Non-Commutative Quiver Algebras and Their Geometric Realizations

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Abstract

The main goal of this paper is to study the (commutative) geometric objects whose properties correspond to those of the noncommutative algebras. Specifically we are interested in the direct deformations of the usual commutative polynomial ring, and we use the tool of quiver algebras and representations to construct vector bundles E_i on projective spaces $\mathbb{P}^n$ whose endormorphism ring $\mathrm{End}(\oplus_i E_i)$ corresponds to these deformations of the polynomial ring. These bundles will turn out to be the symmetric powers of $\mathcal{T}(-1)$ when the algebra is chosen to be the usual polynomial ring. We also derive some properties and exact sequences of these bundles, and in particular there is a short exact sequence showing that $\mathrm{Sym}^k(\mathcal{T}(-1))$ can be generated by $\mathcal{O}$ and $\mathcal{O}(-1)$ in the derived category on $\mathbb{P}^n$ for any $k \leq n$.

1 Properties of Quiver Algebras

1.1 Basics

We now present the basic definitions and properties of quiver algebras, which serve as one of the main tools of this paper. We will mainly follow the notations used in [7], with some differences. A quiver is a directed graph allowing multiple edges and self-looping edges. We let Q_0 denote the set of vertices and Q_1 denote the set of edges. $s, t : Q_1 \to Q_0$ are defined as functions such that, for an edge a,$s(a)$ denotes its source vertex and $t(a)$ denotes its target vertex.

For our purpose, all quivers considered here are finite (i.e. have finitely many vertices and edges) and have a ordering on vertices. That is, there is an injective function $n : Q_0 \to \{0, ..., |Q_0| - 1\}$ such that for all $a \in Q_1$ we have $n(s(a)) < n(t(a))$. In other words, the edges always go from a smaller vertex to a bigger one. Thus cycles and self-looping edges are not allowed.

We next describe the path algebra kQ associated to a quiver. Informally speaking, the path algebra is an associative k algebra generated by all possible paths, with each arrow acting by composition. When the path has length 0, we also need to specify its starting point. Rigorously, kQ is generated by e_q for all $q \in Q_0$ and all $a \in Q_1$ with relations

$$e_q^2 = e_q,\ e_r e_q = 0 \text{ when } r \neq q,\ e_{t(a)}a = ae_{s(a)} = a$$

For two arrows $a, b \in Q_1$, if they are consecutive, i.e. $t(a) = s(b)$, then their product is defined as the arrow ba, i.e. composition of arrows is denoted by prepending. Notice that e=$\sum_{q \in Q_0} e_q$ acts as the identity element in this algebra, since $ae = ea = a$ for all $a \in Q_1$. Since the quiver is assumed to be finite, the path algebra is also finitely generated over k.

We further describe quivers with relations. A relation I is a two-sided ideal of the path algebra kQ generated by linear combinations of paths originating from and ending at the same target. A quiver with algebra is then specified by the pair (Q, I), whose path algebra is defined to be the quotient algebra kQ/I.

As an example, consider the quiver Q drawn below with 3 vertices and 6 arrows. The relation I is a dimension 3 k vector space spanned by $\{y_2x_1 = x_2y_1, z_2y_1 = y_2z_1, x_2z_1 = z_2x_1\}$. Note that this path algebra is not exactly $k[x, y, z]$ in the sense that they are not isomorphic as algebras. In fact this path algebra is not even a commutative algebra, since $y_1z_2 = 0 \neq y_2z_1$. But as we shall see later, this quiver algebra represents the commutative polynomial ring $k[x, y, z]$, in terms of not only the multiplication rule given by commutativity, but also some geometric properties associated to the polynomial ring. Some other non-commutative algebras can be obtained by simply changing the relations in the quiver.

$$\begin{tikzcd}
0 \arrow[r, bend left, "x_1"] \arrow[r, "y_1"] \arrow[r, bend right, "z_1"'] & 1 \arrow[r, bend left, "x_2"] \arrow[r, "y_2"] \arrow[r, bend right, "z_2"'] & 2
\end{tikzcd}$$

1.2 Quiver Representations

We now consider quiver representations, as well as their geometric implications. A representation of a quiver Q assigns to each vertex q a k vector space V_q, and to each edge a a linear transformation ϕ. We assume all such vector spaces V_q have finite dimension. A finite dimensional representation of a quiver with relations (Q, I) has to satisfy further requirements that the composition of these linear transformations has to conform with the relations specified in I. Returning to the previous drawn example, a representation of this quiver would specify three k vector spaces V_0, V_1, V_2 and six k linear transformations $x_1, y_1, z_1 : V_0 \to V_1$ and $x_2, y_2, z_2 : V_1 \to V_2$. These linear transformations must further satisfy the relations that $y_2 \circ x_1 = x_2 \circ y_1, z_2 \circ y_1 = y_2 \circ z_1, x_2 \circ z_1 = z_2 \circ x_1$, as linear transformations from V_0 to V_2.

Morphisms between two representations of a specific quiver Q are a set of linear transformations $\phi_i : V_i \to V_i'$ that need to make all possible squares commute. In other words, for any arrow from any vertex i to any vertex j, if we denote the linear transformations within each representation to be x_i, x_i' respectively, then ϕ_i, ϕ_j must satisfy that $x_i' \circ \phi_i = \phi_j \circ x_i$. In the category theory language, for a fixed quiver with relation (Q, I), the finite dimensional representations of this quiver with morphism defined above form a category $\mathrm{rep}(Q, I)$. This category is abelian, and in particular the kernel and image of quiver representation morphisms can be defined naturally.

Given a morphism of (possibly with relations) quiver representation $\phi : V \to V'$, $\ker \phi$ as a representation of (Q, I) is defined as having $(\ker \phi)_q = \ker(\phi_q) \subset V_q$ at each vertex q. Each arrow morphism x from q_1 to q_2 is simply defined as the restriction $x|_{\ker \phi_{q_1}}$. To see this definition is well-defined, note that for any $v \in \ker \phi_{q_1}$ we have $\phi_{q_2}\big(x(v)\big) = x'\big(\phi_{q_1}(v)\big) = x'(0) = 0$ since x' is a linear transformation. So $x(v) \in \ker \phi_{q_2}$. We defined similarly the image of a quiver representation morphism to be $(\mathrm{Im}\, \phi)_q = \mathrm{Im}(\phi_q) \subset V_q$, and by a similar diagram chasing argument we see that it is well-defined as well. Hence we can define exact sequence, cohomology and derived category in this category. Furthermore, it follows directly from the definition for ker and Im above that a sequence of quiver representation morphisms is exact if and only if it is exact on each component, i.e. the corresponding sequence of vector spaces on each vertex is exact.

