The Lagrangian Averaged Navier-Stokes equations with rough initial data

Nathan Pennington

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2010

Approved by

Advisor: Professor Michael Taylor
Reader: Professor Patrick Eberlein
Reader: Professor Joseph Cima
Reader: Professor Jason Metcalfe
Reader: Professor Mark Williams

ABSTRACT

NATHAN PENNINGTON: The Lagrangian Averaged Navier-Stokes equations with rough initial data

(Under the direction of Professor Michael Taylor)

Turbulent fluid flow is governed by the Navier-Stokes equations, given in their incompressible formulation as

$$\partial_t u + (u \cdot \nabla)u - \nu \triangle u = -\nabla p, \tag{0.0.1}$$

where the incompressibility condition requires div $u = 0$, ν is a constant greater than zero due to the viscosity of the fluid and u is the velocity field of the fluid.

Because of the difficulty of working with the Navier-Stokes equations, several different approximations of the Navier-Stokes equations have been developed. One recently derived approxmation is the Lagrangian Averaged Navier-Stokes equations, which are given in their incompressible, isotropic form as

$$\partial_t u + (u \cdot \nabla)u + \text{div } \tau^\alpha(u) - \nu \triangle u = -(1 - \alpha^2 \triangle)^{-1} \nabla p. \tag{0.0.2}$$

This thesis will focus on three main areas. First, we seek local solutions to the Lagrangian Averaged Navier-Stokes equations with initial data in Sobolev space $H^{r,p}(\mathbb{R}^n)$ with the goal of minimizing r. We generate these results by following the program of [**6**]

for the Navier-Stokes equations. Following results of [**8**], we are able to turn the local solution into a global solution for the $n = 3$, $p = 2$ case.

Secondly, we seek solutions to the Lagrangian Averaged Navier-Stokes equations for initial data in Besov space $B^r_{p,q}$, again following the broad outline of [**6**]. Finally, we get a global result for Besov spaces in the $p = 2$ case and a qualitatively different local result for general p by modifying the results in [**18**] for the homogeneous generalized Navier-Stokes equations.

ACKNOWLEDGMENTS

First, I would like to thank my family and friends for their help making it through this grand adventure. Specifically, I would like to thank my dad for his preparing me for the dissertation process. I'd also like to thank the A Team, Team Book Club, Team Four Corners, and Team Big 12 for succeeding in the difficult task of keeping me sane for the last several years.

Secondly, I would like to thank my advisor, Dr. Taylor, for teaching me more mathematics than I would have believed possible. I would also like to thank my committee, Dr. Cima, Dr. Eberlein, Dr. Metcalfe, Dr. Williams, and Dr. Wogen, for all their constructive suggestions and advice, and thank the entire math department for their support during my graduate school tenure.

CONTENTS

CHAPTER 1

Introduction

1.1. Introduction

The incompressible Navier-Stokes equations govern the motion of incompressible fluids and are given by

$$\partial_t u + (u \cdot \nabla)u - \nu A u = 0, \tag{1.1.1}$$

where A is (essentially) the Laplacian, ν is a constant greater than zero due to the viscosity of the fluid, and u is a velocity field, which means $u(t, x)$ is the velocity of the particle of fluid located at position x a time of t units after the fluid is put in motion. These equations are derived from the Euler Equations, and setting $\nu = 0$ would recover the Euler Equations.

The Navier-Stokes equations govern the behavior of many physical phenomena, including ocean currents, the weather and water flowing through a pipe. The question of global existence for the Navier-Stokes equations is one of the most significant remaining open problems in mathematics, evidenced by its naming by the Clay Mathematics Institute in 2000 to be one of the seven Millennium Prize Problems.

Because of the intractability of the Navier-Stokes equations, several different equations that approximate the Navier-Stokes equations have been studied. A recently derived approximating equation is the Lagrangian Averaged Navier-Stokes equations (LANS). The LANS equations come from the Lagrangian Averaged Euler (LAE) equations in the same way that the Navier-Stokes equations come from the Euler equations. Like the Euler equations, the LAE equations are the geodesics of a specific functional. For the Euler equation, this is the Energy functional. For the LAE equations, the functional is derived via an averaging process, with the averaging occurring at the level of the initial data. For an exhaustive treatment of this process, see [**14**], [**15**], [**12**] and [**9**]. For the convenience of the reader, we briefly summarize a special case of the derivation in Section 2.1. In [**13**] and [**21**], the authors describe the physical implications of the averaging process that generates the LANS equations and discuss the numerical improvements use of the LANS equation provides over more common approximation techniques of the Navier-Stokes equations.

Like the Navier-Stokes equations, the LANS equations have both a compressible and an incompressible formulation. The compressible LANS equations are derived and studied in [**10**]. The incompressible LANS equations exist most generally in the anisotropic form, and are derived and studied in [**9**]. We will consider a special case of these anisotropic equations called the isotropic incompressible LANS equations. One form of the incompressible, isotropic LANS equations on a region without boundary is

$$\begin{aligned} &\partial_t u + (u \cdot \nabla)u + \text{div } \tau^\alpha u = -(1-\alpha^2\triangle)^{-1}\text{grad } p + \nu\triangle u \\ &\text{div } u = 0, \ u(t,x)|_{t=0} = \varphi(x), \end{aligned} \tag{1.1.2}$$

where $\alpha > 0$ and φ is the initial data. The Reynolds stress τ^α is given by

$$\tau^\alpha u = \alpha^2(1-\alpha^2\triangle)^{-1}[Def(u)\cdot Rot(u)] \tag{1.1.3}$$

where $Rot(u) = (\nabla u - \nabla u^T)/2$ is the antisymmetric part of the velocity gradiant and $Def(u) = (\nabla u + \nabla u^T)/2$. Lastly, $(1-\alpha^2\triangle)$ is the Helmholtz operator.

Setting $\nu = 1$, we write (1.1.2) as

$$\begin{aligned} &\partial_t u - Au + P^\alpha(\text{div } \cdot (u\otimes u) + \text{div } \tau^\alpha u) = 0, \\ &x \in \mathbb{R}^n, \ n \geq 2, \ t \geq 0, \ u(0) = \varphi = P^\alpha \varphi \end{aligned} \tag{1.1.4}$$

where $u = u(t) : \mathbb{R}^n \to \mathbb{R}^n$, $A = P^\alpha \triangle$, $u \otimes u$ is the tensor with jk-components $u_j u_k$ and $\text{div} \cdot (u \otimes u)$ is the vector with j-component $\sum_k \partial_k (u_j u_k)$. P^α is the Stokes Projector defined as

$$P^\alpha(w) = w - (1 - \alpha^2 \triangle)^{-1} \text{grad } f \tag{1.1.5}$$

where f is a solution of the Stokes problem: Given w, there is a unique v and a unique (up to additive constants) function f such that

$$(1 - \alpha^2 \triangle) v + \text{grad } f = (1 - \alpha^2 \triangle) w \tag{1.1.6}$$

with div $v = 0$. For a more explicit treatment of the Stokes Projector, see Theorem 4 of [**15**].

The averaging process has a smoothing effect on the resulting PDE. In [**8**], this smoothing is exploited to show the existence of a global solution to (1.1.2) for initial data of any size in $H^{3,2}(\mathbb{R}^3)$. This is in stark contrast to the case for the Navier-Stokes equations, where discovery of such a global existence result is one of the great remaining open problems in mathematics.

Our work here has two main goals. First, we seek local solutions to the LANS equations outside the L^2 setting, specifically by assuming our initial data is in Sobolev space $H^{r,p}$ or in Besov space $B^r_{p,q}$. Secondly, we seek to minimize the assumed regularity of the initial data. We will begin by mirroring the approach used for the Navier-Stokes equations in [**6**], which will give short time solutions in the class of weighted continuous functions in time and the class of integral norms in time. Later we will follow the work of [**18**] and get similar results, but only for Besov spaces.

The paper is organized as follows. In Chapter 2 we briefly describe the derivation of (1.1.2), including the construction of the inner product whose geodesics satisfy (1.1.2). Chapter 3 considers the case of initial data in Sobolev spaces $H^{r,p}(\mathbb{R}^n)$ and Chapter 4 considers initial data Besov spaces $B^r_{p,q}$. In Chapter 5 we give a proof extending local results in Besov space $B^r_{2,q}$ to global results, and in Chapter 6 we get additional Besov space results that are qualitatively different from those obtained in Chapter 4.

We conclude this introduction with special cases of the theorems proven throughout the paper. Our first two sets of results are analogous to those proven in [**6**]. Our first result is a special case of Theorem 3.2.1, and comes from setting $b' = 1$ in (3.4.18).

THEOREM 1.1.1. *For any $\varphi = P^\alpha\varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.1.2) *such that*

$$u \in \overline{C}_{r,p} \cap \dot{C}_{a;k,c} \tag{1.1.7}$$

provided the parameters (with $r = n/p + b$) satisfy (3.4.19)*. If $\|\varphi\|_{r,p}$ is sufficiently small, $T = \infty$.*

The function spaces $\overline{C}$ and $\cot C$ are defined in Section 3.2. We record two additional special cases that illustrate our "best" result.

THEOREM 1.1.2. *Let $r = n/p$, $n < p$, and $n \geq 2$. Then for any $\varphi = P^\alpha\varphi \in H^{r,p}(\mathbb{R}^n)$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.1.2) *such that*

$$u \in \overline{C}_{r,p} \cap \dot{C}_{(1-n/p)/2;1,p}. \tag{1.1.8}$$

This case emphasizes that we can achieve a local existence result for regularity arbitrarily close to zero if we allow sufficiently large p. We contrast this with the result from

[**6**], which gives local existence for the standard Navier-Stokes equations with initial data in $H^{n/p-1,p}(\mathbb{R}^n)$.

We also record the result in the special case $n = 3$ and $p = 2$, which requires a different choice of parameters.

THEOREM 1.1.3. *For any* $\varphi = P^\alpha\varphi \in H^{3/2,2}(\mathbb{R}^3)$ *there is a* $T = T(\varphi) > 0$ *and a unique solution to* (1.1.2) *such that*

$$u \in \overline{C}_{3/2,2} \cap \dot{C}_{1/4;2,2}. \tag{1.1.9}$$

In Theorem 3.6.1, we extend this special case to a global existence result. For the details, see section 3.6. We compare this with the result in [**8**], which holds for initial data in $H^{3,2}$.

Our next series of results generate solutions to (1.1.2) in a slightly different functional setting. Like our previous results, we begin with a special case of the main Theorem (Theorem 3.7.1) obtained by setting $b' = 1$ in (3.9.4).

THEOREM 1.1.4. *For any* $\varphi = P^\alpha\varphi \in H^{r,p}$ *there is a* $T = T(\varphi) > 0$ *and a unique solution to* (1.1.2) *such that*

$$u \in BC([0,T) : H^{r,p}) \cap L^a((0,T) : H^{k,c}) \tag{1.1.10}$$

provided the parameters (with $r = n/p + b$*) satisfy* (3.9.5)*. If* $\|\varphi\|_{r,p}$ *is sufficiently small, then* $T = \infty$*. Lastly, we have that solutions depend continuously on the initial data.*

We also state a further special case.

THEOREM 1.1.5. *Let* $r = n/p$ *and assume* $p > 5/4$ *and* $2 \leq n \leq 5/4 + p$*. Then for any* $\varphi = P^\alpha \varphi \in H^{r,p}(\mathbb{R}^n)$ *there is a* $T = T(\varphi) > 0$ *and a unique solution to* (1.1.2) *such that*

$$u \in BC([0,T) : H^{r,p}) \cap L^a((0,T) : H^{k,n}) \tag{1.1.11}$$

where $k = 5p/4 + 1$ *and* $a = 8p/5$*.*

We note that the case $n = 3$ and $p = 2$ satisfies these conditions, and that Theorem 3.6.1 extends the result in this case to a global solution.

Our next set of results are similar to the first two sets in that they are analogous to the results in [**6**]. This time, in addition to changing the equation under consideration from the Navier-Stokes equations to the LANS equations, we also change our initial data from Sobolev spaces to Besov spaces. From Section 4.1 we have a special case of Theorem 4.3.1:

THEOREM 1.1.6. *With the parameters satisfying* (4.4.4)*, for any* $\varphi = P^\alpha \varphi \in B^r_{p,q}$ *there is a* $T = T(\varphi) > 0$ *and a unique solution to* (1.1.2) *such that*

$$u \in \bar{C}_{r,p,q} \cap \dot{C}_{a;k,b,c}. \tag{1.1.12}$$

and Theorem 4.5.1:

THEOREM 1.1.7. *Provided the parameters satisfy* (4.7.4)*, given* $u_0 \in B^r_{p,q}$ *with* $r = n/p + b$ *there exists a* $T > 0$ *and a unique solution* u *to* (1.1.2) *such that*

$$u \in BC([0,T) : B^r_{p,q}) \cap L^\sigma((0,T) : B^s_{\tilde{p},q}). \tag{1.1.13}$$

In Section 5.1 we prove that local solutions in certain Besov spaces are actually global solutions. We state Theorem 5.2.1:

THEOREM 1.1.8. *Let u be a solution to* (1.1.2) *with initial data $u_0 \in \tilde{B}^r_{2,q}$ where $r > 2$ such that*

$$u \in BC([0,T) : B^r_{2,q}) \cap Y, \tag{1.1.14}$$

where Y is either $L^\sigma(B^{1+n/2}_{2,q})$ or $C_{a;1+n/2,2,q}$ with $0 \leq a < 1$ and $1 \leq \sigma$. Then the local solution is a global solution.

Alternatively, with X either $L^\sigma(B^{1+n/2+\varepsilon}_{2,q})$ or $C_{a;1+n/2+\varepsilon,2,q}$ and $n - r - \varepsilon < 0$, we get that the local solution is a global solution with no restriction on r.

In Section 5.2, we prove another local existence result with initial data in Besov space using an alternate construction. We record here the Theorem to be proven:

THEOREM 1.1.9. *Let $4 < q \leq \infty$. Let $2 \leq p < \infty$. Let $r = n/p + 2/q$, and let $u_0 \in B^r_{p,q}$. Also assume $2 < r + 2/q$. Then there exists a $T = T(u_0) > 0$ and a unique solution u of* (1.1.2) *such that*

$$u \in X \cap Z \tag{1.1.15}$$

where

$$X = C([0,T) : B^r_{p,q}), \;\; Z = L^q\left((0,T) : B^{r+2/q}_{p,q}\right). \tag{1.1.16}$$

We also note that for $u_0, v_0 \in B^r_{p,q}$, the corresponding solutions $u(t), v(t)$ will satisfy $\|u - v\|_{X \cap Z} \leq C\|u_0 - v_0\|_{B^r_{p,q}}$.

CHAPTER 2

Lagrangian Averaging

2.1. Derivation of the Lagrangian Averaged Navier-Stokes equations

For the convenience of the reader, we recall from [**8**], [**9**], [**14**], and references therein the derivation of (1.1.2). Section 2.2 gives an analytic derivation the bilinear form that defines our averaging. Section 2.3 provides the underlying geometry necessary to recast the bilinear form as a Riemannian metric. In Section 2.4, the Lagrangian Averaged Euler Equations are derived as geodesics of this Riemannian metric.

2.2. Lagrangian Averaging

In this section, we follow [**8**] and [**9**] in describing the Lagrangian Averaging procedure. We begin with a bounded region M in $\mathbb{R}^3$ with boundary ∂M and let, for $s > 5/2$, D^s_μ denote group of volume preserving diffeomorphisms of M with H^s regularity. See section 2.3 for a more thorough description of this group. We let

$$X^s = \{u \in H^s(M) | \text{div } u = 0, u \cdot n = 0 \text{ on } \partial M\} \tag{2.2.1}$$

We let S denote the unit sphere in X^s, and for any $u_0 \in X^s$, we let $u(t, x)$ denote the corresponding solution to the Euler equations with initial velocity u_0. We define

$$u_0^\varepsilon = u_0 + \varepsilon w \tag{2.2.2}$$

where $w \in S$ and $\varepsilon \in [0, \alpha]$ where α is a small positive number. We let $u^\epsilon(t, x)$ denote the solution to the Euler equations with initial velocity u_0^ε. We remark that u^ε also depends on w, but we suppress this in the notation.

Now let $\eta(t, x)$ be the Lagrangian flow of $u(t, x)$, which means for each t, $\eta(t) = \eta_t :$ $M \to M$, $\eta(0, x) = x$, and η satisfies

$$\partial_t \eta(t, x) = u(t, \eta(t, x)). \tag{2.2.3}$$

We define η^ε similarly, so

$$\partial_t \eta^\varepsilon(t,x) = u^\varepsilon(t, \eta^\varepsilon(t,x)). \tag{2.2.4}$$

We next define $\xi^\varepsilon(t,x)$ to be the function that satisfies

$$\eta^\varepsilon_t = \xi^\varepsilon_t \circ \eta_t, \tag{2.2.5}$$

which means for each t, $\xi^\varepsilon_t : M \to M$. Note that since $\eta^0(t,x) = \eta(t,x)$, we have that $\xi^0(t,x) = x$ for all $t \geq 0$. ξ^ε is called the Lagrangian fluctuation volume-preserving diffeomorphism.

We next define the Eulerian velocity fluctuation about u by

$$u'(t,x) = \frac{d}{d\varepsilon} u^\varepsilon(t,x)\Big|_{\varepsilon=0} \tag{2.2.6}$$

and define the Lagrangian fluctuation by

$$\xi'(t,x) = \frac{d}{d\varepsilon} \xi^\varepsilon(t,x)\Big|_{\varepsilon=0}. \tag{2.2.7}$$

We similarly define

$$u''(t,x) = \frac{d^2}{d^2\varepsilon} u^\varepsilon(t,x)\Big|_{\varepsilon=0} \tag{2.2.8}$$

and

$$\xi''(t,x) = \frac{d^2}{d^2\varepsilon} \xi^\varepsilon(t,x)\Big|_{\varepsilon=0}. \tag{2.2.9}$$

Lastly, we remark that by (2.2.4), we have

$$\frac{d}{d\varepsilon} \eta^\varepsilon(t,x)\Big|_{\varepsilon=0} = \xi'(t, \eta(t,x)). \tag{2.2.10}$$

Differentiating (2.2.5) with respect to t gives

$$u^\varepsilon(t,\eta^\varepsilon(t,x)) = (\partial_t\xi^\varepsilon)(t,\eta(t,x)) + \nabla\xi^\varepsilon(t,\eta(t,x))\cdot u(t,\eta(t,x)) \tag{2.2.11}$$

where ∇ is the space-gradient and we used (2.2.3) and (2.2.4). Differentiating (2.2.11) with respect to ε and evaluating at $\varepsilon = 0$ gives

$$\begin{aligned} &u'(t,\eta(t,x)) + (\nabla u)(t,\eta(t,x))\cdot\xi'(t,\eta(t,x)) \\ =&\partial_t\xi'(t,\eta(t,x)) + \nabla\xi'(t,\eta(t,x))\cdot u(t,\eta(t,x)) \end{aligned} \tag{2.2.12}$$

where we used (2.2.10).

Writing this result in a more compact form, we get

$$u' = \partial_t\xi' + (u\cdot\nabla)\xi' - (\xi'\cdot\nabla)u. \tag{2.2.13}$$

By a similar calculation, we get

$$u'' = \partial_t\xi'' + (u\cdot\nabla)\xi'' - 2(\xi'\cdot\nabla)u' - \nabla\nabla u(\xi',\xi') \tag{2.2.14}$$

where $\nabla\nabla u(\xi',\xi')$ is given in coordinates by

$$\nabla\nabla u(\xi',\xi') = u^i_{,jk}\xi'^j\xi'^k, \tag{2.2.15}$$

where subscripts indicate coordinate derivatives and superscripts indicate component functions.

With this framework, we now define our averaging operators. Following [**9**], we have a probability measure m on the unit sphere S in X^s, and we define

$$< f >= \frac{1}{\alpha}\int_0^\alpha\int_S f(\varepsilon,w)\mu d\varepsilon. \tag{2.2.16}$$

We call this the average of the function f. Next we define the averaged action operator $\bar{S}$ by

$$\bar{S} = < \frac{1}{2}\int_0^T \int_M |\partial_t \eta^\varepsilon|^2 dxdt > . \tag{2.2.17}$$

Before making use of these averaged quantities, we note that by expanding u^ε about $\varepsilon = 0$, we get

$$u^\varepsilon(t,x) = u(t,x) + \varepsilon u'(t,x) + \frac{1}{2}\varepsilon^2 u''(t,x) + O(\varepsilon^3). \tag{2.2.18}$$

To proceed, we make two assumptions. First, we assume

$$\partial_t \xi' + (u\cdot\nabla)\xi' - (\xi'\cdot\nabla)u = 0 \tag{2.2.19}$$

and secondly we assume that

$$\partial_t \xi'' + (u\cdot\nabla)\xi'' = 0. \tag{2.2.20}$$

These assumptions are called the generalized Taylor hypothesis to order $O(\varepsilon^2)$. See equation 4.5 in [**8**] and equation (18) in [**9**] for a more thorough treatment of these assumptions.

Applying the first assumption to (2.2.13) gives $u' = 0$ and applying both assumptions to (2.2.14) gives $u'' = -\nabla\nabla u(\xi', \xi')$. Combining this with the power series expansion (2.2.18) we get

$$u^\varepsilon = u(t,x) - \frac{1}{2}\varepsilon^2 \nabla\nabla u(\xi', \xi'). \tag{2.2.21}$$

Recalling that $\partial_t \eta^\varepsilon = u^\varepsilon$ and using (2.2.21) to evaluate (2.2.17), we get

$$\frac{1}{2\alpha}\int_0^\alpha \int_S \int_0^T \int_M |u(t,x) - \frac{1}{2}\varepsilon^2 \nabla\nabla u(\xi', \xi') + O(\varepsilon^3)|^2 dxdt\mu d\varepsilon. \tag{2.2.22}$$

The integrand can be re-written as

$$\begin{aligned}
&|u(t,x) - \frac{1}{2}\varepsilon^2\nabla\nabla u(\xi',\xi') + O(\varepsilon^3)|^2 \\
&=(u,u) - \varepsilon^2(u,\varepsilon^2\nabla\nabla u) + \frac{\varepsilon^4}{4}(\nabla\nabla u,\nabla\nabla u) + O(\varepsilon^3) \qquad (2.2.23)\\
&=(u,u) - \varepsilon^2(\nabla\nabla u(\xi',\xi'),u) + O(\varepsilon^3)
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the inner product on M. Integrating with respect to ε, we get

$$\int_S\int_0^T\int_M[(u,u) - \alpha^2(\nabla\nabla u(\xi',\xi'),u) + O(\alpha^3)]dxdt\mu. \qquad (2.2.24)$$

Next, we re-write $\nabla\nabla u(\xi',\xi')$ as $\nabla\nabla u : F$, where F is defined by $F = \xi'\otimes\xi'$. Because F has no dependance on ε, we have

$$\int_S F\mu =< F > . \qquad (2.2.25)$$

We finally note that F is the only term in (2.2.24) with dependance on ω, which means (2.2.24) becomes

$$\frac{1}{2}\int_0^T\int_M[(u,u) - \alpha^2(\nabla\nabla u :< F >,u) + O(\alpha^3)]dxdt. \qquad (2.2.26)$$

To derive the isotropic version of the Lagrangian Averaged Euler Equations, we make the assumption that $< F >$ is equal to the identity matrix, and thus

$$\nabla\nabla u(\xi'\xi') :< F >= \triangle u. \qquad (2.2.27)$$

Using (2.2.27) and truncating to $O(\alpha^2)$, $\bar{S}$ becomes

$$\bar{S} = \frac{1}{2}\int_0^T\int_M[(u,u) - \alpha^2(\triangle u,u)]dxdt. \qquad (2.2.28)$$

In subsequent sections we will use this operator to derive the Lagrangian Averaged Euler equations, but first we require some geometry.

2.3. Manifold Structure on Groups of Diffeomorphisms

In this section we outline the construction of a manifold structure on subgroups of the topological group of diffeomorphisms. Unless otherwise indicated, this construction (and additional details) can be found in [**4**]. We begin with a compact Riemannian manifold M and a vector bundle $\pi : E \to M$. For $s \geq 0$ we define $H^s(E)$ to be the set of all sections r such that $r \in H^s(M, E)$, where we recall $r \in H^s(M, E)$ if $r \in H^s(U, E)$ for each coordinate chart U. By the Sobolev imbedding theorem, if $k \geq 0$, n is the dimension of M and $s > n/2 + k$, then $H^s(E) \subset C^k(E)$ which means each element $r \in H^s(E)$ is defined pointwise. Similarly, for $s > n/2 + k$ we define $H^s(M, N)$ to be the space of mappings from M to N that are H^s in each coordinate chart.

Next, we assume N is compact and has no boundary. Then for any $f \in H^s(M, N)$ we define the tangent space at f by

$$T_f H^s(M, N) = \{g \in H^s(M, TN) : \pi \circ g = f\} \tag{2.3.1}$$

where $\pi : TN \to N$ is the projection map from the tangent space of N onto N. Then $TH^s(M, N)$ is defined by

$$TH^s(M, N) = \bigcup_f T_f H^s(M, N). \tag{2.3.2}$$

We note that the map g_f defined by $g_f(x) = 0 \in T_{f(x)}N$ is an element of $T_f H^s(M, N)$.

To give $H^s(M, N)$ a manifold structure, we will construct an exponential map. First, we choose a point $y \in N$. Then we have an exponential map $\exp_y : T_y N \to N$. Because N is compact and has no boundary, this map can be extended to a map $\exp : TN \to N$.

Next, for any $f \in H^s(M,N)$, we define $\exp_f : T_f H^s(M,N) \to H^s(M,N)$ by

$$(2.3.3) \qquad (\exp_f g)(m) = \exp(g(m)),$$

where $m \in M$. interested We note that

$$(2.3.4) \qquad \exp_f g_f(x) = \exp(0_{f(x)}) = f(x)$$

where $0_{f(x)}$ indicates the origin in $T_{f(x)}M$. Thus exp_f provides a chart structure from some neighborhood of g_f onto a neighborhood of f. This gives $H^s(M,N)$ a manifold structure.

Setting $M = N$ and defining C^1 to be the set of C^1 diffeomorphisms of M, we define $D^s = H^s(M,M) \bigcap C^1$. D^s can be shown (see [**4**]) to be a topological group with the group operation being function composition on the right, and this operation is C^∞.

With e defined as the identity diffeomorphism, we have that $X \in T_e D^s$ is equivalent to the condition that $X(m) \in T_m M$ for all $m \in M$, which means $T_e D^s$ the space of all H^s vector fields on M. Since right multiplication (function composition on the right) is smooth, right invariant vector fields exist, and a right Lie Bracket can be defined at e by viewing elements $X \in T_e D^s$ as vector fields on M.

This procedure can be extended to manifolds M that are not compact and do have a boundary, and it can also be shown that D^s has many of the natural properties one would expect of a Lie Group, in particular that vector fields on D^s have flows that are one-parameter subgroups of D^s. These details can be found in [**4**].

2.4. Geometric Derivation of the Lagrangian Averaged Euler Equations

In this section we construct a functional on paths through a particular subgroup of the topological group D^s constructed in the previous section. This product will be similar to (2.2.28), and our ultimate goal will be to derive an equation for the critical points of this functional. We call this equation the Lagrangian Averaged Euler equation.

As in the previous section, we will assume M is a compact Reimannian manifold without boundary with the metric denoted by $g(\cdot,\cdot)$. D^s is the topological group described in the previous section. We follow the arguments used in [**14**] to address this issue in a more general setting. We let D^s_μ (where μ is a volume element on M) denote the space of volume preserving diffeomorphisms of M, and observe that this is a closed subgroup of D^s. The volume preserving assumption gives that $T_eD^s_\mu$ is the space of divergence free vector fields on M, where e is the identity map.

We begin with some notation. For any $X \in T_eD^s_\mu$, we define $\tilde{X}$ to be the 1-form dual to X. Next, we define the operator $\tilde{S}$ by $\tilde{S}(\omega) = (d+\delta)\omega$, where ω is a differential form, d is the exterior derivative, and δ is the L^2 adjoint of d. Then we define S by $S(X) = \tilde{S}(\tilde{X})$ where X is a vector field.

With our notation established, we let $X, Y \in T_eD^s_\mu$ and define a bilinear form on the fiber $T_eD^s_\mu$ by

$$\begin{aligned} < X, Y >_e &= \int_M g(X(x),Y(x)) + g(S(X)(x),S(Y)(x))d\mu \\ &= \int_M g(X(x),Y(x)) + \alpha^2 g((S^*S)(X)(x),Y(x))d\mu. \end{aligned} \tag{2.4.1}$$

We note that $S^* = S$ (where $*$ denotes the formal L^2 adjoint) and we set $S^*S = -\triangle$ where $\triangle$ denotes the Hodge Laplacian viewed as an operator on vector fields instead of on forms (see Chapter 2, section 10, of [**16**]). We re-write (2.4.1) as

$$< X, Y >_e = \int_M g((1 - \alpha^2 \triangle) X(x), Y(x)) d\mu. \tag{2.4.2}$$

Having defined the form on the fiber $T_e D^s_\mu$, we define the form on the fiber $T_\varphi D^s_\mu$ by

$$< X, Y >_\varphi = < X \circ \varphi^{-1}, Y \circ \varphi^{-1} >_e \tag{2.4.3}$$

for any $\phi \in D^s_\mu$. Since $(1 - \alpha^2 \triangle)$ is a self-adjoint positive operator on divergence free L^2 vector fields, this construction defines a right-invariant metric on D^s_μ.

