \in \Theta_\kappa(\tau) \mid f(-\lambda) = -f(\lambda)\}$

**Theorem 5.1.** *Let  $\kappa \geq 2$  and suppose that  $\text{Im } \eta < 0$ ,  $\text{Im } \tau > 0$ . Then  $T_{\kappa,0}(\tau)$  maps  $E_{\kappa,0,\eta}(\tau - 2\eta\kappa)$  to  $E_{\kappa,0,\eta}(\tau)$ .*

This theorem is based on the identity

$$\theta_{j,\kappa}(\lambda, \tau) = \frac{i}{\sqrt{4i\eta}} \int_{2\eta\mathbb{R}} e^{-\frac{i\pi}{4\eta}(\lambda+\mu)^2} \theta_{j,\kappa}(-\mu, \tau - 2\eta\kappa) d\mu, \quad j \in \mathbb{Z}/2\kappa\mathbb{Z},$$

which gives the action of  $T_{\kappa,0}(\tau)$  on the basis  $\theta_j - \theta_{-j}$ ,  $j = 1, \dots, \kappa - 1$ , of  $\Theta_\kappa(\tau - 2\eta\kappa)$ .

Let us now turn to the case of general  $m$ . To compare with the classical limit we consider the qKZB operator for the quotient  $v$  of the dependent function by  $\prod_{j=1}^m \theta(\lambda + 2\eta j, \tau)$ , i.e., we set

$$T_{\kappa,m}(\tau) = \phi_m(\tau)^{-1} \circ T(z = 0, \tau, p = \tau - 2\eta\kappa) \circ \phi_m(\tau - 2\eta\kappa),$$

where  $\phi_m(\tau)$  is the operator of multiplication by the function  $\lambda \mapsto \prod_{j=1}^m \theta(\lambda + 2\eta j, \tau)$ .

*Example.* If  $m = 1$ , the qKZB operator for  $v$  is  $v \mapsto T_{\kappa,1}(\tau)v$  is

$$T_{\kappa,1}(\tau)v(\lambda) = \alpha(\lambda) \int_{2\eta\mathbb{R}} V(\lambda, \mu, \tau, \tau - 2\eta\kappa) \alpha(\mu) v(-\mu) d\mu,$$

with kernel

$$V(\lambda, \mu, \tau, \sigma) = c e^{-\frac{\pi i \lambda \mu}{2\eta}} \int_\gamma \Omega_{2\eta}(t, \tau, \sigma) \frac{\theta(\lambda + t, \tau) \theta(\mu + t, \sigma)}{\theta(t - 2\eta, \tau) \theta(\lambda + 2\eta, \tau) \theta(t - 2\eta, \sigma) \theta(\mu + 2\eta, \sigma)} dt,$$

for some  $c = c(\tau, \sigma)$  independent of  $\lambda, \mu$ . The integration cycle is depicted in Fig. 1.

Let  $E_{\kappa,2m,\eta}(\tau)$  be the space of holomorphic functions so that

- (i)  $v(\lambda + 2r + 2s\tau) = \exp(4\pi i \eta m(m+1)s - 2\pi i \kappa(s^2\tau + s\lambda))v(\lambda)$ ,
- (ii)  $v(-\lambda) = (-1)^{m+1} \prod_{j=1}^m \frac{\theta(\lambda + 2\eta j, \tau)}{\theta(\lambda - 2\eta j, \tau)} v(\lambda)$ ,
- (iii)  $v$  is divisible by  $\prod_{j=0}^m \theta(\lambda - 2\eta j, \tau)$  in the ring of holomorphic functions.

Alternatively (and more simply),  $E_{\kappa,2m,\eta}(\tau)$  is the space of functions of the form  $\prod_{j=0}^m \theta(\lambda - 2\eta j, \tau) \varphi(\lambda)$ , with  $\varphi \in \Theta_{\kappa-2m-2}(\tau)^W$ . In particular,  $E_{\kappa,2m,\eta}(\tau)$  has the same dimension as the space  $E_{\kappa,2m}(\tau)$  appearing in the differential case. Let  $E_{\kappa,2m,\eta} = \cup_{\tau \in H_+} E_{\kappa,2m,\eta}(\tau)$ . It is naturally a holomorphic vector bundle over  $H_+$ .

