Global Existence in a Coupled Wave System

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1 Introduction and Background

1.1 Problem History

We will consider the Strauss conjecture for a system of coupled wave equations. The original setting of the Strauss conjecture was in the following wave equation:

$$\begin{cases} \Box u = u^p \\ u(0,\cdot) = f_u, \quad \partial_t u(0,\cdot) = g_u. \end{cases} \tag{1}$$

Here $\Box = \partial_t^2 - \sum_{i=1}^3 \partial_{x_i}^2$ and u is a function in $\mathbb{R} \times \mathbb{R}^n$, which we view as n spatial dimensions and a time dimension. The Strauss conjecture is that there is some $p_c \in \mathbb{R}$ such that $p > p_c$ implies existence of a global (both in space and in time) solution for sufficiently small initial data, while $p < p_c$ implies existence of arbitrarily small initial data that result in finite time blow-up of solutions. In [4], the conjecture was proven in three spatial dimensions, with $p_c = 1 + \sqrt{2}$.

Our result (which we will describe later) is on a coupled version of this equation:

$$\begin{cases} \Box u = v^p, \quad \Box v = u^q \\ u(0,\cdot) = f_u, \quad \partial_t u(0,\cdot) = g_u \\ v(0,\cdot) = f_v, \quad \partial_t v(0,\cdot) = g_v. \end{cases} \tag{2}$$

Note that we can get small-data solutions if $p, q > 1 + \sqrt{2}$ by the methods for solving (1). However, it turns out that if one exponent is sufficiently large, the other exponent can be smaller than $1+\sqrt{2}$. The authors of [1] showed global existence given sufficiently small data when $p, q > 2$, $1 > \frac{2+p+1/q}{pq-1}$, and $1 > \frac{2+q+1/p}{pq-1}$.

We will prove the same result, but in a more general setting. In particular, we use a method that is robust under geometric perturbations of the $\Box$ operator. The weighted Strichartz estimate we rely on is developed in [7], which is an extension of that from [3] and [6].

1.2 Notation

We will use $A \lesssim B$ as shorthand for $A \leq CB$ where C is a positive constant independent of important parameters. We write $r = |x| = \sqrt{\sum_{i=1}^3 x_i^2}$ for the radius of x in $\mathbb{R}^3$ (ignoring the

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time component). If $Z = \{z_1, z_2, ..., z_k\}$ is a set of operators and f is a function, we will define $\|Z^{\leq j} f\| = \sum_{|\alpha|\leq j} \|Z^\alpha f\|$ for all multi-indices α.

1.3 Sobolev Spaces

The methods used in this paper rely heavily on Sobolev spaces and estimates on these spaces. We will follow the treatment of this subject from [8]. Consider the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx. \tag{3}$$

We use the Fourier transform to define a space called the Sobolev space. For $s \in \mathbb{R}$, define $\Lambda^s = \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f$. Now we define $H^s(\mathbb{R}^n) = \Lambda^{-s}(L^2(\mathbb{R}^n))$ with the norm $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}$. We also define the homogeneous Sobolev norm by $\|f\|_{\dot{H}^s} = \||\xi|^s \hat{f}\|_{L^2}$.

If $s \in \mathbb{N}$, we have an equivalent definition:

$$\|f\|_{H^s} = \left(\sum_{|\alpha|\leq s} \|\partial^\alpha f\|_{L^2}^2 \right)^{1/2}. \tag{4}$$

We will need two basic properties of Sobolev spaces:

- When $a < b$ and $a, b \in \mathbb{R}$, we have $\|f\|_{H^a} \lesssim \|f\|_{H^b}$.
- If $f \in C^k$, $\|f\|_{H^s} \lesssim \|\partial^j f\|_{H^{s-j}}$ for any integer $j \leq k$.

1.4 General Approach

The way that we attack the problem relies on two types of inequalities. The first is a weighted Strichartz estimate, which effectively gives a lower bound on the rate at which energy leaves a region. The weighted Strichartz estimates we use hold generally for equations $\Box u = F$ and are roughly of the form

$$\|\Psi_1(r)u\|_{L_t^p L_r^p L_\omega^2} \lesssim E_0 + \|\Psi_2(r)F\|_{L_t^1 L_r^1 L_\omega^2} \tag{5}$$

where Ψ_1 and Ψ_2 are decaying functions of the radius and E_0 is a nonnegative real number, which depends solely on the initial conditions. Note that we are using a mixed-norm notation, which is defined as follows:

$$\|f\|_{L^p_t L^r_q L^s_\omega} = \left[\int \left(\int \left[\int |f(t, r\omega)|^s d\omega_{\mathbf{S}^2} \right]^{q/s} r^2 dr \right)^{p/q} dt \right]^{1/p} \tag{6}$$

The other inequalities we use relate different L^p-spaces and Sobolev spaces. In particular, we will use Hölder's inequality

$$\|fg\|_{L^t} \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{t}, \tag{7}$$

and a special case of the Sobolev embedding theorem in $\mathbb{R}^n$

$$\|f\|_{L^p} \lesssim \|f\|_{H^s} \quad \frac{1}{2} - \frac{1}{p} \leq \frac{s}{n}. \tag{8}$$

Additionally, note that we have $\|f\|_{H^0} = \|f\|_{L^2}$.

The method begins by applying the weighted Strichartz estimate to both of the equations in the system and summing. Ignoring the angular components for now, we get something of the form

$$\|\Psi_{1,1}(r)u\|_{L^q_t L^q_r} + \|\Psi_{1,2}(r)v\|_{L^p_t L^p_r} \lesssim E_0 + \|\Psi_{2,1}(r)v^p\|_{L^1_t L^1_r} + \|\Psi_{2,2}(r)u^q\|_{L^1_t L^1_r}. \tag{9}$$

With some work, it will be possible to choose parameters such that $\Psi_{2,1}(r) = \Psi_{1,2}(r)^p$ and $\Psi_{2,2}(r) = \Psi_{1,1}(r)^q$. Then the second term on the right side will be the pth power of the second term on the left side, and the third term on the right side will be the qth power of the first term on the left side. This will allow us to set up a function iteration: $u_{-1} \equiv v_{-1} \equiv 0$, $\Box u_j = v^p_{j-1}$, and $\Box v_j = u^q_{j-1}$ (where j ranges over nonnegative integers), with all iterates having the same initial data. Our goal is to show that this sequence of tuples of functions converges to a solution to the wave system. From (9), with the proper choice of parameters we can get an expression similar to

$$\|\Psi_{1,1}(r)u_j\|_{L^q_t L^q_r} + \|\Psi_{1,2}(r)v_j\|_{L^p_t L^p_r} \lesssim E_0 + \left(\|\Psi_{1,1}(r)u_{j-1}\|_{L^q_t L^q_r}\right)^q + \left(\|\Psi_{1,2}(r)v_{j-1}\|_{L^p_t L^p_r}\right)^p \tag{10}$$

Note that u_{-1} and v_{-1} are both 0 when measured in these norms. It turns out that we will be able to get the same inequality with the angular components of the norms included. This will allow us to establish a constant bound on the norms of u_j and v_j (with the Ψ weights) given small initial data. From this, we will apply (5) on the equation $\Box(u_{j+1} - u_j) = (v_j)^p - (v_{j-1})^p$. Using the constant bounds and with some manipulation of the norms, we will be able to show that $u_j - u_{j-1}$ measured in a suitable norm decays geometrically with j (we will skip the details for now), and similarly for $v_j - v_{j-1}$. This will give that the function sequence is Cauchy. And as we will be using a norm from a complete space, we then have convergence as desired.

2 Flat Problem

Here, we will demonstrate existence of global solutions to the wave system in flat space. This is an adaptation of the methods from [3]. The goal is to provide a concrete example of an application of the above approach. In a later section, we will generalize this approach to show existence in a class of asymptotically flat geometries. Because flat space falls into this class, the result in this section will be a corollary of the later result, but as the proof is easier to follow in flat space, we give it first as preparation for the main result.

