Nonsymmetric Difference Whittaker Functions and Double Affine Hecke Algebras

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Abstract

DANIEL ORR: Nonsymmetric Difference Whittaker Functions and Double Affine Hecke Algebras
(Under the direction of Ivan Cherednik)

This dissertation is devoted to a new theory of nonsymmetric difference Whittaker functions and the corresponding Toda-Dunkl operators for arbitrary reduced irreducible root systems. The nonsymmetric Whittaker functions are obtained as limits of (global) spherical functions under a variant of a limiting procedure due to Ruijsenaars and Etingof. Under this procedure, the Toda-Dunkl operators are realized as limits of difference-reflection Dunkl operators. We give a direct and constructive proof of the existence of these limits. We show that the nonsymmetric Whittaker function solves the eigenvalue problem for Toda-Dunkl operators and admits an explicit expansion in terms of the level-one affine Demazure characters.

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Introduction

This dissertation is devoted to a new theory of nonsymmetric difference Whittaker functions and the corresponding Toda-Dunkl operators for arbitrary reduced irreducible root systems, generalizing the A_1–case considered in [**15, 16, 17**]. Our approach is based on a new technique involving W–spinors, which can be thought of as functions $\{f_w\}$ indexed by the elements of the (finite) Weyl group W with the natural action of W on the indices. This technique has important links to the classical harmonic analysis on symmetric spaces and the theory of spherical, Whittaker, and Bessel functions. For instance, W–spinors arise in the study of nonsymmetric or singular symmetric solutions of symmetric systems such as the Quantum Many-Body Problem; see [**6, 15, 31**].

The theory of nonsymmetric (global) spherical functions from [**9**] is the starting point of our approach; these functions are denoted by $G(X, \Lambda)$ in Theorem 3.2.1. We introduce the nonsymmetric difference Whittaker function Ω as the limit of G under a nonsymmetric variant of a limiting procedure due to Ruijsenaars [**32**] and Etingof [**18**]. The function Ω is a quadratic-type generating function for the nonsymmetric Macdonald polynomials $\overline{E}_b$ (at $t = 0$) for b in the weight lattice P. Moreover, the values of Ω at $X = q^c$ for $c \in P$ coincide with $\overline{E}_c(\Lambda)$ up to an explicit factor. See Theorem 4.4.1, the main result of this dissertation, and also Proposition 4.3.1.

Symmetric variants of Ω were introduced and studied in [**13**]. These symmetric Whittaker functions $\mathcal{W}$ solve the (generalized) difference Toda eigenvalue problem and, moreover, they simultaneously generalize the Whittaker functions from the classical harmonic analysis on symmetric spaces [**23, 37**] and their p–adic counterparts from [**4**]. They are expressed in terms of $\overline{E}_b$ for antidominant b only (such $\overline{E}_b$ are W–invariant).

We show that the function Ω solves the eigenvalue problem for the Toda-Dunkl operators $\widehat{Y}_b$ ($b \in P$), which we introduce as limits of the Dunkl difference-reflection operators. Establishing the existence of the Toda-Dunkl operators is one of the central developments of this dissertation. Proposition 5.2.1 provides a direct and constructive justification of the existence of $\widehat{Y}_b$ via the nonsymmetric Ruijsenaars-Etingof procedure; its proof provides formulas for basic spinor Dunkl operators, including those for the minuscule weights (which are involved even for root systems of type A—see Section 5.5 for some examples).

The nonsymmetric Whittaker function Ω leads an indirect justification of the existence of Toda-Dunkl operators (see the Remark following Theorem 4.4.1). However, this approach is inconvenient for finding explicit formulas and does not clarify the structure of these operators.

Via symmetrization of Ω and $\widehat{Y}_b$, we recover the symmetric Whittaker functions and Toda operators from [**13**]; see (4.22) and Proposition 5.4.1.

0.1. Origins

In order to motivate our approach, let us consider the case of GL_N in more detail. In this setting, the difference Toda Hamiltonian is the operator

$$H = \sum_{i=1}^{N-1} (1 - X_{i+1}X_i^{-1})\,\Gamma_i + \Gamma_N, \tag{0.1}$$

acting on functions F of the variables $X_1, \ldots, X_N \in \mathbb{C}^*$, where Γ_i is the translation operator given by

$$\Gamma_i(F)(X_1, \ldots, X_N) = F(X_1, \ldots, qX_j, \ldots X_N)$$

and $q \in \mathbb{C}^*$. The operator H is equivalent (by an explicit gauge transformation) to Ruijsenaars' quantum relativistic Toda Hamiltonian modeling a system of N particles on a line with exponential nearest-neighbor interactions [**18, 32**].

The operator H is a certain limit of the Macdonald difference operator

$$L = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t^{1/2}X_i - t^{-1/2}X_j}{X_i - X_j}\, \Gamma_i \quad (t \in \mathbb{C}^*)$$

when $t \to 0$. More precisely, before taking the limit one conjugates L as follows. For any $a = (a_1, \dots, a_N) \in \mathbb{Z}^N = P$, we set

$$X_a = X_1^{a_1} \cdots X_N^{a_N} \quad \text{and} \quad \Gamma_a = \Gamma_1^{a_1} \cdots \Gamma_N^{a_N}.$$

Now let $\rho = (\frac{N-1}{2}, \frac{N-3}{2}, \cdots, -\frac{N-1}{2})$ and consider the operator

$$\varkappa(L) = (X_{k\rho}\, \Gamma_{-k\rho})\, L\, (X_{k\rho}\, \Gamma_{-k\rho})^{-1}, \tag{0.2}$$

where we impose the relation $t = q^k$. (In order for this to make sense, we should require that $k\rho \in \mathbb{Z}^N$; however, the result $\varkappa(L)$, when expressed in terms of q and t, depends neither on this restriction nor on the specific choice of k.) A straightforward calculation then shows that

$$RE(L) := \lim_{t \to 0} \varkappa(L) = H. \tag{0.3}$$

This limiting procedure is due to Ruijsenaars [**32**] and Etingof [**18**].

In [**7**], Cherednik used double affine Hecke algebras (DAHAs) to realize L (and its analogues for arbitrary root systems) as symmetrizations of *Dunkl operators*. The latter are pairwise commutative difference-reflection operators indexed by $a \in P$; for GL_N, they are denoted Y_a for $a \in \mathbb{Z}^N$. If $\omega_1 = (1, 0, \dots, 0)$, then one has

$$L = \sum_{a \in W(\omega_1)} Y_{-a},$$

upon the restriction to $W-$invariant functions. Here $W = S_N$ is the symmetric group.

The symmetric (global) spherical function $F(X,\Lambda)$ from [**9**] solves the Macdonald eigenvalue problem. In the case of GL_N, the eigenvalue problem reads:

$$L(F(X,\Lambda)) = (\Lambda_1 + \cdots + \Lambda_N)F(X,\Lambda),$$

where $\Lambda = (\Lambda_1, \cdots, \Lambda_N) \in (\mathbb{C}^*)^N$. In [**13**], it was shown (for arbitrary root systems) that the limit

$$\mathcal{W}(X,\Lambda) := \lim_{t\to 0} X_{k\rho}\Gamma_{-k\rho}(F(X,\Lambda)) \tag{0.4}$$

exists and solves the corresponding Toda eigenvalue problem.

Correspondingly, the nonsymmetric spherical function $G(X,\Lambda)$ from [**9**] is a solution to the Dunkl eigenvalue problem:

$$Y_a(G(X,\Lambda)) = \Lambda_a^{-1}G(X,\Lambda) \quad (a \in \mathbb{Z}^N),$$

where we define Λ_a as above. The main objective of this dissertation is to extend the limits (0.3) and (0.4) to the nonsymmetric setting (for arbitrary root systems).

0.2. Perspectives

The results of this dissertation suggest the following topics for future research:

Representation theory of nil-DAHA. In our construction of the nonsymmetric Whittaker function Ω and the Toda-Dunkl operators $\widehat{Y}_b$, a certain degeneration of the DAHA as $t \to 0$ plays a fundamental role. The resulting algebra is called nil-DAHA; see Definition 2.4.2. The function Ω admits an alternate characterization as the kernel of an integral transform between irreducible nil-DAHA modules. The image of this transform, the so-called *spinor-polynomial representation*, is a new addition to the representation theory of DAHA. In the rank-one case, the spinor-polynomial representation was given its proper representation-theoretic interpretation as a (sub-)induced nil-DAHA module in [**16**]. Extending this description of the spinor-polynomial representation to arbitrary root systems and developing a general classification of induced nil-DAHA modules are interesting problems for future research. In the general representation theory DAHA, the nil-DAHA is expected to play a role analogous to that of crystal bases in the representation theory of quantum groups.

Analytic theory of nonsymmetric Whittaker functions. The study of asymptotic expansions of symmetric (global) Whittaker functions—the analog of Harish-Chandra's expansion of spherical functions on real semisimple Lie groups [**24**]—was initiated by Cherednik in [**13**]. More generally, the symmetric (global) spherical functions have been studied from the same point of view in [**36**]. While convergent expansions are known to exist in the symmetric setting, the exact expansion coefficients are complicated and only indirectly described. The asymptotic theory of *nonsymmetric* (spherical and Whittaker) functions is expected to lead to new insights in this direction. In addition to their own fundamental importance, the asymptotic expansions of symmetric Whittaker functions are particularly relevant in several of the applications discussed below.

Applications of symmetric Whittaker functions. The symmetric difference Whittaker functions are known to have many applications, including the quantum $K-$theory of flag varieties [**3, 22**], the theory of $q-$Whittaker processes [**1**], and generalized Rogers-Ramanujan identities related to the representation theory of affine Lie algebras [**14**]. Furthermore, they exhibit important connections to Whittaker vectors in the representation theory of quantum groups [**18, 19, 34**]. The works [**20, 21**] provide an extensive treatment of the GL_N-case, touching upon some of these applications (and more). The theory of nonsymmetric Whittaker functions and Toda-Dunkl operators developed in this dissertation is expected to enrich these directions of research.

0.3. Outline

Let us describe the contents of this dissertation in more detail. In Chapter 1, we gather basic facts about root systems, Weyl groups, and double affine Hecke algebras. Chapter 2 is devoted to the nonsymmetric Macdonald polynomials, their construction using intertwiners, and their behavior under certain limits. The global spherical functions, which extend the nonsymmetric Macdonald polynomials and are essential for the construction of the nonsymmetric difference Whittaker function Ω, are introduced in Chapter 3. In Chapter 4, we formulate our main result and prove the existence of Ω, leading to an indirect proof of the existence of the Toda-Dunkl operators. Finally, in Chapter 5, we provide a direct and constructive proof of the existence of these operators. This proof involves certain combinatorial properties of reduced expressions in the Weyl group and is quite interesting in its own right.

CHAPTER 1

Double affine Hecke algebras

1.1. Root systems

Fix an integer $n \geq 1$. For vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $(x, y) = x_1 y_1 + \cdots + x_n y_n$ be their dot product. Given $\alpha \in \mathbb{R}^n \setminus \{0\}$, let H_α denote the hyperplane in $\mathbb{R}^n$ orthogonal to α. The reflection through H_α can be expressed as

$$s_\alpha(x) = x - (x, \alpha^\vee)\alpha,$$

where $\alpha^\vee := 2\alpha/(\alpha, \alpha)$.

A *root system* in $\mathbb{R}^n$ is a finite subset $R \subset \mathbb{R}^n \setminus \{0\}$ satisfying the following axioms:

$$R \text{ spans } \mathbb{R}^n, \tag{1.1}$$

$$(\alpha, \beta^\vee) \in \mathbb{Z} \text{ for all } \alpha, \beta \in R, \tag{1.2}$$

$$s_\alpha(\beta) \in R \text{ for all } \alpha, \beta \in R. \tag{1.3}$$

Elements of R are called *roots*. Note that $\alpha \in R$ implies that $-\alpha = s_\alpha(\alpha) \in R$. We will assume, in addition, that R is *reduced* and *irreducible*. We say that R is reduced if for any $\alpha \in R$, one has $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$. We say that R is irreducible if it is not possible to partition R into two nonempty, mutually orthogonal subsets.

We refer to [**2**] for the classification of root systems and for proofs of the following basic properties of R.

There are at most two possible lengths of roots in R. When there are two distinct root lengths, we refer to roots as being *short* and *long*; otherwise, all roots are called both short and long.

The *Weyl group* W is the subgroup of $O(n, \mathbb{R})$ generated by the reflections $\{s_\alpha\}_{\alpha \in R}$.

For any $c \in \mathbb{R}^*$, the set cR is again root system having the same Weyl group as R. Note that the quantity $(\alpha, \beta^\vee)$ is invariant under simultaneous scaling of α and β. We assume that $(\alpha, \alpha) = 2$ for *short* roots α. For any $\alpha \in R$, we set

$$\nu_\alpha = (\alpha, \alpha)/2, \quad \nu_R = \{\nu_\alpha : \alpha \in R\}.$$

Then ν_R is one of the following sets: $\{1\}, \{1, 2\}$, or $\{1, 3\}$.

A *chamber* of R is a connected component of $\mathbb{R}^n \setminus \cup_{\alpha \in R} H_\alpha$. The Weyl group W acts simply transitively on the set of chambers. The choice of a chamber $\mathcal{C}$ gives rise to a partition of R into disjoint subsets $R = R_+ \cup R_-$, where for any $x \in \mathcal{C}$

$$R_+ = \{\alpha \in R : (x, \alpha) > 0\}, \quad R_- = -R_+.$$

Chambers are in bijection with *bases* of R. The latter are, by definition, subsets $\Delta \subset R$ having the property that each root can be written uniquely as a sum

$$\sum_{\alpha \in \Delta} n_\alpha \alpha \tag{1.4}$$

with either all $n_\alpha \geq 0$ or all $n_\alpha \leq 0$. Given a chamber $\mathcal{C}$, determining the partition $R = R_+ \cup R_-$, the associated base Δ is the set of all $\alpha \in R_+$ which cannot be written as $\alpha = \beta + \gamma$ for some $\beta, \gamma \in R_+$. Then R_+ (resp. R_-) consists of all roots represented as sums (1.4) such that $n_\alpha \geq 0$ (resp. $n_\alpha \leq 0$) for all $\alpha \in \Delta$. A base of R is also a basis of $\mathbb{R}^n$ and therefore any base has cardinality equal to n.

From this point on, we fix a chamber $\mathcal{C}$ and the corresponding base Δ. Write $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ and set $s_i = s_{\alpha_i}$. The α_i are called *simple roots*, and the s_i *simple reflections*. The Weyl group W is generated by the simple reflections. In fact, W admits an explicit presentation as a *Coxeter group*—namely, W is generated by $s_1, \ldots, s_n$ subject to the defining relations

$$s_i^2 = 1 \quad (1 \le i \le n), \tag{1.5}$$

$$(s_i s_j)^{m_{ij}} = 1 \quad (1 \le i \ne j \le n), \tag{1.6}$$

where $\pi - \pi/m_{ij}$ is the angle between α_i and α_j. Explicitly, $m_{ij} = 2, 3, 4$, or 6 as $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee) = 0, 1, 2$, or 3, respectively. The relations (1.6), which are typically referred to as the *braid relations*, can be written as

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots , \tag{1.7}$$

where both sides consist of m_{ij} factors.

In particular, any $w \in W$ can be written as a product

$$w = s_{j_l} \cdots s_{j_1}, \tag{1.8}$$

where $j_1, \ldots, j_l \in \{1, \ldots, n\}$. Define the *length* of w, denoted $l(w)$, to be the smallest l for which an expression of the form (1.8) exists. An expression (1.8) having $l = l(w)$ will be called a *reduced expression* for w. The length function on W is independent of the choice of the chamber $\mathcal{C}$, which justifies the notation $l(w)$.

The group W has a unique element w_0 of longest length. This element is also uniquely determined by the condition $w_0(\mathcal{C}) = -\mathcal{C}$. The length of w_0 is the cardinality of R_+.

The *dual root system* is defined as $R^\vee = \{\alpha^\vee : \alpha \in R\}$. One readily verifies that $R^\vee$ is a root system in $\mathbb{R}^n$ with base $\{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ and the same Weyl group as R. Elements of $R^\vee$ are called *coroots*.

The *root lattice* Q and *coroot lattice* $Q^\vee$ are the (additive) subgroups of $\mathbb{R}^n$ generated by R and $R^\vee$, respectively. Explicitly,

$$Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i, \quad Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee.$$

Let

$$Q_{\pm} = \bigoplus_{i=1}^{n} \mathbb{Z}_{\pm}\alpha_i, \quad Q^{\vee}_{\pm} = \bigoplus_{i=1}^{n} \mathbb{Z}_{\pm}\alpha^{\vee}_i,$$

where $\mathbb{Z}_{\pm} = \{m \in \mathbb{Z} : \pm m \geq 0\}$. We also set $\mathbb{Z}_{>0} = \{m \in \mathbb{Z} : m > 0\}$.

The *weight lattice* P is dual to the coroot lattice $Q^{\vee}$ and the *coweight lattice* $P^{\vee}$ is dual to the root lattice Q:

$$P = \{b \in \mathbb{R}^n : (b, \alpha^{\vee}) \in \mathbb{Z}, \forall \alpha \in R\}, \quad P^{\vee} = \{b \in \mathbb{R}^n : (b, \alpha) \in \mathbb{Z}, \forall \alpha \in R\}.$$

In terms of the *fundamental weights* $\omega_1, \ldots, \omega_n$ and *fundamental coweights* $\omega^{\vee}_1, \ldots, \omega^{\vee}_n$ determined by

$$(\omega_i, \alpha^{\vee}_j) = \delta_{ij}, \quad (\omega^{\vee}_i, \alpha_j) = \delta_{ij},$$

one has

$$P = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i, \quad P^{\vee} = \bigoplus_{i=1}^{n} \mathbb{Z}\omega^{\vee}_i.$$

Let

$$P_{\pm} = \bigoplus_{i=1}^{n} \mathbb{Z}_{\pm}\omega_i, \quad P^{\vee}_{\pm} = \bigoplus_{i=1}^{n} \mathbb{Z}_{\pm}\omega^{\vee}_i.$$

Elements of P (resp. $P^{\vee}$) are called *dominant weights* (resp. *dominant coweights*). We refer to elements of P_- and $P^{\vee}_-$ as *antidominant*. An element $b \in P$ (resp. $P^{\vee}$) belonging to a chamber (i.e., $(b, \alpha) \neq 0$ for all $\alpha \in R$) is called a *regular (co)weight*.

Let

$$\rho = \sum_{\nu \in \nu_R} \rho_\nu, \quad \rho_\nu = \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \nu_\alpha = \nu}} \alpha = \sum_{\substack{1 \leq i \leq n \\ \nu_i = \nu}} \omega_i,$$

$$\rho^{\vee} = \sum_{\nu \in \nu_R} \rho^{\vee}_\nu, \quad \rho^{\vee}_\nu = \nu^{-1}\rho_\nu = \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \nu_\alpha = \nu}} \alpha^{\vee} = \sum_{\substack{1 \leq i \leq n \\ \nu_i = \nu}} \omega^{\vee}_i.$$

For any $b \in P$, let b_+ (resp. b_-) denote the unique dominant (resp. antidominant) weight in the orbit $W(b)$. For $x, y \in \mathbb{R}^n$, we write $x \leq y$ to mean that $y - x \in Q_+$, and we write $x < y$ if in addition $x \neq y$. For any $b \in P$, one has $b_- \leq b \leq b_+$.

We define a partial ordering $\preceq$ on P as follows:

$$b \preceq c \iff b_- \leq c_- \text{ and if } b_- = c_-, \text{ then } b \leq c. \tag{1.9}$$

Note that $b_- = c_-$ means that b, c belong to the same W–orbit. We write $b \prec c$ if $b \preceq c$ and $b \neq c$. Note that $\preceq$ and $\leq$ agree on P_-. Within each W–orbit of P, the antidominant weight is minimal with respect to $\preceq$, and the dominant weight is maximal.

We will also use the *Bruhat ordering* on W, which can be defined as follows. Write $w \to w'$ if there exists some $\alpha \in R$ such that $w = s_\alpha w'$ and $l(w) > l(w')$. Then the Bruhat ordering is the partial ordering $\geq$ generated by these relations. In other words, $w \geq w'$ if there exists a chain $w = w_1 \to w_2 \to \cdots \to w_m = w'$.