Now that we have constructed a right-invariant metric, our goal is to find geodesics for this metric. For any smooth curve $v : [a, b] \to D^s_\mu$, we define a curve $u : [a, b] \to T_e D^s_\mu$ as follows. For each t,

$$\frac{d}{dt} v(t) = \dot{v}_t : M \to TM \tag{2.4.4}$$

where $\dot{v}_t(x) \in T_{v_t(x)} M$ and $v_t = v(t) \in D^s_\mu$. We then define $u(t) = u_t$ by

$$u_t(x) = \dot{v}_t(v_t^{-1}(x)) \tag{2.4.5}$$

where v_t^{-1} denotes the inverse of the diffeomorphism v_t. We recall that, for each t, u_t is a vector field on M.

With this construction, the Euler-Poincare Reduction Theorem (Theorem 2.5.1) gives that v is a geodesic of (2.4.3) if u is an extreme point of the reduced action functional L defined by

$$L(u) = \frac{1}{2} \int_a^b < u(t), u(t) >_e dt. \tag{2.4.6}$$

To derive a formula for the extreme points of the functional L, we begin by choosing a fixed end-point (f.e.p.) variation f of v. We recall that an f.e.p. variation is a smooth map $f : (-\varepsilon, \varepsilon) \times [a, b] \to M$ with the property that $f(s,t) = v_s(t) \in D^s_\mu$ for each s and $f(0,t) = v(t)$. The fixed end point condition means $f(s,a) = v(a)$ and $f(s,b) = v(b)$ for each s, which in turn implies

$$\frac{d}{ds} f(s,a) = \frac{d}{ds} f(s,b) = 0 \tag{2.4.7}$$

for any s. We define $\frac{d}{ds} f(s,t)|_{s=0} = \delta v(t) \in T_{v(t)} D^s_\mu$ and (2.4.7) gives $\delta v(a) = \delta v(b) = 0$.

Then we define a variation h of u by $h(s,t) = u_s(t) \in T_e D^s_\mu$ where $h(0,t) = u(t)$. From Proposition 5.1 and Theorem 5.2 of [**7**], we have

$$\delta u(t) = \frac{d}{ds} h(s,t)|_{s=0} = \partial_t(\delta v \circ v^{-1})(t) + [u, \delta v \circ v^{-1}]_e(t). \tag{2.4.8}$$

Using this framework, the reduced action functional becomes

$$L(u(s)) = \frac{1}{2} \int_a^b < u_s(t), u_s(t) >_e dt \tag{2.4.9}$$

and we have

$$\begin{aligned} \frac{d}{ds} L(u(s)) \bigg|_{s=0} &= \int_a^b < \delta u(t), u(t) >_e dt \\ &= \int_a^b \int_M g((1 - \alpha^2 \triangle) u(t,x), \delta u(t,x)) d\mu dt \\ &= \int_a^b \int_M g((1 - \alpha^2 \triangle) u(t,x), \partial_t(\delta v \circ v^{-1})(t,x)) d\mu dt \\ &+ \int_a^b \int_M g((1 - \alpha^2 \triangle) u(t,x), [u, \delta v \circ v^{-1}]_e(t,x)) d\mu dt. \end{aligned} \tag{2.4.10}$$

To deal with the first term, we use integration by parts and the properties of fixed end point variations to get

$$\begin{aligned} &\int_a^b \int_M g((1 - \alpha^2 \triangle) u(t,x), \partial_t(\delta v \circ v^{-1})(t,x)) d\mu dt \\ = &\int_a^b \int_M -g(\partial_t (1 - \alpha^2 \triangle) u(t,x), (\delta v \circ v^{-1})(t,x)) d\mu dt. \end{aligned} \tag{2.4.11}$$

For the second term in (2.4.10), we use Proposition 2.6.2 and the fact that Def $(X) = \frac{1}{2}\nabla X + (\nabla X)^t$ to get

$$\begin{aligned} g(Bu, \mathcal{L}_u \delta v \circ v^{-1}) &= g(\mathcal{L}_u^* Bu, \delta v \circ v^{-1}) \\ &= -g(\mathcal{L}_u Bu + (\nabla u + (\nabla u)^t + (\text{div } u)I)Bu, \delta v \circ v^{-1}) \\ &= -g(\nabla_u Bu - \nabla_{Bu} u + \nabla_{Bu} u + (\nabla u)^t Bu, \delta v \circ v^{-1}) \end{aligned} \tag{2.4.12}$$

where we set $B = (1 - \alpha^2 \triangle)$ and we used the assumption that div $u = 0$. Using (2.4.11) and (2.4.12) in (2.4.10), we get

$$\frac{d}{ds} L(u(s))\bigg|_{s=0} = -\int_a^b < \partial_t Bu + \nabla_u Bu + (\nabla u)^t Bu, \delta v \circ v^{-1} >_e dt. \tag{2.4.13}$$

This gives that u is an extreme point of the functional only if

$$\partial_t u + P^\alpha B^{-1}[\nabla_u Bu + (\nabla u)^t Bu] = 0 \tag{2.4.14}$$

which implies

$$\partial_t u + B^{-1}[\nabla_u Bu + (\nabla u)^t Bu] = -B^{-1}\text{grad } p \tag{2.4.15}$$

where we have used Proposition 2 of [**14**] in an analogous fashion to the use of the Hodge decomposition for the classical Euler equations (see 17.1 of [**16**]). This is our first form of the Lagrangian Averaged Euler equations.

In [**14**], this process is applied to more general M (including considering boundary data) and several additional geometric and analytical results, including existence of the critical points, are obtained.

Specializing to the case where M is a region in $\mathbb{R}^n$, we see that the averaged action operator $\bar{S}$ (see (2.2.17)) coincides with our reduced Lagrangian L. We conclude this section describing the form the Lagrangian Averaged Euler equations take in this context.

Since $(\nabla u)^t u = \mathrm{grad}(\frac{1}{2}|u|^2)$, (2.4.15) becomes

$$\partial_t u + B^{-1}[\nabla_u Bu + \mathrm{grad}(\frac{1}{2}|u|^2) - \alpha^2(\nabla u)^t(\triangle u)] = -B^{-1}\mathrm{grad}\ p. \tag{2.4.16}$$

Combining the two terms involving the gradient and relabeling the pressure accordingly, we have

$$\partial_t u + B^{-1}[\nabla_u Bu - \alpha^2(\nabla u)^t(\triangle u)] = -B^{-1}\mathrm{grad}\ p, \tag{2.4.17}$$

where β^α is defined by (2.6.10).

Next, we use Proposition 2.6.3 and get

$$\partial_t u + B^{-1}[B(\nabla_u u) + \mathrm{div}\ (\beta^\alpha(u)) + \alpha^2\mathrm{grad}\ g] = -B^{-1}\mathrm{grad}\ p. \tag{2.4.18}$$

Relabeling the pressure to include the new gradient term, this simplifies to

$$\partial_t u + \nabla_u u + (1-\alpha^2)^{-1}\mathrm{div}\ (\beta^\alpha(u)) = -(1-\alpha^2\triangle)^{-1}\mathrm{grad}\ p. \tag{2.4.19}$$

Our last observation is that div $((\nabla u)^t \cdot (\nabla u)^t)$ is (up to a constant) equal to the gradient of the scalar function Trace$((\nabla u)^t \cdot (\nabla u)^t)$, so (2.4.19) becomes

$$\partial_t u + \nabla_u u + \mathrm{div}\ (\tau^\alpha(u)) = -(1-\alpha^2\triangle)^{-1}\mathrm{grad}\ p \tag{2.4.20}$$

where the pressure term has again been modified and τ^α is defined in Section 1.1 (absorbing the constant into α). To get the Lagrangian Averaged Navier Stokes equations in the form of (1.1.2), we simply add the viscosity term.

2.5. Euler-Poincare Reduction Theorem

THEOREM 2.5.1. *Let G be a topological group which admits smooth manifold structure with smooth right translation and let $L : TG \to \mathbb{R}$ be a right invariant Lagrangian. Let $\mathbf{g} = T_eG$ and let $l : \mathbf{g} \to \mathbb{R}$ be the restriction of L to $\mathbf{g}$. For a curve $\eta(t)$ through G, define a curve $u(t)$ through $\mathbf{g}$ by $u(t) = \dot{\eta}(t) \circ (\eta(t))^{-1}$. Then the following are equivalent:*

(1) *the curve $\eta(t)$ satisfies the Euler-Lagrange equations on G.*

(2) *the curve $\eta(t)$ is an extreme point of the action functional*

$$S(\eta) = \int L(\eta(t), \dot{\eta}(t))dt \tag{2.5.1}$$

for fixed end point variations.

(3) *the curve $u(t)$ satisfies the Euler-Poincare equations*

$$\frac{d}{dt}\frac{\partial l}{\partial u} = -ad_u^* \frac{\partial l}{\partial u} \tag{2.5.2}$$

where the coadjoint action ad_u^* is defined by

$$< ad_u^* v, w > = < v, [u, w] > \tag{2.5.3}$$

where u, v and w are in $\mathbf{g}$, where $< \cdot, \cdot >$ is the metric on $\mathbf{g}$ and the bracket is a right lie bracket.

(4) *the curve $u(t)$ is an extremum of the reduced action functional*

$$s(u) = \int l(u(t))dt \tag{2.5.4}$$

for variations of the form

$$\frac{\partial}{\partial \varepsilon} u(\varepsilon, t) = \frac{d}{dt}(\frac{\partial \eta}{\partial \varepsilon} \circ \eta^{-1}) + [\frac{\partial \eta}{\partial \varepsilon}, u]. \tag{2.5.5}$$

2.6. Differential Geometry Computations

We begin this section with the computation of the L^2 adjoint of the Levi-Civita connection.

PROPOSITION 2.6.1. *Let X, Y be vector fields on a Riemannian manifold M with Levi-Civita connection ∇. Then*

$$(\nabla_X)^* Y = -\nabla_X Y - (div\ X)Y \tag{2.6.1}$$

where $(\nabla_X)^*$ denotes the L^2 adjoint of ∇_X as an operator on vector fields.

To prove this, we begin with a compactly supported vector field Z, and we have

$$\begin{aligned}\int g((\nabla_X)^* Y, Z) dV &= \int g(Y, \nabla_X Z) dV \\ &= \int Xg(Y, Z) - g(\nabla_X Y, Z)\end{aligned} \tag{2.6.2}$$

where the last equality is the zero-torsion condition of the metric. We observe that

$$\int Xg(Y, Z) dV = \int X(g_{jk} Y^j Z^k) dV = -\int \text{div } X(g_{jk} Y^j Z^k) dV \tag{2.6.3}$$

where the last equality is an application of integration by parts. Using (2.6.3) in (2.6.2), we get

$$\int g(Y, \nabla_X Z) + g(\nabla_X Y, Z) dV = -\int g((\text{div } X)Y, Z) dV \tag{2.6.4}$$

which proves the Proposition.

Our next calculation is of the adjoint of the Lie Derivative on vector fields.

PROPOSITION 2.6.2. *Let X, Y be vector fields on a Riemannian manifold M. Then*

$$\mathcal{L}_X^* Y = -\mathcal{L}_X Y - (TX)Y \tag{2.6.5}$$

where $\mathcal{L}_X^*$ *denotes the* L^2 *adjoint of* $\mathcal{L}_X$*,* T *is the operator defined by* $TX = (div\ X)Y - 2Def\,(X)Y$ *and* $Def\,(X)$ *is a tensor of type* $(1,1)$ *given by*

$$\frac{1}{2}(\mathcal{L}_X g)(Y,Z) = g(Def\,(X)Y,Z). \tag{2.6.6}$$

To prove this, we will need the fact that

$$\mathcal{L}_X g(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \tag{2.6.7}$$

This is equation (3.31) in Chapter 2 of [**16**]. Using (2.6.4) and (2.6.7), we have that

$$\begin{aligned}
&\int g(T(X)Y,Z)dV = \int g(-(\text{div } X)Y,Z) - g(2\text{Def } (X)Y,Z)dV \\
&= \int g(Y,\nabla_X Z) + g(\nabla_X Y,Z) - g(\nabla_Y X,Z) - g(Y,\nabla_Z X)dV \\
&= \int g(Y,\nabla_X Z - \nabla_Z X) + g(Z,\nabla_X Y - \nabla_Y X)dV \\
&= \int g(Y,[X,Z]) + g([X,Y],Z)dV = \int g(Y,\mathcal{L}_X Z) + g(\mathcal{L}_X Y,Z)dV
\end{aligned} \tag{2.6.8}$$

which proves the proposition.

Our next set of results apply to the special case $M = \mathbb{R}^n$.

Proposition 2.6.3. *Let* $B = (1-\alpha^2\triangle)$ *for some* $\alpha > 0$ *and let* $u : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $div\ u = 0$*. Then*

$$\nabla_u Bu - \alpha^2(\nabla u)^t(\triangle u) = B(\nabla_u u) + div\ (\beta^\alpha(u)) + \alpha^2 grad\ f \tag{2.6.9}$$

for some function f *to be specified and for*

$$\beta^\alpha(u) = \alpha^2(\nabla u \nabla u + \nabla u(\nabla u)^t - (\nabla u)^t \nabla u). \tag{2.6.10}$$

To prove this, we begin with the observation that

$$
\begin{aligned}
B(\nabla_u u) &= \nabla_u u - \alpha^2(\nabla_{\triangle u} u + \nabla_u(\triangle u) + 2\sum_k \nabla_{\partial_k u}\partial_k u) \\
&= \nabla_u Bu - \alpha^2(\nabla_{\triangle u} u + 2\sum_k \nabla_{\partial_k u}\partial_k u).
\end{aligned}
\tag{2.6.11}
$$

In indices, the last term is

$$
-\alpha^2(\sum_{j,k} u^i_j u^j_{kk} + 2(u^j_k u^i_{jk}))
\tag{2.6.12}
$$

where lower indices denote partial derivatives and upper indices denote coordinate functions.

We next write div $(\beta^\alpha(u))$ in indices and get

$$
\alpha^2 \sum_{j,k} u^i_{jk}u^j_k + u^i_k u^j_{jk} - u^k_{ij}u^k_j - u^k_i u^k_{jj} + u^i_{jk}u^j_k + u^i_j u^j_{kk}.
\tag{2.6.13}
$$

Since div $u = 0$, we have

$$
\partial_k \mathrm{div}\ u = \partial_k \sum_j u^j_j = \sum_j u^j_{jk} = 0
\tag{2.6.14}
$$

which means

$$
\sum_k u^i_k \sum_j u^j_{jk} = 0
\tag{2.6.15}
$$

for any i, so (2.6.13) becomes

$$
\alpha^2 \sum_{j,k} 2u^i_{jk}u^j_k - u^k_{ij}u^k_j - u^k_i u^k_{jj} + u^i_j u^j_{kk}.
\tag{2.6.16}
$$

Next, we define $f = \frac{1}{2}\sum_j |\partial_j u|^2$ and we have that the i^{th} coordinate of grad f is

$$
\partial_i(\frac{1}{2}\sum_j |\partial_j u|^2) = \partial_i(\frac{1}{2}\sum_j\sum_k (u^k_j)^2) = \sum_{j,k} u^k_{ij}u^k_j.
\tag{2.6.17}
$$

Noting that the i^{th} coordinate of $(\nabla u)^t(\triangle u)$ is given by $\sum_{j,k} u_i^k u_{jj}^k$, we have

$$\text{div } (\beta^\alpha(u)) = \alpha^2[-\text{grad } f - (\nabla u)^t(\triangle u) + v] \tag{2.6.18}$$

where v is the vector with i^{th} component

$$\sum_{j,k} 2u_{jk}^i u_k^j + u_j^i u_{kk}^j. \tag{2.6.19}$$

Adding (2.6.11) and (2.6.18) gives

$$B(\nabla_u u) + \text{div } (\beta^\alpha(u)) = \nabla_u Bu - \alpha^2 \text{grad } f - \alpha^2 (\nabla u)^t(\triangle u) \tag{2.6.20}$$

which proves the Proposition.

CHAPTER 3

Sobolev Space solutions to LANS

3.1. Sobolev space solutions to LANS

In this section we assume the initial data φ is in a Sobolev space $H^{r,p}(\mathbb{R}^n)$ and obtain solutions in two different spaces of functions. In Section 3.2 we define the first of these spaces and state our first theorem, a local existence result. In (3.3) we obtain several necessary supporting results, and in (3.4) we prove the theorem. In (3.6) we extend the local solution to a global solution in the special case of initial data in $H^{n/2,2}(\mathbb{R}^n)$ where $n \geq 3$. In (3.7) we define a new functions space and state our second theorem, which is proven in (3.8) and (3.9).

3.2. Solutions in the class of weighted continuous functions in time

We begin with a brief reminder of the definition of Sobolev spaces. For a positive integer k, the Sobolev space $H^{k,p}$ is defined by

$$H^{k,p} = \{f \in L^p : D^\alpha f \in L^p \text{ for all } |\alpha| \leq k\}. \tag{3.2.1}$$

For integers $k < 0$, the space $H^{k,p}$ is defined as the dual to the space $H^{-k,p'}$, where p' is the Holder conjugate exponent to p. For any non-integer s, the Sobolev space $H^{s,p}$ is defined via interpolation. See Chapter sf of [**16**] for a more thorough definition of Sobolev spaces.

Now we define a more unusual space. Fixing $0 < T \leq \infty$, for any $k \geq 0$, we define the space

$$C^T_{k;s,q} = \{f \in C((0,T) : H^{s,q}) : \|f\|_{k;s,q} < \infty\} \tag{3.2.2}$$

where

$$\|f\|_{k;s,q} = \sup\{t^k \|f(t)\|_{s,q} : t \in (0,T)\}. \tag{3.2.3}$$

$\dot{C}^T_{k;s,q}$ denotes the subspace of $C^T_{k;s,q}$ consisting of f such that

$$\lim_{t \to 0^+} t^k f(t) = 0 \text{ (in } H^{s,q}). \tag{3.2.4}$$

If $k = 0$, we write $\overline{C}^T_{s,q}$ for $BC([0,T) : H^{s,q})$, the space of bounded, continuous functions from $[0,T)$ to $H^{s,q}$.

We will typically write $C^T_{k;s,q}$ and $\overline{C}^T_{s,q}$ as $C_{k;s,q}$ and $\overline{C}_{s,q}$, respectively, suppressing the T dependance.

We now state our first theorem in its full generality.

THEOREM 3.2.1. *For any $\varphi = P^\alpha \varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.1.2) *such that*

$$u \in \overline{C}_{r,p} \cap \dot{C}_{a;k,c} \tag{3.2.5}$$

provided there exist a real number b' such that the list of conditions (3.4.18) *is satisfied (where $r = n/p + b$). If $\|\varphi\|_{r,p}$ is sufficiently small, $T = \infty$.*

This is similar to Theorem 2.1 in [**6**].

Before proving the theorem, we do some preliminary work. We begin by using Duhamel's principle to write (1.1.4) into the integral equation

$$u = \Gamma\varphi - G \cdot P^\alpha(\operatorname{div} \cdot (u \otimes u + \tau^\alpha(u))) \tag{3.2.6}$$

with

$$(\Gamma\varphi)(t) = e^{tA}\varphi, \tag{3.2.7}$$

where A agrees with $\triangle$ when restricted to $P^\alpha H^{s,p}$, and

$$G \cdot g(t) = \int_0^t e^{(t-s)A} \cdot g(s) ds. \tag{3.2.8}$$

Our plan is to construct a contraction mapping based on (3.2.6), but first we prove some results regarding Γ, G, and the Reynolds stress term τ^α.

3.3. Basic Results

We begin this section by examining the Reynolds stress term.

LEMMA 3.3.1. *Given any $r \in [1, \infty)$, $1 < q, p < \infty$, and $q = \frac{np}{2n - s'p}$ where $0 \le s' \le r - 1$ and $s'p < n$, we have div $\tau^\alpha : H^{r,p} \to H^{r,q}$. Specifically, we have the estimate*

$$\|div\ \tau^\alpha(u)\|_{r,q} \le C\|u\|_{r,p}^2 \tag{3.3.1}$$

We begin by recalling that a differential operator P of order m satisfies

$$P : H^{s,p} \to H^{s-m,p}, \tag{3.3.2}$$

which means for any $u \in H^{r,p}$,

$$\|\alpha^2(1 - \alpha^2\triangle)^{-1}u\|_{r+2,p} \le C\|u\|_{r,p} \tag{3.3.3}$$

and

$$\|\nabla u\|_{r-1,p} \le C\|u\|_{r,p}. \tag{3.3.4}$$

So we have

$$\begin{aligned} \|\tau^\alpha(u)\|_{r+1,q} &= \|\alpha^2(1 - \alpha^2\triangle)^{-1}[Def(u) \cdot Rot(u)]\|_{r+1,q} \\ &\le C\|[Def(u) \cdot Rot(u)]\|_{r-1,q}. \end{aligned} \tag{3.3.5}$$

Recalling the definitions of $Def(u)$ and $Rot(u)$, we have that

$$Def(u) \cdot Rot(u) = (\nabla u \nabla u + \nabla u \nabla u^T + \nabla u^T \nabla u + \nabla u^T \nabla u^T)/4. \tag{3.3.6}$$

Observing that $\|\nabla u\|_{k,q} = \|\nabla u^T\|_{k,q}$ for any k, q and applying Proposition 3.10.8, we get

$$\begin{aligned} \|\nabla u \nabla u\|_{r-1,q} &\leq C\|\nabla u\|^2_{r-1,p} \\ \|\nabla u \nabla u^T\|_{r-1,q} &\leq C\|\nabla u\|^2_{r-1,p} \\ \|\nabla u^T \nabla u\|_{r-1,q} &\leq C\|\nabla u\|^2_{r-1,p} \\ \|\nabla u^T \nabla u^T\|_{r-1,q} &\leq C\|\nabla u\|^2_{r-1,p} \end{aligned} \tag{3.3.7}$$

provided that $r \geq 1$.

Using (3.3.5) and (3.3.7) we have

$$\begin{aligned} \|\tau^\alpha(u)\|_{r+1,q} &\leq C\|[Def(u) \cdot Rot(u)]\|_{r-1,q} \\ &\leq C\|\nabla u\|^2_{r-1,p} \\ &\leq C\|u\|^2_{r,p}. \end{aligned} \tag{3.3.8}$$

Since the divergence is a degree one differential operator, we get from (3.3.8) that

$$\|\text{div } \tau^\alpha(u)\|_{r,q} \leq \|\tau^\alpha(u)\|_{r+1,q} \leq C\|u\|^2_{r,p} \tag{3.3.9}$$

which proves the lemma.

This immediately gives

$$t^{2a}\|\text{div } \tau^\alpha(u)\|_{r,q} \leq C(t^a\|u\|_{r,p})^2 \tag{3.3.10}$$

which proves the following corollary.

Corollary 3.3.2. *div* $\tau^\alpha : \dot{C}_{a;r,p} \to \dot{C}_{2a;r,q}$, *with the estimate* $\|div\ \tau^\alpha(u)\|_{2a;r,q} \leq C\|u\|^2_{a;r,p}$.

Our next task is establish some properties of the operator V^α defined by

$$V^\alpha(u,v) = \text{div } u \otimes v + \text{div } \tau^\alpha(u,v) \tag{3.3.11}$$

where $\tau^\alpha(u,v) = \alpha^2(1-\alpha^2\triangle)^{-1}(Def(u)) \cdot (Rot(v))$. Abusing notation, we will write $V^\alpha(u,u) = V^\alpha(u)$. We also observe that V^α is linear in each of its arguments. Using that the divergence is a degree one differential operator, we have

$$\|\text{div } (u \otimes v)\|_{b-1,q} \leq C\|u \otimes v\|_{b,q} \leq C\|u\|_{b,p}\|v\|_{b,p} \tag{3.3.12}$$

provided $b \geq 1$, $1 < q, p < \infty$, and $q = \frac{np}{2n-s'p}$ where $0 \leq s' \leq b$ and $s'p < n$. Slightly modifying Lemma 3.10.5 and Corollary 3.3.2, we have

$$\|\text{div } \tau^\alpha(u,v)\|_{b-1,q} \leq C\|\text{div } \tau^\alpha(u,v)\|_{b,q} \leq C\|u\|_{b,p}\|v\|_{b,p} \tag{3.3.13}$$

provided $b \geq 1$, $1 < q, p < \infty$, and $q = \frac{np}{2n-s'p}$ where $0 \leq s' \leq b-1$ and $s'p < m$.

Replacing u in the above calculation with $t^a u$, we get the following proposition.

Proposition 3.3.3. *Let* $a \geq 0$, $b \geq 1$, $1 < q, p < \infty$, *and* $q = \frac{np}{2n-s'p}$ *where* $0 \leq s' \leq b-1$ *and* $s'p < n$. *Then*

$$V^\alpha : \dot{C}_{a;b,p} \times \dot{C}_{a;b,p} \to \dot{C}_{2a;b-1,q} \tag{3.3.14}$$

with the estimate

$$\|V^\alpha(u,v)\|_{2a;b-1,q} \leq \|u\|_{a;b,p}\|v\|_{a;b,p}. \tag{3.3.15}$$

Next, we observe that

$$V^\alpha(u) - V^\alpha(v) = -(V^\alpha(u, u-v) + V^\alpha(u-v, v)). \tag{3.3.16}$$

Using Proposition 3.3.3, we have

$$\|V^\alpha(u, u-v)\|_{b-1,q} \le \|u\|_{b,p}\|u-v\|_{b,p} \tag{3.3.17}$$

and

$$\|V^\alpha(u-v, v)\|_{b-1,q} \le \|v\|_{b,p}\|u-v\|_{b,p}. \tag{3.3.18}$$

These estimates give that

$$\|V^\alpha(u(s)) - V^\alpha(v(s))\|_{b-1,q} \le C\left(\|u(s)\|_{b,p} + \|v(s)\|_{b,p}\right)\|u(s)-v(s)\|_{b,p}. \tag{3.3.19}$$

Multiplying both sides by t^a and distributing through the right hand side, we get

$$\|V^\alpha(u) - V^\alpha(v)\|_{a;b-1,q} \le C(\|u\|_{a/2;b,p} + \|v\|_{a/2;b,p})\|u-v\|_{a/2;b,p}. \tag{3.3.20}$$

The above calculation proves the following corollary to Proposition 3.3.3.

Corollary 3.3.4. *With the same assumptions on the parameters as in Proposition* 3.3.3*, we have that if* $u, v \in \dot{C}_{a/2;b,q}$ *then*

$$\|V^\alpha(u(s)) - V^\alpha(v(s))\|_{a;b-1,q} \le C(\|u\|_{a/2;b,p} + \|v\|_{a/2;b,p})\|u-v\|_{a/2;b,p}. \tag{3.3.21}$$

Our next topic is the operator Γ.

Proposition 3.3.5. *Let* $s' \le s''$, $1 < q' \le q'' < \infty$*, and define* $k'' = (n/q' - n/q'' + s'' - s')/2$*. Then* $\Gamma : H^{s',q'} \to \dot{C}_{k'';s'',q''}$*, provided* $k'' > 0$*.*

This is an immediate consequence of Lemma 3.11.3.

We now turn our attention to the operator G. Assuming $s' \leq s''$, $q' \leq q''$, and $u \in \dot{C}_{k';s',q'}$, we formally calculate

$$
\begin{aligned}
\|G \cdot u\|_{s'',q''} &= \|\int_0^t e^{(t-s)A}u(s)ds\|_{s'',q''} \\
&\leq C\int_0^t \|e^{(t-s)A}u(s)\|_{s'',q''}ds \\
&\leq C\int_0^t (t-s)^{-(s''-s'+m/q'-m/q'')/2}\|u(s)\|_{s',q'}ds \\
&\leq C\int_0^t (t-s)^z s^{-k'}s^{k'}\|u(s)\|_{s',q'}ds \\
&\leq Ct^{z-k'+1}\|u\|_{k';s',q'}
\end{aligned}
\tag{3.3.22}
$$

where $z = -(s'' - s' + n/q' - n/q'')/2$, the third line uses Lemma 3.11.2, the fourth that $u \in \dot{C}_{k';s',q'}$,and the last line uses Proposition 3.10.1. This result will hold provided $0 \leq (s'' - s' + n/q' - n/q'')/2 < 1$ and $k' < 1$, and this leads to our first result involving G.

PROPOSITION 3.3.6. *With $s' \leq s''$, $q' \leq q''$ and setting $k'' = k' - 1 + (s'' - s' + n/q' - n/q'')/2$, G continuously maps $\dot{C}^T_{k';s',q'}$ into $\dot{C}^T_{k'';s'',q''}$ with $0 \leq (s''-s'+m/q'-m/q'')/2 < 1$ and $k' < 1$ with the estimate*

$$\|G \cdot u\|_{k'';s'',q''} \leq C\|u\|_{k';s',q'}. \tag{3.3.23}$$

3.4. Proof of Theorem 3.2.1

To prove Theorem 3.2.1, we begin by constructing the nonlinear map

$$\Phi u = \Gamma\varphi - G \cdot P^\alpha(\text{div }(u \otimes u) + \text{div }\tau^\alpha u). \tag{3.4.1}$$

Our goal is to show that this map is a contraction on an appropriate function space. Using (3.3.11), Φ can be re-written as

$$\Phi u = \Gamma\varphi - G \cdot P^\alpha(V^\alpha(u)). \tag{3.4.2}$$

Beginning with initial data $\varphi \in H^{r,p}(\mathbb{R}^n)$ where $r = \frac{n}{p} + b$, we construct the space

$$E_{T,M} = \{v \in \bar{C}_{r,p} \cap \dot{C}_{a;k,c} : \|v - \Gamma\varphi\|_{0;r,p} + \|v\|_{a;k,c} \leq M\}, \tag{3.4.3}$$

recalling that the definition of $\bar{C}_{r,p}$ and $\dot{C}_{a;k,c}$ requires a choice of T. Our goal will be to show that Φ is a contraction on this space for appropriate choices of parameters.