Before stating the theorem, we need to define the set of vector fields $Z = \{\partial_i, \Omega_{ij} = x_i\partial_j - x_j\partial_i\}$. These are needed to describe the sense in which the initial data are small.

The theorem we will prove is as follows:

Theorem 1 *Given any $p, q \in \mathbb{R}^+$ with $p, q > 2$, $1 > \frac{2+p+1/q}{pq-1}$, and $1 > \frac{2+q+1/p}{pq-1}$, and any $f_u, g_u, f_v, g_v \in C_c^\infty$, there exists some $\epsilon > 0$ such that*

$$\begin{cases} \Box u = v^p, \quad \Box v = u^q \\ u(0,\cdot) = f_u, \quad \partial_t u(0,\cdot) = g_u \\ v(0,\cdot) = f_v, \quad \partial_t v(0,\cdot) = g_v \end{cases} \tag{11}$$

has global solutions u and v given $E_2 < \epsilon$, where $E_k = \|Z^{\leq k} f_u\|_{\dot{H}^{\gamma_1}(\mathbb{R}^3)} + \|Z^{\leq k} g_u\|_{\dot{H}^{\gamma_1 - 1}(\mathbb{R}^3)} + \|Z^{\leq k} f_v\|_{\dot{H}^{\gamma_2}(\mathbb{R}^3)} + \|Z^{\leq k} g_v\|_{\dot{H}^{\gamma_2 - 1}(\mathbb{R}^3)}$, in which $\gamma_1 = \frac{7+4p-3pq}{2-2pq}$ and $\gamma_2 = \frac{7+4q-3pq}{2-2pq}$.

This theorem only allows us to go below the Strauss exponent in $n = 3$, since the condition $p, q > 2$ is independent of the dimension, and the Strauss exponent is always at most 2 when $n \geq 4$. To prove this, we rely on a weighted Strichartz estimate from [3]:

Theorem 2 *Let u solve the Minkowski wave equation*

$$\begin{cases} \Box u = F, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{cases} \tag{12}$$

Then, for $2 \leq p \leq \infty$, and γ satisfying

$$\frac{1}{2} - \frac{1}{p} < \gamma < \frac{3}{2} - \frac{1}{p}, \quad \frac{1}{2} < 1 - \gamma < \frac{3}{2}, \tag{13}$$

we have the following estimate:

$$\|r^{\frac{3}{2} - \frac{4}{p} - \gamma} u\|_{L_t^p L_r^p L_\omega^2(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} + \|r^{-\frac{1}{2}-\gamma} F\|_{L_t^1 L_r^1 L_\omega^2(\mathbb{R}^+ \times \mathbb{R}^3)}. \tag{14}$$

We apply the theorem to both equations in the system, giving two inequalities that look like (14), and sum. For now, we are going to ignore the angular component of the norms; we will deal with them later. The sum of the inequalities is

$$\begin{aligned}\|r^{\frac{3}{2}-\frac{4}{q}-\gamma_1}u\|_{L_t^q L_r^q} + \|r^{\frac{3}{2}-\frac{4}{p}-\gamma_2}v\|_{L_t^p L_r^p} &\lesssim \|f_u\|_{\dot{H}^{\gamma_1}(\mathbb{R}^3)} + \|g_u\|_{\dot{H}^{\gamma_1-1}(\mathbb{R}^3)} \\ &+ \|f_v\|_{\dot{H}^{\gamma_2}(\mathbb{R}^3)} + \|g_v\|_{\dot{H}^{\gamma_2-1}(\mathbb{R}^3)} + \|r^{-\frac{1}{2}-\gamma_1}v^p\|_{L_t^1 L_r^1} + \|r^{-\frac{1}{2}-\gamma_2}u^q\|_{L_t^1 L_r^1}. \quad (15)\end{aligned}$$

The first four terms on the right side sum to the constant E_0, so we have

$$\|r^{\frac{3}{2}-\frac{4}{q}-\gamma_1}u\|_{L_t^q L_r^q} + \|r^{\frac{3}{2}-\frac{4}{p}-\gamma_2}v\|_{L_t^p L_r^p} \lesssim E_0 + \|r^{-\frac{1}{2}-\gamma_1}v^p\|_{L_t^1 L_r^1} + \|r^{-\frac{1}{2}-\gamma_2}u^q\|_{L_t^1 L_r^1}. \quad (16)$$

Now we have reached the central idea of the method. If given our p and q we can find some appropriate γ_1 and γ_2, we can write the v^p term on the right side as a power of the v term on the left, and similarly for the u^q and u terms. The powers of r will need to be weighted in such a way that they match. This will allow us to set up the iteration of functions.

2.1 Flat Problem - Choosing Parameters

Now we will find the appropriate parameters (when they exist). Because of the hypotheses from (13), we need

$$\frac{1}{2} - \frac{1}{q} < \gamma_1 < \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{p} < \gamma_2 < \frac{1}{2} \quad (17)$$

and to get the exponents on r to match, we need

$$p(\frac{3}{2} - \frac{4}{p} - \gamma_2) = -\frac{1}{2} - \gamma_1, \quad q(\frac{3}{2} - \frac{4}{q} - \gamma_1) = -\frac{1}{2} - \gamma_2. \quad (18)$$

Solving (18) for γ_1 gives $\gamma_1 = \frac{7+4p-3pq}{2-2pq}$, and similarly $\gamma_2 = \frac{7+4q-3pq}{2-2pq}$. Substituting for γ_1 in the first half of the first inequality from (17) and performing some manipulations gives $1 > \frac{2+p+1/q}{pq-1}$, and we can get an analogous inequality by swapping the roles of p and q (and γ_1 and γ_2). Therefore, our parameters need to satisfy

$$1 > \frac{2+p+1/q}{pq-1}, \quad 1 > \frac{2+q+1/p}{pq-1}. \quad (19)$$

This pair of inequalities appeared in related problems in [2], [5], and [1]. We also require $p > 2$ and $q > 2$. Because of this, the hypotheses from (13) of the types $1 - \gamma < \frac{3}{2}$ and $\gamma < \frac{3}{2} - \frac{1}{p}$ are redundant. Note that under the constraint $p = q$, the inequalities give $p > 1 + \sqrt{2}$. In this regard, this theorem is a generalization of Fritz John's classic result [4] that $\Box u = u^p$ with small initial conditions has solutions in $\mathbb{R}^3$ if $p > 1 + \sqrt{2}$.

We do still have to handle the hypotheses $\gamma_1, \gamma_2 < \frac{1}{2}$. This inequality is actually not true for all p, q which satisfy (19) - given any fixed p, there is some q satisfying (19) such that $\gamma_1 \geq \frac{1}{2}$ or $\gamma_2 \geq \frac{1}{2}$. However, it is always possible to find some $\tilde{q} \leq q$ that works and rewrite as $\Box v = u^{\tilde{q}} u^{q-\tilde{q}}$, which admits a solution (as we will discuss later).

Now that we have chosen our parameters to deal with the exponents of r, we will abbreviate our notation. We write $\alpha_2 = -(\frac{3}{2} - \frac{4}{p} - \gamma_2)$ and $\alpha_1 = -(\frac{3}{2} - \frac{4}{q} - \gamma_1)$. This allows us to rewrite (16) as

$$\|r^{-\alpha_1} u\|_{L_t^q L_r^q} + \|r^{-\alpha_2} v\|_{L_t^p L_r^p} \lesssim E_0 + \|r^{-p\alpha_2} v^p\|_{L_t^1 L_r^1} + \|r^{-q\alpha_1} u^q\|_{L_t^1 L_r^1}. \tag{20}$$

2.2 Flat Problem - Angular Component

We have set up the parameters such that the radial and time components of the norms align. We still need to work with the angular components. Ideally, we would like to have the result that $\|f^p\|_{L_t^1 L_r^1 L_\omega^2} \leq \|f\|^p_{L_t^p L_r^p L_\omega^2}$. This would allow us to set up the iteration immediately, as the quantities on the left side would already be powers of the quantities on the right side. This result is not quite true, though. We will instead work with some operators on u, which will allow us to apply Sobolev embeddings.

