CONFORMAL PERTURBATIONS AND LOCAL SMOOTHING

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ABSTRACT

Dylan Muckerman: Conformal Perturbations and Local Smoothing
(Under the direction of Hans Christianson)

The purpose of this paper is to study the effect of conformal perturbations on the local smoothing effect for the Schrödinger equation on surfaces of revolution. The paper [CW13] studied the Schrödinger equation on surfaces of revolution with one trapped orbit. The dynamics near this trapping were unstable, but degenerately so. Beginning from the metric g from these papers, we consider the perturbed metric $g_s = e^{sf}g$, where f is a smooth, compactly supported function. If s is small enough and finitely many derivatives of f satisfy an appropriate bound, then we show that a local smoothing estimate still holds.

Dedicated to Gerd and Mushroom

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CHAPTER 1

Introduction

In this thesis we discuss the effect of conformal perturbtions on local smoothing of the Schrödinger equation on surfaces of revolution.

The local smoothing effect was introduced and first studied for the Korteweg-de Vries equation in [Kat83]. It was studied for the Schrödinger equation in the papers [CS88], [Sjö87], [Veg88], and [KY89]. In [Sjö87] and [Veg88], it was used to prove almost everywhere pointwise convergence of solutions to the Schrödinger equation to their initial data as $t \to 0$.

The paper of [Doi96] showed the connection between the local smoothing effect and the geometry of the underlying manifold, by showing that the full local smoothing effect of 1/2 of a derivative holds if and only if the manifold has no trapped sets.

Following this were a number of results on local smoothing in the presence of trapped sets, including [Bur04], [Chr07], [Chr08], [Chr11], [Dat09], [CW13], [CM14], and [Chrar]. In these papers it was shown that while the full local smoothing effect of 1/2 does not hold, there are many conditions under which a smaller degree of local smoothing does hold. Very broadly speaking, the less stable the trapping, the greater the degree of local smoothing.

In particular, the results in [CW13] and [CM14] concern the degree to which the local smoothing effect for the Schrödinger equation holds on a surface of revolution with a finite number of trapped orbits for which the dynamics near the trapped set are unstable, but degenerately so. Because of this degeneracy, the trapping is not stable under conformal perturbations. Hence there is a possibility of changes in the trapped set and therefore the local smoothing.

In this thesis we will give a condition on the perturbation which ensures that some degree of local smoothing still holds, though it is not as great a local smoothing effect as holds on the unperturbed manifold.

We begin by giving some background on the Schrödinger equation and pseudodifferential operators. In particular, we develop a pseudodifferential calculus suited to our needs. This calculus can be thought of as a hybrid of the classical and semiclassical calculuses.

We then give an overview of local smoothing in Euclidean space. Particular attention is given to using a positive commutator argument to prove local smoothing. This argument will form the initial basis for our main proof.

After that we turn to local smoothing in the presence of trapping. We state the results here in some detail, as our result is concerned with a very similar setting.

We finally give the statement and proof of our main result. The proof works by emulating the proofs in [CW13] and [CM14], though it should be noted that the reduction to one dimension used in those papers is no longer available to us. In proving the local smoothing estimate away from the region of the trapping on the unperturbed manifold, we are able to emulate the previous proofs very closely, using the positive commutator argument. This is also the case in the positive commutator argument used to reduce the proof to a microlocal resolvent estimate.

The proof of the microlocal resolvent estimate is broadly based again on the proofs in [CW13] and [CM14], but we make some changes to the calculus used to avoid the marginal calculus. The cost of this is that our estimates are very likely not sharp. However, this provides us with more room to absorb the many additional terms coming from the perturbation.

CHAPTER 2

Background

2.1 Schrödinger equation

Let M be a Riemannian manifold with metric g and Laplace-Beltrami operator Δ_g. Let D_t denote $\frac{1}{i}\partial_t$. The Schrödinger equation is

$$\begin{cases} (D_t - \Delta_g)u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

A very important property of solutions to the Schrödinger equation is that they have constant L^2 norm. We prove this in Euclidean space for u_0 in the Schwartz class of functions

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \ : \ |x^\alpha \partial^\beta f| \leq M_{\alpha\beta}\}.$$

That is, f and all of its derivatives decay more quickly than any polynomial.

We can write the solution u explicitly using the Fourier transform. Let $\hat{u}(\xi, t)$ denote the Fourier transform in x of $u(x,t)$. Then

$$\hat{u} = e^{i|\xi|^2 t}\hat{u}_0.$$

Schwartz class is preserved by the Fourier transform, hence $\hat{u}_0 \in \mathcal{S}$. Furthermore, for every t, $e^{i|\xi|^2 t}$ and all of its derivatives in ξ grow at most polynomially. Therefore $\hat{u} \in \mathcal{S}$ and hence $u \in \mathcal{S}$, for each t.

We begin by letting

$$E(t) = \|u\|_{L^2}^2$$

and calculating

$$\begin{aligned} E'(t) &= \int_{\mathbb{R}^n} (\partial_t u)\overline{u} + u(\partial_t \overline{u})\, dx \\ &= \operatorname{Re} \int_{\mathbb{R}^n} (\partial_t u)\overline{u}\, dx. \end{aligned}$$

We then use the fact that u solves the Schrödinger equation to conclude that this equals

$$\begin{aligned} E'(t) &= \operatorname{Re} \int_{\mathbb{R}^n} i\Delta(u)\overline{u}\, dx \\ &= \operatorname{Re} -i \int_{\mathbb{R}^n} |\nabla u|^2\, dx \\ &= 0, \end{aligned}$$

where we have also made use of integration by parts. This proves that $E(t) = E(0)$ as long as u_0 (and hence u) is in $\mathcal{S}$. Using the fact that $\mathcal{S}$ is dense in L^2 then allows us to extend this argument to all of L^2.

This result can be strengthened to show that all of the H^s Sobolev norms are constant for solutions to the Schrödinger equation. We introduce the useful notation

$$\langle \xi \rangle = (1 + \xi^2)^{1/2}$$

and let

$$\Lambda^s u = \mathcal{F}^{-1}(\langle \xi \rangle^s\, \hat{u}),$$

where $\hat{u}$ denotes the Fourier transform (in space) of u and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Recall that the H^s norms are defined as

$$\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}.$$

Let $e^{it\Delta}$ denote the Schrödinger propagator, which we define using the Fourier transform, by

$$e^{it\Delta}u = \mathcal{F}^{-1}\left(e^{it|\xi|^2}\hat{u}\right).$$

We need to commute the operators Λ^s and $e^{it\Delta}$. To see that they commute we note that both Λ^s and $e^{it\Delta}$ are both Fourier multipliers and hence

$$\begin{aligned}
\Lambda^s e^{it\Delta}u &= \mathcal{F}^{-1}\left(\langle\xi\rangle^s \mathcal{F}\left(\mathcal{F}^{-1}\left(e^{it|\xi|^2}\hat{u}\right)\right)\right) \\
&= \mathcal{F}^{-1}\left(\langle\xi\rangle^s e^{it|\xi|^2}\hat{u}\right) \\
&= \mathcal{F}^{-1}\left(e^{it|\xi|^2}\langle\xi\rangle^s \hat{u}\right) \\
&= \mathcal{F}^{-1}\left(e^{it|\xi|^2}\mathcal{F}\left(\mathcal{F}^{-1}\left(\langle\xi\rangle^s \hat{u}\right)\right)\right) \\
&= e^{it\Delta}\Lambda^s u.
\end{aligned}$$

We can then combine this with the earlier conservation of L^2 norm to find

$$\begin{aligned}
\|u\|_{H^s} &= \|\Lambda^s e^{it\Delta}u_0\|_{L^2} \\
&= \|e^{it\Delta}(\Lambda^s u_0)\|_{L^2} \\
&= \|\Lambda^s u_0\|_{L^2} \\
&= \|u_0\|_{H^s},
\end{aligned}$$

and thus the H^s norms are conserved for solutions to the Schrödinger equation.

2.1.1 Notations and Conventions

We will use C to denote a large constant which may change from line to line. We will similarly use c to denote a small positive constant which may change from line to line.

2.2 Pseudodifferential Operators

Our outline of pseudodifferential operators will follow the presentation of [Zwo12], [Tay81], and [Tay13].

2.2.1 Basic definitions

In the above definition of Sobolev spaces, we made use of an operator defined by multiplication conjugated by the Fourier transform. Writing this out explicitly, we find

$$\Lambda^s u(x) = \frac{1}{(2\pi)^n} \iint e^{i\langle x-y,\xi\rangle} \langle \xi \rangle^s u(y)\, dy d\xi.$$

Replacing the function $\langle \xi \rangle^s$ with a more general function leads to a very useful class of operators.

We will work with the *symbol classes* S^m_ρ, $\rho \geq 0$ originally defined in [Hör66], given by

$$S^m_\rho = \{a \in C^\infty(\mathbb{R} \times \mathbb{R} \times S^1 \times \mathbb{Z}) \ : \ |\partial_\xi^\alpha \partial_x^\beta \partial_\theta^\gamma \partial_\eta^\delta a| \leq C_{\alpha,\beta,\delta,\gamma} \langle \xi \rangle^{m-|\alpha|\rho} \langle \eta \rangle^{-|\delta|\rho}\},$$

where ∂_η denotes a difference operator in η. In particular, we will work with a symbol supported only where $|\xi| \leq C|\eta|$, allowing us to transfer decay in $|\eta|$ to decay in $|\xi|$.

Define

$$a^w u = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{S^1} \sum_\eta e^{i\langle x-\tilde{x},\xi\rangle + i\left\langle \theta-\tilde{\theta},\eta\right\rangle} a\left(\frac{x+\tilde{x}}{2}, \frac{\theta+\tilde{\theta}}{2}, \xi, \eta\right) u(\tilde{x},\tilde{\theta})\, d\tilde{\theta} d\tilde{x} d\xi$$

The operator a^w is a *pseudodifferential operator* obtained from taking the *Weyl quantization* of a. It should be noted that the Weyl quantization is just one choice of many quantizations. The function a is said to be the *symbol* of the operator.

2.2.2 Symbol calculus

We review a few essential theorems of the symbol calculus.

Theorem 2.2.1 (Calderon-Vaillancourt Theorem). *If $a \in S_0^0$ then the operator a^w is bounded as an operator from L^2 to L^2.*

This theorem is originally due to [CV71]. See Theorem 4.23 in [Zwo12] for another proof. In fact, a more general theorem holds.

Theorem 2.2.2. *If $a \in S_0^m$ then the operator $a^w(x, D)$ is bounded as an operator from H^{s+m} to H^s.*

Quantization does not commute with composition. That is to say, the composition of two pseudodifferential operators is not the quantization of the product of their symbols. In fact, it is not immediately obvious that the composition of two pseudodifferential operators is a pseudodifferential operator. In fact, the following theorem holds.

Theorem 2.2.3 (Theorem 4.18 in [Zwo12]). *Let $a \in S_\rho^m$, $b \in S_\rho^{\tilde{m}}$. Let*

$$A(D) = \frac{1}{2}(\langle D_\xi, D_y \rangle - \langle D_x, D_\eta \rangle).$$

Then

$$a^w(x, D) \circ b^w(x, D) = c^w(x, D)$$

for

$$c = a\#b := \sum_{k=0}^{N} \frac{i^k}{k!} A(D)^k a(x, \xi) b(y, \eta)\Big|_{x=y, \xi=\eta} + r,$$

where r is a symbol in $S_\rho^{m+\tilde{m}-N\rho}$. Furthermore, the symbol c is in the class $S^{m+\tilde{m}}$.

In particular,

Corollary 2.2.4. *Let $a \in S^m_\rho$, $b \in S^{\tilde{m}}_\rho$. Then*

$$a\#b = ab + \frac{1}{2i}\{a, b\} + r,$$

where $r \in S^{m+\tilde{m}-2\rho}_\rho$.

This can be seen from the symbol expansion for the commutator of a^w and b^w using Theorem 2.2.3. Due to the symmetry of the Weyl quantization, the following holds.

Corollary 2.2.5. *Let $a \in S^m_\rho$, $b \in S^{\tilde{m}}_\rho$. Then the commutator*

$$[a^w(x, D), b^w(x, D)] = c^w(x, D),$$

where

$$c = \frac{1}{i}\{a, b\} + r,$$

and $r \in S^{m+\tilde{m}-3\rho}_\rho$.

Note that we gain 3 in the symbol class of the remainder term, rather than the gain of 2 we may naively expect. See Theorem 4.12 in [Zwo12].

Another useful feature of the Weyl quantization is the following theorem.

Theorem 2.2.6. *Let a be a real symbol. Then the operator a^w is essentially self-adjoint.*

A final result we require is the Gårding inequality.

Theorem 2.2.7. *Let $a \in S^m_\rho$ with $0 \leq \rho \leq 1$ and suppose*

$$\operatorname{Re} a \geq C|(\xi, \eta)|^m$$

for $|(\xi, \eta)|$ large. Then for any $s \in \mathbb{R}$ there exist C_1, C_2 such that for all $u \in H^{m/2}$,

$$\operatorname{Re}\langle a^w u, u\rangle \geq C_1\|u\|^2_{H^{m/2}} - C_2\|u\|^2_{H^s}.$$

See Chapter 7, Theorem 6.1 in [Tay13] for a proof.

CHAPTER 3

Local smoothing in Euclidean space

3.1 Background and motivation

In Euclidean space, the local smoothing result for the Schrödinger equation states that on average in time, and locally in space, solutions to the Schrödinger equation gain half a derivative compared to their initial data. More precisely, for every $T > 0$ there exists $C_T > 0$ such that if u solves

$$\begin{cases} (D_t - \Delta)u = 0 \\ u|_{t=0} = u_0, \end{cases}$$

then

$$\int_0^T \| \langle r \rangle^{-3/2} \partial_r u\|^2 + \| \langle r \rangle^{-1/2} r^{-1} \nabla_{S^{n-1}} u\|^2 \, dt \le C_T \|u_0\|^2_{H^{1/2}},$$

for all $u_0 \in H^{1/2}$. Note that the spatial weights are not sharp.

Local smoothing for the linear Schrödinger equation was first studied by [CS88], [Sjö87], [Veg88], and [KY89]. Both [Sjö87] and [Veg88] made use of this inequality to prove that solutions of the Schrödinger equation converge pointwise almost everywhere to their initial data as $t \to 0$.

We give a simple proof of this result, similar to those in [Tao06] and [CW13]. As in our above statement of the theorem, we make use of polar coodinates. We begin with a simplified version of the argument which does not quite work. In seeing where it fails, we see the appropriate properties the commutant should have.