1.2. Affine Weyl groups

Denote elements of $\mathbb{R}^n \times \mathbb{R}$ by $[x, \zeta]$ for $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$. The *(twisted) affine root system*[1] associated to R is the following subset of $\mathbb{R}^n \times \mathbb{R}$:

$$\widetilde{R} = \{[\alpha, \nu_\alpha j] : \alpha \in R,\ j \in \mathbb{Z}\},$$

Recall that $\nu_\alpha = (\alpha, \alpha)/2$ and $\nu_\alpha = 1$ when α is a short root. Elements of $\widetilde{R}$ are called *affine roots*. We extend the dot product on $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}$ trivially:

$$([x, \zeta], [y, \xi]) := (x, y).$$

For any affine root $\widetilde{\alpha} = [\alpha, \nu_\alpha j]$, we set $\widetilde{\alpha}^\vee = 2\widetilde{\alpha}/(\widetilde{\alpha}, \widetilde{\alpha})$ and $\nu_{\widetilde{\alpha}} = (\widetilde{\alpha}, \widetilde{\alpha})/2 = \nu_\alpha$.

For any affine root $\widetilde{\alpha} = [\alpha, \nu_\alpha j]$, consider the hyperplane

$$H_{\widetilde{\alpha}} := \{x \in \mathbb{R}^n : (x, \alpha) + \nu_\alpha j = 0\}.$$

[1] These are the real roots of the following affine root systems from [**26**]:

R	A_n	B_n	C_n	D_n	$E_{6,7,8}$	F_4	G_2
$\widetilde{R}$	$A_n^{(1)}$	$D_{n+1}^{(2)}$	$A_{2n-1}^{(2)}$	$D_n^{(1)}$	$E_{6,7,8}^{(1)}$	$E_6^{(2)}$	$D_4^{(3)}$

We note that $(x,\alpha)+\nu_\alpha j = \nu_\alpha((x,\alpha^\vee)+j)$, so $H_{\widetilde{\alpha}}$ is equivalently described by the equation $(x,\alpha^\vee)+j=0$. Let $s_{\widetilde{\alpha}}$ denote the reflection in $\mathbb{R}^n$ through the hyperplane $H_{\widetilde{\alpha}}$. This reflection is given explicitly by the formula

$$s_{\widetilde{\alpha}}(\!(x)\!) := x - ((x,\alpha^\vee)+j)\alpha = s_\alpha(x) - j\alpha. \tag{1.10}$$

The *affine Weyl group* $\widetilde{W}$ is the group of affine transformations of $\mathbb{R}^n$ generated by the reflections $\{s_{\widetilde{\alpha}}\}_{\widetilde{\alpha}\in\widetilde{R}}$. Due to (1.10), one has

$$s_\alpha s_{\widetilde{\alpha}} = \tau_{-j\alpha},$$

where τ_y for $y\in\mathbb{R}^n$ denotes the translation $\tau_y(\!(x)\!) = x+y$. One has $\widetilde{W} = W \ltimes Q$, where we identify Q with the subgroup $\{\tau_b : b\in Q\}\subset\widetilde{W}$ and $w\tau_b w^{-1} = \tau_{w(b)}$ for any $w\in W$ and $b\in Q$. We simply write b in place of τ_b from now on.

The affine Weyl group naturally acts on the space of affine linear functions on $\mathbb{R}^n$, which we identify with $\mathbb{R}^n\times\mathbb{R}$ as follows. Given any $[y,\zeta]\in\mathbb{R}^n\times\mathbb{R}$, we form the affine linear function $x\mapsto (x,y)+\zeta$. Under this identification, the action $\widetilde{w}(f)(x) = f(\widetilde{w}^{-1}(x))$ of $\widetilde{w} = wb\in\widetilde{W}$ on an affine linear function f, is given by

$$wb([y,\zeta]) = [w(y), \zeta - (b,x)]. \tag{1.11}$$

This action preserves $\widetilde{R}$, since

$$wb([\alpha,\nu_\alpha j]) = [w(\alpha), \nu_\alpha(j-(b,\alpha^\vee))]. \tag{1.12}$$

The actions of $\widetilde{W}$ on $\mathbb{R}^n\times\mathbb{R}$ and $\mathbb{R}^n$ defined above are compatible in the following sense. Define

$$([x,\zeta],[y,\xi]+d) := (x,y)+\zeta. \tag{1.13}$$

In other words, we enlarge $\mathbb{R}^n \times \mathbb{R}$ by adding a linearly independent element[2] d and we extend the pairing by setting $([0,1], d) = 1$. Then one has

$$(1.14) \qquad (\widetilde{w}([x,\zeta]), \widetilde{w}(\!(y)\!) + d) = ([x,\zeta], y + d).$$

Connected components of $\mathbb{R}^n \setminus \bigcup_{\widetilde{\alpha}\in\widetilde{R}} H_{\widetilde{\alpha}}$ are called *alcoves*. The affine Weyl group $\widetilde{W}$ acts simply transitively on the set of alcoves. The *fundamental alcove* $\mathcal{A}$ is determined by the inequalities $0 < (x, \alpha^\vee) < 1$ for all $\alpha \in R_+$. The inequalities $(x, \alpha_i^\vee) > 0$ for $i = 1, \ldots, n$ and $(x, \vartheta) < 1$ are sufficient to describe $\mathcal{A}$. Here ϑ is the highest *short* root of R, i.e., the unique maximal short root with respect to the partial ordering $\leq$. One has $\vartheta^\vee = \vartheta$ and ϑ is the highest root in $R^\vee$. One may also characterize ϑ as the unique short root lying in P_+.

The choice of an alcove determines a disjoint union $\widetilde{R} = \widetilde{R}_+ \cup \widetilde{R}_-$, where for any x in the chosen alcove $\widetilde{R}_+ = \{[\alpha, \nu_\alpha j] : (x, \alpha^\vee) + j > 0\}$ and $\widetilde{R}_- = -\widetilde{R}_+$. From this point on, we choose the fundamental alcove, which gives

$$(1.15) \qquad \widetilde{R}_+ = \{[\alpha, \nu_\alpha j] : \alpha \in R_+, j \geq 0 \ \text{ or } \ \alpha \in R_-, j > 0\}, \quad \widetilde{R}_- = -\widetilde{R}_+.$$

We identify R with the subset $R \times \{0\} \subset \widetilde{R}$.

The base of $\widetilde{R}$ determined by $\mathcal{A}$ consists of the simple roots $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$, where $\alpha_0 = [-\vartheta, 1]$. Every affine root can be written uniquely as a sum $\widetilde{\alpha} = \sum_{i=0}^n n_i \alpha_i$ for $n_i \in \mathbb{Z}$, and one has $\widetilde{\alpha} \in \widetilde{R}_+$ (resp. $\widetilde{\alpha} \in \widetilde{R}_-$) if and only if all $n_i \geq 0$ (resp. all $n_i \leq 0$).

Let $s_0 = s_{\alpha_0}$. The affine Weyl group $\widetilde{W}$ is generated by $\{s_0, s_1, \ldots, s_n\}$ subject to the defining relations

$$(1.16) \qquad s_i^2 = 1 \quad (0 \leq i \leq n),$$

$$(1.17) \qquad (s_i s_j)^{m_{ij}} = 1 \quad (0 \leq i \neq j \leq n),$$

[2] In the notation of [**26**], the element d corresponds to Λ_0 and $[0,1]$ to the imaginary root δ. The affine action $\widetilde{w}(\!(y)\!)$ is the so-called level-one action (modulo δ).

where $m_{ij} = 2, 3, 4$, or 6 as $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee) = 0, 1, 2$, or 3, respectively. Thus $\widetilde{W}$ is a Coxeter group, and the Weyl group W is the subgroup of $\widetilde{W}$ generated by $\{s_1, \ldots, s_n\}$.

We define the length $l(\widetilde{w})$ as the smallest l for which there exists an expression $\widetilde{w} = s_{j_l} \cdots s_{j_1}$ where $j_1, \ldots, j_l \in \{0, \ldots, n\}$. For $\widetilde{w} = w \in W$, this definition agrees with the one given in the previous section.

Another important characterization of the length function $l(\widetilde{w})$ is the following. One has $l(\widetilde{w}) = |\lambda(\widetilde{w})|$, where

$$\lambda(\widetilde{w}) := \{\widetilde{\alpha} \in \widetilde{R}_+ : \widetilde{w}(\alpha) \in \widetilde{R}_-\} = \widetilde{R}_+ \cap \widetilde{w}^{-1}(\widetilde{R}_-).$$

The following refinement of $\lambda(\widetilde{w})$ gives more information:

$$\lambda(\widetilde{w}) = \bigcup_{\nu \in \nu_R} \lambda_\nu(\widetilde{w}), \quad \text{where} \quad \lambda_\nu(\widetilde{w}) = \{\alpha \in \lambda(\widetilde{w}) : \nu_\alpha = \nu\}. \tag{1.18}$$

Define $l_\nu(\widetilde{w}) = |\lambda_\nu(\widetilde{w})|$. Then $l_\nu(\widetilde{w})$ is equal to the number of s_j with $\nu_{\alpha_j} = \nu$ in any reduced expression for $\widetilde{w}$.

Consider the larger group $\widehat{W} = W \ltimes P$, which is called the *extended affine Weyl group*. The affine Weyl group $\widetilde{W}$ is a normal subgroup of $\widehat{W}$, and one has a natural isomorphism $\widehat{W}/\widetilde{W} \cong P/Q$. The affine action of $\widetilde{W}$ on $\mathbb{R}^n$ extends to $\widehat{W}$ via the translations $\tau_b(\!(x)\!) = x + b$ for $b \in P$, and $\widehat{W}$ acts on $\mathbb{R}^n \times \mathbb{R}$ preserving $\widetilde{R}$ by (1.11). The compatibility condition (1.14) continues to hold for these actions of $\widehat{W}$.

We extend the length function to $\widehat{W}$ by defining $\lambda_\nu(\widehat{w}) = \widetilde{R}_+ \cap \widehat{w}^{-1}(\widetilde{R}_-)$, $\lambda(\widehat{w}) = \cup_{\nu \in \nu_R} \lambda_\nu(\widehat{w})$, $l_\nu(\widehat{w}) = |\lambda_\nu(\widehat{w})|$, and $l(\widehat{w}) = |\lambda(\widehat{w})|$ for any $\widehat{w} \in \widehat{W}$.

In contrast to $\widetilde{W}$, the group $\widehat{W}$ is not a Coxeter group; it has elements of length zero other than the identity element id. Let Π denote the subgroup of $\widehat{W}$ consisting of all length zero elements. The group Π can also be characterized as the stabilizer in $\widehat{W}$ of the fundamental alcove. Hence the composition $\Pi \hookrightarrow \widehat{W} \to \widehat{W}/\widetilde{W}$ gives rise to an isomorphism $\Pi \cong \widehat{W}/\widetilde{W}$, since $\widetilde{W}$ acts simply transitively on the set of alcoves.

Therefore, one has isomorphisms $\Pi \cong \widehat{W}/\widetilde{W} \cong P/Q$. The nonzero elements in P/Q are in bijection with the fundamental weights ω_r satisfying $(\omega_r, \vartheta) = 1$, which are the *minuscule* fundamental weights. Note that $(\omega_r, \vartheta) = 1$ is equivalent to $\omega_r \in \mathcal{A}$. Let

$$O' := \{r : (\omega_r, \vartheta) = 1\}, \quad O := O' \cup \{0\}.$$

Under the composite isomorphism $P/Q \cong \Pi$, the coset $\omega_r + Q$ corresponds to the element $\pi_r \in \widehat{W}$ defined by

$$\pi_r := \omega_r u_r^{-1},$$

where u_r the unique shortest element in W such that $u_r(\omega_r) \in P_-$. Explicitly, $u_r = w_0 w_0^{\omega_r}$, where $w_0^{\omega_r}$ is the unique longest element in the stabilizer of ω_r in W.

Since Π preserves the fundamental alcove $\mathcal{A}$, it permutes the simple roots $\{\alpha_0, \ldots, \alpha_n\}$, and consequently

$$\pi_r s_i \pi_r^{-1} = s_j, \quad \text{where} \ \ \pi_r(\alpha_i) = \alpha_j. \tag{1.19}$$

We note that for $r \in O$, one has $\pi_r(\alpha_0) = \alpha_r$ and $\pi_r^{-1} = \pi_{r^*}, u_r^{-1} = u_{r^*}$, where r^* is determined from the relation $-w_0(\alpha_r) = \alpha_{r^*}$.

Any $\widehat{w} \in \widehat{W}$ can be written uniquely as $\widehat{w} = \pi_r \widetilde{w}$, where $r \in O$ and $\widetilde{w} \in \widetilde{W}$. One has $l(\widehat{w}) = l(\widetilde{w})$. We call $\widehat{w} = \pi_r s_{j_l} \cdots s_{j_1}$ a reduced expression if and only $l = l(\widetilde{w}) = l(\widehat{w})$.

Given any $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ and any reduced expression $\widetilde{w} = s_{j_l} \cdots s_{j_1} \in \widetilde{W}$, one obtains an ordering of the λ–set:

$$\lambda(\widehat{w}) = \{\ \widetilde{\alpha}^1 = \alpha_{j_1},\ \widetilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}),\ \ldots\ ,\ \widetilde{\alpha}^l = \widetilde{w}^{-1} s_{j_l}(\alpha_{j_l})\ \}. \tag{1.20}$$

We will call (1.20) the λ–sequence associated with the given reduced expression for $\widehat{w}$. Such sequences are exactly those in $\widetilde{R}_+$ satisfying properties (i, ii) of the following lemma.

Lemma 1.2.1 ([**12**]). *Given $\widehat{w} \in \widehat{W}$ and a reduced expression $\widehat{w} = \pi_r s_{j_l} \cdots s_{j_1}$, form the λ–sequence using (1.20).*

(i) If $\widetilde{\alpha} = \widetilde{\alpha}^q + \widetilde{\alpha}^r \in \widetilde{R}_+$, then $\widetilde{\alpha} = \widetilde{\alpha}^p$ for some p between q and r. The same holds if $\widetilde{\alpha} = c_1 \widetilde{\alpha}^q + c_2 \widetilde{\alpha}^r \in \widetilde{R}_+$ for positive rational c_1, c_2.

(ii) If $\lambda(\widehat{w}) \ni \widetilde{\alpha} = \widetilde{\beta} + \widetilde{\gamma}$ for $\widetilde{\beta}, \widetilde{\gamma} \in \widetilde{R}_+ \cup [0, \mathbb{Z}_+]$, then at least one of $\widetilde{\beta}, \widetilde{\gamma}$ belongs to $\lambda(\widehat{w})$ and exactly one of $\widetilde{\beta}, \widetilde{\gamma}$ comes before $\widetilde{\alpha}$ in $\lambda(\widehat{w})$.

For arbitrary $b \in P$, we set $\pi_b := b u_b^{-1} \in \widehat{W}$, where u_b is defined to be the unique shortest element of W satisfying $u_b(b) = b_-$. Thus $\pi_{\omega_r} = \pi_r$ and $u_{\omega_r} = u_r$ for $r \in O'$. The element π_b can be characterized as the unique minimum length representative of the coset of b in $\widehat{W}/\widetilde{W}$.

Lemma 1.2.2. *One has the following explicit descriptions of λ–sets:*

$$\lambda(b) = \left\{ [\alpha, \nu_\alpha j] \ : \ \begin{array}{ll} 0 \le j < (b, \alpha^\vee) & \textit{if} \ \ \alpha \in R_+ \\ 0 < j \le (b, \alpha^\vee) & \textit{if} \ \ \alpha \in R_- \end{array} \right\}, \tag{1.21}$$

$$\lambda(\pi_b) = \left\{ [\alpha, \nu_\alpha j] \ : \ \alpha \in R_- \ \ \textit{and} \ \ \begin{array}{ll} 0 < j < (b_-, \alpha^\vee) & \textit{if} \ \ u_b^{-1}(\alpha) \in R_+ \\ 0 < j \le (b_-, \alpha^\vee) & \textit{if} \ \ u_b^{-1}(\alpha) \in R_- \end{array} \right\}, \tag{1.22}$$

$$\lambda(u_b) = \{ \ \alpha \in R_+ : \ (b, \alpha) > 0 \, \}. \tag{1.23}$$

Proof. Using (1.12), it is straightforward to verify (1.21) and (1.22). For a proof of (1.23), see [**30**, (2.4.4)]. □

Using (1.21), one sees that $l_\nu(b) = l_\nu(w(b))$ for any $w \in W$ and $b \in P$. Hence

$$l_\nu(b) = l_\nu(b_+) = 2(b_+, \rho_\nu^\vee). \tag{1.24}$$

We will need some further properties of the reflections $s_{\widetilde{\alpha}}$ and their λ–sequences.

Lemma 1.2.3. *Let $\widetilde{\alpha} \in \widetilde{R}_+$.*

(i) If $\widetilde{\beta} \in \lambda(s_{\widetilde{\alpha}}) \setminus \{\widetilde{\alpha}\}$, then $\widetilde{\beta}' = -s_{\widetilde{\alpha}}(\widetilde{\beta})$ belongs to $\lambda(s_{\widetilde{\alpha}})$ and $\widetilde{\alpha}$ lies between $\widetilde{\beta}$ and $\widetilde{\beta}'$ in any ordering of $\lambda(s_{\widetilde{\alpha}})$ via (1.20).

(ii) There exists a reduced expression of the form

$$s_{\widetilde{\alpha}} = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}, \quad \textit{where} \ \ 0 \le j_1, \ldots, j_p, m \le n, \tag{1.25}$$

and $j_1, \ldots, j_p, m \ge 1$ if $\widetilde{\alpha} = \alpha \in R_+$.

(iii) Construct the λ–sequence $\lambda(s_{\widetilde{\alpha}})$ using (1.20) and a reduced expression of the form (1.25). Then one has $-s_{\widetilde{\alpha}}(\widetilde{\alpha}^i) = \widetilde{\alpha}^{l+1-i}$ for any $1 \le i \le l = l(s_{\widetilde{\alpha}})$. In particular, $\widetilde{\alpha} = \widetilde{\alpha}^{p+1}$.

PROOF. (i) Clearly, $\widetilde{\beta}' \in \lambda(s_{\widetilde{\alpha}})$ and $\widetilde{\beta} + \widetilde{\beta}'$ is a positive integer multiple of $\widetilde{\alpha}$. Hence Lemma 1.2.1(ii) gives the claim.

(ii) We argue by induction of $l(s_{\widetilde{\alpha}})$, which must be odd. The claim is trivial when $l(s_{\widetilde{\alpha}}) = 1$. If $l(s_{\widetilde{\alpha}}) \ge 3$, find some α_i $(0 \le i \le n)$ such that $l(s_{\widetilde{\alpha}} s_i) = l(s_i s_{\widetilde{\alpha}}) < l(s_{\widetilde{\alpha}})$. Then $s_{\widetilde{\alpha}}(\alpha_i) \in \widetilde{R}_-$ and $s_{\widetilde{\alpha}}(\alpha_i) \ne -\alpha_i$. Hence $s_i s_{\widetilde{\alpha}}(\alpha_i) < 0$ and consequently $l(s_i s_{\widetilde{\alpha}} s_i) < l(s_i s_{\widetilde{\alpha}}) < l(s_{\widetilde{\alpha}})$. By induction $s_i s_{\widetilde{\alpha}} s_i = s_{s_i(\widetilde{\alpha})}$ has a reduced expression of the form (1.25). We multiply both sides of this expression by s_i to complete the argument.

(iii) This is immediate from (1.20). □

Finally, we need a formula for the length $l_\nu(s_\alpha)$ of non-affine reflections.

LEMMA 1.2.4. *Let $\alpha \in R_+$ and*

$$\delta_{\alpha,\nu} = \delta_{\nu_\alpha,\nu},\ \ \eta_{\alpha,\nu} = \begin{cases} \nu & \text{if } \nu_\alpha = 1 = \nu_{\text{sht}} \text{ and } \nu = \nu_{\text{lng}}, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$l_\nu(s_\alpha) = 2\frac{(\alpha, \rho_\nu)}{\nu_\alpha \eta_{\alpha,\nu}} - \delta_{\alpha,\nu}. \tag{1.26}$$

More explicitly, one has $l_\nu(s_\alpha) = 2(\alpha^\vee, \rho_\nu) - \delta_{\alpha,\nu}$ for long α and $l_\nu(s_\alpha) = 2(\alpha, \rho_\nu^\vee) - \delta_{\alpha,\nu}$ for short α.