Specifically, recall the set of operators $Z = \{\partial_i, \Omega_{ij} = x_i \partial_j - x_j \partial_i\}$, where $i, j \in \{1, 2, 3\}$. These operators commute with $\Box$ [8]. When we apply two Z operators to $|v|^p$, we always obtain sums of terms of the form $|v|^{p-2} |Z^{\leq 1} v|^2$ or $|v|^{p-1} |Z^{\leq 2} v|$ (at least, up to unimportant constants). To see this, note that $Z(Z(|v|^p))$ has terms that look roughly like $Z(|Zv||v|^{p-1})$ by the chain rule, and then applying the second Z results in terms of the above two types, by the product rule. Also, note that we can assume u and v are twice differentiable, because they solve (12).

Now we wish to bound the quantity $\|r^{-p\alpha_2} Z^{\leq 2} v^p\|_{L_t^1 L_r^1 L_\omega^2}$. We will use the triangle inequality to break the inside up into the individual terms of $Z^{\leq 2} v^p$. As discussed above, we have to deal with terms of two types. In the below, we will write out only the angular components of the norms.

For terms of type $\||v|^{p-1} Z^{\leq 2} v\|_{L_\omega^2}$, we apply Hölder's inequality (in which we may take liberties with infinities, as in $\frac{1}{2} + \frac{1}{\infty} = \frac{1}{2}$) to obtain

$$\||v|^{p-1} Z^{\leq 2} v\|_{L_\omega^2} \lesssim \|v\|^{p-1}_{L_\omega^\infty} \|Z^{\leq 2} v\|_{L_\omega^2}. \tag{21}$$

Now we apply a Sobolev embedding on the L^∞ term. Because $\frac{1}{2} < 1$ (we are working on the 2-dimensional space $\mathbb{S}^2$), we have

$$\|v\|_{L^\infty_\omega} \lesssim \|v\|_{H^2_\omega} \lesssim \|Z^{\leq 2}v\|_{H^0_\omega} = \|Z^{\leq 2}v\|_{L^2_\omega}. \tag{22}$$

The first inequality and the equality are basic facts about relationships between Sobolev norms and L^p norms. The second inequality is due to the second property from Section 1.3. Now we substitute back into (21) and obtain

$$\||v|^{p-1}Z^{\leq 2}v\|_{L^2_\omega} \lesssim \|Z^{\leq 2}v\|^{p-1}_{L^2_\omega}\|Z^{\leq 2}v\|_{L^2_\omega} = \|Z^{\leq 2}v\|^p_{L^2_\omega} \tag{23}$$

as desired. Then we work with terms of type $\||v|^{p-2}|Z^{\leq 1}v|^2\|$. First we apply Hölder's inequality twice:

$$\||v|^{p-2}|Z^{\leq 1}v|^2\|_{L^2_\omega} \lesssim \|v\|^{p-2}_{L^\infty_\omega}\||Z^{\leq 1}v||Z^{\leq 1}v|\|_{L^2_\omega} \lesssim \|v\|^{p-2}_{L^\infty_\omega}\|Z^{\leq 1}v\|^2_{L^4_\omega}. \tag{24}$$

Now we apply a Sobolev embedding to the term in L^4. Because $\frac{1}{2} - \frac{1}{4} \leq \frac{1}{2}$, we have

$$\|Z^{\leq 1}v\|_{L^4_\omega} \lesssim \|Z^{\leq 1}v\|_{H^1_\omega} \lesssim \|Z^{\leq 2}v\|_{H^0_\omega} = \|Z^{\leq 2}v\|_{L^2} \tag{25}$$

in which the first inequality is the embedding, and the rest is basic manipulation of Sobolev norms. Substituting back into (24), then, gives

$$\||v|^{p-2}|Z^{\leq 1}v|\|^2_{L^2_\omega} \lesssim \|Z^{\leq 2}v\|^{p-2}_{L^2_\omega}\|Z^{\leq 2}v\|^2_{L^2_\omega} = \|Z^{\leq 2}v\|^p_{L^2_\omega}. \tag{26}$$

There are finitely many terms in $Z^{\leq 2}v^p$, all of whose norms are bounded above by $\|Z^{\leq 2}v\|^p_{L^2_\omega}$. Reintroducing the radial and time components of the norm, we obtain the desired inequality:

$$\|r^{-p\alpha_2}Z^{\leq 2}v^p\|_{L^1_tL^1_rL^2_\omega} \lesssim \|r^{-\alpha_2}Z^{\leq 2}v\|^p_{L^p_tL^p_rL^2_\omega}. \tag{27}$$

By the symmetry of $(v, p, -\alpha_2)$ and $(u, q, -\alpha_1)$, we may apply the same argument to obtain

$$\|r^{-q\alpha_1}Z^{\leq 2}u^q\|_{L^1_tL^1_rL^2_\omega} \lesssim \|r^{-\alpha_1}Z^{\leq 2}u\|^q_{L^q_tL^q_rL^2_\omega}. \tag{28}$$

2.3 Flat Problem - Iteration

Now we have the tools we need to set up a function iteration. We will follow the sketch outlined in Section 1.4, but in more detail. Consider the sequence of functions given by

$$\begin{cases} u_{-1} \equiv v_{-1} \equiv 0, \\ \Box u_j = v^p_{j-1}, \quad \Box v_j = u^q_{j-1}, \\ u_j(0,\cdot) = f_u, \quad \partial_t u_j(0,\cdot) = g_u, \\ v_j(0,\cdot) = f_v, \quad \partial_t v_j(0,\cdot) = g_v. \end{cases} \tag{29}$$

Here j ranges over the nonnegative integers. Applying (14) and the commutativity of $\Box$ and Z to $Z^{\leq 2}\Box u = Z^{\leq 2}v^p$ and $Z^{\leq 2}\Box v = Z^{\leq 2}u^q$, we have

$$\|r^{-\alpha_1}Z^{\leq 2}u_j\|_{L_t^qL_r^qL_\omega^2} + \|r^{-\alpha_2}Z^{\leq 2}v_j\|_{L_t^pL_r^pL_\omega^2} \lesssim E_2 + \|r^{-p\alpha_2}Z^{\leq 2}v_{j-1}^p\|_{L_t^1L_r^1L_\omega^2} + \|r^{-q\alpha_1}Z^{\leq 2}u_{j-1}^q\|_{L_t^1L_r^1L_\omega^2}. \quad (30)$$

Now that we have applied the vector fields, we have shown that we can pull exponents out of the right side to obtain

$$\|r^{-\alpha_1}Z^{\leq 2}u_j\|_{L_t^qL_r^qL_\omega^2} + \|r^{-\alpha_2}Z^{\leq 2}v_j\|_{L_t^pL_r^pL_\omega^2} \leq C\left(E_2 + \|r^{-\alpha_2}Z^{\leq 2}v_{j-1}\|^p_{L_t^pL_r^pL_\omega^2} + \|r^{-\alpha_1}Z^{\leq 2}u_{j-1}\|^q_{L_t^qL_r^qL_\omega^2}\right). \quad (31)$$

Note the similarity between the terms on the left and the terms on the right: in fact, the second term on the right is the pth power of the second on the left, and the third term on the right is the qth power of the first on the left. Using this, we will show by induction that there exists a sufficiently small ϵ such that $E_2 < \epsilon$ implies $\|r^{-\alpha_1}Z^{\leq 2}u_j\|_{L_t^qL_r^qL_\omega^2} \leq 3C\epsilon$ and $\|r^{-\alpha_2}Z^{\leq 2}v_j\|_{L_t^pL_r^pL_\omega^2} \leq 3C\epsilon$ for all $j \geq -1$. In particular, we will choose $\epsilon = \min(\frac{1}{(3C)^{(q/(q-1))}}, \frac{1}{(3C)^{(p/(p-1))}})$