According to the composition rule of arrows given above, it is also natural to consider the category of finitely generated left modules over the path algebra $\mathrm{mod}(kQ/I)$. Given a quiver with relations (Q, I) and arrows $a : q_0 \to q_1$ and $b : q_1 \to q_2$ and any element $m \in M \in \mathrm{mod}(kQ/I)$, m acted by a and then b is naturally defined as $(ba)m$, since arrows are composed by prepending. Note that we do not get this naturality if we consider the right modules instead.

We next state the proposition that the category $\mathrm{rep}(Q, I)$ is equivalent to the finitely generated left modules of the path algebra $\mathrm{mod}(kQ/I)$. It is stated in [5] (Proposition 1.8) and we give a brief description of the correspondence. Given $V \in \mathrm{rep}(Q, I)$, we let $M = \sum_i V_{q_i}$ be the direct sum of all components, and each arrow a acts on the component $s(a)$ only, i.e. $a(v) = \phi_a(v|_{V_{s(a)}})$, where $v|_{V_{s(a)}}$ denote the projection of v onto the subspace $V_{s(a)}$. Conversely, given a left module M over kQ/I, we construct a representation by letting $V_q = e_q M = \{e_q m, m \in M\}$. Given an arrow $a : q_0 \to q_1$, the morphism corresponding to a is defined as $\phi_a : V_{q_0} \to V_{q_1}, \phi_a(e_{q_0} m) = (a e_{q_0})m = (e_{q_1} a)m = e_{q_1}(am) \in V_{q_1}$. From now on we will use these two concepts interchangeably.

1.3 Projective Modules

Given a path algebra with relations $A = kQ/I$, now we consider its left modules P_i which are in fact left submodules of A as a left module over itself, defined as $P_i = Ae_i, i \in \{0, 1, ..., |Q_0| - 1\}$. As a k vector space, P_i is generated by all paths having vertex i as source, subjecting to relations I. As a representation, the j^{th} component of P_i is $(P_i)_j = e_j A e_i$, generated by all effective paths starting at vertex i and ending at vertex j.

It follows immediately that $(P_i)_j = 0$ for $i > j$ and $(P_i)_i = k \forall i$. Also since e_i are orthogonal idempotents with $\sum_i e_i$ being the identity element of A, $A = \oplus_i P_i$ as left module over A.

For a concrete example, consider the quiver with relations representing $k[x, y, z]$ above. As representations, P_2 has 0 on vertex $0, 1$ and k on vertex 2. P_1 has 0 on vertex 0, k on vertex 1 and k^3 on vertex 2, with which $\mathrm{Im}(x_2), \mathrm{Im}(y_2), \mathrm{Im}(z_2)$ are independent vectors. The diagram for P_0 is drawn below.

$$\begin{tikzcd} k \arrow[r, bend left, "x_1"] \arrow[r, "y_1"] \arrow[r, bend right, "z_1"'] & k^3 \arrow[r, bend left, "x_2"] \arrow[r, "y_2"] \arrow[r, bend right, "z_2"'] & k^6 \end{tikzcd}$$

If we choose the basis for second component to be $\{x, y, z\}$ and third component to be $\{x^2, xy, xz, y^2, yz, z^2\}$, then for example the linear transformation y_2 sends x to xy, y to y^2, z to yz, since all relations are assumed to be the ones for commutative polynomial ring. With a different set of three quadratic relations, these linear transformations will change, but the dimension for each vector space remains constant.

These modules satisfy $\forall i, \mathrm{Hom}(P_i, M) \cong e_i M = M_i$, and are therefore projective ([7]), since the exactness is preserved component-wise, from the definition of kernel and image above. For a brief proof of the statement, notice that $\forall j > i, (P_i)_j$ as a vector space is generated by $a_{ij} m_i$, where a_{ij} is a path from i to j, and m_i is the generator of $(P_i)_i \cong k$. Besides, such elements form a basis of $(P_i)_j$ as a_{ij} range from all effective paths. (paths modulo relations) So any element in $(P_i)_j$ can be written as $\sum_n a_n m_i$ where each a_n is a different effective path from i to j. So given an element $\phi \in \mathrm{Hom}(P_i, M)$ we have $\phi(\sum_n a_n m_i) = \sum_n a_n \phi(m_i)$. So all such homomorphisms are parametrized by M_i. From this argument we have in particular

Proposition 1.1. $\mathrm{Hom}(P_i, P_j) \cong (P(j))_i$.

Remark. *It is 0 for $i < j$, and* $\mathrm{Hom}(P_{i+1}, P_i) \cong (P(i))_{i+1}$ *as a vector space is generated by* *all arrows from i to $i+1$. Similarly* $\mathrm{Hom}(P_{i+2}, P_{i+1})$ *is generated by arrows from $i+1$ to $i+2$, and so on.*

Furthermore, the composition map of homomorphisms $\mathrm{Hom}(P_{i+1}, P_i) \otimes \mathrm{Hom}(P_{i+2}, P_{i+1}) \to \mathrm{Hom}(P_{i+2}, P_i)$ corresponds to the composition of arrows in the quiver. If there are quadratic relations in the quiver, then this map is in general not bijective. We also state the generalized version of this, which will be useful afterwards.

Proposition 1.2. *There is a map* $\mathrm{Hom}(P_{i+1}, P_i) \otimes ... \otimes \mathrm{Hom}(P_{i+n}, P_{i+n-1}) \to \mathrm{Hom}(P_{i+n}, P_i)$*, where each* $\mathrm{Hom}(P_{j+1}, P_j)$ *as a vector space is generated by arrows at level j. If we assign each arrow to be $\{x_j, y_j, z_j ...\}$, then the composition map corresponds to the composition of n degree 1 monomials into degree n monomials in the non-commutative algebra.*

Let $N = |Q_0|$. $\langle P_{N-1}, ..., P_0 \rangle$[1] form a full and strong exceptional collection (for definition, see [7]) in the derived category $\mathbb{D}^b(\mathrm{mod}(kQ/I))$. Besides the already shown relations for Hom, it is also easy to check that $\mathrm{Ext}^l(P_i, P_j) = 0$ for all i, j, l since all P_i are projective. One major goal of this paper is to find a projective variety X corresponding to a given quiver, which have vector bundles E_i in the bounded derived category of coherent sheaves $\mathbb{D}^b(\mathrm{coh}(X))$, with additional properties that $\mathrm{Hom}(E_i, E_j) \cong \mathrm{Hom}(P_j, P_i)$, with the same composition rules (in terms of the map of tensor product defined above) as the quiver algebra after choosing bases.