PROOF. We use the formula

$$\rho_\nu - w(\rho_\nu) = \sum_{\beta \in \lambda_\nu(w)} \beta. \tag{1.27}$$

as follows: $2(\rho_\nu, \alpha^\vee) = (\rho_\nu - s_\alpha(\rho_\nu), \alpha^\vee) = \delta_{\alpha,\nu} + \sum_{\beta \in \lambda_\nu(s_\alpha)} \eta_{\alpha,\beta} = \delta_{\alpha,\nu} + \eta_{\alpha,\nu} l_\nu(s_\alpha)$. □

1.3. Affine Hecke algebras

Let $\{t_\alpha^{1/2}\}_{\alpha\in R}$ be a family of indeterminates satisfying $t_{w(\alpha)}^{1/2} = t_\alpha^{1/2}$ for all $\alpha \in R$ and $w \in W$. We understand $t_\alpha^{1/2}$ as a (formal) fractional power of t_α, e.g., $(t_\alpha^{1/2})^2 = t_\alpha$. Let $\mathbb{Q}_t$ be the field of rational functions in the indeterminates $\{t_\alpha^{1/2}\}_{\alpha\in R}$. We write $t_{\widetilde{\alpha}}^{1/2} = t_\alpha^{1/2}$ for $\widetilde{\alpha} = [\alpha, \nu_\alpha j]$ and $t_i^{1/2} = t_{\alpha_i}^{1/2}$ for $i = 0, \ldots, n$.

The *affine Hecke algebra* $\widetilde{\mathcal{H}}$ is the algebra generated over $\mathbb{Q}_t$ by $\{T_0, T_1, \ldots, T_n\}$ subject to the relations

$$(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0 \quad (i = 0, \cdots, n), \tag{1.28}$$

$$T_iT_jT_i\cdots = T_jT_iT_j\cdots \quad (0 \le i \ne j \le n), \tag{1.29}$$

where the braid relations (1.29) match those from $\widetilde{W}$, i.e., they contain exactly m_{ij} factors on each side. We note that if the parameters $t_i^{1/2}$ are specialized to $t_i^{1/2} = 1$ for all $i = 0, \ldots, n$, then $\widetilde{\mathcal{H}}$ can be naturally identified with the group algebra $\mathbb{Q}[\widetilde{W}]$.

Given a reduced expression $\widetilde{w} = s_{j_\ell}\cdots s_{j_1} \in \widetilde{W}$, define $T_{\widetilde{w}} := T_{j_\ell}\cdots T_{j_1}$. Since the T_i satisfy the same braid relations as the s_i, the definition of $T_{\widetilde{w}}$ is independent of the reduced decomposition for $\widetilde{w}$ (see, e.g., [**2**, Chapter IV, §1, Proposition 5]). The elements $T_{\widetilde{w}}$ form a basis for $\widetilde{\mathcal{H}}$ over $\mathbb{Q}_t$.

Corresponding to $\widehat{W}$, we define the *extended affine Hecke algebra* $\widehat{\mathcal{H}}$ by adjoining the group Π to $\widetilde{\mathcal{H}}$ with the additional relations

$$\pi_r T_i \pi_r^{-1} = T_j, \quad \text{where} \quad \pi_r(\alpha_i) = \alpha_j \quad (r \in O,\ 0 \le i \le n). \tag{1.30}$$

More precisely, $\widehat{\mathcal{H}}$ is defined as the tensor product $\widehat{\mathcal{H}} = \mathbb{Q}_t[\Pi] \otimes_{\mathbb{Q}_t} \widetilde{\mathcal{H}}$ with the multiplication between the tensor factors determined by the relations (1.30); this algebra is commonly referred to as the *smash product* of Π and $\widetilde{\mathcal{H}}$.

Given a reduced expression $\widehat{w} = \pi_r s_{j_\ell}\cdots s_{j_1} \in \widehat{W}$, define $T_{\widehat{w}} := \pi_r T_{j_\ell}\cdots T_{j_1}$, which again does not depend on the reduced expression for $\widehat{w}$. The elements $T_{\widehat{w}}$ form a basis for $\widehat{\mathcal{H}}$ and satisfy

$$T_{\widehat{v}} T_{\widehat{w}} = T_{\widehat{v}\widehat{w}}, \quad \text{provided} \quad \ell(\widehat{v}\widehat{w}) = \ell(\widehat{v}) + \ell(\widehat{w}). \tag{1.31}$$

In particular, for $b, c \in P_+$ or $b, c \in P_-$, one has $T_b T_c = T_{b+c}$ and hence T_b and T_c commute; here we use (1.24).

We define elements Y_b ($b \in P$) in $\widehat{\mathcal{H}}$ as follows. Any $b \in P$ can be expressed as $b = b_1 - b_2$ for $b_1, b_2 \in P_+$. Define $Y_b := T_{b_1} T_{b_2}^{-1}$ for any such expression; it is easy to see that Y_b is independent of the choice of b_1, b_2. In particular, $Y_b = T_b$ whenever $b \in P_+$. The $\{Y_b : b \in P\}$ generate a commutative subalgebra naturally isomorphic to the group algebra $\mathbb{Q}_t[P]$; denote this subalgebra by $\mathbb{Q}_t[Y]$.

One has the relations (see [**28**]):

$$T_i Y_b = Y_{s_i(b)} T_i + (t_i^{1/2} - t_i^{-1/2}) \frac{Y_{s_i(b)} - Y_b}{Y_{-\alpha_i} - 1}, \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad b \in P. \tag{1.32}$$

It is easy to see that the quotient $(Y_{s_i(b)} - Y_b)/(Y_{-\alpha_i} - 1)$ belongs to $\mathbb{Q}_t[Y]$. Particular cases of (1.32) include:

$$T_i Y_b = Y_b T_i \quad \text{if} \quad (b, \alpha_i^\vee) = 0 \quad \text{and} \quad i > 0, \tag{1.33}$$

$$T_i^{-1} Y_b T_i^{-1} = Y_{s_i(b)} \quad \text{if} \quad (b, \alpha_i^\vee) = 1 \quad \text{and} \quad i > 0. \tag{1.34}$$

Proposition 1.3.1 ([**28**]). *The sets*

$$\{Y_b T_w : b \in P,\, w \in W\} \quad \textit{and} \quad \{T_w Y_b : b \in P,\, w \in W\}$$

are bases of $\widehat{\mathcal{H}}$.

The Y_b provide a convenient description of the center of $\widehat{\mathcal{H}}$:

Proposition 1.3.2 ([**28**]). *The center of* $\widehat{\mathcal{H}}$ *is* $\mathbb{Q}_t[Y]^W$, *where we let* $w(Y_b) = Y_{w(b)}$ *for* $w \in W$ *and* $b \in P$.

For any $b \in P$ and a reduced expression $b = \pi_r s_{j_l} \cdots s_{j_1}$, one has $Y_b = \pi_r T_{j_l}^{\epsilon_l} \cdots T_{j_1}^{\epsilon_1}$, where

$$(1.35) \qquad \epsilon_p = \begin{cases} +1 & \text{if } \alpha^p > 0, \\ -1 & \text{if } \alpha^p < 0. \end{cases}$$

and $\widetilde{\alpha}^p = [\alpha^p, \nu_{\alpha^p} j]$ are from (1.20); see, e.g., [**30**, (3.2.10)] for a proof of this fact. The total number of factors $T_j^{\pm 1}$ with $\nu_j = \nu$ in this product is $l_\nu(b) = 2(b_+, \rho_\nu^\vee)$.

1.4. Double affine Hecke algebras

Let $\widehat{\mathcal{H}}$ be the extended affine Hecke algebra defined in the previous section. From now on, we denote $\widehat{\mathcal{H}}$ by $\mathcal{H}_Y$.

Let $q^{1/2m}$ be an indeterminate, where m is the least positive integer with the property that $m(P,P) \subset \mathbb{Z}$, and let $\mathbb{Q}_{q,t}$ be the field of rational functions in the indeterminates $q^{1/2m}$ and $\{t_\alpha^{1/2}\}_{\alpha \in R}$.

It is convenient to introduce additional parameters $\{k_\alpha\}_{\alpha \in R}$, where $k_{w(\alpha)} = k_\alpha$ for all $\alpha \in R$ and $w \in W$, and to impose the relation

$$t_\alpha = q_\alpha^{k_\alpha}.$$

As above, we write $k_\nu = k_\alpha$ provided $\nu = \nu_\alpha$. We also set

$$(1.36) \qquad \rho_k = \sum_{\nu \in \nu_R} k_\nu \rho_\nu.$$

Thus

$$q^{(\rho_k, b)} = q^{(\sum_\nu k_\nu \rho_\nu, b)} = \prod_\nu t_\nu^{(\rho_\nu^\vee, b)}.$$

For any ring A, let $A[X]$ be the group algebra of P over A spanned by elements $\{X_b : b \in P\}$ satisfying the relations

$$X_a X_b = X_{a+b} \quad (a, b \in P).$$

More generally, for any $\widetilde{b} = [b, j]$ where $b \in P$ and $j \in \mathbb{Z}$, we set $X_{\widetilde{b}} = q^j X_b$.

We let $\widehat{W}$ act on $A[X]$ by

$$\widehat{w}(X_b) = X_{\widehat{w}(b)},$$

where we use the action of $\widehat{W}$ on $\mathbb{R}^n \times \mathbb{R}$ from (1.11), treating b as $[b, 0]$.

Definition 1.4.1. The *double affine Hecke algebra* $\mathcal{HH}$ *is the* $\mathbb{Q}_{q,t}$–algebra generated by $\mathcal{H}_Y$ and $\mathbb{Q}_{q,t}[X]$ subject to the following additional relations:

$$T_i X_b = X_{s_i(b)} T_i + (t_i^{1/2} - t_i^{-1/2}) \frac{X_{s_i(b)} - X_b}{X_{\alpha_i} - 1} \quad (i \geq 0,\ b \in P), \tag{1.37}$$

$$\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} \quad (r \in O,\ b \in P). \tag{1.38}$$

Proposition 1.4.2 ([**7**]). *The monomials* $\{Y_a T_w X_b : w \in W,\ a, b \in P\}$ *form a basis for* $\mathcal{HH}$ *over* $\mathbb{Q}_{q,t}$.

There exists a unique $\mathbb{Q}_{q,t}$–linear anti-involution φ of $\mathcal{HH}$ satisfying:

$$\varphi : T_i \mapsto T_i \, (1 \leq i \leq n),\ \ X_b \mapsto Y_{-b},\ \ Y_b \mapsto X_{-b}. \tag{1.39}$$

See [**5**]. We call φ the *duality anti-involution.* For $H \in \mathcal{HH}$, we often write $H^\varphi := \varphi(H)$. Using $Y_\vartheta = T_0 T_{s_\vartheta}$ and $Y_{\omega_r} = \pi_r T_{u_r}$, one finds that

$$\varphi(T_0) = T_{s_\vartheta}^{-1} X_\vartheta^{-1}, \quad \varphi(\pi_r) = T_{u_r^{-1}}^{-1} X_{\omega_r}^{-1} = X_{\omega_{r^*}} T_{u_r} = \varphi(\pi_{r^*}^{-1}). \tag{1.40}$$

1.5. Polynomial representation

The *polynomial representation* of $\mathcal{HH}$ is the induced module

$$\mathcal{V} := \mathrm{Ind}_{\mathcal{H}_Y}^{\mathcal{HH}}(\mathbb{Q}_{q,t}) = \mathcal{HH} \otimes_{\mathcal{H}_Y} \mathbb{Q}_{q,t},$$

where $\mathbb{Q}_{q,t}$ carries the action of $\mathcal{H}_Y$ defined by $T_i(1) = t_i^{1/2}$ $(i \geq 0)$ and $\pi_r(1) = 1$ $(r \in O)$.

Proposition 1.4.2 gives rise to an isomorphism $\mathcal{V} \cong \mathbb{Q}_{q,t}[X]$ of $\mathbb{Q}_{q,t}$–vector spaces. Under this identification, elements of $\mathcal{HH}$ act by *difference-reflection operators*, which are by definition operators of the form

$$\sum_{w\in W,\, b\in P} g_{b,w}\,\Gamma_b\, w, \quad g_{b,w}\in \mathbb{Q}_{q,t}(X). \tag{1.41}$$

Here $\mathbb{Q}_{q,t}(X)$ is the field of rational functions in the X_b ($b\in P$), all but finitely many $g_{b,w}$ are zero, and Γ_b ($b\in P$) are the operators

$$\Gamma_b(X_c) = q^{(b,c)}X_c. \tag{1.42}$$

We observe that the action of Γ_{-b} coincides with that of $b\in\widehat{W}$.

Let us permanently identify $\mathcal{V}\cong\mathbb{Q}_{q,t}[X]$. We now describe the action of $\boldsymbol{\mathcal{H}}$ in $\mathcal{V}$ explicitly. For any $H\in\boldsymbol{\mathcal{H}}$, we continue to denote by H the corresponding endomorphism of $\mathcal{V}$.

Due to the relation (1.37), the action of T_i ($i\geq 0$) is given by the *Demazure-Lusztig operator*:

$$T_i = t_i^{1/2}s_i + \frac{t_i^{1/2}-t_i^{-1/2}}{X_{\alpha_i}-1}(s_i-1).$$

The action of π_r ($r\in O$) is given by $\pi_r=\Gamma_{-\omega_r}u_r^{-1}$, and the X_b ($b\in P$) act by multiplication operators.

The Y_b ($b\in P$) act by the *difference Dunkl operators*. One can describe these operators explicitly as follows. Let $b=\pi_r s_{j_l}\cdots s_{j_1}$ be a reduced decomposition and recall the definition of ϵ_p from (1.35). Then

$$Y_b = \pi_r T_{j_l}^{\epsilon_l}\cdots T_{j_1}^{\epsilon_1} = \Gamma_{-b}\, G_{\widetilde{\alpha}^l}^{\mathrm{sgn}(\epsilon_l)}\cdots G_{\widetilde{\alpha}^1}^{\mathrm{sgn}(\epsilon_1)}, \tag{1.43}$$

where $\mathrm{sgn}(\pm 1) = \pm$ and

$$G^+_{\widetilde{\alpha}} = t^{1/2}_{\widetilde{\alpha}} + \frac{t^{1/2}_{\widetilde{\alpha}} - t^{-1/2}_{\widetilde{\alpha}}}{X^{-1}_{\widetilde{\alpha}} - 1}(1 - s_{\widetilde{\alpha}}) = t^{-1/2}_{\widetilde{\alpha}}(f_{\widetilde{\alpha}} + g_{\widetilde{\alpha}}\, s_{\widetilde{\alpha}}), \tag{1.44}$$

$$\text{where} \quad f_{\widetilde{\alpha}} = \frac{t_{\widetilde{\alpha}} X^{-1}_{\widetilde{\alpha}} - 1}{X^{-1}_{\widetilde{\alpha}} - 1}, \quad g_{\widetilde{\alpha}} = \frac{t_{\widetilde{\alpha}} - 1}{1 - X^{-1}_{\widetilde{\alpha}}};$$

$$G^-_{\widetilde{\alpha}} = t^{-1/2}_{\widetilde{\alpha}} + \frac{t^{1/2}_{\widetilde{\alpha}} - t^{-1/2}_{\widetilde{\alpha}}}{1 - X_{\widetilde{\alpha}}}(1 - s_{\widetilde{\alpha}}) = t^{-1/2}_{\widetilde{\alpha}}(f_{\widetilde{\alpha}} - s_{\widetilde{\alpha}}\, g_{\widetilde{\alpha}}). \tag{1.45}$$

We note that

$$G^+_{\alpha_i} = s_i\, T_i, \quad G^+_{-\alpha_i} = T_i\, s_i, \quad G^-_{\alpha_i} = s_i\, T^{-1}_i, \quad \text{and} \quad G^-_{-\alpha_i} = T^{-1}_i\, s_i.$$

We also set $\ddot{G}^{\pm}_{\widetilde{\alpha}} := t^{1/2}_{\widetilde{\alpha}} G^{\pm}_{\widetilde{\alpha}}$, so that

$$\ddot{Y}_b := q^{(b_+, \rho_k)} Y_b = \Gamma_{-b}\, \ddot{G}^{\mathrm{sgn}(\epsilon_l)}_{\widetilde{\alpha}^l} \cdots \ddot{G}^{\mathrm{sgn}(\epsilon_1)}_{\widetilde{\alpha}^1}. \tag{1.46}$$

Let $\mathcal{D}$ denote the algebra of all difference-reflection operators (1.41). Its defining relations are as follows:

$$q^{(a,b)} X_a \Gamma_b = \Gamma_b X_a, \;\; w X_a = X_{w(a)} w, \;\; w \Gamma_b = \Gamma_{w(b)} w, \quad \text{for} \;\; w \in W, \;\; a, b \in P.$$

By *difference operators* we mean elements of the subalgebra of $\mathcal{D}$ generated by $\mathbb{Q}_{q,t}(X)$ and Γ_b $(b \in P)$. There is a natural linear map

$$\mathrm{Red} : \sum_{w \in W,\, b \in P} g_{b,w}(X)\, \Gamma_b\, w \mapsto \sum_{w \in W,\, b \in P} g_{b,w}\, \Gamma_b, \tag{1.47}$$

sending difference-reflection operators to difference operators. Clearly, Red is *not* a homomorphism of algebras.

For $f \in \mathbb{Q}_{q,t}[X]^W$, let

$$\mathcal{L}_f := f(Y_{\omega_1}, \ldots, Y_{\omega_n}) = \sum_{w \in W,\, b \in P} g_{b,w}\, \Gamma_b\, w, \quad g_{b,w} \in \mathbb{Q}_{q,t}(X),$$

$$L_f := \mathrm{Red}(\mathcal{L}_f) = \sum_{w \in W,\, b \in P} g_{b,w}\, \Gamma_b. \tag{1.48}$$

By Proposition 1.3.2, $f(Y)$ is central in $\mathcal{H}_Y$. Hence $\mathcal{L}_f$ and L_f preserve $\mathcal{V}^W$ and coincide upon the restriction to this space. Moreover, the L_f are $W-$invariant difference operators, i.e., $wL_fw^{-1} = L_f$ for any $w \in W$. For $a \in P_+$, we define $\mathcal{L}_a := \mathcal{L}_f$ and $L_a := L_f$ for $f = \sum_{w \in W/W_a} X_{-w(a)}$, where W_a is the stabilizer of a in W.

CHAPTER 2

Nonsymmetric Macdonald polynomials

Recall that $X_{[b,j]} = q^j X_b$, for any $b \in P$ and $j \in \mathbb{Z}$. We also set $X_b(q^x) := q^{(b,x)}$ and $Y_{[b,j]} := q^{-j} Y_b$, so that $Y_{[b,j]}^{-1} = \varphi(X_{[b,j]})$.

2.1. Inner product

Let

$$\mu = \mu(X) := \prod_{\widetilde{\alpha} \in \widetilde{R}_+} \frac{1 - X_{\widetilde{\alpha}}}{1 - t_{\widetilde{\alpha}} X_{\widetilde{\alpha}}}. \tag{2.1}$$

Using the identity

$$(1 - t_{\widetilde{\alpha}} X_{\widetilde{\alpha}})^{-1} = 1 + t_{\widetilde{\alpha}} X_{\widetilde{\alpha}} + t_{\widetilde{\alpha}}^2 X_{2\widetilde{\alpha}} + \cdots ,$$

we expand μ as a formal Laurent series in the variables X_{α_i} $(i = 1, \ldots, n)$ with coefficients in the ring $\mathbb{Z}[t_\nu][[q]]$. We extend the action of $\widehat{W}$ on $\mathbb{Q}_{q,t}[X]$ to Laurent series

$$f = \sum_{b \in P} c_b X_b, \quad \text{where} \quad c_b \in \mathbb{Q}[t_\nu][[q]][q^{-1}], \tag{2.2}$$

by setting $\widehat{w}(f) = \sum_{b \in P} c_b X_{\widehat{w}(b)}$.