The base case is trivial, since $u_{-1} \equiv 0$ and $v_{-1} \equiv 0$ imply that $\|r^{-\alpha_1}Z^{\leq 2}u_{-1}\|_{L_t^qL_r^qL_\omega^2} = \|r^{-\alpha_2}Z^{\leq 2}v_{-1}\|_{L_t^pL_r^pL_\omega^2} = 0 \leq 3C\epsilon$. For the induction step, assume $\|r^{-\alpha_1}Z^{\leq 2}u_{j-1}\|_{L_t^qL_r^qL_\omega^2} \leq 3C\epsilon$ and $\|r^{-\alpha_2}Z^{\leq 2}v_{j-1}\|_{L_t^pL_r^pL_\omega^2} \leq 3C\epsilon$. Then, applying (31) and the inductive hypothesis, we have

$$\|r^{-\alpha_1}Z^{\leq 2}u_j\|_{L_t^qL_r^qL_\omega^2} + \|r^{-\alpha_2}Z^{\leq 2}v_j\|_{L_t^pL_r^pL_\omega^2} \leq C(E_2 + (3C\epsilon)^p + (3C\epsilon)^q) \quad (32)$$

for all $j \geq 0$. As $\|r^{-\alpha_1}Z^{\leq 2}u_j\|_{L_t^qL_r^qL_\omega^2}$ and $\|r^{-\alpha_2}Z^{\leq 2}v_j\|_{L_t^pL_r^pL_\omega^2}$ are both nonnegative, bounding the right side above by $3C\epsilon$ suffices, which occurs if we can bound each of $(3C\epsilon)^p$ and $(3C\epsilon)^q$ above by ϵ. And rearrangement of the definition of ϵ gives that $(3C\epsilon)^q < \epsilon$ and $(3C\epsilon)^p < \epsilon$. $E_2 < \epsilon$ then gives that the right side is in fact less than $C(\epsilon + \epsilon + \epsilon) = 3C\epsilon$, completing the induction. Therefore, we have shown a constant upper bound on the norms. Our final goal will be to show that the sequences are Cauchy, implying convergence.

2.4 Flat Convergence

We have shown that the terms in the sequence are bounded. Now we want to show that the sequence is Cauchy. Our eventual goal is to show that the differences between consecutive iterates decrease geometrically. First, we apply (14) to $\Box(u_{j+1} - u_j) = v_j^p - v_{j-1}^p$. Note that we have no E_2 term because successive iterates have the same initial conditions, and so the E_2 terms will cancel.

We claim:

$$\|r^{-\alpha_1}(|u_{j+1}-u_j|)\|_{L_t^qL_r^qL_\omega^2} \lesssim \|r^{-p\alpha_2}(|v_j^p - v_{j-1}^p|)\|_{L_t^1L_r^1L_\omega^2}$$
$$\lesssim \|r^{-(p-1)\alpha_2}(|v_j|^{p-1}+|v_{j-1}|^{p-1})r^{-\alpha_2}(|v_j-v_{j-1}|)\|_{L_t^1L_r^1L_\omega^2}. \quad (33)$$

To show this, consider $v_j^p - v_{j-1}^p = \int_0^1 \partial_s((sv_j-(1-s)v_{j-1})^p)ds = p\int_0^1(sv_j+(1-s)v_{j-1})^{p-1}(v_j - v_{j-1})ds$. The factor $(sv_j+(1-s)v_{j-1})^{p-1}$ is of order $|v_j|^{p-1}+|v_{j-1}|^{p-1}$, because each cross term that looks like $v_j^{p-1-q}v_{j-1}^q$ is dominated by the larger of $|v_j|^{p-1}$ and $|v_{j-1}|^{p-1}$. Now we use Hölder's inequality again. For the L^1 norms, we use $\frac{p-1}{p}+\frac{1}{p}=1$. We break the L^2 angular component into an L^2 and an L^∞.

$$\lesssim \left(\|(r^{-\alpha_2}|v_j|)^{p-1}\|_{L_t^{\frac{p}{p-1}}L_r^{\frac{p}{p-1}}L_\omega^\infty} + \|(r^{-\alpha_2}|v_{j-1}|)^{p-1}\|_{L_t^{\frac{p}{p-1}}L_r^{\frac{p}{p-1}}L_\omega^\infty}\right)\|r^{-\alpha_2}(|v_j-v_{j-1}|)\|_{L_t^pL_r^pL_\omega^2} \quad (34)$$

Note that the L^2 part of the product looks like the left hand side of (33), but applied to an earlier iterate (and swapping v for u). Our goal is to control the L^∞ part of the product. Now, we can apply two Z operators as before, and pull the $p-1$ exponents outside of the norms (which changes $L^{\frac{p}{p-1}}$ to L^p). This gives

$$\lesssim \left(\|r^{-\alpha_2}Z^{\le 2}v_j\|^{p-1}_{L_t^pL_r^pL_\omega^2} + \|r^{-\alpha_2}Z^{\le 2}v_{j-1}\|^{p-1}_{L_t^pL_r^pL_\omega^2})\right)\|r^{-\alpha_2}|v_j-v_{j-1}|\|_{L_t^pL_r^pL_\omega^2}, \quad (35)$$

which, as we have shown the boundedness of the functions in the iteration, gives

$$\lesssim \left((\|r^{-\alpha_2}Z^{\le 2}v_j\|_{L_t^pL_r^pL_\omega^2})^{p-1} + (\|r^{-\alpha_2}Z^{\le 2}v_{j-1}\|_{L_t^pL_r^pL_\omega^2})^{p-1}\right)\|r^{-\alpha_2}(|v_j-v_{j-1}|)\|_{L_t^pL_r^pL_\omega^2}$$
$$\le C'2(3C\epsilon)^{p-1}\|r^{-\alpha_2}(|v_j-v_{j-1}|)\|_{L_t^pL_r^pL_\omega^2}. \quad (36)$$

So from (33) to (36), we have shown

$$\|r^{-\alpha_1}|u_{j+1}-u_j|\|_{L_t^qL_r^qL_\omega^2} \le C'2(3C\epsilon)^{p-1}\|r^{-\alpha_2}(|v_j-v_{j-1}|)\|_{L_t^pL_r^pL_\omega^2} \quad (37)$$

for some fixed $C'>0$. So an appropriately small choice of ϵ will give

$$\|r^{-\alpha_1}|u_{j+1}-u_j|\|_{L_t^qL_r^qL_\omega^2} \le \frac{1}{2}\|r^{-\alpha_2}|v_j-v_{j-1}|\|_{L_t^pL_r^pL_\omega^2} \quad (38)$$

after which we note that reindexing and swapping the roles of u and v, we can also get

$$\|r^{-\alpha_2}(|v_j-v_{j-1}|)\|_{L_t^pL_r^pL_\omega^2} \le \frac{1}{2}\|r^{-\alpha_1}(|u_{j-1}-u_{j-2}|)\|_{L_t^qL_r^qL_\omega^2}, \quad (39)$$

which at last gives

$$\|r^{-\alpha_1}(|u_{j+1}-u_j|)\|_{L_t^qL_r^qL_\omega^2} \le \frac{1}{4}\|r^{-\alpha_1}(|u_{j-1}-u_{j-2}|)\|_{L_t^qL_r^qL_\omega^2}. \tag{40}$$

The same argument can be made to show that the difference between consecutive v_j decreases geometrically. The fact that the indices decrease by 2 rather than 1 is not an issue; we can bound separately the norms of $u_j - u_{j-1}$ for even and odd j. So this gives that the sequence is Cauchy, and thus converges. Defining $u^* = \lim_{j\to\infty} u_j$ and $v^* = \lim_{j\to\infty} v_j$ and taking the limit on the left and right sides of $\Box u_j = v_{j-1}^p$ and $\Box v_j = u_{j-1}^q$ gives $\Box u^* = (v^*)^p$ and $\Box v^* = (u^*)^q$. Note that we are interchanging a limit of a sequence of functions with the box operator; this is in fact justified (see chapters 3 and 5 of [8]). So $u = u^*, v = v^*$ gives a global solution to the wave system as desired.