To show Φ is a contraction, will will use the mapping properties of G to send each component space of $E_{T,M}$ into an intermediate space, and then use Proposition 3.3.3. Our intermediate space will be of the form $\dot{C}_{2a;k-b',\bar{c}}$.

Our first task is to show that Φ maps $E_{T,M}$ into $E_{T,M}$. To do this, we need to estimate

$$\|\Phi(u) - \Gamma\varphi\|_{0;r,p} = \|G \cdot P^\alpha V^\alpha(u)\|_{0;r,p} \tag{3.4.4}$$

and

$$\|\Phi(u)\|_{a;k,c} = \|\Gamma\varphi - G \cdot P^\alpha V^\alpha(u)\|_{a;k,c}. \tag{3.4.5}$$

To estimate (3.4.4), we note that by Proposition 3.3.6 and that P^α is a projection, we have that

$$\|GV^\alpha u\|_{0;\frac{n}{p}+b,p} \leq C\|V^\alpha u\|_{2a;k-b',\bar{c}} \tag{3.4.6}$$

will hold provided

$$0 = 2a - 1 + \left(\frac{n}{p} + b - (k - b') + \frac{n}{\bar{c}} - \frac{n}{p}\right)/2$$

$$2a < 1$$

$$0 \le (n/p + b - (k - b') + n/\bar{c} - n/p)/2 < 1 \tag{3.4.7}$$

$$k - b' \le n/p + b$$

$$\bar{c} \le p.$$

Proposition 3.3.3 gives

$$\|V^\alpha u\|_{2a;k-b',\bar{c}} \le C\|u\|^2_{a;k,c} \tag{3.4.8}$$

provided

$$k, b' \ge 1,$$

$$c > 1,$$

$$\bar{c} = \frac{nc}{2n - s'c} \tag{3.4.9}$$

$$0 \le s' \le k - 1$$

$$s'c < m.$$

These combine to give our estimate on (3.4.4). To estimate $\|G \cdot P^\alpha V^\alpha(u)\|_{a;k,c}$, we have

$$\|GV^\alpha u\|_{a;k,c} \le C\|V^\alpha u\|_{2a;k-b',\bar{c}} \le C\|u\|^2_{a;k,c} \tag{3.4.10}$$

will hold provided

$$
(3.4.11)\qquad
\begin{aligned}
a &= 2a - 1 + \left(k - (k - b') + \frac{n}{\bar{c}} - \frac{n}{c}\right)/2 \\
2a &< 1 \\
\bar{c} &\leq c \\
0 &\leq (k - (k - b') + n/\bar{c} - n/c)/2 < 1.
\end{aligned}
$$

Using (3.4.6) and (3.4.10), we have

$$
(3.4.12)\qquad \|\Phi(u) - \Gamma\varphi\|_{0;r,p} + \|\Phi(u)\|_{a;k,c} \leq C\|u\|^2_{a;k,c} + \|\Gamma\varphi\|_{a;k,c}.
$$

By assumption, $u \in E_{T,M}$, so $\|u\|^2_{a;k,c} \leq M^2$. So our last task is to estimate $\|\Gamma\varphi\|_{a;k,c}$. From Proposition 3.3.5, we have that

$$
(3.4.13)\qquad \Gamma : \dot{C}_{0;\frac{n}{p}+b,p} \to \dot{C}_{a;k,c}
$$

if $a > 0$, $k \geq \frac{n}{p} + b$, $c \leq p$, and

$$
(3.4.14)\qquad a = \left(\frac{n}{p} - \frac{n}{c} + k - \left(\frac{n}{p} + b\right)\right)/2
$$

which simplifies to

$$
(3.4.15)\qquad 2a = k - \frac{n}{c} - b.
$$

Because $\Gamma\varphi \in \dot{C}_{a;k,c}$, there exists a T, depending only on M and the norm of the initial data φ, such that $\|\Gamma\varphi\|_{a;k,c} \leq M/2$. So by choosing a sufficiently small M and an appropriate T, we have that $\Phi : E_{T,M} \to E_{T,M}$.

Now we seek to show that Φ is a contraction map. Let $u, v \in E^{T,M}$. Then by Corollary 3.3.4 we have

$$
\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_{r,p} &= \|G(P^\alpha V^\alpha u - P^\alpha V^\alpha(v))\|_{r,p} \\
&\le C\|V^\alpha u - V^\alpha(v)\|_{2a;k-b',\bar{c}} \\
&\le C(\|u\|_{a;k,c} + \|v\|_{a;k,c})\|u - v\|_{a;k,c} \\
&\le CM\|u - v\|_{a;k,c},
\end{aligned} \tag{3.4.16}
$$

and similarly we have

$$
\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_{a;k,c} &= \|G(V^\alpha u - V^\alpha(v))\|_{a;k,c} \\
&\le C\|V^\alpha u - V^\alpha(v)\|_{2a;k-b',\bar{c}} \\
&\le C(\|u\|_{a;k,c} + \|v\|_{a;k,c})\|u - v\|_{a;k,c} \\
&\le CM\|u - v\|_{a;k,c}.
\end{aligned} \tag{3.4.17}
$$

So for a sufficiently small choice of M, we can choose a T such that Φ sends $E_{T,M}$ into itself and is a contraction on $E_{T,M}$. So by the contraction mapping principle, we have a unique fixed point $u \in E_{T,M}$ provided our parameters satisfy all the requisite inequalities. Combining and simplifying these inequalities, and allowing $s' = k - 2 - b + b'$ to define

s', we get the following list of restrictions on the parameters:

$$
\begin{aligned}
&1 < p \le c < \infty \\
&s' := k - 2 - b + b' \\
&k \ge 1,\ b' \ge 1,\ s'c < n \\
&0 < 2a = k - n/c - b < 1 \\
&0 \le s' \le k - 1 \\
&1 < \frac{nc}{2n - s'c} \le p \\
&1 \ge b' - b \\
&1 \le b' + \frac{n}{c} - s' < 2 \\
&2 - 2b' + s' \le \frac{n}{p} \le 2 - b' + s'.
\end{aligned}
\tag{3.4.18}
$$

This is not optimal, because of the presence of the "extra" parameter b'. However, this version does make it easy to ascertain certain bounds on the original parameters. For example, the second and seventh conditions require that $b \ge 0$, which provides a lower bound of n/p on the regularity of our initial data.

To eliminate the extra parameters b', we remark that the conditions force $1 \le b' < 2$, and our optimal case ($b = 0$) requires $b' = 1$. So setting $b' = 1$, we let $k = 1 + b + s'$

define s', and our list of conditions becomes

$$\begin{aligned}
&1 < p \le c < \infty \\
&b \ge 0 \\
&s' := k - 1 - b \\
&k \ge 1,\ s'c < n \\
&0 < 2a = k - n/c - b < 1 \\
&1 < \frac{nc}{2n - s'c} \le p \\
&0 \le \frac{n}{c} - s' < 1 \\
&s' \le \frac{n}{p} \le 1 + s'.
\end{aligned} \tag{3.4.19}$$

To get Theorem 1.1.2, we choose $p > n$, $b = 0$, $k = 1$, $c = p$ and $a = 1 - n/p$. To get Theorem 1.1.3, we choose $p = c = k = 2$, $n = 3$, $b = 0$, and $a = 1/4$.

To get global existence for small initial data, we observe that the above calculations for $G \cdot V^\alpha$ only required an assumption that M be small. We also note that

$$\begin{aligned}
\|\Gamma(t)\varphi\|_{r,p} &\le \|\varphi\|_{r,p} \\
\|\Gamma(t)\varphi\|_{a;k,c} &\le \|\varphi\|_{r,p}
\end{aligned} \tag{3.4.20}$$

holds for any $t \ge 0$. So provided $\|\varphi\|_{r,p}$ is sufficiently small, Φ is a contraction on $E_{T,M}$ for any $T > 0$, which gives global existence of the solution.

Continuous dependance on the initial data is also relatively straightforward. Given data u_0 and v_0 with corresponding solutions $u(t)$ and $v(t)$, we have

$$u(t) - v(t) = \Gamma(u_0 - v_0) - \int_0^t e^{(t-s)\triangle} \left(V^\alpha(u(s)) - V^\alpha(v(s))\right) ds, \tag{3.4.21}$$

which implies

$$\begin{aligned}\|u-v\|_{0;r,p}+\|u-v\|_{a;k,c} \le \|u_0-v_0\|_{r,p}+\|\Gamma(u_0-v_0)\|_{a;k,c} \\ +\|C(\|u\|_{a;k,c}+\|v\|_{a;k,c})\|u-v\|_{a;k,c}.\end{aligned} \tag{3.4.22}$$

Since $\|\Gamma(u_0-v_0)\|_{a;k,c}+\|C(\|u\|_{a;k,c}+\|v\|_{a;k,c})$ can be made arbitrarily small, we get that $\|u-v\|_{0;r,p}+\|u-v\|_{a;k,c}$ is arbitrarily small provided $\|u_0-v_0\|_{r,q}$ is sufficiently small.

We remark that the preceding argument is easily modified to fit different functional settings. All that is necessary is establishing supporting results similar to those of the previous section.

3.5. Special cases of Theorem 3.2.1

We begin by remarking that with $n \ge p$, choosing $k = r + 1/4$, $c = p$, $a = 1/8$, $b' = 1$ and $s' = n/p - 3/4$ satisfies (3.4.18). Using these choices for the parameters, let $\varphi \in H^{r,p}(\mathbb{R}^n)$ be our chosen initial data and let

$$u \in BC([0,T) : H^{r,p}) \cap \dot{C}_{1/8;r+1/4,p} \tag{3.5.1}$$

be the solution to (1.1.2) given by Theorem 3.2.1. Then, for any $0 < t' < T$, define $\varphi' = u(t')$. Viewing φ' as "new" initial data, applying Theorem 3.2.1 gives the existence of a solution v to (1.1.2) such that

$$v \in BC([0,T) : H^{r+1/4,p}) \cap \dot{C}_{1/8;r+1/2,p} \tag{3.5.2}$$

where $v(0) = \varphi' = u(t')$. Because

$$BC([0,T) : H^{r+1/4,p}) \cap \dot{C}_{1/8;r+1/2,p} \subset BC([0,T) : H^{r,p}) \cap \dot{C}_{1/8;r+1/4,p}, \tag{3.5.3}$$

uniqueness of our solution gives that u and v are the same solution. Each iteration of this process results in a "gain" of one-quarter of a derivative, and thus for any $0 < t' < T$, $u(t') \in H^{s,p}$ for any $s \in \mathbb{R}$. We record this as a corollary of Theorem 3.2.1, but first we remark that for $n < p$, choosing $k = b + 1$, $c = p$, $a = (1 - n/p)/2$, $b' = 1$ and $s' = 0$ also satisfies (3.4.18), so we have the same result for the $n < p$ case.

Corollary 3.5.1. *Let $\varphi \in H^{r,p}(\mathbb{R}^n)$, with $n \geq p$. If $n \geq p$, let*

$$u \in BC([0,T) : H^{r,p}) \cap \dot{C}_{1/8;r+1/4,p} \tag{3.5.4}$$

be the solution to (1.1.2) *given by Theorem* 3.2.1. *Then for any $0 < t < T$, $u(t) \in H^{s,p}(\mathbb{R}^n)$ for any real s. If $n < p$, let*

$$v \in BC([0,T) : H^{r,p}) \cap \dot{C}_{(1-n/p)/2;b+1,p} \tag{3.5.5}$$

be the solution to (1.1.2) *given by Theorem* 3.2.1. *Then for any $0 < t < T$, $v(t) \in H^{s,p}(\mathbb{R}^n)$ for any real s.*

We conclude by remarking that Theorem 1.1.2 and Theorem 1.1.3 are special cases of this corollary.

3.6. Global Existence in Sobolev space

In this section we extend the result from Theorem 1.1.3 to a global existence result. We recall that Theorem 1.1.3 says that, given $\varphi \in H^{3/2,2}(\mathbb{R}^3)$, there exists a T and a unique solution u to (1.1.2) such that

$$u \in \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2}, \tag{3.6.1}$$

where T depends only on $\|\varphi\|_{H^{3/2,2}(\mathbb{R}^3)}$ and we again recall that the definition of C implies a choice of T. Extending this to a global existence result follows from the following result.

THEOREM 3.6.1. *Let* $\phi \in H^{3/2,2}(\mathbb{R}^3)$ *and let*

$$u \in \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2} \tag{3.6.2}$$

be the unique solution to (1.1.2) *with initial data* φ *on the time strip* $[0, T)$. *Then there exists a real number* M *such that*

$$\|u(t)\|_{H^{2,2}(\mathbb{R}^3)} \leq M \tag{3.6.3}$$

for any $t \in (0, T)$.

We mimic the approach used in part (d) of Section 5 of [**8**]. We begin the proof by recalling (1.1.2):

$$\partial_t u + (u \cdot \nabla)u + \text{div } \tau^\alpha u = -(1 - \alpha^2 \triangle)^{-1} \nabla p + \nu \triangle \tag{3.6.4}$$

and stating an equivalent form (see Section 3 of [**8**])

$$\begin{aligned} &\partial_t (1 - \alpha^2 \triangle)u + \nabla_u [(1 - \alpha^2 \triangle)u] - \alpha^2 (\nabla u)^T \cdot \triangle u \\ = &- (1 - \alpha^2 \triangle)Au - \nabla p. \end{aligned} \tag{3.6.5}$$

To start, we take the L^2 product of (3.6.5) with u. We get

$$I_1 + I_2 + I_3 = J_1 + J_2 \tag{3.6.6}$$

where

$$
\begin{aligned}
I_1 &= (\partial_t(1-\alpha^2\triangle)u, u) \\
I_2 &= (\nabla_u u, u) \\
I_3 &= -\alpha^2\left((\nabla_u \triangle u, u) + ((\nabla u)^T \cdot \triangle u, u)\right) \\
J_1 &= -((1-\alpha^2\triangle)(Au), u) \\
J_2 &= (\nabla p, u).
\end{aligned} \tag{3.6.7}
$$

We start with I_1, which becomes

$$
\begin{aligned}
I_1 &= (\partial_t u, u) - \alpha^2(\triangle\partial_t u, u) \\
&= \frac{1}{2}\partial_t(\|u\|_{L^2}^2 + \alpha^2\|A^{1/2}u\|_{L^2}^2),
\end{aligned} \tag{3.6.8}
$$

where we used integration by parts and that $A = -\triangle$. Next, we have that

$$
\begin{aligned}
I_2 = (\nabla_u u, u) &= \int u_i u_j \partial_i u_j \\
&= \frac{1}{2}\int u_i\partial_i(|u|^2) = -\frac{1}{2}\int |u|^2 \text{div } u = 0,
\end{aligned} \tag{3.6.9}
$$

where we again used integration by parts and the summation convention. For I_3, we begin with

$$
\begin{aligned}
\nabla_u \triangle u \cdot u + (\nabla u)^T \cdot \triangle u \cdot u &= u_i u_j \partial_i \triangle u_j + u_j \triangle u_i \partial_j u_i \\
&= u_i u_j \partial_j \triangle u_i + u_j \triangle u_i \partial_j u_i \\
&= u_j(\partial_j(u_i \triangle u_i)),
\end{aligned} \tag{3.6.10}
$$

so I_3 becomes

$$
I_3 = -\alpha^2((\nabla_u \triangle u, u) + ((\nabla u)^T \cdot \triangle u, u)) = \alpha^2(\text{div } u, u \cdot \triangle u) = 0 \tag{3.6.11}
$$

where we again used integration by parts and the fact that div $u = 0$. For J_2, we easily see that

$$(\nabla p, u) = -(p, \text{div } u) = 0, \tag{3.6.12}$$

and for J_1 that

$$J_1 = -((1 - \alpha^2 \triangle)(Au), u) = -(A^{1/2}u, A^{1/2}u) - \alpha^2 (Au, Au). \tag{3.6.13}$$

Putting all of this back into (3.6.5), we get

$$\frac{1}{2}\partial_t(\|u(t)\|_{L^2}^2 + \alpha^2\|u(t)\|_{\dot{H}^{1,2}}^2) \leq -(\|A^{1/2}u(t)\|_{L^2}^2 + \alpha^2\|Au(t)\|_{L^2}^2), \tag{3.6.14}$$

where $\dot{H}$ denotes the homogeneous Sobolev norm. This proves that $\|u(t)\|_{H^{1,2}}$ is decreasing in time.

For our next estimate, we will apply A to (1.1.2) and take the L^2 product with Au to get

$$(\partial_t Au, Au) + (A^2 u, Au) + (AP^\alpha(\nabla_u u + \text{div } \tau^\alpha u, Au) = 0. \tag{3.6.15}$$

The first piece satisfies

$$(\partial_t Au, Au) = \frac{1}{2}\partial_t\|Au\|_{L^2}^2 \tag{3.6.16}$$

and the second satisfies

$$(A^2 u, Au) = (A^{3/2}u, A^{3/2}) = \|A^{3/2}\|_{L^2}^2. \tag{3.6.17}$$

To handle the last term of (3.6.15), we write it as

$$(AP^\alpha(\nabla_u u), Au) + (AP^\alpha \text{div } \tau^\alpha u, Au) = K_1 + K_2. \tag{3.6.18}$$

To proceed, we will need two inequalities. The first is the well known Sobolev embedding:

$$\|u\|_{L^\infty} \le C\|u\|_{H^{k,2}} \tag{3.6.19}$$

provided $2k > 3$. The second is called a Ladyzhenskaya inequality, and is (5.3) in [**8**]:

$$\|u\|_{\dot{H}^i} \le C\|u\|_{L^2}^{1-i/m}\|u\|_{\dot{H}^m}^{i/m}, \tag{3.6.20}$$

where $\dot{H}$ is the homogeneous Sobolev space.

Starting with K_1, we have

$$\begin{aligned}(AP^\alpha(\nabla_u u), Au) &= (A^{1/2}(\nabla u \cdot u), A^{3/2}u)\\ &\le C\|A^{3/2}u\|_{L^2}(\|(A^{1/2}\nabla u)u\|_{L^2} + \|(A^{1/2}u)\nabla u\|_{L^2})\\ &\le C\|u\|_{\dot{H}^3}(\|u\|_{L^\infty}\|A^{1/2}\nabla u\|_{L^2} + \|A^{1/2}u\|_{L^\infty}\|\nabla u\|_{L^2})\\ &\le C\|u\|_{\dot{H}^3}\left(\|u\|_{L^\infty}\|u\|_{\dot{H}^2} + \|A^{1/2}u\|_{L^\infty}\|u\|_{\dot{H}^1}\right).\end{aligned} \tag{3.6.21}$$

By Sobolev embedding and Proposition 3.10.7, we have

$$\begin{aligned}\|u\|_{L^\infty} \le C\|u\|_{H^{k_1}} \le C(\|u\|_{L^2} + \|u\|_{\dot{H}^{k_1}}) \le C(\|u\|_{H^1} + \|u\|_{\dot{H}^{k_1}})\\ \|A^{1/2}u\|_{L^\infty} \le C\|u\|_{H^{k_2}} \le C(\|\nabla u\|_{L^2} + \|u\|_{\dot{H}^{k_2}}) \le C(\|u\|_{H^1} + \|u\|_{\dot{H}^{k_2}})\end{aligned} \tag{3.6.22}$$

where $k_1 = 3/2 + \varepsilon$ and $k_2 = 5/2 + \delta$ for positive numbers ε and δ. So (3.6.21) becomes

$$\begin{aligned}(AP^\alpha(\nabla_u u), Au) \le C\|u\|_{\dot{H}^3}\|u\|_{\dot{H}^2}\|u\|_{H^1} + C\|u\|_{\dot{H}^3}\|u\|_{\dot{H}^2}\|u\|_{\dot{H}^{k_1}}\\ + C\|u\|_{\dot{H}^3}\|u\|_{H^1}\|u\|_{\dot{H}^{k_2}} + C\|u\|_{\dot{H}^3}\|u\|_{H^1}^2.\end{aligned} \tag{3.6.23}$$

By (3.6.20), we have

$$
\begin{aligned}
&\|u\|_{\dot H^2} = \|\nabla u\|_{\dot H^1} \le C\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{\dot H^2}^{1/2} \le C\|u\|_{\dot H^1}^{1/2}\|u\|_{\dot H^3}^{1/2}\\
&\|u\|_{\dot H^{k_1}} = \|\nabla u\|_{\dot H^{k_1-1}} \le C\|u\|_{\dot H^1}^{1-(k_1-1)/2}\|u\|_{\dot H^3}^{(k_1-1)/2}\\
&\|u\|_{\dot H^{k_2}} = \|\nabla u\|_{\dot H^{k_2-1}} \le C\|u\|_{\dot H^1}^{1-(k_2-1)/2}\|u\|_{\dot H^3}^{(k_2-1)/2}.
\end{aligned}
\tag{3.6.24}
$$

Applying (3.6.24) to (3.6.23), we have

$$
\begin{aligned}
(AP^\alpha(\nabla_u u), Au) \le& C\|u\|_{\dot H^3}^{1+k_1/2}\|u\|_{\dot H^1}^{2-k_1/2} + C\|u\|_{\dot H^3}^{(k_2+1)/2}\|u\|_{\dot H^1}^{(k_2+3)/2}\\
&+ C\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2.
\end{aligned}
\tag{3.6.25}
$$

Choosing $\varepsilon = \delta = 1/4$, we get

$$
\begin{aligned}
(AP^\alpha(\nabla_u u), Au) \le& C\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8})\\
&+ C\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2,
\end{aligned}
\tag{3.6.26}
$$

which finishes our K_1 estimate. For K_2, we have

$$
(A(\text{div } \tau^\alpha)(u), Au) \le \|u\|_{\dot H^2}\|A(\text{div } \tau^\alpha)(u)\|_{L^2}. \tag{3.6.27}
$$

To estimate the second term, we remark that it is sufficient to consider $A(1-\alpha^2\triangle)^{-1}\text{div } (\nabla u \cdot \nabla u)$, and we have

$$
\begin{aligned}
\|A(1-\alpha^2\triangle)^{-1}\text{div } (\nabla u\cdot\nabla u)\|_{L^2} &\le \|\text{div } (\nabla u\cdot\nabla u)\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{\dot H^2}.
\end{aligned}
\tag{3.6.28}
$$

Plugging this back into (3.6.27) and using (3.6.22) and (3.6.24) gives

$$
\begin{aligned}
(A(\text{div } \tau^\alpha)(u), Au) &\le C\|u\|_{\dot H^2}^2\|\nabla u\|_{L^\infty}\\
&\le C(\|u\|_{\dot H^2}^2\|u\|_{H^1} + \|u\|_{\dot H^2}^2\|u\|_{\dot H^{k_2}})\\
&\le C(\|u\|_{\dot H^3}\|u\|_{H^1}^2 + \|u\|_{\dot H^3}^{15/8}\|u\|_{H^1}^{23/8}).
\end{aligned}
\tag{3.6.29}
$$

Combining (3.6.26) and (3.6.29) gives

$$
\begin{aligned}
(AP^\alpha(\nabla_u u + \text{div } \tau^\alpha u), Au) \leq & C\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \\
& + C\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2.
\end{aligned} \tag{3.6.30}
$$

Applying Young's inequality (Proposition 3.10.3) for products with $p = 16/15$ and $p' = 16$, we get

$$
\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \leq C\varepsilon\|u\|_{\dot H^3}^2 + \frac{C}{\varepsilon}(\|u\|_{H^1}^{18} + \|u\|_{H^1}^{46}). \tag{3.6.31}
$$

Choosing $\varepsilon = (4C)^{-1}$, (3.6.31) becomes

$$
\|u\|_{\dot H^3}^{15/8}(\|u\|_{\dot H^1}^{9/8} + \|u\|_{\dot H^1}^{23/8}) \leq \frac{1}{4}\|u\|_{\dot H^3}^2 + C(\|u\|_{H^1}^{18} + \|u\|_{H^1}^{46}). \tag{3.6.32}
$$

Similarly, Young's inequality gives

$$
\|u\|_{\dot H^3}^{3/2}\|u\|_{H^1}^{3/2} + C\|u\|_{\dot H^3}\|u\|_{H^1}^2 \leq \frac{1}{4}\|u\|_{\dot H^3}^2 + C(\|u\|_{H^1}^6 + \|u\|_{H^1}^4). \tag{3.6.33}
$$

Using (3.6.32) and (3.6.33) in (3.6.30) gives

$$
(AP^\alpha(\nabla_u u + \text{div } \tau^\alpha u), Au) \leq \frac{1}{2}\|u\|_{\dot H^3}^2 + C(\|u\|_{H^1}^{18} + \|u\|_{H^1}^{46} + \|u\|_{H^1}^6 + \|u\|_{H^1}^4). \tag{3.6.34}
$$

Finally, using (3.6.16), (3.6.17) and (3.6.34) in (3.6.15) gives

$$
\begin{aligned}
\frac{1}{2}\partial_t\|u(t)\|_{\dot H^2}^2 &\leq \frac{-1}{2}\|u(t)\|_{\dot H^3}^2 + C(\|u(t)\|_{H^1}^{18} + \|u(t)\|_{H^1}^{46} + \|u\|_{H^1}^6 + \|u\|_{H^1}^4) \\
&\leq \frac{-1}{2}\|u(t)\|_{\dot H^3}^2 + C(\|\varphi\|_{H^1}^{18} + \|\varphi\|_{H^1}^{46} + \|\varphi\|_{H^1}^6 + \|\varphi\|_{H^1}^4),
\end{aligned} \tag{3.6.35}
$$

where the last line used (3.6.14). So, for any t such that

$$
\|u(t)\|_{\dot H^3} \geq C(\|\varphi\|_{H^1}^{18} + \|\varphi\|_{H^1}^{46} + \|\varphi\|_{H^1}^6 + \|\varphi\|_{H^1}^4)^{1/2}, \tag{3.6.36}
$$

we get that $\|u(t)\|_{\dot{H}^2}$ is decreasing as a function of time at t. So our last task is to show that $\|u\|_{\dot{H}^2}$ is bounded provided

$$\|u(t)\|_{\dot{H}^3} < C(\|\varphi\|_{H^1}^{18} + \|\varphi\|_{H^1}^{46} + \|\varphi\|_{H^1}^{6} + \|\varphi\|_{H^1}^{4})^{1/2}. \tag{3.6.37}$$

To handle this case, we again use (3.6.20), and get

$$\|u(t)\|_{\dot{H}^2} \leq C\|u(t)\|_{L^2}^{1/3}\|u(t)\|_{\dot{H}^3}^{2/3} \leq C\|\varphi\|_{L^2}^{1/3}(\|\varphi\|_{H^1}^{10}+\|\varphi\|_{H^1}^{2}+\|\varphi\|_{H^1}^{6}+\|\varphi\|_{H^1}^{4})^{1/3}. \tag{3.6.38}$$

Since the right hand side has no time dependence, we get that $\|u(t)\|_{\dot{H}^2}$ is bounded independent of time. Combining this with (3.6.14), we finally get

$$u \in L^\infty([0,T], \bar{C}_{3/2,2} \cap \dot{C}_{1/4;2,2}), \tag{3.6.39}$$

which proves the Theorem.

3.7. Solutions in the class of integral norms in time

We now seek to solve (1.1.2) in a different space. We fix $T > 0$ and let $\mathbb{M}((0,T) : \mathbb{E})$ be the set of measurable functions defined on $(0,T)$ with values in the space $\mathbb{E}$. Then we define

$$L^\sigma((0,T) : H^{s,q}) = \{f \in \mathbb{M}((0,T) : H^{s,q}) : (\int_0^T \|f(t)\|_{s,q}^\sigma dt)^{1/\sigma} < \infty\}. \tag{3.7.1}$$

We now state our second theorem.

Theorem 3.7.1. *For any $\varphi = P^\alpha\varphi \in H^{r,p}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.1.2) *such that*

$$u \in BC([0,T) : H^{r,p}) \cap L^a((0,T) : H^{k,c}) \tag{3.7.2}$$

provided the parameters (with $r = n/p + b$) satisfy (3.9.4). *If $\|\varphi\|_{r,p}$ is sufficiently small, then $T = \infty$. Lastly, we have that solutions depend continuously on the initial data.*

This is similar to Theorem 3.1 in [**6**].

We will use the same basic strategy as we used in the previous argument, and we begin with some supporting results.

3.8. Supporting Results

Our first result is Lemma 3.2 in [**6**] and involves the operator Γ.

PROPOSITION 3.8.1. *Let $1 < q_0 \le q_1 < \infty$, $s_0 \le s_1$, and assume $0 < (s_1 - s_0 + n/q_0 - n/q_1)/2 = 1/\sigma \le 1/q_0$. Then Γ maps H^{s_0,q_0} continuously into $L^\sigma((0,\infty) : H^{s_1,q_1})$, with the estimate*

$$\left(\int_0^\infty \|\Gamma u\|^\sigma_{H^{s_1,q_1}}\right)^{1/\sigma} \le C\|u\|_{H^{s_0,q_0}}. \tag{3.8.1}$$

To begin the proof, we first observe that $(s_1 - s_0 + n/q_0 - n/q_1)/2 = 1/\sigma < 1$ implies $s_1 - s_0 < 2$. So without loss of generality, we assume $s_0 = 0$ and $s_1 \in [0, 2)$. To finish the proof, we need the following Lemma.