Recall that $r\partial_r = x \cdot \partial_x$. Then

$$[r\partial_r, \Delta] = [x_1\partial_{x_1} + \ldots + x_n\partial_{x_n}, \partial^2_{x_1} + \ldots + \partial^2_{x_n}].$$

Most of the terms in this commutator vanish, as

$$[x_k \partial_{x_k}, \partial_{x_j}^2] = 0$$

for $j \neq k$. Thus

$$\begin{aligned}
[r\partial_r, \Delta] &= \sum_{k=1}^{n} [x_k \partial_{x_k}, \partial_{x_k}^2] \\
&= \sum_{k=1}^{n} x_k(\partial_{x_k}^3) - \partial_{x_k}^2(x_k \partial_{x_k}) \\
&= \sum_{k=1}^{n} x_k \partial_{x_k}^3 - \partial_{x_k}(\partial_{x_k} + x_k \partial_{x_k}^2) \\
&= \sum_{k=1}^{n} x_k \partial_{x_k}^3 - 2\partial_{x_k}^2 + x_k \partial_{x_k}^3 \\
&= -2\Delta.
\end{aligned}$$

In order to make use of integration by parts, we will assume $u \in \mathcal{S}$. The result can then be concluded for all $u \in H^{1/2}$ using a density argument. We have

$$\begin{aligned}
0 &= \int_0^T \langle r\partial_r(D_t - \Delta)u, u\rangle - \langle r\partial_r u, (D_t - \Delta)u\rangle \; dt \\
&= \int_0^T \langle [r\partial_r, -\Delta]u, u\rangle \; dt + i \left. \langle r\partial_r u, u\rangle \right|_{t=0}^{T}.
\end{aligned}$$

Rearranging and using the computation of the commutator from above, we find

$$\int_0^T \langle -\Delta u, u\rangle \; dt \leq \left| \left. \langle r\partial_r u, u\rangle \right|_{t=0}^{T} \right|.$$

The left hand side is

$$\int_0^T \|u\|_{\dot{H}^1}^2 \, dt,$$

where $dotH^1$ denotes the homogeneous Sobolev space. The hope with the right hand side would be to average the derivative over the inner product and bound the entire inner product by $\|u\|_{H^{1/2}}^2$ evaluated at $t = 0$ and $t = T$. We would then use the fact that the $H^{\tilde{s}}$ norms are bounded for the Schrödinger equation: There exists $C > 0$ such that

$$\|u\|_{H^{\tilde{s}}} \leq C\|u_0\|_{H^{\tilde{s}}}.$$

We would then conclude an upper bound of $\|u_0\|_{H^{1/2}}^2$. The problem is that r is not a bounded operator on $H^{\tilde{s}}$.

3.2 Positive Commutator argument

To fix this argument, we replace the unbounded commutant $r\partial_r$ with the bounded $\langle r\rangle^{-1} r\partial_r$. Near 0 this is approximately the same as our earlier commutant, so we expect that this should recover the same result near 0. We need to compute $[\langle r\rangle^{-1} r\partial_r, \Delta]$. We do this in a few pieces. First

$$\begin{aligned}
[\langle r\rangle^{-1} r\partial_r, \partial_r^2] &= \langle r\rangle^{-1} r\partial_r^3 - \partial_r^2(\langle r\rangle^{-1} r\partial_r) \\
&= \langle r\rangle^{-1} r\partial_r^3 - \partial_r^2(\langle r\rangle^{-1} r)\partial_r - 2\partial_r(\langle r\rangle^{-1} r)\partial_r^2 - \langle r\rangle^{-1} r\partial_r^3 \\
&= -\partial_r(\langle r\rangle^{-3})\partial_r - 2\langle r\rangle^{-3}\partial_r^2 \\
&= 3r\langle r\rangle^{-5}\partial_r - 2\langle r\rangle^{-3}\partial_r^2.
\end{aligned}$$

Next

$$\begin{aligned}
\left[\langle r\rangle^{-1} r\partial_r, \frac{n-1}{r}\partial_r\right] &= (n-1)\left[(\langle r\rangle^{-1} r\partial_r)(\frac{1}{r}\partial_r) - \frac{1}{r}\partial_r(\langle r\rangle^{-1} r\partial_r)\right] \\
&= (n-1)\left[-\langle r\rangle^{-1} r^{-1} + \langle r\rangle^{-1}\partial_r^2 - \frac{1}{r}\langle r\rangle^{-3}\partial_r - \langle r\rangle^{-1}\partial_r^2\right] \\
&= (n-1)\left[-\langle r\rangle^{-1} r^{-1}(1 + \langle r\rangle^{-2})\partial_r\right].
\end{aligned}$$

Finally,

$$[\langle r\rangle^{-1} r\partial_r, r^{-2}\Delta_{S^{n-1}}] = \langle r\rangle^{-1} r\partial_r(r^{-2}\Delta_{S^{n-1}}), -r^{-2}\Delta_{S^{n-1}}(\langle r\rangle^{-1} r\partial_r)$$
$$= -2\langle r\rangle^{-1} r^{-2}\Delta_{S^{n-1}}.$$

So we find

$$\begin{aligned} 0 &= \int_0^T \left\langle \langle r\rangle^{-1} r\partial_r(D_t - \Delta)u, u\right\rangle - \left\langle \langle r\rangle^{-1} r\partial_r u, (D_t-\Delta)u\right\rangle \, dt \\ &= \int_0^T \left\langle [\langle r\rangle^{-1} r\partial_r, -\Delta)u], u\right\rangle \, dt + i\left\langle \langle r\rangle^{-1} r\partial_r u, u\right\rangle\Big|_{t=0}^T. \end{aligned}$$

We arrange this and use our computation of the commutator to find

$$\begin{aligned} &\int_0^T \Big[\left\langle \left(3r\langle r\rangle^{-5}\partial_r - 2\langle r\rangle^{-3}\partial_r^2\right)u, u\right\rangle \\ &\qquad + \left\langle \left((n-1)\left[-\langle r\rangle^{-1} r^{-1}(1+\langle r\rangle^{-2})\partial_r\right] - 2\langle r\rangle^{-1} r^{-2}\Delta_{S^{n-1}}\right)u, u\right\rangle \Big] \, dt \\ &\leq \left| \left\langle \langle r\rangle^{-1} r\partial_r u, u\right\rangle\Big|_{t=0}^T \right|. \end{aligned}$$

We work first on the upper bound. We have

$$\begin{aligned} \left|\left\langle \langle r\rangle^{-1} r\partial_r u, u\right\rangle\right| &= \left|\left\langle \langle D_r\rangle^{-1/2}\langle r\rangle^{-1} r\partial_r u, \langle D_r\rangle^{1/2} u\right\rangle\right| \\ &\leq \|\langle D_r\rangle^{-1/2}\langle r\rangle^{-1} r\partial_r u\|_{L^2}\|\langle D_r\rangle^{1/2} u\|_{L^2}. \end{aligned}$$

We immediately have $\|\langle D_r\rangle^{1/2} u\|_{L^2} \leq C\|u\|_{H^{1/2}}$. The other term can be bound by consideration of symbol classes. The symbol $\langle r\rangle^{-1} r$ is bounded, as are all of its derivatives, so it's in S^0, and viewed as a pseudodifferential operator, $\langle r\rangle^{-1} r$ is in Ψ^0. The operators $r\langle r\rangle^{-1}\partial_r$ and $\langle D_r\rangle^{-1/2}$ are in Ψ^1 and $\Psi^{-1/2}$ respectively. The pseudodifferential calculus then tells us that the composition of these three operators is in the class $\Psi^{1/2}$, which then implies that this composition is bounded as an operator from $H^{1/2}$ to L^2, so

$$\| \langle D_r \rangle^{-1/2} \langle r \rangle^{-1} r\partial_r u\|_{L^2} \leq \|u\|_{H^{1/2}}.$$

We have thus shown that

$$\left| \left\langle \langle r \rangle^{-1} r\partial_r u, u \right\rangle \Big|_{t=0}^{T} \right| \leq \|u\|^2_{H^{1/2}} \Big|_{t=0} + \|u\|^2_{H^{1/2}} \Big|_{t=T}.$$

Because u solves the Schrödinger equation, it has H^r norm controlled by the H^r norm of the initial data and thus

$$\left| \left\langle \langle r \rangle^{-1} r\partial_r u, u \right\rangle \Big|_{t=0}^{T} \right| \leq C\|u_0\|^2_{H^{1/2}}.$$

Next we will work on the lower bound. The easiest part is

$$\int_0^T \left\langle -2 \langle r \rangle^{-1} r^{-2} \Delta_{S^{n-1}} u, u \right\rangle \, dt = \int_0^T 2\| \langle r \rangle^{-1/2} r^{-1} \nabla_{S^{n-1}} u\|^2 \, dt.$$

Next we have

$$\begin{aligned}
\int_0^T \left\langle -2 \langle r \rangle^{-3} \partial_r^2 u, u \right\rangle \, dt &= 2 \int_0^T \left\langle \partial_r u, \partial_r (\langle r \rangle^{-3} u) \right\rangle \, dt \\
&= 2 \int_0^T \left\langle \partial_r u, \langle r \rangle^{-3} \partial_r u \right\rangle + \left\langle \partial_r u, (-3) r \langle r \rangle^{-5} u, u \right\rangle \, dt \\
&= \int_0^T 2\| \langle r \rangle^{-3/2} \partial_r u\|^2 \, dt + \left\langle \partial_r u, (-3) r \langle r \rangle^{-5} u, u \right\rangle \, dt.
\end{aligned}$$

The first of these terms stays in our lower bound. The other term will be moved to the upper bound and then bounded by the $H^{1/2}$ norm using the technique of "averaging" the derivative over both sides of the inner product:

$$\begin{aligned}\left|\int_0^T \left\langle 3r\langle r\rangle^{-5}\partial_r u, u\right\rangle\, dt\right| &= \left|\int_0^T \left\langle \langle D_r\rangle^{-1/2} 3r\langle r\rangle^{-5}\partial_r u, \langle D_r\rangle^{1/2} u\right\rangle\, dt\right| \\ &\le \int_0^T \|\langle D_r\rangle^{-1/2} 3r\langle r\rangle^{-5}\partial_r u\| \|\langle D_r\rangle^{1/2} u\|\, dt \\ &\le C\int_0^T \|u\|_{H^{1/2}}^2\, dt \\ &\le C\int_0^T \|u_0\|_{H^{1/2}}^2\, dt \\ &\le CT\|u_0\|_{H^{1/2}}^2.\end{aligned}$$

Next we have

$$\begin{aligned}&\left|\int_0^T \left\langle -\langle r\rangle^{-1} r^{-1}(1+\langle r\rangle^{-2})\partial_r u, u\right\rangle\, dt\right| \\ &= \left|\int_0^T \left\langle \langle D_r\rangle^{-1/2}\left(-\langle r\rangle^{-1} r^{-1}(1+\langle r\rangle^{-2})\partial_r u\right), \langle D_r\rangle^{1/2} u\right\rangle\, dt\right| \\ &\le C\int_0^T \|u\|_{H^{1/2}}^2\, dt \\ &\le CT\|u_0\|_{H^{1/2}}.\end{aligned}$$

In the two preceding strings of inequalities we have made use of the pseudodifferential calculus to bound the terms involving $\langle D_r\rangle^{-1/2}$, just as we did in proving the initial upper bound. All together, this gives the following local smoothing inequality:

$$\int_0^T \|\langle r\rangle^{-3/2}\partial_r u\|^2 + \|\langle r\rangle^{-1/2} r^{-1}\nabla_{S^{n-1}} u\|^2\, dt \le C_T\|u_0\|_{H^{1/2}}^2.$$

CHAPTER 4

Local smoothing in the presence of trapping

One perspective on the local smoothing effect is that it arises from the dispersive nature of the Schrödinger equation. In particular, high frequency parts of solutions to the Schrödinger equation have higher velocity. By looking locally at solutions, we see "less" of the high frequency part of our solution, and this is what is responsible for the local smoothing. In other words, the high frequency parts of the solution go off to infinity very quickly. In Euclidean space, where the geodesics are straight lines, this is very easy to visualize, and the above result makes it rigorous. On the opposite extreme, we can consider the Schrödinger equation on the simplest compact manifold, S^1.

Let $k \in \mathbb{Z}$ and consider the function $e_k(\theta) = e^{ik\theta}$. Because

$$\partial_\theta^2 e_k = -k^2 e_k,$$

this is an eigenfunction with eigenvalue $-k^2$.

Let

$$u_k = e^{-itk^2} e_k(\theta).$$

Then

$$D_t u_k = -k^2 e^{-itk^2} e_k(\theta) = e^{itk} \Delta e_k(\theta) = \Delta u_k,$$

and hence u_k is a solution to the Schrödinger equation with initial data $e_k(x)$.

Conveniently, $e_k(\theta)$ is already written as a Fourier series, where the k-th coefficient is 1 and all other coefficients are 0. We compute

$$\|e_k\|_{H^r} = \| \langle k \rangle^r \hat{e}_k \|_{L^2} = \langle k \rangle^r$$

Let g be a smooth, non-vanishing function. Then

$$\begin{aligned}\|gu_k\|_{H^1} &\geq \|\partial_\theta(gu_k)\|_{L^2} \\ &\geq \|g\partial_\theta u_k\|_{L^2} - \|(\partial_\theta g)u_k\|_{L^2}.\end{aligned}$$

Because u is smooth and S^1 is compact, we know

$$\|(\partial_\theta g)u_k\|_{L^2} \leq \max(\partial_\theta g) \leq C_g.$$

On the other hand since g is non-vanishing,

$$\|g\partial_\theta u_k\|_{L^2} \geq \min|g|\|\partial_\theta u_k\|_{L^2} = (\min|g|)k.$$

By taking k large enough we can ensure

$$\|gu_k\|_{H^1} \geq Ck$$

for some C which may be on g. Thus, for $r < 1$, there is no hope of achieving a bound of the form

$$\int_0^T \|gu_k\|_{H^1}^2\,dt \leq C\|e_k\|_{H^r}^2$$

for all k.

This agrees with our heuristic argument: The high frequency parts of the solution cannot escape to infinity, so they continue to contribute to the H^r norms.

4.1 Necessity of a Loss

Many possibilities exist between Euclidean space and compact manifolds. According to our heuristic argument, the important property of Euclidean space is that every geodesic goes to infinity. In other words, there are no *trapped* geodesics, where a trapped geodesic is a complete geodesic that remains in a compact set for all time.

The relationship between trapping and local smoothing was explored in [Doi96]: On asymptotically Euclidean manifolds, solutions to the Schrödinger equation exhibit 1/2 of a derivative of local smoothing if and only if the manifold has no trapped geodesics.

The next question which arises is to what degree the local smoothing effect still holds when a trapped set exists.

The results in [Bur04], [Chr07], [Chr08], [Chr11], and [Dat09] showed that in the presence of non-degenerate hyperbolic trapping, for any $\epsilon > 0$, there is local smoothing of $1/2 - \epsilon$ derivatives for the Schrödinger equation.

4.2 Surfaces of Revolution

In [CW13], local smoothing is studied on surfaces of revolution that have periodic geodesics which are unstable, but degenerately so. In other words, the curvature vanishes to degree $2m-2$ at the geodesic, where $m \geq 2$. The surfaces studied are given by rotating the curve

$$A(x) = (1 + x^{2m})^{1/2m},$$

where $m \geq 2$. The local smoothing effect is then

$$\int_0^T \| \langle x \rangle^{-3/2} u\|_{H^1}^2 \, dt \leq C(\| \langle D_\theta \rangle^{m/(m+1)} u_0\|_{L^2}^2 + \| \langle D_x \rangle^{1/2} u_0\|_{L^2}^2).$$

In other words, we gain the full 1/2 of a derivative of local smoothing in the x direction, but we only gain $1/(m+1)$ derivatives of local smoothing in the θ direction. Note that as the trapping becomes more stable, the local smoothing gained in the θ direction goes to 0.