We still need to resolve the case where p or q is too large. As we mentioned before, it is possible to have them sufficiently large that the conditions (17) fail, but (19) still holds. In this case, we can always find $\tilde{p}$ and $\tilde{q}$ such that $(\tilde{p},\tilde{q})$ satisfies (19) and $p \ge \tilde{p},\ q \ge \tilde{q}$. Then we have $\Box u_j = v_{j-1}^{p-\tilde{p}} v_{j-1}^{\tilde{p}}$ (and analogously for $\Box v$). Once we reach equations (23) and (26), we use Hölder's inequality to pull out excess copies of v on the left sides, putting them in L^∞. Then we use a Sobolev embedding, applying $Z^{\le 2}$, which moves the copies from L^∞ to L^2. This is exactly what is needed on the right sides of (23) and (26).

3 Geometry Problem

3.1 Introduction

Now that we have given an example of the proof in a relatively specific case, we will examine a more general case. It turns out that the approach is robust in small geometric perturbations on the $\Box$ operator (and in fact allows for different perturbations in the two equations in the system). Our conditions on the geometry (which we will discuss formally later) are that it supports a local energy estimate and that it "looks like" flat space sufficiently far away from the origin, i.e. that the perturbations decay sufficiently quickly.

Because we are using different hypotheses, we will need to use a different weighted Strichartz estimate, as Theorem 2 no longer holds. We will use an estimate from [7]. The proof is beyond the scope of this report, so we will cite it without justification. However, to state it, we do need to define the following norm:

$$\|u\|_{\ell_q^s A} = \|\phi_j(x)u(t,x)\|_{\ell_q^s A} = \left\| \left(2^{js}\|\phi_j(x)u(t,x)\|_A\right) \right\|_{l^q_{j\ge 0}},$$

for any norm A and for a ϕ such that

$$\sum_{j\geq 0} \phi_j^2(x) = 1, \quad \text{supp } \phi_j \subset \{\langle x\rangle \approx 2^j\}.$$

where $\langle x\rangle = \sqrt{1+x^2}$. Intuitively, the norm is a summation over annuli supported away from the origin.

3.2 Geometry Problem Statement

Consider the coupled system of equations

$$\begin{cases} P_1 u = F_p(v), \quad P_2 v = F_q(u), \\ u(0,\cdot) = f_u, \quad \partial_t u(0,\cdot) = g_u, \\ v(0,\cdot) = f_v, \quad \partial_t v(0,\cdot) = g_v, \end{cases} \tag{41}$$

in some spacetime M. $F_p(v)$ is any function of v such that

$$\sum_{j=0}^{2} |v|^j |\partial_v^j F_p(v)| \lesssim |v|^p \tag{42}$$

and analogously for $F_q(u)$. We assume $p, q \in \mathbb{R}$ and $p, q > 2$. P_1 and P_2 are variations of $\Box = \partial_t^2 - \Delta$ in the spacetime geometry and with some asymptotically decaying perturbation. More precisely, $P_k u = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \partial_{x_\alpha} g_k^{\alpha\beta} \partial_{x_\beta} u + \sum_{\alpha=0}^{3} b_k^\alpha \partial_{x_\alpha} u + c_k u$, with restrictions on g_k, b_k, and c_k that will be discussed later in this section. Note that we are using a convention that $x_0 = t$ to write the summations more conveniently.

Our hypotheses are the hypotheses in [7], with some simplifications due to the fact that we know we are working in $\mathbb{R}^3$. We will need to assume a local energy estimate for use near the origin and some bounds on the radial behavior of the metric's perturbation, which are needed for the weighted Strichartz estimate and for commuting with Ω. As we will just assume the weighted Strichartz estimate here, we will state the hypotheses on the metric and on P without discussion. Also note that instead of v^p we are using $F_p(v)$ and similarly for u and q. The function v^p is one example of a function satisfying the condition for $F_p(v)$, so this is in fact a generalization.

The spacetime M is either $M = \mathbb{R}^+ \times \mathbb{R}^3$ or $M = \mathbb{R}^+ \times (\mathbb{R}^3 \backslash K)$ where $K \subset \{x : |x| < R_0\}$ and has a smooth boundary. We will not need to refer to K again, because the weighted Strichartz estimate already takes it into account.

We assume that g_k (for $k = 1, 2$) is a Lorentzian metric inducing the operator P_k and that g_k can be decomposed as

$$g_k = m + g_{k,0}(t, r) + g_{k,1}(t, x), \quad k = 1, 2, \tag{43}$$

where $m = diag(-1,1,1,1)$ denotes the Minkowski metric. Note that g_1 and g_2 need not be the same. The perturbations decay as follows:

$$\|\partial^\mu_{t,x} g_{k,i,\alpha\beta}\|_{\ell_1^{i+|\mu|} L^\infty_{t,x}} = \mathcal{O}(1), \quad i = 0,1, \quad k = 1,2 \quad |\mu| \leq 3. \tag{44}$$

The lower-order perturbations b_k and c_k in P_k also decay:

$$\|\partial^\mu_{t,x} b_k^\alpha\|_{\ell_1^{1+|\mu|} L^\infty_{t,x}} + \|\partial^\mu_{t,x} c_k\|_{\ell_1^{2+|\mu|} L^\infty_{t,x}} = \mathcal{O}(1), \quad k = 1,2, \quad |\mu| \leq 2. \tag{45}$$

And to get commutativity with Ω, we assume

$$m + g_{k,0} = (-1+\tilde{g}_{k,00}(t,r))dt^2 + 2\tilde{g}_{k,01}(t,r)dtdr + (1+\tilde{g}_{k,11}(t,r))dr^2 + (1+\tilde{g}_{k,22}(t,r))r^2 d\omega^2_{\mathbb{S}^2}. \tag{46}$$

and $\|\partial^\mu_{t,x}\tilde{g}_{k,\alpha\beta}\|_{l_1^\mu L^\infty_{t,x}} = \mathcal{O}(1)$ for $|\mu| \leq 3$ Finally, we have the local energy estimate. Recall that R_0 is the radius outside of which all points of $\mathbb{R} \times \mathbb{R}^n$ are guaranteed to be in M. The hypothesis is that there exists $R_1 > R_0$ so that if u is a solution to the linear wave equation $Pu = F$, then

$$\begin{aligned} &\|\partial\partial^\mu u\|_{L^\infty_t L^2_x} + \|(1-\chi)\partial\partial^\mu u\|_{\ell_\infty^{-\frac{1}{2}} L^2_{t,x}} + \|\partial^\mu u\|_{\ell_\infty^{-\frac{3}{2}} L^2_{t,x}} \\ &\qquad \lesssim \|u(0,\cdot)\|_{H^{|\mu|+1}} + \|\partial_t u(0,\cdot)\|_{H^{|\mu|}} + \sum_{|\nu|\leq|\mu|} \|\partial^\nu F\|_{L^1_t L^2_x} \end{aligned} \tag{47}$$

for all $|\mu| \leq 2$. χ is any smooth function that is 1 on $B_{R_1/2} := \{|x| \leq \frac{R_1}{2}\}$ and supported in B_{R_1}.

This hypothesis controls three terms on the left side. The first term is the maximum energy that exists at any time. The second term is a localized energy term, which needs the $1-\chi$ factor to avoid any rays that are trapped near the origin due to the geometry. The third term is a lower-order term. As we are citing the weighted Strichartz estimate without proof, we will use this hypothesis only sparingly - we will need to use the control on the third term later, but we will not need to deal with the first two terms.

Now we can state the main theorem.

Theorem 3 *Suppose* $p,q \in \mathbb{R}^+$ *with* $p,q > 2$, $1 > \frac{2+p+1/q}{pq-1}$, $1 > \frac{2+q+1/p}{pq-1}$ *and* $f_u, g_u, f_v, g_v \in C_c^\infty$. *Consider the system* (41) *with the associated spacetime* M, *operators* P_1 *and* P_2, *and functions* F_p *and* F_q. *Suppose* (43), (44), (45), (46), *and* (47) *hold. Then there exists some* $\epsilon > 0$ *such that* $\|f_u\|_{H^3} + \|f_v\|_{H^3} + \|g_u\|_{H^2} + \|g_v\|_{H^2} < \epsilon$ *implies the existence of a global solution to* (41).