2 Deformation of Complex Polynomial Ring

Throughout the rest of this paper, the base field is fixed to be $k = \mathbb{C}$. We first study non-commutative algebras that in some sense resemble (which will be made rigorous later on) the usual commutative polynomial rings. In other words, given one of such algebra A, we need to find a complex projective variety X and vector bundles $\{E_0, ..., E_n\}$ that

[1] Notice that the order here is reversed due to our definition.

have the same morphism structure as the projective modules $\{P_n, ..., P_0\}$ of the associated quiver algebra, i.e. $A = \mathrm{End}(\oplus_i E_i)$. In fact, Orlov in [7] has already given a general recipe to find such varieties given an arbitrary quiver algebra, and the vector bundles will be exceptional. However, the resulting variety from the iterative procedure is often too complex and the vector bundles will not in general be full in $\mathbb{D}^b(\mathrm{coh}(X))$. This section aims for another direction in the particular case of deformations of the usual polynomial ring, that still in some sense resemble the commutative one. We will fix the base space X to be $\mathbb{P}^n$. The bundles will no longer be exceptional, but they have the same morphism structures and are in some sense 'full'. As a byproduct, we will obtain some properties regarding to vector bundles $\mathrm{Sym}^k \mathcal{T}_{\mathbb{P}^n}(-1)$.

The construction in this whole chapter can be summarized in the following theorem

Main Goal. *For a non-commutative polynomial ring A with $n+1$ variables, there exists $n+1$ vector bundles $\{E_0, ..., E_n\}$ on $\mathbb{P}^n$ such that their endomorphism ring (with usual grading) is isomorphic to the quiver algebra associated with A. In other words, the morphism at each level $\mathrm{Hom}(E_i, E_{i+1})$ is $n+1$ dimensional, and after choosing a suitable basis, the composition satisfies the relations in the non-commutative polynomial ring. Furthermore, these bundles satisfy $Ext(E_i, E_j) = 0$ for all $i < j$.*

2.1 Degree 2 Case

The complex projective space $\mathbb{P}^2$ is defined to be the set of dimension 1 subspace of $\mathbb{C}^3$, i.e. $\mathbb{P}^2 = (\mathbb{C}^3 - \{0\})/\sim$, where the equivalence relation is defined to be $a \sim b$ iff $\exists \lambda \in \mathbb{C}$ such that $a = \lambda b$. As mentioned before, $\mathbb{P}^2$ corresponds to the quiver with relations A_2 (see 1.1) associated to the commutative polynomial ring $\mathbb{C}[x, y, z]$, in the sense that there are line bundles (viewed as coherent sheaves) $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ such that the Hom between them as vector spaces are isomorphic to the Hom between the projective modules P_i, which also preserves the composition rules after choosing a suitable basis. Furthermore $\{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\}$ is a full and strong exceptional collection in $\mathbb{D}^b(\mathrm{coh}(\mathbb{P}^2))$ due to the Beĭlinson spectral sequence. ([4]) We also have $A_2 = \mathrm{End}(\oplus_i \mathcal{O}(i))$. But these line bundles don't generalize nicely when the composition rule changes to be non-commutative.

The non-commutative deformations of $\mathbb{P}^2$ can be realized through a certain class of quotient algebras of the non-commutative complex free algebra $\mathbb{C}\langle x, y, z\rangle$ by three quadratic relations, with all variables having weight 1. Here we assume that all relations are homogeneous. We also assume that the resulting quotient algebras are 'well-behaved', in the sense of being Artin-Schelter regular (For the definition please see [2, 9]). [2] Example of such non-commutative algebras include the 3 dimensional quantum polynomial ring (with relations $\langle xy - pyx, yz - qzy, rxz - zx\rangle$ with $p, q, r \in \mathbb{C}$ nonzero) and Sklyanin algebra. (with relations $ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cx^2$ with $a, b, c \in \mathbb{C}$ nonzero)

The procedure here will be similar to [7], From now on we denote $P_{i,j}$ as the j^{th} projective module associated with quivers with relations A_i defined below. We define A_0 to be the path algebra associated to the quiver with a single point and no arrows. A_1 is the path algebra of quiver with two vertices and three arrows in between. A_2 is the one with 3 vertices and 6 arrows, with relations coming from the non-commutative algebra.

Here we also explicitly write out all projective modules $P_{i,j}$. We define vector spaces $U_1 = \mathrm{Hom}_{A_2}(P_{2,1}, P_{2,0}) \cong \mathbb{C}^3$ and $U_2 = \mathrm{Hom}_{A_2}(P_{2,2}, P_{2,1}) \cong \mathbb{C}^3$. We also define $W_{1,2} = \mathrm{Hom}_{A_2}(P_{2,2}, P_{2,0})$ which is 6 dimensional, and $V = \mathrm{Hom}_{A_1}(P_{1,1}, P_{1,0})$. From Proposition 2.2 below, V will be naturally isomorphic to U_2. Since $\mathrm{Hom}(P_i, M) = M_i$ for all i, we can

[2] Since this paper deals mainly with specific examples of non-commutative algebras rather than their general properties, we don't care too much about the actual definition of Artin-Schelter regular algebras.

write each component using U, W defined above.

$$\begin{aligned} P_{0,0} :& \mathbb{C} \\ P_{1,0} : \mathbb{C} \rightsquigarrow & V \\ P_{1,1} : 0 \rightsquigarrow & \mathbb{C} \\ P_{2,0} : \mathbb{C} \rightsquigarrow U_1 \rightsquigarrow & W_{1,2} \\ P_{2,1} : 0 \rightsquigarrow \mathbb{C} \rightsquigarrow & U_2 \\ P_{2,2} : 0 \rightsquigarrow 0 \rightsquigarrow & \mathbb{C} \end{aligned} \tag{1}$$

Here we use the symbol $\rightsquigarrow$ for special meanings. Namely it is an abbreviation of three inner morphisms. For example, in $P_{1,0}$, V is 3 dimensional, and the squiggly arrow denotes the three morphisms, each sending the generator of $\mathbb{C}$ to a vector in V, and the three image vectors will be independent in V.