In [CM14], a similar result is proven for a similar class of manifolds. Here the curve being

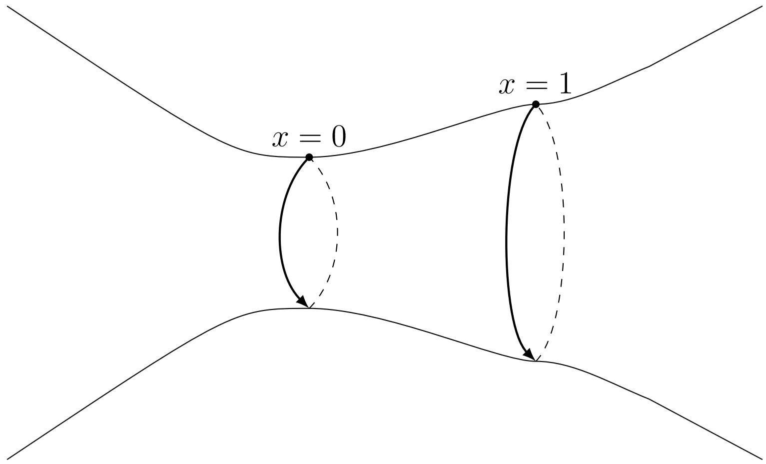

Figure 4.1: A piece of the manifold with trapped geodesics at $x = 0$ and $x = 1$.

rotated can be written explicitly as

$$A^2(x) = 1 + \int_0^x y^{2m_1-1}(y-1)^{2m_2}/(1+y^2)^{m_1+m_2-1}\,dy,$$

where m_1 and m_2 are positive integers. To make things more clear, note that

$$A^2 \sim \begin{cases} 1 + x^{2m_1}, \quad x \sim 0 \\ C_1 + c_2(x-1)^{2m_2+1}, \quad x \sim 1 \\ x^2, \quad |x| \to \infty. \end{cases}$$

The point is that the manifold has two periodic geodesics. The one at $x = 0$ is the type studied in [CW13]. For this manifold, the local smoothing result states that for solutions of the Schrödinger equation, there exists $C > 0$ such that

$$\begin{aligned} \int_0^T \| \langle x \rangle^{-1} \partial_x u \|^2 + \| \langle x \rangle^{-3/2} \partial_\theta u \|^2 \, dt \\ \leq C_T \left(\| \langle D_\theta \rangle^{\beta(m_1,m_2)} u_0 \|^2 + \| \langle D_x \rangle^{1/2} u_0 \|^2 \right), \end{aligned}$$

where

$$\beta(m_1, m_2) = \max\left(\frac{m_1}{m_1+1}, \frac{2m_2+1}{2m_2+3}\right).$$

The meaning of $\beta(m_1, m_2)$ is that the overall degree of local smoothing is determined by whichever trapped geodesic gives us worse local smoothing.

It should be noted that the results of [CW13] and [CM14] are sharp and show that no better (lower) power of $\langle D_\theta \rangle$ is possible.

Finally, [Chrar] gives details of the connection between resolvent estimates for the Laplacian and local smoothing, and a detailed exposition of how the results obtained in [CW13] and [CM14] can be combined via "gluing" to prove local smoothing results for a wide variety of warped product manifolds.

Similar results are also available for localized energy estimates for the wave equation on surfaces of revolution with degenerate trapping in [BCMPar].

CHAPTER 5

Conformal perturbations of surfaces of revolution

The previous results mentioned above essentially complete the study of local smoothing for the Schrödinger equation on surfaces of revolution (and warped product manifolds in general). All of these results are essentially 1 dimensional, thanks to the decomposition into Fourier modes. The degree of local smoothing for the Schrödinger equation on higher dimensional manifolds with trapping is largely unknown with a few exceptions. These exceptions are the case of stable trapping ([Chrar]), the case of non-degenerate hyperbolic trapping (see [Bur04], [Chr07], [Chr08], [Chr11], and [Dat09]), and [Gou12]. We study local smoothing on surfaces which are conformal perturbations of surfaces of revolution.

Recall that a surface of revolution is the manifold $M = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$ endowed with the metric

$$g_0 = dx^2 + A^2(x) d\theta^2,$$

where $A > 0$. We consider conformal perturbations of this metric in which the metric is of the form

$$g_s = e^{sf(x,\theta)} g_0,$$

where $f(x, \theta)$ is a smooth function, compactly supported in x. Note that

$$\Delta_{g_s} = e^{-sf} \Delta_{g_0}.$$

We will work with the function A given in [CW13]. Note that if f depends only on x, then after the perturbation our surface retains its rotational symmetry and so is still a surface of revolution, though it is impractical to write down its metric explicitly in the standard form for surfaces of revolution.

If our perturbation function f has appropriate conditions placed on it, one expects that it will have little effect on the dynamics near the trapped set and thus little effect on the local smoothing. In fact, it is reasonable to expect that the perturbation could make the dynamics less stable and thus lead to greater local smoothing, though this is beyond our scope.

Theorem 5.0.1. *Let $\epsilon > 0$ and let $M = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$ endowed with the metric*

$$g = e^{sf(x,\theta)}(dx^2 + A^2(x)d\theta^2),$$

where

$$A(x) = (1 + x^{2m})^{1/2m}$$

and $f \in C^\infty(M)$ is compactly supported in x and satisfies

$$|\partial_x^j \partial_\theta^k f| \leq C|x|^{2m-1}$$

for x small and $j, k \leq N$ for sufficiently large $N = N(m, \epsilon)$ where $j + k \geq 1$. Suppose also that $s > 0$ is sufficiently small. Let

$$r = \frac{m}{m+1} + \epsilon.$$

Then there exists $C_T > 0$ such that

$$\int_0^T \| \langle x \rangle^{-1} \partial_x u\|^2 + \| \langle x \rangle^{-3/2} \partial_\theta u\|^2 \, dt \leq C_T(\|u_0\|^2_{H_x^{1/2}} + \|u_0\|^2_{H_\theta^r})$$

for all u solving the Schrödinger equation

$$\begin{cases} (D_t - \Delta_g)u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

Remark. In the unperturbed case, there is a gain of

$$\frac{1}{m+1}$$

derivatives, whereas in our case there is the gain of

$$\frac{1}{m+1} - \epsilon$$

derivatives.

This is because we have chosen to avoid the marginal calculus used in [CW13], in order to ensure gains (in terms of θ derivatives) in symbol expansions, so that the many extra terms introduced by the factor e^{-sf} are easier to control.

Remark. Note that we do not require any bound on f itself, only on its derivatives.

Remark. The intuitive reason for our condition on derivatives of f is that in general the degenerate trapping found in the unperturbed manifold is unstable under perturbation, and could potentially be perturbed into much worse trapping, for which the result would not hold.

We also note that non-degenerate hyperbolic trapping is stable under perturbation, so there is no corresponding result in that situation. In fact, the case of non-degenerate hyperbolic trapping has been explored in much greater generality (see [Chr11]). In addition, non-degenerate hyperbolic trapping can be defined independent of coordinates, so our methods which depend heavily on explicitly coordinates would not apply.

The Laplacian Δ_{g_0} on the unperturbed metric is given by

$$\Delta_{g_0} = \partial_x^2 + A^{-2}(x)\partial_\theta^2 + A^{-1}(x)A'(x)\partial_x.$$

Define

$$L_1 : L^2(X, dVol) \to L^2(X, dxd\theta)$$

by

$$L_1 u(x,\theta) = A^{1/2}(x) u(x,\theta)$$

and define

$$L_2 : L^2(e^{sf} dx d\theta) \to L^2(dx d\theta)$$

by

$$L_2 u(x,\theta) = e^{sf/2} u.$$

Let $\tilde{\Delta} = L_2 L_1 \Delta_g L_1^{-1} L_2^{-1}$. Let

$$V_1(x) = \frac{1}{2} A''(x) A^{-1}(x) - \frac{1}{4} (A'(x))^2 A^{-2}(x)$$

We compute $\tilde{\Delta}$ explicitly and find

$$\begin{aligned}
\tilde{\Delta} u &= e^{-sf/2} \left(\partial_x^2 + A^{-2} \partial_\theta^2 - V_1(x) \right) e^{-sf/2} \\
&= e^{-sf} \left(\partial_x^2 + A^{-2} \partial_\theta^2 \right) \\
&+ e^{-sf} (-s f_x \partial_x - A^{-2} s f_\theta \partial_\theta - (s/2) f_{xx} + ((s/2) f_x)^2 - A^{-2} (s/2) f_{\theta\theta} + A^{-2} ((s/2) f_\theta)^2) \\
&- e^{-sf} V_1(x).
\end{aligned}$$

We note that

$$(e^{-sf} (\xi^2 + A^{-2}(x) \eta^2 + V_1(x)))^w = -\tilde{\Delta}$$

Let

$$Q = -(e^{-sf} (\xi^2 + A^{-2} \eta^2))^w$$

and

$$R = e^{-sf} (-s f_x \partial_x - A^{-2} s f_\theta \partial_\theta - s f_{xx} + (s f_x)^2 - A^{-2} s f_{\theta\theta} + A^{-2} (s f_\theta)^2)$$

so that

$$Q = e^{-sf}(\partial_x^2 + A^{-2}\partial_\theta^2) + R.$$

Then Q is essentially self-adjoint and R consists of the lower order parts of the operator.

Below we will commute with an operator B involving only 1 derivative. Commuting B and $e^{-sf}V_1(x)$ will produce a bounded function and no derivatives, or in other words an L^2 bounded operator. This can then easily absorbed into the upper bound of $\|u_0\|^2_{H^{1/2}}$, as will be done with many other remainder terms below. Thus proving the result for Q will prove the result for $\tilde{\Delta}$. Conjugating back then proves the result for Δ_g. For this reason, we will leave out $V_1(x)$ in the computations below and work with Q.

5.1 Positive Commutator

We begin by making the same positive commutator argument as in [CW13]. By commuting the operator we are interested in, Q, with an appropriate operator B we are able to prove the local smoothing estimate away from $x = 0$.

We have

$$Q = e^{-sf}(\partial_x^2 + A^{-2}\partial_\theta^2) + R.$$

For our commutant we choose

$$B = \arctan(x)\partial_x.$$

We begin by commuting the two operators to find

$$
\begin{aligned}
[Q,B] &= e^{-sf}(\partial_x^2 + A^{-2}\partial_\theta^2)[\arctan(x)\partial_x] \\
&\quad - \arctan(x)\partial_x[e^{-sf}(\partial_x^2 + A^{-2}\partial_\theta^2)] + [R,B] \\
&= e^{-sf}\left[\arctan(x)\partial_x^3 + 2\langle x\rangle^{-2}\partial_x^2 - \frac{2x}{(1+x^2)^2}\partial_x\right] \\
&\quad + e^{-sf}A^{-2}\arctan(x)\partial_\theta^2\partial_x \\
&\quad + \arctan(x)sf_x e^{-sf}(\partial_x^2 + A^{-2}\partial_\theta^2) \\
&\quad + \arctan(x)e^{-sf}(-\partial_x^3 - A^{-2}\partial_\theta^2\partial_x + 2A'A^{-3}\partial_\theta^2) + [R,B] \\
&= e^{-sf}\Bigg[2\langle x\rangle^{-2}\partial_x^2 - \frac{2x}{(1+x^2)^2}\partial_x + sf_x\arctan(x)(\partial_x^2 + A^{-2}\partial_\theta^2) \\
&\quad + \arctan(x)2A'A^{-3}\partial_\theta^2\Bigg] + [R,B].
\end{aligned} \tag{5.1}
$$

Now that we are done with the preliminary computations, we begin the argument proper by assuming that u satisfies the Schrödinger equation

$$
\begin{cases}
(D_t - Q)u = 0, \\
u(0,x,\theta) = u_0.
\end{cases}
$$

Using that, we write down the following expression which equals 0:

$$0 = \int_0^T \langle B(D_t - Q)u, u\rangle - \langle Bu, (D_t - Q)u\rangle \, dt.$$

In order to make our commutator term appear, we next need to integrate by parts in the second term and obtain

$$0 = \int_0^T \langle B(D_t - Q)u, u\rangle - \langle (D_t - Q)Bu, u\rangle \, dt + i\,\langle Bu, u\rangle\big|_0^T .$$

We combine the terms involving D_t, Q, and B. This results in

$$0 = \int_0^T \langle B(D_t - Q) - (D_t - Q)Bu, u\rangle \, dt + i \langle Bu, u\rangle|_0^T .$$

Finally, we note that these combined terms are precisely the commutator we computed above, and we end up with the equation

$$0 = \int_0^T \langle [Q, B]u, u\rangle + i \langle Bu, u\rangle|_0^T .$$

Next we write out the commutator and move the terms we are interested in bounding below to the left hand side. The terms we are interested in bounding below are those which appear most similar to the terms in the final local smoothing estimate. In particular, they are the terms which involve 2 derivatives, but do not contain a factor of s coming from the perturbation. This results in the equation

$$\begin{aligned}
&\int_0^T \left\langle -e^{-sf} 2 \langle x\rangle^{-2} \partial_x^2 u, u\right\rangle - \left\langle e^{-sf} \arctan(x) 2A' A^{-3} \partial_\theta^2 u, u\right\rangle \, dt \\
&\quad = -\int_0^T \left\langle e^{-sf} \left[\frac{2x}{(1+x^2)^2} \partial_x + s f_x \arctan(x)(\partial_x^2 + A^{-2}\partial_\theta^2) + [R, B] \right] u, u \right\rangle \, dt \\
&\quad\quad - i \langle Bu, u\rangle|_0^T .
\end{aligned}$$

We begin by working on the left hand side. Our goal is to obtain something that can be used as a lower bound in a local smoothing estimate. To that end, we will need to split the derivatives over both sides of the inner product and obtain something that can be bounded below by a norm of involving a derivative of x and a function which decays at infinity. We will do that by using integration by parts to move one derivative to the other side of the inner product. In the process, lower order terms will be obtained when the derivative hits functions other than u. These will be moved to the right hand side and then absorbed into the upper bound. Starting with the term involving derivatives of x, we first integrate by parts to find

$$-\left\langle e^{-sf}2\left\langle x\right\rangle^{-2}\partial_x^2 u, u\right\rangle = \left\langle \partial_x u, (\partial_x[2e^{-sf}\left\langle x\right\rangle^{-2}u]\right\rangle.$$

Next we use the product rule to find that this equals

$$\|e^{-sf/2}\left\langle x\right\rangle^{-1}\partial_x u\|^2 + \left\langle \partial_x u, \left(-2sf_x e^{-sf}\left\langle x\right\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u\right\rangle.$$

We move the second term to the right hand side and bound it above. First we note that the function

$$-2sf_x e^{-sf}\left\langle x\right\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}$$

and all of its derivatives are bounded. We can then split the ∂_x across both parts of the inner product an obtain an upper bound of $C\|u\|_{H^{1/2}}^2$ as follows: First we apply the operator $\left\langle D_x\right\rangle^{1/2}\left\langle D_x\right\rangle^{-1/2}$, and then we use integration by parts. This term then equals

$$\left\langle \left\langle D_x\right\rangle^{-1/2}\partial_x u, \left(\left\langle D_x\right\rangle^{1/2}((-2sf_x e^{-sf}\left\langle x\right\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u)\right\rangle.$$

Using the Cauchy-Schwarz inequality we are able to bound this from above by

$$C\|\left\langle D_x\right\rangle^{-1/2}\partial_x u\|_{L^2}\|\left(\left\langle D_x\right\rangle^{1/2}((-2sf_x e^{-sf}\left\langle x\right\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u)\|_{L^2}$$

Both of these terms are bounded by $C\|u\|_{H^{1/2}}$, giving us the total bound above by $C\|u\|_{H^{1/2}}^2$ as desired.