**Theorem 5.2.** *Let  $m, \kappa \in \mathbb{Z}_{\geq 0}$ ,  $\kappa \geq 2m + 2$  and suppose that  $\text{Im } \eta < 0$ ,  $\text{Im } \tau > 0$ . Then  $T_{\kappa,m}(\tau)$  maps  $E_{\kappa,2m,\eta}(\tau - 2\eta\kappa)$  to  $E_{\kappa,2m,\eta}(\tau)$ .*

*Proof:* This theorem is a corollary of the results of [FV4]. We give here the proof in the simplest case  $m = 1$ . The proof of the general case is similar. Let  $v \in E_{\kappa, 2, \eta}(\tau - 2\eta\kappa)$ , and set  $\tilde{v} = T_{\kappa, 1}(\tau)v$ . Properties (i), (ii) for  $\tilde{v}$  can be checked straightforwardly, by using the identities

$$\theta(\lambda + 2, \tau) = \theta(\lambda, \tau), \quad \theta(\lambda + 2\tau, \tau) = e^{-4\pi i(\lambda + \tau)}\theta(\lambda, \tau), \quad \theta(-\lambda, \tau) = -\theta(\lambda, \tau),$$

obeyed by  $\theta$  and translating the integration variable in the integral over  $\mu$ . The latter involves moving the integration contour, which presents no problem as the vanishing condition (iii) for  $v$  guarantees that the integrand has no poles. Let us check that  $\tilde{v}$  is holomorphic and obeys (iii). As the zeros of  $\theta(\lambda, \tau)$  are simple and on the lattice  $\mathbb{Z} + \tau\mathbb{Z}$ ,  $\tilde{v}$  is regular except possibly for simple poles at  $-2\eta + \mathbb{Z} + \tau\mathbb{Z}$ .

We claim that  $\tilde{v}$  vanishes at  $\lambda = r + s\tau$  and at  $\lambda = 2\eta + r + s\tau$  for all  $r, s \in \mathbb{Z}$ . Then (ii) implies that  $\tilde{v}$  is regular at the points  $-2\eta + \mathbb{Z} + \tau\mathbb{Z}$  (and thus everywhere), and that  $\tilde{v}$  is divisible by  $\theta(\lambda, \tau)\theta(\lambda - 2\eta, \tau)$ .

Since  $\tilde{v}$  obeys (i), it is sufficient to prove the claim for  $r, s \in \{0, 1\}$

It follows from (ii) that  $\tilde{v}(0) = 0$  and, in conjunction with (i), also  $\tilde{v}(r + s\tau) = 0$ ,  $r, s \in \{0, \pm 1\}$ . For example, we have

$$\tilde{v}(-\tau) = \frac{\theta(\tau + 2\eta, \tau)}{\theta(\tau - 2\eta, \tau)} \tilde{v}(\tau) = \frac{e^{-8\pi i\eta}\theta(-\tau + 2\eta, \tau)}{\theta(\tau - 2\eta, \tau)} \tilde{v}(\tau) = -e^{-8\pi i\eta}\tilde{v}(\tau).$$

On the other hand, (i) implies  $\tilde{v}(-\tau) = e^{-8\pi i\eta}\tilde{v}(\tau)$ , so  $\tilde{v}(\tau) = 0$ .

Let us check that  $\tilde{v}(2\eta)$  vanishes. By using the functional equation (6) for  $\Omega_{2\eta}$ , we obtain

$$\begin{aligned} V(2\eta, \mu, \tau, \sigma) &= c e^{-\pi i\mu} \int \Omega_{2\eta}(t, \tau, \sigma) \frac{\theta(t + 2\eta, \tau)\theta(\mu + t, \sigma)}{\theta(t - 2\eta, \tau)\theta(4\eta, \tau)\theta(t - 2\eta, \sigma)\theta(\mu + 2\eta, \sigma)} dt \\ &= c e^{-\pi i\mu - 4\pi i\eta} \int \Omega_{2\eta}(t + \sigma, \tau, \sigma) \frac{\theta(\mu + t, \sigma)}{\theta(4\eta, \tau)\theta(t - 2\eta, \sigma)\theta(\mu + 2\eta, \sigma)} dt \\ &= c e^{-\pi i\mu - 4\pi i\eta} \int \Omega_{2\eta}(t, \tau, \sigma) \frac{\theta(\mu + t - \sigma, \sigma)}{\theta(4\eta, \tau)\theta(t - 2\eta - \sigma, \sigma)\theta(\mu + 2\eta, \sigma)} dt \\ &= c e^{\pi i\mu} \int \Omega_{2\eta}(t, \tau, \sigma) \frac{\theta(\mu + t, \sigma)}{\theta(4\eta, \tau)\theta(t - 2\eta, \sigma)\theta(\mu + 2\eta, \sigma)} dt \\ &= \frac{\theta'(0, \tau)}{\theta(4\eta, \tau)} \operatorname{res}_{\lambda = -2\eta} V(\lambda, \mu, \tau, \sigma). \end{aligned}$$