As we can see, there are some relations between these projective modules. For example, intuitively $P_{1,1}$ can be obtained from $P_{2,2}$ by cutting the first vertex off. Since $V \cong U_2$ naturally, $P_{1,0}$ can also be obtained this way from $P_{2,1}$. This is made precise in the following propositions.

Proposition 2.1. *Let A be an acyclic quiver with relations, which has vetices numbering from 0 to N. Let $\tilde{A}$ be the quiver which has vertex 0 deleted, in the graph theory sense, while keeping all remaining relations that do not contain vertex 0. That is, the vertex 0 and all edges originating from it are removed from the quiver, without changing the remaining relations. Then there is an inclusion $i_* : \mathrm{rep}(\tilde{A}) \to \mathrm{rep}(A)$ which further satisfies $\mathrm{Hom}(M, N) \cong \mathrm{Hom}(i_*M, i_*N)$ for all $M, N \in \mathrm{rep}(\tilde{A})$. There is also a surjective map $\pi : \mathrm{rep}(A) \to \mathrm{rep}(\tilde{A})$.*

Remark. *In our construction, we always have $\widetilde{(A_i)} = A_{i-1}$ for any valid i. We will also have $\pi(P_{i,j}) = P_{(i-1),(j-1)}$ for all $i \geq j > 0$, where $P_{i,j}$ is an A_i representation and $P_{(i-1),(j-1)}$ an A_{i-1} representation.*

Proof. There is a straightforward construction. Given a representation $M \in \mathrm{rep}(\tilde{A})$, we construct its image $i_*M \in \mathrm{rep}(A)$ by assigning 0 to the newly added vertex 0 of A, and assigning all morphisms originating from the vertex 0 as 0. Obviously this is a representation for A, and it satisfies all the relations newly added from vertex 0, since all morphisms connecting to it are already 0, and linear combinations of 0 morphisms are still 0.

For the surjection π, given a representation M' of A, we construct its image $\pi(M') \in \mathrm{rep}(\tilde{A})$ by deleting the vector space associated with vertex 0 of A as well as all linear transformations from it. □

The next proposition will also be very useful.

Proposition 2.2. *Using the same notation as last proposition, suppose* $M, N \in \mathrm{rep}(A)$ *satisfies* $M_0 = 0$*. Then* $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{\tilde{A}}(\pi(M), \pi(N))$ *are naturally isomorphic.*

Remark. *This would give* $U_2 = \mathrm{Hom}_{A_2}(P_{2,2}, P_{2,1}) \cong \mathrm{Hom}_{A_1}(P_{1,1}, P_{1,0}) = V$ *by the remark above. In general we would have* $\mathrm{Hom}_{A_i}(P_{i,j}, P_{i,k}) \cong \mathrm{Hom}_{A_{i-1}}(P_{(i-1),(j-1)}, P_{(i-1),(k-1)})$ *for any* $j, k > 0$*.*

Proof. Since $M_0 = 0$, any inner morphisms of M originating from M_0 must be the zero morphism. Any $\phi \in \mathrm{Hom}_A(M,N)$ must also satisfy $\phi_0 = 0$. So given such ϕ, we construct $\phi' \in \mathrm{Hom}_{\tilde{A}}(\pi(M), \pi(N))$ by keeping all other $\phi_i, i \neq 0$ the same. Conversely, the inverse map from $\mathrm{Hom}_{\tilde{A}}(\pi(M), \pi(N))$ to $\mathrm{Hom}_A(M,N)$ is defined by setting the morphism at vertex 0 to be zero morphism. This representation satisfies all commuting square relations pertaining vertex 0, since both sides will be zero morphism as $M_0 = 0$. □

Due to Proposition 2.1, $\pi(P_{2,0})$ can be viewed as a A_1 representation. Applying Proposition 2.1 to (1) we can write out $\pi(P_{2,0}) = U_1 \rightsquigarrow W$. Since $P_{1,0}$ and $P_{1,1}$ form a full exceptional collection in $\mathrm{rep}\, A_1$, $\pi(P_{2,0})$ can be generated by these two projective modules. In fact there is a map $U_1 \otimes U_2 \to W_{1,2}$ due to Proposition 1.2. The kernel can therefore be identified as $I_{1,2}$, generated by the relations in the non-commutative algebra. Then we can write out an exact bicomplex

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d, rightsquigarrow] & U_1 \arrow[r] \arrow[d, rightsquigarrow] & U_1 \arrow[r] \arrow[d, rightsquigarrow] & 0 \\ 0 \arrow[r] & I_{1,2} \arrow[r] & U_1 \otimes U_2 \arrow[r] & W_{1,2} \arrow[r] & 0 \end{tikzcd} \tag{*}$$

Or written in projective modules as written down in (1)

$$0 \to I_{1,2} \otimes P_{1,1} \to U_1 \otimes P_{1,0} \to \pi(P_{2,0}) \to 0$$

In the commutative case, U_1 and U_2 can be identified as U, and W is generated by degree 2 monomials, which is isomorphic to $\mathrm{Sym}^2(U)$. In this case I is isomorphic to the exterior product $\Lambda^2 U$. It is easy to see that there is a fully faithful functor $\Phi : \mathbb{D}^b \operatorname{rep}(A_1) \to \mathbb{D}^b(\operatorname{coh}(\mathbb{P}^2))$ that sends $P_{1,0}$ to $\mathcal{T}(-1)$ and $P_{1,1}$ to $\mathcal{O}$, because $\{\mathcal{O}, \mathcal{T}(-1)\}$ is a mutation of the dual of $\{\mathcal{O}, \mathcal{O}(1)\}$. Now we can identify the base projective space as $\mathbb{P}(U_2)$. Then, in the above exact sequence, the representation $0 \rightsquigarrow I_{1,2}$ should be sent to $I_{1,2} \otimes \mathcal{O}_{\mathbb{P}(U_2)}$ and $U_1 \rightsquigarrow U_1 \otimes U_2$ sent to $U_1 \otimes \mathcal{T}_{\mathbb{P}(U_2)}(-1)$. Then we can write out a sequence of sheaves on $\mathbb{P}^2$

$$0 \to I_{1,2} \otimes \mathcal{O} \to U_1 \otimes \mathcal{T}(-1) \to \mathcal{F} \to 0$$

This sequence also appears in [7], and the cokernel is again a vector bundle on $\mathbb{P}(U_2)$, defined as $\mathcal{F}$. When the relations I are the commutative ones, we can identify U_2 with U_1 by identifying the corresponding basis vectors x_1 with x_2, etc, and the vector space W becomes $\mathrm{Sym}^2 U_2$. Then $\mathcal{F}$ is isomorphic to $\mathrm{Sym}^2\big(\mathcal{T}_{\mathbb{P}(U_2)}(-1)\big)$([7]).