Next we move on to the term involving derivatives of θ and proceed similarly. We have

$$\begin{aligned}&\left\langle e^{-sf}\arctan(x)A'A^{-3}\partial_\theta^2 u, u\right\rangle = \\ &\quad \left\langle e^{-sf}\arctan(x)x^{2m-1}(1+x^{2m})^{-1/m-1}\partial_\theta u, \partial_\theta u\right\rangle \\ &\quad - s\left\langle f_\theta e^{-sf}\arctan(x)x^{2m-1}(1+x^{2m})^{-1/m-1}\partial_\theta u, u\right\rangle.\end{aligned}$$

The term involving only a single θ derivative is moved to the right hand side and bounded above by $\|u\|^2_{H^{1/2}_{x,\theta}}$, just as we did for the terms involving only a single x derivative above.

Thus far we have proven the inequality

$$\begin{aligned}
&\int_0^T \|e^{-sf/2} \langle x \rangle^{-1} \partial_x u\|^2_{L^2} + \left\langle e^{-sf} \arctan(x) x^{2m-1}(1+x^{2m})^{-1/m-1} \partial_\theta u, \partial_\theta u \right\rangle \, dt \qquad (5.2) \\
&\quad \leq |\langle Bu, u\rangle| \Big|_0 + |\langle Bu, u\rangle| \Big|_T \\
&\quad + \int_0^T \Bigg(\left| \left\langle \partial_x u, \left(-2sf_x e^{-sf} \langle x \rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2} \right) u \right\rangle \right| \\
&\qquad + \left| \left\langle sf_\theta e^{-sf} \arctan(x) A' A^{-3} \partial_\theta u, u \right\rangle \right| \\
&\qquad - \left\langle e^{-sf} \left[\frac{2x}{(1+x^2)^2} \partial_x + sf_x \arctan(x)(\partial_x^2 + A^{-2}\partial_\theta^2) + [R, B] \right] u, u \right\rangle \, dt \Bigg)
\end{aligned}$$

The first term on the left hand side is already written as a norm. For the second term, we need to do a bit of work before it can be bounded below by a norm. Note that

$$\left\langle e^{-sf} |x|^{2m} \langle x \rangle^{-2m-3} \partial_\theta u, \partial_\theta u \right\rangle \leq C \left\langle e^{-sf} \arctan(x) x^{2m-1}(1+x^{2m})^{-1/m-1} \partial_\theta u, \partial_\theta u \right\rangle,$$

So we may bound (5.2) below by

$$c \int_0^T \|e^{-sf} \langle x \rangle^{-1} \partial_x u\|^2_{L^2} + \|e^{-sf} |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u\|^2_{L^2} \, dt, \qquad (5.3)$$

for some $c > 0$.

Finally, we can drop the factors of e^{-sf} by using the fact that f is compactly supported and hence e^{-sf} is bounded below by some $c > 0$. Thus the lower bound is

$$c \int_0^T \| \langle x \rangle^{-1} \partial_x u\|^2_{L^2} + \| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u\|^2_{L^2} \, dt. \qquad (5.4)$$

So far we have shown

$$\begin{aligned}
c\int_0^T \|e^{-sf/2}\langle x\rangle^{-1}\partial_x u\|_{L^2}^2 + \|e^{-sf/2}|x|^m\langle x\rangle^{-m-3/2}\partial_\theta u\|_{L^2}^2\,dt \\
&\leq |\langle Bu,u\rangle|\Big|_0^T \\
&+\int_0^T\left(\left|\left\langle \partial_x u, \left(-2sf_x e^{-sf}\langle x\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u\right\rangle\right|\right. \\
&\quad + \left|\left\langle sf_\theta e^{-sf}\arctan(x)A'A^{-3}\partial_\theta u, u\right\rangle\right| \\
&\quad \left.- \left\langle e^{-sf}\left[\frac{2x}{(1+x^2)^2}\partial_x + sf_x\arctan(x)(\partial_x^2 + A^{-2}\partial_\theta^2) + [R,B]\right]u,u\right\rangle\right)dt
\end{aligned}$$

In the end, our upper bound will be $C\|u_0\|_{H^{1/2}}^2$. To start, we write out B and apply $\langle D_x\rangle^{-1/2}\langle D_x\rangle^{1/2}$ as we did above, to find

$$|\langle Bu,u\rangle| = \left\langle \langle D_x\rangle^{-1/2}\partial_x u, \langle D_x\rangle^{1/2}\arctan(x)u\right\rangle \leq \|\langle D_x\rangle^{-1/2}\partial_x u\|_{L^2}\|\langle D_x\rangle^{1/2}\arctan(x)u\|_{L^2}.$$

The operator $\langle D_x\rangle^{-1/2}\partial_x$ is bounded as an operator from $H^{1/2}\to L^2$, so

$$\|\langle D_x\rangle^{-1/2}\partial_x u\|_{L^2} \leq C\|u\|_{H^{1/2}}.$$

The operator $\langle D_x\rangle^{1/2}\arctan(x)$ is a composition of operators in the classes $\Psi^{1/2}$ and Ψ^0, so it is in $\Psi^{1/2}$, and thus

$$\|\langle D_x\rangle^{1/2}\arctan(x)u\|_{L^2} \leq C\|u\|_{H^{1/2}}.$$

We apply this to the inner product $\langle Bu,u\rangle$ evaluated at $t=0$ and $t=T$ to find

$$|\langle Bu,u\rangle|\Big|_0^T \leq C(\|u(T,\cdot)\|_{H^{1/2}}^2 + \|u(0,\cdot)\|_{H^{1/2}}^2) \leq C\|u_0\|_{H^{1/2}}^2,$$

where we have used the fact that there exists C such that $\|u\|_{H^r}\leq C\|u_0\|_{H^r}$ independent of time.

For the next term we use the same trick of applying $\langle D_x\rangle^{-1/2}\langle D_x\rangle^{1/2}$ to "average" one derivative over both sides of the inner product,

$$
\begin{aligned}
&\left|\left\langle \partial_x u, \left(-2sf_x e^{-sf}\langle x\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u\right\rangle\right| \\
&\quad = \left|\left\langle \langle D_x\rangle^{-1/2}\partial_x u, \langle D_x\rangle^{1/2}\left(-2sf_x e^{-sf}\langle x\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u\right\rangle\right|.
\end{aligned}
$$

Using the same argument we see that this is controlled by $\|u_0\|^2_{H^{1/2}}$. Then

$$
\int_0^T \left|\left\langle \partial_x u, \left(-2sf_x e^{-sf}\langle x\rangle^{-2} - \frac{4xe^{-sf}}{(1+x^2)^2}\right)u\right\rangle\right| dt \le C\int_0^T \|u\|^2_{H^{1/2}}\,dt \le CT\|u_0\|^2_{H^{1/2}}.
$$

The same argument works as well for each of our terms that involve only one derivative. Because R consists of terms with at most 1 derivative and B involes only 1 derivative, the commutator $[R,B]$ consists of terms involving only a single derivative. The above argument then shows

$$
\int_0^T |\langle [R,B]u,u\rangle|\,dt \le CT\|u_0\|^2_{H^{1/2}}.
$$

That leaves us with an upper bound of

$$
C(T+1)\|u_0\|^2_{H^{1/2}} + \int_0^T \left|\left\langle e^{-sf}(sf_x \arctan(x)(\partial_x^2 + A^{-2}\partial_\theta^2)\right\rangle\right|\,dt
$$

The strategy for dealing with these terms with two derivatives is to make use of the fact that s is small to absorb them into the lower bound. We work first with the term involving x derivatives. We begin by splitting the two derivatives over the two halves of the inner product using integration by parts to find

$$
\int_0^T \left|\left\langle se^{-sf}f_x\arctan(x)\partial_x^2 u, u\right\rangle\right|\,dt = \int_0^T \left|\left\langle s\partial_x(e^{-sf}f_x\arctan(x)u), \partial_x u\right\rangle\right|\,dt
$$

Next we apply the derivative using the product rule to find that this equals

$$\int_0^T \left|\left\langle se^{-sf}\left(-s(f_x)^2 \arctan(x) + f_{xx}\arctan(x) + f_x \langle x\rangle^{-2} + f_x \arctan(x)\partial_x\right) u, \partial_x u\right\rangle\right| \, dt.$$

For the terms where ∂_x has hit something other than u, we are left with an inner product involving only a total of one derivative, and we can use our technique of "averaging" this derivative to bound this by the $H^{1/2}$ norm. This gives us an upper bound of

$$CT\|u_0\|_{H^{1/2}}^2 + \int_0^T \left\langle se^{-sf} f_x \arctan(x)\partial_x u, \partial_x u\right\rangle \, dt.$$

By making use of the fact that f is compactly supported, we can then bound this above by

$$CT\|u_0\|_{H^{1/2}}^2 + Cs\int_0^T \| \langle x\rangle^{-1} \partial_x u\|_{L^2}^2$$

The second of these terms may be moved to the lower bound and absorbed, provided that s is sufficiently small.

We next use essentially the same argument for the term involving two θ derivatives. One difference is that the lower bound involving ∂_θ vanishes near $x = 0$, so there will be a requirement on f_x in order to absorb our term involving s into the lower bound. We begin by using integration by parts and the technique of averaging derivatives to write

$$\begin{aligned}
&\int_0^T \left|\left\langle e^{-sf} s f_x \arctan(x) A^{-2}\partial_\theta^2)u, u\right\rangle\right| \, dt \\
&\quad = s\int_0^T \left|\left\langle \partial_\theta(e^{-sf} f_x \arctan(x) A^{-2}u), \partial_\theta u\right\rangle\right| \, dt \\
&\quad = s\int_0^T \left|\left\langle e^{-sf} \arctan(x) A^{-2}(sf_\theta f_x + f_{x\theta} + f_x\partial_\theta)u, \partial_\theta u\right\rangle\right| \, dt \\
&\quad \le CT\|u_0\|_{H^{1/2}}^2 + \int_0^T s\left|\left\langle e^{-sf} f_x \arctan(x) A^{-2}\partial_\theta u, \partial_\theta u\right\rangle\right| \, dt.
\end{aligned}$$

Next we suppose that

$$|f_x| \leq C|x|^{2m-1}$$

in a neighborhood of $x = 0$. Then using also the fact that f is compactly supported, we have

$$|sf_x \arctan(x)| \leq Cs|x|^{2m} \langle x \rangle^{-2m-3},$$

and thus

$$\int_0^T s \left| \left\langle e^{-sf} f_x \arctan(x) A^{-2} \partial_\theta u, \partial_\theta u \right\rangle \right| \, dt \leq Cs \int_0^T \| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u \|_{L^2}^2 \, dt.$$

By choosing s sufficiently small we may absorb this into the lower bound.

We thus have the estimate

$$\int_0^T \| \langle x \rangle^{-1} \partial_x u \|_{L^2}^2 + \| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u \|_{L^2}^2 \, dt \leq C_T \| u_0 \|_{H^{1/2}}^2 \tag{5.5}$$

This estimate shows that the local smoothing is perfect away from the $x = 0$, and that we have perfect local smoothing in the x direction. Next we will work on the local smoothing in the θ direction and near $x = 0$.

5.2 Estimating in the Frequency Domain

Our plan is to split the function u up based on whether D_x or $\langle D_\theta \rangle$ is larger, writing $u = u_1 + u_2$, so that u_2 satisfies the bound

$$\| \langle D_\theta \rangle u_2 \|_{L^2} \leq \| \partial_x u_2 \|_{L^2}.$$

We give an outline of the proof before proceeding with the proof. First we attempt to repeat the above argument using u_2 in place of u. Because u_2 is only approximately a solution to the Schrödinger equation, there will be additional error terms. The lower bound of

$$\int_0^T \| \langle x \rangle^{-1} \partial_x u_2 \|_{L^2}^2 \, dt$$

can be bounded from below by

$$\int_0^T \| \langle x \rangle^{-1} \langle D_\theta \rangle u_2 \|_{L^2}^2 \, dt$$

This gives us a lower bound in the θ direction away from $x = 0$. However, it is only for u_2, and the upper bound will involve a term other than $\|u_0\|_{H^{1/2}}^2$, due to the fact that u_2 does not solve the Schrödinger equation. The other term in the upper bound will essentially be

$$\int_0^T \| \langle x \rangle^{-2} D_\theta u_1 \|_{L^2}^2 \, dt.$$

Thus, we will have reduced the problem to finding an upper bound for u_1, which will be the subject of the remaining sections.

Let $\psi(\tau)$ be a bump function with $\psi(\tau) = 0$ for $|\tau| > 2$ and $\psi(\tau) = 1$ for $|\tau| < 1$. We define the operator $\psi(D_x / \langle D_\theta \rangle)$ as a Fourier multiplier. Let $\hat{u}(t, \xi, \eta)$ denote the Fourier transform of u in x and θ. Because $\theta \in S^1$, η takes integer values. Let $\mathcal{F}$ denote also this Fourier transform:

$$(\mathcal{F}u)(\xi, \eta) = \int_{\mathbb{R}} \int_{S^1} e^{-ix\xi} e^{-i\theta\eta} u(x, \theta) \, d\theta dx.$$

Let $\mathcal{F}^{-1}$ denote the inverse. Note that $\mathcal{F}^{-1}$ involves an integral in ξ but a sum in η:

$$(\mathcal{F}^{-1}v)(x, \theta) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \sum_{\eta \in \mathbb{Z}} e^{ix\xi} e^{i\theta\eta} v(\xi, \eta) \, d\xi$$

We then define

$$\psi(D_x / \langle D_\theta \rangle) u = \mathcal{F}^{-1}(\psi(\xi / \langle \eta \rangle) \hat{u}).$$

Again suppose u solves

$$\begin{cases} (D_t - Q)u = 0, \\ u(0, x, \theta) = u_0. \end{cases}$$

We will consider $u_1 = \psi(D_x / \langle D_\theta \rangle)u$ and $u_2 = (1 - \psi(D_x / \langle D_\theta \rangle))u$.

While u_2 is not a solution to the Schrödinger equation, we will show that it is close enough to a solution for our purposes. We have

$$\begin{aligned} (D_t - Q)u_2 &= (D_t - Q)[(1 - \psi(D_x / \langle D_\theta \rangle))u] \\ &= (1 - \psi(D_x / \langle D_\theta \rangle))(D_t - Q)u + [Q, \psi(D_x / \langle D_\theta \rangle)]u \\ &= [Q, \psi(D_x / \langle D_\theta \rangle)]u. \end{aligned}$$

Because this is a commutator term, we know that it is lower order. When the time comes to use this, we will show precisely what is meant by lower order here.