In particular, flat space satisfies the hypotheses. Therefore, Theorem 1 is a special case of Theorem 3. Additionally, $1 > \frac{2+p+1/q}{pq-1}$ and $1 > \frac{2+q+1/p}{pq-1}$ are sharp, because [1] showed that they are sharp in the flat case.

3.3 Estimates

As in the flat case, a weighted Strichartz estimate is the key estimate. We only require a specific case of the estimate, in which $n = 3$ and some parameters are already set, so we will quote this version below. We will not prove this or either of the subsequent estimates. Proofs can be found in [7].

Theorem 4 *Consider the wave equation $Pu = F$, and suppose* (43), (44), (45), *and* (47) *hold. Define $E(u, \gamma, P) = \|u(0,\cdot)\|_{H^3} + \|\partial_t u(0,\cdot)\|_{H^2} + \|\psi_R Z^{\le 2} u(0,\cdot)\|_{\dot{H}^s} + \|\psi_R Z^{\le 2}\partial_t u(0,\cdot)\|_{\dot{H}^{s-1}} + \|\psi_R Z^{\le 1} Pu(0,\cdot)\|_{\dot{H}^{s-1}}$. Then there exists $R > R_1$ so that for any smooth ψ_R that is identically 1 on B_{2R}^c and vanishes on B_R, we have*

$$\|\psi_R Z^{\le k} u\|_{L^p_{t,r}L^2_\omega} \lesssim E(u, \gamma, P) + \|\psi_R^p Z^{\le k} Pu\|_{L^1_{t,r}L^2_\omega} + \|\partial^{\le k} Pu\|_{L^1_t L^2_x} \tag{48}$$

for any $p \in (2, \infty)$, $s \in (\frac{1}{2} - \frac{1}{p}, \frac{1}{2})$, and $k \in \{0, 2\}$.

The case where $k = 2$ is proven in [7], and the case where $k = 0$ requires only slight modifications in that proof. Note that we only bound the first two terms of $E(u, \gamma, P)$ above by ϵ in Theorem 3. This is because the other terms in $E(u, \gamma, P)$ either have three derivatives with a negative Sobolev exponent, or two derivatives with a Sobolev exponent less than 1, and so are lower-order.

We also have some weighted Sobolev estimates, which mainly serve to move from one L^p space to another, at the cost of derivatives:

$$\|r^\beta u\|_{L^q_r L^\infty_\omega(r\ge R+1)} \lesssim \sum_{|\mu|\le 2} \|r^{\beta - 2/p + 2/q} Z^\mu u\|_{L^p_r L^2_\omega(r \ge R)}, \tag{49}$$

$$\|r^\beta u\|_{L^q_r L^4_\omega(r\ge R+1)} \lesssim \sum_{|\mu|\le 1} \|r^{\beta - 2/p + 2/q} Z^\mu u\|_{L^p_r L^2_\omega(r \ge R)}. \tag{50}$$

3.4 Global Existence

Now we demonstrate global existence given sufficiently small initial data. We follow the same general structure of showing that the iterates are bounded in some norm and then applying the weighted Strichartz estimate to the difference between consecutive iterates to show that the sequence is Cauchy. However, this section is considerably more technical, because we need to account for the geometry near the origin. The general strategy is to split into regions inside and outside of a ball about the origin, and handle them separately, since the local energy estimate suffices inside of the ball and the weighted Strichartz estimate suffices outside of the ball.

This is a modification of the proof that was given in [7]. We will go through the proof in more detail here. We start by defining two norms.

$$\|u\|_{X_{k,\alpha,q}} = \|r^{-\alpha}\psi_R Z^{\leq k}u\|_{L_t^q L_r^q L_\omega^2} + \|\partial^{\leq k}u\|_{\ell_\infty^{-\frac{3}{2}} L_t^2 L_r^2 L_\omega^2} + \|\partial^{\leq k}\partial u\|_{L_t^\infty L_r^2 L_\omega^2},$$

$$\|g\|_{N_{k,\alpha,q}} = \|r^{-q\alpha}\psi_R^q Z^{\leq k}g\|_{L_t^1 L_r^1 L_\omega^2} + \|Z^{\leq k}g\|_{L_t^1 L_r^2 L_\omega^2}.$$

Note that, as each term has either $Z^{\leq k}$ or $\partial^{\leq k}$, we have $\|Z^{\leq 1}u\|_{X_{k,\alpha,q}} \lesssim \|u\|_{X_{k+1,\alpha,q}}$ and similarly for N. Summing (47) and (48) for both wave equations gives

$$\|u\|_{X_{k,\alpha_1,q}} + \|v\|_{X_{k,\alpha_2,p}} \lesssim E(u,\gamma_1,P_1) + E(v,\gamma_2,P_2) + \|P_1 u\|_{N_{k,\alpha_2,p}} + \|P_2 v\|_{N_{k,\alpha_1,q}}. \tag{51}$$

Now, we will give two lemmas which are sufficient to prove the result. These lemmas are also from [7]. As the proofs do not require extra background material and demonstrate concretely the important theme of treating the ball near the origin separately, we will give detailed proofs here. We will prove the result assuming the lemmas and then return to them. The differences between this segment and the analogous segment in the flat case are minor; we are demonstrating boundedness and convergence in different norms, but the structure of the proof is identical.

Lemma 1 *Suppose $u \in C^2$ and $p > 2$. Then $\|u^p\|_{N_{2,\alpha,p}} \lesssim \|u\|^p_{X_{2,\alpha,p}}$.*

Lemma 2 *Suppose $u_j, u_{j-1} \in C^2$ and $p > 2$. Then*
$\|u_j^p - u_{j-1}^p\|_{N_{0,\alpha,p}} \lesssim \left(\|u_j\|^{p-1}_{X_{2,\alpha,p}} + \|u_{j-1}\|^{p-1}_{X_{2,\alpha,p}}\right)\|u_j - u_{j-1}\|_{X_{0,\alpha,p}}$.

Comparing to the flat case, Lemma 1 and Lemma 2 are roughly analogous to (27) and (35) respectively.

To prove the result, recall the iteration used. Define $\alpha_1 = -(\frac{3}{2} - \frac{4}{q} - \gamma_1)$ and $\alpha_2 = -(\frac{3}{2} - \frac{4}{p} - \gamma_2)$, just as in the flat case. Note that this numerology is why we get the same restrictions on p and q as in the flat case. From the iteration, (51), and Lemma 1, we get

$$\begin{aligned}\|u_j\|_{X_{2,\alpha_1,q}} + \|v_j\|_{X_{2,\alpha_2,p}} &\lesssim E + \|v_{j-1}^p\|_{N_{2,\alpha_2,p}} + \|u_{j-1}^q\|_{N_{2,\alpha_1,q}} \\ &\lesssim E + \|v_{j-1}\|^p_{X_{2,\alpha_2,p}} + \|u_{j-1}\|^q_{X_{2,\alpha_1,q}},\end{aligned} \tag{52}$$

where $E = E(u,\gamma_1,P_1) + E(v,\gamma_2,P_2)$.

We can use this to show that the iterates are bounded in X_2 for sufficiently small E. Suppose the multiplicative constant in (48) is C and the one in (52) is C'. Now, if $\|u_{j-1}\|_{X_{2,\alpha_1,q}} + \|v_{j-1}\|_{X_{2,\alpha_2,p}} \leq 3CE$, then so is $\|u_j\|_{X_{2,\alpha_1,q}} + \|v_j\|_{X_{2,\alpha_2,p}}$, provided $C'(3CE)^{p-1} \leq \frac{1}{3}$ and $C'(3CE)^{q-1} \leq \frac{1}{3}$. Thus, by induction, for sufficiently small E the sequences $\|u_j\|_{X_{2,\alpha_1,q}}$ and $\|v_j\|_{X_{2,\alpha_2,p}}$ are both bounded.