For any Laurent series f of the form (2.2) (with coefficients in any ring), let $\langle f \rangle := c_0$ be the constant term of f. Clearly, $\langle \mu \rangle$ is invertible in $\mathbb{Q}[t_\nu][[q]]$.

Let $\mu_\circ = \mu / \langle \mu \rangle$. Then $\mu_\circ$ has coefficients in $\mathbb{Q}(q, t_\nu)$, the field of rational functions in q and t_ν $(\nu \in \nu_R)$; see, e.g., [**30**, (5.2.10)] for a proof of this fact.

Let $* : \mathcal{V} \to \mathcal{V}$ be the $\mathbb{Q}$–linear involution defined by

$$X_b^* = X_{-b}, \quad (q^{1/2m})^* = q^{-1/2m}, \quad (t_\nu^{1/2})^* = t_\nu^{-1/2},$$

Then one has $\mu_\circ^* = \mu_\circ$ (where we extend $*$ to Laurent series as above), while this does not hold for μ.

For $f, g \in \mathcal{V}$, we define the inner product

$$\langle f, g\rangle = \langle f g^* \mu_\circ \rangle \in \mathbb{Q}_{q,t}.$$

Clearly, $\langle f, g\rangle$ is linear in f, $*$–linear in g, and satisfies $\langle f, g\rangle = \langle g, f\rangle^*$. We observe that if $f, g \in \mathbb{Q}(q, t_\nu)[X]$, then $\langle f, g\rangle \in \mathbb{Q}(q, t_\nu)$. It is straightforward to verify:

Lemma 2.1.1. *For any nonzero $f \in \mathcal{V}$, one has $\langle f, f\rangle \neq 0$. In particular, the restriction of $\langle\ ,\ \rangle$ to any nonzero subspace of $\mathcal{V}$ is nondegenerate.*

Definition 2.1.2. The *nonsymmetric Macdonald polynomials* are the unique elements $\{E_b : b \in P\}$ of $\mathbb{Q}(q, t_\nu)[X]$ satisfying the following two conditions:

$$E_b = X_b + \sum_{c \succ b} p_{bc} X_c, \quad \text{where} \;\; p_{bc} \in \mathbb{Q}(q, t_\nu), \tag{2.3}$$

$$\langle E_b, X_c\rangle = 0 \;\; \text{for} \;\; c \succ b. \tag{2.4}$$

By Lemma 2.1.1, we can apply the Gram-Schmidt process to the finite-dimensional subspaces $\mathbb{Q}(q, t_\nu)[X_c : c \succeq b] \subset \mathbb{Q}(q, t_\nu)[X]$ to construct the E_b. This justifies their existence and uniqueness.

2.2. Orthogonality

An immediate consequence of Definition 2.1.2 is that $\langle E_b, E_c\rangle = 0$ whenever $c \succ b$. In this section, we prove that the E_b are pairwise orthogonal and we give the formula for their norms.

There is a unique anti-involution $\star$ of $\mathcal{H}$ satisfying

$$\star : T_i \mapsto T_i^{-1} \ (i \geq 0),\ X_b \mapsto X_b^{-1},\ \pi_r \mapsto \pi_r^{-1},\ q^{1/2m} = q^{-1/2m},\ t_\nu^{1/2} \mapsto t_\nu^{-1/2}.$$

This is easy to check using Definition 1.4.1. One also has $Y_b^\star = Y_b^{-1}$.

Proposition 2.2.1 ([**7**]). *The representation $\mathcal{V}$ is $\star$-unitary. That is,*

$$\langle H(f), g\rangle = \langle f, H^\star(g)\rangle, \quad \text{for} \quad f, g \in \mathcal{V} \quad \text{and} \quad H \in \mathcal{H}.$$

For any $b \in P$, we set

$$b_\sharp := \pi_b((-\rho_k)) = b - u_b^{-1}(\rho_k).$$

Recall that u_b is the unique element of W of shortest length such that $u_b(b) = b_-$, $\pi_b = bu_b^{-1}$, and ρ_k is from (1.36).

Lemma 2.2.2. *For any $a, b \in P$, one has*

$$Y_a(X_b) \equiv q^{-(a,b_\sharp)} X_b \mod \mathbb{Q}_{q,t}[X_c : c \succ b]. \tag{2.5}$$

We will prove Lemma 2.2.2 at the end of this section. For now, we use it to deduce the following.

Corollary 2.2.3. *(i) For any $a, b \in P$, one has*

$$Y_a(E_b) = q^{-(a,b_\sharp)} E_b. \tag{2.6}$$

(ii) If $b \neq c$, then $\langle E_b, E_c\rangle = 0$.

Proof. (i) For any fixed $a \in P$, one combines Lemma 2.2.2 with Proposition 2.2.1 to see that $\{q^{(a,b_\sharp)} Y_a(E_b) : b \in P\}$ satisfy the conditions (2.3) and (2.4). Since these conditions determine the E_b uniquely, the claim follows.

(ii) For any $a \in P$, Proposition 2.2.1 gives that

$$q^{(a,b_\sharp)}\langle E_b, E_c\rangle = \langle Y_a(E_b), E_c\rangle = \langle E_b, Y_a^{-1}(E_c)\rangle = q^{-(a,c_\sharp)}\langle E_b, E_c\rangle. \tag{2.7}$$

When $b \neq c$, there clearly exists a such that $q^{(a,b_\sharp)} \neq q^{(a,c_\sharp)}$ and hence (2.7) implies $\langle E_b, E_c\rangle = 0$. $\square$

The values of $\langle E_b, E_b\rangle$ are given by the following theorem, which was stated by Macdonald in [**29**] (for $k_\nu \in \mathbb{Z}_{>0}$) and proved by Cherednik in [**8**].

THEOREM 2.2.4 ([**8**]). *For any $b, c \in P$, one has*

$$\langle E_b, E_c\rangle = \delta_{bc} \prod_{[\alpha,\nu_\alpha j]\in\lambda(\pi_b)} \frac{(1-q_\alpha^j t_\alpha^{-1} X_{-\alpha}(q^{\rho_k}))(t_\alpha^{-1} - q_\alpha^j X_{-\alpha}(q^{\rho_k}))}{(1-q_\alpha^j X_{-\alpha}(q^{\rho_k}))^2}. \tag{2.8}$$

We conclude this section with a proof of Lemma 2.2.2.

PROOF OF LEMMA 2.2.2. It suffices to prove the claim for $a \in P_+ \cup P_-$. We consider only $a \in P_+$, the proof for $a \in P_-$ being similar. We use (1.46) to write

$$Y_a = q^{-(a,\rho_k)}\Gamma_{-a}\ddot{G}^+_{\widetilde{\alpha}^l}\cdots\ddot{G}^+_{\widetilde{\alpha}^1},$$

for any reduced expression $a = \pi_r s_{j_l}\cdots s_{j_1}$. Modulo $\mathbb{Q}_{q,t}[X_c : c \succ b]$, one has

$$\ddot{G}^+_{\widetilde{\alpha}}(X_b) \equiv \begin{cases} X_b & \text{if } (b,\alpha) > 0, \\ t_{\widetilde{\alpha}} X_b & \text{otherwise.} \end{cases}$$

It follows that $\ddot{G}^+_{\widetilde{\alpha}^l}\cdots\ddot{G}^+_{\widetilde{\alpha}^1}(X_b) \equiv (\prod_{\substack{\widetilde{\alpha}\in\lambda(a)\\(b,\alpha)\le 0}} t_{\widetilde{\alpha}})X_b$, and due to (1.21) and (1.23) one has

$$\prod_{\substack{\widetilde{\alpha}\in\lambda(a)\\(b,\alpha)\le 0}} t_{\widetilde{\alpha}} = q^{(a,\,\rho_k + u_b^{-1}(\rho_k))}.$$

This gives (2.5) for $a \in P_+$. $\square$

2.3. Intertwiners

The *intertwiners* are the elements

$$\Psi_i := T_i + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1} \quad (i = 1, \ldots, n),$$

$$\Psi_0 := q^{-1} X_\vartheta T_0^{-1} + \frac{t_0^{1/2} - t_0^{-1/2}}{Y_{\alpha_0}^{-1} - 1},$$

$$\Pi_r := q^{-(\omega_r,\omega_r)/2} X_{\omega_r}\pi_r \quad (r \in O'), \quad \Pi_0 := 1.$$

Recall that $Y_{[b,j]} = q^{-j}Y_b$ for $b \in P$ and $j \in \mathbb{Z}$. Strictly speaking, the Ψ_i belong to a localization of $\mathcal{H}$, but since we will apply the intertwiners to the nonsymmetric Macdonald polynomials only, we will not be concerned with this point.

Due to (2.6), the action of Ψ_i on E_b is given by Ψ_i^b, where

$$\Psi_i^b := T_i + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i}(q^{b_\sharp}) - 1} \quad (i = 1, \ldots, n), \quad \Psi_0^b := q^{-1}X_\vartheta T_0^{-1} + \frac{t_0^{1/2} - t_0^{-1/2}}{X_{\alpha_0}(q^{b_\sharp}) - 1}.$$

Recall that $X_{\widetilde{\alpha}}(q^c) = q^{(\alpha,c)+\nu_\alpha j}$. The following proposition describes the action of the intertwiners on the nonsymmetric Macdonald polynomials. Recall the definition of d in the extended pairing (1.13).

Proposition 2.3.1 ([**11**]). *Let $b \in P$.*

(i) If $(b + d, \alpha_i) > 0$ for some $i = 0, \ldots, n$, then

$$q^{-\frac{(c,c)}{2}} E_c = q^{-\frac{(b,b)}{2}} t_i^{1/2} \Psi_i^b(E_b), \quad \textit{where} \quad c = s_i(\!(b)\!). \tag{2.9}$$

(ii) If $(b + d, \alpha_i) = 0$ for some $i = 0, \ldots, n$, then

$$\tau_+(T_i)(E_b) = t_i^{1/2} E_b. \tag{2.10}$$

(iii) For any $r \in O$, one has

$$q^{-\frac{(c,c)}{2}} E_c = q^{-\frac{(b,b)}{2}} \Pi_r(E_b), \quad \textit{where} \quad c = \pi_r(\!(b)\!). \tag{2.11}$$

We note that for $i > 0$, (2.10) is equivalent to $s_i(E_b) = E_b$.

Starting from $E_0 = 1$, Proposition 2.3.1 can be used to construct E_b for any $b \in P$, as follows. For any reduced expression $\pi_b = \pi_r s_{j_l} \cdots s_{j_1}$, form $\lambda(\pi_b) = \{\widetilde{\alpha}^1, \ldots, \widetilde{\alpha}^l\}$ using (1.20) and set $b_1 = 0$ and $b_p = s_{j_{p-1}} \cdots s_{j_1}(\!(0)\!)$ for $p = 2, \ldots, l$. Then

$$(b_p + d, \alpha_{j_p}) = (d, \widetilde{\alpha}^p) > 0, \quad \text{by (1.22)},$$

and hence

$$q^{-(b,b)/2}E_b = t^{l(\pi_b)/2}\,\Pi_r\,\Psi_{j_l}^{b_l}\cdots\Psi_{j_1}^{b_1}(1), \quad \text{where} \quad t^{l(\pi_b)/2} = \prod_{\nu} t_{\nu}^{l_\nu(\pi_b)/2}. \tag{2.12}$$

COROLLARY 2.3.2. *The coefficients of the polynomial*

$$\prod_{[\alpha,\nu_\alpha j]\in\lambda(\pi_b)} (1 - q_\alpha^j X_{-\alpha}(q^{\rho_k}))\ E_b \tag{2.13}$$

belong to $\mathbb{Z}[q, t_\nu]$.

PROOF. We take a reduced expression $\pi_b = \pi_r s_{j_l}\cdots s_{j_1}$ and use (2.12). We use the following property of the elements $\{\pi_b : b \in P\}$: for any $0 \le i \le n$, one has $(b + d, \alpha_i) \neq 0$ if and only if $\pi_{s_i(\!(b)\!)} = s_i\pi_b$; see, e.g., [**10**, (1.20)] or [**30**, (2.4.14)]. In particular, $\pi_{b^p} = s_{j_{p-1}}\cdots s_{j_1}$ and therefore

$$(\widetilde{\alpha}^p, -\rho_k + d) = (\alpha_{j_p}, \pi_{b^p}(\!(-\rho_k)\!) + d).$$

Hence the product in (2.13) clears all denominators in (2.12). □

2.4. Limits

Corollary 2.3.2 implies that the E_b are well defined when $t_\nu = 0$ for all $\nu \in \nu_R$. We denote by $\overline{E}_b$ the image of E_b under this specialization. It also follows from Corollary 2.3.2 that the coefficients of $\overline{E}_b$ belong to $\mathbb{Z}[q]$.

REMARK. As a matter of fact, the coefficients of $\overline{E}_b$ are known to lie in $\mathbb{Z}_+[q]$. This follows from results of Ion [**25**] and Sanderson [**33**], which identify $\{\overline{E}_b\}$ with the characters of level-one Demazure modules for the (twisted) affine Lie algebra associated to R (i.e., for the affine Lie algebra having $\widetilde{R}$ as its system of real roots). The strategy employed in [**25, 33**] is to establish a connection between the intertwiner construction of E_b via Proposition 2.3.1 (as $t_\nu = 0$) and the Demazure character formula, which was proved for any Kac-Moody Lie algebra by Kumar [**27**].

Let us consider more systematically the behavior of $\mathcal{H}$ and $\mathcal{V}$ under the specialization $t_\nu = 0$. Let $\ddot{\mathbb{Q}}_{q,t}$ be the subring of $\mathbb{Q}_{q,t}$ consisting of those rational functions that are well defined when $t_\nu^{1/2} = 0$ for all $\nu \in \nu_R$.

The definition of $\mathcal{H}$ given in Definition 1.4.1, while standard in the literature, is not suited to this specialization. We therefore introduce the following normalization:

$$\ddot{T}_i := t_i^{1/2} T_i, \qquad \ddot{T}_i' := t_i^{1/2} T_i^{-1} = \ddot{T}_i - (t_i - 1).$$

Note that the same normalization is used for both T_i and T_i^{-1}, so that $\ddot{T}_i \ddot{T}_i' = t_i$. This ensures that $\ddot{T}_i$ and $\ddot{T}_i'$ are well defined in $\mathcal{V}$ when $t_i = 0$.

The $\ddot{T}_i$ ($i = 0, \ldots, n$) satisfy the braid relations[1] for T_i given in (1.29). Consequently, the elements $\ddot{T}_{\widehat{w}} := \pi_r \ddot{T}_{j_l} \cdots \ddot{T}_{j_1}$, where $\widehat{w} = \pi_r s_{j_l} \cdots s_{j_1}$ is any reduced expression in $\widehat{W}$, are well defined. The quadratic relations for $\ddot{T}_i$ read: $(\ddot{T}_i - t_i)(\ddot{T}_i + 1) = 0$.

Correspondingly, we define $\ddot{Y}_b := q^{(b_+, \rho_k)} Y_b$ for any $b \in P$. Note that $\ddot{Y}_b \ddot{Y}_{-b} = q^{2(b_+, \rho_k)}$.

Definition 2.4.1. Let $\ddot{\mathcal{H}}$ be the $\ddot{\mathbb{Q}}_{q,t}$–subalgebra of $\mathcal{H}$ generated by the elements

$$X_a \ (b \in P), \quad \ddot{T}_{\widehat{w}} \ (\widehat{w} \in \widehat{W}), \quad \ddot{Y}_b \ (b \in P).$$

It is straightforward to check that the algebra $\ddot{\mathcal{H}}$ is generated over $\ddot{\mathbb{Q}}_{q,t}$ by

$$X_a \ (b \in P), \quad \ddot{T}_i \ (i \geq 0), \quad \text{and} \quad \Pi,$$

subject to the defining relations:

$$\begin{aligned}
&(\ddot{T}_i - t_i)(\ddot{T}_i + 1) = 0, && \ddot{T}_i \ddot{T}_j \ddot{T}_i \cdots = \ddot{T}_j \ddot{T}_i \ddot{T}_j \cdots, \\
&\ddot{T}_i X_b = X_b X_{\alpha_i}^{-1} \ddot{T}_i' \ \text{ if } \ (b, \alpha_i^\vee) = 1, && \ddot{T}_i X_b = X_b \ddot{T}_i \ \text{ if } \ (b, \alpha_i^\vee) = 0, \\
&\pi_r \ddot{T}_i \pi_r^{-1} = \ddot{T}_j \ \text{ if } \ \pi_r(\alpha_i) = \alpha_j, && \pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)},
\end{aligned}$$

[1] This is due to the fact that $m_{ij} = 2, 3, 4$, or 6, and when $m_{ij} = 3$, α_i and α_j have the same length and hence $t_i = t_j$.

where $0 \le i, j \le n$, $b \in P$, $r \in O$, and the braid relations contain m_{ij} factors on each side.

We observe that the restriction to $\ddot{\mathcal{H}}$ of the $\mathcal{H}$–action in the polynomial representation $\mathcal{V}$ preserves $\ddot{\mathcal{V}} := \ddot{\mathbb{Q}}_{q,t}[X]$.

DEFINITION 2.4.2. The *nil-DAHA* $\overline{\mathcal{H}}$ is defined as the specialization of $\ddot{\mathcal{H}}$ at $t_\nu^{1/2} = 0$ for all $\nu \in \nu_R$. More precisely, let $\mathbb{Q}_q := \mathbb{Q}(q^{1/2m})$ and define $\overline{\mathcal{H}}$ to be the $\mathbb{Q}_q$–algebra

$$\overline{\mathcal{H}} := \mathbb{Q}_q \otimes_{\ddot{\mathbb{Q}}_{q,t}} \ddot{\mathcal{H}}, \tag{2.14}$$

where the tensor product structure is defined using the homomorphism $\ddot{\mathbb{Q}}_{q,t} \to \mathbb{Q}_q$ fixing $\mathbb{Q}_q$ and sending $t_\nu^{1/2} \mapsto 0$ for all $\nu \in \nu_R$.

Specializing $\ddot{\mathcal{V}}$ in the same manner, we obtain an $\overline{\mathcal{H}}$–module $\overline{\mathcal{V}} := \mathbb{Q}_q[X]$.

We denote the images of $\ddot{T}_i$ and $\ddot{Y}_b$ in $\overline{\mathcal{H}}$ by $\overline{T}_i$ and $\overline{Y}_b$, respectively. Using (2.6), we arrive at

$$\overline{Y}_a(\overline{E}_b) = \begin{cases} q^{-(a,b)}\overline{E}_b, & \text{if } u_b(a) = a_-, \\ 0, & \text{otherwise.} \end{cases}$$

If $(b, \alpha_i) > 0$ and $1 \le i \le n$, then the intertwiner $t_i^{1/2}\Psi_i^b$ becomes $\overline{T}_i + 1$ as $t_i = 0$. This has the following consequence.

PROPOSITION 2.4.3. *If $b \in P_-$, then $\overline{E}_b$ is W–invariant.*

PROOF. For $1 \le i \le n$ and any $f \in \overline{\mathcal{V}}$, one has $s_i(f) = f$ if and only if $\overline{T}_i(f) = 0$. When $b \in P_-$, Proposition 2.3.1 gives that $\overline{E}_b = (\overline{T}_i + 1)\overline{E}_{s_i(b)}$ for any $i = 1, \ldots, n$. Hence $\overline{T}_i(\overline{E}_b) = 0$, because $\overline{T}_i(\overline{T}_i + 1) = 0$. $\square$

Next we consider the orthogonality relations (2.8) as $t_\nu = 0$. We set

$$\overline{\mu} := \mu(t_\nu = 0) = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} (1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1}), \quad \overline{\mu}_\circ := \overline{\mu}/\langle \overline{\mu} \rangle.$$

To state the counterpart of (2.8) as $t_\nu = 0$, we will need the limits $\overline{E}_b^\dagger$ of the E_b as $t_\nu \to \infty$. We will justify the existence of these limits below. More generally, we set $\overline{f}^\dagger := \lim_{t_\nu\to\infty} f$ for any Laurent polynomial or series depending on q, t_ν, provided the existence of this limit. Then one has

$$\overline{(f^*)} = (\overline{f}^\dagger)^*, \quad \overline{(f^*)}^\dagger = (\overline{f})^*. \tag{2.15}$$

Using this notation, (2.8) reads as follows for $t_\nu = 0$:

$$\langle \overline{E}_b, \overline{E}_c\rangle := \langle \overline{E}_b(\overline{E}_c^\dagger)^*\overline{\mu}_\circ\rangle = \delta_{bc}\prod_{[\alpha,j]}(1-q_\alpha^j), \tag{2.16}$$

where the product runs over all $[-\alpha, \nu_\alpha j] \in \lambda(\pi_b)$ for simple $\alpha = \alpha_i \in R_+$.