Letting $B = \arctan(x)\partial_x$ as above we repeat the positive commutator argument from above. We begin by simply expanding the commutator to find

$$\int_0^T \langle [Q, B]u_2, u_2 \rangle \, dt = \int_0^T \langle QBu_2, u_2 \rangle - \langle BQu_2, u_2 \rangle \, dt$$

Next we want to have both Q's be applied to u_2 so that we can use what we know about u_2 and the Schrödinger equation.

We then proceed as in the calculations following (5.1) to find

$$\begin{aligned} \int_0^T \langle [Q, B]u_2, u_2 \rangle \, dt = \int_0^T \Big[\langle Bu_2, (D_t - Q)u_2 \rangle - \langle B(D_t - Q)u_2, u_2 \rangle \Big] \, dt \\ + i \langle Bu_2, u_2 \rangle \Big|_{t=0}^{T}. \end{aligned}$$

Our lower bound will come from the left hand side of the equality, while the right hand side will need to be bounded from above, in a manner similar to the preceding section.

Next we consider

$$\int_0^T \langle Bu_2, (D_t - Q)u_2\rangle \, dt.$$

We write this as

$$\int_0^T \left\langle \langle x\rangle^{-1} Bu_2, \langle x\rangle (D_t - Q)u_2\right\rangle \, dt \le C\int_0^T (\| \langle x\rangle^{-1} Bu_2\|_{L^2}^2 + \| \langle x\rangle (D_t - Q)u_2\|_{L^2}^2) \, dt.$$

$$\int_0^T \| \langle x\rangle^{-1} Bu_2\|^2 \, dt \le C\int_0^T \| \langle x\rangle^{-1} (1 - \psi(D_x/\langle D_\theta\rangle))\partial_x u\|^2 + \|[\psi(D_x/\langle D_\theta\rangle), \partial_x]u\|^2 \, dt.$$

Note that $[\psi(D_x/\langle D_\theta\rangle), \partial_x]$ is an L^2 bounded operator. Using the inequality we proved in the previous section, we then know

$$\int_0^T \| \langle x\rangle^{-1} Bu_2\|^2 \, dt \le \int_0^T \| \langle x\rangle^{-1} \partial_x u\|^2 \, dt + C\|u\|_{L^2}^2 \le CT\|u_0\|_{H^{1/2}}^2.$$

Recall that from the above,

$$\begin{aligned}
(D_t - Q)u_2 &= [Q, \psi(D_x/\langle D_\theta\rangle)]u \\
&= [e^{-sf}(D_x^2 + A^{-2}D_\theta^2) + R, \psi(D_x/\langle D_\theta\rangle)]u \\
&= [e^{-sf}, \psi(D_x/\langle D_\theta\rangle)](D_x^2 + A^{-2}D_\theta^2) + e^{-sf}[D_x^2 + A^{-2}D_\theta^2, \psi(D_x/\langle D_\theta\rangle)] \\
&\quad + [R, \psi(D_x/\langle D_\theta\rangle)].
\end{aligned}$$

To bound the first of these terms, we make note of the commutator terms. We gain many things from commuting e^{-sf} with $\psi(D_x/\langle D_\theta\rangle)$. Because f has compact support in x, we have decay in x as quickly as we like. Because of the ψ term, we will be working in the region where $D_x \sim \langle D_\theta\rangle$, and we will gain a power of D_x or $\langle D_\theta\rangle$. Thus

$$\| \langle x\rangle [e^{-sf}, \psi(D_x/\langle D_\theta\rangle)](D_x^2 + A^{-2}D_\theta^2)u\| \le C\| \langle x\rangle^{-1} D_x u\|.$$

We may then bound $\int_0^T \| \langle x \rangle^{-1} D_x u\|^2 \, dt$ by $CT\|u_0\|^2_{H^{1/2}}$ as we did before.

Next we note that

$$[D_x^2 + A^{-2} D_\theta^2, \psi(D_x / \langle D_\theta \rangle)] = [A^{-2}(x), \psi(D_x / \langle D_\theta \rangle)] D_\theta^2$$

We have

$$\langle x \rangle [A^{-2}(x), \psi(D_x / \langle D_\theta \rangle)] D_\theta^2 = L \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle),$$

where L is L^2-bounded and $\tilde{\psi} \in C_0^\infty$ equals 1 on $\operatorname{supp} \psi$. Then

$$\begin{aligned}\int_0^T \| \langle x \rangle [A^{-2}(x), \psi(D_x / \langle D_\theta \rangle)] D_\theta^2 u\|^2 \, dt &= \int_0^T \|L \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) u\|^2 \, dt \\ &\leq C \int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) u\|^2 \, dt.\end{aligned}$$

Controlling this will be the subject of the next section.

The term

$$\int_0^T \langle B(D_t - Q) u_2, u_2 \rangle \; dt = \int_0^T \langle (D_t - Q) u_2, B^* u_2 \rangle \; dt$$

is controlled in exactly the same fashion.

Thus far we have shown

$$\int_0^T \langle [Q, B] u_2, u_2 \rangle \; dt \leq CT\|u_0\|^2_{H^{1/2}} + \int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) u\|^2 \, dt.$$

Next we use our expansion of $[Q, B]$ given in (5.1) above:

$$\begin{aligned}[Q, B] = e^{-sf} \bigg[& 2 \langle x \rangle^{-2} \partial_x^2 - \frac{2x}{(1 + x^2)^2} \partial_x - s f_x \arctan(x) (\partial_x^2 + A^{-2}(x) \partial_\theta^2) \\ & + \arctan(x) 2 A' A^{-3} \partial_\theta^2 \bigg] + [R, B]\end{aligned}$$

As in the previous section we have

$$\int_0^T |\langle [R,B]u_2, u_2\rangle|\,dt \le CT\|u_0\|^2_{H^{1/2}}.$$

Next we write

$$\begin{aligned}\int_0^T \left\langle -2e^{-sf}\langle x\rangle^{-2}\partial_x^2 u_2, u_2\right\rangle\,dt &= 2\int_0^T \left\langle e^{-sf/2}\langle x\rangle^{-1}\partial_x u_2, \partial_x(e^{-sf/2}\langle x\rangle^{-1}u_2)\right\rangle\,dt\\ &= 2\int_0^T \|e^{-sf/2}\langle x\rangle^{-1}\partial_x u_2\|^2\,dt\\ &\quad+\int_0^T \left\langle e^{-sf/2}\langle x\rangle^{-1}\partial_x u_2, \partial_x(e^{-sf/2}\langle x\rangle^{-1})u_2\right\rangle\,dt.\end{aligned}$$

Note that

$$\begin{aligned}&\int_0^T \left|\left\langle e^{-sf/2}\langle x\rangle^{-1}\partial_x u_2, \partial_x(e^{-sf/2}\langle x\rangle^{-1})u_2\right\rangle\right|\,dt\\ &\quad\le C\int_0^T \|\langle D_x\rangle^{-1/2}(e^{-sf/2}\langle x\rangle^{-1}\partial_x u_2)\|\|\langle D_x\rangle^{1/2}[(\partial_x e^{-sf/2}\langle x\rangle^{-1}]u_2\|\,dt\\ &\quad\le C\int_0^T \|u\|^2_{H^{1/2}}\,dt\\ &\quad\le CT\|u_0\|^2_{H^{1/2}}.\end{aligned}$$

The following term is taken care of similarly:

$$\begin{aligned}&\int_0^T -\left\langle e^{-sf}\arctan(x)2A'A^{-3}\partial_\theta^2 u_2, u_2\right\rangle\,dt\\ &\quad=\int_0^T \left\langle \arctan(x)2A'A^{-3}\partial_\theta u_2, \partial_\theta(e^{-sf}u_2)\right\rangle\,dt\\ &\quad=\int_0^T \left\langle e^{-sf}\arctan(x)2A'A^{-3}\partial_\theta u_2, (-sf_\theta+\partial_\theta)u_2)\right\rangle\,dt.\end{aligned}$$

Then for the term involving only one derivative we may again bound it from above by

$CT\|u_0\|^2_{H^{1/2}}$. The other term is

$$\int_0^T \left\langle e^{-sf}\arctan(x) 2A'A^{-3}\partial_\theta u_2, \partial_\theta u_2\right\rangle\, dt \geq c\int_0^T \||x|^m \langle x\rangle^{-m-3/2}\partial_\theta u\|^2\, dt.$$

The next term we bound from above is

$$\int_0^T \left\langle 2x\langle x\rangle^{-4}\partial_x u_2, u_2\right\rangle\, dt \leq CT\|u_0\|^2_{H^{1/2}},$$

again by using our technique of "averaging" half a derivative across the inner product.

The remaining terms can be controlled by using our bound from the previous section. First we have

$$\begin{aligned}\int_0^T \left|\left\langle se^{-sf}f_x\arctan(x)\partial_x^2 u_2, u_2\right\rangle\right|\, dt &= \int_0^T s\left|\left\langle \partial_x u_2, \partial_x(e^{-sf}f_x\arctan(x)u_2)\right\rangle\right|\, dt\\ &= \int_0^T s\left|\left\langle \partial_x u_2, e^{-sf}f_x\arctan(x)\partial_x u_2\right\rangle\right|\, dt\\ &+ \int_0^T s\left|\left\langle \partial_x u_2, (\partial_x(e^{-sf}f_x\arctan(x)))u_2\right\rangle\right|\, dt\end{aligned}$$

The second term here can be bounded by $CT\|u_0\|^2_{H^{1/2}}$ again by averaging the derivative. For the first term, we instead note that

$$\begin{aligned}\int_0^T s\left|\left\langle \partial_x u_2, e^{-sf}f_x\arctan(x)\partial_x u_2\right\rangle\right|\, dt &\leq C\int_0^T \|\langle x\rangle^{-1}\partial_x u\|^2\, dt\\ &\leq CT\|u_0\|^2_{H^{1/2}},\end{aligned}$$

where we have used the bound proven in the previous section.

The only remaining term is

$$\begin{aligned}
&\int_0^T s \left|\left\langle e^{-sf} f_x \arctan(x) A^{-2} \partial_\theta^2 u_2, u_2 \right\rangle\right| \, dt \\
&\qquad = \int_0^T s \left|\left\langle e^{-sf} f_x \arctan(x) A^{-2} \partial_\theta u_2, \partial_\theta u_2 \right\rangle\right| \, dt \\
&\qquad\quad + \int_0^T s \left|\left\langle s f_\theta e^{-sf} f_x \arctan(x) A^{-2} \partial_\theta u_2, u_2 \right\rangle\right| \, dt
\end{aligned}$$

The second term is again bounded by $CT\|u_0\|_{H^{1/2}}^2$ by averaging the derivative over the inner product. For the first term we have

$$\begin{aligned}
&\int_0^T s \left|\left\langle e^{-sf} f_x \arctan(x) A^{-2} \partial_\theta u_2, \partial_\theta u_2 \right\rangle\right| \, dt \\
&\qquad \leq Cs \int_0^T \| |x|^m \langle x \rangle^{-m-3/2} \partial_\theta u \|^2 \, dt \\
&\qquad \leq CT \|u_0\|_{H^{1/2}}^2,
\end{aligned}$$

where we have used our condition that $|f_x| \leq x^{2m-1}$ for x near 0, as well as the bound proven in the previous section.

Putting all of this together we end up with

$$\begin{aligned}
&\int_0^T \| \langle x \rangle^{-1} \partial_x u_2 \|_{L^2}^2 + \| |x|^m \langle x \rangle^{m-3/2} \partial_\theta u_2 \|_{L^2}^2 \, dt \\
&\qquad \leq C_T \|u_0\|_{H^{1/2}}^2 + C \int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) u \|_{L^2}^2 \, dt.
\end{aligned}$$

Finally we make use of the support property of $\psi(D_x/\langle D_\theta \rangle)u$. This function cuts $u_2 = 1 - \psi(D_x/\langle D_\theta \rangle)u$ off to where $\langle D_\theta \rangle \leq \partial_x$, so we have

$$\| \langle x \rangle^{-1} \langle D_\theta \rangle u_2 \| \leq C \| \langle x \rangle^{-1} \partial_x u_2 \|.$$

Using this lower bound we see that

$$\int_0^T \| \langle x \rangle^{-1} \langle D_\theta \rangle u_2\|_{L^2}^2 \, dt \leq C_T \|u_0\|_{H^{1/2}}^2 + \int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) u\|_{L^2}^2 \, dt.$$

Furthermore let $\chi(x) \equiv 1$ near 0 and have compact support. Then

$$\begin{aligned}&\int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) u\|_{L^2}^2 \, dt \\ &\quad \leq C \int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) \chi(x) u\|_{L^2}^2 + \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle)(1 - \chi(x)) u\|_{L^2}^2 \, dt\end{aligned}$$

The second of these terms can then be bounded using the bound from the previous section:

$$\begin{aligned}\int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle)(1 - \chi(x)) u\|^2 \, dt &\leq C \int_0^T \langle x \rangle^{-2} (1 - \chi(x)) D_\theta u\|^2 \, dt \\ &\leq \int_0^T \||x| \langle x \rangle^{m-3/2} \partial_\theta u\|^2 \, dt \\ &\leq C_T \|u_0\|_{H^{1/2}}^2.\end{aligned}$$

Thus to finish this part of our estimate need only bound

$$\int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) \chi(x) u\|_{L^2}^2 \, dt.$$

Note that by bounding this, we will also bound

$$\int_0^T \| \langle x \rangle^{-2} D_\theta \chi(x) u_1\|_{L^2}^2 \, dt,$$

which will then complete the local smoothing estimate. We begin this process in the next section.

5.3 High Frequency Estimate

We have proven our local smoothing estimate outside of a region that is "small" in both space and frequency. This suggests that it will be profitable to work microlocally. To that end, we wish to show that bounding

$$\int_0^T \| \langle x \rangle^{-2} D_\theta \tilde{\psi}(D_x / \langle D_\theta \rangle) \chi(x) u\|_{L^2}^2 \, dt$$

is equivalent to proving a bound of the form

$$\|(Q + \tau)\psi\chi u\| \geq \| \langle D_\theta \rangle^{\tilde{r}} \psi\chi u\|,$$

for u microlocalized near $(x, \xi / \langle \eta \rangle) = 0$. We do this by using a "TT^*" argument.

The operator to which we apply the argument will be $F(t)$. Define the operator $F(t)$ by

$$F(t)g = \chi(x)\psi(D_x / \langle D_\theta \rangle) e^{itQ} g(x, \theta).$$

We need to determine for which values of r, with $0 \leq r \leq 1$, we have a bounded map $F : L^2_x L^2_\theta \to L^2([0, T]) L^2_x H^r_\theta$.