LEMMA 3.8.2. *Define the quasi-linear operator T by*

$$(Tf)(t) = t^{-1/\sigma_q}\|f\|_{L^q}, \tag{3.8.2}$$

where σ_q is defined by

$$0 < (s_1 + n/q - n/q_1)/2 = 1/\sigma_q. \tag{3.8.3}$$

Then

$$T : L^q(\mathbb{R}^n) \to L_{\sigma_q,\infty}(I), \tag{3.8.4}$$

where $L_{1/\sigma_q,\infty}$ is weak-L^{σ_q} space (defined in section 3.12), and $I = [0,\infty)$.

To prove the Lemma, using results from 3.12, we have

$$\lambda_{Tf}(\tau) = m(\{t : (Tf)(t) > \tau\}) = \tau^{-\sigma_q}\|f\|_{L^q}^{\sigma_q}. \tag{3.8.5}$$

Then

$$\tau^{\sigma_q}\lambda_{Tf}(\tau) = \|f\|_{L^q}^{\sigma_q}, \tag{3.8.6}$$

which proves the lemma.

To finish the proposition, we define the quasi-linear operator K by

$$(Kf)(t) = \|e^{t\triangle}f\|_{H^{s_1,q_1}}. \tag{3.8.7}$$

Using the heat kernel estimate (Lemma 3.11.2), we have that

$$(Kf)(t) = \|e^{t\triangle}f\|_{H^{s_1,q_1}} \leq Ct^{-1/\sigma_q}\|f\|_{L^q} = C(Tf)(t), \tag{3.8.8}$$

so using the Lemma, we have that K is a quasi-linear map that satisfies

$$K : L^q(\mathbb{R}^n) \to L_{1/\sigma_q,\infty}(I). \tag{3.8.9}$$

For the last piece of the argument, we turn to interpolation. First, since $q_0 > 1$, we have a q', q'' such that $1 < q' < q_0 < q''$. Now define our interpolative variable θ by

$$1/q_0 = \frac{1-\theta}{q'} + \frac{\theta}{q''}. \tag{3.8.10}$$

Since $1/\sigma_q = C + n/q$ (where C is a real number independent of q), we have

$$(3.8.11)\qquad \frac{1}{\sigma} = C + \frac{n}{q_0} = (1-\theta)(C + n/q') + \theta(C + n/q'') = \frac{1-\theta}{\sigma_{q'}} + \frac{\theta}{\sigma_{q''}}.$$

Applying Proposition 3.12.1 (and recalling that L_{pp} is standard L^p space) finishes the proof.

We next establish a corollary.

Corollary 3.8.3. *For any $\varepsilon > 0$, there exists a T which depends only on ε and $\|u\|_{H^{s_0,q_0}}$ such that*

$$(3.8.12)\qquad \left(\int_0^T \|e^{t\triangle}u\|^{\sigma}_{H^{s_1,q_1}}dt\right)^{1/\sigma} \leq \varepsilon,$$

for all $0 < t < T$.

This follows from dominated convergence and Proposition 3.11.3.

Next, we consider the operator V^α on our integral norm space.

Proposition 3.8.4. *Let $b \geq 1$, $1 < q, p < \infty$, and $q = \frac{np}{2n - s'p}$ where $0 \leq s' \leq b-1$ and $s'p < m$. Then*

$$(3.8.13)\qquad V^\alpha : L^\sigma((0,T) : H^{b,p}) \to L^{\sigma/2}((0,T) : H^{b-1,q})$$

with the estimate

$$(3.8.14)\qquad \left(\int_0^T \|V^\alpha(u(s))\|^{\sigma/2}_{b-1,q}ds\right)^{2/\sigma} \leq \left(\int_0^T \|u(s)\|^{\sigma}_{b,p}ds\right)^{2/\sigma}.$$

This follows directly from Proposition 3.3.3. We also have that

$$
\begin{aligned}
&\left(\int_0^T \|V^\alpha(u(s)) - V^\alpha(v(s))\|_{b-1,q}^{\sigma/2} ds\right)^{2/\sigma} \\
&\leq \left(\int_0^T (\|v(s)\|_{b,p} + \|u(s)\|_{b,p})^{\sigma/2}(\|v(s) - u(s)\|_{b,p})^{\sigma/2} ds\right)^{2/\sigma} \\
&\leq \left(\int_0^T (\|v(s)\|_{b,p} + \|u(s)\|_{b,p})^{\sigma} ds\right)^{2/\sigma} \left(\int_0^T \|v(s) - u(s)\|_{b,p}^{\sigma} ds\right)^{2/\sigma}
\end{aligned}
\tag{3.8.15}
$$

where we used Holder's inequality and Minkowski's inequality. This gives an analog to Corollary 3.3.4.

Corollary 3.8.5. *With the same assumptions on the parameters as in Proposition 3.8.4, we have that if $u, v \in L^\sigma((0,T) : H^{b,p})$ then*

$$
\begin{aligned}
&\left(\int_0^T \|V^\alpha(u(s)) - V^\alpha(v(s))\|_{b-1,q}^{\sigma/2} ds\right)^{2/\sigma} \\
&\leq \left(\int_0^T (\|v(s)\|_{b,p} + \|u(s)\|_{b,p})^{\sigma} ds\right)^{2/\sigma} \left(\int_0^T \|v(s) - u(s)\|_{b,p}^{\sigma} ds\right)^{2/\sigma}.
\end{aligned}
\tag{3.8.16}
$$

Our next set of results involve the operator G.

Proposition 3.8.6. *Let $1 \leq q' \leq q'' < \infty$, $s' \leq s''$, $1 < \sigma' < \sigma'' < \infty$, and let $1/\sigma' - 1/\sigma'' = 1 - (s'' - s' + n/q' - n/q'')/2$. Then for any $T \in (0, \infty]$, G maps $L^{\sigma'}((0,T) : H^{s',q'})$ continuously into $L^{\sigma''}((0,T) : H^{s'',q''})$.*

Using Proposition 3.3.6, we observe that

$$
\|Gu(t)\|_{s'',q''} \leq C \int_0^T |t-s|^{(s''-s'+m/q'-m/q'')/2} \|u(s)\|_{s',q'} ds = C I_r f(t) \tag{3.8.17}
$$

where $1/r = (s'' - s' + m/p' - m/p'')/2$, I_r is defined as in Theorem 3.10.2, and $f(t) = \|u(t)\|_{s',q'}$. Then using the Hardy-Littlewood-Sobolev Theorem (Theorem 3.10.2) with

$n = 1$ and $1/\sigma' - 1/\sigma'' = 1 - 1/r$, we have

$$(3.8.18) \qquad (\int_0^T \|Gu\|_{s'',q''}^{\sigma''} dt)^{1/\sigma''} \le C\|I_r f\|_{L^{\sigma''}(I)} \le C\|f\|_{L^{\sigma'}(I)}$$

where $f(t) = \|u(t)\|_{s',q'}$. This completes the proof.

Our next result also involves the operator G, but its proof is significantly more complicated.

Proposition 3.8.7. *Let $1 < q' \le q'' < \infty$, $s' \le s''$ and assume $1/q'' \le 1/\sigma = 1 - (s'' - s' + n/q' - n/q'')/2 \le 1$. Then G maps $L^\sigma((0,T) : H^{s',q'})$ continuously into $BC([0,T) : H^{s'',q''})$.*

To prove this, we begin with a lemma.

Lemma 3.8.8. *Define H by $(Hf)(s,x) = e^{s\triangle} f(s,x)$. Then*

$$(3.8.19) \qquad H : L^\sigma((0,T) : H^{s',q'} \to L^1((0,T) : H^{s'',q''}),$$

where the parameters are as in the proposition.

Our first step is to recall that, by (3.11.3), we have

$$(3.8.20) \qquad \begin{aligned} \int f(x) e^{s\triangle}(g(x)dx &= (4\pi s)^{-n/2} \int\int f(x) e^{|x-y|^2/4s} g(y)dydx \\ &= -(4\pi s)^{-n/2} \int\int f(x) e^{|y-x|^2/4s} g(y) \\ &= -\int e^{s\triangle} f(y)g(y)dy. \end{aligned}$$

We remark that this property is shared by the operator $(1-\triangle)^{k/2}$ for any real number k.

Next, we recall that the dual of the space $L^q((0,T):L^p)$ is $L^{\bar{q}}((0,T):L^{\bar{p}})$ (where $\bar{q}$ and $\bar{p}$ denote the conjugate exponents to q and p and $1 \le p,q < \infty$). Then, for any $g \in L^\infty((0,T):H^{-s'',\bar{q}''})$, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} g(s,x) e^{s\triangle}(1-\triangle)^{s''/2} f(s,x)\,dx\,ds \\
\le& C\int_0^T \int_{\mathbb{R}^m} e^{s\triangle}(1-\triangle)^{(s''-s')/2} g(s,x)(1-\triangle)^{s'/2} f(s,x)\,dx\,ds \\
\le& C\int_0^T \|e^{s\triangle} g(s)\|_{H^{s''-s',\bar{q}'}} \|f(s)\|_{H^{s',q'}}\,ds \qquad (3.8.21)\\
\le& C\Big(\int_0^T \|e^{s\triangle} g(s)\|^{\bar{\sigma}}_{H^{s''-s',\bar{q}'}}\,ds\Big)^{1/\bar{\sigma}}\Big(\int_0^T \|f(s)\|^{\sigma}_{H^{s',q'}}\,ds\Big)^{1/\sigma} \\
\le& C\sup_s \|g(s)\|_{\bar{q}''}\Big(\int_0^T \|f(s)\|^{\sigma}_{H^{s',q'}}\,ds\Big)^{1/\sigma},
\end{aligned}
$$

where the last line is a slight generalization of Proposition 3.8.1. Since g is an arbitrary element of the dual space of $L^1((0,T):H^{s'',q''})$, we have

$$
\int_0^T \|e^{s\triangle} f(s)\|_{H^{s'',q''}}\,ds \le C\Big(\int_0^T \|f(s)\|^{\sigma}_{H^{s',q'}}\,ds\Big)^{1/\sigma} \qquad (3.8.22)
$$

which completes the Lemma.

Returning to the proposition, making liberal use of the change of variables formula and using (3.8.22), we have

$$
\begin{aligned}
\|G\cdot f(t)\|_{s'',q''} &= \Big\|\int_0^t e^{(t-s)\triangle} f(s)\,ds\Big\|_{s'',q''} \\
&\le \int_0^t \|e^{(t-s)\triangle} f(s)\|_{s'',q''}\,ds \\
&= \int_0^t \|e^{s\triangle} f(t-s)\|_{s'',q''}\,ds \qquad (3.8.23)\\
&\le C\Big(\int_0^t \|f(t-s)\|^{\sigma}_{s',q'}\,ds\Big)^{1/\sigma} \\
&\le C\Big(\int_0^T \|f(s)\|^{\sigma}_{s',q'}\,ds\Big)^{1/\sigma},
\end{aligned}
$$

which proves the proposition.

3.9. Proof of Theorem 3.7.1

As in Section 3.4, we begin with the nonlinear map

$$\Phi u = \Gamma\varphi - G \cdot P^{\alpha}(V^{\alpha}(u)) \tag{3.9.1}$$

and the space $F_{T,M}$ defined to be the space of all

$$v \in BC([0,T) : H^{r,p}) \cap L^{a}((0,T) : H^{k,c}) \tag{3.9.2}$$

such that

$$\sup_{0\le t\le T} \|v(t) - \Gamma\varphi\|_{r,p} + \left(\int_0^T \|v(s)\|_{k,c}^{a} ds\right)^{1/a} \le M. \tag{3.9.3}$$

Using the same argument used in Section 3.4, we get that Φ will be a contraction mapping provided the following list of conditions is satisfied

$$\begin{aligned}
&1 < \bar{c} \le p \le c < \infty\\
&s' := k - 2 + b' - b\\
&k \ge 1,\ b' \ge 1,\ s'c < n\\
&0 < 2/a = k - n/c - b < 1\\
&0 \le s' \le k - 1\\
&\bar{c} = \frac{nc}{2n - s'c}\\
&1 \ge b' - b\\
&k - b' \le \frac{n}{p} + b \le k\\
&a/2 \le p \le a.
\end{aligned} \tag{3.9.4}$$

We observe that as in the previous case, these conditions require that $b \geq 0$. We also record the simplified list that arises from setting $b' = 1$:

$$
\begin{aligned}
&1 < \bar{c} \leq p \leq c < \infty \\
&s' := k - 1 - b \\
&k \geq 1, \;\; s'c < n \\
&0 < 2/a = k - n/c - b < 1 \\
&0 \leq k - 1 - b \\
&k = 1 + b + \frac{2n}{c} - n\bar{c} \\
&k - 1 \leq \frac{n}{p} + b \leq k \\
&a/2 \leq p \leq a.
\end{aligned} \tag{3.9.5}
$$

We record the result for the special case $p = 2$, $n = 3$.

THEOREM 3.9.1. *For any $\varphi = P^\alpha \varphi \in H^{3/2,2}$ there is a unique global solution to* (1.1.2) *such that*

$$u \in BC([0,T) : H^{3/2,2}) \cap L^{5/2}((0,T) : H^{2,5/2}). \tag{3.9.6}$$

The local solution follows by choosing $p = k = c = 2$, $n = 3$, $b = 0$, $b' = s' = 1$ and $a = 4$.

The local result extends to a global result via an argument similar to the one used in Section 3.6, and the continuous dependance on the initial data follows from the argument in 3.4.

3.10. Appendix

Our first result involves the Gamma function. Recall $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

PROPOSITION 3.10.1. *Define*

$$B(x,y) = \int_0^t s^{x-1}(t-s)^{y-1} ds. \tag{3.10.1}$$

Then

$$B(x,y) = t^{x+y-1} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.10.2}$$

From equation $(A.23)$ and $(A.24)$ in appendix A of chapter 3 from [**16**], we get that

$$\int_0^1 (1-s)^{x-1} s^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.10.3}$$

The proposition follows from a straightforward change of variables calculation.

The next result we state is called the Hardy-Littlewood-Sobolev Theorem.

THEOREM 3.10.2. *If* $r > 1$ *and* $1/r = 1 - (1/p - 1/q)$ *for some* $1 < p < q < \infty$, *then*

$$\|I_r(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{3.10.4}$$

where

$$I_r f(x) = \int_{\mathbb{R}^n} |x-y|^{-n/r} f(y) dy. \tag{3.10.5}$$

We also state Young's inequality for integrals and for products.

THEOREM 3.10.3. *Let* $1 \leq p, q, r \leq \infty$ *and* $r^{-1} + q^{-1} = p^{-1} + 1$. *If* $f \in L^r$ *and* $g \in L^q$, *then* $f * g \in L^p$ *with the estimate*

$$\|f * g\|_p \leq \|f\|_r \|g\|_q. \tag{3.10.6}$$

This is Theorem 8.9 in [**5**].

PROPOSITION 3.10.4. *Let $1/p + 1/q = 1$. Then if a, b are positive real numbers, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{3.10.7}$$

We remark that this also gives, for any $\varepsilon > 0$,

$$(\varepsilon a)\frac{b}{\varepsilon} = ab \leq \frac{(\varepsilon a)^p}{p} + \frac{b^q}{\varepsilon^q q}. \tag{3.10.8}$$

We now turn our attention to Sobolev space results. Our first result is a straightforward imbedding theorem.

LEMMA 3.10.5. *$H^{k+\epsilon,p} \to H^{k,p}$ for all k, $1 < p < \infty$, and $\varepsilon > 0$.*

For the case where $k \geq 0$, see [**16**], chapter 13, equation 6.9. The proof for the negative k case follows by viewing the space $H^{-k,p}$ as the dual of the space $H^{k,p'}$, where $k > 0$ and p' is the unique real number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

Our next result is the Sobolev imbedding theorem.

PROPOSITION 3.10.6. *For $sp < m$ where $1 \leq p < \infty$ and $s > 0$ we have*

$$H^{s,p}(\mathbb{R}^m) \subset L^{mp/(m-sp)}(\mathbb{R}^m). \tag{3.10.9}$$

with the estimate $\|u\|_{mp/(m-sp)} \leq C\|u\|s, p$.

For $sp > m$, we have

$$H^{s,p}(\mathbb{R}^m) \subset L^{\infty}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \tag{3.10.10}$$

with the estimate $\|u\|_{L^{\infty}(\mathbb{R}^n)} \leq C\|u\|_{H^{s,p}(\mathbb{R}^n)}$.

For a proof, see Chapter 13, sections 2 and 6 in [**16**]. Our next result is another Sobolev space result.

PROPOSITION 3.10.7. *Let $s > 0$. Then*

$$\|u\|_{H^{s,2}} \leq \|u\|_{L^2} + \|u\|_{\dot{H}^{s,2}}, \tag{3.10.11}$$

where $\dot{H}$ denotes the homogenous Sobolev space.

To prove this, we need to bound $(1+|\xi|^2)^s$. For $|\xi| \leq 1$, we immediately have

$$(1+|\xi|^2)^s \leq 2^s \leq C(1+|\xi|^{2s}), \tag{3.10.12}$$

for a sufficiently large C. For $|\xi| > 1$,

$$(1+|\xi|^2)^s \leq C|\xi|^{2s} \leq C(1+|\xi|^{2s}). \tag{3.10.13}$$

So we have, for all $|\xi|$, that

$$(1+|\xi|^2)^s \leq C((1+|\xi|^2)^k + |\xi|^{2s}). \tag{3.10.14}$$

Using (3.10.14), we have

$$\int (1+|\xi|^2)^s \hat{u}^2 \leq C\left(\int \hat{u}^2 + \int |\xi|^{2s}\hat{u}^2\right), \tag{3.10.15}$$

and applying Plancherel's theorem finishes the proposition.

We next consider a Moser-type estimate for Sobolev spaces

PROPOSITION 3.10.8. *Let $u \in H^{s,p}$ for any $s > 0$, $1 < p \leq \infty$. Then*

$$\|u^2\|_{s,r} \leq C\|u\|_{s,p}^2 \tag{3.10.16}$$

provided $r = \frac{mp}{2m-s'p}$ and $s' \leq s$, $s'p < m$.

This is a straightforward consequence of the following proposition, called the Christ-Weinstein estimate.

PROPOSITION 3.10.9. *Let $s > 0$, $1 < r, p_1, p_2, q_1, q_2 < \infty$, and suppose that $r^{-1} = p_i^{-1} + q_i^{-1}$ for $i = 1, 2$. Let $f \in L^{p_1}$, $D^s f \in L^{p_2}$, $g \in L^{q_2}$ and $D^s g \in L^{q_1}$. Then*

$$\|(fg)\|_{s,r} \leq C\|f\|_{p_1}\|g\|_{s,q_1} + C\|f\|_{s,p_2}\|g\|_{q_2}. \tag{3.10.17}$$

When $0 < s < 1$ this is Proposition 3.3 in [**3**]. The general case can be found in Proposition 1.1 of Chapter 2 of [**17**].

To prove Proposition 3.10.8, we choose $p_1 = q_2$ and $q_1 = p_2 = p$ in Proposition 3.10.9 and we have

$$\|(uv)\|_{s,r} \leq C\|u\|_{p_1}\|v\|_{s,p} + C\|u\|_{s,p}\|v\|_{p_1}. \tag{3.10.18}$$

By Proposition 3.10.6, $\|u\|_{p_1} \leq C\|u\|_{s',p}$ provided $p_1 = mp/(m - s'p)$ and $s'p < m$. So we have

$$\|(uv)\|_{s,r} \leq C\|u\|_{s',p}\|v\|_{s,p} + C\|u\|_{s,p}\|v\|_{s',p}. \tag{3.10.19}$$

So for any $s' \leq s$ and $s'p < m$, Sobolev Imbedding gives

$$\|u^2\|_{s,r} \leq C\|u\|_{s,p}^2 \tag{3.10.20}$$

provided $r = \frac{mp}{2m-s'p}$ and $s' \leq s$, $s'p < m$.

3.11. Semigroup properties of the heat kernel

In this section we outline some properties of the the operator $e^{t\triangle}$. We begin by stating two different ways this operator can be defined. We have a pseudo-differential operator type definition

$$e^{t\triangle}u(x) = \int e^{-t|\xi|^2}\hat{u}(\xi)e^{ix\cdot\xi}d\xi \tag{3.11.1}$$

and a definition involving the "heat kernel"

$$e^{t\triangle}\delta(x) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}. \tag{3.11.2}$$

With this definition, we have

$$e^{t\triangle}u(x) = u * e^{t\triangle}\delta(x). \tag{3.11.3}$$

These are equations 5.17 and 5.10 from Chapter 3 of [**16**].

Our first task is to describe the action of $e^{t\triangle}$ on $L^p(\mathbb{R}^n)$ spaces.

LEMMA 3.11.1. *Let* $1 \leq q \leq p \leq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\|e^{t\triangle}u\|_{s,p} \leq Ct^{-(n/q-n/p)/2}\|u\|_{s,q}. \tag{3.11.4}$$

Recalling that $\int_{\mathbb{R}^n} e^{t\triangle}\delta(x) = 1$, we calculate

$$\begin{aligned}
\|\,(4\pi t)^{-n/2}\, e^{-|x|^2/4t}\|_r &= (4\pi t)^{-n/2}\left(\int_{\mathbb{R}^n} e^{-r|x|^2/4t}dx\right)^{1/r} \\
&\leq (4\pi t)^{-n/2}\left(\int_{\mathbb{R}^n} e^{-|x|^2/4t}dx\right)^{1/r} \\
&\leq (4\pi t)^{-n/2}\,(4\pi t)^{n/2r} \\
&\leq (4\pi t)^{-(1-1/r)/2}.
\end{aligned} \tag{3.11.5}$$

This estimate combined with Young's inequality (Theorem 3.10.3) proves the Lemma.

Our second Lemma requires some preliminary work. We begin by considering the operator $e^{z\triangle}$ defined on the right half of the complex plane. Then for each $t > 0$ on the real axis, we let γ_t be the circle centered at t of radius at for any $0 < a < 1$. By Cauchy's integral formula, with $P(z)f = e^{z\triangle}f$, we have

$$\triangle^k P(t)f(x) = P^{(k)}(t)f(x) = C\int_{\gamma_t} \frac{1}{(z-t)^{k+1}} P(z)f(x)dz, \tag{3.11.6}$$

where $P^{(k)}$ denotes the k^{th} time derivative of P. By Minkowski's integral inequality, we have

$$\| \int_{\gamma_t} \frac{1}{(z-t)^{k+1}} P(z)f(\cdot)dz\|_p \leq C\int_{\gamma_t} \frac{1}{|z-t|^{k+1}} \|P(z)f(\cdot)\|_p dz. \tag{3.11.7}$$

For $z \in \gamma_t$, we have $(1-a)t \leq |z| \leq (1+a)t$, so using a calculation similar to (3.11.5) we have

$$\begin{aligned} \|\triangle^k P(t)f(x)\|_p &\leq Ct^{-(n/q-n/p)/2}\|f\|_q \int_{\gamma_t} \frac{1}{(at)^{k+1}} dz \\ &\leq Ct^{-(n/q-n/p)/2-k}\|f\|_q. \end{aligned} \tag{3.11.8}$$

So for any integer k, we have

$$\|e^{t\triangle}f\|_{2k,p} \leq Ct^{-(2k+n/q-n/p)/2}\|f\|_q. \tag{3.11.9}$$

Interpolating between (3.11.5) and (3.11.9) we have

$$\|e^{t\triangle}f\|_{s'',p} \leq Ct^{-(s''-s'+n/q-n/p)/2}\|f\|_{s',q} \tag{3.11.10}$$

for $0 \leq s' \leq s'' \leq 2k$. But since this holds for any integer k, we have almost shown the following lemma.

Lemma 3.11.2. *For $-\infty < s' \leq s'' < \infty$ and $q'' \geq q'$, we have*

$$\|e^{t\triangle}u\|_{s'',q''} \leq Ct^{-(m/q'-m/q'')/2-(s''-s')/2}\|u\|_{s',q'}. \tag{3.11.11}$$

To prove the Lemma, we need to address the case where s' and s'' are negative. To this end, let J_s be a convolution operator whose Fourier Transform is given by

$$\hat{J}_s(|\xi|) = (1 + |\xi|^2)^{s/2} \tag{3.11.12}$$

and we recall that

$$\|J_r u\|_{s,p} = \|u\|_{r+s,p}. \tag{3.11.13}$$

Then for $s'' < 0$, we have

$$\begin{aligned} \|e^{t\triangle}u\|_{s'',q''} &= \|e^{t\triangle}J_{-2s''}u\|_{-s'',p} \\ &\le Ct^{-(m/q'-m/q'')/2-(-s''-(s'-2s''))/2}\|J_{-2s''}u\|_{s'-2s'',q'} \\ &\le Ct^{-(m/q'-m/q'')/2-(s''-s')/2}\|u\|_{s''',q'}, \end{aligned} \tag{3.11.14}$$

which finishes the Lemma.

For notational convenience, we set $r = (m/q' - m/q'')/2 + (s'' - s')/2$ and observe that for $r \ge 0$, the previous results show that $t^r e^{t\triangle}$ is a uniformly bounded set of linear operators from $H^{s',q'}$ into $H^{s'',q''}$ for small t. Our next task is to describe the behavior of this operator as t tends to 0.

Proposition 3.11.3. *As t tends to 0, $t^r e^{t\triangle}u$ tends to 0 in $H^{s'',q''}$ for any $u \in H^{s',q'}$, provided $r > 0$, $q'' \ge q'$ and $s'' \ge s'$. More specifically, with the parameters fixed, for any $\varepsilon > 0$ there exists a $T > 0$ which depends only on $\|u\|_{s',q'}$ such that for all $0 < t < T$,*

$$\|t^r e^{t\triangle}u\|_{s'',q''} < \varepsilon. \tag{3.11.15}$$

We first show this under the assumption that $u \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions. For $u \in \mathcal{S}$ we have that $u \in L^q$ for all $q \ge 1$, so in particular $u \in L^{q''}$. Using Lemma 3.11.1 we have

$$t^r \|e^{t\triangle} u\|_{s'',q''} \le t^r t^{-(n/q''-n/q'')/2} \|u\|_{s'',q''}. \tag{3.11.16}$$

Since $r > (\frac{n}{q''} - \frac{n}{q'})/2$, this proves the result for $u \in \mathcal{S}$.

Now let $\varphi \in H^{s',q'}$ be arbitrary. Since $\mathcal{S}$ is dense in $H^{s',q'}$ we choose an approximating sequence $\varphi_n \in \mathcal{S}$ and get

$$\begin{aligned} \|t^r e^{t\triangle}\varphi\|_{s'',q''} &\le \|t^r e^{t\triangle}(\varphi - \varphi_n)\|_{s'',q''} + \|t^r e^{t\triangle}\varphi_n\|_{s'',q''} \\ &\le \|\varphi - \varphi_n\|_{s',q'} + \|t^r e^{t\triangle}\varphi_n\|_{s'',q''}. \end{aligned} \tag{3.11.17}$$

Since φ_n approximates φ in the $H^{s',q'}$ norm, the first term can be made arbitrarily small by choosing a sufficiently large n. By (3.11.16), $|t^r e^{t\triangle}\varphi_n\|_{s'',q''}$ can be made arbitrarily small by choosing a sufficiently small t, which finishes the proof of Proposition 3.11.3.

3.12. Lorentz spaces and Weak-L^p

We begin by defining Lorentz spaces. Given a measure μ and a measurable function f, we define

$$m(\sigma, f) = \mu(\{x : |f(x)| > \sigma\}). \tag{3.12.1}$$

We next define the decreasing rearrangement of f, denoted f^*, by

$$f^*(t) = \inf\{\sigma : m(\sigma, f) \le t\}. \tag{3.12.2}$$

Then we say f is in the Lorentz space L_{pr} if and only if

$$\|f\|_{L_{pr}} := \left(\int_0^\infty (t^{1/p} f^*(t))^r dt/t \right)^{1/r} < \infty \tag{3.12.3}$$

where $1 \le p \le \infty$ and $1 \le r < \infty$ with the usual modification if $r = \infty$. We remark that $L_{pp} = L^p$, in the sense of equivalent norms.

We next recall the definition of weak-L^p space. For a function f defined on an interval I, the distribution function of f is defined by

$$\lambda_f(\tau) = m(\{t \in I : |f(t)| > \tau\}). \tag{3.12.4}$$

Then we say f is in weak-L^p if

$$\lambda_f(\tau) \leq C/\tau^p. \tag{3.12.5}$$

We remark that Weak-L^p space is a special case of Lorentz space L_{pq} where $q = \infty$, and that, if $0 \leq f(t) \leq g(t)$ for all $t \in I$, then

$$\|f\|_{L_{q,\infty}} \leq \|g\|_{L_{q,\infty}} \tag{3.12.6}$$

where $\|\cdot\|_{L_{q,\infty}}$ is the weak-L^q norm.

The following is Theorem 5.3.1 in [**1**].

THEOREM 3.12.1. *Let p_0, p_1, q_0, q_1 and q be positive, possibly infinite numbers and let $1/p = (1-\theta)/p_0 + \theta/p_1$ where $0 < \theta < 1$. Then, if $p_0 \neq p_1$,*

$$(L_{p_0q_0}, L_{p_1q_1})_{\theta,q} = L_{pq}. \tag{3.12.7}$$

CHAPTER 4

Besov Space solutions to LANS

4.1. Solutions to LANS in Besov Spaces

In this section we mirror the results of the previous section. Instead of assuming our initial data is in a Sobolev space, we assume it is in a Besov space $B^r_{p,q}$. In (4.2) we give a brief derivation of Besov spaces and list some foundational results. In (4.3) we define Continuous-in-time Besov spaces, state our existence theorem, and prove some supporting results necessary. In 4.4) we prove the theorem stated in the previous subsection. In (4.5) we define the Integral-in-time Besov spaces and state our second local existence theorem. In (4.6) we prove supporting results and in (4.7) we prove the Theorem 4.5.1.