Since $U_1 \rightsquigarrow W$ is the preimage of $P_{2,0}$ under π, we can extend the faithful functor $\Phi : \mathbb{D}^b \operatorname{rep}(A_2) \to \mathbb{D}^b(\operatorname{coh}(\mathbb{P}^2))$ such that $\Phi(P_{2,0}) = \mathcal{F}, \Phi(P_{2,1}) = \mathcal{T}(-1), \Phi(P_{2,2}) = \mathcal{O}$. This differs from Orlov's construction in [7], in that he defined these bundles not on $\mathbb{P}^2$, but $\mathbb{P}(\mathcal{F}^\vee)$, a $\mathbb{P}^2$ bundle over $\mathbb{P}^2$. In the commutative case this is isomorphic to $\mathrm{Hilb}^2(\mathbb{P}^2)$, the Hilbert scheme of 2 points on $\mathbb{P}^2$. In fact, the bundles defined here $(\mathcal{O}, \mathcal{T}(-1), \mathcal{F})$ are precisely the derived pushforward of the three bundles Orlov defined, which coincides with the standard pushforward.

Orlov's construction has the further advantage that the bundles will be exceptional on the new variety, while $\mathcal{F}$ is not necessarily exceptional as a $\mathbb{P}^2$ bundle. In particular, computation shows that $\mathrm{Ext}^1(\mathrm{Sym}^2(\mathcal{T}(-1)), \mathrm{Sym}^2(\mathcal{T}(-1)))$ is 10 dimensional. However, the variety he defined is much larger, in the sense that $\mathbb{D}^b(\mathrm{coh}(\mathbb{P}(\mathcal{F}^\vee)))$ has a semi-orthogonal decomposition of 9 objects, ([7], Proposition 3.4) while $\mathbb{D}^b\,\mathrm{rep}(A_2)$ only has 3. Furthermore, when the dimension is larger, the variety, defined as nested projective bundles of multiple layers, will quickly become too complicated to be effectively studied. For $\mathbb{P}^3$ it will already be a bundle over $\mathrm{Hilb}^2(\mathbb{P}^3)$ in the commutative case. In the following section we will fix the base space to be $\mathbb{P}^n$ and construct (not necessarily exceptional) bundles that have the same homomorphism structures.

2.2 Degree n case

We will start with $n = 3$ case, and provide a general formula for higher dimensions. We will follow the similar construction above in Proposition 2.1 and 2.2. In the case of $\mathbb{P}^3$ we are concerned only with direct deformation of the usual commutative ring $\mathbb{C}[x, y, z, w]$. That is, there are still $C_2^4 = 6$ independent relations, all of which are still quadratic.

We also change the definition of the quivers A_i to adapt to the base change. Let A_1 be the quiver with 2 vertices and 4 arrows between them, and A_2 be the quiver with 3 vertices and 8 arrows, with 4 arrows on each level, together with relations corresponding to the non-commutative algebra. For example, if we denote first 4 arrows by $\{x_1, y_1, z_1, w_1\}$ and next 4 by $\{x_2, y_2, z_2, w_2\}$, then the relation $xy+yz = w^2$ in the actual non-commutative algebra would be translated to the relation in quiver as $x_1y_2 + y_1z_2 - w_1w_2 = 0$.

We further define A_3 to be the quiver with 4 vertices and 12 arrows, 4 on each level, with relations now defined one both The projective modules $P_{i,j}$ are defined as the j^{th} projective module of A_i, similar to the previous section. Notice that Proposition 2.1 and 2.2 still holds, and using the same notation in Proposition 2.1 we still have $\tilde{A}_i = A_{i-1}$.

Similar to the previous section we defined $U_i = \mathrm{Hom}_{A_3}(P_{3,i}, P_{3,(i-1)}) = P_{3,(i-1)}(i)$, the i^{th} component of $P_{3,(i-1)}$, because of Proposition 1.1. Due to Proposition 2.2 we also have that $P_{3,(i-1)}(i)$ and $P_{2,(i-2)}(i-1)$ are naturally isomorphic for $i \geq 2$, and so on. We also define $W_{i,j} = \mathrm{Hom}_{A_3}(P_{3,j}, P_{3,(i-1)}) = P_{3,(i-1)}(j)$ for $0 < i < j$. Based on these we can write out explicitly all projective modules as follows. We still use $\rightsquigarrow$ to denote multiple inner morphisms, here abbreviating 4 arrows.

$$P_{0,0} : \mathbb{C}$$

$$P_{1,0} : \mathbb{C} \rightsquigarrow U_3$$

$$P_{1,1} : 0 \rightsquigarrow \mathbb{C}$$

$$P_{2,0} : \mathbb{C} \rightsquigarrow U_2 \rightsquigarrow W_{2,3}$$

$$P_{2,1} : 0 \rightsquigarrow \mathbb{C} \rightsquigarrow U_3$$

$$P_{2,2} : 0 \rightsquigarrow 0 \rightsquigarrow \mathbb{C}$$

$$P_{3,0} : \mathbb{C} \rightsquigarrow U_1 \rightsquigarrow W_{1,2} \rightsquigarrow W_{1,3}$$

$$P_{3,1} : 0 \rightsquigarrow \mathbb{C} \rightsquigarrow U_2 \rightsquigarrow W_{2,3}$$

$$P_{3,2} : 0 \rightsquigarrow 0 \rightsquigarrow \mathbb{C} \rightsquigarrow U_3$$

$$P_{3,3} : 0 \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow \mathbb{C}$$

Notice that all modules of A_1 and A_2 have the same form as in previous section after shifting all subscripts by 1, except that now $\rightsquigarrow$ is slightly different, encompassing one more inner morphism. Also notice that although the dimension of all U and W has changed, the map $U_i \otimes U_{i+1} \to W_{i,(i+1)}$ is still surjective due to Proposition 1.2, with kernel defined as $I_{i,(i+1)}$, representing the degree 2 relations at level i. In particular, if we consider $\pi(P_{2,0})$ as an A_1 representation, there is a bicomplex similar to (*)

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, squiggly] & U_2 \arrow[r] \arrow[d, squiggly] & U_2 \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & I_{2,3} \arrow[r] & U_2 \otimes U_3 \arrow[r] & W_{2,3} \arrow[r] & 0
\end{tikzcd}$$