We will use this boundedness to show that the iterates are Cauchy (in X_0), and thus converge to a solution. In the below, we consider the difference between successive iterates in X_0. Apply (51) and then Lemma 2:

$$\|u_j - u_{j-1}\|_{X_{0,\alpha_1,q}} \lesssim \|v_{j-1}^p - v_{j-2}^p\|_{N_{0,\alpha_2,p}} \lesssim \left(\|v_{j-1}\|_{X_{2,\alpha_2,p}}^{p-1} + \|v_{j-2}\|_{X_{2,\alpha_2,p}}^{p-1}\right) \|v_{j-1} - v_{j-2}\|_{X_{0,\alpha_2,p}}.$$

As in the flat case, there is no E term because the initial data for successive iterates cancel. The factor in X_2 is bounded above by a constant, as shown earlier. By modifying the choice of E in the boundedness argument, we can make that constant arbitrarily small. In particular, we can guarantee $\|u_j - u_{j-1}\|_{X_{0,\alpha_1,q}} \leq \frac{1}{2}\|v_{j-1} - v_{j-2}\|_{X_{0,\alpha_2,p}}$. And applying the same argument to the right side, we can obtain $\|u_j - u_{j-1}\|_{X_{0,\alpha_1,q}} \leq \frac{1}{4}\|u_{j-2} - u_{j-3}\|_{X_{0,\alpha_1,q}}$.

Therefore, the difference of successive iterates in $\{u_j\}$ decreases geometrically in the even terms and in the odd terms (in X_0). This is sufficient to guarantee that it is Cauchy. The proof for $\{v_j\}$ is identical up to a change of variable names. Therefore, we have global existence for sufficiently small data.

3.5 Proof of Lemmas

Now we return to proving Lemma 1 and Lemma 2. This will suffice to prove the theorem. To do so, we will apply another pair of lemmas:

Lemma 3 *Suppose $g \in C^2$, $f \in C^1$, and $p > 2$. Then $\|g^{p-2}f^2\|_{N_{0,\alpha,p}} \lesssim \|g\|_{X_{2,\alpha,p}}^{p-2}\|f\|_{X_{1,\alpha,p}}^2$.*

Lemma 4 *Suppose $g \in C^2$, $f \in C^0$, and $p > 2$. Then $\|g^{p-1}f\|_{N_{0,\alpha,p}} \lesssim \|g\|_{X_{2,\alpha,p}}^{p-1}\|f\|_{X_{0,\alpha,p}}$.*

These are analogous to (26) and (23). From these lemmas, we can show Lemma 1 and Lemma 2 as follows:

$$\|u^p\|_{N_{2,\alpha,p}} = \|Z^{\leq 2}u^p\|_{N_{0,\alpha,p}} \lesssim \||u|^{p-1}|Z^{\leq 2}u|\|_{N_{0,\alpha,p}} + \||u|^{p-2}|Z^{\leq 1}u|^2\|_{N_{0,\alpha,p}} \lesssim \|u\|_{X_{2,\alpha,p}}^{p-1}\|Z^{\leq 2}u\|_{X_{0,\alpha,p}} + \|u\|_{X_{2,\alpha,p}}^{p-2}\|Z^{\leq 1}u\|_{X_{1,\alpha,p}}^2 \lesssim \|u\|_{X_{2,\alpha,p}}^p, \quad (53)$$

$$\|u_j^p - u_{j-1}^p\|_{N_{0,\alpha,p}} \lesssim \|(u_j + u_{j-1})^{p-1}(u_j - u_{j-1})\|_{N_{0,\alpha,p}} \quad (54)$$

$$\lesssim \left(\|u_j\|_{X_{2,\alpha,p}}^{p-1} + \|u_{j-1}\|_{X_{2,\alpha,p}}^{p-1}\right)\|u_j - u_{j-1}\|_{X_{0,\alpha,p}}. \quad (55)$$

So now it remains to show Lemma 3 and Lemma 4. As the N norm has two terms, we will show that each of the terms separately is bounded above in the appropriate product of X norms.

First we handle the portions in $L_t^1 L_r^1 L_\omega^2$, applying Hölder's inequality in each case and then moving into the X norm via Sobolev embeddings. This is a rough analogue of (34).

$$\|r^{-\alpha p}\psi_R^p g^{p-1} f\|_{L^1_t L^1_r L^2_\omega} \lesssim \|r^{-\alpha}\psi_R g\|^{p-1}_{L^p_t L^p_r L^\infty_\omega} \|r^{-\alpha}\psi_R f\|_{L^p_t L^p_r L^2_\omega} \lesssim \|g\|^{p-1}_{X_{2,\alpha,p}} \|f\|_{X_{0,\alpha,p}},$$

$$\|r^{-\alpha p}\psi_R^p g^{p-2} f\|_{L^1_t L^1_r L^2_\omega} \lesssim \|r^{-\alpha}\psi_R g\|^{p-2}_{L^p_t L^p_r L^\infty_\omega} \|r^{-\alpha}\psi_R f\|^2_{L^p_t L^p_r L^4_\omega} \lesssim \|g\|^{p-2}_{X_{2,\alpha,p}} \|f\|^2_{X_{1,\alpha,p}}.$$

Finally, we work on the $L^1_t L^2_r L^2_\omega$ term of the N norm. We use the weighted Strichartz estimate to control the term outside of a ball about the origin, and we use the local energy estimate to work with the inside.

Let us assume $4 + \frac{6}{p} - 2q \geq 0$ and $4 + \frac{6}{q} - 2p \geq 0$. If not, we use Hölder's inequality to get a term of the form $\|u\|^j_{L^\infty_{t,x}}$ for some real j, which is bounded above by the second term of X_2 using Sobolev embeddings. Then we can obtain copies of u in X_2 as needed, and similarly for v. This allows us to recover equations Lemma 3 and Lemma 4. Now that we can make that assumption, using Hölder's inequality and (49) and working over radii at least $2R+1$ gives

$$\begin{aligned}
\|g^{p-1} f\|_{L^1_t L^2_{r\geq 2R+1} L^2_\omega} &\lesssim \|r^{\frac{\alpha}{p-1}} g\|^{p-1}_{L^p_t L^{\frac{2p(p-1)}{(p-2)}}_{r\geq 2R+1} L^\infty_\omega} \|r^{-\alpha} f\|_{L^p_t L^p_{r\geq 2R+1} L^2_\omega} && (56)\\
&\lesssim \|r^{\frac{\alpha}{p-1} - \frac{2}{p} + \frac{(p-2)}{p(p-1)}} Z^{\leq 2} g\|^{p-1}_{L^p_t L^p_{r\geq 2R} L^\infty_\omega} \|f\|_{X_{0,\alpha,p}} && (57)\\
&\lesssim \|r^{-\alpha} Z^{\leq 2} g\|^{p-1}_{L^p_t L^p_{r\geq 2R} L^\infty_\omega} \|f\|_{X_{0,\alpha,p}} \lesssim \|g\|^{p-1}_{X_{2,\alpha,p}} \|f\|_{X_{0,\alpha,p}}. && (58)
\end{aligned}$$

The application of (49) gets us from the first line to the second. We modify the power of r to get from the second line to the third. We can do this as long as the power does not decrease, because we are working outside of a ball about the origin. And in fact an elementary calculation verifies that $-\alpha$ is at least as large as $\frac{\alpha}{p-1} - \frac{2}{p} + \frac{p-2}{p(p-1)}$ given our assumption on p and q.

Similarly, using Hölder's inequality, (49), and (50), we have

$$\begin{aligned}
\|g^{p-2} f^2\|_{L^1_t L^2_{r\geq 2R+1} L^2_\omega} &\lesssim \|r^{\frac{2}{p-2}(\alpha - \frac{2}{p} + \frac{1}{2})} g\|^{p-2}_{L^p_t L^\infty_{r\geq 2R+1} L^\infty_\omega} \|r^{-\alpha + \frac{2}{p} - \frac{1}{2}} f\|^2_{L^p_t L^4_{r\geq 2R+1} L^4_\omega} && (59)\\
&\lesssim \|r^{-\alpha + \frac{2}{p}} g\|^{p-2}_{L^p_t L^\infty_{r\geq 2R+1} L^\infty_\omega} \|r^{-\alpha + \frac{2}{p} - \frac{1}{2}} f\|^2_{L^p_t L^4_{r\geq 2R+1} L^4_\omega} && (60)\\
&\lesssim \|r^{-\alpha} Z^{\leq 2} g\|^{p-2}_{L^p_t L^p_{r\geq 2R} L^2_\omega} \|r^{-\alpha} Z^{\leq 1} f\|^2_{L^p_t L^p_{r\geq 2R} L^2_\omega} && (61)\\
&\lesssim \|g\|^{p-2}_{X_{2,\alpha,p}} \|f\|^2_{X_{1,\alpha,p}}. && (62)
\end{aligned}$$

From the first line to the second line, we have increased the powers on r so that we can use the weighted Sobolev embeddings to move into suitable norms. As above, we are in fact increasing the exponent given our assumption on p and q. Thus, we do in fact have Lemma 3 and Lemma 4 outside of B_{2R+1}.