4.2. Besov Space

We now turn our attention to finding solutions of (1.1.2) with initial data in inhomogeneous Besov Spaces. The inhomogeneous Besov space $B^s_{p,q}$ (with $s \geq 0$) is a Banach space with the norm

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}} \tag{4.2.1}$$

where $\dot{B}^s_{p,q}$ is the homogeneous Besov space, which we now define. Starting with a positive function $\phi := \phi_0 \in \mathcal{S}(\mathbb{R}^n)$ supported on the annulus $1/2 \leq |\xi| \leq 2$, we define, for $j \in \mathbb{Z}$, $\phi_j(\xi) = \phi_0(2^{-j}\xi)$ and observe that ϕ_j is supported on the annulus A_j, where $A_j = \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. Our next step is to define $\psi_j \in \mathcal{S}$ by

$$\hat{\psi}_j(\xi) = \frac{\phi_j(\xi)}{\sum_k \phi_k(\xi)}. \tag{4.2.2}$$

Since the Fourier Transform is invertible on $\mathcal{S}$, this uniquely identifies ψ_j.

LEMMA 4.2.1. *With ψ_j and ϕ_j defined as above, we have that*

$$
\begin{aligned}
&\hat{\psi}_j(\xi) = \hat{\psi}_0(2^{-j}\xi),\\
&supp\ \hat{\psi}_j \subset A_j,\\
&|D^\beta \hat{\psi}_j(\xi)| \le C_\beta 2^{-j|\beta|},\\
&\psi_j(x) = 2^{jn}\psi_0(2^j x),\\
&\sum_{k=-\infty}^{\infty} \hat{\psi}_k(\xi) = 1 \ \mathit{for}\ \xi \neq 0.
\end{aligned} \tag{4.2.3}
$$

The proof of the first relation follows the construction of ϕ_0 and that

$$\sum_k \phi_k(\xi) = \sum_k \phi_k(2^j\xi) \tag{4.2.4}$$

holds for any j. The second follows from the fact that ϕ_j is supported on A_j, the third is an immediate consequence of the first, and to get the fourth relation we note that

$$
\begin{aligned}
\psi(2^j x) = \mathcal{F}^{-1}(\hat{\psi}_0(2^j x)) &= \int \frac{\phi_0(\xi)}{\sum_k \phi_k(\xi)} e^{2^j ix\cdot\xi} d\xi\\
&= 2^{-jn} \int \frac{\phi_0(2^{-j}\xi)}{\sum_k \phi_k(2^{-j}\xi)} e^{ix\cdot\xi} d\xi\\
&= 2^{-jn}\psi_j(x).
\end{aligned} \tag{4.2.5}
$$

To get the last relation, we observe that A_j and A_k are disjoint if $|j-k| \ge 2$. So for $\xi \in A_i$, we have by the second relation that

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \hat{\psi}_k(\xi) &= \sum_{k=i-2}^{i+2} \hat{\psi}_k(\xi)\\
&= \frac{\sum_{k=i-2}^{i+2} \phi_k(\xi)}{\sum_{k=-\infty}^{\infty} \phi_k(\xi)} = 1.
\end{aligned} \tag{4.2.6}
$$

With the Lemma finished, we define the operator $\triangle_j$ by

$$\triangle_j f = \psi_j * f \tag{4.2.7}$$

and define

$$S_j = \sum_{k=-\infty}^{j-1} \triangle_j. \tag{4.2.8}$$

We record some useful facts about these operators.

Lemma 4.2.2. *With $\triangle_j f = \psi_j * f$ and $S_j = \sum_{k=\infty}^{j-1} \triangle_j$ we have*

$$\begin{aligned} &\triangle_j \triangle_k = 0 \text{ if } |j-k| \geq 2, \\ &\triangle_j(\triangle_k f \triangle_i g) = 0 \text{ if } |j-k| \geq 4 \text{ and } |i-k| \leq 1, \\ &\triangle_j(S_{k-1} f \triangle_k g) = 0 \text{ if } |j-k| \geq 3. \end{aligned} \tag{4.2.9}$$

To show the first equality, we have

$$\mathcal{F}(\triangle_j \triangle_k f) = \hat{\psi}_j \hat{\psi}_k \hat{f}. \tag{4.2.10}$$

Because $\hat{\psi}_j$ is supported on A_j and $\hat{\psi}_k$ is supported on A_k, we get that $\mathcal{F}(\triangle_j \triangle_k f) \equiv 0$ provided $A_j \cap A_k$ is empty, which holds provided $|j-k| \geq 2$. To show the second equality, we have

$$\mathcal{F}(\triangle_j(\triangle_k f \triangle_i g))(x) = \hat{\psi}_j(x)(\hat{\psi}_k \hat{f} * \hat{\psi}_i \hat{g})(x). \tag{4.2.11}$$

We have

$$\hat{\psi}_j(x)(\hat{\psi}_k \hat{f} * \hat{\psi}_i \hat{g})(x) = \int \hat{\psi}_j(x) \hat{\psi}_k(x-y) \hat{f}(x-y) \hat{\psi}_i(y) \hat{g}(y) dy. \tag{4.2.12}$$

We note that the integrand of (4.2.12) is supported on

$$\begin{aligned} &2^{j-1} \leq |x| \leq 2^{j+1} \\ &2^{i-1} \leq |y| \leq 2^{i+1} \\ &2^{k-1} \leq |x-y| \leq 2^{k+1}. \end{aligned} \tag{4.2.13}$$

Since $|i-k| \leq 1$, this support is empty if $|j-k| \geq 4$.

With these properties in hand, we now define the homogeneous Besov spaces. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ we define the homogeneous Besov space $\dot{B}^s_{p,q}$ to be the Banach space with norm

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j=-\infty}^{\infty} (2^{js} \|\triangle_j f\|_p)^q \right)^{1/q} \tag{4.2.14}$$

with the usual modification when $q = \infty$. As stated in the beginning of this section, Besov spaces are the normed Banach spaces defined be the norm

$$\|f\|_{B^s_{p,q}} = \|f\|_p + \|f\|_{\dot{B}^s_{p,q}}, \tag{4.2.15}$$

for $s \geq 0$. For $s > 0$, we define $B^{-s}_{p',q'}$ to be the dual of the space $B^s_{p,q}$, where p', q' are the Holder-conjugates to p, q.

We conclude by remarking that by switching the order of the L^p and l^q norms in the definition of Besov spaces, we get a new space called the Triebel-Lizorkin spaces $F^s_{p,q}$. Explicitly, for $s \geq 0$, the Tribel-Lizorkin norm is defined by

$$\|f\|_{F^s_{p,q}} = \|f\|_p + \| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} |\triangle_j u(\cdot)|^q \right)^{1/q} \|_p. \tag{4.2.16}$$

For $s < 0$, the space $F^s_{p,q}$ is defined to be the dual of the space $F^{-s}_{p',q'}$. We also remark that $H^{s,p} = F^s_{p,2}$.

4.3. Weighted continuous in time Besov Spaces

Fixing $0 < T \leq \infty$, for any $k \geq 0$, we define the space

$$C^T_{k;s,p,q} = \{f \in C((0,T) : B^s_{p,q}) : ||f||_{k;s,p,q} < \infty\} \tag{4.3.1}$$

where

$$\|f\|_{k;s,p,q} = \sup\{t^k \|f(t)\|_{B^s_{p,q}} : t \in (0,T)\}. \tag{4.3.2}$$

$\dot{C}^T_{k;s,p,q}$ denotes the subspace of $C^T_{k;s,p,q}$ consisting of f such that

$$\lim_{t\to 0^+} t^k f(t) = 0 \text{ (in } B^s_{p,q}). \tag{4.3.3}$$

If $k = 0$, we write $\overline{C}^T_{s,p,q}$ for $BC([0,T) : B^s_{p,q})$, the space of bounded, continuous functions from $[0,T)$ to $B^s_{p,q}$.

We will typically write $C^T_{k;s,p,q}$ and $\overline{C}^T_{s,p,q}$ as $C_{k;s,p,q}$ and $\overline{C}_{s,p,q}$, respectively, suppressing the T dependance. We finish this section by stating the existence result and proving some supporting results.

THEOREM 4.3.1. *For any $\varphi = P^\alpha \varphi \in B^r_{p,q}$ there is a $T = T(\varphi) > 0$ and a unique solution to* (1.1.2) *such that*

$$u \in \bar{C}_{r,p,q} \cap \dot{C}_{a;k,b,c}, \tag{4.3.4}$$

provided there exist real numbers b' and s' such that (4.4.3)

Our first calculation is an analog of Lemma 3.3.1.

LEMMA 4.3.2. *Given $r \geq 1$, $1 \leq q < \infty$ and $p, p' \in (1,\infty)$ where*

$$\begin{aligned} p' &= \frac{np}{2n - s'p} \\ s' &= n(2/p - 1/p') \\ 0 &\leq s' \leq r - 1 \\ p &\leq 2p', \end{aligned} \tag{4.3.5}$$

we have div $\tau^\alpha : B^r_{p,q} \to B^r_{p',q}$*. Specifically, we have the estimate*

$$\| div\ \tau^\alpha(u)\|_{B^r_{p',q}} \le C\|u\|^2_{B^r_{p,q}}. \tag{4.3.6}$$

We have by Proposition 4.8.3 that

$$\begin{aligned}\|\text{div}\ \tau^\alpha(u)\|_{B^r_{p',q}} &\le C\|\tau^\alpha(u)\|_{B^{r+1}_{p',q}} \\ &\le C\|Def(u)\cdot Rot(u)\|_{B^{r-1}_{p',q}} \\ &\le C\|\nabla u\|^2_{B^{r-1}_{p,q}} \le C\|u\|^2_{B^r_{p,q}}.\end{aligned} \tag{4.3.7}$$

This Lemma has an immediate corollary.

Corollary 4.3.3. *div* $\tau^\alpha : \dot{C}_{a;r,p,q} \to \dot{C}_{2a;r,p',q}$*, with the estimate*

$$\| div\ \tau^\alpha(u)\|_{2a;r,p',q} \le C\|u\|^2_{a;r,p,q}. \tag{4.3.8}$$

Next, we record some results for the operator V^α.

Proposition 4.3.4. *With the parameters as in Lemma* 4.3.2*, we have*

$$V^\alpha : \dot{C}_{a;s,p,q} \times \dot{C}_{a;s,p,q} \to \dot{C}_{2a;s-1,p',q} \tag{4.3.9}$$

with the estimate

$$\|V^\alpha(u,v)\|_{2a;s-1,p',q} \le \|u\|_{a;s,p,q}\|v\|_{a;s,p,q}. \tag{4.3.10}$$

This follows from a calculation parallel to the one used to prove Proposition 3.3.3 with Proposition 4.8.1 replacing the Sobolev embedding results.

COROLLARY 4.3.5. *With the same assumptions on the parameters as in Proposition 4.3.4, we have that*

$$\begin{aligned} &\|V^\alpha(u(s)) - V^\alpha(v(s))\|_{a;s-1,p',q} \\ &\le C(\|u\|_{a/2;s,p,q} + \|v\|_{a/2;s,p,q})\|u-v\|_{a/2;s,p,q}. \end{aligned} \tag{4.3.11}$$

This proof also directly follows the proof for Corollary 3.3.4.

For the remainder of the section, we impose the following list of restrictions on our parameters. We have

$$\begin{aligned} &-\infty < s_0 \le s_1 < \infty \\ &1 \le q \le \infty \\ &1 \le p_0 \le p_1 < \infty \\ &\sigma = s_1 - s_0 + n(1/p_0 - 1/p_1). \end{aligned} \tag{4.3.12}$$

The following proposition is an immediate consequence of Corollary 4.9.3.

PROPOSITION 4.3.6. *Provided the parameters satisfy* (4.3.12), *we have that* $\Gamma : B^{s_0}_{p_0,q} \to \dot{C}_{\sigma/2;s_1,p_1,q}$.

Lastly, we turn our attention to the operator G. Using Proposition 4.8.4 and Proposition 4.9.1, we have

$$\begin{aligned} \|G\cdot u\|_{B^{s_1}_{p_1,q}} &\le C\int_0^t (t-s)^{-\sigma/2}\|u\|_{B^{s_0}_{p_0,q}}ds \\ &\le C\|u\|_{k;s_0,p_0,q}\int_0^t (t-s)^{-\sigma/2}t^{-k} \\ &\le Ct^{-\sigma/2-k+1}\|u\|_{k;s_0,p_0,q} \end{aligned} \tag{4.3.13}$$

where the last inequality used Proposition 3.10.1. We record this as a proposition.

PROPOSITION 4.3.7. *With parameters as specified in* (4.3.12), $0 < \sigma/2 < 1$, $0 \leq k_0 < 1$, *and* $k_1 = k_0 + \sigma/2 - 1$, *we have*

$$\|G \cdot u\|_{k_1;s_1,p_1,q} \leq C\|u\|_{k_0;s_0,p_0,q}. \tag{4.3.14}$$

4.4. Proof of Theorem 4.3.1

As usual, we begin with the nonlinear map

$$\Phi u = \Gamma\varphi - G \cdot P^{\alpha}(V^{\alpha}(u)), \tag{4.4.1}$$

initial data $u_0 \in B^r_{p,q}$ and define

$$E_{T,M} = \{v \in \bar{C}_{r,p,q} \cap \dot{C}_{a;s,\tilde{p},q} : \|v - \Gamma\varphi\|_{0;r,p,q} + \|v\|_{a;s,\tilde{p},q} \leq M\}. \tag{4.4.2}$$

To proceed, we observe that the restrictions on our parameters forced by the results of the previous section are identical to those from Section 3.3. So as in Section 3.4, we will use an intermediate space $\dot{C}_{2a;s-b',p',q}$, and using our now standard argument, we get that Theorem 4.3.1 will hold provided the parameters satisfy the following list of conditions:

$$
\begin{aligned}
&1 < p' \le p \le \tilde{p} \\
&1 \le q \le \infty \\
&s \ge 1,\ b' \ge 1,\ s'\tilde{p} < n \\
&0 < 2a = s - n/\tilde{p} - b < 1 \\
&0 \le s' \le s - 1 \\
&1 < \frac{n\tilde{p}}{2n - s'\tilde{p}c} < \infty \\
&s = 2 - b' + b + s' \\
&1 \ge b' - b \\
&1 \le b' + \frac{n}{\tilde{p}} - s' < 2 \\
&2 - 2b' + s' \le \frac{n}{p} \le 2 - b' + s',
\end{aligned}
\tag{4.4.3}
$$

and the case where we fix $b' = 1$:

$$1 < p' \le p \le \tilde{p}$$

$$s' = s - 1 + b$$

$$1 \le q \le \infty$$

$$s \ge 1, \; s'\tilde{p} < n$$

$$0 < 2a = s - n/\tilde{p} - b < 1 \tag{4.4.4}$$

$$0 \le s' \le s - 1$$

$$1 < \frac{n\tilde{p}}{2n - s'\tilde{p}c} < \infty$$

$$0 \le \frac{n}{\tilde{p}} - s' < 1$$

$$s' \le \frac{n}{p} \le 1 + s'.$$

4.5. Integral-in-time Besov spaces

We now consider the integral norms in time with Besov spaces instead of Sobolev spaces defining the "inside" space. We fix $T > 0$ and let $\mathbb{M}((0,T) : \mathbb{E})$ be the set of measurable functions defined on $(0,T)$ with values in the space $\mathbb{E}$. Then we define

$$L^\sigma((0,T) : B^s_{p,q}) = \{f \in \mathbb{M}((0,T) : B^s_{p,q}) : (\int_0^T \|f(t)\|^\sigma_{B^s_{p,q}} dt)^{1/\sigma} < \infty\}. \tag{4.5.1}$$

We will prove the following Theorem.

Theorem 4.5.1. *Given $u_0 \in B^r_{p,q}$ with $r = n/p + b$ there exists a $T > 0$ and a unique solution u to* (1.1.2) *such that*

$$u \in BC([0,T) : B^r_{p,q}) \cap L^\sigma((0,T) : B^s_{\tilde{p},q}), \tag{4.5.2}$$

provided there exist real numbers b' and s' (4.7.3) holds.

4.6. Besov Integral-in-time results

Our first result is similar to Proposition 3.8.1.

PROPOSITION 4.6.1. *Let $1 < p_0 \le p_1 < \infty$, $1 \le q < \infty$, $-\infty < s_0 \le s_1 < \infty$, and assume $0 < (s_1 - s_0 + n/p_0 - n/p_1)/2 = 1/\sigma$. Then Γ maps $B^{s_0}_{p_0,q_0}$ continuously into $L^\sigma((0,\infty) : B^{s_1}_{p_1,q_1})$ with the estimate*

$$\|\Gamma u\|_{L^\sigma((0,\infty):B^{s_1}_{p_1,q_1})} \le C\|u\|_{B^{s_0}_{p_0,q_0}}. \tag{4.6.1}$$

The proof is similar to Proposition 3.8.1. The two main distinctions are that, because of Theorem 4.10.1, we interpolate using s_0 instead of p_0. Also from Theorem 4.10.1, we do not require $p_0 \le \sigma$, as we did in Proposition 3.8.1.

Our next result is analogous to Proposition 3.8.6.

PROPOSITION 4.6.2. *Given $1 \le p_0 \le p_1 < \infty$, $1 \le q < \infty$, $-\infty < s_0 \le s_1 < \infty$, $1 < \sigma_0 < \sigma_1 < \infty$ and $1/\sigma_0 - 1/\sigma_1 = 1 - (s_1 - s_0 + n/p_0 - n/p_1)/2$, for any $T \in (0,\infty]$, G sends $L^{\sigma_0}((0,T) : B^{s_0}_{p_0,q_0})$ into $L^{\sigma_1}((0,T) : B^{s_1}_{p_1,q_1})$ with the estimate*

$$\|G \cdot u\|_{L^{\sigma_1}((0,T):B^{s_1}_{p_1,q_1})} \le C\|u\|_{L^{\sigma_0}((0,T):B^{s_0}_{p_0,q_0})}. \tag{4.6.2}$$

The proof is similar to the proof of Proposition 3.8.6 and is omitted.

PROPOSITION 4.6.3. *$1 < p_0 \le p_1 < \infty$, $1 \le q < \infty$, $-\infty < s_0 \le s_1 < \infty$, and assume $1/p_1 \le 1/\sigma = 1 - (s_1 - s_0 + n/p_0 - n/p_1)/2 =$. Then G maps $L^\sigma((0,T) : B^{s_0}_{p_0,q_0})$ continuously into $BC([0,T) : B^{s_1}_{p_1,q_1})$ with the estimate*

$$\sup_{t\in[0,T)} \|G \cdot u(t)\|_{B^{s_1}_{p_1,q_1}} \le C\|u\|_{L^\sigma((0,T):B^{s_0}_{p_0,q_0})}. \tag{4.6.3}$$

Using notation established in proof of Proposition 3.8.7, we first seek to show that

$$H : L^{\sigma}((0,T) : B^{s_0}_{p_0,q_0}) \to L^1((0,T) : B^{s_1}_{p_1,q_1}). \tag{4.6.4}$$

The proof follows the proof of Proposition 3.8.7.

We conclude this section with results involving the operator V^{α}. The proofs directly follow the techniques used to prove Proposition 3.8.4 and Corollary 3.8.5.

PROPOSITION 4.6.4. *With the parameters s, p, p' and q as in Proposition 4.3.4, we have*

$$V^{\alpha} : L^{\sigma}((0,T) : B^{s}_{p,q}) \to L^{\sigma/2}((0,T) : B^{s-1}_{p,q}) \tag{4.6.5}$$

with the estimate

$$\left(\int_0^T \|V^{\alpha}(u(s))\|^{\sigma/2}_{B^{s-1}_{p',q}} ds\right)^{2/\sigma} \leq \left(\int_0^T \|u(s)\|^{\sigma}_{B^{s}_{p,q}} ds\right)^{2/\sigma}. \tag{4.6.6}$$

COROLLARY 4.6.5. *If $u, v \in L^{\sigma}((0,T) : B^{s}_{p,q})$, then*

$$\begin{aligned}&\left(\int_0^T \|V^{\alpha}(u(s)) - V^{\alpha}(v(s))\|^{\sigma/2}_{B^{s-1}_{p,q}} ds\right)^{2/\sigma}\\ &\leq \left(\int_0^T \|v(s) + u(s)\|^{\sigma}_{B^{s}_{p,q}} ds\right)^{2/\sigma} \left(\int_0^T \|v(s) - u(s)\|^{\sigma}_{B^{s}_{p,q}} ds\right)^{2/\sigma}.\end{aligned} \tag{4.6.7}$$

4.7. Proof of Theorem 4.5.1

As usual, we begin by defining a Banach space $X_{T,M}$ to be the set of all $u \in BC([0,T) : B^{r}_{p,q}) \cap L^{\sigma}((0,T) : B^{s}_{\tilde{p},q})$ such that

$$\sup_t \|u(t) - \Gamma u_0\|_{B^{r}_{p,q}} + \|u\|_{\sigma;s,\tilde{p},q} \leq M \tag{4.7.1}$$

and we define the operator Φ by

$$\Phi u(t) = \Gamma\varphi + G(V^{\alpha}(u(t))) \tag{4.7.2}$$

where $\varphi \in B^r_{p,q}$.

Again, we have the same restrictions on our parameters as in the previous integral in time situation. We record these parameters in this situation:

$$
\begin{aligned}
&1 < p' \le p \le \tilde{p} < \infty \\
&1 \le q \le \infty \\
&s \ge 1,\ b' \ge 1,\ s'\tilde{p} < n \\
&0 < 2/\sigma = s - n/\tilde{p} - b < 1 \\
&0 \le s' \le s - 1 \\
&p' = \frac{n\tilde{p}}{2n - s'\tilde{p}} \\
&s = 2 - b' + b + s' \\
&1 \ge b' - b \\
&s - b' \le \frac{n}{p} + b \le s \\
&\sigma/2 \le p \le \sigma,
\end{aligned}
\tag{4.7.3}
$$

and as usual we record the case where $b' = 1$:

$$
\begin{aligned}
&1 < p' \le p \le \tilde{p} < \infty \\
&1 \le q \le \infty \\
&s \ge 1,\ \ s'\tilde{p} < n \\
&0 < 2/\sigma = s - n/\tilde{p} - b < 1 \\
&0 \le s' \le s - 1 \qquad (4.7.4)\\
&p' = \frac{n\tilde{p}}{2n - s'\tilde{p}} \\
&s' = s - 1 - b \\
&s - 1' \le \frac{n}{p} + b \le s \\
&\sigma/2 \le p \le \sigma.
\end{aligned}
$$

4.8. Besov Space Results

We list here several results involving Besov Spaces. Our first is an embedding result.

Proposition 4.8.1. *Assume that $\beta \in \mathbb{R}$ and $p, q \in [1,\infty]$. Then if $1 \le q_1 \le q_2 \le \infty$ we have that $\dot{B}^{\beta}_{p,q_1} \subset \dot{B}^{\beta}_{p,q_2}$ with the estimate*

$$\|f\|_{\dot{B}^{\beta}_{p,q_2}} \le C\|f\|_{\dot{B}^{\beta}_{p,q_1}}. \tag{4.8.1}$$

If $1 \le p_1 \le p_2 \le \infty$ and $\beta_1 = \beta_2 + n(1/p_1 - 1/p_2)$, then $\dot{B}^{\beta_1}_{p_1,q}(\mathbb{R}^n) \subset \dot{B}^{\beta_2}_{p_2,q}(\mathbb{R}^n)$ with the estimate

$$\|f\|_{\dot{B}^{\beta_2}_{p_2,q}} \le C\|f\|_{\dot{B}^{\beta_1}_{p_1,q}}. \tag{4.8.2}$$

These results, with the same restrictions on the parameters, hold for the inhomogeneous case. In addition, we have that if $\beta_1 \leq \beta_2$ *then* $B^{\beta_2}_{p,q} \subset B^{\beta_1}_{p,q}$, *with the estimate*

$$\|f\|_{B^{\beta_1}_{p,q}} \leq C\|f\|_{B^{\beta_2}_{p,q}}. \tag{4.8.3}$$

Lastly, we note from the definition of the inhomogeneous Besov space that

$$\|f\|_{L^p} \leq \|f\|_{B^s_{pq}} \tag{4.8.4}$$

for any $1 \leq p, q \leq \infty$, $s \geq 0$. *The last two results do not hold for the homogeneous case.*

We also have the following Moser-type estimate. This is Lemma 2.2 in [**2**].

Proposition 4.8.2. *Let* $s > 0$ *and* $q \in [1, \infty]$. *Then we have*

$$\|fg\|_{\dot{B}^s_{p,q}} \leq C(\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{\dot{B}^s_{r_2,q}}) \tag{4.8.5}$$

and

$$\|fg\|_{B^s_{p,q}} \leq C(\|f\|_{L^{p_1}}\|g\|_{B^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{B^s_{r_2,q}}) \tag{4.8.6}$$

where $p_i, r_i \in [1, \infty]$ *and* $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$.

Using Proposition 4.8.2 and Proposition 4.8.1, we have

$$\begin{aligned} \|u^2\|_{B^s_{p,q}} &\leq C\|u\|_{L^{p_2}}\|u\|_{B^s_{p_1,q}} \\ &\leq C\|u\|_{B^0_{p_2,q}}\|u\|_{B^s_{p_1,q}} \\ &\leq C\|u\|_{B^{s'}_{p_1,q}}\|u\|_{B^s_{p_1,q}} \\ &\leq C\|u\|^2_{B^s_{p_1,q}} \end{aligned} \tag{4.8.7}$$

where $s' \geq 0$ satisfies $p = \frac{np_1}{2n - s'p_1}$ and $n(2/p_1 - 1/p) \leq s$. We record this as a proposition.

PROPOSITION 4.8.3. *Let $s > 0$ and $p, p_1, q \in [1, \infty]$. Then*

$$\|u^2\|_{B^s_{p,q}} \leq C\|u\|^2_{B^s_{p_1,q}} \tag{4.8.8}$$

where $p_1 \leq 2p$, $s \geq n(2/p_1 - 1/p) = s'$, and $p = \frac{np_1}{2n - s'p_1}$.

Our next Proposition is reminiscent of the Minkowski integral inequality.

PROPOSITION 4.8.4. *Let $f(s) \in B^s_{p,q}$ for all $s \in (0, t)$ for some $t > 0$. Then*

$$\| \int_0^t f(\cdot, s) ds \|_{B^s_{pq}} \leq C \int_0^t \|f(\cdot, x)\|_{B^s_{pq}} ds. \tag{4.8.9}$$

Showing this for the homogeneous Besov space follows immediately from applying Minkowski's integral and summation inequalities. It then follows for the inhomogeneous Besov norm by applying Minkowski's integral inequality to the L^p component of the Besov norm.

Next, we formally establish an isometry result. This is Theorem 8 on page 67 of [**11**]

THEOREM 4.8.5. *Let $I_r = (1 - \triangle)^{r/2}$. Then*

$$I_r : B^s_{pq} \to B^{r-s}_{pq} \tag{4.8.10}$$

is an isomorphism.

We conclude this section by stating the Bernstein inequality.

THEOREM 4.8.6. *Let $\alpha \geq 0$ and $1 \leq p \leq q \leq \infty$. Then if supp $\hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq K2^j\}$ for some $K > 0$ and some integer j, then*

$$\|\Lambda^\alpha f\|_q \leq C 2^{j\alpha + jn(1/p - 1/q)} \|f\|_p. \tag{4.8.11}$$

In addition, if supp $\hat{f} \subset \{\xi \in \mathbb{R}^n : K_1 2^j \le |\xi| \le K_2 2^j\}$ *for some* $K_1, K_2 > 0$ *and some integer* j, *then*

$$\tilde{C} 2^{j\alpha + jn(1/p-1/q)} \|f\|_p \le \|\Lambda^\alpha f\|_q \le C 2^{j\alpha + jn(1/p-1/q)} \|f\|_p. \tag{4.8.12}$$

This is Proposition 2.3 in [**18**].

4.9. Heat Kernel in Besov Space

We recall that the heat kernel is defined by

$$e^{t\triangle} f = f * e^{t\triangle}\delta \tag{4.9.1}$$

where

$$e^{t\triangle}\delta(x) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}. \tag{4.9.2}$$

Our task is to determine the action of the heat kernel on inhomogeneous Besov spaces. We begin with an alternate construction of Besov spaces. This can be found in Chapter 3 of [**11**].

We let $\phi_j \in \mathcal{S}$ be a sequence of “test functions” such that $\hat{\phi}_j(\xi) \neq 0$ if and only if $\xi \in R_j = \{2^{j-1} \le |\xi| \le 2^{j+1}\}$,

$$|D^\beta \hat{\phi}_j(\xi)| \le C_\beta 2^{-j|\beta|} \tag{4.9.3}$$

for any multi-index β, and for small ε,

$$|\hat{\phi}_j(\xi)| \ge C_\varepsilon > 0 \tag{4.9.4}$$

for $\xi \in R_{j\varepsilon}$ where

$$R_{j\varepsilon} = \{(2-\varepsilon)^{-1} 2^j \le |\xi| \le (2-\varepsilon) 2^j\}. \tag{4.9.5}$$

Equations (4.9.3) and (4.9.4) are the main distinctions between our two constructions of Besov spaces. We also require

$$\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) = 1 \tag{4.9.6}$$

for $\xi \neq 0$.