In this calculation the change of variable  $t \mapsto t - \sigma$  was used. For this our choice of  $t$ -integration contour is essential, since it implies that one does not encounter poles when one deforms it back to the original position. For general  $m$  this identity is part III of Theorem 26 in [FV4]. Thus

$$\tilde{v}(2\eta) = \frac{\theta'(0, \tau)}{\theta(4\eta, \tau)} \operatorname{res}_{\lambda = -2\eta} \tilde{v}(\lambda).$$

But it follows from (ii) that

$$\tilde{v}(2\eta) = -\frac{\theta'(0, \tau)}{\theta(4\eta, \tau)} \operatorname{res}_{\lambda=-2\eta} \tilde{v}(\lambda),$$

so  $\tilde{v}(2\eta) = 0$ . The same argument may be applied to  $2\eta + r + s\tau$  with  $r, s \in \{0, \pm 1\}$  (or even for general  $r, s$ ). We have

$$V(2\eta + r + s\tau, \tau, \sigma) = \frac{\theta'(0, \tau)}{\theta(4\eta, \tau)} e^{2\pi i s \sigma} \operatorname{res}_{\lambda=-2\eta+r+s\tau} V(\lambda, \mu, \tau, \sigma).$$

This implies that

$$\tilde{v}(2\eta + r + s\tau) = e^{-4\pi i \eta \kappa s} \frac{\theta'(0, \tau)}{\theta(4\eta, \tau)} \operatorname{res}_{\lambda=-2\eta+r+s\tau} \tilde{v}(\lambda).$$

On the other hand, using (ii) and (i) we obtain the same equation but with the opposite sign, so that both sides vanish. Thus  $\tilde{v}(2\eta + r + s\tau) = 0$  and  $\tilde{v}$  is regular at the potential singularities  $\lambda = -2\eta + r + s\tau$ ,  $r, s \in \mathbb{Z}$ .  $\square$

A more direct reformulation of this theorem is the following.

**Corollary 5.3.** *Let  $m, \kappa \in \mathbb{Z}_{\geq 0}$ ,  $\kappa \geq 2m + 2$  and suppose that  $\operatorname{Im} \eta < 0$ ,  $\operatorname{Im} \tau > 0$ . Let, for  $t \in \mathbb{C}^m$ ,*

$$\omega_m(t, \lambda, \tau) = \prod_{1 \leq i < j \leq m} \frac{\theta(t_i - t_j, \tau)}{\theta(t_i - t_j + 2\eta, \tau)} \prod_{j=1}^m \frac{\theta(\lambda + t_j, \tau)}{\theta(t_j - 2\eta m, \tau)}.$$

Introduce the integral kernel

$$M(\lambda, \mu, \tau, p) = \frac{e^{-\frac{\pi i}{4\eta}(\lambda + \mu)^2} u_0(\lambda, \mu, \tau, p) \theta(\mu, p)}{\prod_{j=-m}^m \theta(\lambda - 2\eta j, \tau)},$$

where

$$\begin{aligned} u_0(\lambda, \mu, \tau, p) &= \int \prod_{i=1}^m \Omega_{2\eta m}(t_i, \tau, p) \prod_{1 \leq i < j \leq m} \Omega_{-2\eta}(t_i - t_j, \tau, p) \\ &\quad \times \omega_m(t, \lambda, \tau) \omega_m(t, \mu, p) dt_1 \cdots dt_m. \end{aligned}$$

The integration is over a torus as in 2.2.