Or as exact sequence of projective modules

$$0 \to I_{2,3} \otimes P_{1,1} \to U_2 \otimes P_{1,0} \to \pi(P_{2,0}) \to 0$$

Again in the commutative case, $W_{2,3}$ is isomorphic to $\mathrm{Sym}^2(U)$ and $I_{2,3}$ isomorphic to $\Lambda^2 U$. From this sequence we get an exact sequence of sheaves on $\mathbb{P}(U_3) \cong \mathbb{P}^3$

$$0 \to I_{2,3} \otimes \mathcal{O}_{\mathbb{P}(U_3)} \to U_2 \otimes \mathcal{T}_{\mathbb{P}(U_3)}(-1) \to \mathcal{F}_1 \to 0$$

In other words, the functor now sends $P_{2,2}$ to $\mathcal{O}$, $P_{2,1}$ to $\mathcal{T}(-1)$, $P_{2,0}$ to $\mathcal{F}_1$ as sheaves on $\mathbb{P}^3$, with $\mathcal{F}_1$ defined by the above sequence. Similar from above, in the commutative case $\mathcal{F}_1$ is isomorphic to $\mathrm{Sym}^2 \mathcal{T}_{\mathbb{P}^3}(-1)$, after identifying the basis vectors of U_2 with U_3.

Our next step is to move up a level, and try to resolve $\pi(P_{3,0})$ by all $P_{2,i}$. To do this we need to consider the resolution of $W_{1,3}$. We consider the commutative case first, where $W_{1,3}$ is now degree 3 monomials and isomorphic to S^3U, where S denotes symmetric product. By [1] there exists a complex

$$0 \to \Lambda^3 U \to \Lambda^2 U \otimes U \to U \otimes S^2 U \to S^3 U \to 0$$

The analogy of this sequence in the general non-commutative setting becomes

$$0 \to I_{1,3} \to I_{1,2} \otimes U_3 \to U_1 \otimes W_{2,3} \to W_{1,3} \to 0$$

Here $I_{1,3}$ is the non-commutative analogy of the higher wedge power $\Lambda^3 U$, but it lacks an natural definition since $W_{1,3}$ is not in general simply the symmetric power. If we add this layer of sequence to the previous one, we get a complete description of the projective modules as

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, squiggly] & 0 \arrow[r] \arrow[d, squiggly] & U_1 \arrow[r] \arrow[d, squiggly] & U_1 \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d, squiggly] & I_{1,2} \arrow[r] \arrow[d, squiggly] & U_1 \otimes U_2 \arrow[r] \arrow[d, squiggly] & W_{1,2} \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & I_{1,3} \arrow[r] & I_{1,2} \otimes U_3 \arrow[r] & U_1 \otimes W_{2,3} \arrow[r] & W_{1,3} \arrow[r] & 0
\end{tikzcd}$$

with exact horizontal arrows. Or more concisely

$$0 \to I_{1,3} \otimes P_{2,2} \to I_{1,2} \otimes P_{2,1} \to U_1 \otimes P_{2,0} \to \pi(P_{3,0}) \to 0$$

Translating this exact sequence to sheaves on $\mathbb{P}^3$ using the $\mathcal{F}_1$ defined above, we get a sequence

$$0 \to I_{1,3} \otimes \mathcal{O} \to I_{1,2} \otimes \mathcal{T}(-1) \to U_1 \otimes \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

and in the commutative case $\mathcal{F}_2$ is isomorphic to $\mathrm{Sym}^3\mathcal{T}(-1)$.

The construction for general $\mathbb{P}^n$ is entirely analogous to the above case. After fixing the dimension n, we define $A_i, i \in \{1, ..., n\}$ to be the quiver with i vertices and $(n+1)$ arrows between each adjacent two. We again assume all relations are quadratic and independent. There are in total C_2^{n+1} relations as in the commutative polynomial ring $\mathbb{C}[x_0, ..., x_n]$. The vector bundles constructed will be $\{\mathcal{O}, \mathcal{T}(-1), \mathcal{F}_1, ..., \mathcal{F}_{n-1}\}$ with each $\mathcal{F}_i$ defined iteratively. In the commutative case we have $\mathcal{F}_i \cong \mathrm{Sym}^{i+1}(\mathcal{T}_{\mathbb{P}^n}(-1))$. The projective modules $P_{i,j}$ are defined in the similar fashion as above.

At each level, we use $P_{k,k}, ..., P_{k,0}$ to resolve $\pi(P_{(k+1),0})$. The inner morphisms of these projective modules will be denoted as $\rightsquigarrow$. At the last level, each $P_{(n-1),k}$ has form

$$0 \rightsquigarrow \ldots \rightsquigarrow 0 \rightsquigarrow \mathbb{C} \rightsquigarrow U_{(k+2)} \rightsquigarrow W_{(k+2),(k+3)} \rightsquigarrow \ldots \rightsquigarrow W_(k+2), n \rightsquigarrow 0$$

Also notice that for $i < j$, $P_{i,k}$ is simply the last i^{th} component of $P_{j,k+(j-i)}$. This will result in a 2 dimensional sequence similar to a bicomplex

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, squiggly] & 0 \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & U_1 \arrow[r] \arrow[d, squiggly] & U_1 \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & \vdots \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & \ldots \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d, squiggly] & 0 \arrow[r] \arrow[d, squiggly] & \ldots \arrow[r] \arrow[d, squiggly] & U_1 \otimes W_{2,k} \arrow[r] \arrow[d, squiggly] & W_{1,k} \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & \vdots \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & \ldots \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & \vdots \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d, squiggly] & I_{1,(n-1)} \arrow[r] \arrow[d, squiggly] & \ldots \arrow[r] \arrow[d, squiggly] & U_1 \otimes W_{2,(n-1)} \arrow[r] \arrow[d, squiggly] & W_{1,(n-1)} \arrow[r] \arrow[d, squiggly] & 0 \\
0 \arrow[r] & I_{1,n} \arrow[r] & I_{1,(n-1)} \otimes U_n \arrow[r] & \ldots \arrow[r] & U_1 \otimes W_{2,n} \arrow[r] & W_{1,n} \arrow[r] & 0
\end{tikzcd}
\qquad (**)
$$

with horizontal arrows all exact. Written as each $P_{(n-1),i}$ it is

$$0 \to I_{1,n} \otimes P_{(n-1),(n-1)} \to I_{1,(n-1)} \otimes P_{(n-1),(n-2)} \to \ldots \to U_1 \otimes P_{(n-1),0} \to \pi(P_{n,0}) \to 0$$