We define another test function $\phi \in \mathcal{S}$ such that $\hat{\phi}(\xi) \neq 0$ if $\xi \in \{|\xi| \leq 1\}$ and for small ε,

$$|\hat{\phi}(\xi)| \geq C_\varepsilon > 0 \tag{4.9.7}$$

for $\xi \in \{|\xi| \leq 1 - \varepsilon\}$.

We remark that ϕ can be chosen such that

$$\hat{\phi}(\xi) = \sum_{j=-\infty}^{-1} \hat{\phi}_j(\xi) \tag{4.9.8}$$

We define the Besov space B^s_{pq} to be the normed space defined by the (quasi)-norm

$$\|f\|_{B^s_{pq}} = \|\phi * f\|_{L^p} + \left(\sum_{j=0}^{\infty} (2^{js} \|\phi_j * f\|_{L^p})^q \right)^{1/q} \tag{4.9.9}$$

where $s \in \mathbb{R}$, $1 \leq p \leq \infty$, and $0 < q \leq \infty$. For $1 \leq q \leq \infty$, this is a Banach space.

We also have the homogeneous Besov space $\dot{B}^s_{pq}$ defined by the (quasi-) norm

$$\|f\|_{\dot{B}^s_{pq}} = \left(\sum_{j=-\infty}^{\infty} (2^{js} \|\phi_j * f\|_{L^p})^q \right)^{1/q}. \tag{4.9.10}$$

Proposition 4.9.1. *Let $1 \leq p' \leq p'' < \infty$, $-\infty < s' \leq s'' < \infty$, and let $0 < q < \infty$. Then*

$$\|e^t \triangle u\|_{B^{s''}_{p'',q}} \leq C t^{-(s''-s'+n/p'-n/p'')/2} \|u\|_{B^{s'}_{p',q}}. \tag{4.9.11}$$

We start with the following lemma.

LEMMA 4.9.2. *Let $I_s = (1 - \triangle)^{s/2}$. Then, for any $j > 0$, we have*

$$2^{js}\phi_j \equiv= I_s\phi_j, \tag{4.9.12}$$

where $f \equiv g$ means that there exists a C_1, C_2 such that

$$C_1 f \leq g \leq C_2 f. \tag{4.9.13}$$

Since $\phi_j \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwarz space of rapidly decreasing functions, we know that $\mathcal{F}^*(\mathcal{F}(\phi_j)) = \phi_j$, where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^*$ denotes the inverse Fourier Transform. Because $I_s : \mathcal{S} \to \mathcal{S}$, we have the same result for $I_s\phi_j$. To exploit this, we begin by computing $\mathcal{F}(I_s\phi_j)$, and we get

$$\mathcal{F}(I_s\phi_j)(\xi) = (1 + |\xi|^2)^{s/2}\hat{\phi}_j. \tag{4.9.14}$$

Applying $\mathcal{F}^*$, we have

$$\mathcal{F}^*(\mathcal{F}(I_s\phi_j))(x) = \int (1 + |\xi|^2)^{s/2}\hat{\phi}_j(\xi)e^{ix\cdot\xi}d\xi. \tag{4.9.15}$$

Since $\hat{\phi}_j$ is supported on the annulus $2^{j-1} \leq \xi \leq 2^{j+1}$, we have

$$\int (1 + |\xi|^2)^{s/2}\hat{\phi}_j(\xi)e^{ix\cdot\xi}d\xi \equiv \int 2^{js}\hat{\phi}_j(\xi)e^{ix\cdot\xi}d\xi = 2^{js}\phi_j. \tag{4.9.16}$$

Combining these results, we get

$$I_s\phi_j = \mathcal{F}^*(\mathcal{F}(I_s\phi_J)) \equiv 2^{js}\phi_j, \tag{4.9.17}$$

which finishes the Lemma.

To prove the proposition, we use the Lemma and Sobolev space heat kernel estimates to get

$$
\begin{aligned}
\|e^{t\triangle}u\|_{B^{s''}_{p'',q}} &= \|\phi * e^{t\triangle}u\|_{L^{p''}} + \left(\sum (2^{js'}\|2^{j(s''-s')}\phi_j * e^{t\triangle}u\|_{L^{p''}})^q\right)^{1/q} \\
&\le t^{(n/p'-n/p'')/2}\|\phi * u\|_{L^{p'}} + \left(\sum (2^{js'}\|\phi_j * e^{t\triangle}u\|_{H^{s''-s',p''}})^q\right)^{1/q} \\
&\le t^{-(n/p'-n/p'')/2}\|\phi * u\|_{L^{p'}} + t^{\sigma}\left(\sum (2^{js'}\|\phi_j * u\|_{L^{p'}})^q\right)^{1/q} \\
&\le t^{\sigma}\|u\|_{B^{s'}_{p',q}}.
\end{aligned}
\tag{4.9.18}
$$

where $\sigma = -(s'' - s' + n/p' - n/p'')/2$.

We have an immediate corollary, similar to Proposition 3.11.3.

Corollary 4.9.3. *With the parameters as in Proposition* 4.9.1, *we have*

$$
\lim_{t\to 0} t^{\gamma/2}\|e^{t\triangle}f\|_{\dot{B}^{\alpha+\beta}_{r,q_1}} = 0 \tag{4.9.19}
$$

provided $\sigma > 0$.

The proof is analogous to the proof of Proposition 3.11.3.

4.10. Besov Interpolation Results

We have the following Besov space interpolation result. This is part of Theorem 6.4.5 in [**1**].

Theorem 4.10.1. *Let* $0 < \theta < 1$ *and*

$$
\begin{aligned}
s^* &= (1-\theta)s_0 + \theta s_1, \\
1/p^* &= \frac{(1-\theta)}{p_0} + \frac{\theta}{p_1}, \\
1/q^* &= \frac{(1-\theta)}{q_0} + \frac{\theta}{q_1}.
\end{aligned}
\tag{4.10.1}
$$

Then we have

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,r} = B^{s^*}_{p,r} \tag{4.10.2}$$

where $s_0 \neq s_1$, $1 \leq p \leq \infty$, *and* $1 \leq r, q_0, q_1 \leq \infty$. *We also have*

$$(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{\theta,p^*} = B^{s^*}_{p^*,q^*} \tag{4.10.3}$$

where $s_0 \neq s_1$, $p^* = q^*$, *and* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$.

We conclude this section with Theorem 3.11.8 from [**1**], but first we recall a definition. If A, B are quasi-normed spaces, we say $T : A \to B$ is a quasi-normed linear operator if $\|T(a_0 + a_1)\|_B \leq c(\|a_0\|_A + \|a_1\|_A)$ where $c \geq 1$.

Theorem 4.10.2. *Let* A_i, B_i *be quasi-normed spaces where* $i = 0, 1$. *Suppose that there exists a quasi-linear operator* T *such that* $T : A_i \to B_i$. *Then*

$$T : (A_0, A_1)_{\theta,r} \to (B_0, B_1)_{\theta,r} \tag{4.10.4}$$

where $0 < \theta < 1$.

CHAPTER 5

A Global Existence Result in Besov space

5.1. Global Existence with arbitrary initial data when $p = 2$

In our previous work with initial data in Sobolev spaces, we were able to get a global existence result in the $p = 2$ case to the LANS equations. This was accomplished via higher regularity energy estimates. To get a similar result in the Besov setting requires significantly more work, and the goal of this section is to prove the necessary results. This argument is inspired by the work of [**18**]. In Section asf, we will modify the results of [**18**] in a setting more closely aligned with that paper's original intent.

5.2. An alternate construction of Besov spaces

Our goal in this section is to follow the approach of [**18**] to get solutions to (1.1.2) in the integral norm space. We begin by stating the Theorem we wish to prove.

THEOREM 5.2.1. *Let u be a local solution to* (1.1.2) *with initial data $u_0 \in \tilde{B}^r_{2,q}$ such that*

$$u \in BC([0,T) : B^r_{2,q}) \cap Y, \tag{5.2.1}$$

where $r > 2$ and Y is either $L^\sigma(B^{1+n/2}_{2,q})$ or $C_{a;1+n/2,2,q}$ with $0 \leq a < 1$ and $1 \leq \sigma$. Then the local solution is a global solution.

Alternatively, assuming $n < r + \varepsilon$, a local solution of (1.1.2) *with initial data $u_0 \in \tilde{B}^r_{2,q}$ such that*

$$u \in BC([0,T) : B^r_{2,q}) \cap X, \tag{5.2.2}$$

where X is either $L^\sigma(B^{1+n/2+\varepsilon}_{2,q})$ or $C_{a;1+n/2+\varepsilon,2,q}$, can be extended to a global solution.

To prove the theorem, we begin with *a priori* estimates. We start by recalling aspects of our construction of Besov spaces from Section 4.2. We have that our functions ψ_j satisfy

$$\begin{gathered}\hat{\psi}_j(\xi) = \hat{\psi}_0(2^{-j}\xi), \\ \operatorname{supp} \hat{\psi}_j \subset R_j,\end{gathered} \tag{5.2.3}$$

where $R_j = \{2^{j-1} < \xi < 2^{j+1}\}$. We also have that

$$\sum_{j=-\infty}^{\infty} \hat{\psi}_j(\xi) = 1, \tag{5.2.4}$$

provided $\xi \neq 0$. Our first new function is Ψ, defined by

$$\hat{\Psi}(\xi) = 1 - \sum_{k=0}^{\infty} \hat{\psi}_j(\xi). \tag{5.2.5}$$

We remark that $\hat{\Psi}(\xi) = 1$ for $\xi = 0$, and $\hat{\Psi}(\xi) = 0$ for $\xi > 2$. Next, we define $\triangle_j u$ by

$$\triangle_j u = \psi * u, \tag{5.2.6}$$

and for notational convenience, we define $S_k = \sum_{j=0}^{k} \triangle_j$. Finally, we define the inhomogeneous Besov space by the norm

$$\|u\|_{B^s_{p,q}} = \|\Psi * u\|_{L^p} + \left(\sum_{j=0}^{\infty} (2^{js} \|\triangle_j u\|^q_{L^p}\right)^{1/q}, \tag{5.2.7}$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

In analogy with the definition of homogenous Besov spaces, we denote the second term in the sum by $\tilde{B}^s_{p,q}$, and define the notation

$$\|u\|_{\tilde{B}^s_{p,q}} = \left(\sum_{j=0}^{\infty} (2^{js} \|\triangle_j u\|^q_{L^p}\right)^{1/q}. \tag{5.2.8}$$

Recalling that homogenous Besov spaces are defined by

$$\|u\|_{\dot{B}^s_{p,q}} = \left(\sum_{j=-\infty}^{\infty} (2^{js} \|\triangle_j u\|^q_{L^p} \right)^{1/q}, \tag{5.2.9}$$

we remark that we can regard our new spaces as "half" of the homogenous Besov space $\dot{B}^r_{p,q}$.

We pause here to comment on our strategy. To achieve our global existence argument, we need bounds on the two separate pieces of the Besov norm. We will begin working with the more complicated piece, which is $\tilde{B}^r_{p,q}$. We use arguments inspired by the calculations in [**18**].

With our notation established, we record two facts.

LEMMA 5.2.2. *There exists an $M > 0$ such that if $|j-k| \geq M$, then*

$$\begin{gathered} \triangle_j \triangle_k f \equiv 0 \\ \triangle_j (S_{k-5} f \triangle_{k+1} g) \equiv 0. \end{gathered} \tag{5.2.10}$$

These two facts follow from applying the Fourier Transform and recalling that $\hat{\psi}_j$ is supported on $2^{j-1} \leq |\xi| \leq 2^{j+1}$. We mention another result with the opposite emphasis.

LEMMA 5.2.3. *If k and l are close together, and j is much greater than k, then*

$$\triangle_j (\triangle_k f \triangle_l g) \equiv 0. \tag{5.2.11}$$

Applying the Fourier Transform, we have

$$\mathcal{F} \triangle_j (\triangle_k f \triangle_l g)(x) = \hat{\psi}_j(x) \int \hat{\psi}_k(y) \hat{f}(y) \hat{\psi}_l(x-y) \hat{g}(x-y) dy. \tag{5.2.12}$$

The expression

$$\hat{\psi}_j(x) \hat{\psi}_k(y) \hat{f}(y) \hat{\psi}_l(x-y) \hat{g}(x-y) \tag{5.2.13}$$

will be equal to zero provided any of the following conditions are not satisfied:

$$
\begin{aligned}
&2^{j-1} \leq |x| \leq 2^{j+1} \\
&2^{k-1} \leq |y| \leq 2^{k+1} \qquad (5.2.14)\\
&2^{l-1} \leq |x-y| \leq 2^{l+1}.
\end{aligned}
$$

If k and l are close together and if j is much larger than k, then $|x-y|$ will be of an order similar to 2^j, which will violate the third condition. This proves the Lemma.

Next, we recall Bony's notion of paraproduct (see 0.17 in Chapter 2 of [**17**]). We have that $fg = T_f g + T_g f + R(f,g)$, where

$$
T_f g = \sum_k (S_{k-5} f) \triangle_{k+1} g \qquad (5.2.15)
$$

and

$$
R(f,g) = \sum_k \Big(\sum_{l=k-5}^{k+5} \triangle_l f \Big)(\triangle_k g). \qquad (5.2.16)
$$

Using Bony's paraproduct and Lemmas 5.2.2 and 5.2.3, for some M, we have that

$$
\begin{aligned}
\triangle_j(fg) \leq &\sum_{|j-k|\leq M} \triangle_j (S_{k-5} f \triangle_{k+1} g) + \sum_{|j-k|\leq M} \triangle_j (S_{k-5} g \triangle_{k+1} f) \\
&+ \sum_{k \geq j-M} \triangle_j \Big(\triangle_k g \sum_{l=k-5}^{k+5} \triangle_{k+l} f \Big) \qquad (5.2.17)\\
&= I + II + III.
\end{aligned}
$$

Applying Young's inequality and then Holder's inequality, we have

$$
\begin{aligned}
\|I\|_p &\leq C \sum_{|j-k|\leq M} \|\psi_j\|_1 \|S_{k-5} f \triangle_{k+1} g\|_p \qquad (5.2.18)\\
&\leq C \sum_{|j-k|\leq M} \|S_{k-5} f\|_\infty \|\triangle_{k+1} g\|_p,
\end{aligned}
$$

and similarly

$$\|II\|_p \leq C \sum_{|j-k|\leq M} \|S_{k-5}g\|_\infty \|\triangle_{k+1} f\|_p. \tag{5.2.19}$$

$$\|III\|_p \leq C \sum_{k\geq j-M} (\sum_{l=k-5}^{k+5} \|\triangle_k g\|_p \|\triangle_l f\|_\infty). \tag{5.2.20}$$

We end this section with the statement of our main Proposition.

PROPOSITION 5.2.4. *Any solution u to* 1.1.2 *satisfies*

$$\frac{d}{dt}\|u\|^q_{\tilde{B}^r_{2,q}} \leq C\|u\|_{\tilde{B}^{1+n/2}_{2,q}}\|u\|^q_{\tilde{B}^r_{2,q}}, \tag{5.2.21}$$

provided $r > 2$ or

$$\frac{d}{dt}\|u\|^q_{\tilde{B}^r_{2,q}} \leq C\|u\|_{\tilde{B}^{1+n/2+\varepsilon}_{2,q}}\|u\|^q_{\tilde{B}^r_{2,q}}, \tag{5.2.22}$$

provided $2 < r + \varepsilon < 0$.

We remark that the $r > 2$ restriction is due to the sum in I_3 not being finite. See (5.5.6) for the technical necessity of $r > 2$.

This proposition is very complicated, and its proof will be broken up over the next several sections. We begin with a statement of the LANS equations:

$$\begin{aligned} &\partial_t(1-\alpha^2\triangle)u + \nabla_u(1-\alpha^2\triangle)u - \alpha^2(\nabla u)^T \cdot \triangle u \\ =& -\nu(1-\alpha^2\triangle)Au - \nabla p. \end{aligned} \tag{5.2.23}$$

Applying $\triangle_j$ to both sides and taking the L^2 inner product with $2\triangle_j u$, we get

$$I_1 + I_2 + I_3 + I_4 = I_5 \tag{5.2.24}$$

where

$$(5.2.25)\qquad \begin{aligned} I_1 &= (\partial_t(1-\alpha^2\triangle)\triangle_j u, \triangle_j u)_{L^2}, \\ I_2 &= (\triangle_j(\nabla_u(1-\alpha^2\triangle)u, \triangle_j u)_{L^2}, \\ I_3 &= -\alpha^2(\triangle_j((\nabla u)^T \cdot \triangle u, \triangle_j u)_{L^2}, \\ I_4 &= \nu((1-\alpha^2\triangle)A\triangle_j u, \triangle_j u)_{L^2}, \\ I_5 &= -(\nabla\triangle_j p, \triangle_j u)_{L^2}. \end{aligned}$$

Applying integration by parts to I_1 gives that

$$(5.2.26)\qquad I_1 = \frac{1}{2}\partial_t(\|\triangle_j u\|_{L^2}^2 + \alpha^2\|A^{1/2}\triangle_j u\|_{L^2}^2).$$

Since div $u = 0$, applying integration by parts to I_5 gives that $I_5 = 0$. We also have that

$$(5.2.27)\qquad I_4 = \nu[(A^{1/2}\triangle_j u, A^{1/2}\triangle_j u) + \alpha^2(A\triangle_j u, A\triangle_j u)] \geq 0.$$

Plugging these results back into 5.2.24, we have

$$(5.2.28)\qquad \frac{1}{2}\frac{d}{dt}(\|\triangle_j u\|_{L^2}^2 + \alpha^2\|A^{1/2}\triangle_j u\|_{L^2}^2) \leq \ |I_2| + |I_3|.$$

To proceed, we need to estimate I_2 and I_3.

5.3. I_2 and I_3 estimates

We begin with I_2, and we have by Holder's inequality that

$$(5.3.1)\qquad |I_2| \leq C\|\triangle_j u\|_2\|\triangle_j(\nabla_u(1-\alpha^2\triangle)u)\|_2.$$

To estimate the second term, we use (5.2.18), (5.2.19), and (5.2.20) to write

$$(5.3.2)\qquad \triangle_j(\nabla_u(1-\alpha^2\triangle)u) = J_1 + J_2 + J_3$$

where

$$
\begin{aligned}
J_1 &= \sum_{|j-k|\le M} \triangle_j((S_{k-5}u\cdot\nabla)\triangle_{k+1}u) - \alpha^2 \sum_{|j-k|\le M} \triangle_j((S_{k-5}u\cdot\nabla)\triangle_{k+1}\triangle u)\\
J_2 &= \sum_{|j-k\le M} \triangle_j((\triangle_{k+1}u\cdot\nabla)S_{k-5}u) - \alpha^2 \sum_{|j-k|\le M} \triangle_j((\triangle_{k+1}u\cdot\nabla)S_{k-5}\triangle u)\\
J_3 &= \sum_{k\ge j-M} \triangle_j\left(\triangle_k u\sum_{l=-5}^{l=5}\nabla\triangle_{k+l}u\right) - \alpha^2\sum_{k\ge j-M}\triangle_j\left(\triangle_k u\sum_{l=-5}^{l=5}\nabla\triangle_{k+l}\triangle u\right).
\end{aligned}
\tag{5.3.3}
$$

For notational convenience, we define $J_{i,j}$, with $j=1,2$, to be the j^{th} of the two summations in J_i. We begin with $J_{1,2}$. Recalling that ψ_j is the convolution kernel for $\triangle_j$, we have by integration by parts and the incompressibility condition that

$$
\begin{aligned}
&\int \psi_j(x-y)((S_{k-5}u(y))\nabla\left(\triangle\triangle_{k+1}u(y)\right)dy\\
=&\int -\left(\nabla(\psi_j(x-y))(S_{k-5}u(y)) + \psi_j(x-y)\mathrm{div}\ (S_{k-5}u(y))\right)\triangle\triangle_{k+1}u(y)dy\\
=&\int -\nabla(\psi_j(x-y))(S_{k-5}u(y))\triangle\triangle_{k+1}u(y)dy\\
=&\int -\triangle(\nabla\psi_j(x-y)\cdot(S_{k-5}u(y)))\triangle_{k+1}u(y)dy.
\end{aligned}
\tag{5.3.4}
$$

Next, we use the product rule to distribute the Laplacian through the product, apply Young's inequality, and then take the L^∞ norm of the pieces involving $S_{k-5}u$ and its derivatives. Recalling that the L^1 norm of ψ_j and its derivatives are independent of j, we have

$$
\begin{aligned}
&\|\sum_{|j-k|\le M}\triangle_j((S_{k-5}u\cdot\nabla)\triangle_{k+1}\triangle u)\|_{L^2}\\
\le& C\sum_{|j-k|\le M}(\|S_{k-5}u\|_{L^\infty} + \|\nabla S_{k-5}u\|_{L^\infty} + \|\triangle S_{k-5}u\|_{L^\infty})\|\triangle_{k+1}u\|_{L^2}.
\end{aligned}
\tag{5.3.5}
$$

Recalling the definition of S_{k-5} and using Bernstein's inequality (Theorem 4.8.6) we get

$$
\begin{aligned}
&\|\sum_{|j-k|\le M}\triangle_j((S_{k-5}u\cdot\nabla)\triangle_{k+1}\triangle u)\|_{L^2}\\
\le& C\sum_{|j-k|\le M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(2+n/2)m}\|\triangle_m u\|_{L^2}.
\end{aligned}
\tag{5.3.6}
$$

For $J_{1,1}$, a similar computation gives

$$
\begin{aligned}
&\|\triangle_j((S_{k-5}u\cdot\nabla)\triangle_{k+1}u)\|_{L^2}\\
\le& C\|S_{k-5}u\|_{L^\infty}\|\triangle_{k+1}u\|_{L^2}\\
\le& C\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{nm/2}\|\triangle_m u\|_{L^2}.
\end{aligned}
\tag{5.3.7}
$$

So we finally get that J_1 satisfies

$$
|J_1|\le C\sum_{|j-k|\le M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(2+n/2)m}\|\triangle_m u\|_{L^2}. \tag{5.3.8}
$$

J_2 satisfies the same estimate, so we turn to J_3. Using integration by parts and Young's inequality, we have

$$
\|\triangle_j((\triangle_k u)\nabla\triangle_{k+l}\triangle u)\|_{L^2}\le C\|\triangle_k u\|_{L^2}\|\triangle\triangle_{k+l}u\|_{L^\infty}. \tag{5.3.9}
$$

Bernstein's inequality gives

$$
\|\triangle_j((\triangle_k u)\nabla\triangle_{k+l}\triangle u)\|_{L^2}\le C2^{(2+n/p)(k+l)}\|\triangle_k u\|_{L^2}\|\triangle_{k+l}u\|_{L^2}. \tag{5.3.10}
$$

Applying this to $J_{3,2}$, we get

$$
\|J_{3,2}\|_{L^2}\le C\sum_{k\ge j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5}2^{(2+n/p)(k+l)}\|\triangle_{k+l}u\|_{L^2} \tag{5.3.11}
$$

Similar calculations on $J_{3,1}$ gives

$$\|J_{3,1}\|_{L^2} \le C \sum_{k\ge j-M} \|\triangle_k u\|_{L^2} \sum_{l=-5}^{l=5} 2^{(n/p)(k+l)j} \|\triangle_{k+l} u\|_{L^2}. \tag{5.3.12}$$

So J_3 satisfies

$$|J_3| \le C \sum_{k\ge j-M} \|\triangle_k u\|_{L^2} \sum_{l=-5}^{l=5} 2^{(2+n/p)(k+l)} \|\triangle_{k+l} u\|_{L^2}. \tag{5.3.13}$$

So we finally estimate I_2 by

$$\begin{aligned} |I_2| &= C\|\triangle_j u\|_{L^2} \|J_1 + J_2 + J_3\|_{L^2} \\ &\le C\|\triangle_j u\|_{L^2} \sum_{|j-k|\le M} \|\triangle_{k+1} u\|_{L^2} \sum_{m<k-5} 2^{(2+n/2)m} \|\triangle_m u\|_{L^2} \\ &+C\|\triangle_j u\|_{L^2} \sum_{k\ge j-M} \|\triangle_k u\|_{L^2} \sum_{l=-5}^{l=5} 2^{(2+n/p)(k+l)} \|\triangle_{k+l} u\|_{L^2}. \end{aligned} \tag{5.3.14}$$

The estimation of I_3 is similar, so the details will be omitted. The key difference between the two estimates is that, in the case of I_2, integrating the gradient term by parts and applying the incompressibility condition essentially removed one of the three derivatives from I_2. Because the gradient term in I_3 is $(\nabla u)^T$ instead of ∇u, this does not work when estimating I_3, and the result is the presence of an extra derivative. The estimate for I_3 is

$$\begin{aligned} |I_3| &\le C\|\triangle_j u\|_{L^2} \sum_{|j-k|\le M} \|\triangle_{k+1} u\|_{L^2} \sum_{m<k-5} 2^{(3+n/2)m} \|\triangle_m u\|_{L^2} \\ &+C\|\triangle_j u\|_{L^2} \sum_{k\ge j-M} \|\triangle_k u\|_{L^2} \sum_{l=-5}^{l=5} 2^{(3+n/p)(k+l)} \|\triangle_{k+l} u\|_{L^2}. \end{aligned} \tag{5.3.15}$$

Using (5.3.14) and (5.3.15) in (5.2.28) we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2 + \alpha^2\|A^{1/2}\triangle_j u\|_{L^2}^2\right)\\
&\leq C\|\triangle_j u\|_{L^2}\sum_{|j-k|\leq M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(3+n/2)m}\|\triangle_m u\|_{L^2}\\
&+C\|\triangle_j u\|_{L^2}\sum_{k\geq j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5}2^{(3+n/p)(k+l)}\|\triangle_{k+l}u\|_{L^2}.
\end{aligned}
\tag{5.3.16}
$$

In the next section, we work on the left-hand side of (5.3.16).

5.4. Exploiting the LANS term

We remark that

$$
\mathcal{F}(A^{1/2}\triangle_j u)(\xi) = |\xi|\hat{\psi}_j(\xi)\hat{u}(\xi). \tag{5.4.1}
$$

On the support of $\hat{\psi}_j$, we have that $C2^j \leq |\xi| \leq C'2^j$, so by Plancherel's Theorem we have

$$
C2^{2j}\|\triangle_j u\|_{L^2} \leq \|A^{1/2}\triangle_j u\|_{L^2}^2 \leq C'2^{2j}\|\triangle_j u\|_{L^2}. \tag{5.4.2}
$$

Applying this to (5.3.16), we get

$$
\begin{aligned}
&(1+2^{2j})\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2\right)\\
&\leq C\|\triangle_j u\|_{L^2}\sum_{|j-k|\leq M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(3+n/2)m}\|\triangle_m u\|_{L^2}\\
&+C\|\triangle_j u\|_{L^2}\sum_{k\geq j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5}2^{(3+n/2)(k+l)}\|\triangle_{k+l}u\|_{L^2}.
\end{aligned}
\tag{5.4.3}
$$

If $\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2\right) \leq 0$, then (vacuously) $2^{2j}\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2\right)$ is smaller than the right hand side of (5.4.3). Alternatively, if $\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2\right) > 0$, then we still have that $2^{2j}\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2\right)$ is smaller than the right hand side of (5.3.16). So we have

$$
\begin{aligned}
&2^{2j}\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}^2\right) \\
&\qquad \le C\|\triangle_j u\|_{L^2}\sum_{|j-k|\le M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(3+n/2)m}\|\triangle_m u\|_{L^2} \qquad (5.4.4)\\
&+C\|\triangle_j u\|_{L^2}\sum_{k\ge j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5}2^{(3+n/2)(k+l)}\|\triangle_{k+l}u\|_{L^2}.
\end{aligned}
$$

Computing the time derivative, this becomes

$$
\begin{aligned}
2^{2j}\frac{d}{dt}\left(\|\triangle_j u\|_{L^2}\right) &\le C\sum_{|j-k|\le M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(3+n/2)m}\|\triangle_m u\|_{L^2} \qquad (5.4.5)\\
&+C\sum_{k\ge j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5}2^{(3+n/p)(k+l)}\|\triangle_{k+l}u\|_{L^2}.
\end{aligned}
$$

Next, we multiply both sides by $q2^{rjq}\|\triangle_j u\|_{L^2}^{q-1}$ and sum over j, which gives

$$
\frac{d}{dt}\|u\|_{\tilde{B}^r_{2,q}}^q \le K_1 + K_2, \qquad (5.4.6)
$$

where

$$
K_1 = Cq\sum_{j\ge 0}2^{rjq}2^{-2j}\|\triangle_j u\|_{L^2}^{q-1}\sum_{|j-k|\le M}\|\triangle_{k+1}u\|_{L^2}\sum_{m<k-5}2^{(3+n/2)m}\|\triangle_m u\|_{L^2}, \qquad (5.4.7)
$$

and

$$
K_2 = Cq\sum_{j\ge 0}2^{rjq}2^{-2j}\|\triangle_j u\|_{L^2}^{q-1}\sum_{k\ge j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5}2^{(3+n/p)(k+l)}\|\triangle_{k+l}u\|_{L^2}. \qquad (5.4.8)
$$

We remark that the 2^{-2j} term in K_1 and K_2 is the result of applying this procedure to the LANS equations instead of the Navier-Stokes equations.