We have

$$F^* g = \int_0^T e^{-i\tilde{t}Q} \psi(D_x / \langle D_\theta \rangle) \chi(x) \tilde{g} \, d\tilde{t}$$

and we need to show

$$F^* : L^2([0, T]) L^2_x H^{-r}_\theta \to L^2_x L^2_\theta.$$

Then

$$F F^* \tilde{g} = \chi(x)\psi(D_x / \langle D_\theta \rangle) \int_0^T e^{i(t - \tilde{t})Q} \psi(D_x / \langle D_\theta \rangle) \chi(x) \tilde{g} \, d\tilde{t}$$

and we need to show

$$FF^* : L^2([0,T])L^2_x H^{-r}_\theta \to L^2([0,T])L^2_x H^r_\theta.$$

We split this expression into two. Let

$$v_1 = \int_0^t e^{i(t-\tilde{t})Q}\psi(D_x/\langle D_\theta\rangle)\chi(x)\tilde{g}\,d\tilde{t}$$

and

$$v_2 = \int_t^T e^{i(t-\tilde{t})Q}\psi(D_x/\langle D_\theta\rangle)\chi(x)\tilde{g}\,d\tilde{t}.$$

Then

$$FF^*\tilde{g} = \chi(x)\psi(D_x/\langle D_\theta\rangle)(v_1+v_2).$$

We need to show

$$\|\chi(x)\psi(D_x/\langle D_\theta\rangle)v_j\|_{L^2_t L^2_x H^r_\theta} \leq C\|\tilde{g}\|_{L^2_t L^2_x H^{-r}_\theta}$$

for $j = 1, 2$, where we require some assumptions on $\tilde{g}$ which will be included in the statement of our theorem below.

Note that

$$(D_t + Q)v_1 = -i\psi(D_x/\langle D_\theta\rangle)\chi(x)\tilde{g}$$

and

$$(D_t + Q)v_2 = i\psi(D_x/\langle D_\theta\rangle)\chi(x)\tilde{g}.$$

Let $\hat{\cdot}$ denote the Fourier transform in time. Then

$$(\tau + Q)\hat{v}_j = (-1)^j i\chi\psi\hat{\tilde{g}},$$

If we can prove the bound

$$\|\chi\psi\hat{v}_j\|_{L^2_\tau L^2_x H^r_\theta} \leq C\|\tilde{g}\|_{L^2_\tau L^2_x H^{-r}_\theta},$$

we will have shown that $FF^* : L^2_t L^2_x H^{-r}_\theta \to L^2_t L^2_x H^r_\theta$ (and thus $F : L^2_x L^2_\theta \to L^2_t L^2_x H^r_\theta$) is a bounded operator. To that end, we need to bound the operator

$$\chi(x)\psi(D_x/\langle D_\theta\rangle)(Q+\tau)^{-1}\psi(D_x/\langle D_\theta\rangle)\chi(x)$$

in the $L^2_x H^r_\theta \to L^2_x H^{-r}_\theta$ operator norm, uniformly in τ.

This is equivalent to showing that there exists C such that

$$\|\langle D_\theta\rangle^{2r} u\|_{L^2_{x,\theta}} \leq C\|(Q+\tau)u\|_{L^2_{x,\theta}}.$$

Suppose that $|\tau| \geq C|\eta|^2$. Then by ellipticity we have

$$\|(Q+\tau)u\| \geq c\|\langle D_\theta\rangle^2 u\|.$$

Hence we need only work where $|\tau| \leq C|\eta|^2$.

Proving this estimate will be the subject of the next section. Proving it will bound $\chi(x)\psi(D_x/\langle D_\theta\rangle)e^{itQ}g$ in $L^2_t L^2_x H^r_\theta$, but we are ultimately interested in bounding it in $L^2_t L^2_x H^1_\theta$. To do so, we apply the bound to $\langle D_\theta\rangle^{1-r} g$, so we will ultimately end up with the bound

$$\int_0^T \|\langle D_\theta\rangle \chi(x)\psi(D_x/\langle D_\theta\rangle)u\|^2_{L^2_{x,\theta}}\, dt \leq C\|u_0\|^2_{H^{1-r}}.$$

5.4 Microlocal proof of the Resolvent Estimate

We state and prove the aforementioned resolvent estimate in order to finish the proof of the main theorem.

Theorem 5.4.1. *Let $\epsilon > 0$ and let $p = \xi^2 + \eta^2 A^{-2}$. Suppose $f(x, \theta)$ is a compactly supported, smooth function such that*

$$|\partial_x^j \partial_\theta^k f| \leq C|x|^{2m-1}$$

for x small and $j, k \leq N$ for sufficiently large $N = N(m, \epsilon)$ and $j + k \geq 1$. Suppose also that s is sufficiently small. Then there exists $c > 0$ such that

$$\|((e^{-sf}p)^w + \tau)u\|_{L^2_{x,\theta}} \geq c\| \langle D_\theta \rangle^{2/(m+1)-\epsilon} u\|_{L^2_{x,\theta}},$$

for all τ, provided that u satisfies the following microlocal support properties: We require that u be of the form

$$u = b^w \tilde{u},$$

where b has symbol supported in the region where $|(x, \xi/\eta)| \leq \delta/2$ for some sufficiently small δ and $|\eta| \geq M$ for some sufficiently large M.

Broadly speaking, our proof uses a commutator argument. The basic structure is to make use of the fact that to highest order, the symbol of the commutator of two pseudodifferential operators is given by applying the Hamiltonian vector field of the one symbol to the other symbol. Recall that $Q = (e^{-sf}p)^w$ denotes the operator we are interested in. We define a symbol $a \in S^0_\epsilon$ such that $H_{e^{-sf}p}a$ has the required lower bound. Ignoring the issues of error terms coming from the pseudodifferential calculus for the moment, we will consider the quantity

$$\langle [Q + \tau, a^w]u, u \rangle ,$$

where a is yet to be determined.

Roughly, this is bounded from above by applying absolute values, expanding the commutator, and integrating by parts to give us an upper bound (for now) of

$$C \left| \langle a^w u, (Q + \tau)u \rangle \right| .$$

As we stated above, the lower bound makes use of the fact that to highest order, $[Q, a^w] = (H_{e^{-sf}p}a)^w$. We seek a such that the resulting symbol $H_{e^{-sf}p}a$ is of the form $\eta^{-\epsilon}(\xi^2 + \eta^2 x^{2m})$ at least where x and $\xi\eta^{-1}$ are small and η is large. We may then use a lower bound on this operator to achieve a lower bound of

$$\| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|^2 \leq |\langle a^w u, (Q+\tau)u\rangle| \, .$$

The bulk of the proof is concerned with dealing with the error terms that turn up in applying the pseudodifferential calculus and terms that appear due to the presence of e^{-sf}. It is in the process of absorbing these terms that the need for conditions on f become clear.

The operator we require a lower bound on is $(e^{-sf}p)^w + \tau$. To define the symbol of our commutant a, we first define

$$\Lambda(t) = \int_0^t \langle \tilde{t} \rangle^{-1-\epsilon_0} \, d\tilde{t},$$

where $\epsilon_0 > 0$ is sufficiently small. The important facts about $\Lambda(t)$ is that it is a symbol of order 0, and $\Lambda(t) \sim t$ near 0.

We will also make use of cutoff functions $\chi(t)$ and $\tilde{\chi}(t)$. Let $\chi(t)$ be a smooth, compactly supported function such that $\chi(t) \equiv 1$ for $|t| \leq \delta/2$ and $\chi(t) \equiv 0$ for $|t| \geq \delta$. Let $\tilde{\chi}(t)$ be a smooth function such that $\tilde{\chi}(t) \equiv 0$ for $|t| \leq M$ and $\tilde{\chi}(t) \equiv 1$ for $|t| \geq 2M$.

Let

$$a = \chi(x)\chi(\xi\eta^{-1})\Lambda(x)\Lambda(\xi\eta^{-\epsilon})\tilde{\chi}(\eta)$$

and note that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\theta^\gamma \partial_\eta^\delta a| \leq C_{\alpha,\beta,\gamma,\delta} \langle \xi \rangle^{-\epsilon\beta} \langle \eta \rangle^{-\epsilon\delta} \, ,$$

where we have used the fact that $\chi(\xi\eta^{-1})$ cuts off to where $|\xi| \leq |\eta|$. Because of this inequality, $a \in S^0_\epsilon$.

Theorem 5.4.2. *Let p, a as above. Then for any $\epsilon > 0$ there exists $c > 0$ such that*

$$\left\langle e^{-sf}(H_{e^{-sf}p}a)^w u, u\right\rangle \geq c\left\langle (D_\theta)^{-\epsilon}(D_x^2 + D_\theta^2 x^{2m})u, u\right\rangle - \mathcal{O}(\| \langle D_\theta\rangle^{-\epsilon/2})u\|^2).$$

for all u microsupported where $|x| \leq \delta$, $|\xi\eta^{-1}| \leq \delta$, and $|\eta| \geq M$ for some large M.

Proof. We begin by computing $H_{e^{-sf}p}a$. First recall that

$$p(x, \xi, \theta, \eta) = \xi^2 + A^{-2}(x)\eta^2$$

and

$$a(x, \xi, \theta, \eta) = \chi(x)\chi(\xi\eta^{-1})\Lambda(x)\Lambda(\xi\eta^{-\epsilon})\tilde{\chi}(\eta).$$

Note that a does not depend on θ, so no $(e^{-sf}p)_\eta a_\theta$ term will appear. We next compute the necessary derivatives. Recall that the notation for the η derivative actually refers to a difference operator in η.

$$\begin{aligned}
(e^{-sf}p)_x &= -2e^{-sf}A^{-3}A'\eta^2 - sf_x e^{-sf}p, \\
(e^{-sf}p)_\xi &= 2e^{-sf}\xi, \\
(e^{-sf}p)_\theta &= -sf_\theta e^{-sf}p \\
a_x &= [\chi'(x)\chi(\xi\eta^{-1})\Lambda(x)\Lambda(\xi\eta^{-\epsilon}) + \chi(x)\chi(\xi\eta^{-1})\Lambda'(x)\Lambda(\xi\eta^{-\epsilon})]\tilde{\chi}(\eta), \\
a_\xi &= [\eta^{-1}\chi(x)\chi'(\xi\eta^{-1})\Lambda(x)\Lambda(\xi\eta^{-\epsilon}) + \eta^{-\epsilon}\chi(x)\chi(\xi\eta^{-1})\Lambda(x)\Lambda'(\xi\eta^{-\epsilon})]\tilde{\chi}(\eta) \\
a_\eta &= \Lambda(x)(\Lambda(\xi(\eta+1)^{-\epsilon}) - \Lambda(\xi\eta^{-\epsilon}))\chi(x)\chi(\xi\eta^{-1})\tilde{\chi}(\eta) \\
&\quad + \Lambda(x)\Lambda(\xi(\eta+1)^{-\epsilon})\chi(x)\left[\chi(\xi(\eta+1)^{-1})\tilde{\chi}(\eta+1) - \chi(\xi\eta^{-1})\tilde{\chi}(\eta)\right]
\end{aligned}$$

Using this computation we write down $H_{e^{-sf}p}a$ and split it into two parts. We are only interested in the behavior of $H_{e^{-sf}p}a$ where x is small, $\xi\eta^{-1}$ is small, and η is large, so we separate out the terms of $H_{e^{-sf}p}a$ where derivatives or difference operators hit χ. These terms are supported where x is large, ξ is large relative to η, or η is small. Because we have already proven our local smoothing estimate in these regions, there is no need to apply our resolvent estimate there.

We have

$$\begin{aligned}
H_{e^{-sf}p}a =& \Bigg[2e^{-sf}\xi\Bigg(\chi'(x)\chi(\xi\eta^{-1})\Lambda(x)\Lambda(\xi\eta^{-\epsilon}) + \chi(x)\chi(\xi\eta^{-1})\Lambda'(x)\Lambda(\xi\eta^{-\epsilon})\Bigg) \\
& - (-2e^{-sf}A^{-3}A'\eta^2 - sf_x e^{-sf}p)\Bigg(\chi(x)\chi'(\xi\eta^{-1})\eta^{-1}\Lambda(x)\Lambda(\xi\eta^{-\epsilon}) \\
& \quad + \eta^{-\epsilon}\chi(x)\chi(\xi\eta^{-1})\Lambda(x)\Lambda'(\xi\eta^{-\epsilon})\Bigg) \\
& + (-sf_\theta e^{-sf}p)\Lambda(x)\Bigg(\Lambda(\xi(\eta+1)^{-\epsilon}) - \Lambda(\xi\eta^{-\epsilon})\Bigg)\chi(x)\chi(\xi\eta^{-1})\Bigg](\tilde{\chi}(\eta)) \\
& + (-sf_\theta e^{-sf}p)\Lambda(x)\Lambda(\xi(\eta+1)^{-\epsilon})\chi(x)\left[\chi(\xi(\eta+1)^{-1})\tilde{\chi}(\eta+1) - \chi(\xi\eta^{-1})\tilde{\chi}(\eta)\right].
\end{aligned}$$

We collect the terms involving derivatives of χ or $\tilde{\chi}$ and write

$$\begin{aligned}
H_{e^{-sf}p}a =& \Bigg[2\xi\Lambda'(x)\Lambda(\xi\eta^{-\epsilon}) - 2A^{-3}A'\eta^{2-\epsilon}\Lambda(x)\Lambda'(\xi\eta^{-\epsilon}) \\
& - sf_x p\eta^{-\epsilon}\Lambda(x)\Lambda'(\xi\eta^{-\epsilon}) - sf_\theta p\Lambda(x)\Bigg(\Lambda(\xi(\eta+1)^{-\epsilon}) - \Lambda(\xi\eta^{-\epsilon})\Bigg)\Bigg] \\
& \times e^{-sf}\chi(x)\chi(\xi\eta^{-1})(\tilde{\chi}(\eta)) \\
& + r,
\end{aligned}$$

where

$$\operatorname{supp} r \subset \{|x| \geq \delta/2\} \cup \{|\xi| \geq \delta|\eta|/2\} \cup \{|\eta| \leq 2M\}.$$

We use g to denote the part of $H_{e^{-sf}p}a$ to which we devote most of our efforts. Let

$$g = (2\xi\Lambda'(x)\Lambda(\xi\eta^{-\epsilon}) + 2\eta^{2-\epsilon}A^{-3}(x)A'(x)\Lambda(x)\Lambda'(\xi\eta^{-\epsilon}))e^{-sf}\chi(x)\chi(\xi\eta^{-1})\tilde{\chi}(\eta).$$

We use $\tilde{g}$ to denote the terms in which derivatives have hit e^{-sf}:

$$\begin{aligned}\tilde{g} =& \Big[- sf_x p\eta^{-\epsilon}\Lambda(x)\Lambda'(\xi\eta^{-\epsilon}) \\ & - sf_\theta p\Lambda(x)\Big(\Lambda(\xi(\eta+1)^{-\epsilon}) - \Lambda(\xi\eta^{-\epsilon})\Big)\Big] e^{-sf}\chi(x)\chi(\xi\eta^{-1})\tilde{\chi}(\eta),\end{aligned} \tag{5.6}$$

so

$$H_{e^{-sf}p}a = g + \tilde{g} + r.$$

Our goal is, roughly, to show that $\tilde{g}$ can be absorbed into g and that g is bounded below by a small multiple of $\eta^{-\epsilon}(\xi^2 + \eta^2 x^{2m})$.