Formula (2.16) provides an indirect justification of the existence of the limits $\overline{E}_b^\dagger$ for any $b \in P$. We now give a constructive justification of this fact using the intertwiners. We will also show in Corollary 2.4.5 that the coefficients of $\overline{E}_b^\dagger$ belong to $\mathbb{Z}[q^{-1}]$.

To this end, we set

$$\begin{aligned}
\ddot{T}_i^\dagger &:= t_i^{-1/2}T_i, & (\ddot{T}_i^\dagger)' &:= t_i^{-1/2}T_i^{-1},\\
\overline{T}_i^\dagger &:= \ddot{T}_i^\dagger(t_i=\infty), & (\overline{T}_i^\dagger)' &:= (\ddot{T}_i^\dagger)'(t_i=\infty) = \overline{T}_i^\dagger - 1,
\end{aligned}$$

and, correspondingly, $\ddot{Y}_a^\dagger := q^{-(a_+,\rho_k)}Y_a$. It is then straightforward to see that $\overline{Y}_a^\dagger := \lim_{t_\nu\to\infty}\ddot{Y}_a^\dagger$ is well defined and that (2.6) gives

$$\overline{Y}_a^\dagger(\overline{E}_b^\dagger) = \begin{cases} q^{-(a,b)}\overline{E}_b & \text{if } u_b(a) = a_+,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 2.4.4. *(i) For $b \in P_-$,*

$$\overline{E}_b^\dagger \;=\; q^{(\rho,b)}(\overline{T}'_{\pi_\rho}(\overline{E}_{-w_0(b)}))^*, \tag{2.17}$$

where $\overline{T}'_{\pi_\rho} := \overline{T}'_{j_1}\cdots\overline{T}'_{j_l}\pi_r^{-1}$ is defined for any reduced expression $\pi_\rho = \pi_r s_{j_l}\cdots s_{j_1}$ (and does not depend on the choice of the reduced expression).

(ii) If $(b,\alpha_i)<0$ and $1\le i\le n$, then

$$(2.18)\qquad \overline{E}^{\dagger}_{s_i(b)} \;=\; \begin{cases}(1-q^{(b,\alpha_i)})^{-1}(\overline{T}^{\dagger}_i)'(\overline{E}^{\dagger}_b) & \textit{if } (u_b(\alpha_i),\rho^{\vee})=1,\\ (\overline{T}^{\dagger}_i)'(\overline{E}^{\dagger}_b) & \textit{if } (u_b(\alpha_i),\rho^{\vee})>1.\end{cases}$$

Proof. (i) Let $b\in P_-$. We will use the formula

$$(2.19)\qquad E^*_b=\prod_{\nu\in\nu_R} t_{\nu}^{l_{\nu}(u_b)-l_{\nu}(w_0)/2}\,T_{w_0}(E_{-w_0(b)})$$

from [**11**, (3.3.24)].

We prove (2.17) by renormalizing (2.19) as follows. Note that $q^{(c,\,w_0(b)+\rho_k)}Y_c^{-1}$ acts as the identity on $E_{-w_0(b)}$ for any $c\in P$. Taking $c=c_+$, so that $l_\nu(c)=2(c,\rho^{\vee}_{\nu})$, one therefore has

$$(2.20)\qquad E^*_b=q^{(c,w_0(b))}\prod_{\nu} t_{\nu}^{-l_{\nu}(w_0)/2+l_{\nu}(c)/2}\,T_{w_0}Y_c^{-1}(E_{-w_0(b)}).$$

Specializing further to $c=\rho$, we have $Y_\rho=T_{\pi_\rho}T_{w_0}$ and (2.20) becomes

$$E^*_b=q^{-(\rho,b)}\prod_{\nu} t_{\nu}^{l_{\nu}(\pi_\rho)/2}T^{-1}_{\pi_\rho}(E_{-w_0(b)})=q^{-(\rho,b)}\ddot{T}'_{\pi_\rho}(E_{-w_0(b)}),$$

where by definition $\ddot{T}'_{\pi_\rho}=\ddot{T}'_{j_1}\cdots\ddot{T}'_{j_l}\pi_r^{-1}$ for any reduced decomposition $\pi_\rho=\pi_r s_{j_l}\cdots s_{j_1}$. Moving $*$ to the right-hand side and taking $t_\nu\to\infty$, we obtain (2.17).

(ii) This follows from a modification of (2.9). It is convenient to use the normalized intertwiners $G_i:=\psi_i^{-1}\Psi_i$ for

$$\Psi_i=\tau_+(T_i)+\frac{t_i^{1/2}-t_i^{-1/2}}{Y_{\alpha_i}^{-1}-1},\quad \psi_i=t_i^{1/2}+\frac{t_i^{1/2}-t_i^{-1/2}}{Y_{\alpha_i}^{-1}-1}.$$

For simplicity, let us take here $1\le i\le n$. In addition to the braid relations, the normalized intertwiners satisfy $G_i^2=1$. Hence $\Psi_i^{-1}=\psi_i^{-1}\Psi_i\psi_i^{-1}$. Now, when $(b,\alpha_i)<0$, (2.9) gives $E_{s_i(b)}=t_i^{-1/2}\Psi_i^{-1}(E_b)$. Applying $\Psi_i^{-1}=\psi_i^{-1}\Psi_i\psi_i^{-1}$ to E_b, the first ψ_i^{-1}

produces

$$\Big(t_i^{1/2}+\frac{t_i^{1/2}-t_i^{-1/2}}{q^{(\alpha_i,b_\sharp)}-1}\Big)^{-1}=t_i^{1/2}\frac{q^{(\alpha_i,b_\sharp)}-1}{q^{(\alpha_i,b_\sharp)}t_i-1}.$$

The second (left) ψ_i^{-1} produces the factor

$$\Big(t_i^{1/2}+\frac{t_i^{1/2}-t_i^{-1/2}}{q^{-(\alpha_i,b_\sharp)}-1}\Big)^{-1}=t_i^{-1/2}\frac{q^{-(\alpha_i,b_\sharp)}-1}{q^{-(\alpha_i,b_\sharp)}-t_i^{-1}}=t_i^{-1/2}\frac{q^{(\alpha_i,b_\sharp)}-1}{q^{(\alpha_i,b_\sharp)}t_i^{-1}-1}.$$

Multiplying these two factors and taking $t_\nu\to\infty$, one arrives at (2.18). Note that $u_b(\alpha_i)>0$ and hence $q^{(\alpha_i,b_\sharp)}$ contains nonpositive powers of t_ν and at least one t_i^{-1}.

□

Corollary 2.4.5. *The polynomials $\overline{E}_b^\dagger$ are well defined for any $b\in P$. Moreover, the coefficients of $\overline{E}_b^\dagger$ belong to $\mathbb{Z}[q^{-1}]$ for any $b\in P$.*

Proof. The existence of $\overline{E}_b^\dagger$ for any $b\in P$ follows from (2.17) and (2.18). More precisely, (2.17) gives the existence of $\overline{E}_b^\dagger$ for $b\in P_-$, and (2.18) allows one to construct $\overline{E}_b^\dagger$ for any $b\in P$ starting from $b_-\in P_-$.

By Corollary 2.3.2, the denominators of the coefficients in E_b are of products of factors of the form

$$(1-q^j\prod_\nu t_\nu^{m_\nu}),\quad\text{where}\ \ j,\sum_\nu m_\nu>0.$$

Since we already know that $\overline{E}_b^\dagger$ exists, we may set $t=t_\nu$ for all ν when calculating the limits of the coefficients. As polynomials in t, the denominators of E_b then have leading terms of the form $\pm q^r t^s$ where $r,s>0$, and no higher power of t can appear in the corresponding numerators. Thus the coefficients of $\overline{E}_b^\dagger$ must belong to $\mathbb{Z}[q^{\pm1}]$.

Using (2.17), it is easy to see that $\overline{E}_b^\dagger$ has coefficients in $\mathbb{Z}[q^{-1}]$ for $b\in P_-$. Then (2.18) shows that this holds for arbitrary $b\in P$. □

CHAPTER 3

Spherical and Whittaker functions

In this chapter, we mainly regard the parameters q and t_ν as nonzero complex numbers. Recall that $t_\nu = q_\nu^{k_\nu}$. When discussing the convergence of infinite series, we will assume in addition that $|q| < 1$.

3.1. Gaussian

By the *Gaussian* we mean the Laurent series

$$\widetilde{\gamma} = \sum_{b \in P} q^{(b,b)/2} X_b. \tag{3.1}$$

Multiplication by $\widetilde{\gamma}$ preserves the space of Laurent series with coefficients in $\mathbb{Q}[t][[q^{1/2m}]]$. Recall that $X_b(q^x) = q^{(b,x)}$. Regarding X_b as a function of $x \in \mathbb{C}^n$ in this way, the Gaussian $\widetilde{\gamma}$ converges uniformly on compact subsets of $\mathbb{C}^n$, provided $|q| < 1$. We will write $\widetilde{\gamma}_x$ whenever we want to regard $\widetilde{\gamma}$ as an entire function of $x \in \mathbb{C}^n$.

The main property the Gaussian that we will need is the following:

$$\Gamma_a(\widetilde{\gamma}) = q^{-(a,a)/2} X_a^{-1} \widetilde{\gamma} \quad \text{for} \quad a \in P. \tag{3.2}$$

In particular, one has

$$\Gamma_{-\rho_k}(\widetilde{\gamma}) = q^{-(\rho_k,\rho_k)/2} X_{\rho_k} \widetilde{\gamma}, \tag{3.3}$$

provided $\rho_k \in P$ (e.g., when $k_\nu \in \mathbb{Z}$).

3.2. Global spherical function

We use the notation $\widetilde{\gamma}_\lambda$ for the Gaussian defined in terms of the variable $\lambda \in \mathbb{C}^n$. Correspondingly, we write $\Lambda = q^\lambda$ and $\Lambda_b(q^\lambda) = q^{(b,\lambda)}$. We will use superscripts when applying operators from the polynomial representation of $\mathcal{H}$ to functions of x or λ. For instance, we write T_i^λ for the action of T_i on the variables λ. When no superscript is used, such operators are understood to act on the variables x.

We will also use the normalization constant

$$\widetilde{\gamma}(q^{\rho_k}) = \sum_{b\in P} q^{\frac{(b,b)}{2}+(b,\rho_k)}. \tag{3.4}$$

When $\rho_k \in P$, one has

$$\widetilde{\gamma}(q^{\rho_k}) = q^{-(\rho_k,\rho_k)/2}\widetilde{\gamma}(1), \quad \text{where} \quad \widetilde{\gamma}(1) := \sum_{b\in P} q^{(b,b)/2}. \tag{3.5}$$

The function $G(X,\Lambda)$ defined in the following theorem is called the *global non-symmetric spherical function.*[1]

THEOREM 3.2.1 ([**9**]). *(i) The series*

$$\Xi(X,\Lambda;q,t) := \sum_{b\in P} q^{(b_\sharp,b_\sharp)/2-(\rho_k,\rho_k)/2}\ \frac{E_b^*(X)\,E_b(\Lambda)}{\langle E_b,E_b\rangle} \tag{3.6}$$

converges in the ring of formal Laurent series in X,Λ with coefficients in $\mathbb{Q}[t][[q^{\frac{1}{2m}}]]$. When $|q|<1$, Ξ converges to an entire function of x,λ, provided t_ν are chosen so that all the E_b are well defined (by Proposition 2.3.2, the conditions $|t_\nu|<1$ are sufficient). Accordingly, $G(X,\Lambda)$ defined via

$$\frac{\widetilde{\gamma}_x\,\widetilde{\gamma}_\lambda}{\widetilde{\gamma}(q^{\rho_k})}G(X,\Lambda) := \Xi(X,\Lambda;q,t) \tag{3.7}$$

is a meromorphic function of X,Λ and it is holomorphic where $\widetilde{\gamma}_x\widetilde{\gamma}_\lambda \neq 0$.

(ii) The function $G(X,\Lambda)$ satisfies $G(X,\Lambda)=G(\Lambda,X)$ and

$$H^x(G(X,\Lambda)) = (\varphi(H))^\lambda(G(X,\Lambda)) \quad \textit{for} \quad H\in\mathcal{H}, \tag{3.8}$$

[1] This terminology reflects connections to Harish-Chandra's theory of spherical functions on real semisimple Lie groups; cf. [**24, 36**].

in terms of the anti-involution φ from (1.39). More concretely, one has

$$T_i^x(G(X,\Lambda)) = T_i^\lambda(G(X,\Lambda)) \quad \textit{for} \quad 1 \le i \le n, \tag{3.9}$$

$$Y_a(G(X,\Lambda)) = \Lambda_a^{-1} G(X,\Lambda) \quad \textit{and} \quad X_a^{-1} G(X,\Lambda) = Y_a^\lambda(G(X,\Lambda)) \quad \textit{for} \quad a \in P. \tag{3.10}$$

(iii) The function $G(X,\Lambda)$ extends the nonsymmetric Macdonald polynomials as follows. For any $b \in P$, one has

$$G(X, q^{b_\sharp}) = \frac{E_b(X)}{E_b(q^{-\rho_k})} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \frac{1 - q^{(\rho_k,\alpha)+j\nu_\alpha}}{1 - t_\alpha^{-1} q^{(\rho_k,\alpha)+j\nu_\alpha}}. \tag{3.11}$$

Remark. The convergence of the series Ξ from (3.6) as an entire function can be justified using the following estimate. For any compact subset $K \subset \mathbb{C}^n$, there exists a constant $C = C_K > 0$ such that

$$|E_b(q^x)| \le C^{|b|}, \quad \text{for all} \quad x \in K \quad \text{and} \quad b \in P,$$

where $|b|^2 = (b,b)$. This estimate can be demonstrated using the intertwiner recurrence for the E_b from Proposition 2.3.1; see, e.g., [**35**, Proposition 5.13].

3.3. Symmetrization

Define the symmetrizer

$$\mathcal{P} := \sum_{w \in W} \prod_{\nu \in \nu_R} t_\nu^{l_\nu(w)/2} T_w. \tag{3.12}$$

Then one has (see, e.g., [**30**, (5.5.9)])

$$(T_i - t_i^{1/2})\mathcal{P} = 0 = \mathcal{P}(T_i - t_i^{1/2}),$$

and hence $\mathcal{P} : \mathbb{Q}_{q,t}[X] \to \mathbb{Q}_{q,t}[X]^W$. As above, we will write $\mathcal{P}^x$ or $\mathcal{P}^\lambda$ to distinguish between the variables x and λ.

The *global symmetric spherical function* is defined as

$$F(X,\Lambda) := \mathcal{P}^x(G(X,\Lambda)) = \mathcal{P}^\lambda(G(X,\Lambda)). \tag{3.13}$$

The second equality in (3.13) holds due to (3.9). The function $F(X,\Lambda)$ is W–invariant in both X and Λ, satisfies $F(X,\Lambda) = F(\Lambda, X)$, and for any $f \in \mathbb{Q}_{q,t}[X]^W$, one has

$$L_f(F(X,\Lambda)) = f(\Lambda^{-1})F(X,\Lambda), \quad f(X^{-1})F(X,\Lambda) = L_f^\lambda(G(X,\Lambda)). \tag{3.14}$$

Here $f(X^{-1})$ means we replace X_b by X_{-b}, and similarly for $f(\Lambda^{-1})$. For these and further properties of $F(X,\Lambda)$, we refer to [**9, 13**].

3.4. Whittaker limit

We assume that $|q| < 1$ and $k_\nu \in \mathbb{Z}_{>0}$ in this section. For any difference operator L and any function $F(X)$, set

$$\varkappa(L) := (X_{\rho_k}\Gamma_{-\rho_k})\, L\, (X_{\rho_k}\Gamma_{-\rho_k})^{-1}, \quad \varkappa(F) := X_{\rho_k}\Gamma_{-\rho_k}(F). \tag{3.15}$$

Definition 3.4.1. The *Ruijsenaars-Etingof limiting procedure* is defined by

$$RE(L) := \lim_{k\to\infty} \varkappa(L), \quad RE(F) := \lim_{k\to\infty} \varkappa(F), \tag{3.16}$$

where we take $k_\nu \in \mathbb{Z}_{>0}$ and $k \to \infty$ means that $k_\nu \to \infty$ for all $\nu \in \nu_R$; equivalently, $t_\nu \to 0$ for all $\nu \in \nu_R$.

This limiting procedure is a natural extension of (0.3) to arbitrary root systems. The definition (3.16) first appeared in [**13**], where it was shown that the limit

$$\mathcal{W}(X,\Lambda) := RE(F(X,\Lambda)) \tag{3.17}$$

exists for $F(X,\Lambda)$ from (3.13). The function $\mathcal{W}(X,\Lambda)$ is called the *global symmetric Whittaker function*. We note that $\mathcal{W}(X,\Lambda)$ is W–invariant in Λ but no longer in X.

CHAPTER 4

Nonsymmetric Whittaker function

In this chapter we come to the main objects of this dissertation: the nonsymmetric Ruijsenaars-Etingof limiting procedure RE^{δ}, the global nonsymmetric Whittaker function Ω, and the Toda-Dunkl operators $\widehat{Y}_b$. In order to construct these objects, we first introduce the notion of $W-$spinors.

When discussing the convergence of limits and series, we assume that q is a nonzero complex number and that $|q| < 1$.

4.1. W–spinors

Given a vector space V over any field, let $\mathcal{F}(W, V)$ denote the space of functions from W to V. This space carries a natural action of W given by

$$(\delta(w)f)(u) := f(w^{-1}u), \quad \text{for} \quad f \in \mathcal{F}(W, V), \; w, u \in W.$$

If V is also an algebra over the base field, then $\mathcal{F}(W, V)$ inherits this structure via pointwise multiplication, and W then acts by algebra automorphisms.

Consider the $\mathbb{Q}_{q,t}$–algebra $\mathcal{F}(W, \mathcal{V})$; in [**15**], elements of this algebra are called $W-$*spinors*. For any $w \in W$, denote by ζ_w the characteristic function $\zeta_w(u) = \delta_{wu}$. These are pairwise orthogonal idempotents in $\mathcal{F}(W, \mathcal{V})$, and any element in $\mathcal{F}(W, \mathcal{V})$ can be written uniquely as

$$f = \sum_{w \in W} f_w \, \zeta_w, \quad \text{where} \quad f_w := f(w) \in \mathcal{V}.$$

We refer to f_w as the w–component of f. We observe that $\delta(w)(\zeta_v) = \zeta_{wv}$ and hence for any $f \in \mathcal{F}(W, \mathcal{V})$:

$$(\delta(v)f)_w = f_{v^{-1}w}. \tag{4.1}$$

One has a natural embedding of algebras $\delta : \mathcal{V} \to \mathcal{F}(W, \mathcal{V})$ given by

$$\delta(F) := \sum_{w\in W} F\,\zeta_w. \tag{4.2}$$

The image of δ is the space of W–invariants of $\mathcal{F}(W, \mathcal{V})$, which will be denoted by $\mathcal{F}^\delta(W, \mathcal{V})$.

We define another algebra embedding $\varrho : \mathcal{V} \to \mathcal{F}(W, \mathcal{V})$ by

$$\varrho(F) := \sum_{w\in W} w^{-1}(F)\,\zeta_w. \tag{4.3}$$

Thus $F \in \mathcal{V}$ is W–invariant if and only if $\varrho(F) = \delta(F)$. For arbitrary $F \in \mathcal{V}$ we may also write $F^\varrho := \varrho(F)$ and $F^\delta := \delta(F)$. When no superscript is used, we take the image F^ϱ under the embedding ϱ by default.