5.5. Estimating K_1 and K_2

We begin by re-writing K_1 as

$$K_1 = Cq \sum_{j\geq 0} 2^{rjq} \|\triangle_j u\|_{L^2}^{q-1} \sum_{k=-M}^{M} \|\triangle_{j+k+1} u\|_{L^2} \sum_{m<j+k-5} 2^{(3+n/2)m-2j} \|\triangle_m u\|_{L^2}. \tag{5.5.1}$$

We start by working on the last summation. We have by Holder's inequality that

$$\begin{aligned}
&\sum_{m<j+k-5} 2^{(1+n/2)m} 2^{2m-2j} \|\triangle_m u\|_2 \\
&\leq C \left(\sum_{m<j+k-5} 2^{(1+n/2)qm} \|\triangle_m u\|_2^q \right)^{1/q} \left(\sum_{m<j+k-5} 2^{2q'(m-(j+k-5)} 2^{2(k-5)q'} \right)^{1/q'} \qquad (5.5.2)\\
&\leq C \|u\|_{\tilde{B}^{1+n/2}_{2,q}},
\end{aligned}$$

where q' is the Holder conjugate exponent to q.

Returning to K_1, we have

$$|K_1| \leq C \|u\|_{\tilde{B}^{1+n/2}_{2,q}} \sum_{j\geq 0} 2^{rjq} \|\triangle_j u\|_{L^2}^q \sum_{k=-M}^{M} \|\triangle_{j+k+1} u\|_{L^2}. \tag{5.5.3}$$

For the remaining summation, we have

(5.5.4)

$$\begin{aligned}
&\sum_{j\geq 0} 2^{rjq} \|\triangle_j u\|_{L^2}^{q-1} \sum_{k=-M}^{M} \|\triangle_{j+k+1} u\|_{L^2} \\
&= \sum_{j\geq 0} 2^{rj(q-1)} \|\triangle_j u\|_{L^2}^{q-1} \sum_{k=-M}^{M} 2^{r(j+k+1)} 2^{-r(k+1)} \|\triangle_{j+k+1} u\|_{L^2} \\
&\leq C \left(\sum_{j\geq 0} 2^{rj(q-1)q'} \|\triangle_j u\|_{L^2}^{q'(q-1)} \right)^{q'} \left(\sum_j (\sum_{k=-M}^{M} 2^{r(j+k+1)} 2^{-r(k+1)} \|\triangle_{j+k+1} u\|_{L^2})^q \right)^q \\
&\leq C \|u\|_{\tilde{B}^r_{2,q}}^{q-1} \|u\|_{B^{\tilde{r}}_{2,q}} \leq C \|u\|_{\tilde{B}^r_{2,q}}^q,
\end{aligned}$$

where q' is again the Holder conjugate exponent to q. So we finally bound K_1 by

$$K_1 \leq C \|u\|_{\tilde{B}^r_{2,q}}^q \|u\|_{\tilde{B}^{1+n/2}_{2,q}}. \tag{5.5.5}$$

Now we bound K_2. We have

(5.5.6)

$$\begin{aligned}K_2 &= Cq\sum_{j\geq 0} 2^{rjq}2^{-2j}\|\triangle_j u\|_{L^2}^{q-1}\sum_{k\geq j-M}\|\triangle_k u\|_{L^2}\sum_{l=-5}^{l=5} 2^{(3+n/p)(k+l)}\|\triangle_{k+l}u\|_{L^2}\\ &\leq \sum_j 2^{rj(q-1)}\|\triangle_j u\|_{L^2}^{q-1}\sum_{k>-M}\sum_{l=-5}^{5} 2^{(r-2)j}2^{(3+n/2)(j+k+l)}\|\triangle_{j+k}u\|_{L^2}\|\triangle_{j+k+l}u\|_{L^2}.\end{aligned}$$

Using Holder's inequality, we get

$$|K_2| \leq C\|u\|_{B^r_{2,q}}^{q-1}\tilde{K}, \tag{5.5.7}$$

where

$$\tilde{K} = \left(\sum_j\left(\sum_{k>-M}\sum_{l=-5}^{5} 2^{(r-2)j}2^{(3+n/2)(j+k+l)}\|\triangle_{j+k}u\|_{L^2}\|\triangle_{j+k+l}u\|_{L^2}\right)^q\right)^{1/q}. \tag{5.5.8}$$

Working on the exponents of 2, we have

$$\begin{aligned}2^{(r-2)j}2^{(3+n/2)(j+k+l)} &= 2^{r(j+k)}2^{-rk}2^{-2(j+k+l)}2^{2(k+l)}2^{(3+n/2)(j+k+l)}\\ &= 2^{r(j+k)}2^{(1+n/2)(j+k+l)}2^{k(2-r)}2^{2l}.\end{aligned} \tag{5.5.9}$$

Since the l-summation is finite, we replace the 2^{2l} term with a constant, and we have

(5.5.10)

$$\begin{aligned}\tilde{K} &\leq C\left(\sum_j\left(\sum_{k>-M}2^{k(2-r)}\sum_{l=-5}^{5}(2^{r(j+k)}\|\triangle_{j+k}u\|_2)(2^{(1+n/2)(j+k+l)}\|\triangle_{j+k+l}u\|_{L^2})\right)^q\right)^{1/q}\\ &\leq C\|u\|_{B^{1+n/2}_{2,q}}\left(\sum_j\left(\sum_{k>-M}2^{k(2-r)}\sum_{l=-5}^{5}(2^{r(j+k)}\|\triangle_{j+k}u\|_2)\right)^q\right)^{1/q}.\end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned}|\tilde{K}| &\leq C\|u\|_{B^{1+n/2}_{2,q}}\sum_{k>-M}2^{k(2-r)}\left(\sum_j(2^{r(j+k)}\|\triangle_{j+k}u\|_2)^q\right)^{1/q}\\ &\leq C\|u\|_{B^{1+n/2}_{2,q}}\|u\|_{B^r_{2,q}}\sum_{k>-M}2^{k(2-r)}.\end{aligned} \tag{5.5.11}$$

This last sum will be finite provided $r > 2$. We remark that the restriction that $r > 2$ could be lifted by allowing for additional regularity in the $B^{1+n/2}_{2,q}$ term. Specifically, by choosing ε such that $2 < r + \varepsilon$, then we would have

$$|\tilde{K}| \leq \|u\|_{B^{1+n/2+\varepsilon}_{2,q}} \|u\|_{B^r_{2,q}}. \tag{5.5.12}$$

So finally plugging back into the K_2 estimate, we get that

$$|K_2| \leq C\|u\|^q_{B^r_{2,q}} \|u\|_{B^{1+n/p}_{2,q}} \tag{5.5.13}$$

provided $r > 2$ and

$$|K_2| \leq C\|u\|^q_{B^r_{2,q}} \|u\|_{B^{1+n/2+\varepsilon}_{2,q}} \tag{5.5.14}$$

provided $2 < r + \varepsilon$.

Plugging the K_1 and K_2 estimates into (5.4.6), we finally get

$$\frac{d}{dt}\|u\|^q_{\tilde{B}^r_{2,q}} \leq C\|u\|^q_{\tilde{B}^r_{2,q}} \|u\|_{\tilde{B}^{1+n/2}_{2,q}} \tag{5.5.15}$$

for $r > 2$ and

$$\frac{d}{dt}\|u\|^q_{\tilde{B}^r_{2,q}} \leq C\|u\|^q_{\tilde{B}^r_{2,q}} \|u\|_{\tilde{B}^{1+n/2+\varepsilon}_{2,q}} \tag{5.5.16}$$

for $2 - r - \varepsilon < 0$. This proves Proposition 5.2.4.

5.6. Proof of Global Existence

In this section, we will work under the assumption that $r > 2$, which allows the use of the first relation in Proposition 5.2.4. The proof for the case with the other restriction is similar.

First, we re-write (5.5.16) as

$$\frac{d}{dt}\|u\|_{\tilde{B}^r_{2,q}} \leq C\|u\|_{\tilde{B}^r_{2,q}}\|u\|_{\tilde{B}^{1+n/2}_{2,q}}. \tag{5.6.1}$$

Applying Gronwall's inequality to (5.6.1), we get

$$\|u(t)\|_{\tilde{B}^r_{2,q}} \leq C\|u_0\|_{\tilde{B}^r_{2,q}} \exp(C\int_0^T \|u(s)\|_{\tilde{B}^{1+n/2}_{2,q}} ds). \tag{5.6.2}$$

If $0 \leq a < 1$, then

$$\int_0^T t^{-a}t^a\|u(s)\|_{\tilde{B}^{1+n/2}_{2,q}} ds) \leq C\|u\|_{a;1+n/2,2,q}. \tag{5.6.3}$$

Similarly, if $\sigma \geq 1$, then

$$\int_0^T \|u(s)\|_{\tilde{B}^{1+n/2}_{2,q}} ds) \leq C\|u\|_{L^\sigma(B^{1+n/2}_{2,q})}. \tag{5.6.4}$$

Allowing Y to represent either $L^\sigma(B^{1+n/2}_{2,q})$ or $C_{a;1+n/2,2,q}$, equation (5.6.2) gives

$$\|u(t)\|_{\tilde{B}^r_{2,q}} \leq C\|u_0\|_{B^r_{2,q}} e^{C\|u\|_Y}. \tag{5.6.5}$$

This calculation leads is the hardest part of the following Proposition..

Proposition 5.6.1. *Let u be a solution to* (1.1.2) *with initial data $u_0 \in \tilde{B}^r_{2,q}$ where $r > 2$ such that*

$$u \in BC([0,T) : B^r_{2,q}) \cap Y, \tag{5.6.6}$$

where Y is either $L^\sigma(B^{1+n/2}_{2,q})$ or $C_{a;1+n/2,2,q}$. Then

$$\|u(t)\|_{B^r_{2,q}} \leq M \tag{5.6.7}$$

where $M = C\|u_0\|_{B^r_{2,q}} \exp(C\|u\|_Y)$.

Alternatively, with X either $L^\sigma(B^{1+n/2+\varepsilon}_{2,q})$ or $C_{a;1+n/2+\varepsilon,2,q}$ and $n-r-\varepsilon<0$, we get

$$u \in BC([0,T) : B^r_{2,q}) \cap X, \tag{5.6.8}$$

with no restriction on r.

By our construction of Besov spaces, we need to show that

$$\|\Psi * u(t)\|_{L^2} \leq M \tag{5.6.9}$$

and

$$\|u(t)\|_{\tilde{B}^{n/2}_{2,q}} \leq M. \tag{5.6.10}$$

To prove this, we begin by using results from Section 3.6 to get that

$$\|\Psi * u(t)\|_{L^2} \leq \|u(t)\|_{L^2} \leq M, \tag{5.6.11}$$

which proves (5.6.9). We get (5.6.10) as an immediate consequence of (5.6.2). This Proposition immediately gives Theorem 5.2.1.

CHAPTER 6

An alternative approach to Besov space

6.1. Alternative approach to Besov spaces

In this chapter, we consider an alternative approach to a local existence result in Besov spaces. This method modifies the work of [**18**], where the author considered generalized Navier-Stokes equations in homogenous Besov spaces. Because our non-linearity is the sum of a degree one operator and a degree zero operator, the LANS equation is ill-suited to homogeneous solution space methods. Thus, our work here is to adapt [**18**] to the LANS equation and to adapt homogeneous results to inhomogeneous ones.

6.2. Supporting results for arbitrary data

In this section we establish some supporting results for the existence result in the next section.

Proposition 6.2.1. *Let* $4 < q \leq \infty$ *and* $2 \leq p < \infty$. *Let* $r = n/p + 2/q$ *and* $2 < r + 6/q$. *Assume* $u_0 \in \tilde{B}^r_{p,q}$, *and let* u *be a solution to* 1.1.2 *with initial datum* u_0. *Set*

$$B(t) = \|u\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})}. \tag{6.2.1}$$

Then for any $T > 0$,

$$B(t) \leq C \sum_j (1 - E_j(qT)) 2^{rjq} \|\triangle_j u_0\|^q_p + C \int_0^T B^2(t)dt \tag{6.2.2}$$

where $E_j(t) = \exp(-C2^{2j}t)$.

To prove this, we begin by recalling equation (6.5.17), namely,

$$\begin{aligned}
&\frac{d}{dt}\|\triangle_j u\|_p + C2^{2j}\|\triangle_j u\|_p \\
\le& C\sum_{|j-k|\le M} 2^{k+1}\|\triangle_{k+1}u\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m u\|_p \\
&+C\sum_{k>j-M} 2^{(1+n/p)k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l}\|\triangle_{k+l}u\|_p.
\end{aligned} \tag{6.2.3}$$

Converting this into an integral equation, we get

$$\begin{aligned}
\|\triangle_j u(\cdot,t)\|_p \le& CE_j(t)\|\triangle_j u_0\|_p \\
&+C\int_0^t E_j(t-s)N_1 ds \\
&+C\int_0^t E_j(t-s)N_2 ds,
\end{aligned} \tag{6.2.4}$$

where

$$\begin{aligned}
&E_j(t) = \exp(-C2^{2j}t), \\
&N_1 = \sum_{|j-k|\le M} 2^{k+1}\|\triangle_{k+1}u\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m u\|_p, \\
&N_2 = \sum_{k>j-M} 2^{(1+n/p)k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l}\|\triangle_{k+l}u\|_p.
\end{aligned} \tag{6.2.5}$$

Multiplying both sides by $2^{(r+2/q)j}$, raising both sides to the q^{th} power, summing over j and integrating over $(0,T)$, we get

$$B(t) \le M_1 + M_2 + M_3 \tag{6.2.6}$$

where

$$
\begin{aligned}
M_1 &\equiv C\int_0^T \sum_j E_j^q(t) 2^{(r+2/q)jq} \|\triangle_j u_0\|_p^q dt \\
M_2 &\equiv C\int_0^T \sum_j 2^{(r+2/q)jq} M_{21}(t) dt \\
M_3 &\equiv C\int_0^T \sum_j 2^{(r+2/q)jq} M_{31}(t) dt
\end{aligned}
\tag{6.2.7}
$$

and

$$
\begin{aligned}
M_{21}(t) &= \left(\int_0^t E_j(t-s)N_1\right)^q \\
M_{31}(t) &= \left(\int_0^t E_j(t-s)N_2 ds\right)^q.
\end{aligned}
\tag{6.2.8}
$$

Our task is to estimate M_1, M_2 and M_3. We begin with M_1, and observe that by direct computation

$$
\int_0^T E_j^q(t)dt = C\int_0^T \exp(-C2^{2j}t)dt = C2^{-2j}(1-E_j(qT)).
\tag{6.2.9}
$$

Using this, we bound M_1 by

$$
\begin{aligned}
M_1(t) &\le C\sum_j 2^{(r+2/q)jq}(2^{-2j}(1-E_j(qT))\|\triangle_j u_0\|_p^q \\
&\le C\sum_j (1-E_j(qT))2^{rjq}\|\triangle_j u_0\|_p^q.
\end{aligned}
\tag{6.2.10}
$$

To estimate M_2, we begin with M_{21}. For $q>2$, since $\frac{q-2}{q}+\frac{2}{q}=1$, Holder's inequality gives

$$
\begin{aligned}
\left(\int_0^t E_j(t-s)N_1 ds\right)^q &\le \left(\int_0^t E_j^{\frac{q}{q-2}}(t-s)ds\right)^{q-2}\left(\int_0^t N_1^{q/2}ds\right)^2 \\
&\le C2^{-2j(q-2)}\left(1-E_j(\frac{qt}{q-2})\right)^{q-2}\left(\int_0^t N_1^{q/2}ds\right)^2 \\
&\le C2^{-2j(q-2)}\left(\int_0^t N_1^{q/2}ds\right)^2
\end{aligned}
\tag{6.2.11}
$$

where in the last line we used that $E_j(t) \leq 1$ for $t \geq 0$. Applying Holder's inequality to the remaining integral, we get

$$
\begin{aligned}
&2^{-2j(q-2)} \left(\int_0^t N_1^{q/2} ds \right)^2 \\
&\leq 2^{-2j(q-2)} C \int_0^t \left(\sum_{k=-5}^{5} 2^{j+k} \|\triangle_{j+k} u\|_p \right)^q ds \int_0^t \left(\sum_{m<j+k-5} 2^{(1+n/p)m} \|\triangle_m u\|_p \right)^q ds \\
&\leq C \int_0^t \left(\sum_{k=-5}^{5} \|\triangle_{j+k} u\|_p \right)^q ds \int_0^t \left(\sum_{m<j+k-5} 2^{(1+n/p)m} 2^{(j+k)(4/q-1)} \|\triangle_m u\|_p \right)^q ds.
\end{aligned} \tag{6.2.12}
$$

Working on the last integral, we have

$$
\begin{aligned}
&\sum_{m<j+k-5} 2^{(1+n/p)m} 2^{(j+k)(4/q-1)} \|\triangle_m u\|_p \\
&= \sum_{m<j+k-5} 2^{(4/q+n/p)m} 2^{(j+k)(4/q-1)} 2^{(1-4/q)m} \|\triangle_m u\|_p \\
&\leq C \|u\|_{\tilde{B}^{4/q+n/p}_{p,q}} \left(\sum_{m<j+k-5} 2^{(1-4/q)(m-j+k-5)q'} \right)^{1/q} \leq C \|u\|_{\tilde{B}^{4/q+n/p}_{p,q}}.
\end{aligned} \tag{6.2.13}
$$

We remark that this requires $q > 4$. Plugging this back into (6.2.12), we have

$$
2^{-2j(q-2)} \left(\int_0^t N_1^{q/2} ds \right)^2 \leq C \int_0^t \|u\|^q_{\tilde{B}^{4/q+n/p}_{p,q}} \int_0^t \left(\sum_{k=-5}^{5} \|\triangle_{j+k} u\|_p \right)^q ds, \tag{6.2.14}
$$

and we use this to bound M_{21} by

$$
M_{21} \leq C B(t) \int_0^t \left(\sum_{k=-5}^{5} \|\triangle_{j+k} u\|_p \right)^q ds, \tag{6.2.15}
$$

where we set $r = n/p + 2/q$. Using (6.2.15), we bound M_2 by

$$
\begin{aligned}
M_2 &\leq C \int_0^T B(t) \left(\sum_j 2^{(r+2/q)jq} \int_0^t \left(\sum_{k=-5}^{5} \|\triangle_{j+k} u\|_p \right)^q ds \right) dt \\
&\leq C \int_0^T B(t)^2 dt.
\end{aligned} \tag{6.2.16}
$$

Now we consider M_3, and we start with M_{31}. As in the computation of M_{21}, we have

$$M_{31}(t) \leq C 2^{-2j(q-2)} \left(\int_0^t N_2^{q/2} ds \right)^2 . \tag{6.2.17}$$

Then

$$\begin{aligned} N_2 &\leq C \sum_{k>j-M} 2^k \|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{(r+2/q)(j+k)} 2^{(1-4/q)(j+k)} \|\triangle_{k+l} u\|_p \\ &\leq C \sum_{k>j-M} 2^k 2^{(1-4/q)k} \|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{(r+2/q)(k+j)} \|\triangle_{k+l} u\|_p \\ &\leq C \|u\|_{\tilde{B}^{r+2/q}_{p,q}} \sum_{k>j-M} 2^k 2^{(1-4/q)k} \|\triangle_k u\|_p. \end{aligned} \tag{6.2.18}$$

Applying this to (6.2.17) and applying Holder's inequality, we have

$$M_{31} \leq C 2^{-2j(q-2)} B(t) \int_0^t \left(\sum_{k>j-M} 2^k 2^{(1-4/q)k} \|\triangle_k u\|_p \right)^q ds. \tag{6.2.19}$$

With this, we bound M_3 with

(6.2.20)

$$\begin{aligned} M_3 &\leq C \int_0^T \left(\sum_j 2^{(r+2/q)jq} 2^{-2j(q-2)} B(t) \int_0^t \left(\sum_{k>j-M} 2^k 2^{(1-4/q)k} \|\triangle_k u\|_p \right)^q ds \right) dt \\ &\leq C \int_0^T \left(B(t) \int_0^t \left(\sum_j 2^{(r+2/q-2+4/q)j} \sum_{k>j-M} 2^k 2^{(1-4/q)k} \|\triangle_k u\|_p \right)^q ds \right) dt \\ &\leq C \int_0^T \left(B(t) \int_0^t \left(\sum_{k>-M} 2^{k(2-r-6/q)} \sum_j 2^{(r+2/q)(j+k)} \|\triangle_{j+k} u\|_p \right)^q ds \right) dt \\ &\leq C \int_0^T \left(B(t) \int_0^t \|u\| q_{\tilde{B}^{r+2/q}_{p,q}} ds \left(\sum_{k>-M} 2^{k(2-r-6/q)} \right)^q \right) dt \\ &\leq C \int_0^T B(t)^2 dt, \end{aligned}$$

provided

$$\sum_k 2^{k(2-r-6/q)} \tag{6.2.21}$$

is finite, which will hold provided $2 < r + 6/q$.

Combining (6.2.10), (6.2.16), and (6.2.20) gives the Proposition. Next, we do a similar calculation, this time for the operator Ψ.

PROPOSITION 6.2.2. *Let $2 \le p < \infty$ and $1 \le q < \infty$. Define $r = n/p + 2/q$, and assume $2 < r + 6/q$. Then if u solves* (1.1.2), *then*

$$\|\Psi * u(t)\|_p \le C\|\Psi * u_0\|_p + C\|u\|^2_{L^q(B^{r+2/q}_{p,q})}. \tag{6.2.22}$$

We mirror the construction used in the proof of Proposition 6.5.1, only instead of applying $\triangle_j$ to both sides of (1.1.2), we apply Ψ. We get

$$\frac{d}{dt}\|\Psi * u\|_p \le \quad C\sum_k 2^{(1+n/p)k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l}\|\triangle_{k+l}u\|_p. \tag{6.2.23}$$

This expression is simpler than the one found in (6.5.17) because the application of Ψ annihilates the paraproduct pieces (5.2.18) and (5.2.19).

Integrating both sides we get

$$\|\Psi * u(t)\|_p \le C\|\Psi * u_0\|_p + C\int_0^T \sum_k 2^{(1+n/p)k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l}\|\triangle_{k+l}u\|_p dt. \tag{6.2.24}$$

Working on the summations, we have

$$\begin{aligned} &\sum_k 2^{(1+n/p)k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l}\|\triangle_{k+l}u\|_p \\ \le &C\sum_k 2^{(1+n/(2p))k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{(1+n/(2p))(k+l)}\|\triangle_{k+l}u\|_p \\ \le &\|u\|_{B^{1+n/(2p)}_{p,q}} \sum_k 2^{k(1+n/(2p))}\|\triangle_k u\|_p. \end{aligned} \tag{6.2.25}$$

We observe that, by basic algebra, $2 < n/p+8/q$ implies $1+n/(2p) < n/p+4/q$. So, using Besov Embedding, Young's inequality, and the requirement that $r+6/q = n/p+8/q > 2$, we have

$$\sum_k 2^{(1+n/p)k}\|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l}\|\triangle_{k+l}u\|_p$$

$$\leq C\|u\|_{B^{r+2/q}_{p,q}} \sum_k 2^{k(1+n/(2p)-r-2/q)}2^{k(r+2/q)}\|\triangle_k u\|_p \tag{6.2.26}$$

$$\leq C\|u\|^2_{B^{r+2/q}_{p,q}}.$$

Applying this to (6.2.24), and again applying Young's inequality, we have

$$\|\Psi * u(t)\|_p \leq C\|\Psi * u_0\|_p + C\int_0^T \|u\|^2_{B^{r+2/q}_{p,q}} dt$$

$$\leq C\|\Psi * u_0\|_p + C\|u\|_{L^q(B^{r+2/q}_{p,q})}\|u\|_{L^{q'}(B^{r+2/q}_{p,q})} \tag{6.2.27}$$

Since our time interval $I = (0, T)$ is finite and $2 < q$, we finally get

$$\|\Psi * u(t)\|_p \leq C\|\Psi * u_0\|_p + C\|u\|^2_{L^q(B^{r+2/q}_{p,q})}. \tag{6.2.28}$$

This finishes the Proposition.

6.3. Results for the operator F

In this section, we let $F = F(v, w)$ be an operator and assume F satisfies

$$\partial_t F - AF = -P^\alpha(v \cdot \nabla)w - P^\alpha(\mathrm{div}\ \tau^\alpha(v, w)). \tag{6.3.1}$$

We prove the following proposition.

PROPOSITION 6.3.1. *Let $4 < q \leq \infty$, $2 \leq p < \infty$, and $r = n/p + 2/q$, and $2 < r + 6/q$. Assume $F_0 \in \tilde{B}^r_{p,q}$ (where we recall $F_0 = F(x, 0)$) and*

$$v, w \in L^q((0, T) : \tilde{B}^{r+2/q}_{p,q}) \tag{6.3.2}$$

for some $T > 0$. Then any solution of (6.5.34) *satisfies*

$$\|F\|^q_{L^q((0,T):\tilde{B}^{r+2/q}_{p,q})} \leq C\sum_j (1 - E_j(qT))2^{rjq}\|\triangle_j F_0\|^q_p + C\int_0^T \|v\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})}\|w\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})}dt \tag{6.3.3}$$

We start with equation (6.5.37):

$$\frac{d}{dt}\|\triangle_j F\|_p + C2^{2j}\|\triangle_j F\|_p \leq I_1 + I_2 + I_3 \tag{6.3.4}$$

where

$$I_1 = C\sum_{|j-k|\leq M} 2^{k+1}\|\triangle_{k+1}w\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m v\|_p = \tilde{I}_1 \tag{6.3.5}$$

$$I_2 = C\sum_{|j-k|\leq M} 2^{k+1}\|\triangle_{k+1}v\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m w\|_p = \tilde{I}_2 \tag{6.3.6}$$

and

$$I_3 = C\sum_{k>j-M} 2^{(1+n/p)k}\|\triangle_j v\|_p \sum_{l=k-5}^{k+5} 2^{k+l}\|\triangle_{k+l}w\|_p = \tilde{I}_3. \tag{6.3.7}$$

Re-writing (6.3.4) as an integral equation, we have

$$\|\triangle_j F\|_p \leq CE_j(t)\|\triangle_j F_0\|_p + C\int_0^t E_j(t-s)(I_1 + I_2 + I_3)ds. \tag{6.3.8}$$

Multiplying by $2^{(r+2/q)j}$, raising both sides to the q^{th} power, summing over j and integrating over the interval $(0, T)$, we get

$$\|F\|^q_{L^q((0,T):\tilde{B}^{r+2/q}_{p,q}} \leq H_1 + H_2 + H_3 + H_4 \tag{6.3.9}$$

where

$$(6.3.10)\qquad \begin{aligned} H_1 &= C\int_0^T \sum_j E_j^q(t) 2^{(r+2/q)jq} \|\triangle_j F_0\|_p^q dt,\\ H_2 &= C\int_0^T \sum_j 2^{(r+2/q)jq} H_{21}(t)dt\\ H_3 &= C\int_0^T \sum_j 2^{(r+2/q)jq} H_{31}(t)dt\\ H_4 &= C\int_0^T \sum_j 2^{(r+2/q)jq} H_{41}(t)dt, \end{aligned}$$

and

$$(6.3.11)\qquad \begin{aligned} H_{21}(t) &= \left(\int_0^t E_j(t-s) I_1 ds\right)^q\\ H_{31}(t) &= \left(\int_0^t E_j(t-s) I_2 ds\right)^q\\ H_{41}(t) &= \left(\int_0^t E_j(t-s) I_3 ds\right)^q. \end{aligned}$$

Observing that the H_i are similar to the M_j from the previous proposition, we get

$$(6.3.12)\qquad \begin{aligned} H_1 &\le C\sum_j (1-E_j(qT)) 2^{rjq} \|\triangle_j F_0\|_p^q\\ H_2 &\le C\int_0^T \|v\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})} \|w\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})} dt\\ H_3 &\le C\int_0^T \|v\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})} \|w\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})} dt\\ H_4 &\le C\int_0^T \|w\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})} \|v\|^q_{L^q((0,t):\tilde{B}^{r+2/q}_{p,q})} dt, \end{aligned}$$

which proves the Proposition.

PROPOSITION 6.3.2. *Let $2 \le p < \infty$ and $1 \le q < \infty$. Define $r = n/p + 2/q$, and assume $2 < r + 2/q$. Then if u solves* (1.1.2), *then*

$$(6.3.13)\qquad \|\Psi * F(t)\|_p \le C\|\Psi * F_0\|_p + C\|v\|_{L^q(B^{r+2/q}_{p,q})} \|w\|_{L^q(B^{r+2/q}_{p,q})}.$$

As in Proposition 6.2.2, applying Ψ to our operator equation gives the following modified version of (6.5.37):

$$\frac{d}{dt}\|\Psi * F\|_p \leq C \sum_{k>j-M} 2^{(1+n/p)k}\|\triangle_j v\|_p \sum_{l=k-5}^{k+5} 2^{k+l}\|\triangle_{k+l} w\|_p. \tag{6.3.14}$$

Applying the argument used for Proposition 6.2.2 gives

$$\|\Psi * F\|_p \leq C\|\Psi * F_0\|_p + C\|v\|_{L^q(B^{r+2/q}_{p,q})}\|w\|_{L^q(B^{r+2/q}_{p,q})}. \tag{6.3.15}$$

6.4. Local Existence with arbitrary initial data

In this section we prove our local existence theorem.