We begin by bounding $\tilde{g}$ from above. Because we will only apply this result to functions which are microlocally supported in the region where $\chi(x) = 1$, $\chi(\xi\eta^{-1}) = 1$, and $\tilde{\chi}(\eta) = 1$, we omit the χ and $\tilde{\chi}$ factors. We start with the first term in (5.6):

$$\begin{aligned}|sf_x p\eta^{-\epsilon}\Lambda(x)\Lambda'(\xi\eta^{-\epsilon})e^{-sf}| &\leq C|sf_x\eta^{-\epsilon}(\xi^2 + \eta^2 A^{-2}(x))\Lambda(x)\Lambda'(\xi\eta^{-\epsilon})| \\ &\leq C|sf_x\eta^{2-\epsilon}\Lambda(x)\Lambda'(\xi\eta^{-\epsilon})|,\end{aligned}$$

where we have used the fact that $|\xi| \leq C\eta$.

For the next term in (5.6) we first use the mean value theorem to note that

$$\begin{aligned}\left|\Lambda(\xi(\eta+1)^{-\epsilon}) - \Lambda(\xi\eta^{-\epsilon})\right| &= \left|\int_{\xi\eta^{-\epsilon}}^{\xi(\eta+1)^{-\epsilon}} \langle t\rangle^{-1-\epsilon_0}\, dt\right| \\ &\leq C|\xi|\left|(\eta+1)^{-\epsilon} - \eta^{-\epsilon}\right|\left\langle \xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0} \\ &\leq C|\xi||\eta^{-1-\epsilon}|\left\langle \xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0}.\end{aligned}$$

Thus

$$\begin{aligned}
&\left| -sf_\theta p\Lambda(x)\Big(\Lambda(\xi(\eta+1)^{-\epsilon}) - \Lambda(\xi\eta^{-\epsilon})\Big)\Big]e^{-sf}\right| \\
&\qquad \le C\left|sf_\theta\eta^{-1-\epsilon}(\xi^2+\eta^2A^{-2}(x))\xi\Lambda(x)\right|\left\langle\xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0} \\
&\qquad \le C|sf_\theta\eta^{2-\epsilon}\Lambda(x)\left\langle\xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0}|,
\end{aligned}$$

so

$$|\tilde{g}| \le |s|(|f_x|+|f_\theta|)|\eta^{2-\epsilon}\Lambda(x)|\left\langle\xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0}. \tag{5.7}$$

We need to write g in a more useful form. To get started, we recall that the definition of Λ is

$$\Lambda(t) = \int_0^t \left\langle\tilde{t}\right\rangle^{-1-\epsilon_0}\, d\tilde{t},$$

so $\Lambda'(t) = \langle t\rangle^{-1-\epsilon_0}$, and

$$g = (2\xi\left\langle x\right\rangle^{-1-\epsilon_0}\Lambda(\xi\eta^{-\epsilon}) + 2\eta^{2-\epsilon}A^{-3}(x)A'(x)\Lambda(x)\left\langle\xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0})e^{-sf}\chi(x)\chi(\xi\eta^{-1})\tilde{\chi}(\eta).$$

We will assume throughout that $|x| \le \delta/2$, $|\xi\eta^{-1}| \le \delta/2$, and $|\eta| \ge M$ because we will be applying our operators to functions microlocally cutoff near here. In this region, $\chi(x) = 1$, $\chi(\xi\eta^{-1}) = 1$, and $\tilde{\chi}(\eta) = 1$.

We first break g up into two parts:

$$\begin{aligned}
g &= (2\xi\left\langle x\right\rangle^{-1-\epsilon_0}\Lambda(\xi\eta^{-\epsilon}) + 2\eta^{2-\epsilon}A^{-3}(x)A'(x)\Lambda(x)\left\langle\xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0})e^{-sf} \\
&= g_1 + g_2,
\end{aligned}$$

where

$$g_1 = 2\xi \langle x \rangle^{-1-\epsilon_0} \Lambda(\xi\eta^{-\epsilon})e^{-sf},$$
$$g_2 = 2\eta^{2-\epsilon} A^{-3}(x)A'(x)\Lambda(x) \left\langle \xi\eta^{-\epsilon} \right\rangle^{-1-\epsilon_0} e^{-sf}.$$

Before bounding g from below, we note how $\tilde{g}$ may be absorbed into g. From (5.7) we see that

$$\tilde{g} \le C|s|g_2$$

as long as $|f_x| \le CA'(x)$ and $|f_\theta| \le CA'(x)$. This is satisfied as long as $|f_x| \le C|x|^{2m-1}$ and $|f_\theta| \le C|x|^{2m-1}$.

We will consider two cases. In the first case we make the assumption that $\xi\eta^{-\epsilon} \le \delta$. We are working where Λ is only applied to small quantities, and for $|t|$ small $\Lambda(t) = t + \mathcal{O}(t^3)$ and $\langle t \rangle^{-1-\epsilon} = 1 + \mathcal{O}(t^2)$.

We write out g_1. Because η is relatively large and ξ is relatively small, the most important term will end up being $2\eta^{-\epsilon}\xi^2 e^{-sf}$. Below we will show how the other terms may be absorbed. We separate out this term by writing

$$\begin{aligned} g_1 &= (2\xi(1 + \mathcal{O}(x^2))(\xi\eta^{-\epsilon} + \mathcal{O}((\xi\eta^{-\epsilon})^3)e^{-sf} \\ &= 2\xi^2\eta^{-\epsilon}(1 + \mathcal{O}(x^2))(1 + \mathcal{O}((\xi\eta^{-\epsilon})^2))e^{-sf} \\ &= 2\eta^{-\epsilon}\xi^2 e^{-sf} + \mathcal{O}(x^2\xi^2\eta^{-\epsilon}) + \mathcal{O}(\xi^4\eta^{-3\epsilon}) + \mathcal{O}(\xi^4\eta^{-3\epsilon}x^2) \\ &= 2\eta^{-\epsilon}\xi^2 e^{-sf} + \xi^2\eta^{-\epsilon}\left(\mathcal{O}(x^2) + \mathcal{O}((\xi\eta^{-\epsilon})^2)\right) \end{aligned}$$

where we have used the fact that there exists C such that $e^{-sf(x,\theta)} \le C$.

Because $|x| \le \delta$ and $|\xi\eta^{-\epsilon}| \le \delta$, we then have

$$g_1 = 2\eta^{-\epsilon}\xi^2 e^{-sf}(1 + \mathcal{O}(\delta^2)).$$

Similarly, for g_2, the most important term in the expansion will be $2\eta^{-\epsilon}(\eta x^m)^2 e^{-sf}$. Recall that $A(x) = (1 + x^{2m})^{1/2m}$. Here we use Taylor's theorem to expand $A^{-3}(x)A'(x)$ and write

$$\begin{aligned}
g_2 &= 2\eta^{2-\epsilon}A^{-3}(x)A'(x)(x + \mathcal{O}(x^3))(1 + \mathcal{O}((\xi\eta^{-\epsilon})^2)))e^{-sf} \\
&= 2\eta^{2-\epsilon}(x^{2m-1} + \mathcal{O}(x^{4m-1}))x(1 + \mathcal{O}(x^2))(1 + (\mathcal{O}((\xi\eta^{-\epsilon})^2)))e^{-sf} \\
&= 2\eta^{-\epsilon}(\eta x^m)^2 e^{-sf} + 2\eta^{2-\epsilon}x^{2m}\left((\mathcal{O}(x^{4m}) + \mathcal{O}((x^m\xi\eta^{-\epsilon})^2)\right).
\end{aligned}$$

Again using that $|x| \leq \delta$ and $|\xi\eta^{-\epsilon}| \leq \delta$, we have

$$g_2 = 2\eta^{-\epsilon}(\eta x^m)^2 e^{-sf}(1 + \mathcal{O}(\delta^2)).$$

Because $|\tilde{g}| \leq C|s|g_2$, we then have $g_2 + \tilde{g} = g_2(1 + \mathcal{O}(s))$, hence

$$g_2 + \tilde{g} = 2\eta^{-\epsilon}(\eta x^m)^2 e^{-sf}(1 + \mathcal{O}(\delta^2) + \mathcal{O}(s))$$

We can thus write

$$g + \tilde{g} = 2e^{-sf}\eta^{-\epsilon}(\xi^2 + \eta^2 x^{2m})(1 + \mathcal{O}(\delta^2) + \mathcal{O}(s))$$

as long as $|\xi\eta^{-\epsilon}| \leq \delta$.

We move on to our other case, where $|\xi\eta^{-\epsilon}| \geq \delta$. Our cutoff functions still allow us to assume that $|x| \leq \delta$, $|\xi\eta^{-1}| \leq \delta$, and $|\eta|$ is large.

In this region, we will show that $g + \tilde{g}$ is elliptic. We will consider two cases, based on the size of x relative to the size of $\xi\eta^{-\epsilon}$.

We first note that $g_1, g_2 \geq 0$. Also note using the bound on $\tilde{g}$ given by (5.7) that

$$g_1 + g_2 + \tilde{g} = g_1 + g_2(1 + \mathcal{O}(s)) \geq g_1 + (1 - C|s|)g_2 \geq c(g_1 + g_2).$$

Hence showing that g is elliptic will show that $g + \tilde{g}$ is elliptic.

Suppose $|x|^{1+\epsilon_0} \geq |\xi\eta^{-\epsilon}| \geq \delta$.

$$\begin{aligned}
g_2 &= 2e^{-sf}\eta^{2-\epsilon}A^{-3}(x)A'(x)\Lambda(x)\left\langle \xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0} \\
&= 2e^{-sf}\eta^{2-\epsilon}x^{2m}(1+\mathcal{O}(x^{2m}))(1+\mathcal{O}(x^2))\left\langle \xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0} \\
&\geq ce^{-sf}\eta^{2-\epsilon}x^{2m}(1+\mathcal{O}(x^2))\left\langle \xi\eta^{-\epsilon}\right\rangle^{-1-\epsilon_0} \\
&\geq c\eta^{2-\epsilon}x^{2m}|\xi\eta^{-\epsilon}|^{-1-\epsilon_0} \\
&\geq c\eta^{2-\epsilon}x^{2m}|x|^{-(1+\epsilon_0)^2} \\
&\geq c\eta^{2-\epsilon}.
\end{aligned}$$

On the other hand, if $|\xi\eta^{-\epsilon}| \geq |x|^{1+\epsilon_0}$, but still $|\xi\eta^{-\epsilon}| \geq \delta$, then

$$\begin{aligned}
g_1 &= 2e^{-sf}\xi\left\langle x\right\rangle^{-1-\epsilon_0}\Lambda(\xi\eta^{-\epsilon}) \\
&\geq c\xi\Lambda(\xi\eta^{-\epsilon}) \\
&\geq c\eta^{\epsilon}.
\end{aligned}$$

Hence $g \geq c\eta^{\epsilon}$.

In either case, we find that in this region $g \geq c\eta^{\epsilon}$ and hence $\tilde{g} + g \geq c\eta^{\epsilon}$.

Considering both cases, there thus exists a $\sigma > 0$ such that if

$$\langle (g+\tilde{g})^w u, u\rangle \geq \sigma \| \left\langle D_\theta\right\rangle^{\epsilon/2} u\|^2$$

then u is microlocally supported only in the region where $|\xi\eta^{-\epsilon}| \leq \delta$. We have shown that there exists $c > 0$ such that here we may write

$$g + \tilde{g} \geq c\eta^{-\epsilon}(\xi^2 + \eta^2 x^{2m})(1 + \mathcal{O}(\delta^2) + \mathcal{O}(s)).$$

This then allows us to write

$$g+\tilde{g}=\eta^{-\epsilon}(\xi^2+\eta^2x^{2m})K^2,$$

where K is a strictly positive symbol. Because we are using the Weyl quantization here, this quantizes as

$$\operatorname{Op}^w(K)^*(D_\theta)^{-\epsilon}(D_x^2+D_\theta^2x^{2m})\operatorname{Op}^w(K)+\mathcal{O}(\langle D_\theta\rangle^{-\epsilon}).$$

Thus, for u microsupported in this region,

$$\langle (g+\tilde{g})^w u,u\rangle \geq \left\langle (D_\theta)^{-\epsilon}(D_x^2+D_\theta^2x^{2m})u,u\right\rangle - \|\mathcal{O}(\langle D_\theta\rangle^{-\epsilon/2})u\|^2$$

We thus have

$$\left\langle (H_{e^{-sf}p}a)^w u,u\right\rangle \geq \left\langle (D_\theta)^{-\epsilon}(D_x^2+D_\theta^2x^{2m})u,u\right\rangle - \mathcal{O}(\|\left\langle D_\theta\right\rangle^{-\epsilon/2}u\|^2).$$

□

Proof of Theorem 5.4.1. In the symbol calculus, the commutator $[(e^{-sf}p)^w,a^w]$ has principal symbol $H_{e^{-sf}p}a$, but we will still need to control the remaining terms. Let

$$R_1=[(e^{-sf}p)^w,a^w]-(H_{e^{-sf}p}a)^w.$$

Then we have

$$[(e^{-sf}p)^w+\tau,a^w]=(H_{e^{-sf}p}a)^w+R_1.$$

Applying this to u and taking an inner product with u we find the equality

$$\left\langle [(e^{-sf}p)^w+\tau,a^w]u,u\right\rangle = \left\langle (H_{e^{-sf}p}a)^w u,u\right\rangle + \left\langle R_1u,u\right\rangle.$$

We can then apply the above Theorem 5.4.2 to find

$$\left|\left\langle[(e^{-sf}p)^w + \tau, a^w]u, u\right\rangle\right| \geq c\left\langle\langle D_\theta\rangle^{-\epsilon}(D_x^2 + D_\theta^2 x^{2m})u, u\right\rangle - C\|\langle D_\theta\rangle^{-\epsilon/2}u\|^2 \tag{5.8}$$
$$- |\langle R_1 u, u\rangle|$$

Our goal is to bound $\langle R_1 u, u\rangle$ from above in such a way that it can be absorbed into the term $c\left\langle\langle D_\theta\rangle^{-\epsilon}(D_x^2 + D_\theta^2 x^{2m})u, u\right\rangle$.