Generally, any endomorphism of $\mathcal{V}$ acts pointwise in $\mathcal{F}(W, \mathcal{V})$. For instance, given a translation Γ_b, we set

$$\Gamma_b(f)(u) := \Gamma_b(f(u)).$$

We define

$$\delta(\Gamma_b) = \Gamma_b^\delta := \sum_{w\in W} \Gamma_b\,\zeta_w, \tag{4.4}$$

$$\varrho(\Gamma_b) = \Gamma_b^\varrho := \sum_{w\in W} \Gamma_{w^{-1}(b)}\,\zeta_w, \tag{4.5}$$

where we let ζ_w act by multiplication in $\mathcal{F}(W, \mathcal{V})$. Similarly, we let X_b act by pointwise multiplication:

$$X_b(f)(u) := X_b(f(u)).$$

Remark. All of the constructions above can be applied to more general spaces of functions other than $\mathcal{V}$ (as long as the spaces are preserved by W). For instance, one may replace $\mathcal{V}$ by the field of rational functions $\mathbb{Q}_{q,t}(X)$, or more generally by meromorphic functions of $X \in (\mathbb{C}^*)^n$. We will apply the above constructions in such contexts without further comment. When considering a function $F(X)$, we take by default the image F^{ϱ} under the embedding ϱ, unless otherwise specified.

Recall from Section 1.5 that $\mathcal{D}$ denotes the algebra of difference-reflection operators over $\mathbb{Q}_{q,t}$. We define a map from $\mathcal{D}$ to $\mathrm{End}_{\mathbb{Q}_{q,t}}(\mathcal{F}(W, \mathbb{Q}_{q,t}(X)))$ by

$$\phi : g\,\Gamma_b\, w \;\mapsto\; \varrho(g)\,\varrho(\Gamma_b)\,\delta(w), \quad \text{where} \;\; g \in \mathbb{Q}_{q,t}(X),\; b \in P,\; w \in W.$$

It is then straightforward verify the following:

Lemma 4.1.1. *The map $\phi : \mathcal{D} \to \mathrm{End}_{\mathbb{Q}_{q,t}}(\mathcal{F}(W, \mathbb{Q}_{q,t}(X)))$ is a homomorphism of algebras.*

We obtain an action of $\mathcal{H\!\!H}$ in $\mathcal{F}(W, \mathbb{Q}_{q,t}(X))$ by composing ϕ with the polynomial representation, the latter being viewed as a homomorphism $\mathcal{H\!\!H} \to \mathcal{D}$.

4.2. Nonsymmetric limiting procedure

For a difference-reflection operator $\mathcal{L}$ and a function $F(X)$, let

$$\begin{aligned}(4.6)\qquad \varkappa^{\delta}(\mathcal{L}) &:= \delta(X_{\rho_k}\Gamma_{-\rho_k})\,\phi(\mathcal{L})\,\delta(X_{\rho_k}\Gamma_{-\rho_k})^{-1},\\ \varkappa^{\delta}(F) &:= \delta(X_{\rho_k}\Gamma_{-\rho_k})\big(\varrho(F)\big).\end{aligned}$$

For instance, one has

$$(4.7)\qquad \varkappa^{\delta}(X_b) = \sum_{w\in W}\prod_{\nu} t_{\nu}^{-(\rho_{\nu}^{\vee}, w^{-1}(b))}\, X_{w^{-1}(b)}\,\zeta_w,$$

$$(4.8)\qquad \varkappa^{\delta}(\Gamma_b) = \sum_{w\in W}\prod_{\nu} t_{\nu}^{-(\rho_{\nu}^{\vee}, w^{-1}(b))}\, \Gamma_{w^{-1}(b)}\,\zeta_w,$$

$$(4.9)\qquad \varkappa^{\delta}(w) = \delta(w) \;\; \text{for} \;\; w \in W.$$

We observe that

$$\varkappa^\delta(v\mathcal{L}w) = v\,\varkappa^\delta(\mathcal{L})\,w \quad \text{for} \quad v,w\in W, \tag{4.10}$$

for any difference-reflection operator $\mathcal{L}$.

Definition 4.2.1. The *nonsymmetric Ruijsenaars-Etingof procedure* is defined by

$$\mathit{RE}^\delta(\mathcal{L}) := \lim_{k\to\infty} \varkappa^\delta(\mathcal{L}), \quad \mathit{RE}^\delta(F) := \lim_{k\to\infty} \varkappa^\delta(F), \tag{4.11}$$

where $k_\nu\in\mathbb{Z}_+$ and $k\to\infty$ means that $k_\nu\to\infty$ for all $\nu\in\nu_R$; equivalently, $t_\nu\to 0$.

This limiting procedure was defined in [**15**] for the root system A_1.

4.3. Calculating the limit

Recall that we assume $|q|<1$. We also take $k_\nu\in\mathbb{Z}_+$ in this section.

The *global nonsymmetric difference Whittaker function* is defined as

$$\Omega(X,\Lambda) := \mathit{RE}^\delta(G(X,\Lambda)). \tag{4.12}$$

The following proposition justifies the existence $\Omega(X,\Lambda)$.

Proposition 4.3.1. *The limit, as $k\to\infty$, of the series $\Gamma^\delta_{-\rho_k}(\Xi(X,\Lambda;q,t))$ exists; here $\Xi(X,\Lambda;q,t)$ is the series from (3.6). Accordingly,*

$$\Omega(X,\Lambda) = \frac{\widetilde{\gamma}(1)}{\widetilde{\gamma}_x\,\widetilde{\gamma}_\lambda} \sum_{b\in P} q^{(b,b)/2}\, \frac{\overline{E}_b(\Lambda)}{\langle \overline{E}_b, \overline{E}_b\rangle} \sum_{w\in W} a_{b,w}\, X_{-b_-}\,\zeta_w, \tag{4.13}$$

where $\widetilde{\gamma}(1) := \sum_{b\in P} q^{(b,b)/2}$ and $a_{b,w}$ is the limit, as all $t_\nu\to 0$, of the coefficient of $X_{-w(b_-)}$ in E_b^*. In particular, one has $a_{b,w}\in\mathbb{Z}[q]$ and $a_{b,u_b^{-1}}=1$, $a_{b,\mathrm{id}}=\delta_{b,b_-}$.

Proof. First, one has

$$\delta(X_{\rho_k}\Gamma_{-\rho_k})\,(\widetilde{\gamma}_x)^{-1} = q^{(\rho_k,\rho_k)/2}\,(\widetilde{\gamma}_x)^{-1}\,\delta(\Gamma_{-\rho_k}),$$

as operators. (Due to the W–invariance of $\widetilde{\gamma}_x$, we omit ϱ here.) Using (3.5), we arrive that factor $\widetilde{\gamma}(1)$ in (4.13). It now suffices to consider the limit of

$$q^{-(b_-,\rho_k)}\,\Gamma^{\delta}_{-\rho_k}(\varrho(E^*_b(X))),$$

or, equivalently, the limit as $t_\nu \to 0$ of

$$q^{-(b_-,\rho_k)}\,\Gamma_{-\rho_k}(w^{-1}(E^*_b))\,\zeta_w \quad \text{for all} \quad w \in W. \tag{4.14}$$

Using (2.15) and Proposition 2.4.4, one sees that this limit exists and has the form $a_{b,w}\,X_{-b_-}\,\zeta_w$ for $a_{b,w}$ as claimed. By (2.15), $\overline{E^*_b} = (\overline{E}^\dagger_b)^*$. Hence Corollary 2.4.5 implies that $a_{b,w} \in \mathbb{Z}[q]$. □

Remark. Alternatively, one can set

$$G'(X,\Lambda) = G(X,\Lambda)\frac{\widetilde{\gamma}(X)q^{\frac{(x,x)}{2}}}{\widetilde{\gamma}(q^{\rho_k})q^{\frac{(\rho_k,\rho_k)}{2}}}, \quad \Omega'(X,\Lambda) = RE^{\delta}(G'(X,\Lambda)),$$

and take $\Re k_\nu \to \infty$ for *complex* k_ν. Then $\widetilde{\gamma}_x q^{\frac{(x,x)}{2}}\Omega(X,\Lambda) = \Omega'(X,\Lambda)$. Using G' instead of G somewhat simplifies the calculation of the limit and does not influence the corresponding operators acting on this function (which are studied below), since $\widetilde{\gamma}_x q^{\frac{(x,x)}{2}}$ is $\widehat{W}$–invariant.

4.4. Main theorem

We now come to our main result, a counterpart of Theorem 3.2.1 for the global nonsymmetric Whittaker function $\Omega(X,\Lambda)$. Recall the definition of the algebra $\ddot{\mathcal{H}}$ from Definition 2.4.1 and the anti-involution φ from (1.39).

Theorem 4.4.1. *(i) The operators $RE^{\delta}(H^{\varphi})$ acting in $\mathcal{F}(W,\mathcal{V})$ are well defined for $H \in \ddot{\mathcal{H}}$. For instance, the following operators are well defined:*

$$\widehat{Y}_b := RE^{\delta}(Y_b), \quad \widehat{X}_b := RE^{\delta}(\widetilde{X}_b) \ \ \textit{for} \ \ \widetilde{X}_b := \ddot{Y}^{\varphi}_{-b} = t^{(b_+,\rho^\vee)}X_b, \tag{4.15}$$

$$\widehat{T}_i := RE^{\delta}(\ddot{T}_i) \ \ \textit{for} \ \ i>0, \quad \widehat{T}_0 := RE^{\delta}(\ddot{T}^{\varphi}_0), \quad \widehat{\pi}_r := RE^{\delta}(\pi^{\varphi}_r) \ \ \textit{for} \ \ r \in O'.$$

(ii) The function $\Omega(X, \Lambda)$ satisfies

$$\widehat{\pi}_r(\Omega(X,\Lambda)) = \pi_r^\lambda(\Omega(X,\Lambda)) \quad \textit{for} \;\; r \in O, \tag{4.16}$$

$$\widehat{T}_i(\Omega(X,\Lambda)) = T_i^\lambda(\Omega(X,\Lambda)) \quad \textit{for} \;\; 0 \le i \le n, \tag{4.17}$$

and the following relations corresponding to (3.10):

$$\widehat{Y}_a(\Omega(X,\Lambda)) = \Lambda_a^{-1}\Omega(X,\Lambda) \quad \textit{and} \quad \overline{Y}_a^\lambda(\Omega(X,\Lambda)) = \widehat{X}_{-a}\Omega(X,\Lambda) \quad \textit{for} \;\; a \in P. \tag{4.18}$$

(iii) Let $f(q^{\widetilde{c}}) := f_{u_c^{-1}}(q^{c_-})$ for any $f = \sum_{w\in W} f_w\,\zeta_w$ and $c \in P$. Then

$$\Omega(q^{\widetilde{c}}, \Lambda) = \overline{E}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1-q_i^j}. \tag{4.19}$$

Equivalently, one has (where $x^2 = (x,x)$)

$$\sum_{b\in P} q^{(b_- - c_-)^2/2}\, \frac{\overline{E}_b(\Lambda)}{\langle \overline{E}_b, \overline{E}_b\rangle}\, a_{b,u_c^{-1}} = \widetilde{\gamma}_\lambda\, \overline{E}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1-q_i^j}. \tag{4.20}$$

Proof. Chapter 5 is devoted to a direct and constructive proof of (i). We also sketch an indirect proof of (i) in the remark below. Assuming that (i) holds, (ii) and (iii) are direct consequences of Theorem 3.2.1 and Proposition 4.3.1. For (iii), one uses the formula for $E_b(q^{-\rho_k})$ from [**11**, (3.3.16)]. $\square$

We call the operators $\widehat{Y}_b$ $(b \in P)$ the *Toda-Dunkl operators.*

Remark. The existence of $\Omega(X, \Lambda)$, which was demonstrated in Proposition 4.3.1, provides an indirect proof of Theorem 4.4.1(i). Let us sketch this argument for the operator $\widehat{Y}_b$. One uses (3.10) as follows:

$$\begin{aligned} Y_b(q^{\frac{(c_\sharp,c_\sharp)}{2}-(\rho_k,\rho_k)} E_c^*(X)\,(\widetilde{\gamma}_x)^{-1}) &= \langle Y_b(G(X,\Lambda))\, E_c^*(\Lambda)\, \widetilde{\gamma}_\lambda\, \mu_\circ(\Lambda)\rangle, \\ &= \langle \Lambda_b^{-1} G(X,\Lambda)\, E_c^*(\Lambda)\, \widetilde{\gamma}_\lambda\, \mu_\circ(\Lambda)\rangle. \end{aligned} \tag{4.21}$$

Applying $\varkappa^\delta$ and taking $t_\nu \to 0$, the right-hand side of (4.21) is well defined. It follows that the action of $RE^\delta(Y_b)$ is well defined on

$$RE^\delta(q^{-(c_-,\rho_k)-\frac{(\rho_k,\rho_k)}{2}} E_c^*(X)\,(\widetilde{\gamma}_x)^{-1}) = (\sum_w a_{c,w}\, X_{-c_-}\,\zeta_w)\,(\widetilde{\gamma}_x)^{-1}.$$

In general, one obtains that the action of the operators from (i) are well defined when they are applied to linear combinations of $X_b\,\zeta_w\,(\widetilde{\gamma}_x)^{-1}$ for regular $b \in P_+$ and $w \in W$. The operators $\varkappa^\delta(H^\varphi)$ for $H \in \ddot{\mathcal{H}}$ have rational coefficients; nevertheless, this property is sufficient to see that their coefficients are well defined in the limit $t_\nu \to 0$.

4.5. Symmetrization

The symmetric Whittaker function $\mathcal{W}(X,\Lambda)$ from (3.17) is the symmetrization of $\Omega(X,\Lambda)$. More precisely, one has

$$\delta(\mathcal{W}(X,\Lambda)) = \sum_{w\in W} \widehat{T}_w(\Omega(X,\Lambda)) = \sum_{w\in W} \overline{T}_w(\Omega(X,\Lambda)). \tag{4.22}$$

In particular, all W–components of the right-hand side coincide; see also (5.26) below.

Explicitly, one has

$$\mathcal{W}(X,\Lambda) = \frac{\widetilde{\gamma}(1)}{\widetilde{\gamma}_x\,\widetilde{\gamma}_\lambda} \sum_{b\in P_-} q^{(b,b)/2} \frac{X_b^{-1}\,\overline{E}_b(\Lambda)}{\prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee,b)}(1-q_i^j)}.$$

We recall that $\overline{E}_b$ is W–invariant[1] for $b \in P_-$, by Proposition 2.4.3.

For $c \in P_-$, one has

$$\mathcal{W}(q^c,\Lambda) = \overline{E}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1-q_i^j},$$

which is equivalent to

$$\sum_{b\in P_-} \frac{q^{(b-c)^2/2}\,\overline{E}_b(\Lambda)}{\prod_{i=1}^n \prod_{j=1}^{-(\alpha_i^\vee,b)}(1-q_i^j)} = \widetilde{\gamma}_\lambda\,\overline{E}_c(\Lambda) \prod_{i=1}^n \prod_{j=1}^\infty \frac{1}{1-q_i^j}. \tag{4.23}$$

[1] For $b \in P_-$, $\overline{E}_b$ coincides with the symmetric Macdonald polynomial $P_b(t_\nu = 0)$.

We observe that formula (4.23) results from (4.20). Indeed, when $c \in P_-$, one has $u_c = \mathrm{id}$ and the coefficient $a_{b,u_c^{-1}}$ is nonzero only for $b \in P_-$; in this case, one has $a_{b,\mathrm{id}} = 1$ and hence the summation in (4.23) ranges over $b \in P_-$.

CHAPTER 5

Toda-Dunkl operators

Our main aim in this chapter is to provide a direct and constructive proof of the existence of the operators from Theorem 4.4.1. We also consider symmetrizations of the Toda-Dunkl operators. In Proposition 5.4.1, we show that these symmetrizations coincide with *RE* limits of the symmetric operators L_f from (1.48). Finally, at the end of the chapter we provide some explicit examples of the Toda-Dunkl operators for the root systems A_1, A_2, and B_2.

5.1. Preparations

Recall the explicit description of Y_b in terms of $G^{\pm}_{\widetilde{\alpha}}$ given by (1.43), (1.44), and (1.45). Recall that $\ddot{G}^{\pm}_{\widetilde{\alpha}} := t^{1/2}_{\widetilde{\alpha}} G^{\pm}_{\widetilde{\alpha}}$ and

$$\ddot{Y}_b = \Gamma_{-b}\, \ddot{G}^{\mathrm{sgn}(\epsilon_l)}_{\widetilde{\alpha}^l} \cdots \ddot{G}^{\mathrm{sgn}(\epsilon_1)}_{\widetilde{\alpha}^1}, \tag{5.1}$$

where the ϵ_p are given by (1.35).

Given $u \in W$ and a reduced expression $u = s_{j_l} \cdots s_{j_1}$, form $\lambda(u) = \{\alpha^1, \dots, \alpha^l\}$ using (1.20) and write $s^i = s_{\alpha^i}$. We will consider products of the form

$$\varkappa^{\delta}(\ddot{G}^{+}_{\pm\alpha^p} \cdots \ddot{G}^{+}_{\pm\alpha^r}), \quad \text{for} \quad 1 \le r \le p \le l, \tag{5.2}$$

and similar products for $\ddot{G}^{-}_{\alpha}$. We expand such products by choosing from each $\ddot{G}^{\pm}_{\alpha}$ either f_α or $g_\alpha\, s_\alpha$.

For any $f \in \mathcal{F}(W, \mathbb{Q}_{q,t}(X))$, $w \in W$, and $\nu \in \nu_R$, we define $\mathrm{ord}_w^\nu(f)$ to be the order of the w-component f_w with respect to t_ν. For instance, if $\widetilde{\alpha} = [\alpha, \nu_\alpha j]$, then

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_{\widetilde{\alpha}})) = \begin{cases} 0, & \text{if } w^{-1}(\alpha) > 0, \\ \delta_{\nu,\nu_\alpha}, & \text{if } w^{-1}(\alpha) < 0, \end{cases} \tag{5.3}$$

$$\mathrm{ord}_w^\nu(\varkappa^\delta(g_{\widetilde{\alpha}})) = \begin{cases} 0, & \text{if } w^{-1}(\alpha) > 0, \\ -(\rho_\nu^\vee, w^{-1}(\alpha)), & \text{if } w^{-1}(\alpha) < 0. \end{cases} \tag{5.4}$$

The second line in (5.3) follows from the fact that $(\rho_\nu^\vee, w^{-1}(\alpha)) \neq 0$ for all $w \in W$ provided $\nu = \nu_\alpha$.

For any $f, g \in \mathcal{F}(W, \mathbb{Q}_{q,t}(X))$ and $w, v \in W$, one has

$$\mathrm{ord}_w^\nu(fg) = \mathrm{ord}_w^\nu(f) + \mathrm{ord}_w^\nu(g), \tag{5.5}$$

$$\mathrm{ord}_w^\nu(\delta(v)f) = \mathrm{ord}_{v^{-1}w}^\nu(f), \tag{5.6}$$

due to (4.1).

For $f \in \mathcal{F}(W, \mathbb{Q}_{q,t}(X))$ and $w, v \in W$, we define

$$\mathrm{ord}_w^\nu(f\, v) := \mathrm{ord}_w^\nu(f), \quad {}^{\checkmark}\mathrm{ord}_w^\nu(v\, f) := \mathrm{ord}_w^\nu(f). \tag{5.7}$$

The following proposition will be our main tool in the proof of the existence of the Toda-Dunkl operators $\widehat{Y}_b$.

PROPOSITION 5.1.1. *Let $u \in W$, choose a reduced expression $u = s_{j_l} \cdots s_{j_1}$, and let $1 \leq r \leq p \leq l$.*

(i) The ord_w^ν of any product in the expansion of $\varkappa^\delta(\ddot{G}^+_{\alpha^p} \cdots \ddot{G}^+_{\alpha^r})$ is bounded below by $\mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^r}))$.