THEOREM 6.4.1. *Let $4 < q \leq \infty$. Let $2 \leq p < \infty$. Let $r = n/p + 2/q$, and let $u_0 \in B^r_{p,q}$. Also assume $2 < r + 2/q$. Then there exists a $T = T(u_0) > 0$ and a unique solution u of* (1.1.2) *such that*

$$u \in X \cap Z \tag{6.4.1}$$

where

$$X = C([0,T) : B^r_{p,q}),\ \ Z = L^q\left((0,T) : B^{r+2/q}_{p,q}\right). \tag{6.4.2}$$

We also note that for $u_0, v_0 \in B^r_{p,q}$, the corresponding solutions $u(t), v(t)$ will satisfy $\|u - v\|_{X\cap Z} \leq C\|u_0 - v_0\|_{B^r_{p,q}}$.

First, we will prove that the map $\Phi(u) = \bar{u} + F(u,u)$ is a contraction on the space $D = \{u \in Z : \|u\|_Z \leq R\}$, where $\bar{u}(t,x) = \Gamma(t)u_0(x)$ and $F(u,u) = G \cdot V^\alpha(u)$. As usual, we decompose the Z-norm as

$$\|u\|_Z \leq \left(\int_0^T \|\Psi * u(t)\|_p^q dt\right)^{1/q} + \left(\int_0^T \|u(t)\|^q_{\tilde{B}^{r+2/q}_{p,q}} dt\right)^{1/q}. \tag{6.4.3}$$

We will denote the first piece as $\bar{Z}$ and the second as $\tilde{Z}$.

Observing that

$$\partial_t \bar{u} - \triangle \bar{u} = 0 \tag{6.4.4}$$

and that $\bar{u}(0, x) = u_0(x)$, a slight modification to the proof of Proposition 6.2.1 gives that

$$\begin{aligned} \|\bar{u}\|_{\tilde{Z}}^q &\leq C \sum_j (1 - E_j(qT)) 2^{rjq} \|\triangle_j \bar{u}\|_p^q \\ &\leq C \sum_j 2^{rjq} \|\triangle_j u_0\|_p^q \leq \|u_0\|_{B_{p,q}^r}^q. \end{aligned} \tag{6.4.5}$$

This shows that $\|\bar{u}\|_{\tilde{Z}}$ is finite. We remark that applying the Dominated Convergence Theorem to the first inequality in (6.4.5) gives that $\|\bar{u}\|_{\tilde{Z}} \to 0$ as $T \to 0$.

Applying Proposition 3.8.1, we have

$$\|\bar{u}\|_{\bar{Z}} \leq C \|u\|_{H^{-2/q,p}} \leq C \|u_0\|_{B_{p,q}^r}, \tag{6.4.6}$$

provided $q > p$. By Corollary 3.8.3, we have that $\|\bar{u}\|_{\bar{Z}}$ can be made arbitrarily small by choosing a sufficiently small T. Combining these two results, we get that $\|\bar{u}\|_Z$ is finite and tends to zero as T tends to zero.

Now we consider F. We observe that F satisfies

$$\partial_t F - \nabla F = V^\alpha(u) \tag{6.4.7}$$

and $F(u, u)(x, 0) = 0$, so Proposition 6.3.1 gives

$$\begin{aligned} \|F\|_{\tilde{Z}}^q &\leq C \int_0^T \|u\|_{L^q((0,t):\tilde{B}_{p,q}^{r+2/q})}^2 dt \\ &\leq CT \|u\|_Z^2 \leq CTR^2 \end{aligned} \tag{6.4.8}$$

for any $u \in D$.

Similarly, applying Proposition 6.3.2 gives

$$\|\Psi * F\|_{\bar{Z}} \leq C \int_0^T \|u\|_Z^2 dt \leq CT\|u\|_Z^2 \leq CTR^2. \tag{6.4.9}$$

So for sufficiently small R and T, $\Phi : Z \to Z$. Now we show that Φ is a contraction. We note that

$$\Phi u - \Phi v = F(u,u) - F(v,v) = -(F(u,u-v) + F(u-v,v)). \tag{6.4.10}$$

We remark that $F(u,u-v)$ satisfies

$$\partial_t F - AF = V^\alpha(u,u-v) \tag{6.4.11}$$

and $F(u,u-v)(x,0) = 0$, so using Proposition 6.3.1 and Proposition 6.3.2 we have

$$\|F(u,u-v)\|_Z^q \leq C\|u\|_Z^q \|u-v\|_Z^q. \tag{6.4.12}$$

Obtaining a similar bound for $F(u-v,v)$ and combining the results, we have

$$\|\Phi u - \Phi v\|_Z \leq C(\|u\|_Z + \|v\|_Z)\|u-v\|_Z \leq CR\|u-v\|_Z. \tag{6.4.13}$$

So for small enough R, Φ is a contraction on D for sufficiently small T. To finish the theorem, we need to show that the solution $u \in C([0,T) : B^r_{p,q})$.

We start with Proposition 6.2.2 and get

$$\|\Psi * u(t)\|_p \leq C\|\Psi * u_0\|_p + C\|u\|^2_{L^q(B^{r+2/q}_{p,q})}. \tag{6.4.14}$$

For the second piece of the Besov norm, we bound E_j by 1 and (6.2.4) becomes

$$\begin{aligned}\|\triangle_j u(\cdot,t)\|_p \leq & C\|\triangle_j u_0\|_p + C\int_0^t E_j(t-s)2^{(2+n/p)j}\|\triangle_j u(\cdot,s)\|_p^2 ds \\ & + C\int_0^t E_j(t-s)\|\triangle_j u(\cdot,s)\|_p \sum_{m<j-5} 2^{(1+n/p)m+j}\|\triangle_m u(\cdot,s)\|_p ds.\end{aligned} \tag{6.4.15}$$

Multiplying both sides by of (6.4.15) by 2^{rj}, raising both sides to the q^{th} power and summing over j gives

$$\|u(t)\|^q_{\tilde{B}^r_{p,q}} \leq C\|u_0\|_{\tilde{B}^r_{p,q}} + N_1 + N_2 \tag{6.4.16}$$

where

$$\begin{aligned} N_1 &\equiv C\sum_j 2^{(2+n/p+r)jq} N_{11}(t) \\ N_2 &\equiv C\sum_j 2^{rjq} N_{21}(t) \end{aligned} \tag{6.4.17}$$

and

$$\begin{aligned} N_{11}(t) &= \left(\int_0^t E_j(t-s)\|\triangle_j u\|_p^2 ds\right)^q \\ N_{21}(t) &= \left(\int_0^t E_j(t-s)\|\triangle_j u\|_p \sum_{m<j-5} 2^{(1+n/p)m+j\|\triangle_m u\|_p}\right)^q . \end{aligned} \tag{6.4.18}$$

Estimating these terms as in Proposition 6.2.1, we eventually get

$$\|u(t)\|^q_{\tilde{B}^r_{p,q}} \leq C\|u_0\|_{\tilde{B}^r_{p,q}} + \|u\|_Z \tag{6.4.19}$$

which proves $u \in C([0,T) : B^r_{p,q})$.

Dependance on the initial data follows from the standard argument.

6.5. Additional Besov computations

In this section we prove two additional Besov space results.

PROPOSITION 6.5.1. *Any solution u of* (1.1.2) *satisfies*

$$\frac{d}{dt}\|u\|^q_{\tilde{B}^r_{p,q}} + Cq\|u\|^q_{\tilde{B}^{r+2/q}_{p,q}} \leq Cq\|u\|^q_{\tilde{B}^{r+2/q}_{p,q}}\|u\|_{\tilde{B}^{n/p}_{p,q}}, \tag{6.5.1}$$

provided $r \in \mathbb{R}$, $n \geq 2$, $1 \leq q < \infty$ and $2 \leq p < \infty$ and $2 - r - 2/q < 0$.

This is similar to Theorem 4.1 in [**18**].

We begin by writing (1.1.2) as

$$\partial_t u - Au + P^\alpha(u\cdot\nabla)u + P^\alpha(\text{div }\tau^\alpha u) = 0, \tag{6.5.2}$$

and then, for any $j \geq 0$, apply $\triangle_j$ to (6.5.2) to get

$$\partial_t\triangle_j u + (u\cdot\nabla)\triangle_j u - A\triangle_j u = -[P^\alpha\triangle_j, u\cdot\triangle]u - P^\alpha\triangle_j(\text{div }\tau^\alpha u) \tag{6.5.3}$$

where $[\cdot,\cdot]$ represents the commutator. This differs from equation (4.9) in [**18**] only in the presence of the term involving τ^α. Following the argument used in [**18**], we eventually get

$$\begin{aligned} &\frac{d}{dt}\|\triangle_j u\|_p^p + C2^{2j}\|\triangle_j u\|_p^p \\ \leq& C\|\triangle_j u\|_p^{p-1}(H+J) \end{aligned} \tag{6.5.4}$$

where

$$H = \|[P^\alpha\triangle_j, u\cdot\nabla]u\|_p \tag{6.5.5}$$

and

$$J = \|P^\alpha\triangle_j(\text{div }\tau^\alpha u)\|_p. \tag{6.5.6}$$

Computing the time derivative and canceling the common $\|\triangle_j u\|_p^{p-1}$ factor, we have

$$\begin{aligned} &\frac{d}{dt}\|\triangle_j u\|_p + C2^{2j}\|\triangle_j u\|_p \\ \leq& C(H+J) \end{aligned} \tag{6.5.7}$$

Since the calculations required to estimate H and J are similar (and the estimate for H is essentially identical to the one in [**18**]), we will only estimate J. We remark that to estimate J, it is sufficient to estimate $\|P^{\alpha}\triangle_j(\text{div }(1-\alpha^2\triangle)^{-1}(\nabla u\cdot\nabla u))\|_p$, and we have

$$\|P^{\alpha}\triangle_j(\text{div }(1-\alpha^2\triangle)^{-1}(\nabla u\cdot\nabla u))\|_p C \leq \|\triangle_j(\nabla u\cdot\nabla u)\|_p. \tag{6.5.8}$$

Using (5.2.18), (5.2.19), and (5.2.20), we get

$$\|\triangle_j(\nabla_u \nabla u)\|_p \leq J_1 + J_2 \tag{6.5.9}$$

where

$$|J_1| \leq C\|\sum_{|j-k|\leq M} \triangle_j S_{k-5}(\nabla u)\triangle_{k+1}\nabla u\|_p \tag{6.5.10}$$

and

$$|J_2| \leq C\|\sum_{k\geq j-M} \triangle_j\triangle_k(\nabla u)\sum_{l=-5}^{5}\triangle_{k+l}\nabla u\|_p. \tag{6.5.11}$$

Applying Young's inequality and Bernstein's inequality, we get

$$\begin{aligned} |J_1| &\leq C\sum_{|j-k|\leq M}\|\nabla S_{k-5}u\|_\infty\|\nabla\triangle_{k+1}u\|_p \\ &\leq C\sum_{|j-k|\leq M} 2^{k+1}\|\triangle_{k+1}u\|_p\sum_{m<k-5}2^{(1+n/p)m}\|\triangle_m u\|_p \end{aligned} \tag{6.5.12}$$

and

$$\begin{aligned} |J_2| &\leq C\sum_{k>j-M}\|\nabla\triangle_k u\|_\infty\sum_{l=-5}^{5}\|\nabla\triangle_{k+l}u\|_p \\ &\leq C\sum_{k>j-M}2^{(1+n/p)k}\|\triangle_k u\|_p\sum_{l=-5}^{5}2^{k+l}\|\triangle_{k+l}u\|_p. \end{aligned} \tag{6.5.13}$$

Combining the estimates on J_1 and J_2 gives

$$
\begin{aligned}
|J| \le & C \sum_{|j-k|\le M} 2^{k+1} \|\triangle_{k+1} u\|_p \sum_{m<k-5} 2^{(1+n/p)m} \|\triangle_m u\|_p \\
& + C \sum_{k>j-M} 2^{(1+n/p)k} \|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l} \|\triangle_{k+l} u\|_p .
\end{aligned} \tag{6.5.14}
$$

Using methods similar to those in Section 5.3, we get

$$
\begin{aligned}
|H| \le & C \sum_{|j-k|\le M} \|\triangle_{k+1} u\|_p \sum_{m<k-5} 2^{(1+n/p)m} \|\triangle_m u\|_p \\
& + C \sum_{k>j-M} 2^{(1+n/p)k} \|\triangle_k u\|_p \sum_{l=-5}^{5} \|\triangle_{k+l} u\|_p .
\end{aligned} \tag{6.5.15}
$$

Since $j \ge 0$, we have

$$
\begin{aligned}
|H| + |J| \le & C \sum_{|j-k|\le M} 2^{k+1} \|\triangle_{k+1} u\|_p \sum_{m<k-5} 2^{(1+n/p)m} \|\triangle_m u\|_p \\
& + C \sum_{k>j-M} 2^{(1+n/p)k} \|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l} \|\triangle_{k+l} u\|_p .
\end{aligned} \tag{6.5.16}
$$

Applying this to (6.5.7) gives

$$
\begin{aligned}
& \frac{d}{dt} \|\triangle_j u\|_p + C 2^{2j} \|\triangle_j u\|_p \\
\le & C \sum_{|j-k|\le M} 2^{k+1} \|\triangle_{k+1} u\|_p \sum_{m<k-5} 2^{(1+n/p)m} \|\triangle_m u\|_p \\
& + C \sum_{k>j-M} 2^{(1+n/p)k} \|\triangle_k u\|_p \sum_{l=-5}^{5} 2^{k+l} \|\triangle_{k+l} u\|_p .
\end{aligned} \tag{6.5.17}
$$

Multiplying both sides by $q 2^{rjq} \|\triangle_j u\|_p^{q-1}$ and summing over j, we get

$$
\frac{d}{dt} \|u\|^q_{\tilde{B}^r_{p,q}} + Cq \|u\|^q_{\tilde{B}^{r+2/q}_{p,q}} \le I_1 + I_2 \tag{6.5.18}
$$

where

$$
I_1 = C \sum_j 2^{rjq} \|\triangle_j u\|_p^{q-1} \sum_{k=-M}^{M} 2^{j+k+1} \|\triangle_{j+k+1} u\|_p \sum_{m<j+k-5} 2^{(1+n/p)m} \|\triangle_m u\|_p \tag{6.5.19}
$$

and

$$I_2 = C\sum_j 2^{rjq}\|\triangle_j u\|_p^{q-1}\sum_{k>j-M}2^{(1+n/p)k}\|\triangle_k u\|_p\sum_{l=-5}^{5}2^{k+l}\|\triangle_{k+l}u\|_p. \tag{6.5.20}$$

We start with I_1. Manipulating the powers of 2, we have

$$2^{rjq}2^{j+k+1}2^{(1+n/p)m} = 2^{(r+2/q)jq}2^{-(j+k+l)}2^{nm/p}2^{m-(j+k-5)}2^{C}, \tag{6.5.21}$$

where C is a fixed finite number resulting from manipulating 2^j. Applying this to I_1, we have

$$I_1 \leq C\sum_j 2^{(r+2/q)jq}\|\triangle_j u\|_p^{q-1}\left(\sum_{k=-M}^{M}\|\triangle_{j+k+1}u\|_p I_{1,1}\right), \tag{6.5.22}$$

where

$$I_{1,1} = \sum_{m<j+k-5}2^{mn/p}2^{m-(j+k-5)}\|\triangle_m u\|_p. \tag{6.5.23}$$

Using Holder's inequality for sums, we have

$$\begin{aligned} I_{1,1} &\leq \left(\sum_{m<k+j-5}2^{(n/p)qm}\|\triangle_m u\|_p^q\right)^{1/q}\left(\sum_{m<j+k-5}2^{(m-(j+k-5))q'}\right)^{1/q'} \\ &\leq C\|u\|_{\tilde{B}^{n/p}_{p,q}}. \end{aligned} \tag{6.5.24}$$

Returning to I_1, we have

$$\begin{aligned} I_1 &\leq C\|u\|_{\tilde{B}^{n/p}_{p,q}}\sum_j 2^{(r+2/q)j(q-1)}\|\triangle_j u\|_p^{q-1}\left(\sum_{k=-M}^{M}2^{(r+2/q)(j+k+1)}\|\triangle_{j+k+1}u\|_p\right) \\ &\leq C\|u\|_{\tilde{B}^{n/p}_{p,q}}\|u\|^{q-1}_{\tilde{B}^{r+2/q}_{p,q}}\left(\sum_j\left(\sum_{k=-5}^{5}2^{(r+2/q)(j+k+1)}\|\triangle_{j+k+1}u\|_p\right)^q\right)^{1/q} \\ &\leq C\|u\|_{\tilde{B}^{n/p}_{p,q}}\|u\|^{q}_{\tilde{B}^{r+2/q}_{p,q}}. \end{aligned} \tag{6.5.25}$$

Now we work on I_2. Manipulations similar to those used in (5.5.9) gives

$$I_2 \leq C \sum_j 2^{(r+2/q)j(q-1)} \|\triangle_j u\|_p^{q-1} I_{2,1} \tag{6.5.26}$$

where

$$I_{2,1} = \sum_{k>M} 2^{k(2-r-2/q)} 2^{(r+2/q)(j+k)} \|\triangle_{(j+k)} u\|_p \sum_{l=-5}^{5} 2^{(n/p)(j+k+l)} \|\triangle_{j+k+l} u\|_p. \tag{6.5.27}$$

Applying Holder's inequality gives

$$I_2 \leq C \|u\|^{q-1}_{\tilde{B}^{r+2/q}_{p,q}} \|I_{2,1}\|_{l^q}. \tag{6.5.28}$$

To compute $\|I_{2,1}\|_{l^q}$, we use Minkowski's inequality for sums and get

$$I_{2,1} \leq C \sum_{k>-M} 2^{k(2-r-2/q)} \sum_{l=-5}^{5} \left(\sum_j \left(2^{(r+2/q)(j+k)} \|\triangle_{(j+k)} u\|_p I_{2,2}\right)^q \right)^{1/q} \tag{6.5.29}$$

where $I_{2,2}$ is defined by

$$I_{2,2} = 2^{(n/p)(j+k+l)} \|\triangle_{j+k+l} u\|_p \leq C \|u\|_{\tilde{B}^{n/p}_{p,\infty}} \leq C \|u\|_{\tilde{B}^{n/p}_{p,q}}, \tag{6.5.30}$$

and the last line applied Besov embedding (Theorem 4.8.1). So finally we bound $\|I_{2,1}\|_{l^q}$ by

$$\|I_{2,1}\|_{l^q} \leq C \|u\|_{\tilde{B}^{n/p}_{p,q}} \|u\|_{\tilde{B}^{1+2/q}_{p,q}} \sum_k 2^{k(2-r-2/q)}. \tag{6.5.31}$$

The last sum will be finite provided $2 < r - 2/q$, and we bound I_2 by

$$I_2 \leq C \|u\|_{\tilde{B}^{n/p}_{p,q}} \|u\|^q_{\tilde{B}^{1+2/q}_{p,q}}. \tag{6.5.32}$$

Using the bounds on I_1 and I_2 in (6.5.18), we finally get

$$\frac{d}{dt} \|u\|^q_{\tilde{B}^r_{p,q}} + Cq \|u\|^q_{\tilde{B}^{r+2/q}_{p,q}} \leq C \|u\|_{\tilde{B}^{n/p}_{p,q}} \|u\|^q_{\tilde{B}^{r+2/q}_{p,q}} \tag{6.5.33}$$

which proves Proposition 6.5.1.

Our second estimate is an operator estimate. We recall our previous constructions for the operator F. We let $F = F(v, w)$ be an operator and assume F satisfies

$$\partial_t F - AF = -P^\alpha(v \cdot \nabla)w - P^\alpha(\operatorname{div} \tau^\alpha(v, w)). \tag{6.5.34}$$

We have the following Proposition, which is similar to Theorem 4.3 in [**18**].

Proposition 6.5.2. *Let $r \in \mathbb{R}$ and $q \in [1, \infty)$. Assume v, w are in*

$$L^\infty([0,T) : \tilde{B}^r_{p,q}) \cap L^q([0,T) : \tilde{B}^{r+2/q}_{p,q}) \tag{6.5.35}$$

for $0 < T \leq \infty$. Then any solution F of (6.5.34) satisfies

$$\begin{aligned}&\frac{d}{dt}\|F\|^q_{\tilde{B}^r_{p,q}} + Cq\|F\|^q_{\tilde{B}^{r+2/q}_{p,q}} \\ \leq &Cq\left(\|w\|^q_{\tilde{B}^{r+2/q}_{p,q}}\|v\|^q_{\tilde{B}^{n/p}_{p,q}} + \|v\|^q_{\tilde{B}^{r+2/q}_{p,q}}\|w\|^q_{\tilde{B}^{n/p}_{p,q}}\right).\end{aligned} \tag{6.5.36}$$

We again observe that it is sufficient to consider $(1 - \alpha^2\triangle)^{-1}(\nabla v \cdot \nabla w)$ in place of $\tau^\alpha(v, w)$. Following [**18**], we have

$$\begin{aligned}&\frac{d}{dt}\|\triangle_j F\|_p + C2^{2j}\|\triangle_j F\|_p \\ \leq &C\|\triangle_j(v \cdot \nabla)w\|_p + C\|\triangle_j(\nabla v \cdot \nabla w)\|_p.\end{aligned} \tag{6.5.37}$$

The estimates for $\|\triangle_j(v \cdot \nabla)w\|_p$ and $\|\triangle_j(\nabla v \cdot \nabla w)\|_p$ are similar, so we estimate $\|\triangle_j(\nabla v \cdot \nabla w)\|_p$. Using (5.2.18), (5.2.19), and (5.2.20), we have

$$\|\triangle_j(\nabla v \cdot \nabla w)\|_p \leq I_1 + I_2 + I_3 \tag{6.5.38}$$

where

$$I_1 = C\sum_{|j-k|\leq M}\|S_{k-5}\nabla v\|_\infty\|\triangle_{k+1}\nabla w\|_p \tag{6.5.39}$$

$$I_2 = C \sum_{|j-k|\le M} \|S_{k-5}\nabla w\|_\infty \|\triangle_{k+1}\nabla v\|_p \tag{6.5.40}$$

and

$$I_3 = C \sum_{k>j-M} \|\triangle_k \nabla v\|_\infty (\sum_{l=k-5}^{k+5} \|\triangle_{k+l}\nabla w\|_p). \tag{6.5.41}$$

Applying Bernstein's inequality (Theorem 4.8.6) gives

$$I_1 \le C \sum_{|j-k|\le M} 2^{k+1}\|\triangle_{k+1} w\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m v\|_p = \tilde{I}_1 \tag{6.5.42}$$

$$I_2 \le C \sum_{|j-k|\le M} 2^{k+1}\|\triangle_{k+1} v\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m w\|_p = \tilde{I}_2 \tag{6.5.43}$$

and

$$I_3 \le C \sum_{k>j-M} 2^{(1+n/p)k}\|\triangle_j v\|_p \sum_{l=k-5}^{k+5} 2^{k+l}\|\triangle_{k+l} w\|_p = \tilde{I}_3. \tag{6.5.44}$$

We define the right hand sides of (6.5.42), (6.5.43), and (6.5.44) as $\tilde{I}_1$, $\tilde{I}_2$, and $\tilde{I}_3$, respectively. Next, we have

$$\|\triangle_j (v\cdot\nabla) w\|_p \le H_1 + H_2 + H_3 \tag{6.5.45}$$

where

$$\begin{aligned} H_1 &= C \sum_{|j-k|\le M} 2^{k+1}\|\triangle_{k+1} w\|_p \sum_{m<k-5} 2^{(n/p)m}\|\triangle_m v\|_p \\ H_2 &= C \sum_{|j-k|\le M} \|\triangle_{k+1} v\|_p \sum_{m<k-5} 2^{(1+n/p)m}\|\triangle_m w\|_p \\ H_3 &= C \sum_{k>j-M} 2^{(1+n/p)k}\|\triangle_j v\|_p \sum_{l=k-5}^{k+5} \|\triangle_{k+l} w\|_p. \end{aligned} \tag{6.5.46}$$

Remarking that $H_i \leq \tilde{I}_i$, we have our bound for (6.5.38). Using this bound in (6.5.37), multiplying by $q2^{rjq}\|\triangle F\|_p^{q-1}$, and summing over j gives

$$\frac{d}{dt}\|F\|^q_{\dot{B}^r_{p,q}} + Cq\|F\|^q_{\dot{B}^{r+2/q}_{p,q}} \leq J_1 + J_2 + J_3 \tag{6.5.47}$$

where

$$\begin{aligned}
J_1 &= Cq\sum_j 2^{rjq}\|\triangle_j F\|_p^{q-1}\tilde{I}_1,\\
J_2 &= Cq\sum_j 2^{rjq}\|\triangle_j F\|_p^{q-1}\tilde{I}_2,\\
J_3 &= Cq\sum_j 2^{rjq}\|\triangle_j F\|_p^{q-1}\tilde{I}_3.
\end{aligned} \tag{6.5.48}$$

From here, we mimic the argument used for Proposition 6.5.1 and get

$$\begin{aligned}
&\frac{d}{dt}\|F\|^q_{\dot{B}^r_{p,q}} + Cq\|F\|^q_{\dot{B}^{r+2/q}_{p,q}}\\
&\leq \left(\|v\|_{\tilde{B}^{n/p}_{p,q}}\|w\|_{\tilde{B}^{r+2/q}_{p,q}} + \|w\|_{\tilde{B}^{n/p}_{p,q}}\|v\|_{\tilde{B}^{r+2/q}_{p,q}}\right)\|F\|^{q-1}_{\tilde{B}^{r+2/q}_{p,q}}.
\end{aligned} \tag{6.5.49}$$

Applying Proposition (3.10.4) gives

$$\begin{aligned}
&\frac{d}{dt}\|F\|^q_{\dot{B}^r_{p,q}} + Cq\|F\|^q_{\dot{B}^{r+2/q}_{p,q}}\\
\leq&\|v\|^q_{\tilde{B}^{n/p}_{p,q}}\|w\|^q_{\tilde{B}^{r+2/q}_{p,q}} + \|w\|^q_{\tilde{B}^{n/p}_{p,q}}\|v\|^q_{\tilde{B}^{r+2/q}_{p,q}}
\end{aligned} \tag{6.5.50}$$

which proves Proposition 6.5.2.

BIBLIOGRAPHY

1. J. Bergh and J. Lofstrom, *Interpolation Spaces*, Springer-Verlag Berlin Heidelberg New York, 1976.

2. D. Chae, *Local existence and blow-up criterion for the Euler equations in the Besov spaces*, Asymptotic Analysis **38** (2004), 339–358.

3. F. Christ and M. Weinstein, *Dispersion of Small Amplitude Solutions of the Generalized Korteweg-de Vries Equation*, Journal of Functional Analysis **100** (1991), 87–109.

4. D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, The Annals of Mathematics **92** (1970), no. 1.

5. G. Folland, *Real Analysis*, John Wiley and Sons, 1984.

6. T. Kato and G. Ponce, *The Navier-Stokes equation with weak initial data*, Internation Mathematics Research Notices **10** (1994).

7. D. Bloch, P.S. Krishnaprasad, J.E. Marsden and T.S. Ratiu, *Euler-Poincare Equations and Double Bracket Dissipation*, Communications in Mathematical Physics **175** (1996), 1–42.

8. J. Marsden and S. Shkoller, *Global well-posedness for the Lagrangian averaged Navier-Stokes equations on bounded domains*, Phil. Trans. R. Soc. Lond. (2001), no. 359, 1449–1468.

9. ______, *The Anisotropic Lagrangian Averaged Euler and Navier-Stokes equations*, Arch. Rational Mech. Anal. (2003), no. 166, 27–46.

10. H. S. Bhat, R. C. Fetecau, J. E. Marsden, K. Mohseni, and M. West, *Lagrangain Averaging for Compressible Fluids*, Society for Industrial and Applied Mathematics **3** (2005), no. 4, 818–837.

11. J. Peetre, *New Thoughts on Besov Spaces*, Duke University Mathematics Series, 1976.

12. J. Marsden, T. Ratiu, and S. Shkoller, *The geometry and analysis of the averaged Euler equations and a new diffeomorphism group*, Geometric and Functional Analysis **10** (2000).

13. K. Mohensi, B. Kosovic, S. Shkoller and J. Marsden, *Numberical simulations of the Lagrangian averaged Navier-Stokes equations for homogeneous isotropic turbulence*, Phys. Fluids **15** (2003).

14. S. Shkoller, *On incompressible averaged Lagrangian hydrodynamics*, E-print, (1999), http://xyz.lanl.gov/abs/math.AP/9908109.

15. ______, *Analysis on groups of diffeomorphisms of manifolds with boundary and the averaged motion of a fluid*, Journal of Differential Geometry (2000), no. 55, 145–191.

16. Michael Taylor, *Partial Differential Equations*, Springer-Verlag New York, Inc., 1996.

17. ______, *Tools for PDE*, Mathematical Surveys and Monographs, Vol. 81, American Mathematical Society, Providence RI, 2000.

18. J. Wu, *Lower Bounds for an Integral Involving Fractional Laplacians and Generalized Navier-Stokes Equations in Besov Spaces*, Communications in Mathematical Physics **263** (2005), 803–831.

19. ______, *Solutions of the two dimensional quasi-geostrophic equation in hlder spaces*, Nonlinear Analysis **62** (2005), 579–594.

20. B. Yuan and B. Zhang, *Blow-up criterion of strong solutions to the Navier-Stokes equations in Besov spaces with negative indices*, Journal of Differential Equations **242** (2007), 1–10.

21. S. Chen, D.D. Holm, L. Margolin, R. Zhang, *Direct numerical simulations of the Navier-Stokes alpha model*, Physica D. **133** (1999).