Now we work inside of B_{2R+1}. We begin with an L^p Sobolev inequality:

$$\|g\|_{L^\infty_x} \leq \|\partial^{\leq 1} g\|_{L^6}. \tag{63}$$

This holds due to the following embedding and the fact that $6 > 3$:

$$\|g\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\partial^{\leq m} g\|_{L^a(\mathbb{R}^n)}, \quad a > \frac{n}{m}, \tag{64}$$

where we use $n = 3$ and $m = 1$. Now, picking $\tilde{q}$ so that $\frac{1}{4} = \frac{\frac{1}{p-1}}{\tilde{q}} + \frac{1-\frac{1}{p-1}}{6}$, we can apply Hölder's inequality to the right side to get

$$\|g\|_{L^\infty_x} \lesssim \|\partial^{\leq 1} g\|_{L^{\tilde{q}}}^{\frac{1}{p-1}} \|\partial^{\leq 1} g\|_{L^6}^{1-\frac{1}{p-1}}. \tag{65}$$

Then Sobolev embeddings give

$$\|g\|_{L^\infty_x} \lesssim \|\partial^{\leq 1} g\|_{H^1}^{\frac{1}{p-1}} \|\partial^{\leq 1} \partial g\|_{L^2}^{1-\frac{1}{p-1}}. \tag{66}$$

provided $0 < \tilde{q} < 6$ (to give the Sobolev embedding for the $L^{\tilde{q}}$ piece). This condition in $\tilde{q}$ is always satisfied when $2 < p$.

From (66) and Hölder's inequality, we get

$$\|g\|_{L_t^{2(p-1)} L^\infty_{r\leq 2R+1} L^\infty_\omega} \lesssim \|\partial^{\leq 1} g\|_{L_t^2 H^1_{r\leq 2R+2}}^{\frac{1}{p-1}} \|\partial^{\leq 1} \partial g\|_{L_t^\infty L^2_{r\leq 2R+2} L^2_\omega}^{1-\frac{1}{p-1}} \lesssim \|g\|_{X_{2,\alpha,p}}. \tag{67}$$

In particular, the bound is by the second term of $X_{2,\alpha,p}$. We can ignore the ℓ summation because we are in a compact set. Now, from (67), we have

$$\|g^{p-1} f\|_{L^1_t L^2_{r\leq 2R+1} L^2_\omega} \lesssim \|g\|^{p-1}_{L_t^{2(p-1)} L^\infty_{r\leq 2R+1} L^\infty_\omega} \|f\|_{L^2_t L^2_{r\leq 2R+1} L^2_\omega} \lesssim \|g\|^{p-1}_{X_{2,\alpha,p}} \|f\|_{X_{0,\alpha,p}}$$

and

$$\|g^{p-2} f^2\|_{L^1_t L^2_{r\leq 2R+1} L^2_\omega} \lesssim \|g\|^{p-2}_{L_t^\infty L^\infty_{r\leq 2R+1} L^\infty_\omega} \|f\|^2_{L^2_t L^4_{r\leq 2R+1} L^4_\omega} \lesssim \|g\|^{p-2}_{X_{2,\alpha,p}} \|f\|^2_{X_{1,\alpha,p}}$$

as desired, showing Lemma 3 and Lemma 4 inside of B_{2R+1}.

4 Appendix - Discussion of a Weighted Strichartz Estimate

Here we give a brief discussion (though not a proof) of (2), the weighted Strichartz estimate from [3] used for the flat problem. It is an interpolation between two estimates.

The first estimate is due to the trace lemma, which applied on the unit sphere is the following:

$$\sup_{r>0} r^{\frac{3}{2}-s}\left(\int_{\mathbb{S}^2}|v(r\omega)|^2 d\sigma(\omega)\right)^{\frac{1}{2}} \lesssim \|v\|_{\dot{H}^s}, \quad \frac{1}{2}<s<\frac{3}{2}. \tag{68}$$

Now, we will introduce the notation $e^{it|D|}\phi$, which means the function whose Fourier transform is $e^{it|\xi|}\widehat{\phi}$. We can obtain from (68)

$$\||x|^{-\alpha}e^{it|D|}\phi\|_{L^\infty_r L^2_\omega} \lesssim \|\phi\|_{\dot{H}^{\frac{n}{2}+\alpha}(\mathbb{R}^3)}, \quad -1<\alpha<0, \tag{69}$$

by replacing the supremum with an L^∞ and introducing α and $e^{it|D|}$. This is one of the two estimates that gives the interpolation. In the flat case, the second estimate (which follows) comes from some manipulation of (68) for the Fourier transform of v, which we will not detail here. The weighted Strichartz estimate we used for the version with a background geometry is a hypothesis. It holds in flat space and has been proven on several geometries, such as for the Kerr black hole backgrounds. See [7] for a discussion of several backgrounds on which the local energy estimate holds.

$$\||x|^{-s}e^{it|D|}\phi\|_{L^2(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \||D|^{s-\frac{1}{2}}\|_{L^2(\mathbb{R}^3)}, \quad \frac{1}{2}<s<\frac{3}{2}. \tag{70}$$

A method of interpolation between these two yields, for $2\leq q\leq\infty$,

$$\||x|^{\frac{n}{2}-\frac{n+1}{q}-\gamma}e^{it|D|}\phi\|_{L^q_t L^q_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)}, \quad \frac{1}{2}-\frac{1}{q}<\gamma<\frac{n}{2}-\frac{1}{q}. \tag{71}$$

If $q=\infty$, we recover (69), and if $q=2$, we recover (70). So we have the left side of the estimate. The right side comes from some manipulation of (71) and a duality argument which we will not discuss here.

References

[1] Daniele Del Santo, Vladimir Georgiev, and Enzo Mitidieri. Global existence of the solutions and formation of singularities for a class of hyperbolic systems. In *Geometrical Optics and Related Topics*, pages 117–140. Springer, 1997.

[2] Vladimir Georgiev, Hiroyuki Takamura, and Zhou Yi. The lifespan of solutions to nonlinear systems of a high-dimensional wave equation. *Nonlinear Analysis: Theory, Methods & Applications*, 64(10):2215–2250, 2006.

[3] Kunio Hidano, Jason Metcalfe, Hart Smith, Christopher Sogge, and Yi Zhou. On abstract strichartz estimates and the strauss conjecture for nontrapping obstacles. *Transactions of the American Mathematical Society*, 362(5):2789–2809, 2010.

[4] Fritz John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Mathematica*, 28(1-3):235–268, 1979.

[5] Hideo Kubo and Masahito Ohta. Critical blowup for systems of semilinear wave equations in low space dimensions. *Journal of mathematical analysis and applications*, 240(2):340–360, 1999.

[6] Hans Lindblad, Jason Metcalfe, Christopher D Sogge, Mihai Tohaneanu, and Chengbo Wang. The strauss conjecture on kerr black hole backgrounds. *Mathematische Annalen*, 359(3-4):637–661, 2014.

[7] Jason Metcalfe and Chengbo Wang. The Strauss conjecture on asymptotically flat space-times. *arXiv preprint arXiv:1605.02157*, 2016.

[8] Sigmund Selberg. Lecture notes. math 632, pde. *Lecture Notes on Nonlinear Wave Equations. Johns Hopkins University*, 2001.