(ii) The ord_w^ν of any product in the expansion of $\varkappa^\delta(\ddot{G}^+_{-\alpha^r} \cdots \ddot{G}^+_{-\alpha^p})$ is bounded below by $\mathrm{ord}_w^\nu(\varkappa^\delta(f_{-\alpha^r} \cdots f_{-\alpha^p}))$.

(iii) The ${}^{\checkmark}\mathrm{ord}_w^\nu$ *of any product in the expansion of* $\varkappa^\delta(\ddot{G}^-_{\alpha^p}\cdots\ddot{G}^-_{\alpha^r})$ *is bounded below by* $\mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^p}\cdots f_{\alpha^r}))$.

(iv) The ${}^{\checkmark}\mathrm{ord}_w^\nu$ *of any product in the expansion of* $\varkappa^\delta(\ddot{G}^-_{-\alpha^r}\cdots\ddot{G}^-_{-\alpha^p})$ *is bounded below by* $\mathrm{ord}_w^\nu(\varkappa^\delta(f_{-\alpha^r}\cdots f_{-\alpha^p}))$.

Proof. We will prove (i) only; the other statements can be proved by similar arguments. The proof is based on the following lemma.

Lemma 5.1.2. *Let $u \in W$ and choose a reduced expression $u = s_{j_l}\cdots s_{j_1}$. Then for any $l \geq i \geq r \geq 1$, one has*

$$\mathrm{ord}_w^\nu(\varkappa^\delta(g_{\alpha^i}f_{s^i(\alpha^{i-1})}\cdots f_{s^i(\alpha^r)})) \geq \mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^i}\cdots f_{\alpha^r})), \quad \textit{where} \quad s^i := s_{\alpha^i}. \tag{5.8}$$

Proof of Lemma 5.1.2. Write $\alpha = \alpha^i$ (so $s_\alpha = s^i$) and take $\beta = \alpha^k$ for any $i > k \geq r$. Using (5.3), one has

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_\beta)) \leq \mathrm{ord}_w^\nu(\varkappa^\delta(f_{s_\alpha(\beta)}))$$

unless

$$\nu = \nu_\beta, \quad w^{-1}(\beta) < 0, \quad \text{and} \quad w^{-1}(s_\alpha(\beta)) > 0. \tag{5.9}$$

An equivalent description of (5.9) is

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_\beta)) = 1 \quad \text{and} \quad \mathrm{ord}_w^\nu(\varkappa^\delta(f_{s_\alpha(\beta)})) = 0, \tag{5.10}$$

Assuming (5.9) holds, there are two cases to consider: either $w^{-1}(\alpha) > 0$ or $w^{-1}(\alpha) < 0$.

Suppose $w^{-1}(\alpha) > 0$. Then $\mathrm{ord}_w^\nu(\varkappa^\delta(g_\alpha)) = \mathrm{ord}_w^\nu(\varkappa^\delta(f_\alpha)) = 0$. If (5.10) occurs, then one must have $(\beta, \alpha) < 0$. Hence $s_\alpha(\beta)$ belongs to $\lambda(u)$ and by Lemma 1.2.1, one has $s_\alpha(\beta) = \alpha^j$ where $i > j > k$. Therefore, the application of s_α to the product $f_{\alpha^{i-1}}\cdots f_{\alpha^r}$ reverses the positions of the factors $f_{s_\alpha(\beta)}$ and f_β for all pairs $\{\beta, s_\alpha(\beta)\}$, where β satisfies (5.10); the ord_w^ν of any other factors in this product can only increase upon the application of s_α. This proves (5.8) when $w^{-1}(\alpha) > 0$.

It remains to consider the case when $w^{-1}(\alpha) < 0$. We note that $w^{-1}(s_\alpha(\beta)) = s_{w^{-1}(\alpha)}(w^{-1}(\beta))$. By (1.26), one has

$$l_\nu(s_{w^{-1}(\alpha)}) \leq -2(\rho_\nu^\vee, w^{-1}(\alpha)) - \delta_{\nu,\nu_\alpha}. \tag{5.11}$$

(The only case when (5.11) is not an equality is $\nu_\alpha = \nu_{\mathrm{lng}}$ and $\nu = \nu_{\mathrm{sht}}$.) Combining this with (5.4) and (5.3) yields

$$\mathrm{ord}_w^\nu(\varkappa^\delta(g_\alpha)) \geq \mathrm{ord}_w^\nu(\varkappa^\delta(f_\alpha)) + \frac{l_\nu(s_{w^{-1}(\alpha)}) - \delta_{\nu,\nu_\alpha}}{2}. \tag{5.12}$$

Using Lemma 1.2.1(ii), one sees that

$$\frac{l_\nu(s_{w^{-1}(\alpha)}) - \delta_{\nu,\nu_\alpha}}{2}$$

is the maximum possible number of β satisfying (5.9). In other words, (5.12) compensates for all drops in the order coming from (5.10) when applying s_α to the product $f_{\alpha^{i-1}} \cdots f_{\alpha^r}$. This establishes (5.8). □

Now we return to the proof of Proposition 5.1.1(i). We argue by induction on the number of factors of the form $g_\alpha\, s_\alpha$ chosen to form a particular product in the expansion—the base case being the product when no such factors are chosen, i.e., $\mathcal{P}^\emptyset := \varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^r})$.

Let us first consider some particular cases. Suppose that just one factor of the form $g_\alpha\, s_\alpha$, say $g_{\alpha^i}\, s^i$, is chosen. In other words, take the product

$$\mathcal{P}^i := \varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}}\, g_{\alpha^i}\, s^i\, f_{\alpha^{i-1}} \cdots f_{\alpha^r}).$$

Due to (5.7),

$$\begin{aligned}\mathrm{ord}_w^\nu(\mathcal{P}^i) &= \mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}}\, g_{\alpha^i}\, f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^r)})) \\ &= \mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}})) + \mathrm{ord}_w^\nu(\varkappa^\delta(g_{\alpha^i}\, f_{s_{\alpha^i}(\alpha^{i-1})} \cdots f_{s_{\alpha^i}(\alpha^r)})),\end{aligned}$$

Then (5.8) gives $\mathrm{ord}_w^\nu(\mathcal{P}^i) \geq \mathrm{ord}_w^\nu(\mathcal{P}^\emptyset)$, as claimed.

Now consider the case when two factors of $g_\alpha\, s_\alpha$ are chosen:

$$\mathcal{P}^{ij} := \varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}}\, g_{\alpha^i}\, s^i\, f_{\alpha^{i-1}} \cdots f_{\alpha^{j+1}}\, g_{\alpha^j}\, s^j\, f_{\alpha^{j-1}} \cdots f_{\alpha^r}). \tag{5.13}$$

Due to (5.7),

$$\begin{aligned} \operatorname{ord}^\nu_w(\mathcal{P}^{ij}) = \operatorname{ord}^\nu_w(\varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}}\, g_{\alpha^i}\, f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^{j+1})} \\ \times g_{s^i(\alpha^j)}\, f_{s^i s^j(\alpha^{j-1})} \cdots f_{s^i s^j(\alpha^r)})). \end{aligned} \tag{5.14}$$

Apply (5.6) and (5.8) as follows:

$$\begin{aligned} \operatorname{ord}^\nu_w(\varkappa^\delta(g_{s^i(\alpha^j)}\, f_{s^i s^j(\alpha^{j-1})} \cdots f_{s^i s^j(\alpha^r)})) &= \operatorname{ord}^\nu_{s^i w}(\varkappa^\delta(g_{\alpha^j}\, f_{s^j(\alpha^{j-1})} \cdots f_{s^j(\alpha^r)})) \\ \geq \operatorname{ord}^\nu_{s^i w}(\varkappa^\delta(f_{\alpha^j}\, f_{\alpha^{j-1}} \cdots f_{\alpha^r})) &= \operatorname{ord}^\nu_w(\varkappa^\delta(f_{s^i(\alpha^j)}\, f_{s^i(\alpha^{j-1})} \cdots f_{s^i(\alpha^r)})). \end{aligned}$$

Returning to (5.14), one then has

$$\begin{aligned} \operatorname{ord}^\nu_w(\mathcal{P}^{ij}) &\geq \operatorname{ord}^\nu_w(\varkappa^\delta(f_{\alpha^p} \cdots f_{\alpha^{i+1}}\, g_{\alpha^i}\, f_{s^i(\alpha^{i-1})} \cdots f_{s^i(\alpha^r)})) \\ &= \operatorname{ord}^\nu_w(\mathcal{P}^i) \geq \operatorname{ord}^\nu_w(\mathcal{P}^\emptyset). \end{aligned} \tag{5.15}$$

In general, for any decreasing sequence $p \geq i_1 > i_2 > \cdots > i_m \geq r$, we set

$$\mathcal{P}^{i_1 \ldots i_m} := \varkappa^\delta(h_p \cdots h_r),$$

where $h_i = g_{\alpha^i}\, s^i$ whenever $i \in \{i_1, \ldots, i_m\}$ and $h_i = f_{\alpha^i}$ otherwise. The same reasoning used to arrive at (5.15) shows that

$$\operatorname{ord}^\nu_w(\mathcal{P}^{i_1 \ldots i_m}) \geq \operatorname{ord}^\nu_w(\mathcal{P}^{i_1 \ldots i_{m-1}}), \tag{5.16}$$

which gives the induction step. $\square$

5.2. Limits of Dunkl operators

Now we are ready to prove the existence of the Toda-Dunkl operators.

THEOREM 5.2.1. *The operators $\widehat{Y}_b = \mathit{RE}^\delta(Y_b)$ exist for all $b \in P$.*

PROOF. We will break the proof into steps, proving that $\widehat{Y}_b$ exists for the following choices of b:

(1) $b = \omega_r$ $(r \in O')$,

(2) b equal to a short positive root,

(3) $b = -\omega_r$ $(r \in O')$,

(4) b equal to a short negative root.

These steps are sufficient to prove the theorem, because P is generated by Q together with the minuscule weights, and Q is generated by the short roots. For a proof of the latter assertion, see [**26**, Exercise 6.9].

For (1) and (2), we consider first any $b \in P_+$. Write $b = \pi_r \widetilde{w} = \pi_r s_{j_l} \cdots s_{j_1}$ $(l = l(b))$ in $\widehat{W}$ and form $\widetilde{\alpha}^p$ $(1 \le p \le l)$ from (1.20). Since $b \in P_+$, one has $l_\nu(b) = 2(b, \rho_\nu^\vee)$ and $Y_b = T_b = \pi_r T_{j_l} \cdots T_{j_1}$. Hence $Y_b = q^{-(b,\rho_k)} \Gamma_{-b} \ddot{G}^+_{\widetilde{\alpha}^l} \cdots \ddot{G}^+_{\widetilde{\alpha}^1}$. Using (4.8), we can write

$$\varkappa^\delta(Y_b) = \sum_{w \in W} q^{-(b,\rho_k - w(\rho_k))} \Gamma_{-w^{-1}(b)} \zeta_w \varkappa^\delta(\ddot{G}^+_{\widetilde{\alpha}^l} \cdots \ddot{G}^+_{\widetilde{\alpha}^1}).$$

We claim that

$$\xi^\delta(Y_b) := \sum_{w \in W} q^{-(b,\rho_k - w(\rho_k))} \Gamma_{-w^{-1}(b)} \zeta_w \varkappa^\delta(f_{\widetilde{\alpha}^l} \cdots f_{\widetilde{\alpha}^1}) \tag{5.17}$$

is regular at $t_\nu = 0$ for any $b \in P_+$. Indeed, one has

$$q^{-(b,\rho_k - w(\rho_k))} = \prod_\nu t_\nu^{-(b,\rho_\nu^\vee - w(\rho_\nu^\vee))}$$

and the exponents $(b, \rho_\nu^\vee - w(\rho_\nu^\vee))$ count the number of $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \lambda_\nu(b)$ such that $w^{-1}(\alpha) < 0$. This follows from (1.21) and the following counterpart of (1.27):

$$\rho_\nu^\vee - w(\rho_\nu^\vee) = \sum_{\alpha \in \lambda_\nu(w^{-1})} \alpha^\vee. \tag{5.18}$$

Hence the regularity of $\xi^\delta(Y_b)$ is immediate from (5.3).

(1) Let $b = \omega_r$ for $r \in O'$; recall that $\omega_r = \pi_r u_r$. Using Proposition 5.1.1, where we take $u = u_r$, the regularity of $\varkappa^\delta(Y_{\omega_r})$ follows from that of $\xi^\delta(Y_{\omega_r})$.

(2) Suppose $b = \alpha$ is any short positive root. We use the following lemma.

Lemma 5.2.2. *For any short $\alpha \in R_+$, there exists a reduced expression*

$$s_\vartheta = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$$

such that $\alpha = s_{j_r} \cdots s_{j_1}(\vartheta)$ for some $0 \le r \le p$.

Proof of Lemma 5.2.2. We can write $\vartheta = s_{j_1} \cdots s_{j_r}(\alpha)$ by choosing $\alpha_{j_1}, \ldots, \alpha_{j_r}$ such that

$$(s_{j_i} \cdots s_{j_r}(\alpha), \alpha_{j_{i+1}}^\vee) < 0, \quad 1 \le i \le r.$$

Here we are using the characterization of ϑ as the unique short root lying in P_+. If $\beta \in R_+$ is not a simple root, then $0 < s_i(\beta) < \beta$ for at least one i. Thus we can find $\alpha_{j_{r+1}}, \cdots, \alpha_{j_p}$ and α_m such that $\alpha = s_{j_{r+1}} \cdots s_{j_p}(\alpha_m)$ and

$$(s_{j_{i+1}} \cdots s_{j_p}(\alpha_m), \alpha_{j_i}^\vee) = -1, \quad 1 \le i \le p. \tag{5.19}$$

These inner products must equal -1 because α_m is short. Hence $\vartheta = s_{j_1} \cdots s_{j_p}(\alpha_m)$ and $p \le (\vartheta, \rho^\vee) - 1$. The expression $s_\vartheta = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$ must be reduced, because

$$l(s_\vartheta) = l(\vartheta) - l(s_0) = 2(\vartheta, \rho^\vee) - 1;$$

i.e., we must have $p = (\vartheta, \rho^\vee) - 1$. $\square$

Let $s_\vartheta = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$ be a reduced expression as in Lemma 5.2.2. Let $l = l(s_\vartheta) = 2p + 1$ and construct $\lambda(s_\vartheta) = \{\alpha^1, \ldots, \alpha^l\}$ using this reduced expression.

Note that $\vartheta = s_0 s_\vartheta$ and $l(\vartheta) = l(s_\vartheta) + 1$. Accordingly, $\lambda(\vartheta) = \lambda(s_\vartheta) \cup \{[\vartheta, 1]\}$.

Due to (5.19) and (1.34), one has

$$Y_\alpha = (T_{j_r}^{-1} \cdots T_{j_1}^{-1})\, T_0\, (T_{j_1} \cdots T_{j_p} T_m T_{j_p} \cdots T_{j_{r+1}}).$$

Hence, for $v = s_{j_r} \cdots s_{j_1}$, we can write

$$v^{-1}\, Y_\alpha\, v = q^{-(\vartheta,\, \rho_k)}\, \ddot{G}^-_{\alpha^r} \cdots \ddot{G}^-_{\alpha^1}\, \Gamma_{-\vartheta}\, \ddot{G}^+_{[\vartheta,1]}\, \ddot{G}^+_{\alpha^l} \cdots \ddot{G}^+_{\alpha^{r+1}}.$$

We note that by (4.10) one has

$$\varkappa^\delta(v^{-1}\, Y_\alpha\, v) = v^{-1}\, \varkappa^\delta(Y_\alpha)\, v.$$

Hence it suffices to prove that $\varkappa^\delta(v^{-1}\, Y_\alpha\, v)$ is regular at $t_\nu = 0$.

By Proposition 5.1.1(i, iii), it is enough to consider

$$q^{-(\vartheta,\, \rho_k)} \varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1}\, \Gamma_{-\vartheta}\, \ddot{G}^+_{[\vartheta,1]}\, f_{\alpha^l} \cdots f_{\alpha^{r+1}}).$$

We expand this product by choosing either $f_{[\vartheta,1]}$ or $g_{[\vartheta,1]}\, s_{[\vartheta,1]}$ from $\ddot{G}^+_{[\vartheta,1]}$.

Choosing $f_{[\vartheta,1]}$ from $\ddot{G}^+_{[\vartheta,1]}$, we arrive at $\xi^\delta(Y_\vartheta)$, which is known to be regular at $t_\nu = 0$.

Thus it remains to choose $g_{[\vartheta,1]}\, s_{[\vartheta,1]}$. This yields

$$q^{-(\vartheta,\, \rho_k)} \varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1}\, g_{[\vartheta,-1]}\, s_\vartheta\, f_{\alpha^l} \cdots f_{\alpha^{r+1}}),$$

where we have used that $\Gamma_{-\vartheta}\, g_{[\vartheta,1]}\, s_{[\vartheta,1]} = g_{[\vartheta,-1]}\, s_\vartheta$. According to (5.7), when calculating ord^ν_w, one must move s_ϑ to the right:

$$s_\vartheta\, (f_{\alpha^l} \cdots f_{\alpha^{r+1}}) = (f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})\, s_\vartheta,$$

where we have used Lemma 1.2.4(iii). By (5.4), we need to show that

$$\mathrm{ord}^\nu_w(\varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) \geq \begin{cases} (\vartheta, \rho^\vee_\nu), & \text{if } w^{-1}(\vartheta) > 0, \\ (\vartheta, \rho^\vee_\nu + w(\rho^\vee_\nu)), & \text{if } w^{-1}(\vartheta) < 0, \end{cases} \tag{5.20}$$

for any $0 \leq r \leq p$.

To this end, assume first that $w^{-1}(\vartheta) > 0$. Clearly we have (where $\delta_{\nu,\alpha} = \delta_{\nu,\nu_\alpha}$)

$$\mathrm{ord}^\nu_w(\varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^r})) = \sum_{i=1}^r \delta_{\nu,\alpha^i}.$$

For the remaining factors in the left-hand side of (5.20), one has

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_{-\alpha^{r+1}} \cdots f_{-\alpha^{l-r}})) \geq \delta_{\nu,\vartheta} + \sum_{i=r+1}^{p} \delta_{\nu,\alpha^i}.$$

This can be seen as follows. First, $\alpha^{p+1} = \vartheta$ and hence, by (5.3), $\mathrm{ord}_w^\nu(\varkappa^\delta(f_{-\alpha^{p+1}})) = \delta_{\nu,\vartheta}$. Second, for each $r+1 \leq i \leq p$, at least one of $w^{-1}(\alpha^i)$ or $w^{-1}(\alpha^{l-i+1})$ must be positive. This follows from Lemma 1.2.1(ii), Lemma 1.2.4(iii), and the assumption $w^{-1}(\vartheta) > 0$. Therefore, altogether one has

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) \geq \delta_{\nu,\vartheta} + \sum_{i=1}^{p} \delta_{\nu,\alpha^i}. \tag{5.21}$$

Finally, using Lemma 1.2.4(i), one finds that the right-hand side of (5.21) is exactly $(\vartheta, \rho_\nu^\vee)$.

Now assume $w^{-1}(\vartheta) < 0$. One has

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) = \sum_{\substack{1 \leq i \leq l-r \\ w^{-1}(\alpha^i) > 0}} \delta_{\nu,\alpha^i} \tag{5.22}$$

and

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1})) = \sum_{\substack{1 \leq i \leq r \\ w^{-1}(\alpha^i) < 0}} \delta_{\nu,\alpha^i} \geq \sum_{\substack{l-r+1 \leq i \leq l \\ w^{-1}(\alpha^i) > 0}} \delta_{\nu,\alpha^i}. \tag{5.23}$$

The inequality in (5.23) follows from Lemma 1.2.1(ii), Lemma 1.2.4(iii), and the assumption that $w^{-1}(\vartheta) < 0$. In particular, if $w^{-1}(\alpha^{l-i+1}) > 0$ for $1 \leq i \leq p$, then necessarily $w^{-1}(\alpha^i) < 0$. Putting (5.22) and (5.23) together, one has

$$\mathrm{ord}_w^\nu(\varkappa^\delta(f_{\alpha^r} \cdots f_{\alpha^1} f_{-\alpha^1} \cdots f_{-\alpha^{l-r}})) \geq \sum_{\substack{1 \leq i \leq l \\ w^{-1}(\alpha^i) > 0}} \delta_{\nu,\alpha^i}.$$

Finally, to get (5.20), we observe that

$$\rho_\nu^\vee + w(\rho_\nu^\vee) = \sum_{\substack{\alpha > 0,\, \nu_\alpha = \nu \\ w^{-1}(\alpha) > 0}} \alpha^\vee \tag{5.24}$$

and consequently $(\vartheta, \rho_\nu^\vee + w(\rho_\nu^\vee))$ is exactly the number of $\widetilde{\alpha} \in \lambda_\nu(\vartheta)$ with $w^{-1}(\alpha) > 0$. Note that since $w^{-1}(\vartheta) < 0$, such $\widetilde{\alpha}$ must belong to $\lambda_\nu(s_\vartheta) \setminus \{\vartheta\}$.