At the previous k^{th} level, moving from A_{k-1} to A_k, the 2 dimensional complex is simply (**) truncated at right corner with size $k \times k$ and then shift the subscripts correspondingly. The sequence of projective modules also looks very similar to the bottom level

$$0 \to I_{k,n} \otimes P_{k,k} \to I_{1,(k-1)} \otimes P_{(k-1),(k-2)} \to \ldots \to U_1 \otimes P_{(k-1),0} \to \pi(P_{k,0}) \to 0$$

In the commutative case, after identifying each U_i we get the sequence for $S^kU = \mathrm{Sym}^kU$ ([1])

$$0 \to \Lambda^kU \to \Lambda^{k-1}U \otimes U \to \Lambda^{k-2}U \otimes S^2U \ldots \to \Lambda^2U \otimes S^{k-2}U \to U \otimes S^{k-1}U \to S^kU \to 0$$

From (**) we also get the iterative definition for $\mathcal{F}_i$ as coherent sheaves on $\mathbb{P}^n$ via exact sequences

$$0 \to I_{(n-1),n} \otimes \mathcal{O} \to U_{n-1} \otimes \mathcal{T}(-1) \to \mathcal{F}_1 \to 0$$

$$0 \to I_{(n-2),n} \otimes \mathcal{O} \to I_{(n-2),(n-1)} \otimes \mathcal{T}(-1) \to U_{n-2} \otimes \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

$$\vdots$$

$$0 \to I_{1,n} \otimes \mathcal{O} \to I_{1,(n-1)} \otimes \mathcal{T}(-1) \to I_{1,(n-2)} \otimes \mathcal{F}_1 \to \ldots \to U_1 \otimes \mathcal{F}_{n-2} \to \mathcal{F}_{n-1} \to 0 \quad (***)$$

with $\dim I_{1,k} = \dim \Lambda^k U = C_k^{n+1}$. As said above, in the commutative case we have $\mathcal{F}_i \cong \mathrm{Sym}^{i+1}(\mathcal{T}_{\mathbb{P}^n}(-1))$, and the $n-1$ sequences above give a resolution of $\mathrm{Sym}^{i+1}(\mathcal{T}_{\mathbb{P}^n}(-1))$ using those of lesser degree.

We also get some additional properties of these symmetric powers of twisted tangent bundles. If we denote $E_0 = \mathcal{O}, E_1 = \mathcal{T}(-1), E_i = \mathcal{F}_{i-1}$, then we have $\mathrm{Hom}(E_i, E_j) = \mathrm{Hom}(P_j, P_i)$, which is by definition $W_{i+1,j}$ if $i+1 < j$, and U_{i+1} if $i+1 = j$. Furthermore, since they are the derived pushdown of bundles Orlov constructed, they still have the "strong" property even though they are no longer exceptional. In other words we have $\mathrm{Ext}^i(\mathcal{F}_j, \mathcal{F}_k) = 0, \forall i > 0, j < k$ and $\mathrm{Ext}^i(\mathcal{T}(-1), \mathcal{F}_j) = 0, \forall i > 0$. This is in particular true for $\mathrm{Sym}^{i+1}(\mathcal{T}_{\mathbb{P}^n}(-1))$.

2.3 Further Description of the Sheaves

The sheaves constructed above also have geometric relationships. In the case of $\mathbb{P}^2$ there is a bicomplex on $\mathbb{P}(U_2)$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & U_1 \otimes \mathcal{O}(-1) \arrow[r, equal] \arrow[d] & U_1 \otimes \mathcal{O}(-1) \arrow[d] & \\
0 \arrow[r] & I_{1,2} \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & U_1 \otimes U_2 \otimes \mathcal{O} \arrow[r] \arrow[d] & W_{1,2} \otimes \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{1,2} \otimes \mathcal{O} \arrow[r] & U_1 \otimes \mathcal{T}(-1) \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The bottom left square is obviously commutative. By the proposition below we shall see that the right two squares are also commutative. The rows are also easily seen to be all exact.

The second column is exact, since it can be obtained by tensoring the Euler sequence on $\mathbb{P}(U_2)$ with U_1. In fact this whole bicomplex is exact, which will be proved below. Notice that this sequence is also valid for any $\mathbb{P}^n$, with $\mathcal{O}, \mathcal{T}(-1), \mathcal{F}_1$ defined from (***).

On $\mathbb{P}^3$ there is a larger bicomplex (on $\mathbb{P}(U_3)$)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & I_{1,2} \otimes \mathcal{O}(-1) \arrow[r] \arrow[d] & U_1 \otimes U_2 \otimes \mathcal{O}(-1) \arrow[r] \arrow[d] & W_{1,2} \otimes \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{1,3} \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & I_{1,2} \otimes U_3 \otimes \mathcal{O} \arrow[r] \arrow[d] & U_1 \otimes W_{2,3} \otimes \mathcal{O} \arrow[r] \arrow[d] & W_{1,3} \otimes \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{1,3} \otimes \mathcal{O} \arrow[r] & I_{1,2} \otimes \mathcal{T}(-1) \arrow[r] \arrow[d] & U_1 \otimes \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

again with exact rows. The third column is obtained by tensoring the second column in $\mathbb{P}^2$ diagram above with U_1. From the commutativity of the previous bicomplex, we see that all squares are commutative, with possible exception of rightmost two squares. Again by the proposition below they turn out to be exact.

Continuing in this fashion inductively, we see that there is a bicomplex on $\mathbb{P}^n$ (again the commutative squares will be satisfied)

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & 0 \arrow[r] & I_{1,(n-1)} \otimes \mathcal{O}(-1) \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & W_{1,(n-1)} \otimes \mathcal{O}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{1,n} \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & I_{1,(n-1)} \otimes U_n \otimes \mathcal{O} \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & W_{1,n} \otimes \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{1,n} \otimes \mathcal{O} \arrow[r] & I_{1,(n-1)} \otimes \mathcal{T}(-1) \arrow[r] \arrow[d] & \dots \arrow[r] & \mathcal{F}_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 & & 0 &
\end{tikzcd}
\tag{$\dagger$}
$$

the second row of which is obtained from the bottom row of (**), and the bottom row from (***). The first row is obtained by tensoring the previous level of (**) with $\mathcal{O}(-1)$.