The commutator $[(e^{-sf}p)^w, a^w]$ has symbol given by

$$\sum_{k=0}^{N} \frac{i^k}{k!}\sigma(D)^k \left[p(x,\xi,\theta,\eta)a(\tilde{x},\tilde{\xi},\tilde{\theta},\tilde{\eta}) - (e^{-sf}p)(\tilde{x},\tilde{\xi},\tilde{\theta},\tilde{\eta})a(x,\xi,\theta,\eta)\right]\Bigg|_{\text{diag}} + \mathcal{O}(\langle\eta\rangle^{-\epsilon N}),$$

where $\Big|_{\text{diag}}$ denotes evaluation along the diagonal, i.e. $x = \tilde{x}$, $\xi = \tilde{\xi}$, $\theta = \tilde{\theta}$, and $\eta = \tilde{\eta}$. The bound on the error term is a result of the symbol class we are working in. Recall also from Theorem 2.2.3 that

$$A(D) = \frac{1}{2}\left(\langle(D_\xi, D_\eta), (D_{\tilde{x}}, D_{\tilde{\theta}})\rangle - \left\langle(D_x, D_\theta), (D_{\tilde{\xi}}, D_{\tilde{\eta}}\right\rangle\right).$$

The first non-zero term in this expansion is $H_{e^{-sf}p}a$. Because we are using the Weyl calculus, there are no even terms. Therefore, when applying this symbol expansion to write down R_1, the first term is

$$\frac{i^3}{3!}A(D)^3\left[(e^{-sf}p)(x,\xi,\theta,\eta)a(\tilde{x},\tilde{\xi},\tilde{\eta}) - (e^{-sf}p)(\tilde{x},\tilde{\xi},\tilde{\theta},\tilde{\eta})a(x,\xi,\eta)\right]\Bigg|_{\text{diag}}.$$

Before expanding $A(D)^3$, we note that a does not depend on θ, so there will be no terms involving θ derivatives of a, and thus no terms involving η derivatives of $e^{-sf}p$. We have to consider every combination of x, ξ, and θ as the derivative we will be applying to $e^{-sf}p$. Because of the symbol class of a and p we know that we will gain 3 powers of $\eta^{-\epsilon}$. We also know that if any derivative hits the term e^{-sf} then we will have gained a derivative of f and a factor of s. When this is the case, we can bound above by

$$C|sf_*\eta^{2-\epsilon}\Lambda(x)|, \tag{5.9}$$

where f_* denotes some (first, second, or third) derivative of f. As long as we require $|f_*| \leq Cx^{2m-1}$ we find that (5.9) is bounded above by $C|s||\eta^{2-\epsilon}|x^{2m}$, and will be no problem to absorb.

The remaining term occurs when three x derivatives all hit p. This term is

$$Ce^{-sf}(D_x^3 p)(D_\xi^3 a) = Ce^{-sf}(A^{-2})'''(x)\eta^{2-3\epsilon}\Lambda(x)\Lambda'''(\xi\eta^{-\epsilon})\chi(x)\chi(\xi\eta^{-1})(1-\chi(\eta)) + r_2,$$

where

$$\operatorname{supp} r_2 \subset \{|x| \geq \delta\} \cup \{|\xi| \geq \delta|\eta|\} \cup \{|\eta| \geq \delta\}.$$

Because we will be applying our estimate only on functions microlocally supported away from the support of r_2, it will pose no problem to absorb this term. For now, we simply carry this term along.

We first note that

$$|(A^{-2})'''| \leq C|x|^{2m-3}.$$

Next note that

$$|\Lambda(x)| \leq |x|.$$

Hence

$$C(A^{-2})'''(x)\eta^{2-3\epsilon}\Lambda(x)\Lambda'''(\xi\eta^{-\epsilon})\chi(x)\chi(\xi\eta^{-1})(1-\chi(\eta)) \leq Cx^{2m-2}\eta^{2-3\epsilon}.$$

Next we need to control the symbol $Cx^{2m-2}\eta^{2-3\epsilon}$.

When $|x|^{-1} \leq c_0|\eta|^\epsilon$, we have

$$|Cx^{2m-2}\eta^{2-3\epsilon}| \leq Cc_0x^{2m}\eta^{2-\epsilon},$$

which can be absorbed as long as c_0 is small enough.

On the other hand, if $|x| \leq (c_0)^{-1}|\eta|^{-\epsilon}$ then

$$|Cx^{2m-2}\eta^{2-3\epsilon}| \leq C(c_0)^{-2m+2}|\eta|^{(1-2m)\epsilon}.$$

Because $m \geq 2$ and $|\eta|$ is large, $|\eta|^{(1-2m)}$ is bounded. We may then absorb this term into $\| \langle D_\theta \rangle^{-\epsilon/2} u\|_{L^2}^2$.

We may similarly bound $A(D)^k pa$ for higher powers. Note that for every further term we write down in the expansion, the power of η in the remainder term is improved. This follows from the symbol class of $e^{-sf}p$ and a.

Before bounding the remainder term in this symbol expansion, we make a couple of notes. We know that $e^{-sf}p$ satisfies the inequalities

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\theta^\gamma \partial_\eta^\delta (e^{-sf}p)| \leq C_{\alpha,\beta,\gamma,\delta} \langle \eta \rangle^{2-\epsilon(\beta+\delta)}.$$

As we stated before, a satisfies the inequalities

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\theta^\gamma \partial_\eta^\delta a| \leq C_{\alpha,\beta,\gamma,\delta} \langle \eta \rangle^{-\epsilon(\beta+\delta)}.$$

Let E_N denote the remainder term obtained after expanding the first N terms of the commutator $[P, a^w]$. Using our hybrid calculus, we know that E_N has symbol in the class $S_\epsilon^{2-N\epsilon}$, hence

$$\|E_N u\|_{L^2_{x,\theta}} \leq \| \langle D_\theta \rangle^{2-\epsilon N} u\|_{L^2_{x,\theta}}.$$

By taking N large enough we will be able to absorb this term into our final lower bound.

Combining all this, we have shown

$$| \left\langle e^{-sf} R_1 u, u \right\rangle | \leq C\|u\|^2 + C(c_0 + |s|)\|(\langle D_\theta \rangle^{1-\epsilon/2} x^m)u\|^2,$$

where c_0 is small. Note that

$$c\left\langle \langle D_\theta\rangle^{-\epsilon}(D_x^2+D_\theta^2x^{2m})u,u\right\rangle - C(c_0+|s|)\|(\langle D_\theta\rangle^{1-\epsilon/2}x^m)u\|^2 \geq \tilde{c}\left\langle \langle D_\theta\rangle^{-\epsilon}(D_x^2+D_\theta^2x^{2m})u,u\right\rangle,$$

where $\tilde{c}>0$ is smaller than c.

In total we have found

$$\left|\left\langle [((e^{-sf}p)^w+\tau),a^w]u\right\rangle\right| \geq c\left\langle \langle D_\theta\rangle^{-\epsilon}(D_x^2+D_\theta^2x^{2m})u,u\right\rangle - C\|u\|^2.$$

We will now bound the left hand side from above. We expand the commutator to find

$$\left|\left\langle [((e^{-sf}p)^w+\tau),a^w]u,u\right\rangle\right| \leq \left|\left\langle ((e^{-sf}p)^w+\tau)a^wu,u\right\rangle\right| + \left|\left\langle a^w((e^{-sf}p)^w+\tau)u,u\right\rangle\right|.$$

Because a^w and $(e^{-sf}p)^w+\tau$ are self-adjoint, we can combine these terms and obtain the bound

$$\left|\left\langle [((e^{-sf}p)^w+\tau),a^w]u,u\right\rangle\right| \leq 2\left|\left\langle ((e^{-sf}p)^w+\tau)u,a^wu\right\rangle\right|.$$

We now apply the identity operator, in the form $\langle D_\theta\rangle^{1/(m+1)-\epsilon/2}\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2}$, to the right hand side:

$$\begin{aligned}\left|\left\langle [((e^{-sf}p)^w+\tau),a^w]u,u\right\rangle\right| &\leq 2\left|\left\langle \langle D_\theta\rangle^{-1/(m+1)+\epsilon/2}((e^{-sf}p)^w+\tau)u,\langle D_\theta\rangle^{1/(m+1)-\epsilon/2}a^wu\right\rangle\right| \\ &\leq C\|\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2}((e^{-sf}p)^w+\tau)u\|_{L^2}\|\langle D_\theta\rangle^{1/(m+1)-\epsilon/2}a^wu\|_{L^2}\end{aligned}$$

We need to commute these newly introduced operators past the other operators:

$$\begin{aligned}&C\|\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2}((e^{-sf}p)^w+\tau)u\|_{L^2}\|\langle D_\theta\rangle^{1/(m+1)-\epsilon/2}a^wu\|_{L^2} \\ &\leq C\left(\|((e^{-sf}p)^w+\tau)\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2}u\|_{L^2}^2+\|[\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2},((e^{-sf}p)^w+\tau)]u\|_{L^2}^2\right) \\ &\quad\times\left(\|a^w\langle D_\theta\rangle^{1/(m+1)-\epsilon/2}u\|_{L^2}^2+\|[\langle D_\theta\rangle^{1/(m+1)-\epsilon/2},a^w]u\|_{L^2}^2\right)\end{aligned} \tag{5.10}$$

To bound these terms we note that

$$\|[\langle D_\theta\rangle^{1/(m+1)-\epsilon/2}, a^w]u\|_{L^2} \le C\| \langle D_\theta\rangle^{1/(m+1)-\epsilon/2} u\|_{L^2}.$$

Furthermore if we expand the commutator $[\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2}, (e^{-sf}p)^w + \tau)]$ using the symbol calculus and the condition on derivatives of f we find that we can bound the first N terms by

$$C|s||x|^m \langle \eta\rangle^{1-\epsilon/2},$$

and thanks to the gains in powers of $\eta^{-\epsilon}$ we can guarantee that the remainder term has bounded symbol.

Thus we can bound (5.10) from above by

$$\begin{aligned}
&C\Big(\|((e^{-sf}p)^w + \tau)\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2} u\| + |s|\|(\|C|x|^m \langle D_\theta\rangle^{1-\epsilon/2})^w u\| + \|u\|\Big) \\
&\qquad \times \| \langle D_\theta\rangle^{1/(m+1)-\epsilon/2} u\| \\
&\quad \le \|((e^{-sf}p)^w + \tau)\langle D_\theta\rangle^{-1/(m+1)+\epsilon/2} u\|\| \langle D_\theta\rangle^{1/(m+1)-\epsilon/2} u\| \\
&\qquad + c_1\| \langle D_\theta\rangle^{1/(m+1)-\epsilon/2} u\|^2 + Cc_1^{-1}\|u\|^2 + (Cc_1^{-1}|s|)\||x|^m \langle D_\theta\rangle^{1-\epsilon/2})^w u\|,
\end{aligned}$$

where $c_1 > 0$ is very small. The latter three terms will be absorbed, while the first will become our upper bound. Note similarly to before that

$$\begin{aligned}
&c\left\langle \langle D_\theta\rangle^{-\epsilon} (D_x^2 + D_\theta^2 x^{2m})u, u\right\rangle - (Cc_1^{-1}|s|)\||x|^m \langle D_\theta\rangle^{1-\epsilon/2})^w u\| \\
&\quad \ge c_0 \left\langle \langle D_\theta\rangle^{-\epsilon} (D_x^2 + D_\theta^2 x^{2m})u, u\right\rangle,
\end{aligned}$$

where $c_0 > 0$ is slightly smaller than c, as long as s is sufficiently small.

Also note that since we are working microlocally where η is large,

$$\|u\|^2 \ll c_1\| \langle D_\theta\rangle^{1/(m+1)-\epsilon/2} u\|^2.$$

We thus have the bound

$$\begin{aligned}
&\|((e^{-sf}p)^w + \tau) \langle D_\theta \rangle^{-1/(m+1)+\epsilon/2} u\| \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\| \\
&\geq c \left\langle \langle D_\theta \rangle^{-\epsilon} (D_x^2 + D_\theta^2 x^{2m}) u, u \right\rangle - 2c_1 \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|^2.
\end{aligned}$$

Applying Lemma 5.4.3 we then find

$$\begin{aligned}
&\|((e^{-sf}p)^w + \tau) \langle D_\theta \rangle^{-1/(m+1)+\epsilon/2} u\| \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\| \\
&\geq c \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|^2 - 2c_1 \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|^2 \\
&\geq c \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|^2,
\end{aligned}$$

as long as c_1 is sufficiently small. Dividing through by $\| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|$ we have the inequality

$$\|((e^{-sf}p)^w + \tau) \langle D_\theta \rangle^{-1/(m+1)+\epsilon/2} u\| \geq c \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|.$$

Finally, plugging in $\langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u$ we end up with the inequality

$$\|((e^{-sf}p)^w + \tau) u\| \geq c \| \langle D_\theta \rangle^{2/(m+1)-\epsilon} u\|,$$

which proves the theorem. $\square$

The following Lemma and its proof follows Lemma A.2 in [CW13].

Lemma 5.4.3. *There exists $c > 0$ such that*

$$\left\langle \langle D_\theta \rangle^{-\epsilon} (-\partial_x^2 - \partial_\theta^2 x^{2m}) u, u \right\rangle \geq c \| \langle D_\theta \rangle^{1/(m+1)-\epsilon/2} u\|^2$$

for all $u \in \mathcal{S}$ with microlocal support where $\eta > 0$.

This lemma depends on the following result on the anharmonic oscillator (See [RS78]).

Theorem 5.4.4. *Let $P = -\partial_x^2 + x^{2m}$ with $m \in \mathbb{Z}_{\geq 2}$. Then as an operator on L^2 with domain $\mathcal{S}$, P is essentially self-adjoint and has pure point spectrum with eigenvalues $\lambda_j \to \infty$. Every eigenfunction is in $\mathcal{S}$ and furthermore $\lambda_0 > 0$.*

Proof of Lemma 5.4.3. Letting $\hat{u}$ denote the Fourier transform in only θ, we note

$$\left\langle (-\partial_x^2 - \partial_\theta^2 x^{2m})u, u \right\rangle = \left\langle (-\partial_x^2 + \eta^2 x^{2m}\hat{u}, \hat{u} \right\rangle .$$

This inner product is

$$\sum_{\eta \in \mathbb{Z}} \left(\int ((-\partial_x^2 + \eta^2 x^{2m})\hat{u})\overline{\hat{u}}\, dx \right).$$

We make the change of variables $x = \eta^{-1/(m+1)}\tilde{x}$ to obtain

$$\begin{aligned} &\sum_{\eta \in \mathbb{Z}} \eta^{-1/(m+1)} \left(\int ((\eta^{2/(m+1)}\partial_{\tilde{x}}^2 + \eta^{2-2m/(m+1)}\tilde{x}^{2m})\hat{u})\overline{\hat{u}}\, d\tilde{x} \right) \\ &\qquad \geq c \sum_{\eta \in \mathbb{Z}} \eta^{1/(m+1)} \left(\int (\partial_x^2 + \tilde{x}^{2m})\hat{u}\overline{\hat{u}}\, d\tilde{x} \right). \end{aligned}$$

We then apply Lemma A.1 from [CW13] to find

$$\begin{aligned} \sum_{\eta \in \mathbb{Z}} \eta^{1/(m+1)} \left(\int (\partial_x^2 + \tilde{x}^{2m})\hat{u}\overline{\hat{u}}\, d\tilde{x} \right) &\geq c \sum_{\eta \in \mathbb{Z}} \int \eta^{1/(m+1)} \hat{u}\overline{\hat{u}}\, d\tilde{x} \\ &= c \sum_{\eta \in \mathbb{Z}} \int \eta^{2/(m+1)} \hat{u}\overline{\hat{u}}\, dx \\ &\geq c \left\langle \langle D_\theta \rangle^{2/(m+1)} u, u \right\rangle, \end{aligned}$$

where we have used the fact that this will only be applied to $\hat{u}$ supported away from $\eta = 0$.

To achieve the result with $\langle D_\theta \rangle^{-\epsilon}$ in front of the operator, we apply the inequality we've just proven to the function $\langle D_\theta \rangle^{-\epsilon/2}$. Because $\langle D_\theta \rangle^{-\epsilon/2}$ is self-adjoint and commutes with the operator $\langle D_\theta \rangle^{-\epsilon} (-\partial_x^2 - \partial_\theta^2 x^{2m})$, this proves the lemma. ∎

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