This completes the proof of (5.20) and hence the proof of (2) as well.

Remark. The relation $T_i^{-1} Y_b T_i^{-1} = Y_{s_i(b)}$ from (1.34), which was used at the beginning of (2), is valid only when $(b, \alpha_i^\vee) = 1$. In particular, it does not hold for $b = \alpha_m$ and $i = m$. In this case,

$$T_m^{-1} Y_{\alpha_m} T_m^{-1} = Y_{\alpha_m}^{-1} + (t_m^{1/2} - t_m^{-1/2}) T_m^{-1}. \tag{5.25}$$

One cannot use (5.25) to pass from Y_{α_m} to $Y_{\alpha_m}^{-1}$ in a way that is compatible with the limit $t_\nu \to 0$. Nevertheless, we can reach $Y_{\alpha_m}^{-1}$, along with all the operators corresponding to negative short roots, by starting from Y_ϑ^{-1}. This is carried out in (4) below.

Before (3) and (4), let us make some general remarks about Y_{-b} for arbitrary $b \in P_+$. Write $b = \pi_r s_{j_l} \cdots s_{j_1}$ $(l = l(b))$ and construct $\lambda(b) = \{\widetilde{\alpha}^1, \cdots, \widetilde{\alpha}^l\}$ using this reduced expression. Then $(-b) = \pi_r^{-1} s_{\pi_r(j_1)} \cdots s_{\pi_r(j_l)}$, which is a reduced expression.

For $1 \le p \le l$, let

$$\widetilde{\beta}^p = -b(\widetilde{\alpha}^{l-p+1}) = s_{\pi_r(j_l)} \cdots s_{\pi_r(j_{l-p+2})}(\alpha_{\pi_r(j_{l-p+1})}),$$

so that $\lambda(-b) = \{\widetilde{\beta}^1, \cdots, \widetilde{\beta}^l\}$. We can write

$$Y_{-b} = q^{-(b,\rho_k)} \Gamma_b \, \ddot{G}^-_{\widetilde{\beta}^l} \cdots \ddot{G}^-_{\widetilde{\beta}^1} = \ddot{G}^-_{-\widetilde{\alpha}^1} \cdots \ddot{G}^-_{-\widetilde{\alpha}^l} \, q^{-(b,\rho_k)} \, \Gamma_b.$$

Hence

$$\varkappa^\delta(Y_{-b}) = \varkappa^\delta(\ddot{G}^-_{-\widetilde{\alpha}^1} \cdots \ddot{G}^-_{-\widetilde{\alpha}^l}) \sum_{w \in W} q^{-(b,\, \rho_k + w(\rho_k))} \, \Gamma_{w^{-1}(b)} \, \zeta_w.$$

We claim that

$$\xi^\delta(Y_{-b}) := \varkappa^\delta(f_{-\widetilde{\alpha}^1} \cdots f_{-\widetilde{\alpha}^l}) \sum_{w \in W} q^{-(b,\, \rho_k + w(\rho_k))} \, \Gamma_{w^{-1}(b)} \, \zeta_w.$$

is regular at $t_\nu = 0$. The proof is similar to that for $\xi^\delta(Y_b)$ from (5.17) that was given before step (1). One uses (5.24) instead of (5.18).

(3) In the case of $b = \omega_r$ $(r \in O')$, the regularity of $\varkappa^\delta(Y_{-b})$ is immediate from that of $\xi^\delta(Y_{-b})$, due to Proposition 5.1.1(ii).

(4) For b equal to any negative short root $-\alpha$ $(\alpha \in R_+)$, the proof is similar to (2). Use Lemma 5.2.2 to choose a reduced expression $s_\vartheta = s_{j_1} \cdots s_{j_p} s_m s_{j_p} \cdots s_{j_1}$ such that

$$s_{j_r} \cdots s_{j_1}(-\vartheta) = -\alpha$$

for some $0 \le r \le p$. Then, starting from $Y_\vartheta^{-1} = T_{s_\vartheta}^{-1} T_0^{-1}$, we use (5.19) and (1.34) to get

$$\begin{aligned} &Y^{-1}_{s_{j_1}(\vartheta)} = T_{j_1} Y_\vartheta^{-1} T_{j_1}, \quad Y^{-1}_{s_{j_2} s_{j_1}(\vartheta)} = T_{j_2} T_{j_1} Y_\vartheta^{-1} T_{j_1} T_{j_2}, \quad \ldots \quad , \\ &Y_\alpha^{-1} = T_{j_r} \cdots T_{j_1} Y_\vartheta^{-1} T_{j_1} \cdots T_{j_r}. \end{aligned}$$

Then the regularity of $\varkappa^\delta(Y_{-\alpha})$ can be shown using Proposition 5.1.1(ii, iv) as in (2).

The proof of Theorem 5.2.1 is now complete. $\square$

5.3. Remaining operators

We consider the remaining operators from Theorem 4.4.1(i).

Proposition 5.3.1. *(i) The operator* $\widehat{T}_i = RE^\delta(\ddot{T}_i^\varphi)$ *exists for* $i = 0, \ldots, n$. *Moreover,*

$$\widehat{T}_i = RE^\delta(\ddot{T}_i) = \sum_{\substack{w \in W \text{ s.t.} \\ w^{-1}(\alpha_i) < 0}} \zeta_w\,(s_i - 1) \quad \textit{for} \quad i > 0. \tag{5.26}$$

(ii) For any $b \in P$,

$$\widehat{X}_b = RE^\delta(\widetilde{X}_b) = \sum_{\substack{w \in W \text{ s.t.} \\ w^{-1}(b) = b_+}} X_{b_+}\,\zeta_w. \tag{5.27}$$

(iii) For any $r \in O'$,

$$\widehat{\pi}_r^{-1} = RE^\delta(\varphi(\pi_r^{-1})) = \sum_{w\in W}\Big(X_{w^{-1}(\omega_r)} \prod_{\substack{\alpha\in\lambda(u_r)\ \text{s.t.}\\ (w^{-1}(\alpha),\rho^\vee)=1}} (1 - X^{-1}_{w^{-1}(\alpha)})\,\zeta_w\Big)\, u_r^{-1}$$

$$+ \sum_{v<u_r^{-1}} \big(\sum_{w\in W} f_{v,w}\,\zeta_w\big)\, v \quad \textit{for certain } f_{v,w} \in \mathbb{Q}_q[X]. \tag{5.28}$$

Proof. (i) Using $\varkappa^\delta(s_i) = s_i$ and (4.7), we readily arrive at (5.26) for $i > 0$. The case of $i = 0$ is significantly more involved. We have $\varphi(\ddot{T}_0) = t_0^{1/2} T_{s_\vartheta}^{-1} X_\vartheta^{-1}$. Write $s_\vartheta = s_{j_l}\cdots s_{j_1} = s_{j_1}\cdots s_{j_l}$ $(l = l(s_\vartheta))$. Let $\alpha^p = s_{j_1}\cdots s_{j_{p-1}}(\alpha_{j_p}) \in \lambda(s_\vartheta)$ for $p = 1,\ldots,l$.

Now

$$\ddot{T}_0^\varphi = t_0^{1/2}\prod_\nu t_\nu^{-l_\nu(s_\vartheta)/2}\,\ddot{G}^-_{-\alpha^1}\cdots\ddot{G}^-_{-\alpha^l}\, s_\vartheta\, X_\vartheta^{-1}. \tag{5.29}$$

By Lemma 1.2.4(i), one has $l_\nu(s_\vartheta) = 2(\vartheta,\rho_\nu^\vee) - \delta_{\nu,\vartheta}$. Hence

$$t_0^{1/2}\prod_\nu t_\nu^{-l_\nu(s_\vartheta)/2} = \prod_\nu t_\nu^{-(\vartheta,\rho_\nu^\vee)+\delta_{\nu,\vartheta}}.$$

Returning to (5.29), we have

$$\varkappa^\delta(\ddot{T}_0^\varphi) = \varkappa^\delta(\ddot{G}^-_{-\alpha^1}\cdots\ddot{G}^-_{-\alpha^l})\sum_{w\in W} t_{\text{sht}}\, q^{-(\vartheta,\,\rho_k + w(\rho_k))}\, X_{w^{-1}(\vartheta)}\,\zeta_w\, s_\vartheta. \tag{5.30}$$

By Proposition 5.1.1(iv),

$$^{\checkmark}\text{ord}_w^\nu(\varkappa^\delta(\ddot{G}^-_{-\alpha^1}\cdots\ddot{G}^-_{-\alpha^l})) \geq \text{ord}_w^\nu(\varkappa^\delta(f_{-\alpha^1}\cdots f_{-\alpha^l})) = \sum_{\alpha\in\lambda(s_\vartheta)\cap(R_+\setminus\lambda(w^{-1}))}\delta_{\alpha,\nu}.$$

The claim now follows from (5.24), the description of the sets $\lambda_\nu(\vartheta)$ from (1.21), and $\vartheta = s_0 s_\vartheta$. The t_{sht} factor in (5.30) accounts for the case when $w^{-1}(\vartheta) > 0$, because $\lambda(s_\vartheta) = \lambda(\vartheta)\setminus\{[\vartheta,1]\}$.

(ii) By definition, $\widetilde{X}_b = q^{(b_+,\,\rho_k)} X_b$; hence

$$\varkappa^\delta(\widetilde{X}_b) = \sum_{w\in W} q^{(b_+ - w^{-1}(b),\,\rho_k)} X_{w^{-1}(b)}\,\zeta_w.$$

Now (5.27) follows due to the fact that $b_+ \geq w^{-1}(b)$ for all $w \in W$.

(iii) Recall that $\varphi(\pi_r^{-1}) = X_{\omega_r} T_{u_r^{-1}}$. Let $u_r = s_{j_l} \cdots s_{j_1}$ be a reduced decomposition. Construct $\lambda(u_r) = \{\alpha^1, \ldots, \alpha^l\}$ using this decomposition. Then

$$\varphi(\pi_r^{-1}) = q^{-(\omega_r,\, \rho_k)} X_{\omega_r} \ddot{G}^+_{-\alpha^1} \cdots \ddot{G}^+_{-\alpha^l}\, u_r^{-1}.$$

We have used here that $l_\nu(u_r) = l_\nu(\omega_r) = 2(\omega_r, \rho_\nu^\vee)$. Hence

$$\varkappa^\delta(\varphi(\pi_r^{-1})) = \big(\sum_{w \in W} q^{-(\omega_r,\, \rho_k + w(\rho_k))}\, X_{w^{-1}(\omega_r)}\, \zeta_w\big)\, \varkappa^\delta(\ddot{G}^+_{-\alpha^1} \cdots \ddot{G}^+_{-\alpha^l})\, u_r^{-1}.$$

Now, by (5.24) and Proposition 5.1.1(ii), the limit $RE^\delta(\varphi(\pi_r^{-1}))$ exists. Then (5.28) follows readily. $\square$

5.4. Symmetrization

Recall the definition of RE from (3.15) and the operators $\mathcal{L}_f$ and L_f from (1.48).

PROPOSITION 5.4.1. *For any $f \in \ddot{\mathbb{Q}}_{q,t}[X]^W$, one has*

$$RE^\delta(\mathcal{L}_f) = RE(L_f), \tag{5.31}$$

upon the restriction to $\mathcal{F}^\delta(W, \overline{\mathcal{V}})$, the space of $W-$invariants of $\mathcal{F}(W, \overline{\mathcal{V}})$.

PROOF. By Proposition 1.3.2, $RE^\delta(\mathcal{L}_f)$ commutes with $RE^\delta(\ddot{T}_i)$ from (5.26) for all $i = 1, \ldots, n$. Now we have the following:

LEMMA 5.4.2. *An element $g \in \mathcal{F}(W, \overline{\mathcal{V}})$ belongs to $\mathcal{F}^\delta(W, \overline{\mathcal{V}})$ if and only if $\widehat{T}_i(g) = 0$ for all $i = 1, \ldots, n$.*

PROOF. Applying (5.26) to $g = \sum_{w \in W} g_w\, \zeta_w$ gives

$$\widehat{T}_i(g) = \sum_{\substack{w \in W \text{ s.t} \\ w^{-1}(\alpha_i) < 0}} (g_{s_i w} - g_w)\, \zeta_w.$$

This vanishes provided $g_{s_i w} = g_w$ whenever $w^{-1}(\alpha_i) < 0$; but the latter condition is always met by exactly one of $w, s_i w$. $\square$

It follows that $RE^{\delta}(\mathcal{L}_f)$ preserves $\mathcal{F}^{\delta}(W,\overline{\mathcal{V}})$. Therefore, upon the restriction to $\mathcal{F}^{\delta}(W,\overline{\mathcal{V}})$, this operator has the form $\sum_{w\in W} L\,\zeta_w$ for some fixed difference operator L. By considering the id–component of $RE^{\delta}(\mathcal{L}_f)$ one sees that $L = RE(L_f)$. □

5.5. Examples

For the root system A_1, one has

$$\widehat{Y}_\omega = \Gamma^{\varrho}_{-\omega}\Big((\zeta_{\mathrm{id}} + (1-X_\alpha^{-1})\zeta_s) + (-\zeta_{\mathrm{id}} + X_\alpha^{-1}\zeta_s)\, s\Big)$$
$$\widehat{Y}_\omega^{-1} = \widehat{Y}_{-\omega} = \Big((1-X_\alpha^{-1})\zeta_{\mathrm{id}} + \zeta_s\Big)\Gamma^{\varrho}_{\omega} + \Big(\zeta_{\mathrm{id}} - X_\alpha^{-1}\zeta_s\Big)\Gamma^{\varrho}_{-\omega}\, s$$

in terms of the fundamental weight ω and corresponding simple root α and simple reflection s. See (4.5) for the definition of Γ^{ϱ}_b. Upon the restriction to $\mathcal{F}^{\delta}(W,\overline{\mathcal{V}})$,

$$\widehat{Y}_\omega + \widehat{Y}_\omega^{-1} = \Big((1-X_\alpha^{-1})\Gamma_\omega + \Gamma_{-\omega}\Big),$$

which is a special case of Proposition 5.4.1. See [**15**] and [**16, 17**] for a complete treatment of the rank-one case, including a construction of the Toda-Dunkl operators in terms of (sub-)induced nil-DAHA module in [**17**].

For the root system A_2, one has

$$\begin{aligned}
\widehat{Y}_{\omega_1} = \Gamma^{\varrho}_{-\omega_1}\Big(&\\
&(\zeta_{\mathrm{id}} + (1-X_{\alpha_1}^{-1})\zeta_{s_1} + \zeta_{s_2} + (1-X_{\alpha_2}^{-1})\zeta_{s_1s_2} + (1-X_{\alpha_1}^{-1})\zeta_{s_2s_1} + (1-X_{\alpha_2}^{-1})\zeta_{s_1s_2s_1})\,\mathrm{id}\\
&+(-\zeta_{\mathrm{id}} + X_{\alpha_1}^{-1}\zeta_{s_1} - \zeta_{s_2} - (1-X_{\alpha_1}^{-1})\zeta_{s_2s_1} + X_{\alpha_2}^{-1}\zeta_{s_1s_2s_1})\, s_1\\
&+(\zeta_{s_2} + X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_1s_2} - X_{\alpha_1}^{-1}\zeta_{s_2s_1} - X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_1s_2s_1})\, s_1s_2\\
&+(-(\zeta_{s_1}+\zeta_{s_2}) + (1-X_{\alpha_1}^{-1})X_{\alpha_2}^{-1}\zeta_{s_1s_2} + X_{\alpha_1}^{-1}\zeta_{s_2s_1} + X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_1s_2s_1})\, s_1s_2s_1\Big).
\end{aligned}$$

The operator $\widehat{Y}_{\omega_2}$ is obtained by interchanging the indices 1 and 2 of ω_i, s_i, and α_i in the above formula. The operators $\widehat{Y}_{\omega_i}$ are invertible (their inverses are $\widehat{Y}_{-\omega_i}$), and one has the following special case of Proposition 5.4.1:

$$\widehat{Y}_{\omega_1}^{-1} + \widehat{Y}_{\omega_1}\widehat{Y}_{\omega_2}^{-1} + \widehat{Y}_{\omega_2} = (1 - X_{\alpha_1}^{-1})\Gamma_{\omega_1} + (1 - X_{\alpha_2}^{-1})\Gamma_{-\omega_1+\omega_2} + \Gamma_{-\omega_2} = \mathit{RE}(L_{\omega_1}),$$

upon the restriction to $\mathcal{F}^{\delta}(W, \overline{\mathcal{V}})$; cf. (0.1).

For the root system B_2, with α_1 long and α_2 short, the fundamental weight ω_2 is minuscule, while $\omega_1 = \vartheta$ is not. One has

$$\begin{aligned}
\widehat{Y}_{\omega_2} = \Gamma^{\varrho}_{-\omega_2}\Big(&(\zeta_{\mathrm{id}} + \zeta_{s_1} + (1 - X_{\alpha_1}^{-1})(\zeta_{s_2s_1} + \zeta_{s_1s_2s_1}) \\
&\qquad + (1 - X_{\alpha_2}^{-1})(\zeta_{s_2} + \zeta_{s_1s_2} + \zeta_{s_2s_1s_2} + \zeta_{s_1s_2s_1s_2}))\,\mathrm{id} \\
&+ (-(\zeta_{\mathrm{id}} + \zeta_{s_1}) + X_{\alpha_2}^{-1}(\zeta_{s_2} + \zeta_{s_1s_2s_1s_2}) - (1 - X_{\alpha_2}^{-1})\zeta_{s_1s_2} \\
&\qquad - (1 - X_{\alpha_1}^{-1})\zeta_{s_1s_2s_1})\, s_2 \\
&+ (\zeta_{s_1} + (1 - X_{\alpha_2}^{-1})\zeta_{s_1s_2} + X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_2s_1} - X_{\alpha_1}^{-1}\zeta_{s_1s_2s_1} \\
&\qquad + X_{\alpha_1+\alpha_2}^{-1}(1 - X_{\alpha_2}^{-1})\zeta_{s_2s_1s_2} - X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_1s_2s_1s_2})\, s_2s_1 \\
&+ (-(\zeta_{s_1} + \zeta_{s_2}) - (1 - X_{\alpha_2}^{-1})\zeta_{s_1s_2} + X_{\alpha_1}^{-1}(1 - X_{\alpha_2}^{-1})\zeta_{s_2s_1} \\
&\qquad + X_{\alpha_1+2\alpha_2}^{-1}\zeta_{s_2s_1s_2} + X_{\alpha_1}^{-1}\zeta_{s_1s_2s_1})\, s_2s_1s_2 \\
&+ (-(\zeta_{s_1} + \zeta_{s_2s_1}) + X_{\alpha_2}^{-1}\zeta_{s_1s_2} + X_{\alpha_2}^{-1}(1 - X_{\alpha_1}^{-1})\zeta_{s_2s_1s_2} \\
&\qquad + X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_1s_2s_1} - X_{\alpha_1+\alpha_2}^{-1}(1 - X_{\alpha_2}^{-1})\zeta_{s_1s_2s_1s_2})\, s_1s_2s_1 \\
&+ (\zeta_{s_1} - X_{\alpha_2}^{-1}\zeta_{s_1s_2} - X_{\alpha_1+\alpha_2}^{-1}\zeta_{s_1s_2s_1} + X_{\alpha_1+2\alpha_2}^{-1}\zeta_{s_1s_2s_1s_2})\, s_1s_2s_1s_2\Big).
\end{aligned}$$

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