We now show the exactness of all columns by induction. The inner rows of ($\dagger$) are of the form

$$0 \to I_{1,j} \otimes W_{(j+1),(n-1)} \otimes \mathcal{O}(-1) \to I_{1,j} \otimes W_{(j+1),n} \otimes \mathcal{O} \to I_{1,j} \otimes \mathcal{F}_{n-1-j} \to 0$$

which can be obtained from the last column of of the bicomplex for $\mathbb{P}^j$. So we only need to prove the following

Proposition 2.3. *For any n, suppose all squares (with possible exception of the rightmost two) are commutative. If all columns of ($\dagger$) are well-defined and exact except possibly for* *the last column, then it is also well-defined and exact. It also satisfies the commutativity requirements.*

Proof. We will prove this statement by diagram chasing. For convenience we label the whole diagram as follows

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] \arrow[d] & J \arrow[r, "g"] \arrow[d, "u"] & G \arrow[r, "d"] \arrow[d, "r"] & D \arrow[r, "a"] \arrow[d, "p"] & A \arrow[r] \arrow[d, dashed, "\alpha"] & 0 \\
\dots \arrow[r] \arrow[d] & K \arrow[r, "h"] \arrow[d, "v"] & H \arrow[r, "e"] \arrow[d, "s"] & E \arrow[r, "b"] \arrow[d, "q"] & B \arrow[r] \arrow[d, dashed, "\beta"] & 0 \\
\dots \arrow[r] \arrow[d] & L \arrow[r, "i"] \arrow[d] & I \arrow[r, "f"] \arrow[d] & F \arrow[r, "c"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Our task is to show that maps α, β are well-defined and exact, given that all other rows and columns in this diagram are exact. The commutativity will also follow directly from the proof. Note that in the case of $\mathbb{P}^2$ and $\mathbb{P}^3$ bicomplex, we can set some terms to be 0, and the argument here still remains valid.

For any element (here section of a sheaf) $x \in B$, we define $\beta(x)$ as follows. Since b is surjective, choose $y \in E$ such that $b(y) = x$. Then define $\beta(x) = \beta(b(y)) = c(q(y))$. To show such β is well-defined, consider another y' such that $b(y') = b(y) = x$. Then $y - y' \in \ker b = \operatorname{Im} e$. So we can write $y' = y + e(z)$ for some $z \in H$. Then $c(q(y')) = c(q(y)) + c(q(e(z))) = c(q(y)) + c(f(s(z))) = c(q(y))$ since $c \circ f = 0$. So choosing a different y' yields the same result. The map α is defined in an analogous manner, i.e. given any element $x \in A$, choose $y \in D$ such that $a(y) = x$ and let $\alpha(x) = b(p(y))$. By the essentially same argument we see that α is well-defined as well, and the whole bicomplex is commutative by our definition of α and β.

Now we show that β is surjective. given any $x \in C$, since both q and c are surjective, there exists $y \in E$ such that $c(q(y)) = x$. Then by definition above, $\beta(b(y)) = x$. So $b(y)$ is a preimage.

For $\beta \circ \alpha = 0$, given any $x \in A$ there exists $y \in D$ such that $\beta(\alpha(x)) = \beta(\alpha(a(y))) = c(q(p(y))) = 0$ since $q \circ p = 0$. Proceeding to the exactness at B, consider $x \in \ker \beta$. Choose $y \in E$ such that $b(y) = x$. Then $c(q(y)) = \beta(b(y)) = 0$. So $q(y) \in \ker c = \operatorname{Im} f$. So there exists $z \in I$ such that $q(y) = f(z)$. Since s is surjective, there exists $w \in H$ such that $s(w) = z$. Now consider $y' = y - e(w)$. We have $b(y') = b(y) = x$ since $b \circ e = 0$. We also have $q(y') = q(y) - e(q(w)) = q(y) - f(s(w)) = 0$. Now $y' \in \ker q = \operatorname{Im} p$. So there exists $z \in D$ such that $p(z) = y'$. So $\alpha(a(z)) = b(p(z)) = x$. So x belongs to the image of α, with preimage $a(z)$.

Now we prove the injectivity of α. Let $x \in A$ such that $\alpha(x) = 0$. Since a is surjective there is $y \in D$ such that $a(y) = x$. Then $b(p(y)) = \alpha(a(x)) = 0$. So $p(y) \in \ker b = \operatorname{Im} e$. So there exists $z \in H$ such that $e(z) = p(y)$. Now $f(s(z)) = q(e(z)) = q(p(y)) = 0$. So $s(z) \in \ker f = \operatorname{Im} i$. Since v is surjective, there exists $w \in K$ such that $i(v(w)) = s(z)$. Consider now $z' = z - h(w) \in H$. We have $e(z') = e(z) = p(y)$ since $e \circ h = 0$. Also $s(z') = s(z) - s(h(w)) = s(z) - i(v(w)) = 0$. So $z' \in \ker s = \operatorname{Im} r$. So there exists $t \in G$ such that $r(t) = z'$. Then we have $p(d(t)) = e(r(t)) = e(z') = p(y)$. Since p is injective we have $d(t) = y$. Therefore $x = a(y) = d(a(t)) = 0$ since $a \circ d = 0$. So α is injective. $\square$

This proof shows that the $n + 1$ objects we constructed can be generated by $\mathcal{O}$ and $\mathcal{O}(-1)$ in the derived category. In particular, the collection we here constructed (symmetric powers of $\mathcal{T}(-1)$) is not as nice as the fully exceptional and strong collection $\{\mathcal{O}, ..., \mathcal{O}(n)\}$. Not only do they cease to be exceptional, but also they are not full any more. But they do have the advantage of being able to adapt to non-commutative algebras.

3 Further Remarks

The above sequence produces a way to classify non-commutative algebras by some canonical forms. Take the example of the sequence

$$0 \to W_{1,2} \otimes \mathcal{O}(-1) \to W_{1,3} \otimes \mathcal{O} \to \mathcal{F}_2 \to 0$$

with underlying space to be $\mathbb{P}^3$. Then $W_{1,2}$ is 10 dimensional and $W_{1,3}$ 20 dimensional. Then the sequence is determined by the first morphism, which is a 10×20 matrix with coefficients in degree 1 polynomial of 4 variables. After modding out certain equivalences we should get a canonical matrix for all non-commutative deformations of the polynomial ring $\mathbb{C}[x, y, z, w]$. This should also correspond to other means of classifying non-commutative algebras, for example the quotient graded module category qgr (see [9] definition 4.3 for details). The relation between this two concepts is yet unknown. In particular, it would be of interest to study whether two algebras with same qgr category would produce equivalent matrix in the morphism, and vice versa.

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