Then the integral operator

$$M(\tau)\phi(\lambda) = \int_{2\eta\mathbb{R}} M(\lambda, \mu, \tau, \tau - 2\eta\kappa)\phi(-\mu) d\mu$$

maps  $\Theta_{\kappa-2m-2}(\tau - 2\eta\kappa)^W$  to  $\Theta_{\kappa-2m-2}(\tau)^W$ .

5.3. **Remark.** A section  $v$  of  $E_{\kappa,2m,\eta}$  is called *projectively horizontal* if it obeys the qKZB equation  $T_{\kappa,m}(\tau)v(\tau - 2\eta\kappa) = C(\tau)v(\tau)$  up to a scalar factor  $C(\tau)$ . For  $m = 0$  projectively horizontal sections are given by odd theta functions as in the differential case, see (14). In a sequel [FV3] to this paper, we show that for  $m = 1$  (and conjecturally for higher  $m$  as well), projectively horizontal sections are again given by elliptic hypergeometric integrals.

5.4. **Remark.** The compatibility of the difference operator  $T_{\kappa,m}(\tau)$  with the  $SL(2, \mathbb{Z})$  action can be better understood in terms of a discrete connection on a space with an  $SL(3, \mathbb{Z})$ -action. This will be discussed in [FV3].

#### REFERENCES

- [B] D. Bernard, *On the Wess–Zumino–Witten model on the torus*, Nucl. Phys. B303 (1988), 77–93; *On the Wess–Zumino–Witten model on Riemann surfaces*, Nucl. Phys. B309 (1988), 145–174
- [F] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, Proceedings of the International Congress of Mathematicians, Zürich 1994, p. 1247–1255, Birkhäuser, 1994; *Elliptic quantum groups*, Proceedings of the International Congress of Mathematical Physics, Paris 1994, 211–218, International Press 1995
- [FV1] G. Felder and A. Varchenko, *Integral representation of solutions of the elliptic Knizhnik–Zamolodchikov–Bernard equation*, Int. Math. Res. Notices, No. 5(1995), 221–233
- [FV2] G. Felder and A. Varchenko, *On representations of the elliptic quantum group  $E_{\tau,\eta}(sl_2)$* , Commun. Math. Phys. 181 (1996), 741–761
- [FV3] G. Felder and A. Varchenko, *Quantum KZB heat equation, modular transformations and  $SL(3, \mathbb{Z})$* , in preparation
- [FV4] G. Felder and A. Varchenko, *Resonance relations for solutions of the elliptic QKZB equations, fusion rules, and eigenvectors of transfer matrices of restricted interaction-round-a-face models*, Commun. Contemp. Math. 1 (1999), no. 3, 335–403
- [FV5] G. Felder and A. Varchenko, *The elliptic gamma function and  $SL(3, \mathbb{Z}) \tilde{\times} \mathbb{Z}^3$* , math/9907061
- [FTV1] G. Felder, V. Tarasov and A. Varchenko, *Solutions of the elliptic qKZB equations and Bethe ansatz I*, Amer. Math. Soc. Transl. 180 (1997), 45–75
- [FTV2] G. Felder, V. Tarasov and A. Varchenko, *Monodromy of solutions of the elliptic Knizhnik–Zamolodchikov–Bernard difference equations*, q-alg/9705017, to appear in Int. J. Mod. Phys.
- [FW] G. Felder and C. Wierczkowski, *Conformal blocks on elliptic curves and the Knizhnik–Zamolodchikov–Bernard equation*, Commun. Math. Phys. 176 (1996), 133–162
- [FR] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Commun. Math. Phys. 146 (1992), 1–60
- [LP] S. Lukyanov and Ya. Pugai, *Multi-point Local Height Probabilities in the Integrable RSOS Model*, Nucl. Phys. B473 (1996), 631–658
- [MV] E. Mukhin and A. Varchenko, *Solutions of the qKZB equation in tensor products of finite dimensional modules over the elliptic quantum group  $E_{\tau,\eta}sl_2$* , Fields Institute Communications 24 (1999), 385–396
- [T] T. Takebe, *A system of difference equations with elliptic coefficients and Bethe vectors*, Commun. Math. Phys. 183 (1997), 161–182
- [TUY] A. Tsuchiya, K. Ueno and Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Adv. Stud. Pure Math. 19 (1989), 459–566